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KK-Theory for Banach Algebras and Proper Groupoids

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Abstract

In analogy to the definition of the assembly map of Baum-Connes one can construct a homomorphism $\mu_{\mathcal{A}}^{B}$ from $\mathrm{K}^{\mathrm{top}}(G, B)$ to $\mathrm{K}_0(\mathcal{A}(G, B))$, where G is a locally compact group, B is a G-C*-algebra and $\mathcal{A}(G)$ is an unconditional completion of $\mathcal{C}_c(G)$, that is, a completion with respect to a norm $\|\cdot\|_{\mathcal{A}}$ such that $\|f\|_{\mathcal{A}}$ only depends on the function $g \mapsto |f(g)|$. Is $\mu_{\mathcal{A}}^{B}$ an isomorphism? This question was raised by Vincent Lafforgue, who has also given affirmative answers in many important cases. Moreover, he considered the more general situation where the group G is replaced by a locally compact Hausdorff groupoid $\mathcal{G}$.

In the present thesis the setting is generalised further, taking B to be a non-degenerate $\mathcal{G}$-Banach algebra instead of a $\mathcal{G}$-C*-algebra. The main result asserts that the map $\mu_{\mathcal{A}}^{B}$ is split surjective if the $\mathcal{G}$-Banach algebra B is proper (and $\mathcal{A}(\mathcal{G})$ satisfies some mild condition). The proof rests on the following generalised version of the Green-Julg theorem: If $\mathcal{G}$ is proper and B is a $\mathcal{G}$-Banach algebra (and $\mathcal{A}(\mathcal{G})$ satisfies some mild condition), then $\mathrm{KK}_{\mathcal{G}}^{\mathrm{ban}}(\mathcal{C}_0(X),\ B)$ is naturally isomorphic to $\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G}, B))$, where X denotes the unit space of $\mathcal{G}$.

Building on the work of Lafforgue, the necessary tools to show these results are systematically developed, rounding out some parts of Lafforgue's $\mathrm{KK}^{\mathrm{ban}}$-theory. In particular, a Banach algebra version of RKK is introduced and the functoriality of the groupoid version of $\mathrm{KK}^{\mathrm{ban}}$ under generalised morphisms of groupoids is proved.

Zusammenfassung

Analog zur Definition der Assembly-Abbildung von Baum-Connes kann man auch einen Homomorphismus $\mu_{\mathcal{A}}^{B}$ von $\mathrm{K}^{\mathrm{top}}(G, B)$ nach $\mathrm{K}_0(\mathcal{A}(G, B))$ konstruieren, wobei G eine lokalkompakte Gruppe, B eine G-C*-Algebra und $\mathcal{A}(G)$ eine unbedingte Vervollständigung von $\mathcal{C}_c(G)$ ist, wobei letzteres eine Vervollständigung bezüglich einer Norm $\|\cdot\|_{\mathcal{A}}$ mit der Eigenschaft ist, daß $\|f\|_{\mathcal{A}}$ nur von der Betragsfunktion $g \mapsto |f(g)|$ abhängt. Ist $\mu_{\mathcal{A}}^{B}$ ein Isomorphismus? Vincent Lafforgue, der diese Vermutung als erster in dieser Allgemeinheit behandelt hat, konnte sie bereits in vielen wichtigen Fällen bestätigen. Er ging auch die allgemeinere Situation an, in welcher er die Gruppe G durch ein lokalkompaktes Gruppoid $\mathcal{G}$ ersetzt hat.

Die vorliegende Arbeit geht noch einen Schritt weiter, indem statt $\mathcal{G}$-C*-Algebren nicht-entartete $\mathcal{G}$-Banachalgebren betrachtet werden. Als Hauptresultat wird bewiesen, daß der Homomorphismus $\mu_{\mathcal{A}}^{B}$ surjektiv ist und einen natürlichen Schnitt hat, falls die $\mathcal{G}$-Banachalgebra B eigentlich (und die Vervollständigung $\mathcal{A}(\mathcal{G})$ nicht zu exotisch ist). Die wichtigste Zutat zum Beweis dieses Hauptsatzes ist die folgende Verallgemeinerung des Satzes von Green-Julg: Wenn $\mathcal{G}$ eigentlich und B eine $\mathcal{G}$-Banachalgebra ist (und $\mathcal{A}(\mathcal{G})$ wiederum gewissen schwachen Bedingungen genügt), dann gibt es einen natürlichen Isomorphismus zwischen $\mathrm{KK}_{\mathcal{G}}^{\mathrm{ban}}(\mathcal{C}_0(X),\ B)$ und $\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G}, B))$, wobei X den Einheitenraum von $\mathcal{G}$ bezeichne.

Ausgehend von den Arbeiten von Vincent Lafforgue werden die für die Beweise der genannten Sätze notwendigen Hilfsmittel systematisch zusammengetragen, wobei einige gundlegenden Bereiche seiner $\mathrm{KK}^{\mathrm{ban}}$-Theorie ausgebaut werden. So wird etwa eine Variante der RKK-Theorie für Banachalgebren entwickelt und gezeigt, daß die Gruppoid-Version der $\mathrm{KK}^{\mathrm{ban}}$-Theorie unter verallgemeinerten Morphismen von Gruppoiden funktoriell ist.

Contents

Introduction

The Bost conjecture

Let G be a locally compact Hausdorff group and let $\underline{\mathrm{E}}\mathrm{G}$ denote the classifying space for proper actions of G on locally compact Hausdorff spaces. For every G-C*-algebra B one defines $\mathrm{K}_*^{\mathrm{top}}(G,B)$ to be the group $\lim_{\rightarrow}\mathrm{KK}_G(\mathcal{C}_0(Z),\ B)$ where the limit is taken over the G-equivariant and G-compact subsets Z of $\underline{\mathrm{E}}\mathrm{G}$. In [BCH94], Baum, Connes and Higson define a homomorphism

$$\mu_r^B\colon\ \mathrm{K}_*^{\mathrm{top}}(G,B)\to\mathrm{K}_*(B\rtimes_r G)\,,$$

where $B\rtimes_r G$ denotes the reduced crossed product of B by G. We say that G satisfies the *Baum-Connes conjecture with coefficients* if μ_r^B is a bijection for all G-C^*-algebras B. The Baum-Connes conjecture has been proved for a large number of groups; the main method to prove the injectivity of the Baum-Connes map, the "Dirac-dual-Dirac" method of Kasparov, makes use of Kasparov's equivariant KK-theory for C*-algebras (see [Kas95]).

Formidable progress was achieved by Vincent Lafforgue by the introduction of his bivariant K-theory $\mathrm{KK}^{\mathrm{ban}}$ for general Banach algebras in [Laf02]. In that article he puts forward the following variant of the Baum-Connes conjecture: Let the Banach algebra $\mathcal{A}(G)$ be an unconditional completion of the convolution algebra $\mathcal{C}_c(G)$, i.e., a completion for a norm on $\mathcal{C}_c(G)$ such that $\|f\|$ only depends on $g\mapsto|f(g)|$; the most prominent example of such a completion is $\mathrm{L}^1(G)$. If B is a G-Banach algebra, i.e., a Banach algebra on which G acts continuously by isometries, then Lafforgue defines the Banach algebra $\mathcal{A}(G,B)$, in complete analogy with $\mathrm{L}^1(G,B)$, as a completion of $\mathcal{C}_c(G,B)$. For G-C*-algebras B he then constructs a homomorphism

$$\mu_{\mathcal{A}}^B\colon\ \mathrm{K}_*^{\mathrm{top}}(G,B)\to\mathrm{K}_*(\mathcal{A}(G,B))\,.$$

One can now ask whether $\mu_{\mathcal{A}}^B$ is an isomorphism (this generalises a conjecture of Jean-Benôit Bost[1] which is the special case $B=\mathbb{C}$ and $\mathcal{A}(G)=\mathrm{L}^1(G)$). Using his bivariant K-theory $\mathrm{KK}^{\mathrm{ban}}$, Lafforgue was able to show that for G in a large class of groups $\mu_{\mathcal{A}}^B$ is an isomorphism for all G-C*-algebras B and all unconditional completions $\mathcal{A}(G)$. By comparing the K-theories of $\mathcal{A}(G)$ and $\mathrm{C}_r^*(G)$ he could thus prove the Baum-Connes conjecture for many groups G.

There is an obvious version of the Bost conjecture for general Banach algebras: Let B be a G-Banach algebra and $\mathcal{A}(G)$ be an unconditional completion of $\mathcal{C}_c(G)$. Define $\mathrm{K}_*^{\mathrm{top,ban}}(G,B):=\lim_{\rightarrow}\mathrm{KK^{ban}}_G(\mathcal{C}_0(Z),\ B)$, where the limit is again taken over the G-equivariant and G-compact subsets Z of $\underline{\mathrm{E}}\mathrm{G}$. Then there is a homomorphism[2]

$$\mu_{\mathcal{A}}^B\colon\ \mathrm{K}_*^{\mathrm{top,ban}}(G,B)\to\mathrm{K}_*(\mathcal{A}(G,B))\,.$$

Is $\mu_{\mathcal{A}}^B$ an isomorphism?

[1] See the acknowledgements at the end of the introduction of [Laf02].

[2] Note that for G-C*-algebras B, the two versions of $\mu_{\mathcal{A}}^B$ have different domains of definition. Because we are going to concentrate on general Banach algebras, $\mu_{\mathcal{A}}^B$ will always denote the second, the Banach algebra version, in later chapters.

Although Lafforgue has carried out most of his basic constructions for general Banach algebras, most notably the definition of his bivariant K-theory, important arguments in [Laf02] only work for C^*-algebras. For instance, it is proved that $\mu_{\mathcal{A}}^B$ is an isomorphism for all proper G-C^*-algebras B. But the proof rests on the fact proved in [CEM01] that μ_r^B is an isomorphism for such algebras and hence this proof cannot serve as a model for an analogous result for more general Banach algebras.

One aim of the present work is to make it possible to prove Banach algebra results using only Banach algebra techniques. A central tool, which is not available (and not necessary) in the C^*-algebra world, is a very useful sufficient condition for the homotopy of $\mathrm{KK}^{\mathrm{ban}}$-cycles[3]: Homomorphisms between certain cycles which are isomorphisms in the C^*-algebra world have only dense image in the Banach algebra world; we state and prove a condition which tells us that nevertheless these homomorphisms often induce homotopies between the cycles. A first application of this tool is the systematic treatment of the invariance of K-theory of Banach algebras under Morita equivalences[4] by the introduction of so-called Morita morphisms between Banach algebras.

Expanding the the purely Banach algebraic theory will probably also prove useful when attacking C^*-algebra problems. For example, if one considers generalisations of iterated crossed products of C^*-algebras (as used in [CE01] to prove permanence properties of the Baum-Connes conjecture), then the first step of a stepwise "unconditional descent" would lead out of the category of C^*-algebras.

Groupoids and the Green-Julg theorem

A proper G-C^*-algebra B is a G-C^*-algebra which is at the same time a $\mathcal{C}_0(X)$-algebra for some proper G-space X such that the actions of G and $\mathcal{C}_0(X)$ on B are compatible. We can think of such an algebra as a C^*-algebra on which the transformation groupoid $X \rtimes G$ acts. Since X is a proper G-space, the groupoid $X \rtimes G$ is proper.

For this reason it is natural to consider actions of (proper) groupoids on Banach algebras. Lafforgue has recently translated most of his concepts and results into the framework of actions of groupoids (see [Laf06]). In his article, the fundamental concept is the notion of an upper semi-continuous field of Banach algebras, and if $\mathcal{G}$ is a topological groupoid, then a $\mathcal{G}$-Banach algebra is in particular an upper semi-continuous field of Banach algebras over the unit space $\mathcal{G}^{(0)}$ of $\mathcal{G}$. Lafforgue constructs a bivariant K-theory for $\mathcal{G}$-Banach algebras. The present thesis gives a rather detailed and systematic account of this construction, including a proof of the functoriality under generalised morphisms of groupoids in the sense of Le Gall (see [LG94]), which is only mentioned in [Laf06]. From this functoriality we deduce:

Theorem. *Let $\mathcal{G}$ and $\mathcal{H}$ be locally compact Hausdorff groupoids carrying Haar systems. Let Ω be an equivalence between $\mathcal{G}$ and $\mathcal{H}$. Let A and B be $\mathcal{H}$-Banach algebras. Then*

$$\mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(A,\ B) \ \cong\ \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\Omega^*A,\ \Omega^*B)\,.$$

Here, Ω^*A denotes the pull-back of A along Ω, which could also be denoted as the induced algebra $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} A$. We also show that equivalence is preserved under the descent construction defined in [Laf06]: The Banach algebra $\mathcal{A}(\mathcal{H}, A)$ is Morita equivalent to $\mathcal{A}(\mathcal{G}, \Omega^*A)$, where $\mathcal{A}(\mathcal{G})$ and $\mathcal{A}(\mathcal{H})$ are unconditional completions that are compatible in a certain sense: This applies in particular to $\mathrm{L}^1(\mathcal{G})$ and $\mathrm{L}^1(\mathcal{H})$.

[3] The underlying construction is used in special cases already in [Laf02] and more explicitly in [Laf04].

[4] The invariance was proved in the unpublished note [Laf04]; our result is somewhat more general.

Recall that upper semi-continuous fields of C*-algebras over some locally compact space X can alternatively be described as $\mathcal{C}_0(X)$-C*-algebras. This is no longer completely true for Banach algebras and we clarify the subtle differences between the two concepts. For $\mathcal{C}_0(X)$-Banach algebras we define an equivariant bivariant K-theory that we call $\mathrm{RKK}^{\mathrm{ban}}_G(\mathcal{C}_0(X); A, B)$, and compare it to the equivariant KK-theory for groupoids defined in [Laf06]. Both theories have their natural applications; the descent construction might serve as an example: We show that it not only takes values in $\mathrm{KK}^{\mathrm{ban}}(\mathcal{A}(\mathcal{G}, A),\ \mathcal{A}(\mathcal{G}, B))$, but is a homomorphism

$$j_{\mathcal{A}}\colon\ \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A,\ B)\ \to\ \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G}, A),\ \mathcal{A}(\mathcal{G}, B)),$$

where $\mathcal{G}$ is a locally compact Hausdorff groupoid with unit space X which carries a Haar system, $\mathcal{A}(\mathcal{G})$ is an unconditional completion of $\mathcal{C}_c(\mathcal{G})$ and A and B are $\mathcal{G}$-Banach algebras.

We also use the $\mathrm{RKK}^{\mathrm{ban}}$-theory as the right-hand side in the following variant of the Green-Julg theorem. The C*-algebraic version of this theorem is proved in [Tu99].

Theorem. [5] *Let $\mathcal{G}$ be a proper locally compact Hausdorff groupoid with unit space X and which carries a left Haar system. Let $\mathcal{A}(\mathcal{G})$ be an unconditional completion of $\mathcal{C}_c(\mathcal{G})$ (satisfying some mild conditions). Then for all non-degenerate $\mathcal{G}$-Banach algebras B we have an isomorphism*

$$\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X),\ B)\ \cong\ \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G}, B)).$$

If $X/\mathcal{G}$ is compact, we therefore get an isomorphism

$$\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X),\ B)\ \cong\ \mathrm{KK}^{\mathrm{ban}}(\mathbb{C},\ \mathcal{A}(\mathcal{G}, B)) \cong \mathrm{K}_0(\mathcal{A}(\mathcal{G}, B)).$$

Note that if $\mathcal{G}$ is a compact group G and X is a one-point space, then this theorem says that $\mathrm{K}_0^G(B)$ is isomorphic to $\mathrm{K}_0(\mathcal{A}(G, B))$. For $\mathcal{A}(G) = \mathrm{L}^1(G)$ this is a form of the Green-Julg theorem (compare [Jul81]).

As a consequence of the generalised Green-Julg theorem we can prove the following positive partial answer to the Bost-conjecture for proper Banach algebras. To this end we introduce the notion of a proper $\mathcal{G}$-Banach algebra for locally compact Hausdorff groupoids $\mathcal{G}$ and show:

Theorem. [6] *Let B be a non-degenerate proper $\mathcal{G}$-Banach algebra and let $\mathcal{A}(\mathcal{G})$ be an unconditional completion of $\mathcal{C}_c(\mathcal{G})$ (again satisfying some mild regularity condition). Then the homomorphism*

$$\mu^B_{\mathcal{A}}\colon\ \mathrm{K}^{\mathrm{top,ban}}(\mathcal{G},\ B) \to \mathrm{K}_0(\mathcal{A}(\mathcal{G}, B))$$

is split surjective.

Possible further developments

Expansion

With the toolbox put together in this thesis it should be easier to translate further results for C*-algebras into the language of Banach algebras. For some results it might even be possible to use the brute force method to translate proofs word for word. On the other hand, even proofs of simple facts for $\mathrm{KK}^{\mathrm{ban}}$ can be much more technical than their C*-algebraic counterparts; in particular it can be tiresome if Kasparov cycles that should be isomorphic are only contained densely into one another, making it necessary to construct homotopies.

[5] See Theorem 7.1.9.

[6] See Theorem 8.4.4.

Abstraction

To keep this thesis comprehensible without losing precision and completeness, I have tried to be as systematic as possible (even at the risk of being a bit wordy from time to time). An outcome of being systematic is a certain amount of repetition which might have been avoided by a higher degree of abstraction. However, an elaboration of the necessary categorial concepts would be extensive and too much of a diversion, so I decided to just sketch a possible general construction for now:

The definition of equivariant bivariant K-theory for Banach algebras is presented in the first chapter of this thesis; in the second chapter the construction is repeated for $\mathcal{C}_0(X)$-Banach algebras; and in the third chapter for upper semi-continuous fields of Banach algebras. The underlying blue-print is always the same: Start with a category which is enriched over the category of Banach spaces so that the morphism sets are Banach spaces and the composition is bilinear and contractive (e.g. take the category of G-Banach spaces and continuous linear maps between them, where G is some locally compact group). Distinguish a certain class of morphisms (the G-equivariant contractive linear maps in our example). There should be an associative tensor product compatible with the distinguished morphisms (the projective tensor product of G-Banach spaces) which has a unit (the trivial G-Banach space $\mathbb{C}$). This data could be called a *monoidal Banach category*. Functors between such categories which are compatible with the tensor product could be called *monoidal Banach functors*.

Using the tensor product of such a category, one can define algebras (the G-Banach algebras in our example) and homomorphisms between them (they should be distinguished morphisms — in our example they are the G-equivariant contractive homomorphisms of G-Banach algebras). Similarly, one can define modules and pairs, etc. (e.g. G-Banach modules and G-Banach pairs over G-Banach algebras) and homomorphisms and linear operators between them. To define “generalised Kasparov cycles” in such a setting you need some additional information, most prominently a definition of “compact operators”. You also need some notion of direct image under homomorphisms of algebras and a homotopy relation. The so-constructed variant of KK-theory should be compatible with monoidal Banach functors that respect compact operators, etc.

The exposition in each of the first three chapters of this thesis follows the same fundamental plan. First the underlying monoidal Banach category is introduced. Then the induced categories of algebras, modules and pairs are defined. In a third step, the additional information is given, for instance the compact operators are defined. Finally, the resulting version of KK-theory is derived.

There are several instances of monoidal Banach functors giving homomorphisms of KK-type groups, and we also use a standardised scheme to define them: They are first introduced as functors between the underlying monoidal Banach categories, then it is shown how they induce functors between the derived categories of algebras, of pairs, of modules and of KK-cycles.

A precise abstract treatment of monoidal Banach categories would make it necessary to keep track of a large number of natural isomorphisms and natural transformations that come with the categories and functors, e.g. the natural isomorphism that is needed for a correct statement of the associativity of the tensor product of a monoidal category. This might better be done in a separate exposition.

Connection to kk

It was remarked already in [Laf02] that it would be desirable to connect $\mathrm{KK}^{\mathrm{ban}}$ to Cuntz’ kk-theory defined in [Cun97], and recent work of Cuntz[7] strongly indicates that there is indeed a way to turn cycles for $\mathrm{KK}^{\mathrm{ban}}(A, B)$ into elements of $\mathrm{kk}(A, B)$. Because kk has a number of advantageous features, this would pave the way for a considerable transfer of the techniques and results for C*-algebras into the realm of Banach algebras. For example, the “Dirac-dual-Dirac” method makes use of the Kasparov

[7] J. Cuntz, personal communication, 2006.

product, and kk possesses a product. So far, the product in $\mathrm{KK}^{\mathrm{ban}}$ is only defined for very special elementary cases (such as the action of $\mathrm{KK}^{\mathrm{ban}}$ on K-theory and the product between $\mathrm{KK}^{\mathrm{ban}}$-cycles and Morita equivalences) and it is not clear whether it could be constructed for general $\mathrm{KK}^{\mathrm{ban}}$-cycles at all. Moreover, the "algebraic" definition of kk and its computational features should make it easier to find algebraic proofs of results which might only have rather technical analytic proofs in the world of $\mathrm{KK}^{\mathrm{ban}}$.

Organisation of this work

The first chapter recalls the definition of Banach pairs and of $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$. The basic concepts are introduced rather systematically, one cornerstone being the notion of a (concurrent) homomorphism of Banach pairs (which appears only implicitly in [Laf02]). This new notion also plays a prominent rôle in the statement of the above-mentioned sufficient condition for homotopy of $\mathrm{KK}^{\mathrm{ban}}$-cycles, which is proved in the first chapter and is used (in several variants) about thirty times throughout this thesis. The third important part of the first chapter introduces the notion of Morita morphisms between Banach algebras, generalising both (homotopy classes of) homomorphisms and Morita equivalences of Banach algebras.

The second chapter examines what happens if one adds a compatible non-degenerate action of the Banach algebra $\mathcal{C}_0(X)$ to all the definitions of the first chapter, where X is a locally compact Hausdorff space. Because the first chapter is rather detailed, the second chapter merely summarises the necessary changes. The Banach algebras carrying a compatible action of $\mathcal{C}_0(X)$ are called $\mathcal{C}_0(X)$-Banach algebras, and the resulting bivariant K-theory for $\mathcal{C}_0(X)$-Banach algebras, defined in Chapter 2, is called $\mathrm{RKK}_G^{\mathrm{ban}}\left(\mathcal{C}_0(X);\ A,\ B\right)$.

Technically more demanding than the study of $\mathcal{C}_0(X)$-Banach algebras is the study of upper semi-continuous fields of Banach algebras which we undertake in Chapter 3. This chapter comprises a systematic development of the $\mathrm{KK}^{\mathrm{ban}}$-theory for Banach algebras equipped with actions of groupoids, as introduced in [Laf06].

The notions of upper semi-continuous fields of Banach algebras over X and of $\mathcal{C}_0(X)$-Banach algebras are really very close, and Chapter 4 explores how the two concepts are related to each other. It might be worth mentioning that unlike upper semi-continuous fields of C^*-algebras, upper semi-continuous fields of Banach algebras are more special than $\mathcal{C}_0(X)$-Banach algebras; they correspond to so-called "locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach algebras", as discussed in Chapter 4.

Chapters 5 and 6 address the descent and generalised morphisms of groupoids. The exposition of the descent is more systematic than in [Laf06], giving quite a lot of the technical details of the proofs, and the definition of the $\mathrm{RKK}^{\mathrm{ban}}$-theory allows us to obtain results that are a little more precise. We also show that $\mathrm{KK}^{\mathrm{ban}}$ is functorial under generalised morphisms of groupoids and that (Morita) equivalence of groupoids is compatible with the descent map.

In Chapter 7 we use the theory presented in the first six chapters to show the generalised version of the Green-Julg theorem mentioned above. The proof demands a fair amount of technical care. We divide the proof into two parts: Split surjectivity and split injectivity. This is worth mentioning here because the surjectivity part of the proof needs fewer technical conditions on the unconditional completion that is involved.

In the final chapter we use the split surjectivity part of the generalised Green-Julg theorem to prove the split surjectivity of the Bost homomorphism for proper coefficients. For the formulation of this result we first say what proper $\mathcal{G}$-Banach algebras are in the case that $\mathcal{G}$ is a locally compact Hausdorff groupoid.

The appendices collect technical results and proofs which were banned from the main part of the thesis to increase readability. A noteworthy example is the proof of the fact that the projective tensor product over $\mathcal{C}_0(X)$ of locally $\mathcal{C}_0(X)$-convex Banach spaces is again locally $\mathcal{C}_0(X)$-convex.

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Notational conventions

Throughout this work, all normed spaces and Banach spaces are complex (and so are all Banach algebras, Banach modules, etc.). A linear map T between normed spaces is called *contractive* if $\|T\| \leq 1$. If E is a normed space and E_0 is a subset of E then we write $\mathrm{cl}\,(E_0)$ for the closed linear span of E_0 in E.

If $k \in \mathbb{N}$ and $E_1, \ldots, E_k$ and F are Banach spaces, then the set of continuous k-linear maps from $E_1 \times \cdots \times E_k$ to F is denoted by $\mathrm{M}\,(E_1, \ldots, E_k; F)$. Endowed with the norm $\mu \mapsto \|\mu\| = \sup_{\|e_i\| \leq 1} \|\mu(e_1, \ldots, e_k)\|_F$, it is itself a Banach space. A map $\mu \in \mathrm{M}\,(E_1, \ldots, E_k; F)$ is called *non-degenerate* if the span of its image is a dense subset of F.

If E is a Banach space and X is a locally compact Hausdorff space, then we write EX for the Banach space $\mathcal{C}_0(X, E)$ of continuous functions from X to E vanishing at infinity. We regard this as a closed subspace of the space $\mathcal{C}_b(X, E)$ of all bounded continuous functions from X to E, equipped with the norm $f \mapsto \|f\|_\infty = \sup_{x \in X} \|f(x)\|_E$.

Chapter 1

KK-Theory for Banach Algebras

The equivariant and bivariant K-theory $\mathrm{KK}^{\mathrm{ban}}$ for Banach algebras defined by Vincent Lafforgue is modelled after the KK-theory for C*-algebras as introduced by Kasparov. The cycles for the KK-theory for C*-algebras are given by operators on graded equivariant Hilbert modules, the corresponding notion for Banach algebras which is used to define cycles for $\mathrm{KK}^{\mathrm{ban}}$ is the notion of a graded equivariant Banach pair.

In this chapter we present Lafforgue's theory in some detail. We first discuss elementary notions such as Banach algebras, Banach modules and the balanced tensor product. In a second step, Banach pairs are introduced along with the linear and compact operators between them. It is worth mentioning that there is an additional type of morphisms between Banach pairs, generalising the homomorphisms (with coefficient maps) between Hilbert modules; we coin the term "concurrent homomorphisms" for them.

On our way to the definition of $\mathrm{KK}^{\mathrm{ban}}$ (finally given in Section 1.8) we also define gradings and group actions. To show how these definitions fit into the general scheme sketched in the introduction and to have a model for similar definitions in the later chapters we define gradings first on Banach spaces, then on Banach algebras, Banach modules, and, finally, on Banach pairs. The same systematic approach is repeated for actions of locally compact Hausdorff groups.

As a technical tool which will prove very helpful throughout this thesis we prove in Section 1.9 a sufficient condition for the homotopy of $\mathrm{KK}^{\mathrm{ban}}$-cycles; this condition is then used to systematically present and extend a result of V. Lafforgue that says that the K-theory of Banach algebras is invariant under Morita equivalences.

A general reference for the first part of this chapter is [Laf02], the last two sections are partly based on ideas appearing in [Laf04].

1.1 Banach algebras and Banach modules

1.1.1 Banach algebras

For us, a *Banach algebra* B is a Banach space B endowed with a bilinear associative multiplication such that $\|bc\| \leq \|b\| \, \|c\|$. It is called *unital* if it has a unit of norm one. In this work, a homomorphism of Banach algebras will always be contractive. A Banach algebra B is called *non-degenerate* if the span of $B \cdot B$ is dense in B.

Let B be a Banach algebra. We define the unitalisation $\widetilde{B}$ of B to be the unital Banach algebra given by the following data: The underlying Banach space is $B \oplus \mathbb{C}$ with the norm $\|(b, \lambda)\|_{\widetilde{B}} :=$

$\|b\|_B + |\lambda|$ for every $b \in B$, $\lambda \in \mathbb{C}$. The multiplication is given by $(a, \lambda) \cdot (b, \mu) := (ab + \lambda b + \mu a, \lambda\mu)$ for every $a, b \in B$, $\lambda, \mu \in \mathbb{C}$. Note that the unit element of $\widetilde{B}$ is given by $(0, 1)$. Moreover, B is canonically contained in $\widetilde{B}$ as a closed two-sided ideal.

If B and C are Banach algebras and $\theta\colon B \to C$ is a homomorphism of Banach algebras, then the unitalisation $\widetilde{\theta}$ of θ is the canonical unital homomorphism $(\theta, \mathrm{Id}_{\mathbb{C}})$ from $\widetilde{B}$ to $\widetilde{C}$.

If B is a Banach algebra, then a net $(u_\lambda)_{\lambda \in \Lambda}$ in B is called a *left approximate identity* for B if $\lim_{\lambda \in \Lambda} u_\lambda b = b$ for all $b \in B$. It is *bounded* (by one), if $\|u_\lambda\| \leq 1$ for every $\lambda \in \Lambda$. Analogously, we define a (bounded) right approximate identity. A (bounded) approximate identity is a (bounded) left approximate identity which is at the same time a right approximate identity. Note that B is non-degenerate if B has an approximate identity.

1.1.2 Banach modules

Definition 1.1.1 (Banach module). Let B be a Banach algebra. A *right Banach B-module E* is a Banach space which is at the same time a right B-module satisfying the norm-condition $\|eb\| \leq \|e\|\,\|b\|$ for all $b \in B$ and $e \in E$. We write E_B to emphasise the fact that E is a right B-module.

In the same manner we define left Banach A-modules ${}_AE$ and Banach A-B-bimodules ${}_AE_B$ for Banach algebras A and B. If B is a Banach algebra, then we can regard B as a Banach B-B-bimodule (called the *standard Banach B-B-bimodule*). In the following we are going to concentrate on right Banach modules; the left-handed analogues of the definitions and propositions are immediate.

Definition 1.1.2 ($\mathrm{L}_B(E, F)$). Let B be a Banach algebra and let E_B and F_B be Banach B-modules. Then $\mathrm{L}_B(E, F)$ is defined as the set of $\mathbb{C}$-linear continuous maps from E to F satisfying

$$\forall e \in E \;\forall b \in B: \quad T(eb) = (T(e))b,$$

i.e., the elements of $\mathrm{L}_B(E, F)$ are B-linear. We write $\mathrm{L}_B(E)$ for $\mathrm{L}_B(E, E)$. In the case of left Banach B-modules we write ${}_B\,\mathrm{L}(E, F)$ rather than $\mathrm{L}_B(E, F)$.

Note that the set $\mathrm{L}_B(E, F)$ is a Banach space (being a closed subspace of $\mathrm{L}_{\mathbb{C}}(E, F)$) and that the composition of such B-linear continuous operators is again B-linear and continuous. The space $\mathrm{L}_B(E)$ is hence a unital Banach algebra.

Between Banach modules there is also a second type of morphisms:

Definition 1.1.3 (Homomorphism with coefficient maps). Let B and B' be Banach algebras and let E_B and $E'_{B'}$ be Banach modules over B and B', respectively. A *homomorphism Φ (of right Banach modules) with coefficient map φ* from E_B to $E'_{B'}$ is a pair (Φ, φ) such that $\Phi\colon E \to E'$ is $\mathbb{C}$-linear and *contractive*, $\varphi\colon B \to B'$ is a homomorphism of Banach algebras and

$$\forall e \in E \;\forall b \in B: \quad \Phi(eb) = \Phi(e)\varphi(b).$$

We also write Φ_φ for the pair (Φ, φ). In the case $B = B'$ a homomorphism with coefficient map Id_B is just a contractive B-linear map.

Remark 1.1.4. The main objective of requiring homomorphisms of Banach modules to be contractive rather than just continuous is to align them with homomorphisms of Banach algebras and homomorphisms of Hilbert modules. I consider it beneficial for the intellectual hygiene to put these kinds of homomorphisms into a single box, whereas the continuous B-linear maps between Banach B-modules are akin to (and generalisations of) continuous $\mathbb{C}$-linear maps between Banach spaces and (adjointable) operators between Hilbert modules. We will label the first kind of morphisms "homomorphisms" to distinguish them from the second kind, which we prefer to call "operators".

The definition of homomorphisms with coefficient maps extends naturally to Banach bimodules. There we have to consider triples consisting of a linear map between the modules and two coefficient maps.

Definition 1.1.5 (Non-degenerate Banach module). Let B be a Banach algebra. A right Banach B-module E is called non-degenerate[1] if the span of EB is dense in E.

Proposition 1.1.6 ([Rie67], Proposition 3.4). *Let B be a Banach algebra with bounded approximate identity $(u_\lambda)_{\lambda\in\Lambda}$. Let E be a right Banach B-module. Then the following are equivalent:*

1. *E is non-degenerate;*
2. *$\forall e \in E:\ e = \lim_{\lambda\in\Lambda} eu_\lambda$;*
3. *$\forall e \in E\ \exists f \in E\ \exists b \in B:\ e = fb$.*

1.1.3 Tensor products of Banach modules

Let A, B and C be Banach algebras, let E be a Banach A-B-bimodule and let F be a Banach B-C-bimodule.

Definition 1.1.7 (Balanced bilinear maps). Let G be a Banach A-C-bimodule.

- The space ${}_A\mathrm{M}(E, F; G)$ is defined to be the set of all $\beta \in \mathrm{M}(E, F; G)$ such that
$$\forall e \in E, f \in F, a \in A:\ \beta(ae, f) = a\beta(e, f).$$
- The space $\mathrm{M}_C(E, F; G)$ is defined to be the set of all $\beta \in \mathrm{M}(E, F; G)$ such that
$$\forall e \in E, f \in F, c \in C:\ \beta(e, fc) = \beta(e, f)c.$$
- The space $\mathrm{M}^{\mathrm{bal}}(E, F; G)$ is defined to be the set of all $\beta \in \mathrm{M}(E, F; G)$ which are B-*balanced*:
$$\forall e \in E, f \in F, b \in B:\ \beta(eb, f) = \beta(e, bf).$$

One can combine these notations to define ${}_A\mathrm{M}_C(E, F; G)$, ${}_A\mathrm{M}_C^{\mathrm{bal}}(E, F; G)$, etc. All the mentioned sets are Banach spaces when endowed with the canonical vector space structures and norms.

Definition 1.1.8 (Balanced tensor product). A (projective) *balanced tensor product* of the bimodules E and F is a Banach A-C-bimodule $E \otimes_B F$ together with an element π of ${}_A\mathrm{M}_C^{\mathrm{bal}}(E, F; E \otimes_B F)$ of norm ≤ 1 such that, for every Banach A-C-bimodule G and every $\mu \in {}_A\mathrm{M}_C^{\mathrm{bal}}(E, F; G)$, there is a unique $\hat\mu \in {}_A\mathrm{L}_C(E \otimes_B F, G)$ such that
$$\mu = \hat\mu \circ \pi$$
and $\|\mu\| = \|\hat\mu\|$.

That such a balanced tensor product exists can be shown by forming a quotient of the usual projective tensor product[2]; uniqueness follows from general nonsense. It is easy to show that the balanced tensor product is associative.

[1] "Essential" in Rieffel's article [Rie67]; see also [Laf02], page 11.
[2] See [Laf02], page 12.

Lemma 1.1.9. *If F is non-degenerate from the right, then so is $E \otimes_B F$.*

Definition 1.1.10 (Tensor product of linear operators). Let E' be a Banach A-B-bimodule and let F' be a Banach B-C-bimodule and $S \in {}_A\mathrm{L}_B(E, E')$ and $T \in {}_B\mathrm{L}_C(F, F')$. Then there is a unique element $S \otimes T$ in ${}_A\mathrm{L}_C(E \otimes E',\ F \otimes F')$ such that

$$(S \otimes T)(e \otimes f) = S(e) \otimes T(f)$$

for all $e \in E$ and $f \in F$. We have $\|S \otimes T\| \leq \|S\|\,\|T\|$.

Definition 1.1.11 (Tensor product of homomorphisms). Let A', B', C' be Banach algebras and let E' be a Banach A'-B'-bimodule and F' be a Banach B'-C'-bimodule. Let $\varphi\colon A \to A'$, $\psi\colon B \to B'$ and $\theta\colon C \to C'$ be homomorphisms of Banach algebras. Let ${}_\varphi\Phi_\psi\colon {}_AE_B \to {}_{A'}E'_{B'}$ and ${}_\psi\Psi_\theta\colon {}_BF_C \to {}_{B'}F'_{C'}$ be homomorphisms with coefficient maps. Then there is a unique homomorphism $\Phi \otimes \Psi$ of Banach bimodules from $E \otimes_B F$ to $E' \otimes_{B'} F'$ with coefficient maps φ and θ such that

$$(\Phi \otimes \Psi)(e \otimes f) = \Phi(e) \otimes \Psi(f)$$

for all $e \in E$ and $f \in F$.

1.1.4 The pushout

Note that, if B is a Banach algebra, then every Banach B-module is also a Banach $\widetilde{B}$-module, where $\widetilde{B}$ is the unitalisation of B, and vice versa. The same is true for Banach bimodules.

Definition 1.1.12 (The pushout of Banach modules). [3] Let B, B' be Banach algebras and let E be a Banach B-module. If $\psi\colon B \to B'$ is a morphism of Banach algebras, then define the pushout $\psi_*(E)$ of E along ψ to be the Banach B'-module $E \otimes_{\widetilde{\psi}} \widetilde{B'}$ (regarding E as a right Banach $\widetilde{B}$-module and $\widetilde{B'}$ as a Banach $\widetilde{B}$-B' bimodule via $\widetilde{\psi}$).

Definition 1.1.13 (The pushout of linear operators). Let B, B' be Banach algebras, let $\psi\colon B \to B'$ be a morphism of Banach algebras, and let E and F be Banach B-modules. If $T \in \mathrm{L}_B(E, F)$, then define $\psi_*(T) \in \mathrm{L}_{B'}(\psi_*(E),\ \psi_*(F))$ by

$$\psi_*(T)(e \otimes (b' + \lambda 1)) := T(e) \otimes (b' + \lambda 1)$$

for every $e \in E$, $b' \in B'$ and $\lambda \in \mathbb{C}$. In other words we define $\psi_*(T)$ to be $T \otimes \mathrm{Id}_{\widetilde{B'}}$.

Proposition 1.1.14. *The map ψ_* defines a functor from the category of Banach B-modules to the category of Banach B'-modules which is linear and contractive on the morphism sets.*

Proposition 1.1.15 (Functorial properties of the pushout). [4]

- *Let B be a Banach algebra. Then the functor $(\mathrm{Id}_B)_*$ is naturally isometrically isomorphic to the identity functor on the category of Banach B-modules.*
- *Let B, B', B'' be Banach algebras and let $\psi\colon B \to B'$, $\psi'\colon B' \to B''$ be homomorphisms. Then $\psi'_* \circ \psi_*$ and $(\psi' \circ \psi)_*$ are naturally isometrically isomorphic functors from the category of Banach B-modules to the category of Banach B''-modules.*

[3] What we call "pushout" is called "image directe" in [Laf02].
[4] See [Laf02], Lemme 1.1.1.

Proposition 1.1.16 (The pushout of a non-degenerate Banach module). [5] *Let B, B' be Banach algebras, let $\psi\colon B \to B'$ be a homomorphism and let E be a non-degenerate Banach B-module. Then $\psi_*(E)$ is a non-degenerate Banach B'-module.*

Proof. Because E is non-degenerate we know that EBB is dense in E. Let $e \in E$, $b, c \in B$ and $b' + \lambda 1 \in \widetilde{B'}$. Then

$$(ebc) \otimes_{\widetilde{\psi}} (b' + \lambda 1) = \left(e \otimes_{\widetilde{\psi}} \psi(b)\right) \underbrace{\psi(c)(b' + \lambda 1)}_{\in B'}.$$

By this we know that the subspace $(E \otimes \psi(B))B'$ is dense in $\psi_*(E) = E \otimes \widetilde{B'}$, so $\psi_*(E)$ is non-degenerate. $\square$

1.2 Banach pairs

Definition 1.2.1 ((Banach) B-pair). Let B be a Banach algebra. Then a *(Banach) B-pair* E is a pair $E = (E^<, E^>)$, where $E^<$ is a left Banach B-module and $E^>$ is a right Banach B-module, endowed with a bilinear bracket $\langle \cdot, \cdot \rangle_E \colon E^< \times E^> \to B$ satisfying the following conditions:

- $\forall b \in B\ \forall e^< \in E^<\ \forall e^> \in E^> :\ \langle be^<, e^> \rangle_E = b \langle e^<, e^> \rangle_E$.
- $\forall b \in B\ \forall e^< \in E^<\ \forall e^> \in E^> :\ \langle e^<, e^> b \rangle_E = \langle e^<, e^> \rangle_E b$.
- $\forall e^< \in E^<\ \forall e^> \in E^> :\ \|\langle e^<, e^> \rangle_E\| \leq \|e^<\|\,\|e^>\|$.

We will often omit the index of the bracket and simply write $\langle \cdot, \cdot \rangle$. Sometimes, if we want to stress the algebra B into which the bracket maps we even write $\langle \cdot, \cdot \rangle_B$.

Definition 1.2.2 (Non-degenerate). Let B be a Banach algebra. A Banach B-pair $E = (E^<, E^>)$ is called *non-degenerate* if $E^<$ is a non-degenerate left Banach B-module and $E^>$ is a non-degenerate right Banach B-module.

Note that in[6] [Laf02] a Banach B-pair is required to be non-degenerate by definition.

1.2.1 Linear, compact and finite rank operators

Definition 1.2.3 (Linear operator between B-pairs). [7] Let $E = (E^<, E^>)$ and $F = (F^<, F^>)$ be B-pairs.

- A *linear operator* from E to F is a pair $T = (T^<, T^>)$, with $T^< \in {}_B\mathrm{L}(F^<, E^<)$ and $T^> \in \mathrm{L}_B(E^>, F^>)$, satisfying

$$\forall f^< \in F^<\ \forall e^> \in E^> :\ \langle f^<, T^> e^> \rangle_F = \langle T^< f^<, e^> \rangle_E.$$

- The set of all linear operators from E to F will be denoted by $\mathrm{L}_B(E, F)$.

[5] See [Laf02], page 12.
[6] Compare [Laf02], Définition 1.1.3.
[7] Linear operators are called “morphismes de B-paires” in [Laf02].

- If $T = (T^<, T^>) \in \mathrm{L}_B(E, F)$, then we define
$$\|T\|_{\mathrm{L}_B(E,F)} := \max\left\{\left\|T^<\right\|, \left\|T^>\right\|\right\}.$$
With this norm $\mathrm{L}_B(E, F)$ is a Banach space.
- If G is another B pair, $T \in \mathrm{L}_B(E, F)$, and $S \in \mathrm{L}_B(F, G)$, then
$$S \circ T := \left(T^< \circ S^<, S^> \circ T^>\right) \in \mathrm{L}_B(E, G).$$
We have $\|S \circ T\| \leq \|S\| \, \|T\|$.
- We set $\mathrm{L}_B(E) := \mathrm{L}_B(E, E)$. The pair $(\mathrm{Id}_{E^<}, \mathrm{Id}_{E^>})$ is an element of $\mathrm{L}(E)$ that we denote by Id_E. It is the unit of the Banach algebra $\mathrm{L}_B(E)$.

From time to time we will use the following convention which obscures things a little bit but leads to some handy formulae: If B is a Banach algebra and E, F are a B-pairs, then we write for every $T = (T^<, T^>) \in \mathrm{L}(E, F)$:

$$f^< T := T^<(f^<) \qquad \text{and} \qquad Te^> := T^>(e^>) \tag{1.1}$$

for every $f^< \in F^<$ and every $e^> \in E^>$. The fact that $T \in \mathrm{L}(E, F)$ can then be expressed via the formula
$$\forall f^< \in F^< \, \forall e^> \in E^> : \, \langle f^<, Te^> \rangle_F = \langle f^< T, e^> \rangle_E.$$

Definition 1.2.4 (Finite rank operator). Let E and F be B-pairs. For every $f^> \in F^>$ and every $e^< \in E^<$, we define $\left|f^>\right\rangle\left\langle e^<\right| \in \mathrm{L}(E, F)$ by
$$\left|f^>\right\rangle\left\langle e^<\right|^<(f^<) := \langle f^<, f^> \rangle e^< \quad \text{for all } f^< \in F^<,$$
and
$$\left|f^>\right\rangle\left\langle e^<\right|^>(e^>) := f^> \langle e^<, e^> \rangle \quad \text{for all } e^> \in E^>.$$
The span in $\mathrm{L}(E, F)$ of all such operators is denoted by $\mathcal{F}(E, F)$. An element of $\mathcal{F}(E, F)$ is called an operator of finite rank. We set $\mathcal{F}(E) := \mathcal{F}(E, E)$.

Using the notation introduced in (1.1), we can write the above formulae as
$$f^< \left|f^>\right\rangle\left\langle e^<\right| = \langle f^<, f^> \rangle e^< \quad \text{and} \quad \left|f^>\right\rangle\left\langle e^<\right| e^> = f^> \langle e^<, e^> \rangle.$$

Proposition 1.2.5. *Let E, F and G be B-pairs. Then*

- *The map $\left|\cdot\right\rangle\left\langle\cdot\right| \colon F^> \times E^< \to \mathrm{L}(E, F)$ is bilinear, of norm ≤ 1, and B-balanced.*
- *If $f^> \in F^>$, $e^< \in E^<$ and $T \in \mathrm{L}(F, G)$ then*
$$T \circ \left|f^>\right\rangle \left\langle e^<\right| = \left|T^>(f^>)\right\rangle \left\langle e^<\right| \overset{1.1}{=} \left|Tf^>\right\rangle \left\langle e^<\right|.$$
- *If $g^> \in G^>$, $f^< \in F^<$ and $S \in \mathrm{L}(E, F)$ then*
$$\left|g^>\right\rangle \left\langle f^<\right| \circ T = \left|g^>\right\rangle \left\langle T^<(f^<)\right| \overset{1.1}{=} \left|g^>\right\rangle \left\langle f^< T\right|.$$
- *If $S \in \mathcal{F}(E, F)$ and $T \in \mathrm{L}(F, G)$ then $T \circ S \in \mathcal{F}(E, G)$.*
- *If $S \in \mathrm{L}(E, F)$ and $T \in \mathcal{F}(F, G)$ then $T \circ S \in \mathcal{F}(E, G)$.*
- *$\mathcal{F}(E)$ is an ideal of $\mathrm{L}(E)$.*

Definition 1.2.6 (Compact operator). Let E and F be B-pairs. The closure of the finite rank operators $\mathcal{F}(E, F)$ in $\mathrm{L}(E, F)$ is denoted by $\mathrm{K}(E, F)$. An element of $\mathrm{K}(E, F)$ is called a *compact*[8] operator. We set $\mathrm{K}(E) := \mathrm{K}(E, E)$.

Proposition 1.2.7. *Let E, F and G be B-pairs.*

- *If $S \in \mathrm{K}(E, F)$ and $T \in \mathrm{L}(F, G)$, then $T \circ S \in \mathrm{K}(E, G)$.*
- *If $S \in \mathrm{L}(E, F)$ and $T \in \mathrm{K}(F, G)$, then $T \circ S \in \mathrm{K}(E, G)$.*
- *$\mathrm{K}(E)$ is an ideal of $\mathrm{L}(E)$.*

Definition 1.2.8 (Banach A-B-pair). Let A and B be Banach algebras. A *Banach A-B-pair*[9] E is a B-pair endowed with a homomorphism $\pi_A\colon A \to \mathrm{L}_B(E)$. In other words, $E^<$ is a Banach B-A-bimodule, $E^>$ is a Banach A-B-bimodule and

$$\forall a \in A, e^< \in E^<, e^> \in E^> :\ \langle e^< a, e^> \rangle_B = \langle e^<, a e^> \rangle_B.$$

Note that the situation of the preceding definition is not symmetric as there is no A-valued bracket around. It should be pointed out that a Banach A-B-pair is called non-degenerate in this work if it is a non-degenerate B-pair; we do not require the A-action to be non-degenerate in this case.

Let B be a Banach algebra. If E is a Banach B-pair, then E is a Banach $\mathrm{L}(E)$-B-pair and a Banach $\mathrm{K}(E)$-B-pair. And if we consider B as a right as well as a left B-module then the pair (B, B) with the multiplication of B as bracket is called the *standard B-pair*. We will denote it by $\underline{B}$ or, usually, simply by B. The B-pair B with the obvious additional structure is a Banach B-B-pair.

1.2.2 Concurrent homomorphisms

Definition 1.2.9 (Concurrent homomorphism of B-pairs). Let B, B' be Banach algebras, let E be a B-pair and E' a B'-pair. A *concurrent homomorphism* Ψ from E to E' is a pair $\Psi = (\Psi^<, \Psi^>)$ together with a so-called coefficient map ψ of Ψ, where

- $\Psi^<\colon E^< \to E'^<$ is $\mathbb{C}$-linear and contractive,
- $\Psi^>\colon E^> \to E'^>$ is $\mathbb{C}$-linear and contractive,
- $\psi\colon B \to B'$ is a (contractive) homomorphism of Banach algebras,

such that

1. $\forall b \in B, e^< \in E^< :\ \Psi^<(b e^<) = \psi(b)\Psi^<(e^<)$, i.e., $\Psi^<$ is a homomorphism of left Banach modules with coefficient map ψ,
2. $\forall b \in B, e^> \in E^> :\ \Psi^>(e^> b) = \Psi^>(e^>)\psi(b)$, i.e., $\Psi^>$ is a homomorphism of right Banach modules with coefficient map ψ,
3. $\forall e^< \in E^<, e^> \in E^> : \qquad \psi\left(\langle e^<, e^> \rangle_B\right) = \langle \Psi^<(e^<), \Psi^>(e^>) \rangle_{B'}$.

To indicate the coefficient map we write Ψ_ψ for Ψ.

[8]Conceptually, it would be better to call such operators "approximable".
[9]These are called "(A, B)-bimodule de Banach" in [Laf02].

Remark 1.2.10. The concurrent homomorphisms of Banach pairs generalise the homomorphisms of Hilbert modules. The term “concurrent” is chosen to (further) distinguish the homomorphisms of this type from the linear operators: The homomorphisms consist of two “arrows” pointing in the same direction whereas the linear operators consist of two “arrows” pointing in opposite directions. The word “concurrent” could be translated into “nebenläufig” in German (as opposed to “gegenläufig”) and perhaps to “dirigé” in French.

If B and B' are Banach algebras and $\psi\colon B \to B'$ is a contractive homomorphism, then $(\psi, \psi)_\psi$ is a concurrent homomorphism from $\underline{B}$ to $\underline{B'}$.

Definition 1.2.11 (Concurrent homomorphism of A-B-pairs). Let A, B, A', B' be Banach algebras, let E be an A-B-pair and E' an A'-B'-pair. A concurrent homomorphism Ψ from E to E' is a pair $\Psi = (\Psi^<, \Psi^>)$ together with two coefficient maps ϕ and ψ of Ψ, where

- $\Psi^<\colon E^< \to E'^<$ is $\mathbb{C}$-linear and contractive,
- $\Psi^>\colon E^> \to E'^>$ is $\mathbb{C}$-linear and contractive,
- $\phi\colon A \to A'$ and $\psi\colon B \to B'$ are contractive homomorphisms,

such that

1. $\forall a \in A, b \in B, e^< \in E^< : \qquad \Psi^<(be^<) = \psi(b)\Psi^<(e^<) \quad \wedge \quad \Psi^<(e^<a) = \Psi^<(e^<)\phi(a)$,
2. $\forall a \in A, b \in B, e^> \in E^> : \qquad \Psi^>(e^>b) = \Psi^>(e^>)\psi(b) \quad \wedge \quad \Psi^>(ae^>) = \phi(a)\Psi^>(e^>)$,
3. $\forall e^< \in E^<, e^> \in E^> : \qquad \psi\left(\langle e^<, e^>\rangle_B\right) = \langle \Psi^<(e^<), \Psi^>(e^>)\rangle_{B'}$.

To indicate the coefficient maps we write ${}_\phi\Psi_\psi$ for Ψ.

1.3 Sums, tensor products and the pushout

1.3.1 Sums of Banach pairs

Definition 1.3.1 (Sum of Banach pairs). Let B be a Banach algebra and let E_1, E_2 be Banach B-pairs. Then we define the sum $E_1 \oplus E_2$ of E_1 and E_2 to be the Banach B-pair $(E_1^< \oplus E_2^<,\ E_1^> \oplus E_2^>)$, where the left-hand side is endowed with the norm $(e_1^<, e_2^<) \mapsto \|e_1^<\| + \|e_2^<\|$ and the canonical left B-action; the right-hand side carries the norm $(e_1^>, e_2^>) \mapsto \|e_1^>\| + \|e_2^>\|$ and the canonical right B-action; the bracket is given by $\langle (e_1^<, e_2^<),\ (e_1^>, e_2^>)\rangle := \langle e_1^<, e_1^>\rangle + \langle e_2^<, e_2^>\rangle$.

Note that this is not the categorial sum in the category of Banach B-pairs and linear operators (in this case, one should rather take the $\sup$-norm on the left-hand side); it is the sum in the category of Banach pairs and homomorphisms with coefficient maps. More precisely, it is the universal object for pairs of homomorphisms into E_1 and E_2 with identical coefficient map. Note that the sum is associative and commutative up to isomorphism.

Definition 1.3.2 (Sum of linear operators). Let B be a Banach algebra and let E_1, E_2, F_1, F_2 be Banach B-pairs. Let $T_1 \in \mathrm{L}_B(E_1, F_1)$ and $T_2 \in \mathrm{L}_B(E_2, F_2)$. Then we define

$$T_1 \oplus T_2 := \left(T_1^< \oplus T_2^<,\ T_1^> \oplus T_2^>\right) \in \mathrm{L}_B\left(E_1 \oplus E_2,\ F_1 \oplus F_2\right).$$

This operator satisfies $\|T_1 \oplus T_2\| = \max\{\|T_1\|,\ \|T_2\|\}$.

Similarly one can define the sum of concurrent homomorphisms.

1.3.2 The balanced tensor product of Banach pairs

Definition 1.3.3 (The balanced tensor product of Banach pairs). Let A, B, C be Banach algebras and let E be a Banach A-B-pair and F a Banach B-C-pair. Then we define a Banach A-C-pair $E \otimes_B F$ by

- $(E \otimes_B F)^> := E^> \otimes_B F^>$,
- $(E \otimes_B F)^< := F^< \otimes_B E^<$,
- $\langle \cdot, \cdot \rangle : F^< \otimes_B E^< \times E^> \otimes_B F^> \to C,\ (f^< \otimes e^<, e^> \otimes f^>) \mapsto \langle f^<,\ \langle e^<, e^> \rangle f^> \rangle$.

Note that the balanced tensor product is compatible with the sum of Banach pairs. If E is just a B-pair, then we can take $A := \mathbb{C}$ to make it an A-B-pair. Then the preceding definition gives us a $\mathbb{C}$-C-pair, i.e., we get just a C-pair.

From the corresponding result for Banach modules (Lemma 1.1.9) we can easily deduce:

Proposition 1.3.4. *Let B, C be Banach algebras and let E be a Banach B-pair and let F be a Banach B-C-pair. If F is non-degenerate, then so is $E \otimes_B F$.*

Definition 1.3.5 (Tensor product of concurrent homomorphisms). Let A, B, C, A', B', C' be Banach algebras and let ${}_AE_B$, ${}_BF_C$, ${}_{A'}E'_{B'}$ and ${}_{B'}F'_{C'}$ be Banach pairs. Let $\varphi\colon A \to A'$, $\psi\colon B \to B'$ and $\theta\colon C \to C'$ be homomorphisms of Banach algebras. Let ${}_\varphi\Phi_\psi\colon {}_AE_B \to {}_{A'}E'_{B'}$ and ${}_\psi\Psi_\theta\colon {}_BF_C \to {}_{B'}F'_{C'}$ be concurrent homomorphisms with coefficient maps. Then

$$\Phi \otimes \Psi := \left(\Psi^< \otimes \Phi^<,\ \Phi^> \otimes \Psi^>\right)$$

is a concurrent homomorphism from $E \otimes_B F$ to $E' \otimes_{B'} F'$ with coefficient maps φ and θ, where the left- and the right-hand side, being tensor products of homomorphisms of Banach modules, are defined in 1.1.11.

1.3.3 Operators of the type $T \otimes 1$

Let A, B and C be Banach algebras and let E, E' be Banach B-pairs and F a Banach B-C-pair.

Definition 1.3.6. For every $T \in \mathrm{L}_B(E, E')$, define $T \otimes 1 \in \mathrm{L}_C\left(E \otimes_B F,\ E' \otimes_B F\right)$ to be

$$T \otimes 1 = \left(\mathrm{Id}_{F^<} \otimes T^<,\ T^> \otimes \mathrm{Id}_{F^>}\right).$$

The assignment $T \mapsto T \otimes 1$ is a functor from the category of Banach B-pairs to the category of Banach C-pairs, linear and contractive on the spaces of morphisms.

Proof. Let $e'^< \in E'^<$, $e^> \in E^>$, $f^< \in F^<$ and $f^> \in F^>$. Then

$$\begin{aligned}
\left\langle \left(\mathrm{Id}_{F^<} \otimes T^<\right)(f^< \otimes e'^<),\ e^> \otimes f^> \right\rangle &= \left\langle f^< \otimes T^<(e'^<),\ e^> \otimes f^> \right\rangle \\
&= \left\langle f^<,\ \left\langle T^< e'^<,\ e^> \right\rangle f^> \right\rangle \\
&= \left\langle f^<,\ \left\langle e'^<,\ T^> e^> \right\rangle f^> \right\rangle \\
&= \left\langle f^< \otimes e'^<,\ T^>(e^>) \otimes f^> \right\rangle \\
&= \left\langle f^< \otimes e'^<,\ \left(T^> \otimes \mathrm{Id}_{F^>}\right)\left(e^> \otimes f^>\right) \right\rangle.
\end{aligned}$$

So $T \otimes 1 \in \mathrm{L}_C\left(E \otimes_B F,\ E' \otimes_B F\right)$. □

Proposition 1.3.7. *Let the action $\pi_B\colon B \to \mathrm{L}_C(F)$ on B of F satisfy $\pi_B(B) \subseteq \mathrm{K}_C(F)$. Assume that E or E' is non-degenerate. If $T \in \mathrm{K}_B(E, E')$, then $T \otimes 1 \in \mathrm{K}_C\left(E \otimes_B F,\ E' \otimes_B F\right)$.*

Proof. It suffices to show the assertion for $T = \left|e'^{>}\right\rangle\left\langle be^{<}\right|$ for all $e'^{>} \in E'^{>}$, $e^{<} \in E^{<}$ and $b \in B$, because the function $T \mapsto T \otimes 1$ is linear and continuous and the span of all operators T of the given form is dense in $\mathrm{K}_B(E, E')$; to prove the latter one uses that E or E' is non-degenerate (note that we can also write $T = \left|e'^{>}b\right\rangle\left\langle e^{<}\right|$).

We now express $T \otimes 1$ as the composition of three operators, one of them being compact. To this end we define

$$M_{\langle e^<|} := \left(f^< \mapsto f^< \otimes e^<,\ (e^> \otimes f^>) \mapsto \langle e^<, e^>\rangle f^>\right) \in \mathrm{L}_C\left(E \otimes_B F, F\right)$$

and

$$M_{|e'^>\rangle} := \left(f^< \otimes e'^< \mapsto f^<\langle e'^<, e'^>\rangle,\ f^> \mapsto e'^> \otimes f^>\right) \in \mathrm{L}_C\left(F, E' \otimes_B F\right).$$

The operator $M_{\langle e^<|}$ can be regarded as a kind of annihilation operator (at least on the ket-side), the operator $M_{|e'^>\rangle}$ can be regarded as a creation operator (on the ket-side).

We have

$$\begin{aligned}(T^> \otimes 1)(e^> \otimes f^>) &= (e'^> b\langle e^<, e^>\rangle) \otimes f^> = e'^> \otimes (b\langle e^<, e^>\rangle f^>)\\ &= \left[M^>_{|e'^>\rangle} \circ \pi_B(b)^> \circ M^>_{\langle e^<|}\right]\left(e^> \otimes f^>\right)\end{aligned}$$

for all $e^> \in E^>$ and $f^> \in F^>$ and

$$\begin{aligned}(1 \otimes T^<)(f^< \otimes e'^<) &= f^< \otimes \langle e'^<, e'^>\rangle be^< = (f^<\langle e'^<, e'^>\rangle b) \otimes e^<\\ &= \left[M^<_{\langle e^<|} \circ \pi_B(b)^< \circ M^<_{|e'^>\rangle}\right]\left(f^< \otimes e'^<\right)\end{aligned}$$

for all $f^< \in F^<$ and $e'^< \in E'^<$.

Together, this yields

$$T \otimes 1 = M_{|e'^>\rangle} \circ \pi_B(b) \circ M_{\langle e^<|}.$$

Now $\pi_B(b)$ is compact, so $T \otimes 1$ is compact. $\square$

Corollary 1.3.8. *Let E be an A-B-pair such that A acts on E by compact operators and B acts on F by compact operators. If E is B-non-degenerate, then A acts on $E \otimes_B F$ by compact operators.*

1.3.4 The pushout

Let B, B' be Banach algebras and let $\psi\colon B \to B'$ be a homomorphism.

Definition 1.3.9 (The pushout of a pair). For all B-pairs E, define the *pushout* $\psi_*(E)$ of E along ψ to be the B'-pair

$$\psi_*(E) := E \otimes_{\tilde\psi} \underline{\widetilde{B'}} = \left(\widetilde{B'} \otimes_{\tilde\psi} E^<,\ E^> \otimes_{\tilde\psi} \widetilde{B'}\right) = \left(\psi_*(E^<),\ \psi_*(E^>)\right).$$

Note that this is indeed a B'-pair and not only a $\widetilde{B'}$-pair because the bracket of $\psi_*(E)$ takes its values in the ideal B' of $\widetilde{B'}$.

Definition 1.3.10 (The pushout of a linear operator). Let E, F be B-pairs. For all $T \in \mathrm{L}_B(E,F)$, define $\psi_*(T) \in \mathrm{L}_{B'}\left(\psi_*(E),\ \psi_*(F)\right)$ by

$$\psi_*(T) := \left(\psi_*(T^<),\ \psi_*(T^>)\right) = \left(\mathrm{Id}_{\widetilde{B'}} \otimes T^<,\ T^> \otimes \mathrm{Id}_{\widetilde{B'}}\right) = T \otimes 1.$$

The map ψ_* defines a functor from the category of Banach B-pairs to the category of Banach B'-pairs that is linear and contractive on the morphism sets. It is compatible with the sum of Banach B-pairs.

Proposition 1.3.11 (Functorial properties of the pushout). [10]

- *The functor* $(\mathrm{Id}_B)_*$ *is naturally isometrically equivalent to the identity functor on the category of Banach* B*-pairs in the following sense: Define for every* B*-pair* E *the homomorphism of pairs with coefficient map* Id_B

$$\eta_E = (\eta_E^<, \eta_E^>)\colon E \otimes_{\widetilde{B}} \widetilde{B} \to E$$

by $\widetilde{b} \otimes e^< \mapsto \widetilde{b}e^<$ *and* $e^> \otimes \widetilde{b} \mapsto e^>\widetilde{b}$*, where* $\widetilde{b} \in \widetilde{B}$*,* $e^< \in E^<$ *and* $e^> \in E^>$*. If* E *and* F *are* B*-pairs and* $T \in \mathrm{L}_B(E,F)$*, then*

$$(\mathrm{Id}_B)_*(T)^> \circ \eta_E^> = \eta_F^> \circ T^> \quad \textit{and} \quad (\mathrm{Id}_B)_*(T)^< \circ \eta_F^< = \eta_E^< \circ T^<,$$

i.e., η_E *and* η_F *intertwine* $(\mathrm{Id}_B)_*(T) = T \otimes 1$ *and* T*.*

- *Let* B'' *be another Banach algebra and let* $\psi'\colon B' \to B''$ *be another homomorphism. Then* $\psi'_* \circ \psi_*$ *and* $(\psi' \circ \psi)_*$ *are naturally isometrically equivalent functors from the category of Banach* B*-pairs to the category of Banach* B''*-pairs.*

Proof. This follows from the analogous Proposition 1.1.15 for Banach modules. □

From the analogous Proposition 1.1.16 for Banach modules we get:

Proposition 1.3.12 (The pushout of a non-degenerate Banach pair). *If* E *is a non-degenerate Banach* B*-pair, then* $\psi_*(E)$ *is a non-degenerate Banach* B'*-pair.*

Proposition 1.3.13. [11] *Let* E *and* F *be Banach* B*-pairs. Then for all* $T \in \mathrm{K}_B(E,F)$ *the operator* $\psi_*(T) = T \otimes 1$ *is contained in* $\mathrm{K}_{B'}\left(\psi_*(E),\ \psi_*(F)\right)$*.*

Proof. We give two arguments for this simple fact: The first is that it suffices to show the result for T of the form $\left|f^>\right\rangle\left\langle e^<\right|$ with $e^< \in E^<$ and $f^> \in F^>$. In this case the operator $\left|f^>\right\rangle\left\langle e^<\right| \otimes 1$ equals $\left|f^> \otimes 1_{\widetilde{B'}}\right\rangle\left\langle 1_{\widetilde{B'}} \otimes e^<\right|$ and is therefore compact.

The other argument uses Proposition 1.3.7. It is easy to show that B acts by compact operators on $\widetilde{B'}$ if we regard $\widetilde{B'}$ as a $\widetilde{B'}$-pair. It follows that $\mathrm{K}_B(E) \otimes 1 \subseteq \mathrm{K}_{\widetilde{B'}}\left(\psi_*(E), \psi_*(F)\right)$ by Proposition 1.3.7, and $\mathrm{K}_{\widetilde{B'}}\left(\psi_*(E), \psi_*(F)\right) = \mathrm{K}_{B'}\left(\psi_*(E), \psi_*(F)\right)$. □

[10] See [Laf02], page 15.
[11] See [Laf02], page 16.

1.4 The multiplier algebra

Let B be a Banach algebra.

Definition 1.4.1 (The multiplier algebra $\mathrm{M}(B)$). The unital Banach algebra $\mathrm{L}(\underline{B})$ is called the *multiplier algebra of* B and will be denoted by $\mathrm{M}(B)$.

One usually defines the multiplier algebra of B as the algebra of (continuous) double centralisers, and in fact, that is what we have done here as well. To see this, let $T = (T^<, T^>)$ be an element of the algebra $\mathrm{M}(B) = \mathrm{L}\left(\underline{B}\right)$. Then

1. $\forall a, b \in B:\ T^<(ab) = aT^<(b)$, i.e., $T^<$ is a right centraliser,
2. $\forall a, b \in B:\ T^>(ba) = T^>(b)a$, i.e., $T^>$ is a left centraliser,
3. $\forall a, b \in B:\ aT^>(b) = T^<(a)b$, i.e., T is a double centraliser.

Using the notation introduced in 1.1 we can rewrite the three formulae $(ab)T = a(bT)$, $T(ba) = (Tb)a$ and $a(Tb) = (aT)b$ for all $a, b \in B$. This constitutes three of the possible number of eight laws of associativity between B and $\mathrm{L}(\underline{B})$. The laws $R(ST) = (RS)T$ and $a(bc) = (ab)c$ are trivially satisfied. The way the composition of operators is defined guarantees the laws $(ST)b = S(Tb)$ and $b(ST) = (bS)T$. The only law that is left to check is $(Sb)T = S(bT)$, what can be paraphrased by $T^< \circ S^> = S^> \circ T^<$. As we will see below, this law does not hold in general, but we can give simple conditions on B under which it is true.

Lemma 1.4.2. *If B is non-degenerate,*[12] *then we have*

$$\forall S, T \in \mathrm{M}(B):\ T^< \circ S^> = S^> \circ T^<$$

or, equivalently,

$$\forall S, T \in \mathrm{M}(B), b \in B:\ (Sb)T = S(bT).$$

Proof. For all $b, c \in B$ and $S, T \in \mathrm{M}(B)$ we have

$$(T^< \circ S^>)(bc) = T^<\left(S^>(bc)\right) = T^<\left(S^>(b)c\right) = S^>(b)T^<(c) = S^>(bT^<(c)) = (S^> \circ T^<)(bc).$$

Thus $T^< \circ S^>$ equals $S^> \circ T^<$ on BB. The rest follows from linearity and continuity of $T^< \circ S^>$ and $S^> \circ T^<$ and the fact that B is non-degenerate. $\square$

That the condition that B is non-degenerate cannot simply be dropped can be seen from the following example.

Example 1.4.3. Let E be a Banach space. Equipped with the trivial product it is a Banach algebra. Every pair of $\mathbb{C}$-linear continuous maps from E to E gives an element of $\mathrm{M}(E)$. So if E is of dimension more than one, the above equality fails in general.

If B is a C*-algebra, then B is isomorphic to $\mathrm{K}_B(B)$ and B is "contained" in the multiplier algebra. If we model the multiplier algebra as $\mathrm{L}_B(B)$, then we can rephrase this as follows: The canonical homomorphism from B to $\mathrm{L}_B(B)$ is an isomorphism onto its image $\mathrm{K}_B(B)$. This is no longer true for general Banach algebras: The canonical homomorphism does not need to be injective and its image does not need to be $\mathrm{K}_B(B)$. However there are some relations between B and $\mathrm{K}_B(B)$ that we are going to state now.

[12]The lemma is also true if B has no annihilators (as defined right after 1.4.4). The current and more relevant version of the lemma has been suggested by Ralf Meyer.

Definition 1.4.4. We define a contractive homomorphism

$$\psi_B \colon B \to \mathrm{M}(B),\ b \mapsto (c \mapsto cb,\ c \mapsto bc).$$

If we view $\underline{B}$ as a B-B-bipair, then the action $B \to \mathrm{L}_B(\underline{B})$ is precisely given by ψ_B. We call the elements of the kernel of ψ_B the *annihilators* of B and say that B has *no annihilators* if ψ_B is injective. If B has a bounded approximate identity then ψ_B is isometric and hence injective. The homomorphism ψ_B is an isomorphism precisely if B is unital. The image of ψ_B is a two-sided ideal of $\mathrm{M}(B)$. More precisely we have

$$T \circ \psi_B(b) = \psi_B(T^{>}(b)) = \psi_B(Tb) \quad \text{and} \quad \psi_B(b) \circ T = \psi_B(T^{<}(b)) = \psi_B(bT)$$

for all $b \in B$ and $T \in \mathrm{M}(B)$.

Proposition 1.4.5. [13]

1. $\forall b, c \in B:\ |b\rangle\langle c| = \psi_B(bc)$.
2. $\mathrm{K}_B(\underline{B})$ *is contained in* $\overline{\psi_B(B)}$.
3. *If* B *is non-degenerate, then* $\mathrm{K}_B(\underline{B}) = \overline{\psi_B(B)}$.
4. *If* B *is non-degenerate and* ψ_B *is isometric, then* ψ_B *is an isomorphism from* B *onto* $\mathrm{K}_B(\underline{B})$.

1.5 Graded Banach pairs

1.5.1 Graded Banach spaces

Definition 1.5.1 (Graded Banach space). Let E be a Banach space. A *grading automorphism* σ_E of E is an isometric linear endomorphism of E such that $\sigma_E^2 = \mathrm{Id}_E$. A *graded Banach space* is a Banach space endowed with a grading automorphism.

Definition 1.5.2 (Homogeneous element, degree). If E is a graded Banach space with grading automorphism σ_E, then we define $E_0 := \{e \in E :\ \sigma_E(e) = e\}$ and $E_1 := \{e \in E :\ \sigma_E(e) = -e\}$. The elements of E_0 are called even, the elements of E_1 are called odd. The elements of $E_0 \cup E_1$ are called homogeneous. If $e \in E \setminus \{0\}$ is homogeneous, then we define the degree $\deg e$ of e to be $0 \in \mathbb{Z}_2$ if $e \in E_0$ and $1 \in \mathbb{Z}_2$ if $e \in E_1$. Note that $E = E_0 \oplus E_1$.

Definition 1.5.3 (Odd and even operators). Let E, F be graded Banach spaces with grading automorphisms σ_E and σ_F, respectively. On $\mathrm{L}(E, F)$ define a grading operator $\sigma_{\mathrm{L}(E,F)}$ by $T \mapsto \sigma_F \circ T \circ \sigma_E$. A linear operator $T \in \mathrm{L}(E, F)$ is then called *graded* or *even* if

$$T \circ \sigma_E = \sigma_F \circ T,$$

or, equivalently, if $T(E_0) \subseteq F_0$ and $T(E_1) \subseteq F_1$. It is called *odd* if $T \circ \sigma_E = -\sigma_F \circ T$ or, equivalently, if $T(E_0) \subseteq F_1$ and $T(E_1) \subseteq F_0$. The set of all even and of all odd elements of $\mathrm{L}(E, F)$ will be denoted by $\mathrm{L}^{\mathrm{even}}(E, F)$ and $\mathrm{L}^{\mathrm{odd}}(E, F)$, respectively.

[13] Compare the more general version Lemme 1.1.6 of [Laf02].

Definition 1.5.4 (Odd and even bilinear maps). Let E_1, E_2 and F be graded Banach spaces with grading automorphism σ_{E_1}, σ_{E_2} and σ_F, respectively. On the Banach space $\mathrm{M}(E_1, E_2; F)$ define a grading automorphism $\sigma_{\mathrm{M}(E_1,E_2;F)}$ by setting

$$\sigma_{\mathrm{M}(E_1,E_2;F)}(\mu)\;(e_1, e_2) := \sigma_F\left(\mu\left(\sigma_{E_1}(e_1), \sigma_{E_2}(e_2)\right)\right)$$

for all $\mu \in \mathrm{M}(E_1, E_2; F)$, $e_1 \in E_1$, and $e_2 \in E_2$. An element $\mu \in \mathrm{M}(E_1, E_2; F)$ is consequently called *graded* or *even* if

$$\sigma_F\left(\mu(e_2, e_2)\right) = \mu\left(\sigma_{E_1}(e_1), \sigma_{E_2}(e_2)\right)$$

for all $e_1 \in E_1$ and $e_2 \in E_2$. It is called *odd* if the same equality is true with a minus sign.

Note that $\mu \in \mathrm{M}(E_1, E_2; F)$ is graded if and only if $\mu(e_1, e_2)$ is homogeneous for all homogeneous $e_1 \in E_1$ and $e_2 \in E_2$ with

$$\deg \mu(e_1, e_2) = \deg e_1 \;+\; \deg e_2.$$

Definition 1.5.5 (The graded tensor product). Let E_1 and E_2 be graded Banach spaces with grading automorphism σ_{E_1} and σ_{E_2}, respectively. Then the graded tensor product of E_1 and E_2 is defined as the projective tensor product $E_1 \otimes E_2$ with the grading operator $\sigma_{E_1} \otimes \sigma_{E_2}$ which is also called the diagonal grading operator. It has the universal property for graded continuous bilinear maps.

Note that the graded tensor product is associative.

1.5.2 Graded Banach algebras

Definition and Lemma 1.5.6 (Graded Banach algebra). Let B be a Banach algebra with a linear grading automorphism σ_B. Then the following are equivalent:

1. σ_B is multiplicative, i.e., σ_B is a Banach algebra automorphism.
2. The product of B is even with respect to σ_B.
3. If $a, b \in B$ are homogeneous then ab is homogeneous and $\deg ab = \deg a + \deg b$.

If one (and therefore all) of these conditions is (are) satisfied, then we call σ_B a *grading automorphism* of the Banach algebra B and B a *graded Banach algebra*.

Example 1.5.7. Let B be a Banach algebra. Then the identity of B is a grading automorphism. If we endow B with this grading automorphism, then we call it *trivially graded*.

Example 1.5.8. Let E be a graded Banach space. Then $\mathrm{L}(E)$ is a graded Banach algebra.

1.5.3 Graded Banach modules

Definition 1.5.9 (Graded Banach module). Let B be a graded Banach algebra with grading automorphism σ_B. Let E be a right Banach B-module. A *grading automorphism σ_E of E* is an automorphism of E with coefficient map σ_B such that $\sigma_E^2 = \mathrm{Id}_E$. A *graded right Banach B-module* is a right Banach B-module endowed with a grading automorphism. Similarly, graded left Banach modules and graded Banach bimodules are defined.

One can characterise grading automorphisms of graded Banach modules just as we have done for grading automorphisms for Banach algebras in 1.5.6.

Example 1.5.10. Let B be a graded Banach algebra. Then B is also a graded Banach B-B-bimodule.

Definition and Lemma 1.5.11 (Odd and even operators). If E and F are graded right Banach B-modules, then $\mathrm{L}_B(E,F)$ is a graded subspace of $\mathrm{L}(E,F)$. In particular, $\mathrm{L}_B(E)$ is a graded subalgebra of $\mathrm{L}(E)$. The set of all even and of all odd elements of $\mathrm{L}_B(E,F)$ will be denoted by $\mathrm{L}_B^{\mathrm{even}}(E,F)$ and $\mathrm{L}_B^{\mathrm{odd}}(E,F)$, respectively.

Definition 1.5.12 (Graded homomorphism). Let B and B' be graded Banach algebras with grading automorphisms σ_B and $\sigma_{B'}$, respectively. Let E_B and $E'_{B'}$ be graded Banach modules with grading operators σ_E and $\sigma_{E'}$. A homomorphism $\Psi\colon E \to E'$ with coefficient map $\psi\colon B \to B'$ is called *graded* if

$$(\sigma_{E'})_{\sigma_{B'}} \circ \Psi_\psi = \Psi_\psi \circ (\sigma_E)_{\sigma_B}$$

or, equivalently, if Ψ and ψ are graded maps.

Definition 1.5.13 (Graded sum of Banach modules). Let B be a graded Banach algebra, and let E_1 and E_2 be graded Banach B-modules. On the sum $E_1 \oplus E_2$ define the grading automorphism $\sigma_{E_1 \oplus E_2} := \sigma_{E_1} \oplus \sigma_{E_2}$.

Definition 1.5.14 (Graded tensor product of Banach modules). Let A, B and C be graded Banach algebras, and let ${}_AE_B$ and ${}_BF_C$ be graded Banach bimodules. On the balanced tensor product $E \otimes_B F$ define the grading automorphism $\sigma_{E\otimes_B F} := \sigma_E \otimes \sigma_F$. With this grading automorphism, the balanced tensor product has the universal property for continuous graded balanced bilinear maps and is called the *graded balanced tensor product* of E and F.

Note that the automorphism $\sigma_E \otimes \sigma_F$ is the tensor product of homomorphisms with coefficient maps defined in 1.1.11. The graded balanced tensor product is compatible with the graded sum.

Definition 1.5.15 (The graded pushout of Banach modules). Let B and B' be graded Banach algebras, and let E be a right graded Banach B-module. Let $\psi\colon B \to B'$ be a graded homomorphism of Banach algebras. Extend the grading automorphism $\sigma_{B'}$ on B' to a grading automorphism of the unitalisation $\widetilde{B'}$ by letting the unit 1 be even. Define the *graded pushout* $\psi_*(E)$ of E to be the right graded Banach B'-module $E \otimes_{\widetilde{\psi}} \widetilde{B'}$.

The map ψ_* defines a functor from the category of graded Banach B-modules to the category of graded Banach B'-modules which is linear, contractive and even on the morphism sets. The functoriality properties of the pushout listed in Proposition 1.1.15 carry over to the graded case. Also, the graded pushout is compatible with the sum of Banach B-modules.

1.5.4 Graded Banach pairs

Definition 1.5.16 (Graded Banach pair). Let B be a graded Banach algebra with grading automorphism σ_B. Let $E = (E^<, E^>)$ be a Banach B-pair. A *grading automorphism* of E is a concurrent automorphism $\sigma_E = (\sigma_E^<, \sigma_E^>)$ with coefficient map σ_B such that $\sigma_E^2 = \mathrm{Id}_E$. A *graded Banach B-pair* is a Banach B-pair endowed with a grading automorphism. Similarly one defines graded Banach A-B-pairs if A is another graded Banach algebra.

Note that in particular the left and right parts of σ_E are grading automorphisms of $E^<$ and $E^>$, respectively. We hence write $\sigma_E^>$ or $\sigma_{E^>}$, interchangeably.

Example 1.5.17. Let B be a graded Banach algebra. Then $\underline{B}$ is a graded Banach B-pair (and also a graded Banach B-B-pair).

Definition 1.5.18 (Odd and even operators). Let E and F be graded Banach B-pairs with grading automorphisms σ_E and σ_F. Then we define a grading on the Banach space $\mathrm{L}_B(E,F)$ by setting

$$\sigma_{\mathrm{L}_B(E,F)}(T) := \left(\sigma_E^< \circ T^< \circ \sigma_F^<,\ \sigma_F^> \circ T^> \circ \sigma_E^>\right) = \left(\sigma_{B\,\mathrm{L}(F^<,E^<)}(T^<),\ \sigma_{\mathrm{L}_B(E^>,F^>)}(T^>)\right)$$

for all $T \in \mathrm{L}_B(E,F)$. The set of all even and of all odd elements of $\mathrm{L}_B(E,F)$ will be denoted by $\mathrm{L}_B^{\text{even}}(E,F)$ and $\mathrm{L}_B^{\text{odd}}(E,F)$, respectively.

Note that composition of operators is an even bilinear map. This also means that $\mathrm{L}_B(E)$ is a graded Banach algebra for every graded Banach B-pair E.

Lemma 1.5.19. *Let E and F be graded Banach B-pairs. Then $\mathrm{K}_B(E,F)$ is a graded subspace of $\mathrm{L}_B(E,F)$. The bilinear map $(f^>, e^<) \mapsto \left|f^>\right\rangle\left\langle e^<\right|$ from $F^> \times E^<$ to $\mathrm{K}_B(E,F)$ is even. In particular, $\mathrm{K}_B(E)$ is a graded Banach algebra.*

Note that, building on the respective notions for Banach modules, there are obvious definitions of graded concurrent homomorphism between graded Banach pairs, of the graded sum, the graded tensor product and the graded pushout of graded Banach pairs; these notions are pairwise compatible.

1.6 Group actions

Let G be a locally compact Hausdorff group.

1.6.1 G-Banach spaces

Definition 1.6.1 (G-Banach space). Let E be a Banach space. We call E a *G-Banach space* if it is equipped with a strongly continuous G-action $\eta\colon G \to \mathrm{L}(E)$ by isometries.

We will usually write se instead of $\eta_s(e)$ for all $s \in G$ and $e \in E$.

Definition 1.6.2. Let E and F be G-Banach spaces. Then we define an action by isometries (which is not necessarily continuous) on $\mathrm{L}(E,F)$ by setting

$$(sT)(e) := s(T(s^{-1}e))$$

for all $s \in G$, $e \in E$, and $T \in \mathrm{L}(E,F)$.

Definition 1.6.3 (G-equivariant linear operator). Let E and F be G-Banach spaces. An element T of $\mathrm{L}(E,F)$ is called *G-equivariant* if

$$s(T(e)) = T(s(e))$$

for all $s \in G$ and $e \in E$, i.e., if T is invariant under the G-action on $\mathrm{L}(E,F)$.

Definition 1.6.4 (G-equivariant bilinear maps). Let E_1, E_2 and F be G-Banach spaces, and let $\mu\colon E_1 \times E_2 \to F$ be in $\mathrm{M}(E_1,E_2;F)$. Then μ is called *G-equivariant* if

$$\mu(se_1,\ se_2) = s\mu(e_1,e_2)$$

for all $e_1 \in E_1$, $e_2 \in E_2$, and $s \in G$.

Definition 1.6.5 (The G-tensor product). Let E_1 and E_2 be G-Banach spaces. Then $E_1 \otimes E_2$ is a G-Banach space with the action given by

$$s(e_1 \otimes e_2) := (se_1) \otimes (se_2)$$

for all $s \in G$ and $e_1 \in E_1$, $e_2 \in E_2$. The tensor product map from $E_1 \times E_2$ to $E_1 \otimes E_2$ is then G-equivariant by definition and if F is a G-Banach space and $\mu \in \mathrm{M}(E_1, E_2; F)$ then μ is G-equivariant if and only if $\hat{\mu}\colon E_1 \otimes E_2 \to F$ is G-equivariant.

1.6.2 G-Banach algebras and G-Banach modules

Definition 1.6.6 (G-Banach algebra). An action of G on a Banach algebra B is a strongly continuous homomorphism of G into $\mathrm{Aut}(B)$. A Banach algebra endowed with an action of G is called a G-Banach algebra.

Definition 1.6.7 (G-equivariant homomorphism of Banach algebras). Let B and B' be G-Banach algebras. A homomorphism of Banach algebras $\psi\colon B \to B'$ is called G-equivariant if $\psi(sb) = s(\psi(b))$ for all $b \in B$ and $s \in G$.

Definition 1.6.8 (G-Banach module). Let B be a G-Banach algebra. Then a *G-Banach B-module* is a G-Banach space which is at the same time a Banach B-module such that the module action of B on E is G-equivariant.

Note that this can also be expressed as follows: If β is the action of G on B and η is a strongly continuous action of G on the Banach B-module E, then E is a G-Banach B-module if and only if η_s is a homomorphism with coefficient map β_s from E_B onto itself.

Lemma 1.6.9. *Let B be a G-Banach algebra, and let E_B and F_B be G-Banach B-modules. Then the set $\mathrm{L}_B(E, F)$ of B-linear operators is a G-invariant subspace of $\mathrm{L}(E, F)$. So G acts on $\mathrm{L}_B(E, F)$. The composition of B-linear operators is G-equivariant.*

Definition 1.6.10 (G-equivariant homomorphism of Banach modules). Let B and B' be G-Banach algebras, and let E_B and $E'_{B'}$ be right G-Banach modules. A homomorphism $\Psi_\psi\colon E_B \to E'_{B'}$ (with coefficient map ψ) is called *G-equivariant* if ψ and Ψ are both G-equivariant maps.

Definition 1.6.11 (The equivariant balanced tensor product of G-Banach modules). Let A, B and C be G-Banach algebras and let ${}_AE_B$ and ${}_BF_C$ be G-Banach bimodules. Then we define a G-action on the balanced tensor product $E \otimes_B F$ by setting $s(e \otimes f) := (se) \otimes (sf)$ for all $s \in G$, $e \in E$ and $f \in F$. This is well-defined by 1.1.11 and it is easy to see that this defines a strongly continuous action of G on $E \otimes_B F$. With this action, $E \otimes_B F$ has the universal property for continuous G-equivariant balanced bilinear maps and will be called the *G-equivariant balanced tensor product* of E and F.

Definition 1.6.12 (The equivariant pushout of G-Banach modules). Let B and B' be G-Banach algebras and let E be a right G-Banach B-module. Let $\psi\colon B \to B'$ be an equivariant homomorphism of G-Banach algebras. Extend the action of G on B' to an action on the unitalisation $\widetilde{B'}$ by letting G act trivially on the unit 1. Define the *equivariant pushout* $\psi_*(E)$ of E to be the right G-Banach B'-module $E \otimes_{\tilde{\psi}} \widetilde{B'}$.

1.6.3 G-Banach pairs

Definition 1.6.13 (G-Banach B-pair). Let B be a G-Banach algebra. A *G-Banach B-pair* is a Banach B-pair $(E^<, E^>)$ such that $E^<$ and $E^>$ are G-Banach B-modules and the bracket is G-equivariant.

Similarly, G-Banach A-B-pairs are defined if A and B are G-Banach algebras.

Definition 1.6.14 (The action on linear operators). Let E and F be G-Banach B-pairs with action η^E and η^F, respectively. Then we define an action of G on $\mathrm{L}_B(E,F)$ by

$$sT := (sT^<, sT^>) = \left(\eta_s^{E^<} \circ T^< \circ \eta_{s^{-1}}^{F^<},\ \eta_s^{F^>} \circ T^> \circ \eta_{s^{-1}}^{E^>}\right)$$

for all $T \in \mathrm{L}_B(E,F)$ and all $s \in G$ (this is an action by isometries, but it does not have to be continuous in any interesting sense). Composition of operators is equivariant.

Definition 1.6.15 (G-equivariant concurrent homomorphism). Let B and B' be G-Banach algebras and let E_B and $E'_{B'}$ be G-Banach pairs. A concurrent homomorphism Ψ_ψ from E_B to $E'_{B'}$ is called G-equivariant if ψ , the left part $\Psi^<\colon E^< \to E'^<$ and the right part $\Psi^>\colon E^> \to E'^>$ are G-equivariant. A similar definition can be made for G-Banach pairs that carry additional left actions of G-Banach algebras.

Proposition 1.6.16. *Let E and F be G-Banach B-pairs. Then $\mathrm{K}_B(E,F)$ is a G-invariant subspace of $\mathrm{L}_B(E,F)$. The bilinear map $(f^>, e^<) \mapsto |f^>\rangle\langle e^<|$ from $F^> \times E^<$ to $\mathrm{K}_B(E,F)$ is equivariant.*

Proposition 1.6.17. *The action of G on $\mathrm{K}_B(E,F)$ is strongly continuous and thus $\mathrm{K}_B(E)$ is a G-Banach algebra.*

Proof. Let $f^> \in F^>$ and $e^< \in E^<$. Now the map $s \mapsto (sf^>, se^<)$ is continuous and so is the map $(\tilde{f}^>, \tilde{e}^<) \mapsto |\tilde{f}^>\rangle\langle\tilde{e}^<|$. Now $s|f^>\rangle\langle e^<| = |sf^>\rangle\langle se^<|$ for all $s \in G$, so $s \mapsto s|f^>\rangle\langle e^<|$ is continuous as a composition of continuous maps. So for every finite-rank operator T, the map $s \mapsto sT$ is continuous. But the space of all finite-rank operators is dense in $\mathrm{K}_B(E,F)$ and the action is by isometries, so the action is strongly continuous. □

Definition 1.6.18 (The equivariant sum of G-Banach pairs). Let B be a G-Banach algebra and let E_1 and E_2 be G-Banach B-pairs. Then the obvious action of G on $E_1 \oplus E_2$ makes it a G-Banach B-pair.

Definition 1.6.19 (The equivariant balanced tensor of G-Banach pairs). Let A, B and C be G-Banach algebras and let ${}_AE_B$ and ${}_BF_C$ be G-Banach pairs. Then we define the *G-equivariant balanced tensor product* of E and F to be the Banach A-C-pair $E \otimes_B F = (F^< \otimes_B E^<,\ E^> \otimes_B F^>)$ taking the G-equivariant tensor product of Banach modules on both sides.

The definition of the *G-equivariant pushout of G-Banach pairs* is just as simple minded. The functoriality properties of the pushout given in Proposition 1.1.15 carry over to the equivariant case. The equivariant tensor product and the equivariant pushout are both compatible with the sum of G-Banach pairs.

1.6.4 Group actions and gradings

Definition 1.6.20 (Graded G-Banach space). A graded G-Banach space E is a graded Banach space E together with a strongly continuous action of G on E which commutes with the grading automorphism.

Remark 1.6.21. Let E be a G-Banach space and let σ_E be a grading operator on the Banach space E. Then the action of G and σ_E commute if and only if they give rise to a strongly continuous action of $G \times \mathbb{Z}_2$ on E. Hence all the notions that we have for G-actions carry over to G-actions on graded spaces, graded algebras, etc.

Let us elaborate on two highlights:

Proposition 1.6.22. *If E is a graded G-Banach space, then the subspaces of odd and even elements are invariant under the action of G. If F is another graded G-Banach space, then the spaces of odd and even operators from E to F are invariant under the action of G on $\mathrm{L}(E, F)$. Similar things are true for graded G-Banach modules and graded G-Banach pairs.*

Proposition 1.6.23. *Let E be a graded G-Banach B-pair. Then $\mathrm{K}_B(E)$ is a graded G-Banach algebra.*

1.7 Example: Trivial bundles over X

Let X be a locally compact Hausdorff space.

Definition 1.7.1 (The Banach space EX). Let E be a Banach space. Then we define EX as the Banach space $\mathcal{C}_0(X, E)$ of continuous functions from X to E that vanish at infinity. For all $x \in X$, we define $\mathrm{ev}_x^E \colon EX \to E,\ \xi \mapsto \xi(x)$. It is a contractive linear map.

Definition 1.7.2 (The Banach algebra BX). Let B be a Banach algebra. Then $BX = \mathcal{C}_0(X, B)$ is a Banach algebra with the pointwise product. For all $x \in X$, the map $\mathrm{ev}_x^B \colon BX \to B$ is a homomorphism of Banach algebras.

Lemma 1.7.3. *If B is a non-degenerate Banach algebra, then BX is non-degenerate as well.*

Proof. Let Γ be the subspace of BX spanned by all products $\beta\beta'$ with $\beta, \beta' \in BX$. Then Γ is closed under the multiplication with functions in $\mathcal{C}_c(X)$. Moreover, if $x \in X$, then $\{\gamma(x) :\ \gamma \in \Gamma\}$ is dense in B. A short argument using partitions of unity shows that this suffices for Γ to be dense in BX. □

Definition 1.7.4 (The Banach BX-module EX). Let B be a Banach algebra and let E be a Banach B-module. Then $EX = \mathcal{C}_0(X, E)$ is a Banach BX-module. For all $x \in X$, the map $\mathrm{ev}_x^E \colon EX \to E$ is an equivariant homomorphism with coefficient map ev_x^B.

If A is another Banach algebra and E is a Banach A-B-bimodule, then EX is a Banach AX-BX-bimodule.

As above, one proves:

Lemma 1.7.5. *Let B be a Banach algebra and E a non-degenerate right Banach B-module. Then EX is a non-degenerate Banach BX-module.*

Proposition 1.7.6. *Let A and B be Banach algebras and let ${}_A E_B$ be a B-non-degenerate Banach A-B-bimodule. Then the Banach AX-BX-bimodule ${}_{AX} EX_{BX}$ has the property*

$$\mathrm{ev}_{x,*}^{B}(EX) \cong \mathrm{ev}_{x}^{A,*}(E)$$

as AX-B-bimodules for every $x \in X$.

Example 1.7.7. Let A, B be Banach algebras and let E be a Banach A-B-bimodule. Then $E[0,1]$ is a Banach $A[0,1]$-$B[0,1]$-bimodule. For all $t \in [0,1]$, we have $\mathrm{ev}_{t,*}^{B}(E[0,1]) \cong \mathrm{ev}_{t}^{A,*}(E)$.

Definition 1.7.8 (The Banach BX-pair EX). Let A, B be Banach algebras and let E be a Banach A-B-pair. Then $EX := (E^{<}X,\ E^{>}X)$ is a Banach AX-BX-pair when equipped with the pointwise bracket.

Proposition 1.7.9. *Let A and B be Banach algebras and let ${}_A E_B$ be a B-non-degenerate Banach A-B-pair. Then the Banach AX-BX-pair ${}_{AX} EX_{BX}$ has the property*

$$\mathrm{ev}_{x,*}^{B}(EX) \cong \mathrm{ev}_{x}^{A,*}(E)$$

as AX-B-pairs for every $x \in X$.

Proposition 1.7.10. *Let B be a Banach algebra and let E and F be Banach B-pairs. Then*

$$\mathrm{K}_B(E,F)X \cong \mathrm{K}_{BX}(EX, FX).$$

Proof. First we define an isometric linear map from $\mathrm{K}_B(E,F)X$ to $\mathrm{K}_{BX}(EX,FX)$. We do this by showing that the isometric homomorphism of Banach algebras

$$\begin{array}{cccc} \Psi & : & \mathcal{C}_b(X, \mathrm{L}_B(E,F)) & \to & \mathrm{L}_{BX}(EX,FX), \\ & & T & \mapsto & \big(\eta^{<} \mapsto (x \mapsto T(x)^{<}\eta^{<}(x)), \quad \xi^{>} \mapsto (x \mapsto T(x)^{>}\xi^{>}(x))\big) \end{array}$$

maps $\mathcal{C}_0(X, \mathrm{K}_B(E,F))$ to $\mathrm{K}_{BX}(EX,FX)$. Since Ψ is isometric, it suffices to show that Ψ maps a dense subset of $\mathcal{C}_0(X, \mathrm{K}_B(E,F))$ into $\mathrm{K}_{BX}(EX,FX)$. By the use of a partition of unity we can show that for a subset S of $\mathcal{C}_0(X, \mathrm{K}_B(E,F))$ to be dense it is enough to be pointwise dense, i.e., it suffices that, for every $x \in X$, the set $\{s(x) : s \in S\}$ is dense in $\mathrm{K}_B(E,F)$. Take S to be the span of all functions of the form $x \mapsto \chi_1(x)\chi_2(x)\,|f^{>}\rangle\langle e^{<}|$ where χ_1, χ_2 run through $\mathcal{C}_c(X)$, $e^{<}$ runs through $E^{<}$ and $f^{>}$ runs through $F^{>}$. Now

$$\Psi\left(x \mapsto \chi_1(x)\chi_2(x)\big|f^{>}\big\rangle\big\langle e^{<}\big|\right) = \big|x \mapsto \chi_1(x)f^{>}\big\rangle\big\langle x \mapsto \chi_2(x)e^{<}\big| \quad \in \quad \mathrm{K}_{BX}(EX,FX).$$

So Ψ maps $\mathrm{K}_B(E,F)X$ isometrically into $\mathrm{K}_{BX}(EX,FX)$. To show that the image is dense let $\xi^{<} \in E^{<}X$ and $\eta^{>} \in F^{>}X$. Then $x \mapsto \big|\eta^{>}(x)\big\rangle\big\langle\xi^{<}(x)\big|$ is in $\mathrm{K}_B(E,F)X$ and

$$\Psi\left(x \mapsto \big|\eta^{>}(x)\big\rangle\big\langle\xi^{<}(x)\big|\right) = \big|\eta^{>}\big\rangle\big\langle\xi^{<}\big|.$$

So all finite rank operators are in the (closed) image of Ψ, so $\Psi(\mathrm{K}_B(E,F)X) = \mathrm{K}_{BX}(EX,FX)$. ∎

Remark 1.7.11 (Gradings and group actions). Let G be a locally compact Hausdorff group. If the Banach spaces, Banach algebras, etc. in the preceding definitions are all graded or G-equivariant, then all the constructions are compatible with these structures.

To be more precise, let E be a graded Banach space with grading operator σ_E; then EX is graded with grading operator $\xi \mapsto (x \mapsto \sigma_E(\xi(x)))$. Similarly, if E is a G-Banach space, then a standard argument shows that the pointwise action of G on EX is strongly continuous, so EX is a G-Banach space.

1.8 Equivariant KK-theory

The equivariant KK-theory $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$ was introduced in [Laf02], Définition 1.2.2. The exposition there is very clear but also somewhat brief; we try to follow a very systematic and elaborate approach here to be able to easily refer to this section later on when we generalise the definitions in the subsequent chapters.

Let G be a locally compact Hausdorff group. Most of the following definitions and propositions concerning the G-equivariant KK-theory $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$ make sense for graded G-Banach algebras A and B. However, we restrict our attention to the case that A and B are trivially graded. Nevertheless, we formulate most definitions and statements in a way that makes it easy to construct the suitable generalisations to the graded case.

1.8.1 $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}$-cycles

Definition 1.8.1 ($\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}$-cycle). [14] Let A and B be G-Banach algebras. A $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}$-*cycle* from A to B is a pair (E, T) such that E is a non-degenerate graded G-Banach A-B-pair (i.e., E is a non-degenerate graded G-Banach B-pair together with an even G-equivariant homomorphism $\pi_A \colon A \to \mathrm{L}_B(E)$) and T is an odd element of $\mathrm{L}_B(E)$ such that[15]

$$[\pi_A(a), T], \quad \pi_A(a)\,(\mathrm{Id} - T^2) \quad \in \quad \mathrm{K}_B(E)$$

and

$$s \mapsto \pi_A(a)\,(T - sT) \quad \in \quad \mathcal{C}\,(G, \mathrm{K}_B(E))$$

for all $a \in A$. We write $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$ for the class of all $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}$-cycles from A to B. If G is trivial, we just write $\mathbb{E}^{\mathrm{ban}}(A, B)$.

Definition 1.8.2 (The sum of $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}$-cycles). Let A and B be G-Banach algebras. If (E_1, T_1) and (E_2, T_2) are elements of $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$, then we define $(E_1, T_1) \oplus (E_2, T_2) := (E_1 \oplus E_2, T_1 \oplus T_2)$. It is an element of $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$.

Definition 1.8.3 (The inverse of a $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}$-cycle). Let A and B be G-Banach algebras. If (E, T) is in $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$, then we define $-(E, T)$ to be (E, T), but equipped with the opposite grading. It is an element of $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$.

Definition 1.8.4 (The pullback of a $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}$-cycle). Let A, B and C be G-Banach algebras. Let $(E, T) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(B, C)$ and $\varphi \colon A \to B$ be a G-equivariant homomorphism. Then we define $\varphi^*(E, T)$ to be just the cycle (E, T) with the exception that the left B-action π_B on E is replaced by the A-action $\pi_B \circ \varphi$.

Definition 1.8.5 (The pushout of a $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}$-cycle). Let A, B and C be G-Banach algebras. Let (E, T) be an element of $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$ and let $\theta \colon B \to C$ be an equivariant homomorphism from B to C. Then the pushout $\theta_*(E, T)$ of (E, T) along θ is defined as $(\theta_*(E), T \otimes 1)$ where $\theta_*(E)$ is the pair $\left(\widetilde{C} \otimes_{\widetilde{B}} E^<,\ E^> \otimes_{\widetilde{B}} \widetilde{C}\right)$ with the diagonal grading operator and the diagonal G-action and the A-action given by $a \mapsto a \otimes 1$.

[14] Alternative names could perhaps be “generalised Kasparov cycles” or “Kasparov-Lafforgue cycles”.

[15] Later on, we will often identify $\pi_A(a)$ with a; for instance, we will write $[a, T]$ instead of $[\pi_A(a), T]$.

Proof. We have to check that $\theta_*(E,T)$ is indeed in $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A,C)$. Clearly, $\theta_*(E)$ is a graded G-A-C-bimodule and $T\otimes 1$ is an odd element of $\mathrm{L}_C(\theta_*(E))$. Let a be a homogeneous element of A. Then

$$[\pi_A(a)\otimes 1, T\otimes 1] = (\pi_A(a)\otimes 1)\,(T\otimes 1) - (-1)^{\deg a}(T\otimes 1)\,(\pi_A(a)\otimes 1) = [\pi_A(a),T]\otimes 1.$$

From Proposition 1.3.13 it follows that this is compact. Similarly, $(\pi_A(a)\otimes 1)\left(\mathrm{Id}-(T\otimes 1)^2\right)\in \mathrm{K}_C\left(\theta_*(E)\right)$. For all $s\in S$, we have, using some obvious abbreviations:

$$\begin{aligned}(\pi_A(a)\otimes 1)\ \ (T\otimes 1 - s(T\otimes 1)) &= (\pi_A(a)\otimes 1)\ \ (T\otimes 1 - T\otimes s1)\\ &= (\pi_A(a)(T-sT))\ \otimes\ 1 \quad \in \quad \mathrm{K}_C\left(\theta_*(E)\right)\end{aligned}$$

because $s1=1$. Now the map $S\mapsto S\otimes 1$ is a linear and contractive map from $\mathrm{L}_B(E)$ to $\mathrm{L}_C(\theta_*(E))$, so the map $s\mapsto(\pi_A(a)\otimes 1)\ \ (T\otimes 1 - s(T\otimes 1))$ is continuous as the composition of continuous maps. □

1.8.2 Morphisms between $\mathrm{KK}^{\mathrm{ban}}$-cycles

Let A, A' and B, B' be G-Banach algebras. Let $\varphi\colon A\to A'$ and $\psi\colon B\to B'$ be G-equivariant morphisms of Banach algebras.

Definition 1.8.6 (Morphism of $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}$-cycles). Let (E,T) and (E',T') be elements of $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$ and $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A',B')$, respectively. Then a *morphism* from (E,T) to (E',T') with coefficient maps φ and ψ is a homomorphism $\Phi=(\Phi^<,\Phi^>)$ of graded G-Banach A-B-pairs from ${}_AE_B$ to ${}_{A'}E'_{B'}$ with coefficient maps φ and ψ which intertwines T and T', i.e.,

$$T'^<\circ\Phi^< = \Phi^<\circ T^< \quad\text{and}\quad T'^>\circ\Phi^> = \Phi^>\circ T^>.$$

The class $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$ together with the morphisms of cycles (with Id_A and Id_B as coefficient maps) forms a category. This gives us a notion of *isomorphic* $\mathrm{KK}^{\mathrm{ban}}$*-cycles* in $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$.

Proposition 1.8.7 (Associativity of the sum of cycles). *If (E_1,T_1), (E_2,T_2), and (E_3,T_3) are in $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$, then there is a natural isomorphism*

$$(E_1,T_1)\oplus((E_2,T_2)\oplus(E_3,T_3))\ \cong\ ((E_1,T_1)\oplus(E_2,T_2))\oplus(E_3,T_3).$$

Proposition 1.8.8 (Functoriality of the pushout). *Let C and D be G-Banach algebras and let $\varphi\colon B\to C$ and $\psi\colon C\to D$ be G-equivariant homomorphisms. Let $(E,T)\in\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$. Then there is a natural isomorphism*

$$(\psi\circ\varphi)_*(E,T)\ \cong\ \psi_*\left(\varphi_*(E,T)\right)\quad\in\ \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A,D).$$

Moreover, $\mathrm{Id}_{B,*}(E,T)\cong(E,T)\in\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$*, naturally.*

Note that the pullback and the pushout of $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}$-cycles are compatible with the addition of cycles (up to isomorphism).

1.8.3 Homotopies

Let A, B be graded G-Banach algebras.

Definition 1.8.9 (Homotopies). A *homotopy* between cycles (E_0, T_0) and (E_1, T_1) in $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$ is a cycle (E, T) in $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B[0,1])$ such that $\mathrm{ev}_{0,*}(E, T)$ is isomorphic to (E_0, T_0) and $\mathrm{ev}_{1,*}(E, T)$ is isomorphic to (E_1, T_1). If such a homotopy exists, then (E_0, T_0) and (E_1, T_1) are called *homotopic*. We will denote by $\sim$ the equivalence relation on $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B[0,1])$ generated by homotopy.

Remark 1.8.10. It is easy to see that homotopy is reflexive and symmetric. In the case of C^*-algebras and ordinary Kasparov cycles the homotopy relation is also transitive, but I was not able to show this in the Banach algebra situation, and the article [Laf02] does not elaborate this point. Indeed, there is evidence that homotopy is not transitive in general (see the discussion in Section 4.8.1), but the equivalence relation generated by homotopy is good enough to make all the definitions work.

Definition 1.8.11 ($\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$). The class of all $\sim$-equivalence classes in $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$ is denoted by $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$. The addition of cycles induces a law of composition on $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$ making it an abelian group.[16]

That $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$ is an abelian group was proved in [Laf02], Lemme 1.2.5; the following result is Proposition 1.2.6 of the same article.

Proposition 1.8.12 (Functoriality of $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$). *Let A' and B' be G-Banach algebras. Let $\varphi\colon A' \to A$ and $\psi\colon B \to B'$ be equivariant homomorphisms. If $(E, T) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$, then the homotopy class of $\psi_*(E, T)$ and $\varphi^*(E, T)$ depends only on the homotopy class of (E, T). We hence get homomorphisms*

$$\varphi^*(\cdot)\colon\ \mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B) \to \mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A', B) \quad \textit{and} \quad \psi_*(\cdot)\colon\ \mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B) \to \mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B').$$

Note that $\varphi^*(\cdot)$ and $\psi_*(\cdot)$ commute.

1.8.4 Basic properties of $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$

In [Laf02] it is shown[17] that $\mathrm{KK}^{\mathrm{ban}}(\mathbb{C}, B) \cong \mathrm{K}_0(B)$ for all non-degenerate Banach algebras B, and an action of $\mathrm{KK}^{\mathrm{ban}}$ on the K-theory is constructed (which could be interpreted as a product $\mathrm{KK}^{\mathrm{ban}}(\mathbb{C}, A) \times \mathrm{KK}^{\mathrm{ban}}(A, B) \to \mathrm{KK}^{\mathrm{ban}}(\mathbb{C}, B)$ with B non-degenerate).

In the same article, Lafforgue introduces the notion of an "unconditional completion" of $\mathcal{C}_c(G)$, usually called $\mathcal{A}(G)$: It is a completion for a so-called unconditional norm on $\mathcal{C}_c(G)$, i.e., a norm which makes $\mathcal{C}_c(G)$ a normed algebra and satisfies $\|f_1\| \leq \|f_2\|$ for all $f_1, f_2 \in \mathcal{C}_c(G)$ with $|f_1(g)| \leq |f_2(g)|$ for all $g \in G$. A main example is $\mathrm{L}^1(G)$. We are going to define unconditional completions in the context of groupoids in Chapter 5 and refer to [Laf02] for the construction of the "crossed product" $\mathcal{A}(G, B)$, where B is a G-Banach algebra, and the descent homomorphism $j_{\mathcal{A}}\colon\ \mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B) \to \mathrm{KK}^{\mathrm{ban}}\left(\mathcal{A}(G, A),\ \mathcal{A}(G, B)\right)$ in the group case.

[16] At least if we restrict the cardinality of a dense subset of the involved Banach modules by some fixed cardinality to obtain a set $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$ rather than just a class.

[17] See Théorème 1.2.8 and Proposition 1.2.9

1.9 A sufficient condition for homotopy

The sufficient condition for homotopy of $\mathrm{KK}^{\mathrm{ban}}_{\mathrm{G}}$-cycles that we put forward in this section is already present in a rudimentary form in[18] [Laf02] and more explicitly in the unpublished note [Laf04]. Here, we state and prove it in full generality and give some abstract background which might perhaps lead to further developments and is for now just reflected in some fancy notation. The condition itself is fundamental to large parts of this work because it is the main technical tool to construct homotopies.

Theorem 1.9.1 (Sufficient condition for homotopy of $\mathrm{KK}^{\mathrm{ban}}_{\mathrm{G}}$-cycles). *Let G be a locally compact Hausdorff group and let A and B be G-Banach algebras. Let $(E,T),(E',T')$ be in $\mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(A,B)$. If there is a morphism Φ from (E,T) to (E',T') (with coefficient maps Id_A and Id_B) such that*

1. $\forall a \in A: \ [a,(T,T')] = ([a,T],\ [a,T']) \in \mathrm{K}(\Phi,\Phi)\,,$
2. $\forall a \in A: \ a((T,T')^2-1) = \Big(a(T^2-1),\ a(T'^2-1)\Big) \in \mathrm{K}(\Phi,\Phi)\,,$
3. $\forall a \in A\ \forall g \in G: \ a\,(g(T,T')-(T,T')) = (a(gT-T),\ a(gT'-T')) \in \mathrm{K}(\Phi,\Phi)\,,$

then $(E,T) \sim (E',T')$; here $\mathrm{K}(\Phi,\Phi)$ denotes the set of all pairs of operators $(S,S') \in \mathrm{L}(E)\times\mathrm{L}(E')$ such that

$$\forall \varepsilon > 0\ \exists n \in \mathbb{N}\ \exists e_1^<,\ldots,e_n^< \in E^<, e_1^>,\ldots,e_n^> \in E^> :$$
$$\left\| S - \sum_{i=1}^n \left|e_i^>\right\rangle\left\langle e_i^<\right| \right\| \leq \varepsilon \quad \text{and} \quad \left\| S' - \sum_{i=1}^n \left|\Phi^>(e_i^>)\right\rangle\left\langle \Phi^<(e_i^<)\right| \right\| \leq \varepsilon.$$

Moreover, if $T=0$ and $T'=0$, then the homotopy can be chosen to have trivial operator as well.

1.9.1 Some useful categories

Definition 1.9.2 (The category $\mathbf{Hom}(\mathbf{BanSp})$). The objects of the category $\mathbf{Hom}(\mathbf{BanSp})$ are the contractive linear maps $\rho\colon E \to E'$ between Banach spaces. If $\rho\colon E \to E'$ and $\sigma\colon F \to F'$ are such maps, then a morphism from ρ to σ is a pair $(T,T') \in \mathrm{L}(E,F)\times\mathrm{L}(E',F')$ satisfying $\sigma\circ T = T'\circ\rho$, i.e., the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\rho"] \arrow[d, "T"'] & E' \arrow[d, "T'"] \\ F \arrow[r, "\sigma"] & F' \end{tikzcd}$$

The set of all morphisms from ρ to σ will be denoted by $\mathrm{L}(\rho,\sigma)$; it actually has a canonical Banach space structure. The composition of morphisms is defined componentwise.

Definition 1.9.3 (The category of homomorphisms of Banach modules). Let $\psi\colon B \to B'$ be a homomorphism of Banach algebras. Then the objects of the category $\mathbf{Mod}_\psi(\mathbf{Hom}(\mathbf{BanSp}))$ are the homomorphisms $\Phi_\psi\colon E_B \to E'_{B'}$ of Banach modules (with coefficient map ψ). If $\Phi_\psi\colon E_B \to E'_{B'}$ and $\Psi_\psi\colon F_B \to F'_{B'}$ are two such homomorphisms, then a morphism from Φ to Ψ is a pair $(T,T') \in \mathrm{L}_B(E,F)\times\mathrm{L}_{B'}(E',F')$ satisfying

$$\Psi\circ T = T'\circ\Phi.$$

The morphism set will be denoted by $\mathrm{L}_\psi(\Phi,\Psi)$, being a Banach space in a canonical way. The composition is defined componentwise.

[18] For example, there is an argument using mapping cylinders on page 24.

Definition 1.9.4 (The category of homomorphisms of Banach pairs). Let $\psi\colon B \to B'$ be a homomorphism of Banach algebras. Then the objects of the category $\mathbf{Pair}_\psi\,(\mathbf{Hom}(\mathbf{BanSp}))$ are the homomorphisms $\Phi_\psi\colon E_B \to E'_{B'}$ of Banach pairs (with coefficient map ψ). If $\Phi_\psi\colon E_B \to E'_{B'}$ and $\Psi_\psi\colon F_B \to F'_{B'}$ are two such homomorphisms, then a morphism from Φ to Ψ is a pair $(T, T') \in \mathrm{L}_B(E, F) \times \mathrm{L}_{B'}(E', F')$ satisfying

$$\Psi^> \circ T^> = T'^> \circ \Phi^> \quad \text{and} \quad T'^< \circ \Psi^< = \Phi^< \circ T^<.$$

This means that the following diagrams are commutative:

$$\begin{tikzcd}
E^> \arrow[r, "\Phi^>"] \arrow[d, "T^>"'] & E'^> \arrow[d, "T'^>"] \\
F^> \arrow[r, "\Psi^>"] & F'^>
\end{tikzcd}
\qquad
\begin{tikzcd}
E^< \arrow[r, "\Phi^<"] & E'^< \\
F^< \arrow[r, "\Psi^<"] \arrow[u, "T^<"] & F'^< \arrow[u, "T'^<"']
\end{tikzcd}$$

The morphism set will be denoted by $\mathrm{L}_\psi\,(\Phi, \Psi)$, which is a Banach space. The composition is defined componentwise.

Remark 1.9.5 (A categorial interpretation). There is a good and systematic reason for the notation chosen in the preceding definitions:

To arrive at the notion of a Banach pair, one starts with the category of Banach spaces. It has an associative tensor product (the projective tensor product in this case) which allows us to build from it the category of Banach algebras. As a next step, one considers the Banach modules, and from them one constructs the Banach pairs. The main ingredient is the category of Banach spaces and its tensor product. The underlying categorial concept is the notion of a "monoidal Banach category", a monoidal category enriched over the category of Banach spaces.

If we take the category $\mathbf{Hom}(\mathbf{BanSp})$ as a starting point and if we imitate the construction of the category of Banach algebras from the category **BanSp** of Banach spaces, then the analogous category of "Banach algebras" constructed from $\mathbf{Hom}(\mathbf{BanSp})$ is the category $\mathbf{Alg}\,(\mathbf{Hom}(\mathbf{BanSp}))$ of homomorphisms ψ of Banach algebras. The category $\mathbf{Mod}_\psi\,(\mathbf{Hom}(\mathbf{BanSp}))$ of "Banach ψ-modules" constructed from $\mathbf{Hom}(\mathbf{BanSp})$ is the category of homomorphisms of Banach modules with coefficient map ψ. And the category $\mathbf{Pair}_\psi\,(\mathbf{Hom}(\mathbf{BanSp}))$of "Banach ψ-pairs" stemming from $\mathbf{Hom}(\mathbf{BanSp})$ is the category of homomorphisms of (ordinary) Banach pairs, again with coefficient map ψ.

If Φ_ψ and Ψ_ψ are objects of this category, i.e., if they are homomorphisms of Banach pairs, then it makes sense to talk about ψ-linear operators between them, because we can regard ψ as an "algebra". The definition one gets from this is the definition of $\mathrm{L}_\psi\,(\Phi, \Psi)$ given above.

Remark 1.9.6. The above definitions can also be made for the category of *graded* Banach spaces. In this case, you should substitute "graded Banach algebra" for "Banach algebra" and "graded homomorphism" for homomorphism. Also, the definitions can be adapted to *G-equivariant* and *graded G-equivariant* Banach spaces (where G is a locally compact Hausdorff group). Instead of writing down all the definitions to the bitter end, we confine ourselves to pointing out that in all the cases we just impose the additional conditions (being graded, etc.) on the homomorphisms (= the objects of the categories) but not on the pairs of operators (= the morphisms). Instead, we get a grading (or a G-action) on the morphism sets. In particular, this allows us to talk about odd or even elements in $\mathrm{L}_\psi\,(\Phi, \Psi)$ (= pairs of odd / even operators) if ψ, Φ and Ψ are graded.

1.9.2 $\mathrm{KK}^{\mathrm{ban}}_{\mathrm{G}}$-cycles of homomorphisms of Banach algebras

Although the categorial viewpoint sketched above gives us a systematic background to construct the "ψ-linear operators", it does not tell us how to construct the "compact operators" between homomorphisms of Banach pairs. However, there is the following natural choice:

Definition 1.9.7 (The space $\mathrm{K}_\psi(\Phi, \Psi)$). Let B and B' be Banach algebras and $\psi\colon B \to B'$ a morphism. Let $\Phi\colon E \to E'$ and $\Psi\colon F \to F'$ be homomorphisms of pairs with coefficient map ψ. For all $f^> \in F^>$ and $e^< \in E^<$, the pair

$$\left(\left|f^>\right\rangle\left\langle e^<\right|,\ \left|\Psi^>(f^>)\right\rangle\left\langle \Phi^<(e^<)\right|\right)$$

is contained in $\mathrm{L}_\psi(\Phi, \Psi)$. Denote by $\mathrm{K}_\psi(\Phi, \Psi)$ (or just $\mathrm{K}(\Phi, \Psi)$) the closed linear span of all such operators, writing $\mathrm{K}_\psi(\Phi)$ or $\mathrm{K}_\psi(\Phi\colon E \to E')$ for $\mathrm{K}_\psi(\Phi, \Phi)$.

Remark 1.9.8. Note that if $(T, T') \in \mathrm{K}_\psi(\Phi, \Psi)$, then $T \in \mathrm{K}_B(E, F)$ and $T' \in \mathrm{K}_{B'}(E', F')$. However, the condition of being in $\mathrm{K}_\psi(\Phi, \Psi)$ is (a priori) stronger than the condition of being in $\mathrm{K}_B(E, F) \times \mathrm{K}_{B'}(E', F') \cap \mathrm{L}_\psi(\Phi, \Psi)$ as it means that the approximation of T and T' by finite rank operators can be done simultaneously:

$$\begin{gathered}\forall \varepsilon > 0\ \exists n \in \mathbb{N}\ \exists f_1^>, \dots, f_n^> \in F^>,\ e_1^<, \dots, e_n^< \in E^< :\\ \left\| T - \sum_{i=1}^n \left|f_i^>\right\rangle\left\langle e_i^<\right| \right\| < \varepsilon \quad \wedge \quad \left\| T' - \sum_{i=1}^n \left|\Psi^>\left(f_i^>\right)\right\rangle\left\langle \Phi^<\left(e_i^<\right)\right| \right\| < \varepsilon.\end{gathered}$$

Proposition 1.9.9. *Let $\Xi\colon G \to G'$ be another homomorphism of pairs with coefficient map ψ (with G and G' being Banach pairs). Then $\mathrm{L}(\Psi, \Xi) \circ \mathrm{K}(\Phi, \Psi) \subseteq \mathrm{K}(\Phi, \Xi)$ and likewise $\mathrm{K}(\Psi, \Xi) \circ \mathrm{L}(\Phi, \Psi) \subseteq \mathrm{K}(\Phi, \Xi)$.*

Proof. Let $(T, T') \in \mathrm{L}(\Psi, \Xi)$. Then the map $(T, T') \circ \cdot\colon \mathrm{L}(\Phi, \Psi) \to \mathrm{L}(\Phi, \Xi)$ is linear and continuous, so it suffices to show that $(T, T') \circ \left(\left|f^>\right\rangle\left\langle e^<\right|,\ \left|\Psi^>(f^>)\right\rangle\left\langle \Phi^<(e^<)\right|\right)$ is contained in $\mathrm{K}(\Phi, \Xi)$ for all $f^> \in F^>$ and $e^< \in E^<$. But $T \circ \left|f^>\right\rangle\left\langle e^<\right| = \left|T^>(f^>)\right\rangle\left\langle e^<\right|$ and

$$T' \circ \left|\Psi^>(f^>)\right\rangle\left\langle \Phi^<(e^<)\right| = \left|T'^>\left(\Psi^>(f^>)\right)\right\rangle\left\langle \Phi^<(e^<)\right| = \left|\Xi^>\left(T^>(f^>)\right)\right\rangle\left\langle \Phi^<(e^<)\right|$$

because $(T, T') \in \mathrm{L}(\Psi, \Xi)$. So we are done with the first inclusion. The second inclusion can be proved similarly. □

Definition 1.9.10 (The class $\mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(\varphi, \psi)$). Let $\varphi\colon A \to A'$ and $\psi\colon B \to B'$ be G-equivariant homomorphisms of G-Banach algebras. A $\mathrm{KK}^{\mathrm{ban}}$-cycle from φ to ψ is a pair $(\Phi\colon E \to E',\ (T, T'))$ such that E is a non-degenerate graded G-Banach A-B-pair, E' is a non-degenerate graded G-Banach A'-B'-pair, Φ is an even G-equivariant homomorphism from ${}_AE_B$ to ${}_{A'}E'_{B'}$ with coefficient maps φ and ψ, and $(T, T') \in \mathrm{L}_\psi(\Phi, \Phi)$ is a pair of odd linear operators such that

1. $\forall a \in A:\ [a, (T, T')] = ([a, T],\ [\varphi(a), T']) \in \mathrm{K}(\Phi, \Phi)$;
2. $\forall a \in A:\ a((T, T')^2 - 1) = \left(a(T^2 - 1),\ \varphi(a)(T'^2 - 1)\right) \in \mathrm{K}(\Phi, \Phi)$;
3. $\forall a \in A:\ g \mapsto a\left(g(T, T') - (T, T')\right) = \left(a(gT - T),\ \varphi(a)(gT' - T')\right) \in \mathcal{C}\left(G,\ \mathrm{K}(\Phi, \Phi)\right)$.

The class of all such cycles will be denoted by $\mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(\varphi, \psi)$.

Remark 1.9.11. If $(\Phi\colon E \to E',\ (T, T'))$ is an element of $\mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(\varphi, \psi)$, then $(E, T) \in \mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(A, B)$ and $\varphi^*(E', T') \in \mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(A, B')$, and, if (E', T') is itself a $\mathrm{KK}^{\mathrm{ban}}$-cycle (which is automatic if φ is surjective), then Φ is a morphism of $\mathrm{KK}^{\mathrm{ban}}$-cycles from (E, T) to (E', T'). But not all morphisms of $\mathrm{KK}^{\mathrm{ban}}$-cycles seem to give $\mathrm{KK}^{\mathrm{ban}}$-cycles of morphisms. Being a cycle is a regularity condition which ensures that a morphism of $\mathrm{KK}^{\mathrm{ban}}$-cycles induces a homotopy as we shall see below.

Remark 1.9.12. Now that we have defined the class $\mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(\varphi, \psi)$, it is a natural question to ask what $\mathrm{KK}^{\mathrm{ban}}_{\mathrm{G}}(\varphi, \psi)$ could be. To answer this, one could define morphisms between elements of $\mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(\varphi, \psi)$ providing us with a notion of isomorphic cycles. In a second step, one should define the pushout of cycles making it possible to define homotopies (using the homomorphism $\psi[0,1]\colon B[0,1] \to B'[0,1], \psi[0,1](\beta)(t) = \psi(\beta(t))$ as a starting point). With a little bit of luck one ends up with an abelian group $\mathrm{KK}^{\mathrm{ban}}_{\mathrm{G}}(\varphi, \psi)$ which is somehow related to $\mathrm{KK}^{\mathrm{ban}}_{\mathrm{G}}(A, B)$ and $\mathrm{KK}^{\mathrm{ban}}_{\mathrm{G}}(A', B')$.

In this work, we just need the $\mathrm{KK}^{\mathrm{ban}}$-cycles in $\mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(\varphi, \psi)$ as a source of (ordinary) homotopies and do not pursue these considerations any further.

1.9.3 Mapping cylinders

Let G be a locally compact Hausdorff group. In the following paragraphs we are going to consider graded G-Banach spaces and even G-equivariant linear maps between them. Of course all definitions and results also apply, in a simpler form, to plain Banach spaces and linear maps. These simple definitions and results are contained as subcases in the following (just let the group G and the grading be trivial).

Mapping cylinders of contractive linear maps between graded G-Banach spaces

Definition 1.9.13 (The mapping cylinder of linear maps between Banach spaces). Let E and E' be graded G-Banach spaces with grading automorphisms σ_E and $\sigma_{E'}$ and let $\rho \in \mathrm{L}(E, E')$ be contractive, even and G-equivariant. Let $\mathrm{ev}_0^{E'}: E'[0,1] \to E'$ be evaluation at zero. Then the mapping cylinder $\mathrm{Z}\,(\rho)$ is the fibre product of $\rho\colon E \to E'$ and $\mathrm{ev}_0^{E'}: E'[0,1] \to E'$:

$$\begin{tikzcd}
\mathrm{Z}\,(\rho) \arrow[r] \arrow[d] & E'[0,1] \arrow[d, "\mathrm{ev}_0^{E'}"] \\
E \arrow[r, "\rho"'] & E'
\end{tikzcd}$$

So $\mathrm{Z}\,(\rho)$ is the graded G-Banach space $\{(e, \xi') \in E \times E'[0,1] :\ \xi'(0) = \rho(e)\} \subseteq E \times E'[0,1]$ with the norm $\|(e, \xi')\| = \max\{\|e\|, \|\xi'\|_\infty\}$; the grading operator on $\mathrm{Z}\,(\rho)$ sends a pair (e, ξ') to the pair $(\sigma_E(e),\ t \mapsto \sigma_{E'}(\xi'(t)))$; the G-action is given by $g(e, \xi') := (ge,\ t \mapsto g(\xi'(t)))$ for all $g \in G$.

Definition 1.9.14 (The mapping cylinder construction as a functor). One can regard the mapping cylinder construction as a functor from the category of graded contractive linear G-equivariant maps to the category of graded G-Banach spaces in the following way:

Let $\rho\colon E \to E'$ and $\sigma\colon F \to F'$ be graded contractive linear G-equivariant maps between graded G-Banach spaces. Let $(T, T') \in \mathrm{L}(\rho, \sigma)$, which means that $T \in \mathrm{L}(E, F)$ and $T' \in \mathrm{L}(E', F')$ such that $\sigma \circ T = T' \circ \rho$. To make the mapping cylinder construction a functor one defines

$$\mathrm{Z}\left(T, T'\right) :\ \mathrm{Z}\,(\rho) \to \mathrm{Z}\,(\sigma)\,,\ (e, \xi') \mapsto \left(T(e),\ t \mapsto T'(\xi'(t))\right).$$

Then $\mathrm{Z}(T,T') \in \mathrm{L}(\mathrm{Z}(\rho), \mathrm{Z}(\sigma))$. The so-defined functor is linear and contractive on the morphism sets. It respects the canonical grading automorphisms and the G-actions on $\mathrm{L}(\rho,\sigma)$ and $\mathrm{L}(\mathrm{Z}(\rho), \mathrm{Z}(\sigma))$.

Definition 1.9.15. There is a canonical action of $\mathcal{C}[0,1]$ on $\mathrm{Z}(\rho)$; it is given by

$$\chi \cdot (e,\xi') = \big(\chi(0)e,\ \chi\xi'\big)$$

for all $\chi \in \mathcal{C}[0,1]$, $(e,\xi') \in \mathrm{Z}(\rho)$.

Mapping cylinders of homomorphisms of graded G-Banach algebras

Definition 1.9.16 (The mapping cylinder of a homomorphism of Banach algebras). Let B and B' be graded G-Banach algebras and let $\psi\colon B \to B'$ be a graded equivariant homomorphism. Then the mapping cylinder $\mathrm{Z}(\psi)$ of ψ is a graded G-Banach algebra with the componentwise product.

Lemma 1.9.17. *Let B and B' be Banach algebras and let $\psi\colon B \to B'$ be a morphism. Then $\mathrm{Z}(\psi)$ is non-degenerate if B and B' are non-degenerate.*

Proof. Let B and B' be non-degenerate. Write S for the span of $\mathrm{Z}(\psi)\ \mathrm{Z}(\psi)$. We have to show that S is dense in $\mathrm{Z}(\psi)$. Let $(b,\beta') \in \mathrm{Z}(\psi)$, i.e., $b \in B$, $\beta' \in B'[0,1]$ and $\psi(b) = \beta'(0)$. Let $\varepsilon > 0$. Find c in the span of BB such that $\|b-c\| \leq \varepsilon/2$. Let $\gamma' := (t \mapsto \psi(c)) \in B'[0,1]$. Note that $(c,\ \gamma') \in S$. Find a neighbourhood U of 0 in $[0,1]$ such that $\|\beta'(t) - \beta'(0)\| \leq \varepsilon/2$ for all $t \in U$. Because $B'[0,1]$ is non-degenerate, we can find some $\tilde{\beta}'$ in the span of $B'[0,1]\ B'[0,1]$ such that $\|\beta' - \tilde{\beta}'\|_\infty \leq \varepsilon$. Find a function $\chi \in \mathcal{C}[0,1]$ with the following properties: $\chi(0) = 1$, $0 \leq \chi \leq 1$, $\operatorname{supp}\chi \subseteq U$. Then $(0,(1-\chi)\tilde{\beta}')$ is in S. Also $(c,\chi\gamma')$ is in S. So we have

$$\Big(0,(1-\chi)\tilde{\beta}'\Big) + \big(c,\chi\gamma'\big) = \Big(c,(1-\chi)\tilde{\beta}' + \chi\gamma'\Big) \in S.$$

Note that

$$\Big\|(b,\beta') - \Big(c,(1-\chi)\tilde{\beta}' + \chi\gamma'\Big)\Big\| = \max\Big\{\|b-c\|\,,\ \Big\|\beta' - (1-\chi)\tilde{\beta}' - \chi\gamma'\Big\|_\infty\Big\}.$$

The first term is $\leq \varepsilon/2$ by the choice of c. If $t \in U$ then

$$\begin{aligned}
\Big\|\beta'(t) - (1-\chi(t))\tilde{\beta}'(t) - \chi(t)\gamma'(t)\Big\| &\leq (1-\chi(t))\Big\|\beta'(t) - \tilde{\beta}'(t)\Big\| + \chi(t)\Big\|\beta'(t) - \gamma'(t)\Big\| \\
&\leq (1-\chi(t))\Big\|\beta' - \tilde{\beta}'\Big\|_\infty + \chi(t)\Big(\Big\|\beta'(t) - b\Big\| + \Big\|b - c\Big\|\Big) \\
&\leq (1-\chi(t))\varepsilon + \chi(t)\varepsilon = \varepsilon.
\end{aligned}$$

If $t \notin U$, then $\chi(t) = 0$ so $\Big\|\beta'(t) - (1-\chi(t))\tilde{\beta}'(t) - \chi(t)\gamma'(t)\Big\| \leq \varepsilon$ as well. So all in all we get

$$\Big\|(b,\beta') - \Big(c,(1-\chi)\tilde{\beta}' + \chi\gamma'\Big)\Big\| \leq \varepsilon$$

and hence we are done. □

Remark 1.9.18. In Chapter 2 we are going to introduce the notion of a $\mathcal{C}[0,1]$-Banach space. The $\mathcal{C}[0,1]$-action on the mapping cylinder $\mathrm{Z}(\rho)$ for a contractive linear map ρ between Banach spaces actually makes $\mathrm{Z}(\rho)$ a $\mathcal{C}[0,1]$-Banach space and the mapping cylinder construction is a functor with values in the category of $\mathcal{C}[0,1]$-Banach spaces and $\mathcal{C}[0,1]$-linear maps. Moreover, the mapping cylinder of a homomorphism of Banach algebras is a so-called $\mathcal{C}[0,1]$-Banach algebra. Later on, we will define the notion of the fibres of such a $\mathcal{C}[0,1]$-Banach algebra, and the fibre of the above $\mathrm{Z}(\psi)$ at 0 is isomorphic to B and the fibre at $t \in]0,1]$ is isomorphic to B'. It will also follow that $\mathrm{Z}(\psi)$ is non-degenerate if and only if its fibres are non-degenerate. But this result, though not very deep, is still quite far away, so I decided to include the above non-systematic proof of Lemma 1.9.17.

Lemma 1.9.19. *For every Banach algebra B, the mapping cylinder of Id_B is isomorphic to $B[0,1]$.*

Mapping cylinders of homomorphisms between graded G-Banach modules

Definition 1.9.20 (The mapping cylinder of a homomorphism of Banach modules). Let E_B and $E'_{B'}$ be graded G-Banach modules and let $\Phi_\psi\colon E_B \to E'_{B'}$ be a graded G-equivariant homomorphism. Then the mapping cylinder $\mathrm{Z}(\Phi)$ of Φ is a graded G-Banach $\mathrm{Z}(\psi)$-module with the componentwise action of $\mathrm{Z}(\psi)$.

Remark 1.9.21. Conceptually, the mapping cylinder $\mathrm{Z}(\Phi)$ is a fibre product: Let $\mathrm{ev}_0\colon B'[0,1] \to B'$ and $\mathrm{Ev}_0\colon E'[0,1] \to E'$ be the evaluation maps at zero. Then $(\mathrm{Ev}_0)_{\mathrm{ev}_0}$ is a graded G-equivariant homomorphism from $F'[0,1]_{B'[0,1]}$ to $F'_{B'}$ and $\mathrm{Z}(\Psi)$ is the fibre product of $\Phi_\psi\colon E_B \to E'_{B'}$ and $(\mathrm{Ev}_0)_{\mathrm{ev}_0}\colon E'[0,1]_{B'[0,1]} \to E'_{B'}$:

$$\begin{tikzcd}
\mathrm{Z}(\Phi) \arrow[r] \arrow[d] & E'[0,1] \arrow[d, "\mathrm{Ev}_0"] \\
E \arrow[r, "\Phi"] & E'
\end{tikzcd}$$

Analogously to Lemma 1.9.17 one proves:

Lemma 1.9.22. *Let E_B and $E'_{B'}$ be Banach modules and let $\Phi_\psi\colon E_B \to E'_{B'}$ be a homomorphism. If E_B and $E'_{B'}$ are non-degenerate, then $\mathrm{Z}(\Phi)$ is a non-degenerate Banach $\mathrm{Z}(\psi)$-module.*

Proposition 1.9.23. *Let E_B and $E'_{B'}$ be non-degenerate graded right G-Banach modules and let $\Phi_\varphi\colon E_B \to E'_{B'}$ be a graded G-equivariant homomorphism. On $\mathrm{Z}(\varphi)$ define the evaluation homomorphisms $\mathrm{ev}_0\colon\ \mathrm{Z}(\varphi) \to B,\ (b,\beta') \mapsto b$ and $\mathrm{ev}_1\colon\ \mathrm{Z}(\varphi) \to B',\ (b,\beta') \mapsto \beta'(1)$. Then*

$$\mathrm{ev}_{0,*}(\mathrm{Z}(\Phi)) \cong E \quad \textit{and} \quad \mathrm{ev}_{1,*}(\mathrm{Z}(\Phi)) \cong E'.$$

Proof. Define

$$\Psi^0_{\mathrm{Id}_C}\colon\ \mathrm{Z}(\Phi) \otimes_{\widetilde{\mathrm{ev}_0}} \widetilde{B} \to E,\ (e,\xi') \otimes \widetilde{b} \mapsto e\widetilde{b}.$$

This is a contractive graded G-equivariant homomorphism. Define

$$\Xi^0_{\mathrm{Id}_C}\colon E \to \mathrm{Z}(\Phi) \otimes_{\widetilde{\mathrm{ev}_0}} \widetilde{B},\ e \mapsto (e,\ t \mapsto \Phi(e)) \otimes 1.$$

This too is a contractive graded G-equivariant homomorphism. We have $\Psi^0 \circ \Xi^0 = \mathrm{Id}_E$, so Ψ^0 is surjective. Let $\tau := \sum_{n\in\mathbb{N}} (e_n, \xi'_n) \otimes \widetilde{b}_n \in \mathrm{Z}(\Phi) \otimes \widetilde{B}$. Since E_B is non-degenerate, we can show that

$$\chi\tau = \chi(0)\tau \tag{1.2}$$

for every $\chi \in \mathcal{C}([0,1])$. Let U be a neighbourhood of 0 in $[0,1]$. Find $\chi \in \mathcal{C}([0,1])$ such that $0 \le \chi \le 1$, $\chi(0) = 1$ and $\operatorname{supp}\chi \subseteq U$. Then

$$\sum_{n\in\mathbb{N}} (e_n, \xi'_n) \otimes \widetilde{b}_n = \sum_{n\in\mathbb{N}} \chi(e_n, \xi'_n) \otimes \widetilde{b}_n = \sum_{n\in\mathbb{N}} (e_n, \chi\xi'_n) \otimes \widetilde{b}_n.$$

Then

$$\begin{aligned}\|T\| &\leq \left\|\sum_{n\in\mathbb{N}}(e_n,\chi\xi'_n)\otimes\widetilde{b}_n\right\| = \left\|\sum_{n\in\mathbb{N}}\left(e_n\widetilde{b}_n,\chi\xi'_n\widetilde{\varphi}(\widetilde{b}_n)\right)\otimes 1\right\| \\ &= \left\|\left(\sum_{n\in\mathbb{N}}(e_n\widetilde{b}_n,\chi\xi'_n\widetilde{\varphi}(\widetilde{b}_n)\right)\otimes 1\right\| \leq \left\|\left(\sum_{n\in\mathbb{N}}e_n\widetilde{b}_n,\ \sum_{n\in\mathbb{N}}\chi\xi'_n\widetilde{\varphi}(\widetilde{b}_n)\right)\right\|.\end{aligned}$$

Since

$$\left\|\sum_{n\in\mathbb{N}}\chi\xi'_n\widetilde{\varphi}(\widetilde{b}_n)(0)\right\| = \left\|\varphi\left(\sum_{n\in\mathbb{N}}e_n\widetilde{b}_n\right)\right\| \leq \left\|\sum_{n\in\mathbb{N}}e_n\widetilde{b}_n\right\|$$

and U can be chosen arbitrarily small, we get

$$\left\|T\right\| \leq \left\|\sum_{n\in\mathbb{N}}e_n\widetilde{b}_n\right\| = \left\|\Psi^0(T)\right\| \leq \left\|T\right\|,$$

so Ψ^0 is isometric. It follows that Ψ^0 is an isomorphism.

We still have to show (1.2). Let $S\in\mathrm{Z}(\Phi)$, $s\in\mathrm{Z}(\varphi)$ and $\widetilde{b}\in\widetilde{B}$. Let $\chi\in\mathcal{C}([0,1])$. Then

$$\chi Ss\otimes\widetilde{b} = S(\chi s)\otimes\widetilde{b} = S\otimes\chi(0)\,\mathrm{ev}_0(s)\widetilde{b} = Ss\otimes\chi(0)\widetilde{b} = \chi\left(Ss\otimes\widetilde{b}\right).$$

Because $\mathrm{Z}(\Phi)$ is non-degenerate we have this equality for all $\tau\in\mathrm{Z}(\Phi)\otimes\widetilde{B}$.
The second assertion is shown similarly. □

Proposition 1.9.24. *Let E_B, F_B, $E'_{B'}$, $F'_{B'}$ be right Banach modules and let $\Phi_\psi\colon E_B\to E'_{B'}$ and $\Psi_\psi\colon F_B\to F'_{B'}$ be concurrent homomorphisms. Let $(T,T')\in\mathrm{L}(\Phi,\Psi)$. Then $\mathrm{Z}(T,T')$ as defined in 1.9.14 is in $\mathrm{L}_{\mathrm{Z}(\psi)}(\mathrm{Z}(\Phi),\mathrm{Z}(\Psi))$.*

Proof. If $(b,\beta')\in\mathrm{Z}(\psi)$ and $(e,\xi')\in\mathrm{Z}(\Phi)$, then

$$\begin{aligned}\mathrm{Z}\left(T,T'\right)\left((b,\beta')\cdot(e,\xi')\right) &= \mathrm{Z}\left(T,T'\right)\left(be,\beta'\xi'\right) = \left(T(be),\ t\mapsto T'\left(\beta'(t)\xi'(t)\right)\right) \\ &= \left(bT(e),\ t\mapsto\beta'(t)T'\left(\xi'(t)\right)\right) = (b,\beta')\left(T(e),\ t\mapsto T'(\xi'(t))\right) \\ &= (b,\beta')\left(\mathrm{Z}\left(T,T'\right)(e,\xi')\right).\end{aligned}$$

So the linear operator $\mathrm{Z}(T,T')$ is $\mathrm{Z}(\psi)$-linear. □

Mapping cylinders of homomorphisms between graded G-Banach pairs

Definition 1.9.25 (The mapping cylinder of a homomorphism of Banach pairs). Let E_B and $E'_{B'}$ be graded G-Banach pairs and let $\Phi_\psi\colon E_B\to E'_{B'}$ be a graded G-equivariant concurrent homomorphism. Then the mapping cylinder $\mathrm{Z}(\Phi)$ of Φ is defined to be the graded G-Banach $\mathrm{Z}(\psi)$-pair $(\mathrm{Z}(\Phi^<),\ \mathrm{Z}(\Phi^>))$ with the componentwise bracket

$$\mathrm{Z}\left(\Phi^<\right)\times\mathrm{Z}\left(\Phi^>\right)\mathrm{Z}(\psi),\ \left((e^<,\xi'^<),(e^>,\xi'^>)\right)\mapsto\left(\langle e^<,e^>\rangle,\ \langle\xi'^<,\xi'^>\rangle\right).$$

The mapping cylinder $\mathrm{Z}(\Phi)$ can be realised as a fibre product, compare Remark 1.9.21. From the corresponding Lemma 1.9.22 and Proposition 1.9.23 for Banach modules we can deduce the following two facts.

Lemma 1.9.26. *Let E_B and $E'_{B'}$ be Banach pairs and let $\Phi_\psi\colon E_B \to E'_{B'}$ be a concurrent homomorphism. If E_B and $E'_{B'}$ are non-degenerate Banach pairs, then $\mathrm{Z}(\Phi)$ is a non-degenerate Banach $\mathrm{Z}(\psi)$-pair.*

Proposition 1.9.27. *Let E_B and $E'_{B'}$ be non-degenerate graded G-Banach pairs and let $\Phi_\psi\colon E_B \to E'_{B'}$ be a graded G-equivariant concurrent homomorphism. Let $\mathrm{ev}_0\colon\ \mathrm{Z}(\psi) \to B,\ (b,f) \mapsto b$ and $\mathrm{ev}_1\colon\ \mathrm{Z}(\psi) \to B',\ (b,f) \mapsto f(1)$. Then*

$$\mathrm{ev}_{0,*}(\mathrm{Z}(\Phi)) \cong E \quad \textit{and} \quad \mathrm{ev}_{1,*}(\mathrm{Z}(\Phi)) \cong E'.$$

Definition 1.9.28 (The mapping cylinder construction as a functor). Let E_B, F_B, $E'_{B'}$, $F'_{B'}$ be non-degenerate Banach pairs and let $\Phi_\psi\colon E_B \to E'_{B'}$ and $\Psi_\psi\colon F_B \to F'_{B'}$ be concurrent homomorphisms. Let $(T,T') \in \mathrm{L}_\psi(\Phi,\Psi)$, i.e., $T \in \mathrm{L}_B(E,F)$, $T' \in \mathrm{L}_{B'}(E',F')$ and $\Psi^> \circ T^> = T'^> \circ \Phi^>$ and $T'^< \circ \Psi^< = \Phi^< \circ T^<$. As stated above, this implies $(T^>, T'^>) \in \mathrm{L}_\psi(\Phi^>,\Psi^>)$ and $(T^<, T'^<) \in \mathrm{L}_\psi(\Psi^<,\Phi^<)$. We have

$$\mathrm{Z}\left(T^>, T'^>\right) \in \mathrm{L}_{\mathrm{Z}(\psi)}\left(\mathrm{Z}\left(\Phi^>\right), \mathrm{Z}\left(\Psi^>\right)\right) \quad \text{and} \quad \mathrm{Z}\left(T^<, T'^<\right) \in \mathrm{L}_{\mathrm{Z}(\psi)}\left(\mathrm{Z}\left(\Psi^<\right), \mathrm{Z}\left(\Phi^<\right)\right).$$

Define $\mathrm{Z}(T,T') \in \mathrm{L}_{\mathrm{Z}(\psi)}(\mathrm{Z}(\Phi), \mathrm{Z}(\Psi))$ to be the pair $(\mathrm{Z}(T^<, T'^<),\ \mathrm{Z}(T^>, T'^>))$.

The maps $\Phi \mapsto \mathrm{Z}(\Phi)$ and $(T,T') \mapsto \mathrm{Z}(T,T')$ define a functor from the category of graded G-equivariant homomorphisms of Banach pairs with coefficient map ψ to the category of Banach $\mathrm{Z}(\psi)$-pairs. It is linear, even, G-equivariant and contractive on the morphisms sets.

Definition 1.9.29 (Mapping cylinders and left actions on pairs). Let ${}_AE_B$ and ${}_{A'}E'_{B'}$ be graded G-Banach pairs and let ${}_\varphi\Phi_\psi\colon E \to E'$ be a graded G-equivariant concurrent homomorphism. Then the mapping cylinder $\mathrm{Z}(\Phi) = (\mathrm{Z}(\Phi^<),\ \mathrm{Z}(\Phi^>))$ of Φ is a graded G-Banach $\mathrm{Z}(\varphi)$-$\mathrm{Z}(\psi)$-pair when equipped with the componentwise action of $\mathrm{Z}(\varphi)$.

Proposition 1.9.30. *Let ${}_AE_B$ and ${}_{A'}E'{}_{B'}$ be non-degenerate graded G-Banach pairs and let ${}_\varphi\Phi_\psi$ be a graded G-equivariant concurrent homomorphism between them. Let $\mathrm{ev}_0\colon\ \mathrm{Z}(\psi) \to B,\ (b,\beta') \mapsto b$ and $\mathrm{ev}_1\colon\ \mathrm{Z}(\psi) \to B',\ (b,\beta') \mapsto \beta'(1)$ and $\iota_A\colon A \to \mathrm{Z}(\varphi),\ a \mapsto (a, t \mapsto \varphi(a))$. Then*

$${}_A\left(\iota_A^*\left(\mathrm{ev}_{0,*}(\mathrm{Z}(\Phi))\right)\right)_B \cong {}_AE_B \quad \textit{and} \quad {}_{A'}\left(\mathrm{ev}_{1,*}(\mathrm{Z}(\Phi))\right)_{B'} \cong {}_{A'}E'{}_{B'}.$$

Proof. In view of Proposition 1.9.27 the first assertion is obvious. For the second, we just have to specify the action of A' on $\mathrm{ev}_{1,*}(\mathrm{Z}(\Phi))$. Let χ be some function in $\mathcal{C}[0,1]$ such that $\chi(1) = 1$, $0 \leq \chi \leq 1$ and $\chi(0) = 0$. Then define

$$a'(f,g) \otimes \widetilde{b'} := (0, \chi a' g) \otimes \widetilde{b'}$$

for all $a' \in A'$, $(f,g) \in \mathrm{Z}(\Phi^>)$ and $\widetilde{b'} \in \widetilde{B'}$. This can easily be shown to be a well-defined action of A' on $\mathrm{Z}(\Phi^>)$ and we can do the same for $\mathrm{Z}(\Phi^<)$. □

Mapping cylinders and compact operators

Proposition 1.9.31. *Let E_B, F_B, $E'_{B'}$, $F'_{B'}$ be non-degenerate Banach pairs and let $\Phi_\psi\colon E_B \to E'_{B'}$ and $\Psi_\psi\colon F_B \to F'_{B'}$ be concurrent homomorphisms. Let $(T,T') \in \mathrm{L}_\psi(\Phi,\Psi)$. Then the following are equivalent:*

1. $(T,T') \in \mathrm{K}_\psi(\Phi,\Psi)$;
2. $\mathrm{Z}(T,T') \in \mathrm{K}_{\mathrm{Z}(\psi)}(\mathrm{Z}(\Phi),\mathrm{Z}(\Psi))$.

Proof. 1. $\Rightarrow$ 2.: Since the map $(T,T') \mapsto \mathrm{Z}(T,T')$ is linear and contractive, it suffices to consider the case that (T,T') is of the form $\left(\left|f^>\right\rangle\left\langle e^<\right|,\ \left|\Psi^>(f^>)\right\rangle\left\langle\Phi^<(e^<)\right|\right)$ for some $f^> \in F^>$ and $e^< \in E^<$. What is $Z := \mathrm{Z}(T,T')$? Let $(e^>,\xi^>) \in \mathrm{Z}(\Phi^>)$. Then

$$Z^>(e^>,\xi'^>) = \left(T^> e^>,\ t \mapsto T'^>(\xi'^>(t))\right) = \left(f^>\langle e^<, e^>\rangle,\ t \mapsto \Psi^>(f^>)\langle\Phi^<(e^<),\ \xi'^>(t)\rangle\right).$$

Define $\tilde{f}^> := (f^>,\ t\mapsto \Psi^>(f^>)) \in \mathrm{Z}(\Psi^>)$ and $\tilde{e}^< := (e^<,\ t\mapsto\Phi^<(e^<)) \in \mathrm{Z}(\Phi^<)$. Then we have shown that $Z^> = \left|\tilde{f}^>\right\rangle\left\langle\tilde{e}^<\right|^>$. The analogous formula holds for the left-hand side, so $Z = \left|\tilde{f}^>\right\rangle\left\langle\tilde{e}^<\right|$. In particular, Z is compact.

2. $\Rightarrow$ 1.: Let $Z := \mathrm{Z}(T,T')$ be compact. Let $\varepsilon > 0$. Find $n \in \mathbb{N}$ and $(f_1^>,\eta_1'^>),\ldots,(f_n^>,\eta_n'^>) \in \mathrm{Z}(\Psi^>)$, $(e_1^<,\xi_1'^<),\ldots,(e_n^<,\xi_n'^<) \in \mathrm{Z}(\Phi^<)$ such that

$$\left\| Z - \sum_{i=1}^n \left|(f_i^>,\eta_i'^>)\right\rangle\left\langle(e_i^<,\xi_i'^<)\right| \right\| \le \varepsilon.$$

Define

$$(S,S') := \sum_{i=1}^n \left(\left|f_i^>\right\rangle\left\langle e_i^<\right|,\ \left|\Psi^>(f_i^>)\right\rangle\left\langle\Phi^<(e_i^<)\right|\right).$$

We show $\|(T-S,T'-S')\| \le \varepsilon$, i.e., we show $\|T-S\| \le \varepsilon$ and $\|T'-S'\| \le \varepsilon$: Let $e^> \in E^>$. Define $\xi'^>(t) := \Phi^>(e^>)$ for all $t \in [0,1]$. Then $(e^>,\xi'^>) \in \mathrm{Z}(\Phi^>)$. Now

$$\begin{aligned} &\left(Z - \sum_{i=1}^n \left|(f_i^>,\eta_i'^>)\right\rangle\left\langle(e_i^<,\xi_i'^<)\right|\right)^> (e^>,\xi'^>) \\ = &\left((T-S)^>(e^>),\ t\mapsto \left(T' - \sum_{i=1}^n \left|\eta_i'^>(t)\right\rangle\left\langle\xi_i'^<(t)\right|\right)^> (\xi'^>(t))\right). \end{aligned}$$

The norm of this expression is $\le \varepsilon\,\|(e^>,\xi'^>)\| = \varepsilon\,\|e^>\|$. So in particular $\|(T-S)^>(e^>)\| \le \varepsilon\,\|e^>\|$ and hence $\|(T-S)^>\| \le \varepsilon$. After applying a similar argument to the left-hand side we get $\|T-S\| \le \varepsilon$.

Let $e'^> \in E'^>$. Let $t_0 \in]0,1]$. Find a function $\chi_{t_0} \in \mathcal{C}[0,1]$ such that $0 \le \chi_{t_0} \le 1$ and $\chi_{t_0} = 1$ and $\chi_{t_0}(0) = 0$. Then $(0, t\mapsto \chi_{t_0}(t)e'^>)$ is in $\mathrm{Z}(\Phi)^>$ and

$$\begin{aligned} &\left(Z - \sum_{i=1}^n \left|(f_i^>,\eta_i'^>)\right\rangle\left\langle(e_i^<,\xi_i'^<)\right|\right)^> (0, t\mapsto\chi_{t_0}(t)e'^>) \\ = &\left(0,\ t\mapsto \chi_{t_0}(t)\left(T' - \sum_{i=1}^n \left|\eta_i'^>(t)\right\rangle\left\langle\xi_i'^<(t)\right|\right)^> (e'^>).\right) \end{aligned}$$

The norm of this expression is $\leq \varepsilon \left\| e'^{>} \right\|$, so in particular

$$\begin{aligned} & \left\| T'^{>}(e'^{>}) - \sum_{i=1}^{n} \left|\eta_i'^{>}(t_0)\right\rangle\left\langle\xi_i'^{<}(t_0)\right|^{>}(e'^{>}) \right\| \\ = \; & \left\| \left(T' - \sum_{i=1}^{n} \left|\eta_i'^{>}(t_0)\right\rangle\left\langle\xi_i'^{<}(t_0)\right| \right)^{>} (e'^{>}) \right\| \leq \varepsilon \left\| e'^{>} \right\| . \end{aligned}$$

The map $t_0 \mapsto \sum_{i=1}^{n} \left|\eta_i'^{>}(t_0)\right\rangle\left\langle\xi_i'^{<}(t_0)\right|^{>}(e'^{>})$ depends continuously on $t_0 \in [0,1]$, so we also get the inequality in zero:

$$\left\| T'^{>}(e'^{>}) - \sum_{i=1}^{n} \left|\eta_i'^{>}(0)\right\rangle\left\langle\xi_i'^{<}(0)\right|^{>}(e'^{>}) \right\| \leq \varepsilon \left\| e'^{>} \right\| .$$

Now $\eta_i'^{>}(0) = \Psi^{>}(f_i^{>})$ and $\xi_i'^{<}(0) = \Phi^{<}(e_i^{<})$. It follows that

$$\left\| T'^{>}(e'^{>}) - S'^{>}(e'^{>}) \right\| \leq \varepsilon \left\| e'^{>} \right\|$$

for all $e'^{>} \in E'^{>}$, and hence $\|T'^{>} - S'^{>}\| \leq \varepsilon$. After a similar argumentation for the left-hand side we arrive at $\|T' - S'\| \leq \varepsilon$. □

Mapping cylinders and $\mathrm{KK}^{\mathrm{ban}}$-cycles

Theorem 1.9.32. *Let A, B, A', B' be G-Banach algebras and let $\varphi\colon A \to A'$ and $\psi\colon B \to B'$ be equivariant homomorphisms of Banach algebras. Let $(\Phi\colon E \to E',\ (T,T')) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(\varphi,\psi)$. Write ι_A for the canonical injection $A \to \mathrm{Z}(\varphi)$. Then*

$$\iota_A^* \left(\mathrm{Z}(\Phi),\ \mathrm{Z}\left(T,T'\right) \right) \quad \in \quad \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}\left(A,\ \mathrm{Z}(\psi)\right).$$

If we write ev_0 for the canonical map $\mathrm{Z}(\psi) \to B$ and ev_t for the map $\mathrm{Z}(\psi) \to B',\ (b,\beta') \mapsto \beta'(t)$ for all $t \in]0,1]$, then

$$\mathrm{ev}_{0,*}\left(\iota_A^* \left(\mathrm{Z}(\Phi),\ \mathrm{Z}\left(T,T'\right) \right) \right) \cong (E,T)$$

and

$$\mathrm{ev}_{t,*}\left(\iota_A^* \left(\mathrm{Z}(\Phi),\ \mathrm{Z}\left(T,T'\right) \right) \right) \cong \varphi^*(E',T')$$

for all $t \in]0,1]$.

Proof. First of all, $\mathrm{Z}(\Phi)$ is a non-degenerate graded G-Banach $\mathrm{Z}(\psi)$-pair that carries a left even action of $\mathrm{Z}(\varphi)$, and hence it also carries a left even action of A. The operator $\mathrm{Z}(T,T')$ is odd. Let $a \in A$. Then $(a, t \mapsto \varphi(a)) \in \mathrm{Z}(\varphi)$, and the action of $a \in A$ is given as multiplication by $(a, t \mapsto \varphi(a))$. Now

$$\left[(a, t \mapsto \varphi(a)),\ \mathrm{Z}\left(T,T'\right)\right] = \mathrm{Z}\left([a,T],\ \left[\varphi(a),T'\right]\right) \in \mathrm{K}_{\mathrm{Z}(\psi)}\left(\mathrm{Z}(\Phi)\right)$$

and

$$(a, t \mapsto \varphi(a)) \left(\mathrm{Z}\left(T,T'\right)^2 - 1 \right) = \mathrm{Z}\left(a(T^2-1),\ \varphi(a)(T'^2-1) \right) \in \mathrm{K}_{\mathrm{Z}(\psi)}\left(\mathrm{Z}(\Phi)\right)$$

for all $a \in A$. For all $g \in G$ and $a \in A$, we have

$$(a, \varphi(a)) \left(g\,\mathrm{Z}\left(T,T'\right) - \mathrm{Z}\left(T,T'\right) \right) = \mathrm{Z}\left(a(gT-T),\ \varphi(a)(gT'-T') \right) \in \mathrm{K}_{\mathrm{Z}(\psi)}\left(\mathrm{Z}(\Phi)\right)$$

and, because the map $\mathrm{Z}(\cdot)$ is continuous, this expression depends continuously on $g \in G$. Hence $\iota_A^*\left(\mathrm{Z}(\Phi),\ \mathrm{Z}(T,T')\right) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}\left(A,\ \mathrm{Z}(\psi)\right)$. □

Remark 1.9.33. If $(E',T') \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A',B')$ (which is automatic if φ is surjective), then one could possibly show that $(\mathrm{Z}(\Phi),\ \mathrm{Z}(T,T')) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(\mathrm{Z}(\varphi),\ \mathrm{Z}(\psi))$. However, we have confined ourselves to the somewhat simpler object $\iota_A^*(\mathrm{Z}(\Phi),\ \mathrm{Z}(T,T'))$, because we are only interested in the case that $\varphi = \mathrm{Id}_A$ and $A' = A$ and use this machinery to construct homotopies as in the following proposition.

Proposition 1.9.34. *Let A, B, A', B' be G-Banach algebras and let $\varphi\colon A \to A'$ and $\psi\colon B \to B'$ be equivariant morphisms of Banach algebras. Let $(\Phi\colon E \to E',\ (T,T')) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(\varphi,\psi)$. Write ι_A for the canonical injection $A \to \mathrm{Z}(\varphi)$ and $p_{B'[0,1]}$ for the canonical map $\mathrm{Z}(\psi) \to B'[0,1]$. Then*

$$\left(p_{B'[0,1]}\right)_*\left(\iota_A^*\left(\mathrm{Z}(\Phi),\ \mathrm{Z}\left(T,T'\right)\right)\right) \quad \in \quad \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}\left(A,\ B'[0,1]\right).$$

This is a homotopy

$$\psi_*(E,T) \sim \varphi^*(E',T').$$

Proof. The first assertion follows from the fact that $\iota_A^*(\mathrm{Z}(\Phi),\ \mathrm{Z}(T,T')) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A,\ \mathrm{Z}(\psi))$. For the second, we have to calculate the fibres of $\left(p_{B'[0,1]}\right)_*(\iota_A^*(\mathrm{Z}(\Phi),\ \mathrm{Z}(T,T')))$ at 0 and 1. We have

$$\begin{aligned}
\left(p_{B'[0,1]}\right)_*\left(\iota_A^*\left(\mathrm{Z}(\Phi),\ \mathrm{Z}\left(T,T'\right)\right)\right)_t &= \left(\iota_A^*\left(\mathrm{Z}(\Phi),\ \mathrm{Z}\left(T,T'\right)\right) \otimes_{p_{B'[0,1]}} \widetilde{B'[0,1]}\right) \otimes_{\mathrm{ev}_t^{B'}} \tilde{B}' \\
&\cong \iota_A^*\left(\mathrm{Z}(\Phi),\ \mathrm{Z}\left(T,T'\right)\right) \otimes_{\mathrm{ev}_t^{B'} \circ p_{B'[0,1]}} \tilde{B}' \\
&= \left(\mathrm{ev}_t^{B'} \circ p_{B'[0,1]}\right)_*\left(\iota_A^*\left(\mathrm{Z}(\Phi),\ \mathrm{Z}\left(T,T'\right)\right)\right)
\end{aligned}$$

for all $t \in [0,1]$. If we write ev_0 for the canonical map $\mathrm{Z}(\psi) \to B$, then $\mathrm{ev}_0^{B'} \circ p_{B'[0,1]} = \psi \circ \mathrm{ev}_0$ and hence we can deduce that

$$\left(\mathrm{ev}_t^{B'} \circ p_{B'[0,1]}\right)_*\left(\iota_A^*\left(\mathrm{Z}(\Phi),\ \mathrm{Z}\left(T,T'\right)\right)\right) = \psi_*\left(\mathrm{ev}_{0,*}\left(\iota_A^*\left(\mathrm{Z}(\Phi),\ \mathrm{Z}\left(T,T'\right)\right)\right)\right) \cong \psi_*(E,T).$$

On the other hand, if $\mathrm{ev}_t\colon\ \mathrm{Z}(\psi) \to B',\ (b,\beta) \mapsto \beta(t)$, then $\mathrm{ev}_t^{B'} \circ p_{B'[0,1]} = \mathrm{ev}_t$ for all $t \in]0,1]$, so

$$\left(\mathrm{ev}_t^{B'} \circ p_{B'[0,1]}\right)_*\left(\iota_A^*\left(\mathrm{Z}(\Phi),\ \mathrm{Z}\left(T,T'\right)\right)\right) = \mathrm{ev}_{t,*}\left(\iota_A^*\left(\mathrm{Z}(\Phi),\ \mathrm{Z}\left(T,T'\right)\right)\right) \cong \varphi^*(E',T'). \qquad \square$$

Corollary 1.9.35. *Let A and B be G-Banach algebras and let $(\Phi\colon E \to E',\ (T,T'))$ be an element of $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(\mathrm{Id}_A,\mathrm{Id}_B)$. Then $(E,T),(E',T') \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$ and $(E,T) \sim (E',T')$.*

1.10 Morita theory and $\mathrm{KK}^{\mathrm{ban}}$

V. Lafforgue proves in his unpublished note [Laf04] that the K-theory of Banach algebras is invariant under Morita equivalence. He also introduces a rather flexible notion of Morita equivalence and gives a version of the above sufficient condition for homotopy. The present section is dedicated to a systematic study of the relation between Morita equivalences and $\mathrm{KK}^{\mathrm{ban}}$, building on Lafforgue's notion of "flèches de Morita", called "Morita cycles" in this work. A category of "Morita morphisms" is introduced which acts on $\mathrm{KK}^{\mathrm{ban}}$ from the right. Morita equivalences give isomorphisms in this category, so $\mathrm{KK}^{\mathrm{ban}}$ is invariant under Morita equivalences at least in the second component. Although our main interest is the non-equivariant situation, the equivariant case comes for free by adding the word "equivariant" to all the definitions and propositions, so we include it.

Let G be a locally compact Hausdorff group.

1.10.1 Morita equivalences

Definition 1.10.1 (Full Banach pair). Let B be a Banach algebra and let E be a Banach B-pair. Then E is called *full* if the span of $\langle E^<, E^>\rangle$ is dense in B.

Definition 1.10.2 ((Equivariant) Morita equivalence). Let A, B be G-Banach algebras. A (G-equivariant) Morita equivalence between A and B is a pair $\left({}_BE^<_A, {}_AE^>_B\right)$ endowed with an equivariant bilinear bracket $\langle\cdot,\cdot\rangle_B \colon E^< \times E^> \to B$ and an equivariant bilinear bracket ${}_A\langle\cdot,\cdot\rangle \colon E^> \times E^< \to A$ satisfying the following conditions:

1. $(E^<, E^>)$ with $\langle\cdot,\cdot\rangle_B$ is an A-B-pair.
2. $(E^>, E^<)$ with ${}_A\langle\cdot,\cdot\rangle$ is a B-A-pair.
3. The two brackets are compatible:
$$\langle e^<, e^>\rangle_B f^< = e^<{}_A\langle e^>, f^<\rangle \quad \text{and} \quad e^>\langle f^<, f^>\rangle_B = {}_A\langle e^>, f^<\rangle f^>.$$
for all $e^<, f^< \in E^<$ and $e^>, f^> \in E^>$.
4. The pairs $(E^<, E^>)$ and $(E^>, E^<)$ are full and non-degenerate.

A and B are called Morita equivalent if there is a Morita equivalence between A and B.

If B is a non-degenerate G-Banach algebra, then the standard B-pair $\underline{B} = (B, B)$ with the obvious additional structure is a G-equivariant Morita equivalence between B and itself. Conversely, if A and B are Morita equivalent, then A and B are non-degenerate.

If A and B are G-Banach algebras and E is a Morita equivalence from A to B, then $\overline{E} = (E^>, E^<)$ is a Morita equivalence from B to A, called the *inverse Morita equivalence*. And finally, if A, B, C are G-Banach algebras, E is a Morita equivalence from A to B and F is a Morita equivalence from B to C, then $E \otimes_B F$ with the obvious operations is a Morita equivalence from A to C (use Proposition 1.3.4 to see that $E \otimes_B F$ is non-degenerate).

Gathering these facts we can conclude:

Proposition 1.10.3. *G-equivariant Morita equivalence is an equivalence relation on the class of non-degenerate G-Banach algebras.*

Proposition 1.10.4. *Let E be a full and non-degenerate G-Banach B-pair. Then E is a G-equivariant Morita equivalence between $\mathrm{K}(E)$ and B.*

Proof. Obviously, $(E^>, E^<)$ is a full Banach $\mathrm{K}(E)$-pair. The question is whether it is non-degenerate. But this follows easily because $(E^<, E^>)$ is a full and non-degenerate B-pair. □

Corollary 1.10.5. *Let E be a full and non-degenerate B-pair. Then $\mathrm{K}(E)$ is non-degenerate.*

Remark 1.10.6. It is not clear which further regularity conditions are satisfied by the algebra $\mathrm{K}(E)$, even if B is a rather nice algebra. There are examples of Banach spaces E where the closure $F(E)$ of the algebra of finite rank operators on E (which we call $\mathrm{K}(E)$ in this thesis) has no bounded approximate identity. We can even find Banach spaces E where the canonical map from the π-tensor product $F(E) \otimes F(E)$ to $F(E)$ is not surjective.[19]

[19] See [Pis00].

1.10.2 Corners and the linking algebra

Let A be a non-degenerate G-Banach algebra and let p be a G-invariant projection in $\mathrm{M}(A)$. Then pAp is a G-Banach subalgebra of A. Under which circumstances is pAp Morita equivalent to A? A natural choice for the Morita equivalence is (pA, Ap). The right action of pAp, the left action of A and the pAp-valued and A-valued brackets are all given by the product on A. Since A is non-degenerate, we have

$$\mathrm{cl}\,(pA \cdot Ap) = p\,\mathrm{cl}\,(AA)\,p = pAp.$$

and

$$\mathrm{cl}\,(A \cdot Ap) = \mathrm{cl}\,(A \cdot A)\,p = Ap \quad \text{and} \quad \mathrm{cl}\,(pA \cdot A) = p\,\mathrm{cl}\,(AA) = pA.$$

So the pAp-valued bracket is full and the left A-action is non-degenerate. We just need a criterion for the A-valued bracket to be full and the pAp-action to be non-degenerate. It is easy to see that both conditions are equivalent to the following property of p:

Definition 1.10.7 (Full projection). Let A be a Banach algebra. Then a projection p in $\mathrm{M}(A)$ is called *full* if $\mathrm{cl}\,(ApA) = A$.

So we can formulate the following fact:

Proposition 1.10.8. *Let A be a non-degenerate G-Banach algebra and let $p \in \mathrm{M}(A)$ be a G-invariant full projection. Then pAp is a non-degenerate G-Banach algebra and (pA, Ap) is a G-equivariant Morita equivalence from A to pAp.*

Definition 1.10.9 ((Full, complementary) corner). Let A be a non-degenerate G-Banach algebra. A *corner* of A is a subalgebra B of A such there is a G-invariant idempotent $p \in \mathrm{M}(A)$ with $pAp = B$. A corner is said to be *full* if there is a full G-invariant idempotent $p \in \mathrm{M}(A)$ with $pAp = B$. Two corners B and C are *(full) complementary* if there are (full) G-invariant idempotents $p, q \in \mathrm{M}(A)$ such that $p + q = 1$ and $B = pAp$ and $C = qAq$.

By using the transitivity of being Morita equivalent we get the following consequence:

Corollary 1.10.10. *Let B and C be full complementary corners of a non-degenerate G-Banach algebra A. Then B and C are G-equivariantly Morita equivalent to A and hence to each other.*

There is also a direct construction of a Morita equivalence between pAp and qAq, namely (qAp, pAq) with the obvious operations.

Definition 1.10.11 (Linking algebra). Let A and B be G-Banach algebras and let $E = (E^<, E^>)$ be an equivariant Morita equivalence between A and B. Define the *linking algebra*

$$L := \begin{pmatrix} A & E^> \\ E^< & B \end{pmatrix}$$

to be the following G-Banach algebra: The underlying G-Banach space is the direct sum $A \oplus E^> \oplus E^< \oplus B$; the product is given by the operations on A, B and E if we write the elements of L as matrices according to the pattern suggested by our notation.

The linking algebra is non-degenerate and we find A and B as full complementary corners in L.

1.10.3 Morita cycles and Morita morphisms

Definition 1.10.12 (Morita cycle). [20] Let A and B be non-degenerate G-Banach algebras. Then a *Morita cycle*[21] F from A to B is a non-degenerate G-Banach A-B-pair F such that A acts on F by compact operators, i.e., if $\pi_A\colon A \to \mathrm{L}_B(F)$ is the action of A on F, then (π_A is G-equivariant and) $\pi_A(A) \subseteq \mathrm{K}_B(F)$. The class of all Morita cycles from A to B is denoted by $\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$.

Morita cycles are hence exactly the trivially graded elements of $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$ with zero-operator. We can thus apply almost all the definitions we have made for $\mathrm{KK}^{\mathrm{ban}}$-cycles also to Morita cycles (morphisms between them, pullback, push-forward, homotopy, etc.). The extra conditions (trivial grading, zero operator) are compatible with almost all of the constructions and will usually make them simpler.

Definition 1.10.13 (Various elementary constructions). Let A, A', B, B', C, D be non-degenerate G-Banach algebras.

1. Let $F \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$ and $F' \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A',B')$. A morphism Ψ between F and F' is a concurrent G-equivariant homomorphism ${}_{\varphi}\Psi_{\psi}$ from ${}_AF_B$ to ${}_{A'}F'_{B'}$. If we are only considering morphisms Ψ between elements of $\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$, we will usually impose the conditions $\varphi = \mathrm{Id}_A$ and $\psi = \mathrm{Id}_B$.

2. If $F, F' \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$, then F and F' are called *isomorphic* if there is a concurrent isomorphism of G-Banach A-B-pairs with trivial coefficient maps between them.

3. Let $F_1, F_2 \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$. Then $F_1 \oplus F_2$ is also in $\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$. The so-defined operation is associative and commutative up to isomorphism. Moreover, the zero-pair $0 = (0,0) \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$ is a neutral element in $\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$ (up to isomorphism).

4. If $\vartheta\colon A \to B$ and $\psi\colon C \to D$ are equivariant homomorphisms and $F \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(B,C)$, then
$$\vartheta^*(F) \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A,C) \quad \text{and} \quad \psi_*(F) \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(B,D)$$
and the maps $\vartheta^*(\cdot)$ and $\psi_*(\cdot)$ commute and are additive up to isomorphism.

Also the notion of homotopy carries over to Morita cycles, and the use of this notion seems to give a picture of Morita cycles which is even more conceptual than the one presented in [Laf04].

Definition 1.10.14 (Homotopy). Let A and B be non-degenerate G-Banach algebras and $F_0, F_1 \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$. Then a *homotopy* from F_0 to F_1 is an $\mathcal{F} \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A,B[0,1])$ such that $\mathrm{ev}_{0,*}(\mathcal{F}) \cong F_0$ and $\mathrm{ev}_{1,*}(\mathcal{F}) \cong F_1$. If such a homotopy exists, F_0 and F_1 are called *homotopic*. The equivalence relation on $\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$ generated by homotopy will be denoted by $\sim_{\mathrm{h}}$.

It is easy to show (e.g. using Proposition 1.7.10 and Proposition 1.7.9) that homotopy is a reflexive and symmetric relation on $\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$. But just as for $\mathrm{KK}^{\mathrm{ban}}$-cycles, I was not able to prove transitivity. However, using the relation generated by homotopy is just as good.

Definition 1.10.15 (Morita morphism, $\mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$). Let A and B be non-degenerate G-Banach algebras. Then we define
$$\mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(A,B) := \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A,B)/\sim_{\mathrm{h}}.$$
The elements of $\mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$, i.e., the homotopy classes of Morita cycles from A to B, are called *Morita morphisms from A to B*.

[20] Compare [Laf04], definition 2.2.
[21] In French they are called “flèches de Morita”.

The addition of cycles lifts to a well-defined abelian law of composition of Morita morphisms with neutral element $[0]_{\sim_{\mathrm{h}}}$. A straightforward argument shows that homotopy is also compatible with the pullback and pushout of cycles; more precisely:
If A, B, C and D are non-degenerate G-Banach algebras, $F_0, F_1 \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(B, C)$, $\vartheta\colon A \to B$ and $\psi\colon C \to D$ are homomorphisms of G-Banach algebras, then

$$F_0 \sim_{\mathrm{h}} F_1 \quad \Rightarrow \quad \vartheta^*(F_0) \sim_{\mathrm{h}} \vartheta^*(F_1) \ \wedge \ \psi_*(F_0) \sim_{\mathrm{h}} \psi_*(F_1).$$

We therefore have additive maps

$$\vartheta^*(\cdot)\colon \operatorname{Mor}_{\mathrm{G}}^{\mathrm{ban}}(B, C) \to \operatorname{Mor}_{\mathrm{G}}^{\mathrm{ban}}(A, C) \quad \text{and} \quad \psi_*(\cdot)\colon \operatorname{Mor}_{\mathrm{G}}^{\mathrm{ban}}(B, C) \to \operatorname{Mor}_{\mathrm{G}}^{\mathrm{ban}}(A, C).$$

Using Proposition 1.3.7 we can define the composition of Morita cycles as follows:

Definition 1.10.16 (Composition of Morita cycles). [22] Let A, B, C be non-degenerate G-Banach algebras and ${}_AE_B \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$, ${}_BF_C \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(B, C)$. Then

$${}_AE \ \otimes_B \ F_C \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A, C)$$

is called the *composition* of Morita cycles.

The composition of Morita cycles is biadditive up to isomorphism. It is also associative up to isomorphism since the tensor product of pairs is. An interesting question is whether we have left or right identities for this tensor product:

If B is a non-degenerate G-Banach algebra, then ${}_B\underline{B}_B$ is a Morita cycle (the homomorphism[23] $\psi_B\colon B \to \mathrm{L}_B(\underline{B})$ satisfies $\psi_B(B) \subseteq \mathrm{K}_B(\underline{B})$). However, it does not in general act identically on cycles, neither on the left nor on the right. [24] So the isomorphism classes of Morita cycles are not a veritable category (not even mentioning the set-theoretic difficulties). To overcome this problem we switch to homotopy classes, i.e., to Morita morphisms.

Definition and Proposition 1.10.17 (Composition of Morita morphisms). Let A, B, C be non-degenerate G-Banach algebras. The composition of cycles $\otimes_B\colon \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A, B) \times \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(B, C) \to \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A, C)$ lifts to a biadditive associative law of composition on the level of Morita morphisms which we are going to denote by $\otimes_B$ or by $\circ$ (with the order of the factors reversed).

Proof. Let A, B, C be non-degenerate G-Banach algebras. Let $E_0, E_1 \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$ and $F_0, F_1 \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(B, C)$. Let E be a homotopy from E_0 to E_1 and F a homotopy from F_0 to F_1.

First we show that $\mathcal{F} := E_0 \otimes_B F \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A, C[0,1])$ is a homotopy from $E_0 \otimes_B F_0$ to $E_0 \otimes_B F_1$. This is almost trivial since

$$\mathrm{ev}_{i,*}^C(\mathcal{F}) \ = \ \Big(E_0 \otimes_B F\Big) \otimes_{\mathrm{ev}_i^C} \widetilde{C} \ \cong \ E_0 \otimes_B \Big(F \otimes_{\mathrm{ev}_i^C} \widetilde{C}\Big) \ = \ E_0 \otimes_B \mathrm{ev}_{i,*}^C(F) \ \cong \ E_0 \otimes_B F_i$$

for all $i \in \{0, 1\}$.

[22] Compare [Laf04], Proposition 2.6.

[23] See Definition 1.4.4.

[24] An exception are, by definition, cycles (E, T) such that the underlying Banach modules are B-induced in the sense of [Grø96], i.e., $E^> \otimes_B B \cong E^>$ and $B \otimes_B E^< \cong E^<$. If B has a bounded approximate identity, then every non-degenerate B-pair is automatically B-induced in this sense.

Now we show that $\mathcal{F}' := E \otimes_{B[0,1]} F_1[0,1] \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A, C[0,1])$ is a homotopy from $E_0 \otimes_B F_1$ to $E_1 \otimes_B F_1$:

$$\begin{aligned}\mathrm{ev}_{i,*}^{C}\left(\mathcal{F}'\right) &= \mathrm{ev}_{i,*}^{C}\left(E \otimes_{B[0,1]} F_1[0,1]\right) = E \otimes_{B[0,1]} \mathrm{ev}_{i,*}^{C}\left(F_1[0,1]\right) \\ &\overset{1.7.9}{\cong} E \otimes_{B[0,1]} \mathrm{ev}_i^{B,*}\left(F_1\right) \cong \mathrm{ev}_{i,*}^{B}(E) \otimes_B F_1 \cong E_i \otimes_B F_1\end{aligned}$$

for all $i \in \{0,1\}$. □

The remainder of this section is primarily concerned with the proof of the following result:

Theorem 1.10.18. *The non-degenerate G-Banach algebras together with the Morita morphisms form a category (apart from the fact that the morphism classes might not be sets). If A is a non-degenerate G-Banach algebra, then the identity morphism on A is given by the equivalence class of ${}_A\underline{A}_A$.*

We have already proved that the composition is associative. What is missing is the statement about the identity morphisms. We are actually going to show a little bit more, and to formulate this, we define:

Definition 1.10.19 ($\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(\varphi)$, $\mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(\varphi)$). Let A and B be non-degenerate G-Banach algebras and let $\varphi\colon A \to B$ be a G-equivariant homomorphism. Then A acts on $\underline{B}_B$ from the left via φ and the so-constructed Morita cycle will be denoted by $\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(\varphi)$ and its homotopy class by $\mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(\varphi)$ or simply by $[\varphi]$.

Theorem 1.10.20. *The map $\varphi \mapsto \mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(\varphi)$ is a functor from the category of non-degenerate G-Banach algebras and equivariant homomorphisms to the category of non-degenerate G-Banach algebras and Morita morphisms. It has the following property:*

If A, B, C are non-degenerate G-Banach algebras and $\varphi\colon A \to B$, $\psi\colon B \to C$ are equivariant homomorphisms, and if $f \in \mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$, $g \in \mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(B,C)$ are Morita morphisms, then

$$f \otimes_B \mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(\psi) = \psi_*(f) \quad \textit{and} \quad \mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(\varphi) \otimes_B g = \varphi^*(g). \tag{1.3}$$

Before we come to the proof of Theorem 1.10.18 and Theorem 1.10.20 note that the most important thing to prove is Equation (1.3):
Let A, B, C be non-degenerate G-Banach algebras. Let $\varphi\colon A \to B$ and $\psi\colon B \to C$ be homomorphisms of G-Banach algebras. Then $\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(\psi \circ \varphi) = \varphi^*\left(\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(\psi)\right)$. It follows that

$$\mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(\psi \circ \varphi) = \varphi^*(\mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(\psi)) \overset{(1.3)}{=} \mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(\varphi) \otimes_B \mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(\psi).$$

So Equation (1.3) implies that $\mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(\cdot)$ is functorial. And using $\varphi = \mathrm{Id}$ or $\psi = \mathrm{Id}$ one can also deduce the missing bit of Theorem 1.10.18 from Equation (1.3). The first part of the equation is proved in Lemma 1.10.22, the second part in Lemma 1.10.24.

The main technical tool is the following sufficient condition for homotopy. It is Theorem 1.9.1 in the case that the involved operators T and T' vanish, which corresponds to the case of Morita cycles.

Proposition 1.10.21 (Sufficient condition for homotopy for Morita cycles). [25] *Let G be a locally compact Hausdorff group and let A and B be non-degenerate G-Banach algebras. Let F, F' be elements of $\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$ with A-actions π and π'. If there is a morphism Φ from F to F' (with coefficient maps Id_A and Id_B) such that $(\pi(a), \pi'(a)) \in \mathrm{K}(\Phi,\Phi)$ for all $a \in A$, then F and F' are*

[25] Compare Proposition 2.10 of [Laf04].

homotopic; here, as above, $\mathrm{K}(\Phi, \Phi)$ *denotes the set of all pairs of operators* $(S, S') \in \mathrm{L}(E) \times \mathrm{L}(E')$ *such that*

$$\forall \varepsilon > 0\ \exists n \in \mathbb{N}\ \exists e_1^<, \ldots, e_n^< \in E^<, e_1^>, \ldots, e_n^> \in E^> :$$
$$\left\| S - \sum_{i=1}^n \left| e_i^> \right\rangle \left\langle e_i^< \right| \right\| \leq \varepsilon \quad \textit{and} \quad \left\| S' - \sum_{i=1}^n \left| \Phi^>(e_i^>) \right\rangle \left\langle \Phi^<(e_i^<) \right| \right\| \leq \varepsilon.$$

Lemma 1.10.22. *Let* A*,* B *and* C *be non-degenerate* G*-Banach algebras,* $F \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$ *and* $\psi\colon B \to C$ *a* G*-equivariant homomorphism. Then*

$$\psi_*(F) \sim_h F \otimes_B \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(\psi).$$

Proof. Recall that $\psi_*(F) = F \otimes_{\tilde{\psi}} \widetilde{C}$. Let π be the action of A on F. We give an equivariant concurrent homomorphism Φ from $F \otimes_B C$ to $F \otimes_B \widetilde{C}$ which satisfies the sufficient condition for homotopy given above. It is simply defined by

$$\Phi^> \colon F^> \otimes_B C \to F^> \otimes_B \widetilde{C},\ f^> \otimes c \mapsto f^> \otimes c$$

and analogously for $\Phi^<$. It is clear that this defines an equivariant concurrent homomorphism with coefficient maps Id_A and Id_B. Let $a \in A$. We have to show that $\left(\pi(a) \otimes 1_C,\ \pi(a) \otimes 1_{\widetilde{C}}\right)$ is contained in $\mathrm{K}(\Phi, \Phi)$. We do this by showing the following more general result:

$$\forall S \in \mathrm{K}_B(F): \quad \left(S \otimes 1_C,\ S \otimes 1_{\widetilde{C}}\right) \in \mathrm{K}(\Phi, \Phi). \tag{1.4}$$

Because the map that sends $S \in \mathrm{L}_B(F)$ to $\left(S \otimes 1_C,\ S \otimes 1_{\widetilde{C}}\right)$ is linear and contractive, it suffices to show (1.4) in the case that S is a rank one operator. Because F is non-degenerate, it even suffices to consider the case that $S = \left| f^> b^> \right\rangle \left\langle b^< f^< \right|$ for $f^> \in F^>$, $f^< \in F^<$ and $b^>, b^< \in B$. Now

$$\begin{aligned}
(S \otimes 1_C)^>(f'^> \otimes c) &= \left(\left| f^> b^> \right\rangle \left\langle b^< f^< \right| \otimes 1_C\right)(f'^> \otimes c) \\
&= f^> b^> b^< \left\langle f^<,\ f'^> \right\rangle \otimes c \\
&= f^> \otimes \varphi\left(b^>\right) \left\langle \varphi\left(b^<\right) \otimes f^<,\ f'^> \otimes c \right\rangle \\
&= \left| f^> \otimes \varphi\left(b^>\right) \right\rangle \left\langle \varphi\left(b^<\right) \otimes f^< \right|^> (f'^> \otimes c)
\end{aligned}$$

for all $c \in C$ and $f'^> \in F^>$. This and a similar calculation for the right-hand side show

$$S \otimes 1_C = \left| f^> \otimes \varphi\left(b^>\right) \right\rangle \left\langle \varphi\left(b^<\right) \otimes f^< \right| \in \mathrm{K}_C\left(F \otimes_B C\right).$$

The same calculation for $\tilde{C}$ instead of C results in

$$S \otimes 1_{\widetilde{C}} = \left| \Phi^>(f^> \otimes \varphi\left(b^>\right)) \right\rangle \left\langle \Phi^<(\varphi\left(b^<\right) \otimes f^<) \right| \in \mathrm{K}_C\left(F \otimes_B \widetilde{C}\right).$$

So trivially $\left(S \otimes 1_C,\ S \otimes 1_{\widetilde{C}}\right) \in \mathrm{K}(\Phi, \Phi)$. $\square$

Lemma 1.10.23. *Let* A *and* B *be non-degenerate* G*-Banach algebras and* $F \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$*. Define* $\overline{AF} := \left(\overline{F^< A},\ \overline{AF^>}\right) := \left(\mathrm{cl}\left(F^< A\right), \mathrm{cl}\left(AF^>\right)\right)$ *which is a* G*-Banach* A*-*B*-pair. Then* $A \otimes_A F, \overline{AF} \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$ *and*

$$A \otimes_A F \sim_h \overline{AF} \sim_h F.$$

Note that $A \otimes_A F$ *and* $\overline{AF}$ *are* A*-non-degenerate, so every Morita morphism is homotopic to a Morita morphism with non-degenerate left action.*

Proof. Let π be the action of A on F. We are going to define concurrent homomorphisms of G-Banach A-B-pairs from $A \otimes_A F$ to $\overline{AF}$ and from $\overline{AF}$ to F which satisfy the Condition (1.4). On the way we are going to show that $\overline{AF}$ is indeed a Morita morphism.

Define

$$\Phi^>\colon A \otimes_A F^> \to \overline{AF^>},\ a \otimes f^> \mapsto af^>$$

and similarly for the left-hand side. This clearly gives an equivariant concurrent homomorphism with trivial coefficient maps. Let ι denote the obvious concurrent homomorphism from $\overline{AF}$ to F.

Since A is non-degenerate, it suffices to show Condition (1.4) for abc instead of a where $a, b, c \in A$. Let $\varepsilon > 0$. Since $\pi(b)$ is compact, we can find $n \in \mathbb{N}$ and $f_1^>, \dots, f_n^> \in F^>$ and $f_1^<, \dots, f_n^< \in F^<$ such that

$$\left\| \pi(b) - \sum_{i=1}^{n} \left|f_i^>\right\rangle\left\langle f_i^<\right| \right\| \|a\| \, \|c\| \leq \varepsilon.$$

Define

$$\begin{aligned}
S &:= \sum_{i=1}^{n} \left|a \otimes f_i^>\right\rangle\left\langle f_i^< \otimes c\right| && \in \mathrm{K}_B\left(A \otimes_A F\right), \\
S' &:= \sum_{i=1}^{n} \left|af_i^>\right\rangle\left\langle f_i^< c\right| && \in \mathrm{K}_B\left(\overline{AF}\right), \\
S'' &:= \sum_{i=1}^{n} \left|af_i^>\right\rangle\left\langle f_i^< c\right| && \in \mathrm{K}_B\left(F\right).
\end{aligned}$$

If $d \in A$ and $f^> \in F^>$, then

$$\begin{aligned}
S^>(d \otimes f^>) &= \sum_{i=1}^{n} \left(a \otimes f_i^>\right) \left\langle f_i^<,\ \pi(\langle c, d\rangle) f^> \right\rangle \\
&= a \otimes \left(\sum_{i=1}^{n} \left|f_i^>\right\rangle\left\langle f_i^<\right| \right) \pi(\langle c, d\rangle) f^> \\
&= \left(M_{|a\rangle} \circ \left(\sum_{i=1}^{n} \left|f_i^>\right\rangle\left\langle f_i^<\right| \right) \circ M_{\langle c|} \right)^> (d \otimes f^>),
\end{aligned}$$

where $M_{|a\rangle} \in \mathrm{L}_B\left(F,\ A \otimes_A F\right)$ and $M_{\langle c|} \in \mathrm{L}_B\left(A \otimes_A F, F\right)$ are defined as in the proof of Proposition 1.3.7. This and a similar calculation for the left-hand side show

$$S = M_{|a\rangle} \circ \left(\sum_{i=1}^{n} \left|f_i^>\right\rangle\left\langle f_i^<\right| \right) \circ M_{\langle c|}.$$

Let ϕ be the action of A on $A \otimes_A F$. Then for every $d \in A$ and $f^> \in F^>$:

$$\phi(abc)(d \otimes f^>) = (abcd) \otimes f^> = a \otimes bcdf^> = \left(M_{|a\rangle} \circ \pi(b) \circ M_{\langle c|}\right)(d \otimes f^>).$$

Similarly for the left-hand side. So

$$\phi(abc) = M_{|a\rangle} \circ \pi(b) \circ M_{\langle c|}.$$

Hence

$$\begin{aligned}\|\phi(abc) - S\| &= \left\| M_{|a\rangle} \circ \left(\pi(b) - \sum_{i=1}^{n} \left|f_i^{>}\right\rangle\left\langle f_i^{<}\right| \right) \circ M_{\langle c|} \right\| \\ &\leq \|a\| \left\| \pi(b) - \sum_{i=1}^{n} \left|f_i^{>}\right\rangle\left\langle f_i^{<}\right| \right\| \|c\| \leq \varepsilon.\end{aligned}$$

Let π_0 denote the action of A on $\overline{AF}$. Note that

$$\left\|\pi_0(abc) - S'\right\| \leq \left\|\pi(abc) - S''\right\|.$$

Now

$$S'' = \sum_{i=1}^{n} \left|af_i^{>}\right\rangle\left\langle f_i^{<}c\right| = \pi(a) \sum_{i=1}^{n} \left|f_i^{>}\right\rangle\left\langle f_i^{<}\right| \pi(c)$$

and hence

$$\begin{aligned}\left\|\pi(abc) - S''\right\| &= \left\| \pi(abc) - \sum_{i=1}^{n} \left|af_i^{>}\right\rangle\left\langle f_i^{<}c\right| \right\| = \left\| \pi(a) \left(\pi(b) - \sum_{i=1}^{n} \left|f_i^{>}\right\rangle\left\langle f_i^{<}\right| \right) \pi(c) \right\| \\ &\leq \|a\| \left\| \pi(b) - \sum_{i=1}^{n} \left|f_i^{>}\right\rangle\left\langle f_i^{<}\right| \right\| \|c\| \leq \varepsilon.\end{aligned}$$

From this it also follows that

$$\left\|\pi_0(abc) - S'\right\| \leq \varepsilon$$

and hence that $\pi_0(abc) \in \mathrm{K}_B\left(\overline{AF}\right)$. So in particular $\overline{AF} \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$. □

Lemma 1.10.24. *Let A, B and C be non-degenerate G-Banach algebras, $F \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(B, C)$ and $\varphi\colon A \to B$ a G-equivariant homomorphism. Then*

$$\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(\varphi) \otimes_B F \ \sim_h \ \varphi^*(F).$$

Proof. Note that

$$\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(\varphi) \otimes_B F = \varphi^*\left(B \otimes_B F\right).$$

We have already shown that F and $B \otimes_B F$ are homotopic elements of $\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(B, C)$. So by Lemma 1.10.23, $\varphi^*(F)$ and $\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(\varphi) \otimes_B F$ are homotopic elements of $\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A, C)$. □

1.10.4 Morita equivalences induce Morita isomorphisms

We are going to call the isomorphisms in the category of Morita morphisms *Morita isomorphisms*.

Proposition 1.10.25. *Let A and B be non-degenerate G-Banach algebras and let E be a Morita equivalence between A and B. Then E, regarded as a G-Banach A-B-pair with trivial grading, is in $\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$. Let $\mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(E)$ or $[E]$ denote the Morita morphism associated to E.*

Proof. We have to show that the image of the A-action $\pi\colon A \to \mathrm{L}_B(E)$ is contained in $\mathrm{K}_B(E)$. Since π is continuous and linear and since ${}_A\langle E^>, E^<\rangle$ is dense in A, it suffices to check that $\pi\left({}_A\langle e^>,\ e^<\rangle\right) \in \mathrm{K}_B(E)$ for all $e^> \in E^>$ and $e^< \in E^<$. If $x^> \in E^>$, then

$$\pi\left({}_A\left\langle e^>,\ e^<\right\rangle\right)^>(x^>) = {}_A\left\langle e^>, e^<\right\rangle x^> = e^>\left\langle e^<, x^>\right\rangle_B = \left|e^>\right\rangle\!\left\langle e^<\right|^>(x^>).$$

Similarly on the left-hand side. Hence

$$\pi\left({}_A\left\langle e^>,\ e^<\right\rangle\right) = \left|e^>\right\rangle\!\left\langle e^<\right| \in \mathrm{K}_B(E). \tag{1.5}$$

□

Lemma 1.10.26. *Let A, B be non-degenerate G-Banach algebras and let E and E' be Morita equivalences from A to B. Assume that ${}_{\mathrm{Id}_A}\theta_{\mathrm{Id}_B}\colon {}_AE_B \to {}_AE'_B$ is a concurrent morphism of Morita equivalences (meaning that it is an equivariant morphism of Morita cycles that also preserves the left bracket). Then*

$$[E] = [E'] \in \mathrm{Mor}^{\mathrm{ban}}_{\mathrm{G}}(A, B).$$

Proof. We use Condition (1.4). Let π and π' be the action of A on E and on E', respectively. Since ${}_A\langle E^>,\ E^<\rangle$ is dense in A, it suffices to consider only such $a \in A$ which are of the form ${}_A\langle e^>,\ e^<\rangle$ for some $e^> \in E^>$ and $e^< \in E^<$. We have seen in Equation (1.5) that $\pi(a) = \left|e^>\right\rangle\!\left\langle e^<\right| \in \mathrm{K}_B(E)$. Now

$$\pi'(a) = \pi'\left({}_A\left\langle e^>,\ e^<\right\rangle\right) = \pi'\left({}_A\left\langle \theta^>(e^>),\ \theta^<(e^<)\right\rangle\right) \in \mathrm{K}_B(E').$$

So Condition (1.4) is trivially satisfied. □

Theorem 1.10.27. *Let A and B be non-degenerate G-Banach algebras and let E be a Morita equivalence between A and B. Then the Morita morphism $[E]$ is an* isomorphism *with inverse $[E]^{-1} = \left[\overline{E}\right]$.*

Proof. Write ${}_A\langle,\rangle : E^> \times E^< \to A$ for the left bracket and $\langle,\rangle_B : E^< \times E^> \to B$ for the right bracket of the Morita equivalence E.

Note that the composition of Morita morphisms given by Morita equivalences is the Morita morphism given by the composition of the equivalences. We will thus show that the Morita equivalence $F := {}_AE \otimes_B \overline{E}_A$ gives the identity Morita morphism, and we will do so by providing an equivariant concurrent homomorphism θ from $F = {}_AE \otimes_B \overline{E}_A$ to the Morita equivalence $F' := {}_A\underline{A}_A$. We have $F^> = E^> \otimes_B E^< = F^<$. Note that $E^> \otimes_B E^<$ is itself a Banach algebra when equipped with the multiplication that is given on elementary tensors by the formula

$$(e^> \otimes e^<) \cdot (e'^> \otimes e'^<) := e^> \otimes \langle e^<, e'^>\rangle_B e'^<$$

for all $e^>, e'^> \in E^>$ and $e^<, e'^< \in E^<$. Write μ for this product on $E^> \otimes_B E^<$. Note that

$$e^> \otimes e^<{}_A\langle e'^>, e'^<\rangle = e^> \otimes \langle e^<, e'^>\rangle_B e'^< = e^>\langle e^<, e'^>\rangle_B \otimes e'^< = {}_A\langle e^<, e^>\rangle e'^> \otimes e'^<.$$

We define

$$\zeta\colon E^> \otimes_B E^< \to A,\ e^> \otimes e^< \mapsto {}_A\left\langle e^>,\ e^<\right\rangle.$$

This is a homomorphism of Banach algebras:

$$\begin{aligned} & \zeta\left((e^> \otimes e^<) \cdot (e'^< \otimes e'^<)\right) = \zeta\left(e^> \otimes e^<{}_A\langle e'^>, e'^<\rangle\right) \\ = \quad & {}_A\left\langle e^>,\ e^<{}_A\langle e'^>, e'^<\rangle\right\rangle = {}_A\langle e^>, e^<\rangle{}_A\langle e'^>, e'^<\rangle \end{aligned}$$

for all $e^{>}, e'^{>} \in E^{>}$ and $e^{<}, e'^{<} \in E^{<}$. The right bracket and the left bracket from $F^{<} \times F^{>}$ to A are both given by $\zeta \circ \mu$. We check that $\theta := (\zeta,\ \zeta)$ is a G-equivariant concurrent homomorphism with coefficient map Id_A on both sides. Note that

$$\zeta(a(e^{>} \otimes e^{<})) = \zeta((ae^{>}) \otimes e^{<}) = {}_A\big\langle ae^{>},\ e^{<}\big\rangle = a\ {}_A\big\langle e^{>},\ e^{<}\big\rangle = a\zeta(e^{>} \otimes e^{<})$$

for all $a \in A$, $e^{<} \in E^{<}$, and $e^{>} \in E^{>}$, so ζ is A-linear on the left. Similarly on the right-hand side. Moreover,

$$_A\langle s,\ t\rangle = \langle s,\ t\rangle_A = \zeta\,(s \cdot t) = \zeta(s) \cdot \zeta(t) \in A$$

for all $s, t \in E^{>} \otimes_B E^{<}$. As ζ is G-equivariant, θ is indeed an equivariant concurrent homomorphism. □

1.10.5 The action of Morita morphisms on $\mathrm{KK}^{\mathrm{ban}}_{\mathrm{G}}$

Definition and Proposition 1.10.28. Let A, B and C be non-degenerate G-Banach algebras. Let (E, T) be an element of $\mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(A, B)$ and F an element of $\mathbb{M}^{\mathrm{ban}}_{\mathrm{G}}(B, C)$. Then we define

$$\mu_F(E, T) := (E, T) \otimes_B F := (E \otimes_A F,\ T \otimes 1) \in \mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(A, C).$$

Proof. We have to show that $(E, T) \otimes_B F$ is indeed in $\mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(A, C)$. Let $\pi_A\colon A \to \mathrm{L}_B(E)$ be the action of A. Recall from Proposition 1.3.7 that operators of the form "compact tensor one" are compact because B acts on F by compact operators.

1. The operator $T \otimes 1$ is odd.
2. If $a \in A$, then $[(\pi_A(a) \otimes 1), T \otimes 1] = [\pi_A(a), T] \otimes 1 \in \mathrm{K}_C(E \otimes_B F)$.
3. If $a \in A$, then
$$(\pi_A(a) \otimes 1)\left(\mathrm{Id}_{E \otimes_B F} - T^2 \otimes 1\right) = \left(\pi_A(a)(\mathrm{Id}_E - T^2)\right) \otimes 1 \in \mathrm{K}_C\left(E \otimes_B F\right).$$
4. If $a \in A$ and $g \in G$ then
$$(\pi_A(a) \otimes 1)\left(g(T \otimes 1) - T \otimes 1\right) = (\pi_A(a)\left(gT - T\right)) \otimes 1 \in \mathrm{K}_C\left(E \otimes_B F\right).$$
   Moreover, this expression depends continuously on $g \in G$.

□

Definition and Proposition 1.10.29. Let A, B, C be non-degenerate G-Banach algebras. Then the product $\otimes_B\colon\ \mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(A, B) \times \mathbb{M}^{\mathrm{ban}}_{\mathrm{G}}(B, C) \to \mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(A, C)$ is compatible with the respective homotopy relations, so it lifts to a product

$$\otimes_B\colon\ \mathrm{KK}^{\mathrm{ban}}_{\mathrm{G}}(A, B) \times \mathrm{Mor}^{\mathrm{ban}}_{\mathrm{G}}(B, C) \to \mathrm{KK}^{\mathrm{ban}}_{\mathrm{G}}(A, C).$$

Proof. We split the proof into two parts and treat the compatibility in the first and in the second component separately:

1. Let $(E_0, T_0), (E_1, T_1) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$ be homotopic and $F \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(B, C)$. We show that $(E_0, T_0) \otimes_B F$ and $(E_1, T_1) \otimes_B F$ are homotopic in $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, C)$: Find a homotopy $(E, T) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B[0,1])$ such that $\mathrm{ev}_{0,*}^{B}(E, T) \cong (E_0, T_0)$ and $\mathrm{ev}_{0,*}^{B}(E, T) \cong (E_1, T_1)$. The $\mathrm{KK}^{\mathrm{ban}}$-cycle $(E, T) \otimes_{B[0,1]} F[0,1] \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, C[0,1])$ is the homotopy from $(E_0, T_0) \otimes_B F$ to $(E_1, T_1) \otimes_B F$ we are looking for:

$$\begin{aligned} \mathrm{ev}_{t,*}^{C}\left(E \otimes_{B[0,1]} F[0,1]\right) &= \mathrm{ev}_{t,*}^{C}\left(E \otimes_{B[0,1]} F[0,1]\right) = E \otimes_{B[0,1]} \mathrm{ev}_{t,*}^{C}\left(F[0,1]\right) \\ &\cong E \otimes_{B[0,1]} \mathrm{ev}_{t}^{B,*}\left(F\right) \cong \mathrm{ev}_{t,*}^{B}(E) \otimes_B F \end{aligned}$$

   for all $t \in [0,1]$, and these isomorphisms of the underlying pairs are compatible with the respective operators.

2. Let $F_0, F_1 \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(B, C)$ be homotopic and $(E, T) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$. We show that $(E,T) \otimes_B F_0$ and $(E, T) \otimes_B F_1$ are homotopic elements of $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, C)$: Let $F \in \mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}(B, C[0,1])$ be a homotopy from F_0 to F_1. Then

$$\mathrm{ev}_{i,*}^{C}\left((E, T) \otimes_B F\right) \cong (E, T) \otimes_B \mathrm{ev}_{i,*}^{C}(F) \cong (E, T) \otimes_B F_i$$

   as elements of $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, C)$ for all $i \in \{0, 1\}$. Hence $(E,T) \otimes_B F$ is a homotopy from $(E,T) \otimes_B F_0$ to $(E, T) \otimes_B F_1$. □

The action of $\mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}$ on $\mathrm{KK}^{\mathrm{ban}}$ has the following properties:

Proposition 1.10.30. *Let A, B, C, D be non-degenerate G-Banach algebras.*

1. *Let $x, y \in \mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$ and $f \in \mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(B, C)$. Then*

$$(x \oplus y) \otimes_B f = (x \otimes_B f) \oplus (y \otimes_B f).$$

2. *Let $x \in \mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$ and $f, f' \in \mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(B, C)$. Then*

$$x \otimes_B (f \oplus f') = (x \otimes_B f) \oplus (x \otimes_B f').$$

3. *Let $x \in \mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$, $f \in \mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(B, C)$ and $f' \in \mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(C, D)$. Then*[26]

$$x \otimes_B (f \otimes_B f') = (x \otimes_B f) \otimes_C f'.$$

4. *Let $x \in \mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(B, C)$, $f \in \mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(C, D)$, and $\varphi \colon A \to B$ a homomorphism of G-Banach algebras. Then*

$$\varphi^{*}(x \otimes_B f) = \varphi^{*}(x) \otimes_B f.$$

5. *Let $x \in \mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$ and $\psi \colon B \to C$ a homomorphism of G-Banach algebras. Then*

$$x \otimes_B [\psi] = \psi_{*}(x).$$

[26]Compare Proposition 2.9 in [Laf04].

Proof. The properties 1. to 4. are already true on the level of $\mathrm{KK}^{\mathrm{ban}}$-cycles (at least up to isomorphism). We omit their straightforward proofs. We prove property 5.:
Let $x = [(E,T)]$ with $(E,T) \in \mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(A,B)$. We show that $(E \otimes_B C,\ T \otimes 1_C)$ is homotopic to $(E \otimes_B \widetilde{C},\ T \otimes 1_{\widetilde{C}})$ using the sufficient condition given in Theorem 1.9.1.
Remember that we have proved 5. in the case $T = 0$ in Lemma 1.10.22. Define $\Phi\colon E \otimes_B C \to E \otimes_B \widetilde{C}$ as in the proof of Lemma 1.10.22 (with E instead of F). Now we use Equation (1.4) to show that $\left(T \otimes 1_C,\ T \otimes 1_{\widetilde{C}}\right)$ satisfies the hypotheses of Theorem 1.9.1. Let π be the action of A on E.

Let $a \in A$. Then $[\pi(a) \otimes 1_C,\ T \otimes 1_C] = [\pi(a), T] \otimes 1_C$ and the same is true for $1_{\widetilde{C}}$. Letting $S := [\pi(a), T]$ in Equation (1.4) we can conclude that

$$\left([\pi(a) \otimes 1_C,\ T \otimes 1_C],\ \left[\pi(a) \otimes 1_{\widetilde{C}},\ T \otimes 1_{\widetilde{C}}\right]\right) \in \mathrm{K}(\Phi, \Phi).$$

For the other two conditions of Theorem 1.9.1 proceed analogously. □

Note that 1. implies that Morita morphisms act as group homomorphisms on $\mathrm{KK}^{\mathrm{ban}}_{\mathrm{G}}$, whereas 5. implies that the identity morphism acts identically, which, together with 3. implies that Morita isomorphisms act as group isomorphisms on $\mathrm{KK}^{\mathrm{ban}}_{\mathrm{G}}$. Now Theorem 1.10.27 tells us that Morita equivalences induce Morita isomorphisms, so we can deduce the following theorem:

Theorem 1.10.31. [27] *Let A, B, C be non-degenerate G-Banach algebras and let E be a Morita equivalence from B to C. Then $\cdot \otimes_B [E]$ is an isomorphism from $\mathrm{KK}^{\mathrm{ban}}_{\mathrm{G}}(A,B)$ to $\mathrm{KK}^{\mathrm{ban}}_{\mathrm{G}}(A,C)$ with inverse $\cdot \otimes_B \left[\overline{E}\right]$.*

Remark 1.10.32 (Graded Morita morphisms). The Morita cycles presented in this work are $\mathrm{KK}^{\mathrm{ban}}$-cycles with trivial operator and trivial grading. The second condition can be deleted, and if A and B are non-degenerate (trivially graded) Banach algebras, then a graded Morita cycle F from A to B can be thought of as a pair (F_+, F_-) of non-graded Morita cycles from A to B. The advantage of this more general setting is that we can define a structure of an abelian group on the Morita morphisms, making the theory a bit more systematic. We confine ourselves to non-graded Morita cycles because we do not need the graded ones in the rest of the work and we want to avoid further technical difficulties: The suitable equivalence relation on the graded Morita cycles would no longer be the equivalence relation generated by homotopy, but also cycles of the form (F, F), where F is a non-graded Morita cycle, should be equivalent to zero; this is automatic in the case of $\mathrm{KK}^{\mathrm{ban}}$-cycles as degenerate cycles are homotopic to zero, but the homotopy used in this case can only be constructed if non-zero operators are allowed.

[27] Compare Théorème 1.4 in [Laf04], the corresponding result for K_0.

Chapter 2

KK-Theory for $\mathcal{C}_0(X)$-Banach Algebras

Let X be a locally compact Hausdorff space. The notion of a $\mathcal{C}_0(X)$-C*-algebra is well-known in the literature, and it has already been generalised to the concept of a $\mathcal{C}_0(X)$-Banach algebra.[1] For $\mathcal{C}_0(X)$-C*-algebras there is a natural variant of KK-theory called RKK. This chapter is dedicated to the development of an analogous theory for $\mathcal{C}_0(X)$-Banach algebras. This can be thought of as an intermediate step between $\mathrm{KK}^{\mathrm{ban}}$ as defined in the first chapter and the variant of $\mathrm{KK}^{\mathrm{ban}}$ for fields of Banach algebras that we are going to define in the third chapter (following the path of [Laf06]). The $\mathrm{RKK}^{\mathrm{ban}}$-theory defined in the present chapter is really just a straightforward generalisation of $\mathrm{KK}^{\mathrm{ban}}$: The introduction to $\mathrm{KK}^{\mathrm{ban}}$ in the first chapter has been rather detailed to enable us to say that the reader should just browse through the first chapter and add an action of $\mathcal{C}_0(X)$ everywhere. All results from the first chapter carry over, especially the sufficient condition for homotopy and the theory of Morita morphisms.

The starting point for our definition of RKK is the following observation: If A and B are $\mathcal{C}_0(X)$-C*-algebras and (E,T) is a cycle for $\mathrm{RKK}(A,B)$, then E carries a canonical action of $\mathcal{C}_0(X)$ defined through the identification $E \cong E \otimes_B B$ (just let $\mathcal{C}_0(X)$ act on the second factor). This action is the unique action of $\mathcal{C}_0(X)$ on E that is compatible with the module action of B. The usual condition on a RKK-cycle, namely that $(\chi a)(eb) = (ae)(\chi b)$ for all $a \in A$, $e \in E$, $b \in B$ and $\chi \in \mathcal{C}_0(X)$, then just means that the actions of $\mathcal{C}_0(X)$ on A and E should be compatible. So E is what could be called a $\mathcal{C}_0(X)$-Hilbert A-B-module. The corner stone for the definition of $\mathrm{RKK}^{\mathrm{ban}}$ should hence be the notion of a $\mathcal{C}_0(X)$-Banach A-B-pair (if A and B are $\mathcal{C}_0(X)$-Banach algebras). The fundamental notion underlying all this is of course a notion of a $\mathcal{C}_0(X)$-Banach space, which turns out to be rather simple:

2.1 $\mathcal{C}_0(X)$-Banach spaces

Definition 2.1.1 (The category of $\mathcal{C}_0(X)$-Banach spaces). A $\mathcal{C}_0(X)$-Banach space is by definition a non-degenerate Banach $\mathcal{C}_0(X)$-module. If E and F are $\mathcal{C}_0(X)$-Banach spaces, then we take the bounded linear $\mathcal{C}_0(X)$-linear maps from E to F as morphisms from E to F. We are going to denote the morphisms from E to F by $\mathrm{L}^{\mathcal{C}_0(X)}(E,F)$.

Example 2.1.2. Let E be a Banach space. Then $EX = \mathcal{C}_0(X, E)$ is a $\mathcal{C}_0(X)$-Banach space with the canonical action of $\mathcal{C}_0(X)$.

[1] See [Bla96].

Definition 2.1.3 (The product of $\mathcal{C}_0(X)$-Banach spaces). Let E_1 and E_2 be $\mathcal{C}_0(X)$-Banach spaces. Let $E_1 \times E_2$ be the product Banach space (with the sup-norm). We define an action of $\mathcal{C}_0(X)$ on E by $\varphi(e_1, e_2) := (\varphi e_1, \varphi e_2)$ for all $\varphi \in \mathcal{C}_0(X)$, $e_1 \in E_1$ and $e_2 \in E_2$. Then $E_1 \times E_2$ is a $\mathcal{C}_0(X)$-Banach space.

There is also an obvious notion of the sum $E_1 \oplus E_2$ of $\mathcal{C}_0(X)$-Banach spaces E_1 and E_2 using the sum-norm. It is compatible with the $\mathcal{C}_0(X)$-tensor product that we are going to define next.

Definition 2.1.4 ($\mathcal{C}_0(X)$-bilinear). Let E_1, E_2, F be $\mathcal{C}_0(X)$-Banach spaces. An element $\mu \in \mathrm{M}(E_1, E_2; F)$ is called $\mathcal{C}_0(X)$-bilinear if μ is $\mathcal{C}_0(X)$-linear in every component. The (closed) subspace of $\mathrm{M}(E_1, E_2; F)$ formed by the $\mathcal{C}_0(X)$-bilinear maps will be denoted by $\mathrm{M}^{\mathcal{C}_0(X)}(E_1, E_2; F)$.

Definition and Proposition 2.1.5 ($\mathcal{C}_0(X)$-tensor product). Let E_1 and E_2 be $\mathcal{C}_0(X)$-Banach spaces. Consider E_1 and E_2 as Banach $\mathcal{C}_0(X)$-$\mathcal{C}_0(X)$-bimodules. Then we can form the (projective) balanced tensor product $E_1 \otimes_{\mathcal{C}_0(X)} E_2$, being itself a $\mathcal{C}_0(X)$-Banach space. It has the obvious universal property for continuous $\mathcal{C}_0(X)$-bilinear maps. We will denote the $\mathcal{C}_0(X)$-tensor product of E_1 and E_2 by $E_1 \otimes_{\mathcal{C}_0(X)} E_2$.

2.2 $\mathcal{C}_0(X)$-Banach algebras, modules and pairs

2.2.1 $\mathcal{C}_0(X)$-Banach algebras

Definition 2.2.1 ($\mathcal{C}_0(X)$-Banach algebra). A $\mathcal{C}_0(X)$-Banach algebra B is a Banach algebra B which is at the same time a $\mathcal{C}_0(X)$-Banach space such that the multiplication of B is $\mathcal{C}_0(X)$-bilinear.

We discuss an alternative definition of a $\mathcal{C}_0(X)$-Banach algebra using the so-called structure homomorphism in Appendix E.1.

Definition 2.2.2 (Homomorphism of $\mathcal{C}_0(X)$-Banach algebras). Let A and B be $\mathcal{C}_0(X)$-Banach algebras. A homomorphism of $\mathcal{C}_0(X)$-Banach algebras $\varphi\colon A \to B$ is a homomorphism φ of Banach algebras which is at the same time a homomorphism of $\mathcal{C}_0(X)$-Banach spaces (i.e., it is $\mathcal{C}_0(X)$-linear).

Definition 2.2.3 (The fibrewise unitalisation). Let B be a $\mathcal{C}_0(X)$-Banach algebra. Then we define the fibrewise unitalisation of B to be $B \oplus \mathcal{C}_0(X)$. The norm on $B \oplus \mathcal{C}_0(X)$ is the sum-norm and the product is given by

$$(b, \varphi) \cdot (c, \psi) := (bc + \psi b + \varphi c,\ \varphi\psi)$$

for all $b, c \in B$, $\varphi, \psi \in \mathcal{C}_0(X)$. The action of $\mathcal{C}_0(X)$ on $B \oplus \mathcal{C}_0(X)$ is given componentwise. Note that B is contained as a $\mathcal{C}_0(X)$-invariant ideal in $B \oplus \mathcal{C}_0(X)$ and that $B \oplus \mathcal{C}_0(X)$ is non-degenerate, it even has a bounded approximate unit.

2.2.2 $\mathcal{C}_0(X)$-Banach modules

Definition 2.2.4 ($\mathcal{C}_0(X)$-Banach module). Let B, C be $\mathcal{C}_0(X)$-Banach algebras. Then a $\mathcal{C}_0(X)$-Banach B-module is a Banach B-module E which is at the same time a $\mathcal{C}_0(X)$-Banach space such that the module action is $\mathcal{C}_0(X)$-bilinear. Analogously we define $\mathcal{C}_0(X)$-Banach B-C-bimodules.

Lemma 2.2.5. *If B is a $\mathcal{C}_0(X)$-Banach algebra and E is a non-degenerate Banach B-module, then there is at most one $\mathcal{C}_0(X)$-structure on E such that E is a $\mathcal{C}_0(X)$-Banach B-module.*

Proof. Let E have a $\mathcal{C}_0(X)$-Banach B-module structure. Then for all $\varphi \in \mathcal{C}_0(X)$, $e \in E$ and $b \in B$, we have $(eb)\varphi = e(b\varphi)$, so on EB the $\mathcal{C}_0(X)$-action is known from the $\mathcal{C}_0(X)$-action on B. By linearity and continuity it is known on E. □

Lemma 2.2.6. *Let B be a $\mathcal{C}_0(X)$-Banach algebra and let E be a right B-induced Banach B-module in the sense of [Grø96], i.e., assume that $E \otimes_B B \cong E$, canonically. Then there exists a (unique) $\mathcal{C}_0(X)$-structure on E such that E is a $\mathcal{C}_0(X)$-Banach B-module.*

Proof. We have $E \cong E \otimes_B B$ so we can let $\mathcal{C}_0(X)$ act on the factor B of the tensor product to get an action on E. □

Definition 2.2.7 ($\mathrm{L}_B^{\mathcal{C}_0(X)}(E,F)$). Let B be a $\mathcal{C}_0(X)$-Banach algebra and let E, F be $\mathcal{C}_0(X)$-Banach B-modules. Then we write $\mathrm{L}_B^{\mathcal{C}_0(X)}(E,F)$ for the subspace of $\mathrm{L}_B(E,F)$ of operators which are also $\mathcal{C}_0(X)$-linear.

Lemma 2.2.8. *Let B be a $\mathcal{C}_0(X)$-Banach algebra and let E, F be $\mathcal{C}_0(X)$-Banach B-modules such that E* is non-degenerate. *Then all elements of $\mathrm{L}_B(E,F)$ are automatically $\mathcal{C}_0(X)$-linear, i.e., we have*

$$\mathrm{L}_B(E,F) = \mathrm{L}_B^{\mathcal{C}_0(X)}(E,F).$$

Proof. Let $e \in E$, $b \in B$ and $\varphi \in \mathcal{C}_0(X)$. Then

$$T\left(\varphi(eb)\right) = T\left(e(\varphi b)\right) = T(e)(\varphi b) = \varphi(T(e)b) = \varphi T(eb).$$

Since EB is dense in E we have $T(\varphi e) = \varphi T(e)$ for all $e \in E$. □

Lemma 2.2.9. *Let E be a $\mathcal{C}_0(X)$-Banach B-module. Then for every $\varphi \in \mathcal{C}_0(X)$, the map $e \mapsto \varphi e$ on E is in $\mathrm{L}_B^{\mathcal{C}_0(X)}(E)$.*

The definition of homomorphisms with coefficient maps between $\mathcal{C}_0(X)$-Banach modules is the obvious variation of the basic Definition 1.1.3, requiring all maps to be $\mathcal{C}_0(X)$-linear.

The balanced $\mathcal{C}_0(X)$-tensor product of $\mathcal{C}_0(X)$-Banach modules

Let A, B, C be $\mathcal{C}_0(X)$-Banach algebras, let E be a $\mathcal{C}_0(X)$-Banach A-B-bimodule, let F be a $\mathcal{C}_0(X)$-Banach B-C-bimodule and let G be a $\mathcal{C}_0(X)$-Banach A-C-bimodule.

Definition 2.2.10 (Balanced $\mathcal{C}_0(X)$-bilinear maps). The set of all balanced bilinear maps from $E\times F$ to G that are also $\mathcal{C}_0(X)$-bilinear will be denoted by $\mathrm{M}^{\mathrm{bal},\mathcal{C}_0(X)}(E,F;G)$. In the same spirit we use the notation ${}_A\mathrm{M}_C^{\mathrm{bal},\mathcal{C}_0(X)}(E,F;G)$, etc.

Lemma 2.2.11. *Let $\mu \in {}_A\mathrm{M}_C^{\mathrm{bal}}(E,F;G)$. If E is B-non-degenerate and F is C-non-degenerate, then μ is automatically $\mathcal{C}_0(X)$-multilinear.*

Definition 2.2.12 (The balanced $\mathcal{C}_0(X)$-tensor product of Banach modules). The *balanced $\mathcal{C}_0(X)$-tensor product $E \otimes_B^{\mathcal{C}_0(X)} F$* of E and F over B is defined to be the universal object for the balanced $\mathcal{C}_0(X)$-multilinear maps on $E \times F$. It can be obtained by taking $E \otimes_B F$ and dividing out elements of the form $e\varphi \otimes f - e \otimes \varphi f$. Alternatively (and more conceptually) it can be constructed by taking the $\mathcal{C}_0(X)$-tensor product $E \otimes^{\mathcal{C}_0(X)} F$ as a substitute for the projective tensor product (of Banach spaces) and proceed exactly as in the construction of the usual balanced tensor product.

Proposition 2.2.13. *If in the preceding definition E or F is B-non-degenerate, then the usual balanced tensor product and the balanced $\mathcal{C}_0(X)$-tensor product agree:*

$$E \otimes_B^{\mathcal{C}_0(X)} F = E \otimes_B F.$$

The pushout

Definition 2.2.14 (The pushout). Let B and B' be $\mathcal{C}_0(X)$-Banach algebras and let $\psi\colon B \to B'$ be a $\mathcal{C}_0(X)$-linear homomorphism. Let E be a right $\mathcal{C}_0(X)$-Banach B-module. Note that E is also a Banach $B \oplus \mathcal{C}_0(X)$-module and ψ can be extended to a morphism from $B \oplus \mathcal{C}_0(X)$ to $B' \oplus \mathcal{C}_0(X)$. Now we define

$$\psi_*(E) := E \otimes_{B\oplus\mathcal{C}_0(X)} (B' \oplus \mathcal{C}_0(X)),$$

being a $\mathcal{C}_0(X)$-Banach B'-module. If E is a non-degenerate Banach B-module, then one could take the tensor product over B instead of $B \oplus \mathcal{C}_0(X)$ and $\psi_*(E)$ is non-degenerate as a Banach B'-module.

Proof. By definition, $\psi_*(E)$ is a Banach $B' \oplus \mathcal{C}_0(X)$-module, so it also is a $\mathcal{C}_0(X)$-Banach B'-module. If E is non-degenerate then the bilinear map $(e, (b', \chi)) \mapsto e \otimes (b', \chi)$ from $E \times (B' \oplus \mathcal{C}_0(X))$ to $E \otimes_B (B' \oplus \mathcal{C}_0(X))$ is not only B-balanced but automatically $B \oplus \mathcal{C}_0(X)$-balanced. Hence the tensor products over B and over $B \oplus \mathcal{C}_0(X)$ agree. The fact that $\psi_*(E)$ is non-degenerate as a B'-module follows as in the case of the ordinary pushout. $\square$

Lemma 2.2.15. *Let B be a $\mathcal{C}_0(X)$-Banach algebra and let E be a $\mathcal{C}_0(X)$-Banach B-module. Then the map $e \otimes (b, f) \mapsto eb + ef$ induces an isometric isomorphism of $\mathcal{C}_0(X)$-Banach B-modules*

$$\mathrm{Id}_{B,*}(E) = E \otimes_{B\oplus\mathcal{C}_0(X)} (B \oplus \mathcal{C}_0(X)) \;\cong\; E.$$

Proof. Denote the map by Φ, being a $\mathcal{C}_0(X)$-linear homomorphism with coefficient map Id_B. We show that it is injective and a quotient map.

To see that it is injective let t be an element of its kernel. We show that $t\chi = 0$ for all $\chi \in \mathcal{C}_0(X)$. As this is also true for an approximate unit in $\mathcal{C}_0(X)$, this shows $t = 0$. Represent t as $\sum_{n\in\mathbb{N}} e_n \otimes (b_n, f_n)$ with $e_n \in E$, $b_n \in B$ and $f_n \in \mathcal{C}_0(X)$. Then $0 = \Phi(t) = \sum_{n\in\mathbb{N}} e_n b_n + e_n f_n$. Now

$$\begin{aligned} t\chi &= \sum_{n\in\mathbb{N}} e_n \otimes [(b_n, f_n)(0, \chi)] = \sum_{n\in\mathbb{N}} e_n(b_n, f_n) \otimes (0, \chi) \\ &= \left[\sum_{n\in\mathbb{N}} e_n b_n + e_n f_n\right] \otimes (0, \chi) = 0 \otimes (0, \chi) = 0. \end{aligned}$$

To see that Φ is a quotient map let $e \in E$ and $\varepsilon > 0$. By Cohen's Factorisation Theorem we can find $e' \in E$ and $f \in \mathcal{C}_0(X)$ such that $e'f = e$, $\|e - e'\| < \varepsilon$ and $\|f\| \leq 1$. Let $t := e' \otimes (0, f) \in \mathrm{Id}_{B,*}(E)$. Then $\|t\| \leq \|e'\|\,\|f\| \leq \|e\| + \varepsilon$ and $\Phi(t) = e'f = e$. So Φ is a quotient map. $\square$

Because this construction is clearly natural in B, we get the first part of the following proposition. The second part is proved similarly.

Proposition 2.2.16 (Functorial properties of the pushout).

- *Let B be a $\mathcal{C}_0(X)$-Banach algebra. Then the functor $(\mathrm{Id}_B)_*$ is naturally isometrically isomorphic to the identity functor on the category of $\mathcal{C}_0(X)$-Banach B-modules.*

- *Let B, B', B'' be $\mathcal{C}_0(X)$-Banach algebras and let $\psi\colon B \to B'$, $\psi'\colon B' \to B''$ be homomorphisms. Then $\psi'_* \circ \psi_*$ and $(\psi' \circ \psi)_*$ are naturally isometrically isomorphic functors from the category of $\mathcal{C}_0(X)$-Banach B-modules to the category of $\mathcal{C}_0(X)$-Banach B''-modules.*

2.2.3 $\mathcal{C}_0(X)$-Banach pairs

Definition 2.2.17 ($\mathcal{C}_0(X)$-Banach pair). Let B be a $\mathcal{C}_0(X)$-Banach algebra. A $\mathcal{C}_0(X)$-Banach B-pair E is a B-pair E such that $E^<$ and $E^>$ are $\mathcal{C}_0(X)$-Banach B-modules and such that the inner product is $\mathcal{C}_0(X)$-bilinear. If A is another $\mathcal{C}_0(X)$-Banach algebra, then a Banach A-B-pair E is a $\mathcal{C}_0(X)$-Banach A-B-pair if it is a $\mathcal{C}_0(X)$-Banach B-pair and the actions of A on $E^<$ and $E^>$ are $\mathcal{C}_0(X)$-bilinear.

Example 2.2.18. Let B be a $\mathcal{C}_0(X)$-Banach algebra. Then $\underline{B}$ is a $\mathcal{C}_0(X)$-Banach B-pair.

The following lemmas are the Banach pair versions of Lemma 2.2.5 and Lemma 2.2.6 for Banach modules.

Lemma 2.2.19. *If B is a $\mathcal{C}_0(X)$-Banach algebra and E is a non-degenerate Banach B-pair such that $E^<$ and $E^>$ are $\mathcal{C}_0(X)$-Banach B-modules, then the inner product is automatically $\mathcal{C}_0(X)$-bilinear.*

Lemma 2.2.20. *Let B be a $\mathcal{C}_0(X)$-Banach algebra and let E be a B-induced Banach B-pair, i.e., $B \otimes_B E^< \cong E^<$ and $E^> \otimes_B B \cong E^>$. Then there exists a unique $\mathcal{C}_0(X)$-action on E such that E becomes a $\mathcal{C}_0(X)$-Banach B-pair.*

Definition 2.2.21 (Linear operators between $\mathcal{C}_0(X)$-Banach pairs). Let E and F be $\mathcal{C}_0(X)$-Banach B-pairs. Then an element T of $\mathrm{L}_B(E,F)$ is called $\mathcal{C}_0(X)$-linear if $T^<$ and $T^>$ are $\mathcal{C}_0(X)$-linear. The subspace of all $\mathcal{C}_0(X)$-linear maps in $\mathrm{L}_B(E,F)$ is denoted by $\mathrm{L}_B^{\mathcal{C}_0(X)}(E,F)$.

Lemma 2.2.22. *Let E and F be $\mathcal{C}_0(X)$-Banach B-pairs. If E and F are non-degenerate, then $\mathrm{L}_B(E,F) = \mathrm{L}_B^{\mathcal{C}_0(X)}(E,F)$, i.e., $\mathcal{C}_0(X)$-linearity is automatic.*

Lemma 2.2.23. *Let E be a $\mathcal{C}_0(X)$-Banach B-pair. Then for every $\varphi \in \mathcal{C}_0(X)$, the pair of maps $(e^< \mapsto \varphi e^<,\ e^> \mapsto \varphi e^>)$ is in $\mathrm{L}_B^{\mathcal{C}_0(X)}(E)$.*

As in the case of $\mathcal{C}_0(X)$-Banach modules the definition of concurrent homomorphisms with coefficient maps between $\mathcal{C}_0(X)$-Banach pairs is the obvious variation of the basic Definitions 1.2.9 and 1.2.11, requiring all maps to be $\mathcal{C}_0(X)$-linear.

Compact operators between $\mathcal{C}_0(X)$-Banach pairs

Proposition 2.2.24. *Let E and F be $\mathcal{C}_0(X)$-Banach B-pairs. Then $\mathrm{K}_B(E,F)$ is always contained in $\mathrm{L}_B^{\mathcal{C}_0(X)}(E,F)$, i.e., $\mathcal{C}_0(X)$-linearity is automatic for compact operators.*

Proof. Let $f^> \in F^>$ and $e^< \in E^<$. Let $T := \left|f^>\right\rangle\left\langle e^<\right|$. To show that $T^>$ is $\mathcal{C}_0(X)$-linear let $e^> \in E^>$ and $\varphi \in \mathcal{C}_0(X)$. Then

$$T^>(\varphi e^>) = f^>\langle e^<,\, \varphi e^>\rangle = f^>(\varphi\langle e^<,\, e^>\rangle) = \varphi(f^>\langle e^<,\, e^>\rangle) = \varphi T^>(e^>).$$

Similarly one shows that $T^<$ is $\mathcal{C}_0(X)$-linear. Now the set of all $\mathcal{C}_0(X)$-linear elements in $\mathrm{L}_B(E,F)$ is a closed subspace, so it contains the whole of $\mathrm{K}_B(E,F)$. □

Proposition 2.2.25. *Let E and F be $\mathcal{C}_0(X)$-Banach B-pairs. Then $\mathrm{K}_B(E,F)$ is a $\mathcal{C}_0(X)$-Banach space. The canonical bilinear map from $F^> \times E^< \to \mathrm{K}_B(E,F)$ is $\mathcal{C}_0(X)$-bilinear.*

Proof. We have to show that $\mathrm{K}_B(E,F)$ is invariant under the $\mathcal{C}_0(X)$-action and that $\mathrm{K}_B(E,F)$ is a non-degenerate Banach $\mathcal{C}_0(X)$-module.

Let $f^> \in F^>$ and $e^< \in E^<$. Let $\varphi \in \mathcal{C}_0(X)$. Then

$$\varphi\left(|f^>\rangle\langle e^<|\right)(e^>) = \varphi\left(f^>\langle e^<, e^>\rangle\right) = |\varphi f^>\rangle\langle e^<|(e^>)$$

for all $e^> \in E^>$. Similarly on the left-hand side. So

$$\varphi\left(|f^>\rangle\langle e^<|\right) = |\varphi f^>\rangle\langle e^<| = |f^>\rangle\langle \varphi e^<|.$$

By linearity and continuity we can conclude that $\mathrm{K}_B(E,F)$ is invariant under the $\mathcal{C}_0(X)$-action. We also see that $|f^>\rangle\langle e^<|$ can be approximated by elements of the form $\varphi|f^>\rangle\langle e^<|$, so $\mathrm{K}_B(E,F)$ is a non-degenerate $\mathcal{C}_0(X)$-module. □

Proposition 2.2.26. *Let E, F and G be $\mathcal{C}_0(X)$-Banach B-pairs. Then the composition of elements of $\mathrm{K}_B(F,G)$ and $\mathrm{K}_B(E,F)$ is $\mathcal{C}_0(X)$-bilinear and $\mathrm{K}_B(E)$ is a $\mathcal{C}_0(X)$-Banach algebra.*

Definition 2.2.27 (Locally compact operator). Let E and F be $\mathcal{C}_0(X)$-Banach B-pairs. Then $T \in \mathrm{L}_B^{\mathcal{C}_0(X)}(E,F)$ is called *locally compact* if χT is compact for all $\chi \in \mathcal{C}_0(X)$.

If T is in $\mathrm{L}_B(E,F)$ such that χT is compact for all $\chi \in \mathcal{C}_0(X)$, then T is automatically $\mathcal{C}_0(X)$-linear. Moreover, it suffices to check $\chi T \in \mathrm{K}_B(E,F)$ for all $\chi \in \mathcal{C}_c(X)$. The bounded locally compact operators form a closed subset of $\mathrm{L}_B^{\mathcal{C}_0(X)}(E,F)$.

Balanced tensor product and the pushout

The definition of the balanced tensor product of $\mathcal{C}_0(X)$-Banach pairs is the obvious result of pairing Definition 1.3.3, the definition of the balanced product of ordinary Banach pairs, and Definition 2.2.12, the definition of the balanced $\mathcal{C}_0(X)$-tensor product of Banach modules. If all the Banach pairs are non-degenerate, then one does not even need to take the $\mathcal{C}_0(X)$-tensor product, the ordinary balanced tensor product does the job.

Similar things can be said about the pushout: Just take the definition of the pushout of Banach pairs (Definition 1.3.9) and pair it with the definition of the pushout of $\mathcal{C}_0(X)$-Banach modules (Definition 2.2.14) to get the definition of the pushout of a $\mathcal{C}_0(X)$-Banach pair under a homomorphism of $\mathcal{C}_0(X)$-Banach algebras. It has the desired functorial properties (compare Proposition 1.3.11 and Proposition 2.2.16).

2.3 The pullback

Let X and Y be locally compact Hausdorff spaces and $p\colon Y \to X$ be continuous.

2.3.1 The pullback of $\mathcal{C}_0(X)$-Banach spaces

Definition 2.3.1 (The pullback). For every $\mathcal{C}_0(X)$-Banach space E, we define

$$p^*(E) := \theta_*(E) := E \otimes^{\mathcal{C}_0(X)} \mathcal{C}_0(Y)$$

being a $\mathcal{C}_0(Y)$-Banach space, where $\theta\colon \mathcal{C}_0(X) \to \mathcal{C}_b(Y),\ \varphi \mapsto \varphi \circ p$.

If E and F are $\mathcal{C}_0(X)$-Banach spaces and $T \in \mathrm{L}_{\mathcal{C}_0(X)}(E,F)$, then we define

$$p^*(T) := T \otimes 1\colon E \otimes^{\mathcal{C}_0(X)} \mathcal{C}_0(Y) \to F \otimes^{\mathcal{C}_0(X)} \mathcal{C}_0(Y),\ e \otimes \chi \mapsto T(e) \otimes \chi.$$

The so defined map is a functor from the category of $\mathcal{C}_0(X)$-Banach spaces to the category of $\mathcal{C}_0(Y)$-Banach spaces, linear and contractive on the morphism sets.

Example 2.3.2. We have

$$p^*\left(\mathcal{C}_0(X)\right) = \mathcal{C}_0(X) \otimes^{\mathcal{C}_0(X)} \mathcal{C}_0(Y) \cong \mathcal{C}_0(Y)$$

as $\mathcal{C}_0(Y)$-Banach spaces where the isomorphism is given by the product.

Remark 2.3.3. In the proof of the following proposition we use some machinery which we just want to sketch here to avoid yet another appendix: If B and B' are $\mathcal{C}_0(X)$-Banach algebras, then the $\mathcal{C}_0(X)$-tensor product $B \otimes^{\mathcal{C}_0(X)} B'$ carries a canonical $\mathcal{C}_0(X)$-Banach *algebra* structure. If E_B and $E'_{B'}$ are $\mathcal{C}_0(X)$-Banach modules, then $E \otimes^{\mathcal{C}_0(X)} E'$ is a $\mathcal{C}_0(X)$-Banach $B \otimes^{\mathcal{C}_0(X)} B'$-module in a canonical way. If E and E' are non-degenerate, then so is $E \otimes^{\mathcal{C}_0(X)} E'$. And finally, if ${}_BF$ and ${}_{B'}F'$ are non-degenerate left $\mathcal{C}_0(X)$-Banach modules, then

$$(E \otimes_B F) \otimes_{\mathcal{C}_0(X)} \left(E' \otimes_{B'} F'\right) \cong \left(E \otimes^{\mathcal{C}_0(X)} E'\right) \otimes_{B \otimes^{\mathcal{C}_0(X)} B'} \left(F \otimes^{\mathcal{C}_0(X)} F'\right).$$

Proposition 2.3.4. *The functor $p^*(\cdot)$ commutes with the tensor product: If E_1 and E_2 are $\mathcal{C}_0(X)$-Banach spaces, then there is a natural isometric isomorphism*

$$p^*(E_1) \otimes^{\mathcal{C}_0(Y)} p^*(E_2) \cong p^*\left(E_1 \otimes^{\mathcal{C}_0(X)} E_2\right).$$

Proof. Define a map $\mathfrak{m}^{p^*}_{E_1,E_2} \colon p^*(E_1) \otimes^{\mathcal{C}_0(Y)} p^*(E_2) \to p^*\left(E_1 \otimes^{\mathcal{C}_0(X)} E_2\right)$ sending $(e_1 \otimes \varphi_1) \otimes (e_2 \otimes \varphi_2)$ to $(e_1 \otimes e_2) \otimes (\varphi_1\varphi_2)$. Now we use Remark 2.3.3: $\mathcal{C}_0(X) \otimes^{\mathcal{C}_0(X)} \mathcal{C}_0(Y) \cong \mathcal{C}_0(Y)$ is also isomorphic to $\mathcal{C}_0(Y)$ as a $\mathcal{C}_0(X)$-Banach algebra and it follows that

$$\begin{aligned} p^*(E_1) \otimes^{\mathcal{C}_0(Y)} p^*(E_2) &= \left(E_1 \otimes^{\mathcal{C}_0(X)} \mathcal{C}_0(Y)\right) \otimes_{\mathcal{C}_0(X) \otimes^{\mathcal{C}_0(X)} \mathcal{C}_0(Y)} \left(E_2 \otimes^{\mathcal{C}_0(X)} \mathcal{C}_0(Y)\right) \\ &\cong \left(E_1 \otimes^{\mathcal{C}_0(X)} E_2\right) \otimes^{\mathcal{C}_0(X)} \left(\mathcal{C}_0(Y) \otimes^{\mathcal{C}_0(Y)} \mathcal{C}_0(Y)\right) \\ &\cong \left(E_1 \otimes^{\mathcal{C}_0(X)} E_2\right) \otimes^{\mathcal{C}_0(X)} \mathcal{C}_0(Y) = p^*\left(E_1 \otimes^{\mathcal{C}_0(X)} E_2\right). \end{aligned}$$

The composition of these isomorphisms is $\mathfrak{m}^{p^*}_{E_1,E_2}$. It is natural: Let F_1, F_2 be some other $\mathcal{C}_0(X)$-Banach spaces and $T_i \in \mathrm{L}^{\mathcal{C}_0(X)}(E_i, F_i)$. Then for all $e_1 \in E_1$, $e_2 \in E_2$ and $\chi_1, \chi_2 \in \mathcal{C}_0(X)$:

$$\begin{aligned} & \mathfrak{m}^{p^*}_{F_1,F_2}\left((p^*(T_1) \otimes p^*(T_2))\left((e_1 \otimes \chi_1) \otimes (e_2 \otimes \chi_2)\right)\right) \\ = \; & \mathfrak{m}^{p^*}_{F_1,F_2}\left(p^*(T_1)(e_1 \otimes \chi_1) \; \otimes \; p^*(T_2)(e_2 \otimes \chi_2)\right) \\ = \; & \mathfrak{m}^{p^*}_{F_1,F_2}\left(T_1(e_1) \otimes \chi_1, \; T_2(e_2) \otimes \chi_2\right) \\ = \; & (T_1(e_1) \otimes T_2(e_2)) \otimes (\chi_1\chi_2) \\ = \; & ((T_1 \otimes T_2)(e_1 \otimes e_2)) \otimes (\chi_1\chi_2) \\ = \; & p^*\left(T_1 \otimes T_2\right)\left((e_1 \otimes e_2) \otimes (\chi_1\chi_2)\right) \\ = \; & p^*\left(T_1 \otimes T_2\right)\left(\mathfrak{m}^{p^*}_{E_1,E_2}\left((e_1 \otimes \chi_1) \otimes (e_2 \otimes \chi_2)\right)\right). \end{aligned}$$

In short $\mathfrak{m}^{p^*}_{F_1,F_2} \circ (p^*(T_1) \otimes p^*(T_2)) = p^*\left(T_1 \otimes T_2\right) \circ \mathfrak{m}^{p^*}_{E_1,E_2}$, so the isomorphism is natural. □

Definition 2.3.5. Let E_1, E_2 and F be $\mathcal{C}_0(X)$-Banach spaces and let $\mu\colon E_1 \times E_2 \to F$ be $\mathcal{C}_0(X)$-bilinear and continuous. Then the map

$$p^*(\mu)\colon p^*(E_1) \times p^*(E_2) \to p^*(F),\ (e_1 \otimes \chi_1,\ e_2 \otimes \chi_2) \mapsto \mu(e_1, e_2) \otimes \chi_1\chi_2$$

is a $\mathcal{C}_0(Y)$-bilinear continuous map such that $\|p^*(\mu)\| \leq \|\mu\|$.

If we identify $p^*(E_1) \otimes^{\mathcal{C}_0(Y)} p^*(E_2)$ and $p^*(E_1 \otimes^{\mathcal{C}_0(X)} E_2)$, then we have

$$\widehat{p^*(\mu)} = p^*\left(\hat{\mu}\right).$$

Proposition 2.3.6 (Preservation of associativity). *Let E_1, E_2, E_3, F_1, F_2 and G be $\mathcal{C}_0(X)$-Banach spaces. Let $\mu_1 \in \mathrm{M}^{\mathcal{C}_0(X)}(E_1, E_2; F_1)$, $\mu_2 \in \mathrm{M}^{\mathcal{C}_0(X)}(E_2, E_3; F_2)$, $\nu_1 \in \mathrm{M}^{\mathcal{C}_0(X)}(F_1, E_3; G)$, and $\nu_2 \in \mathrm{M}^{\mathcal{C}_0(X)}(E_1, F_2; G)$. Assume that*

$$\hat{\nu_1} \circ (\hat{\mu_1} \otimes \mathrm{Id}_{E_3}) = \hat{\nu_2} \circ (\mathrm{Id}_{E_1} \otimes \hat{\mu_2}).$$

Then the same law holds after applying the pullback functor:

$$\widehat{p^*\left(\nu_1\right)} \circ \left(\widehat{p^*\left(\mu_1\right)} \otimes \mathrm{Id}_{p^*(E_3)}\right) = \widehat{p^*\left(\nu_2\right)} \circ \left(\mathrm{Id}_{p^*(E_1)} \otimes \widehat{p^*\left(\mu_2\right)}\right).$$

Proposition 2.3.7.

1. *If $X = Y$ and $p = \mathrm{Id}_X$, then p^* is naturally isomorphic to the identity functor on the category of $\mathcal{C}_0(X)$-Banach spaces, the natural transformation being linear and isometric and compatible with the tensor product.*

2. *If Z is another locally compact Hausdorff space and $q\colon Z \to Y$ is continuous, then $q^* \circ p^*$ and $(p \circ q)^*$ are naturally isomorphic, the natural transformation being linear and isometric and compatible with the tensor product.*

Proof. We just give the isomorphisms and leave it to the reader to check naturality and the other properties.

1. The natural isomorphism $\mathrm{Id}_X^*(E) = E \otimes^{\mathcal{C}_0(X)} \mathcal{C}_0(X) \cong E$ is given by the module action.

2. The natural isomorphism $q^*\left(p^*(E)\right) \cong \left(p \circ q\right)^*(E)$ is defined as the map that sends $e \otimes \chi \otimes \chi'$ to $e \otimes (\chi \circ q)\chi'$, so it is the tensor product of Id_E and the canonical isomorphism from $\mathcal{C}_0(Y) \otimes^{\mathcal{C}_0(Y)} \mathcal{C}_0(Z)$ to $\mathcal{C}_0(Z)$ and hence it is an isomorphism. □

Proposition 2.3.8. *If $T \in \mathrm{L}_{\mathcal{C}_0(X)}(E, F)$ has dense image, then so has $p^*(T)$.*

Proposition 2.3.9. *Let E_1, E_2 and F be $\mathcal{C}_0(X)$-Banach spaces and let $\mu\colon E_1 \times E_2 \to F$ be $\mathcal{C}_0(X)$-bilinear and continuous. Let $p\colon Y \to X$ be continuous. If μ is non-degenerate, then so is $p^*(\mu)$.*

Proof. If μ is non-degenerate, then $\hat{\mu}$ has dense image. Then also $p^*(\hat{\mu})$ has dense image, so $\widehat{p^*(\mu)}$ has dense image, too. This just means that $p^*(\mu)$ is non-degenerate. □

2.3.2 The pullback of Banach algebras, etc.

Let B be a $\mathcal{C}_0(X)$-Banach algebra with product μ. Then p^*B is a $\mathcal{C}_0(Y)$-Banach algebra with multiplication $p^*\mu$. If B' is another $\mathcal{C}_0(X)$-Banach algebra and $\psi\colon B \to B'$ is a homomorphism of $\mathcal{C}_0(X)$-Banach algebras, then $p^*\psi$ is a homomorphism of $\mathcal{C}_0(Y)$-Banach algebras from p^*B to p^*B'.

If E is a right $\mathcal{C}_0(X)$-Banach B-module, then p^*E is a right $\mathcal{C}_0(Y)$-Banach p^*B-module. If F is another $\mathcal{C}_0(X)$-Banach B-module, then $p^*(E \oplus F) \cong (p^*E) \oplus (p^*F)$. Similar things can be said about left Banach modules. If $T \in \mathrm{L}_B^{\mathcal{C}_0(X)}(E,F)$, then $p^*T \in \mathrm{L}_{p^*B}^{\mathcal{C}_0(Y)}(p^*E, p^*F)$.

If E is a right $\mathcal{C}_0(X)$-Banach B-module and F is a left $\mathcal{C}_0(X)$-Banach B-module, then

$$p^*\left(E \otimes_B^{\mathcal{C}_0(X)} F\right) \cong p^*E \otimes_{p^*B}^{\mathcal{C}_0(Y)} p^*F.$$

If $\widetilde{B}$ is the Banach algebra $B \oplus \mathcal{C}_0(X)$, then $p^*(\widetilde{B}) \cong \widetilde{p^*B}$. Finally, if $\psi\colon B \to B'$ is a $\mathcal{C}_0(X)$-linear homomorphism of Banach algebras, then $p^*\left(\psi_*(E)\right) \cong (p^*\psi)_* p^*E$ for all right $\mathcal{C}_0(X)$-Banach B-modules.

If $E = (E^<, E^>)$ is a $\mathcal{C}_0(X)$-Banach B-pair, then $p^*E = (p^*E^<,\, p^*E^>)$ is a $\mathcal{C}_0(Y)$-Banach p^*B-pair in a canonical way. The pullback along p is compatible with linear operators, homomorphisms, the balanced tensor product, the direct sum and the pushout (just as for Banach modules).

The pullback of a compact operator is not compact in general. However, we have the following result:

Proposition 2.3.10. *Let E and F be $\mathcal{C}_0(X)$-Banach B-pairs over some $\mathcal{C}_0(X)$-Banach algebra B. Let T be a locally compact bounded operator from E to F. Then p^*T is a locally compact bounded operator from p^*E to p^*F satisfying $\|p^*T\| \leq \|T\|$.*

Proof. Let $\chi \in \mathcal{C}_0(Y)$. Find $\chi_1, \chi_2 \in \mathcal{C}_0(Y)$ such that $\chi = \chi_1\chi_2$. Let $e^< \in E^<$ and $f^> \in F^>$. Then

$$\chi p^* \big|f^>\big\rangle\big\langle e^<\big|^>(e^> \otimes \varphi) = f^>\langle e^<, e^>\rangle \otimes \chi_1\chi_2\varphi = \big|f^> \otimes \chi_1\big\rangle\big\langle e^< \otimes \chi_2\big|^>(e^> \otimes \varphi)$$

for all $e^> \in E^>$ and $\varphi \in \mathcal{C}_0(Y)$ (and similarly for the left-hand side). It follows that $\chi p^*\big|f^>\big\rangle\big\langle e^<\big| = \big|f^> \otimes \chi_1\big\rangle\big\langle e^< \otimes \chi_2\big|$. In particular, $p^*\big|f^>\big\rangle\big\langle e^<\big|$ is locally compact. It follows that p^*S is locally compact whenever S is compact.

Let T be locally compact. Because $\mathcal{C}_0(Y)$ is a non-degenerate $\mathcal{C}_0(X)$-module, we can factorise every element of $\mathcal{C}_0(Y)$ in a product of an element of $\mathcal{C}_0(X)$ and an element of $\mathcal{C}_0(Y)$. If $\chi \in \mathcal{C}_0(Y)$ and $\chi' \in \mathcal{C}_0(X)$, then $(\chi\chi')p^*T = \chi p^*(\chi' T)$, which is compact because $\chi' T$ is compact. Hence p^*T is locally compact. □

2.4 Gradings and group actions

Definition 2.4.1 (Graded $\mathcal{C}_0(X)$-Banach space). A *graded $\mathcal{C}_0(X)$-Banach space* is a $\mathcal{C}_0(X)$-Banach space E endowed with a grading automorphism commuting with the $\mathcal{C}_0(X)$-action.

Let G be a locally compact Hausdorff group that acts continuously on X. Note that $\mathcal{C}_0(X)$ is a G-Banach algebra when equipped with the G-action $(g\chi)(x) := \chi(g^{-1}x)$, $\chi \in \mathcal{C}_0(X)$, $g \in G$, $x \in X$.

Definition 2.4.2 (G-$\mathcal{C}_0(X)$-Banach space). A *G-$\mathcal{C}_0(X)$-Banach space* is a G-Banach space E which is at the same time a $\mathcal{C}_0(X)$-Banach space such that the actions of G and $\mathcal{C}_0(X)$ are compatible in the following sense:

$$g(\chi e) = (g\chi)(ge), \quad \chi \in \mathcal{C}_0(X), g \in G, e \in E,$$

i.e., the product $\mathcal{C}_0(X) \times E \to E$ is G-equivariant.

From these definitions we also get an obvious definition for a graded G-$\mathcal{C}_0(X)$-Banach space. Taking this as a starting point one can define graded $\mathcal{C}_0(X)$-Banach algebras, homomorphisms of graded $\mathcal{C}_0(X)$-Banach algebras, graded $\mathcal{C}_0(X)$-Banach modules, G-$\mathcal{C}_0(X)$-Banach algebras, graded G-$\mathcal{C}_0(X)$-Banach pairs, etc. All constructions we have made for graded and equivariant structures in Chapter 1 are compatible with the additional $\mathcal{C}_0(X)$-structure; we skip the details.

Also, the pullback along G-equivariant maps between locally compact Hausdorff spaces on which G acts is compatible with the additional G-action on G-$\mathcal{C}_0(X)$-Banach spaces, etc.

Remark 2.4.3. The way we have defined the pullback for $\mathcal{C}_0(X)$-Banach spaces is not really compatible with the pullback of upper semi-continuous fields of Banach spaces that we are going to meet later; in fact, to obtain the same structure one has to consider locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces and the pullback has to be adjusted so that we stay in the same category.

We will see that in the context of upper semi-continuous fields of Banach spaces an action of a groupoid can be modelled using the pullback. This is not possible in the setting of $\mathcal{C}_0(X)$-Banach spaces, at least not in an obvious way (apart from the fact that we can shift everything to the category of upper semi-continuous fields, do the modelling there, and transfer everything back to $\mathcal{C}_0(X)$-Banach spaces using the functors introduced in Chapter 4).

I would like to thank Ralf Meyer for providing me with the following example which shows that the above construction of the pullback really is not suitable to model actions of groupoids. There might be a better choice of the involved tensor product which remedies the problem, but we do not venture into this.

Example 2.4.4. Let G be a discrete group and define $E := l^1(G)$. Then $E = l^1(G)$ carries a canonical action of G, namely $(g\xi)(h) = \xi(g^{-1}h)$ for all $\xi \in l^1(G)$ and $g, h \in G$. Let $p\colon G \to \{*\}$ be the projection onto the one-point space (being the range and source map of G regarded as a groupoid). Can the action of G on E be encoded in a continuous map from p^*E to p^*E? Note that p^*E is the projective tensor product $c_0(G) \otimes^\pi l^1(G)$ which can be identified with $l^1(G,\ c_0(G))$. If $f \in c_0(G)$ has finite support and $\xi \in l^1(G)$, then the map we are looking for should send $f \otimes \xi$ to $\sum_{g\in G} f(g)\delta_g \otimes g\xi$.

Let $\xi = \delta_{e_G}$ be the indicator function of the identity element e_G of G. Then

$$\sum_{g\in G} f(g)\delta_g \otimes g\delta_{e_G} = \sum_{g\in G} f(g)\delta_g \otimes \delta_g$$

for all $f \in c_0(G)$ with finite support. If we identify $c_0(G) \otimes^\pi l^1(G)$ with $l^1(G,\ c_0(G))$, then this element corresponds to $g \mapsto f(g)\delta_g$. The norm of $f \otimes \delta_{e_G}$ is $\|f\|_\infty \|\delta_{e_G}\|_1 = \|f\|_\infty$, the norm of $\sum_{g\in G} f(g)\delta_g \otimes \delta_g$ is equal to $\|g \mapsto f(g)\delta_g\|_1 = \sum_{g\in G} \|f(g)\delta_g\|_\infty = \|f\|_1$. This is true for all $f \in c_0(G)$ with finite support. Obviously, the map we are looking for is not isometric and, more dramatically, cannot be extended to a continuous map on p^*E. The reason is of course that we have taken the "wrong" tensor product; for the injective tensor product everything would work fine in this particular situation. So far I have not checked whether the injective tensor product leads to a theory that works smoothly in general.

2.5 $\mathrm{RKK}_G^{\mathrm{ban}}\left(\mathcal{C}_0(X); A, B\right)$

2.5.1 Definition

Definition 2.5.1 ($\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}\left(\mathcal{C}_0(X); A, B\right)$). Let A and B be G-$\mathcal{C}_0(X)$-Banach algebras. Then the class $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}\left(\mathcal{C}_0(X); A, B\right)$ is defined to be the class of pairs (E, T) such that E is a non-degenerate graded G-$\mathcal{C}_0(X)$-Banach A-B-pair and, if we forget the $\mathcal{C}_0(X)$-structure, the pair (E, T) is an element of $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$.

The constructions from Section 1.8 are obviously compatible with the additional $\mathcal{C}_0(X)$-structure, so we can form the sum of $\mathrm{KK}^{\mathrm{ban}}$-cycles and take their pushout along homomorphisms of G-$\mathcal{C}_0(X)$-Banach algebras. We also have a $\mathcal{C}_0(X)$-linear notion of morphisms of $\mathrm{KK}^{\mathrm{ban}}$-cycles, giving us a $\mathcal{C}_0(X)$-linear version of isomorphisms of $\mathrm{KK}^{\mathrm{ban}}$-cycles. Hence also the notion of homotopy makes sense in the $\mathcal{C}_0(X)$-setting so we can formulate the following definition:

Definition 2.5.2 ($\mathrm{RKK}_G^{\mathrm{ban}}\left(\mathcal{C}_0(X); A, B\right)$). The class of all homotopy classes in $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}\left(\mathcal{C}_0(X); A, B\right)$ is denoted by $\mathrm{RKK}_G^{\mathrm{ban}}\left(\mathcal{C}_0(X); A, B\right)$. The addition of cycles induces a law of composition on $\mathrm{RKK}_G^{\mathrm{ban}}\left(\mathcal{C}_0(X); A, B\right)$ making it an abelian group.

The fact that the composition on $\mathrm{RKK}_G^{\mathrm{ban}}\left(\mathcal{C}_0(X); A, B\right)$ has inverses can be proved just as in the case without the $\mathcal{C}_0(X)$-structure, i.e., Lemme 1.2.5 of [Laf02] and its proof are compatible with the additional $\mathcal{C}_0(X)$-module action. There is an obvious forgetful group homomorphism

$$\mathrm{RKK}_G^{\mathrm{ban}}\left(\mathcal{C}_0(X); A, B\right) \;\to\; \mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)\,.$$

2.5.2 The pullback of $\mathrm{RKK}_G^{\mathrm{ban}}$-cycles

In this paragraph let Y be another locally compact Hausdorff G-space and $p\colon Y \to X$ be continuous and G-equivariant.

Let E be a $\mathcal{C}_0(X)$-Banach A-B-pair over $\mathcal{C}_0(X)$-Banach algebras A and B. Let $T \in \mathrm{L}_B(E)$. Then $[a, T]$ is compact for all $a \in A$ if (and only if) $[a, T]$ is locally compact for all $a \in A$: Let $a \in A$. Find $\chi \in \mathcal{C}_0(X)$ and $a' \in A$ such that $a = \chi a'$. If $[a', T]$ is locally compact, then $[a, T] = \chi[a', T]$ is compact. It follows that we can replace the condition that $[a, T]$ is compact in the definition of cycles for $\mathrm{RKK}^{\mathrm{ban}}$ with the condition that these operators are locally compact. The same is true for the other compactness conditions in the definition of $\mathrm{RKK}^{\mathrm{ban}}$. Hence we have the following lemma:

Lemma 2.5.3. *Let A and B be G-$\mathcal{C}_0(X)$-Banach algebras and $(E, T) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}\left(\mathcal{C}_0(X);\ A,\ B\right)$. Then*

$$p^*(E, T) = (p^*E,\ p^*T) \;\in\; \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}\left(\mathcal{C}_0(Y);\ p^*A,\ p^*B\right).$$

Proof. The pair p^*E carries a canonical grading and p^*T surely is an odd linear operator on p^*E for this grading. Let $a \in A$ and $\chi \in \mathcal{C}_0(Y)$. Then $[a \otimes \chi,\ p^*T] = \chi\left([a, T] \otimes 1\right) = \chi p^*[a, T]$. Now $p^*[a, T]$ is locally compact (and $\chi p^*[a, T]$ is compact).

Similar arguments are valid for the other compactness conditions. □

Let B be a G-$\mathcal{C}_0(X)$-Banach algebra. Let ϕ_B be the canonical homomorphism from $p^*\left(B[0,1]\right)$ to $(p^*B)[0,1]$ which sends $\beta \otimes \chi$ to $t \mapsto \beta(t) \otimes \chi$. This map might not be an isomorphism, but it nevertheless satisfies $p^*(\mathrm{ev}_t^B) = \mathrm{ev}_t^{p^*B} \circ \phi_B$ for all $t \in [0,1]$. If $(E, T) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}\left(\mathcal{C}_0(X);\ A,\ B[0,1]\right)$, then $p^*(E, T) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}\left(\mathcal{C}_0(Y);\ p^*A,\ p^*(B[0,1])\right)$. It follows that $(\phi_B)_*(p^*(E, T))$ is an element of $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}\left(\mathcal{C}_0(Y);\ p^*A,\ (p^*B)[0,1]\right)$. The functoriality of the pushout shows that this is a homotopy between $p^*(E_0, T_0)$ and $p^*(E_1, T_1)$ where $(E_i, T_i) = \mathrm{ev}_{i,*}^B(E, T)$ for all $i \in \{0, 1\}$. Hence the pullbacks of homotopic elements are homotopic. We can therefore say:

Proposition 2.5.4. *Let A and B be G-$\mathcal{C}_0(X)$-Banach algebras. Then the pullback along p induces a homomorphism*

$$p^*\colon \mathrm{RKK}_G^{\mathrm{ban}}\left(\mathcal{C}_0(X);\ A,\ B\right) \to \mathrm{RKK}_G^{\mathrm{ban}}\left(\mathcal{C}_0(Y);\ p^*A,\ p^*B\right).$$

2.6 Homotopy and Morita equivalence

2.6.1 The sufficient condition for homotopy

All the constructions of Section 1.9, in particular the sufficient condition for homotopy presented in Theorem 1.9.1, are compatible with an additional $\mathcal{C}_0(X)$-structure. We explicitly state one definition for further reference:

Definition 2.6.1 (The class $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(\mathcal{C}_0(X);\varphi,\psi)$). Let $\varphi\colon A \to A'$ and $\psi\colon B \to B'$ be G-equivariant $\mathcal{C}_0(X)$-linear homomorphisms of G-$\mathcal{C}_0(X)$-Banach algebras. A $\mathrm{KK}^{\mathrm{ban}}$-cycle from φ to ψ is a pair $(\Phi\colon E \to E',\ (T,T'))$ such that E is a non-degenerate graded G-$\mathcal{C}_0(X)$-Banach A-B-pair, E' is a non-degenerate graded G-$\mathcal{C}_0(X)$-Banach A'-B'-pair, Φ is an even G-equivariant $\mathcal{C}_0(X)$-linear homomorphism from ${}_AE_B$ to ${}_{A'}E'_{B'}$ with coefficient maps φ and ψ and $(T,T') \in \mathrm{L}_{\psi}^{\mathcal{C}_0(X)}\left(\Phi,\Phi\right)$ is a pair of odd linear operators such that[2]

1. $\forall a \in A:\ [a,(T,T')] = ([a,T],\ [\psi(a),T']) \in \mathrm{K}\left(\Phi,\Phi\right);$
2. $\forall a \in A:\ a((T,T')^2-1) = \Big(a(T^2-1),\ \psi(a)(T'^2-1)\Big) \in \mathrm{K}\left(\Phi,\Phi\right);$
3. $\forall a \in A:\ g \mapsto a\left(g(T,T')-(T,T')\right) = \left(a(gT-T),\ \psi(a)(gT'-T')\right) \in \mathcal{C}\left(G,\ \mathrm{K}\left(\Phi,\Phi\right)\right).$

The class of all such cycles will be denoted by $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}\left(\mathcal{C}_0(X);\varphi,\psi\right)$.

Note that it is not necessary to introduce the notation $\mathrm{K}^{\mathcal{C}_0(X)}\left(\Phi,\Phi\right)$ in the preceding definition (i.e., imposing the extra condition of $\mathcal{C}_0(X)$-linearity on the compact operators) since compact operators are automatically $\mathcal{C}_0(X)$-linear. Moreover, the condition on T and T' to be $\mathcal{C}_0(X)$-linear is also automatic because E and E' are non-degenerate.

We now state the new version of the sufficient condition for homotopy of cycles:

Theorem 2.6.2 (Sufficient condition for homotopy of $\mathrm{RKK}_G^{\mathrm{ban}}$-cycles). *Let G be a locally compact Hausdorff group acting on the locally compact Hausdorff space X. Let A and B be G-$\mathcal{C}_0(X)$-Banach algebras and let $(E,T),(E',T')$ be elements of $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}\left(\mathcal{C}_0(X);A,B\right)$. If there is a G-equivariant $\mathcal{C}_0(X)$-linear morphism Φ from (E,T) to (E',T') (with coefficient maps Id_A and Id_B) such that*

1. $\forall a \in A:\ [a,(T,T')] = ([a,T],\ [a,T']) \in \mathrm{K}\left(\Phi,\Phi\right),$
2. $\forall a \in A:\ a((T,T')^2-1) = \Big(a(T^2-1),\ a(T'^2-1)\Big) \in \mathrm{K}\left(\Phi,\Phi\right),$
3. $\forall a \in A\ \forall g \in G:\ a\left(g(T,T')-(T,T')\right) = \left(a(gT-T),\ a(gT'-T')\right) \in \mathrm{K}\left(\Phi,\Phi\right),$

then (E,T) and (E',T') are homotopic (and thus give the same elements of $\mathrm{RKK}_G^{\mathrm{ban}}\left(\mathcal{C}_0(X);A,B\right)$). If $T=0$ and $T'=0$, then the homotopy can be chosen to have trivial operator as well.

[2] See Theorem 1.9.1 or Definition 1.9.7 for a definition of $\mathrm{K}\left(\Phi,\Phi\right)$.

2.6.2 Morita theory and $\mathrm{RKK}_G^{\mathrm{ban}}$

Also the definitions and constructions of Section 1.10 are compatible with the additional $\mathcal{C}_0(X)$-structure.

Definition 2.6.3 ($\mathcal{C}_0(X)$-linear Morita cycle). Let A and B be non-degenerate G-$\mathcal{C}_0(X)$-Banach algebras. Then a *$\mathcal{C}_0(X)$-linear Morita cycle* F from A to B is a non-degenerate G-$\mathcal{C}_0(X)$-Banach A-B-pair F such that A acts on F by compact operators. The class of all Morita cycles from A to B is denoted by $\mathbb{M}_{\mathrm{G}}^{\mathrm{ban}}\left(\mathcal{C}_0(X); A, B\right)$.

Similarly, a G-$\mathcal{C}_0(X)$-Morita equivalence of non-degenerate G-$\mathcal{C}_0(X)$-Banach algebras is an equivariant Morita equivalence which also carries a compatible $\mathcal{C}_0(X)$-structure such that all structure maps are $\mathcal{C}_0(X)$-bilinear.

All the tensor products that appear in Section 1.10 should be made $\mathcal{C}_0(X)$-tensor products to fit into the $\mathcal{C}_0(X)$-setting, but this is automatic because at least one of the involved modules (or pairs) will always be non-degenerate (see Proposition 2.2.13).

After having made all the necessary changes in Chapter 1, we end up with the following version of Theorem 1.10.31:

Theorem 2.6.4. *Let A, B, C be non-degenerate G-$\mathcal{C}_0(X)$-Banach algebras and let E be a G-equivariant $\mathcal{C}_0(X)$-linear Morita equivalence from B to C. Then $\cdot \otimes_B [E]$ is an isomorphism*

$$\mathrm{RKK}_G^{\mathrm{ban}}\left(\mathcal{C}_0(X); A, B\right) \;\cong\; \mathrm{RKK}_G^{\mathrm{ban}}\left(\mathcal{C}_0(X); A, C\right)$$

with inverse $\cdot \otimes_B \left[\overline{E}\right]$.

2.7 The pushforward

Let G be a locally compact Hausdorff group and X and Y be locally compact Hausdorff spaces on which G acts continuously. Let $p\colon Y \to X$ be a continuous G-equivariant map. Then the map $p^*\colon \varphi \mapsto \varphi \circ p$ is a continuous homomorphism from $\mathcal{C}_0(X)$ to $\mathcal{C}_b(Y)$ which is non-degenerate in the sense that $p^*\left(\mathcal{C}_0(X)\right)\mathcal{C}_0(Y)$ is dense in $\mathcal{C}_0(Y)$. It follows that we can turn every G-$\mathcal{C}_0(Y)$-Banach space into a G-$\mathcal{C}_0(X)$-Banach space:

Definition 2.7.1. Let E be a G-$\mathcal{C}_0(Y)$-Banach space. Then we define an action of $\mathcal{C}_0(X)$ on E by $\varphi e := (\varphi \circ p)e$ for all $e \in E$ and $\varphi \in \mathcal{C}_0(X)$. With this action E is a G-$\mathcal{C}_0(X)$-Banach space which we call p_*E.

Every $\mathcal{C}_0(Y)$-linear map between $\mathcal{C}_0(Y)$-Banach spaces is also $\mathcal{C}_0(X)$-linear, so we get a functor p_* from the category of $\mathcal{C}_0(Y)$-Banach spaces to the category of $\mathcal{C}_0(X)$-Banach spaces. Similarly, $\mathcal{C}_0(X)$-bilinearity is weaker than $\mathcal{C}_0(Y)$-bilinearity. So $\mathcal{C}_0(Y)$-Banach algebras are also $\mathcal{C}_0(X)$-Banach algebras and the same is true for Banach modules and Banach pairs. The result is a forgetful map on the level of $\mathrm{KK}^{\mathrm{ban}}$-cycles: If A and B are G-$\mathcal{C}_0(Y)$-Banach algebras and (E,T) is in $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(\mathcal{C}_0(Y); A, B)$, then $p_*(E,T) = (p_*E,\ p_*T)$ is in $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(\mathcal{C}_0(X);\ p_*A,\ p_*B)$. Sometimes we regard A and B simply also as G-$\mathcal{C}_0(X)$-Banach algebras without renaming them, so we write $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(\mathcal{C}_0(X);\ A,\ B)$ instead of $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(\mathcal{C}_0(X);\ p_*A,\ p_*B)$, etc. This construction respects direct sums, pushouts (the pairs are non-degenerate!) and homotopies. Hence:

Proposition 2.7.2. *If $p\colon Y \to X$ is G-equivariant and continuous, then there is a canonical "forgetful" homomorphism*

$$p_*\colon\ \mathrm{RKK}_G^{\mathrm{ban}}(\mathcal{C}_0(Y); A, B) \to \mathrm{RKK}_G^{\mathrm{ban}}(\mathcal{C}_0(X);\ p_*A,\ p_*B).$$

Note that this applies in particular to the case that X is just a single point; then $\mathcal{C}_0(X)$ is isomorphic to $\mathbb{C}$ and $\mathrm{RKK}_G^{\mathrm{ban}}(\mathcal{C}_0(X); p_*A, p_*B) = \mathrm{RKK}_G^{\mathrm{ban}}(\mathcal{C}_0(X); A, B)$ is equal to $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$.

2.8 Special case: X compact

Let G be a locally compact Hausdorff group and X be a *compact* Hausdorff space on which G acts. Let A be a non-degenerate G-Banach algebra and let B be a non-degenerate G-$\mathcal{C}(X)$-Banach algebra. Then the projective tensor product $A \otimes \mathcal{C}(X)$ is a non-degenerate G-$\mathcal{C}(X)$-Banach algebra.

Remember that there is a canonical forgetful homomorphism

$$\mathrm{RKK}_G^{\mathrm{ban}}\left(\mathcal{C}(X); A \otimes \mathcal{C}(X), B\right) \;\to\; \mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}\left(A \otimes \mathcal{C}(X),\ B\right).$$

Secondly, there is a canonical homomorphism j_A of G-Banach algebras from A to $A \otimes \mathcal{C}(X)$, namely the map $a \mapsto a \otimes 1$. This gives a group homomorphism from $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A \otimes \mathcal{C}(X); B)$ to $\mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$. Let

$$\kappa\colon\ \mathrm{RKK}_G^{\mathrm{ban}}\left(\mathcal{C}(X); A \otimes \mathcal{C}(X),\ B\right) \to \mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}\left(A, B\right)$$

be the composition of these two homomorphisms.

Proposition 2.8.1. *The homomorphism κ is an isomorphism.*

Proof. We first prove surjectivity: Let $(E, T) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$. Instead of defining a $\mathcal{C}(X)$-structure on E, which we do not know how to do, we define a structure on the cycle $(E \otimes_B B,\ T \otimes 1) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$. Note that $(E \otimes_B B,\ T \otimes 1) = (E, T) \otimes_B \mathrm{Mor}_{\mathrm{G}}^{\mathrm{ban}}(\mathrm{Id}_B)$, so it is homotopic to (E, T). On $E^> \otimes_B B$ we define the $\mathcal{C}(X)$-structure as in Lemma 2.2.6, i.e., if $e^> \in E^>$ and $b \in B$ and $\varphi \in \mathcal{C}(X)$, then $\varphi(e^> \otimes b) := e^> \otimes (\varphi b)$. This makes $E^> \otimes_B B$ a right G-$\mathcal{C}(X)$-Banach B-module. We proceed similarly on the left-hand side. It is easy to see that $E \otimes_B B$ is a G-$\mathcal{C}(X)$-Banach B-pair with this $\mathcal{C}(X)$-action. The operator $T \otimes 1$ is clearly $\mathcal{C}(X)$-linear (which is automatic anyway, because $E \otimes_B B$ is non-degenerate).

Now we have to define an action of $A \otimes \mathcal{C}(X)$ on $E \otimes_B B$: If $a \in A$, $\chi \in \mathcal{C}(X)$, $e^> \in E^>$ and $b \in B$ then we define $(a \otimes \chi)(e^> \otimes b) := (ae^>) \otimes (\chi b)$. This gives an action of $A \otimes \mathcal{C}(X)$ on $E^> \otimes_B B$ making it a G-$\mathcal{C}(X)$-Banach $A \otimes \mathcal{C}(X)$-B-bimodule. A similar definition can be made for the left-hand side. We check that $A \otimes \mathcal{C}(X)$ acts on $E \otimes_B B$ by elements of $\mathrm{L}_B(E \otimes_B B)$. Let therefore be $a \in A$, $\chi \in \mathcal{C}(X)$, $e^< \in E^<$, $e^> \in E^>$ and $b^<, b^> \in B$. Then

$$\begin{aligned} \left\langle b^< \otimes e^<,\ (a \otimes \chi)(e^> \otimes b^>)\right\rangle &= \left\langle b^< \otimes e^<,\ (ae^>) \otimes (\chi b^>)\right\rangle = b^< \left\langle e^<,\ ae^>\right\rangle (\chi b^>) \\ &= (\chi b^<) \left\langle e^< a,\ e^>\right\rangle b^> = \left\langle (b^< \otimes e^<)(a \otimes \chi),\ e^> \otimes b^>\right\rangle. \end{aligned}$$

By trilinearity and continuity of both sides this equation can be extended from the elementary tensors to all of $A \otimes \mathcal{C}(X)$, $B \otimes_B E^<$ and $E^> \otimes_B B$. So $E \otimes_B B$ is in $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(\mathcal{C}(X); A \otimes \mathcal{C}(X), B)$. Applying κ to it means forgetting the $\mathcal{C}(X)$-structure and reducing the $A \otimes \mathcal{C}(X)$-action back to the A-action on $E \otimes_B B$, so we are back where we started. Hence κ is surjective.

The same argument shows that κ is injective: Let (E_0, T_0) and (E_1, T_1) be elements of the class $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(\mathcal{C}(X); A \otimes \mathcal{C}(X), B)$ such that $\kappa(E_0, T_0)$ and $\kappa(E_1, T_1)$ are homotopic in $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B)$. Without loss of generality we can assume that $\kappa(E_0, T_0)$ and $\kappa(E_1, T_1)$ can be connected through a single homotopy (otherwise we use the surjectivity to find inverse images of the intermediate steps and proceed step by step). Let $(E, T) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(\mathcal{C}(X); A \otimes \mathcal{C}(X),\ B[0,1])$ be such that $\kappa(E, T) \in \mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A, B[0,1])$ is a homotopy from $\kappa(E_0, T_0)$ to $\kappa(E_1, T_1)$. Now $\mathrm{ev}_{i,*}^B(E, T)$ is contained in

$\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(\mathcal{C}(X); A \otimes \mathcal{C}(X),\ B)$ for all $i \in \{0,1\}$ and $\kappa(\mathrm{ev}_{i,*}^{B}(E,T))$ is isomorphic (in $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(A,B)$) to (E_i, T_i). Now E_i is a non-degenerate B-pair, so the $\mathcal{C}(X)$-structure on E is unique.[3] Hence the isomorphism between $\kappa(\mathrm{ev}_{i,*}^{B}(E,T))$ and (E_i,T_i) must be $\mathcal{C}(X)$-linear. Also the action of $A \otimes \mathcal{C}(X)$ is uniquely determined by the actions of A and $\mathcal{C}(X)$, so the isomorphism from $\kappa(\mathrm{ev}_{i,*}^{B}(E,T))$ and (E_i,T_i) must also respect this structure. In other words, it is an isomorphism of cycles in $\mathbb{E}_{\mathrm{G}}^{\mathrm{ban}}(\mathcal{C}(X); A \otimes \mathcal{C}(X), B)$. So (E_0,T_0) and (E_1,T_1) are homotopic. Hence κ is injective. $\square$

If we take A to be $\mathbb{C}$ with the trivial G-action, then $A \otimes \mathcal{C}(X)$ is isomorphic to $\mathcal{C}(X)$. The proposition then reduces to the following statement:

Corollary 2.8.2. *Let B be a non-degenerate G-$\mathcal{C}(X)$-Banach algebra. If X is compact, then*

$$\mathrm{RKK}_{G}^{\mathrm{ban}}\left(\mathcal{C}(X);\ \mathcal{C}(X), B\right) \quad \cong \quad \mathrm{KK}_{\mathrm{G}}^{\mathrm{ban}}\left(\mathbb{C}, B\right).$$

[3]See Lemma 2.2.5.

Chapter 3

$\mathbb{KK}$-Theory for Fields of Banach algebras and Groupoids

To define the action of a groupoid $\mathcal{G}$ on a Banach algebra B it is inevitable to have some sort of bundle structure over the unit space of $\mathcal{G}$ on B. There are different ways to formalise the notion of a bundle of Banach algebras over some base space X.

First, one could consider a continuous surjection $p\colon B \to X$ where B is some topological space (the total space of the bundle) carrying some of extra structure which makes sure that, in particular, the fibres $p^{-1}(\{x\})$ are Banach algebras. This bundle point of view was taken in[1] [FD88].

Second, it is possible to concentrate on the space of continuous sections rather than on the total space. If X is a locally compact Hausdorff space, then the continuous sections vanishing at infinity of a bundle of Banach algebras over X form a Banach algebra with a non-degenerate action of $\mathcal{C}_0(X)$. So $\mathcal{C}_0(X)$-Banach algebras can serve as a starting point for a formalisation of what a bundle of Banach algebras over X should be.

Third, one could start with a family of Banach algebras $(B_x)_{x\in X}$, corresponding to the fibres in the bundle picture, and say what the "continuous sections" should be (each such section ξ being a function defined on X such that $\xi(x) \in B_x$). This leads to the definition of an upper semi-continuous field of Banach algebras over X (generalising the notion of a continuous field of [Dix64]). The field picture is the one that V. Lafforgue has devised in [Laf06] to define actions of groupoids on Banach algebras, and we want to systematically develop his theory in the present chapter, adding a number of technical details.

It would be interesting to compare the field picture with the bundle picture in our context, but it seems advisable to exclude the bundle picture totally because this thesis is already rather voluminous. On the other hand, the $\mathcal{C}_0(X)$-Banach algebra picture appears quite natural in applications and is obviously not very challenging on the technical level, so I decided to introduce it and to compare it to the field picture.[2] Unfortunately, the $\mathcal{C}_0(X)$-Banach algebra picture seems not to be suitable to formalise actions of general locally compact groupoids,[3] making it necessary to head for the realm of fields of Banach algebras.

Technically, the basic notion underlying the whole theory is the notion of an upper semi-continuous (u.s.c.) field of Banach spaces. We define tensor products of such fields, which allows us to define fields of Banach algebras, modules and pairs. Moreover, we define pullbacks of u.s.c. fields of

[1] See Definition 13.4 in [FD88]; see also 13.18 of the same book for a comparison of bundles and fields of Banach spaces.
[2] See Chapter 2 and Chapter 4, respectively.
[3] See Example 2.4.4.

Banach spaces, which allows us to define actions of groupoids on fields of Banach spaces, algebras, etc. The other important ingredient that we need for a version of $\mathrm{KK}^{\mathrm{ban}}$-theory in this setting is the definition of compact operators on (fields of) Banach pairs; in this chapter, we define what we call "locally compact operators" instead, the main reason for this being that we do not assume the base space to be locally compact. Later on we will see that if the base space is locally compact, then there also is a canonical notion of compact operators which can be used instead.[4]

The resulting $\mathrm{KK}^{\mathrm{ban}}$-theory generalises the theory introduced in the first chapter (just take the base space to be a single point). However, it does not generalise $\mathrm{RKK}^{\mathrm{ban}}$ as introduced in the second chapter.More precisely, not every $\mathcal{C}_0(X)$-Banach algebra comes from a u.s.c. field of Banach algebras over X. In Chapter 4 we are going to compare the two situations in greater detail.

The main tools for $\mathrm{KK}^{\mathrm{ban}}$ as developed in the first chapter for ordinary Banach algebras generalise to the $\mathrm{KK}^{\mathrm{ban}}$-theory for fields of Banach algebras presented in this chapter. In particular, Section 3.7 explains how to translate the sufficient condition 1.9.1 for homotopy of cycles to the setting of fields of Banach algebras equipped with groupoid actions. Furthermore, one can also define equivariant Morita morphisms for fields of Banach algebras, and the respective results of the first chapter carry over in full generality (this is summarised in Section 3.8).

Note that there is an additional section in this chapter, namely Section 3.2, which discusses the simple and basic notion of a monotone completion. This section is not needed for the development of $\mathrm{KK}^{\mathrm{ban}}$ for fields of Banach algebras, but it is needed at several points in the remaining chapters. It is too short to deserve to be made into an entire chapter and does not have a canonical place somewhere else in this thesis, so I have put this section at the first place where all the required definitions are available.

3.1 Upper semi-continuous fields of Banach spaces

Before defining what an upper semi-continuous field of Banach spaces is, we introduce some useful vocabulary. Some of the definitions even make sense for families of Banach spaces over a set (without any topology on the base space). For example, this is the natural place to say what a selection is. In a second step, we discuss families of Banach spaces over a topological space, which enables us to talk about the support of a selection or a selection being locally bounded. The latter notion is already rather useful and turns out to be the technical heart of a lot of arguments for upper semi-continuous fields of Banach spaces.

Subsection 3.1.3 then gives an elaborate introduction to upper semi-continuous fields of Banach spaces. After that we finally define u.s.c. fields of Banach algebras, Banach modules and Banach pairs as well as locally compact operators.

3.1.1 Families of Banach spaces over a set

Let X be a set. A family of Banach spaces over X is a family $(E_x)_{x\in X}$ such that E_x is a Banach space for all $x \in X$.

Definition 3.1.1 (Selections). Let E be a family of Banach spaces over X.

1. An element ξ of the complex vector space $\prod_{x\in X} E_x$ is called a *selection* of E.
2. For every selection ξ of E we define $|\xi| : X \to \mathbb{R}_{\geq 0},\ x \mapsto \|\xi(x)\|_{E_x}$.

[4] See Section 4.7.1, in particular Proposition 4.7.5.

3. For every selection ξ of E define $\|\xi\|_\infty := \||\xi|\|_\infty = \sup_{x\in X} \|\xi(x)\|_{E_x} \in [0,\infty]$.
4. For every $x_0 \in X$ define the map $\mathrm{ev}^E_{x_0}$ on $\prod_{x\in X} E_x$ to be the projection map onto E_{x_0}.

If E is a family of Banach spaces over X, then the subspace of bounded selections of E is a Banach space in its canonical norm.

Definition 3.1.2 (Total subset). Let X be a set and E be a family of Banach spaces over X. Let Γ_0 be a subset of $\prod_{x\in X} E_x$. Let $\langle\Gamma_0\rangle$ be the linear subspace generated by Γ_0. Then Γ_0 is called *total* if for every $x \in X$ the space $\{\mathrm{ev}^E_x\,\xi :\ \xi \in \langle\Gamma_0\rangle\}$ is dense in E_x.

Definition 3.1.3 (Families of linear maps). Let E and F be families of Banach spaces over X. Then a family of bounded linear maps from E to F is a family $(T_x)_{x\in X}$ such that $T_x \in \mathrm{L}(E_x,\ F_x)$ for all $x \in X$, i.e., a selection of the family $(\mathrm{L}(E_x,\ F_x))_{x\in X}$ of Banach spaces over X.

Definition 3.1.4 (Composition). Let E, F, G be families of Banach spaces over X. If S is a family of bounded linear maps from E to F and T is a family of bounded linear maps from F to G, then their composition $T\circ S := (T_x \circ S_x)_{x\in X}$ is a family of bounded linear maps from E to G. If S and T are bounded, then the family $T \circ S$ is also bounded with $\|T\circ S\| \le \|T\|\,\|S\|$.

Definition 3.1.5 (Evaluation). Let E and F be families of Banach spaces over X. If $(T_x)_{x\in X}$ is a family of bounded linear maps from E to F and ξ is a selection of E, then we define a selection of F as follows:

$$T\circ\xi\colon x \mapsto T_x(\xi(x)).$$

The map $\xi \mapsto T\circ\xi$ defines a linear map from the selections of E to the selections of F. If T is bounded, then $\xi \mapsto T\circ\xi$ is a continuous linear map from the bounded selections of E to the bounded selections of F, bounded by $\|T\|_\infty$.

Definition 3.1.6 (The internal product and and the internal sum). Let E and F be families of Banach spaces over X. Then the *internal product* $E\times_X F$ of E and F is the family $(E_x\times F_x)_{x\in X}$ over X where we take the sup-norm on the fibres. Analogously, the *internal sum* $E\oplus_X F$ of E and F is the family $(E_x\oplus F_x)_{x\in X}$ over X where we take the sum-norm on the fibres.

Definition 3.1.7 (Families of continuous bilinear maps). Let E, F and G be families of Banach spaces over X. Then a family of continuous bilinear maps from $E\times_X F$ to G is a family $(\mu_x)_{x\in X}$ such that μ_x is a continuous bilinear map from $E_x\times F_x$ to G_x for all $x\in X$, i.e., μ is a selection in the family $(\mathrm{M}(E_x,F_x;G_x))_{x\in X}$ of Banach spaces over X. We say that μ is *non-degenerate* if μ_x is non-degenerate for all $x\in X$, i.e., the image of μ_x spans a dense subset of G_x.

Definition 3.1.8 (Evaluation). Let E, F and G be families of Banach spaces over X and let μ be a family of continuous bilinear maps from $E\times_X F$ to G. If ξ is a selection of E and η is a selection of F, then we define a selection of G as follows

$$\mu\circ(\xi,\eta)\colon x\mapsto \mu_x\left(\xi(x),\eta(x)\right).$$

The evaluation map $(\xi,\eta)\mapsto \mu\circ(\xi,\eta)$ is bilinear. If μ is bounded, then the evaluation map is a continuous bilinear map when restricted to the bounded selections; it is bounded by $\|\mu\|$.

Definition 3.1.9 (The internal tensor product). Let E and F be families of Banach spaces over X. Define $E\otimes_X F$ to be the family $(E_x\otimes F_x)_{x\in X}$, where $\otimes$ denotes the projective tensor product of Banach spaces. It is universal for bounded families of continuous linear maps. This tensor product is associative since the projective tensor product of Banach spaces is associative.

If E, E', F and F' are families of Banach spaces over X, S is a family of continuous linear maps from E to E' and T is a family continuous linear maps from F to F', then we define $S \otimes_X T$ to be the family $(S_x \otimes T_x)_{x \in X}$; it is a family of continuous linear maps from $E \otimes_X F$ to $E' \otimes_X F'$. If S and T are bounded then so is $S \otimes T$ and we have $\|S \otimes_X T\| \leq \|S\| \, \|T\|$.

3.1.2 Families of Banach spaces over a topological space

For the rest of Section 3.1, let X be a *topological space*.

Definition 3.1.10 (Locally bounded selection). Let E be a family of Banach spaces over X.

1. A selection ξ of E is called *locally bounded* if every point in X has neighbourhood on which ξ is bounded. The space of locally bounded selections of E will be denoted by $\Sigma(X, E)$.

2. The set of all bounded selections of E will be denoted by $\Sigma_b(X, E)$.

3. A selection ξ of E is said to *vanish at infinity* if for all $\varepsilon > 0$ there is a compact subset $K \subseteq X$ such that $|\xi|(x) = \|\xi(x)\|_{E_x} \leq \varepsilon$ for all $x \in X \setminus K$. The set of all locally bounded selections of E vanishing at infinity is denoted by $\Sigma_0(X, E)$.

Note that $\Sigma_0(X, E) \subseteq \Sigma_b(X, E)$, and both spaces are Banach spaces.

Definition 3.1.11 (The support of a selection). Let X be a topological space and let E be a family of Banach spaces over X. Let ξ be a selection of E. Then the support of ξ is defined as

$$\operatorname{supp} \xi := \overline{\{x \in X : \xi(x) \neq 0\}}.$$

The following definition will only be of interest for us if the underlying space X is locally compact and Hausdorff. Nonetheless, it also makes sense for general topological spaces.

Definition 3.1.12 (Selections of compact support). We define $\Sigma_c(X, E)$ to be the space of all (locally) bounded selections of E which have compact support.

Note that any locally bounded selection with compact support is bounded. Moreover,

$$\Sigma_c(X, E) \subseteq \Sigma_0(X, E) \subseteq \Sigma_b(X, E) \subseteq \Sigma(X, E).$$

Definition 3.1.13 (Local approximation). Let X be a topological space and let E be a family of Banach spaces over X. Let Γ be a subset of the space $\prod_{x \in X} E_x$ of all selections of E.

1. If $\xi \in \prod_{x \in X} E_x$ and $x_0 \in X$, then we say that ξ is *approximable near x_0 by elements of* Γ if for all $\varepsilon > 0$ there is an $\eta \in \Gamma$ and an open neighbourhood U of x_0 in X such that $\|\eta(u) - \xi(u)\| \leq \varepsilon$ for all $u \in U$.

2. If $\xi \in \prod_{x \in X} E_X$, then we say that ξ is *locally approximable by elements of* Γ if ξ is approximable near x_0 by elements of Γ for all $x_0 \in X$.

3. We define $\overline{\Gamma}$ to be the set of selections of E which are locally approximable by elements of Γ.

Proposition 3.1.14. *Let X be a topological space and let E be a family of Banach spaces over X. Let Γ, Δ be subsets of the space $\prod_{x \in X} E_x$ of all selections of E. Then*

1. *If $\Gamma \subseteq \Delta$, then $\overline{\Gamma} \subseteq \overline{\Delta}$.*

2. $\overline{\overline{\Gamma}} = \overline{\Gamma}$.

3. $\overline{\emptyset} = \emptyset$.

4. *If Γ is a linear subspace of $\prod_{x\in X} \mathbb{E}_x$, then $\overline{\Gamma}$ is a linear subspace as well.*

Proof.

1. Obvious from the definition.

2. From 1. it follows that $\overline{\Gamma} \subseteq \overline{\overline{\Gamma}}$. Let ξ be an element of $\overline{\overline{\Gamma}}$. Let $x_0 \in X$ and $\varepsilon > 0$. Find a neighbourhood U_1 of x_0 in X and a selection $\eta_1 \in \overline{\Gamma}$ such that $\sup_{x\in U_1} \|\xi(x) - \eta_1(x)\| \le \varepsilon/2$. Now find a neighbourhood U_2 of x_0 in X and some $\eta_2 \in \Gamma$ such that $\sup_{x\in U_2} \|\eta_1(x) - \eta_2(x)\| \le \varepsilon/2$. Let U be $U_1 \cap U_2$. Then $\sup_{x\in U} \|\xi(x) - \eta_2(x)\| \le \varepsilon$, so $\xi \in \overline{\Gamma}$.

3. Obvious from the definition.

4. Let $\xi_1, \xi_2 \in \overline{\Gamma}$, $x_0 \in X$ and $\varepsilon > 0$. Find neighbourhoods U_1 and U_2 of x_0 in X and $\eta_1, \eta_2 \in \Gamma$ such that $\sup_{x\in U_i} \|\xi_i(x) - \eta_i(x)\| < \varepsilon/2$ for all $i \in \{1, 2\}$. Define $U := U_1 \cap U_2$. Let $x \in U$. Then

$$\|(\xi_1 + \xi_2)(x) - (\eta_1 + \eta_2)(x)\| \le \|\xi_1(x) - \eta_1(x)\| + \|\xi_2(x) - \eta_2(x)\| \le \varepsilon.$$

   As $\eta_1 + \eta_2$ belongs to Γ this shows that $\xi_1 + \xi_2 \in \overline{\Gamma}$. Similarly one shows that $\overline{\Gamma}$ is closed under scalar multiplication. □

Remark 3.1.15. What we have called closure is not a proper closure operator since, in general, it fails to satisfy the condition $\overline{\Gamma \cup \Delta} = \overline{\Gamma} \cup \overline{\Delta}$. To see this let $X := \{1, 2\}$ be a discrete space with two points. Let E_0 be a non-trivial Banach space and $e, f \in E_0$ with $e \neq f$. Let E be $(E_0)_{x\in X}$ and let Γ and Δ be the sets containing only the constant selection which sends every $x \in X$ to e and f, respectively. Then $\overline{\Gamma} = \Gamma$ and $\overline{\Delta} = \Delta$. On the other hand, the selection which sends 1 to e and 2 to f is in $\overline{\Gamma \cup \Delta}$.

Lemma 3.1.16. *Let X be a topological space and let E be a family of Banach spaces over X. Then the space $\Sigma(X, E)$ satisfies $\overline{\Sigma(X, E)} = \Sigma(X, E)$.*

Proof. Let ξ be a selection of E which lies in $\overline{\Sigma(X, E)}$. Let $x_0 \in X$. We show that ξ is bounded near x_0. Let $\varepsilon := 1$. Find some $\eta \in \Sigma(X, E)$ and some neighbourhood U_1 of x_0 such that $\sup_{x\in U_1} \|\xi(x) - \eta(x)\| \le \varepsilon = 1$. Find some neighbourhood U_2 of x_0 such that η is bounded on U_2. Define $U := U_1 \cap U_2$. Let $x \in X$. Then $\|\xi(x)\| \le \|\eta(x)\| + \|\xi(x) - \eta(x)\| \le \sup_{u\in U} \|\eta(u)\| + \varepsilon$. So ξ is bounded on U. □

Locally bounded families of linear and bilinear maps

If E and F are families of Banach spaces over X, then it is natural to consider the locally bounded families of linear maps as morphisms between them, i.e., the locally bounded selections in the family $(\mathrm{L}\,(E_x, F_x))_{x\in X}$. It is easy to see that the composition of locally bounded families of linear maps is again locally bounded.

Locally bounded families of linear maps have the following continuity property:

Lemma 3.1.17. *Let T be a locally bounded family of linear operators from E to F.*

1. *If ξ is a locally bounded selection of E, then $T \circ \xi$ is a locally bounded selection of F, in other words, we have*
$$T \circ \Sigma(X, E) \subseteq \Sigma(X, F).$$

2. *If Γ is a subset of $\Sigma(X, E)$ and $\xi \in \Sigma(X, E)$ is locally approximable by elements of Γ, then $T \circ \xi$ is locally approximable by elements of $T \circ \Gamma = \{T \circ \gamma : \gamma \in \Gamma\}$, in other words, we have*
$$T \circ \overline{\Gamma} \subseteq \overline{T \circ \Gamma}.$$

Proof. We only prove the second assertion. Let $\varepsilon > 0$ and $x_0 \in X$. Find a neighbourhood U of x_0 in X such that T is bounded by some constant $C > 0$ on U. Find a neighbourhood V of x_0 in X and an element $\eta \in \Gamma$ such that $\|\xi(x) - \gamma(x)\| \leq \varepsilon / C$ for all $x \in V$. If $x \in U \cap V$, then
$$\big\|T_x(\xi(x)) - T_x(\eta(x))\big\| \leq \|T_x\| \, \big\|\xi(x) - \eta(x)\big\| \leq \|T_x\| \, \varepsilon / C \leq \varepsilon.$$
So $T \circ \eta$ is sufficiently close to $T \circ \xi$ near x. □

Similarly, one can consider locally bounded families of bilinear maps: If E, F and G are families of Banach spaces over X, then a locally bounded family of bilinear maps from $E \times_X F$ to G is a locally bounded selection in $(\mathrm{M}\,(E_x \times F_x; G_x))_{x \in X}$. It has continuity properties that are analogous to those given above for locally bounded families of linear operators:

Lemma 3.1.18. *Let μ be a locally bounded field of bilinear maps from $E \times_X F$ to G.*

1. *If $\xi \in \Sigma(X, E)$ and $\eta \in \Sigma(X, F)$, then $\mu \circ (\xi, \eta) \in \Sigma(X, G)$.*

2. *Let $\Gamma \subseteq \Sigma(X, E)$, $\Delta \subseteq \Sigma(X, F)$, $\xi \in \Sigma(X, E)$ and $\eta \in \Sigma(X, F)$. Assume that ξ and η are locally approximable by elements of Γ and Δ, respectively. Then $\mu \circ (\xi, \eta)$ is locally approximable by elements of $\{\mu \circ (\gamma, \delta) : \gamma \in \Gamma, \delta \in \Delta\}$.*

Proof. Again, we only proof 2.: Let $\varepsilon > 0$ and $x_0 \in X$. Since μ is locally bounded near x_0, we can find a neighbourhood U_μ of x_0 in X and a constant $C_\mu \geq 0$ such that μ is bounded by C_μ on U_μ.

Since ξ and η are locally bounded, we can find neighbourhoods U_ξ and U_η of x_0 in X such that ξ is bounded on U_ξ by some constant C_ξ and η is bounded on U_η by C_η.

Since ξ and η are approximable near x_0 by elements of Γ and Δ, respectively, we can find a neighbourhood $U \subseteq U_\xi \cap U_\eta$ of x_0 in X and elements $\gamma \in \Gamma$ and $\delta \in \Delta$ such that
$$C_\mu C_\eta \sup_{u \in U} \|\xi(u) - \gamma(u)\| \leq \varepsilon/3, \quad C_\mu C_\xi \sup_{u \in U} \|\eta(u) - \delta(u)\| \leq \varepsilon/3$$
as well as
$$C_\mu \sup_{u \in U} \|\xi(u) - \gamma(u)\| \, \|\eta(u) - \delta(u)\| \leq \varepsilon/3.$$
For all $u \in U \cap U_\mu$, we have
$$\begin{aligned}
& \|\mu_u(\xi(u), \eta(u)) - \mu_u(\gamma(u), \delta(u))\| \\
\leq \;& \|\mu_u\| \Big(\|\xi(u) - \gamma(u)\| \, \|\eta(u)\| + \|\xi(u)\| \, \|\eta(u) - \delta(u)\| + \|\xi(u) - \gamma(u)\| \, \|\eta(u) - \delta(u)\| \Big) \\
\leq \;& C_\mu \Big(C_\eta \|\xi(u) - \gamma(u)\| + C_\xi \|\eta(u) - \delta(u)\| + \|\xi(u) - \gamma(u)\| \, \|\eta(u) - \delta(u)\| \Big) \\
\leq \;& \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$
So $\mu \circ (\gamma, \delta)$ is sufficiently close to $\mu \circ (\xi, \eta)$ on $U \cap U_\mu$. □

3.1.3 Upper semi-continuous fields of Banach spaces

Definition and basic properties

Definition 3.1.19 (Upper semi-continuous field of Banach spaces). An *upper semi-continuous field of Banach spaces* over the topological space X is a pair $E = ((E_x)_{x\in X},\, \Gamma)$, where $(E_x)_{x\in X}$ is a family of Banach spaces and $\Gamma \subseteq \prod_{x\in X} E_x$ is a set of selections, which has the following properties:

(C1) Γ is a complex linear subspace of $\prod_{x\in X} E_x$;

(C2) for all $x \in X$, the evaluation map $\mathrm{ev}_x \colon \Gamma \to E_x,\ \xi \mapsto \xi(x)$, has dense image;

(C3) for all $\xi \in \Gamma$, the map $|\xi| : X \to \mathbb{R}_{\geq 0},\ x \mapsto \|\xi(x)\|_{E_x}$, is upper semi-continuous;

(C4) if $\xi \in \prod_{x\in X} E_x$ and if, for all $x_0 \in X$ and all $\varepsilon > 0$, there is an element $\gamma \in \Gamma$ and a neighbourhood U of x_0 in X such that for all $x \in U$ we have $\|\xi(x) - \gamma(x)\|_{E_x} \leq \varepsilon$, then also ξ belongs to Γ.

Condition (C4) just says that a selection which can be approximated locally by elements of Γ is itself in Γ, i.e. $\overline{\Gamma} = \Gamma$. Note that all elements of Γ are locally bounded by (C3). Instead of "upper semi-continuous field" we will usually say "u.s.c. field" or just "field" of Banach spaces. If $|\xi|$ is continuous for all $\xi \in \Gamma$, then we call E a *continuous field of Banach spaces*. However, we are not going to use this notion in this thesis very often.

Sections

Definition 3.1.20 (Sections). Let $E = ((E_x)_{x\in X},\, \Gamma)$ be a u.s.c field of Banach spaces.

1. The elements of Γ are called the *sections of* E. We will also write $\Gamma(X, E)$ for Γ.
2. The Banach space of bounded sections is denoted by $\Gamma_b(X, E)$.
3. The Banach space of all sections of E vanishing at infinity is denoted by $\Gamma_0(X, E)$.
4. The linear space of all sections of E with compact support is denoted by $\Gamma_c(X, E)$.

Note that
$$\Gamma_c(X, E) \subseteq \Gamma_0(X, E) \subseteq \Gamma_b(X, E) \subseteq \Gamma(X, E).$$

Example 3.1.21 (Constant fields). Let E be a Banach space. For every $x \in X$, set $E_x := E$ and let Γ be the space $\mathcal{C}(X, E)$ of all continuous maps from X to E. Then this gives a continuous field E_X of Banach spaces, called the *constant* field over X with fibre E.

Example 3.1.22 (Mapping cylinders). Let E and F be Banach spaces and let $T \in \mathrm{L}_{\leq 1}(E, F)$ be a contractive operator. Then the *mapping cylinder* $\mathrm{Z}(T)$ of T, considered as a u.s.c. field of Banach spaces, is the field $\left((G_x)_{x\in[0,1]},\, \Gamma\right)$ such that $G_0 = E$, $G_x = F$ for all $x > 0$ and $\Gamma = E \oplus \mathcal{C}_0(]0,1], F)$ (where an element $e \in E$ corresponds to the section $\xi(0) = e$ and $\xi(x) = T(e)$ for $x > 0$).

Example 3.1.23 (Fields over discrete spaces). Let X be a discrete topological space and let E be a u.s.c. field of Banach spaces over X. Then $\Gamma(X, E) = \prod_{x\in X} E_x$. This follows from conditions (C2) and (C4). Vice versa, if E is just a family of Banach spaces over a set X, then equipping X with the discrete topology makes E into a continuous field of Banach spaces with sections $\prod_{x\in X} E_x$.

Proposition 3.1.24. [5] *Let E be a u.s.c. field of Banach spaces and let $x_0 \in X$. If $\xi \in \Gamma(X, E)$ is a section of E and $\chi \in \mathcal{C}(X)$, then $\chi\xi \in \Gamma(X, E)$.*

Lemma 3.1.25. [6] *Let X be uniformisable and let E be a u.s.c. field of Banach spaces over X. Then for all $x \in X$, the evaluation map $\mathrm{ev}_x \colon \Gamma(X, E) \to E_x$ has not only dense image but is a metric surjection when restricted to the bounded sections. If X is locally compact Hausdorff, then this is also true for the sections vanishing at infinity.*

Proof. Let $x_0 \in X$. Let $e \in E$ and $\varepsilon > 0$. Find a $\xi \in \Gamma(X, E)$ such that $\|e - \xi(x_0)\|_{E_{x_0}} \leq \varepsilon/2$. Find a neighbourhood U of x_0 such that $\|\xi(x)\|_{E_x} \leq \|\xi(x_0)\|_{x_0} + \varepsilon/2$ for all $x \in U$. Find a continuous function φ on X such that $0 \leq \varphi(x) \leq 1$ for all $x \in X$, $\varphi(x_0) = 1$ and $\varphi(x) = 0$ for $x \neq U$. Let $\eta := \varphi\xi$. Note that $\eta \in \Gamma(X, E)$. Then $\|\eta(x)\| \leq \|\xi(x_0)\| + \varepsilon/2 \leq \|e\| + \varepsilon$ for all $x \in X$, so $\|\eta\|_\infty \leq \|e\| + \varepsilon$. On the other hand, we have $\eta(x_0) = \varphi(x_0)\xi(x_0) = \xi(x_0)$. We can now apply Corollary E.3.2 to see that ev_{x_0} is indeed a metric surjection.

If X is locally compact Hausdorff, then the same argument goes through with $\Gamma_0(X, E)$ instead of $\Gamma_b(X, E)$. The neighbourhood U can be chosen to be compact. □

Total subsets

Proposition 3.1.26. [7] *Let $(E_x)_{x\in X}$ be a family of Banach spaces and $\Lambda \subseteq \prod_{x\in X} E_x$. Let $\langle\Lambda\rangle$ be the complex linear subspace of $\prod_{x\in X} E_x$ generated by Λ. If $\langle\Lambda\rangle$ satisfies condition (C1), (C2), and (C3) (with $\langle\Lambda\rangle$ instead of Γ) then there is a unique subset Γ of $\prod_{x\in X} E_x$ containing Λ and satisfying (C1), (C2), (C3), (C4). This set is given by*

$$\Gamma = \left\{ \xi \in \prod_{x\in X} E_x : \; \forall x_0 \in X, \; \varepsilon > 0 \; \exists \eta \in \overline{\langle\Lambda\rangle}, \; x_0 \in U \overset{\text{open}}{\subseteq} X \; \forall x \in U : \; \|\eta(x) - \xi(x)\| \leq \varepsilon \right\},$$

i.e., the closure $\overline{\langle\Lambda\rangle}$ in the sense of Definition 3.1.13.

Proof. To see existence we have to check that the elements of Γ (defined as above) satisfy (C1)-(C4): Firstly, the closure of the total linear subspace $\langle\Lambda\rangle$ is indeed total and linear, so (C1) and (C2) are trivial. We also already know that taking the closure a second time does not change anything anymore, so (C4) is also true. What is left to show is (C3). Let $\xi \in \Gamma$ and $x_0 \in X$. Let $\varepsilon > 0$. Find a neighbourhood U_1 of x_0 in X and a selection $\eta \in \langle\Lambda\rangle$ such that $\sup_{x\in U_1} \|\xi(x) - \eta(x)\| < \varepsilon/3$. Now η has an upper semi-continuous modulus function, so we can find a neighbourhood U_2 of x_0 in X such that $\|\eta(x)\| \leq \|\eta(x_0)\| + \varepsilon/3$ for all $x \in U_2$. Define $U := U_1 \cap U_2$. Let $x \in X$. Then

$$\|\xi(x)\| \;\leq\; \|\xi(x) - \eta(x)\| + \|\eta(x) - \eta(x_0)\| + \|\eta(x_0) - \xi(x_0)\| + \|\xi(x_0)\| \;\leq\; \|\xi(x_0)\| + \varepsilon.$$

Hence ξ has an upper semi-continuous modulus function.

To prove uniqueness assume that Γ' is another subspace of $\prod_{x\in X} E_x$ containing Λ and satisfying (C1)-(C4). Since Γ' is a vector space it contains $\langle\Lambda\rangle$, and since it satisfies (C4) it contains Γ. To see the reverse inclusion let ξ be an element of Γ'. Let $x_0 \in X$ and $\varepsilon > 0$. Find a selection $\eta \in \langle\Lambda\rangle$ such that $\|\xi(x_0) - \eta(x_0)\| < \varepsilon/2$. This is possible because $\langle\Lambda\rangle$ satisfies (C2). Now $\xi - \eta \in \Gamma'$, which implies that its modulus function is upper semi-continuous. We can therefore find a neighbourhood U of x_0 in X such that $\|\xi(x) - \eta(x)\| \leq \|\xi(x_0) - \eta(x_0)\| + \varepsilon/2 \leq \varepsilon$ for all $x \in U$. This implies that $\xi \in \overline{\langle\Lambda\rangle} = \Gamma$. □

[5] See [Laf06], Proposition 1.1.3, and compare Proposition IX.10.1.9 in [Dix64].
[6] Compare [Laf06], Proposition 1.1.6, and Proposition IX.10.1.10 in [Dix64].
[7] See [Laf06], Proposition 1.1.4, and compare Proposition IX.10.2.3 in [Dix64].

Proposition 3.1.27. *Let E be a u.s.c. field of Banach spaces over locally compact Hausdorff X. Let Λ be a subset of $\Gamma_0(X,E)$ which is invariant under multiplication with elements of $\mathcal{C}_c(X)$. Then Λ is dense in the Banach space $\Gamma_0(X,E)$ if and only if it is total.*

Proof. If Λ is dense in $\Gamma_0(X,E)$, then it is clearly fibrewise dense as the evaluation maps are metric surjections.

Let Λ be fibrewise dense. Let $\varepsilon > 0$. Let $\xi \in \Gamma_0(X,E)$. Without loss of generality we can assume that ξ has compact support. For every $x \in X$, find an element $\lambda^x \in \Lambda$ such that $\|\lambda^x(x) - \xi(x)\| \leq \varepsilon/2$. Since the function of $y \mapsto \|\lambda^x(y) - \xi(y)\|$ is upper semi-continuous we can find a neighbourhood U_x of x such that $\|\lambda^x(y) - \xi(y)\| \leq \varepsilon$ for all $y \in U_x$.

Let $K := \operatorname{supp} \xi$ be a the compact support of ξ and let L be a compact neighbourhood of K. Then $\{U_x : x \in L\}$ is an open cover of L, so we can find a finite set $S \subseteq L$ such that $\{U_s : s \in S\}$ is a cover of L. Let $(\chi^s)_{s\in S}$ be a continuous partition of unity on K subordinate to this cover consisting of non-negative elements of $\mathcal{C}_c(X)$ supported in L such that their sum does not exceed 1 on L. Define

$$\lambda := \sum_{s\in S} \chi^s \lambda^s \in M.$$

For every $y \in X$, we have $\|\xi(y) - \lambda(y)\| \leq \varepsilon$: If $y \notin L$, then $\lambda(y) = 0$ and $\xi(y) = 0$. If $y \in L \setminus K$, then $\xi(y) = 0$ and for every $s \in S$:

$$\|\chi^s(y)\lambda^s(y)\| \leq \chi^s(y) \|\lambda^s(y) - \xi(y)\| \leq \chi^s(y)\varepsilon$$

so $\|\lambda(y)\| \leq \varepsilon$. If $y \in K$, then similarly $\|\xi(y) - \lambda(y)\| \leq \varepsilon$. □

Continuous fields of linear maps

A continuous field of linear maps between fields of Banach spaces is a locally bounded family of linear maps which sends sections to sections. Here is the stand-alone version of the definition:

Definition 3.1.28 (Continuous field of linear maps). Let E and F be u.s.c. fields of Banach spaces. Then a *continuous field of linear maps* from E to F is a family $(T_x)_{x\in X}$ such that

1. $T_x \in \mathrm{L}(E_x, F_x)$ for all $x \in X$;
2. $\forall \xi \in \Gamma(X,E): \quad T \circ \xi \colon x \mapsto T_x(\xi(x)) \in \Gamma(X,F)$;
3. the function $x \mapsto \|T_x\|$ is locally bounded[8] on X.

The set of all continuous fields of linear maps from E to F will be denoted by $\mathrm{L}^{\mathbf{loc}}(E,F)$. The subset of (globally) bounded continuous fields of linear maps from E to F is denoted by $\mathrm{L}(E,F)$.

We call an element $T \in \mathrm{L}^{\mathbf{loc}}(E,F)$ a *continuous* field because we think of property 2. as a continuity property of T. Although T is a locally bounded selection of the family $(\mathrm{L}(E_x,F_x))_{x\in X}$ of Banach spaces over X, it wont be generally true that $|T|$ is upper semi-continuous. So the space $\mathrm{L}^{\mathbf{loc}}(E,F)$ will generally not define a u.s.c. field of Banach spaces.

Note that the composition of continuous fields of linear maps is again continuous, the same applies to bounded continuous fields of linear maps. We hence have several choices for the morphisms of the

[8] In [Laf06] continuous fields of linear maps are defined leaving out our third condition (Définition 1.1.7), however, Proposition 1.1.9 of the same article states that Condition 3. is automatic if X is metrisable. A more general result along these lines is proved in Appendix E.2.

u.s.c. fields of Banach spaces over X: The continuous fields of linear maps, the bounded continuous fields and the continuous fields bounded by one. We hence also get three different notions of an isomorphism of u.s.c. fields of Banach spaces; we will call these isomorphisms "continuous", "bounded" and "isometric".

Example 3.1.29. Let E and F be constant fields over X with fibres E_0 and F_0, respectively. Then the condition on a locally bounded family of operators $(T_x)_{x\in X}$, where $T_x \in \mathrm{L}(E_0, F_0)$, for being a continuous field of linear maps reads

$$\forall \xi \in \mathcal{C}(X, E_0): \quad T \circ \xi \in \mathcal{C}(X, F_0).$$

This is the case if and only if the family T is strongly continuous.

Proposition 3.1.30 (A test for continuity). *Let E and F be u.s.c. fields of Banach spaces over X. Let $(T_x)_{x\in X}$ be a locally bounded family (not necessary continuous) of linear maps from E to F. Then T is a continuous field of linear maps from E to F if and only if $T \circ \xi \in \Gamma(X, F)$ for all elements ξ of some total subset $\Lambda \subseteq \Gamma(X, E)$.*

Proof. Since T is a family of linear maps it takes the span $\langle\Lambda\rangle$ into $\Gamma(X, E)$. Since T is locally bounded, it is continuous with respect to the closure operator defined in 3.1.13 by Lemma 3.1.17, so $\Gamma(X, E) = \overline{\langle\Lambda\rangle}$ is mapped into $\overline{\Gamma(Y, F)} = \Gamma(Y, F)$. □

Let E and F be u.s.c. fields of Banach spaces over X. Then $\mathrm{L}^{\mathbf{loc}}(E, F)$ carries a canonical vector space structure. Moreover, it is a $\mathcal{C}(X)$-module and the map $\mathrm{L}^{\mathbf{loc}}(E, F) \times \Gamma(X, E) \to \Gamma(X, F)$ is $\mathcal{C}(X)$-bilinear.

Proposition 3.1.31. *Let E and F be u.s.c. fields of Banach spaces over X. Then the space $\mathrm{L}^{\mathbf{loc}}(E, F)$ of selections in $(\mathrm{L}(E_x, F_x))_{x\in X}$ is closed with respect to the closure operator defined in 3.1.13, i.e., if $T = (T_x)_{x\in X}$ is a family of continuous linear maps from E to F which can be locally approximated by elements of $\mathrm{L}^{\mathbf{loc}}(E, F)$, then T is itself in $\mathrm{L}^{\mathbf{loc}}(E, F)$.*

Proof. Let T be a family of continuous linear maps from E to F which can be locally approximated by elements of $\mathrm{L}^{\mathbf{loc}}(E, F)$. Then T is locally bounded because it can be approximated locally by locally bounded selections (see Lemma 3.1.16).

Now let $\xi \in \Gamma(X, E)$. We show that $T \circ \xi \in \Gamma(X, F)$ by using condition (C4) of the definition of a u.s.c. field of Banach spaces. So let $x_0 \in X$ and $\varepsilon > 0$. Since $x \mapsto \|\xi(x)\|_{E_x}$ is upper semi-continuous it is locally bounded and we can find a constant $C \geq 0$ and a neighbourhood U_ξ of x_0 in X such that $\|\xi(u)\|_{E_u} \leq C$ for all $u \in U_\xi$. We can find a neighbourhood $U \subseteq U_\xi$ of x_0 in X and a continuous field of linear maps $(S_x)_{x\in X}$ from E to F such that $C\,\|T_u - S_u\| \leq \varepsilon$ for all $u \in U$. We now compare $T \circ \xi$ to $S \circ \xi \in \Gamma(X, F)$ on U:

$$\begin{aligned}
\|(T\circ\xi)(u) - (S\circ\xi)(u)\|_{F_u} &\leq \|T_u(\xi(u)) - S_u(\xi(u))\|_{F_u} \\
&\leq \|T_u - S_u\|\,\|\xi(u)\|_{E_u} \leq C\,\|T_u - S_u\| \leq \varepsilon.
\end{aligned}$$

for all $u \in U$. It follows that $T \circ \xi \in \Gamma(X, E)$ by (C4), and hence T is continuous. □

Proposition 3.1.32. *If E and F are u.s.c. fields of Banach spaces over X, then $\mathrm{L}(E, F)$ is a Banach space. The evaluation map $\mathrm{L}(E, F) \times \Gamma_b(X, E) \to \Gamma_b(X, F)$ is bilinear and bounded by 1.*

If $T \in \mathrm{L}(E, F)$, then it is easy to show that $T \circ \xi \in \Gamma_0(X, F)$ for all $\xi \in \Gamma_0(X, F)$. We we also have a continuous bilinear map $\mathrm{L}(E, F) \times \Gamma_0(X, E) \to \Gamma_0(X, F)$.

Continuous fields of bilinear maps

In this subsection, let E, F, G be u.s.c. fields of Banach spaces over the topological space X.

Definition 3.1.33 (The internal product and the internal sum). The *internal product* $E \times_X F$ of E and F is the upper semi-continuous field of Banach spaces over X given by the following data: The underlying family of Banach spaces is just the family $E \times_X F = (E_x \times F_x)_{x \in X}$, and the space of sections is

$$\Gamma := \{x \mapsto (\xi(x), \eta(x)) : \xi \in \Gamma(X, E), \eta \in \Gamma(X, F)\}.$$

The set Γ satisfies condition (C1) - (C4), thus it defines the structure of a u.s.c. field of Banach spaces on $E \times_X F$. Similarly we define the *internal sum* $E \oplus_X F$ of E and F over X, the difference being that we take the sum-norm instead of the sup-norm on the fibres.

Definition 3.1.34 (Continuous fields of bilinear maps). A continuous field of bilinear maps from $E \times_X F$ to G is a family $(\mu_y)_{y \in Y}$ of continuous bilinear maps $\mu_y \in \mathrm{M}(E_{\gamma(y)}, F_{\gamma(y)}; G_y)$ for all $y \in Y$ such that

1. $\forall \xi \in \Gamma(X, E)\ \forall \eta \in \Gamma(X, F): \quad x \mapsto \mu_x(\xi(x),\ \eta(x)) \ \in\ \Gamma(X, G)$.
2. μ is locally bounded.

We write $\mathrm{M}^{\mathbf{loc}}(E, F; G)$ for the linear space of all continuous fields of bilinear maps from $E \times_X F$ to G. The linear space of (globally) bounded elements of $\mathrm{M}^{\mathbf{loc}}(E, F; G)$ is denoted by $\mathrm{M}(E, F; G)$.

Analogously to the case of continuous fields of linear maps we have:

Proposition 3.1.35 (A test for continuity). *Let μ be a locally bounded family of bilinear maps from $E \times_X F$ to G. Then μ is continuous if and only if there is a total linear subspace $\Gamma_0 \subseteq \Gamma(X, E \times_X F)$ such that $\mu \circ \zeta \in \Gamma(X, G)$ for all $\zeta \in \Gamma_0$.*

Definition and Proposition 3.1.36 (Internal tensor product). [9]We define $E \otimes_X F$ to be the following u.s.c. field of Banach spaces over X: The underlying family of Banach spaces is what we have already called $E \otimes_X F$, i.e., for all $x \in X$ the fibre over x is $E_x \otimes^\pi F_x$. To define the sections of $E \otimes_X F$, let Λ be the $\mathbb{C}$-linear span of all selections of the family $E \otimes_X F$ given by $x \mapsto \xi(x) \otimes \eta(x)$, where ξ runs through $\Gamma(X, E)$ and η runs through $\Gamma(X, F)$. Then Λ satisfies conditions (C1), (C2) and (C3) so by the use of Proposition 3.1.26 we get the structure of a u.s.c. field of Banach spaces on $E \otimes_X F$. There is a canonical contractive continuous field of bilinear maps $\pi = (\pi_x)_{x \in X}$ from $E \times_X F$ to $E \otimes_X F$.

Proof based on an argument of V. Lafforgue. We check that Λ satisfies the conditions (C1), (C2) and (C3). Firstly, Λ is a linear subspace of $\prod_{x \in X} E_x \otimes F_x$ by definition. Condition (C2) is also obvious. For (C3) we have to show: For all $x_0 \in X$ and all $\zeta \in \Lambda$:

$$\limsup_{x \to x_0} \|\zeta(x)\|_{E_x \otimes F_x} \leq \|\zeta(x_0)\|_{E_{x_0} \otimes F_{x_0}}.$$

So let $x_0 \in X$. Define the bilinear maps

$$\theta_{x_0} \colon \Gamma(X, E) \times \Gamma(X, F) \to E_{x_0} \otimes F_{x_0},\ (\xi, \eta) \mapsto \xi(x_0) \otimes \eta(x_0),$$

and

$$\theta \colon \Gamma(X, E) \times \Gamma(X, F) \to \prod_{x \in X} E_x \otimes F_x,\ (\xi, \eta) \mapsto [x \mapsto \xi(x) \otimes \eta(x)].$$

[9]Compare Proposition 1.1.19 in [Laf06].

Since both maps are $\mathbb{C}$-bilinear, they give linear maps $\Gamma(X,E) \otimes^{\mathrm{alg}} \Gamma(X,F)$ to $E_{x_0} \otimes F_{x_0}$ and to $\prod_{x\in X} E_x \otimes F_x$, respectively. Call them $\hat{\theta}_{x_0}$ and $\hat{\theta}$. The image of $\hat{\theta}$ is Λ. On Λ define the semi-norm

$$\|\zeta\|_{x_0,\lim} := \limsup_{x\to x_0} \|\zeta(x)\|_{E_x\otimes F_x}, \qquad \zeta \in \Lambda.$$

For every $\xi \in \Gamma(X,E)$ and every $\eta \in \Gamma(X,F)$, we have

$$\begin{aligned} \|\theta(\xi,\eta)\|_{x_0,\lim} &= \limsup_{x\to x_0} \|\xi(x)\otimes\eta(x)\|_{E_x\otimes F_x} \\ &\leq \limsup_{x\to x_0} \|\xi(x)\|_{E_x} \|\eta(x)\|_{F_x} \leq \|\xi(x_0)\|_{E_{x_0}} \|\eta(x_0)\|_{F_{x_0}}. \end{aligned}$$

This implies that there is a bilinear map μ from $E_{x_0} \times F_{x_0}$ to the Hausdorff completion $\overline{\Lambda}$ of Λ with respect to the above norm such that $\|\mu\| \leq 1$ and $\mu(\xi(x_0),\eta(x_0)) = \iota(\theta(\xi,\eta))$, where ι denotes the canonical map from Λ to its completion. From the universal property of the projective tensor product we know that there is a unique continuous linear map $\hat{\mu}$ from $E_{x_0} \otimes F_{x_0}$ to $\overline{\Lambda}$ such that $\hat{\mu}(e\otimes f) = \mu(e,f)$ for all $e \in E_{x_0}$ and $f \in F_{x_0}$, and $\|\hat{\mu}\| \leq 1$. We have

$$\hat{\mu}\left(\theta_{x_0}(\xi,\eta)\right) = \hat{\mu}(\xi(x_0)\otimes\eta(x_0)) = \mu(\xi(x_0),\eta(x_0)) = \iota(\theta(\xi,\eta))$$

for all $\xi \in \Gamma(X,E)$ and $\eta \in \Gamma(X,F)$, so we also have $\hat{\mu}(\hat{\theta}_{x_0}(\omega)) = \iota(\hat{\theta}(\omega))$ for all $\omega \in \Gamma(X,E) \otimes^{\mathrm{alg}} \Gamma(X,F)$. From this it follows that $\hat{\mu}(\zeta(x_0)) = \iota(\zeta)$ for all $\zeta \in \Lambda$. By $\|\hat{\mu}\| \leq 1$ it follows that

$$\|\zeta\|_{x_0,\lim} = \|\iota(\zeta)\|_{\overline{\Gamma_0}} \leq \|\zeta(x_0)\|_{E_{x_0}\otimes F_{x_0}}.$$

But this is exactly what we wanted to show. □

Proposition 3.1.37 (Universal property). *For every (bounded / contractive) continuous field μ of bilinear maps from $E \times_X F$ to G, there is a unique (bounded / contractive) continuous field $\widehat{\mu}$ of linear maps from $E \otimes_X F$ to G such that the following diagram commutes*

$$\begin{tikzcd} E \times_X F \arrow[r, "\mu"] \arrow[d, "\pi"] & G \\ E \otimes_X F \arrow[ur, dashed, "\widehat{\mu}"] & \end{tikzcd}$$

The family $\widehat{\mu}$ is given by $(\widehat{\mu_x})_{x\in X}$.

Corollary 3.1.38. *Let E, E', F and F' be u.s.c. fields of Banach spaces over X. For all (bounded / contractive) continuous fields S of linear maps from E to E' and T from F to F', there is a unique (bounded / contractive) continuous field $S \otimes T$ of linear maps from $E \otimes_X F$ to $E' \otimes_Y F'$ such that the following diagram commutes*

$$\begin{tikzcd} E \times_X F \arrow[r, "S\times_X T"] \arrow[d] & E' \times_X F' \arrow[d] \\ E \otimes_X F \arrow[r, dashed, "S\otimes T"] & E' \otimes_X F' \end{tikzcd}$$

This assignment is functorial.

3.1.4 Fields of Banach algebras

Definition 3.1.39 (Field of Banach algebras). A *u.s.c. field of Banach algebras over X* is an upper semi-continuous field A of Banach spaces over X together with a continuous field of bilinear maps $\mu\colon A \times_X A \to A$ such that (A_x, μ_x) is a Banach algebra for all $x \in X$. In particular, this means that μ is bounded by 1. A field of Banach algebras A over X (with multiplication μ) is called *non-degenerate* if μ_x is non-degenerate for all $x \in X$, i.e., the span of $A_x A_x$ is dense in A_x.

Example 3.1.40 (Constant fields of Banach algebras). Let A be a Banach algebra with multiplication μ. Then the constant field A_X as defined in Example 3.1.21, together with the multiplication $(\mu)_{x\in X}$, is a continuous field of Banach algebras called the *constant field* over X with fibre A.

Definition 3.1.41 (Homomorphism of fields of Banach algebras). Let A and B be u.s.c. fields of Banach algebras over X. Then a *homomorphism (of fields of Banach algebras) from A to B* is a continuous field of homomorphisms of Banach algebras from A to B, i.e., a continuous field $(\varphi_x)_{x\in X}$ of linear maps from A to B such that φ_x is a (contractive) homomorphism of Banach algebras from A_x to B_x. In particular, such a φ is bounded by 1.

Definition 3.1.42 (Fibrewise unitalisation of a field of Banach algebras). Let B be a u.s.c. field of Banach algebras over X. Then we define the *fibrewise unitalisation*

$$\widetilde{B} = B \oplus \mathbb{C}_X = \left(\widetilde{B_x}\right)_{x\in X}$$

to be the following u.s.c. field of Banach algebras: For all $x \in X$, the fibre of $\widetilde{B}$ is the unitalisation $\widetilde{B_x}$ of the fibre B_x of B. The sections of $\widetilde{B}$ are $\Gamma(X, B) \oplus \mathcal{C}(X)$.

3.1.5 Fields of Banach modules

Let A, B and C be u.s.c. fields of Banach algebras over X.

Definition 3.1.43 (Field of Banach modules). A *right Banach B-module* is an upper semi-continuous field E of Banach spaces over X together with a continuous field of bilinear maps $\mu^E\colon E \times_X B \to E$ such that, for all $x \in X$, E_x is a right Banach B_x-module with the B_x-action μ^E_x. In particular, this means that μ^E is bounded by 1. The module E is called non-degenerate if μ^E_x is non-degenerate for all $x \in X$, i.e., the span of $E_x B_x$ is dense in E_x.

Left Banach A-modules and Banach A-B-bimodules are defined similarly.

Definition 3.1.44 (Linear operator between fields of Banach modules). Let E and F be right Banach B-modules. Then a *B-linear field of operators from E to F* (or just a *B-linear operator from E to F*) is a continuous field T of linear maps from E to F such that T_x is B_x-linear (on the right) for all $x \in X$. We denote the space of all such T by $\mathrm{L}_B^{\mathbf{loc}}(E, F)$.

As usual, the field T is called bounded if $\|T\| := \sup_{x\in X} \|T_x\| < \infty$, i.e., if T is a bounded continuous field of linear maps. We denote the bounded B-linear operators from E to F by $\mathrm{L}_B(E, F)$.

Definition 3.1.45 (Homomorphism between fields of Banach modules). Let B' be another field of Banach algebras over X and let $\psi\colon B \to B'$ be a continuous field of homomorphisms. Let E be a right Banach B-module and E' be a right Banach B'-module. Then a homomorphism Φ_ψ (of u.s.c. fields of Banach modules) from E_B to $E'_{B'}$ with coefficient map ψ is a *contractive* continuous field Φ of linear maps from E to E' such that Φ_x is a homomorphism with coefficient map ψ_x from $(E_x)_{B_x}$ to $(E'_x)_{B'_x}$.

An analogous definition can be made for left Banach modules and Banach bimodules.

Definition 3.1.46 (Field of balanced bilinear maps). Let E_1 be a right Banach B-module and E_2 a left Banach B-module. Let F be a u.s.c. field of Banach spaces over X. A continuous field μ of bilinear maps from $E_1 \times_X E_2$ to F is called B-*balanced* if $\mu_x \colon (E_1)_x \times (E_2)_x \to F_x$ is B_x-balanced for all $x \in X$.

The following definition is analogous to 3.1.36 (and what has to be shown can also proved in much the same way). See also Proposition 1.1.19 of [Laf06].

Definition 3.1.47 (The balanced tensor product of fields of Banach modules). Let E be a right Banach B-module and F a left Banach B-module. Define the B-balanced tensor product $E \otimes_B F$ of E and F to be the following u.s.c. field of Banach spaces: For all $x \in X$, the fibre at x is $E_x \otimes_{B_x} F_x$; to define the sections of $E \otimes_B F$, let Λ be the $\mathbb{C}$-linear span of all selections of the family $E \otimes_B F$ given by $x \mapsto \xi(x) \otimes \eta(x)$, where ξ runs through $\Gamma(X, E)$ and η runs through $\Gamma(X, F)$. Then Λ satisfies conditions (C1), (C2) and (C3) so by the use of Proposition 3.1.26 we get the structure of a u.s.c. field of Banach spaces on $E \otimes_B F$.

There is a canonical contractive continuous field of bilinear maps $\pi = (\pi_x)_{x \in X}$ from $E \times_X F$ to $E \otimes_B F$ and a canonical fibrewise surjective and open contractive continuous field of linear maps from $E \otimes_X F$ to $E \otimes_B F$. The field $E \otimes_B F$ has the universal property for continuous fields of B-balanced bilinear maps. If F is not only a left Banach B-module but a Banach B-C-bimodule, then $E \otimes_B F$ is a right Banach C-module in an obvious way.

Definition 3.1.48. Let E and E' be right Banach B-modules and F a Banach B-C-bimodule. For all $T \in \mathrm{L}_B^{\mathbf{loc}}(E, E')$ define $T \otimes 1 \in \mathrm{L}_C^{\mathbf{loc}}(E \otimes_B F,\ E' \otimes_B F)$ as the family $(T_x \otimes_{B_x} \mathrm{Id}_{F_x})_{x \in X}$.

Note that the assignment $T \mapsto T \otimes 1$ is linear and functorial. If T is bounded, then $\|T \otimes 1\| \leq \|T\|$.

Definition 3.1.49 (The pushout of fields of Banach modules). Let B' be a u.s.c. field of Banach algebras and $\psi \colon B \to B'$ a continuous field of homomorphisms. Let E be a right Banach B-module. Then $\psi(E) := E \otimes_{\widetilde{B}} \widetilde{B'}$ is a right Banach B'-module, called the *pushout of* E *along* ψ. The fibre of $\psi(E)$ at x is $\psi_x(E_x)$.

The pushout has the usual functorial properties, compare Proposition 1.3.11.

3.1.6 Fields of Banach pairs

Let A and B be u.s.c. fields of Banach algebras over X.

Definition 3.1.50 (Field of Banach pairs). A *Banach* B-*pair* is a pair $E = (E^<, E^>)$ such that $E^<$ is a left Banach B-module and $E^>$ is a right Banach B-module, together with a contractive continuous field of bilinear maps $\langle , \rangle \colon E^< \times_X E^> \to B$, B-linear on the left and on the right. E is called non-degenerate if $E^<$ and $E^>$ are non-degenerate Banach B-modules.

Define $E_x := (E_x^<, E_x^>)$ which is a B_x-pair when equipped with the bracket $\langle , \rangle_x$.

Definition 3.1.51 (Linear operator between fields of Banach pairs). Let E and F be Banach B-pairs. Then a *continuous field of* B-*linear operators from* E *to* F (or just a B-*linear operator from* E *to* F) is a pair $(T^<, T^>)$ where $T^>$ is a continuous field of B-linear operators from $E^>$ to $F^>$ and $T^<$ is a continuous field of B-linear operators from $F^<$ to $E^<$ such that $T_x := (T_x^<, T_x^>)$ is in $\mathrm{L}_{B_x}(E_x, F_x)$ for all $x \in X$. We denote the linear space of all such T by $\mathrm{L}_B^{\mathbf{loc}}(E, F)$.

A B-linear operator from E to F is called *bounded* if $T^<$ and $T^>$ are bounded. The space of all bounded B-linear operators from E to F will be denoted by $\mathrm{L}_B(E,F)$. It is a Banach space when equipped with the obvious operations and the norm $\|T\| := \max\{\|T^<\|, \|T^>\|\} = \sup_{x\in X}\|T_x\|$.

The condition on a pair of B-linear operators $(T^<, T^>)$ presented in the preceding definition can be conveniently written as

$$\langle,\rangle_E \circ \left(T^< \times_X \mathrm{Id}_{E^>}\right) = \langle,\rangle_F \circ \left(\mathrm{Id}_{F^<} \times_X T^>\right),$$

where the two sides represent fields of maps from $F^< \times_X E^>$ to B.

Definition 3.1.52 (Homomorphism between fields of Banach pairs). Let B and B' be u.s.c. fields of Banach algebras over X and let $\psi\colon B \to B'$ be a continuous field of homomorphisms of Banach algebras. Let E_B and $E'_{B'}$ be Banach pairs. Then a *continuous field* Φ *of homomorphisms from* E *to* E' *with coefficient map* ψ is a pair $(\Phi^<, \Phi^>)$ where $\Phi^>$ is a continuous field of homomorphisms from $E^>$ to $E'^>$ and $\Phi^<$ is a continuous field of homomorphisms from $E^<$ to $E'^<$, both with coefficient map ψ, such that $\Phi_x := (\Phi_x^<, \Phi_x^>)$ is a homomorphism with coefficient map ψ_x from the pair E_{xB_x} to the pair $E'_{xB'_x}$.

Note that the composition of linear operators is again a linear operator and the composition of homomorphisms is again a homomorphism.

Definition 3.1.53 (Banach A-B-pair). A Banach A-B-pair $E = (E^<, E^>)$ is a Banach B-pair E such that $E^<$ is a Banach B-A-bimodule and $E^>$ is a Banach A-B-bimodule and the bracket $\langle,\rangle\colon E^< \times_X E^> \to B$ is A-balanced (which means that for all $x \in X$ the map $\langle,\rangle_x\colon E_x^< \times E_x^> \to B_x$ is A_x-balanced).

There is an obvious notion of a homomorphism with coefficient maps between Banach A-B-pairs.

Using the definition of the balanced tensor product of fields of Banach modules (Definition 3.1.47) we can define the balanced tensor product of fields of Banach pairs, just as in Definition 1.3.3, the definition of the ordinary balanced tensor product of Banach pairs. Similarly, we can define the pushout of fields of Banach pairs along continuous fields of homomorphisms between u.s.c. fieds of Banach algebras. It has the usual functorial properties, compare Proposition 1.3.11.

Locally compact operators

Definition 3.1.54 (Rank one operator). Let E and F be Banach B-pairs. Then we define for all $\xi^< \in \Gamma(X, E^<)$ and all $\eta^> \in \Gamma(X, F^>)$ the continuous field of operators $\left|\eta^>\right\rangle\left\langle\xi^<\right| := \left(\left|\eta^>\right\rangle\left\langle\xi^<\right|_x\right)_{x\in X} \in \mathrm{L}_B^{\mathbf{loc}}(E,F)$ by

$$\left|\eta^>\right\rangle\left\langle\xi^<\right|_x := \left|\eta^>(x)\right\rangle\left\langle\xi^<(x)\right| \in \mathrm{K}_{B_x}(E_x, F_x)$$

for all $x \in X$.

If $\xi^<$ and $\eta^>$ are bounded then $\left|\eta^>\right\rangle\left\langle\xi^<\right|$ is bounded by $\|\xi^<\|\,\|\eta^>\|$. If $\xi^<$ and $\eta^>$ vanish at infinity, then so does $\left|\eta^>\right\rangle\left\langle\xi^<\right|$.

Definition 3.1.55 (Locally compact Operator). [10] Let E and F be Banach B-pairs. A continuous field T of B-linear operators is called *locally compact* if for all $x \in X$ and all $\varepsilon > 0$ there is an open neighbourhood U of x, an $n \in \mathbb{N}$ and $\xi_1^<, \ldots, \xi_n^< \in \Gamma(X, E^<)$ and $\eta_1^>, \ldots, \eta_n^> \in \Gamma(X, F^>)$ such that $\left\|T_u - \sum_{i=1}^n \left|\eta_i^>(u)\right\rangle\left\langle\xi_i^<(u)\right|\right\| \le \varepsilon$ for all $u \in U$. The space of all locally compact operators from E to F is denoted by $\mathrm{K}_B^{\mathbf{loc}}(E,F)$.

[10] V. Lafforgue calls such operators “partout compact” in [Laf06].

In other words: If $\mathcal{F}$ denotes the linear span of the operators of the form $\left|\eta^>\right\rangle\left\langle\xi^<\right|$, with $\xi^< \in \Gamma(X, E^<)$ and $\eta^> \in \Gamma(X, F^>)$, in the space $\mathrm{L}_B^{\mathbf{loc}}(E, F)$, then $\mathrm{K}_B^{\mathbf{loc}}(E, F)$ is the space of all operators that are locally approximable by elements of $\mathcal{F}$, i.e., $\mathrm{K}_B^{\mathbf{loc}}(E, F) = \overline{\mathcal{F}}$ in the sense of Definition 3.1.13.

Lemma 3.1.56. *Let E, F and G be Banach B-pairs. Then $\mathrm{L}_B^{\mathbf{loc}}(F, G) \circ \mathrm{K}_B^{\mathbf{loc}}(E, F) \subseteq \mathrm{K}_B^{\mathbf{loc}}(E, G)$ and $\mathrm{K}_B^{\mathbf{loc}}(F, G) \circ \mathrm{L}_B^{\mathbf{loc}}(E, F) \subseteq \mathrm{K}_B^{\mathbf{loc}}(E, G)$.*

Proof. Let $S \in \mathrm{K}_B^{\mathbf{loc}}(E, F)$ and $T \in \mathrm{L}_B^{\mathbf{loc}}(F, G)$. Let $\varepsilon > 0$ and $x_0 \in X$. Because T is locally bounded, we can find a neighbourhood U_T of x_0 in X and a constant $C_T > 0$ such that $\|T_u\| \leq C_T$ for all $u \in U_T$.

Because S is locally compact we can find a neighbourhood U_S of x_0 in X, an $n \in \mathbb{N}$ and $\xi_1^<, \dots, \xi_n^< \in \Gamma_0(X, E^<)$ and $\eta_1^>, \dots, \eta_n^> \in \Gamma_0(X, F^>)$ such that

$$\left\| S_u - \sum_{i=1}^n \left|\eta_i^>(u)\right\rangle\left\langle\xi_i^<(u)\right| \right\| \leq \frac{\varepsilon}{C_T}$$

for all $u \in U_S$. Note that $T \circ \left|\eta_i^>\right\rangle\left\langle\xi_i^<\right| = \left|(T^> \circ \eta_i^>)\right\rangle\left\langle\xi_i^<\right|$ for all $i \in \{1, \dots, n\}$ (with $T^> \circ \eta_i^> \in \Gamma_0(X, G^>)$), and

$$\begin{aligned} &\left\| (T \circ S)_u - \sum_{i=1}^n \left| \left(T^> \circ \eta_i^>\right)(u)\right\rangle\left\langle\xi_i^<(u)\right| \right\| \\ = \ &\left\| T_u \circ \left(S_u - \sum_{i=1}^n \left|\eta_i^>(u)\right\rangle\left\langle\xi_i^<(u)\right| \right) \right\| \leq \|T_u\| \cdot \frac{\varepsilon}{C_T} \leq \varepsilon \end{aligned}$$

for all $u \in U_S \cap U_T$. Hence $T \circ S$ is locally compact. Similarly one shows the other assertion. $\square$

Example 3.1.57. Let B be a non-degenerate u.s.c. field of Banach algebras over X. Then $\Gamma(X, B)$ acts by locally compact operators on the Banach B-pair (B, B).

Operators of the form $T \otimes 1$

Operators of the form $T \otimes 1$ for fields of Banach modules where defined in 3.1.48. From this definition, we get a straightforward generalisation for fields of Banach pairs:

Definition 3.1.58. Let E and E' be Banach B-pairs and F a Banach B-C-pair. For all operators $T \in \mathrm{L}_B^{\mathbf{loc}}(E, E')$ define $T \otimes 1 \in \mathrm{L}_C^{\mathbf{loc}}\left(E \otimes_B F,\ E' \otimes_B F\right)$ as $(1 \otimes T^<,\ T^> \otimes 1)$.

The assignment $T \mapsto T \otimes 1$ is linear and functorial, and if T is bounded, then $\|T \otimes 1\| \leq \|T\|$.

Proposition 3.1.59. *Let E and E' be Banach B-pairs and F a Banach B-C-pair. Assume that $\Gamma(X, B)$ acts on F by locally compact operators, call the action $\pi\colon \Gamma(X, B) \to \mathrm{K}_C^{\mathbf{loc}}(F)$. Assume moreover that E or E' is non-degenerate. Then*

$$T \in \mathrm{K}_B^{\mathbf{loc}}(E, E') \ \Rightarrow\ T \otimes 1 \in \mathrm{K}_C^{\mathbf{loc}}\left(E \otimes_B F,\ E' \otimes_B F\right).$$

Proof. Let $x_0 \in X$ and $\varepsilon > 0$. Assume that E is non-degenerate. Then $\Gamma(X, E)$ is non-degenerate in the following sense: The space of all sections of the form $x \mapsto \xi^>(x)\beta(x)$, with $\xi^> \in \Gamma(X, E^>)$ and $\beta \in \Gamma(X, B)$, spans a total subset of $\Gamma(X, E^>)$ (and similarly for $\Gamma(X, E^<)$). From the upper semi-continuity of the sections it follows that for all $\xi^< \in \Gamma(X, E^<)$, all $x_0 \in X$ and all $\delta > 0$ there is an $m \in \mathbb{N}$ and $\beta_1, \ldots, \beta_m \in \Gamma(X, B)$, $\xi_1^<, \ldots, \xi_m^< \in \Gamma(X, E^<)$ and a neighbourhood U of x_0 in X such that $\|\xi^<(u) - \sum_{i=1}^m \beta_i(u)\xi_i^<(u)\| \leq \delta$.

From this it follows that we can find a neighbourhood V of x_0 in X and $n \in \mathbb{N}$, $\xi_1'^>, \ldots \xi_n'^> \in \Gamma(X, E'^>)$, $\beta_1, \ldots, \beta_n \in \Gamma(X, B)$ and $\xi_1^<, \ldots, \xi_n^< \in \Gamma(X, E^<)$ such that for all $v \in V$:

$$\left\| T(v) - \sum_{j=1}^n \left|\xi_j'^>(v)\right\rangle\left\langle\beta_j(v)\xi_j^<(v)\right| \right\| \leq \varepsilon.$$

Now for all $x \in X$

$$\left|\xi_j'^>(x)\right\rangle\left\langle\beta_j(x)\xi_j^<(x)\right| \otimes 1 = M_{|\xi_j'^>(x)\rangle} \circ \pi(\beta_j)_x \circ M_{\langle\xi_j^<(x)|}$$

as in the proof of Proposition 1.3.7. By Lemma 3.1.56 the continuous field of operators $M_{|\xi_j'^>\rangle} \circ \pi(\beta_j) \circ M_{\langle\xi_j^<|}$ is locally compact because $\pi(\beta_j)$ is locally compact. We have

$$\left\| (T \otimes 1)(v) - \sum_{j=1}^n M_{|\xi_j'^>(v)\rangle} \circ \pi(\beta_j)_v \circ M_{\langle\xi_j^<(v)|} \right\| \leq \varepsilon$$

for all $v \in V$. Hence $T \otimes 1$ is locally compact as well. □

As in Proposition 1.3.13 one proves

Proposition 3.1.60. *Let B' be another u.s.c. field of Banach algebras and $\psi\colon B \to B'$ a continuous field of homomorphisms. Let E and F be Banach B-pairs. For all operators $T \in \mathrm{K}_B^{\mathbf{loc}}(E, F)$, the operator $\psi_*(T) = T \otimes 1$ is contained in $\mathrm{K}_C^{\mathbf{loc}}(\psi_*(E),\ \psi_*(F))$.*

3.2 Monotone completions

In [Laf02] and [Laf06] the notion of an unconditional completion[11] was introduced which is a special case of what we propose to call a *monotone* completion. The article [Laf02] provides us with some interesting examples of monotone completions which are not unconditional completions,[12] but we also meet and need this more general notion in two situations in this thesis, namely in Subsection 7.2.3 and in Section 7.3. It therefore seems advisable to dedicate an entire and separate section to the introduction of this basic notion.

In Section 3.2, let X be a locally compact Hausdorff space.

Definition 3.2.1 (Monotone (semi-)norm, monotone completion). A semi-norm $\|\cdot\|_{\mathcal{H}}$ on $\mathcal{C}_c(X)$ is called *monotone* if the following condition holds:

$$\forall \varphi_1, \varphi_2 \in \mathcal{C}_c(X): \quad (\forall x \in X:\ |\varphi_1(x)| \leq |\varphi_2(x)|) \Rightarrow \|\varphi_1\|_{\mathcal{H}} \leq \|\varphi_2\|_{\mathcal{H}}. \tag{3.1}$$

Let $\mathcal{H}(X)$ denote the (Hausdorff-)completion of $\mathcal{C}_c(X)$ with respect to this semi-norm; this Banach space is called a *monotone completion* of $\mathcal{C}_c(X)$.

[11] Unconditional completions are discussed in extenso in Section 5.2.

[12] For example $H^2(G, A)$ defined after Lemme 1.6.5 or the "normalised" completions $\mathrm{L}^{p,l}_{\mathrm{norm}}(G, A)$ appearing in 4.5.

By "let $\mathcal{H}(X)$ be a monotone completion of $\mathcal{C}_c(X)$" we mean in the sequel "let $\|\cdot\|_{\mathcal{H}}$ be a monotone semi-norm on $\mathcal{C}_c(X)$ and let $\mathcal{H}(X)$ denote its completion". If $\|\cdot\|_{\mathcal{H}}$ is a norm we can think of $\mathcal{C}_c(X)$ as a subspace of $\mathcal{H}(X)$. For the rest of the section, let $\mathcal{H}(X)$ be a monotone completion of $\mathcal{C}_c(X)$.

In [Laf06], unconditional norms on $\mathcal{C}_c(\mathcal{G})$ are extended to the non-negative upper semi-continuous functions with compact support on $\mathcal{G}$ (where $\mathcal{G}$ is a locally compact Hausdorff groupoid with Haar measure); this however is not sufficient because we want to apply unconditional norms also to the absolute value of continuous fields of operators (with compact support), which are not upper semi-continuous in general. This problem can be overcome very easily by extending unconditional norms or, more generally, monotone semi-norms to an even larger class of functions:

Definition 3.2.2 (The extension of a monotone semi-norm). Let $\mathcal{F}_c(X)$ be the set of all (locally) *bounded* functions $\varphi\colon X \to \mathbb{R}$ with compact support. Let $\mathcal{F}_c^+(X)$ be the set of elements of $\mathcal{F}_c(X)$ which are non-negative. Define

$$\|\varphi\|_{\mathcal{H}} := \inf\{\|\psi\|_{\mathcal{H}} :\ \psi \in \mathcal{C}_c(X),\ \psi \geq \varphi\}$$

for all $\varphi \in \mathcal{F}_c^+(X)$.

Note that by Property (3.1) the new semi-norm agrees on $\mathcal{C}_c^+(X)$ with the semi-norm we started with. We now deduce some computational rules for the extension:

Lemma 3.2.3. *The following holds for all $\varphi_1, \varphi_2, \varphi \in \mathcal{F}_c^+(X)$ and all $c \geq 0$:*

1. $\varphi_1 + \varphi_2 \in \mathcal{F}_c^+(X)$ *and* $\|\varphi_1 + \varphi_2\|_{\mathcal{H}} \leq \|\varphi_1\|_{\mathcal{H}} + \|\varphi_2\|_{\mathcal{H}}$*;*
2. $c\varphi \in \mathcal{F}_c^+(X)$ *and* $\|c\varphi\|_{\mathcal{H}} = c\,\|\varphi\|_{\mathcal{H}}$*;*
3. *if* $\varphi_1 \leq \varphi_2$*, then* $\|\varphi_1\|_{\mathcal{H}} \leq \|\varphi_2\|_{\mathcal{H}}$.

Proof. 1. $\varphi_1+\varphi_2$ is obviously bounded, non-negative and has compact support. If $\psi_1, \psi_2 \in \mathcal{C}_c(X)$ are such that $\varphi_i \leq \psi_i$, then $\varphi_1 + \varphi_2 \leq \psi_1 + \psi_2$, and hence

$$\|\varphi_1 + \varphi_2\|_{\mathcal{H}} \leq \|\psi_1 + \psi_2\|_{\mathcal{H}} \leq \|\psi_1\|_{\mathcal{H}} + \|\psi_2\|_{\mathcal{H}}.$$

Taking the infimum on the right-hand side we get the desired inequality.

2. Proceed as in 1. to show $\|c\varphi\|_{\mathcal{H}} \leq c\,\|\varphi\|_{\mathcal{H}}$. By symmetry the we get equality.

3. This is trivial. □

For the rest of the section, let E and F be a u.s.c. fields of Banach spaces over X.

Definition 3.2.4 ($\mathcal{H}(X, E)$). We define the following semi-norm on $\Gamma_c(X, E)$:

$$\|\xi\|_{\mathcal{H}} := \left\| x \mapsto \|\xi(x)\|_{E_x} \right\|_{\mathcal{H}}.$$

The Hausdorff completion of $\Gamma_c(X, E)$ with respect to this semi-norm will be denoted by $\mathcal{H}(X, E)$.

Note that the function $x \mapsto \|\xi(x)\|$ appearing in the preceding definition is not necessarily continuous. However, it has compact support and is non-negative upper semi-continuous, so we can apply the extended semi-norm on $\mathcal{F}_c^+(X)$ to it.

If E is the trivial bundle over X with fibre E_0, then $\Gamma_c(X, E)$ is $\mathcal{C}_c(X, E_0)$. The completion $\mathcal{H}(X, E)$ of $\mathcal{C}_c(X, E_0)$ could hence also be denoted as $\mathcal{H}(X, E_0)$ and might be considered as a sort of tensor product of $\mathcal{H}(X)$ and E_0. If in particular $E_0 = \mathbb{C}$, then $\mathcal{H}(X, E) = \mathcal{H}(X, \mathbb{C}) = \mathcal{H}(X)$.

Definition 3.2.5 ($\mathcal{H}(X,T)$). Let T be a bounded continuous field of linear maps from E to F. Then $\xi \mapsto T \circ \xi$ is a linear map from $\Gamma_c(X,E)$ to $\Gamma_c(X,F)$ such that $\|T \circ \xi\|_{\mathcal{H}} \leq \|T\| \, \|\xi\|_{\mathcal{H}}$. Hence T induces a canonical continuous linear map from $\mathcal{H}(X,E)$ to $\mathcal{H}(X,F)$ with norm $\leq \|T\|$.

This way, we define a functor from the category of u.s.c. fields of Banach spaces over X to the category of Banach spaces, which is linear and contractive on the morphism sets.

Proposition 3.2.6. *The canonical map from $\Gamma_c(X,E)$ to $\mathcal{H}(X,E)$ is continuous if we take the inductive limit topology on $\Gamma_c(X,E)$ and the norm topology on $\mathcal{H}(X,E)$.*

Proof. Let $K \subseteq X$ be compact. We just have to show that the map $\iota_K\colon \Gamma_K(X,E) \to \mathcal{H}(X,E)$ is continuous for the rest follows from the universal property of the inductive limit topology. Find a function $\chi \in \mathcal{C}_c(X)$ such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on K. Define $C_K := \|\chi\|_{\mathcal{H}}$. Let $\xi \in \Gamma_K(X,E)$. Then

$$\|\xi(x)\|_{E_x} \leq \sup_{x' \in X} \left\|\xi(x')\right\|_{E_x} \chi(x)$$

for all $x \in X$, so

$$\|\iota_K(\xi)\|_{\mathcal{H}(\mathcal{G},E)} = \|x \mapsto \|\xi(x)\|\|_{\mathcal{H}} \leq \|x \mapsto \chi(x)\|_{\mathcal{H}} \, \|\xi\|_K = C_K \, \|\xi\|_K \, . \qquad \square$$

Corollary 3.2.7. *If Ξ is dense in $\Gamma_c(X,E)$ for the inductive limit topology, then its canonical image in $\mathcal{H}(X,E)$ is dense for the norm topology.*

3.3 The pullback

In this section let X and Y be topological spaces and let $p\colon Y \to X$ be continuous.

3.3.1 The pullback of fields of Banach spaces

Definition 3.3.1 (The pullback). [13] Let E be a u.s.c. field of Banach spaces over X. Then we define a u.s.c. field $p^*(E)$ of Banach spaces over Y as follows: The underlying family of Banach spaces is $(E_{p(y)})_{y \in Y}$. Let

$$\Lambda := \{\xi \circ p : \ \xi \in \Gamma(X,E)\}.$$

Then Λ is a subspace of $\prod_{y \in Y} E_{p(y)}$ satisfying (C1), (C2), and (C3). By Proposition 3.1.26, the set $\Gamma := \overline{\Lambda}$ is the unique subset of $\prod_{y \in Y} E_{p(y)}$ containing Λ and satisfying (C1)-(C4). Let $p^*(E)$ be $\big((E_{p(y)})_{y \in Y}, \ \Gamma\big)$.

Example 3.3.2. Let E_0 be a Banach space. Consider E_0 as a continuous field of Banach spaces over a one-point set $\{*\}$ and let $p\colon Y \to \{*\}$ be the projection map. Then $p^*(E_0)$ is the constant field with fibre E_0 over Y.

Definition and Lemma 3.3.3 (The pullback as a functor). Let E and F be u.s.c. fields of Banach spaces over X and let T be a continuous field of linear maps from E to F. Define

$$p^*(T)_y := T_{p(y)} \in \mathrm{L}\left(E_{p(y)}, \ F_{p(y)}\right)$$

for all $y \in Y$. Then $p^*(T)$ is a continuous field of linear maps from $p^*(E)$ to $p^*(F)$. If T is bounded, then so is p^*T with $\|p^*T\| \leq \|T\|$. The assignment $T \mapsto p^*(T)$ is a functor from the category of fields of Banach spaces over X to the category of fields of Banach spaces over Y.

[13] See [Laf06], page 3.

Proof. Proposition 3.1.30 allows us to check the continuity of $p^*(T)$ just on the total subset $\{\xi \circ p : \xi \in \Gamma(X, E)\}$. So let $\xi \in \Gamma(X, E)$. Then $(p^*(T) \circ (\xi \circ p))(y) = T_{p(y)}(\xi(p(y))) = ((T \circ \xi) \circ p)(y)$ for all $y \in Y$, i.e., $p^*(T) \circ (\xi \circ p) = (T \circ \xi) \circ p$. Because $T \circ \xi \in \Gamma(X, F)$, we have $(T \circ \xi) \circ p \in \Gamma(Y, p^*F)$ by definition, so T is continuous. □

Proposition 3.3.4 (Composition and pullback). *Let Z be another topological space and let $q\colon Z \to Y$ be continuous. Let E be a u.s.c. field of Banach spaces over X. Then the u.s.c. fields q^*p^*E and $(p \circ q)^*E$ of Banach spaces over Z are identical. The same is true for the pullback of continuous fields of linear maps.*

Proof. Let $z \in Z$. Then $(q^*p^*E)_z = (p^*E)_{q(z)} = E_{p(q(z))} = ((p \circ q)^*E)_z$. So the fibres of the two fields agree. We have to check that also the set of sections are the same.

Let $\Lambda := \{\xi \circ p : \ \xi \in \Gamma(X, \ E)\}$ and $M := \{\xi \circ p \circ q : \ \xi \in \Gamma(X, \ E)\}$. Define $M' := \{\eta \circ q : \xi \in \Gamma(X, \ p^*E)\}$. Then $M \subseteq M'$ because $\Lambda \subseteq \Gamma(X, \ p^*(E))$ by the definition of $\Gamma(X, \ p^*(E))$. Let $\eta \in \Gamma(X, \ p^*(E))$. We show that $\zeta := \eta \circ q \in M'$ is locally approximable by elements of M. Let $z_0 \in Z$ and $\varepsilon > 0$. Since η is in $\Gamma(X, \ p^*E) = \overline{\Lambda}$, we can find a neighbourhood V of $q(z_0)$ and a $\eta_0 \in \Lambda$ such that the norm of $\eta - \eta_0$ is less than ε on V. Since q is continuous, the set $W := q^{-1}(V)$ is a neighbourhood of z_0. Define $\zeta_0 := \eta_0 \circ q \in M$. Then for all $z \in W$ we have

$$\|\zeta(z) - \zeta_0(z)\| = \|\eta(q(z)) - \eta_0(q(z))\| \overset{q(z)\in V}{<} \varepsilon.$$ □

Corollary 3.3.5. *Suppose that E is a u.s.c. field of Banach spaces over X. Let $x_0 \in p(Y) \subseteq X$. Then $p^*(E)|_{p^{-1}(\{x_0\})}$ is a constant field over $p^{-1}(\{x_0\})$ with fibre E_{x_0}.*

Proof. This follows from the identity $p \circ \iota_{p^{-1}(\{x_0\})} = \iota_{\{x_0\}} \circ p|_{p^{-1}(\{x_0\})}$, where ι_* stands for the respective inclusion maps. □

Corollary 3.3.6. *The pullback of a constant field is constant with the same fibre.*

Proposition 3.3.7 (Pullback and product). *Let E and F be u.s.c. fields of Banach spaces over X. Then the internal product $p^*(E) \times_Y p^*(F)$ and $p^*\left(E \times_X F\right)$ are identical.*

Proof. Let $y \in Y$. Then $(p^*E \times_Y p^*F)_y = (p^*E)_y \times (p^*F)_y = E_{p(y)} \times F_{p(y)} = (E \times_X F)_{p(y)}$. So the fibres are equal. The sets of sections are also the same because the set

$$\{y \mapsto (\xi(p(y)), \eta(p(y))) : \ \xi \in \Gamma(X, E), \eta \in \Gamma(X, F)\}$$

is total and contained both in $\Gamma\left(Y, \ p^*E \times_Y p^*F\right)$ and in $\Gamma\left(Y, \ p^*(E \otimes_X F)\right)$. □

Definition and Lemma 3.3.8 (Pullback and bilinear maps). Let E, F, G be u.s.c. fields of Banach spaces over X. If μ is a continuous field of bilinear maps from $E \times_X F$ to G, then the family $p^*(\mu) := (\mu_{p(y)})_{y \in Y}$ is a continuous field of bilinear maps from $p^*(E) \times_Y p^*(F) = p^*(E \times_X F)$ to $p^*(G)$. If μ is bounded, then so is $p^*\mu$ with $\|p^*\mu\| \leq \|\mu\|$.

Proof. $p^*(\mu)$ is obviously locally bounded. Let $\xi \in \Gamma(X, E)$ and $\eta \in \Gamma(X, F)$. Then $\xi \circ p \in \Gamma(Y, p^*E)$ and $\eta \circ p \in \Gamma(Y, p^*F)$. Using Proposition 3.1.35, the test for continuity of bilinear maps, it suffices to show that $y \mapsto p^*(\mu)_y(\xi(p(y)), \eta(p(y))) \in \Gamma(Y, p^*(G))$. Now

$$p^*(\mu)_y(\xi(p(y)), \eta(p(y))) = \mu_{p(y)}(\xi(p(y)), \eta(p(y))) = \left(\mu \circ (\xi, \eta)\right)(p(y))$$

for all $y \in Y$. Since $\mu \circ (\xi, \eta)$ is in $\Gamma(X, G)$, we are done. □

Remark 3.3.9. If μ in the preceding definition is non-degenerate (i.e., the image of μ_y spans a dense subset of G_y for all $y \in Y$), then $p^*(\mu)$ is non-degenerate as well.

Proposition 3.3.10 (Pullback and tensor products). *Let E and F be u.s.c. fields of Banach spaces over X. Then $p^*(E) \otimes_Y p^*(F)$ and $p^*(E \otimes_X F)$ are identical. The analogous statement is true for the tensor product and the pullback of continuous fields of linear maps.*

Proof. The underlying families of Banach spaces are in both cases $(E_{p(y)} \otimes F_{p(y)})_{y \in Y}$. We have to show that also the sets of sections agree. Let $\xi \in \Gamma(X, E)$ and $\eta \in \Gamma(X, F)$. Then $\xi \circ p \in \Gamma(Y,\ p^*E)$ and $\eta \circ p \in \Gamma(Y,\ p^*F)$ and hence $y \mapsto \xi(p(y)) \otimes \eta(p(y)) \in \Gamma(Y,\ p^*E \otimes_Y p^*F)$. On the other hand, $x \mapsto \xi(x) \otimes \eta(x) \in \Gamma(X,\ E \otimes_X F)$ and hence $y \mapsto \xi(p(y)) \otimes \eta(p(y)) \in \Gamma(Y,\ p^*(E \otimes_X F))$. Note that the span of such selections is total, so we have found a total set of selections that are sections for both fields, so the fields are equal. □

Proposition 3.3.11 (Pullback and linearisations). *Let E, F, G be u.s.c. fields of Banach spaces over X. Let μ be a continuous field of bilinear maps from $E \times_X F$ to G. Then $\widehat{p^*(\mu)} = p^*(\widehat{\mu})$ as families of linear maps from $p^*(E \otimes_X F) = p^*E \otimes_Y p^*F$ to p^*G.*

Proof. Let $y \in Y$. Then $p^*(\mu)_y = \mu_{p(y)}$ by definition. Hence $\widehat{p^*(\mu)}_y = \widehat{\mu_{p(y)}}$. On the other hand, $p^*(\widehat{\mu}) = \widehat{\mu}_{p(y)} = \widehat{\mu_{p(y)}}$. □

Proposition 3.3.12. *Let E, F, E', F' be u.s.c. fields of Banach spaces over X. Let S be a continuous field of linear maps from E to E' and T be a continuous field of linear maps from F to F'. Then*

$$p^*(S \otimes T) = p^*(S) \otimes p^*(T).$$

Proof. Let $y \in Y$. Then $p^*(S \otimes T)_y = (S \otimes T)_{p(y)} = S_{p(y)} \otimes T_{p(y)}$ and $(p^*(S) \otimes p^*(T))_y = p^*(S)_y \otimes p^*(T)_y = S_{p(y)} \otimes T_{p(y)}$. □

Proposition 3.3.13 (Preservation of associativity). *Let E_1, E_2, E_3, F_1, F_2 and G be u.s.c. fields of Banach spaces over X. Let $\mu_1 \in \mathrm{M}^{\mathbf{loc}}(E_1, E_2; F_1)$, $\mu_2 \in \mathrm{M}^{\mathbf{loc}}(E_2, E_3; F_2)$, $\nu_1 \in \mathrm{M}^{\mathbf{loc}}(F_1, E_3; G)$, and $\nu_2 \in \mathrm{M}^{\mathbf{loc}}(E_1, F_2; G)$. Assume that*

$$\hat{\nu_1} \circ (\hat{\mu_1} \otimes \mathrm{Id}_{E_3}) = \hat{\nu_2} \circ (\mathrm{Id}_{E_1} \otimes \hat{\mu_2})$$

which could be regarded as a formulation of a very general associativity law. Then the same law holds after applying the functor $p^*(\cdot)$:

$$\widehat{p^*(\nu_1)} \circ \left(\widehat{p^*(\mu_1)} \otimes \mathrm{Id}_{p^*E_3}\right) = \widehat{p^*(\nu_2)} \circ \left(\mathrm{Id}_{p^*E_1} \otimes \widehat{p^*(\mu_2)}\right).$$

3.3.2 The pullback of fields of Banach algebras and Banach modules

Because the pullback construction preserves associativity, we can pull back algebras and modules and obtain algebras and modules again:

Definition 3.3.14 (The pullback of a field of Banach algebras). Let A be a field of Banach algebras over X with multiplication μ. Then we equip $p^*(A)$ with the multiplication $p^*(\mu)$ to give a field of Banach algebras over Y. If A is non-degenerate, then $p^*(A)$ is non-degenerate as well.

Let A and B be fields of Banach algebras over X and $\varphi\colon A \to B$ a homomorphism. Then $p^*(\varphi)$ is a homomorphism of fields of Banach algebras from $p^*(A)$ to $p^*(B)$, and this defines a functor from the category of fields of Banach algebras over X to those over Y.
If Z is another topological space and $q\colon Z \to Y$ is continuous, then

$$(p \circ q)^*(A) = q^*(p^*(A))$$

as fields of Banach algebras over Z (compare 3.3.4). This is also true for homomorphisms in the sense that $(p \circ q)^*(\varphi) = q^*(p^*(\varphi))$ if $\varphi\colon A \to B$ is a homomorphism of fields of Banach algebras over X.

Proposition 3.3.15. *Let A be a u.s.c. field of Banach algebras over X. Then the fibrewise unitalisation commutes with the pullback, i.e., we have $\widetilde{p^*A} = p^*(\widetilde{A})$.*

Definition 3.3.16 (The pullback of a field of Banach modules). Let A be a field of Banach algebras over X. Let E be a left Banach A-module with A-action μ^E. Then we equip $p^*(E)$ with the $p^*(A)$-action $p^*(\mu^E)\colon p^*(A) \times_Y p^*(E) \to p^*(E)$ to give a Banach $p^*(A)$-module. If E is non-degenerate, then $p^*(E)$ is non-degenerate as well.

The pullback of fields of bimodules is defined similarly. The pullback of A-linear operators gives p^*A-linear operators and also the pullback of homomorphisms with coefficient maps gives homomorphisms with coefficient maps. The pullback is functorial with respect to both homomorphisms and linear operators. Moreover, the pullback of fields of Banach modules, linear operators and homomorphisms is compatible with the composition of continuous maps: If Z is another topological space and $q\colon Z \to Y$ is continuous, then $(p \circ q)^*(E) = q^*(p^*(E))$ as Banach $(p \circ q)^*A$-modules (compare 3.3.4).

Lemma 3.3.17. *Let B be a u.s.c. field of Banach algebras over X. Let E_B, ${}_BF$ be Banach B-modules and G a field of Banach spaces over X. Let μ be a B-balanced continuous field of bilinear maps from $E \times_X F$ to G. Then $p^*\mu$ is p^*B-balanced.*

As in Proposition 3.3.10, the corresponding result for fields of Banach spaces, one proves:

Proposition 3.3.18. *Let A, B, C be u.s.c. fields of Banach algebras over X. Let ${}_AE_B$ and ${}_BF_C$ be Banach bimodules. Then $p^*(E \otimes_B F) = (p^*E) \otimes_{p^*B} (p^*F)$ as Banach p^*A-p^*C-bimodules. The analogous statement is true for the pullback and the tensor product of homomorphisms.*

Proposition 3.3.19. *Let B and B' be u.s.c. fields of Banach algebras over X and let $\psi\colon B \to B'$ be a continuous field of homomorphisms. Let E be a Banach B-module. Then $(p^*\psi)_* (p^*E) = p^* (\psi_* E)$.*

3.3.3 The pullback of fields of Banach pairs

Definition 3.3.20 (The pullback of a field of Banach pairs). Let B be a field of Banach algebras over X and let $E = (E^<, E^>)$ be a Banach B-pair. Then $p^*(E) := (p^*(E^<),\ p^*(E^>))$ is a Banach $p^*(B)$-pair when equipped with the obvious bracket.

This defines a functor from the category of Banach B-pairs to the category of Banach $p^*(B)$-pairs, linear and contractive on the spaces of linear operators. As for Banach modules, the pullback of a homomorphism is a homomorphism and the pullback commutes with the tensor product and the pushout.

We now study how the pullback and locally compact operators are related.

Lemma 3.3.21. *Let E and F be Banach B-pairs. If $\eta^> \in \Gamma(X, F^>)$ and $\xi^< \in \Gamma(X, E^<)$, then*

$$\left(p^* \left|\eta^>\right\rangle\left\langle\xi^<\right|\right)_y = \left|\eta^>(p(y))\right\rangle\left\langle\xi^<(p(y))\right|$$

for all $y \in Y$.

Proposition 3.3.22. *Let E and F be Banach B-pairs and let T be a B-linear operator from E to F. If T is locally compact, then so is $p^*(T)\colon p^*(E) \to p^*(F)$. Conversely, every operator $\tilde{T} \in \mathrm{K}^{\mathbf{loc}}_{p^*B}(p^*E,\ p^*F)$ can be locally approximated by operators of the form p^*T with $T \in \mathrm{K}^{\mathbf{loc}}_B(E, F)$.*

Proof. Let T be locally compact. Let $y_0 \in Y$. Let $\varepsilon > 0$. Find a neighbourhood U of $x_0 := p(y_0)$ in X and $n \in \mathbb{N}$ and $\xi_1^<, \ldots, \xi_n^< \in \Gamma(X, E^<)$, $\eta_1^>, \ldots, \eta_n^> \in \Gamma(X, F^>)$ such that

$$\left\| T_u - \sum_{i=1}^n \left|\eta_i^>(u)\right\rangle\left\langle\xi_i^<(u)\right| \right\| \leq \varepsilon$$

for all $u \in U$. Let $V := p^*(V)$. Then V is a neighbourhood of y_0 in Y. For all $i \in \{1, \ldots, n\}$, the sections $\xi_i^< \circ p$ and $\eta_i^> \circ p$ belong to $\Gamma(Y, p^*E^<)$ and $\Gamma(Y, p^*F^>)$, respectively. Let $v \in V$ and define $u := p(v) \in U$. Then

$$\left\| p^*(T)_v - \sum_{i=1}^n \left|\eta_i^>(p(v))\right\rangle\left\langle\xi_i^<(p(v))\right| \right\| = \left\| T_u - \sum_{i=1}^n \left|\eta_i^>(u)\right\rangle\left\langle\xi_i^<(u)\right| \right\| \leq \varepsilon.$$

Hence $p^*(T)$ is locally compact.

Now let $\tilde{T} \in \mathrm{K}^{\mathbf{loc}}_{p^*B}(p^*E,\ p^*F)$. Without loss of generality we can assume that $\tilde{T}$ is of the form $\left|\tilde{\eta}^>\right\rangle\left\langle\tilde{\xi}^<\right|$ with $\tilde{\eta}^> \in \Gamma(Y, p^*F^>)$ and $\tilde{\xi}^< \in \Gamma(Y, p^*E^<)$. Let $y_0 \in Y$ and $\varepsilon > 0$. Find a neighbourhood V_η of y_0 in Y such that $\tilde{\eta}^>$ is bounded on V_η by some constant $C_\eta > 0$. Find an analogous neighbourhood V_ξ for $\tilde{\xi}^<$ and the constant $C_\xi > 0$. Find a neighbourhood V contained in $V_\eta \cap V_\xi$ and $\eta^> \in \Gamma(X, F^>)$, $\xi^< \in \Gamma(X, E^<)$ such that $\|\tilde{\eta}^>(v) - \eta^>(p(v))\| \leq \varepsilon/(3C_\eta)$ and $\|\tilde{\xi}^<(v) - \xi^<(p(v))\| \leq \varepsilon/(3C_\xi)$ and $\|\tilde{\eta}^>(v) - \eta^>(p(v))\|\ \|\tilde{\xi}^<(v) - \xi^<(p(v))\| \leq \varepsilon/3$ for all $v \in V$. Then

$$\left\| \left|\tilde{\eta}^>(v)\right\rangle\left\langle\tilde{\xi}^<(v)\right| - \left|\eta^>(p(v))\right\rangle\left\langle\xi^<(p(v))\right| \right\| \quad \leq \quad \varepsilon$$

for all $v \in V$. $\square$

3.4 Groupoids

3.4.1 Some notation and examples

A *groupoid*[14] is a small category such that every morphism is invertible. If $\mathcal{G}$ is a groupoid, then we will denote the set of composable pairs of morphisms by $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ or $\mathcal{G} * \mathcal{G}$, and the set of identity morphisms by $\mathcal{G}^{(0)} \subseteq \mathcal{G}$. The set $\mathcal{G}^{(0)}$, called the unit space, can also be regarded as the set of objects of $\mathcal{G}$. The range and source maps $\mathcal{G} \to \mathcal{G}^{(0)}$ will be denoted by $r_\mathcal{G}$ and $s_\mathcal{G}$ (or r and s if $\mathcal{G}$ is understood).

Often we will think of $\mathcal{G}^{(0)}$ as being a set that is not a subset of $\mathcal{G}$ but a distinct set on which the groupoid "acts". If X is a set and $\mathcal{G}$ is a groupoid such that $\mathcal{G}^{(0)} = X$, then we say that $\mathcal{G}$ is a groupoid

[14] See [LG99], section 2.1.

over X. The map that sends some $x \in X$ to the associated identity morphism in $\mathcal{G}$ will usually be called ϵ. In calculations, we will usually omit the map ϵ.

Let $\mathcal{G}$ be a groupoid. If K and L are subsets of $\mathcal{G}^{(0)}$, then $\mathcal{G}^L := \{\gamma \in \mathcal{G} : \ r(\gamma) \in L\}$, $\mathcal{G}_K := \{\gamma \in \mathcal{G} : \ s(\gamma) \in K\}$ and $\mathcal{G}_K^L := \mathcal{G}^L \cap \mathcal{G}_K$. If $g, h \in \mathcal{G}^{(0)}$, then $\mathcal{G}_g := \mathcal{G}_{\{g\}}$, $\mathcal{G}^h := \mathcal{G}^{\{h\}}$ and $\mathcal{G}_g^h := \mathcal{G}_g \cap \mathcal{G}^h = \{\gamma \in \mathcal{G} : \ r(\gamma) = h, s(\gamma) = g\}$.

A *topological groupoid*[15] $\mathcal{G}$ is a groupoid which is at the same time a topological space such that the composition, inversion and the range and source maps are continuous. If $\mathcal{G}$ is a groupoid over a set X, then we also have to assume that X is a topological space and the map $\epsilon\colon X \to \mathcal{G}$ is continuous.

Example 3.4.1. Let X be a topological space. Then we define the structure of a topological groupoid on X by setting $r := s := \mathrm{Id}_X$ (so there are only units).

Example 3.4.2. Let G be a topological group. Then G can be regarded as a topological groupoid if we let r and s be the projection on the identity element of G.

Example 3.4.3. Let X be a topological space. Then we define the structure of a topological groupoid on $X \times X$ by setting

$$(X \times X)^{(0)} := X \quad \text{and} \quad \epsilon\colon X \to X \times X,\ x \mapsto (x, x),$$

$$r\colon X \times X \to X,\ (y, x) \mapsto y \quad \text{and} \quad s\colon X \times X \to X,\ (y, x) \mapsto x,$$

$$\forall x, y, z \in X :\ (z, y) \circ (y, x) := (z, x) \quad \text{and} \quad (y, x)^{-1} = (x, y).$$

Note that r and s are open maps.

Example 3.4.4. Let X and Z be topological spaces and let $p\colon X \to Z$ be a continuous map. Extending the preceding example we define the structure of a topological groupoid on $X \times_Z X = X \times_p X$ by setting

$$(X \times_Z X)^{(0)} := X \quad \text{and} \quad \epsilon\colon X \to X \times_Z X,\ x \mapsto (x, x),$$

$$r\colon X \times_Z X \to X,\ (y, x) \mapsto y \quad \text{and} \quad s\colon X \times_Z X \to X,\ (y, x) \mapsto x,$$

$$\forall x, y, z \in X, p(x) = p(y) = p(z) :\ (z, y) \circ (y, x) := (z, x) \quad \text{and} \quad (y, x)^{-1} = (x, y).$$

If p is open, then Lemma 3.4.5 guaranties that r and s are open, too.

Lemma 3.4.5. *Let X, Y and Z be topological spaces. Let $f_X\colon X \to Z$ and $f_Y\colon Y \to Z$ be continuous maps. Let $X \times_Z Y$ be the fibre product $\{(x, y) \in X \times Y |\ f_X(x) = f_Y(y)\}$ of X and Y over Z. If f_Y is open (and surjective), then the canonical projection $p_X\colon X \times_Z Y \to X$ is open (and surjective).*

Proof. Let $(x, y) \in X \times_Z Y$. Let U be a neighbourhood of (x, y) in $X \times_Z Y$. Then there are $U_X \subseteq X$ and $U_Y \subseteq Y$ such that $(U_X \times U_Y) \cap X \times_Z Y \ \subseteq \ U$. Since f_Y is open, we know that $f_Y(U_Y)$ is a neighbourhood of $f_Y(y)$. Since $f_Y(y) = f_X(x)$ and f_X is continuous, we know that $U_X' := f_X^{-1}(f_Y(U_Y))$ is a neighbourhood of x. So $U_X \cap U_X'$ is also a neighbourhood of x. Let x' be an element of this neighbourhood. Then $f_X(x') \in f_Y(U_y)$, and hence we can find an $y' \in U_Y$ such that $f_X(x') = f_Y(y')$. Note that $(x, y) \in U$. But this means that $p_X(U)$ contains $U_X \cap U_X'$ and is hence a neighbourhood of x. So p_X is open. $\square$

[15] See [LG99], section 2.1.

Definition 3.4.6 (Strict morphism). [16] Let $\mathcal{G}$ and $\mathcal{H}$ be topological groupoids. Then a *strict morphism* f from $\mathcal{G}$ to $\mathcal{H}$ is a continuous map from $\mathcal{G}$ to $\mathcal{H}$ which also is a homomorphism of groupoids (i.e., a functor).

The topological groupoids, together with the strict morphisms, form a category.

Example 3.4.7. Let X be a topological space and let $\mathcal{G}$ be a topological groupoid over X. Then there is a canonical strict morphism from $\mathcal{G}$ to the groupoid $X \times X$ introduced in Example 3.4.3, namely the map that sends a $\gamma \in \mathcal{G}$ to the pair $(r(\gamma), s(\gamma))$.

3.4.2 $\mathcal{G}$-Banach spaces

For the rest of Section 3.4, let $\mathcal{G}$ be a topological groupoid with unit space $\mathcal{G}^{(0)} = X$.

Definition 3.4.8 ($\mathcal{G}$-Banach space). A $\mathcal{G}$-Banach space E is a u.s.c. field E of Banach spaces over $\mathcal{G}^{(0)}$ together with an isometric isomorphism $\alpha\colon s^*(E) \to r^*(E)$ such that

1. $\forall g \in \mathcal{G}^{(0)} : \ \alpha_g = \mathrm{Id}_{E_g}$;
2. $\forall (\gamma, \gamma') \in \mathcal{G} * \mathcal{G} : \ \alpha_{\gamma\circ\gamma'} = \alpha_\gamma \circ \alpha_{\gamma'}$;
3. $\forall \gamma \in \mathcal{G} : \ \alpha_{\gamma^{-1}} = \alpha_\gamma^{-1}$.

The Axioms 1. and 3. follow from Axiom 2. They are just stated to give a clearer impression of a $\mathcal{G}$-Banach space. The second axiom can also be stated as $\mu^*(\alpha) = \pi_1^*(\alpha) \circ \pi_2^*(\alpha)$ where $\mu\colon \mathcal{G} * \mathcal{G} \to \mathcal{G}$ is the composition in $\mathcal{G}$ and $\pi_i\colon \mathcal{G} * \mathcal{G} \to \mathcal{G}$ is the projection onto the ith coordinate.

Example 3.4.9. Let X be a topological space. If we regard X as a groupoid with unit space X, then every u.s.c. field of Banach spaces over X is, canonically, an X-Banach space (and every X-Banach space is, trivially, a u.s.c. field over X).

Definition 3.4.10 ($\mathcal{G}$-equivariant fields of linear maps). Let E and F be $\mathcal{G}$-Banach spaces with actions α and β, respectively. A *$\mathcal{G}$-equivariant continuous field of linear maps from E to F* is a continuous field $(T_x)_{x\in X}$ of linear maps from E to F such that the following diagram commutes

$$\begin{tikzcd} s^*(E) \arrow[r, "s^*(T)"] \arrow[d, "\alpha"] & s^*(F) \arrow[d, "\beta"] \\ r^*(E) \arrow[r, "r^*(T)"] & r^*(F) \end{tikzcd}$$

This means that $T_{r(\gamma)} \circ \alpha_\gamma = \beta_\gamma \circ T_{s(\gamma)}$ for all $\gamma \in \mathcal{G}$.

Definition 3.4.11 (The product and the sum of $\mathcal{G}$-Banach spaces). Let E and F be $\mathcal{G}$-Banach spaces with actions α and β, respectively. Then $r^*(E \times_X F) = r^*E \times_{\mathcal{G}} r^*F$ and $s^*(E \times_X F) = s^*E \times_{\mathcal{G}} s^*F$. We hence get a continuous field of isomorphisms $\alpha \times_{\mathcal{G}} \beta\colon s^*(E \times_X F) \to r^*(E \times_X F)$. It is an action on $E \times_X F$ which we call the *product action* of α and β. Similarly, we define an action $\alpha \oplus_{\mathcal{G}} \beta$ on $E \oplus_X F$.

[16] See [LG99], Définition 2.1.

Definition 3.4.12 (Equivariant bilinear maps between $\mathcal{G}$-Banach spaces). Let E_1, E_2 and F be $\mathcal{G}$-Banach spaces with $\mathcal{G}$-actions α_1, α_2 and β, respectively. Let $\mu\colon E_1 \times_X E_2 \to F$ be a continuous field of bilinear maps. Then μ is called $\mathcal{G}$-*equivariant* if the following diagram commutes

$$\begin{tikzcd} s^*(E_1 \times_X E_2) \arrow[r, "s^*(\mu)"] \arrow[d, "\alpha_1 \times_{\mathcal{G}} \alpha_2"] & s^*(F) \arrow[d, "\beta"] \\ r^*(E_1 \times_X E_2) \arrow[r, "r^*(\mu)"] & r^*(F) \end{tikzcd}$$

This means that $\gamma\mu_{s(\gamma)}(e_1, e_2) = \mu_{r(\gamma)}(\gamma e_1,\ \gamma e_2)$ for all $\gamma \in \mathcal{G}$ and $e_1 \in (E_1)_{s(\gamma)}$ and $e_2 \in (E_2)_{s(\gamma)}$.

Definition 3.4.13 (The tensor product of $\mathcal{G}$-Banach spaces). Let E and F be $\mathcal{G}$-Banach spaces with actions α and β, respectively. Then we can form the tensor product $E \otimes_X F$ of the continuous fields of Banach spaces E and F. Now

$$s^*(E \otimes_X F) = s^*(E) \otimes_{\mathcal{G}} s^*(F) \quad \text{and} \quad r^*(E \otimes_X F) = r^*(E) \otimes_{\mathcal{G}} r^*(F).$$

Now $\alpha \otimes \beta$ is a continuous field of isometric isomorphisms from $s^*(E) \otimes_{\mathcal{G}} s^*(F)$ to $r^*(E) \otimes_{\mathcal{G}} r^*(F)$. This induces on $E \otimes_X F$ the structure of a $\mathcal{G}$-Banach spaces.

Proof. To see that $\alpha \otimes \beta$ is an action on $E \otimes_X F$ we calculate

$$\begin{aligned} \mu^*(\alpha \otimes \beta) &= \mu^*(\alpha) \otimes \mu^*(\beta) = (\pi_1^*(\alpha) \circ \pi_2^*(\alpha)) \otimes (\pi_1^*(\beta) \circ \pi_2^*(\beta)) \\ &= (\pi_1^*(\alpha) \otimes \pi_1^*(\beta)) \circ (\pi_2^*(\alpha) \otimes \pi_2^*(\beta)) = \pi_1^*(\alpha \otimes \beta) \circ \pi_2^*(\alpha \otimes \beta). \end{aligned}$$

□

Note that $E \otimes_X F$ has the universal property for $\mathcal{G}$-equivariant continuous fields of bilinear maps.

Definition 3.4.14 (The trivial $\mathcal{G}$-Banach space). Let $\mathbb{C}_X$ denote the constant field of Banach spaces over X with fibre $\mathbb{C}$. Note that $s^*(\mathbb{C}_X) = \mathbb{C}_{\mathcal{G}} = r^*(\mathbb{C}_X)$. So $\mathbb{C}_X$ is a $\mathcal{G}$-Banach space if we take $(\mathrm{Id}_{\mathbb{C}})_{\gamma \in \mathcal{G}}$ as the action of $\mathcal{G}$.

3.4.3 $\mathcal{G}$-Banach algebras and $\mathcal{G}$-Banach modules

Definition 3.4.15 ($\mathcal{G}$-Banach algebra). A $\mathcal{G}$-Banach algebra A is a u.s.c. field A of Banach algebras over $\mathcal{G}^{(0)}$ together with a continuous field of isometric Banach algebra isomorphisms between the continuous fields of Banach algebras $s^*(A)$ and $r^*(A)$ which makes A a $\mathcal{G}$-Banach space.

Definition 3.4.16 (Homomorphism of $\mathcal{G}$-Banach algebras). If A and B are $\mathcal{G}$-Banach algebras, then a $\mathcal{G}$-equivariant homomorphism from A to B is a homomorphism of fields of Banach algebras over $\mathcal{G}^{(0)}$ which is at the same time a $\mathcal{G}$-equivariant continuous field of linear maps.

Definition 3.4.17 (Unitalisation). Let A be a $\mathcal{G}$-Banach algebra with $\mathcal{G}$-action α. Let ι denote the canonical action of $\mathcal{G}$ on the constant field $\mathbb{C}_X$. Then we take the action $\alpha \oplus_{\mathcal{G}} \iota$ on the unitalisation $\widetilde{A} = A \oplus_X \mathbb{C}_X$ of A.

Let B be a $\mathcal{G}$-Banach algebra with $\mathcal{G}$-action $\alpha\colon s^*(B) \to r^*(B)$.

Definition 3.4.18 ($\mathcal{G}$-Banach module). A right $\mathcal{G}$-Banach B-module E is a right Banach module E over the u.s.c. field B of Banach algebras over $\mathcal{G}^{(0)}$ together with a continuous field of isometric isomorphisms $\alpha^E\colon s^*(E) \to r^*(E)$ with coefficient map α between the Banach $s^*(B)$-module $s^*(E)$ and the Banach $r^*(B)$-module $r^*(E)$ which makes E a $\mathcal{G}$-Banach space.

Analogously one defines left $\mathcal{G}$-Banach modules and $\mathcal{G}$-Banach bimodules.

Definition 3.4.19 ($\mathcal{G}$-equivariant linear operator). If E and F are $\mathcal{G}$-Banach B-modules, then a $\mathcal{G}$-equivariant B-linear operator from E to F is a B-linear operator between Banach B-modules which also is a $\mathcal{G}$-equivariant continuous field of linear maps.

Analogously one defines $\mathcal{G}$-equivariant homomorphisms with coefficient maps between $\mathcal{G}$-Banach modules and $\mathcal{G}$-Banach bimodules.

The balanced tensor product of $\mathcal{G}$-Banach modules is defined analogously to the tensor product of $\mathcal{G}$-Banach spaces, using that the balanced tensor product commutes with the pullback along r and s. Similarly, the pushout along a continuous equivariant field of homomorphisms of Banach algebras is defined.

3.4.4 $\mathcal{G}$-Banach pairs

Let B be a $\mathcal{G}$-Banach algebra with $\mathcal{G}$-action α.

Definition 3.4.20 ($\mathcal{G}$-Banach B-pair). A $\mathcal{G}$-Banach B-pair E is a Banach B-pair $E = (E^<, E^>)$ together with an isometric isomorphisms $\alpha^E\colon s^*(E) \to r^*(E)$ with coefficient map α between the Banach $s^*(B)$-pair $s^*(E)$ and the Banach $r^*(B)$-pair $r^*(E)$ which makes $E^<$ and $E^>$ into $\mathcal{G}$-Banach spaces.

Remark 3.4.21. In [Laf06] the definition of a $\mathcal{G}$-Banach pair is formulated differently: Quite obviously, the aim of Définition 1.2.4 in [Laf06] is to define the same kind of object that we have defined here, but in [Laf06] the notion of a homomorphism with coefficient maps is missing (or at least it has not been made explicit); hence the definition of a $\mathcal{G}$-Banach pair makes use of continuous fields of linear operators (as we prefer to call them here), which leads to a result which is certainly not intended by the author.
On the other hand, the notation in [Laf06] is a bit simpler as a consequence of this imprecision because thinking of the action of $\mathcal{G}$ on E as an invertible linear operator V from s^*E to r^*E makes it possible to conjugate operators of the form s^*T, where $T \in \mathrm{L}_B(E,E)$, to get an operator $Vs^*TV^{-1} \in \mathrm{L}_{r^*B}(r^*E,\ r^*E)$. In our notation, it is not obvious what the composition of an operator and a concurrent morphism should be. In this particular case, there is not much choice, but we prefer to stay systematic and write $\alpha^{\mathrm{L}(E)}s^*T$ for the operator Vs^*TV^{-1}, see Definition 3.4.23 and 3.4.24 and compare also Definition 3.5.2.

Definition 3.4.22 ($\mathcal{G}$-equivariant operator). If E and F are $\mathcal{G}$-Banach B-pairs, then a $\mathcal{G}$-equivariant B-linear operator from E to F is an B-linear operator $T = (T^<, T^>)$ between the B-pairs E and F such that $T^<\colon F^< \to E^<$ and $T^>\colon E^> \to F^>$ are $\mathcal{G}$-equivariant continuous fields of linear maps.

Similarly define $\mathcal{G}$-equivariant homomorphisms with coefficient maps. The definitions of the balanced equivariant tensor product of $\mathcal{G}$-Banach pairs and the definition and properties of the pushout are straightforward.

3.4.5 The $\mathcal{G}$-action on operators

Let B be a $\mathcal{G}$-Banach algebra with $\mathcal{G}$-action α.

Definition and Proposition 3.4.23 ($\mathcal{G}$-action on fields of linear maps). Let E and F be $\mathcal{G}$-Banach spaces with the respective $\mathcal{G}$-actions α^E and α^F. Let $S \in \mathrm{L}^{\mathbf{loc}}(s^*E, s^*F)$. Then we define

$$\gamma(S_\gamma) := \alpha^F_\gamma \circ S_\gamma \circ \left(\alpha^E_\gamma\right)^{-1} \in \mathrm{L}\left(E_{r(\gamma)},\ F_{r(\gamma)}\right)$$

for all $\gamma \in \mathcal{G}$ and

$$\alpha^{\mathrm{L}(E,F)}(S) := \left(\gamma(S_\gamma)\right)_{\gamma\in\mathcal{G}} \in \mathrm{L}^{\mathbf{loc}}\left(r^*E,\ r^*F\right).$$

$\alpha^{\mathrm{L}(E,F)}$ is a $\mathbb{C}$-linear and $\mathcal{C}_0(\mathcal{G})$-linear bijection, compatible with the composition of fields of linear maps. If S is bounded, then so is $\alpha^{\mathrm{L}(E,F)}(S)$ with the same norm, so the restriction of $\alpha^{\mathrm{L}(E,F)}$ is an isometric bijection $\mathrm{L}(s^*E, s^F) \cong \mathrm{L}(r^*E, r^*F)$.

Proof. We just check that $\alpha^{\mathrm{L}(E,F)}(S)$ is a continuous field of linear maps. Let $\xi^{\in}\Gamma(\mathcal{G}, r^*E)$. Then

$$\alpha^{\mathrm{L}(E,F)}(S)\left(\xi(\gamma)\right) = \alpha^F_\gamma\left(S_\gamma\left(\left(\alpha^E_\gamma\right)^{-1}\left(\xi(\gamma)\right)\right)\right)$$

for all $\gamma \in \mathcal{G}$. Now $\gamma \mapsto (\alpha^E_\gamma)^{-1}(\xi(\gamma))$ is a section of s^*E, so $\gamma \mapsto S_\gamma((\alpha^E_\gamma)^{-1}(\xi(\gamma)))$ is a section of s^*F. It follows that $\gamma \mapsto \alpha^{\mathrm{L}(E,F)}(S)\left(\xi(\gamma)\right)$ is a section of r^*F. Moreover, $\alpha^{\mathrm{L}(E,F)}(S)$ is clearly locally bounded, so it is in $\mathrm{L}^{\mathbf{loc}}\left(r^*E,\ r^*F\right)$. $\square$

If in the preceding definition E and F are not only $\mathcal{G}$-Banach spaces but $\mathcal{G}$-Banach B-modules over some $\mathcal{G}$-Banach algebra B, then $\alpha^{\mathrm{L}(E,F)}$ preserves B-linearity and hence gives bijections

$$\mathrm{L}^{\mathbf{loc}}_{s^*B}\left(s^*E,\ s^*F\right) \cong \mathrm{L}^{\mathbf{loc}}_{r^*B}\left(r^*E,\ r^*F\right) \quad \text{and} \quad \mathrm{L}_{s^*B}\left(s^*E,\ s^*F\right) \cong \mathrm{L}_{r^*B}\left(r^*E,\ r^*F\right).$$

Definition and Proposition 3.4.24 ($\mathcal{G}$-action on operators between pairs). Let E and F be $\mathcal{G}$-Banach B-pairs. Let $S \in \mathrm{L}^{\mathbf{loc}}_{s^*B}(s^*E, s^*F)$. Then we define a $\mathcal{C}(Y)$-linear bijection, compatible with the composition of linear operators,

$$\alpha^{\mathrm{L}(E,F)}(S) := \left(\alpha^{\mathrm{L}(F^<,E^<)}(S^<),\ \alpha^{\mathrm{L}(E^>,F^>)}(S^>)\right) \in \mathrm{L}^{\mathbf{loc}}_{r^*B}\left(r^*E, r^*F\right).$$

If S is bounded, then so is $\alpha^{\mathrm{L}(E,F)}(S)$ and both have the same norm.

Proof. Let $\gamma \in \mathcal{G}$, $e^>_{r(\gamma)} \in E^>_{r(\gamma)}$ and $f^<_{r(\gamma)} \in F^<_{r(\gamma)}$. Then

$$\begin{aligned}
\left\langle \alpha^{\mathrm{L}(E,F)}(S)^<_\gamma f^<_{r(\gamma)},\ e^>_{r(\gamma)}\right\rangle &= \left\langle \gamma S^<_\gamma \gamma^{-1} f^<_{r(\gamma)}, e^>_{r(\gamma)}\right\rangle = \gamma\left\langle S^<_\gamma \gamma^{-1} f^<_{r(\gamma)}, \gamma^{-1} e^>_{r(\gamma)}\right\rangle \\
= \ \gamma\left\langle \gamma^{-1} f^<_{r(\gamma)}, S^>_\gamma \gamma^{-1} e^>_{r(\gamma)}\right\rangle &= \left\langle f^<_{r(\gamma)}, \gamma S^>_\gamma \gamma^{-1} e^>_{r(\gamma)}\right\rangle = \left\langle f^<_{r(\gamma)},\ \alpha^{\mathrm{L}(E,F)}(S)^>_\gamma e^>_{r(\gamma)}\right\rangle.
\end{aligned}$$

$\square$

Proposition 3.4.25. *Let E and F be $\mathcal{G}$-Banach B-pairs. If $\xi^< \in \Gamma(\mathcal{G}, s^*E^<)$ and $\eta^> \in \Gamma(\mathcal{G}, s^*F^>)$, then*

$$\alpha^{\mathrm{L}(E,F)}\left(\left|\eta^>\right\rangle\left\langle\xi^<\right|\right) = \left|\alpha^{F^>} \circ \eta^>\right\rangle\left\langle\alpha^{E^<} \circ \xi^<\right|. \tag{3.2}$$

If $S \in \mathrm{K}^{\mathbf{loc}}_{s^*B}(s^*E, s^*F)$, then $\alpha^{\mathrm{L}(E,F)}(S) \in \mathrm{K}^{\mathbf{loc}}_{r^*B}(r^*E, r^*F)$. Thus $\alpha^{\mathrm{L}(E,F)}$ restricts to a $\mathcal{C}(\mathcal{G})$-linear bijection $\alpha^{\mathrm{K}(E,F)}$ between the spaces of locally compact operators.

Proof. We check formula (3.2): Let $\gamma \in \mathcal{G}$ and $e^{>}_{r(\gamma)} \in E^{>}_{r(\gamma)}$. Then

$$\begin{aligned}\alpha^{\mathrm{L}(E,F)}\left(\left|\eta^{>}\right\rangle\left\langle\xi^{<}\right|\right)^{>}(e^{>}_{r(\gamma)}) &= \gamma\left|\eta^{>}\right\rangle\left\langle\xi^{<}\right|^{>}_{\gamma}\gamma^{-1}e^{>}_{r(\gamma)} = \gamma\left(\eta^{>}(\gamma)\left\langle\xi^{<}(\gamma),\ \gamma^{-1}e^{>}_{r(\gamma)}\right\rangle\right)\\ &= (\gamma\eta^{>}(\gamma))\left\langle\gamma\xi^{<}(\gamma),\ \gamma\gamma^{-1}e^{>}_{r(\gamma)}\right\rangle = \left|\gamma\eta^{>}(\gamma)\right\rangle\left\langle\gamma\xi^{<}(\gamma)\right|^{>}\left(e^{>}_{r(\gamma)}\right).\end{aligned}$$

A similar calculation can be done for the left-hand side, which shows (3.2). □

Remark 3.4.26. In Section 4.7.1 we are going to introduce the set of compact operators $\mathrm{K}_B(E,F) \subseteq \mathrm{K}^{\mathbf{loc}}_B(E,F)$ and discuss in Section 4.8.3 to what extend one can think of $\alpha^{\mathrm{K}(E,F)}$ as an action of $\mathcal{G}$ on $\mathrm{K}_B(E,F)$ (which would make $\mathrm{K}_B(E)$ a $\mathcal{G}$-Banach algebra).

If E is a $\mathcal{G}$-Banach A-B-pair, then the action of A on E regarded as a homomorphism from A to $\mathrm{L}_B(E)$, is $\mathcal{G}$-equivariant in the following sense:

Lemma 3.4.27. *Let E be a $\mathcal{G}$-Banach A-B-pair with A and B being $\mathcal{G}$-Banach algebras. Let $\tilde{a} \in \Gamma(\mathcal{G}, s^*A)$. Then*

$$\alpha^{\mathrm{L}(E)}(\pi_{s^*A}(\tilde{a})) = \pi_{r^*A}\left(\alpha^A \circ \tilde{a}\right)$$

where π_{s^*A} and π_{r^*A} are the actions of s^*A on s^*E and r^*A on r^*E (regarded as homomorphisms into the linear operators) and α^A is the action of $\mathcal{G}$ on A.

Proof. Let $\gamma \in \mathcal{G}$ and $e^{>}_{r(\gamma)} \in E^{>}_{r(\gamma)}$. Then

$$\begin{aligned}\alpha^{\mathrm{L}(E)}(\pi_{s^*A}(\tilde{a}))^{>}_{\gamma}(e^{>}_{r(\gamma)}) &= \alpha^{E^{>}}_{\gamma}\left((\pi_{s^*A}(\tilde{a}))^{>}_{\gamma}\left(\left(\alpha^{E^{>}}_{\gamma}\right)^{-1}(e^{>}_{r(\gamma)})\right)\right)\\ &= \gamma\left(\tilde{a}(s(\gamma))\cdot(\gamma^{-1}e^{>}_{r(\gamma)})\right)\\ &= (\gamma\tilde{a}(s(\gamma)))\cdot(\gamma\gamma^{-1})e^{>}_{r(\gamma)} = \pi_{r^*A}\left(\alpha^A\circ\tilde{a}\right)^{>}_{\gamma}(e^{>}_{r(\gamma)}).\end{aligned}$$

A similar calculation can be done for the left-hand side, yielding $\alpha^{\mathrm{L}(E)}(\pi_{s^*A}(\tilde{a}))_{\gamma} = \pi_{r^*A}\left(\alpha^A \circ \tilde{a}\right)_{\gamma}$ for all $\gamma \in \mathcal{G}$. □

3.5 $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$

Let $\mathcal{G}$ be a topological groupoid with unit space X.

3.5.1 Gradings

Definition 3.5.1 (A graded $\mathcal{G}$-Banach space). Let E be a $\mathcal{G}$-Banach space. Then a *grading automorphism σ_E* of E is a $\mathcal{G}$-equivariant contractive continuous field of linear maps from E to E such that $\sigma_E^2 = \mathrm{Id}_E$. A $\mathcal{G}$-Banach space endowed with a grading automorphism is called a *graded $\mathcal{G}$-Banach space*.

Just as for gradings of ordinary Banach spaces or Banach spaces with group actions we can define the notions of graded (=even) and odd $\mathcal{G}$-equivariant continuous fields of linear maps between graded $\mathcal{G}$-Banach spaces, graded $\mathcal{G}$-Banach algebras, graded $\mathcal{G}$-Banach modules and graded $\mathcal{G}$-Banach pairs. All the above constructions are compatible with this additional structure, e.g., the tensor product or the pullback along a strict morphism of groupoids.

3.5.2 $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}$-cycles

Let A and B be $\mathcal{G}$-Banach algebras.

Definition 3.5.2 ($\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}$-cycle). A $\mathrm{KK}^{\mathrm{ban}}$-*cycle* from A to B is a pair (E,T) such that E is a non-degenerate graded $\mathcal{G}$-A-B-bimodule and T is an odd element of $\mathrm{L}_B(E)$ such that

$$[\pi_A(a), T]\,,\ \pi_A(a)\,(\mathrm{Id} - T^2) \in \mathrm{K}^{\mathbf{loc}}_B(E)$$

for all $a \in \Gamma(X, A)$ and

$$\pi(\tilde{a})\left(\alpha^{\mathrm{L}(E)}(s^*T) - r^*T\right) \in \mathrm{K}^{\mathbf{loc}}_{r^*B}\,(r^*E)$$

for all $\tilde{a} \in \Gamma\,(\mathcal{G}, r^*A)$, where $\alpha^{\mathrm{L}(E)}\colon\ \mathrm{L}^{\mathbf{loc}}_{s^*B}\,(s^*E) \to \mathrm{L}^{\mathbf{loc}}_{r^*B}\,(r^*E)$ denotes the "action" of $\mathcal{G}$ on $\mathrm{L}(E)$ defined in 3.4.24. We write $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$ for the class of all $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}$-cycles from A to B.

Definition 3.5.3 (The sum of $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}$-cycles). If (E_1,T_1) and (E_2,T_2) are elements of $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$, then we define $(E_1,T_1)\oplus(E_2,T_2) := (E_1\oplus E_2, T_1\oplus T_2)$. It is an element of $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$.

Definition 3.5.4 (The inverse of a $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}$-cycle). If (E,T) is in $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$, then we define $-(E,T)$ to be (E,T), but equipped with the opposite grading. This is an element of $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$.

Using the facts that the pushout of locally compact operators is again locally compact (Proposition 3.1.60) and that the pullback commutes with the pushout (Proposition 3.3.19), we can define the pushout for cycles:

Definition 3.5.5 (The pushout of $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}$-cycles). Let B' be another $\mathcal{G}$-Banach algebra and $\psi\colon B \to B'$ a $\mathcal{G}$-equivariant homomorphism from B to B'. Let (E,T) be an element of $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$. Then the pushout $\psi_*(E,T)$ of (E,T) along ψ is defined as $(\psi_*(E), T\otimes 1)$. It is contained in $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A,B')$.

3.5.3 Morphisms between $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}$-cycles

Let A, A' and B, B' be $\mathcal{G}$-Banach algebras. Let $\varphi\colon A \to A'$ and $\psi\colon B \to B'$ be $\mathcal{G}$-equivariant homomorphisms.

Definition 3.5.6. Let (E,T) and (E',T') be elements of $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$ and $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A',B')$, respectively. Then a morphism from (E,T) to (E',T') with coefficient maps φ and ψ is a pair $\Phi = (\Phi^<, \Phi^>)$ such that

- $(\Phi^<, \Phi^>)$ is an equiv. homomorphism of graded Banach pairs with coefficient maps φ and ψ;
- we have
$$T'^< \circ \Phi^< = \Phi^< \circ T^< \quad \text{and} \quad T'^> \circ \Phi^> = \Phi^> \circ T^>.$$

The class $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$, together with the morphisms of cycles (with Id_A and Id_B as coefficient maps), forms a category. This gives us an obvious notion of *isomorphic* $\mathrm{KK}^{\mathrm{ban}}$-*cycles* in $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$. Just as for ordinary $\mathrm{KK}^{\mathrm{ban}}$-cycles, the sum of cycles is associative and the pushout is functorial up to isomorphism (compare Propositions 1.8.7 and 1.8.8).

3.5.4 Homotopies between $\mathrm{KK}^{\mathrm{ban}}$-cycles

The $\mathcal{G}$-Banach algebra $B[0,1]$

Definition 3.5.7 (The $\mathcal{G}$-Banach space $E[0,1]$). Let E be a $\mathcal{G}$-Banach space with $\mathcal{G}$-action $\alpha\colon s^*E \to r^*E$. Then we define the $\mathcal{G}$-Banach space $E[0,1]$ by the following data:

1. the underlying family of Banach spaces is $(E_x[0,1])_{x\in X}$;
2. a section ξ of $E[0,1]$ is continuous if and only if $(x,t) \mapsto \xi(x)(t)$ is a continuous section in $p_1^*(E)$, where $p_1\colon X\times[0,1]\to X$ denotes the projection onto the first component;
3. the action $\alpha[0,1]\colon s^*(E[0,1]) \to r^*(E[0,1])$ is defined by

$$E[0,1]_{s(\gamma)} = E_{s(\gamma)}[0,1] \ni \xi_\gamma \quad\mapsto\quad (t\mapsto \alpha_\gamma(\xi_\gamma(t))) \in E_{r(\gamma)}[0,1].$$

For all $t\in[0,1]$, define the continuous family of linear contractions $\mathrm{ev}_t\colon E[0,1]\to E$ given by $(\mathrm{ev}_t)_x\colon E_x[0,1]\to E_x,\ \xi_x\mapsto \xi_x(t)$ for all $x\in X$.

Proposition 3.5.8. *If B is a $\mathcal{G}$-Banach algebra, then $B[0,1]$ is a $\mathcal{G}$-Banach algebra as well (when equipped with the obvious multiplication). The field $\mathrm{ev}_t\colon B[0,1]\to B$ is a continuous field of homomorphisms in this case. Similar statements hold for Banach modules and pairs.*

Note that $(\mathrm{ev}_{t,*}\,E)_x = (\mathrm{ev}_t)_{x,*}E_x$ for every $\mathcal{G}$-Banach $B[0,1]$-pair E.

Homotopies and $\mathrm{KK}^{\mathrm{ban}}$

Let A, B be $\mathcal{G}$-Banach algebras.

Definition 3.5.9 (Homotopies). A *homotopy* between cycles (E_0,T_0) and (E_1,T_1) in $\mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(A,B)$ is a cycle (E,T) in $\mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(A,B[0,1])$ such that $\mathrm{ev}_{0,*}(E,T)$ is isomorphic to (E_0,T_0) and $\mathrm{ev}_{1,*}(E,T)$ is isomorphic to (E_1,T_1). If such a homotopy exists then (E_0,T_0) and (E_1,T_1) are called *homotopic*. We will denote by $\sim$ the equivalence relation on $\mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(A,B[0,1])$ generated by homotopy (note that homotopy is reflexive and symmetric). The equivalence classes for $\sim$ are called homotopy classes.

Definition and Proposition 3.5.10 ($\mathrm{KK}_{\mathcal{G}}^{\mathrm{ban}}(A,B)$). The class of all homotopy classes in $\mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(A,B)$ is denoted by $\mathrm{KK}_{\mathcal{G}}^{\mathrm{ban}}(A,B)$. The addition of cycles induces a law of composition on $\mathrm{KK}_{\mathcal{G}}^{\mathrm{ban}}(A,B)$ making it an abelian group (at least if we restrict the cardinality of dense subsets of the involved Banach modules by some cardinality to obtain a set $\mathrm{KK}_{\mathcal{G}}^{\mathrm{ban}}(A,B)$ rather than just a class). $\mathrm{KK}_{\mathcal{G}}^{\mathrm{ban}}(A,B)$ is functorial in both variables with respect to $\mathcal{G}$-equivariant continuous fields of homomorphisms of Banach algebras.

The fact that $\mathrm{KK}_{\mathcal{G}}^{\mathrm{ban}}(A,B)$ has inverses should be proved by adjusting Lemme 1.2.5 in [Laf02] to the situation of $\mathcal{G}$-Banach algebras. The above definition is part of Définition-Proposition 1.2.6 in [Laf06]. The functoriality result is analogous to Proposition 1.8.12 for the ordinary $\mathrm{KK}^{\mathrm{ban}}$-groups.

The following Lemma is the obvious generalisation of Lemme 1.2.3 in [Laf02].

Lemma 3.5.11. *Let $(E,T)\in\mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(A,B)$ and assume that $T'\in\mathrm{L}(E)$ is odd bounded operator such that $a(T-T'),\ (T-T')a\in\mathrm{K}_B^{\mathbf{loc}}(E)$ for all $a\in\Gamma_0(X,A)$. Then $(E,T')\in\mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(A,B)$ and there is a homotopy from (E,T) to (E,T').*

Proof. First we prove that (E, T') is a $\mathrm{KK}^{\mathrm{ban}}$-cycle:
Let $a \in \Gamma(X, A)$. Then

$$\begin{aligned} [a, T'] &= aT' - T'a = aT - a(T - T') - Ta + (T - T')a \\ &= [a, T] - a(T - T') + (T - T')a \ \in \ \mathrm{K}_B^{\mathbf{loc}}(E). \end{aligned}$$

Secondly,

$$\begin{aligned} & a(T'^2 - 1) = a((T - (T - T'))^2 - 1) \\ =\ & a(T^2 - T(T - T') - (T - T')T + (T - T')^2 - 1) \\ =\ & a(T^2 - 1) - [a, T](T - T') - Ta(T - T') - a(T - T')T + a(T - T')^2 \ \in \ \mathrm{K}_B^{\mathbf{loc}}(E) \end{aligned}$$

for all $a \in \Gamma(X, A)$. Thirdly, if $\tilde{a} \in \Gamma(\mathcal{G},\ r^*A)$:

$$\begin{aligned} & \tilde{a}\left(r^*T' - \alpha^{\mathrm{L}(E)}(s^*T')\right) \\ =\ & \tilde{a}\left(r^*T - \alpha^{\mathrm{L}(E)}(s^*T)\right) - \tilde{a}r^*(T - T') - \tilde{a}\alpha^{\mathrm{L}(E)}(s^*(T - T')) \in \mathrm{K}_B^{\mathbf{loc}}(E). \end{aligned}$$

The first term is locally compact because (E, T) is a $\mathrm{KK}^{\mathrm{ban}}$-cycle. The second term is locally compact because $\tilde{a}$ can be approximated locally by sections of the form $a \circ r$ with $a \in \Gamma(X, A)$; hence $\tilde{a}r^*(T - T')$ can be approximated locally by operators of the form $(a \circ r)r^*(T - T') = r^*(a(T - T'))$ and such operators are locally compact. The third term can be rewritten as $\alpha^{\mathrm{L}(E)}\left[\left((\alpha^A)^{-1}(\tilde{a})\right) s^*(T - T')\right]$ where α^A is the $\mathcal{G}$-action on A (see Lemma 3.4.27 for a more precise statement). Now $(\alpha^A)^{-1}(\tilde{a})$ is in $\Gamma(\mathcal{G},\ s^*A)$, so by a similar argument as for the second term, $\left((\alpha^A)^{-1}(\tilde{a})\right) s^*(T - T')$ is locally compact. Hence the third term is locally compact.

Now we construct the homotopy: The idea is to connect (E, T) to (E, T') through cycles of the form $(E, (1 - t)T + tT')$ for $t \in [0, 1]$. First note that $E[0, 1]$ is a non-degenerate graded $\mathcal{G}$-Banach $B[0, 1]$-pair and $(E[0, 1],\ T[0, 1])$ is in $\mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(A[0, 1], B[0, 1])$. We can also regard it as an element of $\mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(A, B[0, 1])$. Moreover, if $S \in \mathrm{K}_B^{\mathbf{loc}}(E)$, then $S[0, 1] \in \mathrm{K}_{B[0,1]}^{\mathbf{loc}}(E[0, 1])$. It follows that $a(T' - T)[0, 1] \in \mathrm{K}_{B[0,1]}^{\mathbf{loc}}(E[0, 1])$ for all $a \in \Gamma(X, A)$. The multiplication with $\mathrm{Id}_{[0,1]}$ in every fibre is in $\mathrm{L}_{B[0,1]}(E[0, 1])$, so $\mathrm{Id}_{[0,1]}\, a(T' - T)[0, 1]$ is also in $\mathrm{K}_{B[0,1]}^{\mathbf{loc}}(E[0, 1])$ for all $a \in \Gamma(X, A)$.

Applying the first part of the proof to $T[0, 1]$ and $T[0, 1] + \mathrm{Id}_{[0,1]}(T' - T)[0, 1]$ shows that $\left(E[0, 1],\ T[0, 1] + \mathrm{Id}_{[0,1]}(T' - T)[0, 1]\right)$ is a $\mathrm{KK}^{\mathrm{ban}}$-cycle. For all $t \in [0, 1]$ the pushout along ev_t^B of this cycle is isomorphic to $(E, T + t(T' - T)) = (E, (1 - t)T + tT')$. So we have found a homotopy from (E, T) to (E', T'). □

3.6 $\mathrm{KK}^{\mathrm{ban}}$-cycles and strict morphisms of groupoids

3.6.1 The pullback along strict morphisms

Let $\mathcal{G}$ and $\mathcal{H}$ be topological groupoids and let $f\colon \mathcal{H} \to \mathcal{G}$ be a strict morphism of topological groupoids as defined in 3.4.6.

The pullback of $\mathcal{G}$-Banach spaces

Definition 3.6.1 (The pullback of a $\mathcal{G}$-Banach space). Let E be an $\mathcal{G}$-Banach space with action α. Write f_0 for $f|_{\mathcal{H}^{(0)}}\colon \mathcal{H}^{(0)} \to \mathcal{G}^{(0)}$. Then $f_0^*(E)$ is a u.s.c. field of Banach spaces over $\mathcal{H}^{(0)}$. Now

$s_{\mathcal{G}} \circ f = f_0 \circ s_{\mathcal{H}}$ and $r_{\mathcal{G}} \circ f = f_0 \circ r_{\mathcal{H}}$, so

$$s^*_{\mathcal{H}}(f^*_0(E)) = (f_0 \circ s_{\mathcal{H}})^*(E) = (s_{\mathcal{G}} \circ f)^*(E) = f^*(s^*_{\mathcal{G}}(E))$$

and similarly for the range maps. So $f^*(\alpha)$ is a continuous field of isometric isomorphisms from $s^*_{\mathcal{H}}(f^*_0(E))$ to $r^*_{\mathcal{H}}(f^*_0(E))$. It is an action of $\mathcal{H}$.

The $\mathcal{H}$-Banach space $f^*_0(E)$ with the action $f^*(\alpha)$ is called the *pullback of E along f* and is denoted by $f^*(E)$.

Proof. Let $\mu_{\mathcal{G}}$ and $\mu_{\mathcal{H}}$ denote the composition maps of $\mathcal{G}$ and $\mathcal{H}$, respectively, and write $\pi^{\mathcal{G}}_i \colon \mathcal{G} * \mathcal{G} \to \mathcal{G}$ and $\pi^{\mathcal{H}}_i \colon \mathcal{H} * \mathcal{H} \to \mathcal{H}$ for the respective projections onto the ith component. Let $f * f$ denote the map $\mathcal{H} * \mathcal{H} \to \mathcal{G} * \mathcal{G}$ which sends (η, η') to $(f(\eta), f(\eta'))$. Then $\pi^{\mathcal{G}}_i \circ (f * f) = f \circ \pi^{\mathcal{H}}_i$ for all $i \in \{1, 2\}$ and $\mu_{\mathcal{G}} \circ (f * f) = f \circ \mu_{\mathcal{H}}$. Now

$$\begin{aligned}
\mu^*_{\mathcal{H}}(f^*(\alpha)) &= (f \circ \mu_{\mathcal{H}})^* (\alpha) = (\mu_{\mathcal{G}} \circ (f * f))^* (\alpha) = (f * f)^* \left(\mu^*_{\mathcal{G}}(\alpha)\right) \\
&= (f * f)^* \left(\left(\pi^{\mathcal{G}}_1\right)^* (\alpha) \circ \left(\pi^{\mathcal{G}}_2\right)^* (\alpha) \right) \\
&= (f * f)^* \left(\left(\pi^{\mathcal{G}}_1\right)^* (\alpha) \right) \circ (f * f)^* \left(\left(\pi^{\mathcal{G}}_2\right)^* (\alpha) \right) \\
&= \left(\pi^{\mathcal{G}}_1 \circ (f * f)\right)^* (\alpha) \circ \left(\pi^{\mathcal{G}}_2 \circ (f * f)\right)^* (\alpha) \\
&= \left(f \circ \pi^{\mathcal{H}}_2\right)^* (\alpha) \circ \left(f \circ \pi^{\mathcal{H}}_1\right)^* (\alpha) \\
&= \left(\pi^{\mathcal{H}}_2\right)^* (f^*(\alpha)) \circ \left(\pi^{\mathcal{H}}_1\right)^* (f^*(\alpha)) .
\end{aligned}$$

So $f^*(\alpha)$ is an action. □

Proposition 3.6.2. *The pullback commutes with the tensor product: Let E and F be $\mathcal{G}$-Banach spaces. Then $f^* \left(E \otimes_{\mathcal{G}^{(0)}} F\right) = f^*(E) \otimes_{\mathcal{H}^{(0)}} f^*(F)$ as $\mathcal{H}$-Banach spaces.*

Proof. The identity is true for the underlying u.s.c. fields of Banach spaces. We have to show that the actions of $\mathcal{H}$ on the spaces are the same. Let α and β denote the action of $\mathcal{G}$ on E and F, respectively. Then it follows from the last sentence of Proposition 3.3.10 that $f^* (\alpha \otimes \beta) = f^*(\alpha) \otimes f^*(\beta)$. □

Proposition 3.6.3. *Let E and F be $\mathcal{G}$-Banach spaces and let $T \in \mathrm{L}^{\mathbf{loc}}(E, F)$ be $\mathcal{G}$-equivariant. Then $f^*T \in \mathrm{L}^{\mathbf{loc}}(f^*E, f^*F)$ is $\mathcal{H}$-equivariant.*

Proof. Write α^E and α^F for the $\mathcal{G}$-action on E and F, respectively. From $r^*_{\mathcal{G}}(T) \circ \alpha^E = \alpha^F \circ s^*_{\mathcal{G}}(T)$ we can deduce that

$$\begin{aligned}
r^*_{\mathcal{H}}(f^*(T)) \circ f^*(\alpha^E) &= f^*(r^*_{\mathcal{G}}(T)) \circ f^*(\alpha^E) = f^* \left(r^*_{\mathcal{G}}(T) \circ \alpha^E\right) \\
&= f^* \left(\alpha^F \circ s^*_{\mathcal{G}}(T)\right) = f^*(\alpha^F) \circ f^*(s^*_{\mathcal{G}}(T)) = f^*(\alpha^F) \circ s^*_{\mathcal{H}}(f^*(T)).
\end{aligned}$$

□

An analogous statement is true for equivariant bilinear maps.

Proposition 3.6.4. *The pullback along f is a functor from the category of $\mathcal{G}$-Banach spaces to the category of $\mathcal{H}$-Banach spaces, linear and contractive on the sets of bounded continuous fields of linear maps, and sending equivariant continuous fields of linear maps to equivariant continuous fields.*

Proposition 3.6.5. *Let $\mathcal{K}$ be another topological groupoid and let $g\colon \mathcal{K} \to \mathcal{H}$ be a strict morphism. Then $(f \circ g)^* = g^* \circ f^*$ as functors from the category of $\mathcal{G}$-Banach spaces to the category of $\mathcal{K}$-Banach spaces.*

Proof. Let E be a $\mathcal{G}$-Banach space with $\mathcal{G}$-action α. Then $(f_0 \circ g_0)^*(E) = g_0^*\left(f_0^*(E)\right)$ and $(f \circ g)^*(\alpha) = g^*\left(f^*(\alpha)\right)$. □

Proposition 3.6.6. $\mathrm{Id}_{\mathcal{G}}^*$ *is the identity functor of the category of $\mathcal{G}$-Banach spaces.*

Lemma 3.6.7. *Let E and F be $\mathcal{G}$-Banach spaces. For all $S \in \mathrm{L}^{\mathbf{loc}}\left(s_{\mathcal{G}}^* E,\ s_{\mathcal{G}}^* F\right)$, we have*

$$\alpha^{\mathrm{L}(f^*E, f^*F)}(f^*S) = f^*\left(\alpha^{\mathrm{L}(E,F)}(S)\right) \ \in\ \mathrm{L}^{\mathbf{loc}}\left(f^* r_{\mathcal{G}}^* E,\ f^* r_{\mathcal{G}}^* F\right).$$

Note that $\mathrm{L}^{\mathbf{loc}}\left(f^* r_{\mathcal{G}}^* E,\ f^* r_{\mathcal{G}}^* F\right) = \mathrm{L}^{\mathbf{loc}}\left(r_{\mathcal{H}}^* f^* E,\ r_{\mathcal{H}}^* f^* F\right)$ and similarly for $s_{\mathcal{G}}$ and $s_{\mathcal{H}}$.

The pullback of $\mathcal{G}$-Banach algebras and $\mathcal{G}$-Banach modules

Let B be a $\mathcal{G}$-Banach algebra. Then f^*B is an $\mathcal{H}$-Banach algebra. Also the pullback along f of a $\mathcal{G}$-equivariant homomorphism of Banach algebras is a $\mathcal{H}$-equivariant homomorphism.

If E is a $\mathcal{G}$-Banach B-module, then f^*E is an $\mathcal{H}$-Banach f^*B-module in an obvious way. Similarly for $\mathcal{G}$-Banach bimodules. The pullback along f of a $\mathcal{G}$-equivariant linear operator or of a $\mathcal{G}$-equivariant homomorphism with coefficient maps is an $\mathcal{H}$-equivariant linear operator or an $\mathcal{H}$-equivariant homomorphism with coefficient maps.

The pullback along f respects balanced equivariant bilinear maps and balanced tensor products of equivariant Banach modules. Regarding the pushout of equivariant Banach modules we have the following result:

Proposition 3.6.8. *Let B be a $\mathcal{G}$-Banach algebra and E a right $\mathcal{G}$-Banach B-module. Let B' be another $\mathcal{G}$-Banach algebra and let $\psi\colon B \to B'$ be a $\mathcal{G}$-equivariant homomorphism. Then*

$$f^*\left(\psi_*(E)\right) = \left(f^*(\psi)\right)_*\left(f^*(E)\right)$$

as right $\mathcal{H}$-Banach $f^*(B')$-modules.

The pullback of $\mathcal{G}$-Banach pairs

The functor f^* from the category of $\mathcal{G}$-Banach spaces to the category of $\mathcal{H}$-Banach spaces induces a functor f^* from the category of $\mathcal{G}$-Banach B-pairs to the category of $\mathcal{H}$-Banach $f^*(B)$-pairs. It sends a $\mathcal{G}$-Banach B-pair $E = (E^<,\ E^>)$ to the $\mathcal{H}$-Banach f^*B-pair $f^*(E) = (f^*(E^<),\ f^*(E^>))$. A ($\mathcal{G}$-equivariant) B-linear operator $T = (T^<, T^>)$ is sent to the ($\mathcal{H}$-equivariant) $f^*(B)$-linear operator $f^*(T) = (f^*(T^<), f^*(T^>))$.

One proceeds similarly for $\mathcal{G}$-Banach A-B-pairs and homomorphisms with coefficient maps. The functor respects the tensor product of Banach pairs. Also the pushout of Banach pairs is preserved just as in Proposition 3.6.8.

Lemma 3.6.9. *Let E and F be $\mathcal{G}$-Banach B-pairs. For all $S \in \mathrm{L}^{\mathbf{loc}}_{s_{\mathcal{G}}^* B}\left(s_{\mathcal{G}}^* E,\ s_{\mathcal{G}}^* F\right)$, we have*

$$\alpha^{\mathrm{L}(f^*E, f^*F)}(f^*S) = f^*\left(\alpha^{\mathrm{L}(E,F)}(S)\right).$$

Note that $\mathrm{L}^{\mathbf{loc}}_{f^* r_{\mathcal{G}}^* B}\left(f^* r_{\mathcal{G}}^* E,\ f^* r_{\mathcal{G}}^* F\right) = \mathrm{L}^{\mathbf{loc}}_{r_{\mathcal{H}}^* f^* B}\left(r_{\mathcal{H}}^* f^* E,\ r_{\mathcal{H}}^* f^* F\right)$ and similarly for $s_{\mathcal{G}}$ and $s_{\mathcal{H}}$. The preceding lemma could be interpreted as a way to give meaning to the formula

$$f^*\alpha^{\mathrm{L}(E,F)} = \alpha^{\mathrm{L}(f^*E, f^*F)}.$$

3.6.2 The pullback of $\mathrm{KK}^{\mathrm{ban}}$-cycles along strict morphisms

Let $\mathcal{G}$ and $\mathcal{H}$ be topological groupoids over X and Y, respectively, and let $f\colon \mathcal{H} \to \mathcal{G}$ be a strict morphism of topological groupoids. Let A and B be $\mathcal{G}$-Banach algebras.

Proposition 3.6.10. *Let $(E,T) \in \mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$. Then $f^*(E,T) := (f^*E, f^*T)$ is an element of $\mathbb{E}^{\mathrm{ban}}_{\mathcal{H}}(f^*A, f^*B)$.*

Proof. We already know that f^*E is a non-degenerate $\mathcal{H}$-Banach f^*A-f^*B-pair. If σ_E is the grading automorphism of E, then $f^*\sigma_E = (f^*\sigma_E^<,\ f^*\sigma_E^>)$ is a grading automorphism for f^*E. The operator f^*T is odd for this grading. Let $a \in \Gamma(X,A)$. Then $a \circ f \in \Gamma(Y,\ f^*A)$. Now Proposition 3.3.22 says that the pullback of locally compact operators is again locally compact, so

$$[\pi(a \circ f),\ f^*T] = [f^*(\pi(a)),\ f^*T] = f^*[\pi(a),T] \in \mathrm{K}^{\mathbf{loc}}_{f^*B}(f^*E).$$

Now let $b \in \Gamma(Y,\ f^*Y)$. Let $\varepsilon > 0$ and $y_0 \in Y$. Then we can find an $a \in \Gamma(X,A)$ and a neighbourhood V of y_0 in Y such that $\|T\|\,\|b(v) - a(f(v))\| \le \varepsilon$ for all $v \in V$. For all $v \in V$, we have

$$\begin{aligned} \|[\pi(b),\ f^*T]_v - [\pi(a\circ f),\ f^*T]_v\| &= \|[\pi(b - a\circ f),\ f^*T]_v\| \\ &= \left\|\left[\pi_{A_{f(v)}}(b(v) - a(f(v))),\ T_{f(v)}\right]\right\| \\ &\le \|T\|\,\|b(v) - a(f(v))\| \le \varepsilon. \end{aligned}$$

So $[\pi(b),\ f^*T]$ is locally approximable by locally compact operators, so it is itself locally compact. Analogously one shows that $\pi(b)\left(\mathrm{Id} - f^*T^2\right)$ is locally compact.

Now let $\tilde{a} \in \Gamma\left(\mathcal{G},\ r_{\mathcal{G}}^*A\right)$. Then $\tilde{a} \circ f \in \Gamma\left(\mathcal{H},\ f^*r_{\mathcal{G}}^*A\right) = \Gamma(\mathcal{H},\ r_{\mathcal{H}}^*f^*A)$; note that $f^*r_{\mathcal{G}}^*A = r_{\mathcal{H}}^*f^*A$. Now

$$\begin{aligned} \pi(\tilde{a}\circ f)\left(\alpha^{\mathrm{L}(f^*E)}(s_{\mathcal{H}}^*f^*T) - r_{\mathcal{H}}^*f^*T\right) &= f^*\pi(\tilde{a})\left(\alpha^{\mathrm{L}(f^*E)}\left(f^*s_{\mathcal{G}}^*T\right) - f^*r_{\mathcal{G}}^*T\right) \\ &= f^*\left(\pi(\tilde{a})\left(\alpha^{\mathrm{L}(E)}(s_{\mathcal{G}}^*T) - r_{\mathcal{G}}^*T\right)\right) \\ &\in \mathrm{K}^{\mathbf{loc}}_{f^*r_{\mathcal{G}}^*B}\left(f^*r_{\mathcal{G}}^*E\right) = \mathrm{K}^{\mathbf{loc}}_{r_{\mathcal{H}}^*f^*B}(r_{\mathcal{H}}^*f^*E). \end{aligned}$$

As above, one can extend this to all $\tilde{b} \in \Gamma(\mathcal{H},\ r_{\mathcal{H}}^*f^*A)$ (instead of $\tilde{a}\circ f$).
So $f^*(E,T) \in \mathbb{E}^{\mathrm{ban}}_{\mathcal{H}}(f^*A, f^*B)$. □

The pullback along f respects the direct sum of cycles, the pushout and $f^*(B[0,1]) = (f^*B)[0,1]$. It follows that the pullback also respects homotopies. Hence we get the following theorem:

Theorem 3.6.11. *The pullback along the strict morphism $f\colon \mathcal{H} \to \mathcal{G}$ induces a homomorphism*

$$f^*\colon\ \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A,B) \to \mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(f^*A,\ f^*B).$$

It is natural with respect to $\mathcal{G}$-equivariant homomorphisms in both variables.

3.7 The sufficient condition for homotopy

Let X be a topological space and let $\mathcal{G}$ be a topological groupoid over X. We now reformulate the sufficient condition 1.9.1 for the homotopy of $\mathrm{KK}^{\mathrm{ban}}$-cycles for $\mathcal{G}$-Banach algebras. The notation $\mathrm{K}^{\mathbf{loc}}(r^*\Phi, r^*\Phi)$ will be explained in Definition 3.7.4. This very general form of the sufficient condition will become important in the proof of the injectivity part of the generalised Green-Julg Theorem in Chapter 7 and is going to be proved at the end of this section.

Theorem 3.7.1 (Sufficient condition for homotopy of $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}$-cycles). *Let A and B be $\mathcal{G}$-Banach algebras. Let $(E,T),(E',T')$ be elements of $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$. If there is a morphism Φ from (E,T) to (E',T') (with coefficient maps Id_A and Id_B) such that*

1. $\forall a \in \Gamma(X,A):\ [a,(T,T')] = ([a,T],\ [a,T']) \in \mathrm{K}^{\mathbf{loc}}(\Phi,\Phi)$,
2. $\forall a \in \Gamma(X,A):\ a((T,T')^2-1) = \Big(a(T^2-1),\ a(T'^2-1)\Big) \in \mathrm{K}^{\mathbf{loc}}(\Phi,\Phi)$,
3. $\forall a \in \Gamma(\mathcal{G}, r^*A)\ a\left(\left(\alpha^{\mathrm{L}(E,F)} s^*T,\ \alpha^{\mathrm{L}(E',F')} s^*T'\right) - (r^*T, r^*T')\right) \in \mathrm{K}^{\mathbf{loc}}(r^*\Phi, r^*\Phi)$,

then $(E,T) \sim (E',T')$. Moreover, if $T=0$ and $T'=0$, then the homotopy can be chosen to have trivial operator as well.

3.7.1 Some notation

Definition 3.7.2 ($\mathrm{L}^{\mathbf{loc}}(\rho,\sigma)$). Let $\rho\colon E \to E'$ and $\sigma\colon F \to F'$ be contractive continuous fields of linear maps between u.s.c. fields of Banach spaces over X. Then a morphism from ρ to σ is a pair (T,T') such that $T \in \mathrm{L}^{\mathbf{loc}}(E,F)$ and $T' \in \mathrm{L}^{\mathbf{loc}}(E',F')$ and $\sigma \circ T = T' \circ \rho$. The vector space of all morphisms between ρ and σ is denoted by $\mathrm{L}^{\mathbf{loc}}(\rho,\sigma)$. The Banach space of all pairs in $\mathrm{L}^{\mathbf{loc}}(\rho,\sigma)$ of bounded fields of operators will be called $\mathrm{L}(\rho,\sigma)$.

Just as in Section 1.9.1 and based on the preceding definition one can define morphisms between u.s.c. fields of Banach modules and Banach pairs. We make the last definition explicit:

Definition 3.7.3 ($\mathrm{L}^{\mathbf{loc}}_\psi(\Phi,\Psi)$). Let $\psi\colon B \to B'$ be a continuous field of homomorphisms between u.s.c. fields of Banach algebras over X. Let $\Phi_\psi\colon E_B \to E'_{B'}$ and $\Psi_\psi\colon F_B \to F'_{B'}$ be contractive continuous fields of concurrent homomorphisms with coefficient map ψ between u.s.c. fields of Banach pairs over X. Then the vector space $\mathrm{L}^{\mathbf{loc}}_\psi(\Phi,\Psi)$ of morphisms from Φ_ψ to Ψ_ψ is defined to be the set of pairs (T,T') such that $T \in \mathrm{L}^{\mathbf{loc}}_B(E,F)$, $T' \in \mathrm{L}^{\mathbf{loc}}_{B'}(E',F')$ satisfying

$$\Psi^> \circ T^> = T'^> \circ \Phi^> \quad \text{and} \quad T'^< \circ \Psi^< = \Phi^< \circ T^<.$$

The Banach space $\mathrm{L}_\psi(\Phi,\Psi)$ is the subspace of $\mathrm{L}^{\mathbf{loc}}_\psi(\Phi,\Psi)$ of bounded pairs.

Now we proceed in analogy to Section 1.9.2:

Definition 3.7.4 ($\mathrm{K}^{\mathbf{loc}}(\Phi,\Psi)$). Let $\Phi_\psi\colon E_B \to E'_{B'}$ and $\Psi_\psi\colon F_B \to F'_{B'}$ be as above. Then $\mathrm{K}^{\mathbf{loc}}_\psi(\Phi,\Psi)$ is the vector space of pairs $(T,T') \in \mathrm{L}^{\mathbf{loc}}_B(E,F) \times \mathrm{L}^{\mathbf{loc}}_{B'}(E',F')$ such that for all $\varepsilon > 0$ and all $x \in X$ there is a neighbourhood U of x in X, an $n \in \mathbb{N}$, $\xi_1^<, \dots, \xi_n^< \in \Gamma(X,E^<)$ and $\eta_1^>, \dots, \eta_n^> \in \Gamma(X,F^>)$ such that

$$\left\| T_u - \sum_{i=1}^n \left|\eta_i^>(u)\right\rangle\left\langle\xi_i^<(u)\right| \right\| \le \varepsilon \quad \text{and} \quad \left\| T'_u - \sum_{i=1}^n \left|\Psi_u^>(\eta_i^>(u))\right\rangle\left\langle\Phi_u^<(\xi_i^<(u))\right| \right\|$$

for all $u \in U$.

If $(T,T') \in \mathrm{K}^{\mathbf{loc}}_{\psi}(\Phi,\Psi)$, then $T \in \mathrm{K}^{\mathbf{loc}}_{B}(E,F)$, $T' \in \mathrm{K}^{\mathbf{loc}}_{B'}(E',F')$ and $(T,T') \in \mathrm{L}^{\mathbf{loc}}_{\psi}(\Phi,\Psi)$.

Proposition 3.7.5. *Let $\rho\colon E \to E'$ and $\sigma\colon F \to F'$ be contractive continuous fields of linear maps between u.s.c. fields of Banach spaces over X. Let $(T,T') \in \mathrm{L}^{\mathbf{loc}}(\rho,\sigma)$. Let Y be a topological space and let $p\colon Y \to X$ be continuous. Then $(p^*T,\ p^*T') \in \mathrm{L}^{\mathbf{loc}}(p^*\rho,\ p^*\sigma)$.*

This proposition carries over to fields of Banach pairs and also applies to locally compact operators:

Proposition 3.7.6. *Let $\psi\colon B \to B'$ be a continuous field of homomorphisms between u.s.c. fields of Banach algebras over X. Let $\Phi_{\psi}\colon E_B \to E'_{B'}$ and $\Psi_{\psi}\colon F_B \to F'_{B'}$ be contractive continuous fields of concurrent homomorphisms with coefficient map ψ between u.s.c. fields of Banach pairs over X. Let Y be a topological space and let $p\colon Y \to X$ be continuous. If $(T,T') \in \mathrm{L}^{\mathbf{loc}}_{\psi}(\Phi,\Psi)$, then $(p^*T,\ p^*T') \in \mathrm{L}^{\mathbf{loc}}_{p^*\psi}(p^*\Phi,\ p^*\Psi)$. Moreover, if (T,T') is locally compact, then so is $p^*(T,T') := (p^*T,\ p^*T')$.*

Definition 3.7.7 (The class $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(\varphi,\psi)$). Let $\varphi\colon A \to A'$ and $\psi\colon B \to B'$ be $\mathcal{G}$-equivariant homomorphisms of $\mathcal{G}$-Banach algebras. A $\mathrm{KK}^{\mathrm{ban}}$-cycle from φ to ψ is a pair $(\Phi\colon E \to E',\ (T,T'))$ such that E is a non-degenerate graded $\mathcal{G}$-Banach A-B-pair, E' is a non-degenerate graded $\mathcal{G}$-Banach A'-B'-pair, Φ is an even $\mathcal{G}$-equivariant homomorphism from ${}_AE_B$ to ${}_{A'}E'_{B'}$ with coefficient maps φ and ψ and $(T,T') \in \mathrm{L}_{\psi}(\Phi,\Phi)$ is a pair of odd bounded continuous fields of linear operators such that

1. $\forall a \in \Gamma(X,A):\ [a,(T,T')] = ([a,T],\ [\varphi\circ a,T']) \in \mathrm{K}^{\mathbf{loc}}_{\psi}(\Phi,\Phi)$,
2. $\forall a \in \Gamma(X,A):\ a((T,T')^2-1) = \Big(a(T^2-1),\ (\varphi\circ a)(T'^2-1)\Big) \in \mathrm{K}^{\mathbf{loc}}_{\psi}(\Phi,\Phi)$,
3. $\forall a \in \Gamma(\mathcal{G},r^*A):\ a\left(\left(\alpha^{\mathrm{L}(E,F)}s^*T,\ \alpha^{\mathrm{L}(E',F')}s^*T'\right) - (r^*T,r^*T')\right) \in \mathrm{K}^{\mathbf{loc}}_{r^*\psi}(r^*\Phi,r^*\Phi)$,

The class of all such cycles will be denoted by $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(\varphi,\psi)$.

With this notation we can restate Theorem 3.7.1: If ${}_{\mathrm{Id}_A}\Phi_{\mathrm{Id}_B}$ is a morphism between elements (E,T) and (E',T') of $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$ for $\mathcal{G}$-Banach algebras A and B, then a sufficient condition for (E,T) and (E',T') to be homotopic is that $\Phi \in \mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(\mathrm{Id}_A,\mathrm{Id}_B)$.

3.7.2 Mapping cylinders

Mapping cylinders of contractive fields of linear maps between graded $\mathcal{G}$-Banach spaces

Definition 3.7.8. Let $\rho\colon E \to E'$ be a contractive $\mathcal{G}$-equivariant graded continuous field of linear maps between graded $\mathcal{G}$-Banach spaces. Let $\mathrm{ev}_0^{E'}$ denote the canonical contractive $\mathcal{G}$-equivariant graded continuous field of linear maps from $E'[0,1]$ to E' obtained by evaluation at zero as defined in 3.5.7. Then the mapping cylinder $\mathrm{Z}(\rho)$ of ρ is defined to be the fibre product of ρ and $\mathrm{ev}_0^{E'}$:

$$
\begin{tikzcd}
\mathrm{Z}(\rho) \arrow[r] \arrow[d] & E'[0,1] \arrow[d, "\mathrm{ev}_0^{E'}"] \\
E \arrow[r, "\rho"'] & E'
\end{tikzcd}
$$

In particular, $\mathrm{Z}(\rho)$ is a graded $\mathcal{G}$-Banach space. For all $x \in X$, the fibre $\mathrm{Z}(\rho)_x$ of $\mathrm{Z}(\rho)$ at x is $\mathrm{Z}(\rho_x)$. The sections of $\mathrm{Z}(\rho)$ have the form (ξ,ξ') where ξ is a section of E and ξ' is a section of $E'[0,1]$ such that $(\xi(x),\xi'(x)) \in \mathrm{Z}(\rho_x)$, i.e., $\rho_x(\xi(x)) = \xi'(x)(0)$. The grading automorphism of $\mathrm{Z}(\rho)$ is given fibrewise.

A technical detail that needs to be checked to make sure that this definition makes sense is that there are indeed enough such sections, i.e., that condition (C2) is satisfied: For all $x \in X$, the set $(\xi(x),\ \xi'(x))$ is dense in $\mathrm{Z}(\rho_x)$ if (ξ, ξ') runs through the sections defined above. So let $(e_x, \xi'_x) \in \mathrm{Z}(\rho_x)$.

If $e_x = 0$, then we first find a section $\tilde{\xi}'$ of $E'[0,1]$ such that $\tilde{\xi}'(x)$ is close to ξ'_x. By cutting $\tilde{\xi}'$ down with a function $\chi \in \mathcal{C}[0,1]$ which satisfies $0 \leq \chi \leq 1$ and $\chi(0) = 0$ and $\chi(t) = 1$ for all $t \in [0,1]$ outside some small neighbourhood of 0 one can assume without loss of generality that $\tilde{\xi}'(y)(0) = 0$ for all $y \in X$. Then $(0, \tilde{\xi}')$ satisfies that $(0, \tilde{\xi}')(x)$ is close to (e_x, ξ'_x).

Secondly, if e_x is arbitrary but $\xi'_x(t) = \rho_x(e_x)$ for all $t \in [0,1]$, then it is rather trivial to find a section of $\mathrm{Z}(\rho)$ such that its value at x is close to (e_x, ξ'_x).

Combining these two facts one can treat the general case.

Definition 3.7.9 (The mapping cylinder construction as a functor). Let $\rho\colon E \to E'$ and $\sigma\colon F \to F'$ be contractive $\mathcal{G}$-equivariant graded continuous fields of linear maps between graded $\mathcal{G}$-Banach spaces. Let $(T, T') \in \mathrm{L}^{\mathbf{loc}}(\rho, \sigma)$. Define

$$\mathrm{Z}\left(T, T'\right) := \left(\mathrm{Z}\left(T_x, T'_x\right)\right)_{x \in X}.$$

Then $\mathrm{Z}(T, T') \in \mathrm{L}^{\mathbf{loc}}(\mathrm{Z}(\rho),\ \mathrm{Z}(\sigma))$.

The mapping cylinder construction carries over to graded $\mathcal{G}$-Banach algebras, $\mathcal{G}$-Banach modules and $\mathcal{G}$-Banach pairs. We skip most of the details and give an overview:

If $\psi\colon B \to B'$ is a homomorphism of graded $\mathcal{G}$-Banach algebras, then $\mathrm{Z}(\psi)$ is a graded $\mathcal{G}$-Banach algebra. If B and B' are non-degenerate, then so is $\mathrm{Z}(\psi)$. The mapping cylinder of Id_B is isomorphic to $B[0,1]$.

If $\Phi_\psi\colon E_B \to E'_{B'}$ is a homomorphism of graded $\mathcal{G}$-Banach modules with coefficient map ψ, then $\mathrm{Z}(\Phi)$ is a graded $\mathcal{G}$-Banach $\mathrm{Z}(\psi)$-module. If E_B and $E'_{B'}$ are non-degenerate, then so is $\mathrm{Z}(\Phi)$ and $\mathrm{ev}_{0,*}(\mathrm{Z}(\Phi)) \cong E$ and $\mathrm{ev}_{t,*}(\mathrm{Z}(\Phi)) \cong E'$ for all $t \in]0,1]$. If $\Psi_\psi\colon F_B \to F'_{B'}$ is another homomorphism of graded $\mathcal{G}$-Banach modules with coefficient map ψ and $(T, T') \in \mathrm{L}^{\mathbf{loc}}_\psi(\Phi, \Psi)$, then $\mathrm{Z}(T, T') \in \mathrm{L}^{\mathbf{loc}}_{\mathrm{Z}(\psi)}(\mathrm{Z}(\Phi),\ \mathrm{Z}(\Psi))$.

The same is true for Banach pairs. The main technical result for Banach pairs is the following:

Proposition 3.7.10. *Let $\Phi_\psi\colon E_B \to E'_{B'}$ and $\Psi_\psi\colon F_B \to F'_{B'}$ be concurrent homomorphisms of graded $\mathcal{G}$-Banach pairs. Let $(T, T') \in \mathrm{L}^{\mathbf{loc}}_\psi(\Phi,\ \Psi)$. Then the following are equivalent:*

1. $(T, T') \in \mathrm{K}^{\mathbf{loc}}_\psi(\Phi, \Psi)$*;*
2. $\mathrm{Z}(T, T') \in \mathrm{K}^{\mathbf{loc}}_{\mathrm{Z}(\psi)}(\mathrm{Z}(\Phi),\ \mathrm{Z}(\Psi))$

Proof. 1. $\Rightarrow$ 2.: By straightforward linearity and continuity arguments it suffices to consider the case that (T, T') is of the form $(T, T') = \left(\left|\eta^>\right\rangle\left\langle\xi^<\right|,\ \left|\Psi^> \circ \eta^>\right\rangle\left\langle\Phi^< \circ \xi^<\right|\right)$ for $\eta^> \in \Gamma(X,\ F^>)$ and $\xi^< \in \Gamma(X,\ E^<)$. Define $\tilde{\eta}^>(x) := (\eta^>(x),\ t \mapsto \Psi^>(\eta^>(x))) \in \mathrm{Z}(\Psi^>_x)$ and $\tilde{\xi}^<(x) := (\xi^<(x),\ t \mapsto \Phi^<(\xi^<(x))) \in \mathrm{Z}(\Phi^<_x)$ for all $x \in X$. Then we have $\tilde{\eta}^> \in \Gamma(X,\ \mathrm{Z}(\Psi^>))$ and $\tilde{\xi}^< \in \Gamma(X,\ \mathrm{Z}(\Phi^<))$. Just as in the proof of Proposition 1.9.31 one can now show that $\mathrm{Z}(T, T') = \left|\tilde{\eta}^>\right\rangle\left\langle\tilde{\xi}^<\right|$. So $\mathrm{Z}(T, T')$ is in particular locally compact.

2. $\Rightarrow$ 1.: Let $\mathrm{Z}(T, T')$ be locally compact. Let $\varepsilon > 0$ and $x \in X$. Find a neighbourhood U of x in X and find $n \in \mathbb{N}$ and $(\eta^>_1,\ \eta'^>_1), \ldots, (\eta^>_n,\ \eta'^>_n) \in \mathrm{Z}(\Psi^>)$ and $(\xi^<_1,\ \xi'^<_1), \ldots, (\xi^<_n,\ \xi'^<_n) \in \mathrm{Z}(\Phi^<)$ such that

$$\left\| \mathrm{Z}\left(T, T'\right)_u - \sum_{i=1}^{n} \left|\ \left(\eta^>_i(u),\ \eta'^>_i(u)\right)\ \right\rangle\left\langle\ \left(\xi^<_i(u),\ \xi'^<_i(u)\right)\ \right| \right\| \leq \varepsilon$$

for all $u \in U$. Define

$$(S, S') := \sum_{i=1}^{n} \left(\left|\eta_i^{>}\right\rangle\left\langle\xi_i^{<}\right|,\ \left|\Psi^{>} \circ \eta_i^{>}\right\rangle\left\langle\Phi^{<} \circ \xi_i^{<}\right| \right).$$

In the proof of 1.9.31 it is shown that $\|(T_u, T'_u) - (S_u, S'_u)\| \leq \varepsilon$ for all $u \in U$. Hence (T, T') is locally compact. □

Mapping cylinders and $\mathrm{KK}^{\mathrm{ban}}$-cycles

Theorem 3.7.11. *Let $\varphi\colon A \to A'$ and $\psi\colon B \to B'$ be homomorphisms of $\mathcal{G}$-Banach algebras. Let $(\Phi\colon E \to E',\ (T, T'))$ be an element of $\mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(\varphi, \psi)$. Let $\iota_A\colon A \to \mathrm{Z}(\varphi)$ be the field of canonical injections $(\iota_A)_x = \iota_{A_x}\colon A_x \to \mathrm{Z}(\varphi_x)$ where x runs through X. Then*

$$\iota_A^* \left(\mathrm{Z}(\Phi),\ \mathrm{Z}\left(T, T'\right)\right) \in \mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}\left(A,\ \mathrm{Z}(\psi)\right).$$

If we write ev_0 for the canonical homomorphism $\mathrm{Z}(\psi) \to B$ and ev_t for the homomorphism $\mathrm{Z}(\psi) \to B',\ (b_x, \beta'_x) \mapsto \beta'_x(t)$ for all $t \in]0,1]$, then

$$\mathrm{ev}_{0,*}\left(\iota_A^*\left(\mathrm{Z}(\Phi),\ \mathrm{Z}\left(T, T'\right)\right)\right) \cong (E, T)$$

and

$$\mathrm{ev}_{t,*}\left(\iota_A^*\left(\mathrm{Z}(\Phi),\ \mathrm{Z}\left(T, T'\right)\right)\right) \cong \varphi^*(E', T')$$

for all $t \in]0,1]$.

Proof. The operator $\mathrm{Z}(T, T')$ is indeed bounded and odd on the non-degenerate graded $\mathcal{G}$-Banach $\mathrm{Z}(\psi)$-module $\mathrm{Z}(\Phi)$ which carries a left action of $\mathrm{Z}(\varphi)$. Let $a \in \Gamma(X, A)$. Then $(\iota_A \circ a)(x) = (a(x),\ t \mapsto \varphi_x(a(x)))$ for all $x \in X$. Now

$$\left[(a(x),\ t \mapsto \varphi_x(a(x)))_{x \in X},\ \mathrm{Z}\left(T, T'\right)\right] = \mathrm{Z}\left([a, T],\ \left[\varphi \circ a, T'\right]\right) \in \mathrm{K}^{\mathbf{loc}}_{\mathrm{Z}(\psi)}\left(\mathrm{Z}(\Phi)\right).$$

Similarly, $(a(x),\ t \mapsto \varphi_x(a(x)))_{x \in X}\left(\mathrm{Z}\left(T, T'\right)^2 - 1\right)$ is locally compact. Now let $a \in \Gamma\left(\mathcal{G},\ r^*A\right)$. Then for all $\gamma \in \mathcal{G}$:

$$\begin{aligned}
& \left[a\left(\alpha^{\mathrm{L}(\mathrm{Z}(\Phi),\mathrm{Z}(\Phi))} s^* \,\mathrm{Z}\left(T, T'\right) - r^* \,\mathrm{Z}\left(T, T'\right)\right)\right]_\gamma \\
= \;& a(\gamma)\left(\gamma \,\mathrm{Z}\left(T, T'\right)_{s(\gamma)} - \mathrm{Z}\left(T, T'\right)_{r(\gamma)}\right) \\
= \;& a(\gamma)\left(\mathrm{Z}\left(\gamma T_{s(\gamma)}, \gamma T'_{s(\gamma)}\right) - \mathrm{Z}\left(T_{r(\gamma)}, T'_{r(\gamma)}\right)\right) \\
= \;& \mathrm{Z}\left(a(\gamma)(\gamma T_{s(\gamma)} - T_{r(\gamma)}),\ \varphi_{r(\gamma)}(a(\gamma))(\gamma T'_{s(\gamma)} - T'_{r(\gamma)})\right) \\
= \;& \mathrm{Z}\left(a(\alpha^{\mathrm{L}(E,E)} s^* T - r^* T),\ (\varphi \circ a)(\alpha^{\mathrm{L}(E',E')} s^* T' - r^* T')\right)_\gamma.
\end{aligned}$$

By definition of $\mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(\varphi, \psi)$ the pair $\left(a(\alpha^{\mathrm{L}(E,E)} s^* T - r^* T),\ (\varphi \circ a)(\alpha^{\mathrm{L}(E',E')} s^* T' - r^* T')\right)$ is locally compact, so we are done. □

The following Proposition is proved just as its analogue 1.9.34 for groups instead of groupoids.

Proposition 3.7.12. *Let $\varphi\colon A \to A'$ and $\psi\colon B \to B'$ be homomorphisms of $\mathcal{G}$-Banach algebras. Let $(\Phi\colon E \to E',\ (T,T'))$ be an element of $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(\varphi,\psi)$. Write ι_A for the canonical "injection" $A \to \mathrm{Z}(\varphi)$ and $p_{B'[0,1]}$ for the canonical homomorphism $\mathrm{Z}(\psi) \to B'[0,1]$. Then*

$$\left(p_{B'[0,1]}\right)_* \left(\iota_A^* \left(\mathrm{Z}(\Phi),\ \mathrm{Z}\left(T,T'\right)\right)\right) \quad \in \quad \mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}\left(A,\ B'[0,1]\right).$$

This is a homotopy

$$\psi_*(E,T) \sim \varphi^*(E',T').$$

Theorem 3.7.1 can now be restated as the following corollary:

Corollary 3.7.13. *Let A and B be $\mathcal{G}$-Banach algebras and $(\Phi\colon E \to E',\ (T,T')) \in \mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(\mathrm{Id}_A,\mathrm{Id}_B)$. Then $(E,T),(E',T') \in \mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$ and $(E,T) \sim (E',T')$.*

3.8 Morita theory

Let $\mathcal{G}$ be a topological groupoid over X. The results and definitions of Section 1.10 all carry over to the case of $\mathcal{G}$-Banach algebras:

3.8.1 Morita equivalences, Morita cycles, Morita morphisms

Definition 3.8.1 ($\mathcal{G}$-equivariant Morita equivalence). Let A and B be $\mathcal{G}$-Banach algebras. A *$\mathcal{G}$-equivariant Morita equivalence* between A and B is a pair $\left({}_B E^<_A, {}_A E^>_B\right)$ of $\mathcal{G}$-Banach bimodules endowed with an equivariant continuous field of bilinear maps $\langle\cdot,\cdot\rangle_B\colon E^< \times E^> \to B$ and an equivariant continuous field of bilinear maps ${}_A\langle\cdot,\cdot\rangle\colon E^> \times E^< \to A$ such that for all $x \in X$ the pair $(E^<_x,\ E^>_x)$ with the brackets $\langle\cdot,\cdot\rangle_{B,x}$ and ${}_A\langle\cdot,\cdot\rangle_x$ is a Morita equivalence between A_x and B_x.

This notion of Morita equivalence is an equivalence relation on the class of non-degenerate $\mathcal{G}$-Banach algebras.

Definition 3.8.2 ($\mathcal{G}$-equivariant Morita cycle). Let A and B be non-degenerate $\mathcal{G}$-Banach algebras. Then a *$\mathcal{G}$-equivariant Morita cycle* F from A to B is a non-degenerate $\mathcal{G}$-Banach A-B-pair F such that $\Gamma(X,A)$ acts on F by locally compact operators, i.e., if $\pi_A\colon \Gamma(X,A) \to \mathrm{L}^{\mathbf{loc}}_B(F)$ is the action of $\Gamma(X,A)$ on F, then $\pi_A(\Gamma(X,A)) \subseteq \mathrm{K}^{\mathbf{loc}}_B(F)$. The class of all Morita cycles from A to B is denoted by $\mathbb{M}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$.

Just as in the first chapter the Morita cycles are just the trivially graded $\mathrm{KK}^{\mathrm{ban}}$-cycles with zero operator. There are obvious notions of (iso)morphisms between Morita cycles, the sum of Morita cycles and of the pullback and the pushout of Morita cycles also in the $\mathcal{G}$-equivariant setting (compare Definition 1.10.13). Hence there is also a canonical notion of *homotopy* of $\mathcal{G}$-equivariant Morita cycles. The homotopy classes of $\mathcal{G}$-equivariant Morita cycles are called *$\mathcal{G}$-equivariant Morita morphisms*.

Using Proposition 3.1.59, which says that operators of the form $T \otimes 1$ are locally compact if T is and the left action of the second factor is by locally compact operators, one can define the composition of Morita cycles just as in the first chapter. The composition, the homotopy, the sum, the pullback and the pushout are all pairwise compatible.

From Example 3.1.57 we know that $\Gamma(X,B)$ acts by locally compact operators on the standard Banach B-pair (B,B) if B is non-degenerate. We can therefore make the following definition:

Definition 3.8.3 ($\mathbb{M}_{\mathcal{G}}^{\mathrm{ban}}(\varphi)$, $\mathrm{Mor}_{\mathcal{G}}^{\mathrm{ban}}(\varphi)$). Let A and B be non-degenerate $\mathcal{G}$-Banach algebras and let $\varphi\colon A \to B$ be a $\mathcal{G}$-equivariant homomorphism. Then $\Gamma(X, A)$ acts on $\underline{B}_B$ from the left via φ and the so-constructed Morita cycle will be denoted by $\mathbb{M}_{\mathcal{G}}^{\mathrm{ban}}(\varphi)$ and its homotopy class by $\mathrm{Mor}_{\mathcal{G}}^{\mathrm{ban}}(\varphi)$ or simply by $[\varphi]$.

In particular, the standard B-pair (B, B) is a Morita cycle from B to B for every non-degenerate $\mathcal{G}$-Banach algebra B.

Theorem 3.8.4. *The non-degenerate $\mathcal{G}$-Banach algebras together with the $\mathcal{G}$-equivariant Morita morphisms form a category (apart from the fact that the classes of morphisms are not sets). If B is a non-degenerate $\mathcal{G}$-Banach algebra, then the identity morphism on B is given by the equivalence class of (B, B).*

To prove this one can proceed as in Chapter 1 and show the following lemmas:

Lemma 3.8.5. *Let A, B and C be non-degenerate $\mathcal{G}$-Banach algebras, $F \in \mathbb{M}_{\mathcal{G}}^{\mathrm{ban}}(A, B)$ and $\psi\colon B \to C$ a $\mathcal{G}$-equivariant homomorphism. Then*

$$\psi_*(F) \ \sim_h \ F \otimes_B \mathbb{M}_{\mathcal{G}}^{\mathrm{ban}}(\psi).$$

Lemma 3.8.6. *Let A and B be non-degenerate $\mathcal{G}$-Banach algebras and $F \in \mathbb{M}_{\mathcal{G}}^{\mathrm{ban}}(A, B)$. Define the $\mathcal{G}$-Banach A-B-pair $\overline{AF}$ as $\left(\overline{F_x^< A_x},\ \overline{A_x F_X^>}\right)_{x\in X} := \left(\mathrm{cl}\left(F_x^< A_x\right), \mathrm{cl}\left(A_x F_x^>\right)\right)_{x\in X}$ (the sections being just the sections of F that take their values in $\overline{AF}$) being a $\mathcal{G}$-Banach A-B-pair. Then $A \otimes_A F$, $\overline{AF} \in \mathbb{M}_{\mathcal{G}}^{\mathrm{ban}}(A, B)$ and*

$$A \otimes_A F \sim_h \overline{AF} \sim_h F.$$

Note that $A \otimes_A F$ and $\overline{AF}$ are A-non-degenerate so every Morita morphism is homotopic to a Morita morphism with non-degenerate left action.

Lemma 3.8.7. *Let A, B and C be non-degenerate $\mathcal{G}$-Banach algebras, $F \in \mathbb{M}_{\mathcal{G}}^{\mathrm{ban}}(B, C)$ and $\varphi\colon A \to B$ a $\mathcal{G}$-equivariant homomorphism. Then*

$$\mathbb{M}_{\mathcal{G}}^{\mathrm{ban}}(\varphi) \otimes_B F \ \sim_h \ \varphi^*(F).$$

Proposition 3.8.8 (Morita equivalences are Morita morphisms). *Let A and B be non-degenerate $\mathcal{G}$-Banach algebras and let $E = (E^<, E^>)$ be a $\mathcal{G}$-equivariant Morita equivalence between A and B. Then E, regarded as a $\mathcal{G}$-Banach A-B-pair with trivial grading, is in $\mathbb{M}_{\mathcal{G}}^{\mathrm{ban}}(A, B)$. Let $\mathrm{Mor}_{\mathcal{G}}^{\mathrm{ban}}(E)$ or $[E]$ denote the Morita morphism associated to E.*

Proof. We have to show that $\Gamma(X, A)$ acts on E by locally compact operators. Let $a \in \Gamma(X, A)$, $x_0 \in X$ and $\varepsilon > 0$. Because the A_{x_0}-valued inner product on $(E_{x_0}^>,\ E_{x_0}^<)$ is full, we can find an $n \in \mathbb{N}$ and $\xi_1^>, \dots, \xi_n^> \in \Gamma(X, E^>)$ and $\xi_1^<, \dots, \xi_n^< \in \Gamma(X, E^<)$ such that

$$\left\| a(x_0) - \sum_{i=1}^{n} {}_A\left\langle \xi_i^>(x_0),\ \xi_i^<(x_0)\right\rangle \right\| \leq \varepsilon/2.$$

Now $x \mapsto {}_A\langle \xi_i^>(x),\ \xi_i^<(x)\rangle$, and hence also $x \mapsto a(x) - \sum_{i=1}^{n} {}_A\langle \xi_i^>(x),\ \xi_i^<(x)\rangle$, is a section of A. Because the modulus of sections is upper semi-continuous, we can find a neighbourhood U of x_0 such that $\|a(x) - \sum_{i=1}^{n} {}_A\langle \xi_i^>(x),\ \xi_i^<(x)\rangle\|$ for all $x \in U$. As in the proof of Proposition 1.10.25 one shows that $x \mapsto {}_A\langle \xi_i^>(x),\ \xi_i^<(x)\rangle$ acts on E as the locally compact operator $\left|\xi_i^>\right\rangle\left\langle\xi_i^<\right|$, so we can approximate the action of a on E by the locally compact operator $\sum_{i=1}^{n} \left|\xi_i^>\right\rangle\left\langle\xi_i^<\right|$ up to ε on U. Hence the action of a on E is locally compact. $\square$

As in the case for group actions and Banach algebras one proves:

Lemma 3.8.9. *Let A, B be non-degenerate $\mathcal{G}$-Banach algebras and let E and E' be $\mathcal{G}$-equivariant Morita equivalences between A and B. Assume that ${}_{\mathrm{Id}_A}\theta_{\mathrm{Id}_B} \colon {}_AE_B \to {}_AE'_B$ is a concurrent morphism of Morita equivalences (meaning that it is a equivariant morphism of Morita cycles that also preserves the left bracket). Then*

$$[E] = [E'] \in \mathrm{Mor}^{\mathrm{ban}}_{\mathcal{G}}(A, B).$$

Using this lemma the following theorem is straightforward to show, compare Theorem 1.10.27.

Theorem 3.8.10. *Let A and B be non-degenerate $\mathcal{G}$-Banach algebras and let E be a $\mathcal{G}$-equivariant Morita equivalence between A and B. Then the $\mathcal{G}$-equivariant Morita morphism $[E]$ is an* isomorphism *with inverse $[E]^{-1} = \left[\,\overline{E}\,\right]$.*

3.8.2 The action of Morita morphisms on $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}$

Definition and Proposition 3.8.11. Let A, B and C be non-degenerate $\mathcal{G}$-Banach algebras. Let (E, T) be an element of $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A, B)$ and F an element of $\mathbb{M}^{\mathrm{ban}}_{\mathcal{G}}(B, C)$. Then we define

$$\mu_F(E, T) := (E, T) \otimes_B F := (E \otimes_A F,\ T \otimes 1) \in \mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A, C).$$

Proof. We have to show that $(E, T) \otimes_B F$ is indeed in $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A, C)$. Let $\pi_A \colon \Gamma(X, A) \to \mathrm{L}_B(E)$ be the action of $\Gamma(X, A)$ on E. Recall from Proposition 3.1.59 that operators of the form “locally compact tensor one” are locally compact because $\Gamma(X, B)$ acts on F by locally compact operators.

1. The operator $T \otimes 1$ is odd.

2. If $a \in \Gamma(X, A)$, then $[(\pi_A(a) \otimes 1), T \otimes 1] = [\pi_A(a), T] \otimes 1 \in \mathrm{K}^{\mathbf{loc}}_C(E \otimes_B F)$.

3. If $a \in \Gamma(X, A)$, then

$$(\pi_A(a) \otimes 1)\left(\mathrm{Id}_{E \otimes_B F} - T^2 \otimes 1\right) = \left(\pi_A(a)(\mathrm{Id}_E - T^2)\right) \otimes 1 \in \mathrm{K}^{\mathbf{loc}}_C\left(E \otimes_B F\right).$$

4. We use $r^*\left(E \otimes_B F\right) = r^*E \otimes_{r^*B} r^*F$: If $a \in \Gamma(\mathcal{G},\ r^*A)$, then

$$\begin{aligned}
& (\pi_A(a) \otimes 1)\left(\alpha^{\mathrm{L}(E\otimes F, E\otimes F)}(s^*(T \otimes 1)) - r^*(T \otimes 1)\right) \\
= \;& (\pi_A(a) \otimes 1)\left(\alpha^{\mathrm{L}(E\otimes F, E\otimes F)}(s^*T \otimes 1) - r^*T \otimes 1\right) \\
= \;& (\pi_A(a) \otimes 1)\left(\left[\alpha^{\mathrm{L}(E,E)} s^*T\right] \otimes \left[\alpha^{\mathrm{L}(F,F)} 1\right] - r^*T \otimes 1\right) \\
= \;& (\pi_A(a) \otimes 1)\left(\alpha^{\mathrm{L}(E,E)} s^*T \otimes 1 - r^*T \otimes 1\right) \\
= \;& \left(\pi_A(a)\left(\alpha^{\mathrm{L}(E,E)} s^*T - r^*T\right)\right) \otimes 1 \in \mathrm{K}^{\mathbf{loc}}_{r^*C}\left(r^*(E \otimes_B F)\right).
\end{aligned}$$

□

Just as 1.10.29 one now proves:

Definition and Proposition 3.8.12. Let A, B, C be non-degenerate $\mathcal{G}$-Banach algebras. Then the product $\otimes_B\colon \mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A,B) \times \mathbb{M}^{\mathrm{ban}}_{\mathcal{G}}(B,C) \to \mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A,C)$ is compatible with the respective homotopy relations, so it lifts to a product

$$\otimes_B\colon\ \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A,B) \times \mathrm{Mor}^{\mathrm{ban}}_{\mathcal{G}}(B,C) \to \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A,C).$$

This action of the Morita morphisms on $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}$ is biadditive, associative and compatible with pullback and pushout (compare Proposition 1.10.30). We can therefore conclude

Theorem 3.8.13. *Let A, B, C be non-degenerate $\mathcal{G}$-Banach algebras and let E be a $\mathcal{G}$-equivariant Morita equivalence between B and C. Then $\cdot \otimes_B [E]$ is an isomorphism from $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A,B)$ to $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A,C)$ with inverse $\cdot \otimes_B [\overline{E}]$.*

Chapter 4

$\mathcal{C}_0(X)$-Banach Spaces and Fields over X

Let X be a locally compact Hausdorff space. In the preceding two chapters we have defined two different but very similar notions: The $\mathcal{C}_0(X)$-Banach spaces and the upper semi-continuous fields of Banach spaces over X. We have also seen how these notions can be used to define categories of Banach algebras, Banach modules and Banach pairs, giving two constructions of a $\mathrm{KK}^{\mathrm{ban}}$-theory. The present chapter is dedicated to a comparison of these two points of view.

The central tools are two rather obvious functors $\mathfrak{M}$ and $\mathfrak{F}$: Given a u.s.c. field E of Banach spaces over X one can form the $\mathcal{C}_0(X)$-module $\Gamma_0(X, E)$ of sections vanishing at infinity; we call this module $\mathfrak{M}(E)$. On the other hand, if $\mathcal{E}$ is a $\mathcal{C}_0(X)$-Banach space, then there is a straight-forward notion of a fibre $\mathcal{E}_x$ over x for every point $x \in X$, and these fibres give a field of Banach spaces $(\mathcal{E}_x)_{x\in X}$ which we call $\mathfrak{F}(\mathcal{E})$. It is not hard to check that these functors $\mathfrak{M}$ and $\mathfrak{F}$ descent to the categories of Banach algebras, etc., and give homomorphisms on the level of $\mathrm{KK}^{\mathrm{ban}}$-theory (see Propositions 4.7.10 and 4.7.14).

The composition $\mathfrak{F} \circ \mathfrak{M}$ is naturally equivalent to the identity on the category of u.s.c. fields of Banach spaces over X. Unfortunately, the composition $\mathfrak{M} \circ \mathfrak{F}$ does not give back the original $\mathcal{C}_0(X)$-Banach spaces. We call this composition the Gelfand functor $\mathfrak{G}$. The $\mathcal{C}_0(X)$-Banach spaces which are "invariant" under $\mathfrak{G}$ can be characterised: They are the so-called locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces, a notion which is well-known in the literature.[1] We discuss this notion here and also in Appendix A where the hitherto unknown fact is proved that the projective tensor product over $\mathcal{C}_0(X)$ of two such spaces is again locally $\mathcal{C}_0(X)$-convex.

The main result of this chapter is, as one might have expected, that for locally $\mathcal{C}_0(X)$-convex Banach algebras one really can go back and forth between the two definitions of $\mathrm{KK}^{\mathrm{ban}}$-theory (see Theorem 4.7.20). This is not *completely* trivial because in the definition of $\mathrm{RKK}^{\mathrm{ban}}$, even for locally $\mathcal{C}_0(X)$-convex Banach algebras, the cycles that turn up do not have to be modeled on locally $\mathcal{C}_0(X)$-convex Banach spaces.

4.1 The functor $\mathfrak{M}$: from fields to $\mathcal{C}_0(X)$-Banach spaces

Definition 4.1.1 (The functor $\mathfrak{M}$). Let E be a u.s.c. field of Banach spaces over X. Then

$$\mathfrak{M}(E) := \Gamma_0(X, E)$$

[1] See [Gie82] and [KR89b].

is a $\mathcal{C}_0(X)$-Banach space with the pointwise product. If F is another u.s.c. field of Banach spaces over X and T is a bounded continuous field of linear maps from E to F, then the map

$$\mathfrak{M}(T) : \Gamma_0(X, E) \to \Gamma_0(X, F),\ \xi \mapsto (x \mapsto T_x(\xi(x))),$$

defines an element of $\mathrm{L}^{\mathcal{C}_0(X)}(\mathfrak{M}(E), \mathfrak{M}(F))$ such that $\|\mathfrak{M}(T)\| = \|T\|$.

Proof. We show the statement about the norm: Clearly, $\|\mathfrak{M}(T)\| \leq \|T\|$. To see the opposite inequality, let $\varepsilon > 0$. Then we can find an $x \in X$ such that $\|T_x\| \geq \|T\| - \varepsilon/2$. By definition of the operator norm there is an $e_x \in E_x$ such that $\|e_x\| < 1$ and $\|T(e_x)\| \geq \|T\| - \varepsilon$. Since the map $\xi \mapsto \xi(x)$ is a metric surjection from $\Gamma_0(X, E)$ to E_x, there is an $\xi \in \Gamma_0(X, E)$ such that $\|\xi\| \leq 1$ and $\xi(x) = e_x$. It follows that

$$\|\mathfrak{M}(T)\| \geq \|\mathfrak{M}(T)(\xi)\| \geq \|\mathfrak{M}(T)(\xi)(x)\| = \|T_x(\xi(x))\| = \|T_x(e_x)\| \geq \|T\| - \varepsilon. \qquad \square$$

Proposition 4.1.2. *$\mathfrak{M}$ is a functor from the category of u.s.c. fields of Banach spaces over X and bounded continuous fields of linear maps to the category of $\mathcal{C}_0(X)$-Banach spaces and bounded $\mathcal{C}_0(X)$-linear maps. It is linear and isometric on the morphism sets and compatible with the tensor products.*

Proof. That $\mathfrak{M}$ is a functor and linear on the morphism sets is straightforward to show. We already know that it is isometric. That it is compatible with the tensor products is surprisingly hard to show. This statement is actually equivalent to the statement, proved in Appendix A.2.4, that the $\mathcal{C}_0(X)$-tensor product of locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces is again locally $\mathcal{C}_0(X)$-convex. We show how multiplicativity of the functor $\mathfrak{M}$ follows from this fact, using some results and concepts from Appendix A:

Let E^1 and E^2 be u.s.c. fields of Banach spaces over X. We define a natural isomorphism $\mathfrak{m}^{\mathfrak{M}}_{E^1,E^2}$ from $\mathfrak{M}(E^1) \otimes^{\mathcal{C}_0(X)} \mathfrak{M}(E^2)$ to $\mathfrak{M}(E^1 \otimes_X E^2)$. For all $\xi_1 \in \Gamma_0(X, E^1)$ and $\xi_2 \in \Gamma_0(X, E^2)$ define

$$\mu(\xi_1, \xi_2)(x) := \xi_1(x) \otimes \xi_2(x) \in E^1_x \otimes E^2_x$$

for all $x \in X$. Then $\mu(\xi_1, \xi_2)$ is in $\Gamma_0\left(X,\ E^1 \otimes_X E^2\right)$ by definition of the sections on $E^1 \otimes_X E^2$. Moreover, μ is a contractive bilinear $\mathcal{C}_0(X)$-balanced map from $\Gamma_0(X, E^1) \times \Gamma_0(X, E^2)$ to $\Gamma_0\left(X,\ E^1 \otimes_X E^2\right)$. Hence we have a contractive linear map

$$\mathfrak{m}^{\mathfrak{M}}_{E^1,E^2} : \Gamma_0\left(X, E^1\right) \otimes^{\mathcal{C}_0(X)} \Gamma_0\left(X, E^2\right) \to \Gamma_0(X,\ E^1 \otimes_X E^2),\ \xi_1 \otimes \xi_2 \mapsto \mu(\xi_1, \xi_2). \tag{4.1}$$

This clearly is a natural transformation. Fibrewise (see 4.2.3), this map is an isometric isomorphism. In Proposition A.2.8 we will meet a criterion which tells us that $\mathfrak{m}^{\mathfrak{M}}_{E^1,E^2}$ is an isometric isomorphism if the left-hand side $\Gamma_0\left(X, E^1\right) \otimes^{\mathcal{C}_0(X)} \Gamma_0\left(X, E^2\right)$ is a so-called locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach space. This notion is defined in 4.4.1, and Theorem A.2.15 together with Example A.2.4 shows that the left-hand side is indeed locally $\mathcal{C}_0(X)$-convex. $\square$

We can exploit the fact that $\mathfrak{M}$ is multiplicative, i.e., that it commutes with the tensor product, to define $\mathfrak{M}(\mu)$ for bounded continuous fields μ of multilinear maps. However, it is more natural to define $\mathfrak{M}(\mu)$ directly.

Definition 4.1.3. Let E^1, E^2 and F be u.s.c. fields of Banach spaces over X and let $\mu\colon E^1 \times_X E^2 \to F$ be a bounded continuous field of bilinear maps. Then the map

$$\mathfrak{M}(\mu) : \mathfrak{M}\left(E^1\right) \times \mathfrak{M}\left(E^1\right) \to \mathfrak{M}(F),\ (\xi_1, \xi_2) \mapsto (x \mapsto \mu_x(\xi_1(x), \xi_2(x))$$

is $\mathcal{C}_0(X)$-bilinear and bounded by $\|\mu\|$.

Proposition 4.1.4. *Let E^1, E^2 and F be u.s.c. fields of Banach spaces over X and let $\mu\colon E^1 \times_X E^2 \to F$ be a bounded continuous field of bilinear maps. If we identify $\mathfrak{M}\left(E^1\right) \otimes_{\mathcal{C}_0(X)} \mathfrak{M}\left(E^2\right)$ and $\mathfrak{M}\left(E^1 \otimes_X E^2\right)$, then*

$$\widehat{\mathfrak{M}(\mu)} = \mathfrak{M}\left(\widehat{\mu}\right) : \mathfrak{M}\left(E^1\right) \otimes^{\mathcal{C}_0(X)} \mathfrak{M}\left(E^2\right) \to \mathfrak{M}\left(F\right).$$

Corollary 4.1.5. *Associativity of bilinear maps is preserved under $\mathfrak{M}$.*

A precise statement of how associativity is preserved can be obtained by adopting Proposition 3.3.13 (which says the same for another functor).

4.2 The functor $\mathfrak{F}$: from $\mathcal{C}_0(X)$-Banach spaces to fields

4.2.1 Fibres

Definition 4.2.1 (The fibres of a $\mathcal{C}_0(X)$-Banach space). Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space and $x \in X$. Regard $\mathcal{C}_0(X \setminus \{x\})$ as the closed subalgebra of $\mathcal{C}_0(X)$ of functions vanishing at x. Then $\mathcal{C}_0(X \setminus \{x\})\mathcal{E}$ is a closed subspace of $\mathcal{E}$. Define the fibre $\mathcal{E}_x$ of $\mathcal{E}$ at x to be the quotient Banach space

$$\mathcal{E}_x := \mathcal{E} / \left(\mathcal{C}_0(X \setminus \{x\})\mathcal{E}\right).$$

For all $e \in \mathcal{E}$ we will denote by e_x the corresponding element of the fibre $\mathcal{E}_x$. The canonical projection map from $\mathcal{E}$ onto $\mathcal{E}_x$ will be denoted by $\pi_x^{\mathcal{E}}$ or just by π_x, if the space $\mathcal{E}$ is understood.

The construction and the properties of the fibres of a $\mathcal{C}_0(X)$-Banach space, being a special case of the restriction of a $\mathcal{C}_0(X)$-Banach space to a closed subset of X, is discussed in Appendix A.1. There the following propositions and examples are proved, most of them for the restriction on arbitrary closed subsets $V \subseteq X$ instead of a single point $x \in X$.

Example 4.2.2. Let E be a Banach space. Then $\mathcal{E} := \mathcal{C}_0(X, E)$ is a $\mathcal{C}_0(X)$-Banach space and $\mathcal{E}_x \cong E$ for all $x \in X$. The same is true for $\mathcal{E}' := \mathcal{C}_0(X) \otimes^{\pi} E$.

Definition and Proposition 4.2.3. Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{C}_0(X)$-Banach spaces and $T \in \mathrm{L}^{\mathcal{C}_0(X)}(\mathcal{E}, \mathcal{F})$. Then there is a unique linear operator $T_x \in \mathrm{L}\left(\mathcal{E}_x, \mathcal{F}_x\right)$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "T"] \arrow[d, "\pi_x^{\mathcal{E}}"'] & \mathcal{F} \arrow[d, "\pi_x^{\mathcal{F}}"] \\
\mathcal{E}_x \arrow[r, "T_x"] & \mathcal{F}_x
\end{tikzcd}$$

It satisfies $\|T_x\| \le \|T\|$.

Proposition 4.2.4. *The maps $\mathcal{E} \mapsto \mathcal{E}_x$ and $T \mapsto T_x$ define a functor from the category of $\mathcal{C}_0(X)$-Banach spaces to the category of Banach spaces, linear and contractive on the morphism sets and respecting the tensor product. The maps $\pi_x^{\mathcal{E}}\colon \mathcal{E} \to \mathcal{E}_x$ define a natural transformation if we consider the category of Banach spaces as a subcategory of the category of $\mathcal{C}_0(X)$-Banach spaces.*

Here "respecting the tensor product" means: If $\mathcal{E}_1$ and $\mathcal{E}_2$ are $\mathcal{C}_0(X)$-Banach spaces, then for every $x \in X$ there is a natural isometric isomorphism

$$\left(\mathcal{E}_1 \otimes^{\mathcal{C}_0(X)} \mathcal{E}_2\right)_x \cong (\mathcal{E}_1)_x \otimes (\mathcal{E}_2)_x.$$

There is also the notion of the fibre of bilinear maps: If $\mu\colon \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{F}$ is a $\mathcal{C}_0(X)$-bilinear continuous map between $\mathcal{C}_0(X)$-Banach spaces, then $\mu_x\colon (\mathcal{E}_1)_x \times (\mathcal{E}_2)_x \to (\mathcal{F})_x$ is a bilinear continuous map such that $\mu_x((e_1)_x, (e_2)_x) = (\mu(e_1, e_2))_x$ for all $e_1 \in \mathcal{E}_1$ and $e_2 \in \mathcal{E}_2$.

Proposition 4.2.5. *Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{C}_0(X)$-Banach spaces and $T \in \mathrm{L}^{\mathcal{C}_0(X)}(\mathcal{E}, \mathcal{F})$. Let $x \in X$.*

1. *If T is isometric, then also T_x is isometric.*
2. *If T is surjective and a quotient map, then so is T_x.*
3. *If T has dense image, then so has T_x.*
4. *If T is an isometric isomorphism, then so is T_x.*

4.2.2 Definition of the functor $\mathfrak{F}$

The following lemma is a special case of Lemma A.1.6, the analogous result for the restriction to closed subsets.

Lemma 4.2.6. *Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. For every $x \in X$ and every $e \in \mathcal{E}$, we have*

$$\|e_x\| = \inf\left\{\|\varphi e\| :\ \varphi \in \mathcal{C}_c(X)\ \exists U \subseteq X \textit{ open}:\ \varphi|_U = 1,\ 0 \le \varphi \le 1, x \in U\right\}.$$

We use this lemma to prove:

Proposition 4.2.7. *Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. Then for all $e \in \mathcal{E}$ the function $x \mapsto \|e_x\|$ is upper semi-continuous and vanishes at infinity.*

Proof. Let $e \in \mathcal{E}$.

Upper semi-continuity: Let $x \in X$. Let $\varepsilon > 0$. By Lemma 4.2.6 find a $\psi \in \mathcal{C}_c(X)$ such that ψ equals one on a neighbourhood U of x and such that $0 \le \psi \le 1$ and such that $\|\psi e\| \le \|e_x\| + \varepsilon$. Then for every $y \in U$, we have $\|e_y\| \le \|\psi e\| \le \|e_x\| + \varepsilon$.

Behaviour at infinity: Let $(\chi_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for $\mathcal{C}_0(X)$ such that all χ_λ have compact support. The $(\chi_\lambda)_{\lambda\in\Lambda}$ is also an approximate unit for $\mathcal{E}$. Let $\varepsilon > 0$. Find a $\lambda \in \Lambda$ such that $\|e - \chi_\lambda e\| \le \varepsilon$. Then for every $x \in X \setminus \operatorname{supp}\chi_\lambda$ we have

$$(e - \chi_\lambda e)_x = e_x - \chi_\lambda(x) e_x = e_x$$

and hence

$$\|e_x\| = \|(e - \chi_\lambda e)_x\| \le \|e - \chi_\lambda e\| \le \varepsilon. \qquad \square$$

Definition 4.2.8 (The functor $\mathfrak{F}$). Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. Then

$$\mathfrak{F}(\mathcal{E}) := (\mathcal{E}_x)_{x\in X}$$

is a u.s.c. field of Banach spaces over X if we define $\Gamma_0 := \{x \mapsto e_x :\ e \in \mathcal{E}\}$, noting that Γ_0 satisfies (C1) - (C3), and let the sections of $\mathfrak{F}(\mathcal{E})$ be defined by Γ_0 according to Proposition 3.1.26. If $\mathcal{F}$ is another $\mathcal{C}_0(X)$-Banach space and $T \in \mathrm{L}^{\mathcal{C}_0(X)}(\mathcal{E}, \mathcal{F})$, then we define

$$\mathfrak{F}(T) := (T_x)_{x\in X}\colon (\mathcal{E}_x)_{x\in X} \to (\mathcal{F}_x)_{x\in X}$$

Proposition 4.2.9. *$(T_x)_{x\in X}$ is a continuous field of linear maps from $\mathfrak{F}(\mathcal{E})$ to $\mathfrak{F}(\mathcal{F})$, bounded by $\|T\|$. Moreover, $\mathfrak{F}$ defines a contractive functor from the category of $\mathcal{C}_0(X)$-Banach spaces to the category of continuous fields of Banach spaces over X, linear and contractive on the morphism sets and compatible with the tensor product.*

Proof. $\mathfrak{F}(T)$ is certainly a family of linear maps from $\mathfrak{F}(\mathcal{E})$ to $\mathfrak{F}(\mathcal{F})$, bounded by $\|T\|$. To see that $\mathfrak{F}(T)$ is continuous we can appeal to Proposition 3.1.30 which says that it suffices that a total subset of the sections of $\mathfrak{F}(\mathcal{E})$ is mapped to the sections of $\mathfrak{F}(\mathcal{F})$. We check that for all $e\in\mathcal{E}$ the family $(e_x)_{x\in X}$ is taken to some section of $\mathfrak{F}(\mathcal{F})$; indeed $(\mathfrak{F}(T)\circ(e_x)_{x\in X})(x)=T_x(e_x)=(T(e))_x$ for all $x\in X$, so we get something in $\Gamma(X,\mathcal{F})$.

Compatibility with the tensor product: Let $\mathcal{E}^1$ and $\mathcal{E}^2$ be $\mathcal{C}_0(X)$-Banach spaces. We define a natural isometric isomorphism $\mathfrak{m}^{\mathfrak{F}}_{\mathcal{E}^1,\mathcal{E}^2}$ from $\mathfrak{F}\left(\mathcal{E}^1\right)\otimes_X\mathfrak{F}\left(\mathcal{E}^2\right)$ to $\mathfrak{F}\left(\mathcal{E}^1\otimes^{\mathcal{C}_0(X)}\mathcal{E}^2\right)$. For all $x\in X$, let $\mathfrak{m}^{\mathfrak{F}}_{\mathcal{E}^1,\mathcal{E}^2,x}$ denote the natural isomorphism from $\mathcal{E}^1_x\otimes\mathcal{E}^2_x$ to $(\mathcal{E}^1\otimes^{\mathcal{C}_0(X)}\mathcal{E}^2)_x$. If $e^1\in\mathcal{E}^1$ and $e^2\in\mathcal{E}^2$, then this isomorphism by definition sends $e^1_x\otimes e^2_x$ to $(e^1\otimes e^2)_x$. Consider the family

$$\mathfrak{m}^{\mathfrak{F}}_{\mathcal{E}^1,\mathcal{E}^2}:=\left(\mathfrak{m}^{\mathfrak{F}}_{\mathcal{E}^1,\mathcal{E}^2,x}\right)_{x\in X}. \tag{4.2}$$

It is a family of isometric isomorphisms from $\mathfrak{F}\left(\mathcal{E}^1\right)\otimes_X\mathfrak{F}\left(\mathcal{E}^2\right)$ to $\mathfrak{F}\left(\mathcal{E}_1\otimes^{\mathcal{C}_0(X)}\mathcal{E}^2\right)$. It is continuous because it sends $x\mapsto e^1_x\otimes e^2_x$ to $x\mapsto(e^1\otimes e^2)_x$ which is an element of $\Gamma_0\left(X,\ \mathfrak{F}\left(\mathcal{E}^1\otimes^{\mathcal{C}_0(X)}\mathcal{E}^2\right)\right)$. □

Bilinear maps

Definition 4.2.10. Let $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{F}$ be $\mathcal{C}_0(X)$-Banach spaces. Let $\mu\in\mathrm{M}^{\mathcal{C}_0(X)}(\mathcal{E}_1,\mathcal{E}_2;\mathcal{F})$ be a continuous $\mathcal{C}_0(X)$-bilinear map. Define $\mathfrak{F}(\mu):=(\mu_x)_{x\in X}$, where $\mu_x\colon(\mathcal{E}_1)_x\times(\mathcal{E}_2)_x\to\mathcal{F}_x$. We have $\|\mu_x\|\le\|\mu\|$. Then $\mathfrak{F}(\mu)$ is a continuous field of bilinear maps[2] from $\mathfrak{F}(\mathcal{E}_1)\times_X\mathfrak{F}(\mathcal{E}_2)$ to $\mathfrak{F}(\mathcal{F})$, bounded by $\|\mu\|$.

Proof. For all $x\in X$, we have $\|\mu_x\|\le\|\mu\|$, so $\mathfrak{F}(\mu)$ is bounded by $\|\mu\|$. Let $e_1\in\mathcal{E}_1$ and $e_2\in\mathcal{E}_2$. Then $\xi\colon x\mapsto((e_1)_x,(e_2)_x)$ is a section of the internal product $\mathfrak{F}(\mathcal{E}_1)\times_X\mathfrak{F}(\mathcal{E}_2)$, and it suffices to check $\mathfrak{F}(\mu)\circ\xi\in\Gamma(X,\mathfrak{F}(\mathcal{F}))$ for such a section ξ. We have

$$(\mathfrak{F}(\mu)\circ\xi)(x)=\mathfrak{F}(\mu)((e_1)_x,(e_2)_x)=\mu(e_1,e_2)_x$$

for all $x\in X$, so $\mathfrak{F}(\mu)\circ\xi=\mathfrak{g}_{\mathcal{F}}(\mu(e_1,e_2))\in\Gamma_0(X,\mathfrak{F}(\mathcal{F}))$. □

Proposition 4.2.11. *Let $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{F}$ be $\mathcal{C}_0(X)$-Banach spaces. Let $\mu\in\mathrm{M}^{\mathcal{C}_0(X)}(\mathcal{E}_1,\mathcal{E}_2;\mathcal{F})$. Then under the identification $\mathfrak{F}(\mathcal{E}_1)\times_X\mathfrak{F}(\mathcal{E}_2)=\mathfrak{F}\left(\mathcal{E}_1\otimes^{\mathcal{C}_0(X)}\mathcal{E}_2\right)$ we have*

$$\mathfrak{F}(\hat{\mu})=\widehat{\mathfrak{F}(\mu)}$$

where $\hat{\mu}$ and $\widehat{\mathfrak{F}(\mu)}$ are the linearisations of μ and $\mathfrak{F}(\mu)$, respectively, i.e., the following diagram commutes

$$\begin{tikzcd}
\mathfrak{F}\left(\mathcal{E}_1\otimes^{\mathcal{C}_0(X)}\mathcal{E}_2\right) \arrow[rr, "\cong"] \arrow[dr, "\mathfrak{F}(\hat{\mu})"'] & & \mathfrak{F}(\mathcal{E}_1)\otimes_X\mathfrak{F}(\mathcal{E}_2) \arrow[dl, "\widehat{\mathfrak{F}(\mu)}"] \\
& \mathfrak{F}(\mathcal{F}) &
\end{tikzcd}$$

[2]See Definition 3.1.34.

Proof. Let $e_1 \in \mathcal{E}_1$, $e_2 \in \mathcal{E}_2$ and $x \in X$. We trace the element $(e_1 \otimes e_2)_x$ through the above triangle. It is mapped to $\mu(e_1 \otimes e_2)_x$ by $\mathfrak{F}(\hat{\mu})$ and it corresponds to $(e_1)_x \otimes (e_2)_x$ in the upper right corner. This element is also mapped to $\mu_x((e_1)_x \otimes (e_2)_x) = \mu(e_1 \otimes e_2)_x$. $\square$

Corollary 4.2.12. *Associativity of bilinear maps is preserved under $\mathfrak{F}$.*

4.3 The compositions of $\mathfrak{F}$ and $\mathfrak{M}$ (and the Gelfand functor)

4.3.1 What is $\mathfrak{F}(\mathfrak{M}(E))$?

Theorem 4.3.1. *The functor $E \mapsto \mathfrak{F}(\mathfrak{M}(E))$ is equivalent to the identity functor on the category of u.s.c. fields of Banach spaces; the natural isomorphism between these functors is linear and isometric and compatible with the tensor product.*

Proof. **The definition of the transformation:** Let E be a u.s.c. field of Banach spaces over X. We show that for all $x \in X$ the map $\mathrm{ev}_x \colon \Gamma_0(X, E) \to E_x,\ \xi \mapsto \xi(x)$ induces an isometric isomorphism $J^E_x \colon \mathfrak{M}(E)_x \to E_x$ and that $(J^E_x)_{x \in X}$ is an isometric continuous field of isomorphisms from $\mathfrak{F}(\mathfrak{M}(E))$ onto E.

That ev_x is a metric surjection follows from Lemma 3.1.25. Its kernel is given by $\mathcal{C}_0(X \setminus \{0\})\Gamma_0(X, E)$: This set is certainly contained in the kernel. On the other hand, the kernel is a non-degenerate $\mathcal{C}_0(X \setminus \{0\})$-module, so every element of the kernel can be factorized into an element of $\mathcal{C}_0(X \setminus \{0\})$ and an element of the kernel. So it is obviously contained in $\mathcal{C}_0(X \setminus \{0\})\Gamma_0(X, E)$.

J^E is hence a family of isometric linear isomorphisms. If $\xi \in \Gamma_0(X, E)$, then we have to check that $(\xi_x)_{x \in X}$, as a section in $\mathfrak{F}(\mathfrak{M}(E))$, is mapped to a section of E. Indeed $\mathrm{ev}_x(\xi) = \xi(x)$ and hence $J^E_x(\xi_x) = \xi(x)$ for all $x \in X$. In other words, $(\xi_x)_{x \in X}$ is mapped to ξ, so J^E is a continuous field of linear maps.

Naturality: To see that $E \mapsto J^E$ is natural let E and F be u.s.c. fields of Banach spaces over X and let T be a bounded continuous field of continuous linear maps from E to F. Then $\mathfrak{F}(\mathfrak{M}(T))_x$ sends $\xi_x \in \Gamma_0(X, E)_x$ to $(T \circ \xi)_x \in \Gamma_0(X, F)_x$ for all $x \in X$ and $\xi \in \Gamma_0(X, E)$. But then

$$T_x\left(J^E_x(\xi_x)\right) = T_x(\xi(x)) = (T \circ \xi)(x) = J^F_x\left((T \circ \xi)_x\right) = J^F_x\left(\mathfrak{F}(\mathfrak{M}(T))(\xi_x)\right),$$

so $T \circ J^E = J^F \circ \mathfrak{F}(\mathfrak{M}(T))$. Hence J is natural.

Compatibility with the tensor product: We show that the following diagram commutes for all u.s.c. fields of Banach spaces E_1 and E_2 over X:

$$\begin{tikzcd}
\mathfrak{F}(\mathfrak{M}(E_1)) \otimes_X \mathfrak{F}(\mathfrak{M}(E_2)) \arrow[r, "\cong"] \arrow[d, "J^{E_1} \otimes J^{E_2}"] & \mathfrak{F}\left(\mathfrak{M}(E_1) \otimes^{\mathcal{C}_0(X)} \mathfrak{M}(E_2)\right) \arrow[r, "\cong"] & \mathfrak{F}(\mathfrak{M}(E_1 \otimes_X E_2)) \arrow[d, "J^{E_1 \otimes E_2}"] \\
E_1 \otimes_X E_2 \arrow[rr, "="] & & E_1 \otimes_X E_2
\end{tikzcd}$$

In the fibre over $x \in X$ this means

$$\begin{tikzcd}
\mathfrak{M}(E_1)_x \otimes \mathfrak{M}(E_2)_x \arrow[r, "\cong"] \arrow[d, "J_x^{E_1} \otimes J_x^{E_2}"] & \left(\mathfrak{M}(E_1) \otimes^{\mathcal{C}_0(X)} \mathfrak{M}(E_2)\right)_x \arrow[r, "\cong"] & \mathfrak{M}(E_1 \otimes_X E_2)_x \arrow[d, "J_x^{E_1 \otimes E_2}"] \\
E_{1,x} \otimes E_{2,x} \arrow[rr, "="] & & E_{1,x} \otimes E_{2,x}
\end{tikzcd}$$

This diagram is commutative: if $\xi_1 \in \Gamma_0(X, E_1)$ and $\xi_2 \in \Gamma_0(X, E_2)$, then

$$\begin{tikzcd}
(\xi_1)_x \otimes (\xi_2)_x \arrow[r, mapsto] \arrow[d, mapsto, "J_x^{E_1} \otimes J_x^{E_2}"] & (\xi_1 \otimes \xi_2)_x \arrow[r, mapsto] & (y \mapsto \xi_1(y) \otimes \xi_2(y))_x \arrow[d, mapsto, "J_x^{E_1 \otimes E_2}"] \\
\xi_1(x) \otimes \xi_2(x) \arrow[rr, mapsto, "="] & & \xi_1(x) \otimes \xi_2(x)
\end{tikzcd}$$

□

4.3.2 What is $\mathfrak{M}(\mathfrak{F}(\mathcal{E}))$?

Definition 4.3.2 (The Gelfand functor). Define the functor $\mathfrak{G} := \mathfrak{M} \circ \mathfrak{F}$, which is called the *Gelfand functor*.

There is a natural transformation from the identity functor on the category of $\mathcal{C}_0(X)$-Banach spaces to the Gelfand functor. It is defined as follows:

Definition 4.3.3 (The Gelfand transformation). For all $\mathcal{C}_0(X)$-Banach spaces $\mathcal{E}$ define a map $\mathfrak{g}_\mathcal{E}$ from $\mathcal{E}$ to $\mathfrak{G}(\mathcal{E})$ by $\mathfrak{g}_\mathcal{E}(e) = (e_x)_{x \in X}$ for all $e \in \mathcal{E}$. We will call this map the *Gelfand transformation* of $\mathcal{E}$.

Proposition 4.3.4 (Properties of the Gelfand transformation). $\mathfrak{g}$ *is a natural transformation from the identity functor to* $\mathfrak{G}$*; it is linear and contractive and compatible with the tensor product. Moreover, for all* $\mathcal{C}_0(X)$*-Banach spaces* $\mathcal{E}$ *the space* $\mathfrak{g}_\mathcal{E}(\mathcal{E})$ *is dense in* $\mathfrak{G}(\mathcal{E})$ *so* $\mathfrak{G}(\mathcal{E})$ *can be considered to be the Hausdorff-completion of* $\mathcal{E}$ *with respect to the semi-norm* $e \mapsto \sup_{x \in X} \|e_x\|$.

Proof. First of all, $\mathfrak{g}_\mathcal{E}$ is surely an element of $\mathrm{L}^{\mathcal{C}_0(X)}(\mathcal{E}, \mathfrak{G}(\mathcal{E}))$ of norm ≤ 1.

1. $\mathfrak{g}$ **is a natural transformation:** If $\mathcal{F}$ is another $\mathcal{C}_0(X)$-Banach space and $T \in \mathrm{L}^{\mathcal{C}_0(X)}(\mathcal{E}, \mathcal{F})$, then for all $e \in \mathcal{E}$:

   $$\mathfrak{G}(T)\left(\mathfrak{g}_\mathcal{E}(e)\right) = \mathfrak{M}\left(\mathfrak{F}(T)\right)(e_x)_{x \in X} = (T_x e_x)_{x \in X} = (T(e)_x)_{x \in X} = \mathfrak{g}_\mathcal{F}\left(T(e)\right),$$

   so $\mathfrak{g}$ is a natural transformation.

2. $\mathfrak{G}(\mathcal{E})$ **is a completion of** $\mathcal{E}$**:**

   We have to show that $\Gamma_0(X, \mathfrak{F}(\mathcal{E}))$ is a completion of $\Gamma_0 := \{\mathfrak{g}_\mathcal{E}(e) : e \in \mathcal{E}\}$ in the sup-norm. Now $\Gamma_0(X, \mathfrak{F}(\mathcal{E}))$ is a Banach space containing Γ_0, so we just have to check that this subspace is dense for the sup-norm. Let $\xi \in \Gamma_0\left(X, \mathfrak{F}\left(\mathcal{E}\right)\right)$. Without loss of generality let ξ have compact support. Let $\varepsilon > 0$. For all $x \in X$ we can find a neighbourhood U_x of x in X and an $e^x \in \mathcal{E}$ such that $\|\mathfrak{g}_{\mathcal{E}}(e^x)(y) - \xi(y)\| < \varepsilon$ for all $y \in U_x$. Now the support K of ξ is compact. Hence we can find a finite set $S \subseteq K$ such that $K \subseteq \bigcup_{s\in S} U_s$. Find a continuous partition of unity $(\varphi_s)_{s\in S}$ on K in X subordinate to the cover $(U_s)_{s\in S}$. Define

$$\eta := \sum_{s\in S} \varphi_s \mathfrak{g}_E\left(e^s\right) = \sum_{s\in S} \mathfrak{g}_E\left(\varphi_s e^s\right) \in \Gamma_0.$$

Then $\|\xi - \eta\|_\infty \leq \varepsilon$.

3. $\mathfrak{g}$ **is compatible with the tensor product:** Let $\mathcal{E}^1$ and $\mathcal{E}^2$ be $\mathcal{C}_0(X)$-Banach spaces. We define a natural isometric isomorphism $\mathfrak{m}^{\mathfrak{G}}_{\mathcal{E}^1,\mathcal{E}^2}$ making the following diagram commutative:

$$\begin{tikzcd}
\mathrm{Id}(\mathcal{E}^1) \otimes^{\mathcal{C}_0(X)} \mathrm{Id}(\mathcal{E}^2) \arrow[r, "="] \arrow[d, "\mathfrak{g}_{\mathcal{E}^1} \otimes \mathfrak{g}_{\mathcal{E}^2}"'] & \mathrm{Id}(\mathcal{E}^1 \otimes^{\mathcal{C}_0(X)} \mathcal{E}^2) \arrow[d, "\mathfrak{g}_{\mathcal{E}^1 \otimes \mathcal{E}^2}"] \\
\mathfrak{G}\big(\mathcal{E}^1\big) \otimes^{\mathcal{C}_0(X)} \mathfrak{G}\big(\mathcal{E}^2\big) \arrow[r, "\mathfrak{m}^{\mathfrak{G}}_{\mathcal{E}^1,\mathcal{E}^2}"] & \mathfrak{G}\big(\mathcal{E}^1 \otimes^{\mathcal{C}_0(X)} \mathcal{E}^2\big)
\end{tikzcd}$$

It is given as the composition

$$\mathfrak{m}^{\mathfrak{G}}_{\mathcal{E}^1,\mathcal{E}^2} := \mathfrak{M}\left(\mathfrak{m}^{\mathfrak{F}}_{\mathcal{E}^1,\mathcal{E}^2}\right) \circ \mathfrak{m}^{\mathfrak{M}}_{\mathfrak{F}(\mathcal{E}^1),\mathfrak{F}(\mathcal{E}^2)},$$

where the natural isomorphism

$$\mathfrak{m}^{\mathfrak{M}}_{\mathfrak{F}(\mathcal{E}^1),\mathfrak{F}(\mathcal{E}^2)} \colon \mathfrak{M}\left(\mathfrak{F}\left(\mathcal{E}^1\right)\right) \otimes^{\mathcal{C}_0(X)} \mathfrak{M}\left(\mathfrak{F}\left(\mathcal{E}^2\right)\right) \to \mathfrak{M}\left(\mathfrak{F}\left(\mathcal{E}^1\right) \otimes_X \mathfrak{F}\left(\mathcal{E}^2\right)\right)$$

is defined in Equation (4.1) in the proof of Proposition 4.1.2 and $\mathfrak{m}^{\mathfrak{F}}_{\mathcal{E}^1,\mathcal{E}^2} \colon \mathfrak{F}(\mathcal{E}^1) \otimes_X \mathfrak{F}(\mathcal{E}^2) \to \mathfrak{F}(\mathcal{E}^1 \otimes^{\mathcal{C}_0(X)} \mathcal{E}^2)$ is defined in (4.2) in the proof of Proposition 4.2.9.

Let $\xi^1 \in \mathcal{E}^1$ and $\xi^2 \in \mathcal{E}^2$. Then $\mathfrak{g}_{\mathcal{E}^1}(\xi^1) \colon x \mapsto \xi^1_x$ is an element of $\mathfrak{G}\big(\mathcal{E}^1\big) = \mathfrak{M}\left(\mathfrak{F}\left(\mathcal{E}^1\right)\right)$ and $\mathfrak{g}_{\mathcal{E}^2}(\xi^2) \colon x \mapsto \xi^2_x$ is in $\mathfrak{G}\big(\mathcal{E}^2\big) = \mathfrak{M}\left(\mathfrak{F}\left(\mathcal{E}^2\right)\right)$. The map $\mathfrak{m}^{\mathfrak{M}}_{\mathfrak{F}(\mathcal{E}^1),\mathfrak{F}(\mathcal{E}^2)}$ sends $\mathfrak{g}_{\mathcal{E}^1}\left(\xi^1\right) \otimes \mathfrak{g}_{\mathcal{E}^2}\left(\xi^2\right)$ to $x \mapsto \xi^1_x \otimes \xi^2_x$. And $\mathfrak{M}(\mathfrak{m}^{\mathfrak{F}}_{\mathcal{E}^1,\mathcal{E}^2})$ sends $x \mapsto \xi^1_x \otimes \xi^2_x$ to $x \mapsto (\xi^1 \otimes \xi^2)_x$ which happens to be $\mathfrak{g}_{\mathcal{E}^1 \otimes \mathcal{E}^2}\left(\xi^1 \otimes \xi^2\right)$. Together, this means

$$\mathfrak{m}^{\mathfrak{G}}_{\mathcal{E}^1,\mathcal{E}^2}\left(\mathfrak{g}_{\mathcal{E}^1}\left(\xi^1\right) \otimes \mathfrak{g}_{\mathcal{E}^2}\left(\xi^2\right)\right) = \mathfrak{g}_{\mathcal{E}^1 \otimes \mathcal{E}^2}\left(\xi^1 \otimes \xi^2\right).$$

Since the set of all elements of the form $\mathfrak{g}_{\mathcal{E}^1}\left(\xi^1\right) \otimes \mathfrak{g}_{\mathcal{E}^2}\left(\xi^2\right)$ spans a dense subset of the tensor product $\mathfrak{G}\big(\mathcal{E}^1\big) \otimes^{\mathcal{C}_0(X)} \mathfrak{G}\big(\mathcal{E}^2\big)$, we can conclude that the above square commutes. $\square$

The following example shows, in a rather dramatic case, that $\mathfrak{G}$ is not isomorphic to the identity, i.e., that $\mathfrak{F}$ and $\mathfrak{M}$ are not inverses of each other.

Example 4.3.5. Let X be the unit interval $[0,1]$ and $\mathcal{E} := \mathrm{L}^1(X, \mathbb{C}, \lambda)$, where λ denotes the Lebesgue measure on $[0,1]$. Then $\mathcal{E}$ is a $\mathcal{C}_0(X)$-Banach space with the $\mathcal{C}_0(X)$-module action given by pointwise multiplication. Now $\mathcal{E}_x = 0$ for all $x \in [0,1]$ and hence $\mathfrak{G}(\mathcal{E}) = 0$ and $\mathfrak{g}_{\mathcal{E}} = 0$, so in particular $\mathfrak{g}_{\mathcal{E}}$ is not an isomorphism.

Proof. Let f be an L^1-function on $[0,1]$. In order to show $f_x = 0$ it suffices to consider the case that f is bounded since the bounded L^1-functions are dense in L^1. W.l.o.g. let f be bounded by 1. Let $\varepsilon > 0$. Find an open neighbourhood U of x with measure less then ε. Find a continuous function $\chi \in \mathcal{C}[0,1]$ such that $\chi(x) = 1$, $0 \leq \chi \leq 1$ and $\chi \equiv 0$ outside U. Then $f_x = (\chi f)_x$ and $\|\chi f\|_1 = \int_{[0,1]} |\chi(t) f(t)| \,\mathrm{d}t \leq \|\chi_1\| \leq \varepsilon$. So $\|f_x\| \leq \varepsilon$ for all $\varepsilon > 0$ and hence $f_x = 0$. □

4.4 Locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces

The following notion is discussed extensively in Appendix A.

Definition 4.4.1 (Locally $\mathcal{C}_0(X)$-convex). Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. Then $\mathcal{E}$ is called *locally $\mathcal{C}_0(X)$-convex* if $\|e\| = \sup_{x \in X} \|e_x\|$ for all $e \in \mathcal{E}$, i.e., if the Gelfand transformation is isometric.

In [DG83], Theorem 2.5., it is shown that $\mathcal{E}$ is locally $\mathcal{C}_0(X)$-convex if and only if

$$\forall \chi_1, \chi_2 \in \mathcal{C}_0(X), \chi_1, \chi_2 \geq 0, \chi_1 + \chi_2 \leq 1 \; \forall e_1, e_2 \in \mathcal{E}: \; \|\chi_1 e_2 + \chi_2 e_2\| \leq \max\{\|e_1\|, \|e_2\|\}$$

which justifies the name.

For all $\mathcal{C}_0(X)$-Banach spaces $\mathcal{E}$, locally $\mathcal{C}_0(X)$-convex or not, the $\mathcal{C}_0(X)$-Banach space $\mathfrak{G}(\mathcal{E})$ is locally $\mathcal{C}_0(X)$-convex, and applying the Gelfand functor twice does not change anything anymore. So we can regard the Gelfand functor as a projection functor to the subcategory of locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces (a so-called reflector).

This shows that the category of locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces is isomorphic to the category of u.s.c. fields of Banach spaces over the locally compact Hausdorff space X (with the bounded continuous fields of linear maps as morphisms).

Closed subspaces, *quotients* and *finite products* of locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces are again locally $\mathcal{C}_0(X)$-convex. The same is true for the $\mathcal{C}_0(X)$-*tensor product*, but this seems to be much harder to prove (see Appendix A for the details). Note that it follows that the balanced $\mathcal{C}_0(X)$-tensor product of locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach modules is also locally $\mathcal{C}_0(X)$-convex because it is a quotient of the $\mathcal{C}_0(X)$-tensor product.

It is worth mentioning that the *sum* of locally $\mathcal{C}_0(X)$-convex spaces needs not be locally $\mathcal{C}_0(X)$-convex. However, the Gelfand functor applied to the ordinary sum of two locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces is the (abstract) sum in the category of locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces. This just means switching to an equivalent norm:

Definition 4.4.2 (The locally $\mathcal{C}_0(X)$-convex sum). Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces. Then we define the *locally $\mathcal{C}_0(X)$-convex sum* $\mathcal{E}_1 \oplus^{\text{l.c.}} \mathcal{E}_2$ of $\mathcal{E}_1$ and $\mathcal{E}_2$ to be the ordinary sum $\mathcal{E}_1 \oplus \mathcal{E}_2$ of $\mathcal{C}_0(X)$-Banach spaces with the new norm

$$\|(e_1, e_2)\| := \sup_{x \in X} \left(\|(e_1)_x\| + \|(e_2)_x\|\right)$$

for all $(e_1, e_2) \in \mathcal{E}_1 \oplus \mathcal{E}_2$.

We also adjust the definition of the unitalisation of a $\mathcal{C}_0(X)$-Banach algebra to the locally $\mathcal{C}_0(X)$-convex setting:

Definition 4.4.3 (The locally $\mathcal{C}_0(X)$-convex unitalisation). Let $\mathcal{B}$ be a locally $\mathcal{C}_0(X)$-convex G-$\mathcal{C}_0(X)$-Banach algebra. Then we define the *locally $\mathcal{C}_0(X)$-convex unitalisation* of $\mathcal{B}$ to be $\mathcal{B} \oplus^{\text{l.c.}} \mathcal{C}_0(X)$ which is a fibrewise unital, locally $\mathcal{C}_0(X)$-convex G-$\mathcal{C}_0(X)$-Banach algebra in a canonical way.

In the following we are going to take the locally $\mathcal{C}_0(X)$-convex unitalisation whenever it is necessary without further mentioning it.

4.5 Group actions and gradings

As gradings can be regarded as a special case of group actions, we wont discuss gradings explicitly; it is obvious that gradings of fields of Banach spaces and gradings of $\mathcal{C}_0(X)$-Banach spaces are interchanged by the functors $\mathfrak{F}$ and $\mathfrak{M}$.

Let G be a locally compact Hausdorff group acting continuously on X. Let $G \ltimes X$ be the transformation groupoid (see Definition 6.1.3). We identify[3] $G \ltimes X$ with $G \times X$ in such a way that $r_{G\ltimes X}(g,x) = x$, $s_{G\ltimes X}(g,x) = g^{-1}x$ and $(g,x)(g',x') = (gg',x)$ for all $(g,x),(g',x') \in G \times X$ such that $x' = g^{-1}x$. We write r and s for $r_{G\ltimes X}$ and $s_{G\ltimes X}$, respectively.

4.5.1 Group actions and $\mathfrak{M}$

Definition and Proposition 4.5.1. Let E be a $G \ltimes X$-Banach space (being in particular a u.s.c. field of Banach spaces over X) with action $\alpha\colon s^*E \to r^*E$. For all $g \in G$ and all $\xi \in \mathfrak{M}(E) = \Gamma_0(X,E)$ define

$$(g\xi)(x) := [\alpha \circ (\xi \circ s)]\,(g,x) = \alpha_{(g,x)}\xi(g^{-1}x), \quad x \in X.$$

Then $g\xi \in \mathfrak{M}(E) = \Gamma_0(X,E)$ and $g \mapsto g\xi$ is continuous for all $\xi \in \Gamma_0(X,E)$. This defines the structure of a G-$\mathcal{C}_0(X)$-Banach space on $\mathfrak{M}(E)$.

Proof. $g\xi$ **is in** $\Gamma_0(X,E)$**:** Let $g \in G$ and $\xi \in \Gamma_0(X,E)$. The continuous map $\varphi_g\colon X \to G\ltimes X,\ x \mapsto (g,x)$ satisfies $r \circ \varphi_g = \mathrm{Id}_X$. Now $\xi \circ s \in \Gamma(G \ltimes X,\ s^*E)$ and hence $\alpha \circ (\xi \circ s) \in \Gamma(G \ltimes X,\ r^*E)$. So $g\xi = [\alpha \circ (\xi \circ s)] \circ \varphi_g \in \Gamma(X,\ \varphi_g^* r^*E) = \Gamma(X,E)$. It is easy to see that $g\xi$ vanishes at infinity, i.e., that $g\xi \in \Gamma_0(X,E) = \mathfrak{M}(E)$.
$g \mapsto g\xi$ **defines an action of** G**:** Let $\xi \in \Gamma_0(X,E)$ and g and h be elements of G. Then

$$\begin{aligned} g(h\xi)(x) &= \alpha_{(g,x)}\left((h\xi)(g^{-1}x)\right) = \alpha_{(g,x)}\left(\alpha_{(h,g^{-1}x)}\xi(h^{-1}g^{-1}x)\right) \\ &= \alpha_{(g,x)\cdot(h,g^{-1}x)}\xi((gh)^{-1}x) = \alpha_{(gh,x)}\xi((gh)^{-1}x) = ((gh)\xi)(x) \end{aligned}$$

for all $x \in X$. Moreover, $e_G\xi = \xi$.
The action is continuous: We have $\alpha \circ (\xi \circ s) \in \Gamma_b\,(G \ltimes X,\ r^*E)$. Moreover, if $\chi \in \mathcal{C}_0\,(G)$, then it is easy to see that $\chi \cdot \alpha \circ (\xi \circ s) \in \Gamma_0\,(G \ltimes X,\ r^*E)$. So by Lemma 4.5.2, $g \mapsto \chi(g)g\xi$ is continuous with values in $\Gamma_0(X,E)$. Since this is true for all $\chi \in \mathcal{C}_0(G)$, also the map $g \mapsto g\xi$ is continuous.
Compatibility with the $\mathcal{C}_0(X)$**-action:** Let $\chi \in \mathcal{C}_0(X)$, $g \in G$ and $\xi \in \Gamma_0(X,E)$. Then

$$[g(\chi\xi)](x) = \alpha_{(g,x)}[(\chi\xi)(g^{-1}x)] = \chi(g^{-1}x)\,(g\xi)\,(x)$$

for all $x \in X$, so $g(\chi\xi) = (g\chi)(g\xi)$. □

By elementary means one can show:

Lemma 4.5.2. *Let Y and Y' be locally compact Hausdorff spaces and let $p\colon Y \times Y' \to Y$ and $p'\colon Y \times Y' \to Y'$ be the canonical projections. Let E be a u.s.c. field of Banach spaces over Y'. Then p'^*E is a u.s.c. field of Banach spaces over $Y \times Y'$. Let η be a selection of p'^*E. Then $\eta \in \Gamma_0\,(Y \times Y',\ p'^*E)$ if and only if $\eta_y\colon y' \mapsto \eta(y,y')$ is in $\Gamma_0\,(Y',E)$ for all $y \in Y$ and $y \mapsto \eta_y \in \mathcal{C}_0\,(Y,\ \Gamma_0\,(Y',E))$. Moreover, $\eta \mapsto (y \mapsto \eta_y)$ is an isometric linear isomorphism from $\Gamma_0\,(Y \times Y',\ p'^*E)$ to $\mathcal{C}_0\,(Y,\ \Gamma_0\,(Y',E))$.*

[3] There is another, equivalent way to identify $G\ltimes X$ and $G\times X$ which differs from our convention by the homeomorphism $G \times X \to G \times X,\ (g,x) \mapsto (g, g^{-1}x)$; for technical reasons we prefer our identification.

Proposition 4.5.3. *Let E and F be $G \ltimes X$-Banach spaces and let $T\colon E \to F$ be a bounded equivariant continuous field of linear maps. Then $\mathfrak{M}(T) : \mathfrak{M}(E) \to \mathfrak{M}(F)$ is G-equivariant for the G-actions on $\mathfrak{M}(E)$ and $\mathfrak{M}(F)$ defined above.*

Proof. Write α and β for the $G \ltimes X$-actions on E and on F, respectively. Let $\xi \in \mathfrak{M}(E) = \Gamma_0(X, E)$ and $g \in G$. Then

$$\begin{aligned} [\mathfrak{M}(T)(g\xi)](x) &= T_x((g\xi)(x)) = T_x\left(\alpha_{(g,x)}\xi(g^{-1}x)\right) = \beta_{(g,x)}\left(T_{g^{-1}x}\xi(g^{-1}x)\right) \\ &= \beta_{(g,x)}\left((\mathfrak{M}(T)(\xi))(g^{-1}x)\right) = (g(\mathfrak{M}(T)(\xi)))(x) \end{aligned}$$

for all $x \in X$, so $\mathfrak{M}(T)(g\xi) = g(\mathfrak{M}(T)(\xi))$. Hence $\mathfrak{M}(T)$ is G-equivariant. □

Similarly one proves that $\mathfrak{M}(\mu)$ is G-equivariant if μ is a bounded $G \ltimes X$-equivariant continuous field of bilinear maps. Moreover, if E_1 and E_2 are $G \ltimes X$-Banach spaces and if we identify $\mathfrak{M}(E_1) \otimes^{\mathcal{C}_0(X)} \mathfrak{M}(E_2)$ and $\mathfrak{M}(E_1 \otimes_X E_2)$, then the G-action coming from the $G \ltimes X$-action on $E_1 \otimes_X E_2$ and the tensor product of the actions on $\mathfrak{M}(E_1)$ and $\mathfrak{M}(E_2)$ agree. In other words: $\mathfrak{M}$ is compatible with equivariant tensor products.

4.5.2 Group actions and $\mathfrak{F}$

Definition and Proposition 4.5.4. Let $\mathcal{E}$ be a G-$\mathcal{C}_0(X)$-Banach space. Then we define an action of the groupoid $G \ltimes X$ on the u.s.c. field of Banach spaces $\mathfrak{F}(\mathcal{E}) = (\mathcal{E}_x)_{x\in X}$ as follows: If $g \in G$, then $e \mapsto ge$ is not a $\mathcal{C}_0(X)$-linear map from $\mathcal{E}$ to $\mathcal{E}$, but $\mathcal{C}_0(X)$-linear "with a twist": It maps the fibre $\mathcal{E}_{s(g,x)} = \mathcal{E}_{g^{-1}x}$ isometrically and isomorphically to the fibre $\mathcal{E}_{r(g,x)} = \mathcal{E}_x$. Let $\alpha_{(g,x)}$ denote this isomorphism from $\mathcal{E}_{s(g,x)} = \mathcal{E}_{g^{-1}x}$ to $\mathcal{E}_{r(g,x)} = \mathcal{E}_x$ for every $x \in X$. Then α is a continuous action of $G \ltimes X$ on $\mathfrak{F}(\mathcal{E})$.

Proof. **α is a continuous field of linear maps from $s^*\mathfrak{F}(\mathcal{E})$ to $r^*\mathfrak{F}(\mathcal{E})$:** Let $e \in \mathcal{E}$. Then $x \mapsto e_x$ is by definition a section of $\mathfrak{F}(\mathcal{E})$. Hence $(g, x) \mapsto e_{s(g,x)} = e_{g^{-1}x}$ is a section of $s^*\mathfrak{F}(\mathcal{E})$. This section is mapped by α to the section $(g, x) \mapsto (ge)_x$ of $r^*\mathfrak{F}(\mathcal{E})$. So α maps a total set of sections to sections, so it is continuous.
α is an action of $G \ltimes X$: Let $g, h \in G$ and $x \in X$. Then (h, x) and $(g, h^{-1}x)$ are (typical) composable elements of $G \ltimes X$. If $e \in \mathcal{E}$, then

$$\alpha_{(h,x)}\left(\alpha_{(g,h^{-1}x)}(e_{g^{-1}(h^{-1}x)})\right) = \alpha_{(h,x)}\left((ge)_{h^{-1}x}\right) = (h(ge))_x = ((hg)e)_x = \alpha_{(hg,x)}e_{(hg)^{-1}x}.$$

So $\alpha_{(h,x)}\alpha_{(g,h^{-1}x)} = \alpha_{(hg,x)}$, and α is hence an action of $G \ltimes X$ on $\mathfrak{F}(\mathcal{E})$. □

Proposition 4.5.5. *Let $\mathcal{E}$ and $\mathcal{F}$ be G-$\mathcal{C}_0(X)$-Banach spaces and let $T \in \mathrm{L}^{\mathcal{C}_0(X)}(\mathcal{E}, \mathcal{F})$ be G-equivariant. Then $\mathfrak{F}(T) : \mathfrak{F}(\mathcal{E}) \to \mathfrak{F}(\mathcal{F})$ is $G \ltimes X$-equivariant.*

Proof. Let α and β denote the $G \ltimes X$-actions on $\mathfrak{F}(\mathcal{E})$ and $\mathfrak{F}(\mathcal{F})$, respectively. Let $(g, x) \in G \ltimes X$ and $e \in \mathcal{E}$. Then

$$\begin{aligned} \mathfrak{F}(T)_{r(g,x)}\left(\alpha_{(g,x)}e_{s(g,x)}\right) &= T_x((ge)_x) = (T(ge))_x = (g(Te))_x \\ &= \beta_{(g,x)}(Te)_{s(g,x)} = \beta_{(g,x)}\left(T_{s(g,x)}e_{s(g,x)}\right). \end{aligned}$$

This means $\mathfrak{F}(T)_{r(g,x)} \circ \alpha_{(g,x)} = \beta_{(g,x)}\mathfrak{F}(T)_{s(g,x)}$, in other words: $\mathfrak{F}(T)$ is $G \ltimes X$-equivariant. □

Similarly one shows that $\mathfrak{F}(\mu)$ is $G \ltimes X$-equivariant for G-equivariant bounded $\mathcal{C}_0(X)$-bilinear maps μ. Moreover, $\mathfrak{F}(\mu)$ is compatible with the equivariant tensor product.

4.5.3 Group actions and $\mathfrak{G}$

Proposition 4.5.6. *Let $\mathcal{E}$ be a G-$\mathcal{C}_0(X)$-Banach space. Then the Gelfand functor $\mathcal{G}$ takes $\mathcal{E}$ to a locally $\mathcal{C}_0(X)$-convex G-$\mathcal{C}_0(X)$-Banach space $\mathfrak{G}(\mathcal{E})$. The Gelfand transformation $\mathfrak{g}_{\mathcal{E}}$ is G-equivariant.*

Proof. Let α denote the induced action of $G \ltimes X$ on $\mathfrak{F}(\mathcal{E})$. We have to proof the G-equivariance of the Gelfand transformation. Let $e \in \mathcal{E}$ and $g \in G$. Then $\alpha_{(g,x)} e_{s(g,x)} = (ge)_{r(g,x)}$ for all $x \in X$ by definition. Hence

$$(g\mathfrak{g}_{\mathcal{E}}(e))(x) = \alpha_{(g,x)}\left(\mathfrak{g}_{\mathcal{E}}(e)\,(g^{-1}x)\right) = \alpha_{(g,x)}\left(e_{g^{-1}x}\right) = (ge)_x = \mathfrak{g}_{\mathcal{E}}(ge)\,(x)$$

for all $x \in X$, so $g\left(\mathfrak{g}_{\mathcal{E}}(e)\right) = \mathfrak{g}_{\mathcal{E}}(ge)$, and $\mathfrak{g}_{\mathcal{E}}$ is hence G-equivariant. □

4.6 Algebras, modules and pairs and the functors $\mathfrak{F}$, $\mathfrak{M}$ and $\mathfrak{G}$

Because $\mathfrak{M}$, $\mathfrak{F}$ and $\mathfrak{G}$ are compatible with the (graded equivariant) tensor products we get functors on the "derived" categories of Banach algebras, Banach modules and Banach pairs. They map operators to operators and homomorphisms with coefficient map to homomorphisms with coefficient maps. We omit the details of all these definitions and just give some models and highlights.

4.6.1 The functor $\mathfrak{M}$

Definition 4.6.1. Let B be a u.s.c. field of Banach algebras over X with multiplication μ. Then $\mathfrak{M}(B) = \Gamma_0(X, B)$ is a locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach algebra when equipped with the multiplication $\mathfrak{M}(\mu)$. If B is non-degenerate, then so is $\mathfrak{M}(B)$. If B is carries an action of $G \ltimes X$, then $\mathfrak{M}(B)$ is a G-$\mathcal{C}_0(X)$-Banach algebra. Moreover, $\mathfrak{M}$ is a functor from the $G \ltimes X$-Banach algebras to the locally $\mathcal{C}_0(X)$-convex G-$\mathcal{C}_0(X)$-Banach algebras.

Proposition 4.6.2. *Let B be a $G \times X$-Banach algebra, E a right $G \ltimes X$-Banach B-module and F a left $G \ltimes X$-Banach B-module. Then*

$$\mathfrak{M}(E) \otimes^{\mathcal{C}_0(X)}_{\mathfrak{M}(B)} \mathfrak{M}(F) \cong \mathfrak{M}(E \otimes_B F).$$

Proof. Define

$$\mu\colon \mathfrak{M}(E) \times \mathfrak{M}(F) \to \mathfrak{M}(E \otimes_B F),\ (\xi, \eta) \mapsto (x \mapsto \xi(x) \otimes \eta(x)).$$

This map is well-defined, $\mathbb{C}$-bilinear, $\mathcal{C}_0(X)$-bilinear and contractive. Moreover, if $\beta \in \mathfrak{M}(B) = \Gamma_0(X, B)$, $\xi \in \mathfrak{M}(E)$ and $\eta \in \mathfrak{M}(F)$, then

$$\mu(\xi\beta,\ \eta)(x) = (\xi\beta)(x) \otimes \eta(x) = (\xi(x)\,\beta(x)) \otimes \eta(x) = \xi(x) \otimes (\beta(x)\eta(x)) = \mu(\xi,\ \beta\eta)(x)$$

for all $x \in X$. So μ is $\mathfrak{M}(B)$-balanced. Hence it induces a linear and contractive map

$$\hat{\mu}\colon \mathfrak{M}(E) \otimes^{\mathcal{C}_0(X)}_{\mathfrak{M}(B)} \mathfrak{M}(F) \to \mathfrak{M}(E \otimes_B F).$$

Because $\mathfrak{M}(E)$ and $\mathfrak{M}(F)$ are both locally $\mathcal{C}_0(X)$-convex, so is their $\mathcal{C}_0(X)$-tensor product; because the balanced $\mathcal{C}_0(X)$-tensor product is a quotient of the $\mathcal{C}_0(X)$-tensor product, it is locally $\mathcal{C}_0(X)$-convex as well. We can therefore check that $\hat{\mu}$ is an isomorphism by checking it on the fibres. Let $x \in X$. Then the fibre of $\mathfrak{M}(E) \otimes^{\mathcal{C}_0(X)}_{\mathfrak{M}(B)} \mathfrak{M}(F)$ at x is isomorphic to $\mathfrak{M}(E)_x \otimes_{\mathfrak{M}(B)_x} \mathfrak{M}(F)_x =$

$E_x \otimes_{B_x} F_x$ which happens to be the fibre of $\mathfrak{M}(E \otimes_B F)$ at x. The isometric isomorphism on the fibre over x is induced by $\hat{\mu}$ and thus $\hat{\mu}$ is an isometric isomorphism.

We now check that $\hat{\mu}$ is also G-equivariant by checking that μ is G-equivariant. Write α^E and α^F for the actions of $G \ltimes X$ on E and F, respectively. Let $g \in G$, $\xi \in \Gamma_0(X, E)$ and $\eta \in \Gamma_0(X, F)$. Then

$$\begin{aligned} \mu(g\xi,\ g\eta)(x) &= (g\xi)(x) \otimes (g\eta)(x) = \left[\alpha^E_{(g,x)} \xi(g^{-1}x)\right] \otimes \left[\alpha^F_{(g,x)} \eta(g^{-1}x)\right] \\ &= \left(\alpha^E \otimes \alpha^F\right)_{(g,x)} \left[\xi(g^{-1}x) \otimes \eta(g^{-1}x)\right] = \left[g\left(\mu(\xi,\eta)\right)\right](x) \end{aligned}$$

for all $x \in X$, i.e., $\mu(g\xi,\ g\eta) = g\mu(\xi,\eta)$. So μ and $\hat{\mu}$ are G-equivariant. □

Lemma 4.6.3. *Let B be a $G \ltimes X$-Banach algebra. Then $\mathfrak{M}(\widetilde{B}) = \mathfrak{M}(B) \oplus \mathcal{C}_0(X)$.*

Proposition 4.6.4. *Let B and B' be $G \times X$-Banach algebras and E a right $G \ltimes X$-Banach B-module. Let $\psi\colon B \to B'$ be an equivariant field of homomorphisms. Then*

$$\mathfrak{M}(\psi)_* (\mathfrak{M}(E)) \ = \ \mathfrak{M}(E) \otimes_{\mathfrak{M}(\widetilde{B})} \mathfrak{M}(\widetilde{B'}) \ \cong \ \mathfrak{M}(E \otimes_{\widetilde{B}} \widetilde{B'}) \ = \ \mathfrak{M}(\psi_*(E)),$$

canonically.

4.6.2 The functor $\mathfrak{F}$

Definition 4.6.5. Let $\mathcal{B}$ be a $\mathcal{C}_0(X)$-Banach algebra with multiplication μ. Then $\mathfrak{F}(\mathcal{B})$ is a u.s.c. field of Banach algebras over X when equipped with the multiplication $\mathfrak{F}(\mu) = (\mu_x)_{x \in X}$. If $\mathcal{B}$ is non-degenerate, then so is $\mathfrak{F}(\mathcal{B})$. If $\mathcal{B}$ is a G-$\mathcal{C}_0(X)$-Banach algebra, then $\mathfrak{F}(\mathcal{B})$ is a $G \ltimes X$-Banach algebra. Moreover, $\mathfrak{F}$ is a functor from the G-$\mathcal{C}_0(X)$-Banach algebras to the $G \ltimes X$-Banach algebras.

Proposition 4.6.6. *Let $\mathcal{B}$ be a G-$\mathcal{C}_0(X)$-Banach algebra and let $\mathcal{E}$ be a right G-$\mathcal{C}_0(X)$-Banach $\mathcal{B}$-module and $\mathcal{F}$ a left G-$\mathcal{C}_0(X)$-Banach $\mathcal{B}$-module. Then*

$$\mathfrak{F}(\mathcal{E}) \otimes_{\mathfrak{F}(\mathcal{B})} \mathfrak{F}(\mathcal{F}) \ \cong \ \mathfrak{F}\left(\mathcal{E} \otimes^{\mathcal{C}_0(X)}_{\mathcal{B}} \mathcal{F}\right).$$

Lemma 4.6.7. *Let $\mathcal{B}$ be a G-$\mathcal{C}_0(X)$-Banach algebra. Then $\mathfrak{F}(\mathcal{B} \oplus \mathcal{C}_0(X)) \cong \widetilde{\mathfrak{F}(\mathcal{B})}$.*

Proposition 4.6.8. *Let $\mathcal{B}$ and $\mathcal{B}'$ be G-$\mathcal{C}_0(X)$-Banach algebras and let $\mathcal{E}$ be a right G-$\mathcal{C}_0(X)$-Banach $\mathcal{B}$-module. Let $\psi\colon \mathcal{B} \to \mathcal{B}'$ be a G-equivariant $\mathcal{C}_0(X)$-linear homomorphism. Then*

$$\mathfrak{F}(\psi)_* (\mathfrak{F}(E)) \cong \mathfrak{F}(\psi_*(E)),$$

canonically.

4.6.3 The functor $\mathfrak{G}$

Definition 4.6.9. Let $\mathcal{B}$ be a $\mathcal{C}_0(X)$-Banach algebra with multiplication μ. Then $\mathfrak{G}(\mathcal{B})$ is a locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach algebra when equipped with the multiplication $\mathfrak{G}(\mu)$. If $\mathcal{B}$ is non-degenerate, then so is $\mathfrak{G}(\mathcal{B})$. If $\mathcal{B}$ is a G-$\mathcal{C}_0(X)$-Banach algebra, then so is $\mathfrak{G}(\mathcal{B})$. The map $\mathfrak{g}_{\mathcal{B}}$ is a (graded, G-equivariant) homomorphism from $\mathcal{B}$ to $\mathfrak{G}(\mathcal{B})$ with dense image. Moreover, $\mathfrak{G}$ is a functor from the graded G-$\mathcal{C}_0(X)$-Banach algebras to the graded locally $\mathcal{C}_0(X)$-convex G-$\mathcal{C}_0(X)$-Banach algebras.

Definition 4.6.10. Let $\mathcal{A}$ and $\mathcal{B}$ be graded G-$\mathcal{C}_0(X)$-Banach algebras and let $\mathcal{E} = (\mathcal{E}^<, \mathcal{E}^>)$ be a graded G-$\mathcal{C}_0(X)$-Banach A-B-pair. Then $\mathfrak{G}(\mathcal{E}) = (\mathfrak{G}(\mathcal{E}^<)\,,\ \mathfrak{G}(\mathcal{E}^>))$ is a graded locally $\mathcal{C}_0(X)$-convex G-$\mathcal{C}_0(X)$-Banach $\mathfrak{G}(A)$-$\mathfrak{G}(B)$-pair. The pair $(\mathfrak{g}_{\mathcal{E}^<},\ \mathfrak{g}_{\mathcal{E}^>})$ is a a graded G-equivariant concurrent $\mathcal{C}_0(X)$-linear homomorphism from $\mathcal{E}$ to $\mathfrak{G}(\mathcal{E})$ with coefficient maps $\mathfrak{g}_A$ and $\mathfrak{g}_B$.

Proposition 4.6.11. *Let $\mathcal{B}$ be a graded G-$\mathcal{C}_0(X)$-Banach algebra and let $\mathcal{E}$ be a graded right G-$\mathcal{C}_0(X)$-Banach $\mathcal{B}$-module and $\mathcal{F}$ a graded left G-$\mathcal{C}_0(X)$-Banach $\mathcal{B}$-module. Then*

$$\mathfrak{G}(\mathcal{E}) \otimes_{\mathfrak{G}(\mathcal{B})} \mathfrak{G}(\mathcal{F}) \;\cong\; \mathfrak{G}\Big(\mathcal{E} \otimes_{\mathcal{B}}^{\mathcal{C}_0(X)} \mathcal{F}\Big)\,.$$

Lemma 4.6.12. *If $\mathcal{B}$ is a graded G-$\mathcal{C}_0(X)$-Banach algebra, then $\mathfrak{G}(\mathcal{B} \oplus \mathcal{C}_0(X)) \cong \mathfrak{G}(\mathcal{B}) \oplus^{\text{l.c.}} \mathcal{C}_0(X)$.*

Proposition 4.6.13. *Let $\mathcal{B}$ and $\mathcal{B}'$ be G-$\mathcal{C}_0(X)$-Banach algebras and let $\mathcal{E}$ be a graded right G-$\mathcal{C}_0(X)$-Banach $\mathcal{B}$-module. Let $\psi\colon \mathcal{B} \to \mathcal{B}'$ be an even G-equivariant $\mathcal{C}_0(X)$-linear homomorphism. Then*

$$\mathfrak{G}(\psi)_*\,(\mathfrak{G}(E)) \cong \mathfrak{G}(\psi_*(E))\,,$$

canonically.

4.7 $\mathrm{KK}^{\mathrm{ban}}$, $\mathrm{RKK}^{\mathrm{ban}}$ and the functors $\mathfrak{M}$, $\mathfrak{F}$ and $\mathfrak{G}$

4.7.1 Compact operators on fields of Banach pairs

Let B be a u.s.c. field of Banach algebras over the locally compact Hausdorff space X.

Definition 4.7.1 (Compact operators). Let E and F be Banach B-pairs. A continuous field T of B-linear operators is called *compact* if for all $\varepsilon > 0$ there is an $n \in \mathbb{N}$ and $\xi_1^<, \dots, \xi_n^< \in \Gamma_0(X, E^<)$ and $\eta_1^>, \dots, \eta_n^> \in \Gamma_0(X, F^>)$ such that

$$\left\| T - \sum_{i=1}^n \left|\eta_i^>\right\rangle\left\langle\xi_i^<\right| \right\| = \sup_{x \in X} \left\| T_x - \sum_{i=1}^n \left|\eta_i^>(x)\right\rangle\left\langle\xi_i^<(x)\right| \right\| \le \varepsilon.$$

The compact operators from E to F are denoted by $\mathrm{K}_B(E, F)$.

Note that the sections are taken to be vanishing at infinity. This means that, if T is compact, then $(\|T_x\|)_{x \in X}$ is also vanishing at infinity. It follows that $\mathrm{K}_B(E, F) \subseteq \mathrm{L}_B(E, F)$ and $\mathrm{K}_B(E, F)$ is the closed linear span in $\mathrm{L}_B(E, F)$ of all operators of the form $\left|\eta^>\right\rangle\left\langle\xi^<\right|$. In particular, $\mathrm{K}_B(E, F)$ is a Banach space.

We will now justify the choice of the name “locally compact operator”:

Proposition 4.7.2 (Characterisation of locally compact operators). *Let E and F be Banach B-pairs and let T be a continuous field of B-linear operators from E to F. Then the following are equivalent:*

1. *T is locally compact.*
2. *For all compact subsets K of X and all $\varepsilon > 0$ there is an $n \in \mathbb{N}$ and $\xi_1^<, \dots, \xi_n^< \in \Gamma(X, E^<)$ and $\eta_1^>, \dots, \eta_n^> \in \Gamma(X, F^>)$ such that $\left\|T_k - \sum_{i=1}^n \left|\eta_i^>(k)\right\rangle\left\langle\xi_i^<(k)\right|\right\| \le \varepsilon$ for all $k \in K$.*
3. *For all $x \in X$ and all $\varepsilon > 0$ there is an open neighbourhood U of x and a compact operator $S \in \mathrm{K}_B(E, F)$ such that $\|T_u - S_u\| \le \varepsilon$ for all $u \in U$.*

4. *For all compact subsets $K \subseteq X$ and all $\varepsilon > 0$ there is a operator $S \in \mathrm{K}_B(E, F)$ such that $\|T_k - S_k\| \leq \varepsilon$ for all $k \in K$.*

5. *For all $\varphi \in \mathcal{C}_c(X)$ the field φT is compact.*

Proof. Assume that 1. holds. Let $K \subseteq X$ be a compact subset. Let $\varepsilon > 0$. For all $x \in X$, find U_x, n_x, $\xi^<_{x,1}, \ldots, \xi^<_{x,n_x} \in \Gamma(X, E^<)$ and $\eta^>_{x,1}, \ldots, \eta^>_{x,n_x} \in \Gamma(X, F^>)$ as in the definition of local compactness for T. Then $\{U_x : x \in K\}$ is an open cover of K so we can find a finite subset A of K such that $K \subseteq \bigcup_{a \in A} U_a$. Find a partition of unity $(\chi_a)_{a \in A}$ on K subordinate to the cover $(U_a)_{a \in A}$. Then for all $k \in K$:

$$\left\| T_k - \sum_{a \in A} \chi_a(k) \sum_{i=1}^{n_a} \left| \eta^>_{a,i}(k) \right\rangle \left\langle \xi^<_{a,i}(k) \right| \right\| \leq \varepsilon.$$

This shows 1. $\Rightarrow$ 2..

The same argument shows 3. $\Rightarrow$ 4.. Since X is locally compact the implications 2. $\Rightarrow$ 1. and 4. $\Rightarrow$ 3. are trivial. Moreover, it is clear that 4. implies 2. and 3. implies 1.. Cutting down the sections used in the approximation in 2. easily shows 2. $\Rightarrow$ 4.. So the first four conditions are mutually equivalent. It is straightforward to show 4. $\Leftrightarrow$ 5. (note that if S is compact, then φS is also compact for all $\varphi \in \mathcal{C}_c(X)$). □

Proposition 4.7.3. *Let E and F be Banach B-pairs and let $T\colon E \to F$ be a continuous field of operators. Then T is compact if and only if T is locally compact and $x \mapsto \|T_x\|$ vanishes at infinity.*

Proof. Let T be compact. It is clear from the definitions that T is locally compact. Moreover we have already noted that $x \mapsto \|T_x\|$ vanishes at infinity.

Conversely, let T be locally compact and let $x \mapsto \|T_x\|$ vanish at infinity. Let $\varepsilon > 0$. Find a compact set $K \subseteq X$ such that $\|T_x\| \leq \varepsilon$ for all $x \notin K$. Find a function $\chi \in \mathcal{C}_c(X)$ such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on K. Find a compact operator $S \in \mathrm{K}_B(E, F)$ such that $\|T_l - S_l\| \leq \varepsilon$ for all $l \in \operatorname{supp} \chi$ (using the above characterisation of local compactness). Then also $\|T_l - (\varphi S)_l\| \leq \varepsilon$ for all $l \in \operatorname{supp} \varphi$ and $T_x = (\varphi S)_x = 0$ for all $x \notin \operatorname{supp} \varphi$. Hence $\|T - \varphi S\| \leq \varepsilon$. So T can be approximated by compact operators and is therefore compact. □

Lemma 4.7.4. *Let E_1, E_2 and E_3 be Banach B-pairs. Then we have $\mathrm{L}_B(E_2, E_3) \circ \mathrm{K}_B(E_1, E_2) \subseteq \mathrm{K}_B(E_1, E_3)$ and $\mathrm{K}_B(E_2, E_3) \circ \mathrm{L}_B(E_1, E_2) \subseteq \mathrm{K}_B(E_1, E_3)$.*

Proof. The composition of a compact operator and a bounded linear operator is surely locally compact and vanishes at infinity. Hence it is compact. One can also easily prove this by direct calculation. □

In the definition of $\mathrm{KK}^{\mathrm{ban}}$-cycles in the setting of fields of Banach space (Definition 3.5.2) we have used locally compact operators. If the underlying space X is locally compact Hausdorff, then we can actually use compact operators instead. More precisely, we have the following characterisation of $\mathrm{KK}^{\mathrm{ban}}$-cycles:

Proposition 4.7.5. *Let $\mathcal{G}$ be a locally compact Hausdorff groupoid over X and let A and B be $\mathcal{G}$-Banach algebras. Then a pair (E, T) such that E is a non-degenerate graded $\mathcal{G}$-A-B-bimodule and T is an odd element of $\mathrm{L}_B(E)$ is a $\mathrm{KK}^{\mathrm{ban}}$-cycle from A to B, i.e., an element of $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A, B)$, if and only if*

$$[\pi_A(a), T], \; \pi_A(a) \, (\mathrm{Id} - T^2) \in \mathrm{K}_B(E)$$

for all $a \in \Gamma_0(X, A)$ *and*

$$\pi(\tilde{a}) \left(\alpha^{\mathrm{L}(E)}(s^*T) - r^*T\right) \in \mathrm{K}_{r^*B}\left(r^*E\right)$$

for all $\tilde{a} \in \Gamma_0\left(\mathcal{G}, r^*A\right)$.

Proof. If (E, T) is a $\mathrm{KK}^{\mathrm{ban}}$-cycle, then we know that $[\pi_A(a), T]$ is locally compact for all $a \in \Gamma(X, A)$. In particular this is true if $a \in \Gamma_0(X, A)$. Since T is bounded and $x \mapsto \|\pi_A(a)_x\|$ vanishes at infinity also $x \mapsto [\pi_A(a), T]_x$ vanishes at infinity. So $[\pi_A(a), T]$ is compact. The same argument works for the other operators which have to be shown to be compact. □

4.7.2 $\mathrm{KK}^{\mathrm{ban}}$, $\mathrm{RKK}^{\mathrm{ban}}$ and the functor $\mathfrak{M}$

Proposition 4.7.6. *Let B be a u.s.c. field of Banach algebras and let E and F be Banach B-pairs. Let $\xi^< \in \Gamma_0(X, E^<)$ and $\eta^> \in \Gamma_0(X, F^>)$. Then*

$$\mathfrak{M}\left(\left|\eta^>\right\rangle\left\langle\xi^<\right|\right) = \left|\xi^>\right\rangle\left\langle\eta^<\right| \quad \in \quad \mathrm{K}_{\mathfrak{M}(B)}\left(\mathfrak{M}\left(E\right), \mathfrak{M}\left(F\right)\right).$$

It follows that, if $S \in \mathrm{K}_B(E, F)$, then $\mathfrak{M}\left(S\right) \in \mathrm{K}_{\mathfrak{M}(B)}\left(\mathfrak{M}\left(E\right), \mathfrak{M}\left(F\right)\right)$.

Definition and Proposition 4.7.7 ($\mathfrak{M}$ and $\mathrm{KK}^{\mathrm{ban}}$-cycles). Let A and B be $G \ltimes X$-Banach algebras. Let $(E, T) \in \mathbb{E}^{\mathrm{ban}}_{G\ltimes X}\left(A, B\right)$. Then

$$\mathfrak{M}\left(E, T\right) := \left(\mathfrak{M}\left(E\right),\ \mathfrak{M}\left(T\right)\right) \quad \in \quad \mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}\left(\mathcal{C}_0(X);\ \mathfrak{M}\left(A\right),\ \mathfrak{M}\left(B\right)\right).$$

Proof. First of all $\mathfrak{M}\left(E\right)$ is a graded non-degenerate G-$\mathcal{C}_0(X)$-Banach $\mathfrak{M}\left(A\right)$-$\mathfrak{M}\left(B\right)$-pair. The operator $\mathfrak{M}\left(T\right)$ is odd. If $a \in \mathfrak{M}\left(A\right) = \Gamma_0(X, A)$, then

$$\left[\pi_{\mathfrak{M}(A)}(a),\ \mathfrak{M}\left(T\right)\right] = \left[\mathfrak{M}\left(\pi_A(a)\right),\ \mathfrak{M}\left(T\right)\right] = \mathfrak{M}\left(\left[\pi_A(a), T\right]\right) \ \in\ \mathrm{K}_{\mathfrak{M}(B)}\left(\mathfrak{M}\left(E\right)\right).$$

Similarly $\pi_{\mathfrak{M}(A)}(a)\left(\mathfrak{M}\left(T\right)^2 - 1\right)$ is compact. What is left to check is that

$$g \mapsto \pi_{\mathfrak{M}(A)}(a)\left(g\mathfrak{M}\left(T\right) - \mathfrak{M}\left(T\right)\right)$$

is a continuous map from G into $\mathrm{K}_{\mathfrak{M}(B)}\left(\mathfrak{M}\left(E\right)\right)$. Define as above $\varphi_g\colon X \to G \times X,\ x \mapsto (g, x)$ for all $g \in G$. Then a short calculation shows that $g\mathfrak{M}\left(T\right) = \mathfrak{M}(x \mapsto \left(\alpha^{\mathrm{L}(E)}(s^*T)\right)_{\varphi_g(x)}) = \mathfrak{M}\left(\varphi_g^*(\alpha^{\mathrm{L}(E)}(s^*T))\right)$ where $\alpha^{\mathrm{L}(E)}$ denotes the isomorphism from $\mathrm{L}^{\mathbf{loc}}_{s^*B}(s^*E)$ to $\mathrm{L}^{\mathbf{loc}}_{r^*B}(r^*E)$ induced by the action of $G \ltimes X$ on E (recall that $r\colon G \ltimes X \to X, (g, x) \mapsto x$ and $s\colon G \ltimes X \to X, (g, x) \mapsto g^{-1}x$). It follows that

$$\pi_{\mathfrak{M}(A)}(a)\left(g\mathfrak{M}\left(T\right) - \mathfrak{M}\left(T\right)\right) \ =\ \mathfrak{M}\left(\varphi_g^*\left(\pi_{r^*A}(a \circ r)\left(\alpha^{\mathrm{L}(E)}(s^*T) - r^*T\right)\right)\right).$$

Because by assumption $\pi_{r^*A}(a \circ r)\left(\alpha^{\mathrm{L}(E)}(s^*T) - r^*T\right) \in \mathrm{K}^{\mathbf{loc}}_{r^*B}\left(r^*E\right)$, we have

$$\varphi_g^*\left(\pi_{\tilde{A}}(a \circ r)\left(\alpha^{\mathrm{L}(E)}(s^*T) - r^*T\right)\right) \ \in\ \mathrm{K}^{\mathbf{loc}}_{\varphi_g^*r^*B}\left(\varphi_g^*r^*E\right).$$

Because $r \circ \varphi_g = \mathrm{Id}_X$, we have $\mathrm{K}^{\mathbf{loc}}_{\varphi_g^*r^*B}\left(\varphi_g^*r^*E\right) = \mathrm{K}^{\mathbf{loc}}_B\left(E\right)$. Because a vanishes at infinity, we thus have $\varphi_g^*\left(\pi_{\tilde{A}}(a \circ r)\left(\alpha^{\mathrm{L}(E)}(s^*T) - r^*T\right)\right) \in \mathrm{K}_B(E)$. It follows that

$$\mathfrak{M}\left(\varphi_g^*\left(\pi_{\tilde{A}}(a \circ r)\left(\alpha^{\mathrm{L}(E)}(s^*T) - r^*T\right)\right)\right) \ \in\ \mathrm{K}_{\mathfrak{M}(B)}\left(\mathfrak{M}\left(E\right)\right)$$

for all $g \in G$. For all $\chi \in \mathcal{C}_0(G)$, we have $\chi\pi_{\tilde{A}}(a \circ r)\left(\alpha^{\mathrm{L}(E)}(s^*T) - r^*T\right) \in \mathrm{K}_{r^*B}(r^*E)$. By Lemma 4.7.8 it follows that $g \mapsto \chi(g)\varphi_g^*\left(\pi_{\tilde{A}}(a \circ r)\left(\alpha^{\mathrm{L}(E)}(s^*T) - r^*T\right)\right)$ is in $\mathcal{C}_0\left(G, \mathrm{K}_B(E)\right)$. Hence also $g \mapsto \varphi_g^*\left(\pi_{\tilde{A}}(a \circ r)\left(\alpha^{\mathrm{L}(E)}(s^*T) - r^*T\right)\right)$ is continuous. This implies that also

$$g \mapsto \mathfrak{M}\left(\varphi_g^*\left(\pi_{\tilde{A}}(a \circ r)\left(\alpha^{\mathrm{L}(E)}(s^*T) - r^*T\right)\right)\right)$$

is continuous. □

Using Lemma 4.5.2 one can show:

Lemma 4.7.8. *Let Y and Y' be locally compact Hausdorff spaces and let $p'\colon Y \times Y' \to Y'$ be the canonical projection onto the second component. Let B be a u.s.c. field of Banach algebras over Y' and let E and F be Banach B-pairs. Then p'^*E is a Banach p'^*B-pair. Let T be in $\mathrm{L}_{p'^*B}\left(p'^*E,\ p'^*F\right)$. Then T is compact if and only if for all $y \in Y$ the field $T_y := \left(T_{(y,y')}\right)_{y' \in Y'}$ is in $\mathrm{K}_B(E,F)$ and $y \mapsto T_y$ is in $\mathcal{C}_0\left(Y,\ \mathrm{K}_B(E,F)\right)$.*

Lemma 4.7.9. *Let A be a $G \ltimes X$-Banach algebra. Then $\mathfrak{M}(A)[0,1]$ is isomorphic to $\mathfrak{M}(A[0,1])$.*

Proof. The isomorphism is

$$\Phi\colon \mathfrak{M}(A)[0,1] \to \mathfrak{M}(A[0,1]),\ \xi \mapsto (x \mapsto (t \mapsto (\xi(t))(x))).$$

This is a bijection by the definition of $A[0,1]$. It is obviously isometric and $\mathcal{C}_0(X)$-linear. What is left to check is that it is G-equivariant. Let α denote the action of $G \ltimes X$ on A. If $\xi \in \mathfrak{M}(A)[0,1]$, $g \in G$, $x \in X$, and $t \in [0,1]$, then

$$\begin{aligned}
(\Phi(g\xi)(x))(t) &\overset{\text{def. }\Phi}{=} ((g\xi)(t))(x) \\
&\overset{\text{def. }G\text{-action on }\mathfrak{M}(A)[0,1]}{=} (g(\xi(t)))(x) \\
&\overset{\text{def. }G\text{-action on }\mathfrak{M}(A)}{=} \alpha_{(g,x)}\left[(\xi(t))(g^{-1}x)\right] \\
&\overset{\text{def. }\Phi}{=} \alpha_{(g,x)}\left[(\Phi(\xi)(g^{-1}x))(t)\right] \\
&\overset{\text{def. }G\ltimes X\text{-action on }A[0,1]}{=} \left[(\alpha[0,1])_{(g,x)}(\Phi(\xi)(g^{-1}x))\right](t) \\
&\overset{\text{def. }G\ltimes X\text{-action on }\mathfrak{M}(A[0,1])}{=} \left[(g\Phi(\xi))(x)\right](t)
\end{aligned}$$

So Φ is G-equivariant. □

Because the functor $\mathfrak{M}$ is compatible with the pushout — at least up to a delicate point where it comes to comparing the locally $\mathcal{C}_0(X)$-convex unitalisation and the ordinary unitalisation which we will just leave aside — the functor $\mathfrak{M}$ also respects homotopy. So it lifts from $\mathrm{KK}^{\mathrm{ban}}$-cycles to the level of KK-theory:

Proposition 4.7.10. *Let A and B be $G \ltimes X$-Banach algebras. Then $(E,T) \mapsto \mathfrak{M}(E,T)$ lifts to a group homomorphism*

$$\mathfrak{M}\colon\ \mathrm{KK}^{\mathrm{ban}}_{G\ltimes X}(A,B) \ \to\ \mathrm{RKK}^{\mathrm{ban}}_G\left(\mathcal{C}_0(X);\ \mathfrak{M}(A),\ \mathfrak{M}(B)\right).$$

To show that $\mathfrak{M}$ is a group homomorphism we have to check that it is compatible with direct sum. This is the case at least up to equivalence of norms, so it is true up to homotopy which is certainly sufficient for our purposes.

4.7.3 $\mathrm{KK}^{\mathrm{ban}}$, $\mathrm{RKK}^{\mathrm{ban}}$ and the functor $\mathfrak{F}$

Proposition 4.7.11. *Let $\mathcal{B}$ be a $\mathcal{C}_0(X)$-Banach algebra and let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{C}_0(X)$-Banach $\mathcal{B}$-pairs. Then for all $e^< \in \mathcal{E}^<$ and $f^> \in \mathcal{F}^>$, we have*

$$\mathfrak{F}\left(\left|f^>\right\rangle\left\langle e^<\right|\right) = \left|x \mapsto f^>_x\right\rangle\left\langle x \mapsto e^<_x\right| \quad \in \quad \mathrm{K}_{\mathfrak{F}(\mathcal{B})}\left(\mathfrak{F}(\mathcal{E}), \mathfrak{F}(\mathcal{F})\right).$$

It follows that, if $S \in \mathrm{K}_{\mathcal{B}}(\mathcal{E}, \mathcal{F})$, then $\mathfrak{F}(S) \in \mathrm{K}_{\mathfrak{F}(\mathcal{B})}(\mathfrak{F}(\mathcal{E}), \mathfrak{F}(\mathcal{F}))$.

Definition and Proposition 4.7.12. Let $\mathcal{A}$ and $\mathcal{B}$ be G-$\mathcal{C}_0(X)$-Banach algebras. Let $(\mathcal{E}, T) \in \mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(\mathcal{C}_0(X); \mathcal{A}, \mathcal{B})$. Then

$$\mathfrak{F}(\mathcal{E}, T) := \left(\mathfrak{F}(\mathcal{E}),\ \mathfrak{F}(T)\right) \quad \in \quad \mathbb{E}^{\mathrm{ban}}_{G\ltimes X}\left(\mathfrak{F}(\mathcal{A}),\ \mathfrak{F}(\mathcal{B})\right).$$

Proof. First note that $\mathfrak{F}(\mathcal{E})$ is a graded non-degenerate $G \ltimes X$-Banach $\mathfrak{F}(\mathcal{A})$-$\mathfrak{F}(\mathcal{B})$-pair and $\mathfrak{F}(T)$ is an odd and bounded continuous field of operators on $\mathfrak{F}(\mathcal{E})$. Let $a \in \mathcal{A}$. Then $\mathfrak{g}_{\mathcal{A}}(a) = (a_x)_{x\in X}$ is in $\Gamma_0(X, \mathfrak{F}(\mathcal{A}))$ and the set of sections of this form is dense. Now

$$\left[\mathfrak{g}_{\mathcal{A}}(a),\ \mathfrak{F}(T)\right]_x = [a_x,\ T_x] = \mathfrak{F}([a, T])_x$$

for all $x \in X$, so $[\mathfrak{g}_{\mathcal{A}}(a),\ \mathfrak{F}(T)] = \mathfrak{F}([a, T])$ is compact. Similarly, $\mathfrak{g}_{\mathcal{A}}(a)(\mathfrak{F}(T)^2 - 1)$ can be shown to be compact. Because $\mathfrak{g}_{\mathcal{A}}$ has dense image, this is true for all sections a in $\Gamma_0(X, A)$.

Let $\chi \in \mathcal{C}_0(G)$. Then $\chi \cdot (\mathfrak{g}_{\mathcal{A}}(a) \circ r) \in \Gamma_0(G \ltimes X,\ r^*\mathfrak{F}(\mathcal{A}))$ and the span of such sections is dense in $\Gamma_0(G \ltimes X,\ r^*\mathfrak{F}(\mathcal{A}))$. A short calculation shows that

$$S_{(g,x)} := \left(\chi \cdot (\mathfrak{g}_{\mathcal{A}}(a) \circ r) \cdot \left(\alpha^{\mathrm{L}(\mathfrak{F}(\mathcal{E}))}(s^*\mathfrak{F}(T)) - r^*\mathfrak{F}(T)\right)\right)_{(g,x)} = \mathfrak{F}\left(\chi(g) a \left(gTg^{-1} - T\right)\right)_x$$

for all $(g, x) \in G \ltimes X$. This implies that $g \mapsto \left[x \mapsto S_{(g,x)}\right]$ is in $\mathcal{C}_0\left(G,\ \mathrm{K}_{\mathfrak{F}(\mathcal{B})}(\mathfrak{F}(\mathcal{E}))\right)$. Now Lemma 4.7.8 implies that $\left(S_{(g,x)}\right)_{(g,x)\in G\ltimes X}$ is compact. □

Lemma 4.7.13. *Let $\mathcal{B}$ be a G-$\mathcal{C}_0(X)$-Banach algebra. Then $\mathfrak{F}(\mathcal{B})[0,1]$ is isomorphic to $\mathfrak{F}(\mathcal{B}[0,1])$.*

Proof. The isomorphism is $(\Psi_x)_{x\in X} \colon \mathfrak{F}(\mathcal{B}[0,1]) \to \mathfrak{F}(\mathcal{B})[0,1]$ with

$$\Psi_x \colon \mathfrak{F}(\mathcal{B}[0,1])_x \to \mathfrak{F}(\mathcal{B})_x[0,1],\ \beta_x \mapsto (t \mapsto \beta(t)_x)$$

for all $x \in X$ (where $\beta \in \mathcal{B}[0,1] = \mathcal{C}([0,1], \mathcal{B})$). For all $x \in X$, the map Ψ_x is well-defined, linear, a quotient map and injective (and hence an isomorphism). Ψ is continuous because it takes $x \mapsto \beta_x$ to $x \mapsto (t \mapsto \beta(t)_x) \in \Gamma_0(X, \mathfrak{F}(\mathcal{B})[0,1])$ for all $\beta \in \mathcal{B}[0,1]$ (use Lemma 4.5.2).

Now we check that Ψ is $G \ltimes X$-equivariant. Let α and α' denote the actions of $G \ltimes X$ on $\mathfrak{F}(\mathcal{B})$ and $\mathfrak{F}(\mathcal{B}[0,1])$, respectively. Then $\alpha[0,1]$ denotes the action of $G \ltimes X$ on $\mathfrak{F}(\mathcal{B})[0,1]$. Let $(g, x) \in G \ltimes X$ and $\beta \in \mathcal{B}[0,1]$. Then

$$\begin{aligned}
\alpha[0,1]_{(g,x)}\left[\Psi_x(\beta_x)\right] &\overset{\text{def. }\Psi}{=} \alpha[0,1]_{(g,x)}\left(t \mapsto \beta(t)_x\right) \\
&\overset{\text{def. }\alpha[0,1]}{=} \left(t \mapsto \alpha_{(g,x)}\beta(t)_x\right) \\
&\overset{\text{def. }\alpha}{=} \left(t \mapsto (g(\beta(t)))_{gx}\right) \\
&\overset{\text{def. }G\text{-action on }\mathcal{B}[0,1]}{=} \left(t \mapsto ((g\beta)(t))_{gx}\right) \\
&\overset{\text{def. }\Psi}{=} \Psi_{gx}\left((g\beta)_{gx}\right) \\
&\overset{\text{def. }\alpha'}{=} \Psi_{gx}\left(\alpha'_{(g,x)}(\beta_x)\right).
\end{aligned}$$

So Ψ is equivariant. □

Because the functor $\mathfrak{F}$ is compatible with the pushout, it is also compatible with the homotopy relation. As it is also compatible with the direct sum we get the following result.

Proposition 4.7.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be G-$\mathcal{C}_0(X)$-Banach algebras. Then $(\mathcal{E},T) \mapsto \mathfrak{F}(\mathcal{E},T)$ lifts to a group homomorphism*

$$\mathfrak{F}\colon\ \mathrm{RKK}^{\mathrm{ban}}_G(\mathcal{C}_0(X);\mathcal{A},\mathcal{B}) \ \to\ \mathrm{KK}^{\mathrm{ban}}_{G\ltimes X}(\mathfrak{F}(\mathcal{A})\,,\ \mathfrak{F}(\mathcal{B}))\,.$$

4.7.4 $\mathrm{KK}^{\mathrm{ban}}$, $\mathrm{RKK}^{\mathrm{ban}}$ and the functor $\mathfrak{G}$

Lemma 4.7.15. *Let A and B be $G \ltimes X$-Banach algebras. Then every $(E,T) \in \mathbb{E}^{\mathrm{ban}}_{G\ltimes X}(A,B)$ is isomorphic to $\mathfrak{F}(\mathfrak{M}(E,T))$. It follows that*

$$\mathrm{KK}^{\mathrm{ban}}_{G\ltimes X}(A,\ B) \quad\cong\quad \mathrm{KK}^{\mathrm{ban}}_{G\ltimes X}(\mathfrak{F}(\mathfrak{M}(A))\,,\ \mathfrak{F}(\mathfrak{M}(B)))\,.$$

Lemma 4.7.16. *Let $\mathcal{B}$ be a $\mathcal{C}_0(X)$-Banach algebra and let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{C}_0(X)$-Banach $\mathcal{B}$-modules. Let $e^< \in \mathcal{E}^<$ and $f^> \in \mathcal{F}^>$. Then*

$$\mathfrak{G}\big(\big|f^>\big\rangle\big\langle e^<\big|\big) = \big|\mathfrak{g}_{\mathcal{F}^>}(f^>)\big\rangle\big\langle \mathfrak{g}_{\mathcal{E}^<}(e^<)\big| \quad\in\quad \mathrm{K}_{\mathfrak{G}(\mathcal{B})}(\mathfrak{G}(\mathcal{E})\,,\ \mathfrak{G}(\mathcal{F}))\,.$$

Proposition 4.7.17. *Let $\mathcal{B}$ be a $\mathcal{C}_0(X)$-Banach algebra and let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{C}_0(X)$-Banach $\mathcal{B}$-modules. If $S \in \mathrm{K}_{\mathcal{B}}(\mathcal{E},\mathcal{F})$, then $\mathfrak{G}(S) \in \mathrm{K}_{\mathfrak{G}(\mathcal{B})}(\mathfrak{G}(\mathcal{E})\,,\ \mathfrak{G}(\mathcal{F}))$. Moreover, we have*

$$(S,\ \mathfrak{G}(S)) \quad\in\quad \mathrm{K}(\mathfrak{g}_{\mathcal{E}},\ \mathfrak{g}_{\mathcal{F}})\,.$$

Lemma 4.7.18. *Let $\mathcal{A}$ and $\mathcal{B}$ be G-$\mathcal{C}_0(X)$-Banach algebras and $(\mathcal{E},T) \in \mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(\mathcal{C}_0(X);\mathcal{A},\mathcal{B})$. Then*

$$(\mathfrak{g}_{\mathcal{E}}\colon \mathcal{E} \to \mathfrak{G}(\mathcal{E})\,,\ (T,\mathfrak{G}(T))) \quad\in\quad \mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(\mathcal{C}_0(X);\ \mathfrak{g}_{\mathcal{A}},\ \mathfrak{g}_{\mathcal{B}})\,.$$

Proof. We already know that $\mathfrak{g}_{\mathcal{E}}$ is a graded G-equivariant $\mathcal{C}_0(X)$-linear concurrent homomorphism from $\mathcal{E}$ to $\mathfrak{G}(\mathcal{E})$ with coefficient maps $\mathfrak{g}_{\mathcal{A}}$ and $\mathfrak{g}_{\mathcal{B}}$. Because $\mathfrak{g}_{\mathcal{E}^<}$ and $\mathfrak{g}_{\mathcal{E}^>}$ are natural transformations we can deduce that $\mathfrak{g}_{\mathcal{E}}$ intertwines T and $\mathfrak{G}(T)$, both being odd bounded $\mathcal{C}_0(X)$-linear operators. If $a \in \mathcal{A}$, then

$$\begin{aligned}
[a,\ (T,\mathfrak{G}(T))] &= [(\pi(a),\pi(\mathfrak{g}_{\mathcal{A}}(a))),\ (T,\mathfrak{G}(T))]\\
&= [(\pi(a),\mathfrak{G}(\pi(a)))\,,\ (T,\ \mathfrak{G}(T))] = ([a,T],\ \mathfrak{G}([a,T])) \ \in\ \mathrm{K}(\mathfrak{g}_{\mathcal{E}},\mathfrak{g}_{\mathcal{E}})\,.
\end{aligned}$$

Similarly one shows that

$$a\left((T,\mathfrak{G}(T))^2 - 1\right) \ \in\ \mathrm{K}(\mathfrak{g}_{\mathcal{E}},\mathfrak{g}_{\mathcal{E}})\,.$$

For all $g \in G$, we have

$$a\,(g(T,\mathfrak{G}(T)) - (T,\mathfrak{G}(T))) = (a(gT-T),\ \mathfrak{G}(a(gT-T))) \ \in\ \mathrm{K}(\mathfrak{g}_{\mathcal{E}},\mathfrak{g}_{\mathcal{E}})\,,$$

and this expression depends continuously on g. $\square$

Lemma 4.7.19. *Let $\mathcal{A}$ and $\mathcal{B}$ be locally $\mathcal{C}_0(X)$-convex G-$\mathcal{C}_0(X)$-Banach algebras and $(\mathcal{E},T) \in \mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(\mathcal{C}_0(X);\mathcal{A},\mathcal{B})$. If we identify $\mathfrak{G}(\mathcal{A})$ and $\mathcal{A}$ as well as $\mathfrak{G}(\mathcal{B})$ and $\mathcal{B}$, then*

$$(\mathfrak{g}_{\mathcal{E}}\colon \mathcal{E} \to \mathfrak{G}(\mathcal{E})\,,\ (T,\mathfrak{G}(T))) \quad\in\quad \mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(\mathcal{C}_0(X);\ \mathrm{Id}_{\mathcal{A}},\ \mathrm{Id}_{\mathcal{B}})$$

and $(\mathcal{E},T)$ is homotopic to $(\mathfrak{G}(\mathcal{E})\,,\ \mathfrak{G}(T))$ in $\mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(\mathcal{C}_0(X);\ \mathcal{A},\ \mathcal{B})$.

Theorem 4.7.20. *Let A and B be $G \ltimes X$-Banach algebras. Then $\mathfrak{M}$ is an isomorphism*

$$\mathrm{KK}^{\mathrm{ban}}_{G\ltimes X}(A,B) \quad \cong \quad \mathrm{RKK}^{\mathrm{ban}}_{G}(\mathcal{C}_0(X);\, \Gamma_0(X,A),\, \Gamma_0(X,B))$$

with inverse $\mathfrak{F}$.

Proof. We already know that $\mathfrak{F} \circ \mathfrak{M}$ is the identity on $\mathrm{KK}^{\mathrm{ban}}_{G\ltimes X}(A,B)$. We have to show that the Gelfand functor is the identity on $\mathrm{RKK}^{\mathrm{ban}}_{G}(\mathcal{C}_0(X);\, \mathfrak{M}(A),\, \mathfrak{M}(B))$. Now $\mathcal{A} := \mathfrak{M}(A) = \Gamma_0(X,A)$ and $\mathcal{B} := \mathfrak{M}(B) = \Gamma_0(X,B)$ are locally $\mathcal{C}_0(X)$-convex. If $(\mathcal{E},T) \in \mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(\mathcal{C}_0(X);\, \mathcal{A},\, \mathcal{B})$, then it is homotopic to $\mathfrak{G}(\mathcal{E},T)$ by Lemma 4.7.19. So $\mathfrak{G}$ is surjective on $\mathrm{RKK}^{\mathrm{ban}}_{G}(\mathcal{C}_0(X);\mathcal{A},\mathcal{B})$. Because $\mathfrak{G}(\mathcal{B})[0,1] \cong \mathfrak{G}(\mathcal{B}[0,1])$ and the Gelfand functor commutes with the pushout, we also have that two locally $\mathcal{C}_0(X)$-convex elements of $\mathbb{E}^{\mathrm{ban}}_{\mathrm{G}}(\mathcal{C}_0(X);\mathcal{A},\mathcal{B})$ which are homotopic can also be connected via a locally $\mathcal{C}_0(X)$-convex homotopy. This means that the Gelfand functor is also injective. □

In other words: In the definition of $\mathrm{RKK}^{\mathrm{ban}}$ for locally $\mathcal{C}_0(X)$-convex Banach algebras one can assume without loss of generality that all cycles are locally $\mathcal{C}_0(X)$-convex.

4.8 $\mathrm{K}_B(E)$ as a $\mathcal{G}$-Banach algebra

In this section, let B be a u.s.c. field of Banach algebras over X and let E and F be Banach B-pairs.

4.8.1 $\mathrm{K}_B(E,F)$ as a $\mathcal{C}_0(X)$-Banach space

Lemma 4.8.1. *For all $T \in \mathrm{K}_B(E,F)$ and $\varphi \in \mathcal{C}_b(X)$, we have $\varphi \cdot T \in \mathrm{K}_B(E,F)$. Moreover, $\mathrm{K}_B(E,F)$ is a non-degenerate Banach $\mathcal{C}_0(X)$-module, i.e., it is a $\mathcal{C}_0(X)$-Banach space.*

Proof. Since $\mathrm{K}_B(E,F)$ is a left Banach $\mathrm{L}_B(F)$-module and $\mathcal{C}_b(X)$ can be mapped homomorphically into $\mathrm{L}_B(F)$ as multiplication operators, it follows that $\mathrm{K}_B(E,F)$ is a left Banach $\mathcal{C}_b(X)$-module (with the pointwise product). One can easily show that the elements of $\mathrm{K}_B(E,F)$ with compact support are dense in $\mathrm{K}_B(E,F)$. Hence $\mathrm{K}_B(E,F)$ is a non-degenerate $\mathcal{C}_0(X)$-module. □

The preceding lemma makes it possible to speak of the fibres of $\mathrm{K}_B(E,F)$ as a $\mathcal{C}_0(X)$-Banach space. An immediate conjecture is that the fibre of $\mathrm{K}_B(E,F)$ at $x \in X$ is $\mathrm{K}_{B_x}(E_x,F_x)$. This is true for Hilbert modules and C*-algebras, but it is *false* for general Banach pairs as the following example shows. More precisely, we are going to present a counterexample for the following two statements which hold true for Hilbert modules:

1. For all $T \in \mathrm{K}_B(E,F)$ the function $x \mapsto \|T_x\|$ is upper semi-continuous.

2. For all $x \in X$ the evaluation map induces an isomorphism $(\mathrm{K}_B(E,F))_x \cong \mathrm{K}_{B_x}(E_x,\, F_x)$.

Example 4.8.2. Let $X = \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and let B be the constant field of Banach spaces over $\overline{\mathbb{N}}$ with fibre $\mathbb{C}$. The left and right parts of E and F are also constant fields over $\overline{\mathbb{N}}$, namely $E^{<}$, $F^{<}$ and $F^{>}$ with fibre $\mathbb{C}$, and $E^{>}$ with constant fibre $c_0(\mathbb{N})$. The action of B on E and F are the obvious ones. The bracket on F is the zero-bracket.

For all $n \in \mathbb{N}$ define the map

$$\langle\cdot,\cdot\rangle_n \colon \mathbb{C} \times c_0(\mathbb{N}) \to \mathbb{C},\ (\lambda,x) \mapsto \langle\lambda,x\rangle_n := \lambda x_n.$$

Define

$$\langle\cdot,\cdot\rangle_\infty\colon \mathbb{C}\times c_0(\mathbb{N})\to\mathbb{C},\ (\lambda,x)\mapsto\langle\lambda,x\rangle_\infty:=0.$$

All these maps are $\mathbb{C}$-bilinear. Moreover, the family $(\langle\cdot,\cdot\rangle_n)_{n\in\overline{\mathbb{N}}}$ is a continuous field of bilinear maps on $E^<\times_{\overline{\mathbb{N}}}E^>$. To show this, let $(\lambda^m)_{m\in\overline{\mathbb{N}}}$ be an element of $\Gamma\left(\overline{\mathbb{N}},\ E^<\right)$ and let $(x^m)_{m\in\overline{\mathbb{N}}}$ be an element of $\Gamma\left(\overline{\mathbb{N}},\ E^>\right)$. We have to show that the sequence $(\lambda^n x^n_n)_{n\in\mathbb{N}}$ converges to zero.

Let $\varepsilon>0$. Because $\lambda^\infty x^\infty\in c_0(\mathbb{N})$, we can find an $M\in\mathbb{N}$ such that $|\lambda^\infty x^\infty_m|<\varepsilon/2$ for all $m\geq M$. Because $x^n\to x^\infty$ in $c_0(\mathbb{N})$ and $\lambda^n\to\lambda^\infty$ in $\mathbb{C}$, we can find an $N\in\mathbb{N}$ such that $\|\lambda^\infty x^\infty-\lambda^n x^n\|_\infty<\varepsilon/2$ for all $n\geq N$. Let $n\geq\max\{M,N\}$. Then

$$|\lambda^n x^n_n|\leq|\lambda^\infty x^\infty_n|+|\lambda^\infty x^\infty_n-\lambda^n x^n_n|\leq\varepsilon/2+\varepsilon/2.$$

Note that for $\lambda\in\mathbb{C}$ the map $\langle\lambda,\cdot\rangle_n$ is of norm $|\lambda|$ for all $n\in\mathbb{N}$ and of norm 0 if $n=\infty$. In particular, the family $(\langle 1,\cdot\rangle_n)_{n\in\overline{\mathbb{N}}}$ is not upper semi-continuous in norm. But this family can be written as the right part of a compact field T of operators in $\mathrm{K}_B(E,F)$, namely as $\left|1\right\rangle\left\langle 1\right|^>$, where 1 is a short-hand notation for the constant function on $\overline{\mathbb{N}}$ with value 1. Because the inner product of F is zero, the left-hand part of T is zero. So the norm-function of T is given by the norm-function of the right-hand part. So in this particular case, $T=\left|1\right\rangle\left\langle 1\right|\in\mathrm{K}_B(E,F)$, but the norm function of T is not upper semi-continuous.

The space $\mathrm{K}_{B_\infty}(E_\infty,F_\infty)$ is zero, because the involved inner products vanish. However, the fibre $\mathrm{K}_B(E,F)_\infty$ does not vanish, because the fibre of the element T in ∞ has non-zero norm, namely $\limsup_{n\to\infty}\|T_n\|=1$.

This example should make it possible to construct two homotopies of $\mathrm{KK}^{\mathrm{ban}}$-cycles which cannot be linked in any obvious way, showing that the homotopy relation for $\mathrm{KK}^{\mathrm{ban}}$-cycles is not transitive and we thus have to take equivalence relation generated by homotopy instead.

Although we do not know the fibres of $\mathrm{K}_B(E,F)$ exactly we nevertheless know that $\mathrm{K}_B(E,F)$ as a $\mathcal{C}_0(X)$-Banach space is not too bad:

Proposition 4.8.3. *$\mathrm{K}_B(E,F)$ is a locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach space. In particular, $\mathrm{K}_B(E)$ is a locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach algebra.*

Proof. Let $T=(T_x)_{x\in X}$ be an element of $\mathrm{K}_B(E,F)$. If π_x denotes the quotient map from $\mathrm{K}_B(E,F)$ to its fibre at $x\in X$, then

$$\|\pi_x(T)\|=\limsup_{y\to x}\|T_y\|.$$

It follows that $\sup_{x\in X}\|\pi_x(T)\|=\|T\|$. □

4.8.2 The pullback of $\mathrm{K}_B(E,F)$ along an (open) continuous map

Lemma 4.8.4. *For all $x\in X$, let ψ_x denote the canonical map from $\mathrm{K}_B(E,F)_x$ to $\mathrm{K}_{B_x}(E_x,F_x)$.*[4] *Let Y be a locally compact Hausdorff space and let $p\colon Y\to X$ be continuous. For all $T\in\Gamma\left(Y,\ p^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)\right)$, let*

$$\Psi(T)_y:=\psi_{p(y)}(T(y))\ \in\ \mathrm{K}_{B_{p(y)}}\left(E_{p(y)},\ F_{p(y)}\right)$$

for all $y\in Y$. Then $\Psi(T)\in\mathrm{K}^{\mathbf{loc}}_{p^*B}\left(p^*E,\ p^*F\right)$. The map Ψ is $\mathcal{C}(Y)$-linear. If T is bounded, then so is $\Psi(T)$ with $\|\Psi(T)\|\leq\|T\|$. If T vanishes at infinity, then so does $\Psi(T)$ and is hence compact.

[4] ψ_x is continuous with norm ≤ 1 and has dense image; however, we know that it need not be injective (see Example 4.8.2).

Proof. To check that the map Ψ is well-defined we first approximate T locally in the following way:

Let $y \in Y$ and $\varepsilon > 0$. By the definition of the sections of the pullback $p^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)$ we can approximate T near y by the product of the pullback of a section of $\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)$ and a continuous function of Y. We can even assume that both, the section and the function, have compact support. Using the Gelfand transformation for the $\mathcal{C}_0(X)$-Banach space $\mathrm{K}_B(E,F)$ we can then assume that the section comes from an element of $\mathrm{K}_B(E,F)$. More precisely: We can find a compact operator $S \in \mathrm{K}_B(E,F)$ and a function $\chi \in \mathcal{C}_c(Y)$ such that $0 \le \chi \le 1$ and $\chi \equiv 1$ on a neighbourhood of y and such that $\left\|\chi(T - \mathfrak{g}_{\mathrm{K}(E,F)}(S) \circ p)\right\| \le \varepsilon$.

What is $\Psi(\chi(\mathfrak{g}_{\mathrm{K}(E,F)}(S) \circ p))$? We have

$$\Psi(\chi(\mathfrak{g}_{\mathrm{K}(E,F)}(S) \circ p))_y = \chi(y)\psi_{p(y)}(\mathfrak{g}_{\mathrm{K}(E,F)}(S)(p(y))) = \chi(y)S_{p(y)}$$

for all $y \in Y$. In other words: $\Psi(\chi(\mathfrak{g}_{\mathrm{K}(E,F)}(S) \circ p)) = \chi\, p^*S$, so $\Psi(\chi(\mathfrak{g}_{\mathrm{K}(E,F)}(S) \circ p))$ is, in particular, a continuous field of linear operators. Now p^*S is locally compact as S is (locally) compact (see Proposition 3.3.22), and hence $\Psi(\chi(\mathfrak{g}_{\mathrm{K}(E,F)}(S) \circ p))$ is compact.

Let us check that $\Psi(T)$ really is a continuous field of linear operators: Let $\xi^> \in \Gamma(Y,\ p^*E^>)$. Now $\|\Psi(\chi T)^> \circ \xi^> - (\chi\, p^*S)^> \circ \xi^>\| \le \varepsilon$; because $(\chi\, p^*S)^> \circ \xi^>$ is a section for all $\chi \in \mathcal{C}_c(Y)$, also $\Psi(T)^> \circ \xi^>$ is a section of $p^*F^>$ (use Property (C4) of the definition of a u.s.c. field). Similarly one shows that $\Psi(T)^<$ sends sections to sections. Hence $\Psi(T) \in \mathrm{L}^{\mathbf{loc}}_{p^*B}\left(p^*E,\ p^*F\right)$.

To see that $\Psi(T)$ is compact note that $\|\chi\Psi(T) - \chi p^*S\| \le \varepsilon$, so $\Psi(T)$ can be approximated near y by compact operators, hence $\Psi(T)$ is compact near y. On the other hand, $\|\Psi(T)_y\| \to 0$ for $y \to \infty$, so $\Psi(T)$ is compact. □

Proposition 4.8.5. *Let Y, p and Ψ be as in Lemma 4.8.4. Then the image of $\Gamma_0\left(Y,\ p^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)\right)$ under Ψ is dense in $\mathrm{K}_{p^*B}\left(p^*E,\ p^*F\right)$. If p is open, then Ψ is isometric on the sections vanishing at infinity and we hence have a $\mathcal{C}_0(Y)$-linear isometric isomorphism*

$$\Gamma_0\left(Y,\ p^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)\right) \cong \mathrm{K}_{p^*B}\left(p^*E,\ p^*F\right)$$

and a $\mathcal{C}(Y)$-linear bijection

$$\Gamma\left(Y,\ p^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)\right) \cong \mathrm{K}^{\mathbf{loc}}_{p^*B}\left(p^*E,\ p^*F\right).$$

Proof. Ψ has dense image: Let $\xi^< \in \Gamma_c(Y, p^*E^<)$ and $\eta^> \in \Gamma_c(Y, p^*F^>)$. It suffices to check that $\left|\eta^>\right\rangle\left\langle\xi^<\right|$ can be approximated by elements in the image of Ψ. Moreover, it suffices to check this when $\xi^<$ is of the form $\chi'(\xi_0^< \circ p)$ with $\chi' \in \mathcal{C}_c(Y)$ and $\xi_0^< \in \Gamma_c(X, E^<)$ and $\eta^>$ is of the form $\chi''(\eta_0^> \circ p)$ with $\chi'' \in \mathcal{C}_c(Y)$ and $\eta_0^> \in \Gamma_c(X, F^>)$. But in this case, $S := \left|\eta_0^>\right\rangle\left\langle\xi_0^<\right| \in \mathrm{K}_B(E,F)$ and $\chi'\chi''(\mathfrak{g}_{\mathrm{K}(E,F)}(S) \circ p)$ do the job: $\Psi(\chi'\chi''(\mathfrak{g}_{\mathrm{K}(E,F)}(S) \circ p)) = \chi'\chi'' p^*S$.

Now assume that p is open. We show that for all $T \in \Gamma\left(Y,\ p^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)\right)$ and all $y \in Y$ we have $\|\Psi(T)\|_{\lim,y} = \|T(y)\|$. This shows that Ψ is isometric on the sections vanishing at infinity (even on the bounded sections) and that (the unrestricted) Ψ is a bijection.

Let $T \in \Gamma\left(Y,\ p^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)\right)$ and $y \in Y$. As above, find $S \in \mathrm{K}_B(E,F)$ and $\chi \in \mathcal{C}_c(Y)$ such that $0 \le \chi \le 1$ and $\chi \equiv 1$ on a neighbourhood of y and such that $\left\|\chi(T - \mathfrak{g}_{\mathrm{K}(E,F)}(S) \circ p)\right\| \le \varepsilon$. Then

$$\begin{aligned}
\left\|\Psi(\chi\mathfrak{g}_{\mathrm{K}(E,F)}(S) \circ p)\right\|_{\lim,y} &= \|p^*S\|_{\lim,y} \overset{(\star)}{=} \|S\|_{\lim,p(y)} \\
&= \left\|\mathfrak{g}_{\mathrm{K}(E,F)}(S)(p(y))\right\| = \left\|(\chi\mathfrak{g}_{\mathrm{K}(E,F)}(S) \circ p)(y)\right\|.
\end{aligned}$$

The equality $(\star)$ follows from the fact that p is open. We have

$$\begin{aligned}\Big| \|\Psi(T)\|_{\lim,y} - \|T(y)\| \Big| &\leq \Big| \|\chi\Psi(T)\|_{\lim,y} - \big\|\chi\Psi(\mathfrak{g}_{\mathrm{K}_B(E,F)}(S)\circ p)\big\|_{\lim,y} \Big| \\ &\quad + \Big| \big\|(\chi\mathfrak{g}_{\mathrm{K}_B(E,F)}(S)\circ p)(y)\big\| - \|\chi T(y)\| \Big| \leq \varepsilon + \varepsilon.\end{aligned}$$

Since ε was arbitrary, we have $\|\Psi(T)\|_{\lim,y} = \|T(y)\|$ for all $y \in Y$. □

Note that the proposition says in particular that the fibre of $\mathrm{K}_{p^*B}(p^*E,\ p^*F)$ at $y \in Y$ is isometrically isomorphic to the fibre of $\mathrm{K}_B(E,F)$ at $p(y)$ (if p is open).

Corollary 4.8.6. *If $p = \mathrm{Id}_X$, then the map Ψ gives a $\mathcal{C}_0(X)$-linear isometric isomorphism*

$$\Gamma_0\left(X,\ \mathfrak{F}\left(\mathrm{K}_B(E,F)\right)\right) \cong \mathrm{K}_B\left(E,\ F\right)$$

(namely the inverse of the Gelfand transform) and a $\mathcal{C}(X)$-linear bijection

$$\Gamma\left(X,\ \mathfrak{F}\left(\mathrm{K}_B(E,F)\right)\right) \cong \mathrm{K}_B^{\mathbf{loc}}\left(E,\ F\right).$$

4.8.3 Is $\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)$ a $\mathcal{G}$-Banach space?

In this subsection, let B be a $\mathcal{G}$-Banach algebra and let E and F be $\mathcal{G}$-Banach B-pairs. Let α^B, α^E and α^F denote the $\mathcal{G}$-actions on B, E and F, respectively. Note that we already have an "action" of $\mathcal{G}$ on $\mathrm{K}_B(E,F)$, namely the isomorphism $\alpha^{\mathrm{K}(E,F)}\colon \mathrm{K}^{\mathbf{loc}}_{s^*B}\left(s^*E,\ s^*F\right) \to \mathrm{K}^{\mathbf{loc}}_{r^*B}\left(r^*E,\ r^*F\right)$ defined in Proposition 3.4.25. The restriction clearly is an isometric isomorphism $\mathrm{K}_{s^*B}\left(s^*E,\ s^*F\right) \to \mathrm{K}_{r^*B}\left(r^*E,\ r^*F\right)$. If s and r are *open* maps, then we can identify these spaces (regarded as fields of Banach spaces) with $s^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)$ and $r^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)$, respectively, and can used it to define an action on $\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)$:

Definition and Proposition 4.8.7. Let $\mathcal{G}$ have *open* range and source maps. Let Ψ_s denote the isometric isomorphism $\Gamma_0\left(\mathcal{G},\ s^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)\right) \to \mathrm{K}_{s^*B}\left(s^*E,\ s^*F\right)$ and define Ψ_r analogously. Then there is a unique continuous field of linear maps

$$\alpha^{\mathfrak{F}(\mathrm{K}(E,F))}\colon s^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right) \to r^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)$$

such that the following diagram is commutative

$$\begin{tikzcd}
\Gamma_0\left(\mathcal{G},\ s^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)\right) \arrow[r, "T\mapsto\alpha^{\mathfrak{F}(\mathrm{K}(E,F))}\circ T"] \arrow[d, "\Psi_s"'] & \Gamma_0\left(\mathcal{G},\ r^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)\right) \arrow[d, "\Psi_r"] \\
\mathrm{K}_{s^*B}\left(s^*E,\ s^*F\right) \arrow[r, "\alpha^{\mathrm{K}(E,F)}"] & \mathrm{K}_{r^*B}\left(r^*E,\ r^*F\right)
\end{tikzcd}$$

It is an isometric continuous field of isomorphisms.

Proof. The map $\Psi_r^{-1}\circ\alpha^{\mathrm{K}(E,F)}\circ\Psi_s$ from $\Gamma_0\left(\mathcal{G},\ s^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)\right)$ to $\Gamma_0\left(\mathcal{G},\ r^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)\right)$ is an isometric $\mathcal{C}_0(\mathcal{G})$-linear isomorphism. It therefore comes from an isometric continuous field of isomorphisms from $s^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)$ to $r^*\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)$ which we call $\alpha^{\mathfrak{F}(\mathrm{K}(E,F))}$. □

Conjecture 4.8.8. Let $\mathcal{G}$ have open range and source maps. Then $\alpha^{\mathfrak{F}(\mathrm{K}(E,F))}$ is an action of $\mathcal{G}$ on $\mathfrak{F}\left(\mathrm{K}_B(E,F)\right)$.

We already know that $\alpha := \alpha^{\mathfrak{F}(\mathrm{K}(E,F))}$ is an isometric continuous field of isomorphisms. It remains to show the (algebraic) identity $\alpha_\gamma \circ \alpha_{\gamma'} = \alpha_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \mathcal{G}$ such that $s(\gamma) = r(\gamma')$. This looks fairly innocent, and if it is true, then the proof is probably rather simple. Nevertheless, this question remains open for now, and fortunately, the result is not needed for other parts of this thesis; however, it would make some constructions more systematic, in particular Subsection 5.2.7: The convolution with fields of compact operators would then be closer to the ordinary convolution product. Note that the conjecture implies in particular that $\mathfrak{F}\left(\mathrm{K}_B(E)\right)$ is a $\mathcal{G}$-Banach algebra because the “action” of $\mathcal{G}$ is clearly compatible with the composition.

Chapter 5

The Descent

The descent for locally compact Hausdorff groupoids $\mathcal{G}$ and for $\mathcal{G}$-C*-algebras was first considered in [LG94]; the descent for $\mathcal{G}$-Banach algebras was introduced in [Laf06] in Section 1.3, being a homomorphism

$$j_{\mathcal{A}}\colon\ \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A,B)\ \to\ \mathrm{KK}^{\mathrm{ban}}(\mathcal{A}(\mathcal{G},A),\,\mathcal{A}(\mathcal{G},B)),$$

where A and B are $\mathcal{G}$-Banach algebras and $\mathcal{A}(\mathcal{G},\ A)$ and $\mathcal{A}(\mathcal{G},\ B)$ are completions of $\Gamma_c(\mathcal{G},\ r^*A)$ and $\Gamma_c(\mathcal{G},\ r^*B)$, respectively, for semi-norms which are induced by an unconditional completion[1] $\mathcal{A}(\mathcal{G})$ of $\mathcal{C}_c(\mathcal{G})$. In the present chapter, we improve this homomorphism a little bit, showing that it is indeed a homomorphism

$$j_{\mathcal{A}}\colon\ \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A,B)\ \to\ \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G},A),\ \mathcal{A}(\mathcal{G},B)).$$

Note that we assume in this chapter that the topology on $X/\mathcal{G}$ is locally compact Hausdorff whenever we want to take the extra $\mathcal{C}_0(X/\mathcal{G})$-structure into account. This is automatic if $\mathcal{G}$ is proper. We try to give a rather detailed and systematic treatment of the descent, and this means in particular that we follow two rules:

1. We standardise the formulae for the convolution product; this is done by always using the pull-back along the range map instead of sometimes pulling back along the source map, i.e., we use always $\Gamma_c(\mathcal{G},\ r^*E)$ instead of $\Gamma_c(\mathcal{G},\ s^*E)$. The result is that we can always work with the same convolution formula (5.2). Note that we therefore arrive at a definition of the descent which is slightly different from but equivalent to the one in [Laf06].

2. We try to prove as much as possible on the level of sections with compact support. Most of the definitions make sense already on this level, and algebraic questions and questions concerning the density of certain subsets can be settled in this framework. After forming the (unconditional) completions the corresponding questions can then easily be answered in the setting of Banach algebras.

The last part of the chapter deals with the question what happens if we change the underlying groupoid. Later on we will discuss this question in the framework of generalised morphisms of groupoids[2], but for now we confine ourselves to the case of moving to a subgroupoid of a special kind; actually, this case will later turn out to be rather close to the general case of (Morita) equivalent groupoids.

[1] See Definition 5.2.1.

[2] See Chapter 6, in particular Subsection 6.6.4.

5.1 Convolution and sections with compact support

A topological groupoid $\mathcal{G}$ is called locally compact if $\mathcal{G}$ is locally compact as a topological space, i.e., every point in $\mathcal{G}$ has a compact neighbourhood. In this thesis we will assume that our locally compact groupoids are Hausdorff; in this case it is quite trivial that $\mathcal{G}^{(0)}$ is closed, locally compact and Hausdorff. [3]

Definition 5.1.1 (Haar system). Let $\mathcal{G}$ be a locally compact Hausdorff groupoid. A left Haar system λ on $\mathcal{G}$ is a faithful continuous field[4] $(\lambda^g)_{g\in\mathcal{G}^{(0)}}$ of measures on $\mathcal{G}$ over $\mathcal{G}^{(0)}$ with coefficient map r such that

$$\forall\gamma\in\mathcal{G}\,\forall\varphi\in\mathcal{C}_c(\mathcal{G}):\int_{\gamma'\in\mathcal{G}^{r(\gamma)}}\varphi(\gamma')\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')=\int_{\gamma'\in\mathcal{G}^{s(\gamma)}}\varphi(\gamma\gamma')\,\mathrm{d}\lambda^{s(\gamma)}(\gamma'). \tag{5.1}$$

Note that such a Haar system need not exist. If $\mathcal{G}$ is a locally compact Hausdorff groupoid admitting a Haar system, then it follows from Lemma B.2.4 that its range and source maps are open.

For the rest of this section, let $\mathcal{G}$ be a locally compact Hausdorff groupoid with left Haar system λ. Write X for the unit space $\mathcal{G}^{(0)}$.

5.1.1 Bilinear maps and the convolution product

Definition and Proposition 5.1.2. Let E_1, E_2 and F be $\mathcal{G}$-Banach spaces.[5] Let $\mu\colon E_1\times_X E_2\to F$ be a continuous field of bilinear maps (so that $\mu_x\colon(E_1)_x\times(E_2)_x\to F_x$ for all $x\in X=\mathcal{G}^{(0)}$). We define

$$\mu(\xi_1,\xi_2)(\gamma'):=\int_{\mathcal{G}^{r(\gamma')}}\mu_{r(\gamma')}\left(\xi_1(\gamma),\ \gamma\left(\xi_2(\gamma^{-1}\gamma')\right)\right)\,\mathrm{d}\lambda^{r(\gamma')}(\gamma) \tag{5.2}$$

for all $\xi_1\in\Gamma_c(\mathcal{G},r^*E_1)$, $\xi_2\in\Gamma_c(\mathcal{G},r^*E_2)$ and $\gamma'\in\mathcal{G}$. Then $\mu(\xi_1,\xi_2)$ is in $\Gamma_c(\mathcal{G},r^*F)$ and $(\xi_1,\xi_2)\mapsto\mu(\xi_1,\xi_2)$ defines a separately continuous bilinear map which is non-degenerate if μ is non-degenerate.

If μ is written as a product, then we simply write $\xi_1*\xi_2$ for $\mu(\xi_1,\xi_2)$. If μ is written as a bracket $\langle\cdot,\cdot\rangle$ then we write $\langle\xi_1,\xi_2\rangle$ for $\mu(\xi_1,\xi_2)$.

The proof of 5.1.2 is a refined version of the proof of Proposition 7.1.1 in [LG94]; this proposition states that the above formula makes sense for μ being the multiplication of a $\mathcal{G}$-C*-algebra. We are not only interested in the fact that $\mu(\xi_1,\xi_2)$ is a well-defined element of $\Gamma_c(\mathcal{G},r^*F)$, but also in the continuity and non-degeneracy of the product of sections, and therefore we have to work a little more. In [Laf06] the general Definition 5.1.2 is not stated and the special cases given there are not proved explicitly, although some variant of the proof given here is certainly in the background.

Our proof rests on the following lemma which is proved in Appendix C.1.

Lemma 5.1.3. *Let $\xi_1\in\Gamma(\mathcal{G},r^*E_1)$ and $\xi_2\in\Gamma(\mathcal{G},r^*E_2)$ be sections (with arbitrary support). Then*

$$\tilde{\mu}(\xi_1,\xi_2)(\gamma,\gamma')=\mu_{r(\gamma)}\left(\xi_1(\gamma),\ \alpha_\gamma(\xi_2(\gamma'))\right)$$

[3] As shown in [Tu04], Proposition 2.5, the unit space $\mathcal{G}^{(0)}$ of a locally compact (possibly non-Hausdorff) $\mathcal{G}$ is locally closed in $\mathcal{G}$ and hence locally compact as well.

[4] See Definition B.2.1.

[5] For the definition and some of the basic properties, it suffices to assume that E_1 and F are u.s.c. fields of Banach spaces over $\mathcal{G}^{(0)}$.

is in $\Gamma(\mathcal{G}^{(2)},\ \pi_1^* r^* F)$, where $\pi_1 \colon \mathcal{G}^{(2)} \to \mathcal{G}$ is the projection onto the first coordinate. The map $\tilde{\mu}$ is bilinear and jointly continuous for uniform convergence on compact subsets. The support of $\tilde{\mu}(\xi_1, \xi_2)$ is contained in $\operatorname{supp}\xi_1 \times \operatorname{supp}\xi_2$, so if ξ_1 and ξ_2 have compact support, so has their product. On the sections with compact support, $\tilde{\mu}$ is separately continuous. If μ is non-degenerate, then $\tilde{\mu}$ is non-degenerate in two senses: Firstly, it sends the product of two total subsets to a total subset, and secondly, the set $\Xi := \{\tilde{\mu}(\xi_1, \xi_2) :\ \xi_i \in \Gamma_c(\mathcal{G}, r^* E_i)\}$ spans a subset of $\Gamma_c(\mathcal{G}^{(2)}, \pi_1^* r^* F)$ which is dense for the inductive limit topology.

Proof of 5.1.2. First define the map $\Phi \colon \mathcal{G} *_{r,r} \mathcal{G} \to \mathcal{G}^{(2)} = \mathcal{G} *_{s,r} \mathcal{G},\ (\gamma, \gamma') \mapsto (\gamma, \gamma^{-1}\gamma')$. This is a homeomorphism. Let p_1 and p_2 denote the projections of $\mathcal{G} *_{r,r} \mathcal{G}$ onto the first and second component, respectively. Then $\pi_1 \circ \Phi = p_1$ (quite trivially), and we have $\Phi^*(\pi_1^* r^* F) = p_1^* r^* F = p_2^* r^* F$. The map Φ therefore induces an isomorphism

$$\Gamma_c(\mathcal{G}^{(2)},\ \pi_1^* r^* F) \to \Gamma_c\left(\mathcal{G} *_{r,r} \mathcal{G},\ p_2^* r^* F\right)$$

which sends some η to $(\gamma, \gamma') \mapsto \eta(\gamma, \gamma^{-1}\gamma')$. In particular, it sends our $\tilde{\mu}(\xi_1, \xi_2)$ to

$$(\gamma, \gamma') \mapsto \mu_{r(\gamma)}\left(\xi_1(\gamma),\ \alpha_\gamma\left(\xi_2(\gamma^{-1}\gamma')\right)\right).$$

Note that this is the integrand in the convolution formula and a section of compact support.

Now we define a suitable continuous field of measures on $\mathcal{G} *_{r,r} \mathcal{G}$. Consider the map $p_2 \colon \mathcal{G} *_{r,r} \mathcal{G} \to \mathcal{G}$. Its fibres are of the form $p_2^{-1}(\{\gamma'\}) = \{(\gamma, \gamma') :\ \gamma \in \mathcal{G}, r(\gamma) = r(\gamma')\}$ for each $\gamma' \in \mathcal{G}$. These fibres are homeomorphic to $\mathcal{G}^{r(\gamma')}$, so we can put the measure $\lambda^{r(\gamma')}$ on them. Technically, we are forming the pullback $r^*\lambda$ by r of the continuous field of measures λ on $\mathcal{G}$ with coefficient map r (see Definition B.2.8):

$$\begin{tikzcd}
(\mathcal{G},\ \lambda) \arrow[d, "r"] & (\mathcal{G} *_{r,r} \mathcal{G},\ r^*\lambda) \arrow[l, "p_1"'] \arrow[d, "p_2"] \\
X = \mathcal{G}^{(0)} & \mathcal{G} \arrow[l, "r"']
\end{tikzcd}$$

By Proposition B.3.1 we can deduce that $r^*\lambda$ maps $\Gamma_c(\mathcal{G} *_{r,r} \mathcal{G},\ p_2^* r^* F)$ to $\Gamma_c(\mathcal{G},\ r^* F)$, and this map is onto since λ is faithful and so is $r^*\lambda$. The composition of $\tilde{\mu}$, the isomorphism induced by Φ and $r^*\lambda$ is our convolution product $(\xi_1, \xi_2) \mapsto \mu(\xi_1, \xi_2)$, which is therefore well-defined, separately continuous and non-degenerate if μ is non-degenerate. $\square$

By direct calculation of the involved integrals one can prove:

Proposition 5.1.4 (Preservation of associativity). *Let E_1, E_2, E_3, F_1, F_2 and G be $\mathcal{G}$-Banach spaces. Let $\mu_1 \colon E_1 \times_X E_2 \to F_1$, $\mu_2 \colon E_2 \times_X E_3 \to F_2$, $\nu_1 \colon F_1 \times_X E_3 \to G$ and $\nu_2 \colon E_1 \times_X F_2 \to G$ be continuous fields of bilinear maps. Assume that the following associativity law holds:*

$$(\nu_1)_x\big((\mu_1)_x(e_1, e_2),\ e_3\big) = (\nu_2)_x\big(e_1,\ (\mu_2)_x(e_2, e_3)\big)$$

for all $x \in X = \mathcal{G}^{(0)}$, $e_1 \in (E_1)_x$, $e_2 \in (E_2)_x$, and $e_3 \in (E_3)_x$. If, in addition, μ_2 is $\mathcal{G}$-equivariant, then the same associativity law holds on the level of sections with compact support:

$$\nu_1\big(\mu_1(\xi_1, \xi_2),\ \xi_3\big) = \nu_2\big(\xi_1,\ \mu_2(\xi_2, \xi_3)\big)$$

for all $\xi_1 \in \Gamma_c(\mathcal{G}, r^* E_1)$, $\xi_2 \in \Gamma_c(\mathcal{G}, r^* E_2)$, and $\xi_3 \in \Gamma_c(\mathcal{G}, r^* E_3)$.

5.1.2 Linear maps

Definition and Proposition 5.1.5. Let E and F be $\mathcal{G}$-Banach spaces and T a continuous field of linear maps between them. Then $\xi \mapsto (\gamma \mapsto T_{r(\gamma)}\xi(\gamma))$ defines a continuous linear map, we call it $T * \cdot$ or $\Gamma_c(\mathcal{G}, r^*T)$, from $\Gamma_c(\mathcal{G}, r^*E)$ to $\Gamma_c(\mathcal{G}, r^*F)$.

The notation $T * \cdot$ for the linear map defined in 5.1.5 is justified by the following consideration: If one thinks of $T = (T_g)_{g\in\mathcal{G}^{(0)}}$ as a kind of distribution on $\mathcal{G}$ which assigns to every $g \in \mathcal{G}^{(0)} = X$ the operator T_g with mass 1 and zero to all other elements of $\mathcal{G}$, then the convolution product $T * \xi$ for $\xi \in \Gamma_c(\mathcal{G}, r^*E)$ can be calculated as

$$(T * \xi)(\gamma) = \int_{\mathcal{G}^{r(\gamma)}} T_{\gamma'}\gamma'\xi\left(\gamma'^{-1}\gamma\right)\, \mathrm{d}\lambda^{r(\gamma)}(\gamma')$$

where the integrand is zero for $\gamma' \neq r(\gamma)$. If $\gamma' = r(\gamma)$, then the integrand (and hence the integral) gives $(T * \xi)(\gamma) = T_{r(\gamma)}\xi(\gamma)$ as desired.

Now the questions arises what happens if we let T act on the right on E, a phenomenon which must be discussed because we are going to meet this situation when considering the left-hand side of a Banach pair. In this case we can formally calculate

$$(\xi * T)(\gamma) = \int_{\mathcal{G}^{r(\gamma)}} \xi(\gamma')\gamma' T_{\gamma'^{-1}\gamma}\, \mathrm{d}\lambda^{r(\gamma)}(\gamma').$$

Now the integrand vanishes if $\gamma' \neq \gamma$, whereas the case $\gamma' = \gamma$ yields $(\xi * T)(\gamma) = \xi(\gamma)\gamma T_{s(\gamma)}$. To further evaluate this, it would be desirable to translate the $\mathcal{G}$-action on E into an action on the right. However, we just translate the right action of T back into a left action to get $(\xi * T)(\gamma) = \left[\gamma T_{s(\gamma)}\right]\xi(\gamma) = \gamma\left[T_{s(\gamma)}(\gamma^{-1}\xi(\gamma))\right]$. Of course, this only makes sense if $\mathcal{G}$ acts on E and F (instead of E and F just being continuous fields over $\mathcal{G}^{(0)}$ as above). As a conclusion, we have the following proposition:

Definition and Proposition 5.1.6. Let E and F be $\mathcal{G}$-Banach spaces and T a bounded continuous field of linear maps between them. Then $\xi \mapsto \gamma\left[T_{s(\gamma)}(\gamma^{-1}\xi(\gamma))\right]$ defines a continuous linear map, which we denote by $\cdot * T$, from $\Gamma_c(\mathcal{G}, r^*E)$ to $\Gamma_c(\mathcal{G}, r^*F)$. Note that $T * \cdot = \cdot * T$ if T is $\mathcal{G}$-equivariant.

The interplay of linear and bilinear maps and the descent procedure can be summarized in the following general proposition. It can be proved by direct calculation.

Proposition 5.1.7 (Linear and bilinear maps). *Let E_1, E_2, F, E_1', E_2' and F' be $\mathcal{G}$-Banach spaces. Let $S_i\colon E_i \to E_i'$ for $i = 1, 2$ and $T\colon F \to F'$ be continuous fields of linear maps. Let $\mu\colon E_1 \times_X E_2 \to F$ and $\mu'\colon E_1' \times_X E_2' \to F'$ be continuous fields of bilinear maps. Assume that*

$$\mu' \circ (S_1 \times_X S_2) = T \circ \mu.$$

1. *If S_2 is $\mathcal{G}$-equivariant, then*

$$\mu'\left(S_1 * \xi_1,\ \xi_2 * S_2\right) = T * \mu\left(\xi_1,\ \xi_2\right)$$

for all $\xi_1 \in \Gamma_c(\mathcal{G}, r^*E_1)$ and $\xi_2 \in \Gamma_c(\mathcal{G}, r^*E_2)$.

2. *If the linear map S_1 and the bilinear maps μ and μ' are $\mathcal{G}$-equivariant, then*

$$\mu'\left(S_1 * \xi_1,\ \xi_2 * S_2\right) = \mu\left(\xi_1,\ \xi_2\right) * T$$

for all $\xi_1 \in \Gamma_c(\mathcal{G}, r^*E_1)$ and $\xi_2 \in \Gamma_c(\mathcal{G}, r^*E_2)$.

5.1.3 Banach algebras, modules and pairs

We have seen above how one can lift fields of bilinear maps to bilinear maps between the respective spaces of sections of compact support. This applies in particular to the multiplication of $\mathcal{G}$-Banach algebras and the other fields of bilinear maps that appear in the definition of $\mathcal{G}$-Banach modules and $\mathcal{G}$-Banach pairs. In the preceding paragraph we have discussed the interplay of the lifts of linear and bilinear maps. We now apply these considerations to homomorphisms between $\mathcal{G}$-Banach algebras, $\mathcal{G}$-Banach modules etc.

Banach algebras and Banach modules

As a special case of 5.1.2 we obtain the following result:

Proposition 5.1.8. *Let B be a $\mathcal{G}$-Banach algebra. Then $\Gamma_c(\mathcal{G}, r^*B)$ is an associative algebra with the convolution product*

$$(\xi_1 * \xi_2)(\gamma') := \int_{\mathcal{G}^{r(\gamma')}} \xi_1(\gamma)\, \gamma\xi_2(\gamma^{-1}\gamma')\, \mathrm{d}\lambda^{r(\gamma')}(\gamma)$$

for all $\gamma' \in \mathcal{G}$, $\xi_1, \xi_2 \in \Gamma_c(\mathcal{G}, r^*B)$. If B is non-degenerate, then the linear span of $\Gamma_c(\mathcal{G}, r^*B) * \Gamma_c(\mathcal{G}, r^*B)$ is dense in $\Gamma_c(\mathcal{G}, r^*B)$ for the inductive limit topology.

We can lift homomorphism of $\mathcal{G}$-Banach algebras:

Proposition 5.1.9. *Let B and C be $\mathcal{G}$-Banach algebras and let φ denote a $\mathcal{G}$-equivariant field of homomorphisms between them. Then $\Gamma_c(\mathcal{G},\, r^*\varphi)$ is a continuous homomorphism of algebras from $\Gamma_c(\mathcal{G},\, r^*B)$ to $\Gamma_c(\mathcal{G},\, r^*C)$.*

Proof. Use the first part of Proposition 5.1.7 with $E_1 = E_2 = F = B$, $E_1' = E_2' = F' = C$, $S_1 = S_2 = T = \varphi$ and μ and μ' being the multiplication of B and C, respectively. □

What we have done for $\mathcal{G}$-Banach algebras also applies to $\mathcal{G}$-Banach modules (and equivariant homomorphism between them):

Proposition 5.1.10. *Let B be a $\mathcal{G}$-Banach algebra and E a right $\mathcal{G}$-Banach B-module. Then the right module action of B on E gives rise to a right module action of the algebra $\Gamma_c(\mathcal{G}, r^*B)$ on $\Gamma_c(\mathcal{G}, r^*E)$. If the action of B on E is non-degenerate, then the linear span of $\Gamma_c(\mathcal{G}, r^*E) * \Gamma_c(\mathcal{G}, r^*B)$ is dense in $\Gamma_c(\mathcal{G}, r^*E)$ for the inductive limit topology.*

Proposition 5.1.11. *Let B and B' be $\mathcal{G}$-Banach algebras and let φ denote a $\mathcal{G}$-equivariant field of homomorphisms between them. Let E be a right $\mathcal{G}$-Banach B-module and let E' be a right $\mathcal{G}$-Banach B'-module. Let Φ be a $\mathcal{G}$-equivariant homomorphism from E to E' with coefficient map φ. Then $\Gamma_c(\mathcal{G},\, r^*\Phi)$ is a continuous homomorphism of modules from $\Gamma_c(\mathcal{G},\, r^*E)$ to $\Gamma_c(\mathcal{G},\, r^*E')$ with coefficient map $\Gamma_c(\mathcal{G},\, r^*\varphi)$.*

Proof. Use the first part of Proposition 5.1.7 with $E_1 = F = E$, $E_2 = B$, $E_1' = F' = E'$, $E_2' = B'$, $S_1 = T = \Phi$, $S_2 = \varphi$, and μ and μ' being the module action of B on E and of B' on E', respectively. □

Not only equivariant homomorphisms lift to the level of sections of compact support, but also linear operators. Note that we do not require the linear operators to be equivariant. This makes it necessary to discuss left and right modules separately:

Proposition 5.1.12. *Let B be a $\mathcal{G}$-Banach algebra and let E, E' be right $\mathcal{G}$-Banach B-modules. Let T be a B-linear continuous field of linear operators from E to E' (not necessarily equivariant). Then $\Gamma_c(\mathcal{G}, r^*T)$ is a continuous $\Gamma_c(\mathcal{G}, r^*B)$-linear map from $\Gamma_c(\mathcal{G}, r^*E)$ to $\Gamma_c(\mathcal{G}, r^*E')$.*

Proof. Use the first part of Proposition 5.1.7 with $E_1 = F = E$, $E_2 = E'_2 = B$, $E'_1 = F' = E'$, $S_1 = T$, $S_2 = \mathrm{Id}_B$, and μ and μ' being the module action of B on E and on E', respectively. □

Proposition 5.1.13. *Let B be a $\mathcal{G}$-Banach algebra and let E, E' be left $\mathcal{G}$-Banach B-modules. Let T be a B-linear continuous field of linear operators from E to E' (not necessarily equivariant). Then $\cdot * T$ is a continuous $\Gamma_c(\mathcal{G}, r^*B)$-linear map from $\Gamma_c(\mathcal{G}, r^*E)$ to $\Gamma_c(\mathcal{G}, r^*E')$.*

Proof. Use the second part of Proposition 5.1.7 with $E_2 = F = E$, $E_1 = E'_1 = B$, $E'_2 = F' = E'$, $S_2 = T$, $S_1 = \mathrm{Id}_B$, and μ and μ' being the module action of B on E and on E', respectively. □

Banach pairs

We can also lift the bracket of a $\mathcal{G}$-Banach pair to the level of sections with compact support:

Proposition 5.1.14. *Let B be a $\mathcal{G}$-Banach algebra and let E be a $\mathcal{G}$-Banach B-pair. Then the space $\Gamma_c(\mathcal{G}, r^*E^>)$ is a right $\Gamma_c(\mathcal{G}, r^*B)$-module and $\Gamma_c(\mathcal{G}, r^*E^<)$ is a left $\Gamma_c(\mathcal{G}, r^*B)$-module. Moreover, the bracket of E induces a bilinear map*

$$\langle \cdot, \cdot \rangle_{\Gamma_c(\mathcal{G}, r^*E)} \colon \Gamma_c(\mathcal{G}, r^*E^<) \times \Gamma_c(\mathcal{G}, r^*E^>) \to \Gamma_c(\mathcal{G}, r^*B)$$

which is $\Gamma_c(\mathcal{G}, r^*B)$-linear on the left and on the right.

The following proposition says that the descent of a linear operator between $\mathcal{G}$-Banach pairs is a formally adjoint pair of linear operators between the respective pairs of spaces of sections with compact support. More precisely:

Proposition 5.1.15. *Let B be a $\mathcal{G}$-Banach algebra and let E and F be $\mathcal{G}$-Banach B-pairs. Let $T = (T^<, T^>)$ be an element of $\mathrm{L}_B(E, F)$. Then*

1. *$T^> * \cdot$ is a continuous linear operator from $\Gamma_c(\mathcal{G}, r^*E^>)$ to $\Gamma_c(\mathcal{G}, r^*F^>)$ being $\Gamma_c(\mathcal{G}, r^*B)$-linear on the right;*
2. *$\cdot * T^<$ is a continuous linear operator from $\Gamma_c(\mathcal{G}, r^*F^<)$ to $\Gamma_c(\mathcal{G}, r^*E^<)$ being $\Gamma_c(\mathcal{G}, r^*B)$-linear on the left;*
3. *for all $\xi^> \in \Gamma_c(\mathcal{G}, r^*E^>)$ and all $\eta^< \in \Gamma_c(\mathcal{G}, r^*F^<)$ we have*

$$\left\langle \eta^<, T^> * \xi^> \right\rangle_{\Gamma_c(\mathcal{G}, r^*F)} = \left\langle \eta^< * T^<, \xi^> \right\rangle_{\Gamma_c(\mathcal{G}, r^*E)} \in \Gamma_c(\mathcal{G}, r^*B).$$

Proof.
1. This is Proposition 5.1.12.
2. This is Proposition 5.1.13.

3. Let $\xi^{>} \in \Gamma_c(\mathcal{G},\, r^*E^{>})$ and all $\eta^{<} \in \Gamma_c(\mathcal{G},\, r^*F^{<})$. For all $\gamma \in \mathcal{G}$, we have

$$\begin{aligned}
\left\langle \eta^{<},\, T^{>} * \xi^{>} \right\rangle_{\Gamma_c(\mathcal{G},\, r^*F)}(\gamma) &= \int_{\mathcal{G}^{r(\gamma)}} \left\langle \eta^{<}(\gamma'),\, \gamma'\left(T^{>} * \xi^{>}\right)(\gamma'^{-1}\gamma) \right\rangle \,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= \int_{\mathcal{G}^{r(\gamma)}} \left\langle \eta^{<}(\gamma'),\, \gamma'\left(T_{s(\gamma')}\xi^{>}(\gamma'^{-1}\gamma)\right) \right\rangle \,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= \int_{\mathcal{G}^{r(\gamma)}} \left\langle \eta^{<}(\gamma'),\, \gamma'\left(T_{s(\gamma')}\gamma'^{-1}\gamma'\xi^{>}(\gamma'^{-1}\gamma)\right) \right\rangle \,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= \int_{\mathcal{G}^{r(\gamma)}} \left\langle \gamma'\left(T_{s(\gamma')}\gamma'^{-1}\eta^{<}(\gamma')\right),\, \gamma'\xi^{>}(\gamma'^{-1}\gamma) \right\rangle \,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= \left\langle \eta^{<} * T^{<},\, \xi^{>} \right\rangle_{\Gamma_c(\mathcal{G},\, r^*E)}(\gamma).
\end{aligned}$$

□

5.1.4 The actions of $\mathcal{C}(X)$ and $\mathcal{C}(X/\mathcal{G})$

Definition 5.1.16. Let E be a $\mathcal{G}$-Banach space. Then we define

- a "left" module action of $\mathcal{C}(X)$ on $\Gamma_c(\mathcal{G},\, r^*E)$ by setting
$$(\chi \cdot \xi)(\gamma) = \chi(r(\gamma))\xi(\gamma) \qquad \chi \in \mathcal{C}(X),\ \xi \in \Gamma_c(\mathcal{G},\, r^*E),\ \gamma \in \mathcal{G};$$
- a "right" module action of $\mathcal{C}(X)$ on $\Gamma_c(\mathcal{G},\, r^*E)$ by setting
$$(\xi \cdot \chi)(\gamma) = \chi(s(\gamma))\xi(\gamma) \qquad \chi \in \mathcal{C}(X),\ \xi \in \Gamma_c(\mathcal{G},\, r^*E),\ \gamma \in \mathcal{G};$$
- a module action of $\mathcal{C}(X/\mathcal{G})$ on $\Gamma_c(\mathcal{G},\, r^*E)$ by setting
$$(\chi \cdot \xi)(\gamma) = \chi(\pi(\gamma))\xi(\gamma) \qquad \chi \in \mathcal{C}(X/\mathcal{G}),\ \xi \in \Gamma_c(\mathcal{G},\, r^*E),\ \gamma \in \mathcal{G},$$
where $\pi\colon \mathcal{G} \to X/\mathcal{G}$ denotes the map $\gamma \mapsto [r(\gamma)] = [s(\gamma)]$.

$\Gamma_c(\mathcal{G},\, r^*E)$ is a $\mathcal{C}(X)$-bimodule when equipped with the left and right action.

Note that the action of $\mathcal{C}(X/\mathcal{G})$ is coherent with the left and right action of $\mathcal{C}(X)$ on $\Gamma_c(\mathcal{G},\, r^*E)$ in the sense that pulling back a function $\chi \in \mathcal{C}(X/\mathcal{G})$ to a function in $\mathcal{C}(X)$ and letting it act on $\Gamma_c(\mathcal{G},\, r^*E)$ gives the same action, no matter whether we choose the left or the right action of $\mathcal{C}(X)$.

Let E be a $\mathcal{G}$-Banach space. For all $\xi \in \Gamma_c(\mathcal{G},\, r^*E)$ there is a function $\chi \in \mathcal{C}_c(X)$ such that $\chi\xi = \xi$ and such that $\xi\chi = \xi$; and there is a function $\chi' \in \mathcal{C}_c(X/\mathcal{G})$ such that $\chi'\xi = \xi$. So the actions of $\mathcal{C}_c(X)$ and $\mathcal{C}_c(X/\mathcal{G})$ are non-degenerate in a strong sense.

By direct calculation we get the following formulae.

Proposition 5.1.17. *Let E_1, E_2 and F be $\mathcal{G}$-Banach spaces. Let $\mu\colon E_1 \times_X E_2 \to F$ be a continuous field of bilinear maps. Let $\mathcal{G}$ act on E_2. Then for all $\xi_1 \in \Gamma_c(\mathcal{G},\, r^*E_1)$, $\xi_2 \in \Gamma_c(\mathcal{G},\, r^*E_2)$, $\chi \in \mathcal{C}(X)$ and $\chi' \in \mathcal{C}(X/\mathcal{G})$:*

1. $\chi \cdot \mu(\xi_1,\, \xi_2) = \mu(\chi \cdot \xi_1,\, \xi_2)$,
2. $\mu(\xi_1,\, \xi_2) \cdot \chi = \mu(\xi_1,\, \xi_2 \cdot \chi)$,
3. $\mu(\xi_1 \cdot \chi,\, \xi_2) = \mu(\xi_1,\, \chi \cdot \xi_2)$,
4. $\chi' \cdot \mu(\xi_1,\, \xi_2) = \mu(\chi' \cdot \xi_1,\, \xi_2) = \mu(\xi_1,\, \chi' \cdot \xi_2)$.

5.2 Unconditional completions

Let $\mathcal{G}$ be a locally compact Hausdorff groupoid with left Haar-system λ. Write X for $\mathcal{G}^{(0)}$.

5.2.1 Unconditional norms and fields of Banach spaces

The notion of an unconditional norm for $\mathcal{C}_c(\mathcal{G})$ was first defined in [Laf02] for the group case and in [Laf06] for $\mathcal{G}$ being a groupoid.

Definition 5.2.1. An *unconditional completion* $\mathcal{A}(\mathcal{G})$ of $\mathcal{C}_c(\mathcal{G})$ is a Banach algebra containing $\mathcal{C}_c(\mathcal{G})$ as a dense subalgebra and having the following property

$$\forall f_1, f_2 \in \mathcal{C}_c(\mathcal{G}): \quad \big(\forall \gamma \in \mathcal{G}: \ |f_1(\gamma)| \leq |f_2(\gamma)|\big) \ \Rightarrow \ \|f_1\|_{\mathcal{A}(\mathcal{G})} \leq \|f_2\|_{\mathcal{A}(\mathcal{G})}. \tag{5.3}$$

In this case we say that the norm of $\mathcal{A}(\mathcal{G})$ is unconditional. We also write $\|\cdot\|_{\mathcal{A}}$ for the norm on $\mathcal{A}(\mathcal{G})$.

An unconditional norm is a special case of a monotone norm, compare Section 3.2. In particular, we can extend the norm to a semi-norm on $\mathcal{F}_c^+(\mathcal{G})$.

Examples 5.2.2.

1. For all $\chi \in \mathcal{C}_c(\mathcal{G})$, define

$$\|\chi\|_1 := \sup_{x \in X} \int_{\mathcal{G}^x} |\chi(\gamma)| \, \mathrm{d}\gamma.$$

This is an unconditional norm on $\mathcal{C}_c(\mathcal{G})$ and the corresponding unconditional completion is called $\mathrm{L}^1(\mathcal{G})$.

2. If we define $\chi^*(\gamma) := \overline{\chi(\gamma^{-1})}$ for all $\gamma \in \mathcal{G}$ and $\chi \in \mathcal{C}_c(\mathcal{G})$, then we can define a symmetrised version of the L^1-norm on $\mathcal{C}_c(\mathcal{G})$ by setting

$$\|\chi\| := \max\{\|\chi\|_1, \ \|\chi^*\|_1\}$$

for all $\chi \in \mathcal{C}_c(\mathcal{G})$. In [Ren80], the completion for this norm is called $\mathrm{L}^1(\mathcal{G})$, but we follow [Laf06] and call it $\mathrm{L}^1(\mathcal{G}) \cap \mathrm{L}^1(\mathcal{G})^*$.

3. In [Laf06], Section 3, the following unconditional completion is defined: For all $\chi \in \mathcal{C}_c(\mathcal{G})$, set

$$\|\chi\|_{\mathcal{A}_{\max}(\mathcal{G})} := \Big\|\gamma \mapsto |\chi(\gamma)|\Big\|_{\mathrm{C}_r^*(\mathcal{G})}.$$

Note that $\mathrm{C}_r^*(\mathcal{G})$ itself is very rarely unconditional.

4. If the groupoid $\mathcal{G}$ carries a length function[6] l and $\mathcal{A}(\mathcal{G})$ is an unconditional completion of $\mathcal{C}_c(\mathcal{G})$, then one can define the weighted norm

$$\|\chi\|_{\mathcal{A}_l(\mathcal{G})} := \Big\|\gamma \mapsto e^{l(\gamma)}\chi(\gamma)\Big\|_{\mathcal{A}(\mathcal{G})}$$

for all $\chi \in \mathcal{C}_c(\mathcal{G})$. This gives an unconditional completion $\mathcal{A}_l(\mathcal{G})$.

5. In the fourth chapter of [Laf02] V. Lafforgue defines generalised Schwartz spaces $\mathcal{S}_t^l(G, A)$ on which the convolution product (sometimes) defines a continuous multiplication. After renormalisation of the norm this would also be an example of an unconditional completion.

[6] See Définition 1.2.1 of [Laf06].

6. Let G be a locally compact Hausdorff group acting on some locally compact Hausdorff space X. Define $\mathcal{G} := G \ltimes X$. On $\mathcal{C}_c(\mathcal{G})$ there is the unconditional norm $\|\cdot\|_1$ from the first example, which can be calculated as
$$\|\chi\|_1 = \sup_{x\in X} \int_G |\chi(g,x)| \,\mathrm{d}g$$
for all $\chi \in \mathcal{C}_c(G \ltimes X)$. There is an alternative unconditional norm on $\mathcal{C}_c(G \ltimes X)$ coming from the algebra $\mathrm{L}^1(G,\, \mathcal{C}_0(X))$:
$$\|\chi\|_{\mathrm{L}^1(G,\, \mathcal{C}_0(X))} := \int_G \sup_{x\in X} |\chi(g,x)| \,\mathrm{d}g.$$
Note that we have $\|\chi\|_{\mathrm{L}^1(G,\, \mathcal{C}_0(X))} \leq \|\chi\|_1$ for all $\chi \in \mathcal{C}_c(G \ltimes X)$.

Fix an unconditional completion $\mathcal{A}(\mathcal{G})$ for the rest of this chapter.

If E is a $\mathcal{G}$-Banach space, then r^*E is a u.s.c. field of Banach spaces over $\mathcal{G}$. We can use the construction given in Definition 3.2.4 for general monotone semi-norms:

Definition 5.2.3 (The Banach space $\mathcal{A}(\mathcal{G}, E)$). Let E be a $\mathcal{G}$-Banach space. Then we define the following semi-norm on $\Gamma_c(\mathcal{G},\, r^*E)$:
$$\|\xi\|_{\mathcal{A}} := \left\| \gamma \mapsto \|\xi(\gamma)\|_{E_{r(\gamma)}} \right\|_{\mathcal{A}}.$$
The Hausdorff completion of $\Gamma_c(\mathcal{G},\, r^*E)$ with respect to this semi-norm will be denoted by $\mathcal{A}(\mathcal{G}, E)$ (and not by $\mathcal{A}(\mathcal{G},\, r^*E)$ to save some letters).

Note that the function $\gamma \mapsto \|\xi(\gamma)\|$ is not necessarily continuous but has at least compact support and is non-negative upper semi-continuous, so we can apply the extended norm on $\mathcal{F}_c^+(\mathcal{G})$ to it. If E is the trivial bundle over $\mathcal{G}^{(0)}$ with fibre E_0, then $\Gamma_c(\mathcal{G},\, r^*E)$ is $\mathcal{C}_c(\mathcal{G}, E_0)$ and $\mathcal{A}(\mathcal{G}, E)$ could also be denoted as $\mathcal{A}(\mathcal{G}, E_0)$; in particular, if $E_0 = \mathbb{C}$, then $\mathcal{A}(\mathcal{G}, E) = \mathcal{A}(\mathcal{G}, \mathbb{C}) = \mathcal{A}(\mathcal{G})$.

From the corresponding general result 3.2.6 for monotone completions we can deduce:

Proposition 5.2.4. *Let E be a $\mathcal{G}$-Banach space. Then the canonical map from $\Gamma_c(\mathcal{G},\, r^*E)$ to $\mathcal{A}(\mathcal{G}, E)$ is continuous with respect to the inductive limit topology on $\Gamma_c(\mathcal{G},\, r^*E)$ and the norm topology on $\mathcal{A}(\mathcal{G}, E)$.*

In particular, if Ξ is dense in $\Gamma_c(\mathcal{G}, r^*E)$ for the inductive limit topology, then its canonical image in $\mathcal{A}(\mathcal{G}, E)$ is dense for the norm topology.

5.2.2 Bilinear maps and the convolution product

In addition to the computational rules 3.2.3 for monotone completions we also have the following:

Lemma 5.2.5. *If $\varphi_1 * \varphi_2$ is defined for $\varphi_1, \varphi_2 \in \mathcal{F}_c^+(\mathcal{G})$, then $\varphi_1 * \varphi_2$ is in $\mathcal{F}_c^+(\mathcal{G})$ and $\|\varphi_1 * \varphi_2\|_{\mathcal{A}} \leq \|\varphi_1\|_{\mathcal{A}} \|\varphi_2\|_{\mathcal{A}}$*

Proof. Assume that $\varphi_1 * \varphi_2$ is defined by which we mean that the defining integral exists pointwise. Then the support of $\varphi_1 * \varphi_2$ is compact and the function is bounded by $\|\varphi_1\|_\infty \|\varphi_2\|_\infty \|\lambda(\chi)\|_\infty$ where χ is some function in $\mathcal{C}_c^+(\mathcal{G})$ which is 1 on $\operatorname{supp} \varphi_1$.

Let $\psi_1, \psi_2 \in \mathcal{C}_c(\mathcal{G})$ such that $\varphi_1 \leq \psi_1$ and $\varphi_2 \leq \psi_2$. Then for all $\gamma, \gamma' \in \mathcal{G}$ such that $r(\gamma) = r(\gamma')$:
$$\varphi_1(\gamma)\varphi_2(\gamma^{-1}\gamma') \leq \psi_1(\gamma)\psi_2(\gamma^{-1}\gamma').$$

Since the integral is monotonous, it follows that $(\varphi_1 * \varphi_2)(\gamma') \leq (\psi_1 * \psi_2)(\gamma')$ for all $\gamma' \in \mathcal{G}$. Now $\varphi_1 * \varphi_2$ is bounded, non-negative and of compact support and $\psi_1 * \psi_2$ is, in addition, continuous. It follows that

$$\|\varphi_1 * \varphi_2\|_{\mathcal{A}} \leq \|\psi_1 * \psi_2\|_{\mathcal{A}} \leq \|\psi_1\|_{\mathcal{A}} \|\psi_2\|_{\mathcal{A}}.$$

Taking the infimum on the right-hand side gives $\|\varphi_1 * \varphi_2\|_{\mathcal{A}} \leq \|\varphi_1\|_{\mathcal{A}} \|\varphi_2\|_{\mathcal{A}}$. □

Definition and Proposition 5.2.6. Let E_1, E_2, F be $\mathcal{G}$-Banach spaces and let $\mu\colon E_1 \times_X E_2 \to F$ be a bounded continuous field of bilinear maps. Then for all $\xi_1 \in \Gamma_c(\mathcal{G},\, r^*E_1)$ and $\xi_2 \in \Gamma_c(\mathcal{G},\, r^*E_2)$:

$$\|\mu(\xi_1,\xi_2)\|_{\mathcal{A}(\mathcal{G},F)} \leq \|\mu\|_\infty \|\xi_1\|_{\mathcal{A}(\mathcal{G},E_1)} \|\xi_2\|_{\mathcal{A}(\mathcal{G},E_2)}.$$

So μ lifts to a continuous bilinear map $\mathcal{A}(\mathcal{G},\mu)$ from $\mathcal{A}(\mathcal{G},E_1) \times \mathcal{A}(\mathcal{G},E_2)$ to $\mathcal{A}(\mathcal{G},F)$ (with norm less than or equal to $\|\mu\|_\infty$). If μ is non-degenerate, then so is $\mathcal{A}(\mathcal{G},\mu)$.

Proof. For all $\gamma' \in \mathcal{G}$, we have

$$\begin{aligned}
\left\|\mu(\xi_1,\xi_2)(\gamma')\right\|_{F_{r(\gamma')}} &= \left\|\int_{\mathcal{G}^{r(\gamma')}} \mu_{r(\gamma')}\left(\xi_1(\gamma),\ \gamma\left(\xi_2(\gamma^{-1}\gamma')\right)\right)\, \mathrm{d}\lambda^{r(\gamma')}(\gamma)\right\|_{F_{r(\gamma')}} \\
&\leq \int_{\mathcal{G}^{r(\gamma')}} \left\|\mu_{r(\gamma')}\right\| \|\xi_1(\gamma)\|_{(E_1)_{r(\gamma)}} \left\|\gamma\left(\xi_2(\gamma^{-1}\gamma')\right)\right\|_{(E_2)_{r(\gamma)}}\, \mathrm{d}\lambda^{r(\gamma')}(\gamma) \\
&\leq \|\mu\|_\infty \int_{\mathcal{G}^{r(\gamma')}} \|\xi_1(\gamma)\|_{(E_1)_{r(\gamma)}} \left\|\xi_2(\gamma^{-1}\gamma')\right\|_{(E_2)_{r(\gamma^{-1}\gamma')}}\, \mathrm{d}\lambda^{r(\gamma')}(\gamma) \\
&= \|\mu\|_\infty \left(|\xi_1| * |\xi_2|\right)(\gamma'),
\end{aligned}$$

where we use $|\xi_1|$ to denote $\gamma \mapsto \|\xi_1(\gamma)\|_{(E_1)_{r(\gamma)}}$ and similar for ξ_2. Note that $|\xi_1|$ and $|\xi_2|$ are not only[7] upper semi-continuous but also continuous on the fibres of r in the following sense: For fixed $\gamma' \in \mathcal{G}$, the functions $\gamma \mapsto |\xi_1|(\gamma') = \|\xi_1(\gamma)\|_{(E_1)_{r(\gamma')}}$ and $\gamma \mapsto \left\|\xi_2(\gamma^{-1}\gamma')\right\|_{(E_2)_{s(\gamma)}} = \left\|\gamma(\xi_2(\gamma^{-1}\gamma'))\right\|_{(E_2)_{r(\gamma')}}$ are continuous on $\mathcal{G}^{r(\gamma')}$. So the convolution $|\xi_1| * |\xi_2|$ exists and we can apply Lemma 5.2.5 to derive

$$\|\mu(\xi_1,\xi_2)\|_{\mathcal{A}(\mathcal{G},F)} \ \leq\ \|\mu\|_\infty \left\|\, |\xi_1| * |\xi_2|\, \right\|_{\mathcal{A}} \ \leq\ \|\mu\|_\infty\ \|\xi_1\|_{\mathcal{A}(\mathcal{G},E_1)}\ \|\xi_2\|_{\mathcal{A}(\mathcal{G},E_2)}.$$ □

Proposition 5.2.7 (Preservation of associativity). *Let E_1, E_2, E_3, F_1, F_2 and G be $\mathcal{G}$-Banach spaces. Let $\mu_1\colon E_1 \times_X E_2 \to F_1$, $\mu_2\colon E_2 \times_X E_3 \to F_2$, $\nu_1\colon F_1 \times_X E_3 \to G$ and $\nu_2\colon E_1 \times_X F_2 \to G$ be* bounded *continuous fields of bilinear maps. Assume that the following associativity law holds:*

$$(\nu_1)_x\big((\mu_1)_x(e_1,e_2),\ e_3\big) = (\nu_2)_x\big(e_1,\ (\mu_2)_x(e_2,e_3)\big)$$

for all $x \in X = \mathcal{G}^{(0)}$, $e_1 \in (E_1)_x$, $e_2 \in (E_2)_x$, and $e_3 \in (E_3)_x$. If, in addition, μ_2 is $\mathcal{G}$-equivariant, then the same associativity law holds on the level of the unconditional completions:

$$\mathcal{A}(\mathcal{G},\nu_1)\big(\mathcal{A}(\mathcal{G},\mu_1)(\xi_1,\xi_2),\ \xi_3\big) = \mathcal{A}(\mathcal{G},\nu_2)\big(\xi_1,\ \mathcal{A}(\mathcal{G},\mu_2)(\xi_2,\xi_3)\big)$$

for all $\xi_1 \in \mathcal{A}(\mathcal{G},E_1)$, $\xi_2 \in \mathcal{A}(\mathcal{G},E_2)$, and $\xi_3 \in \mathcal{A}(\mathcal{G},E_3)$.

[7] Actually, upper semi-continuity is enough for the convolution to exist: Since upper semi-continuous functions are Borel measurable and bounded Borel measurable functions with compact support are integrable, the function which appears under the integral in the convolution product is easily seen to be integrable when the involved functions are upper semi-continuous and of compact support. Thomas Timmermann brought this argument to my attention.

5.2.3 Linear maps

Let E and F be $\mathcal{G}$-Banach spaces and let T be a bounded continuous field of linear maps between them. We are now constructing linear maps between $\mathcal{A}(\mathcal{G}, E)$ to $\mathcal{A}(\mathcal{G}, F)$; there are two different ways to do this and both rely on 3.2.5, the corresponding construction for the general case of monotone completions.

Let E and F be $\mathcal{G}$-Banach spaces and T a bounded continuous field of linear maps between them.

Proposition 5.2.8. *We have*

$$\|T * \xi\|_{\mathcal{A}(\mathcal{G},F)} = \left\|\gamma \mapsto T_{r(\gamma)}\left(\xi(\gamma)\right)\right\|_{\mathcal{A}(\mathcal{G},F)} \leq \|T\|_\infty \|\xi\|_{\mathcal{A}(\mathcal{G},E)}$$

for all $\xi \in \Gamma_c(\mathcal{G}, E)$. So $\xi \mapsto r^*T \circ \xi$ defines a continuous linear operator, called $T * \cdot$, $\mathcal{A}(\mathcal{G}, T\cdot)$ or $\mathcal{A}(\mathcal{G}, T)$, from $\mathcal{A}(\mathcal{G}, E)$ to $\mathcal{A}(\mathcal{G}, F)$ of norm less than or equal to $\|T\|_\infty$.

The so-defined map $T \mapsto \mathcal{A}(\mathcal{G}, T)$ makes $E \mapsto \mathcal{A}(\mathcal{G}, E)$ a functor from the $\mathcal{G}$-Banach spaces to the Banach spaces. The same is true for the following "right-hand version" of the construction:

Proposition 5.2.9. *We have*

$$\|\xi * T\|_{\mathcal{A}(\mathcal{G},F)} = \left\|\gamma \mapsto \gamma\left[T_{s(\gamma)}(\gamma^{-1}\xi(\gamma))\right]\right\|_{\mathcal{A}(\mathcal{G},F)} \leq \|T\|_\infty \|\xi\|_{\mathcal{A}(\mathcal{G},E)}$$

for all $\xi \in \Gamma_c(\mathcal{G}, E)$. So $\xi \mapsto (\gamma \mapsto \gamma T_{s(\gamma)}(\gamma^{-1}\xi(\gamma)))$ defines a continuous linear operator, called $\cdot * T$ or $\mathcal{A}(\mathcal{G},\ \cdot T)$, from $\mathcal{A}(\mathcal{G}, E)$ to $\mathcal{A}(\mathcal{G}, F)$ of norm less than or equal to $\|T\|_\infty$.

Note that $\mathcal{A}(\mathcal{G},\ \cdot T) = \mathcal{A}(\mathcal{G},\ T\cdot)$ if T is $\mathcal{G}$-equivariant.

5.2.4 The actions of $\mathcal{C}_0(X)$ and $\mathcal{C}_0(X/\mathcal{G})$

Definition and Proposition 5.2.10. Let E be a $\mathcal{G}$-Banach space. We have

$$\|\chi\xi\|_{\mathcal{A}} \leq \|\chi\|_\infty \|\xi\|_{\mathcal{A}}$$

for all $\chi \in \mathcal{C}_b(X)$ and $\xi \in \Gamma_c(\mathcal{G},\ r^*E)$. So the left action of $\mathcal{C}_b(X)$ on $\Gamma_c(\mathcal{G},\ r^*E)$ can be extended to a left action of $\mathcal{C}_b(X)$ on $\mathcal{A}(\mathcal{G}, E)$. This gives rise to a left action of $\mathcal{C}_0(X)$ on $\mathcal{A}(\mathcal{G}, E)$ which is non-degenerate. The same is true for the right actions of $\mathcal{C}_b(X)$ and $\mathcal{C}_0(X)$ and the actions of $\mathcal{C}_b(X/\mathcal{G})$ and $\mathcal{C}_0(X/\mathcal{G})$.

Proof. For all $\xi \in \Gamma_c(\mathcal{G},\ r^*E)$ and for all $\gamma \in \mathcal{G}$, we have $\|(\chi\xi)(\gamma)\| \leq \|\chi\|_\infty \|\xi(\gamma)\|$. It follows that $\|\chi\xi\|_{\mathcal{A}} \leq \|\chi\|_\infty \|\xi\|_{\mathcal{A}}$. The action of $\mathcal{C}_0(X)$ on $\mathcal{A}(\mathcal{G}, E)$ which we can therefore define is non-degenerate, because the action of $\mathcal{C}_c(X)$ on $\Gamma_c(\mathcal{G},\ r^*E)$ is non-degenerate.

The arguments for the right action of $\mathcal{C}_0(X)$ and the action of $\mathcal{C}_0(X/\mathcal{G})$ are identical. □

Proposition 5.2.11. *1. Let E and F be $\mathcal{G}$-Banach spaces and T a bounded continuous field of linear maps between them. Then $\mathcal{A}(\mathcal{G},\ T\cdot)\colon \mathcal{A}(\mathcal{G}, E) \to \mathcal{A}(\mathcal{G}, F)$ is $\mathcal{C}_0(X/\mathcal{G})$-linear. The same applies to $\mathcal{A}(\mathcal{G},\ \cdot T)$.*

2. Let E_1, E_2, F be $\mathcal{G}$-Banach spaces. Let $\mu\colon E_1 \times_X E_2 \to F$ be a bounded equivariant continuous field of bilinear maps. Then the continuous bilinear map $\mathcal{A}(\mathcal{G}, \mu)$ from $\mathcal{A}(\mathcal{G}, E_1) \times \mathcal{A}(\mathcal{G}, E_2)$ to $\mathcal{A}(\mathcal{G}, F)$ is $\mathcal{C}_0(X/\mathcal{G})$-bilinear.

Similar results hold for the actions of $\mathcal{C}_0(X)$; compare Proposition 5.1.17.

5.2.5 Banach algebras and Banach modules

Proposition 5.2.12. *If B is a $\mathcal{G}$-Banach algebra (with product μ_B), then $\mathcal{A}(\mathcal{G}, B)$ is a $\mathcal{C}_0(X/\mathcal{G})$-Banach algebra (with the convolution product $\mathcal{A}(\mathcal{G}, \mu_B)$). If B is non-degenerate, then so is $\mathcal{A}(\mathcal{G}, B)$. In particular, $\mathcal{A}(\mathcal{G})$ is a non-degenerate $\mathcal{C}_0(X/\mathcal{G})$-Banach algebra.*

If B and C are $\mathcal{G}$-Banach algebras and φ denotes a $\mathcal{G}$-equivariant field of homomorphisms between them, then $\mathcal{A}(\mathcal{G}, \varphi)$ is a continuous homomorphism of $\mathcal{C}_0(X/\mathcal{G})$-Banach algebras from $\mathcal{A}(\mathcal{G}, B)$ to $\mathcal{A}(\mathcal{G}, C)$.

Proposition 5.2.13. *If B is a $\mathcal{G}$-Banach algebra and E is a right / left $\mathcal{G}$-Banach B-module, then $\mathcal{A}(\mathcal{G}, E)$ is a right / left $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G}, B)$-module. If E is non-degenerate, then so is $\mathcal{A}(\mathcal{G}, E)$.*

If B and B' are $\mathcal{G}$-Banach algebras, if φ is a $\mathcal{G}$-equivariant field of homomorphisms between them, if E is a right $\mathcal{G}$-Banach B-module, if E' is a right $\mathcal{G}$-Banach B'-module and if Φ is a $\mathcal{G}$-equivariant homomorphism from E to E' with coefficient map φ, then $\mathcal{A}(\mathcal{G},\ \Phi)$ is a continuous homomorphism of $\mathcal{C}_0(X/\mathcal{G})$-Banach modules from $\mathcal{A}(\mathcal{G}, E)$ to $\mathcal{A}(\mathcal{G}, E')$ with coefficient map $\mathcal{A}(\mathcal{G},\ \varphi)$.

A similar result is true for operators between $\mathcal{G}$-Banach modules:

Proposition 5.2.14. *Let B be a $\mathcal{G}$-Banach algebra and let E, E' be right $\mathcal{G}$-Banach B-modules. Let T be a B-linear continuous field of linear operators from E to E' (not necessarily equivariant). Then $\mathcal{A}(\mathcal{G}, T\cdot) = T * \cdot$ is a continuous $\mathcal{A}(\mathcal{G}, B)$-linear and $\mathcal{C}_0(X/\mathcal{G})$-linear operator from $\mathcal{A}(\mathcal{G}, E)$ to $\mathcal{A}(\mathcal{G}, E')$.*

An analogous statement is true for left $\mathcal{G}$-Banach B-modules (if $T * \cdot$ is replaced with $\cdot * T$).

5.2.6 Banach pairs

Definition and Proposition 5.2.15 (The Banach pair $\mathcal{A}(\mathcal{G}, E)$). Let B be a $\mathcal{G}$-Banach algebra and let E be a $\mathcal{G}$-Banach B-pair. Then $\mathcal{A}(\mathcal{G}, E^>)$ is a right $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G}, B)$-module and $\mathcal{A}(\mathcal{G}, E^<)$ is a left $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G}, B)$-module. Moreover, the bracket of E induces a bilinear map

$$\langle \cdot, \cdot \rangle_{\mathcal{A}(\mathcal{G},E)} \colon \mathcal{A}\left(\mathcal{G}, E^<\right) \times \mathcal{A}\left(\mathcal{G}, E^>\right) \to \mathcal{A}\left(\mathcal{G}, B\right)$$

which is $\mathcal{C}_0(X/\mathcal{G})$-bilinear and $\mathcal{A}(\mathcal{G}, B)$-linear on the left and on the right.

In other words, $(\mathcal{A}(\mathcal{G}, E^<),\ \mathcal{A}(\mathcal{G}, E^>))$ is a $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G}, B)$-pair. We denote it by $\mathcal{A}(\mathcal{G}, E)$.

Proposition 5.2.16. *Let B be a $\mathcal{G}$-Banach algebra and let E and F be $\mathcal{G}$-Banach B-pairs. Let $T = (T^<, T^>)$ be an element of $\mathrm{L}_B(E, F)$. Then*

1. *$T^> * \cdot$ is a $\mathcal{C}_0(X/\mathcal{G})$-linear operator from $\mathcal{A}(\mathcal{G}, E^>)$ to $\mathcal{A}(\mathcal{G}, F^>)$ being $\mathcal{A}(\mathcal{G}, B)$-linear on the right and of norm $\|T^> * cdot\| \leq \|T^>\|$;*

2. *$\cdot * T^<$ is a $\mathcal{C}_0(X/\mathcal{G})$-linear operator from $\mathcal{A}(\mathcal{G}, F^<)$ to $\mathcal{A}(\mathcal{G}, E^<)$ being $\mathcal{A}(\mathcal{G}, B)$-linear on the left and of norm $\|\cdot * T^>\| \leq \|T^<\|$;*

3. *The pair $(\cdot * T^<,\ T^> * \cdot)$ is in $\mathrm{L}^{\mathcal{C}_0(X/\mathcal{G})}_{\mathcal{A}(\mathcal{G},B)}(\mathcal{A}(\mathcal{G}, E),\ \mathcal{A}(\mathcal{G}, F))$ and of norm less than or equal to $\|T\|$. It will be denoted by $\mathcal{A}(\mathcal{G},\ T)$.*

The assignment $E \mapsto \mathcal{A}(\mathcal{G}, E)$ and $T \mapsto \mathcal{A}(\mathcal{G}, T)$ defines a functor from the category of $\mathcal{G}$-Banach B-pairs to the category of $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G}, B)$-pairs.

5.2.7 The convolution with fields of compact operators

This paragraph contains a technical tool for the proof of 5.2.19, namely operators which are given by the convolution with a (locally) compact operator with compact support. More details and the proofs are given in Appendix C.2, compare also Lemme 1.3.5 of [Laf06] which we brake up into several pieces here.
Let E and F be $\mathcal{G}$-Banach spaces and let $S = (S_\gamma)_{\gamma\in\mathcal{G}}$ be a continuous field of linear maps from r^*E to r^*F *with compact support*. For all $\xi \in \Gamma_c(\mathcal{G},\ r^*E)$, define

$$(S * \xi)(\gamma) := \int_{\mathcal{G}^{r(\gamma)}} S_{\gamma'}\ \gamma'\xi(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')$$

and

$$(\xi * S)(\gamma) := \int_{\mathcal{G}^{r(\gamma)}} \xi(\gamma')\,\gamma' S_{\gamma'^{-1}\gamma}\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') = \int_{\mathcal{G}^{r(\gamma)}} \gamma'\left[S_{\gamma'^{-1}\gamma}\left(\gamma'^{-1}\xi(\gamma')\right)\right]\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')$$

for all $\gamma \in \mathcal{G}$. Then $S*\xi, \xi * S \in \Gamma_c(\mathcal{G},\ r^*F)$. For all $\xi \in \Gamma_c(\mathcal{G},\ r^*E)$, we have

$$\|S*\xi\|_{\mathcal{A}(\mathcal{G},F)} \leq \left\|\gamma \mapsto \|S_\gamma\|\right\|_{\mathcal{A}}\ \|\xi\|_{\mathcal{A}(\mathcal{G},E)}$$

and

$$\|\xi * S\|_{\mathcal{A}(\mathcal{G},F)} \leq \|\xi\|_{\mathcal{A}(\mathcal{G},E)}\ \left\|\gamma \mapsto \|S_\gamma\|\right\|_{\mathcal{A}}.$$

In particular, $\xi \mapsto S*\xi$ and $\xi \mapsto \xi * S$ extend to linear and $\mathcal{C}_0(X/\mathcal{G})$-linear continuous maps from $\mathcal{A}(\mathcal{G},E)$ to $\mathcal{A}(\mathcal{G},F)$. If E and F are not only $\mathcal{G}$-Banach spaces but right $\mathcal{G}$-Banach B-modules over some $\mathcal{G}$-Banach algebra B, then $\xi \mapsto S*\xi$ is $\mathcal{A}(\mathcal{G},B)$-linear on the right. An analogous statement is true for left $\mathcal{G}$-Banach modules and $\xi \mapsto \xi * S$.

Definition 5.2.17. Let B be a $\mathcal{G}$-Banach algebra and let E and F be $\mathcal{G}$-Banach B-pairs. Let $S = (S^<, S^>) \in \mathrm{L}_{r^*B}(r^*E,\ r^*F)$ have compact support. Then, for all $\xi^> \in \Gamma_c(\mathcal{G},\ r^*E^>)$ and $\eta^< \in \Gamma_c(\mathcal{G},\ r^*F^<)$, we have

$$\left\langle \eta^<,\ S^> * \xi^> \right\rangle = \left\langle \eta^< * S^<,\ \xi^> \right\rangle.$$

It follows that

$$\hat{S} := \left(\eta^< \mapsto \eta^< * S^<,\ \xi^> \mapsto S^> * \xi^>\right) \in \mathrm{L}_{\mathcal{A}(\mathcal{G},B)}\left(\mathcal{A}(\mathcal{G},E),\ \mathcal{A}(\mathcal{G},F)\right)$$

with

$$\left\|\hat{S}\right\| \leq \max\left\{\left\|\gamma \mapsto \|S^<_\gamma\|\right\|_{\mathcal{A}},\ \left\|\gamma \mapsto \|S^>_\gamma\|\right\|_{\mathcal{A}}\right\} \leq \left\|\ \gamma \mapsto \max\left\{\left\|S^<_\gamma\right\|, \left\|S^>_\gamma\right\|\right\}\ \right\|_{\mathcal{A}}.$$

Proposition 5.2.18. *Let B be a $\mathcal{G}$-Banach algebra and let E and F be $\mathcal{G}$-Banach B-pairs. If S is an element of $\mathrm{K}_{r^*B}(r^*E,\ r^*F)$ with compact support, then $\hat{S}$ is compact, i.e., we have*

$$\hat{S} \in \mathrm{K}_{\mathcal{A}(\mathcal{G},B)}\left(\mathcal{A}(\mathcal{G},E),\ \mathcal{A}(\mathcal{G},F)\right).$$

5.2.8 The descent and $\mathrm{KK}^{\mathrm{ban}}$-cycles

Let A and B be $\mathcal{G}$-Banach algebras. If E is a $\mathcal{G}$-Banach A-B-pair, then there is[8] a canonical action of the $\mathcal{C}_0(X/\mathcal{G})$-Banach algebra $\mathcal{A}(\mathcal{G}, A)$ on the $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G}, B)$-pair $\mathcal{A}(\mathcal{G}, E)$.

Definition and Proposition 5.2.19. [9] Let $(E, T) \in \mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(A, B)$. Then define

$$j_{\mathcal{A}}(E, T) := (\mathcal{A}(\mathcal{G}, E),\ \mathcal{A}(\mathcal{G}, T)) \in \mathbb{E}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G}, A),\ \mathcal{A}(\mathcal{G}, B)\right).$$

Proof. If $\sigma \in \mathrm{L}_B(E)$ is the grading operator for E, then the grading on $\mathcal{A}(\mathcal{G}, E)$ is given by $\mathcal{A}(\mathcal{G}, \sigma)$. Then $\mathcal{A}(\mathcal{G}, T)$ is clearly odd. We have to check the compactness conditions.

1. Let $\alpha \in \Gamma_c(\mathcal{G},\ r^*A)$. It is easy to check by direct computation that $[\alpha,\ \mathcal{A}(\mathcal{G},\ T)]$ acts on $\mathcal{A}(\mathcal{G}, E^>)$ and $\mathcal{A}(\mathcal{G}, E^<)$ by convolution with a continuous field of linear operators, namely with
$$\gamma \mapsto \alpha(\gamma)\gamma T_{s(\gamma)} - T_{r(\gamma)}\alpha(\gamma)\ \in\ \mathrm{L}_{r^*B}(r^*E)_c.$$
Note that this field can be conveniently written as $\alpha * T - T * \alpha$. Now
$$\alpha(\gamma)\gamma T_{s(\gamma)} - T_{r(\gamma)}\alpha(\gamma) \quad = \quad \alpha(\gamma)\left(\gamma T_{s(\gamma)} - T_{r(\gamma)}\right) + \alpha(\gamma)T_{r(\gamma)} - T_{r(\gamma)}\alpha(\gamma)$$
for all $\gamma \in \mathcal{G}$. The term $\gamma \mapsto \alpha(\gamma)(\gamma T_{s(\gamma)} - T_{r(\gamma)})$ is compact by assumption, the second term can be rewritten as $[\alpha, r^*T]$. This operator can be approximated by operators of the form $\chi[\alpha' \circ r,\ r^*T] = \chi r^*[\alpha', T]$ with $\chi \in \mathcal{C}_c(\mathcal{G})$ and $\alpha' \in \Gamma_c(X, A)$. But these operators are compact, so $[\alpha, r^*T]$ is compact as well. So $[\alpha,\ \mathcal{A}(\mathcal{G},\ T)]$ is compact by Proposition 5.2.18.

2. Let $\alpha \in \Gamma_c(\mathcal{G},\ r^*A)$. Also $\alpha(\mathcal{A}(\mathcal{G}, T)^2 - 1)$ acts by convolution with a continuous field of operators, namely with
$$\gamma \mapsto \alpha(\gamma)\left((\gamma T_{s(\gamma)})^2 - 1\right) \in \mathrm{L}_{r^*B}\left(r^*E\right)_c.$$
To show that this is a compact operator we will now transform the field $\gamma \mapsto \alpha(\gamma)(\gamma T_{s(\gamma)})^2$ by adding or subtracting compact operators until we get to $\gamma \mapsto \alpha(\gamma)$. The relation "$\equiv$" will be used, somewhat imprecisely, for "differs by a compact operator":
$$\begin{aligned} \alpha(\gamma)\left(\gamma T_{s(\gamma)}\right)^2 &= \alpha(\gamma)\left(\gamma T_{s(\gamma)} - T_{r(\gamma)}\right)\gamma T_{s(\gamma)} + \alpha(\gamma)T_{r(\gamma)}\gamma T_{s(\gamma)} \\ &\equiv \alpha(\gamma)T_{r(\gamma)}\gamma T_{s(\gamma)} \equiv T_{r(\gamma)}\alpha(\gamma)\gamma T_{s(\gamma)} \\ &\equiv T_{r(\gamma)}\alpha(\gamma)T_{r(\gamma)} \equiv \alpha(\gamma)(T_{r(\gamma)})^2 \equiv \alpha(\gamma) \end{aligned}$$
for all $\gamma \in \mathcal{G}$. So $\alpha(\mathcal{A}(\mathcal{G}, T)^2 - 1)$ is compact by Proposition 5.2.18. □

The following lemmas are proved in Appendix C.3.3.

Lemma 5.2.20. *The map $j_{\mathcal{A}} \colon \mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(A, B) \to \mathbb{E}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G}, A),\ \mathcal{A}(\mathcal{G}, B))$ respects the direct sum of cycles up to homotopy.*

Lemma 5.2.21. *Let A, B and C be $\mathcal{G}$-Banach algebras. Let $\psi \colon B \to C$ be a $\mathcal{G}$-equivariant homomorphism. Let $(E, T) \in \mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(A, B)$. Then*

$$\mathcal{A}\left(\mathcal{G}, \psi\right)_*\left(j_{\mathcal{A}}\left(E, T\right)\right) \sim j_{\mathcal{A}}\left(\psi_*\left(E, T\right)\right)$$

in $\mathbb{E}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G}, A),\ \mathcal{A}(\mathcal{G}, C))$.

[8] Compare Proposition 1.3.3 of [Laf06].

[9] Compare Définition-Proposition 1.3.4 of [Laf06].

Lemma 5.2.22. *Let B be a $\mathcal{G}$-Banach algebra. Define a map*

$$\phi_B \colon \mathcal{A}\left(\mathcal{G},\, B[0,1]\right) \to \mathcal{A}\left(\mathcal{G}, B\right)[0,1]$$

by

$$\left(\phi_B(\beta)(t)\right)(\gamma) = \beta(\gamma)(t) \in B_{r(\gamma)}$$

for all $\beta \in \Gamma_c(\mathcal{G},\, r^* B[0,1])$, $t \in [0,1]$ and $\gamma \in \mathcal{G}$. This is a contractive homomorphism of $\mathcal{C}_0(X/\mathcal{G})$-Banach algebras that satisfies the equation

$$\mathrm{ev}_t^{\mathcal{A}(\mathcal{G},B)} \circ \phi_B = \mathcal{A}\left(\mathcal{G},\, \mathrm{ev}_t^B\right)$$

for all $t \in [0,1]$, where ev_t^B denotes the canonical $\mathcal{G}$-equivariant homomorphism from $B[0,1]$ to B and $\mathrm{ev}_t^{\mathcal{A}(\mathcal{G},B)}$ denotes the canonical morphism from $\mathcal{A}(\mathcal{G},B)[0,1]$ to $\mathcal{A}(\mathcal{G},B)$, both given by evaluation at t.

Proposition 5.2.23. *Let A and B be $\mathcal{G}$-Banach algebras. If $(E,T) \in \mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(A, B[0,1])$ is a homotopy from (E_0,T_0) to (E_1,T_1), then $j_{\mathcal{A}}(E_0,T_0)$ and $j_{\mathcal{A}}(E_1,T_1)$ are homotopic.*

Proof. First note that

$$j_{\mathcal{A}}\left(E,T\right) \in \mathbb{E}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G},A),\ \mathcal{A}(\mathcal{G},B[0,1])\right)$$

by 5.2.19 and hence $\phi_{B,*}(j_{\mathcal{A}}(E,T))$ is an element of $\mathbb{E}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G},A),\ \mathcal{A}(\mathcal{G},B)[0,1])$. The pushout $\mathrm{ev}_{t,*}^{\mathcal{A}(\mathcal{G},B)}(\phi_{B,*}(j_{\mathcal{A}}(E,T)))$ of this cycle along the evaluation map is isomorphic to $\mathcal{A}(\mathcal{G},\,\mathrm{ev}_t^B)_*(j_{\mathcal{A}}(E,T))$ for all $t \in [0,1]$. For all $t \in \{0,1\}$, we have

$$\begin{aligned} j_{\mathcal{A}}\left(E_t,T_t\right) &\cong j_{\mathcal{A}}\left(\mathrm{ev}_{t,*}^B(E,T)\right) \overset{5.2.21}{\sim} \mathcal{A}\left(\mathcal{G},\, \mathrm{ev}_t^B\right)_*\left(j_{\mathcal{A}}(E,T)\right) \\ &\overset{5.2.22}{\cong} \left(\mathrm{ev}_t^{\mathcal{A}(\mathcal{G},B)} \circ \phi_B\right)_*\left(j_{\mathcal{A}}(E,T)\right) \cong \mathrm{ev}_{t,*}^{\mathcal{A}(\mathcal{G},B)}\left(\phi_{B,*}\left(j_{\mathcal{A}}(E,T)\right)\right), \end{aligned}$$

so $\phi_{B,*}(j_{\mathcal{A}}(E,T))$ is a homotopy from $j_{\mathcal{A}}(E_0,T_0)$ to $j_{\mathcal{A}}(E_1,T_1)$. □

Lemma 5.2.24. *Let A, B and C be $\mathcal{G}$-Banach algebras. Let $\theta\colon A \to B$ be a $\mathcal{G}$-equivariant homomorphism. Let $(E,T) \in \mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(B,C)$. Then*

$$\mathcal{A}\left(\mathcal{G},\theta\right)^*\left(j_{\mathcal{A}}\left(E,T\right)\right) = j_{\mathcal{A}}\left(\theta^*\left(E,T\right)\right)$$

in $\mathbb{E}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G},A),\ \mathcal{A}(\mathcal{G},C))$.

Theorem 5.2.25. *Let A and B be $\mathcal{G}$-Banach algebras and $\mathcal{A}(\mathcal{G})$ an unconditional completion of $\mathcal{C}_c(\mathcal{G})$. Then $j_{\mathcal{A}}$ induces a group homomorphism*

$$j_{\mathcal{A}}\colon\ \mathrm{KK}_{\mathcal{G}}^{\mathrm{ban}}\left(A,B\right) \ \to\ \mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G},A),\ \mathcal{A}(\mathcal{G},B)\right).$$

It is natural with respect to $\mathcal{G}$-equivariant homomorphisms in both variables.

5.2.9 The descent and Morita morphisms

Let A and B be non-degenerate $\mathcal{G}$-Banach algebras. If $F \in \mathbb{M}_{\mathcal{G}}^{\mathrm{ban}}(A,B)$ is a $\mathcal{G}$-equivariant Morita cycle, then a close inspection of the definition of the descent of a $\mathrm{KK}^{\mathrm{ban}}$-cycle tells us that $\mathcal{A}(\mathcal{G},F)$ is in $\mathbb{M}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G},A),\ \mathcal{A}(\mathcal{G},B))$, i.e., the descent sends Morita cycles to Morita cycles. Moreover, the descent respects the direct sum and the pushout of Morita cycles (this can be deduced from the fact that the homotopies in the Lemmas 5.2.20 and 5.2.21 can be taken to have zero operator if the involved cycles have zero operator). It follows that homotopic elements of $\mathbb{M}_{\mathcal{G}}^{\mathrm{ban}}(A,B)$ give homotopic elements of $\mathbb{M}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G},A),\ \mathcal{A}(\mathcal{G},B))$. Thus we have

Proposition 5.2.26. *For all non-degenerate $\mathcal{G}$-Banach algebras A and B and all unconditional completions $\mathcal{A}(\mathcal{G})$ of $\mathcal{C}_c(\mathcal{G})$, the descent map $j_{\mathcal{A}}$ induces a homomorphism of monoids*

$$j_{\mathcal{A}}\colon\ \mathrm{Mor}_{\mathcal{G}}^{\mathrm{ban}}(A,B) \to \mathrm{Mor}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G},A),\ \mathcal{A}(\mathcal{G},B)\right).$$

It is natural with respect to $\mathcal{G}$-equivariant homomorphisms in both variables.

The following proposition is proved in Appendix C.3.3.

Proposition 5.2.27. *Let A, B, C be non-degenerate $\mathcal{G}$-Banach algebras. Let $(E,T) \in \mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(A,B)$ be a $\mathrm{KK}^{\mathrm{ban}}$-cycle and $F \in \mathbb{M}_{\mathcal{G}}^{\mathrm{ban}}(B,C)$ be a $\mathcal{G}$-equivariant Morita cycle. Then*

$$j_{\mathcal{A}}\left(E,T\right) \otimes_{\mathcal{A}(\mathcal{G},B)}^{\mathcal{C}_0(X/\mathcal{G})} j_{\mathcal{A}}(F) \sim j_{\mathcal{A}}\left((E,T) \otimes_B F\right)$$

in $\mathbb{E}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G},A),\ \mathcal{A}(\mathcal{G},C))$. If $T=0$, then the homotopy can be taken to have zero operator as well.

Corollary 5.2.28. *The descent is a functor from the category of non-degenerate $\mathcal{G}$-Banach algebras and $\mathcal{G}$-equivariant Morita morphisms to the category of non-degenerate $\mathcal{C}_0(X/\mathcal{G})$-Banach algebras and $\mathcal{C}_0(X/\mathcal{G})$-linear Morita morphisms.*

Proof. Let A, B and C be non-degenerate $\mathcal{G}$-Banach algebras. The identity morphism on A is given by the homotopy class $[A]$ of the standard Banach A-A-pair ${}_AA_A$. We have $j_{\mathcal{A}}({}_AA_A) = {}_{\mathcal{A}(\mathcal{G},A)}\mathcal{A}(\mathcal{G},A)_{\mathcal{A}(\mathcal{G},A)}$, so $[A]$ is mapped to the identity morphism on $\mathcal{A}(\mathcal{G},A)$. Secondly, if $E \in \mathbb{M}_{\mathcal{G}}^{\mathrm{ban}}(A,B)$ and $F \in \mathbb{M}_{\mathcal{G}}^{\mathrm{ban}}(B,C)$, then $j_{\mathcal{A}}(E) \otimes_{\mathcal{A}(\mathcal{G},B)}^{\mathcal{C}_0(X/\mathcal{G})} j_{\mathcal{A}}(F) \sim j_{\mathcal{A}}(E \otimes_B F)$ by the preceding proposition. So $j_{\mathcal{A}}([E]) \otimes_{\mathcal{A}(\mathcal{G},B)}^{\mathcal{C}_0(X/\mathcal{G})} j_{\mathcal{A}}([F]) = j_{\mathcal{A}}([E] \otimes_B [F])$. So $j_{\mathcal{A}}$ is a functor. $\square$

Corollary 5.2.29. *The action of the Morita morphisms on* $\mathrm{KK}^{\mathrm{ban}}$ *is compatible with the descent.*

Because the descent is a functor, it maps isomorphisms to isomorphisms, and from this we know that it maps the homotopy class of a $\mathcal{G}$-equivariant Morita equivalence to an isomorphism. We can actually easily obtain a result that is a bit more precise:

Remark 5.2.30. If A and B are non-degenerate $\mathcal{G}$-Banach algebras and E is a $\mathcal{G}$-equivariant Morita equivalence between A and B, then $\mathcal{A}(\mathcal{G},E)$ is a $\mathcal{C}_0(X/\mathcal{G})$-linear Morita equivalence between the non-degenerate Banach algebras $\mathcal{A}(\mathcal{G},A)$ and $\mathcal{A}(\mathcal{G},B)$.

5.3 The descent and open subgroupoids

5.3.1 The setting

If $\mathcal{H}$ and $\mathcal{G}$ are topological groupoids, $f\colon \mathcal{H} \to \mathcal{G}$ is a strict morphism and A and B are $\mathcal{G}$-Banach algebras, then Theorem 3.6.11 says that we have a homomorphism

$$f^*\colon \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A, B) \to \mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(f^*A,\ f^*B).$$

If $\mathcal{G}$ and $\mathcal{H}$ are locally compact Hausdorff over X and Y, respectively, and carry Haar systems, and if $\mathcal{A}(\mathcal{G})$ and $\mathcal{B}(\mathcal{H})$ are unconditional completions, then we can ask whether the following diagram can be completed

$$\begin{tikzcd}
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A, B) \arrow[r, "f^*"] \arrow[d, "j_{\mathcal{A}}"] & \mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(f^*A,\ f^*B) \arrow[d, "j_{\mathcal{B}}"] \\
\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G}, A),\ \mathcal{A}(\mathcal{G}, B)) \arrow[r, leftrightarrow, dashed, "?"] & \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(Y/\mathcal{H});\ \mathcal{B}(\mathcal{H}, f^*A),\ \mathcal{B}(\mathcal{H}, f^*B))
\end{tikzcd}$$

There is no hope for an affirmative answer if the question is formulated in this generality. However, one can obtain some results if one restricts attention to some special class of strict morphisms. We will do this quite drastically and only consider the case that $\mathcal{H}$ is an *open subgroupoid* of $\mathcal{G}$ (and Y is hence an open subspace of X) and $f = \iota_{\mathcal{H}}$ is the inclusion of $\mathcal{H}$ into $\mathcal{G}$. In this case, $\mathcal{C}_c(\mathcal{H})$ is contained as a subalgebra in $\mathcal{C}_c(\mathcal{G})$ and if $\mathcal{A}(\mathcal{G})$ is an unconditional completion of $\mathcal{C}_c(\mathcal{G})$, then the norm on $\mathcal{C}_c(\mathcal{G})$ restricts to an unconditional norm on $\mathcal{C}_c(\mathcal{H})$. We call the completion of $\mathcal{C}_c(\mathcal{H})$ for this norm $\mathcal{A}(\mathcal{H})$. There is a canonical homomorphism from $\mathcal{A}(\mathcal{H})$ to $\mathcal{A}(\mathcal{G})$.

If A is a $\mathcal{G}$-Banach algebra, then $\iota_{\mathcal{H}}^* A$ is just the restriction of A to Y with the restricted action of $\mathcal{H}$. There is a canonical homomorphism of Banach algebras from $\mathcal{A}(\mathcal{H},\ \iota_{\mathcal{H}}^* A)$ to $\mathcal{A}(\mathcal{G}, A)$. We denote $\mathcal{A}(\mathcal{H},\ \iota_{\mathcal{H}}^* A)$ by $\mathcal{A}(\mathcal{H}, A)$ to save some letters.

Let $p\colon Y/\mathcal{H} \to X/\mathcal{G}$ be the unique map making the following square commutative:

$$\begin{tikzcd}
Y \arrow[r, "\iota_{\mathcal{H}}"] \arrow[d] & X \arrow[d] \\
Y/\mathcal{H} \arrow[r, dashed, "p"] & X/\mathcal{G}
\end{tikzcd}$$

where the vertical arrows are the canonical quotient maps. The map p is continuous. Using Definition 2.7.1 that discusses the change of the base space we can turn every $\mathcal{C}_0(Y/\mathcal{H})$-Banach space into a $\mathcal{C}_0(X/\mathcal{G})$-Banach space and every $\mathcal{C}_0(Y/\mathcal{H})$-Banach algebra into a $\mathcal{C}_0(X/\mathcal{G})$-Banach algebra, etc. As a special case of Proposition 2.7.2 we therefore get a homomorphism

$$p_*\colon \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(Y/\mathcal{H});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{H}, B)) \to \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{H}, B)).$$

The pushout along the canonical map from $\mathcal{A}(\mathcal{H}, B)$ to $\mathcal{A}(\mathcal{G}, B)$, which happens to be $\mathcal{C}_0(X/\mathcal{G})$-linear, gives a homomorphism

$$\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{H}, B)) \to \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{G}, B)).$$

The pullback along the canonical homomorphism from $\mathcal{A}(\mathcal{H}, A)$ to $\mathcal{A}(\mathcal{G}, A)$ in the first component gives a homomorphism

$$\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G}, A),\ \mathcal{A}(\mathcal{G}, B)) \to \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{G}, B)).$$

So we get a diagram:

$$
\begin{tikzcd}
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A, B) \arrow[r, "\iota^*_{\mathcal{H}}"] \arrow[d, "j_{\mathcal{A}}"] & \mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(A|_Y,\ B|_Y) \arrow[d, "j_{\mathcal{A}}"] \\
\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G}, A),\ \mathcal{A}(\mathcal{G}, B)) \arrow[d] & \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(Y/\mathcal{H});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{H}, B)) \arrow[d, "p_*"] \\
\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{G}, B)) & \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{H}, B)) \arrow[l]
\end{tikzcd}
\tag{5.4}
$$

We are now going to show that the above diagram is commutative. For this, we need the following lemma which is proved in Appendix C.2.3.

Lemma 5.3.1. *Let E and F be $\mathcal{G}$-Banach B-pairs. Let $S \in \mathrm{K}_{r^*B}(r^*E,\ r^*F)$ have compact support contained in $\mathcal{H}$. Then the convolution by S as an operator from $\mathcal{A}(\mathcal{G}, E)$ to $\mathcal{A}(\mathcal{G}, F)$, denoted above by $\hat{S}$, is not only in $\mathrm{K}_{\mathcal{A}(\mathcal{G},B)}(\mathcal{A}(\mathcal{G}, E),\ \mathcal{A}(\mathcal{G}, F))$, but can be approximated by sums of operators of the form $\left|\eta^>\right\rangle\left\langle\xi^<\right|$ with $\eta^> \in \Gamma_c(\mathcal{G},\ r^*F^>)$ and $\xi^< \in \Gamma_c(\mathcal{G},\ r^*E^<)$,* both having their support in $\mathcal{H}$.

Theorem 5.3.2. *Diagram (5.4) is commutative.*

Proof. Let (E, T) be in $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A, B)$. We have to trace (E, T) through diagram (5.4) and prove that the two cycles that we get in the lower left corner are homotopic. If we go down and down in the diagram, then we get the cycle $(\mathcal{A}(\mathcal{G}, E), \mathcal{A}(\mathcal{G}, T))$ where we regard $\mathcal{A}(\mathcal{G}, E)$ as a $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{H}, A)$-$\mathcal{A}(\mathcal{G}, B)$-pair. If we start with going right, then we get the cycle $(E|_Y, T|_Y) \in \mathbb{E}^{\mathrm{ban}}_{\mathcal{H}}(A|_Y, B|_Y)$. If we go right and down and down, then we are left with the cycle $(\mathcal{A}(\mathcal{H}, E|_Y),\ \mathcal{A}(\mathcal{H}, T|_Y))$ regarded as a $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{H}, A)$-$\mathcal{A}(\mathcal{H}, B)$-pair. Finally, if we go right-down-down-left, then we get the cycle $(\mathcal{A}(\mathcal{H}, E|_Y) \otimes_{\mathcal{A}(\mathcal{H},B)\oplus\mathcal{C}_0(X/\mathcal{G})} (\mathcal{A}(\mathcal{G}, B) \oplus \mathcal{C}_0(X/\mathcal{G})),\ \mathcal{A}(\mathcal{H}, T|_Y) \otimes 1)$. Into this cycle there is a canonical homomorphism from the cycle $(\mathcal{A}(\mathcal{H}, E|_Y) \otimes_{\mathcal{A}(\mathcal{H},B)} \mathcal{A}(\mathcal{G}, B),\ \mathcal{A}(\mathcal{H}, T|_Y) \otimes 1)$; it induces a homotopy, so we restrict our attention to this simpler $\mathrm{RKK}^{\mathrm{ban}}$-cycle.

We now define a homomorphism Φ from $\mathcal{A}(\mathcal{H}, E|_Y) \otimes_{\mathcal{A}(\mathcal{H},B)} \mathcal{A}(\mathcal{G}, B)$ to $\mathcal{A}(\mathcal{G}, E)$ with coefficient maps $\mathrm{Id}_{\mathcal{A}(\mathcal{H},A)}$ and $\mathrm{Id}_{\mathcal{A}(\mathcal{G},B)}$. Define

$$
\begin{aligned}
\Phi^> \colon \mathcal{A}(\mathcal{H},\ E^>|_Y) \otimes_{\mathcal{A}(\mathcal{H},B)} \mathcal{A}(\mathcal{G}, B) &\to \mathcal{A}(\mathcal{G}, E^>), \\
\xi^> \otimes \beta &\mapsto \xi^> * \beta
\end{aligned}
$$

where we regard $\xi^> \in \mathcal{A}(\mathcal{H},\ E^>|_Y)$ as an element of $\mathcal{A}(\mathcal{G}, E^>)$. Define $\Phi^<$ similarly. By the associativity of the convolution the pair $\Phi := (\Phi^<, \Phi^>)$ is a concurrent homomorphism. It is $\mathcal{C}_0(X/\mathcal{G})$-linear. We show that it induces a homotopy:

Let $\alpha \in \Gamma_c(\mathcal{H}, r^*A)$ and $\varepsilon > 0$. Then $[\alpha,\ \mathcal{A}(\mathcal{G}, T)]$ is given by convolution with the compact continuous field of operators with compact support

$$
\gamma \mapsto \alpha(\gamma)\gamma T_{s(\gamma)} - T_{r(\gamma)}\alpha(\gamma) \in \mathrm{K}_{r^*B}\left(r^*E\right)_c.
$$

The support of this field is actually contained in $\mathcal{H}$ because α is supported in $\mathcal{H}$. By the above lemma we can approximate $[\alpha,\ \mathcal{A}(\mathcal{G}, T)]$ by sums of operators of the form $\left|\eta^>\right\rangle\left\langle\xi^<\right|$ with $\xi^> \in \Gamma_c(\mathcal{G},\ r^*E^>)$ and $\xi^< \in \Gamma_c(\mathcal{G},\ r^*E^<)$, both having their support in $\mathcal{H}$. Because $\mathcal{A}(\mathcal{H}, E^>)$ is a non-degenerate right Banach $\mathcal{A}(\mathcal{H}, B)$-module and $\mathcal{A}(\mathcal{H}, E^<)$ is a non-degenerate left Banach $\mathcal{A}(\mathcal{H}, B)$-module, we can actually approximate $[\alpha,\ \mathcal{A}(\mathcal{G}, T)]$ as follows: We can find an $n \in \mathbb{N}$ and $\xi_1^<, \ldots, \xi_n^< \in \Gamma_c(\mathcal{G},\ r^*E^<)$,

$\xi_1^>, \ldots, \xi_n^> \in \Gamma_c(\mathcal{G}, r^*E^>)$ and $\beta_1^<, \ldots, \beta_n^<, \beta_1^>, \ldots, \beta_n^> \in \Gamma_c(\mathcal{G}, r^*B)$ which all are supported in $\mathcal{H}$ such that

$$\left\| [\alpha, \mathcal{A}(\mathcal{G}, T)] - \sum_{i=1}^n \left|\xi_i^> * \beta_i^>\right\rangle\left\langle \beta_i^< * \xi_i^<\right| \right\| \leq \varepsilon.$$

Note that we can regard the $\xi_i^>$ and the $\xi_i^<$ also as sections living on $\mathcal{H}$. If we do so, we have $\xi_i^> * \beta_i^> = \Phi^>(\xi_i^> \otimes \beta_i^>)$ and $\beta_i^< * \xi_i^< = \Phi^<(\beta_i^< \otimes \xi_i^<)$ for all $i \in \{1, \ldots, n\}$.

The operator $[\alpha, \mathcal{A}(\mathcal{G}, T)] - \sum_{i=1}^n \left|\xi_i^> * \beta_i^>\right\rangle\left\langle \beta_i^< * \xi_i^<\right|$ leaves the subspace $\mathcal{A}(\mathcal{H}, E|_Y)$ invariant. The norm of the restricted operator is of course less than or equal to the norm of the operator itself.

Note that $\left|\xi_i^> \otimes \beta_i^>\right\rangle\left\langle \beta_i^< \otimes \xi_i^<\right| = \left|\xi_i^> * \beta_i^>\right\rangle\left\langle \beta_i^< * \xi_i^<\right| \otimes 1$ and hence

$$\begin{aligned} & \left\| [\alpha \otimes 1, \mathcal{A}(\mathcal{H}, T|_Y) \otimes 1] - \sum_{i=1}^n \left|\xi_i^> \otimes \beta_i^>\right\rangle\left\langle \beta_i^< \otimes \xi_i^<\right| \right\| \\ = & \left\| \left([\alpha, \mathcal{A}(\mathcal{H}, T|_Y)] - \sum_{i=1}^n \left|\xi_i^> * \beta_i^>\right\rangle\left\langle \beta_i^< * \xi_i^<\right| \right) \otimes 1 \right\| \\ \leq & \left\| [\alpha, \mathcal{A}(\mathcal{H}, T|_Y)] - \sum_{i=1}^n \left|\xi_i^> * \beta_i^>\right\rangle\left\langle \beta_i^< * \xi_i^<\right| \right\| \leq \varepsilon. \end{aligned}$$

A similar calculation can be done for $\alpha(\mathcal{A}(\mathcal{G}, T)^2 - 1)$. This shows that Φ induces a homotopy. Hence the above diagram is commutative. □

5.3.2 The descent and Morita equivalence

There is a case when much more can be said about the Diagram (5.4): If U is an open and closed subset of X and $\mathcal{H} = \mathcal{G}_U^U$.

Definition and Proposition 5.3.3. Let A be a $\mathcal{G}$-Banach algebra and U an open and closed subset of $X = \mathcal{G}^{(0)}$. Define continuous linear maps

$$p_U^> \colon \Gamma_c(\mathcal{G}, r^*A) \to \Gamma_c(\mathcal{G}, r^*A), \ \xi \mapsto \xi|_{\mathcal{G}_U}$$

and

$$p_U^< \colon \Gamma_c(\mathcal{G}, r^*A) \to \Gamma_c(\mathcal{G}, r^*A), \ \xi \mapsto \xi|_{\mathcal{G}^U},$$

where the restricted sections should be extended by zero to all of $\mathcal{G}$. Then $(p_U^<)^2 = p_U^<$ and $(p_U^>)^2 = p_U^>$. Moreover, $p_U^<$ is $\Gamma_c(\mathcal{G}, r^*A)$-linear on the right, $p_U^>$ is $\Gamma_c(\mathcal{G}, r^*A)$-linear on the left. Both maps are $\mathcal{C}(X/\mathcal{G})$-linear. Finally, for all $\xi_1, \xi_2 \in \Gamma_c(\mathcal{G}, r^*A)$:

$$p_U^>(\xi_1) * \xi_2 = \xi_1 * p_U^<(\xi_2).$$

So $p_U = (p_U^<, p_U^>)$ could be called an (idempotent $\mathcal{C}(X/\mathcal{G})$-linear) multiplier of $\Gamma_c(\mathcal{G}, r^*A)$. We have

$$p_U \Gamma_c(\mathcal{G}, r^*A) p_U = \Gamma_c(\mathcal{G}_U^U, r^*A).$$

Definition and Proposition 5.3.4. Let A be a $\mathcal{G}$-Banach algebra and U an open and closed subset of $X = \mathcal{G}^{(0)}$. Let $\mathcal{A}(\mathcal{G})$ be an unconditional completion of $\mathcal{C}_c(\mathcal{G})$. Let ι denote the embedding of $\Gamma_c(\mathcal{G}, r^*A)$ into $\mathcal{A}(\mathcal{G}, A)$. Then there is a unique multiplier $P_U = (P_U^<, P_U^>)$ of $\mathcal{A}(\mathcal{G}, A)$ such that

$$\iota \circ p_U^< = P_U^< \circ \iota \quad \text{and} \quad \iota \circ p_U^> = P_U^> \circ \iota.$$

It is idempotent, $\mathcal{C}_0(X/\mathcal{G})$-linear and contractive. We have

$$P_U \mathcal{A}(\mathcal{G}, A) P_U = \mathcal{A}(\mathcal{G}_U^U, A).$$

If we want to stress that the underlying Banach algebra is A, then we write P_U^A for P_U.

Proof. Uniqueness is trivial. To prove existence note that the maps $p_U^<$ and $p_U^>$ are contractive on the level of sections with compact support because $\mathcal{A}(\mathcal{G})$ is unconditional. Hence $p_U^<$ and $p_U^>$ give contractive operators $P_U^<$ and $P_U^>$ on $\mathcal{A}(\mathcal{G}, A)$ such that $\iota \circ p_U^< = P_U^< \circ \iota$ and $\iota \circ p_U^> = P_U^> \circ \iota$. The operators P_U inherit the algebraic properties of the p_U. □

Proposition 5.3.5. *If U is open and closed in X and $\mathcal{H} := \mathcal{G}_U^U$ and $Y := U$, then the homomorphism p_* in Diagram (5.4) is an isomorphism, i.e.,*

$$\mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0(Y/\mathcal{H});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{H}, B)\right) \cong \mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{H}, B)\right).$$

Proof. Note that $Y/\mathcal{H} = U/(\mathcal{G}_U^U)$ can be identified with $U/\mathcal{G} \subseteq X/\mathcal{G}$, i.e., we can think of $Y/\mathcal{H}$ as a closed and open subset of $X/\mathcal{G}$. Let $(E, T) \in \mathbb{E}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{H}, B))$. Let $\chi \in \mathcal{C}_c(X/\mathcal{G})$ such that $\chi|_{Y/\mathcal{H}} = 0$. Then for all $\xi^> \in E^>$ and $\beta \in \mathcal{A}(\mathcal{H}, B)$, we have $(\xi^>\beta)\chi = \xi^>(\beta\chi) = \xi^> 0 = 0$. Because $E^>$ is non-degenerate, it follows that $\xi^>\chi = 0$ for all $\xi^> \in E^>$. So $E^>$ is already a non-degenerate Banach $\mathcal{C}_0(Y/\mathcal{H})$-module, i.e., a $\mathcal{C}_0(Y/\mathcal{H})$-Banach space. The same is true for $E^<$. In other words,

$$\mathbb{E}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{H}, B)\right) \subseteq \mathbb{E}^{\mathrm{ban}}\left(\mathcal{C}_0(Y/\mathcal{H});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{H}, B)\right).$$

The other inclusion is trivial. As the same is true for homotopies we can deduce that p_* is actually the identity homomorphism. □

Definition 5.3.6 (Connected/full subsets). We call two subsets U and V of $\mathcal{G}^{(0)}$ *connected* if

$$\mathcal{G}_U^V \mathcal{G}_V^U = \mathcal{G}_V^V \quad \text{and} \quad \mathcal{G}_V^U \mathcal{G}_U^V = \mathcal{G}_U^U.$$

A subset U is called *full* if it is connected to the whole of $\mathcal{G}$ (which means that $\mathcal{G}^U \mathcal{G}_U = \mathcal{G}$).

Two open subsets U and V are connected if and only if the range and source maps, restricted to $\mathcal{G}_U^V$, are surjective onto V and U, respectively.

In Section 6.2 we are going to meet the construction of the linking groupoid of an equivalence of groupoids: If two groupoids $\mathcal{G}$ and $\mathcal{H}$ are equivalent in the sense of Definition 6.1.28 and $\mathcal{L}$ is the linking groupoid, then $\mathcal{L}^{(0)}$ is the union of $U := \mathcal{G}^{(0)}$ and $V := \mathcal{H}^{(0)}$, both being open, closed, full and connected subsets, and $\mathcal{G} = \mathcal{L}_U^U$ and $\mathcal{H} = \mathcal{L}_V^V$; so the situation discussed in the following theorem is of some relevance. To prove it, we need the following Lemma which is proved in Appendix C.1:

Lemma 5.3.7. *Let U, V and W be open pairwise connected subsets of $\mathcal{G}^{(0)}$. Let E_1, E_2 and F be $\mathcal{G}$-Banach spaces and let $\mu\colon E_1 \times_X E_2 \to F$ be a continuous field of bilinear maps. The map $(\xi_1, \xi_2) \mapsto \mu(\xi_1, \xi_2)$ is a separately continuous bilinear map from $\Gamma_c(\mathcal{G}_V^W, r^*E_1) \times \Gamma_c(\mathcal{G}_U^V, r^*E_2)$ to $\Gamma_c(\mathcal{G}_U^W, r^*F)$, and if μ is non-degenerate, then*

$$\left\{\mu(\xi_1, \xi_2) :\ \xi_1 \in \Gamma_c(\mathcal{G}_V^W, r^*E_1),\ \xi_2 \in \Gamma_c(\mathcal{G}_U^V, r^*E_2)\right\}$$

spans a dense subset of $\Gamma_c(\mathcal{G}_U^W, r^*F)$.

The idea of the proof is to write μ, restricted to $\Gamma_c(\mathcal{G}_V^W,\ r^*E_1) \times \Gamma_c(\mathcal{G}_U^V,\ r^*E_2)$, as a composition of carefully chosen maps, imitating the proof for the special case $U = V = W = \mathcal{G}^{(0)}$ given above.

The lemma has an immediate consequence:

Lemma 5.3.8. *Let A be a non-degenerate $\mathcal{G}$-Banach algebra. Let $\mathcal{A}(\mathcal{G})$ be an unconditional completion of $\mathcal{C}_c(\mathcal{G})$. If U and V are open subsets of $\mathcal{G}^{(0)}$, then let $\mathcal{A}(\mathcal{G}_U^V, A)$ denote the completion of $\Gamma_c(\mathcal{G}_U^V,\ r^*A)$ for the restricted norm. Let U, V and W be open pairwise connected subsets of $\mathcal{G}^{(0)}$. Then $(\xi_1, \xi_2) \mapsto \xi_1 * \xi_2$ induces a non-degenerate contractive bilinear map $\mathcal{A}(\mathcal{G}_V^W, A) \times \mathcal{A}(\mathcal{G}_U^V, A) \to \mathcal{A}(\mathcal{G}_U^W, A)$.*

Theorem 5.3.9. *Let A be a non-degenerate $\mathcal{G}$-Banach algebra and U an open and closed subset of $\mathcal{G}^{(0)}$. Let $\mathcal{A}(\mathcal{G})$ be an unconditional completion of $\mathcal{C}_c(\mathcal{G})$. If U is full, then P_U is full in the sense of Definition 1.10.7, i.e., $\mathcal{A}(\mathcal{G}, A) P_U \mathcal{A}(\mathcal{G}, A)$ is dense in $\mathcal{A}(\mathcal{G}, A)$. In particular, $\mathcal{A}(\mathcal{G}, A)$ and $\mathcal{A}(\mathcal{G}_U^U, A)$ are Morita equivalent $\mathcal{C}_0(X/\mathcal{G})$-Banach algebras:*

$$\mathcal{A}(\mathcal{G}, A) \sim_M \mathcal{A}\left(\mathcal{G}_U^U, A\right).$$

Proof. We show that $p_U^{>}(\Gamma_c(\mathcal{G},\ r^*A)) * p_U^{<}(\Gamma_c(\mathcal{G},\ r^*A))$ is dense in $\Gamma_c(\mathcal{G},\ r^*A)$. But $p_U^{>}(\Gamma_c(\mathcal{G},\ r^*A))$ is the same as $\Gamma_c(\mathcal{G}_U,\ r^*A)$ and $p_U^{<}(\Gamma_c(\mathcal{G},\ r^*A))$ is the same as $\Gamma_c(\mathcal{G}^U,\ r^*A)$, so we are done using Lemma 5.3.8. Explicitly, the Morita equivalence can be obtained as follows: Let $\mathcal{A}(\mathcal{G}_U, A)$ be the completion of $\Gamma_c(\mathcal{G}_U,\ r^*A)$ for the restriction of the unconditional norm on $\Gamma_c(\mathcal{G},\ r^*A)$. Analogously define $\mathcal{A}(\mathcal{G}^U, A)$. Then $(\mathcal{A}(\mathcal{G}^U, A),\ \mathcal{A}(\mathcal{G}_U, A))$ is an equivalence between $\mathcal{A}(\mathcal{G}, A)$ and $\mathcal{A}(\mathcal{G}_U^U, A)$. □

Corollary 5.3.10. *Let U be a full open and closed subset of X, let $\mathcal{H} := \mathcal{G}_U^U$ and $Y := U$. Let B be non-degenerate. Then the lower horizontal arrow in Diagram (5.4) is an isomorphism:*

$$\begin{tikzcd}
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A, B) \arrow[r, "\iota_{\mathcal{H}}^*"] \arrow[d, "j_{\mathcal{A}}"] & \mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(A|_Y,\ B|_Y) \arrow[d, "j_{\mathcal{A}}"] \\
\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G}, A),\ \mathcal{A}(\mathcal{G}, B)) \arrow[d] & \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(Y/\mathcal{H});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{H}, B)) \arrow[d, "\cong"] \\
\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{G}, B)) & \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{H}, B)) \arrow[l, "\cong"']
\end{tikzcd}$$

By inverting the two isomorphisms in this diagram we can construct a homomorphism from $\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G}, A),\ \mathcal{A}(\mathcal{G}, B))$ to $\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(Y/\mathcal{H});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{H}, B))$ making the following diagram commutative:

$$\begin{tikzcd}
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A, B) \arrow[r, "\iota_{\mathcal{H}}^*"] \arrow[d, "j_{\mathcal{A}}"] & \mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(A|_Y,\ B|_Y) \arrow[d, "j_{\mathcal{A}}"] \\
\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G}, A),\ \mathcal{A}(\mathcal{G}, B)) \arrow[r] & \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(Y/\mathcal{H});\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{H}, B))
\end{tikzcd} \tag{5.5}$$

Note that we can identify $\mathcal{C}_0(Y/\mathcal{H})$ and $\mathcal{C}_0(X/\mathcal{G})$ if Y is full: We have already seen that we can think of $Y/\mathcal{H}$ as a closed and open subset of $X/\mathcal{G}$. If Y is full, then it meets every $\mathcal{G}$-orbit, so $Y/\mathcal{H}$ can be identified with $X/\mathcal{G}$.

If A is non-degenerate as well, then it is very likely that the homomorphism

$$\mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G},A),\ \mathcal{A}(\mathcal{G},B)\right) \to \mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0(Y/\mathcal{H});\ \mathcal{A}(\mathcal{H},A),\ \mathcal{A}(\mathcal{H},B)\right)$$

is actually an isomorphism, a statement which is equivalent to the homomorphism

$$\mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G},A),\ \mathcal{A}(\mathcal{G},B)\right) \to \mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{H},A),\ \mathcal{A}(\mathcal{G},B)\right)$$

being an isomorphism. As $\mathcal{A}(\mathcal{G},A)$ and $\mathcal{A}(\mathcal{H},A)$ are Morita equivalent, this could well be true, but we need new methods to show this because we do not have a Kasparov product in the Banach algebra setting.

Note that there is an obvious generalisation of Theorem 5.3.9:

Theorem 5.3.11. *Let A be a non-degenerate $\mathcal{G}$-Banach algebra and let U and V be open and closed connected subsets of $X = \mathcal{G}^{(0)}$. Let $\mathcal{A}(\mathcal{G})$ be an unconditional completion of $\mathcal{C}_c(\mathcal{G})$. Then the pair $(\mathcal{A}(\mathcal{G}_V^U, A),\ \mathcal{A}(\mathcal{G}_U^V, A))$ is a $\mathcal{C}_0(X/\mathcal{G})$-linear Morita equivalence between $\mathcal{A}(\mathcal{G}_V^V, A)$ and $\mathcal{A}(\mathcal{G}_U^U, A)$.*

Note that this Morita equivalence gives an isomorphism

$$\mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\ C,\ \mathcal{A}(\mathcal{G}_V^V,\ A)\right) \cong \mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\ C,\ \mathcal{A}(\mathcal{G}_U^U,\ A)\right)$$

for every $\mathcal{C}_0(X/\mathcal{G})$-Banach algebra C. This construction is transitive in the following sense:

Proposition 5.3.12. *Let U, V, W be open and closed pairwise connected subsets of $X = \mathcal{G}^{(0)}$ and let A be a non-degenerate $\mathcal{G}$-Banach algebra. Let $\mathcal{A}(\mathcal{G})$ be an unconditional completion of $\mathcal{C}_c(\mathcal{G})$. Then the restriction of the multiplication defines a concurrent homomorphism*

$$\left(\mathcal{A}\left(\mathcal{G}_W^V, A\right),\ \mathcal{A}\left(\mathcal{G}_V^W, A\right)\right)\ \otimes_{\mathcal{A}(\mathcal{G}_V^V, A)}\ \left(\mathcal{A}\left(\mathcal{G}_V^U, A\right),\ \mathcal{A}\left(\mathcal{G}_U^W, A\right)\right)\ \to\ \left(\mathcal{A}\left(\mathcal{G}_W^U, A\right),\ \mathcal{A}\left(\mathcal{G}_U^V, A\right)\right)$$

which is a morphism of $\mathcal{C}_0(X/\mathcal{G})$-linear Morita equivalences. It induces a homotopy of Morita cycles, so the two Morita equivalences give the same ($\mathcal{C}_0(X/\mathcal{G})$-linear) Morita morphism.

Proof. This follows from the $\mathcal{C}_0(X/\mathcal{G})$-linear version of Lemma 1.10.26. □

5.4 The descent and local convexity

Definition 5.4.1 (Locally convex unconditional norm). An unconditional norm $\|\cdot\|_{\mathcal{A}}$ on $\mathcal{C}_c(\mathcal{G})$ is called *locally $\mathcal{C}_0(X/\mathcal{G})$-convex* or simply *locally convex* if $\mathcal{A}(\mathcal{G})$ is a locally $\mathcal{C}_0(X/\mathcal{G})$-convex $\mathcal{C}_0(X/\mathcal{G})$-Banach algebra.

Proposition 5.4.2. *Let E be a $\mathcal{G}$-Banach space. If $\mathcal{A}(\mathcal{G})$ is locally $\mathcal{C}_0(X/\mathcal{G})$-convex, then so is $\mathcal{A}(\mathcal{G}, E)$.*

Proof. Let ξ be an element of $\Gamma_c(\mathcal{G},\ r^*E)$. Then for all $\chi \in \mathcal{C}_0(X/\mathcal{G})$:

$$\|\chi\xi\|_{\mathcal{A}(\mathcal{G},E)} = \Big\|\gamma\ \mapsto\ |\chi(\pi(\gamma))|\ \|\xi(\gamma)\|\ \Big\|_{\mathcal{A}} = \Big\|\ |\chi|\ |\xi|\ \Big\|_{\mathcal{A}}.$$

We therefore have for all $x \in X$:

$$\begin{aligned} \left\|(\xi)_{[x]}\right\| &= \inf\left\{\|\chi\xi\|_{\mathcal{A}(\mathcal{G},E)}:\ \chi \in \mathcal{C}_c\left(X/\mathcal{G}\right),\ 0 \le \chi \le 1,\ \chi([x]) = 1\right\} \\ &= \inf\left\{\|\chi\,|\xi|\|_{\mathcal{A}}:\ \chi \in \mathcal{C}_c\left(X/\mathcal{G}\right),\ 0 \le \chi \le 1,\ \chi([x]) = 1\right\} = \left\|(|\xi|)_{[x]}\right\|. \end{aligned}$$

Now the local convexity of $\mathcal{A}(\mathcal{G})$ implies

$$\|\xi\|_{\mathcal{A}(\mathcal{G},E)} = \big\| \, |\xi| \, \big\|_{\mathcal{A}} = \sup_{x \in X} \big\| (|\xi|)_{[x]} \big\| = \sup_{x \in X} \big\| (\xi)_{[x]} \big\|.$$

This identity carries over to all elements of the completion $\mathcal{A}(\mathcal{G}, E)$ of $\Gamma_c(\mathcal{G}, r^*E)$, so $\mathcal{A}(\mathcal{G}, E)$ is locally $\mathcal{C}_0(X/\mathcal{G})$-convex. □

If $\mathcal{A}(\mathcal{G})$ is a locally convex unconditional completion of $\mathcal{C}_c(\mathcal{G})$, then the descent can be considered to be a homomorphism

$$j_{\mathcal{A}} \colon \ \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A, B) \ \to \mathrm{KK}^{\mathrm{ban}}_{X/\mathcal{G}}\left(\mathcal{A}(\mathcal{G}, A), \ \mathcal{A}(\mathcal{G}, B)\right).$$

Chapter 6

Generalised Morphisms of Locally Compact Groupoids

The aim of this chapter is to define a homomorphism[1]

$$\Omega^* \colon \mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(A, B) \to \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\Omega^* A,\ \Omega^* B),$$

where $\mathcal{G}$ and $\mathcal{H}$ are locally compact Hausdorff groupoids (with open range and source maps), Ω is a generalised morphism from $\mathcal{G}$ to $\mathcal{H}$, and A and B are $\mathcal{H}$-Banach algebras. This homomorphism is functorial and generalises the pullback homomorphism along strict morphisms. In particular, it is an isomorphism if Ω is an equivalence of groupoids.

The construction follows the same fundamental plan as the analogous construction for C*-algebras given by Le Gall in [LG94]: If Ω is as above and has anchor maps $\rho\colon \Omega \to \mathcal{G}^{(0)}$ and $\sigma\colon \Omega \to \mathcal{H}^{(0)}$, then we can put Ω in the following commutative triangle

$$\begin{tikzcd}
\rho^*(\mathcal{G}) \arrow[d, "\rho"'] \arrow[dr, "f_\Omega"] & \\
\mathcal{G} \arrow[r, "\Omega"] & \mathcal{H}
\end{tikzcd}$$

where we identify generalised morphisms with their graphs.[2] The locally compact groupoid $\rho^*(\mathcal{G})$ (with unit space Ω) is the pullback of $\mathcal{G}$ along ρ, it would be called $\mathcal{G}_\Omega$ in the notation of [LG99] and $\mathcal{G}[\Omega]$ in [Tu04]. The morphism f_Ω is actually a strict morphism, and the graph of the strict morphism $\rho\colon \rho^*(\mathcal{G}) \to \mathcal{G}$ turns out to be a rather simple equivalence of groupoids. We already know how to pull $\mathcal{H}$-Banach spaces (and $\mathcal{H}$-Banach algebras, etc.) back along f_Ω, which gives us $\rho^*(\mathcal{G})$-Banach spaces (and $\rho^*(\mathcal{G})$-Banach algebras, etc.). What we need is a way of turning $\rho^*(\mathcal{G})$-Banach spaces into $\mathcal{G}$-Banach spaces, i.e., we want to invert the pullback functor ρ^* from the category of $\mathcal{G}$-Banach spaces to the category of $\rho^*(\mathcal{G})$-Banach spaces. This is done in Section[3] 6.5, and the resulting functor is called $\rho_!$.

[1] V. Lafforgue mentions in [Laf06] that such a homomorphism exists without giving any details.
[2] See Diagram (6.3) for a more precise statement.
[3] See the beginning of that section for a more precise statement of what is being constructed.

This way, we construct a functor $\Omega^* := \rho_! \circ f_\Omega^*$ from the category of $\mathcal{H}$-Banach spaces to the category of $\mathcal{G}$-Banach spaces. The functor descends to functors between the respective categories of Banach algebras, modules and pairs, and finally gives us a homomorphism Ω^* between the $\mathrm{KK}^{\mathrm{ban}}$-groups with the above-mentioned properties.[4]

The chapter is organised as follows: The first section recalls the definition of generalised morphisms of groupoids (in the sense of [LG94]) and also the definition of equivalences of groupoids (which are shown to be precisely the generalised isomorphisms). Most of the results are proved somewhere in the literature, especially in [LG94] and [Tu99], or are folklore (in particular, the rather unpleasant matter of showing the continuity of the various operations appearing in the construction of certain groupoids seems to be traditionally regarded as folklore; we introduce the notion of an inner product on a $\mathcal{G}$-spaces to be able to treat these questions without too much ado). As a technical tool we also introduce the linking groupoid of an equivalence of groupoids, in complete analogy to the linking algebra of a Morita equivalence of (Banach) algebras. The linking groupoid can be used to prove that equivalent groupoids have equivalent L^1-algebras; this is actually true in greater generality (with coefficients and for more general unconditional completions).[5] The corresponding theorem for C^*-algebras is well-known in the literature (for instance, see [MRW87]), but to my knowledge, this is not the case for the L^1-version (although it might have been around somewhere as well).

The third section is introduces the pullback of groupoids, which leads to the factorisation result for generalised homomorphisms sketched above (this is inspired heavily by [LG94] and [LG99]). In Section 6.4, we introduce Haar systems on groupoids and on spaces carrying actions of groupoids, and show that these notions are compatible with taking pullbacks or forming the linking groupoid.

Technically, Section 6.5 is the heart of this chapter, introducing the functor $p_!$ between equivariant fields of Banach spaces and showing how it descends to the $\mathrm{KK}^{\mathrm{ban}}$-groups, which is applied to define the pullback along generalised morphisms in the next section. The C^*-version of this construction can be found in [LG94], the Banach algebra version needs some more technical care. The final section relates equivalences of groupoids to induction from closed subgroups of locally compact groups and shows how to obtain a version of a theorem of Green concerning induced algebras.

6.1 Generalised morphisms

6.1.1 $\mathcal{G}$-spaces

We will only consider actions of locally compact Hausdorff groupoids on locally compact Hausdorff spaces. Many of the results and constructions that are collected in this section have analogous counterparts for actions of (possibly non-Hausdorff) locally compact groupoids on locally compact spaces. A general reference for this is [Tu04].

So let $\mathcal{G}$ be a locally compact Hausdorff groupoid.

Definition 6.1.1 ((Free/proper/principal) $\mathcal{G}$-spaces). A *left $\mathcal{G}$-space* is a locally compact Hausdorff space Ω together with a continuous so-called anchor map $\rho\colon \Omega \to \mathcal{G}^{(0)}$ and a continuous map $\mu\colon \mathcal{G} * \Omega \to \Omega$, where $\mathcal{G} * \Omega = \{(\gamma, \omega) \in \mathcal{G} \times \Omega : \ s(\gamma) = \rho(\omega)\}$, such that

1. $\rho(\mu(\gamma, \omega)) = r(\gamma)$ for all $(\gamma, \omega) \in \mathcal{G} * \Omega$;
2. $\mu(\rho(\omega), \omega) = \omega$ for all $\omega \in \Omega$;

[4] Take this with a grain of salt, there is a little twist in the definition of Ω^* for $\mathrm{KK}^{\mathrm{ban}}$-cycles; compare Lemma 6.5.17.

[5] See Theorem 6.6.10 and Section 6.7.

3. $\mu(\gamma \cdot \gamma', \omega) = \mu(\gamma, \mu(\gamma', \omega))$ for all $(\gamma, \gamma') \in \mathcal{G} * \mathcal{G}$ and $(\gamma', \omega) \in \mathcal{G} * \Omega$;

A right $\mathcal{G}$ space is defined similarly (and the anchor map of a right $\mathcal{G}$-space will usually be called σ). The action μ is usually written multiplicatively, i.e., $\mu(\gamma, \omega)$ is denoted by $\gamma \cdot \omega$ or $\gamma\omega$. The action is called *free* if for all $(g, \omega) \in \mathcal{G} * \Omega$ we have $\gamma \cdot \omega = \omega \Rightarrow \gamma \in \mathcal{G}^{(0)}$, i.e., only units have fixed points. The action is called *proper* if the map $(\mu, \mathrm{Id})\colon \mathcal{G} * \Omega \to \Omega \times \Omega,\ (\gamma, \omega) \mapsto (\gamma \cdot \omega, \omega)$ is proper. The space Ω is called a *principal* $\mathcal{G}$-space if it is free and proper.

To get a notion of isomorphic $\mathcal{G}$-spaces we define morphisms of $\mathcal{G}$-spaces as follows:

Definition 6.1.2 (Equivariant maps). Let Ω and Ω' be left $\mathcal{G}$-spaces with anchor maps ρ and ρ', respectively. A continuous map $\tau\colon \Omega \to \Omega'$ is called $\mathcal{G}$-*equivariant* if $\rho'(\tau(\omega)) = \rho(\omega)$ for all $\omega \in \Omega$ and

$$\tau(\gamma \cdot \omega) = \gamma \cdot \tau(\omega)$$

for all $\gamma \in \mathcal{G}$ and $\omega \in \Omega$ such that $s(\gamma) = \rho(\omega)$. In a similar manner one can define equivariant maps between right $\mathcal{G}$-spaces.

The left $\mathcal{G}$-spaces, together with the $\mathcal{G}$-equivariant continuous maps, form a category. The isomorphisms in this category are the $\mathcal{G}$-equivariant homeomorphisms.

Definition 6.1.3 (The crossed product). Let Ω be a left $\mathcal{G}$-space. Then the crossed product groupoid $\mathcal{G} \ltimes \Omega$ is defined as the subgroupoid of $\mathcal{G} \times (\Omega \times \Omega)$ consisting of elements $(\gamma, \omega', \omega)$ such that $s(\gamma) = \rho(\omega)$ and $\omega' = \gamma\omega$. The unit space of $\mathcal{G} \ltimes \Omega$ can be identified with Ω. If $\mathcal{G}$ has open range and source maps, then the range and source maps $\mathcal{G} \ltimes \Omega \to \Omega$ are open as well.[6]

In a similar fashion one defines crossed products for right actions. The map from $\mathcal{G} \ltimes \Omega$ to $\mathcal{G} \times_{r,\rho} \Omega$ given by $(\gamma, \omega', \omega) \mapsto (\gamma, \omega')$ is a homeomorphism, the groupoid $\mathcal{G} \ltimes \Omega$ can thus also be considered as a subspace of $\mathcal{G} \times \Omega$, and this is what we will do most of the time.[7]

Definition 6.1.4 (The quotient $\mathcal{G} \setminus \Omega$). Let Ω be a left $\mathcal{G}$-space. Then we define the quotient space $\mathcal{G} \setminus \Omega$ to be the set $\{[\omega] = \{\gamma\omega : s(\gamma) = \rho(\omega)\} : \omega \in \Omega\}$ of all orbits of the $\mathcal{G}$-action on Ω equipped with the quotient topology.

If $\mathcal{G}$ acts from the right on Ω, then we write $\Omega/\mathcal{G}$ for the quotient space.

Proposition 6.1.5. *The following are equivalent:*

1. $r\colon \mathcal{G} \to \mathcal{G}^{(0)}$ *is open;*

2. *for every left $\mathcal{G}$-space Ω the canonical map $\Omega \to \mathcal{G} \setminus \Omega$ is open.*

Proof. This is a special case of Lemma 2.30 of [Tu04]. □

Proposition 6.1.6. *Let Ω be a left $\mathcal{G}$-space. If Ω is a proper $\mathcal{G}$-space and the quotient map $\Omega \mapsto \mathcal{G} \setminus \Omega$ is open (for example, if $r\colon \mathcal{G} \to \mathcal{G}^{(0)}$ is open), then $\mathcal{G} \setminus \Omega$ is a locally compact Hausdorff space.*

Proof. This follows from Proposition 2.12 of [Tu04]. □

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[6] This is a special case of Lemma 2.24 in [Tu04] and also follows from our Lemma 3.4.5, applied to $\mathcal{G} \ltimes \Omega \cong \mathcal{G} \times_{\mathcal{G}^{(0)}} \Omega$.
[7] Compare [Tu04], 1.1.

Definition 6.1.7 (The flipped $\mathcal{G}$-space). If Ω is a left $\mathcal{G}$-space with anchor map ρ_Ω, then we define Ω^{-1} to be the right $\mathcal{G}$-space with underlying space Ω and the same anchor map $\sigma_{\Omega^{-1}} := \rho_\Omega$ and multiplication from $\Omega^{-1} * \mathcal{G} = \{(\omega^{-1}, \gamma) \in \Omega \times \mathcal{G} : \ \sigma_{\Omega^{-1}}(\omega^{-1}) = \rho_\Omega(\omega) = s(\gamma) = r(\gamma^{-1})\}$ to Ω^{-1} given by $(\omega^{-1}, \gamma) \mapsto (\gamma^{-1} \cdot \omega)^{-1}$. If Ω is proper or free, then so is Ω^{-1}.

Definition 6.1.8 (Products of $\mathcal{G}$-spaces). Let Ω_1 and Ω_2 be left $\mathcal{G}$-spaces. Let ρ_i be the anchor map of Ω_i for each $i \in \{1, 2\}$. Then define

$$\Omega := \Omega_1 * \Omega_2 = \big\{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 : \ \rho_1(\omega_1) = \rho_2(\omega_2)\big\}$$

and $\rho \colon \Omega \to \mathcal{G}^{(0)},\ \omega \mapsto \rho_1(\omega_1) = \rho_2(\omega_2)$. Define the map

$$\mathcal{G} * \Omega \to \Omega,\ (\gamma, \omega) \mapsto (\gamma \cdot \omega_1, \gamma \cdot \omega_2).$$

Then Ω is a left $\mathcal{G}$-space and the just defined action is called the *diagonal action*.

Proposition 6.1.9. *Let Ω_1 and Ω_2 be left $\mathcal{G}$-spaces. If Ω_1* or *Ω_2 is proper, then $\Omega_1 * \Omega_2$ is proper.*

Proof. This is proved in Appendix D.1 on page 291; compare Proposition 2.20 of [Tu04]. □

Definition 6.1.10. Let Ω be a right $\mathcal{G}$-space and Ω' a left $\mathcal{G}$-space. Then define $\Omega \times_{\mathcal{G}} \Omega'$ to be the quotient of $\Omega^{-1} * \Omega'$ by the diagonal (left) action of $\mathcal{G}$.

If the action of $\mathcal{G}$ on Ω *or* Ω' is proper and the canonical map $\Omega \times_{\mathcal{G}^{(0)}} \Omega' \to \Omega \times_{\mathcal{G}} \Omega'$ is open (which is the case if $\mathcal{G}$ has open range and source maps), then $\Omega \times_{\mathcal{G}} \Omega'$ is locally compact Hausdorff.

6.1.2 Principal $\mathcal{G}$-spaces and inner products

In this section let Ω be a left $\mathcal{G}$-space with open anchor map ρ. The map which sends some $\omega \in \Omega$ to its orbit $[\omega] \in \mathcal{G} \setminus \Omega$ will be denoted by σ.

Definition 6.1.11. An *inner product* on Ω is a continuous map $\langle \cdot, \cdot \rangle \colon \Omega \times_\sigma \Omega \to \mathcal{G}$ such that

1. $r(\langle \omega, \omega' \rangle) = \rho(\omega)$ and $s(\langle \omega, \omega' \rangle) = \rho(\omega')$ for all $(\omega, \omega') \in \Omega \times_\sigma \Omega$;
2. $\langle \gamma\omega, \omega' \rangle = \gamma \langle \omega, \omega' \rangle$ for all $(\omega, \omega') \in \Omega \times_\sigma \Omega$ and $\gamma \in \mathcal{G}$ such that $s(\gamma) = \rho(\omega)$;
3. $\langle \omega, \gamma\omega' \rangle = \langle \omega, \omega' \rangle \gamma^{-1}$ for all $(\omega, \omega') \in \Omega \times_\sigma \Omega$ and $\gamma \in \mathcal{G}$ such that $s(\gamma) = \rho(\omega')$;
4. $\langle \omega, \omega \rangle = \rho(\omega)$ for all $\omega \in \Omega$;
5. $\langle \omega', \omega \rangle = \langle \omega, \omega' \rangle^{-1}$ for all $(\omega, \omega') \in \Omega \times_\sigma \Omega$.

Proposition 6.1.12. *An inner product exists on Ω if and only if Ω is a principal $\mathcal{G}$-space, in case of which the inner product of $(\omega, \omega') \in \Omega \times_\sigma \Omega$ is the unique element $\langle \omega, \omega' \rangle$ such that*

$$\omega = \langle \omega, \omega' \rangle \omega'.$$

Proof. This is proved in Appendix D.1 on page 291. □

Proposition 6.1.13. *If Ω is a left principal $\mathcal{G}$ space, then*

$$\mathcal{G} \ltimes \Omega \cong \Omega \times_\sigma \Omega$$

as locally compact Hausdorff groupoids.

Proof. By definition, $\mathcal{G} \ltimes \Omega$ is a subspace of $\mathcal{G} \times (\Omega \times_\sigma \Omega)$, and the strict isomorphism we are looking for is given by the "projection" onto the second component. Alternatively, if we regard $\mathcal{G} \ltimes \Omega$ as $\mathcal{G} \times_{r,\rho} \Omega$, then the isomorphism is given by the map from $\mathcal{G} \ltimes \Omega$ to $\Omega \times_\sigma \Omega$ which sends (γ, ω) to $(\omega,\ \gamma^{-1}\omega)$. □

6.1.3 The groupoid $\Omega^{-1} \times_{\mathcal{G}} \Omega$

Let Ω be a $\mathcal{G}$-space with anchor map ρ. Then by Example 3.4.4 the space $\Omega * \Omega = \Omega \times_{\mathcal{G}_0} \Omega = \Omega \times_\rho \Omega$ carries the structure of a topological groupoid. Because Ω is locally compact and Hausdorff so is $\Omega \times_\rho \Omega$.

In what follows we will define the structure of a locally compact groupoid on the factor space $\Omega^{-1} \times_{\mathcal{G}} \Omega = \mathcal{G} \setminus \Omega * \Omega$. This structure is related to the above-mentioned groupoid structure on $\Omega \times_\rho \Omega$ and can be regarded as the structure of a "quotient groupoid".

We will assume that the locally compact Hausdorff groupoid $\mathcal{G}$ has *open* range and source maps and that Ω is a left *principal* $\mathcal{G}$-space. Then we know in particular that $\Omega^{-1} \times_{\mathcal{G}} \Omega$ is locally compact Hausdorff.

The map which sends some $\omega \in \Omega$ to its orbit $[\omega] \in \mathcal{G} \setminus \Omega$ will again be denoted by σ (note that this map is open by Proposition 6.1.5). The map from $\Omega \times_\sigma \Omega$ to $\mathcal{G}$ which assigns to each (ω, ω') the unique element $\gamma \in \mathcal{G}$ such that $\omega = \gamma\omega'$ will be denoted by $\langle \cdot, \cdot \rangle$. It is the inner product described above (in particular, it is continuous).

Definition and Proposition 6.1.14. The space $\mathcal{H} := \Omega^{-1} \times_{\mathcal{G}} \Omega$ carries the following structure of a locally compact Hausdorff groupoid:

$$\mathcal{H}^{(0)} := \mathcal{G} \setminus \Omega \quad \text{and} \quad \epsilon_{\mathcal{H}} \colon \mathcal{G} \setminus \Omega \to \Omega^{-1} \times_{\mathcal{G}} \Omega,\ [\omega] \mapsto [\omega^{-1}, \omega],$$

$$r_{\mathcal{H}} \colon \Omega^{-1} \times_{\mathcal{G}} \Omega \to \mathcal{G} \setminus \Omega,\ [\omega^{-1}, \omega'] \mapsto [\omega] \quad \text{and} \quad s_{\mathcal{H}} \colon \Omega^{-1} \times_{\mathcal{G}} \Omega \to \mathcal{G} \setminus \Omega,\ [\omega^{-1}, \omega'] \mapsto [\omega'].$$

If ρ is open, then $r_{\mathcal{H}}$ and $s_{\mathcal{H}}$ are open. The composition is defined as follows: Let $(\omega_1, \omega_1'), (\omega_2, \omega_2') \in \Omega \times_{\mathcal{G}^{(0)}} \Omega$ be such that $[\omega_1'] = [\omega_2]$. Then

$$[\omega_1^{-1}, \omega_1'] \circ [(\omega_2)^{-1}, \omega_2'] := [\omega_1^{-1},\ \langle \omega_1', \omega_2 \rangle \omega_2'].$$

It follows that $[\omega^{-1}, \omega']^{-1} = [\omega'^{-1}, \omega]$.

The maps $(\omega, \omega') \mapsto [\omega'^{-1}, \omega]$ and $\omega \mapsto [\omega]$ define a strict morphism q from $\Omega \times_{\mathcal{G}^0} \Omega$ onto $\mathcal{H}$. The locally compact groupoid $\mathcal{H} = \Omega^{-1} \times_{\mathcal{G}} \Omega$ could also be called ${}_{\mathcal{G}}\mathcal{K}(\Omega)$ in analogy with the compact operators on a (left) Hilbert module.

Proof. This is proved in Appendix D.1 on page 292. □

Proposition 6.1.15. *The locally compact Hausdorff groupoid $\mathcal{H} := \Omega^{-1} \times_{\mathcal{G}} \Omega$ acts* freely *and* properly *from the right on Ω.*

The action is defined as follows: The anchor map is σ and if $\omega \in \Omega$ and $[(\omega')^{-1}, \omega''] \in \mathcal{H}$ such that $\sigma(\omega) = \sigma(\omega') = r_{\mathcal{H}}([(\omega')^{-1}, \omega''])$, then $\omega \cdot [(\omega')^{-1}, \omega''] := \langle \omega, \omega' \rangle \omega''$.

The map $\rho \colon \Omega \to \mathcal{G}^{(0)}$ induces a continuous injection $\tilde{\rho}$ from $\Omega/\mathcal{H}$ to $\mathcal{G}^{(0)}$. If ρ is open and surjective, then $\tilde{\rho}$ is homeomorphism.

Proof. This is proved in Appendix D.1 on page 294. □

6.1.4 Bimodules

Let $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$ be locally compact Hausdorff groupoids.

Definition 6.1.16 ($\mathcal{G}$-$\mathcal{H}$-bimodule). A *$\mathcal{G}$-$\mathcal{H}$-bimodule* or $\mathcal{G}$-$\mathcal{H}$-space is a locally compact Hausdorff space Ω which is at the same time a left $\mathcal{G}$-space and a right $\mathcal{H}$-space (with anchor maps $\rho \colon \Omega \to \mathcal{G}^{(0)}$ and $\sigma \colon \Omega \to \mathcal{H}^{(0)}$, respectively) such that the actions commute, i.e.,

1. $\rho(\omega \cdot \eta) = \rho(\omega)$ for all $(\omega, \eta) \in \Omega * \mathcal{H}$;

2. $\sigma(\gamma \cdot \omega) = \sigma(\omega)$ for all $(\gamma, \omega) \in \mathcal{G} * \Omega$; and

3. $\gamma \cdot (\omega \cdot \eta) = (\gamma \cdot \omega) \cdot \eta$ for all $(\gamma, \omega) \in \mathcal{G} * \Omega$ and $(\omega, \eta) \in \Omega * \mathcal{H}$.

Example 6.1.17. Let Ω be a principal left $\mathcal{G}$-space with anchor map ρ, where the range and source maps of $\mathcal{G}$ are open. Let $\mathcal{H} := \Omega^{-1} \times_{\mathcal{G}} \Omega$. Then Ω is a $\mathcal{G}$-$\mathcal{H}$-bimodule when equipped with the $\mathcal{H}$-action defined above.

Definition 6.1.18 (The flipped bimodule). Let Ω be a $\mathcal{G}$-$\mathcal{H}$-bimodule. Then we define an $\mathcal{H}$-$\mathcal{G}$-bimodule Ω^{-1} as follows:

1. The underlying space of Ω^{-1} is simply Ω.

2. The anchor maps are given by $\sigma_{\Omega^{-1}}(\omega^{-1}) := \rho(\omega)$, defining a map from Ω^{-1} to $\mathcal{G}^{(0)}$, and $\rho_{\Omega^{-1}}(\omega^{-1}) := \sigma(\omega)$, defining a map $\Omega^{-1} \to \mathcal{H}^{(0)}$.

3. The left action of $\mathcal{H}$ on Ω^{-1} is given by $\mathcal{H} * \Omega^{-1} \to \Omega^{-1},\ (\eta,\ \omega^{-1}) \mapsto (\omega\eta^{-1})^{-1}$.

4. The right action of $\mathcal{G}$ on Ω^{-1} is given by $\Omega^{-1} * \mathcal{G} \to \Omega^{-1},\ (\omega^{-1},\ \gamma) \mapsto (\gamma^{-1}\omega)^{-1}$.

That the following definition makes sense is proved in Appendix D.1 on page 166.

Definition 6.1.19 (Product of bimodules). Let Ω be a proper right $\mathcal{H}$-space and Ω' an $\mathcal{H}$-$\mathcal{K}$-bimodule. Let $\mathcal{H}$ have open range and source maps. Then the quotient space $\Omega'' := \Omega \times_{\mathcal{H}} \Omega'$ of $\Omega \times_{\mathcal{H}^{(0)}} \Omega'$ is a locally compact Hausdorff space.

1. Define
$$\sigma'' \colon \Omega'' \to \mathcal{K}^{(0)},\ [(\omega, \omega')] \mapsto \sigma'(\omega')$$
where σ' is the right anchor map of Ω'. Define a $\mathcal{K}$-action on Ω'' (with anchor map σ'') by setting
$$[(\omega, \omega')] \cdot \kappa := [(\omega, \omega'\kappa)]$$
for all $(\omega, \omega') \in \Omega \times_{\mathcal{H}^{(0)}} \Omega'$ and $\kappa \in \mathcal{K}$ such that $\sigma'(\omega') = r(\kappa)$.

2. If Ω is not only a proper right $\mathcal{H}$ space but also a $\mathcal{G}$-$\mathcal{H}$-bimodule, then we can define a $\mathcal{G}$-$\mathcal{K}$-bimodule structure on Ω'' by defining
$$\rho'' \colon \Omega'' \to \mathcal{G}^{(0)},\ [(\omega, \omega')] \mapsto \rho(\omega)$$
and
$$\gamma \cdot [(\omega, \omega')] := [(\gamma\omega, \omega')]$$
for all $(\omega, \omega') \in \Omega \times_{\mathcal{H}^{(0)}} \Omega'$ and $\gamma \in \mathcal{G}$ such that $s(\gamma) = \rho(\omega)$.

6.1.5 Principal fibrations, graphs and morphisms

Let $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$ be locally compact Hausdorff groupoids with open range and source maps. The openness of these maps is not a dramatic restriction because our main interest is to be able to treat the case that the groupoids carry Haar systems, and in this case, the range and source maps are automatically open. For the definition of principal fibrations and generalised morphisms, we can thus go back to the definitions of [LG94] instead of the more elaborate concepts[8] of [LG99].

Definition 6.1.20 (Principal fibration). Let $\mathcal{H}$ act on the locally compact Hausdorff space Ω on the right. A map p from Ω to another topological space X is called *principal fibration* with structure groupoid $\mathcal{H}$ if

1. Ω is a principal $\mathcal{H}$-space;
2. p is continuous, open and surjective;
3. p is invariant under the action of $\mathcal{H}$, i.e., $\forall(\omega,\eta) \in \Omega * \mathcal{H}:\ p(\omega) = p(\omega\eta)$.
4. $\mathcal{H}$ acts transitively on each fibre of p, i.e., for all $\omega, \omega' \in \Omega$ such that $p(\omega) = p(\omega')$ there is an $\eta \in \mathcal{H}$ such that $\omega\eta = \omega'$; note that η is unique as Ω is free.

Because p is invariant under the action of $\mathcal{H}$ it induces a continuous map $\tilde{p}\colon \Omega/\mathcal{H} \to X$. Because $\mathcal{H}$ acts transitively on each fibre, $\tilde{p}$ is injective and hence a homeomorphism.

If $p\colon \Omega \to X$ is a principal fibration with structure groupoid $\mathcal{H}$, then there is a canonical continuous $\mathcal{H}$-valued inner product on $\Omega \times_p \Omega$. More precisely, $\Omega \times_p \Omega = \Omega \times_\sigma \Omega$ where $\sigma\colon \Omega \to \Omega/\mathcal{H}$ denotes the quotient map. Since Ω is a principal $\mathcal{H}$ space, we can now take the inner product from $\Omega \times_\sigma \Omega$ to $\mathcal{H}$ which assigns to each (ω, ω') the unique element η of $\mathcal{H}$ such that $\omega\eta = \omega'$. We will denote this element η by $\langle \omega, \omega' \rangle_{\mathcal{H}}$.

A generalised morphism of locally compact Hausdorff groupoids is an isomorphism class of graphs, and such a graph is defined as follows:

Definition 6.1.21 (Graph). A *graph* Ω (of a morphism) from $\mathcal{G}$ to $\mathcal{H}$ is a $\mathcal{G}$-$\mathcal{H}$-bimodule (with anchor maps ρ and σ, say), such that $\rho\colon \Omega \to \mathcal{G}^{(0)}$ is a principal fibration with structure groupoid $\mathcal{H}$.

Proposition 6.1.22. *Let Ω be a graph from $\mathcal{G}$ to $\mathcal{H}$. Since ρ is a principal fibration, there is an inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ from $\Omega \times_\rho \Omega$ to $\mathcal{H}$. It is $\mathcal{G}$-balanced in the sense that*

$$\big\langle \omega,\ \gamma\omega' \big\rangle_{\mathcal{H}} = \big\langle \gamma^{-1}\omega,\ \omega' \big\rangle_{\mathcal{H}} \tag{6.1}$$

for all $\omega, \omega' \in \Omega$ and $\gamma \in \mathcal{G}$ such that $\rho(\omega') = s(\gamma)$ and $r(\gamma) = \rho(\omega')$. It follows that the inner product factors through $\Omega^{-1} \times_{\mathcal{G}} \Omega$ to give a continuous $\mathcal{H}$-$\mathcal{H}$-bimodule map from $\Omega^{-1} \times_{\mathcal{G}} \Omega$ to $\mathcal{H}$.

Proof. The element $\eta = \langle \omega,\ \gamma\omega' \rangle_{\mathcal{H}}$ has the property $\omega\eta = \gamma\omega'$. It follows that $(\gamma^{-1}\omega)\eta = \omega'$, so η has the defining property of $\langle \gamma^{-1}\omega,\ \omega' \rangle_{\mathcal{H}}$. □

Definition 6.1.23 ((Generalised) morphism, equivalence of graphs). Two graphs Ω and Ω' from $\mathcal{G}$ to $\mathcal{H}$ are called *equivalent* if there is a homeomorphism from Ω to Ω' which intertwines the anchor maps and the actions of $\mathcal{G}$ and $\mathcal{H}$, i.e., an isomorphism of $\mathcal{G}$-$\mathcal{H}$-bimodules. A *(generalised) morphism* from $\mathcal{G}$ to $\mathcal{H}$ is simply an equivalence class of graphs. If Ω is a graph, then we denote the corresponding morphism by $[\Omega]$.

[8] For groupoids with open range and source maps, the definitions of [LG99] seem to amount to much the same picture as the one presented in the earlier article. The concepts of [LG94] are somewhat easier to handle, and another reason to use them here is that I was not able to check all the technical details of the more recent article.

Definition and Proposition 6.1.24 (Strict morphisms are morphisms). Let $f\colon \mathcal{G} \to \mathcal{H}$ be a strict morphism of groupoids. Then we define $\mathrm{Graph}(f)$ to be the following graph from $\mathcal{G}$ to $\mathcal{H}$:

$$\mathrm{Graph}(f) := \Omega := \mathcal{G}^{(0)} \times_{\mathcal{H}_0} \mathcal{H},$$

where the fibre product is taken over the maps $f|_{\mathcal{G}^{(0)}}$ and $r\colon \mathcal{H} \to \mathcal{H}^{(0)}$. The anchor maps are given by

$$\rho\colon \Omega \to \mathcal{G}^{(0)},\ (g, \eta) \mapsto g \quad \text{and} \quad \sigma\colon \Omega \to \mathcal{H}^{(0)},\ (g, \eta) \mapsto s(\eta).$$

The action of $\mathcal{G}$ on Ω is given by

$$\gamma(g, \eta) := (r(\gamma),\ f(\gamma)\eta)$$

for all $\gamma \in \mathcal{G}$, $g \in \mathcal{G}^{(0)}$ and $\eta \in \mathcal{H}$ such that $s(\gamma) = g$ and $f(g) = r(\eta)$. The action of $\mathcal{H}$ on Ω is given by multiplication from the right in the second component. The morphism $[\mathrm{Graph}(f)]$ given by $\mathrm{Graph}(f)$ is denoted by $\mathrm{Morph}(f)$.

Proof. Straightforward calculations show that Ω is indeed a bimodule. The map ρ is clearly invariant under the action of $\mathcal{H}$ and open because the range map of $\mathcal{H}$ is open (see Lemma 3.4.5). We have to show that $\mathcal{H}$ acts freely and properly on Ω and transitively on its fibres.

- Let $(g, \eta) \in \Omega$ and $\eta', \eta'' \in \mathcal{H}$ such that $s(\eta) = r(\eta') = r(\eta'')$ and $(g, \eta)\eta' = (g, \eta)\eta''$. Then this means $\eta\eta' = \eta\eta''$ and therefore $\eta' = \eta''$. So $\mathcal{H}$ acts freely on Ω.
- Consider the map from $\Omega * \mathcal{H}$ to $\Omega \times \Omega$ which maps $((g, \eta), \eta')$ to $((g, \eta), (g, \eta\eta'))$. This is composed of maps which are proper such as $g \mapsto (g, g)$ and $(\eta, \eta') \mapsto (\eta, \eta\eta')$, and standard arguments show that it is proper itself; hence the action of $\mathcal{H}$ on Ω is proper.
- Let $g \in \mathcal{G}^{(0)}$ and $\eta, \eta' \in \mathcal{H}$ such that $f(g) = r(\eta) = r(\eta')$. Define $\eta'' := \eta^{-1}\eta'$. Then $\eta\eta'' = \eta(\eta^{-1}\eta') = (\eta\eta^{-1})\eta' = \eta'$. Moreover, $r(\eta'') = s(\eta) = \sigma(g, \eta)$ and $(g, \eta)\eta'' = (g, \eta')$. So $\mathcal{H}$ acts transitively on the fibres of Ω. □

Definition 6.1.25 (Identity morphism). The *identity morphism of* $\mathcal{G}$ is defined as $\mathrm{Morph}(\mathrm{Id}_{\mathcal{G}})$, where $\mathrm{Id}_{\mathcal{G}}$ is the (strict) identity (morphism) on $\mathcal{G}$. It is the equivalence class of the graph $\mathcal{G}$, where we consider $\mathcal{G}$ to be a bimodule over itself, as $\mathcal{G}^{(0)} \times_{\mathcal{G}^{(0)}} \mathcal{G}$ is equivalent to $\mathcal{G}$. For obvious reasons we will denote this morphism also as $\mathrm{Id}_{\mathcal{G}}$.

Definition 6.1.26 (Composition of graphs). Let Ω be a graph from $\mathcal{G}$ to $\mathcal{H}$ and Ω' a graph from $\mathcal{H}$ to $\mathcal{K}$. Then we define on $\Omega'' := \Omega \times_{\mathcal{H}} \Omega'$ the structure of a $\mathcal{G}$-$\mathcal{K}$-bimodule as in 6.1.19. Then this bimodule is a graph from $\mathcal{G}$ to $\mathcal{K}$, called the *composition of* Ω *and* Ω'.

That Ω'' really is a graph is proved in Appendix D.1 on page 295.

The definition of the composition of graphs lifts to equivalence classes. Hence we have also defined the *composition of morphisms*. The locally compact Hausdorff groupoids, together with their morphisms, form a category: Associativity can be shown by a lengthy series of standard arguments. To see that the identity morphisms deserve their name let Ω be a graph from $\mathcal{G}$ to $\mathcal{H}$. Then the left action $\mu_{\mathcal{G}}$ from $\mathcal{G} * \Omega$ to Ω lifts to a continuous map from $\mathcal{G} \times_{\mathcal{G}} \Omega$ to Ω. This map clearly is a morphism of $\mathcal{G}$-$\mathcal{H}$-bimodules. It is inverted by the map $\omega \mapsto [(\rho(\omega), \omega)]$ which is continuous. Similarly one shows that $\Omega \times_{\mathcal{H}} \mathcal{H} \cong \Omega$.

Proposition 6.1.27. *The assignment $f \mapsto \mathrm{Morph}(f)$ is a functor from the category of locally compact Hausdorff groupoids (with open range and source maps) with the strict morphisms as morphisms to the category of locally compact Hausdorff groupoids (with open range and source maps) with all (generalised) morphisms.*

Proof. This is proved in Appendix D.1 on page 296. □

6.1.6 Equivalences

Let $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$ be locally compact Hausdorff groupoids with open range and source maps (to require the range and source maps to be open is a natural condition because we want equivalences to be morphisms).

Definition 6.1.28 ($\mathcal{G}$-$\mathcal{H}$-equivalence). A $\mathcal{G}$-$\mathcal{H}$-bimodule Ω is called a $\mathcal{G}$-$\mathcal{H}$-equivalence bimodule if

1. it is free and proper both as a $\mathcal{G}$- and an $\mathcal{H}$-space;
2. the anchor map $\rho\colon \Omega \to \mathcal{G}^{(0)}$ induces a homeomorphisms from $\Omega/\mathcal{H}$ to $\mathcal{G}^{(0)}$; and
3. the anchor map $\sigma\colon \Omega \to \mathcal{H}^{(0)}$ induces a homeomorphism from $\mathcal{G} \setminus \Omega$ to $\mathcal{H}^{(0)}$.

We call $\mathcal{G}$ and $\mathcal{H}$ *(Morita) equivalent*, and write $\mathcal{G} \sim_{\mathrm{M}} \mathcal{H}$, if such an equivalence exists.

Gathering what we have said above about the groupoid $\Omega^{-1} \times_{\mathcal{G}} \Omega$ we get the following fundamental example of an equivalence of groupoids:

Example 6.1.29. Let Ω be a free proper left $\mathcal{G}$-space with open and surjective anchor map ρ. Then Ω is an equivalence and
$$\mathcal{G} \quad \sim_{\mathrm{M}} \quad \Omega^{-1} \times_{\mathcal{G}} \Omega.$$

Proposition 6.1.30. *Let Ω be a $\mathcal{G}$-$\mathcal{H}$-equivalence. Then the locally compact groupoid $\Omega^{-1} \times_{\mathcal{G}} \Omega$ is strictly isomorphic to $\mathcal{H}$ through an isomorphism that also respects the canonical $\mathcal{H}$-$\mathcal{H}$-bimodule structures on $\Omega^{-1} \times_{\mathcal{G}} \Omega$ and $\mathcal{H}$.*

Proof. This is proved in Appendix D.1 on page 296. $\square$

Corollary 6.1.31. *If Ω is a $\mathcal{G}$-$\mathcal{H}$-equivalence bimodule, then Ω is the graph of an isomorphism from $\mathcal{G}$ to $\mathcal{H}$, the inverse having graph Ω^{-1}.*

The converse of this corollary is also true, so we have:

Proposition 6.1.32. *$\mathcal{G}$ and $\mathcal{H}$ are equivalent if and only if they are isomorphic (in the generalised sense). More precisely: If Ω is a graph of a generalised isomorphism from $\mathcal{G}$ to $\mathcal{H}$ and Ω' is a graph of its inverse from $\mathcal{H}$ to $\mathcal{G}$, then Ω is a $\mathcal{G}$-$\mathcal{H}$-equivalence and Ω^{-1} is isomorphic to Ω' as $\mathcal{H}$-$\mathcal{G}$-bimodules.*

Proof. This is proved in Appendix D.1 on page 297. $\square$

The following corollaries can also easily be obtained from direct calculation.

Corollary 6.1.33. *Let Ω be a $\mathcal{G}$-$\mathcal{H}$-equivalence and Ω' an $\mathcal{H}$-$\mathcal{K}$-equivalence. Then $\Omega'' := \Omega \times_{\mathcal{H}} \Omega'$ is a $\mathcal{G}$-$\mathcal{K}$-equivalence.*

Corollary 6.1.34. *Morita equivalence defines an equivalence relation on the locally compact Hausdorff groupoids with open range and source maps.*

Proposition 6.1.35. *Let Ω be an equivalence from $\mathcal{G}$ to $\mathcal{H}$. Write $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ for the $\mathcal{H}$-valued inner product $\Omega^{-1} \times_{\mathcal{G}^{(0)}} \Omega \to \mathcal{H}$ and ${}_{\mathcal{G}}\langle \cdot, \cdot \rangle$ for the $\mathcal{G}$-valued inner product $\Omega \times_{\mathcal{H}^{(0)}} \Omega^{-1} \to \mathcal{G}$. Then for all $\omega, \omega', \omega'' \in \Omega$ such that $\sigma(\omega) = \sigma(\omega')$ and $\rho(\omega') = \rho(\omega'')$ we have*

$$ {}_{\mathcal{G}}\langle \omega, \omega' \rangle\, \omega'' = \omega\, \langle \omega', \omega'' \rangle_{\mathcal{H}}. \tag{6.2}$$

Proof. We have
$$ {}_{\mathcal{G}}\langle \omega, \omega' \rangle\, \omega'' \langle \omega', \omega'' \rangle_{\mathcal{H}}^{-1} \;=\; {}_{\mathcal{G}}\langle \omega, \omega' \rangle\, \omega'' \langle \omega'', \omega' \rangle_{\mathcal{H}} \;=\; {}_{\mathcal{G}}\langle \omega, \omega' \rangle\, \omega' \;=\; \omega.$$
Multiplying this by $\langle \omega', \omega'' \rangle_{\mathcal{H}}$ on both sides gives (6.2). $\square$

6.2 The linking groupoid

6.2.1 Definition

Let $\mathcal{G}$ and $\mathcal{H}$ be locally compact Hausdorff groupoids with open range and source maps. Let Ω be a $\mathcal{G}$-$\mathcal{H}$-equivalence.

Definition 6.2.1 (The linking groupoid). Let $\mathcal{L}$ be the locally compact Hausdorff space $\mathcal{L} := \mathcal{G} \sqcup \Omega \sqcup \Omega^{-1} \sqcup \mathcal{H}$ and $\mathcal{L}^{(0)} := \mathcal{G}^{(0)} \sqcup \mathcal{H}^{(0)}$. Define the range and source maps of $\mathcal{L}$ as

$$r_{\mathcal{L}}\colon \mathcal{L} \to \mathcal{L}, \left\{ \begin{array}{ccccll} \mathcal{G} & \ni & \gamma & \mapsto & r_{\mathcal{G}}(\gamma) & \in \ \mathcal{G}^{(0)} \\ \Omega & \ni & \omega & \mapsto & \rho(\omega) & \in \ \mathcal{G}^{(0)} \\ \Omega^{-1} & \ni & \omega^{-1} & \mapsto & \rho(\omega^{-1}) = \sigma(x) & \in \ \mathcal{H}^{(0)} \\ \mathcal{H} & \ni & \eta & \mapsto & r_{\mathcal{H}}(\eta) & \in \ \mathcal{H}^{(0)} \end{array} \right\},$$

and

$$s_{\mathcal{L}}\colon \mathcal{L} \to \mathcal{L}, \left\{ \begin{array}{ccccll} \mathcal{G} & \ni & \gamma & \mapsto & s_{\mathcal{G}}(\gamma) & \in \ \mathcal{G}^{(0)} \\ \Omega & \ni & \omega & \mapsto & \sigma(\omega) & \in \ \mathcal{H}^{(0)} \\ \Omega^{-1} & \ni & \omega^{-1} & \mapsto & \sigma(\omega^{-1}) = \rho(\omega) & \in \ \mathcal{G}^{(0)} \\ \mathcal{H} & \ni & \eta & \mapsto & s_{\mathcal{H}}(\eta) & \in \ \mathcal{H}^{(0)} \end{array} \right\}.$$

With these definitions,

$$\mathcal{L} * \mathcal{L} = \mathcal{G}{*}\mathcal{G} \ \sqcup \ \mathcal{G}{*}\Omega \ \sqcup \ \Omega{*}\Omega^{-1} \ \sqcup \ \Omega{*}\mathcal{H} \ \sqcup \ \Omega^{-1}{*}\mathcal{G} \ \sqcup \ \Omega^{-1}{*}\Omega \ \sqcup \ \mathcal{H}{*}\Omega^{-1} \ \sqcup \ \mathcal{H}{*}\mathcal{H}.$$

Define a composition map from $\mathcal{L} * \mathcal{L}$ to $\mathcal{L}$ as the obvious map on the components $\mathcal{G}{*}\mathcal{G}$, $\mathcal{G}{*}\Omega$, $\Omega{*}\mathcal{H}$, $\Omega^{-1}{*}\mathcal{G}$, $\mathcal{H}{*}\Omega^{-1}$, and $\mathcal{H}{*}\mathcal{H}$; on $\Omega^{-1}{*}\Omega$ and $\Omega{*}\Omega^{-1}$ we take the factor map onto $\Omega^{-1} \times_{\mathcal{G}} \Omega$ and $\Omega \times_{\mathcal{H}} \Omega^{-1}$, which we identify with $\mathcal{H}$ and $\mathcal{G}$, respectively. In other words, a $(\omega^{-1}, y) \in \Omega^{-1}{*}\Omega$ is mapped to its inner product $\langle \omega, \omega' \rangle_{\mathcal{H}} \in \mathcal{H}$, which is the unique element η of $\mathcal{H}$ such that $\omega' = \omega\eta$ (and similarly for $\Omega{*}\Omega^{-1}$).

Proposition 6.2.2. *$\mathcal{L}$ is a locally compact Hausdorff groupoid with open range and source maps. The inversion on $\mathcal{L}$ is the map*

$$\mathcal{L} \to \mathcal{L}, \left\{ \begin{array}{ccccll} \mathcal{G} & \ni & \gamma & \mapsto & \gamma^{-1} & \in \ \mathcal{G} \\ \Omega & \ni & \omega & \mapsto & \omega^{-1} & \in \ \Omega^{-1} \\ \Omega^{-1} & \ni & \omega^{-1} & \mapsto & \omega & \in \ \Omega \\ \mathcal{H} & \ni & \eta & \mapsto & \eta^{-1} & \in \ \mathcal{H} \end{array} \right\}.$$

6.2.2 Full subsets

Recall from Definition 5.3.6 that a subset $U \subseteq \mathcal{G}^{(0)}$ of the unit space of a locally compact Hausdorff groupoid $\mathcal{G}$ is called *full* if $\mathcal{G}_U \circ \mathcal{G}^U = \mathcal{G}$, i.e., if every element γ of $\mathcal{G}$ can be written as a product $\gamma_1\gamma_2$ with γ_1 starting in U and γ_2 ending in U.

Proposition 6.2.3. *Let $\mathcal{G}$ be a locally compact Hausdorff groupoid with open range and source maps and $U \subseteq \mathcal{G}^{(0)}$ a full open subset. Then $\mathcal{G}_U^U$ is a locally compact Hausdorff groupoid with open range and source maps and $\mathcal{G}^U$ is a $\mathcal{G}_U^U$-$\mathcal{G}$-equivalence.*

Proof. First of all, $\Omega := \mathcal{G}^U$ is an open subset of $\mathcal{G}$ such that $\mathcal{G}_U^U \subseteq \Omega$. The range map $\rho := r|_\Omega \colon \Omega \to U$ is open and surjective (sice $\mathcal{G}_U^U \subseteq \Omega$). Also the source map $\sigma := s|_\Omega \colon \Omega \to \mathcal{G}^{(0)}$ is open and surjective since U is full. $\mathcal{G}_U^U$ acts from the left and $\mathcal{G}$ acts from the right on Ω by multiplication. The map $(\gamma, \gamma') \mapsto \gamma^{-1}\gamma'$ is a continuous inner product $\Omega \times_\rho \Omega \to \mathcal{G}$, so ρ is a principal fibration with structure groupoid $\mathcal{G}$, and the map $(\gamma, \gamma') \mapsto \gamma\gamma^{-1}$ is a continuous inner product $\Omega \times_\sigma \Omega \to \mathcal{G}_U^U$, so σ is a principal fibration with structure groupoid $\mathcal{G}_U^U$. Hence Ω is an equivalence. □

Corollary 6.2.4. *Let $\mathcal{G}$ and $\mathcal{H}$ be locally compact Hausdorff groupoids with open range and source maps and let Ω be a $\mathcal{G}$-$\mathcal{H}$-equivalence. Form the linking groupoid $\mathcal{L}$ as above. Then $U := \mathcal{G}^{(0)}$ is a full open and closed subset of $\mathcal{L}^{(0)}$ and $\mathcal{L}_U^U$ can be identified with $\mathcal{G}$. So $\mathcal{G}$ is equivalent to $\mathcal{L}$. In a similar fashion, $\mathcal{H}$ is equivalent to $\mathcal{L}$.*

6.3 The pullback of groupoids

Definition 6.3.1 (The pullback of a topological groupoid). [9] Let X and Y be topological spaces, let $\mathcal{G}$ be a topological groupoid over X and let $p\colon Y \to X = \mathcal{G}^{(0)}$ be a continuous map. Then we define $p^*(\mathcal{G})$ to be the fibre product of $Y \times Y$ and $\mathcal{G}$ over $X \times X = \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$, i.e., $p^*(\mathcal{G})$ is defined as the pullback in the following diagram:

$$\begin{tikzcd} p^*(\mathcal{G}) \arrow[r] \arrow[d] & Y \times Y \arrow[d, "p\times p"] \\ \mathcal{G} \arrow[r, "(r,s)"] & X \times X \end{tikzcd}$$

It can be realised as follows:

$$p^*(\mathcal{G}) \cong \{(z, \gamma, y) \in Y \times \mathcal{G} \times Y :\ s(\gamma) = p(y),\ r(\gamma) = p(z)\}$$

and the unit space of $p^*(\mathcal{G})$ can be identified with Y. The source and range function are

$$R\colon p^*(\mathcal{G}) \to Y,\ (z, \gamma, y) \mapsto z, \quad S\colon p^*(\mathcal{G}) \to Y,\ (z, \gamma, y) \mapsto y.$$

Moreover,

$$\epsilon\colon Y \to p^*(\mathcal{G}),\ y \mapsto (y, \epsilon(p(y)), y).$$

The composition is given by

$$(z, \gamma, y) \circ (z', \gamma', y') = (z, \gamma \circ \gamma', y')$$

and is defined if and only if $y = z'$. The inverse is given by $(z, \gamma, y)^{-1} = (y, \gamma^{-1}, z)$.

There is a canonical strict morphism from $p^*(\mathcal{G})$ to $\mathcal{G}$, appearing in the above diagram, which we call $p^{\mathcal{G}}$ or simply p if the context is clear. It is given explicitly by $(z, \gamma, y) \mapsto \gamma$.

Proposition 6.3.2. [10] *If $\mathcal{G}$ and Y are Hausdorff, second countable or locally compact, then so is $p^*(\mathcal{G})$. If r, s and p are open, then so are the maps R and S.*

[9] What we call $p^*(\mathcal{G})$ is called $\mathcal{G}_Y$ in [LG99] and $\mathcal{G}[Y]$ in [Tu04].

[10] See [Tu04], Proposition 2.7 and Lemma 2.24, for more precise results.

Example 6.3.3. Let X and Y be topological spaces and let $p\colon Y \to X$ be continuous. Then X itself can be regarded as a topological groupoid over X as we have seen in Example 3.4.1. We have

$$p^*(X) \cong Y \times_X Y.$$

The isomorphism from $p^*(X)$ to $Y \times_X Y$ sends (y', x, y) to (y', y), where $y, y' \in Y$, $x \in X$ and $p(y') = x = p(y)$.

If $\mathcal{G}$ is a topological groupoid over X and X is closed in $\mathcal{G}$ (which is automatic if $\mathcal{G}$ is Hausdorff), then $p^*(X) = Y \times_X Y$ is contained as a closed subgroupoid in $p^*(\mathcal{G})$.

Proposition 6.3.4. *Let X, Y, Z be topological spaces and let $\mathcal{G}$ be a topological groupoid over X. Assume that $p\colon Y \to X$ and $q\colon Z \to Y$ are continuous maps. Then there is a canonical isomorphism from $q^*(p^*(\mathcal{G}))$ to $(p \circ q)^*(\mathcal{G})$ such that the following diagram commutes:*

$$\begin{tikzcd}
q^*(p^*(\mathcal{G})) \arrow[r] \arrow[d] & (p \circ q)^*(\mathcal{G}) \arrow[ddl] \\
p^*(\mathcal{G}) \arrow[d] & \\
\mathcal{G} &
\end{tikzcd}$$

Proof. The groupoid $q^*(p^*(\mathcal{G}))$ can be realised as

$$\{(z', y', \gamma, y, z) :\ q(z') = y', r(\gamma) = p(y'), s(\gamma) = p(y), q(z) = y\}.$$

The isomorphism to $(p \circ q)^*(\mathcal{G})$ is given by

$$(z', y', \gamma, y, z) \mapsto (z', \gamma, z),$$

whereas its inverse is given by

$$(z', \gamma, z) \mapsto (z', q(z'), \gamma, q(z), z). \qquad \square$$

Proposition 6.3.5. *If $\mathcal{G}$ is a topological groupoid, then $\mathrm{Id}^*_{\mathcal{G}^{(0)}}(\mathcal{G}) \cong \mathcal{G}$ where the isomorphism is given by "Id", the canonical map $\mathrm{Id}^*_{\mathcal{G}^{(0)}}(\mathcal{G}) \to \mathcal{G}$. The inverse is given by $\gamma \mapsto (r(\gamma), \gamma, s(\gamma))$.*

Under certain conditions, the graph of $p\colon p^*(\mathcal{G}) \to \mathcal{G}$ is an equivalence:

Proposition 6.3.6. *Let $\mathcal{G}$ be a locally compact Hausdorff groupoid over X with open range and source maps. Let Y be a locally compact Hausdorff space and let $p\colon Y \to X$ be continuous. The strict morphism $p\colon p^*(\mathcal{G}) \to \mathcal{G}$ has graph*

$$\Omega := p^*(\mathcal{G})^{(0)} \times_{\mathcal{G}^{(0)}} \mathcal{G} = Y \times_{\mathcal{G}^{(0)}} \mathcal{G} = \{(y, \gamma) \in Y \times \mathcal{G} :\ p(y) = r(\gamma)\}.$$

If $p\colon Y \to X$ is open and surjective, then Ω is an equivalence.

Proof. Because Ω is a graph, it is a principal $\mathcal{G}$-space and the map ρ is a surjective and open principal fibration with structure groupoid $\mathcal{G}$. Moreover, $\sigma\colon \Omega \to \mathcal{G}^{(0)}$, $(y, \gamma) \mapsto s(\gamma)$, is open and surjective because p is open and surjective and s is open and surjective. We have to show that σ is a principal fibration with structure groupoid $p^*(\mathcal{G})$.

Define a map

$$\langle \cdot,\cdot \rangle \colon \Omega \times_\sigma \Omega \to p^*(\mathcal{G}),\ ((y_1,\gamma_1),(y_2,\gamma_2)) \mapsto \langle (y_1,\gamma_1),\ (y_2,\gamma_2)\rangle := (y_1, \gamma_1\gamma_2^{-1}, y_2).$$

If $(y_1,\gamma_1),(y_2,\gamma_2) \in \Omega$ with $\sigma(y_1,\gamma_1) = s(\gamma_1) = s(\gamma_2) = \sigma(y_2,\gamma_2)$, then

$$\langle (y_1,\gamma_1),\ (y_2,\gamma_2)\rangle \cdot (y_2,\gamma_2) = (y_1,\gamma_1\gamma_2^{-1},y_2)\cdot(y_2,\gamma_2) = (y_1,\gamma_1\gamma_2^{-1}\gamma_2) = (y_1,\gamma_1).$$

This implies that the fibres of σ are the orbits of the $p^*(\mathcal{G})$-action on Ω, i.e.,

$$\Omega \times_\sigma \Omega = \Omega \times_{p^*(\mathcal{G})\backslash\Omega} \Omega.$$

We show that $\langle\cdot,\cdot\rangle$ is an inner product on Ω in the sense of Definition 6.1.11. To this end we check the properties 2. and 4. of the definition: Let (y_1,γ_1) and (y_2,γ_2) in Ω such that $s(\gamma_1) = s(\gamma_2)$ and $(z,\gamma,y) \in p^*(\mathcal{G})$ such that $y = s((z,\gamma,y)) = \rho((y_1,\gamma)) = y_1$. Then

$$\begin{aligned}\langle (z,\gamma,y)\cdot(y_1,\gamma_1),\ (y_2,\gamma_2)\rangle &= \langle (z,\gamma\gamma_1),\ (y_2,\gamma_2)\rangle = \left(z,\gamma\gamma_1\gamma_2^{-1},y_2\right)\\ &= (z,\gamma,y)\left(y_1,\gamma_1\gamma_2^{-1},y_2\right) = (z,\gamma,y)\,\langle (y_1,\gamma_1),\ (y_2,\gamma_2)\rangle\end{aligned}$$

and

$$\langle (y_1,\gamma_1),\ (y_1,\gamma_1)\rangle = \left(y_1,\gamma_1\gamma_1^{-1},y_1\right) = (y_1,p(y_1),y_1)\,.$$

This shows that $\langle\cdot,\cdot\rangle$ is an inner product. So Ω is a free and proper $p^*(\mathcal{G})$-space and σ is a principal fibration with structure groupoid $p^*(\mathcal{G})$. □

Definition and Proposition 6.3.7 (The strict morphism f_Ω). Let $\mathcal{G}$ and $\mathcal{H}$ be locally compact Hausdorff groupoids with open range and source maps and let Ω be a graph from $\mathcal{G}$ to $\mathcal{H}$. Write $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ for the $\mathcal{H}$-valued inner product from $\Omega\times_\rho\Omega$ to $\mathcal{H}$, i.e., $\langle\omega,\omega'\rangle_{\mathcal{H}}$ is defined to be the unique element of $\mathcal{H}$ such that $\omega'\langle\omega',\omega\rangle_{\mathcal{H}} = \omega$.

Define $f_\Omega(\omega',\gamma,\omega) := \langle\omega',\gamma\omega\rangle_{\mathcal{H}}$ for all $(\omega',\gamma,\omega) \in \rho^*(\mathcal{G})$. Then $f_\Omega\colon \rho^*(\mathcal{G}) \to \mathcal{H}$ is a strict morphism extending $\sigma\colon \Omega = \rho^*(\mathcal{G})^{(0)} \to \mathcal{H}^{(0)}$.

Proof.
- Let $\omega \in \Omega$. Then $f_\Omega(\omega,\rho(\omega),\omega) = \langle\omega,\rho(\omega)\omega\rangle_{\mathcal{H}} = \langle\omega,\omega\rangle_{\mathcal{H}} = \sigma(\omega)$. Hence $f_\Omega|_{\rho^*(\mathcal{G})^{(0)}} = \sigma$.
- Let $(\omega'',\gamma',\omega'),(\omega',\gamma,\omega) \in \rho^*(\mathcal{G})$. Then

  $$\begin{aligned}f_\Omega(\omega'',\gamma',\omega')f_\Omega(\omega',\gamma,\omega) &= \left\langle\omega'',\gamma'\omega'\right\rangle\left\langle\omega',\gamma\omega\right\rangle = \left\langle\omega'',\gamma'\omega'\left\langle\omega',\gamma\omega\right\rangle\right\rangle\\ &= \left\langle\omega'',\gamma'\gamma\omega\right\rangle = f_\Omega(\omega'',\gamma'\gamma,\omega) = f_\Omega\left((\omega'',\gamma',\omega')(\omega',\gamma,\omega)\right).\end{aligned}$$

  So f_Ω is a homomorphism of groupoids.
- Since the inner product is continuous, it follows that f_Ω is continuous. □

Proposition 6.3.8. *Let $\mathcal{G}$ and $\mathcal{H}$ be locally compact Hausdorff groupoids with open range and source maps and let Ω be a graph from $\mathcal{G}$ to $\mathcal{H}$. Then* $\mathrm{Morph}(f_\Omega)$ *makes the following diagram commutative*

(6.3)
$$\begin{tikzcd}
\rho^*(\mathcal{G}) \arrow[d, "\mathrm{Morph}(\rho)"'] \arrow[dr, "\mathrm{Morph}(f_\Omega)"] & \\
\mathcal{G} \arrow[r, "{[\Omega]}"] & \mathcal{H}
\end{tikzcd}$$

Proof. The composition of $\mathrm{Graph}(\rho)$ and Ω is given by $(\Omega \times_{\mathcal{G}^{(0)}} \mathcal{G}) \times_{\mathcal{G}} \Omega \cong \Omega \times_{\mathcal{G}^{(0)}} \Omega$ and the graph of f_Ω is given by $\Omega \times_{\mathcal{H}^{(0)}} \mathcal{H} = \Omega * \mathcal{H}$. Define the map

$$\iota\colon \Omega * \mathcal{H} \to \Omega \times_\rho \Omega,\ (\omega, \eta) \mapsto (\omega, \omega\eta).$$

Note that, since ρ is a principal fibration, $\Omega \times_\rho \Omega = \Omega \times_{\Omega/\mathcal{H}} \Omega$. As Ω is a free and proper $\mathcal{H}$-space, the map ι is a homeomorphism.

The action of $\mathcal{H}$ on $\Omega \times_{\mathcal{G}^{(0)}} \mathcal{H}$ is given by multiplication from the right in the second component. The action of $\mathcal{H}$ on $\Omega \times_{\mathcal{H}^{(0)}} \Omega$ is given by $(\omega, \omega')\eta := (\omega, \omega'\eta)$. Now

$$\iota(\omega, \eta)\eta' = (\omega, \omega\eta)\eta' = (\omega, \omega\eta\eta') = \iota(\omega, \eta\eta')$$

for all $\omega \in \Omega$, $\eta, \eta' \in \mathcal{H}$ such that $\sigma(\omega) = r(\eta)$ and $s(\eta) = r(\eta')$.

The action of $\rho^*(\mathcal{G})$ on $\Omega \times_{\mathcal{G}^{(0)}} \mathcal{H}$ is given by

$$(\omega', \gamma, \omega)(\omega, \eta) = (\omega', f_\Omega(\omega', \gamma, \omega)\eta) = (\omega', \langle \omega', \gamma\omega \rangle \eta).$$

The action of $\rho^*(\mathcal{G})$ on $\Omega \times_{\mathcal{H}^{(0)}} \Omega$ is given by $(\omega', \gamma, \omega)(\omega, \omega'') = (\omega', \gamma\omega'')$. Now

$$(\omega', \gamma, \omega)\iota(\omega, \eta) = (\omega', \gamma, \omega)(\omega, \omega\eta) = (\omega', \gamma\omega\eta)$$

and

$$\iota\left((\omega', \gamma, \omega)(\omega, \eta)\right) = \iota\left(\omega', \langle \omega', \gamma\omega \rangle \eta\right) = (\omega', \omega'\langle \omega', \gamma\omega \rangle \eta).$$

Because $\omega'\langle \omega', \gamma\omega \rangle = \gamma\omega$ by definition, we have thus shown that ι respects the bimodule structure. $\square$

Corollary 6.3.9. *Every generalised morphism can be written as the composition of an equivalence and a strict morphism.*

Remark 6.3.10. The triangle (6.3) can be completed to give the following square:

(6.4)
$$\begin{tikzcd}
\rho^*(\mathcal{G}) \arrow[r, "\mathrm{Morph}(F_\Omega)"] \arrow[d, "\mathrm{Morph}(\rho)"'] \arrow[dr, "\mathrm{Morph}(f_\Omega)"] & \sigma^*(\mathcal{H}) \arrow[d, "\mathrm{Morph}(\sigma)"] \\
\mathcal{G} \arrow[r, "{[\Omega]}"] & \mathcal{H}
\end{tikzcd}$$

where the top arrow is given by the strict morphism

$$F_\Omega\colon \rho^*(\mathcal{G}) \to \sigma^*(\mathcal{H}),\ (\omega', \gamma, \omega) \mapsto (\omega', f_\Omega(\omega', \gamma, \omega), \omega).$$

Ω is an equivalence if and only if F_Ω is a strict isomorphism. Note that this implies that every equivalence can be written as a product of three very special equivalences, namely an strict isomorphism and two equivalences stemming from the pullback construction described above.

Tu[11] has shown that if Ω is an equivalence, then $\rho^*(\mathcal{G}) \cong \mathcal{G} \ltimes (\Omega \rtimes \mathcal{H})$, i.e., $\rho^*(\mathcal{G})$ is the iterated crossed product of groupoids (which we have not defined here). It follows by symmetry that $\sigma^*(\mathcal{H}) \cong (\mathcal{G} \ltimes \Omega) \rtimes \mathcal{H}$. Since the two different iterated crossed products are isomorphic, there is an induced isomorphism between $\rho^*(\mathcal{G})$ and $\sigma^*(\mathcal{H})$, which turns out to be the one we have given above.

[11] See [Tu04], Proof of Proposition 2.29.

6.4 Locally compact groupoids with Haar systems

We have used Haar systems on groupoids already in the preceding chapter when we discussed the descent. We will now analyse how Haar systems behave under the constructions we have introduced above: Can one lift Haar systems to equivalent groupoids, to the pullback of a groupoids or to linking groupoids? To be able to discuss these questions systematically, we will introduce Haar systems not only on groupoids but also on spaces on which groupoids act.

6.4.1 Haar systems

Let $\mathcal{G}$ and $\mathcal{H}$ be locally compact Hausdorff groupoids.

Definition 6.4.1 (Haar system). A left Haar system on a left $\mathcal{G}$-space Ω with (open and) surjective anchor map ρ is a faithful continuous field[12] $(\lambda_\Omega^g)_{g\in\mathcal{G}^{(0)}}$ of measures on Ω over $\mathcal{G}^{(0)}$ with coefficient map ρ such that

$$\forall\gamma\in\mathcal{G}\;\forall\varphi\in\mathcal{C}_c(\Omega):\;\int_{\omega\in\Omega}\varphi(\omega)\,\mathrm{d}\lambda_\Omega^{r(\gamma)}(\omega)=\int_{\omega\in\Omega}\varphi(\gamma\omega)\,\mathrm{d}\lambda_\Omega^{s(\gamma)}(\omega). \tag{6.5}$$

Similarly, right Haar systems are defined.

Definition 6.4.2 (Haar system on $\mathcal{G}$). Using that $\mathcal{G}$ acts on itself on the left, we define a left Haar system on the groupoid $\mathcal{G}$ to be a left Haar system for this action.

Note that such a Haar system need not exist. If $\mathcal{G}$ is a locally compact Hausdorff groupoid admitting a Haar system, then it follows from Lemma B.2.4 that its range and source maps are open.

Example 6.4.3. Let X and Y be locally compact Hausdorff spaces and let $p\colon Y\to X$ be an open continuous map. On $Y\times_X Y$ there is a structure of a locally compact Hausdorff groupoid with unit space Y as we have seen in 3.4.4.

1. Let $(\mu_x)_{x\in X}$ be a faithful continuous field of measures on Y over X with coefficient map p. For every $y\in Y$ and all Borel subsets A of $Y\times_X Y$, define

$$\lambda^y(A):=\mu_{p(y)}(\{y'\in Y:\;(y,y')\in A\})$$

   Then $(\lambda^y)_{y\in Y}$ is a left Haar system on $Y\times_X Y$.

2. Conversely, if λ is a left Haar system on $Y\times_X Y$, then this means $\lambda^y=\lambda^{y'}$ for all $(y,y)\in Y\times_X Y$. If we thus define $\mu_x:=\lambda^y$ for every $y\in Y$ such that $p(y)=x$ and if p is (open and) surjective, then $(\mu_x)_{x\in X}$ is a faithful continuous field of measures on Y over X.

Proof. 1. To see that λ is a continuous field of measures, note that λ is the same as $p^*(\mu)$, where the pullback $p^*(\mu)$ is defined as in B.2.8. This also shows that λ is faithful. Let us check the invariance property (6.5). Let $(y',y)\in Y\times_X Y$ and $\varphi\in\mathcal{C}_c(Y\times_X Y)$. By definition, $r(y',y)=y'$ and $s(y',y)=y$; moreover, the fibre $(Y\times_X Y)^y$ is the set $\{(y,y''):\;y''\in Y,p(y'')=p(y)\}$ and can thus be identified with $Y_{p(y)}$. We have to show

$$\int_{y''\in Y_{p(y)}}\varphi((y',y)\cdot(y,y''))\,\mathrm{d}\lambda^y(y,y'')=\int_{y''\in Y_{p(y')}}\varphi(y',y'')\,\mathrm{d}\lambda^{y'}(y',y'').$$

But this is trivial since both sides are equal to $\int_{y''\in Y_{p(y)}}\varphi(y',y'')\,\mathrm{d}\mu_{p(y)}(y'')$.

[12] See Definition B.2.1.

2. We have to show that μ is a faithful continuous field of measures. It is easy to see that μ is faithful once we have established that it is continuous. To see the latter, let $\chi \in \mathcal{C}_c(Y)$. Define $\tilde{\chi}(y, y') := \chi(y')$ for all $(y, y') \in Y \times_X Y$. Then the support of $\tilde{\chi}$ is proper and $\lambda(\tilde{\chi})(y) = \mu(\chi)(p(y))$. The function $\lambda(\tilde{\chi})$ is continuous and constant on the fibres of p, so $\mu(\chi)$ is continuous. Hence μ is continuous. □

Definition and Proposition 6.4.4 (Haar systems on $\mathcal{H}$ give Haar systems on Ω). Let Ω be a graph from $\mathcal{G}$ to $\mathcal{H}$ and let $\mathcal{H}$ carry a left Haar system $\lambda_{\mathcal{H}}$. Then we define a faithful continuous field of measures $(\lambda_\Omega^g)_{g\in\mathcal{G}^{(0)}}$ on Ω over $\mathcal{G}^{(0)}$ with the canonical projection ρ as coefficient map by

$$\lambda_\Omega^g(\varphi) := \int_{\eta\in\mathcal{H}^{\sigma(\omega)}} \varphi(\omega\eta)\,\mathrm{d}\lambda_{\mathcal{H}}^{\sigma(\omega)}(\eta)$$

for all $g \in \mathcal{G}^{(0)}$ and $\varphi \in \mathcal{C}_c(\Omega)$, where ω is some arbitrary element of Ω such that $\rho(\omega) = g$. This continuous field of measures is a left Haar system on Ω for the action of $\mathcal{G}$.

Proof. First we prove that $\lambda_\Omega(\varphi)$ is well-defined. Note that the defining integral makes sense because the action of $\mathcal{H}$ on Ω is proper; we have to check that it is independent of the choice of ω. Let $\omega, \omega' \in \Omega$ with $\rho(\omega) = g = \rho(\omega')$. Because ρ is a principal fibration with structure groupoid $\mathcal{H}$, we can find a unique element $\tilde{\eta} \in \mathcal{H}$ such that $\omega' = \omega\tilde{\eta}$. Now

$$\int_{\eta\in\mathcal{H}^{\sigma(\omega')}} \varphi(\omega'\eta')\,\mathrm{d}\lambda_{\mathcal{H}}^{\sigma(\omega')}(\eta') = \int_{\eta'\in\mathcal{H}^{\sigma(\omega')}} \varphi(\omega\tilde{\eta}\eta')\,\mathrm{d}\lambda_{\mathcal{H}}^{\sigma(\omega')}(\eta') = \int_{\eta\in\mathcal{H}^{\sigma(\omega)}} \varphi(\omega\eta)\,\mathrm{d}\lambda_{\mathcal{H}}^{\sigma(\omega)}(\eta)$$

by the left invariance of $\lambda_{\mathcal{H}}$. So the integral defining $\lambda_\Omega^g(\varphi)$ is independent of the choice of $\omega \in \rho^{-1}(g)$.

We now show that λ_Ω is continuous. Instead of making all the calculations by hand we are going to give some background information which shows how the Haar system can be obtained systematically. Consider the following diagram

$$\begin{tikzcd}
\mathcal{H} \arrow[d, "r"'] & \Omega * \mathcal{H} \arrow[l, "\pi_2"'] \arrow[r, "\mu"] \arrow[d, "\pi_1"] & \Omega \arrow[d, "\rho"] \\
\mathcal{H}^{(0)} & \Omega \arrow[l, "\sigma"] \arrow[r, "\rho"'] & \mathcal{G}^{(0)}
\end{tikzcd}$$

On $\mathcal{H}$ there is, by assumption, the faithful continuous field $\lambda_{\mathcal{H}}$ of measures over $\mathcal{H}^{(0)}$ with coefficient map r. This induces a faithful continuous field of measures $\lambda_{\Omega*\mathcal{H}} := \sigma^*(\lambda_{\mathcal{H}})$ on $\Omega * \mathcal{H}$ over Ω with coefficient map π_1, the projection onto the first component.[13] Note that for all $\omega \in \Omega$ we have

$$\lambda_{\Omega*\mathcal{H}}^\omega(\varphi) = \int_{\eta\in\mathcal{H}^{\sigma(\omega)}} \varphi(\omega, \eta)\,\mathrm{d}\lambda_{\mathcal{H}}^{\sigma(\omega)}(\eta)$$

for all $\varphi \in \mathcal{C}_c(\Omega * \mathcal{H})$. This integral can be extended to all functions φ on $\Omega * \mathcal{H}$ with proper support (here "proper support" means that for all compact subsets K of Ω the set $\operatorname{supp}\varphi \cap \pi_1^{-1}(K)$ is compact). If $\varphi \in \mathcal{C}_c(\Omega)$, then $\varphi \circ \mu$ is a function on $\Omega * \mathcal{H}$ with proper support (because the action of $\mathcal{H}$ on Ω is proper). Moreover, if $\varphi \in \mathcal{C}_c(\Omega)$, then $\lambda_{\Omega*\mathcal{H}}^\omega(\varphi \circ \mu)$ depends only on $\rho(\omega)$ (that is what we have shown in the first part of the proof). The map $\omega \mapsto \lambda_{\Omega*\mathcal{H}}^\omega(\varphi \circ \mu)$ is continuous on Ω and constant on the fibres of ρ. Hence there is a unique continuous function ψ on $\mathcal{G}^{(0)}$ such that $\lambda_{\Omega*\mathcal{H}}(\varphi\circ\mu) = \psi\circ\rho$. This function has compact support and equals $\lambda_\Omega(\varphi)$. Hence λ_Ω is a continuous field of measures. It is faithful. □

[13] See B.2.8 for the definition of the pullback of a continuous field of measures.

The following proposition is straightforward:

Proposition 6.4.5 (The Haar system on the linking groupoid). *Let $\mathcal{G}$ and $\mathcal{H}$ carry left Haar systems $\lambda_{\mathcal{G}}$ and $\lambda_{\mathcal{H}}$, respectively. Let Ω be a Morita equivalence between $\mathcal{G}$ and $\mathcal{H}$. Then the Haar system on $\mathcal{H}$ induces a left Haar system for the $\mathcal{G}$-action on Ω, and the Haar system on $\mathcal{G}$ induces a left Haar system for the left action of $\mathcal{H}$ on Ω^{-1}. Together, these four Haar systems define a left Haar system on the linking groupoid.*

We have a partial inverse of the construction presented in 6.4.4. There, a Haar system on the "range groupoid" of a graph between groupoids induces a Haar system on the graph. Vice versa, a Haar system on a graph induces a Haar system on the range groupoid, at least in the case of a Morita equivalence:

Definition and Proposition 6.4.6. Let Ω be a Morita equivalence between $\mathcal{G}$ and $\mathcal{H}$ and let λ_Ω be a left Haar system on Ω. Define a left Haar system $\lambda_{\mathcal{H}}$ on $\mathcal{H}$ as follows: If $\omega \in \Omega$, then $\mathcal{H}^{\sigma(\omega)}$ is homeomorphic to $\{\omega' \in \Omega :\ \rho(\omega') = \rho(\omega)\} = \rho^{-1}(\rho(\omega))$. On this fibre we take the Haar measure $\lambda_\Omega^{\rho(\omega)}$. On functions $\varphi \in \mathcal{C}_c(\mathcal{H})$ this amounts to the following integral:

$$\int_{\eta \in \mathcal{H}^{\sigma(\omega)}} \varphi(\eta)\, \mathrm{d}\lambda_{\mathcal{H}}^{\sigma(\omega)}(\eta) \;=\; \int_{\omega' \in \rho^{-1}(\rho(\omega))} f(\langle \omega, \omega' \rangle)\, \mathrm{d}\lambda_\Omega^{\rho(\omega)}(\omega')$$

for all $\omega \in \Omega$, where $\langle \omega, \omega' \rangle$ denotes the unique element η of $\mathcal{H}$ such that $\omega\eta = \omega'$.

Proof. Consider the following diagram

$$\begin{tikzcd}
\Omega \arrow[d, "\rho"'] & \Omega \times_\rho \Omega \arrow[l, "\pi_2"'] \arrow[r, "\langle , \rangle"] \arrow[d, "\pi_1"] & \mathcal{H} \arrow[d, "r_{\mathcal{H}}"] \\
\mathcal{G}^{(0)} & \Omega \arrow[l, "\rho"] \arrow[r, "\sigma"'] & \mathcal{H}^{(0)}
\end{tikzcd}$$

Now proceed as in the proof of 6.4.4. □

6.4.2 Haar systems and pullbacks

Lemma 6.4.7. *Let $\mathcal{G}$ be a topological groupoid acting on the left on a topological space Ω with anchor map ρ. Let $\tilde{\rho}\colon \mathcal{G} * \Omega \to \mathcal{G}^{(0)},\ (\gamma, \omega) \mapsto r(\gamma)$. Then, for every $g \in \mathcal{G}^{(0)}$, there is a canonical homoeomorphism between $\tilde{\rho}^{-1}(g) \subseteq \mathcal{G} * \Omega$ and $r^{-1}(g) \times \rho^{-1}(g) \subseteq \mathcal{G} \times \Omega$.*

Proof. For every $(\gamma, \omega) \in \tilde{\rho}^{-1}(g)$, the element $(\gamma, \gamma\omega)$ is in $r^{-1}(\mathcal{G}) \times \rho^{-1}(g)$. On the other hand, if $(\gamma, \omega') \in r^{-1}(g) \times \rho^{-1}(g)$ then $(\gamma, \gamma^{-1}\omega') \in \tilde{\rho}^{-1}(g)$. The two maps are obviously continuous and inverses of each other. □

Definition and Proposition 6.4.8. Let $\mathcal{G}$ be a locally compact Hausdorff groupoid with left Haar system λ and let Ω be a left Haar $\mathcal{G}$-space with anchor map ρ. Let $\omega \in \Omega$. Then we define a measure μ^ω on $\rho^*(\mathcal{G})$ by

$$\mu^\omega(\varphi) = \int_{\omega' \in \rho^{-1}(\rho(\omega))} \int_{\gamma \in G^{\rho(\omega)}} \varphi(\omega', \gamma, \gamma^{-1}\omega)\, \mathrm{d}\lambda^{\rho(\omega)}(\gamma)\, \mathrm{d}\lambda_\Omega^{\rho(\omega)}(\omega')$$

for all $\varphi \in \mathcal{C}_c(\rho^*(\mathcal{G}))$. The family $(\mu^\omega)_{\omega \in \Omega}$ defines a left Haar system on $\rho^*(\mathcal{G})$.

Note that the measure μ^ω on $\rho^*(\mathcal{G})$ has support

$$R^{-1}(\{\omega\}) = \{(\omega, \gamma, \omega') : \gamma \in \mathcal{G}, \omega' \in \Omega, \rho(\omega) = r(\gamma), \rho(\omega') = s(\gamma)\}$$

that can be identified with $\{(\gamma, \omega') \in \mathcal{G} * \Omega : s(\gamma) = \rho(\omega')\}$ which can, by the preceding lemma, further be identified with $r^{-1}(\rho(\omega)) \times \rho^{-1}(\rho(\omega))$. This space can be equipped with the measure $\lambda^{\rho(\omega)} \times \lambda_\Omega^{\rho(\omega)}$, and this measure corresponds to μ^ω under the identification.

The Haar system defined in the preceding definition could also be obtained by defining a left Haar system on $\mathcal{G} * \Omega$, since $\mathcal{G} * \Omega$ implements a Morita equivalence between $\mathcal{G}$ and $\rho^*(\mathcal{G})$.

Corollary 6.4.9. *Let $\mathcal{G}$ and $\mathcal{H}$ be locally compact Hausdorff groupoids carrying left Haar measures. Let Ω be a graph from $\mathcal{G}$ to $\mathcal{H}$. Then Ω carries a left Haar system by 6.4.4 and hence $\rho^*(\mathcal{G})$ also carries a canonical Haar system.*

6.5 The functor $p_!$

Let Y and X be locally compact Hausdorff spaces and let $p \colon Y \to X$ be continuous, open, and surjective. Let $\mathcal{G}$ be a locally compact Hausdorff groupoid over X. We denote the canonical strict morphism from $p^*(\mathcal{G})$ onto $\mathcal{G}$ also by p. In this section we are going to investigate the relationship between the category of $\mathcal{G}$-Banach spaces and the category of $p^*(\mathcal{G})$-Banach spaces.

If E is a u.s.c. field of Banach spaces over X, then $p^*(E)$ is not only a u.s.c. field of Banach spaces over Y, but also a $Y \times_X Y$-Banach space. As a consequence, a condition on the linear operators between $p^*(\mathcal{G})$-Banach spaces which is natural in our context is $Y \times_X Y$-equivariance. Every continuous field of linear maps between $p^*(\mathcal{G})$-Banach spaces which is $p^*(\mathcal{G})$-equivariant is also $Y \times_X Y$-equivariant (recall that $Y \times_X Y$ can be found as a closed subgroupoid $p^*(X)$ in $p^*(\mathcal{G})$, we just identify some $(y', y) \in Y \times_X Y$ with $(y', p(y), y) \in p^*(\mathcal{G})$). Our goal is to show that the pullback functor p^* implements the following one-to-one correspondences:

1. $\mathcal{G}$-Banach spaces correspond to $p^*(\mathcal{G})$-Banach spaces;
2. continuous fields of linear maps between $\mathcal{G}$-Banach spaces correspond to $Y \times_X Y$-equivariant continuous fields of linear maps between $p^*(\mathcal{G})$-Banach spaces;
3. $\mathcal{G}$-equivariant continuous fields of linear maps correspond to $p^*(\mathcal{G})$-equivariant fields of linear maps.

We reach this goal by defining a functor $p_!$ which inverts p^*; it points in the opposite direction, from the $p^*(\mathcal{G})$-Banach spaces to the $\mathcal{G}$-Banach spaces. The functor $p_!$ is obtained by "factoring out" the action of the $Y \times_X Y$-action on the given $p^*(\mathcal{G})$-Banach space.

For technical reasons, we assume that there exists a faithful continuous field of measures on Y over X with coefficient map p. From Example 6.4.3 we know that this condition is equivalent to the condition that the locally compact Hausdorff groupoid $Y \times_X Y$ admits a left Haar system. Note that such a faithful continuous field of measures on Y (and hence a Haar system on $Y \times_X Y$) exists if $\mathcal{C}_0(Y)$ is separable.[14] In the situation we are interested in, the space Y is actually a graph Ω from $\mathcal{G}$ into some other locally compact Hausdorff groupoid $\mathcal{H}$. We have learned above that such an Ω carries a canonical left Haar system if $\mathcal{H}$ carries a left Haar system, so the existence of a faithful continuous field of measures on $Y = \Omega$ will be automatic in this case.

[14] This can bee deduced from Proposition 3.9 in [Bla96].

6.5.1 The case $\mathcal{G} = X$ and $p^*(\mathcal{G}) = Y \times_X Y$

We will first consider the case that $\mathcal{G}$ is the trivial groupoid X and that $p^*(\mathcal{G})$ is therefore isomorphic to $Y \times_X Y$. The following is proved in Appendix D.2 on page 300; the proof uses the existence of a faithful continuous field of measures on Y over X.

Definition and Proposition 6.5.1 (The u.s.c. field $p_!E$). Let E be a $Y \times_X Y$-Banach space with action α. Assume that there exists a faithful continuous field of measures on Y over X. We define a u.s.c. field of Banach spaces $p_!E$ over X as follows: For every $x \in X$, define

$$(p_!E)_x := \left\{ (e_y)_{y \in Y_x} \,\middle|\, \forall y, y' \in Y_x : \ e_y \in E_y \wedge \alpha_{(y',y)}(e_y) = e_{y'} \right\} \subseteq \prod_{y \in Y_x} E_y,$$

where we take the $\sup$-norm on $\prod_{y \in Y_x} E_y$. Note that $(p_!E)_x$ is a closed linear subspace of the product. Since α is a field of isometries, it follows that the norm of a family $(e_y)_{y \in Y_x} \in (p_!E)_x$ equals the norm of each e_y, $y \in Y_x$; hence $(p_!E)_x$ is isometrically isomorphic to E_y for each $y \in Y_x$ (note that $Y \times_X Y$ acts freely on Y).

To define the structure of a u.s.c. field of Banach spaces over X on $(p_!E_x)_{x \in X}$, we set

$$\Delta := \Delta_E := \left\{ \delta \in \Gamma(Y, E) \,\middle|\, \forall (y, y') \in Y \times_X Y : \ \alpha_{(y',y)}(\delta(y)) = \delta(y') \right\}.$$

In other words: Δ consists of those sections of E which are invariant under the action of $Y \times_X Y$. If $\delta \in \Delta$ and $x \in X$, then define

$$(p_!\delta)(x) := (\delta(y))_{y \in Y_x} \in (p_!E)_x$$

Now

$$\Gamma := \{ p_!(\delta) : \delta \in \Delta \}$$

satisfies conditions (C1)-(C4), so $(p_!E, \ \Gamma)$ is a u.s.c. field of Banach spaces over X.

Definition and Proposition 6.5.2 ($p_!$ for morphisms). Let E and F be $Y \times_X Y$-Banach spaces. Let T be an $Y \times_X Y$-equivariant continuous field of linear maps from E to F. Define for all $x \in X$ and $e = (e_y)_{y \in Y_x} \in (p_!E)_x$:

$$(p_!T)_x(e) := (T_y e_y)_{y \in Y_x} \in (p_!F)_x .$$

Then $p_!T$ is a continuous field of linear maps from $p_!E$ to $p_!F$. If T is bounded, then $\|p_!T\| = \|T\|$.

Proof. Let α and β denote the respective actions of $Y \times_X Y$ on E and F.

First, $p_!T$ is a well-defined family of linear operators because $\beta_{(z,y)}(T_y e_y) = T_z(\alpha_{(z,y)} e_y) = T_z e_z$. The statement about the norm is obvious, so we only have to care about the continuity of $p_!T$. Let $\zeta \in \Gamma(X, \ p_!E) = p_!\Delta_E$. Then there is a $\delta \in \Delta_E$ such that $p_!\delta = \zeta$. Now $T \circ \delta \in \Delta_F$ because $\beta_{(z,y)}(T_y \delta(y)) = T_z(\alpha_{(z,y)} \delta(y)) = T_z \delta(z)$ for all $(z, y) \in Y \times_X Y$. We have

$$(p_!T) \circ p_!\delta = p_!(T \circ \delta),$$

because

$$(p_!T)((p_!\delta)(x)) = (p_!T)(\delta(y))_{y \in Y_x} = (T_y \delta(y))_{y \in Y_x} = (p_!(T \circ \delta))_x$$

for all $x \in X$. In particular, $(p_!T) \circ \zeta \in p_!\Delta_F = \Gamma(X, \ p_!F)$. Hence $p_!T$ is continuous. $\square$

Proposition 6.5.3. *The maps $E \mapsto p_! E$ and $T \mapsto p_! T$ define a functor from the category of $Y \times_X Y$-Banach spaces with the bounded equivariant continuous fields of linear maps to the category of u.s.c. fields of Banach spaces over X, isometric and linear on the morphism sets and respecting the tensor product.*

Proof. This is proved in Appendix D.2 on page 301. □

Proposition 6.5.4. *The functor $p_!$ from the category of $Y \times_X Y$-Banach spaces to the category of X-Banach spaces is an equivalence which inverts p^*; more precisely:*

1. *Define for all $Y \times_X Y$-Banach spaces E and all $y \in Y$ the linear map*

$$I_y^E \colon \left(p^* p_! E\right)_y = \left(p_! E\right)_{p(y)} \to E_y,\ \left(e_z\right)_{z \in Y_{p(y)}} \to e_y.$$

Then

$$I^E \colon p^* p_! E \cong E$$

is a natural isometric isomorphism, compatible with the tensor product (="multiplicative").

2. *For all X-Banach spaces E there is a natural multiplicative isometric isomorphism*

$$J^E \colon p_! p^* E \cong E.$$

To define J^E, let us analyse the action α of $Y \times_X Y$ on $p^* E$ and the fibres of $p_! p^* E$: The action α is the pullback of the trivial action of X on E, so for all $(z, y) \in Y \times_X Y$ we have $p^*(E)_z = E_{p(z)} = E_{p(y)} = p^*(E)_y$ and $\alpha_{(z,y)} = \mathrm{Id}_{E_{p(y)}}$. So if $x \in X$, then the elements of $(p_! p^* E)_x$ are of the form $(e)_{y \in Y_x}$ with $e \in E_x$; so it makes sense to define

$$J_x^E \colon \left(p_! p^* E\right)_x \to E_x,\ (e)_{y \in Y_x} \mapsto e.$$

Proof. This is proved in Appendix D.2 on page 302. □

6.5.2 The functor $p_!$ for general $\mathcal{G}$

Actions of groupoids on fields of Banach spaces are defined using the pullback construction. It is therefore advisable to study the interplay of the functor $p_!$ and the pullback:

Proposition 6.5.5. *The functor $p_!$ commutes with the pullback in the following sense: Let X' and Y' be locally compact Hausdorff spaces and let $p' \colon Y' \to X'$ be continuous, open and surjective. Let $f_Y \colon Y' \to Y$ be a continuous function. Assume that there is a function f_X from X' to X such that the following diagram commutes*

$$\begin{tikzcd} Y' \arrow[r, "f_Y"] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ X' \arrow[r, "f_X"] & X \end{tikzcd}$$

Note that this map from X' to X is unique with this property and that it is continuous. The map $Y' \times_{X'} Y' \to Y \times_X Y$, $(y_2', y_1') \mapsto (f_Y(y_2'), f_Y(y_1'))$, which we also call f_Y, is a continuous strict morphism.

There is a natural isomorphism of u.s.c. fields of Banach spaces over X'

$$f_X^* \left(p_!(E)\right) \cong p_!' \left(f_Y^*(E)\right)$$

for all $Y \times_X Y$-Banach spaces E. This natural transformation is isometric and multiplicative.

Proof. First note that $p^*p_!E$ is naturally isomorphic to E. So $f_Y^*p^*p_!E$ is naturally isomorphic to f_Y^*E. But $f_Y^*p^*p_!E = p'^*f_X^*p_!E$. So f_Y^*E is naturally isomorphic to $p'^*f_X^*p_!E$, and hence $p'_!f_Y^*E$ is naturally isomorphic to $p'_!p'^*f_X^*p_!E$, which is naturally isomorphic to $f_X^*p_!E$. All the isomorphisms are isometric and compatible with the tensor product, so $p'_!f_Y^*E$ is naturally isometrically and multiplicatively isomorphic to $f_X^*p_!E$.

An explicit isomorphism $(f_X^*p_!E)_{x'} = (p_!E)_{f_X(x')}$ to $(p'_!f_Y^*E)_{x'}$ is given by

$$(e_y)_{y\in Y_{f_X(x')}} \mapsto \left(e_{f_Y(y')}\right)_{y'\in Y'_{x'}}. \tag{6.6}$$

□

Definition and Proposition 6.5.6 (The $\mathcal{G}$-action on $p_!E$). Let E be a $p^*(\mathcal{G})$-Banach space with action α. Then we define a $\mathcal{G}$-action on $p_!(E)$ as follows: Let $R, S\colon p^*(\mathcal{G}) = Y \times_{p,r} \mathcal{G} \times_{s,p} Y \to Y$ be the range and source maps. Then the following diagrams commute

$$\begin{tikzcd} p^*(\mathcal{G}) \arrow[r, "R"] \arrow[d, "p"'] & Y \arrow[d, "p"] \\ \mathcal{G} \arrow[r, "r"] & X \end{tikzcd} \qquad \begin{tikzcd} p^*(\mathcal{G}) \arrow[r, "S"] \arrow[d, "p"'] & Y \arrow[d, "p"] \\ \mathcal{G} \arrow[r, "s"] & X \end{tikzcd}$$

This means that $s^*(p_!E) \cong p_!(S^*E)$ and $r^*(p_!E) \cong p_!(R^*E)$. Now $p_!(\alpha)$ is an isometric isomorphism from $p_!(S^*E)$ to $p_!(R^*E)$, and this defines an action $p_!\alpha$ on $p_!E$. It has the property that for all $\gamma \in \mathcal{G}$, $e = (e_y)_{y\in Y_{s(\gamma)}} \in (p_!E)_{s(\gamma)}$, and $y \in Y_{s(\gamma)}$:

$$(p_!\alpha)_\gamma(e) = (\alpha_{(z,\gamma,y)}e_y)_{z\in Y_{r(\gamma)}}. \tag{6.7}$$

Proof. If we know that $p_!\alpha$ satisfies equation (6.7), then we can check fibrewise that $p_!\alpha$ is an action $\mathcal{G}$ (actually, one can take (6.7) to define the action $p_!\alpha$, but then one has to check that this gives a continuous field of isomorphisms which is automatic in our approach): Let therefore $\gamma, \gamma' \in \mathcal{G}$ such that $r(\gamma) = s(\gamma')$. Let $y \in Y_{s(\gamma)}$ and $e = (e_y)_{y\in Y_{s(\gamma)}} \in (p_!E)_{s(\gamma)}$. Then

$$\left((p_!\alpha)_{\gamma'\gamma}(e)\right)_z = \alpha_{(z,\gamma'\gamma,y)}e_y = \alpha_{(z,\gamma,y')}\alpha_{(y',\gamma,y)}e_y = \alpha_{(z,\gamma,y')}\left((p_!\alpha)_\gamma(e)\right)_{y'} = (p_!\alpha)_{\gamma'}\left((p_!\alpha)_\gamma(e)\right)_z$$

for all $z \in Y_{r(\gamma')}$ (here y' is an arbitrary element of Y with $p(y) = r(\gamma) = s(\gamma')$). So $(p_!\alpha)_{\gamma'\gamma} = (p_!\alpha)_{\gamma'}(p_!\alpha)_\gamma$.

To show that the family $p_!\alpha$ indeed satisfies equation (6.7), we make the identifications of fields $p_!(S^*(E)) = s^*(p_!(E))$ and $p_!(R^*(E)) = r^*(p_!(E))$ visible. Let $\gamma \in \mathcal{G}$ and $e = (e_y)_{y\in Y_{s(\gamma)}} \in (p_!E)_{s(\gamma)}$. This e is identified via (6.6) with $(e_y)_{(z,\gamma,y)\in p^*(\mathcal{G})} \in (p_!(S^*(E)))_\gamma$ (use $x' = \gamma$, $y' = (z,\gamma,y)$, $f_X = s$ and $f_Y = S$, so $f_Y(z,\gamma,y) = y$). Now

$$(p_!\alpha)_\gamma\,(e_y)_{(z,\gamma,y)\in p^*(\mathcal{G})} = \left(\alpha_{(z,\gamma,y)}e_y\right)_{(z,\gamma,y)\in p^*(\mathcal{G})} \in (p_!(R^*(E)))_\gamma\,.$$

The identification $p_!(R^*(E)) = r^*(p_!(E))$ maps this to $\left(\alpha_{(z,\gamma,y)}e_y\right)_{z\in r(\gamma)} \in r^*(p_!(E))_\gamma$, where $y \in Y_{s(\gamma)}$ is arbitrary. This shows (6.7). □

Proposition 6.5.7. *If E and F are $p^*(\mathcal{G})$-Banach spaces and $T\colon E \to F$ is a $p^*(\mathcal{G})$-equivariant continuous field of linear maps, then $p_!T\colon p_!E \to p_!F$ is $\mathcal{G}$-equivariant.*

Proof. That T is equivariant means that

$$\begin{tikzcd} S^*(E) \arrow[r, "S^*(T)"] \arrow[d, "\alpha"] & S^*(F) \arrow[d, "\beta"] \\ R^*(E) \arrow[r, "R^*(T)"] & R^*(F) \end{tikzcd}$$

commutes. Since $p_!$ is functorial, where we mean this time by p the map $p\colon p^*(\mathcal{G}) \to \mathcal{G}$, this implies

$$\begin{tikzcd} p_!\left(S^*(E)\right) \arrow[r, "p_!(S^*(T))"] \arrow[d, "p_!\alpha"] & p_!\left(S^*(F)\right) \arrow[d, "p_!\beta"] \\ p_!\left(R^*(E)\right) \arrow[r, "p_!(R^*(T))"] & p_!\left(R^*(F)\right) \end{tikzcd}$$

The identification that was used to define the actions $p_!\alpha$ and $p_!\beta$ is natural by Proposition 6.5.5, so the following square commutes

$$\begin{tikzcd} s^*(p_!(E)) \arrow[r, "s^*(p_!(T))"] \arrow[d, "p_!\alpha"] & s^*(p_!(F)) \arrow[d, "p_!\beta"] \\ r^*(p_!(E)) \arrow[r, "r^*(p_!(T))"] & r^*(p_!(F)) \end{tikzcd}$$

This means that $p_!(T)$ is equivariant. $\square$

Proposition 6.5.8. *The functor $E \mapsto p_!E$ is an isometric multiplicative functor from the category of $p^*(\mathcal{G})$-Banach spaces to the category of $\mathcal{G}$-Banach spaces.*

Proof. We know that it is a well-defined isometric functor. That it is multiplicative follows from the fact that the natural isomorphism in 6.5.5 is multiplicative. $\square$

Theorem 6.5.9. *The functor $p_!$ from the category of $p^*(\mathcal{G})$-Banach spaces to the category of $\mathcal{G}$-Banach spaces is a multiplicative equivalence which inverts p^*.*

Proof. We have to show that the natural transformations $E \mapsto I^E$ and $E \mapsto J^E$ appearing in 6.5.4 are $p^*(\mathcal{G})$- and $\mathcal{G}$-equivariant, respectively.

1. I^E **is** $p^*(\mathcal{G})$**-equivariant:** Let E be a $p^*(\mathcal{G})$-Banach space with action α and let $y \in Y$. Let $e = (e_z)_{z \in Y_{p(y)}} \in (p^*p_!E)_y = (p_!E)_{p(y)}$. Let $(y', \gamma, y) \in p^*(\mathcal{G})$. Note that $p(y') = r(\gamma)$ and $p(y) = s(\gamma)$. We have

$$(p^*p_!\alpha)_{(y',\gamma,y)}\,(e_z)_{z \in Y_{s(\gamma)}} = (p_!\alpha)_\gamma\,(e_z)_{z \in Y_{s(\gamma)}} \overset{(6.7)}{=} \left(\alpha_{(z',\gamma,y)} e_y\right)_{z' \in Y_{r(\gamma)}}.$$

   $I^E_{y'}$ maps this to $\alpha_{(y',\gamma,y)} e_y$, which happens to be $\alpha_{(y',\gamma,y)} I^E_y(e)$. So I^E is equivariant.

2. J^E **is** $\mathcal{G}$**-equivariant:** Let E be a $\mathcal{G}$-Banach space with action α and $x \in X$. Let $e \in E_x$ so that $(e)_{y \in Y_x} \in (p_!p^*E)_x$. Let $\gamma \in \mathcal{G}$ such that $s(\gamma) = x$. Find $y' \in Y$ such that $p(y') = x$. Then

$$(p_!p^*\alpha)_\gamma\,(e)_{y \in Y_x} = \Big((p^*\alpha)_{(z,\gamma,y')}\, e\Big)_{z \in Y_{r(\gamma)}} = (\alpha_\gamma e)_{z \in Y_{r(\gamma)}}.$$

   $J^E_{r(\gamma)}$ maps this to $\alpha_\gamma e = \alpha_\gamma J^E_x((e)_{y \in Y_x})$, so J^E is equivariant.

$\square$

Proposition 6.5.10. *Let Z be another locally compact Hausdorff space and let $q\colon Z \to Y$ be open, continuous and surjective. Assume that there is a faithful continuous field of measures on Z over Y. Then $(p \circ q)_!$ and $p_! \circ q_!$ both invert $(p \circ q)^* = q^* \circ p^*$. So $(p \circ q)_!$ and $p_! \circ q_!$ are naturally multiplicatively isometrically isomorphic as functors from the $(p \circ q)^*(\mathcal{G})$-Banach spaces to the $\mathcal{G}$-Banach spaces.*

Proof. Note that there is a faithful continuous field of measures on Z over X: If μ is a faithful continuous field of measures on Y over X and if ν is a faithful continuous field of measures on Z over Y, then $\varphi \mapsto \mu(\nu(\varphi))$, as a map from $\mathcal{C}_c(Z)$ to $\mathcal{C}_c(X)$, defines a faithful continuous field of measures on Z over X. $\square$

6.5.3 The functor $p_!$ for Banach algebras, etc.

The functor $p_!$ is multiplicative and contractive on the morphism sets. The multiplicativity gives us a way to define the functor also for equivariant fields of bilinear maps. We can therefore also define a $\mathcal{G}$-Banach algebra $p_!A$ for $p^*(\mathcal{G})$-Banach algebras A and $\mathcal{G}$-equivariant homomorphisms $p_!\varphi$ for $p^*(\mathcal{G})$-equivariant homomorphisms of Banach algebras. Similarly, we can define $p_!E$ for $p^*(\mathcal{G})$-Banach modules and $p^*(\mathcal{G})$-equivariant homomorphisms of Banach modules. Moreover, if T is a $Y \times_X Y$-equivariant continuous field of linear operators between $p^*(\mathcal{G})$-Banach modules E_B and F_B, then $p_!T$ is a continuous field of linear operators between $p_!E_{p_!B}$ and $p_!F_{p_!B}$ (where B is some $p^*(\mathcal{G})$-Banach algebra). All this culminates in the following definition:

Definition 6.5.11. Let B be a $p^*(\mathcal{G})$-Banach algebra and let $E = (E^<, E^>)$ be a $p^*(\mathcal{G})$-Banach B-pair. Then $p_!E = (p_!E^<,\ p_!E^>)$ is a $\mathcal{G}$-Banach $p_!B$-pair. If F is another $p^*(\mathcal{G})$-Banach B-pair and $T \in \mathrm{L}^{\mathbf{loc}}_B(E, F)$ is $Y \times_X Y$-equivariant, then $p_!T = (p_!T^<,\ p_!T^>)$ is in $\mathrm{L}^{\mathbf{loc}}_{p_!B}(p_!E, p_!F)$.

This defines a functor form the category of $p^*(\mathcal{G})$-Banach B-pairs to the category of $\mathcal{G}$-Banach $p_!B$-pairs. It inverts the functor p^* and respects grading automorphisms.

As a variant of Proposition 3.3.22 one proves:

Proposition 6.5.12. *Let B be a $p^*(\mathcal{G})$-Banach algebra and let E and F be $p^*(\mathcal{G})$-Banach B-pairs. If $T \in \mathrm{K}^{\mathbf{loc}}_B(E, F)$ is $Y \times_X Y$-equivariant, then $p_!T \in \mathrm{K}^{\mathbf{loc}}_{p_!B}(p_!E,\ p_!F)$.*

It is obvious that the functor $p_!$ is compatible with the direct sum of $p^*(\mathcal{G})$-Banach spaces and of $\mathcal{G}$-Banach spaces and that the same is true for Banach modules and Banach pairs. Because $p_!$ is also compatible with the (balanced) tensor product, we obtain:

Proposition 6.5.13. *Let B and C be $p^*(\mathcal{G})$-Banach algebras and let $\psi\colon B \to C$ be a $p^*(\mathcal{G})$-equivariant homomorphism. Let E be a right $p^*(\mathcal{G})$-Banach B-module. Then $p_!\mathbb{C}_Y$ is isomorphic to $\mathbb{C}_X$, $p_!\tilde{C} = p_!(C \oplus_Y \mathbb{C}_Y)$ is isomorphic to $\widetilde{p_!C} = p_!C \oplus_X \mathbb{C}_X$ and, finally, $p_!(\psi_*(E)) = p_!(E \otimes_{\tilde{B}} \tilde{C})$ is isomorphic to $(p_!\psi)_*(p_!E)$.*

Moreover, $p_!$ is also compatible with the construction of trivial fields over $[0, 1]$; in particular, we have:

Proposition 6.5.14. *Let B be $p^*(\mathcal{G})$-Banach algebra. Then $p_!(B[0, 1])$ is isomorphic to $(p_!B)[0, 1]$. The isomorphism in the fibre over $x \in X$ sends $(\beta_y)_{y\in Y_x} \in p_!(B[0, 1])_x$ to $t \mapsto (\beta_y(t))_{y\in Y_x} \in (p_!B)[0, 1]_x$.*

6.5.4 The functor $p_!$ and $\mathrm{KK}^{\mathrm{ban}}$-cycles

This section is a translation of Section 7.2 in [LG99] into the language of Banach algebras; in particular, the method to make the operator of a $\mathrm{KK}^{\mathrm{ban}}$-cycle equivariant is borrowed from Lemma 7.1 of that article.

Let A and B be $p^*(\mathcal{G})$-Banach algebras. Let $\mathbb{E}^{\mathrm{ban}, Y\times_X Y}_{p^*(\mathcal{G})}(A,\ B)$ be the class of those cycles (E, T) in $\mathbb{E}^{\mathrm{ban}}_{p^*(\mathcal{G})}(A,\ B)$ such that T is $Y \times_X Y$-equivariant. In an obvious manner, we define $\mathrm{KK}^{\mathrm{ban}, Y\times_X Y}_{p^*(\mathcal{G})}(A,\ B)$.

Proposition 6.5.15. *Let* $(E,T) \in \mathbb{E}^{\mathrm{ban},Y\times_X Y}_{p^*(\mathcal{G})}(A,\ B)$. *Then*

$$p_!(E,T) := (p_!E,\ p_!T) \in \mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(p_!A,\ p_!B)\,.$$

Proof. Let $a \in \Gamma(X, p_!A)$. Then we can find a $\tilde{a} \in \Gamma(Y,A)$ which is invariant under the action of $Y \times_X Y$ such that $p_!\tilde{a} = a$. Now

$$[a, p_!T] = [p_!\tilde{a},\ p_!T] = p_!\,[\tilde{a},\ T] \in \mathrm{K}^{\mathbf{loc}}_{p_!B}(p_!E)$$

where we have used the fact that the action of a on $p_!E$ is $p_!$ of the action of $\tilde{a}$ on E. Similarly, $a(p_!T^2-1)$ is locally compact. For the third condition that we have to check use Proposition 6.5.5. $\square$

Up to isomorphism of cycles, $p_!$ inverts p^* as a map from $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(p_!A,\ p_!B)$ to $\mathbb{E}^{\mathrm{ban},Y\times_X Y}_{p^*(\mathcal{G})}(A,\ B)$. And up to isomorphism, $p_!$ commutes with the push-forward and the pullback of cycles. It also commutes with homotopies. We therefore get:

Proposition 6.5.16. *The map* $p_!$ *defines an isomorphism*

$$p_!\colon\ \mathrm{KK}^{\mathrm{ban},Y\times_X Y}_{p^*(\mathcal{G})}(A,\ B) \cong \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(p_!A,\ p_!B)\,,$$

inverting p^*.

Lemma 6.5.17. *Let there exist a faithful continuous field of measures on* Y *over* X *and let* X *be* σ*-compact. Let* $(E,T) \in \mathbb{E}^{\mathrm{ban}}_{p^*(\mathcal{G})}(A,\ B)$. *Then there is an odd* $Y\times_X Y$*-equivariant linear operator* $\tilde{T}$ *on* E *such that* $a(T-\tilde{T})$ *and* $(T-\tilde{T})a$ *are locally compact for all* $a \in \Gamma(Y,A)$. *In particular,* (E,T) *is homotopic to* $(E,\tilde{T})$. *The construction is compatible with the pullback and hence with homotopies of cycles.*

Proof. Let μ be a faithful continuous field of measures on Y over X. Then the locally compact Hausdorff groupoid $Y\times_X Y$ admits a left Haar system. Because X is σ-compact, we can find a cut-off function[15] $c\colon Y \to [0,\infty[$ for $Y\times_X Y$, i.e., a continuous function c on Y such that $\int_{y\in Y_x} c(y)\,\mathrm{d}\mu_x(y) = 1$ for all $x \in X$ and $p^{-1}(K) \cap \operatorname{supp} c$ is compact for all compact $K \subseteq X$.
Define

$$\tilde{T}_y := \int_{y'\in Y_{p(y)}} c(y')\ \alpha_{(y,y')}T_{y'}\alpha_{(y',y)}\,\mathrm{d}\mu_{p(y)}(y')$$

for all $y \in Y$, where α denotes the action of $p^*(\mathcal{G})$ (and hence also of $Y\times_X Y$) on E (actually, the formula makes sense for the right-hand side of the pair E and should be interpreted properly for the left-hand side). This definition is a special case of 7.2.5: The groupoid $Y\times_X Y$ is proper in the sense of Definition 7.1.2. Hence $\tilde{T}$ is a $Y\times_X Y$-equivariant bounded continuous field of linear operators on E. It is obviously odd. Just as in Lemma 7.2.6 one can show that $a(T-\tilde{T})$ and $(T-\tilde{T})a$ are locally compact for all $a \in \Gamma(Y,A)$. $\square$

The preceding lemma implies the following proposition.

Proposition 6.5.18. *The obvious homomorphism from* $\mathrm{KK}^{\mathrm{ban},Y\times_X Y}_{p^*(\mathcal{G})}(A,\ B)$ *to* $\mathrm{KK}^{\mathrm{ban}}_{p^*(\mathcal{G})}(A,\ B)$ *is an isomorphism.*

Corollary 6.5.19. $p_!$ *is a well-defined isomorphism*

$$p_!\colon\ \mathrm{KK}^{\mathrm{ban}}_{p^*(\mathcal{G})}(A,\ B) \cong \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(p_!A,\ p_!B)\,,$$

inverting p^*.

[15] See [Tu99] for a proof for the case that $\mathcal{G}$ is σ-compact. Cut-off functions are also discussed at the beginning of Chapter 7 of this thesis.

6.6 The pullback along generalised morphisms

Let $\mathcal{G}$ and $\mathcal{H}$ be locally compact Hausdorff groupoids (with open range and source maps) carrying left Haar systems. Note that the existence of a left Haar system on $\mathcal{H}$ implies the existence of a left Haar system on each graph from $\mathcal{G}$ to $\mathcal{H}$ by 6.4.4.

6.6.1 The pullback of Banach spaces

Definition 6.6.1. Let Ω be a graph from $\mathcal{G}$ to $\mathcal{H}$ with anchor maps ρ and σ. Then f_Ω as defined in 6.3.7 is a strict morphism from $\rho^*(\mathcal{G})$ to $\mathcal{H}$, which extends $\sigma\colon \Omega \to \mathcal{H}^{(0)}$. For all $\mathcal{H}$-Banach spaces E, define

$$\Omega^*(E) := \rho_! f_\Omega^*(E).$$

This will also be written as $\rho_!\sigma^*E$. The strict homomorphism $f_\Omega\colon \rho^*(\mathcal{G}) \to \mathcal{H}$ is defined in 6.3.7.

If Ω is as above, then $E \mapsto \Omega^*E$ is a functor from the category of $\mathcal{H}$-Banach spaces with the $\mathcal{H}$-equivariant (bounded, contractive) continuous fields of linear maps to the category of $\mathcal{G}$-Banach spaces with the $\mathcal{G}$-equivariant (bounded, contractive) continuous fields of linear maps. It commutes with the tensor product and has the (characterising) property that $\rho^*\Omega^*E$ is naturally isomorphic to $f_\Omega^*(E)$.

Proposition 6.6.2. *Let $\mathcal{K}$ be another locally compact Hausdorff groupoid carrying a left Haar system. Let Ω be a graph from $\mathcal{G}$ to $\mathcal{H}$ and Ω' a graph from $\mathcal{H}$ to $\mathcal{K}$. Then*

$$\Omega^* \circ (\Omega')^* \cong \left(\Omega \times_{\mathcal{H}} \Omega'\right)^*$$

as multiplicative functors from the $\mathcal{K}$-Banach spaces to the $\mathcal{G}$-Banach spaces.

Proof. Let ρ and σ be the anchor maps of Ω and ρ' and σ' those of Ω'. Let π_1 and π_2 denote the projections from $\Omega \times_{\mathcal{H}^{(0)}} \Omega'$ to the first and second component. As ρ' is open and surjective, so is π_1. Write p for the (open and surjective) quotient map from $\Omega \times_{\mathcal{H}^{(0)}} \Omega'$ onto $\Omega'' := \Omega \times_{\mathcal{H}} \Omega'$, and denote the anchor maps of Ω'' by ρ'' and σ''. Consider the diagram

$$\begin{tikzcd}
\Omega \times_{\mathcal{H}^{(0)}} \Omega' \arrow[d, "\pi_1"] \arrow[dr, "\pi_2"] & & \\
\Omega \arrow[d, "\rho"] \arrow[dr, "\sigma"] & \Omega' \arrow[d, "\rho'"] \arrow[dr, "\sigma'"] & \\
\mathcal{G}^{(0)} & \mathcal{H}^{(0)} & \mathcal{K}^{(0)}
\end{tikzcd}$$

This a diagram just for the unit spaces, but of course there is a corresponding commutative diagram also for the groupoids themselves:

$$\begin{tikzcd}
(\rho \circ \pi_1)^*(\mathcal{G}) \arrow[d, "\pi_1"] \arrow[dr, "\tilde{f}_\Omega"] & & \\
\rho^*(\mathcal{G}) \arrow[d, "\rho"] \arrow[dr, "f_\Omega"] & \rho'^*(\mathcal{H}) \arrow[d, "\rho'"] \arrow[dr, "f_{\Omega'}"] & \\
\mathcal{G} & \mathcal{H} & \mathcal{K}
\end{tikzcd}$$

Here the strict morphism $\tilde{f}_\Omega$ is defined as follows: It sends $((\omega_2, \omega_2'),\ \gamma,\ (\omega_1, \omega_1')) \in (\rho \circ \pi_1)^*(\mathcal{G})$ to $(\omega_2',\ f_\Omega(\omega_2, \gamma, \omega_1),\ \omega_1')$. It follows that

$$\Omega^* \circ (\Omega')^* = \rho_! \circ \left(f_\Omega^* \circ \rho_!'\right) \circ f_{\Omega'}^* \cong \rho_! \circ \left((\pi_1)_! \circ \tilde{f}_\Omega^*\right) \circ f_{\Omega'}^* \cong (\rho \circ \pi_1)_! \circ \left(f_{\Omega'} \circ \tilde{f}_\Omega\right)^* .$$

On the other hand, also the following diagrams commute

$$\begin{tikzcd}
& \Omega \times_{\mathcal{H}^{(0)}} \Omega' \arrow[ddl, "\rho\circ\pi_1"'] \arrow[d, "p"] \arrow[ddr, "\sigma'\circ\pi_2"] & \\
& \Omega'' \arrow[dl, "\rho''"'] \arrow[dr, "\sigma''"] & \\
\mathcal{G}^{(0)} & & \mathcal{K}^{(0)}
\end{tikzcd}
\qquad
\begin{tikzcd}
& (\rho\circ\pi_1)^*(\mathcal{G}) = (\rho''\circ p)^*(\mathcal{G}) \arrow[ddl, "\rho\circ\pi_1"'] \arrow[d, "p"] \arrow[ddr, "f_{\Omega'}\circ\tilde{f}_\Omega"] & \\
& \rho''(\mathcal{G}) \arrow[dl, "\rho''"'] \arrow[dr, "f_{\Omega''}"] & \\
\mathcal{G} & & \mathcal{K}
\end{tikzcd}$$

To check that $f_{\Omega'} \circ \tilde{f}_\Omega = f_{\Omega''} \circ p$ let $((\omega_2, \omega_2'), \gamma, (\omega_1, \omega_1'))$ be an element of $(\rho \circ \pi_1)^*(\mathcal{G})$. Then $f_{\Omega''}(p((\omega_2, \omega_2'), \gamma, (\omega_1, \omega_1')))$ is defined to be the unique element $\kappa \in \mathcal{K}$ such that $[\omega_2, \omega_2']\kappa = \gamma[\omega_1, \omega_1']$. Also $f_\Omega(\omega_2, \gamma, \omega_1)$ is the unique element $\eta \in \mathcal{H}$ such that $\omega_2\eta = \gamma\omega_1$ and $f_{\Omega'}(\omega_2', \eta, \omega_1')$ is the unique element $\kappa' \in \mathcal{K}$ such that $\omega_2'\kappa' = \eta\omega_1'$. Now

$$[\omega_2, \omega_2']\kappa' = [\omega_2, \omega_2'\kappa'] = [\omega_2, \eta\omega_1'] = [\omega_2\eta, \omega_1'] = [\gamma\omega_1, \omega_1'] = \gamma[\omega_1, \omega_1'],$$

so $\kappa = \kappa'$, which is what we wanted to verify.

So it follows that

$$\left(\Omega''\right)^* = \rho_!'' \circ f_{\Omega''}^* \cong ((\rho \circ \pi_1)_! \circ p^*) \circ \left(p_! \circ \left(f_{\Omega'} \circ \tilde{f}_\Omega\right)^*\right) \cong (\rho \circ \pi_1)_! \circ \left(f_{\Omega'} \circ \tilde{f}_\Omega\right)^* . \qquad \square$$

Proposition 6.6.3. *Let f be a strict morphism from $\mathcal{G}$ to $\mathcal{H}$. Then $\mathrm{Graph}(f)^* \cong f^*$. In particular we have $\mathcal{G}^* \cong \mathrm{Id}_\mathcal{G}^*$.*

Proof. Write Ω for $\mathrm{Graph}(f) = \mathcal{G}^{(0)} \times_{\mathcal{H}^{(0)}} \mathcal{H}$ and denote the anchor maps of Ω by ρ and σ. Then $\rho^*(\mathcal{G}) = \Omega \times_{\mathcal{G}^{(0)}} \mathcal{G} \times_{\mathcal{G}^{(0)}} \Omega$, and $f_\Omega \colon \rho^*(\mathcal{G}) \to \mathcal{H}$ sends $(g, \eta, \gamma, g', \eta')$ to $\eta^{-1}f(\gamma)\eta'$. If E is an $\mathcal{H}$-Banach space and $g \in \mathcal{G}^{(0)}$, then the fibre $(\Omega^*E)_g$ of Ω^*E at g is, by definition, given by

$$\left\{ \left(e_{(g,\eta)}\right)_{(g,\eta)\in\Omega} \,\middle|\, \forall (g, \eta, g, g, \eta') \in \rho^*(\mathcal{G}) :\ e_{(g,\eta)} \in (\sigma^*E)_{(g,\eta)} \wedge e_{(g,\eta)} = (g, \eta, g, g, \eta')e_{(g,\eta')} \right\}.$$

Analysing the action of $\rho^*(\mathcal{G})$ on $\sigma^*(E)$ gives $(g, \eta, \gamma, g', \eta')e = (\eta^{-1}f(\gamma)\eta')e$ for all elements $(g, \eta, \gamma, g', \eta') \in \rho^*(\mathcal{G})$ and $e \in (\sigma^*(E))_{(g',\eta')} = E_{s(\eta')}$. We can therefore simplify the above expressions:

$$(\Omega^*E)_g = \left\{ \left(e_{(g,\eta)}\right)_{(g,\eta)\in\Omega} \,\middle|\, \forall \eta, \eta' \in \mathcal{H}^{f(g)} :\ e_{(g,\eta)} \in E_{s(\eta)} \wedge e_{(g,\eta)} = \eta^{-1}\eta' e_{(g,\eta')} \right\}.$$

For all $g \in \mathcal{G}^{(0)}$, the fibre of f^*E at g is simply $(f^*E)_g = E_{f(g)}$. If $e \in E_{f(g)}$, then define

$$\Phi_g(e) := \left(\eta^{-1}e\right)_{(g,\eta)\in\Omega} \in (\Omega^*E)_g.$$

This defines an isometric bijection between $(f^*E)_g$ and $(\Omega^*E)_g$; the inverse sends $\left(e_{(g,\eta)}\right)_{(g,\eta)\in\Omega}$ to $e_{(g,f(g))} \in E_{f(g)}$. It can be shown that Φ is a $\mathcal{G}$-equivariant continuous field of isometric linear maps and that this construction is compatible with the tensor product. $\square$

Corollary 6.6.4. *Let Ω be an equivalence between $\mathcal{G}$ and $\mathcal{H}$. Then $E \mapsto \Omega^* E$ is an equivalence of the categories of $\mathcal{H}$-Banach spaces and $\mathcal{G}$-Banach spaces, isometric and linear on the morphism sets of equivariant bounded continuous fields of linear maps and compatible with the tensor product.*

6.6.2 The pullback of $\mathrm{KK}^{\mathrm{ban}}$-cycles along generalised morphisms

For the rest of this chapter, assume that all the unit spaces of the appearing groupoids are σ-compact.

Because the functor Ω^* is compatible with the tensor product, we can define a $\mathcal{G}$-Banach algebra $\Omega^* A$ for every $\mathcal{H}$-Banach A. This defines a functor form the category of $\mathcal{H}$-Banach algebras together with the $\mathcal{H}$-equivariant homomorphisms to the category of $\mathcal{G}$-Banach algebras with the $\mathcal{G}$-equivariant homomorphisms. If Ω is an equivalence, then Ω^* is an equivalence of these categories.

Similar statements are true for Banach modules and equivariant homomorphisms of Banach modules, and for Banach pairs and equivariant homomorphisms of Banach pairs. Note that Ω^* is *not* defined for linear operators between Banach modules or between Banach pairs. The problem is that f_Ω^* makes sense for linear operators, but the resulting operator between, say, $\rho^*\mathcal{G}$-Banach modules is not necessarily $\Omega \times_\rho \Omega$-invariant. So $\rho_!$ of this operator cannot be defined in general.

However, we still get a map on the level of KK-groups because in the intermediate step, we can *make* the operator of the $\mathrm{KK}^{\mathrm{ban}}$-cycle $\Omega \times_\rho \Omega$-invariant (recall that we have assumed $\mathcal{G}^{(0)}$ to be σ-compact). This was done in Lemma 6.5.17, which enables us to define Ω^* on the level of $\mathrm{KK}^{\mathrm{ban}}$-groups.

Definition 6.6.5. Let Ω be a graph from $\mathcal{G}$ to $\mathcal{H}$. Then Theorem 3.6.11 gives a homomorphism

$$f_\Omega^* \colon \mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(A, B) \to \mathrm{KK}^{\mathrm{ban}}_{\rho^*(\mathcal{G})}(f_\Omega^* A,\ f_\Omega^* B).$$

Corollary 6.5.19 gives us an isomorphism

$$\rho_! \colon \mathrm{KK}^{\mathrm{ban}}_{\rho^*(\mathcal{G})}(f_\Omega^* A,\ f_\Omega^* B) \cong \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\Omega^* A,\ \Omega^* B).$$

Define

$$\Omega^* := \rho_! \circ f_\Omega^* \colon \mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(A, B) \to \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\Omega^* A,\ \Omega^* B).$$

A variant of the proof of Proposition 6.6.2, the corresponding statement for Banach spaces, shows:

Proposition 6.6.6. *Let $\mathcal{K}$ be another locally compact Hausdorff groupoid carrying a left Haar system. Let Ω be a graph from $\mathcal{G}$ to $\mathcal{H}$ and Ω' a graph from $\mathcal{H}$ to $\mathcal{K}$. Then*

$$\Omega^* \circ (\Omega')^* = \left(\Omega \times_{\mathcal{H}} \Omega'\right)^* \colon \mathrm{KK}^{\mathrm{ban}}_{\mathcal{K}}(A, B) \to \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\Omega^*\Omega'^* A,\ \Omega^*\Omega'^* B\right).$$

Proposition 6.6.7. *Let $f \colon \mathcal{G} \to \mathcal{H}$ be a strict morphism. Then*

$$f^* = \mathrm{Graph}(f)^* \colon \mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(A, B) \to \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(f^* A,\ f^* B)$$

if we identify $f^* A$ with $\mathrm{Graph}(f)^* A$ and $f^* B$ with $\mathrm{Graph}(f)^* B$ (which is possible according to Proposition 6.6.3).

Proof. This is proved in Appendix D.2 on page 304. □

Corollary 6.6.8. *The homomorphism*

$$\mathcal{G}^* \colon \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A,\ B) \to \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(A,\ B)$$

is the identity.

Corollary 6.6.9. *Let Ω be a Morita equivalence from $\mathcal{G}$ to $\mathcal{H}$. Then*

$$\Omega^*\colon\ \mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(A,B) \cong \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\Omega^*A,\ \Omega^*B)$$

is an isomorphism with inverse map $(\Omega^{-1})^*$.

6.6.3 $\mathrm{KK}^{\mathrm{ban}}$-cycles and the linking groupoid

Let Ω be a Morita equivalence between $\mathcal{G}$ and $\mathcal{H}$ and let A and B be $\mathcal{H}$-Banach algebras. Let $\mathcal{L}$ denote the linking groupoid as defined in Section 6.2.

There are two canonical $\mathcal{L}$-Banach algebras which we can construct from the $\mathcal{H}$-Banach algebra A. Note that an $\mathcal{L}$-Banach algebra is in particular a u.s.c. field of Banach algebras over $\mathcal{L}^{(0)} = \mathcal{G}^{(0)} \sqcup \mathcal{H}^{(0)}$. Now Ω^*A is a $\mathcal{G}$-Banach algebra and hence a u.s.c. field of Banach algebras over $\mathcal{G}^{(0)}$. We form a family of Banach algebras over $\mathcal{L}^{(0)}$ by making Ω^*A and A into a single family over $\mathcal{L}^{(0)}$. It is a $\mathcal{L}$-Banach algebra in a canonical way.

Alternatively, we can use the fact that $\Omega \sqcup \mathcal{H} = \mathcal{L}_{\mathcal{H}^{(0)}}$ is a Morita equivalence between $\mathcal{L}$ and $\mathcal{H}$. Hence $(\Omega \sqcup \mathcal{H})^*A$ is an $\mathcal{L}$-Banach algebra. A straightforward calculation shows the plausible fact that these two constructions give the same $\mathcal{L}$-Banach algebra. We are going to call it $\Omega^*A \sqcup A$.

The pullback along the inclusions $\iota_{\mathcal{G}}$ of $\mathcal{G}$ and $\iota_{\mathcal{H}}$ of $\mathcal{H}$ as open and closed subgroupoids of $\mathcal{L}$ give back Ω^*A and A. The graphs of the inclusions are Morita equivalences such that $\mathrm{Graph}(\iota_{\mathcal{G}})^{-1} \times_{\mathcal{L}} \mathrm{Graph}(\iota_{\mathcal{H}})$ is equivalent to Ω.

So we have isomorphisms

$$\iota_{\mathcal{H}}^*\colon\ \mathrm{KK}^{\mathrm{ban}}_{\mathcal{L}}(\Omega^*A \sqcup A,\ \Omega^*B \sqcup B) \cong \mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(A,\ B)$$

and

$$\iota_{\mathcal{G}}^*\colon\ \mathrm{KK}^{\mathrm{ban}}_{\mathcal{L}}(\Omega^*A \sqcup A,\ \Omega^*B \sqcup B) \cong \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\Omega^*A,\ \Omega^*B)$$

satisfying $\iota_{\mathcal{G}}^* \circ (\iota_{\mathcal{H}}^*)^{-1} = \Omega^*$.

6.6.4 Morita equivalence and descent

Again, let Ω be a Morita equivalence between $\mathcal{G}$ and $\mathcal{H}$ and let A and B be non-degenerate $\mathcal{H}$-Banach algebras. Let $\mathcal{L}$ denote the linking groupoid. Note that we have assumed that $\mathcal{G}$ and $\mathcal{H}$ carry left Haar systems; so there is an induced left Haar system on Ω and also on $\mathcal{L}$. Let $\mathcal{A}(\mathcal{L})$ be an unconditional completion of $\mathcal{C}_c(\mathcal{L})$. This completion also gives unconditional completions $\mathcal{A}(\mathcal{G})$ of $\mathcal{C}_c(\mathcal{G})$ and $\mathcal{A}(\mathcal{H})$ of $\mathcal{C}_c(\mathcal{H})$. Note that $\mathcal{G}^{(0)}$ and $\mathcal{H}^{(0)}$ are open, closed, full and connected subsets of $\mathcal{L}^{(0)}$. From Theorem 5.3.9 we can therefore conclude:

Theorem 6.6.10. *The $\mathcal{C}_0(\mathcal{L}^{(0)}/\mathcal{L})$-Banach algebras $\mathcal{A}(\mathcal{G},\ \Omega^*A)$ and $\mathcal{A}(\mathcal{H},\ A)$ are Morita equivalent.*

A $\mathcal{C}_0(\mathcal{L}^{(0)}/\mathcal{L})$-linear Morita equivalence can be obtained be taking the completions of $\Gamma_c(\Omega,\ \sigma^*A)$ and $\Gamma_c(\Omega^{-1},\ \sigma^*A)$ for the unconditional norm inherited from $\mathcal{C}_c(\mathcal{L})$.

We now come back to the other considerations of Section 5.3, in particular to Diagram (5.5). Note that the notation we have used in this diagram is somewhat different from the notation of the present chapter, in particular the groupoid $\mathcal{G}$ is now called $\mathcal{L}$. The translated version of the diagram (which is flipped to allow it to be typeset properly) is

$$\begin{tikzcd}
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{L}}(\Omega^*A \sqcup A,\ \Omega^*B \sqcup B) \arrow[r, "j_{\mathcal{A}}"] \arrow[d, "\iota_{\mathcal{H}}^*"] & \mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0\left(\mathcal{L}^{(0)}/\mathcal{L}\right);\ \mathcal{A}(\mathcal{L},\ \Omega^*A \sqcup A),\ \mathcal{A}(\mathcal{L},\ \Omega^*B \sqcup B)\right) \arrow[d] \\
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(A,\ B) \arrow[r, "j_{\mathcal{A}}"] & \mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0\left(\mathcal{H}^{(0)}/\mathcal{H}\right);\ \mathcal{A}(\mathcal{H}, A),\ \mathcal{A}(\mathcal{H}, B)\right)
\end{tikzcd}$$

There is a similar diagram for the embedding of $\mathcal{G}$ into $\mathcal{L}$. We now know that the left arrow is an isomorphism, however, we still do not know whether the right arrow is an isomorphism as well (see the discussion following Corollary 5.3.10). If it is, then the following conjecture is true:

Conjecture 6.6.11. There is a canonical isomorphism

$$\mathrm{RKK}^{\mathrm{ban}}\Big(\mathcal{C}_0(\mathcal{H}^{(0)}/\mathcal{H});\ \mathcal{A}(\mathcal{H},\ A),\ \mathcal{A}(\mathcal{H},\ B)\Big) \to \mathrm{RKK}^{\mathrm{ban}}\Big(\mathcal{C}_0(\mathcal{G}^{(0)}/\mathcal{G});\ \mathcal{A}(\mathcal{G},\ \Omega^* A),\ \mathcal{A}(\mathcal{G},\ \Omega^* B)\Big)$$

making the following diagram commutative

$$\begin{tikzcd}
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(A,\ B) \arrow[r, "j_{\mathcal{A}}"] \arrow[d, "\Omega^*"] & \mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0\left(\mathcal{H}^{(0)}/\mathcal{H}\right);\ \mathcal{A}(\mathcal{H},\ A),\ \mathcal{A}(\mathcal{H},\ B)\right) \arrow[d] \\
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\Omega^* A,\ \Omega^* B) \arrow[r, "j_{\mathcal{A}}"] & \mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0\left(\mathcal{G}^{(0)}/\mathcal{G}\right);\ \mathcal{A}(\mathcal{G},\ \Omega^* A),\ \mathcal{A}(\mathcal{G},\ \Omega^* B)\right)
\end{tikzcd}$$

6.7 Examples

6.7.1 Writing pullbacks as induction

If Ω is a graph from $\mathcal{G}$ to $\mathcal{H}$ and B is a $\mathcal{H}$-Banach algebra, then we could call the $\mathcal{G}$-Banach algebra $\Omega^* B$ also $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} B$. In this notation, we have defined a homomorphism from $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(A, B)$ to $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} A,\ \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} B)$ for all $\mathcal{H}$-Banach algebras A and B. We have also shown that $\mathrm{Ind}_{\mathcal{G}}^{\mathcal{G}} B \cong B$ for all $\mathcal{G}$-Banach algebras B and

$$\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} \mathrm{Ind}_{\mathcal{K}}^{\mathcal{H}} B \cong \mathrm{Ind}_{\mathcal{K}}^{\mathcal{G}} B$$

for all $\mathcal{K}$-Banach algebras B (if $\mathcal{K}$ is another locally compact Hausdorff groupoid with Haar system and we are given a graph from $\mathcal{H}$ to $\mathcal{K}$ which we can use to define the induction from $\mathcal{K}$ to $\mathcal{H}$). Additionally, we have seen that the corresponding (functoriality) rules are also true on the level of $\mathrm{KK}^{\mathrm{ban}}$-theory. As a consequence, if Ω is an equivalence between $\mathcal{G}$ and $\mathcal{H}$, then

$$\mathrm{KK}^{\mathrm{ban}}_{\mathcal{H}}(A, B) \cong \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} A,\ \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} B\right)$$

for all $\mathcal{H}$-Banach algebras A and B.

Moreover, if Ω is an equivalence between $\mathcal{G}$ and $\mathcal{H}$ and if $\mathcal{L}$ denotes the linking groupoid and $\mathcal{A}(\mathcal{L})$ is an unconditional completion of $\mathcal{C}_c(\mathcal{L})$, then this also gives unconditional completions $\mathcal{A}(\mathcal{G})$ and $\mathcal{A}(\mathcal{H})$ of $\mathcal{C}_c(\mathcal{G})$ and $\mathcal{C}_c(\mathcal{H})$, respectively. If B is a $\mathcal{H}$-Banach algebra, then $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}$ is a $\mathcal{G}$-Banach algebra and we have shown that

$$\mathcal{A}\left(\mathcal{H}, B\right) \sim_{\mathrm{M}} \mathcal{A}\left(\mathcal{G},\ \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} B\right).$$

In particular, this applies to the unconditional completion $\mathrm{L}^1(\mathcal{L})$ which induces the completions $\mathrm{L}^1(\mathcal{G})$ and $\mathrm{L}^1(\mathcal{H})$ on $\mathcal{C}_c(\mathcal{G})$ and $\mathcal{C}_c(\mathcal{H})$:

$$\mathrm{L}^1\left(\mathcal{H}, B\right) \sim_{\mathrm{M}} \mathrm{L}^1\left(\mathcal{G},\ \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} B\right).$$

6.7.2 The special case of groups and group actions

Let G be a locally compact σ-compact Hausdorff group and let H be a closed subgroup of G. Let $\mathcal{H}$ be H, regarded as a groupoid, and let $\mathcal{G}$ be the transformation groupoid $G \ltimes G/H$ for the left action of G on the quotient space G/H. Then G is an equivalence of the groupoids $\mathcal{G}$ and $\mathcal{H}$. If B is an H-Banach algebra, then B is also an $\mathcal{H}$-Banach algebra (with just one fibre). The $\mathcal{G}$-Banach algebra $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} B$ is a u.s.c. field of Banach algebras over G/H. If we form the algebra $\Gamma_0(G/H,\, \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} B)$ of sections vanishing at infinity of $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} B$, then this Banach algebra carries a G-action (see Definition 4.5.1) and is canonically isomorphic to $\mathrm{Ind}_H^G B$. The construction of $\mathrm{Ind}_H^G B$ is of course much simpler as the construction of the induction functor in the groupoid case, and some of the above results have counterparts for $\mathrm{Ind}_H^G B$ which can be proved directly. However, using the general machinery, we get the following results:

Induction is an isomorphism

$$\mathrm{KK}_H^{\mathrm{ban}}(A,B) \cong \mathrm{KK}_{\mathcal{G}}^{\mathrm{ban}}\left(\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} A,\, \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} B\right) \cong \mathrm{RKK}_G^{\mathrm{ban}}\left(\mathcal{C}_0(G/H);\, \mathrm{Ind}_H^G A,\, \mathrm{Ind}_H^G B\right)$$

for all H-Banach algebras A and B. For the second isomorphism, see Theorem 4.7.20, it is given by $\mathfrak{M}(\cdot)$.

If B is a non-degenerate H-Banach algebra and $\mathcal{A}(\mathcal{L})$ is an unconditional completion of $\mathcal{C}_c(\mathcal{L})$, where $\mathcal{L}$ is the linking groupoid for the equivalence G between $G \ltimes G/H$ and H, then

$$\mathcal{A}(H,B) \sim_{\mathrm{M}} \mathcal{A}\left(\mathcal{G},\, \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} B\right).$$

In particular we have

$$\mathrm{L}^1(H,B) \sim_{\mathrm{M}} \mathrm{L}^1\left(\mathcal{G},\, \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} B\right).$$

Note that the right-hand side is the completion of $\Gamma_c(G \times G/H,\, r_{\mathcal{G}}^* \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} B)$ for the norm

$$\|\beta\|_1 = \sup_{g \in G} \int_{g' \in G} \left\|\beta\left(g', gH\right)\right\| \,\mathrm{d}g'$$

for all $\beta \in \Gamma_c(G \times G/H,\, r_{\mathcal{G}}^* \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} B)$; in general, this is smaller than the norm

$$\|\beta\| = \int_{g' \in G} \sup_{g \in G} \left\|\beta\left(g', gH\right)\right\| \,\mathrm{d}g'.$$

Note that $\sup_{g \in G} \|\beta(g', gH)\| = \|gH \mapsto \beta(g', gH)\|_\infty$ is the norm of $\beta(g', \cdot)$ in $\mathrm{Ind}_H^G B$ for all $g' \in G$. We can regard β as a continuous map from G to $\mathrm{Ind}_H^G B$ having compact support, and the norm of β given above is then the norm in $\mathrm{L}^1(G,\, \mathrm{Ind}_H^G B)$. It is easy to see that the completion of $\Gamma_c(G \times G/H,\, r_{\mathcal{G}}^* \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} B)$ for the second norm will then be (isomorphic to) $\mathrm{L}^1(G,\, \mathrm{Ind}_H^G B)$. Hence we have a canonical homomorphism

$$\mathrm{L}^1\left(G,\, \mathrm{Ind}_H^G B\right) \xrightarrow{\iota} \mathrm{L}^1\left(\mathcal{G},\, \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}} B\right) \sim_{\mathrm{M}} \mathrm{L}^1(H,B).$$

Compare this to Green's theorem[16] for H-C*-algebras B:

$$\mathrm{Ind}_H^G B \rtimes_r G \sim_{\mathrm{M}} B \rtimes_r H.$$

In Example 8.2.7 we will show that the homomorphism ι is an isomorphism in K-theory (it has dense and hereditary image and a nilpotent kernel), so that we have in particular

$$\mathrm{K}_0\left(\mathrm{L}^1\left(G,\, \mathrm{Ind}_H^G B\right)\right) \cong \mathrm{K}_0\left(\mathrm{L}^1(H,B)\right)$$

for all non-degenerate H-Banach algebras B.

[16] See, for example, [EKQR02], Theorem B.2.

Chapter 7

A Generalised Green-Julg Theorem for Proper Groupoids

7.1 The theorem and its generalisation

One version of the theorem of Green-Julg is the following:

Theorem 7.1.1 (Green-Julg). *Let G be a compact Hausdorff group and let B be a G-C*-algebra. Then*

$$\mathrm{K}_0^G(B) \;\cong\; \mathrm{K}_0(B \rtimes_r G).$$

This theorem remains true if we replace the C*-algebra algebra $B \rtimes_r G$ by the Banach algebra $\mathrm{L}^1(G, B)$ on the right-hand side. This chapter deals with a version of this latter formulation for proper groupoids; note that the proper groupoids which are groups (i.e., those which have trivial unit space) are precisely the compact groups.

7.1.1 Proper groupoids

Definition 7.1.2 (Proper groupoid). A locally compact Hausdorff groupoid is called *proper* if the following map is proper:

$$\mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)},\ \gamma \mapsto (r(\gamma), s(\gamma)).$$

Examples 7.1.3.

1. Let G be a locally compact Hausdorff group acting from the left on a locally compact Hausdorff space X. Then the transformation groupoid $G \ltimes X$ is proper if and only if the action of G on X is proper.
2. More generally, if $\mathcal{G}$ is a locally compact Hausdorff groupoid and X is a left $\mathcal{G}$-space, then $\mathcal{G} \ltimes X$ is proper if and only if X is a proper $\mathcal{G}$-space.
3. A locally compact Hausdorff group is proper (as a groupoid) if and only if G is compact.
4. If the range and source maps of a locally compact Hausdorff groupoid $\mathcal{G}$ are equal, the groupoid can be regarded as a bundle of groups. If such a $\mathcal{G}$ is proper, then all the fibres are compact groups.

For the remainder of this chapter, let $\mathcal{G}$ be a locally compact proper Hausdorff groupoid with unit space X and carrying a Haar system λ. Assume moreover that there exists a cut-off function for $\mathcal{G}$. Recall from [Tu04] that there is a cut-off function for $\mathcal{G}$ if $X/\mathcal{G}$ is σ-compact:

Definition 7.1.4 (Cut-off function). [1] A continuous function $c\colon X \to [0,\infty[$ is called *cut-off function* for $\mathcal{G}$ if

1. $\forall x \in X:\ \int_{\mathcal{G}^x} c(s(\gamma))\,\mathrm{d}\lambda^x(\gamma) = 1$;
2. $r\colon \operatorname{supp}(c \circ s) \to X$ is proper.

The latter condition means that $\operatorname{supp} c \cap \mathcal{G}K$ is compact for all compact subsets K of X.

7.1.2 Generalising the Green-Julg theorem

In [Tu99] the following version of the Green-Julg theorem is proved:[2][3]

Theorem 7.1.5 (Tu). *Let $\mathcal{G}$ be σ-compact and let B be a $\mathcal{G}$-C*-algebra. Then there is a canonical isomorphism*

$$\mathrm{KK}_{\mathcal{G}}(\mathcal{C}_0(X),\ B)\ \cong\ \mathrm{KK}_{X/\mathcal{G}}(\mathcal{C}_0(X/\mathcal{G}),\ B \rtimes_r \mathcal{G}). \tag{7.1}$$

In order to translate this theorem into the language of $\mathrm{KK}^{\mathrm{ban}}$ we proceed as follows: We replace the $\mathcal{G}$-C*-algebra B by a $\mathcal{G}$-Banach algebra so the left-hand side of (7.1) should then be replaced by[4] $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X), B)$. The crossed product of B with $\mathcal{G}$ should be replaced by $\mathcal{A}(\mathcal{G}, B)$, where $\mathcal{A}(\mathcal{G})$ is some unconditional completion of $\mathcal{C}_c(\mathcal{G})$. Because $\mathcal{A}(\mathcal{G}, B)$ is not necessarily a locally $\mathcal{C}_0(X/\mathcal{G})$-convex $\mathcal{C}_0(X/\mathcal{G})$-Banach algebra, we have to use RKK-theory on the right-hand side instead of $\mathrm{KK}^{\mathrm{ban}}_{X/\mathcal{G}}$. So the theorem becomes the following conjecture

$$\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X),\ B) \cong \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G}, B)).$$

7.1.3 The plan of attack and an outline of the proof

To prove this conjecture we are going to proceed as follows:

1. We define a homomorphism $J^B_{\mathcal{A}}$ from the left-hand side to the right-hand side.
2. We define a homomorphism $M^B_{\mathcal{A}}$ in the other direction.
3. We show that $J^B_{\mathcal{A}} \circ M^B_{\mathcal{A}} = \mathrm{Id}$ if $\mathcal{A}(\mathcal{G})$ satisfies some (mild) regularity condition.
4. We show that also $M^B_{\mathcal{A}} \circ J^B_{\mathcal{A}} = \mathrm{Id}$ if $\mathcal{A}(\mathcal{G})$ satisfies some additional regularity condition.

Note that already the split surjectivity is an interesting result as it implies the split surjectivity of the Bost-map with proper coefficients for many unconditional completions, as shown in Chapter 8.

To get an idea of the construction of the two homomorphisms let us take a look at the corresponding constructions for C*-algebras that one can use for a proof of Theorem 7.1.5.

[1] Compare [Tu99], Définition 6.7.

[2] Actually, Proposition 6.25 of [Tu99] is more general than cited here: It allows C*-algebras in the first variable that are of a more general form. For now, we confine ourselves to "trivial" coefficients in the first variable.

[3] This theorem also generalises Theorem 5.4 in [KS03].

[4] Actually, it *should* be replaced by $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathbb{C}_X, B)$ where $\mathbb{C}_X$ denotes the constant field over X with fibre $\mathbb{C}$. We will sometimes identify $\mathcal{C}_0(X)$ and $\mathbb{C}_X$ to obtain statements of theorems which look familiar.

The construction of the homomorphism J_r^B in the C*-context, I

How is the homomorphism J_r^B from $\mathrm{KK}_{\mathcal{G}}(\mathcal{C}_0(X),\, B)$ to $\mathrm{KK}_{X/\mathcal{G}}(\mathcal{C}_0(X/\mathcal{G}),\, B \rtimes_r \mathcal{G})$ defined? The descent is at least a homomorphism

$$j_r\colon\ \mathrm{KK}_{\mathcal{G}}(\mathcal{C}_0(X),\, B) \to \mathrm{KK}_{X/\mathcal{G}}(\mathrm{C}_r^*(\mathcal{G}),\, B \rtimes_r \mathcal{G}).$$

To define a homomorphism from $\mathrm{KK}_{X/\mathcal{G}}(\mathrm{C}_r^*(\mathcal{G}),\, B \rtimes_r \mathcal{G})$ to $\mathrm{KK}_{X/\mathcal{G}}(\mathcal{C}_0(X/\mathcal{G}),\, B \rtimes_r \mathcal{G})$ which we can compose with j_r, we define a $\mathcal{C}_0(X/\mathcal{G})$-linear homomorphism of C*-algebras from $\mathcal{C}_0(X/\mathcal{G})$ to $\mathrm{C}_r^*(\mathcal{G})$. To this end we introduce the following simple notion (already in the generality we are going to need later in this chapter).

Definition 7.1.6 (Cut-off pair). A *cut-off pair* for $\mathcal{G}$ is a pair $(c^<, c^>)$ such that

1. $c^< \in \mathcal{C}(X)_{\geq 0}$ with $r\colon\ \mathrm{supp}(c^< \circ s) \to X$ proper;
2. $c^> \in \mathcal{C}(X)_{\geq 0}$ with $r\colon\ \mathrm{supp}(c^> \circ s) \to X$ proper;
3. $\forall x \in X : \int_{\mathcal{G}^x} c^<(s(\gamma))c^>(s(\gamma))\,\mathrm{d}\lambda^x(\gamma) = 1$.

In particular, $x \mapsto c^<(x)c^>(x)$ is a cut-off function. Conversely, if c is a cut-off function for $\mathcal{G}$ and $p, p' \in]1, \infty[$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, then $(c^{1/p'},\ c^{1/p})$ is a cut-off pair. We can extend this to the case $p = 1$ as follows:

Proposition 7.1.7. *If $\mathcal{G}$ is such that $X/\mathcal{G}$ is σ-compact and c is a cut-off function for $\mathcal{G}$, then there exists a function $d \in \mathcal{C}(X)$ with $\|d\|_\infty = 1$ such that (d, c) is a cut-off pair.*

Proof. Let $(K_n)_{n\in\mathbb{N}}$ be an exhausting sequence of compacts in $X/\mathcal{G}$ such that K_n is contained in the interior of K_{n+1} for all $n \in \mathbb{N}$. Define $L_n := \mathrm{supp}\, c\ \cap\ \pi^{-1}(K_n)$ for all $n \in \mathbb{N}$ (where π denotes the canonical surjection from X to $X/\mathcal{G}$). Then the L_n are all compact. Recursively find functions $f_1, f_2, f_3 \ldots$ such that $f_n \in \mathcal{C}_c(\pi^{-1}(K_n))$, $0 \leq f_n \leq 1$ and $f_n|_{L_n} \equiv 1$ and $f_n \subseteq f_{n+1}$ for all $n \in \mathbb{N}$. Define $f := \bigcup_{n\in\mathbb{N}} f_n$. Then this is a well-defined continuous function on X such that $0 \leq f \leq 1$. It satisfies $f|_{\mathrm{supp}\, c} \equiv 1$. Moreover, it satisfies the support condition: Let $K \subseteq X/\mathcal{G}$ be compact. Find an $n \in \mathbb{N}$ such that $K \subseteq K_n$. Then the closed set $\pi^{-1}(K)$ is contained in $\pi^{-1}(K_n)$, so $\pi^{-1}(K) \cap \mathrm{supp}\, f$ is contained in $\pi^{-1}(K_n) \cap \mathrm{supp}\, f = \pi^{-1}(K_n) \cap \mathrm{supp}\, f_n = \mathrm{supp}\, f_n$. Now $\mathrm{supp}\, f_n$ is a compact subset of $\pi^{-1}(K_n)$, so $\pi^{-1}(K) \cap \mathrm{supp}\, f$ is compact as a closed subset of a compact subset. $\square$

On the level of functions with compact support we can define a homomorphism from $\mathcal{C}_c(X/\mathcal{G})$ to $\mathcal{C}_c(\mathcal{G})$ quite generally; it is a delicate question for which completions of $\mathcal{C}_c(\mathcal{G})$ this homomorphism can be extended continuously to $\mathcal{C}_0(X/\mathcal{G})$.

Definition and Proposition 7.1.8. Let $(c^<, c^>)$ be a cut-off pair for $\mathcal{G}$. For all $\chi \in \mathcal{C}_c(X/\mathcal{G})$, define

$$(\varphi(\chi))(\gamma) := c^>(r(\gamma))\chi(\pi(\gamma))c^<(s(\gamma))$$

for all $\gamma \in \mathcal{G}$. Then $\varphi(\chi) \in \mathcal{C}_c(\mathcal{G})$ and φ is a continuous homomorphism of algebras from $\mathcal{C}_c(X/\mathcal{G})$ to $\mathcal{C}_c(\mathcal{G})$ (with the convolution product).

Proof. Let $\pi\colon\ X \to X/\mathcal{G}$ denote the quotient map and let $K \subseteq X/\mathcal{G}$ be the support of χ. Then $K_1 := \mathrm{supp}\, c^< \cap \pi^{-1}(K)$ is compact in X and so is $K_2 := \mathrm{supp}\, c^> \cap \pi^{-1}(K)$. So $\{\gamma \in \mathcal{G} :\ s(\gamma) \in K_1,\ r(\gamma) \in K_2\}$ is compact and contains the support of $\varphi(\chi)$. So $\varphi(\chi) \in \mathcal{C}_c(\mathcal{G})$.

Let $\chi_1, \chi_2 \in \mathcal{C}_c(\mathcal{G})$. Then for all $\gamma \in \mathcal{G}$:

$$
\begin{aligned}
&(\varphi(\chi_1) * \varphi(\chi_2))(\gamma) \\
= &\int_{\mathcal{G}^{r(\gamma)}} c^{>}(r(\gamma'))\,\chi_1(\pi(\gamma'))\,c^{<}(s(\gamma'))\,c^{>}(r(\gamma'^{-1}\gamma))\,\chi_2(\pi(\gamma'^{-1}\gamma))\,c^{<}(s(\gamma'^{-1}\gamma))\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
= &\; c^{>}(r(\gamma))\,(\chi_1\chi_2)(\pi(\gamma))\,c^{>}(s(\gamma)) \underbrace{\int_{\mathcal{G}^{r(\gamma)}} c^{<}(s(\gamma'))\,c^{>}(s(\gamma'))\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')}_{=1} = (\varphi(\chi_1\chi_2))(\gamma).
\end{aligned}
$$

□

In the C*-algebra case the interesting cut-off pair is of course $(c^{\frac{1}{2}},\ c^{\frac{1}{2}})$, where c is a cut-off function for $\mathcal{G}$. In this case,[5] the homomorphism $\varphi\colon \mathcal{C}_c(X/\mathcal{G}) \to \mathcal{C}_c(\mathcal{G})$ preserves the involution and can be extended to a $*$-homomorphism from $\mathcal{C}_0(X/\mathcal{G})$ to $\mathrm{C}^*_r(\mathcal{G})$. The pullback along this $*$-homomorphism gives us the desired homomorphism J^B_r of groups from $\mathrm{KK}_{X/\mathcal{G}}(\mathrm{C}^*_r(\mathcal{G}),\ B \rtimes_r \mathcal{G})$ to $\mathrm{KK}_{X/\mathcal{G}}(\mathcal{C}_0(X/\mathcal{G}),\ B \rtimes_r \mathcal{G})$.

Can the same homomorphism $\varphi\colon \mathcal{C}_c(X/\mathcal{G}) \to \mathcal{C}_c(\mathcal{G})$ be extended to a homomorphism from $\mathcal{C}_0(X/\mathcal{G})$ to $\mathcal{A}(\mathcal{G})$ if $\mathcal{A}(\mathcal{G})$ is an unconditional completion of $\mathcal{C}_c(\mathcal{G})$? This would be needed to accomplish a completely analogous construction for Banach algebras because the Banach algebra descent for the unconditional completion $\mathcal{A}(\mathcal{G})$ is a homomorphism

$$
J^B_{\mathcal{A}}\colon\ \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X),\ B) \to \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{A}(\mathcal{G}),\ \mathcal{A}(\mathcal{G}, B)).
$$

Apparently, φ is not bounded even for rather elementary unconditional completions like $\mathrm{L}^1(\mathcal{G})$ and rather simple cut-off pairs. The construction works for C*-algebras because the choice of the cut-off pair is compatible with the norm on $\mathrm{C}^*_r(\mathcal{G})$ which is defined through the action of $\mathcal{C}_c(\mathcal{G})$ on $\mathrm{L}^2(\mathcal{G})$. We have to find another way to define the homomorphism for our generalised Green-Julg theorem if we do not want to deal with the technical problems that come with unbounded homomorphisms or with the compression of a Banach algebra by an unbounded projection.

The construction of the homomorphism in the C*-context, II

A possible solution is to define the homomorphism $J^B_{\mathcal{A}}$ from the group $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X),\ B)$ to the group $\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G}, B))$ directly (on the level of cycles). We sketch the analogous construction for C*-algebras:[6]

Let $(E, T) \in \mathbb{E}_{\mathcal{G}}(\mathcal{C}_0(X),\ B)$. Without loss of generality one can assume that T is $\mathcal{G}$-equivariant. We are going to define a cycle in $\mathbb{E}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ B \rtimes_r \mathcal{G})$ as follows.

The underlying module is a completion of $\Gamma_c(X, E)$ which we obtain by embedding $\Gamma_c(X, E)$ into $E \rtimes_r \mathcal{G}$: if $e \in \Gamma_c(X, E)$, then $\iota(e) \in \Gamma_c(\mathcal{G},\ r^*E)$ is defined by $\iota(e)(\gamma) = c^{1/2}(r(\gamma))\gamma e(s(\gamma))$ for all $\gamma \in \mathcal{G}$ (where c is some cut-off function for $\mathcal{G}$). It is easy to show that the image of ι is a subspace of $\Gamma_c(\mathcal{G},\ r^*E)$ that is invariant under the action of $\Gamma_c(\mathcal{G},\ r^*B)$; hence we can define a Hilbert $B \rtimes_r \mathcal{G}$-module by taking the closure $\tilde{E}$ in $E \rtimes_r \mathcal{G}$. Let us see what the inner product is. The $B \rtimes_r \mathcal{G}$-valued inner product on $E \rtimes_r \mathcal{G}$ is given for $\xi_1, \xi_2 \in \Gamma_c(\mathcal{G},\ r^*E)$ as follows: It is the element

[5] See Proposition 6.23 in [Tu99] for a proof.

[6] The construction is inspired by the way E_Y is embedded in $\mathrm{C}^*(\Gamma,\ \mathcal{C}_0(Y))$ on page 178 of [KS03] and by the way Theorem 5.4 of this article is proved.

of $\Gamma_c(\mathcal{G},\, r^*B)$ defined as $\gamma \mapsto \int_{\mathcal{G}^{r(\gamma)}} \gamma' \langle \xi_1(\gamma'^{-1}),\ \xi_2(\gamma'^{-1}\gamma)\rangle \,\mathrm{d}\lambda^{r(\gamma)}(\gamma')$. We therefore have

$$\begin{aligned} \big\langle \iota(e_1),\ \iota(e_2)\big\rangle(\gamma) &= \int_{\mathcal{G}^{r(\gamma)}} \gamma' \left\langle c^{1/2}(s(\gamma'))\,\gamma'^{-1} e_1(r(\gamma')),\ c^{1/2}(s(\gamma'))\,\gamma'^{-1}\gamma e_2(s(\gamma))\right\rangle \mathrm{d}\lambda^{r(\gamma)}(\gamma') \\ = \ & \int_{\mathcal{G}^{r(\gamma)}} c^{1/2}(s(\gamma'))\, c^{1/2}(s(\gamma'))\,\big\langle e_1(r(\gamma)),\ \gamma e_2(s(\gamma))\big\rangle \,\mathrm{d}\lambda^{r(\gamma)}(\gamma') = \langle e_1(r(\gamma)),\ \gamma e_2(s(\gamma))\rangle \end{aligned}$$

for all $e_1, e_2 \in \Gamma_c(X, E) \subseteq \tilde{E}$ and $\gamma \in \mathcal{G}$. Note that this does not depend on the particular choice of the cut-off function c. Similarly, the action of $\Gamma_c(\mathcal{G},\, r^*B)$ inherited by $\Gamma_c(X, E)$ is independent of c, so the same applies to the $B \rtimes_r \mathcal{G}$-action on $\tilde{E}$. Because the norm of $\iota(e)$, where $e \in \Gamma_c(X, E)$, just depends on the inner product, it follows that also the norm on $\tilde{E}$ does not depend on c.

Moreover, it is easy to see that $\mathcal{G}$-equivariant operators such as T give canonical operator $\tilde{T}$ on $\tilde{E}$ with $\|\tilde{T}\| \leq \|T\|$. One now shows that $(\tilde{E},\ \tilde{T}) \in \mathbb{E}(\mathcal{C}_0(X/\mathcal{G});\, \mathcal{C}_0(X/\mathcal{G}),\, B \rtimes_r \mathcal{G})$ and that this defines a homomorphism on the level of KK-theory. It is the same as the homomorphism J_r^B that we have constructed above, but the alternative construction can be imitated easier in the Banach algebraic context (see below).

The construction of the inverse homomorphism in the C*-context

The standard procedure to show that $(E, T) \mapsto (\tilde{E}, \tilde{T})$ induces an isomorphism in KK-theory is the following: The first observation is that this construction is compatible with the sum of Kasparov cycles. Secondly, if $E = \mathrm{L}^2(\mathcal{G}, B)$, then one shows that $\tilde{E} \cong B \rtimes_r \mathcal{G}$. One can then reduce to the case that E is of the standard form $\bigoplus_{n=1}^{\infty} \mathrm{L}^2(\mathcal{G}, B)$ (and therefore $\tilde{E} \cong \bigoplus_{n=1}^{\infty} B \rtimes_r \mathcal{G}$) using a suitable form of the stabilisation theorem.

This procedure is not viable in the Banach algebra context, but there is another way in the C*-algebra context to show that $(E, T) \mapsto (\tilde{E}, \tilde{T})$ induces an isomorphism, namely by construction of an inverse homomorphism M_r^B: The space $\mathrm{L}^2(\mathcal{G}, B)$ is, by definition, a (right) Hilbert B-module. It also carries an action of $\mathcal{C}_0(X)$ and an action of $\mathcal{G}$. In other words, it is a $\mathcal{G}$-Hilbert B-module. On the other hand, it also carries a left $B \rtimes_r \mathcal{G}$-action (by definition of $B \rtimes_r \mathcal{G}$) making it a bimodule. The idea is now very simple: If $(\mathcal{E}, \mathcal{T})$ is a cycle in $\mathbb{E}(\mathcal{C}_0(X/\mathcal{G});\, \mathcal{C}_0(X/\mathcal{G}),\, B \rtimes_r \mathcal{G})$, then $\mathcal{E} \otimes_{B \rtimes_r \mathcal{G}} \mathrm{L}^2(\mathcal{G}, B)$ is a $\mathcal{G}$-Hilbert B-module, where B and $\mathcal{G}$ are acting only on the second factor, and

$$\left(\mathcal{E} \otimes_{B \rtimes_r \mathcal{G}} \mathrm{L}^2(\mathcal{G}, B),\ \mathcal{T} \otimes 1\right) \ \in\ \mathbb{E}_{\mathcal{G}}(\mathcal{C}_0(X);\ B)$$

with the extra feature that $\mathcal{T} \otimes 1$ is $\mathcal{G}$-equivariant. This surely defines a homomorphism on the level of KK-theory, we call it M_r^B.

To check that the two homomorphisms are really inverses of each other, one checks that for each $\mathcal{G}$-Hilbert B-module E we have $\tilde{E} \otimes_{B \rtimes_r \mathcal{G}} \mathrm{L}^2(\mathcal{G}, B) \cong E$, which boils down to the isomorphism $(E \rtimes_r \mathcal{G}) \otimes_{B \rtimes_r \mathcal{G}} \mathrm{L}^2(\mathcal{G}, B) \cong \mathrm{L}^2(\mathcal{G}, E)$ and is quite straightforward to show.[7] The construction for linear operators is compatible with this isomorphism.

On the other hand, if $\mathcal{E}$ is a $\mathcal{C}_0(X/\mathcal{G})$-Hilbert $B \rtimes_r \mathcal{G}$-module, then $\widetilde{\mathcal{E} \otimes_{B \rtimes_r \mathcal{G}} \mathrm{L}^2(\mathcal{G}, B)} \cong \mathcal{E} \otimes_{B \rtimes_r \mathcal{G}} \widetilde{\mathrm{L}^2(\mathcal{G}, B)}$ with $\widetilde{\mathrm{L}^2(\mathcal{G}, B)} \cong B \rtimes_r \mathcal{G}$. Also these isomorphisms are compatible with the constructions for the linear operators.

[7] Compare the proof of Proposition 6.24 of [Tu99].

The theorem in the Banach algebra context

The homomorphism $J^B_{\mathcal{A}}$ from $\mathrm{KK}^{\mathrm{ban}}{}_{\mathcal{G}}(\mathcal{C}_0(X),\ B)$ to $\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G},B))$ is defined similarly to the homomorphism J^B_r in the C*-case (using our second construction). This time, we embed $\Gamma_c(X,E)$ into $\mathcal{A}(\mathcal{G},E)$ (again using a cut-off pair $c=(c^<,c^>)$). Unfortunately, the norm that we get on $\Gamma_c(X,E)$ now depends not only on the norm of $\mathcal{A}(\mathcal{G})$, but also on the cut-off pair c. In Section 7.2 we will show that this is not a serious problem because the homomorphism $J^B_{\mathcal{A}}$ turns out to be independent of the choice of c.

The inverse homomorphism $M^B_{\mathcal{A}}$ can be constructed similarly as in the C*-algebra case, but we have to be careful to find a suitable substitute for $\mathrm{L}^2(\mathcal{G},B)$: If $\mathcal{A}(\mathcal{G})$ is $\mathrm{L}^1(\mathcal{G})\cap\mathrm{L}^1(\mathcal{G})^*$, the version of $\mathrm{L}^1(\mathcal{G})$ with the symmetrized norm, then $\mathcal{A}(\mathcal{G})$ acts on the left on $\mathrm{L}^2(\mathcal{G})$. For more general unconditional completions $\mathcal{A}(\mathcal{G})$ (already for the non-symmetrized $\mathrm{L}^1(\mathcal{G})$), this might not be the case. The solution that I suggest in Section 7.3 is the following: Replace $\mathrm{L}^2(\mathcal{G})$ by a general monotone completion $\mathcal{H}(\mathcal{G})$ of $\mathcal{C}_c(\mathcal{G})$ (defined as in the Section 3.2) on which $\mathcal{A}(\mathcal{G})$ acts on the left (and insert in the theorem the extra hypothesis that such a completion should exist). More precisely, being in the world of Banach pairs, we actually need a pair of completions. Examples are $(\mathrm{L}^2(\mathcal{G}),\ \mathrm{L}^2(\mathcal{G}))$, but also $(\mathrm{L}^{p'}(\mathcal{G}),\ \mathrm{L}^p(\mathcal{G}))$ for $p,p'\in]1,\infty[$ with $\frac{1}{p}+\frac{1}{p'}=1$. Another example is $(\mathcal{C}_0(\mathcal{G}),\ \mathrm{L}^1(\mathcal{G}))$ on which $\mathrm{L}^1(\mathcal{G})$ acts. Each such pair $\mathcal{H}(\mathcal{G})$, or rather the version[8] $\mathcal{H}(\mathcal{G},B)$ with coefficients in B, gives a homomorphism from $\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G},B))$ to $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X),\ B)$. This is shown in Section 7.5, where it is also proved that all possible choices of $\mathcal{H}(\mathcal{G})$ give the same homomorphism, which we call $M^B_{\mathcal{A}}$.

To show that the two homomorphisms are inverses of each other we can no longer use that they are inverses already on the level of cycles (up to isomorphism) as in the C*-algebra case. However, we can construct homotopies using our sufficient condition for homotopy of cycles (resulting in a large number of technical considerations, see Sections 7.6 and 7.8). To make this possible, we have to make sure that the monotone completion $\mathcal{H}(\mathcal{G})$ and the cut-off pair c are compatible (for such a cut-off pair we coin the term "$\mathcal{H}(\mathcal{G})$-cut-off pair", see Section 7.7). The theorem we can prove using this technique reads as follows:

Theorem 7.1.9 (Generalised Green-Julg Theorem). *Let $\mathcal{A}(\mathcal{G})$ be an unconditional completion of $\mathcal{C}_c(\mathcal{G})$ such that there exists an equivariant locally convex pair $\mathcal{H}(\mathcal{G})=(\mathcal{H}^<(\mathcal{G}),\ \mathcal{H}^>(\mathcal{G}))$ of monotone completions of $\mathcal{C}_c(\mathcal{G})$ such that $\mathcal{A}(\mathcal{G})$ acts on $\mathcal{H}(\mathcal{G})$ and such that there exists an $\mathcal{H}(\mathcal{G})$-cut-off pair for $\mathcal{G}$. Then there is an isomorphism*

$$J^B_{\mathcal{A}}\colon\ \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X),\ B)\ \cong\ \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G},B)),$$

natural in the non-degenerate $\mathcal{G}$-Banach algebra B.

We will show a partial result which is interesting because it has slightly less restrictive assumptions:

Theorem 7.1.10. *Let $\mathcal{A}(\mathcal{G})$ be an unconditional completion of $\mathcal{C}_c(\mathcal{G})$ such that there exists an equivariant locally convex pair $\mathcal{H}(\mathcal{G})=(\mathcal{H}^<(\mathcal{G}),\ \mathcal{H}^>(\mathcal{G}))$ of monotone completions of $\mathcal{C}_c(\mathcal{G})$ such that $\mathcal{A}(\mathcal{G})$ acts on $\mathcal{H}(\mathcal{G})$. Let there exists a cut-off function for $\mathcal{G}$. Then the natural homomorphism*

$$J^B_{\mathcal{A}}\colon\ \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X),\ B)\ \cong\ \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G},B))$$

is split surjective (with natural split $M^B_{\mathcal{A}}$) for all non-degenerate Banach algebras B.

We will also show that the unconditional completion $\mathrm{L}^1(\mathcal{G})$ and its symmetrised version $\mathrm{L}^1(\mathcal{G})\cap\mathrm{L}^1(\mathcal{G})^*$ satisfy the hypotheses of both theorems if $X/\mathcal{G}$ is σ-compact.

[8]Note that $\mathrm{L}^2(\mathcal{G},B)$ for a $\mathcal{G}$-C*-algebra has two different meanings in our context: It can denote a Hilbert module and also a completion of $\Gamma_c(\mathcal{G},\ r^*B)$ for some unconditional norm. There is a subtle difference between these spaces, and we always mean the second space.

7.2 The homomorphism $J_{\mathcal{A}}^{B}$

7.2.1 Making operators $\mathcal{G}$-equivariant

Before we start with the construction, we want to proof the following fact:

Proposition 7.2.1. *If B is a $\mathcal{G}$-Banach algebra (with $\mathcal{G}$ being proper and allowing a cut-off function), then the operators and homotopies in the definition of $\mathrm{KK}_{\mathcal{G}}^{\mathrm{ban}}(\mathcal{C}_0(X),\ B)$ can be assumed to be $\mathcal{G}$-equivariant.*

The basic idea here, as in the proof of the corresponding result for C*-algebras, is to use the cut-off function and the integration with respect to the Haar system to make given operators equivariant. On a technical level, we do this by integrating fields of operators with compact support; note that we define this integration pointwise:

Definition 7.2.2. Let E and F be $\mathcal{G}$-Banach spaces. If $T \in \mathrm{L}(r^*E,\ r^*F)$ has compact support, then we define, for all $x \in X$,

$$\int_{\mathcal{G}^x} T_\gamma \,\mathrm{d}\lambda^x(\gamma) : E_x \to F_x,\ e_x \mapsto \int_{\mathcal{G}^x} T_\gamma e_x \,\mathrm{d}\lambda^x(\gamma).$$

This is a continuous field of operators from E to F of compact support. The same definition makes sense if T has proper support, i.e., if the support of $(\chi \circ r) \cdot T$ is compact for all $\chi \in \mathcal{C}_c(X)$.

Definition and Lemma 7.2.3. Let B be a $\mathcal{G}$-Banach algebra and let E and F be $\mathcal{G}$-Banach B-pairs. Let $T = (T^<, T^>) \in \mathrm{L}_{r^*B}(r^*E,\ r^*F)$ have compact (proper) support. Then

$$\int_{\mathcal{G}^x} T_\gamma \,\mathrm{d}\lambda^x(\gamma) := \left(\int_{\mathcal{G}^x} T_\gamma^< \,\mathrm{d}\lambda^x(\gamma),\ \int_{\mathcal{G}^x} T_\gamma^> \,\mathrm{d}\lambda^x(\gamma)\right)$$

is a continuous field of linear operators from E to F. It is compact if $T \in \mathrm{K}_{r^*B}(r^*E,\ r^*F)$.

Proof. We just proof the statement about the compactness. Assume that T is compact. First consider the case that $T = \left(\chi(\gamma)\left|f^>_{r(\gamma)}\right\rangle\left\langle e^<_{r(\gamma)}\right|\right)_{\gamma\in\mathcal{G}}$ with $e^< \in \Gamma(X, E^<)$, $f^> \in \Gamma(X, F^>)$ and $\chi \in \mathcal{C}_c(\mathcal{G})$. Then

$$\int_{\mathcal{G}^x} T_\gamma \,\mathrm{d}\lambda^x(\gamma) = \int_{\mathcal{G}^x} \chi(\gamma)\left|f^>_{r(\gamma)}\right\rangle\left\langle e^<_{r(\gamma)}\right| \mathrm{d}\lambda^x(\gamma) = \int_{\mathcal{G}^x} \chi(\gamma)\,\mathrm{d}\lambda^x(\gamma) \cdot \left|f^>_x\right\rangle\left\langle e^<_x\right|$$

for all $x \in X$. So $(\int_{\mathcal{G}^x} T_\gamma \,\mathrm{d}\lambda^x(\gamma))_{x\in X}$ is compact. The linear span of the operators which are of the same form as T is dense in the compact operators with compact support in the inductive limit topology. As the integral is continuous, we are done. □

If the operator T in the preceding lemma is just of proper support and just locally compact, then the integral yields a locally compact operator.

Now we use these definitions to make operators equivariant. *We fix a cut-off function c for $\mathcal{G}$.*

Definition 7.2.4. Let E and F be $\mathcal{G}$-Banach spaces. If $T \in \mathrm{L}(E, F)$ is arbitrary, then

$$T_x^{\mathcal{G}} = \int_{\mathcal{G}^x} c(s(\gamma))\,\gamma T_{s(\gamma)}\,\mathrm{d}\lambda^x(\gamma), \qquad x \in X,$$

is a $\mathcal{G}$-equivariant continuous field of operators from E to F such that $\left\|T^{\mathcal{G}}\right\| \leq \|T\|$. The map $T \mapsto T^{\mathcal{G}}$ is $\mathbb{C}$-linear. If T is already $\mathcal{G}$-equivariant, then $T^{\mathcal{G}} = T$.

Definition and Lemma 7.2.5. Let B be a $\mathcal{G}$-Banach algebra and let E and F be $\mathcal{G}$-Banach B-pairs. Let $T = (T^<, T^>) \in \mathrm{L}_B(E,F)$. Then $T^{\mathcal{G}} := \left((T^<)^{\mathcal{G}},\ (T^>)^{\mathcal{G}}\right)$ is an element of $\mathrm{L}_B(E,F)$. The construction commutes with the pushout: If B' is another $\mathcal{G}$-Banach algebra and $\varphi\colon B \to B'$ is a $\mathcal{G}$-equivariant homomorphism, then

$$\varphi_*\left(T^{\mathcal{G}}\right) = (\varphi_*(T))^{\mathcal{G}} \quad \in \quad \mathrm{L}^{\mathcal{G}}_{B'}\left(\varphi_*(E), \varphi_*(F)\right).$$

Proof. We check only $\varphi_*\left(T^{\mathcal{G}}\right) = (\varphi_*(T))^{\mathcal{G}}$ and only for the right-hand side: Let $x \in X$, $e_x^> \in E_x^>$ and $b'_x \in \widetilde{B'_x}$. Then

$$\begin{aligned}
\varphi_*\left(T^{>,\mathcal{G}}\right)_x (e_x^> \otimes b'_x) &= \left((T^{>,\mathcal{G}})_x(e_x^>)\right) \otimes b'_x = \left(\int_{\mathcal{G}^x} c(s(\gamma))\gamma T^>_{s(\gamma)} e_x^> \,\mathrm{d}\lambda^x(\gamma)\right) \otimes b'_x \\
&= \int_{\mathcal{G}^x} c(s(\gamma))\gamma \varphi_*(T^>)_{s(\gamma)} \left(e_x^> \otimes b'_x\right) \mathrm{d}\lambda^x(\gamma) = \left(\varphi_*(T^>)\right)^{\mathcal{G}}_x \left(e_x^> \otimes b'_x\right).
\end{aligned}$$

□

The following two lemmas show Proposition 7.2.1.

Lemma 7.2.6. *Let $(E,T) \in \mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X),\ B)$. Then $\left(E, T^{\mathcal{G}}\right)$ is in $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X),\ B)$ and homotopic to (E,T).*

Proof. For all $x \in X$, we have

$$\left(T_x^> - (T^>)^{\mathcal{G}}_x\right)(e_x^>) \;=\; \int_{\mathcal{G}^x} c(s(\gamma)) \left(T^>_{r(\gamma)} - \gamma T^>_{s(\gamma)}\right)(e_x^>) \,\mathrm{d}\lambda^x(\gamma).$$

The same is true for the left-hand side. The family $\gamma \mapsto c(s(\gamma))(T_{r(\gamma)} - (\gamma T_{s(\gamma)})$ is locally compact and of proper support, so the integral is locally compact. So T and $T^{\mathcal{G}}$ differ by a locally compact operator. By Lemma 3.5.11, $(E, T^{\mathcal{G}})$ is a $\mathrm{KK}^{\mathrm{ban}}$-cycle and homotopic to (E,T). □

Lemma 7.2.7. *If (E_0, T_0) and (E_1, T_1) are homotopic in $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X),\ B)$ and if T_0 and T_1 are equivariant, then there is an equivariant homotopy between them.*

Proof. Let $(E,T) \in \mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X),\ B[0,1])$ be a homotopy from (E_0,T_0) to (E_1,T_1). Then

$$(E_t, T_t) \;\cong\; (\mathrm{ev}_{t,*}(E,T)) \;=\; (\mathrm{ev}_{t,*}(E,T))^{\mathcal{G}} \;\overset{7.2.5}{=}\; \mathrm{ev}_{t,*}\left(E, T^{\mathcal{G}}\right)$$

for both, $t=0$ and $t=1$. So $(E, T^{\mathcal{G}})$ is a $\mathcal{G}$-equivariant homotopy from (E_0,T_0) to (E_1,T_1). □

7.2.2 The algebraic construction of $J^B_{\mathcal{A}}$ on the level of sections with compact support

Definition and Lemma 7.2.8. Let B be a $\mathcal{G}$-Banach algebra and let E be a $\mathcal{G}$-Banach B-pair. Define the operations

$$(e^>\beta)(x) := \int_{\mathcal{G}^x} \gamma e > (s(\gamma))\gamma\beta(\gamma^{-1}) \,\mathrm{d}\lambda^x(\gamma)$$

and

$$(\beta e^<)(x) := \int_{\mathcal{G}^x} \beta(\gamma)\gamma e^<(s(\gamma)) \,\mathrm{d}\lambda^x(\gamma),$$

where $x \in X$, and the $\Gamma_c(\mathcal{G},\ r^*B)$-valued bracket

$$\langle\langle e^<,\ e^> \rangle\rangle(\gamma) := \left\langle e^<(r(\gamma)),\ \gamma e^>(s(\gamma)) \right\rangle_{E_{r(\gamma)}},$$

where $\gamma \in \mathcal{G}$, for all $e^< \in \Gamma_c(X, E^<)$, $e^> \in \Gamma_c(X, E^>)$ and $\beta \in \Gamma_c(\mathcal{G},\ r^*B)$.

This turns $\Gamma_c(X, E^>)$ into a right $\Gamma_c(\mathcal{G}, r^*B)$-module and $\Gamma_c(X, E^<)$ a left $\Gamma_c(\mathcal{G},\ r^*B)$-module. These module actions are separately continuous, and non-degenerate for the inductive limit topologies if E is non-degenerate. The bracket is $\mathbb{C}$-bilinear and $\Gamma_c(\mathcal{G},\ r^*B)$-linear on the left and on the right. Moreover, it is separately continuous for the inductive limit topologies.

Moreover, there are canonical actions of $\mathcal{C}(X/\mathcal{G})$ on the modules $\Gamma_c(X, E^<)$ and $\Gamma_c(X, E^>)$ given by

$$(\chi e^>)(x) := \chi(\pi(x))e^>(x)$$

for all $\chi \in \mathcal{C}(X/\mathcal{G})$, $e^> \in \Gamma_c(X, E^>)$ and $x \in X$ (and analogously for the left-hand side). The module actions and the bracket are compatible with these actions.

Proof. One can check by direct computation that the above formula give module actions; that these module actions are separately continuous and non-degenerate can be shown by proving that the map $(e^>, \beta) \mapsto \left[\gamma \mapsto \gamma e^>(s(\gamma))\gamma\beta(\gamma^{-1})\right]$ is a separately continuous and non-degenerate bilinear map (and similarly for the left-hand side). We show now, as an example, that the bracket is $\Gamma_c(\mathcal{G},\ r^*B)$-linear on the right. Let therefore $e^< \in \Gamma_c(X, E^<)$, $e^> \in \Gamma_c(X, E^>)$ and $\beta \in \Gamma_c(\mathcal{G},\ r^*B)$. Then for all $\gamma \in \mathcal{G}$:

$$\begin{aligned}
\langle\langle e^<,\ (e^>\beta) \rangle\rangle(\gamma) &= \left\langle e^<(r(\gamma)),\ \gamma\left[\left(e^>\beta\right)(s(\gamma))\right]\right\rangle \\
&= \left\langle e^<(r(\gamma)),\ \gamma \int_{\mathcal{G}^{s(\gamma)}} \gamma' e^>(s(\gamma'))\gamma'\beta(\gamma'^{-1})\,\mathrm{d}\lambda^{s(\gamma)}(\gamma') \right\rangle \\
&= \left\langle e^<(r(\gamma)),\ \int_{\mathcal{G}^{s(\gamma)}} (\gamma\gamma')e^>(s(\gamma'))(\gamma\gamma')\beta(\gamma'^{-1})\,\mathrm{d}\lambda^{s(\gamma)}(\gamma') \right\rangle \\
&= \int_{\mathcal{G}^{s(\gamma)}} \left\langle e^<(r(\gamma)),\ (\gamma\gamma')e^>(s(\gamma')) \right\rangle (\gamma\gamma')\beta(\gamma'^{-1})\,\mathrm{d}\lambda^{s(\gamma)}(\gamma') \\
&= \int_{\mathcal{G}^{r(\gamma)}} \left\langle e^<(r(\gamma)),\ \gamma' e^>(s(\gamma')) \right\rangle \gamma'\beta(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= \left(\langle\langle e^<,\ e^> \rangle\rangle * \beta\right)(\gamma).
\end{aligned}$$

□

Definition 7.2.9. Let E and F be $\mathcal{G}$-Banach B-pairs and let T be a $\mathcal{G}$-equivariant continuous field of operators from E to F. For all $e^> \in \Gamma_c(X, E^>)$, define

$$(T^>e^>)(x) := T_x^> e^>(x)$$

for all $x \in X$. Then $e^> \mapsto T^>e^>$ is $\mathbb{C}$-linear, $\mathcal{C}(X/\mathcal{G})$-linear, $\Gamma_c(\mathcal{G},\ r^*B)$-linear on the right and continuous for the inductive limit topology. The same formula defines an operator $f^< \mapsto T^<f^<$ on the left-hand side. The pair of operators $(f^< \mapsto T^<f^<,\ e^> \mapsto T^>e^>)$ is formally adjoint with respect to the brackets on $(\Gamma_c(X, E^<),\ \Gamma_c(X, E^>))$ and $(\Gamma_c(X, F^<),\ \Gamma_c(X, F^>))$:

$$\langle\langle f^<T^<,\ e^> \rangle\rangle = \langle\langle f^<,\ T^>e^> \rangle\rangle.$$

Proof. We proof right $\Gamma_c(\mathcal{G}, r^*B)$-linearity of $e^> \mapsto T^> e^>$. Let $e^> \in \Gamma_c(X, E^>)$ and $\beta \in \Gamma_c(\mathcal{G}, r^*B)$. Let $x \in X$. Then

$$\begin{aligned}
(T^>(e^>\beta))(x) &= T_x^>(e^>\beta(x)) = T_x^>\left(\int_{\mathcal{G}^x} \gamma e^>(s(\gamma))\gamma\beta(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma)\right)\\
&= \int_{\mathcal{G}^x} T_{r(\gamma)}^>\left(\gamma e^>(s(\gamma))\right)\gamma\beta(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma)\\
&= \int_{\mathcal{G}^x} \gamma T_{s(\gamma)}^>\left(e^>(s(\gamma))\right)\gamma\beta(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma) = ((T^>e^>)\beta)(x).
\end{aligned}$$

Note that we made use of the $\mathcal{G}$-equivariance of $T^>$. □

Definition 7.2.10. Let E and F be $\mathcal{G}$-Banach B-pairs, $f^> \in \Gamma_c(X, F^>)$ and $e^< \in \Gamma_c(X, E^<)$. Define

$$\left|f^>\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle e^<\right|^> : \Gamma_c(X, E^>) \to \Gamma_c(X, F^>),\ e^> \mapsto f^>\left\langle\!\!\left\langle e^<, e^>\right\rangle\!\!\right\rangle$$

and

$$\left|f^>\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle e^<\right|^< : \Gamma_c(X, F^<) \to \Gamma_c(X, E^<),\ f^< \mapsto \left\langle\!\!\left\langle f^<, f^>\right\rangle\!\!\right\rangle e^<.$$

Definition and Lemma 7.2.11. Let E and F be $\mathcal{G}$-Banach B-pairs, $f^> \in \Gamma_c(X, F^>)$ and $e^< \in \Gamma_c(X, E^<)$. Then for all $e^> \in \Gamma_c(X, E^>)$, $f^< \in \Gamma_c(X, F^<)$ and $x \in X$, we have

$$\left(\left|f^>\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle e^<\right|^>(e^>)\right)(x) = \int_{\mathcal{G}^x} \left|\gamma f^>(s(\gamma))\right\rangle\!\left\langle\gamma e^<(s(\gamma))\right|^>(e^>(x))\,\mathrm{d}\lambda^x(\gamma)$$

and

$$\left(\left|f^>\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle e^<\right|^<(f^<)\right)(x) = \int_{\mathcal{G}^x} \left|\gamma f^>(s(\gamma))\right\rangle\!\left\langle\gamma e^<(s(\gamma))\right|^<(f^<(x))\,\mathrm{d}\lambda^x(\gamma).$$

So we define for all $x \in X$:

$$\left|f^>\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle e^<\right|_x := \int_{\mathcal{G}^x} \left|\gamma f^>(s(\gamma))\right\rangle\!\left\langle\gamma e^<(s(\gamma))\right|\,\mathrm{d}\lambda^x(\gamma) \in \mathrm{L}_{B_x}(E_x, F_x).$$

Then $(\left|f^>\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle e^<\right|_x)_{x\in X}$ is a $\mathcal{G}$-equivariant element of $\mathrm{L}_B(E, F)$. By 7.2.3 it is locally compact.

Proof. On the right-hand side we have

$$\begin{aligned}
\left(\left|f^>\right\rangle\!\left\langle e^<\right|^>(e^>)\right)(x) &= \left(f^>\left\langle e^<,\ e^>\right\rangle\right)(x) = \int_{\mathcal{G}^x} \gamma f^>(s(\gamma))\,\gamma\left(\left\langle e^<,\ e^>\right\rangle(\gamma^{-1})\right)\,\mathrm{d}\lambda^x(\gamma)\\
&= \int_{\mathcal{G}^x} \gamma f^>(s(\gamma))\,\gamma\left\langle e^<(r(\gamma^{-1})),\ \gamma^{-1}e^>(s(\gamma^{-1}))\right\rangle\,\mathrm{d}\lambda^x(\gamma)\\
&= \int_{\mathcal{G}^x} \gamma f^>(s(\gamma))\ \left\langle\gamma e^<(s(\gamma)),\ e^>(r(\gamma))\right\rangle\,\mathrm{d}\lambda^x(\gamma)\\
&= \int_{\mathcal{G}^x} \left|\gamma f^>(s(\gamma))\right\rangle\!\left\langle\gamma e^<(s(\gamma))\right|^> e^>(x)\,\mathrm{d}\lambda^x(\gamma).
\end{aligned}$$

The calculation for the left-hand side is similar. □

Note that we have just defined two different objects which carry the name $\left|f^>\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle e^<\right|$: One is the pair of operators $(\left|f^>\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle e^<\right|^<,\ \left|f^>\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle e^<\right|^>)$, the other is the field of operators $(\left|f^>\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle e^<\right|_x)_{x\in X}$. Now 7.2.11 implies in particular that this convention does not lead to much ambiguity. It also gives us a source of locally compact fields of operators from E to F. The following lemma says that this source is rather rich.

Lemma 7.2.12. *Let E and F be $\mathcal{G}$-Banach B-pairs and let $T \in \mathrm{L}_B(E,F)$ be locally compact and $\mathcal{G}$-equivariant. Then for all $\varepsilon > 0$ and all compact subsets $K \subseteq X/\mathcal{G}$ there exists $n \in \mathbb{N}$, and $f_1^>, \dots, f_n^> \in \Gamma_c(X, F^>)$, $e_1^<, \dots, e_n^< \in \Gamma_c(X, E^<)$ such that for all $x \in \pi^{-1}(K)$:*

$$\left\| T_x - \sum_{i=1}^n \left|f_i^>\right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle e_i^<\right|_x \right\|_{\mathrm{L}(E_x,F_x)} \leq \varepsilon.$$

Proof. Let c be a cut-off function for $\mathcal{G}$. Since $\pi\colon X \to X/\mathcal{G}$ is open, the set K is the image under π of a compact subset of X. In other words, $\pi^{-1}(K)$ is the saturation of some compact subset of X. Hence the set $L := \pi^{-1}(K) \cap \operatorname{supp} c$ is compact.

Since T is locally compact, we can find $n \in \mathbb{N}$, $f_1^>, \dots, f_n^> \in \Gamma_c(X, F^>)$, $e_1^<, \dots, e_n^< \in \Gamma_c(X, E^<)$ such that

$$\left\| T_l - \sum_{i=1}^n \left|f_i^>(l)\right\rangle\left\langle e_i^<(l)\right| \right\| \leq \varepsilon$$

for all $l \in L$. Now let $x \in \pi^{-1}(K)$. Then

$$\begin{aligned}
& \left\| T_x - \int_{\mathcal{G}^x} c(s(\gamma)) \sum_{i=1}^n \left|\gamma f_i^>(s(\gamma))\right\rangle\left\langle \gamma e_i^<(s(\gamma))\right| \,\mathrm{d}\lambda^x(\gamma) \right\| \\
= \;& \left\| \int_{\mathcal{G}^x} c(s(\gamma)) \left[\gamma T_{s(\gamma)} - \sum_{i=1}^n \left|\gamma f_i^>(s(\gamma))\right\rangle\left\langle \gamma e_i^<(s(\gamma))\right| \right] \,\mathrm{d}\lambda^x(\gamma) \right\| \\
\leq \;& \int_{\mathcal{G}^x} c(s(\gamma)) \left\| \gamma T_{s(\gamma)} - \sum_{i=1}^n \left|\gamma f_i^>(s(\gamma))\right\rangle\left\langle \gamma e_i^<(s(\gamma))\right| \right\| \,\mathrm{d}\lambda^x(\gamma) \\
\leq \;& \int_{\mathcal{G}^x} c(s(\gamma)) \varepsilon \,\mathrm{d}\lambda^x(\gamma) = \varepsilon.
\end{aligned}$$

Note that for all $i \in \{1, \dots, n\}$ and all $x \in X$ we have

$$\begin{aligned}
& \int_{\mathcal{G}^x} c(s(\gamma)) \left|\gamma f_i^>(s(\gamma))\right\rangle\left\langle \gamma e_i^<(s(\gamma))\right| \,\mathrm{d}\lambda^x(\gamma) \\
= \;& \int_{\mathcal{G}^x} \left|\gamma (c^{1/2} f_i^>)(s(\gamma))\right\rangle\left\langle \gamma (c^{1/2} e_i^<)(s(\gamma))\right| \,\mathrm{d}\lambda^x(\gamma) = \left|c^{1/2} f_i^>\right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle c^{1/2} e_i^<\right|_x,
\end{aligned}$$

so

$$\left\| T_x - \sum_{i=1}^n \left|c^{1/2} f_i^>\right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle c^{1/2} e_i^<\right|_x \right\| \leq \varepsilon$$

for all $x \in \pi^{-1}(K)$. □

Proposition 7.2.13. *Let E and F be $\mathcal{G}$-Banach B-pairs. Then the map*

$$\Gamma_c(X, F^>) \times \Gamma_c(X, E^<) \to \mathrm{L}_B(E,F), \; (f^>, e^<) \mapsto \left(\left|f^>\right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle e^<\right|_x \right)_{x \in X}$$

is bilinear and separately continuous for the inductive limit topologies on $\Gamma_c(X, F^>)$ and $\Gamma_c(X, E^<)$ and the norm topology on $\mathrm{L}_B(E,F)$.

Proof. We show continuity in the second component: Let $f^> \in \Gamma_c(X, F^>)$ be fixed. Define $C := \sup_{x\in X} \int_{\gamma\in\mathcal{G}^x} \|f^>(s(\gamma))\| \,\mathrm{d}\lambda^x(\gamma)$. Then for all $e^< \in \Gamma_c(X, E^<)$:

$$\left\| \left|f^>\right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle e^<\right|_x \right\| \;\leq\; \int_{\mathcal{G}^x} \left\|\gamma f^>(s(\gamma))\right\| \,\left\|\gamma e^<(s(\gamma))\right\| \,\mathrm{d}\lambda^x(\gamma) \;\leq\; C \left\|e^<\right\|_\infty$$

for all $x \in X$. So $e^< \mapsto \left|f^>\right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle e^<\right|$ is continuous even on $\Gamma_0(X, E^<)$ with norm $\leq C$. □

As a consequence, we obtain the following version of Lemma 7.2.12 which we are going to need later on:

Corollary 7.2.14. *Let E and F be non-degenerate $\mathcal{G}$-Banach B-pairs and let $T \in \mathrm{L}_B(E, F)$ be locally compact and $\mathcal{G}$-equivariant. Then for all $\varepsilon > 0$ and all compact subsets $K \subseteq X/\mathcal{G}$, there exists $n \in \mathbb{N}$ and $f_1^>, \ldots, f_n^> \in \Gamma_c(X, F^>)$, $e_1^<, \ldots, e_n^< \in \Gamma_c(X, E^<)$, $\beta_1, \ldots, \beta_n \in \Gamma_c(\mathcal{G}, r^*B)$ such that for all $x \in \pi^{-1}(K)$:*

$$\left\| T_x - \sum_{i=1}^n \left|f_i^> \cdot \beta_i\right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle e_i^<\right|_x \right\|_{\mathrm{L}(E_x, F_x)} \leq \varepsilon.$$

7.2.3 The analytic part of the construction of $J_{\mathcal{A}}^B$

In the C*-world, the right module $\Gamma_c(\mathcal{G},\ r^*B)$-action and the inner product on $\Gamma_c(X, E)$ is sufficient to define the structure $B \rtimes_r \mathcal{G}$-Hilbert module if E is a Hilbert B-module. There can only be one norm on $\Gamma_c(X, E)$ which completes to a Hilbert module and the bracket actually gives such a norm.

In the Banach-world, the situation is more complicated. Let B be a $\mathcal{G}$-Banach algebra and let E and F be $\mathcal{G}$-Banach B-pairs. *Let $\mathcal{A}(\mathcal{G})$ be an unconditional completion of $\mathcal{C}_c(\mathcal{G})$.* As sketched in the introduction to this chapter, every cut-off pair c will give an embedding of $\Gamma_c(X, E)$ into $\mathcal{A}(\mathcal{G}, E)$ and the completion will be a $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G}, B)$-pair with the extended versions of the operations defined above. This construction turns out not to be flexible enough for our purposes, and I propose a simple generalisation: Because the norm on $\mathcal{A}(\mathcal{G}, E)$ comes from an unconditional completion of $\mathcal{C}_c(\mathcal{G})$, the inherited norm on $\Gamma_c(X, E)$ comes from a monotone completion of $\mathcal{C}_c(X)$. This monotone completion is compatible with the norm of $\mathcal{A}(\mathcal{G})$ in a sense that we will now make into a definition.

Compatible pairs of monotone completions of $\mathcal{C}_c(X)$

Definition 7.2.15 (Compatible pair of monotone completions of $\mathcal{C}_c(X)$). Let $\mathcal{D}^<(X)$ and $\mathcal{D}^>(X)$ be monotone completions[9] of $\mathcal{C}_c(X)$. Then the pair $\mathcal{D}(X) := (\mathcal{D}^<(X),\ \mathcal{D}^>(X))$ is called *compatible with* $\mathcal{A}(\mathcal{G})$ if

1. $\forall \chi^< \in \mathcal{C}_c(X),\ \beta \in \mathcal{C}_c(\mathcal{G}) \quad : \ \|\beta\chi^<\|_{\mathcal{D}^<} \leq \|\beta\|_{\mathcal{A}} \|\chi^<\|_{\mathcal{D}^<}$;
2. $\forall \chi^> \in \mathcal{C}_c(X),\ \beta \in \mathcal{C}_c(\mathcal{G}) \quad : \ \|\chi^>\beta\|_{\mathcal{D}^>} \leq \|\chi^>\|_{\mathcal{D}^>} \|\beta\|_{\mathcal{A}}$;
3. $\forall \chi^< \in \mathcal{C}_c(X),\ \chi^> \in \mathcal{C}_c(X) : \ \left\|\left\langle\!\left\langle \chi^<, \chi^> \right\rangle\!\right\rangle\right\|_{\mathcal{A}} \leq \|\chi^<\|_{\mathcal{D}^<} \|\chi^>\|_{\mathcal{D}^>}$.

With the extended bilinear maps, $\mathcal{D}(X)$ is a Banach $\mathcal{A}(\mathcal{G})$-pair.

Note that the action of $\mathcal{C}_0(X/\mathcal{G})$ on $\mathcal{C}_c(X)$ also gives a continuous non-degenerate action of $\mathcal{C}_0(X/\mathcal{G})$ on $\mathcal{D}^<(X)$ and $\mathcal{D}^>(X)$ making it a $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G})$-pair.

[9] Monotone completions are defined in Definition 3.2.1.

Definition 7.2.16 ($\mathcal{D}(X,E)$). Let $\mathcal{D}(X)$ be a pair of monotone completions of $\mathcal{C}_c(X)$, compatible with $\mathcal{A}(\mathcal{G})$, and let $E = (E^<, E^>)$ be a $\mathcal{G}$-Banach B-pair. On $\Gamma_c(X, E^<)$ define the norm $\|\xi^<\|_{\mathcal{D}^<} := \|\, x \mapsto \|\xi^<(x)\|\,\|_{\mathcal{D}^<}$ as in Definition 3.2.4 and define a semi-norm $\|\cdot\|_{\mathcal{D}^>}$ on $\Gamma_c(X, E^>)$ similarly. Then the actions of $\Gamma_c(\mathcal{G},\ r^*B)$ on $\Gamma_c(X, E^<)$ and on $\Gamma_c(X, E^>)$ and the bracket satisfy

$$\left\|\beta\xi^<\right\|_{\mathcal{D}^<} \leq \|\beta\|_{\mathcal{A}} \left\|\xi^<\right\|_{\mathcal{D}^<}, \quad \left\|\xi^>\beta\right\|_{\mathcal{D}^>} \leq \left\|\xi^>\right\|_{\mathcal{D}^>} \|\beta\|_{\mathcal{A}}, \quad \left\|\langle\langle \xi^<,\, \xi^> \rangle\rangle\right\|_{\mathcal{A}} \leq \left\|\xi^<\right\|_{\mathcal{D}^<} \left\|\xi^>\right\|_{\mathcal{D}^>}$$

for all $\beta \in \Gamma_c(\mathcal{G},\ r^*B)$, $\xi^< \in \Gamma_c(X, E^<)$ and $\xi^> \in \Gamma_c(X, E^>)$. As in Definition 3.2.4 write $\mathcal{D}^<(X, E^<)$ for the completion of $\Gamma_c(X, E^<)$ for the semi-norm $\|\cdot\|_{\mathcal{D}^<}$; define $\mathcal{D}^>(X, E^>)$ analogously. With the extensions of the actions of $\Gamma_c(\mathcal{G},\ r^*B)$ and the extension of the bracket,

$$\mathcal{D}(X,E) := \left(\mathcal{D}^<(X, E^<),\ \mathcal{D}^>(X, E^>)\right)$$

is a $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G}, B)$-pair.

For the remainder of this subsection, let $\mathcal{D}(X)$ be a pair of monotone completions of $\mathcal{C}_c(X)$, compatible with $\mathcal{A}(\mathcal{G})$. The construction of linear maps between monotone completions was discussed in 3.2.5.

Definition 7.2.17. Let $T \in \mathrm{L}_B(E,F)$ be $\mathcal{G}$-equivariant. Then $e^> \mapsto T^> e^>$ is a bounded $\mathbb{C}$-linear, $\mathcal{C}_0\left(X/\mathcal{G}\right)$-linear and $\Gamma_c\left(\mathcal{G},\ r^*B\right)$-linear map from $\Gamma_c(X, E^>)$ to $\Gamma_c(X, F^>)$ with norm $\leq \|T^>\|$, so it extends to a bounded $\mathbb{C}$-linear, $\mathcal{C}_0\left(X/\mathcal{G}\right)$-linear and $\mathcal{A}(\mathcal{G}, B)$-linear map $\mathcal{D}(X, T^>)$ from $\mathcal{D}(X, E^>)$ to $\mathcal{D}(X, F^>)$ of the same norm. Similarly, one gets a map $\mathcal{D}(X, T^<)$ from $\mathcal{D}(X, F^<)$ to $\mathcal{D}(X, E^<)$ of norm $\leq \|T^<\|$. Together, this defines a pair

$$\mathcal{D}(X,T) := \left(\mathcal{D}(X, T^<),\ \mathcal{D}(X, T^>)\right) \ \in\ \mathrm{L}_{\mathcal{A}(\mathcal{G},B)}^{\mathcal{C}_0(X/\mathcal{G})}\left(\mathcal{D}(X,E),\ \mathcal{D}(X,F)\right)$$

of norm $\leq \|T\|$.

The assignment $E \mapsto \mathcal{D}(X,E)$ and $T \mapsto \mathcal{D}(X,T)$ is a contractive functor from the category $\mathcal{G}$-Banach B-pairs and bounded $\mathcal{G}$-equivariant operators to the category of $\mathcal{C}_0\left(X/\mathcal{G}\right)$-Banach $\mathcal{A}(\mathcal{G}, B)$-pairs. Similarly one can define $\mathcal{D}(X, \Phi)$ for $\mathcal{G}$-equivariant concurrent homomorphisms.

Lemma 7.2.18. *For all $f^> \in \Gamma_c(X, F^>)$ and $e^< \in \Gamma_c(X, E^<)$, we have*

$$\mathcal{D}\left(X,\ \left(\left|f^>\right\rangle\!\right\rangle\!\left\langle\!\left\langle e^<\right|_x\right)_{x\in X}\right) = \left|f^>\right\rangle\!\right\rangle\!\left\langle\!\left\langle e^<\right|.$$

This lemma follows from 7.2.11 and maybe needs some explanation: The operator $\left|f^>\right\rangle\!\right\rangle\!\left\langle\!\left\langle e^<\right|_x$ on the left-hand side is the element $\int_{\mathcal{G}^x} \left|\gamma f^>(s(\gamma))\right\rangle\left\langle \gamma e^<(s(\gamma))\right| \,\mathrm{d}\lambda^x(\gamma)$ of $\mathrm{L}_{B_x}\left(E_x, F_x\right)$ as defined in 7.2.11. The operator $\left|f^>\right\rangle\!\right\rangle\!\left\langle\!\left\langle e^<\right|$ on the right-hand side is the compact operator from $\mathcal{D}(X,E)$ to $\mathcal{D}(X,F)$ given by $f^>$ and $e^<$. The ambiguous but suggestive notation was chosen to avoid yet another hat or another tilde on top of an operator.

Proposition 7.2.19. *Let $S \in \mathrm{L}_B(E,F)$ be bounded, $\mathcal{G}$-equivariant and locally compact. Then $\mathcal{D}(X,S)$ is locally compact in the sense of Definition 2.2.27, i.e., $\chi\mathcal{D}(X,S)$ is compact for all $\chi \in \mathcal{C}_c\left(X/\mathcal{G}\right)$.*

Proof. In order to show that $\mathcal{D}(X,S)$ is locally compact, it suffices to show that $\mathcal{D}(X,S)$ is compact if $\pi(\operatorname{supp} S)$ is compact. Let $\varepsilon > 0$. Let $K := \pi(\operatorname{supp} S) \subseteq X/\mathcal{G}$. We now approximate $\mathcal{D}(X,S)$ on K by finite rank operators:

By Lemma 7.2.12 we can find $n \in \mathbb{N}$, and $f_1^>, \ldots, f_n^> \in \Gamma_c(X, F^>)$, $e_1^<, \ldots, e_n^< \in \Gamma_c(X, E^<)$ such that for all $x \in \pi^{-1}(K')$ (where K' is some compact neighbourhood of K):

$$\left\| S_x - \sum_{i=1}^n |f_i^>\rangle\rangle\langle\langle e_i^<|_x \right\|_{\mathrm{L}(E_x, F_x)} \leq \varepsilon.$$

Because S vanishes outside $\pi^{-1}(K)$, we can assume without loss of generality that this inequality is true for all $x \in X$. Then

$$\varepsilon \geq \left\| \mathcal{D}(X,S) - \sum_{i=1}^n \mathcal{D}\left(X, \left(|f_i^>\rangle\rangle\langle\langle e_i^<|_x\right)_{x\in X}\right) \right\| = \left\| \mathcal{D}(X,S) - \sum_{i=1}^n |f_i^>\rangle\rangle\langle\langle e_i^<| \right\|.$$

So $\mathcal{D}(X,S)$ is compact. □

Theorem 7.2.20. *Let (E,T) be a cycle in $\mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(\mathcal{C}_0(X),\ B)$ with T $\mathcal{G}$-equivariant. Equip $\mathcal{D}(X,E)$ with the obvious grading operator. Then $\mathcal{D}(X,T)$ is odd and*

$$J_{\mathcal{A},\mathcal{D}}^B(E,T) := (\mathcal{D}(X,E),\ \mathcal{D}(X,T)) \in \mathbb{E}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G})\,;\ \mathcal{C}_0(X/\mathcal{G})\,,\ \mathcal{A}(\mathcal{G},B)\right).$$

Proof. The important property that we have to check is that $\mathcal{D}(X,T)^2 - 1$ is locally compact. But

$$\mathcal{D}(X,T)^2 - 1 = \mathcal{D}(X,\ T^2 - 1),$$

and $T^2 - 1$ is locally compact. Since $T^2 - 1$ is also $\mathcal{G}$-equivariant, we can apply the preceding proposition which implies that $\mathcal{D}(X,\ T^2-1)$ is locally compact. □

Proposition 7.2.21. *Let B and B' be $\mathcal{G}$-Banach algebras and $\varphi\colon B \to B'$ be a $\mathcal{G}$-equivariant morphism. If $(E,T) \in \mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(\mathcal{C}_0(X),\ B)$ with T $\mathcal{G}$-equivariant, then*

$$J_{\mathcal{A},\mathcal{D}}^B(\varphi_*(E,T)) \sim (\mathcal{A}(\mathcal{G},\ \varphi))_*\left(J_{\mathcal{A},\mathcal{D}}^B(E,T)\right).$$

Proof. The pairs underlying the left- and the right-hand side are

$$\mathcal{D}\left(X,\ E \otimes_B \widetilde{B'}\right) \quad \text{and} \quad \mathcal{D}(X,E) \otimes_{\mathcal{A}(\mathcal{G},B)} \widetilde{\mathcal{A}(\mathcal{G},B')},$$

respectively. Straightforward but technical argumentations using our sufficient condition for homotopy show that we can leave away the (fibrewise) unitalisations and reduce to the simpler pairs $\mathcal{D}(X,\ E \otimes_B B')$ and $\mathcal{D}(X,E) \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{A}(\mathcal{G},B')$, the first equipped with $\mathcal{D}(X, T\otimes 1)$, the second with $\mathcal{D}(X,T) \otimes 1$.

We now proceed in three steps: First we define a homomorphism Φ from $\mathcal{D}(X,E) \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{A}(\mathcal{G},B')$ to $\mathcal{D}(X,\ E\otimes_B B')$, second we show that it intertwines $\mathcal{D}(X,T)\otimes 1$ and $\mathcal{D}(X,T\otimes 1)$, and third we prove that Φ induces a homotopy.

1. For all $e^> \in \Gamma_c(X,E^>)$ and all $\beta' \in \Gamma_c(\mathcal{G},\ r^*B')$, define

$$\mu^>(e^>,\ \beta')\,(x) := \int_{\mathcal{G}^x} \gamma e^>(s(\gamma)) \otimes \gamma\beta'(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma)$$

for all $x \in X$. Then $\mu^>(e^>,\ \beta') \in \Gamma_c(X,\ E^> \otimes_B B')$. Moreover, $\mu^>$ is $\Gamma_c(\mathcal{G},\ r^*B)$-balanced and satisfies

$$\left\|\mu^>(e^>,\ \beta')\right\|_{\mathcal{D}^>} \leq \left\|e^>\right\|_{\mathcal{D}^>}\left\|\beta'\right\|_{\mathcal{A}}.$$

Moreover, $\mu^{>}$ is $\mathcal{C}_c(\mathcal{G},\ r^*B')$-linear on the right and $\mathcal{C}_0(X/\mathcal{G})$-bilinear. $\mu^{>}$ can hence be extended to an $\mathcal{A}(\mathcal{G},B)$-balanced contractive bilinear map $\mu^{>}\colon \mathcal{D}^{>}(X,E^{>})\times\mathcal{A}(\mathcal{G},B')\to\mathcal{D}^{>}(X,\ E\otimes_B B')$ which is $\mathcal{A}(\mathcal{G},B')$-linear on the right and $\mathcal{C}_0(X/\mathcal{G})$-bilinear. This gives a contractive linear map $\Phi^{>}\colon \mathcal{D}^{>}(X,E^{>})\otimes_{\mathcal{A}(\mathcal{G},B)}\mathcal{A}(\mathcal{G},B')\to\mathcal{D}^{>}(X,\ E\otimes_B B')$ which is $\mathcal{A}(\mathcal{G},B')$-linear on the right and $\mathcal{C}_0(X/\mathcal{G})$-linear.

For the left-hand side, we define

$$\mu^{<}(\beta',e^{<})(x):=\int_{\mathcal{G}^x}\beta'(\gamma)\otimes\gamma e^{<}(s(\gamma))\,\mathrm{d}\lambda^x(\gamma)$$

for all $\beta'\in\Gamma_c(\mathcal{G},\ r^*B')$, $e^{<}\in\Gamma_c(X,E^{<})$ and $x\in X$. This defines a contractive $\mathcal{A}(\mathcal{G},B')$-linear map $\Phi^{<}$ from $\mathcal{A}(\mathcal{G},B')\otimes_{\mathcal{A}(\mathcal{G},B)}\mathcal{D}(X,E^{<})$ to $\mathcal{D}(X,\ B'\otimes_B E^{<})$ which is $\mathcal{A}(\mathcal{G},B')$-linear on the left and $\mathcal{C}_0(X/\mathcal{G})$-linear. We check that $(\Phi^{<},\Phi^{>})$ is a homomorphism by direct computation:

$$\begin{aligned}
&\left\langle\!\left\langle\Phi^{<}(\beta'^{<}\otimes e^{<}),\ \Phi^{>}(e^{>}\otimes\beta'^{>})\right\rangle\!\right\rangle(\gamma)\\
=\ &\left\langle\mu^{<}(\beta'^{<},e^{<})(r(\gamma)),\ \gamma\mu^{>}(e^{>},\beta'^{>})(s(\gamma))\right\rangle\\
=\ &\left\langle\int_{\mathcal{G}^{r(\gamma)}}\beta'^{<}(\gamma')\otimes\gamma'e^{<}(s(\gamma'))\,\mathrm{d}\lambda^{r(\gamma)}(\gamma'),\ \gamma\int_{\mathcal{G}^{s(\gamma)}}\gamma''e^{>}(s(\gamma''))\otimes\gamma''\beta'^{>}(\gamma''^{-1})\,\mathrm{d}\lambda^{s(\gamma)}(\gamma'')\right\rangle\\
=\ &\left\langle\int_{\mathcal{G}^{r(\gamma)}}\beta'^{<}(\gamma')\otimes\gamma'e^{<}(s(\gamma'))\,\mathrm{d}\lambda^{r(\gamma)}(\gamma'),\ \int_{\mathcal{G}^{r(\gamma)}}\gamma''e^{>}(s(\gamma''))\otimes\gamma''\beta'^{>}(\gamma''^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma'')\right\rangle\\
=\ &\int_{\mathcal{G}^{r(\gamma)}}\int_{\mathcal{G}^{r(\gamma)}}\left\langle\beta'^{<}(\gamma')\otimes\gamma'e^{<}(s(\gamma')),\ \gamma''e^{>}(s(\gamma''))\otimes\gamma''\beta'^{>}(\gamma''^{-1}\gamma)\right\rangle\ \mathrm{d}\lambda^{r(\gamma)}(\gamma')\,\mathrm{d}\lambda^{r(\gamma)}(\gamma'')\\
=\ &\int_{\mathcal{G}^{r(\gamma)}}\int_{\mathcal{G}^{r(\gamma'')}}\beta'^{<}(\gamma')\,\varphi\left(\left\langle\gamma'e^{<}(s(\gamma')),\ \gamma''e^{>}(s(\gamma''))\right\rangle\right)\ \gamma''\beta'^{>}(\gamma''^{-1}\gamma)\ \mathrm{d}\lambda^{r(\gamma'')}(\gamma')\,\mathrm{d}\lambda^{r(\gamma)}(\gamma'')\\
=\ &\int_{\mathcal{G}^{r(\gamma)}}\int_{\mathcal{G}^{r(\gamma'')}}\beta'^{<}(\gamma')\,\gamma'\varphi\left(\left\langle e^{<}(s(\gamma')),\ \gamma'^{-1}\gamma''e^{>}(s(\gamma''))\right\rangle\right)\,\mathrm{d}\lambda^{r(\gamma'')}(\gamma')\,\gamma''\beta'^{>}(\gamma''^{-1}\gamma)\ \mathrm{d}\lambda^{r(\gamma)}(\gamma'')\\
=\ &\int_{\mathcal{G}^{r(\gamma)}}\int_{\mathcal{G}^{r(\gamma'')}}\beta'^{<}(\gamma')\,\gamma'\varphi\left(\left\langle\!\left\langle e^{<},\ e^{>}\right\rangle\!\right\rangle(\gamma'^{-1}\gamma'')\right)\,\mathrm{d}\lambda^{r(\gamma'')}(\gamma')\,\gamma''\beta'^{>}(\gamma''^{-1}\gamma)\ \mathrm{d}\lambda^{r(\gamma)}(\gamma'')\\
=\ &\int_{\mathcal{G}^{r(\gamma)}}\int_{\mathcal{G}^{r(\gamma'')}}\beta'^{<}(\gamma')\,\gamma'\mathcal{A}(\mathcal{G},\varphi)\left(\left\langle\!\left\langle e^{<},\ e^{>}\right\rangle\!\right\rangle\right)(\gamma'^{-1}\gamma'')\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')\,\gamma''\beta'^{>}(\gamma''^{-1}\gamma)\ \mathrm{d}\lambda^{r(\gamma)}(\gamma'')\\
=\ &\int_{\mathcal{G}^{r(\gamma)}}\left(\beta'^{<}*\mathcal{A}(\mathcal{G},\varphi)\left(\left\langle\!\left\langle e^{<},\ e^{>}\right\rangle\!\right\rangle\right)\right)(\gamma'')\,\gamma''\beta'^{>}(\gamma''^{-1}\gamma)\ \mathrm{d}\lambda^{r(\gamma)}(\gamma'')\\
=\ &\left(\beta'^{<}*\mathcal{A}(\mathcal{G},\varphi)\left(\left\langle\!\left\langle e^{<},\ e^{>}\right\rangle\!\right\rangle\right)*\beta'^{>}\right)(\gamma)=\left\langle\beta'^{<}\otimes e^{<},\ e^{>}\otimes\beta'^{>}\right\rangle(\gamma),
\end{aligned}$$

for all $\gamma\in\mathcal{G}$, so $\left\langle\!\left\langle\Phi^{<}(\beta'^{<}\otimes e^{<}),\ \Phi^{>}(e^{>}\otimes\beta'^{>})\right\rangle\!\right\rangle=\left\langle\beta'^{<}\otimes e^{<},\ e^{>}\otimes\beta'^{>}\right\rangle$ for all $\beta'^{<},\beta'^{>}\in\Gamma_c(\mathcal{G},\ r^*B)$, $e^{<}\in\Gamma_c(X,E^{<})$ and $e^{>}\in\Gamma_c(X,E^{>})$.

2. Φ intertwines $\mathcal{D}(X,T)\otimes 1$ and $\mathcal{D}(X,T\otimes 1)$: For all $e^{>}\in\Gamma_c(X,E^{>})$ and all $\beta'\in\Gamma_c(\mathcal{G},\ r^*B')$,

we have

$$\begin{aligned}
\Phi^{>}\left(\left(\mathcal{D}(X,T)^{>}e^{>}\right)\otimes\beta'\right) &= \int_{\mathcal{G}^x}\gamma\left(T^{>}_{s(\gamma)}e^{>}(s(\gamma))\right)\otimes\gamma\beta'(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma)\\
&= \int_{\mathcal{G}^x}T^{>}_x\left(\gamma e^{>}(s(\gamma))\right)\otimes\gamma\beta'(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma)\\
&= \int_{\mathcal{G}^x}(T^{>}\otimes 1)_x\left(\gamma e^{>}(s(\gamma))\ \otimes\ \gamma\beta'(\gamma^{-1})\right)\,\mathrm{d}\lambda^x(\gamma)\\
&= (T^{>}\otimes 1)_x\left(\int_{\mathcal{G}^x}\gamma e^{>}(s(\gamma))\ \otimes\ \gamma\beta'(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma)\right)\\
&= \mathcal{D}\left(X,\ T\otimes 1\right)^{>}\ \left(\Phi^{>}(e^{>}\otimes\beta')\right).
\end{aligned}$$

A similar calculation can be done for the left-hand side.

3. Let S be a $\mathcal{G}$-equivariant and locally compact element of $\mathrm{L}_B(E)$ such that $\pi(\operatorname{supp} S)$ is a compact subset of $X/\mathcal{G}$. We are now going to approximate $\mathcal{D}(X,S)\otimes 1$ and $\mathcal{D}(X,\ S\otimes 1)$ simultaneously by finite rank operators. Let $\varepsilon > 0$. As in the proof of Proposition 7.2.19 and using, in addition, the non-degeneracy of the modules $\Gamma_c(X,F^{>})$ and $\Gamma_c(X,E^{<})$ we can find $n\in\mathbb{N}$, $e_1^{>},\ldots,e_n^{>}\in\Gamma_c(X,E^{>})$, $e_1^{<},\ldots,e_n^{<}\in\Gamma_c(X,E^{<})$ and $\beta_1^{<},\ldots,\beta_n^{<},\beta_1^{>},\ldots,\beta_n^{>}\in\Gamma_c(\mathcal{G},\ r^*B)$ such that

$$\left\|S_x-\sum_{i=1}^n\left|f_i^{>}\beta_i^{>}\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle\beta_i^{<}e_i^{<}\right|_x\right\|\leq\varepsilon$$

for all $x\in X$. It follows that

$$\left\|\mathcal{D}(X,S)-\sum_{i=1}^n\left|f_i^{>}\beta_i^{>}\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle\beta_i^{<}e_i^{<}\right|\right\|\leq\varepsilon$$

and hence $\mathcal{D}(X,S)\otimes 1$ can be approximated by

$$\sum_{i=1}^n\left|f_i^{>}\beta_i^{>}\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle\beta_i^{<}e_i^{<}\right|\otimes 1=\sum_{i=1}^n\left|f_i^{>}\otimes(\varphi\circ\beta_i^{>})\right\rangle\!\left\langle(\varphi\circ\beta_i^{<})\otimes e_i^{<}\right|$$

up to ε.

On the other hand, a long but straightforward calculation shows

$$\mathcal{D}\left(X,\ \left(\left|f_i^{>}\beta_i^{>}\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle\beta_i^{<}e_i^{<}\right|_x\otimes 1\right)_{x\in X}\right)=\left|\Phi^{>}(f_i^{>}\otimes(\varphi\circ\beta_i^{>}))\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle\Phi^{<}((\varphi\circ\beta_i^{<})\otimes e_i^{<})\right|$$

for all i, and because

$$\left\|S_x\otimes 1-\sum_{i=1}^n\left|f_i^{>}\beta_i^{>}\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle\beta_i^{<}e_i^{<}\right|_x\otimes 1\right\|\leq\varepsilon$$

for all $x\in X$, it follows that $\mathcal{D}(X,S\otimes 1)$ can be approximated by

$$\sum_{i=1}^n\left|\Phi^{>}(f_i^{>}\otimes(\varphi\circ\beta_i^{>}))\right\rangle\!\!\left\rangle\!\left\langle\!\!\left\langle\Phi^{<}((\varphi\circ\beta_i^{<})\otimes e_i^{<})\right|$$

up to ε. The homomorphism Φ intertwines these approximations.

Applying these considerations to $S=\chi(T^2-1)$ with $\chi\in\mathcal{C}_c(X/\mathcal{G})$ shows that Φ satisfies the technical conditions for a homomorphism to induce a homotopy, see Theorem 2.6.2. $\square$

As in Proposition 5.2.23 one proves:

Proposition 7.2.22. *Let B be a $\mathcal{G}$-Banach algebras. If $(E,T) \in \mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X), B[0,1])$ is a homotopy from (E_0,T_0) to (E_1,T_1) with T equivariant, then $J^B_{\mathcal{A},\mathcal{D}}(E_0,T_0)$ and $J^B_{\mathcal{A},\mathcal{D}}(E_1,T_1)$ are homotopic.*

Proposition 7.2.23. *Let B be a $\mathcal{G}$-Banach algebra. If $(E_1,T_1),(E_2,T_2) \in \mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X), B)$, then*

$$J^B_{\mathcal{A},\mathcal{D}}\left((E_1,T_1) \oplus (E_2,T_2)\right) \sim J^B_{\mathcal{A},\mathcal{D}}(E_1,T_1) \oplus J^B_{\mathcal{A},\mathcal{D}}(E_2,T_2).$$

Proof. We define a homomorphism $\Phi\colon J^B_{\mathcal{A},\mathcal{D}}\left((E_1,T_1) \oplus (E_2,T_2)\right) \to J^B_{\mathcal{A},\mathcal{D}}(E_1,T_1) \oplus J^B_{\mathcal{A},\mathcal{D}}(E_2,T_2)$: For all $e_1^> \in \Gamma_c(X, E_1^>)$ and $e_2^> \in \Gamma_c(X, E_2^>)$, define

$$\Phi^>(e_1^>, e_2^>)(x) := (e_1^>(x), e_2^>(x)), \quad x \in X.$$

Then $\Phi^>(e_1^>, e_2^>) \in \Gamma_c(X, E_1^> \oplus E_2^>)$. Now $\Phi^>(e_1^>, e_2^>) = (x \mapsto (e_1^>(x), 0)) + (x \mapsto (0, e_2^>(x)))$, so

$$\left\|\Phi^>(e_1^>, e_2^>)\right\|_{\mathcal{D}} \leq \left\|e_1^>\right\|_{\mathcal{D}} + \left\|e_2^>\right\|_{\mathcal{D}}.$$

So $\Phi^>$ can be extended to a contractive, $\mathbb{C}$-linear, $\mathcal{C}_0(X/\mathcal{G})$-linear and $\mathcal{A}(\mathcal{G}, B)$-linear map from $\mathcal{D}(X, E_1^>) \oplus \mathcal{D}(X, E_2^>)$ to $\mathcal{D}(X, E_1^> \oplus E_2^>)$. One can define a similar map $\Phi^<$ for the left-hand side and a short calculation shows that $\Phi = (\Phi^<, \Phi^>)$ is a homomorphism intertwining $\mathcal{D}(X,T_1) \oplus \mathcal{D}(X,T_2)$ and $\mathcal{D}(X, T_1 \oplus T_2)$.

Φ satisfies the conditions of Theorem 2.6.2: the first and the last condition are void, the second is satisfied because $\Phi^>$ and $\Phi^<$ are bijective with continuous inverse (with norm ≤ 2). So Φ induces a homotopy. $\square$

Proposition 7.2.24. *The map $(E,T) \mapsto (\mathcal{D}(X,E),\ \mathcal{D}(X,T))$ gives rise to a group-homomorphism from*

$$J^B_{\mathcal{A},\mathcal{D}}\colon \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathcal{C}_0(X),\ B\right) \to \mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0\left(X/\mathcal{G}\right);\ \mathcal{C}_0\left(X/\mathcal{G}\right),\ \mathcal{A}\left(\mathcal{G}, B\right)\right)$$

which is natural in the non-degenerate $\mathcal{G}$-Banach algebra B.

Definition and Proposition 7.2.25. Let $\mathcal{D}'(X) = (\mathcal{D}'^<(X),\ \mathcal{D}'^>(X))$ be another pair of monotone completions of $\mathcal{C}_c(X)$, compatible with $\mathcal{A}(\mathcal{G})$. Then $J^B_{\mathcal{A},\mathcal{D}} = J^B_{\mathcal{A},\mathcal{D}'}$ as homomorphisms from $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathcal{C}_0(X),\ B\right)$ to $\mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0\left(X/\mathcal{G}\right);\ \mathcal{C}_0\left(X/\mathcal{G}\right),\ \mathcal{A}\left(\mathcal{G}, B\right)\right)$. We hence write $J^B_{\mathcal{A}}$ for this homomorphism.

Proof. We first consider the case that $\|\cdot\|_{\mathcal{D}^<} \leq \|\cdot\|_{\mathcal{D}'^<}$ and $\|\cdot\|_{\mathcal{D}^>} \leq \|\cdot\|_{\mathcal{D}'^>}$. Let (E,T) be a cycle in $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathcal{C}_0(X),\ B\right)$ with $\mathcal{G}$-equivariant T. Then there is a canonical homomorphism $\Phi = (\Phi^<, \Phi^>)$ from $\mathcal{D}'(X,E)$ to $\mathcal{D}(X,E)$ which intertwines $\mathcal{D}'(X,T)$ and $\mathcal{D}(X,T)$. Let $\chi \in \mathcal{C}_c(X/\mathcal{G})$ and define $S := \chi\left(T^2 - 1\right)$. The proof of Proposition 7.2.19 also shows that $(\mathcal{D}'(X,S),\ \mathcal{D}(X,S)) \in \mathrm{K}_{\mathrm{Id}_{\mathcal{A}(\mathcal{G},B)}}(\Phi)$, so $J^B_{\mathcal{A},\mathcal{D}'}(E,T)$ is homotopic to $J^B_{\mathcal{A},\mathcal{D}}(E,T)$ by our sufficient condition for homotopy.

Now let $\mathcal{D}'$ be a general pair of monotone completions of $\mathcal{C}_c(X)$ compatible with $\mathcal{A}(\mathcal{G})$. Define $\|\chi\|_{\mathcal{D}''^<} := \max\{\|\chi\|_{\mathcal{D}^<},\ \|\chi\|_{\mathcal{D}'^<}\}$ and $\|\chi\|_{\mathcal{D}''^>} := \max\{\|\chi\|_{\mathcal{D}^>},\ \|\chi\|_{\mathcal{D}'^>}\}$ for all $\chi \in \mathcal{C}_c(X)$. Then $\mathcal{D}''(X) := (\mathcal{D}''^<(X),\ \mathcal{D}''^>(X))$ is also a pair of monotone completions of $\mathcal{C}_c(X)$ compatible with $\mathcal{A}(\mathcal{G})$. By the first part of the proof we have $J^B_{\mathcal{A},\mathcal{D}} = J^B_{\mathcal{A},\mathcal{D}''} = J^B_{\mathcal{A},\mathcal{D}'}$. $\square$

Existence of compatible pairs of monotone completions

Now that we have seen how compatible pairs of monotone completions can be used to construct the homomorphism $J_{\mathcal{A}}^B$ the natural question is of course whether such pairs exist. We now show that this is the case if $\mathcal{G}$ admits a cut-off function. There are even quite a few of them, for every cut-off pair c we construct a compatible pair of monotone completions that we call $\mathcal{A}^c(X)$. Although the homomorphism $J_{\mathcal{A}}^B$ does not depend on the particular choice of c (as shown above) we are going to need the precise form of the construction later on when we specify certain cut-off pairs to be able to perform calculations on the level of cycles.

So let $c = (c^<, c^>)$ be a cut-off pair for $\mathcal{G}$.

Definition 7.2.26. Let E be a $\mathcal{G}$-Banach B-pair. Define

$$j_{E,c}^<\colon \Gamma_c\left(X, E^<\right) \to \Gamma_c\left(\mathcal{G},\ E^<\right),\ e^< \mapsto \left(\gamma \mapsto c^<(s(\gamma))e^<(r(\gamma))\right)$$

and

$$j_{E,c}^>\colon \Gamma_c\left(X, E^>\right) \to \Gamma_c\left(\mathcal{G},\ E^>\right),\ e^> \mapsto \left(\gamma \mapsto c^>(r(\gamma))\gamma e^>(s(\gamma))\right).$$

One can think of $j_{E,c}^<(e^<)$ as $e^< * c^<$ and of $j_{E,c}^>(e^>)$ as $c^> * e^>$.

The following proposition can be checked by direct calculation.

Proposition 7.2.27. *Let E be a $\mathcal{G}$-Banach B-pair. Then $j_{E,c} = (j_{E,c}^<,\ j_{E,c}^>)$ is a pair of injective maps such that*

1. *$j_{E,c}^<$ is $\mathbb{C}$-linear, $\Gamma_c\left(X/\mathcal{G}\right)$-linear and $\Gamma_c\left(\mathcal{G},\ r^*B\right)$-linear on the left,*
2. *$j_{E,c}^>$ is $\mathbb{C}$-linear, $\Gamma_c\left(X/\mathcal{G}\right)$-linear and $\Gamma_c\left(\mathcal{G},\ r^*B\right)$-linear on the right,*
3. *for all $e^< \in \Gamma_c(X, E^<)$ and $e^> \in \Gamma_c(X, E^>)$, we have*

$$\left\langle j_{E,c}^<(e^<),\ j_{E,c}^>(e^>)\right\rangle_{\Gamma_c(\mathcal{G},r^*B)} = \langle\langle e^<,\ e^>\rangle\rangle.$$

Proposition 7.2.28. *Let E and F be $\mathcal{G}$-Banach B-pairs. Let $T = (T^<, T^>)$ be a $\mathcal{G}$-*equivariant *field of operators from E to F. Then*

$$j_{F,c}^>(T^>e^>) = T^> * j_{E,c}^>(e^>) \quad \textit{and} \quad j_{E,c}^<(f^<T^<) = j_{F,c}^<(f^<) * T^<$$

for all $e^> \in \Gamma_c(X, E^>)$ and $f^< \in \Gamma_c(X, F^<)$.

Proof. We just show this for the right-hand side: Let $e^> \in \Gamma_c(X, E^>)$ and $\gamma \in \mathcal{G}$. Then

$$\begin{aligned} j_{F,c}^>(T^>e^>)\,(\gamma) &= c^>(r(\gamma))\,\gamma(T^>e^>)(s(\gamma)) = c^>(r(\gamma))\,\gamma\left(T_{s(\gamma)}^>e^>(s(\gamma))\right) \\ &= c^>(r(\gamma))\,T_{r(\gamma)}^>\left(\gamma e^>(s(\gamma))\right) = \left(T^> * j_{E,c}^>(e^>)\right)(\gamma). \end{aligned}$$

□

Definition 7.2.29 ($\mathcal{A}^c(X, E)$). Let B be a $\mathcal{G}$-Banach algebra and let E be a $\mathcal{G}$-Banach B-pair. Define a $\mathcal{C}_0\left(X/\mathcal{G}\right)$-Banach $\mathcal{A}(\mathcal{G}, B)$-pair $\mathcal{A}^c(X, E) = (\mathcal{A}^c(X, E^<),\ \mathcal{A}^c(X, E^>))$ by pulling back the norm of $\mathcal{A}(\mathcal{G},E)$ along $j_{E,c}$ and completing $\Gamma_c(X,E)$ for this norm. Alternatively, one could take the closure of the image of $j_{E,c}$. In particular, the norms on the left and the right part are given by

$$\left\|e^<\right\|_{\mathcal{A}^c(X,E^<)} := \left\|j^<_{E,c}(e^<)\right\|_{\mathcal{A}(\mathcal{G},E^<)} = \left\| \gamma \mapsto c^<(s(\gamma)) \left\|e^<(r(\gamma))\right\| \right\|_{\mathcal{A}}$$

and

$$\left\|e^>\right\|_{\mathcal{A}^c(X,E^>)} := \left\|j^>_{E,c}(e^>)\right\|_{\mathcal{A}(\mathcal{G},E^>)} = \left\| \gamma \mapsto c^>(r(\gamma)) \left\|e^>(s(\gamma))\right\| \right\|_{\mathcal{A}}$$

for all $e^< \in \Gamma_c(X,E^<)$ and $e^> \in \Gamma_c(X,E^>)$.

Note that the norms depend on $\mathcal{A}(\mathcal{G})$ as well as on c. The pair $\mathcal{A}^c(X) = ((\mathcal{A}^c)^<(X),\ (\mathcal{A}^c)^>(X))$ is a pair of monotone completions of $\mathcal{C}_c(X)$ compatible with $\mathcal{A}(\mathcal{G})$ and the notation $\mathcal{A}^c(X,E)$ is unambiguous. If $\mathcal{A}(\mathcal{G})$ is locally $\mathcal{C}_0\,(X/\mathcal{G})$-convex, then $\mathcal{A}^c(X,E)$ is locally $\mathcal{C}_0\,(X/\mathcal{G})$-convex. Note that $J^B_{\mathcal{A},\mathcal{A}^c}$ as a homomorphism from $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X),B)$ to $\mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0\,(X/\mathcal{G})\,;\ \mathcal{C}_0\,(X/\mathcal{G})\,,\ \mathcal{A}\,(\mathcal{G},B)\right)$ does not depend on c by 7.2.25; without the detour via more general compatible pairs $\mathcal{D}(X)$ of monotone completions this latter fact seems to be hard to prove.

7.3 Monotone completions as analogues of $\mathrm{L}^2(\mathcal{G},B)$

As sketched in the introduction to this chapter, a possible proof of the C*-algebra version of the generalised Green-Julg theorem makes use of the tensor product with the $\mathcal{G}$-Hilbert B-module $\mathrm{L}^2(\mathcal{G},B)$ which carries a left action of $B \rtimes_r \mathcal{G}$ by locally compact operators. We want to find analogues of the module $\mathrm{L}^2(\mathcal{G},B)$ for the case that B is a $\mathcal{G}$-Banach algebra. Apparently, if B is a $\mathcal{G}$-Banach algebra, it is not sufficient (or not systematic, at least) to just consider pairs of the type $(\mathrm{L}^2(\mathcal{G},B),\ \mathrm{L}^2(\mathcal{G},B))$; we want to treat rather general unconditional completions, so it seems advisable to consider rather general completions of the space $\Gamma_c(\mathcal{G},\ r^*B)$, and our treatment should also cover pairs of the form $(\mathrm{L}^1(\mathcal{G},B),\ \Gamma_0(\mathcal{G},\ B))$ or $(\mathrm{L}^{p'}(\mathcal{G},B),\ \mathrm{L}^p(\mathcal{G},B))$ for $p,p' \in]1,\infty[$ with $1/p + 1/p' = 1$ (compare the precise definitions below).

Our substitute for $\mathrm{L}^2(\mathcal{G})$ is what we call (not very imaginatively) a pair of monotone completions; we will usually denote such a pair by $\mathcal{H}(\mathcal{G})$, and write $\mathcal{H}(\mathcal{G},B)$ for its version with coefficients in B. It seems advisable to even consider pairs of the form $\mathcal{H}(\mathcal{G},E)$ where E is a $\mathcal{G}$-Banach B-pair because this makes the constructions a bit clearer. The important result is that (under certain compatibility conditions) the unconditional completion $\mathcal{A}(\mathcal{G},B)$ acts on $\mathcal{H}(\mathcal{G},B)$ by locally compact operators. To prove this, we need a result concerning the compactness of operators which are given by kernels which is presented in detail in Appendix E.8.

7.3.1 Pairs of monotone completions of $\mathcal{C}_c(\mathcal{G})$

Recall that in this chapter $\mathcal{G}$ denotes a locally compact Hausdorff groupoid with left Haar system λ and X denotes the unit space of $\mathcal{G}$.

Definition 7.3.1 (Pair of monotone completions ($\mathcal{H}(\mathcal{G})$)). A pair of monotone completions of $\mathcal{C}_c(\mathcal{G})$ is a pair $\mathcal{H}(\mathcal{G}) = (\mathcal{H}^<(\mathcal{G}),\ \mathcal{H}^>(\mathcal{G}))$ such that $\mathcal{H}^<(\mathcal{G})$ and $\mathcal{H}^>(\mathcal{G})$ are monotone completions of $\mathcal{C}_c(\mathcal{G})$ and such that the bilinear map

$$\langle\cdot,\cdot\rangle_{\mathcal{C}_c(X)}\colon \mathcal{C}_c(\mathcal{G}) \times \mathcal{C}_c(\mathcal{G}) \to \mathcal{C}_c(X),\ (\varphi^<,\varphi^>) \mapsto \left(x \mapsto \int_{\mathcal{G}^x} \varphi^<(\gamma)\,\varphi^>(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma)\right)$$

satisfies

$$\left\|\langle\varphi^<,\ \varphi^>\rangle_{\mathcal{C}_c(X)}\right\|_\infty \leq \left\|\varphi^<\right\|_{\mathcal{H}^<}\left\|\varphi^>\right\|_{\mathcal{H}^>}$$

for all $\varphi^<, \varphi^> \in \mathcal{C}_c(\mathcal{G})$. In this case $\langle\cdot,\cdot\rangle_{\mathcal{C}_c(X)}$ can be extended to a continuous bilinear map $\langle\cdot,\cdot\rangle_{\mathcal{C}_0(X)}\colon \mathcal{H}^<(\mathcal{G}) \times \mathcal{H}^>(\mathcal{G}) \to \mathcal{C}_0(X)$ which is $\mathcal{C}_0(X)$-bilinear if we consider the following actions of $\mathcal{C}_0(X)$:

$$(\chi\xi^<)(\gamma) := \chi(r(\gamma))\xi^<(\gamma) \quad \text{and} \quad (\xi^>\chi)(\gamma) := \xi^>(\gamma)\chi(s(\gamma))$$

for all $\chi \in \mathcal{C}_0(X)$, $\xi^< \in \mathcal{C}_c(\mathcal{G}) \subseteq \mathcal{H}^<(\mathcal{G})$, $\xi^> \in \mathcal{C}_c(\mathcal{G}) \subseteq \mathcal{H}^>(\mathcal{G})$ and $\gamma \in \mathcal{G}$.

Examples 7.3.2. Let $p \in [1, \infty[$. Define the norm

$$\left\|\chi^<\right\|_{p,r} := \sup_{x\in X}\left(\int_{\mathcal{G}^x}\left|\chi^<(\gamma)\right|^p \,\mathrm{d}\lambda^x(\gamma)\right)^{\frac{1}{p}}$$

for all $\chi^< \in \mathcal{C}_c(\mathcal{G})$. The corresponding monotone completion is called $\mathrm{L}^p_r(\mathcal{G})$. Note that $\mathrm{L}^1(\mathcal{G}) = \mathrm{L}^1_r(\mathcal{G})$. Secondly, define

$$\left\|\chi^>\right\|_{p,s} := \sup_{x\in X}\left(\int_{\mathcal{G}^x}\left|\chi^>(\gamma^{-1})\right|^p \,\mathrm{d}\lambda^x(\gamma)\right)^{\frac{1}{p}}$$

for all $\chi^> \in \mathcal{C}_c(\mathcal{G})$. The corresponding monotone completion is called $\mathrm{L}^p_s(\mathcal{G})$

1. The pairs $(\mathrm{L}^1(\mathcal{G}),\ \mathcal{C}_0(\mathcal{G}))$ and $(\mathcal{C}_0(\mathcal{G}),\ \mathrm{L}^1_s(\mathcal{G}))$ are pairs of monotone completions in the above sense.

2. If $p, p' \in]1, \infty[$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, then $(\mathrm{L}^{p'}_r(\mathcal{G}),\ \mathrm{L}^p_s(\mathcal{G}))$ is also a pair of monotone completions.

3. In particular this applies to $(\mathrm{L}^2_r(\mathcal{G}),\ \mathrm{L}^2_s(\mathcal{G}))$.

Definition 7.3.3 (The pair $\mathcal{H}(\mathcal{G}, E)$). Let B be a $\mathcal{G}$-Banach algebra and let E be a $\mathcal{G}$-Banach B-pair. Let $\mathcal{H}(\mathcal{G})$ be a pair of monotone completions of $\mathcal{C}_c(\mathcal{G})$. Define a right action of $\Gamma(X, B)$ on $\Gamma_c(\mathcal{G},\ r^*E^>)$ by

$$(\xi^>\beta)(\gamma) := \xi^>(\gamma)\gamma\beta(s(\gamma)), \quad \xi^> \in \Gamma_c(\mathcal{G}, r^*E^>), \beta \in \Gamma(X, B), \gamma \in \mathcal{G},$$

and a left action of $\Gamma(X, B)$ on $\Gamma_c(\mathcal{G}, r^*E^<)$ by

$$(\beta\xi^<)(\gamma) := \beta(r(\gamma))\xi^<(\gamma), \quad \beta \in \Gamma(X, B), \xi^< \in \Gamma_c(\mathcal{G},\ r^*E^<), \gamma \in \mathcal{G}.$$

These actions define continuous actions of $\Gamma_0(X, B)$ on $\mathcal{H}^>(\mathcal{G}, E^>)$ (from the right) and $\mathcal{H}^<(\mathcal{G}, E^<)$ (from the left). Define a bilinear map

$$\begin{aligned}
\langle\cdot,\cdot\rangle_{\Gamma_c(X,B)}\colon \Gamma_c(\mathcal{G},\ r^*E^<) \times \Gamma_c(\mathcal{G},\ r^*E^>) &\to \Gamma_c(X, B),\\
(\xi^<, \xi^>) &\mapsto \left(x \mapsto \int_{\mathcal{G}^x}\left\langle\xi^<(\gamma),\ \gamma\xi^>(\gamma^{-1})\right\rangle_{E_{r(\gamma)}} \mathrm{d}\lambda^x(\gamma)\right).
\end{aligned}$$

This map extends to a contractive bracket from $\mathcal{H}^<(\mathcal{G}, E^<) \times \mathcal{H}^>(\mathcal{G}, E^>)$ to $\Gamma_0(X, B)$ which makes $\mathcal{H}(\mathcal{G}, E) := (\mathcal{H}^<(\mathcal{G}, E^<), \mathcal{H}^>(\mathcal{G}, E^>))$ a $\mathcal{C}_0(X)$-Banach $\Gamma_0(X, B)$-pair. If E is non-degenerate, then so is $\mathcal{H}(\mathcal{G}, E)$.

Proof. We just check that the bracket is bilinear to make sure that we have adjusted the definition of the actions of $\Gamma_0(X, B)$ correctly: Let $\beta \in \Gamma_0(X, B)$, $\xi^< \in \Gamma_c(\mathcal{G},\ r^*E^<)$ and $\xi^> \in \Gamma_c(\mathcal{G},\ r^*E^>)$. Then

$$\left\langle \beta\xi^<,\ \xi^> \right\rangle\ (x) = \int_{\mathcal{G}^x} \left\langle (\beta\xi^<)(\gamma),\ \gamma\xi^>(\gamma^{-1}) \right\rangle_{E_{r(\gamma)}}\ \mathrm{d}\lambda^x(\gamma)$$

$$= \int_{\mathcal{G}^x} \left\langle \beta(x)\ \xi^<(\gamma),\ \gamma\xi^>(\gamma^{-1}) \right\rangle_{E_{r(\gamma)}}\ \mathrm{d}\lambda^x(\gamma) = \beta(x)\ \left\langle \xi^<,\ \xi^> \right\rangle (x)$$

and

$$\begin{aligned}\left\langle \xi^<,\ \xi^>\beta \right\rangle\ (x) &= \int_{\mathcal{G}^x} \left\langle \xi^<(\gamma),\ \gamma\left((\xi^>\beta)(\gamma^{-1})\right) \right\rangle_{E_{r(\gamma)}}\ \mathrm{d}\lambda^x(\gamma) \\ &= \int_{\mathcal{G}^x} \left\langle \xi^<(\gamma),\ \gamma\left(\xi^>(\gamma^{-1})\gamma^{-1}\beta\left(s(\gamma^{-1})\right)\right) \right\rangle_{E_{r(\gamma)}}\ \mathrm{d}\lambda^x(\gamma) \\ &= \int_{\mathcal{G}^x} \left\langle \xi^<(\gamma),\ \left(\gamma\xi^>(\gamma^{-1})\right)\beta\,(x) \right\rangle_{E_{r(\gamma)}}\ \mathrm{d}\lambda^x(\gamma) = \left\langle \xi^<,\ \xi^> \right\rangle (x)\ \beta(x)\end{aligned}$$

for all $x \in X$. □

Note that in the preceding definition, the $\mathcal{C}_0(X)$-structures on $\mathcal{H}^<(\mathcal{G}, E^<)$ and $\mathcal{H}^>(\mathcal{G}, E^>)$ are not the same in general: on the left-hand side, it is induced by the range map r, on the right-hand side by the source map s. This implies that the fibre of $\mathcal{H}^<(\mathcal{G}, E^<)$ over some $x \in X$ should be regarded as a completion of $\Gamma_c(\mathcal{G}^x, E^<_x)$, whereas the fibre of $\mathcal{H}^>(\mathcal{G}, E^>)$ over x should be regarded as a completion of $\Gamma_c(\mathcal{G}_x,\ (r^*E)|_{\mathcal{G}_x})$; compare Proposition E.8.5. The difference can of course be remedied by the application of the pullback along the inversion of the groupoid (we formulate this as a general statement about a single monotone completion of $\mathcal{C}_c(\mathcal{G})$ instead of a pair):

Lemma 7.3.4. *Let $\mathcal{H}(\mathcal{G})$ be a monotone completion of $\mathcal{C}_c(\mathcal{G})$. Then also the semi-norm $\|\varphi\|_{\check{\mathcal{H}}} := \left\|\gamma \mapsto \varphi(\gamma^{-1})\right\|_{\mathcal{H}}$ is a monotone semi-norm on $\mathcal{C}_c(\mathcal{G})$. The map $\varphi \mapsto (\gamma \mapsto \varphi(\gamma^{-1}))$ induces an isometric isomorphism from the Banach space $\mathcal{H}(\mathcal{G})$ to $\check{\mathcal{H}}(\mathcal{G})$. It is an isomorphism of $\mathcal{C}_0(X)$-Banach spaces if we take on $\mathcal{H}(\mathcal{G})$ the $\mathcal{C}_0(X)$-action induced by r and on $\check{\mathcal{H}}(\mathcal{G})$ the action induced by s (or vice versa).*

Note that if $\mathcal{H}(\mathcal{G})$ is a pair of monotone completions and if we put the $\mathcal{C}_0(X)$-structure which is induced by the range map on both sides, then the bracket of $\mathcal{H}(\mathcal{G}, E)$ no longer has the shape of a restricted convolution. It thus seems to be more systematic to have different $\mathcal{C}_0(X)$-structures on both sides of the pair.

7.3.2 $\mathcal{A}(\mathcal{G})$ acting on pairs of monotone completions of $\mathcal{C}_c(\mathcal{G})$

Recall that $\mathcal{G}$ denotes a locally compact Hausdorff groupoid with left Haar system λ and X denotes the unit space of $\mathcal{G}$. Let $\mathcal{A}(\mathcal{G})$ be an unconditional completion of $\mathcal{C}_c(\mathcal{G})$.

Definition 7.3.5 ($\mathcal{A}(\mathcal{G})$ acting on $\mathcal{H}(\mathcal{G})$). $\mathcal{A}(\mathcal{G})$ is said to *act on a pair* $\mathcal{H}(\mathcal{G}) = (\mathcal{H}^<(\mathcal{G}),\ \mathcal{H}^>(\mathcal{G}))$ of monotone completions of $\mathcal{C}_c(\mathcal{G})$

$$\left\|\chi * \xi^>\right\|_{\mathcal{H}^>(\mathcal{G})} \leq \|\chi\|_{\mathcal{A}(\mathcal{G})} \left\|\xi^>\right\|_{\mathcal{H}^>(\mathcal{G})}$$

and

$$\left\|\xi^< * \chi\right\|_{\mathcal{H}^<(\mathcal{G})} \leq \left\|\xi^<\right\|_{\mathcal{H}^<(\mathcal{G})} \|\chi\|_{\mathcal{A}(\mathcal{G})}$$

for all $\chi, \xi^<, \xi^> \in \mathcal{C}_c(\mathcal{G})$.

Definition and Proposition 7.3.6 ($\mathcal{A}(\mathcal{G}, A)$ acting on $\mathcal{H}(\mathcal{G}, E)$). Let $\mathcal{A}(\mathcal{G})$ act on the pair of monotone completions $\mathcal{H}(\mathcal{G})$. Let A and B be $\mathcal{G}$-Banach algebras and let E be a $\mathcal{G}$-Banach A-B-pair. For all $a \in \Gamma_c(\mathcal{G},\ r^*A)$, all $\xi^< \in \Gamma_c(\mathcal{G},\ r^*E^<)$ and all $\xi^> \in \Gamma_c(\mathcal{G},\ r^*E^>)$, define

$$(a\,\xi^>)(\gamma) = (a * \xi^>)(\gamma) = \int_{\mathcal{G}^{r(\gamma)}} a(\gamma')\gamma'\xi^>(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')$$

and

$$(\xi^<\,a)(\gamma) = (\xi^< * a)(\gamma) = \int_{\mathcal{G}^{r(\gamma)}} \xi^<(\gamma')\gamma' a(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')$$

for all $\gamma \in \mathcal{G}$. These actions lift to actions of $\mathcal{A}(\mathcal{G}, A)$ on $\mathcal{H}^>(\mathcal{G}, E^>)$ and $\mathcal{H}^<(\mathcal{G}, E^<)$, respectively. Equipped with them, $\mathcal{H}(\mathcal{G}, E)$ (as defined in 7.3.3) becomes a $\mathcal{C}_0(X)$-Banach $\Gamma_0(X, B)$-pair on which $\mathcal{A}(\mathcal{G}, A)$ by elements of $\mathrm{L}^{\mathcal{C}_0(X)}_{\Gamma_0(X,B)}(\mathcal{H}(\mathcal{G}, E))$.

Proof. Let $a \in \Gamma_c(\mathcal{G},\ r^*A)$, $\xi^> \in \Gamma_c(\mathcal{G},\ r^*E^>)$, and $\beta \in \Gamma(X, B)$. Then

$$\begin{aligned}
((a * \xi^>)\beta)(\gamma) &= (a * \xi^>)(\gamma)\gamma\beta(s(\gamma)) \\
&= \int_{\mathcal{G}^{r(\gamma)}} a(\gamma')\gamma'\xi^>(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')\gamma\beta(s(\gamma)) \\
&= \int_{\mathcal{G}^{r(\gamma)}} a(\gamma')\gamma'\xi^>(\gamma'^{-1}\gamma)\gamma'(\gamma'^{-1}\gamma)\beta(s(\gamma'^{-1}\gamma))\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= (a * (\xi^>\beta))(\gamma)
\end{aligned}$$

for all $\gamma \in \mathcal{G}$. This shows that $\Gamma_c(\mathcal{G},\ r^*E^>)$ is a $\Gamma_c(\mathcal{G},\ r^*A)$-$\Gamma(X, B)$-bimodule. Similarly one shows that $\Gamma_c(\mathcal{G},\ r^*E^<)$ is a $\Gamma(X, B)$-$\Gamma_c(\mathcal{G},\ r^*A)$-bimodule. Because the actions are given by convolution and also the bracket is given by (the restriction of) convolution, it is easy to see that $\langle \xi^<,\ a\xi^>\rangle_{\Gamma_c(X,B)} = \langle \xi^< a,\ \xi^>\rangle_{\Gamma_c(X,B)}$.

Because $\mathcal{A}(\mathcal{G})$ acts on $\mathcal{H}(\mathcal{G})$, we have

$$\left\|a\xi^>\right\|_{\mathcal{H}^>} \leq \|a\|_{\mathcal{A}} \left\|\xi^>\right\|_{\mathcal{H}^>} \quad \text{and} \quad \left\|\xi^< a\right\|_{\mathcal{H}^<} \leq \left\|\xi^<\right\|_{\mathcal{H}^<} \|a\|_{\mathcal{A}}$$

and the actions on the sections with compact support lift to actions on the completions. Moreover, these actions are surely by $\mathcal{C}_0(X)$-linear operators. □

Proposition 7.3.7. *Let $\mathcal{H}(\mathcal{G}) = (\mathcal{H}^<(\mathcal{G}),\ \mathcal{H}^>(\mathcal{G}))$ be a pair of monotone completions of $\mathcal{C}_c(\mathcal{G})$ on which $\mathcal{A}(\mathcal{G})$ acts. Let A and B be $\mathcal{G}$-Banach algebras and let E be a $\mathcal{G}$-Banach A-B-pair. If $\Gamma(X, A)$ acts on E* by locally compact operators *and $\mathcal{G}$ is* proper*, then $\mathcal{A}(\mathcal{G}, A)$ acts on $\mathcal{H}(\mathcal{G}, E)$ by* locally compact operators*.*

Proof. Let $a \in \Gamma_c(\mathcal{G},\ r^*A)$. If we can show that the action of a on $\mathcal{H}(\mathcal{G}, E)$, denoted by $\pi(a) \in \mathrm{L}_{\Gamma_0(X,B)}(\mathcal{H}(\mathcal{G}, E))$, is locally compact, then we are done. Let $\chi \in \mathcal{C}_c(X)$. We have to show that $\chi\pi(a)$ is compact. Define

$$k_{(\gamma_1,\gamma_2)} := \chi(s(\gamma_1))\pi_A(a(\gamma_2)) \in \mathrm{L}_{B_{r(\gamma_1)}}\left(E_{r(\gamma_1)}\right)$$

for all $(\gamma_1, \gamma_2) \in \mathcal{G} *_{r,r} \mathcal{G}$. Then the action of $\chi\pi(a)$ on $\Gamma_c\left(\mathcal{G},\ r^*E^>\right)$ is given by

$$\begin{aligned}
(\chi\pi(a))^>(\xi^>)(\gamma) &= \chi(s(\gamma)) \int_{\mathcal{G}^{r(\gamma)}} a(\gamma')\gamma'\xi^>(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= \int_{\mathcal{G}^{r(\gamma)}} \chi(s(\gamma))a(\gamma')\gamma'\xi^>(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= \int_{\mathcal{G}^{r(\gamma)}} k_{(\gamma,\gamma')}\gamma'\xi^>(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')
\end{aligned}$$

for all $\xi^> \in \Gamma_c(\mathcal{G},\ r^*E^>)$ and $\gamma \in \mathcal{G}$.

On the left-hand side, for $\xi^< \in \Gamma_c(\mathcal{G},\ r^*E^<)$, we calculate:

$$\begin{aligned}(\chi\pi(a))^<(\xi^<)(\gamma) &= \chi(r(\gamma))\int_{\mathcal{G}^{r(\gamma)}} \xi^<(\gamma')\gamma' a(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\ &= \int_{\mathcal{G}^{r(\gamma)}} \xi^<(\gamma')\chi(r(\gamma'))\gamma' a(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\ &= \int_{\mathcal{G}^{r(\gamma)}} \xi^<(\gamma')\gamma' k_{(\gamma'^{-1},\gamma'^{-1}\gamma)}\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')\end{aligned}$$

for all $\xi^> \in \Gamma_c(\mathcal{G},\ r^*E^>)$ and $\gamma \in \mathcal{G}$.

The field of operators $(\pi_A(a(\gamma_2)))_{(\gamma_1,\gamma_2)\in\mathcal{G}*_{r,r}\mathcal{G}}$ is locally compact, so the same is true for k. Moreover, the support of k is compact: Since $\mathcal{G}$ is proper, the set $K := \{\gamma \in \mathcal{G} :\ r(\gamma) \in \operatorname{supp}\chi,\ s(\gamma) \in r(\operatorname{supp} a)\}$ is compact. Let $(\gamma_1, \gamma_2) \in \mathcal{G} *_{r,r} \mathcal{G}$. Then $k_{(\gamma_1,\gamma_2)} \neq 0$ implies $\gamma_1 \in K$ and $\gamma_2 \in \operatorname{supp} a$. So (γ_1, γ_2) is contained in $K \times \operatorname{supp} a$. Hence k has compact support. Now the proposition can be deduced from the following lemma. □

Lemma 7.3.8 (Operators given by kernels). *Let $\mathcal{H}(\mathcal{G}) = (\mathcal{H}^<(\mathcal{G}),\ \mathcal{H}^>(\mathcal{G}))$ be a pair of monotone completions of $\mathcal{C}_c(\mathcal{G})$. Let B be a $\mathcal{G}$-Banach algebra and let E be a $\mathcal{G}$-Banach B-pair. Let $k \in \mathrm{L}_{p^*B}(p^*E)$ be a continuous field of operators with compact support, where $p\colon \mathcal{G} *_{r,r} \mathcal{G} \to \mathcal{G}^{(0)} = X,\ (\gamma_1,\gamma_2) \mapsto r(\gamma_1) = r(\gamma_2)$. Define an operator T_k on $\mathcal{H}(\mathcal{G}, E)$ by*

$$T_k^>(\xi^>)(\gamma) := \int_{\mathcal{G}^{r(\gamma)}} k^>_{(\gamma,\gamma')}\gamma'\xi^>(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')$$

and

$$T_k^<(\xi^<)(\gamma) := \int_{\mathcal{G}^{r(\gamma)}} \gamma' k^<_{(\gamma'^{-1},\gamma'^{-1}\gamma)}\,\xi^<(\gamma')\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')$$

for all $\xi^> \in \Gamma_c(\mathcal{G},\ r^*E^>)$, $\xi^< \in \Gamma_c(\mathcal{G},\ r^*E^<)$ and $\gamma \in \mathcal{G}$.

If k is compact then T_k is compact.

Proof. This is Lemma E.8.12 in disguise. On the surface, the formulae in that lemma look different, but this is a consequence of the fact that we have altered the definition of $\mathcal{H}(\mathcal{G}, E)$ by taking a different but equivalent bracket. □

As a corollary of Proposition 7.3.7 we get:

Corollary 7.3.9. *Let $\mathcal{H}(\mathcal{G}) = (\mathcal{H}^<(\mathcal{G}),\ \mathcal{H}^>(\mathcal{G}))$ be a pair of monotone completions of $\mathcal{C}_c(\mathcal{G})$ on which $\mathcal{A}(\mathcal{G})$ acts. Let B be a non-degenerate $\mathcal{G}$-Banach algebra. If $\mathcal{G}$ is proper, then $\mathcal{A}(\mathcal{G}, B)$ acts on $\mathcal{H}(\mathcal{G}, B)$ by locally compact operators.*

7.3.3 $\mathcal{G}$ acting on pairs of monotone completion of $\mathcal{C}_c(\mathcal{G})$

If we are given a pair $\mathcal{H}(\mathcal{G})$ of monotone completions in the above sense and a $\mathcal{G}$-Banach B-pair E, then we want to put an action of $\mathcal{G}$ on $\mathcal{H}(\mathcal{G}, E)$. Technically, we have to replace $\mathcal{H}(\mathcal{G}, E)$ with the u.s.c. field $\mathfrak{F}(\mathcal{H}(\mathcal{G}, E))$ of pairs over X, so it is a natural to assume that all the $\mathcal{C}_0(X)$-Banach spaces that appear are locally $\mathcal{C}_0(X)$-convex. Moreover, we have to make sure that the $\mathcal{G}$-action that we define is isometric. We hence formulate the following definition:

Definition 7.3.10 ((Locally convex, equivariant) pair of monotone completions). Let $\mathcal{H}(\mathcal{G})$ be a pair of monotone completions. Then $\mathcal{H}(\mathcal{G})$ is called *locally convex* if $\mathcal{H}^<(\mathcal{G})$ is a locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach space (with respect to the $\mathcal{C}_0(X)$-action induced by r) and also $\mathcal{H}^>(\mathcal{G})$ is locally $\mathcal{C}_0(X)$-convex (with respect to the action induced by s).

For all $\gamma \in \mathcal{G}$, define a map $\alpha^<_\gamma$ from $\mathcal{C}_c\left(\mathcal{G}^{s(\gamma)}\right)$ to $\mathcal{C}_c\left(\mathcal{G}^{r(\gamma)}\right)$ by

$$\chi^< \mapsto \alpha^<_\gamma(\chi^<) = \gamma\chi^< = \left(\gamma' \mapsto \chi^<(\gamma^{-1}\gamma')\right)$$

and a map $\alpha^>_\gamma$ from $\mathcal{C}_c\left(\mathcal{G}_{s(\gamma)}\right)$ to $\mathcal{C}_c\left(\mathcal{G}_{r(\gamma)}\right)$ by

$$\chi^> \mapsto \alpha^>_\gamma(\chi^>) = \gamma\chi^> = \left(\gamma' \mapsto \chi^>(\gamma'\gamma)\right).$$

If $\mathcal{H}(\mathcal{G})$ is locally convex, then it is called *equivariant* if $\alpha^<$ and $\alpha^>$ are families of isometric maps, i.e., if we have that

$$\left\|\gamma\chi^<\right\|_{\mathcal{H}^<\left(\mathcal{G}^{r(\gamma)}\right)} = \left\|\chi^<\right\|_{\mathcal{H}^<\left(\mathcal{G}^{s(\gamma)}\right)} \quad \text{and} \quad \left\|\gamma\chi^>\right\|_{\mathcal{H}^>\left(\mathcal{G}_{r(\gamma)}\right)} = \left\|\chi^>\right\|_{\mathcal{H}^>\left(\mathcal{G}_{s(\gamma)}\right)}$$

for all $\chi^< \in \mathcal{C}_c(\mathcal{G}^{s(\gamma)})$, $\chi^> \in \mathcal{C}_c(\mathcal{G}_{s(\gamma)})$ and all $\gamma \in \mathcal{G}$.

Examples 7.3.11. All the examples of 7.3.2 are locally $\mathcal{C}_0(X)$-convex and equivariant.

Definition and Proposition 7.3.12 (The $\mathcal{G}$-action on $\mathfrak{F}(\mathcal{H}(\mathcal{G},E))$). Let $\mathcal{H}(\mathcal{G})$ be a locally convex equivariant pair of monotone completions of $\mathcal{C}_c(\mathcal{G})$ and let E be a $\mathcal{G}$-Banach B-pair. Define

$$\alpha^<_\gamma\colon \Gamma_c\left(\mathcal{G}^{s(\gamma)},\ r^*E^<\right) \to \Gamma_c\left(\mathcal{G}^{r(\gamma)},\ r^*E^<\right),\ \xi^< \mapsto \gamma\xi^< := \left(\gamma' \mapsto \gamma\xi^<(\gamma^{-1}\gamma')\right), \tag{7.2}$$

and

$$\alpha^>_\gamma\colon \Gamma_c\left(\mathcal{G}_{s(\gamma)},\ r^*E^>\right) \to \Gamma_c\left(\mathcal{G}_{r(\gamma)},\ r^*E^>\right),\ \xi^> \mapsto \gamma\xi^> := \left(\gamma' \mapsto \xi^>(\gamma'\gamma)\right), \tag{7.3}$$

for all $\gamma \in \mathcal{G}$. Then $\alpha^<_\gamma$ and $\alpha^>_\gamma$ are isometric for all $\gamma \in \mathcal{G}$ and extend to isometric isomorphisms $\mathcal{H}^<(\mathcal{G}^{s(\gamma)},\ r^*E^<) \to \mathcal{H}^<(\mathcal{G}^{r(\gamma)},\ r^*E^<)$ and $\mathcal{H}^>(\mathcal{G}_{s(\gamma)},\ r^*E^>) \to \mathcal{H}^>(\mathcal{G}_{r(\gamma)},\ r^*E^>)$, respectively. The field $(\alpha^<_\gamma,\ \alpha^>_\gamma)_{\gamma\in\mathcal{G}}$ is a continuous field of isomorphisms making $\mathfrak{F}(\mathcal{H}(\mathcal{G},E))$ a $\mathcal{G}$-Banach B-pair.

Proof. We have to check that $(\alpha^<_\gamma)_{\gamma\in\mathcal{G}}$ and $(\alpha^>_\gamma)_{\gamma\in\mathcal{G}}$ are continuous families of isomorphisms. We check this only on the left-hand side, the proof for the right-hand side being analogous. Define

$$\alpha^<\colon \Gamma_c\left(\mathcal{G}\times_{r,s}\mathcal{G},\ Q^*_{r,s}E^<\right) \to \Gamma_c\left(\mathcal{G}\times_{r,r}\mathcal{G},\ Q^*_{r,r}E^<\right),\ \xi^< \mapsto \left[(\gamma,\gamma') \mapsto \gamma\xi^<(\gamma^{-1}\gamma')\right]$$

where $Q_{r,s}\colon \mathcal{G}\times_{r,s}\mathcal{G} \to \mathcal{G}^{(0)},\ (\gamma',\gamma) \mapsto r(\gamma') = s(\gamma)$ and $Q_{r,r}$ is defined analogously. We check that α is isometric for the norms $\|\cdot\|_{s^*\mathcal{H}^<}$ and $\|\cdot\|_{r^*\mathcal{H}^<}$ defined as in Definition E.8.13. If $\xi^< \in \Gamma_c\left(\mathcal{G}\times_{r,s}\mathcal{G},\ Q^*_{r,s}E^<\right)$, then

$$\begin{aligned}
\left\|\alpha^<(\xi^<)\right\|_{r^*\mathcal{H}^<} &= \sup_{\gamma\in\mathcal{G}}\left\|\mathcal{G}^{r(\gamma)} \ni \gamma' \mapsto \left\|\alpha^<(\xi^<)(\gamma,\gamma')\right\|\right\|_{\mathcal{H}^<_{r(\gamma)}}\\
&= \sup_{\gamma\in\mathcal{G}}\left\|\mathcal{G}^{r(\gamma)} \ni \gamma' \mapsto \left\|\gamma\xi^<(\gamma^{-1}\gamma')\right\|\right\|_{\mathcal{H}^<_{r(\gamma)}}\\
&= \sup_{\gamma\in\mathcal{G}}\left\|\mathcal{G}^{r(\gamma)} \ni \gamma' \mapsto \left\|\xi^<(\gamma^{-1}\gamma')\right\|\right\|_{\mathcal{H}^<_{r(\gamma)}}\\
&= \sup_{\gamma\in\mathcal{G}}\left\|\mathcal{G}^{s(\gamma)} \ni \gamma' \mapsto \left\|\xi^<(\gamma')\right\|\right\|_{\mathcal{H}^<_{s(\gamma)}} = \left\|\xi^<\right\|_{s^*\mathcal{H}^<}.
\end{aligned}$$

So $\alpha^<$ is an isometric isomorphism from $s^*\mathcal{H}^<(\mathcal{G}_{r,s}\mathcal{G},\ s^*E^<)$ to $r^*\mathcal{H}^<(\mathcal{G}_{r,r}\mathcal{G},\ r^*E^<)$. Identifying the field $\mathfrak{F}(s^*\mathcal{H}^<(\mathcal{G}_{r,s}\mathcal{G},\ s^*E^<))$ with $s^*\mathfrak{F}(\mathcal{H}^<(\mathcal{G},\ E^<))$ and $\mathfrak{F}(r^*\mathcal{H}^<(\mathcal{G}_{r,r}\mathcal{G},\ r^*E^<))$ with $r^*\mathfrak{F}(\mathcal{H}^<(\mathcal{G},\ E^<))$ (using E.8.14) makes the field $\alpha^<$ an action of $\mathcal{G}$ on the left Banach B-module $\mathfrak{F}(\mathcal{H}^<(\mathcal{G},\ E^<))$.

The proof of the algebraic properties of $\alpha^<$ and $\alpha^>$ is straightforward. We only check explicitly that the bracket and the module action on $\mathfrak{F}(\mathcal{H}(\mathcal{G}, E))$ are $\mathcal{G}$-equivariant. Let $\gamma \in \mathcal{G}$, $\xi^< \in \Gamma_c\left(\mathcal{G}^{s(\gamma)},\ r^*E^<\right)$ and $\xi^> \in \Gamma_c\left(\mathcal{G}_{s(\gamma)},\ r^*E^>\right)$. Then

$$
\begin{aligned}
\langle \gamma\xi^<, \gamma\xi^> \rangle_{r(\gamma)} &= \int_{\mathcal{G}^{r(\gamma)}} \left\langle (\gamma\xi^<)(\gamma'),\ \gamma'((\gamma\xi^>)(\gamma'^{-1}))\right\rangle \,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= \int_{\mathcal{G}^{r(\gamma)}} \left\langle \gamma\xi^<(\gamma^{-1}\gamma'),\ \gamma'\xi^>(\gamma'^{-1}\gamma)\right\rangle \,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= \int_{\mathcal{G}^{r(\gamma)}} \left\langle \gamma\xi^<(\gamma^{-1}\gamma'),\ \gamma\gamma^{-1}\gamma'\xi^>(\gamma'^{-1}\gamma)\right\rangle \,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= \gamma\int_{\mathcal{G}^{r(\gamma)}} \left\langle \xi^<(\gamma^{-1}\gamma'),\ \gamma^{-1}\gamma'\xi^>((\gamma^{-1}\gamma')^{-1})\right\rangle \,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= \gamma\int_{\mathcal{G}^{s(\gamma)}} \left\langle \xi^<(\gamma'),\ \xi^>(\gamma'^{-1})\right\rangle \,\mathrm{d}\lambda^{s(\gamma)}(\gamma') \\
&= \gamma\langle \xi^<, \xi^> \rangle_{s(\gamma)}.
\end{aligned}
$$

This shows that the bracket is equivariant. To see that the actions of B are compatible with the $\mathcal{G}$-actions we calculate for $b \in B_{s(\gamma)}$

$$
\begin{aligned}
\gamma\left(\xi^> b\right) &= \gamma\left(\mathcal{G}_{s(\gamma)} \ni \gamma' \mapsto \xi^>(\gamma')\gamma' b\right) \\
&= \left[\mathcal{G}_{r(\gamma)} \ni \gamma' \mapsto \xi^>\left(\gamma'\gamma\right) \cdot \gamma'(\gamma b)\right] = (\gamma\xi^>)\,(\gamma b)
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma\left(b\xi^<\right) &= \gamma\left(\mathcal{G}^{s(\gamma)} \ni \gamma' \mapsto b\xi^<(\gamma')\right) \\
&= \left[\mathcal{G}^{r(\gamma)} \ni \gamma' \mapsto \gamma b \cdot \gamma\xi^<(\gamma^{-1}\gamma')\right] = (\gamma b)\ \left(\gamma\xi^<\right).
\end{aligned}
$$

□

Corollary 7.3.13. *If $\mathcal{H}(\mathcal{G})$ is a locally convex equivariant pair of monotone completions of $\mathcal{C}_c(\mathcal{G})$, then $\mathfrak{F}(\mathcal{H}(\mathcal{G}))$ is a $\mathcal{G}$-Banach $\mathbb{C}_X$-pair.*

Proposition 7.3.14. *Let $\mathcal{H}(\mathcal{G})$ be a locally convex equivariant pair of monotone completions of $\mathcal{C}_c(\mathcal{G})$. Let B be a $\mathcal{G}$-Banach algebra and let E be a $\mathcal{G}$-Banach B-module. Then the convolution*

$$
\begin{aligned}
\Gamma_c(\mathcal{G},\ r^*E^<) \times \Gamma_c(\mathcal{G},\ r^*E^>) &\to \Gamma_c(\mathcal{G}, r^*B), \\
(\xi^<, \xi^>) &\mapsto \xi^< * \xi^> = \left(\gamma \mapsto \int_{\mathcal{G}^{r(\gamma)}} \left\langle \xi^<(\gamma'),\ \gamma'\xi^>(\gamma'^{-1}\gamma)\right\rangle_{E_{r(\gamma)}} \,\mathrm{d}\lambda^{r(\gamma)}(\gamma')\right)
\end{aligned}
$$

extends to a contractive bilinear map

$$
\mathcal{H}^<(\mathcal{G}, E^<) \times \mathcal{H}^>(\mathcal{G}, E^>) \to \Gamma_0(\mathcal{G}, r^*B),
$$

also written as a convolution product, such that the bracket on $\mathcal{H}(\mathcal{G}, E)$ is the composition of this map and the restriction map from $\Gamma_0(\mathcal{G}, r^*B)$ to $\Gamma_0(X, B)$.

Proof. Let $\xi^< \in \Gamma_c(\mathcal{G}, r^*E^<)$ and $\xi^> \in \Gamma_c(\mathcal{G}, r^*E^>)$. For all $\gamma \in \mathcal{G}$, we have

$$(\xi^< * \xi^>)(\gamma) = \left\langle \xi^<_{r(\gamma)},\ \gamma\xi^>_{s(\gamma)} \right\rangle_{r(\gamma)}$$

and hence

$$\begin{aligned} \left\|(\xi^< * \xi^>)(\gamma)\right\| &= \left\| \left\langle \xi^<_{r(\gamma)},\ \gamma\xi^>_{s(\gamma)} \right\rangle_{r(\gamma)} \right\| \leq \left\|\xi^<_{r(\gamma)}\right\|_{\mathcal{H}^<(\mathcal{G},E^<)_{r(\gamma)}} \left\|\gamma\xi^>_{s(\gamma)}\right\|_{\mathcal{H}^>(\mathcal{G},E^>)_{r(\gamma)}} \\ &= \left\|\xi^<_{r(\gamma)}\right\|_{\mathcal{H}^<(\mathcal{G},E^<)_{r(\gamma)}} \left\|\xi^>_{s(\gamma)}\right\|_{\mathcal{H}^>(\mathcal{G},E^>)_{s(\gamma)}} \leq \left\|\xi^<\right\|_{\mathcal{H}^<(\mathcal{G},E^<)} \left\|\xi^>\right\|_{\mathcal{H}^>(\mathcal{G},E^>)}, \end{aligned}$$

because $\mathcal{H}^>(\mathcal{G})$ is equivariant. Hence the convolution is continuous with norm ≤ 1 and extends to a map $\mathcal{H}^<(\mathcal{G}, E^<) \times \mathcal{H}^>(\mathcal{G}, E^>) \to \Gamma_0(\mathcal{G}, r^*B)$ with the desired properties. $\square$

In the 7.3.6 we have not assumed that $\mathcal{H}(\mathcal{G})$ is locally convex or equivariant. If it is, we can refine the result as follows:

Proposition 7.3.15. *Let $\mathcal{H}(\mathcal{G}) = (\mathcal{H}^<(\mathcal{G}),\ \mathcal{H}^>(\mathcal{G}))$ be a locally convex equivariant pair of monotone completions on which $\mathcal{A}(\mathcal{G})$ acts. Let A and B be $\mathcal{G}$-Banach algebras and let E be a $\mathcal{G}$-Banach A-B-pair. Then $\mathfrak{F}(\mathcal{H}(\mathcal{G}, E))$ is a $\mathcal{G}$-Banach B-pair on which $\mathcal{A}(\mathcal{G}, A)$ acts by bounded equivariant fields of linear operators. If $\mathcal{G}$ is proper and $\Gamma(X, A)$ acts on E by locally compact operators, then the action of $\mathcal{A}(\mathcal{G}, A)$ on $\mathfrak{F}(\mathcal{H}(\mathcal{G}, E))$ is by $\mathcal{G}$-*equivariant *bounded locally compact fields of operators.*

The only thing that we really have to check is that the action of $\mathcal{A}(\mathcal{G}, A)$ is equivariant. This is a consequence of the following lemma:

Lemma 7.3.16. *Let A and B be $\mathcal{G}$-Banach algebras and let E be a $\mathcal{G}$-Banach A-B-pair. Then the action of $\Gamma_c(\mathcal{G}, r^*A)$ on $\Gamma_c(\mathcal{G}, r^*E^<)$ and $\Gamma_c(\mathcal{G}, r^*E^>)$ commutes with the action of $\mathcal{G}$ in the obvious sense.*

Proof. Let $\gamma \in \mathcal{G}$, $a \in \Gamma_c(\mathcal{G}, r^*A)$, $\xi^< \in \Gamma_c(\mathcal{G}, r^*E^<)$, $\xi^> \in \Gamma_c(\mathcal{G}, r^*E^>)$. Then

$$\begin{aligned} \left(\gamma(a * \xi^>)_{s(\gamma)}\right)(\gamma') &= (a * \xi^>)_{s(\gamma)}(\gamma'\gamma) \\ &= \int_{\mathcal{G}^{s(\gamma)}} a(\gamma'')\gamma''\xi^>(\gamma''^{-1}\gamma'\gamma)\,\mathrm{d}\lambda^{s(\gamma)}(\gamma'') \\ &= a * (\gamma\xi^>_{s(\gamma)})(\gamma') \end{aligned}$$

for all $\gamma' \in \mathcal{G}_{r(\gamma)}$. Secondly,

$$\begin{aligned} \left(\gamma(\xi^< * a)_{s(\gamma)}\right)(\gamma') &= \gamma\,(\xi^< * a)_{s(\gamma)}(\gamma^{-1}\gamma') \\ &= \gamma \int_{\mathcal{G}^{s(\gamma)}} \xi^<(\gamma'')\gamma''a(\gamma''^{-1}\gamma^{-1}\gamma')\,\mathrm{d}\lambda^{s(\gamma)}(\gamma'') \\ &= \int_{\mathcal{G}^{s(\gamma)}} \gamma\xi^<(\gamma^{-1}(\gamma\gamma''))\gamma\gamma''a((\gamma\gamma'')^{-1}\gamma')\,\mathrm{d}\lambda^{s(\gamma)}(\gamma'') \\ &= \int_{\mathcal{G}^{r(\gamma)}} \gamma\xi^<(\gamma^{-1}\gamma'')\gamma''a(\gamma''^{-1}\gamma')\,\mathrm{d}\lambda^{r(\gamma)}(\gamma'') \\ &= ((\gamma\xi^<)_{r(\gamma)} * a)(\gamma') \end{aligned}$$

for all $\gamma' \in \mathcal{G}^{r(\gamma)}$. $\square$

Corollary 7.3.17. *Let $\mathcal{H}(\mathcal{G}) = (\mathcal{H}^<(\mathcal{G}),\ \mathcal{H}^>(\mathcal{G}))$ be a locally convex equivariant pair of monotone completions of $\mathcal{C}_c(\mathcal{G})$ on which $\mathcal{A}(\mathcal{G})$ acts. Let B be a non-degenerate $\mathcal{G}$-Banach algebra. If $\mathcal{G}$ is proper, then $\mathcal{A}(\mathcal{G}, B)$ acts on $\mathfrak{F}\mathcal{H}(\mathcal{G}, B)$ by locally compact $\mathcal{G}$-equivariant operators.*

7.4 Regular unconditional completions

For simplicity, we introduce the following abbreviation:

Definition 7.4.1 (Regular unconditional completion). An unconditional completion $\mathcal{A}(\mathcal{G})$ of $\mathcal{C}_c(\mathcal{G})$ is said to be a *regular unconditional completion* if there exists an equivariant pair of locally convex monotone completions of $\mathcal{C}_c(\mathcal{G})$ on which $\mathcal{A}(\mathcal{G})$ acts.

Note that there might exist many different equivariant pairs of monotone completions on which a regular unconditional completion acts, the important part of the definition really is the existence of such a pair, not its particular shape.

Examples 7.4.2. Most examples of unconditional completions that we have come across so far (compare 5.2.2) are regular for rather obvious reasons:

1. The unconditional completion $\mathrm{L}^1(\mathcal{G})$ acts on the pair $\left(\mathrm{L}^1(\mathcal{G}),\ \mathcal{C}_0(\mathcal{G})\right)$.

2. The symmetrised version $\mathrm{L}^1(\mathcal{G})\cap\mathrm{L}^1(\mathcal{G})^*$ is also regular because the norm defining it dominates the norm $\|\cdot\|_1$. Moreover, it acts on the pair $\left(\mathrm{L}^2_r(\mathcal{G}),\ \mathrm{L}^2_s(\mathcal{G})\right)$ (see [Ren80]). It should not be too hard to check that it also acts on $\left(\mathrm{L}^{p'}_r(\mathcal{G}),\ \mathrm{L}^{p}_s(\mathcal{G})\right)$ for all $p, p' \in]1,\infty[$ such that $\frac{1}{p}+\frac{1}{p'}=1$.

3. The completion $\mathcal{A}_{\max}(\mathcal{G})$ acts on $\left(\mathrm{L}^2_r(\mathcal{G}),\ \mathrm{L}^2_s(\mathcal{G})\right)$ by definition.

4. If G is a locally compact Hausdorff group acting on some locally compact Hausdorff space X, then $\mathrm{L}^1(G,\ \mathcal{C}_0(X))$ is a regular completion of $\mathcal{C}_c(G \ltimes X)$ because its norm dominates the norm of the regular completion $\mathrm{L}^1(G \ltimes X)$.

Regularity is essential in our construction of the homomorphism $M^B_{\mathcal{A}}$ down below. It also makes some arguments in the next chapter simpler (but might perhaps be avoided in some instances).

Note that in [Laf02] there is an argument which seems to hold in general but is definitely simpler in the case of regular unconditional completions: The proof of Lemme 1.7.8 uses a concept very similar to regularity, and the subsequent arguments show that $\mathcal{B}(G,B)$ and $\mathrm{C}^*_r(G,B)$ have the same K-theory but do not explain explicitly why $\mathcal{B}(G,B)$ and $\mathcal{A}(G,B)$ have the same K-theory, too. [10] If $\mathcal{A}(G)$ is regular, then one can use the same argument as for $\mathrm{C}^*_r(G,B)$.

The issue recurs in [Laf06], the respective result there is Lemme 1.5.7. Note that there is a very similar statement (for regular completions) in Chapter 8 of this thesis, namely Proposition 8.4.3.

7.5 The (inverse) homomorphism $M^B_{\mathcal{A}}$

Let $\mathcal{G}$ be a proper locally compact Hausdorff groupoid with unit space X and Haar system λ. Let $\mathcal{A}(\mathcal{G})$ be a regular unconditional completion of $\mathcal{C}_c(\mathcal{G})$ acting on the equivariant pair $\mathcal{H}(\mathcal{G})$ of locally convex monotone completions. Let B be a non-degenerate $\mathcal{G}$-Banach algebra.

[10] V. Lafforgue has recently given me an argument why Lemme 1.7.8 is true in general; it consists of a careful estimate showing directly that $\Gamma_c(G,B)$ is always a hereditary subalgebra of $\mathcal{A}(G,B)$.

7.5.1 The first step: The tensor product with $\mathcal{H}(\mathcal{G}, B)$

If E be is a non-degenerate $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G}, B)$-pair, then we can form the tensor product $E \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{H}(\mathcal{G}, B)$. This is a non-degenerate $\mathcal{C}_0(X)$-Banach $\Gamma_0(X, B)$-pair. Actually, this construction defines a functor from the category of non-degenerate $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G}, B)$-pairs with the bounded linear operators to the category of non-degenerate $\mathcal{C}_0(X)$-Banach $\Gamma_0(X, B)$-pairs with the bounded linear operators, linear and contractive on the morphism sets. Because $\mathcal{A}(\mathcal{G}, B)$ acts on $\mathcal{H}(\mathcal{G}, B)$ by locally compact operators, it follows that locally compact operators are mapped to locally compact operators under this functor. We therefore have

Lemma 7.5.1. *If* $(E, T) \in \mathbb{E}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G}, B))$*, then*

$$\left(E \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{H}(\mathcal{G}, B)\,,\ T \otimes 1\right) \in \mathbb{E}^{\mathrm{ban}}(\mathcal{C}_0(X);\ \mathcal{C}_0(X),\ \Gamma_0(X, B))\,.$$

The map $(E, T) \mapsto \left(E \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{H}(\mathcal{G}, B),\ T \otimes 1\right)$ *induces a homomorphism* $\cdot \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{H}(\mathcal{G}, B)$

$$\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G}, B)) \to \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X);\ \mathcal{C}_0(X),\ \Gamma_0(X, B))\,.$$

To verify that we really have a well-defined homomorphism we have to check that $(E, T) \mapsto \left(E \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{H}(\mathcal{G}, B),\ T \otimes 1\right)$ respects the sum of cycles (which is trivial) and that is compatible with homotopy. The latter fact can be proved just as in 1.10.29, i.e., by using the Banach $\mathcal{A}(\mathcal{G}, B)[0, 1]$-$\Gamma_0(X, B)[0, 1]$-pair $\mathcal{H}(\mathcal{G}, B)[0, 1]$.

An alternative picture of the first step

Note that $\mathcal{H}(\mathcal{G}, B)$ is not exactly a Morita cycle from $\mathcal{A}(\mathcal{G}, B)$ to $\Gamma_0(X, B)$, because $\mathcal{A}(\mathcal{G}, B)$ is a $\mathcal{C}_0(X/\mathcal{G})$-Banach algebra and $\Gamma_0(X, B)$ is $\mathcal{C}_0(X)$-Banach algebra. However, we can change the setting a little bit and use the theory that we have provided in the earlier chapters by regarding $\mathcal{H}(\mathcal{G}, B)$ as a $\mathcal{C}_0(X)$-linear Morita cycle.

Let π denote the canonical projection from X to $X/\mathcal{G}$. Recall from Chapter 2 that π^*E is defined as $\mathcal{C}_0(X) \otimes_{\mathcal{C}_0(X/\mathcal{G})} E$ for every $\mathcal{C}_0(X/\mathcal{G})$-Banach space E. If A is a $\mathcal{C}_0(X/\mathcal{G})$-Banach algebra, then π^*A is a $\mathcal{C}_0(X)$-Banach algebra. As a special case we have $\pi^*\mathcal{C}_0(X/\mathcal{G}) = \mathcal{C}_0(X) \otimes_{\mathcal{C}_0(X/\mathcal{G})} \mathcal{C}_0(X/\mathcal{G}) \cong \mathcal{C}_0(X)$ (as $\mathcal{C}_0(X)$-Banach algebras). By what we have shown in Chapter 2 we can now deduce that $(E, T) \mapsto (\pi^*E,\ \pi^*T)$ defines a homomorphism

$$\pi^* \colon \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G}, B)) \to \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X);\ \mathcal{C}_0(X),\ \pi^*\mathcal{A}(\mathcal{G}, B))\,.$$

Combining the given $\mathcal{C}_0(X)$-action on $\mathcal{H}(\mathcal{G}, B)$ with the left action of $\mathcal{A}(\mathcal{G}, B)$ we get a left action of $\pi^*\mathcal{A}(\mathcal{G}, B) = \mathcal{C}_0(X) \otimes_{\mathcal{C}_0(X/\mathcal{G})} \mathcal{A}(\mathcal{G}, B)$ on $\mathcal{H}(\mathcal{G}, B)$. It is also an action by locally compact operators. Now $\mathcal{H}(\mathcal{G}, B)$ is an element of $\mathbb{M}^{\mathrm{ban}}(\mathcal{C}_0(X);\ \mathcal{C}_0(X), \Gamma_0(X, B))$ when equipped with this action. Tensoring with this Morita cycle will thus yield a homomorphism $\otimes_{\pi^*\mathcal{A}(\mathcal{G},B)}\mathcal{H}(\mathcal{G}, B)$

$$\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X);\ \mathcal{C}_0(X),\ \pi^*\mathcal{A}(\mathcal{G}, B)) \to \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X);\ \mathcal{C}_0(X),\ \Gamma_0(X, B))\,.$$

Let E be a non-degenerate $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G}, B)$-pair. Then we can actually compute what the composition of π^* and $\otimes_{\mathcal{A}(\mathcal{G},B)}\mathcal{H}(\mathcal{G}, B)$ does to E:

$$\begin{aligned}
& (\pi^*E) \otimes_{\pi^*\mathcal{A}(\mathcal{G},B)} \mathcal{H}(\mathcal{G}, B) \\
\cong\ & \left(\mathcal{C}_0(X) \otimes_{\mathcal{C}_0(X/\mathcal{G})} E\right) \otimes_{\mathcal{C}_0(X)\otimes_{\mathcal{C}_0(X/\mathcal{G})}\mathcal{A}(\mathcal{G},B)} \left(\mathcal{C}_0(X) \otimes_{\mathcal{C}_0(X)} \mathcal{H}(\mathcal{G}, B)\right) \\
\cong\ & \left(\mathcal{C}_0(X) \otimes_{\mathcal{C}_0(X)} \mathcal{C}_0(X)\right) \otimes_{\mathcal{C}_0(X/\mathcal{G})\otimes_{\mathcal{C}_0(X/\mathcal{G})}\mathcal{C}_0(X)} \left(E \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{H}(\mathcal{G}, B)\right) \\
\cong\ & \mathcal{C}_0(X) \otimes_{\mathcal{C}_0(X)} \left(E \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{H}(\mathcal{G}, B)\right) \\
\cong\ & E \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{H}(\mathcal{G}, B).
\end{aligned}$$

It is easy to check that this isomorphism intertwines $\pi^* T \otimes 1$ and $T \otimes 1$ if (E, T) is a cycle. Hence we have shown that this alternative approach gives the same result as tensoring with $\mathcal{H}(\mathcal{G}, B)$ right away.

7.5.2 The second step: From $\mathcal{C}_0(X)$-Banach spaces to fields

Recall from Chapter 4 that $\mathfrak{F}(\cdot)$ is a functor from the $\mathcal{C}_0(X)$-Banach spaces to the u.s.c. fields of Banach spaces over X that sends a space E to $(E_x)_{x \in X}$ where E_x is the fibre of E over x. We have shown that this functor induces a homomorphism on $\mathrm{KK}^{\mathrm{ban}}$-theory by sending a $\mathrm{RKK}^{\mathrm{ban}}$-cycle (E, T) to $\mathfrak{F}(E, T) = (\mathfrak{F}(E), \mathfrak{F}(T))$. Note that $\mathfrak{F}(\cdot)$ takes bounded locally compact operators to bounded locally compact operators.

Definition 7.5.2. Let E be a $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G}, B)$-pair. Define

$$M_{\mathcal{A},\mathcal{H}}^{B}(E) := \mathfrak{F}\left(E \otimes_{\mathcal{A}(\mathcal{G},B)}^{\mathcal{C}_0(X/\mathcal{G})} \mathcal{H}(\mathcal{G}, B)\right).$$

Then $M_{\mathcal{A},\mathcal{H}}^{B}(E)$ is a of Banach B-pair.

Note that $M_{\mathcal{A},\mathcal{H}}^{B}(\cdot)$ is actually a functor from the $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G}, B)$-pairs to the Banach B-pairs which sends locally compact operators to locally compact operators.

Lemma 7.5.3. *If* $(E, T) \in \mathbb{E}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G}, B))$*, then*[11]

$$M_{\mathcal{A},\mathcal{H}}^{B}(E, T) := \left(M_{\mathcal{A},\mathcal{H}}^{B}(E),\ M_{\mathcal{A},\mathcal{H}}^{B}(T)\right) \in \mathbb{E}_X^{\mathrm{ban}}(\mathbb{C}_X,\ B).$$

The map $(E, T) \mapsto M_{\mathcal{A},\mathcal{H}}^{B}(E, T)$ *induces a homomorphism*

$$M_{\mathcal{A},\mathcal{H}}^{B}(E, T)\colon\ \mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G}, B)) \to \mathrm{KK}_X^{\mathrm{ban}}(\mathbb{C}_X,\ B).$$

7.5.3 The third step: The $\mathcal{G}$-action

Let E be a $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G}, B)$-pair. Then, for all $x \in X$, the fibre of $M_{\mathcal{A},\mathcal{H}}^{B}(E)$ at x can be identified with

$$E_{\pi(x)} \otimes_{\mathcal{A}(\mathcal{G},B)_{\pi(x)}} \mathcal{H}(\mathcal{G}, B)_x.$$

Definition and Proposition 7.5.4. Let E be a $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G}, B)$-pair. For all $\gamma \in \mathcal{G}$, define a map

$$E_{\pi(s(\gamma))} \otimes_{\mathcal{A}(\mathcal{G},B)_{\pi(s(\gamma))}} \mathcal{H}(\mathcal{G}, B)_{s(\gamma)} \to E_{\pi(r(\gamma))} \otimes_{\mathcal{A}(\mathcal{G},B)_{\pi(r(\gamma))}} \mathcal{H}(\mathcal{G}, B)_{r(\gamma)},$$

by $\mathrm{Id} \otimes \alpha_\gamma$ where α denotes the action of $\mathcal{G}$ on $\mathcal{H}(\mathcal{G}, B)$. This defines an action of $\mathcal{G}$ on $M_{\mathcal{A},\mathcal{H}}^{B}(E)$ called $\mathrm{Id} \otimes \alpha$. With this action, $M_{\mathcal{A},\mathcal{H}}^{B}(E)$ is a $\mathcal{G}$-Banach B-pair.

To see that this really is a continuous action we provide a conceptional alternative picture of the construction in the upcoming subsection. For now, we just state the results that we are going to obtain:

Proposition 7.5.5. *Let* E *and* F *be* $\mathcal{C}_0(X/\mathcal{G})$*-Banach* $\mathcal{A}(\mathcal{G}, B)$*-pairs. Let* $T \in \mathrm{L}_{\mathcal{A}(\mathcal{G},B)}^{\mathcal{C}_0(X/\mathcal{G})}(E, F)$*. Then*

$$M_{\mathcal{A},\mathcal{H}}^{B}(T) := \mathfrak{F}(T \otimes 1) \in \mathrm{L}_B\left(M_{\mathcal{A},\mathcal{H}}^{B}(E),\ M_{\mathcal{A},\mathcal{H}}^{B}(F)\right)$$

is $\mathcal{G}$*-equivariant.*

[11] During the technical part of this chapter we will distinguish $\mathcal{C}_0(X)$ and $\mathbb{C}_X$ to have clearer statements.

Hence the maps $E \mapsto M^B_{\mathcal{A},\mathcal{H}}(E)$ and $T \mapsto M^B_{\mathcal{A},\mathcal{H}}(T)$ define a functor from the $\mathcal{C}_0\left(X/\mathcal{G}\right)$-Banach $\mathcal{A}(\mathcal{G},B)$-pairs with the bounded fields of linear operators to the $\mathcal{G}$-Banach B-pairs with the $\mathcal{G}$-equivariant bounded fields of operators. It maps locally compact operators to locally compact operators.

Proposition 7.5.6. *Let $(E,T) \in \mathbb{E}^{\mathrm{ban}}\left(\mathcal{C}_0\left(X/\mathcal{G}\right);\, \mathcal{C}_0\left(X/\mathcal{G}\right),\, \mathcal{A}(\mathcal{G},B)\right)$. Then*

$$M^B_{\mathcal{A},\mathcal{H}}(E,T) := \left(M^B_{\mathcal{A},\mathcal{H}}(E),\ M^B_{\mathcal{A},\mathcal{H}}(T)\right) \in \mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathbb{C}_X,\ B\right)$$

with $\mathcal{G}$-equivariant T. The map $M^B_{\mathcal{A},\mathcal{H}}$ induces a natural homomorphism of groups

$$M^B_{\mathcal{A},\mathcal{H}}\colon\ \mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0\left(X/\mathcal{G}\right);\, \mathcal{C}_0\left(X/\mathcal{G}\right),\, \mathcal{A}(\mathcal{G},B)\right) \to \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathbb{C}_X,\ B\right).$$

To show that this homomorphism is indeed natural is rather straightforward but requires a bit of work. The key ingredient is the obvious homomorphism Φ with coefficient map φ from $\mathcal{H}(\mathcal{G},B)$ to $\mathcal{H}(\mathcal{G},B')$ if φ is a $\mathcal{G}$-equivariant homomorphism from B to B'; one has to show that this homomorphism is compatible with the actions of $\mathcal{A}(\mathcal{G},B)$ and $\mathcal{A}(\mathcal{G},B')$ in the sense that one can approximate the action of some $\beta \in \Gamma_c(\mathcal{G},\ r^*B)$ on $\mathcal{H}(\mathcal{G},B)$ and of $\varphi \circ \beta$ on $\mathcal{H}(\mathcal{G},B')$ simultaneously by finite rank operators. We leave out the details.

7.5.4 An alternative picture of the construction

Recall that we used the name π for the canonical projection from X to $X/\mathcal{G}$. Let π also denote the map from $\mathcal{G}$ to $X/\mathcal{G}$ that maps γ to $\pi(r(\gamma)) = \pi(s(\gamma))$ (which extends the $\pi\colon X \to X/\mathcal{G}$). Regarding $X/\mathcal{G}$ as a locally compact Hausdorff groupoid the map $\pi\colon \mathcal{G} \to X/\mathcal{G}$ is actually a strict morphism of groupoids. If E is a u.s.c. field of Banach spaces over $X/\mathcal{G}$, then π^*E is a $\mathcal{G}$-Banach space (with a rather trivial action). If T is a continuous field of linear maps between u.s.c. fields of Banach spaces over $X/\mathcal{G}$, then π^*T is an $\mathcal{G}$-equivariant continuous field of linear maps between $\mathcal{G}$-Banach spaces. We use these facts to define our "inverse homomorphism":

1. The first step is the map $\mathfrak{F}\left(\cdot\right)$, this time giving a homomorphism

   $$\mathfrak{F}\left(\cdot\right):\ \mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G},B)\right) \to \mathrm{KK}^{\mathrm{ban}}_{X/\mathcal{G}}\left(\mathbb{C}_{X/\mathcal{G}},\ \mathfrak{F}\left(\mathcal{A}(\mathcal{G},B)\right)\right).$$

2. The second step is the pullback homomorphism along π:

   $$\pi^*:\ \mathrm{KK}^{\mathrm{ban}}_{X/\mathcal{G}}\left(\mathbb{C}_{X/\mathcal{G}},\ \mathfrak{F}\left(\mathcal{A}(\mathcal{G},B)\right)\right) \to \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathbb{C}_X,\ \pi^*\mathfrak{F}\left(\mathcal{A}(\mathcal{G},B)\right)\right).$$

   Note that this homomorphism, on the level of cycles, just produces cycles with $\mathcal{G}$-equivariant operator.

3. We have discussed above how $\mathcal{H}(\mathcal{G},B)$ can be regarded as a $\mathcal{C}_0(X)$-linear Morita cycle from $\pi^*\mathcal{A}(\mathcal{G},B)$ to $\Gamma_0(X,B)$. Observe that $\mathfrak{F}\left(\pi^*\mathcal{A}(\mathcal{G},B)\right) \cong \pi^*\mathfrak{F}\left(\mathcal{A}(\mathcal{G},B)\right)$, so we can regard $\mathfrak{F}\left(\mathcal{H}(\mathcal{G},B)\right)$ as a Morita cycle from $\pi^*\mathfrak{F}\left(\mathcal{A}(\mathcal{G},B)\right)$ to $B \cong \mathfrak{F}\left(\Gamma_0(X,B)\right)$. The important point is that this Morita cycle carries an action of $\mathcal{G}$ which makes it a $\mathcal{G}$-equivariant Morita cycle! Hence we get a homomorphism

   $$\otimes_{\pi^*\mathfrak{F}(\mathcal{A}(\mathcal{G},B))}\mathfrak{F}\left(\mathcal{H}(\mathcal{G},B)\right):\ \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathbb{C}_X,\ \pi^*\mathfrak{F}\left(\mathcal{A}(\mathcal{G},B)\right)\right) \to \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathbb{C}_X,\ B\right).$$

   On the level of cycles: If a cycle has a $\mathcal{G}$-equivariant operator, then it stays equivariant under this homomorphism.

The composition of these three homomorphisms gives the desired natural homomorphism

$$M^B_{\mathcal{A},\mathcal{H}}\colon\ \mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0\left(X/\mathcal{G}\right);\, \mathcal{C}_0\left(X/\mathcal{G}\right),\, \mathcal{A}(\mathcal{G},B)\right) \to \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathbb{C}_X,\ B\right)$$

which produces cycles with $\mathcal{G}$-equivariant operators.

7.5.5 The uniqueness of the inverse homomorphism

For every regular unconditional completion $\mathcal{A}(\mathcal{G})$ of $\mathcal{C}_c(\mathcal{G})$, we have a canonical homomorphism from $\mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0\left(X/\mathcal{G}\right);\, \mathcal{C}_0\left(X/\mathcal{G}\right),\, \mathcal{A}(\mathcal{G},B)\right) \to \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathbb{C}_X,\, B\right)$, canonical in the sense that it does not depend on the particular shape of the equivariant pair of monotone completions on which $\mathcal{A}(\mathcal{G})$ acts:

Proposition 7.5.7. *Let $\mathcal{H}'(\mathcal{G}) = (\mathcal{H}'^{<}(\mathcal{G}),\, \mathcal{H}'^{>}(\mathcal{G}))$ be another equivariant pair of locally convex completions of $\mathcal{C}_c(\mathcal{G})$ on which $\mathcal{A}(\mathcal{G})$ acts. Then the natural homomorphisms $M^B_{\mathcal{A},\mathcal{H}}$ and $M^B_{\mathcal{A},\mathcal{H}'}$ are equal. We call this natural homomorphism $M^B_{\mathcal{A}}$.*

Proof. We first consider the case that $\|\cdot\|_{\mathcal{H}^<} \leq \|\cdot\|_{\mathcal{H}'^<}$ and $\|\cdot\|_{\mathcal{H}^>} \leq \|\cdot\|_{\mathcal{H}'^>}$. We then have a canonical homomorphism Φ from $\mathcal{H}'(\mathcal{G},B)$ to $\mathcal{H}(\mathcal{G},B)$ which gives us an equivariant homomorphism $\mathfrak{F}(\Phi)$ from $\mathfrak{F}(\mathcal{H}'(\mathcal{G},B))$ to $\mathfrak{F}(\mathcal{H}(\mathcal{G},B))$. The homomorphism $\mathfrak{F}(\Phi)$ is actually a morphism of equivariant Morita cycles from $\pi^*\mathfrak{F}(\mathcal{A}(\mathcal{G},B))$ to B. A careful revision of the proof that $\pi^*\mathfrak{F}(\mathcal{A}(\mathcal{G},B))$ acts by compact operators on $\mathfrak{F}(\mathcal{H}'(\mathcal{G},B))$ and on $\mathfrak{F}(\mathcal{H}(\mathcal{G},B))$ shows that $\mathfrak{F}(\Phi)$ satisfies the conditions of Theorem 3.7.1 and hence induces a homotopy from $\mathfrak{F}(\mathcal{H}'(\mathcal{G},B))$ to $\mathfrak{F}(\mathcal{H}(\mathcal{G},B))$. So $M^B_{\mathcal{A},\mathcal{H}'} = M^B_{\mathcal{A},\mathcal{H}}$.

Now consider the case that $\mathcal{H}'$ is a general equivariant pair of locally convex completions of $\mathcal{C}_c(\mathcal{G})$ on which $\mathcal{A}(\mathcal{G})$ acts. By taking the maximum of the norms on $\mathcal{H}^<(\mathcal{G})$ and $\mathcal{H}'^<(\mathcal{G})$ we can define an equivariant locally convex completion $\mathcal{H}''^<(\mathcal{G})$ of $\mathcal{C}_c(\mathcal{G})$ on which $\mathcal{A}(\mathcal{G})$ acts; similarly we can define $\mathcal{H}''^>(\mathcal{G})$. The pair $\mathcal{H}''(\mathcal{G}) := (\mathcal{H}''^<(\mathcal{G}),\, \mathcal{H}''^>(\mathcal{G}))$ is a pair of locally convex completions on which $\mathcal{A}(\mathcal{G})$ acts. By the first part of the proof we can conclude $M^B_{\mathcal{A},\mathcal{H}} = M^B_{\mathcal{A},\mathcal{H}''} = M^B_{\mathcal{A},\mathcal{H}'}$. □

7.6 $J^B_{\mathcal{A}} \circ M^B_{\mathcal{A}} = \mathrm{Id}$ on the level of $\mathrm{KK}^{\mathrm{ban}}$

Let $\mathcal{G}$ be a proper locally compact Hausdorff groupoid with unit space X and Haar system λ. Let $\mathcal{A}(\mathcal{G})$ be a regular unconditional completion of $\mathcal{C}_c(\mathcal{G})$. Let B be a non-degenerate $\mathcal{G}$-Banach algebra. Assume that there exists a pair $\mathcal{D}(X)$ of monotone completions of $\mathcal{C}_c(X)$ compatible with $\mathcal{A}(\mathcal{G})$ (this is the case if $\mathcal{G}$ admits a cut-off function which, in turn, is true if $X/\mathcal{G}$ is σ-compact).

Theorem 7.6.1. $J^B_{\mathcal{A}} \circ M^B_{\mathcal{A}} = \mathrm{Id}$ *as an endomorphism of* $\mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\, \mathcal{C}_0(X/\mathcal{G}),\, \mathcal{A}\left(\mathcal{G},B\right)\right)$.

Idea of the proof

Because $\mathcal{A}(\mathcal{G})$ is regular, we can find an equivariant pair $\mathcal{H}(\mathcal{G}) = (\mathcal{H}^<(\mathcal{G}),\, \mathcal{H}^>(\mathcal{G}))$ of monotone completions of $\mathcal{C}_c\left(\mathcal{G}\right)$ on which $\mathcal{A}(\mathcal{G})$ acts. Let $(E,T) \in \mathbb{E}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\, \mathcal{C}_0(X/\mathcal{G}),\, \mathcal{A}\left(\mathcal{G},B\right)\right)$. We have to show that (E,T) is homotopic to $J^B_{\mathcal{A},\mathcal{D}}(M^B_{\mathcal{A},\mathcal{H}}(E,T))$. The obvious strategy is to define a morphism from $J^B_{\mathcal{A},\mathcal{D}}(M^B_{\mathcal{A},\mathcal{H}}(E))$ to E which induces a homotopy; there is a canonical candidate for such a morphism defined on a dense subspace, but this candidate does not extend to a continuous morphism on the entire space: The norms on $J^B_{\mathcal{A},\mathcal{D}}(M^B_{\mathcal{A},\mathcal{H}}(E))$ and E seem to be difficult to compare in general.

We overcome this problem by constructing a pair $\tilde{E} := (\tilde{E}^<, \tilde{E}^>)$ of $\mathbb{C}$-vector spaces which are equipped with compatible $\mathcal{C}_c(X/\mathcal{G})$-module structures and left/right $\Gamma_c\left(\mathcal{G},\, r^*B\right)$-module structures and a bilinear map from $\tilde{E}^< \times \tilde{E}^>$ to $\Gamma_c\left(\mathcal{G},\, r^*B\right)$. On this pair, which could be called a "pre-$\mathcal{A}(\mathcal{G},B)$-pair", we construct a pair of formally adjoint operators $\tilde{T}$. Moreover, we define canonical "homomorphisms" Φ_E from $\tilde{E}$ to E and Ψ_E from $\tilde{E}$ to $J^B_{\mathcal{A},\mathcal{D}}(M^B_{\mathcal{A},\mathcal{H}}(E))$ which intertwine $\tilde{T}$ and T

and $J^B_{\mathcal{A},\mathcal{D}}(M^B_{\mathcal{A},\mathcal{H}}(T))$, respectively:

$$\begin{tikzcd}
 & (\tilde{E},\tilde{T}) \arrow[dl, "\Phi_E"'] \arrow[dr, "\Psi_E"] & \\
(E,T) & & J^B_{\mathcal{A},\mathcal{D}}(M^B_{\mathcal{A},\mathcal{H}}(E,T))
\end{tikzcd}$$

One can think of $\tilde{E}$ as a dense subspace of both, E and $J^B_{\mathcal{A},\mathcal{D}}(M^B_{\mathcal{A},\mathcal{H}}(E))$. Now we put on $\tilde{E}$ the supremum of the semi-norms which are induced by the two homomorphisms, making the homomorphisms continuous. The completion of $\tilde{E}$ together with the continuous extension of $\tilde{T}$ will then be in $\mathbb{E}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\, \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}\left(\mathcal{G},B\right)\right)$ and the two homomorphisms will induce homotopies. Hence also (E,T) and $J^B_{\mathcal{A},\mathcal{D}}(M^B_{\mathcal{A},\mathcal{H}}(E,T))$ are homotopic.

The construction of $\tilde{E}$, Φ_E and Ψ_E

We are going to cut the proof into a series of lemmas and definitions. In this section, let E and F be $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G},B)$-pairs.

Definition 7.6.2 (The pair $\tilde{E}$). Define

$$\tilde{E}^{>} := E^{>} \otimes_{\Gamma_c(\mathcal{G},\, r^*B)} \Gamma_c\left(\mathcal{G},\ r^*B\right)$$

and

$$\tilde{E}^{<} := \Gamma_c\left(\mathcal{G},\ r^*B\right) \otimes_{\Gamma_c(\mathcal{G},\, r^*B)} E^{<}.$$

These vector spaces carry canonical and compatible actions of $\Gamma_c\left(\mathcal{G},\ r^*B\right)$ and $\mathcal{C}_c\left(X/\mathcal{G}\right)$. A bracket on $\tilde{E}$ is defined by

$$\begin{array}{rcl} \langle\cdot,\cdot\rangle\colon \tilde{E}^{<}\times\tilde{E}^{>} & \to & \Gamma_c\left(\mathcal{G},\ r^*B\right), \\ \left\langle \beta^{<}\otimes e^{<},\ e^{>}\otimes\beta^{>}\right\rangle(\gamma) & := & \beta^{<} * \langle e^{<}, e^{>}\rangle * \beta^{>} = \left\langle \beta^{<}e^{<},\ e^{>}\beta^{>}\right\rangle. \end{array}$$

We check that the bracket has indeed its values in $\Gamma_c(\mathcal{G}, r^*B)$: The element $\langle e^{<}, e^{>}\rangle$ is in $\mathcal{A}(\mathcal{G},B)$ by definition, and we now show that the product $\beta^{<} * \beta * \beta^{>}$ is in $\Gamma_c(\mathcal{G},B)$ for all $\beta^{<},\beta^{>} \in \Gamma_c(\mathcal{G},B)$ and $\beta \in \mathcal{A}(\mathcal{G},B)$. If we regard $\beta^{<}$ as an element of $\mathcal{H}^{<}(\mathcal{G},B)$ and $\beta^{>}$ as an element of $\mathcal{H}^{>}(\mathcal{G},B)$, then we can conclude from Proposition 7.3.14 that the map $\beta \mapsto \beta^{<} * \beta * \beta^{>}$ is continuous from $\mathcal{A}(\mathcal{G})$ to $\Gamma_0(\mathcal{G},B)$ because $\mathcal{A}(\mathcal{G})$ acts on $\mathcal{H}(\mathcal{G})$. Moreover, the support of the product $\beta^{<} * \beta * \beta^{>}$ is always contained in the set $\{\gamma \in \mathcal{G} :\ r(\gamma) \in r(\operatorname{supp}\beta^{<}),\ s(\gamma) \in s(\operatorname{supp}\beta^{>})\}$, which is compact because $\mathcal{G}$ is proper. [12]

Definition 7.6.3 (The map Φ_E). Define

$$\Phi_E^{>}\colon \tilde{E}^{>} \to E^{>},\ e^{>}\otimes\beta^{>} \mapsto e^{>}\beta^{>}$$

and

$$\Phi_E^{<}\colon \tilde{E}^{<} \to E^{<},\ \beta^{<}\otimes e^{<} \mapsto \beta^{<}e^{<}.$$

Both maps are clearly $\Gamma_c\left(\mathcal{G},\ r^*B\right)$- and $\mathcal{C}_c\left(X/\mathcal{G}\right)$-linear. The pair $\Phi_E = \left(\Phi_E^{<}, \Phi_E^{>}\right)$ is compatible with the brackets on $\tilde{E}$ and E.

[12] Compare the proof of Lemma 8.2.4.

Definition 7.6.4 (The map Ψ_E). Let $e^> \in E^>$ and $\beta^> \in \Gamma_c(\mathcal{G}, r^*B)$. Since $\beta^>$ has compact support, the function $x \mapsto (e^> \otimes \beta^>)_x = e^>_{\pi(x)} \otimes \beta^>_x$ is in $\Gamma_c\left(X,\ \mathfrak{F}\left(E^> \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{H}^>(\mathcal{G},B)\right)\right)$ which we can regard as an element $\Psi_E^>(e^> \otimes \beta^>)$ of $\mathcal{D}^>\left(X,\ \mathfrak{F}\left(E^> \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{H}^>(\mathcal{G},B)\right)\right)$; here $\pi\colon \mathcal{G} \to X/\mathcal{G}$ denotes the canonical projection. This gives rise to a map $\Psi_E^>$ from $\tilde{E}^>$ to $J_{\mathcal{A},\mathcal{D}}^B(M_{\mathcal{A},\mathcal{H}}^B(E))^>$. Similarly we define

$$\Psi_E^<\left(\beta^< \otimes e^<\right)_x := \beta_x^< \otimes e^<_{\pi(x)} \in \mathcal{H}^<(\mathcal{G},B)_x \otimes_{\mathcal{A}(\mathcal{G},B)_{\pi(x)}} E^<_{\pi(x)}$$

for all $e^< \in E^<$, $\beta^< \in \Gamma_c(\mathcal{G}, r^*B)$ and $x \in X$, giving us a $\Gamma_c(\mathcal{G}, r^*B)$-linear and $\mathcal{C}_c(X/\mathcal{G})$-linear map $\Psi_E^<$ from $\tilde{E}^<$ to $J_{\mathcal{A},\mathcal{D}}^B(M_{\mathcal{A},\mathcal{H}}^B(E))^<$.

Lemma 7.6.5. $\Psi_E = \left(\Psi_E^<, \Psi_E^>\right)$ *is a pair of* $\mathcal{C}_c(X/\mathcal{G})$*-linear and* $\Gamma_c(\mathcal{G}, r^*B)$*-linear maps, compatible with the brackets on* $\tilde{E}$ *and* $J_{\mathcal{A},\mathcal{D}}^B(M_{\mathcal{A},\mathcal{H}}^B(E))$*.*

Proof. Let $e^> \in E^>$ and $\beta^> \in \Gamma_c(\mathcal{G}, r^*B)$. Let $\chi \in \mathcal{C}_c(X/\mathcal{G})$. Then

$$\Psi_E^>\left(\chi e^> \otimes \beta^>\right)_x = (\chi e^>)_{\pi(x)} \otimes \beta^>_x = \chi(\pi(x))\Psi_E^>\left(e^> \otimes \beta^>\right)_x$$

so $\Psi_E^>$ is $\mathcal{C}_c(X/\mathcal{G})$-linear. If $\beta \in \Gamma_c(\mathcal{G}, r^*B)$, then

$$\Psi_E^>\left((e^> \otimes \beta^>)\beta\right)_x \quad = \quad e^>_{\pi(x)} \otimes (\beta^>\beta)_x$$

whereas

$$\begin{aligned}\left(\Psi_E^>(e^> \otimes \beta^>)\beta\right)_x &= \int_{\mathcal{G}^x} \gamma(e^>_{\pi(x)} \otimes \beta^>_{s(\gamma)})\gamma\beta(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma)\\ &= e^>_{\pi(x)} \otimes \int_{\mathcal{G}^x} \gamma\beta^>_{s(\gamma)}\gamma\beta(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma).\end{aligned}$$

Now

$$\left(\gamma\beta^>_{s(\gamma)}\gamma\beta(\gamma^{-1})\right)(\gamma') = \beta^>(\gamma'\gamma)\gamma'(\gamma\beta(\gamma^{-1}))$$

for all $\gamma' \in \mathcal{G}_x$, and

$$\begin{aligned}&\left[\int_{\mathcal{G}^x} \gamma\beta^>_{s(\gamma)}\gamma\beta(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma)\right](\gamma') = \int_{\mathcal{G}^x} \left[\gamma\beta^>_{s(\gamma)}\gamma\beta(\gamma^{-1})\right](\gamma')\,\mathrm{d}\lambda^x(\gamma)\\ = &\int_{\mathcal{G}^x} \beta^>(\gamma'\gamma)(\gamma'\gamma)\beta(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma) = \int_{\mathcal{G}^x} \beta^>(\gamma)\gamma\beta(\gamma^{-1}\gamma')\,\mathrm{d}\lambda^x(\gamma) = (\beta^> * \beta)(\gamma'),\end{aligned}$$

so

$$\left(\Psi_E^>(e^> \otimes \beta^>)\beta\right)_x = e^>_{\pi(x)} \otimes (\beta^> * \beta)_x$$

as well. Hence $\Psi_E^>$ is $\Gamma_c(\mathcal{G},\ r^*B)$-linear. Similar calculations can be done for the left-hand side.

To see that $\Psi_E = \left(\Psi_E^<, \Psi_E^>\right)$ is compatible with the brackets let $e^< \in E^<$, $e^> \in E^>$, $\beta^<, \beta^> \in \Gamma_c(\mathcal{G},\ r^*B)$ and $\gamma \in \mathcal{G}$. Then

$$\begin{aligned}&\left\langle\!\left\langle \Psi_E^<\left(\beta^< \otimes e^<\right),\ \Psi_E^>\left(e^> \otimes \beta^>\right)\right\rangle\!\right\rangle(\gamma)\\ = &\left\langle \Psi_E^<\left(\beta^< \otimes e^<\right)_{r(\gamma)},\ \gamma\Psi_E^>\left(e^> \otimes \beta^>\right)_{s(\gamma)}\right\rangle_{M_{\mathcal{A},\mathcal{H}}^B(E)_{r(\gamma)}}\\ = &\left\langle \beta^<_{r(\gamma)} \otimes e^<_{\pi(\gamma)},\ \gamma\left(e^>_{\pi(\gamma)} \otimes \beta^>_{s(\gamma)}\right)\right\rangle_{M_{\mathcal{A},\mathcal{H}}^B(E)_{r(\gamma)}}\\ = &\left\langle \beta^<_{r(\gamma)},\ \left\langle e^<_{\pi(\gamma)},\ e^>_{\pi(\gamma)}\right\rangle_{E_{\pi(x)}} \gamma\beta^>_{s(\gamma)}\right\rangle_{\mathcal{H}(\mathcal{G},B)_{r(\gamma)}}\\ = &\left\langle \beta^<_{r(\gamma)},\ \left(\langle e^<,\ e^>\rangle_E\right)_{\pi(\gamma)}\gamma\beta^>_{s(\gamma)}\right\rangle_{\mathcal{H}(\mathcal{G},B)_{r(\gamma)}}.\end{aligned}$$

Write $\alpha := \langle e^<, e^> \rangle \in \Gamma_c(\mathcal{G}, r^*B)$. Now

$$\begin{aligned}\left[\alpha_{\pi(\gamma)}\gamma\beta^>_{s(\gamma)}\right](\gamma') &= \int_{\mathcal{G}^{r(\gamma')}} \alpha(\gamma'')\gamma''(\gamma\beta^>_{s(\gamma)})(\gamma''^{-1}\gamma')\,\mathrm{d}\lambda^{r(\gamma')}(\gamma'') \\ &= \int_{\mathcal{G}^{r(\gamma')}} \alpha(\gamma'')\gamma''\beta^>(\gamma''^{-1}\gamma'\gamma)\,\mathrm{d}\lambda^{r(\gamma')}(\gamma'') = (\alpha * \beta^>)(\gamma'\gamma)\end{aligned}$$

for all $\gamma' \in \mathcal{G}_{r(\gamma)}$. So

$$\begin{aligned}&\left\langle \beta^<_{r(\gamma)},\ \left(\langle e^<,\ e^> \rangle_E\right)_{\pi(\gamma)}\gamma\beta^>_{s(\gamma)} \right\rangle_{\mathcal{H}(\mathcal{G},B)_{r(\gamma)}} \\ = &\int_{\mathcal{G}^{r(\gamma)}} \beta^<(\gamma')\gamma'\left[\left(\alpha(\pi(\gamma))\gamma\beta^>_{s(\gamma)}\right)(\gamma'^{-1})\right]\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\ = &\int_{\mathcal{G}^{r(\gamma)}} \beta^<(\gamma')\gamma'\left[(\alpha * \beta^>)(\gamma'^{-1}\gamma)\right]\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\ = &(\beta^< * \alpha * \beta^>)(\gamma).\end{aligned}$$

Hence

$$\begin{aligned}&\left\langle\!\left\langle \Psi^<_E\left(\beta^< \otimes e^<\right),\ \Psi^>_E\left(e^> \otimes \beta^>\right)\right\rangle\!\right\rangle \\ = &\ \beta^< * \alpha * \beta^> = \beta^< * \langle e^<, e^> \rangle * \beta^> \\ = &\ \langle \beta^< \otimes e^<,\ e^> \otimes \beta^> \rangle_{\tilde{E}}.\end{aligned}$$

□

Definition 7.6.6. Let $S \in \mathrm{L}_{\mathcal{A}(\mathcal{G},B)}(E, F)$ be an operator between the $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G}, B)$-pairs E and F. Define

$$\tilde{S}^> \colon \tilde{E}^> \to \tilde{F}^>,\ \xi^> \otimes \beta^> \mapsto S^>(\xi^>) \otimes \beta^>$$

and

$$\tilde{S}^< \colon \tilde{F}^< \to \tilde{E}^<,\ \beta^< \otimes \xi^< \mapsto \beta^< \otimes S^<(\xi^<).$$

Note that $\tilde{S} := \left(\tilde{S}^<, \tilde{S}^>\right)$ is formally adjoint in the following sense:

$$\begin{aligned}&\left\langle \tilde{S}^<\left(\beta^< \otimes \xi^<\right),\ \xi^> \otimes \beta^> \right\rangle = \beta^< * \left\langle S^<(\xi^<),\ \xi^> \right\rangle * \beta^> \\ = &\ \beta^< * \left\langle \xi^<,\ S^>(\xi^>)\right\rangle * \beta^> = \left\langle \beta^< \otimes \xi^<,\ \tilde{S}^>\left(\xi^> \otimes \beta^>\right)\right\rangle\end{aligned}$$

for all $\beta^<, \beta^> \in \Gamma_c(\mathcal{G}, r^*B)$, $\xi^< \in \Gamma_c(X, F^<)$ and $\xi^> \in \Gamma_c(X, E^>)$.

Lemma 7.6.7. *1. The maps Φ_E and Φ_F intertwine $\tilde{S}$ and S in the obvious sense.*

2. The maps Ψ_E and Ψ_F intertwine $\tilde{S}$ and $J^B_{\mathcal{A},\mathcal{D}}\left(M^B_{\mathcal{A},\mathcal{H}}(S)\right)$.

Proof. We only show that $\Psi^>_E$ and $\Psi^>_F$ intertwine $\tilde{S}^>$ and $J^B_{\mathcal{A},\mathcal{D}}(M^B_{\mathcal{A},\mathcal{H}}(S))^>$. The situation for $\Psi^<_E$ and $\Psi^<_F$ is similar, and also the situation for Φ_E and Φ_F is similar and (even) simpler.

Let $\xi^> \in \Gamma_c(X, E^>)$ and $\beta^> \in \Gamma_c\left(\mathcal{G}, r^*B\right)$. Then

$$\begin{aligned}\Psi^>_F\left(\tilde{S}^>(\xi^> \otimes \beta^>)\right)_x &= \Psi^>_F\left(S^>(\xi^>) \otimes \beta^>\right)_x \\ &= S^>(\xi^>)_{\pi(x)} \otimes \beta^>_x = (S^> \otimes 1)_x\left(\xi^>_{\pi(x)} \otimes \beta^>_x\right)\end{aligned}$$

for all $x \in X$. □

Putting a norm on $\tilde{E}$

Definition 7.6.8 (The completion $\overline{E}$ of $\tilde{E}$). If $\tilde{e}^> \in \tilde{E}^>$, then define

$$\left\|\tilde{e}^>\right\| := \max\left\{\left\|\Phi^>_E(\tilde{e}^>)\right\|,\ \left\|\Psi^>_E(\tilde{e}^>)\right\|\right\}.$$

This is a semi-norm on $\tilde{E}^>$. Let $\overline{E}^>$ be the (Hausdorff-) completion of $\tilde{E}^>$ with respect to this semi-norm. In an analogous fashion, define a semi-norm on $\tilde{E}^<$ and call the completion $\overline{E}^<$.

Lemma 7.6.9. *The actions of $\Gamma_c(\mathcal{G}, r^*B)$ and $\mathcal{C}_c(X/\mathcal{G})$ on $\tilde{E}$ extend to non-degenerate actions of $\mathcal{A}(\mathcal{G}, B)$ and $\mathcal{C}_0(X/\mathcal{G})$ on $\overline{E}$. The bracket on $\tilde{E}$ extends to a continuous bracket on $\overline{E}$.*

Proof. If $\tilde{e}^> \in \tilde{E}^>$, $\beta^> \in \Gamma_c(\mathcal{G},\ r^*B)$, and $\chi \in \mathcal{C}_c(X/\mathcal{G})$, then

$$\begin{aligned}\left\|\tilde{e}^>\beta^>\right\| &= \max\left\{\left\|\Phi^>_E(\tilde{e}^>)\beta^>\right\|,\ \left\|\Psi^>_E(\tilde{e}^>)\beta^>\right\|\right\}\\ &\le \max\left\{\left\|\Phi^>_E(\tilde{e}^>)\right\|\left\|\beta^>\right\|_{\mathcal{A}},\ \left\|\Psi^>_E(\tilde{e}^>)\right\|\left\|\beta^>\right\|_{\mathcal{A}}\right\} = \left\|\tilde{e}^>\right\|\left\|\beta^>\right\|_{\mathcal{A}}\end{aligned}$$

and similarly,

$$\left\|\chi\tilde{e}^>\right\| \le \|\chi\|_\infty \left\|\tilde{e}^>\right\|.$$

So the actions of $\Gamma_c(\mathcal{G}, r^*B)$ and $\mathcal{C}_c(X/\mathcal{G})$ on $\tilde{E}^>$ extend to actions of $\mathcal{A}(\mathcal{G}, B)$ and $\mathcal{C}_0(X/\mathcal{G})$ on $\overline{E}^>$. Similarly for $\overline{E}^<$. It is clear that all the actions are non-degenerate.

If $\tilde{e}^< \in \tilde{E}^<$ and $\tilde{e}^> \in \tilde{E}^>$, then

$$\left\|\left\langle\tilde{e}^<, \tilde{e}^>\right\rangle\right\| = \left\|\left\langle\Phi^<(\tilde{e}^<),\ \Phi^>(\tilde{e}^>)\right\rangle\right\| \le \left\|\Phi^<(\tilde{e}^<)\right\|\left\|\Phi^>(\tilde{e}^>)\right\| \le \left\|\tilde{e}^<\right\|\left\|\tilde{e}^>\right\|.$$

So the bracket on $\tilde{E}$ is contractive. □

Definition and Lemma 7.6.10. The map $\Phi^>_E$ extends by continuity to a continuous linear map from $\overline{E}^>$ to E which is $\mathcal{A}(\mathcal{G}, B)$- and $\mathcal{C}_0(X/\mathcal{G})$-linear. Similar things can be said about $\Phi^<_E$, $\Psi^>_E$ and $\Psi^<_E$. We get homomorphisms Φ_E from $\overline{E}$ to E and Ψ_E from $\overline{E}$ to $J^B_{\mathcal{A},\mathcal{D}}(M^B_{\mathcal{A},\mathcal{H}}(E))$.

Definition and Lemma 7.6.11. Let $S \in \mathrm{L}_{\mathcal{A}(\mathcal{G},B)}(E, F)$ as above. Then the map $\tilde{S}^>$ satisfies

$$\left\|\tilde{S}^>(\tilde{e}^>)\right\| \le \left\|S^>\right\|\left\|\tilde{e}^>\right\|$$

for all $\tilde{e}^> \in \tilde{E}^>$ and extends therefore to an operator $\overline{S}^>$ from $\overline{E}^>$ to $\overline{F}^>$. Analogously for $\tilde{S}^<$. We thus get an element $\overline{S} \in \mathrm{L}_{\mathcal{A}(\mathcal{G},B)}\left(\overline{E}, \overline{F}\right)$ of norm $\le \|S\|$. The map $S \mapsto \overline{S}$ is $\mathbb{C}$-linear and functorial. The homomorphisms Φ_E and Φ_F intertwine $\overline{S}$ and S in the obvious sense and the homomorphisms Ψ_E and Ψ_F intertwine $\overline{S}$ and $J^B_{\mathcal{A},\mathcal{D}}(M^B_{\mathcal{A},\mathcal{H}}(S))$.

By direct comparison of the operators one can show:

Lemma 7.6.12. *Let $e^< \in \Gamma_0(X, E^<)$, $f^> \in \Gamma_0(X, F^>)$, $\beta^<, \beta^> \in \Gamma_c(\mathcal{G}, r^*B)$. If*

$$S = \left|f^>\beta^>\right\rangle\left\langle\beta^< e^<\right| \in \mathrm{K}_{\mathcal{A}(\mathcal{G},B)}(E, F),$$

then

$$\overline{S} = \left|f^> \otimes \beta^>\right\rangle\left\langle\beta^< \otimes e^<\right| \in \mathrm{K}_{\mathcal{A}(\mathcal{G},B)}\left(\overline{E}, \overline{F}\right)$$

and

$$J^B_{\mathcal{A},\mathcal{D}}\left(M^B_{\mathcal{A},\mathcal{H}}(S)\right) = \left|\Psi^>_F(f^>\otimes\beta^>)\right\rangle\left\langle\Psi^<_E(\beta^<\otimes e^<)\right| \in \mathrm{K}_{\mathcal{A}(\mathcal{G},B)}\left(J^B_{\mathcal{A},\mathcal{D}}\left(M^B_{\mathcal{A},\mathcal{H}}(E)\right), J^B_{\mathcal{A},\mathcal{D}}\left(M^B_{\mathcal{A},\mathcal{H}}(F)\right)\right).$$

It follows for all $S \in \mathrm{K}_{\mathcal{A}(\mathcal{G},B)}(E, F)$ that $\overline{S}$ and $J^B_{\mathcal{A},\mathcal{D}}(M^B_{\mathcal{A},\mathcal{H}}(S))$ are compact and that $(\overline{S}, S) \in \mathrm{K}(\Phi_E, \Phi_F)$ as well as $(\overline{S},\ J^B_{\mathcal{A},\mathcal{D}}(M^B_{\mathcal{A},\mathcal{H}}(S))) \in \mathrm{K}(\Psi_E, \Psi_F)$.

The proof of Theorem 7.6.1

Let $(E, T) \in \mathbb{E}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G}, B)\right)$. We show that $\left(\overline{E}, \overline{T}\right)$ is homotopic to (E, T) as well as to $J^B_{\mathcal{A},\mathcal{D}}\left(M^B_{\mathcal{A},\mathcal{H}}(E, T)\right)$.

If $\chi \in \mathcal{C}_c(X/\mathcal{G})$ and $S := \chi(T^2 - 1)$, then $(\overline{S}, S)$ is in $\mathrm{K}(\Phi_E)$ and $\left(\overline{S}, J^B_{\mathcal{A},\mathcal{D}}\left(M^B_{\mathcal{A},\mathcal{H}}(S)\right)\right) \in \mathrm{K}\left(\Psi_E\right)$ by Lemma 7.6.12. If follows that $(\overline{E}, \overline{T})$ is in $\mathbb{E}^{\mathrm{ban}}\left(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G}, B)\right)$ and, using Theorem 2.6.2, that it is homotopic to (E, T) as well as to $J^B_{\mathcal{A},\mathcal{D}}\left(M^B_{\mathcal{A},\mathcal{H}}(E, T)\right)$.

7.7 Embedding E into $\mathcal{H}(\mathcal{G}, E)$ as a summand

An important technical step in the proof of the C*-algebraic version of the generalised Green-Julg theorem is the following: If E is a $\mathcal{G}$-Hilbert B-module, then E is a direct summand of $\mathrm{L}^2(\mathcal{G}, E)$. The proof of this observation makes use of a cut-off function for $\mathcal{G}$.

In the Banach algebraic situation, something similar is true: We can embed a $\mathcal{G}$-Banach B-pair E into the pair $\mathcal{H}(\mathcal{G}, E)$, provided that $\mathcal{H}(\mathcal{G})$ is a locally convex equivariant pair of monotone completions of $\mathcal{C}_c(\mathcal{G})$ and provided that there exists a suitable cut-off pair for $\mathcal{G}$. Actually, we are not going to embed E into $\mathcal{H}(\mathcal{G}, E)$, but, which is the technically correct way of rephrasing this, embed $\Gamma_0(X, E)$ in $\mathcal{H}(\mathcal{G}, E)$.

7.7.1 The embedding on the level of sections with compact support

Definition and Proposition 7.7.1. Let $c = (c^<, c^>)$ be a cut-off pair for $\mathcal{G}$. Let E be a $\mathcal{G}$-Banach space. Define

$$\pi^>_E\colon \Gamma_c\left(\mathcal{G},\ r^*E\right) \to \Gamma_c(X, E),\ \left(\pi^>_E(\xi)\right)(x) := \int_{\mathcal{G}^x} c^<(s(\gamma'))\gamma'\xi(\gamma'^{-1})\,\mathrm{d}\lambda^x(\gamma')$$

and

$$\iota^>_E\colon \Gamma_c(X, E) \to \Gamma_c\left(\mathcal{G},\ r^*E\right),\ \left(\iota^>_E(e)\right)(\gamma) := c^>(r(\gamma))\gamma e(s(\gamma)).$$

Then both maps are $\mathbb{C}$-linear, $\mathcal{C}(X)$-linear[13] and continuous for the inductive limit topologies. Moreover, $\pi^>_E \circ \iota^>_E = \mathrm{Id}_{\Gamma_c(X,E)}$ and $P^>_E := \iota^>_E \circ \pi^>_E$ is a projection.

Proof. Let us first consider $\pi^>_E$: Let ξ be an element of $\Gamma_c\left(\mathcal{G},\ r^*E\right)$. Write K for the support of ξ. The support of $\gamma' \mapsto c^<(s(\gamma'))\gamma'\xi(\gamma'^{-1})$ is contained in K^{-1}, so this is a continuous section of compact support. $s(K)$ is a compact subset of X and if $x \notin s(K)$ then $\pi^>_E(\xi)(x) = 0$, so $\pi^>_E(\xi)$ is a section of compact support, too. The map $\pi^>_E$ is clearly $\mathbb{C}$-linear and $\mathcal{C}(X)$-linear. Note that $\left\|\pi^>_E(\xi)\right\|_\infty \leq \|\xi\|_\infty \sup_{x\in s(K)} \int_{\mathcal{G}^x} c^<(s(\gamma))\,\mathrm{d}\lambda^x(\gamma)$, so $\pi^>_E$ is continuous for the inductive limit topology.

Let us now consider $\iota^>_E$: Let e be in $\Gamma_c(X, E)$. Then the support of $\gamma \mapsto c^>(r(\gamma))\gamma e(s(\gamma))$ is compact by the support property of $c^>$. Moreover, it is a continuous section, so $\iota^>_E$ is well-defined. The map $\iota^>_E$ is $\mathcal{C}(X)$-linear and $\mathbb{C}$-linear. From $\left\|\iota^>_E(e)\right\|_\infty \leq \|e\|_\infty \sup_{s(\gamma)\in \operatorname{supp} e} c^>(r(\gamma))$ it is easy to deduce that $\iota^>_E$ is continuous for the inductive limit topology by noting that the support of $\iota^>_E(e)$ depends monotonously on the support of e.

[13] If we take the action $(\xi\chi)(\gamma) = \xi(\gamma)\chi(s(\gamma))$, $\xi \in \Gamma_c\left(\mathcal{G},\ r^*E\right)$, $\chi \in \mathcal{C}(X)$, $\gamma \in \mathcal{G}$.

If $e \in \Gamma_c(X,E)$, then

$$\begin{aligned}\pi_E^{>}\left(\iota_E^{>}(e)\right)(x) &= \pi_E^{>}\left(\gamma \mapsto c^{>}(r(\gamma))\gamma e(s(\gamma))\right)(x)\\ &= \int_{\mathcal{G}^x} c^{<}(s(\gamma'))c^{>}(r(\gamma'^{-1}))\gamma'\gamma'^{-1}e(s(\gamma'^{-1}))\,\mathrm{d}\lambda^x(\gamma) = e(x)\end{aligned}$$

for all $x \in X$, so $\pi_E^{>} \circ \iota_E^{>}$ is the identity. It follows that $\iota_E^{>} \circ \pi_E^{>}$ is an idempotent. □

Definition 7.7.2. Let $c = (c^{<}, c^{>})$ be a cut-off pair for $\mathcal{G}$. Let E be a $\mathcal{G}$-Banach space. Define

$$\pi_E^{<}\colon \Gamma_c\left(\mathcal{G},\ r^*E\right) \to \Gamma_c(X,E),\ \left(\pi_E^{<}(\xi)\right)(x) := \int_{\mathcal{G}^x} c^{>}(s(\gamma'))\xi(\gamma')\,\mathrm{d}\lambda^x(\gamma')$$

and

$$\iota_E^{<}\colon \Gamma_c(X,E) \to \Gamma_c\left(\mathcal{G},\ r^*E\right),\ \left(\iota_E^{<}(e)\right)(\gamma) := c^{<}(s(\gamma))e(r(\gamma)).$$

Then both maps are $\mathbb{C}$-linear, $\mathcal{C}(X)$-linear and continuous for the inductive limit topologies. Moreover, $\pi_E^{<} \circ \iota_E^{<} = \mathrm{Id}_{\Gamma_c(X,E)}$ and $P_E^{<} := \iota_E^{<} \circ \pi_E^{<}$ is a projection.

Proposition 7.7.3. *Let $c = (c^{<}, c^{>})$ be a cut-off pair for $\mathcal{G}$. Let B be a $\mathcal{G}$-Banach algebra and let E be a $\mathcal{G}$-Banach B-pair. The map*

$$\pi_{E^{>}}^{>}\colon \Gamma_c\left(\mathcal{G},\ r^*E^{>}\right) \to \Gamma_c(X,E^{>})$$

is $\Gamma_c(X,B)$-linear and so is the map

$$\iota_{E^{<}}^{<}\colon \Gamma_c(X,E^{<}) \to \Gamma_c\left(\mathcal{G},\ r^*E^{<}\right).$$

The pair $\left(\iota_{E^{<}}^{<},\ \pi_{E^{>}}^{>}\right)$ satisfies

$$\forall e^{<} \in \Gamma_c\left(X,E^{<}\right)\ \forall \xi^{>} \in \Gamma_c\left(\mathcal{G}, r^*E^{>}\right):\quad \left\langle e^{<},\ \pi_{E^{>}}^{>}(\xi^{>})\right\rangle = \left\langle \iota_{E^{<}}^{<}(e^{<}),\ \xi^{>}\right\rangle.$$

A similar formula is true for the pair $\left(\pi_{E^{<}}^{<},\ \iota_{E^{>}}^{>}\right)$ and thus for the pair $\left(P_{E^{<}}^{<},\ P_{E^{>}}^{>}\right)$ which we also denote by P_E.

Proof. Let $e^{<} \in \Gamma_c\left(X,E^{<}\right)$ and $\xi^{>} \in \Gamma_c\left(\mathcal{G},\ r^*E^{>}\right)$. Then

$$\begin{aligned}\left\langle e^{<},\ \pi_{E^{>}}^{>}(\xi^{>})\right\rangle(x) &= \left\langle e^{<}(x),\ \pi_{E^{>}}^{>}(\xi^{>})(x)\right\rangle = \left\langle e^{<}(x),\ \int_{\mathcal{G}^x} c^{<}(s(\gamma))\gamma\xi^{>}(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma)\right\rangle\\ &= \int_{\mathcal{G}^x}\left\langle e^{<}(x),\ c^{<}(s(\gamma))\gamma\xi^{>}(\gamma^{-1})\right\rangle\,\mathrm{d}\lambda^x(\gamma)\\ &= \int_{\mathcal{G}^x}\left\langle c^{<}(s(\gamma))e^{<}(r(\gamma)),\ \gamma\xi^{>}(\gamma^{-1})\right\rangle\,\mathrm{d}\lambda^x(\gamma) = \left\langle \iota_{E^{<}}^{<}(e^{<}),\ \xi^{>}\right\rangle(x)\end{aligned}$$

for all $x \in X$. The calculations for the other pair is similar. □

7.7.2 $\mathcal{H}(\mathcal{G})$-cut-off pairs

Let $\mathcal{H}(\mathcal{G}) = (\mathcal{H}^<(\mathcal{G}),\ \mathcal{H}^>(\mathcal{G}))$ be an equivariant pair of monotone completions of $\mathcal{C}_c(\mathcal{G})$.

Definition 7.7.4 ($\mathcal{H}(\mathcal{G})$-cut-off pair). Let $c = (c^<, c^>)$ be a cut-off pair for $\mathcal{G}$. Then c is called an $\mathcal{H}(\mathcal{G})$*-cut-off pair* if

$$\forall x \in X : \left\| \mathcal{G}_x \ni \gamma \mapsto c^>(r(\gamma)) \right\|_{\mathcal{H}^>(\mathcal{G}_x)} \leq 1 \tag{7.4}$$

and

$$\forall x \in X : \left\| \mathcal{G}^x \ni \gamma \mapsto c^<(s(\gamma)) \right\|_{\mathcal{H}^<(\mathcal{G}^x)} \leq 1. \tag{7.5}$$

Examples 7.7.5. Assume that $X/\mathcal{G}$ is σ-compact. Let c be a cut-off-function for $\mathcal{G}$.

1. The Proposition 7.1.7 gives a $\mathcal{H}(\mathcal{G})$-cut-off pair (c, d) for the pair $\mathcal{H}(\mathcal{G}) = \left(\mathrm{L}^1(\mathcal{G}),\ \mathcal{C}_0\left(\mathcal{G}\right)\right)$.

2. If $p, p' \in]1, \infty[$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, then $\left(c^{\frac{1}{p'}},\ c^{\frac{1}{p}}\right)$ is a $\mathcal{H}(\mathcal{G})$-cut-off pair for the pair $\mathcal{H}(\mathcal{G}) = \left(\mathrm{L}^{p'}_r\left(\mathcal{G}\right),\ \mathrm{L}^p_s\left(\mathcal{G}\right)\right)$.

Lemma 7.7.6. *If $c = (c^<, c^>)$ is an $\mathcal{H}(\mathcal{G})$-cut-off pair, then equality holds in (7.4) and (7.5).*

Proof. Let $x \in X$. Then $\langle c^<_x,\ c^>_x\rangle_x = \int_{\mathcal{G}^x} c^<(s(\gamma))c^>(r(\gamma^{-1}))\,\mathrm{d}\lambda^x(\gamma) = 1$. It follows that

$$1 \leq \left\| \mathcal{G}^x \ni \gamma \mapsto c^<(s(\gamma)) \right\|_{\mathcal{H}^<(\mathcal{G}^x)} \left\| \mathcal{G}_x \ni \gamma \mapsto c^>(r(\gamma)) \right\|_{\mathcal{H}^>(\mathcal{G}_x)}.$$

If c is an $\mathcal{H}(\mathcal{G})$-cut-off pair, then it follows that both norms have got to be one. □

Proposition 7.7.7. *Let $c = (c^<, c^>)$ be a cut-off pair for $\mathcal{G}$. Then c is an $\mathcal{H}(\mathcal{G})$-cut-off pair if and only if*

$$\forall \chi \in \mathcal{C}_c(X) : \left\| \gamma \mapsto c^>(r(\gamma))\chi(s(\gamma)) \right\|_{\mathcal{H}^>(\mathcal{G})} = \|\chi\|_\infty$$

and

$$\forall \chi \in \mathcal{C}_c(X) : \left\| \gamma \mapsto c^<(s(\gamma))\chi(r(\gamma)) \right\|_{\mathcal{H}^<(\mathcal{G})} = \|\chi\|_\infty .$$

Proof. Assume that c is an $\mathcal{H}(\mathcal{G})$-cut-off pair. Let $\chi \in \mathcal{C}_c(X)$. For all $x \in X$, we have

$$\left\| \mathcal{G}_x \ni \gamma \mapsto c^>(r(\gamma))\chi(s(\gamma)) \right\|_{\mathcal{H}^>(\mathcal{G}_x)} = |\chi(x)| \left\| \mathcal{G}^x \ni \gamma \mapsto c^>(r(\gamma)) \right\|_{\mathcal{H}^>(\mathcal{G}_x)} = |\chi(x)| .$$

Since $\mathcal{H}^>(\mathcal{G})$ is locally convex it follows that

$$\begin{aligned} \left\| \gamma \mapsto c^>(r(\gamma))\chi(s(\gamma)) \right\|_{\mathcal{H}^>(\mathcal{G})} &= \sup_{x \in X} \left\| \mathcal{G}_x \ni \gamma \mapsto c^>(r(\gamma))\chi(s(\gamma)) \right\|_{\mathcal{H}^>(\mathcal{G}_x)} \\ &= \sup_{x \in X} |\chi(x)| = \|\chi\|_\infty . \end{aligned}$$

A similar argumentation works for the left-hand side.

To show the reverse implication suppose that the conditions given in the proposition hold. Let $x \in X$. Let $\chi \in \mathcal{C}_c(X)$ be such that $\chi(x) = 1$ and $0 \le \chi \le 1$. Then by assumption

$$\left\| \gamma \mapsto c^<(s(\gamma))\chi(r(\gamma)) \right\|_{\mathcal{H}^<(\mathcal{G})} = \|\chi\|_\infty = 1.$$

Moreover, for all $\gamma \in \mathcal{G}^x$: $c^<(s(\gamma))\chi(r(\gamma)) = c^<(s(\gamma))$ and hence

$$\begin{aligned} \left\| \mathcal{G}^x \ni \gamma \mapsto c^<(s(\gamma)) \right\|_{\mathcal{H}^<(\mathcal{G}^x)} &= \left\| \mathcal{G}^x \ni \gamma \mapsto c^<(s(\gamma))\chi(r(\gamma)) \right\|_{\mathcal{H}^<(\mathcal{G}^x)} \\ &= \left\| \mathcal{G} \ni \gamma \mapsto c^<(s(\gamma))\chi(r(\gamma)) \right\|_{\mathcal{H}^<(\mathcal{G})_x}. \end{aligned}$$

This last norm is the infimum of $\|\mathcal{G} \ni \gamma \mapsto c^<(s(\gamma))\chi(r(\gamma))\chi'(r(\gamma))\|_{\mathcal{H}^<(\mathcal{G})}$ for all $\chi' \in \mathcal{C}_c(X)$ with $\chi'(x) = 1$ and $0 \le \chi \le 1$. But this is 1. A similar argument holds for $c^>$. □

7.7.3 The embedding of $\Gamma_0(X,E)$ into $\mathcal{H}(\mathcal{G},E)$

Let $\mathcal{H}(\mathcal{G}) = (\mathcal{H}^<(\mathcal{G}),\ \mathcal{H}^>(\mathcal{G}))$ be an equivariant pair of monotone completions of $\mathcal{C}_c(\mathcal{G})$.

Proposition 7.7.8. *Let B be a $\mathcal{G}$-Banach algebra and let E be a $\mathcal{G}$-Banach B-pair. Let $c = (c^<, c^>)$ be an $\mathcal{H}(\mathcal{G})$-cut-off pair for $\mathcal{G}$. Then $\pi^>_{E^>} : \Gamma_c(\mathcal{G},\ r^*E^>) \to \Gamma_c(X, E^>)$ satisfies*

$$\left\|\pi^>_{E^>}(\xi^>)\right\|_\infty \le \left\|\xi^>\right\|_{\mathcal{H}^>(\mathcal{G},E^>)}$$

and

$$\left\|\iota^<_{E^<}(e^<)\right\|_{\mathcal{H}^<(\mathcal{G},E^<)} = \left\|e^<\right\|_\infty$$

for all $\xi^> \in \Gamma_c(\mathcal{G},\ r^*E^>)$ and $e^> \in \Gamma_c(X, E^<)$. So we can extend $\pi^>_{E^>}$ to a contractive operator $\mathcal{H}^>(\mathcal{G},E^>)$ from $\Gamma_0(\mathcal{G},\ r^*E^>)$ and $\iota^<_{E^<}$ to an isometric operator from $\Gamma_0(\mathcal{G},\ r^*E^<)$ to $\mathcal{H}^>(\mathcal{G},E^<)$, both $\mathcal{C}_0(X)$-linear and $\Gamma_0(X,B)$. This gives a pair

$$\pi_E := \left(\iota^<_{E^<},\ \pi^>_{E^>}\right) \in \mathrm{L}^{\mathcal{C}_0(X)}_{\Gamma_0(X,B)}\left(\mathcal{H}(\mathcal{G},E),\ \Gamma_0(X,E)\right).$$

Similarly, we can construct a pair

$$\iota_E = \left(\pi^<_{E^<},\ \iota^>_{E^>}\right) \in \mathrm{L}_B\left(\Gamma_0(X,E),\ \mathcal{H}(\mathcal{G},E)\right)$$

of norm ≤ 1. The operators satisfy

$$\pi_E \circ \iota_E = \mathrm{Id}_E\,.$$

We hence get an idempotent

$$P_E := \iota_E \circ \pi_E \in \mathrm{L}_B\left(\mathcal{H}(\mathcal{G},E),\ \mathcal{H}(\mathcal{G},E)\right)$$

of norm ≤ 1.

Proof. Let $\xi^> \in \Gamma_c(\mathcal{G},\ r^*E^>)$. Find a function $\chi \in \mathcal{C}_c(X)$ such that $0 \le \chi \le 1$ and $\chi \equiv 1$ on $s(\operatorname{supp}\xi^>)$. It follows that $\xi\chi = \xi$. We calculate

$$\begin{aligned} \left\|\pi^>_{E^>}(\xi^>)\right\|_\infty &= \left\| x \mapsto \int_{\mathcal{G}^x} c^<(s(\gamma))\gamma\xi^>(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma) \right\|_\infty \\ &\le \left\| x \mapsto \int_{\mathcal{G}^x} c^<(s(\gamma))\chi(r(\gamma))\left\|\xi^>(\gamma^{-1})\right\|\,\mathrm{d}\lambda^x(\gamma) \right\|_\infty \\ &\le \underbrace{\left\| \gamma \mapsto c^<(s(\gamma))\chi(r(\gamma)) \right\|_{\mathcal{H}^<(\mathcal{G})}}_{\le 1} \left\| \gamma \mapsto \left\|\xi^>(\gamma)\right\| \right\|_{\mathcal{H}^>(\mathcal{G})} \\ &\le \|\xi\|_{\mathcal{H}^>(\mathcal{G},E^>)}\,. \end{aligned}$$

Now let $e^< \in \Gamma_c(X, E^<)$. Let $\chi \in \mathcal{C}_c(X)$ be such that $0 \le \chi \le 1$ and $\chi \equiv 1$ on $\operatorname{supp} e^<$. We have

$$\begin{aligned}\left\|\gamma \mapsto c^<(s(\gamma))e^<(r(\gamma))\right\|_{\mathcal{H}^<(\mathcal{G},E^<)} &= \left\|\gamma \mapsto c^<(s(\gamma))\left\|\chi(r(\gamma))e^<(r(\gamma))\right\|\ \right\|_{\mathcal{H}^<(\mathcal{G},E^<)} \\ &\le \left\|e^<\right\|_\infty \left\|\gamma \mapsto c^<(s(\gamma))\chi(r(\gamma))\right\|_{\mathcal{H}^<(\mathcal{G})} \le \left\|e^<\right\|_\infty .\end{aligned}$$

The calculations for ι_E (and hence for P_E) are almost identical. □

Corollary 7.7.9. *If an $\mathcal{H}(\mathcal{G})$-cut-off pair exists, then we can regard E as a summand of $\mathcal{H}(\mathcal{G}, E)$.*

7.8 $M_{\mathcal{A}}^B \circ J_{\mathcal{A}}^B = \mathrm{Id}$ on the level of $\mathrm{KK}^{\mathrm{ban}}$

Let $\mathcal{G}$ be a proper locally compact Hausdorff groupoid with unit space X and Haar system λ. Let $\mathcal{A}(\mathcal{G})$ be a regular unconditional completion of $\mathcal{C}_c(\mathcal{G})$ acting on the equivariant pair $\mathcal{H}(\mathcal{G})$ of locally convex monotone completions. Let B be a non-degenerate $\mathcal{G}$-Banach algebra. Assume that there exists an $\mathcal{H}(\mathcal{G})$-cut-off pair $c = (c^<, c^>)$.

Theorem 7.8.1. $M_{\mathcal{A}}^B \circ J_{\mathcal{A}}^B = \mathrm{Id}$ *as an endomorphism of* $\mathrm{RKK}^{\mathrm{ban}}(\mathcal{C}_0(X/\mathcal{G});\ \mathcal{C}_0(X/\mathcal{G}),\ \mathcal{A}(\mathcal{G}, B))$.

Idea of the proof: If $(E, T) \in \mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(\mathcal{C}_0(X),\ B)$ with $\mathcal{G}$-equivariant T. Then we define a homomorphism Φ_E from $M_{\mathcal{A},\mathcal{H}}^B(J_{\mathcal{A},\mathcal{A}^c}^B(E))$ to E that commutes with the operator $M_{\mathcal{A},\mathcal{H}}^B(J_{\mathcal{A},\mathcal{A}^c}^B(T))$ and T. Note that we use the particular pair $\mathcal{A}^c(X)$ of monotone completions of $\mathcal{C}_c(X)$ here; in our proof it is important that we take $\mathcal{A}^c(X)$ for the $\mathcal{H}(\mathcal{G})$-cut-off pair c to make the calculations work.

The main difficulty of the proof will then be to check that Φ_E really gives a homotopy between $M_{\mathcal{A},\mathcal{H}}^B(J_{\mathcal{A},\mathcal{A}^c}^B(E, T))$ and (E, T); this is carried out at the end of this section.

To define Φ_E we introduce a bilinear contractive map $\mu_E^>$ from $\mathcal{A}^c(X, E^>) \times \mathcal{H}^>(\mathcal{G}, B)$ to $\Gamma_0(X, E^>)$, and similar on the left-hand side, and show that these maps give rise to a homomorphism $\hat{\mu}_E$ from $\mathcal{A}^c(X, E) \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{H}(\mathcal{G}, B)$ to $\Gamma_0(X, E)$ that intertwines $J_{\mathcal{A},\mathcal{A}^c}^B(T) \otimes 1$ and $\mathfrak{M}(T)$. Then $\Phi_E := \mathfrak{F}(\hat{\mu}_E)$ is the homomorphism we are looking for; we just have to show that it is $\mathcal{G}$-equivariant. This part of the construction can and will be carried out for general $\mathcal{G}$-Banach B-modules E and not only for cycles (E, T).

Construction of $\mu_E^>$ and $\hat{\mu}_E^>$: Let E be a $\mathcal{G}$-Banach B-pair. Let $e^> \in \Gamma_c(X, E^>) \subseteq \mathcal{A}^c(X, E^>)$ and $\beta^> \in \Gamma_c(\mathcal{G},\ r^*B) \subseteq \mathcal{H}(\mathcal{G}, B)$. Then $j_{E,c}^>(e^>) \in \Gamma_c(\mathcal{G},\ r^*E^>) \subseteq \mathcal{A}(\mathcal{G}, E^>)$. More generally, if $\xi^> \in \Gamma_c(\mathcal{G},\ r^*E^>) \subseteq \mathcal{A}(\mathcal{G}, E^>)$, then define

$$(\xi^>\beta^>)(\gamma) := (\xi^> * \beta^>)(\gamma) = \int_{\mathcal{G}^{r(\gamma)}} \xi^>(\gamma')\gamma'\beta^>(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')$$

for all $\gamma \in \mathcal{G}$. This defines an element of $\Gamma_c(\mathcal{G},\ r^*E^>) \subseteq \mathcal{H}^>(\mathcal{G}, E^>)$ with $\|\xi^>\beta^>\|_{\mathcal{H}^>(\mathcal{G},E^>)} \le \|\xi^>\|_{\mathcal{A}(\mathcal{G},E^>)} \|\beta^>\|_{\mathcal{H}^>(\mathcal{G},B)}$. So this product extends to a bilinear map $\mathcal{A}(\mathcal{G},\ E^>) \times \mathcal{H}^>(\mathcal{G},\ B) \to \mathcal{H}^>(\mathcal{G},\ E^>)$ which is $\mathcal{A}(\mathcal{G}, B)$-balanced, $\mathcal{C}_0(X/\mathcal{G})$-balanced, and $\Gamma_0(X, B)$-linear as well as $\mathcal{C}_0(X)$-linear ont the right.

Now

$$\begin{aligned}
\left(j^{>}_{E,c}(e^{>})\beta^{>}\right)(\gamma) &= \int_{\mathcal{G}^{r(\gamma)}} (j^{>}_{E,c}(e^{>}))(\gamma')\,\gamma'\beta^{>}(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= \int_{\mathcal{G}^{r(\gamma)}} c^{>}(r(\gamma'))\gamma' e^{>}(s(\gamma'))\,\gamma'\beta^{>}(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= c^{>}(r(\gamma)) \int_{\mathcal{G}^{r(\gamma)}} \gamma' e^{>}(s(\gamma'))\,\gamma'\beta^{>}(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= c^{>}(r(\gamma))\,\gamma \int_{\mathcal{G}^{s(\gamma)}} \gamma' e^{>}(s(\gamma'))\,\gamma'\beta^{>}(\gamma'^{-1})\,\mathrm{d}\lambda^{s(\gamma)}(\gamma')
\end{aligned}$$

for all $\gamma \in \mathcal{G}$. Define

$$\mu^{>}_{E}(e^{>}, \beta^{>})\,(x) := (e^{>}\beta^{>})(x) := \int_{\mathcal{G}^x} \gamma e^{>}(s(\gamma))\,\gamma\beta^{>}(\gamma^{-1})\,\mathrm{d}\lambda^{x}(\gamma)$$

for all $x \in X$. Then $j^{>}_{E,c}(e^{>})\beta^{>} = \iota^{>}_{E}(e^{>}\beta^{>})$ or, equivalently, $e^{>}\beta^{>} = \pi^{>}_{E}\left(j^{>}_{E,c}(e^{>})\beta^{>}\right)$. It follows that

$$\left\|e^{>}\beta^{>}\right\|_{\infty} = \left\|\pi^{>}_{E}\left(j^{>}_{E,c}(e^{>})\beta^{>}\right)\right\|_{\infty} \overset{7.7.8}{\leq} \left\|j^{>}_{E,c}(e^{>})\beta^{>}\right\|_{\mathcal{H}^{>}(\mathcal{G},E^{>})} \leq \left\|e^{>}\right\|_{\mathcal{A}^c(X,E^{>})} \left\|\beta^{>}\right\|_{\mathcal{H}^{>}(\mathcal{G},B)}.$$

So we get a contractive bilinear map $\mu^{>}_{E}\colon \mathcal{A}^c(X,E^{>}) \times \mathcal{H}^{>}(\mathcal{G},B) \to \mathcal{C}_0(X,E^{>})$ which is $\mathcal{A}(\mathcal{G},B)$-balanced, $\mathcal{C}_0(X/\mathcal{G})$-balanced, and $\Gamma_0(X,B)$-linear as well as $\mathcal{C}_0(X)$-linear on the right. This map $\mu^{>}_{E}$ induces a contractive linear map $\hat{\mu}^{>}_{E}\colon \mathcal{A}^c(X,E^{>}) \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{H}^{>}(\mathcal{G},B) \to \Gamma_0(X,E^{>})$ which is $\mathcal{C}_0(X)$-linear and $\Gamma_0(X,B)$-linear on the right.

Construction of $\mu^{<}_{E}$ and $\hat{\mu}^{<}_{E}$: A similar argument on the left-hand side gives a a contractive bilinear map $\mu^{<}_{E}\colon \mathcal{H}^{<}(\mathcal{G},B) \times \mathcal{A}^c(X,E^{<}) \to \mathcal{C}_0(X,E^{<})$ which is $\mathcal{A}(\mathcal{G},B)$-balanced, $\mathcal{C}_0(X/\mathcal{G})$-balanced, and $\Gamma_0(X,B)$-linear as well as $\mathcal{C}_0(X)$-linear on the left. For $\beta^{<} \in \Gamma_c(\mathcal{G},\ r^*B) \subseteq \mathcal{H}^{<}(\mathcal{G},B)$ and $e^{<} \in \Gamma_c(X,E^{<}) \subseteq \mathcal{A}^c(X,E^{<})$, it is given by

$$\mu^{<}_{E}(\beta^{<}, e^{<})\,(x) := (\beta^{<}e^{<})(x) = \int_{\mathcal{G}^x} \beta^{<}(\gamma)\gamma e^{<}(s(\gamma))\,\mathrm{d}\lambda^{x}(\gamma)$$

for all $x \in X$. This induces a contractive linear map $\hat{\mu}^{<}_{E}\colon \mathcal{H}^{<}(\mathcal{G},B) \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{A}^c(X,E^{<}) \to \mathcal{C}_0(X,E^{<})$ which is $\mathcal{C}_0(X)$-linear and $\Gamma_0(X,B)$-linear on the left.

The concurrent homomorphism $\hat{\mu}_E$: We check that $\hat{\mu}_E = (\hat{\mu}^{<}_{E},\ \hat{\mu}^{>}_{E})$ is a homomorphism; this follows almost by construction, but we give a direct proof: Let $\beta^{<}, \beta^{>} \in \Gamma_c(\mathcal{G},\ r^*B)$, $e^{<} \in \Gamma_c(X,E^{<})$ and $e^{>} \in \Gamma_c(X,E^{>})$. We have

$$\left\langle \beta^{<} \otimes e^{<},\ e^{>} \otimes \beta^{>} \right\rangle = \left\langle \beta^{<},\ \langle e^{<},\ e^{>}\rangle \beta^{>} \right\rangle.$$

Now

$$\begin{aligned}
\left(\left\langle e^{<},\, e^{>}\right\rangle \beta^{>}\right)(\gamma) &= \int_{\mathcal{G}^{r(\gamma)}} \left(\langle e^{<}, e^{>}\rangle(\gamma')\right)\, \gamma'\beta^{>}(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= \int_{\mathcal{G}^{r(\gamma)}} \left\langle e^{<}(r(\gamma')),\, \gamma' e^{>}(s(\gamma'))\right\rangle\, \gamma'\beta^{>}(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= \left\langle e^{<}(r(\gamma)),\, \int_{\mathcal{G}^{r(\gamma)}} \gamma' e^{>}(s(\gamma'))\, \gamma'\beta^{>}(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')\right\rangle \\
&= \left\langle e^{<}(r(\gamma)),\, \gamma\int_{\mathcal{G}^{s(\gamma)}} \gamma' e^{>}(s(\gamma'))\, \gamma'\beta^{>}(\gamma'^{-1})\,\mathrm{d}\lambda^{s(\gamma)}(\gamma')\right\rangle \\
&= \left\langle e^{<}(r(\gamma)),\, \gamma(e^{>}\beta^{>})(s(\gamma))\right\rangle
\end{aligned}$$

for all $\gamma \in \mathcal{G}$ and hence

$$\begin{aligned}
\left\langle \beta^{<},\, \langle e^{<},\, e^{>}\rangle\beta^{>}\right\rangle(x) &= \int_{\mathcal{G}^x} \beta^{<}(\gamma)\gamma\left(\left\langle e^{<},\, e^{>}\right\rangle\beta^{>}\right)(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma) \\
&= \int_{\mathcal{G}^x} \beta^{<}(\gamma)\gamma\left\langle e^{<}(s(\gamma)),\, \gamma^{-1}(e^{>}\beta^{>})(r(\gamma))\right\rangle\,\mathrm{d}\lambda^x(\gamma) \\
&= \int_{\mathcal{G}^x} \beta^{<}(\gamma)\left\langle \gamma e^{<}(s(\gamma)),\, (e^{>}\beta^{>})(r(\gamma))\right\rangle\,\mathrm{d}\lambda^x(\gamma) \\
&= \left\langle \int_{\mathcal{G}^x} \beta^{<}(\gamma)\gamma e^{<}(s(\gamma))\,\mathrm{d}\lambda^x(\gamma),\, (e^{>}\beta^{>})(x)\right\rangle \\
&= \langle(\beta^{<}e^{<})(x),\, (e^{>}\beta^{>})(x)\rangle = \langle\beta^{<}e^{<},\, e^{>}\beta^{>}\rangle(x) \\
&= \left\langle \mu_E^{\leq}(\beta^{<}, e^{<}),\, \mu_E^{\geq}(e^{>}, \beta^{>})\right\rangle(x) \\
&= \left\langle \hat{\mu}_E^{\leq}(\beta^{<}\otimes e^{<}),\, \hat{\mu}_E^{\geq}(e^{>}\otimes\beta^{>})\right\rangle(x)
\end{aligned}$$

for all $x \in X$. So $\hat{\mu}_E$ respects the brackets.

The $\mathcal{G}$-equivariant concurrent homomorphism Φ_E: Define $\Phi_E := \mathfrak{F}(\hat{\mu}_E)$, which is a concurrent homomorphism from $M_{\mathcal{H}}(E)$ to $\mathfrak{F}(\Gamma_0(X, E)) \cong E$. We now show that Φ_E is $\mathcal{G}$-equivariant.

Let $\gamma \in \mathcal{G}$, $e^{>} \in \Gamma_c(\pi(\gamma), E^{>}) \subseteq \mathcal{A}^c(X, E^{>})_{\pi(\gamma)}$ and $\beta^{>} \in \Gamma_c(\mathcal{G}_{s(\gamma)}, r^*B) \subseteq \mathcal{H}^{>}(\mathcal{G}, B)_{s(\gamma)}$. Then $e^{>}\otimes\beta^{>} \in \left(\mathcal{A}^c(X, E^{>})\otimes_{\mathcal{A}(\mathcal{G},B)}\mathcal{H}^{>}(\mathcal{G}, B)\right)_{s(\gamma)}$ and

$$\gamma(e^{>}\otimes\beta^{>}) = e^{>}\otimes\left(\gamma' \mapsto \beta^{>}(\gamma'\gamma)\right),$$

so

$$\begin{aligned}
(\Phi_E^{\geq})_{r(\gamma)}\left(\gamma(e^{>}\otimes\beta^{>})\right) &= \int_{\mathcal{G}^{r(\gamma)}} \gamma' e^{>}(s(\gamma'))\gamma'\beta^{>}(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= \int_{\mathcal{G}^{s(\gamma)}} \gamma\gamma' e^{>}(s(\gamma\gamma'))\gamma\gamma'\beta^{>}(\gamma'^{-1})\,\mathrm{d}\lambda^{s(\gamma)}(\gamma') \\
&= \gamma\int_{\mathcal{G}^{s(\gamma)}} \gamma' e^{>}(s(\gamma'))\gamma'\beta^{>}(\gamma'^{-1})\,\mathrm{d}\lambda^{s(\gamma)}(\gamma') \\
&= \gamma\left[(\Phi_E^{\geq})_{s(\gamma)}\left(e^{>}\otimes\beta^{>}\right)\right].
\end{aligned}$$

So $\Phi_E^{\geq}$ is equivariant. Now let $e^{<} \in \Gamma_c(\pi(\gamma), E^{<}) \subseteq \mathcal{A}^c(X, E^{<})_{\pi(x)}$ and $\beta^{<} \in \Gamma_c(\mathcal{G}^{s(\gamma)}, r^*B) \subseteq \mathcal{H}^{<}(\mathcal{G}, B)_{s(\gamma)}$. Then $\beta^{<}\otimes e^{<} \in \left(\mathcal{H}^{<}(\mathcal{G}, B)\otimes_{\mathcal{A}(\mathcal{G},B)}\mathcal{A}^c(X, E^{<})\right)_{s(\gamma)}$ and

$$\gamma(\beta^{<}\otimes e^{<}) = \left(\gamma' \mapsto \gamma'\beta^{<}(\gamma'^{-1}\gamma)\right)\otimes e^{<},$$

so

$$\begin{aligned}
(\Phi^<_E)_{r(\gamma)}\left(\gamma(\beta^< \otimes e^<)\right) &= \int_{\mathcal{G}^{r(\gamma)}} \gamma'\beta^<(\gamma^{-1}\gamma')\gamma' e^<(s(\gamma'))\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\
&= \int_{\mathcal{G}^{s(\gamma)}} \gamma\gamma'\beta^<(\gamma')\gamma\gamma' e^<(s(\gamma\gamma'))\,\mathrm{d}\lambda^{s(\gamma)}(\gamma') \\
&= \gamma \int_{\mathcal{G}^{s(\gamma)}} \gamma'\beta^<(\gamma')\gamma' e^<(s(\gamma'))\,\mathrm{d}\lambda^{s(\gamma)}(\gamma') \\
&= \gamma\left[(\Phi^<_E)_{s(\gamma)}\left(\beta^< \otimes e^<\right)\right].
\end{aligned}$$

Hence also $\Phi^<_E$ is $\mathcal{G}$-equivariant.

$\hat{\mu}_E$ intertwines $\mathcal{A}^c(X,T) \otimes 1$ and $\mathfrak{M}(T)$: Let E and F be a $\mathcal{G}$-Banach B-pair and let $T = (T^<, T^>) \in \mathrm{L}_B(E,F)$ be a $\mathcal{G}$-equivariant operator. We show

$$\hat{\mu}^>_F \circ \left(\mathcal{A}^c(X,T)^> \otimes 1\right) = \mathfrak{M}(T)^> \circ \hat{\mu}^>_E$$

and the analogous equation for the left-hand side. Let $e^> \in \Gamma_c(X, E^>) \subseteq \mathcal{A}^c(X, E^>)$ and $\beta^> \in \Gamma_c(\mathcal{G}, r^*B) \subseteq \mathcal{H}^>(\mathcal{G}, B)$. Then

$$\begin{aligned}
&\hat{\mu}^>_F\left(\left(\mathcal{A}^c(X,T)^> \otimes 1\right)(e^> \otimes \beta^>)\right)(x) = \hat{\mu}^>_F\left((x' \mapsto T^>_{x'} e^>(x')) \otimes \beta^>\right)(x) \\
= &\int_{\mathcal{G}^x} \gamma T^>_{s(\gamma)} e^>(s(\gamma))\gamma\beta^>(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma) = \int_{\mathcal{G}^x} T^>_{r(\gamma)}\gamma e^>(s(\gamma))\gamma\beta^>(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma) \\
= &T^>_x\left(\int_{\mathcal{G}^x} \gamma e^>(s(\gamma))\gamma\beta^>(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma)\right) = T^>_x\left(\hat{\mu}^>_E(e^> \otimes \beta^>)(x)\right) \\
= &\mathfrak{M}(T)^>\left(\hat{\mu}^>_E(e^> \otimes \beta^>)\right)(x)
\end{aligned}$$

for all $x \in X$. A similar calculation goes through on the left-hand side.

Φ_E intertwines $\mathfrak{F}(\mathcal{A}^c(X,T) \otimes 1)$ and T: This follows from the fact that $\mathfrak{F}(\cdot)$ is a functor (on the level of Banach spaces, say).

$\hat{\mu}_E$ induces a homotopy: Now we show that if S is a bounded locally compact $\mathcal{G}$-equivariant operator from E to F, then not only is $\mathcal{A}^c(X,S) \otimes 1$ bounded and locally compact, but the pair $(\mathcal{A}^c(X,S) \otimes 1, \mathfrak{M}(S))$ is a locally compact element of $\mathrm{L}_{\mathrm{Id}}(\hat{\mu}_E, \hat{\mu}_F)$, i.e., we can approximate $\mathcal{A}^c(X,S) \otimes 1$ and $\mathfrak{M}(S)$ simultaneously with finite rank operators. *This is the main technical difficulty of this part of the proof.* Applying this result to the operator $S = T^2 - 1$, where (E,T) is a $\mathrm{KK}^{\mathrm{ban}}$-cycle, shows that $(\mathfrak{M}(E), \mathfrak{M}(T))$ and $(\mathcal{A}^c(X,E) \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{H}(\mathcal{G},B),\ \mathcal{A}^c(X,T) \otimes 1)$ are homotopic elements of $\mathbb{E}^{\mathrm{ban}}(\mathcal{C}_0(X);\ \mathcal{C}_0(X),\ \Gamma_0(X,B))$; for this, we use the sufficient condition for homotopy given in Theorem 2.6.2. It also follows, this time from Theorem 3.7.1, that (E,T) and $M^B_{\mathcal{A},\mathcal{H}}(J^B_{\mathcal{A},\mathcal{A}^c}(E,T))$ are $\mathcal{G}$-equivariantly homotopic (because Φ_E is $\mathcal{G}$-equivariant).

By Corollary 7.2.14 it suffices to consider the case $S = (\left|f^> \cdot \beta\right\rangle\!\rangle\langle\!\langle e^<\big|_x)_{x \in X}$ with $f^> \in \Gamma_c(X, F^>)$, $\beta \in \Gamma_c(\mathcal{G}, r^*B)$ and $e^< \in \Gamma_c(X, E <)$. Let $\chi \in \mathcal{C}_c(X)$. We show that $(\chi\mathcal{A}^c(X,S) \otimes 1, \chi\mathfrak{M}(S))$ is in $\mathrm{K}(\hat{\mu}_E,\ \hat{\mu}_F)$. Let $\varepsilon > 0$. We now concentrate on the right-hand side of the operators because the calculations for the left-hand side are similar and similarly unedifying.

Let $k \in \Gamma_c(\mathcal{G} \times_{r,r} \mathcal{G},\ p^*B)$ with $p\colon \mathcal{G} \times_{r,r} \mathcal{G} \to X, (\gamma_1, \gamma_2) \mapsto r(\gamma_1) = r(\gamma_2)$. Define

$$\tau^>_k\left(e^>\right)(x) := \int_{\mathcal{G}^x}\int_{\mathcal{G}^x} \gamma f^>(s(\gamma))\gamma k(\gamma^{-1}, \gamma^{-1}\gamma')\langle\gamma' e^<(s(\gamma')),\ e^>(x)\rangle\,\mathrm{d}\lambda^x(\gamma')\,\mathrm{d}\lambda^x(\gamma)$$

for all $e^> \in \Gamma_c(X, E^>)$ and $x \in X$. Then $\tau_k^>(e^>) \in \Gamma_c(X, F^>)$.

Define $k_0(\gamma, \gamma') = \chi(s(\gamma))\beta(\gamma')$ for all $(\gamma, \gamma') \in \mathcal{G} \times_{r,r} \mathcal{G}$. Then

$$\begin{aligned}
\tau_{k_0}^> \left(e^>\right)(x) &= \int_{\mathcal{G}^x}\int_{\mathcal{G}^x} \gamma f^>(s(\gamma))\gamma\chi(r(\gamma))\beta(\gamma^{-1}\gamma')\langle\gamma' e^<(s(\gamma')),\ e^>(x)\rangle \,\mathrm{d}\lambda^x(\gamma)\,\mathrm{d}\lambda^x(\gamma') \\
&= \chi(x)\int_{\mathcal{G}^x}\int_{\mathcal{G}^{s(\gamma')}} \gamma'\gamma f^>(s(\gamma))\gamma'\gamma\beta(\gamma^{-1})\,\mathrm{d}\lambda^{s(\gamma')}(\gamma)\langle\gamma' e^<(s(\gamma')),\ e^>(x)\rangle \,\mathrm{d}\lambda^x(\gamma') \\
&= \chi(x)\int_{\mathcal{G}^x} \gamma'\left((f^> \cdot \beta)(s(\gamma'))\right)\langle\gamma' e^<(s(\gamma')),\ e^>(x)\rangle \,\mathrm{d}\lambda^x(\gamma') \\
&= \chi(x)\int_{\mathcal{G}^x} \left|\gamma'(f^> \cdot \beta)(s(\gamma'))\right\rangle\left\langle\gamma' e^<(s(\gamma'))\right|\, e^>(x)\rangle \,\mathrm{d}\lambda^x(\gamma') \\
&= \chi(x)\left|f^> \cdot \beta\right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle e^<\right|_x^> (e^>(x))
\end{aligned}$$

for all $e^> \in \Gamma_c(X, E^>)$ and $x \in X$, so $(\tau_{k_0}^>)_x = \chi(x)\left|f^> \cdot \beta\right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle e^<\right|_x^>$ in this case.

If $k(\gamma, \gamma') = h^>(\gamma)\gamma h^<(\gamma^{-1}\gamma')$ for all $(\gamma, \gamma') \in \mathcal{G} \times_{r,r} \mathcal{G}$, where $h^>, h^< \in \Gamma_c(\mathcal{G}, r^*B)$, then

$$\begin{aligned}
\tau_k^> \left(e^>\right)(x) &= \int_{\mathcal{G}^x}\int_{\mathcal{G}^x} \gamma f^>(s(\gamma))\gamma\left(h^>(\gamma^{-1})\gamma^{-1}h^<(\gamma')\right)\langle\gamma' e^<(s(\gamma')),\ e^>(x)\rangle \,\mathrm{d}\lambda^x(\gamma')\,\mathrm{d}\lambda^x(\gamma) \\
&= \int_{\mathcal{G}^x}\int_{\mathcal{G}^x} \gamma f^>(s(\gamma))\gamma h^>(\gamma^{-1})h^<(\gamma')\gamma'\langle e^<(r(\gamma'^{-1})),\ \gamma'^{-1}e^>(s(\gamma'^{-1}))\rangle \,\mathrm{d}\lambda^x(\gamma')\,\mathrm{d}\lambda^x(\gamma) \\
&= \int_{\mathcal{G}^x} \gamma f^>(s(\gamma))\gamma h^>(\gamma^{-1})\int_{\mathcal{G}^x} h^<(\gamma')\gamma'\langle\!\langle e^<,\ e^>\rangle\!\rangle(\gamma'^{-1})\,\mathrm{d}\lambda^x(\gamma')\,\mathrm{d}\lambda^x(\gamma) \\
&= \int_{\mathcal{G}^x} \gamma f^>(s(\gamma))\gamma h^>(\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma)\ \left\langle h^<,\ \langle\!\langle e^<,\ e^>\rangle\!\rangle\right\rangle(x) \\
&= \left(f^> \cdot h^>\right)(x)\ \left\langle h^<,\ \langle\!\langle e^<,\ e^>\rangle\!\rangle\right\rangle(x) \\
&= \left|\hat\mu_F^>(f^> \otimes h^>)\right\rangle\left\langle\hat\mu_E^<(h^< \otimes e^<)\right|^>(e^>)(x)
\end{aligned}$$

for all $e^> \in \Gamma_c(X, E^>)$ and $x \in X$, so $\tau_k^> = \left|\hat\mu_F^>(f^> \otimes h^>)\right\rangle\left\langle\hat\mu_E^<(h^< \otimes e^<)\right|^>$.

The idea is to approximate $k_0\colon (\gamma, \gamma') \mapsto \chi(s(\gamma))\beta(\gamma')$ by functions of the form $k\colon (\gamma, \gamma') \mapsto \sum_{i=1}^n h_i^>(\gamma)\gamma h_i^<(\gamma^{-1}\gamma')$ (in a sense which we have to specify) so that $\sum_{i=1}^n \left|f^> \otimes h_i^>\right\rangle\left\langle h_i^< \otimes e^<\right|^>$ approximates $\chi\mathcal{A}^c(X,S)\otimes 1 = \chi\left|f^> \cdot \beta\right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle e^<\right|^> \otimes 1$ and $\tau_k^> = \sum_{i=1}^n \left|\hat\mu_F^>(f^> \otimes h_i^>)\right\rangle\left\langle\hat\mu_E^<(h_i^< \otimes e^<)\right|^>$ approximates at the same time $\mathfrak{M}(S) = \tau_{k_0}^> = \mathfrak{M}((\chi(x)\left|f^> \cdot \beta\right\rangle\!\!\left\rangle\!\!\left\langle\!\!\left\langle e^<\right|_x^>)_{x\in X})$. To prove this we will show that $\tau_k^>$ depends continuously (in a sense that we have to specify as well) on the function k.

On $\Gamma_c(\mathcal{G} \times_{r,r} \mathcal{G},\ p^*B)$ we take the inductive limit topology.

If we map $k \in \Gamma_c(\mathcal{G} \times_{r,r} \mathcal{G},\ p^*B)$ to the functions

$$T_k^> \colon \Gamma_c(\mathcal{G},\ r^*B) \to \Gamma_c(\mathcal{G},\ r^*B),\ \xi^> \mapsto \left[\gamma \mapsto \int_{\mathcal{G}^{r(\gamma)}} k(\gamma, \gamma')\gamma'\xi^>(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')\right]$$

and

$$T_k^< \colon \Gamma_c(\mathcal{G},\ r^*B) \to \Gamma_c(\mathcal{G},\ r^*B),\ \xi^< \mapsto \left[\gamma \mapsto \int_{\mathcal{G}^{r(\gamma)}} \xi^<(\gamma')\gamma' k(\gamma'^{-1}, \gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')\right],$$

then $(T_k^<, T_k^>)$ extends continuously to an element of $T_k \in \mathrm{L}_{\Gamma_0(X,B)}(\mathcal{H}(\mathcal{G}, B))$. The operator T_k

depends continuously on k. If $k(\gamma, \gamma) = h^>(\gamma)\gamma h^<(\gamma^{-1}\gamma')$, then

$$\begin{aligned} T_k^>(\xi^>)(\gamma) &= \int_{\mathcal{G}^{r(\gamma)}} h^>(\gamma)\gamma h^<(\gamma^{-1}\gamma')\gamma'\xi^>(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\ &= h^>(\gamma)\gamma \int_{\mathcal{G}^{s(\gamma)}} h^<(\gamma')\gamma'\xi^>(\gamma'^{-1})\,\mathrm{d}\lambda^{s(\gamma)}(\gamma') \\ &= \left(h^>\langle h^<, \xi^>\rangle_{\mathcal{H}(\mathcal{G},B)}\right)(\gamma) = \left(\left|h^>\right\rangle\left\langle h^<\right|^>(\xi^>)\right)(\gamma) \end{aligned}$$

for all $\xi^> \in \Gamma_c(\mathcal{G},\ r^*B)$ and $\gamma \in \mathcal{G}$, so $T_k^> = \left|h^>\right\rangle\left\langle h^<\right|^>$. A similar calculation for the left-hand side shows $T_k = \left|h^>\right\rangle\left\langle h^<\right|$. On the other hand, we have

$$\begin{aligned} T_{k_0}^>(\xi^>)(\gamma) &= \int_{\mathcal{G}^{r(\gamma)}} \chi(s(\gamma))\beta(\gamma')\gamma'\xi^>(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') \\ &= \left(\chi(\beta * \xi^>)\right)(\gamma), \end{aligned}$$

for all $\xi^> \in \Gamma_c(\mathcal{G},\ r^*B)$ and $\gamma \in \mathcal{G}$, so $T_{k_0}^> = \chi\pi(\beta)^>$, where π denotes the action of $\mathcal{A}(\mathcal{G}, B)$ on $\mathcal{H}(\mathcal{G}, B)$. We actually have $T_{k_0} = \chi\pi(\beta)$.

Note that, in the obvious notation,

$$\left|f^> \otimes h^>\right\rangle\left\langle h^< \otimes e^<\right| = \left|f^>\right\rangle\!\!\left.\right\rangle \circ \left|h^>\right\rangle\left\langle h^<\right| \circ \left\langle\!\left\langle e^<\right|\right..$$

As in the proof of Lemma E.8.12, T_k depends continuously on k and we can approximate k_0 in the inductive limit topology by sections of the form $k\colon (\gamma, \gamma') \mapsto \sum_{i=1}^n h_i^>(\gamma)\gamma h_i^<(\gamma^{-1}\gamma')$ so that T_k approximates $T_{k_0} = \chi\pi(\beta)$. Since

$$\left|f^>\right\rangle\!\!\left.\right\rangle \circ \chi\pi(\beta) \circ \left\langle\!\left\langle e^<\right|\right. = \chi\left(\left|f^> \cdot \beta\right\rangle\!\!\left.\right\rangle\left\langle\!\left\langle e^<\right|\right. \otimes 1\right)$$

it follows that

$$\left|f^>\right\rangle\!\!\left.\right\rangle \circ T_k \circ \left\langle\!\left\langle e^<\right|\right. = \sum_{i=1}^n \left|f^> \otimes h_i^>\right\rangle\left\langle h_i^< \otimes e^<\right|$$

approximates $\chi\left(\left|f^> \cdot \beta\right\rangle\!\!\left.\right\rangle\left\langle\!\left\langle e^<\right|\right. \otimes 1\right)$ as desired.

Define

$$L := \{\gamma \in \mathcal{G} :\ r(\gamma) \in \operatorname{supp}\chi \wedge s(\gamma) \in \operatorname{supp} f^>\}$$

and

$$L' := \{\gamma' \in \mathcal{G} :\ r(\gamma') \in \operatorname{supp}\chi \wedge s(\gamma') \in \operatorname{supp} e^<\}.$$

Both of these sets are compact because $\mathcal{G}$ is proper. Find functions $\delta, \delta' \in \mathcal{C}_c(\mathcal{G})$ with $0 \le \delta, \delta' \le 1$ and such that $\delta \equiv 1$ and $\delta' \equiv 1$ on a compact neighbourhood M of L and M' of L', respectively.

Let $k \in \Gamma_c(\mathcal{G} \times_{r,r} \mathcal{G}, p^*B)$ such that $\operatorname{supp} k \subseteq M \times M'$. Then

$$\begin{aligned}
\left\|\tau_k(e^>)(x)\right\| &= \left\|\int_{\mathcal{G}^x}\int_{\mathcal{G}^x} \gamma f^>(s(\gamma))\gamma k(\gamma^{-1}, \gamma^{-1}\gamma')\langle \gamma' e^<(s(\gamma')),\ e^>(x)\rangle \,\mathrm{d}\lambda^x(\gamma')\,\mathrm{d}\lambda^x(\gamma)\right\| \\
&\leq \int_{\mathcal{G}^x}\int_{\mathcal{G}^x} \left\|f^>(s(\gamma))\right\| \left\|k(\gamma^{-1}, \gamma^{-1}\gamma')\right\| \left\|e^<(s(\gamma'))\right\| \left\|e^>(x)\right\| \,\mathrm{d}\lambda^x(\gamma')\,\mathrm{d}\lambda^x(\gamma) \\
&= \int_{\mathcal{G}^x}\int_{\mathcal{G}^x} \left\|f^>(s(\gamma))\right\| \delta(\gamma^{-1})\delta'(\gamma^{-1}\gamma') \left\|k(\gamma^{-1}, \gamma^{-1}\gamma')\right\| \left\|e^<(s(\gamma'))\right\| \left\|e^>(x)\right\| \,\mathrm{d}\lambda^x(\gamma')\,\mathrm{d}\lambda^x(\gamma) \\
&\leq \int_{\mathcal{G}^x}\int_{\mathcal{G}^x} \delta(\gamma^{-1})\delta'(\gamma^{-1}\gamma')\,\mathrm{d}\lambda^x(\gamma')\,\mathrm{d}\lambda^x(\gamma) \left\|f^>\right\|_\infty \|k\|_\infty \left\|e^<\right\|_\infty \left\|e^>\right\|_\infty \\
&= \int_{\mathcal{G}^x} \delta(\gamma^{-1}) \int_{\mathcal{G}^{s(\gamma)}} \delta'(\gamma')\,\mathrm{d}\lambda^{s(\gamma)}(\gamma')\,\mathrm{d}\lambda^x(\gamma) \left\|f^>\right\|_\infty \|k\|_\infty \left\|e^<\right\|_\infty \left\|e^>\right\|_\infty \\
&\leq \|\delta\|_1 \left\|\delta'\right\|_1 \left\|f^>\right\|_\infty \|k\|_\infty \left\|e^<\right\|_\infty \left\|e^>\right\|_\infty
\end{aligned}$$

for all $e^> \in \Gamma_c(X, E^>)$ and $x \in X$; here $\|\cdot\|_1$ denotes the symmetrised version of the L^1-norm. Write $C := \|\delta\|_1 \|\delta'\|_1 \|f^>\|_\infty \|e^<\|_\infty$ then we have shown that

$$\left\|\tau_k^>\right\| \leq C \|k\|_\infty$$

provided that $\operatorname{supp} k \subseteq M \times M'$; a similar result is true for the left-hand side. Since we can approximate k_0 by sections of the form $(\gamma, \gamma') \mapsto \sum_{i=1}^n h_i^>(\gamma)\gamma h_i^<(\gamma^{-1}\gamma')$ which are supported in $M \times_{r,r} M'$ in the sup-norm (which is at the same time and by definition an approximation in the inductive limit topology), we are done.

Chapter 8

The Surjectivity of the Bost Map for Proper Banach Algebras

Let $\mathcal{G}$ be a locally compact Hausdorff groupoid with unit space X. Assume[1] that there is a locally compact classifying space $\underline{\mathrm{E}}\mathcal{G}$ for proper actions of $\mathcal{G}$, unique up to homotopy.

In the first section of this chapter we introduce the group $\mathrm{K}^{\mathrm{top,ban}}(\mathcal{G},\ B)$ for every $\mathcal{G}$-Banach algebra B (this is really just the obvious variant of $\mathrm{K}^{\mathrm{top}}(\mathcal{G}, B)$ for $\mathcal{G}$-C^*-algebras B) and the Banach algebraic version of the Baum-Connes map. Then we prove that this map, called the Bost map, is split surjective if $\mathcal{G}$ is proper. This is a special case and also the main ingredient of the proof of the split surjectivity for general $\mathcal{G}$ and proper B. The notion of a proper $\mathcal{G}$-Banach algebra is introduced in the third section, the exact definition being somewhat technical: The main idea is of course that a proper $\mathcal{G}$-Banach algebra is a $\mathcal{G}$-Banach algebra which is at the same time a $\mathcal{G} \ltimes Z$-algebra, where Z is some proper $\mathcal{G}$-space. The trouble is, that B then is, technically, a u.s.c. field of Banach algebras over X and at the same time a u.s.c. field over Z, and this does not make much sense. The solution that I propose is that a proper $\mathcal{G}$-Banach algebra B is a $\mathcal{G}$-Banach algebra such there exists a proper $\mathcal{G}$-space Z and a $\mathcal{G} \ltimes Z$-Banach algebra $\hat{B}$ which is "practically the same as B", i.e., B is the pushforward of $\hat{B}$ along the anchor map of Z (we define the pushforward in the third section of this chapter; one can think of it as a "partially forgetful map"). This definition of a proper $\mathcal{G}$-Banach algebra makes it necessary to think about the relation of unconditional completions of $\mathcal{C}_c(\mathcal{G})$ and of $\mathcal{C}_c(\mathcal{G} \ltimes Z)$ etc.

The actual proof of the split surjectivity of the Bost map for proper groupoids is then contained in the final section of this thesis. It is inspired by the proof of the corresponding C^*-algebraic result for group actions (see, for example, Proposition 5.11 in [KS03]).

For $\mathcal{G}$-C^*-algebras, the analogous constructions were carried out by V. Lafforgue in [Laf06], where it is also proved that the Bost homomorphism is an isomorphism for all proper $\mathcal{G}$-C^*-algebras (with the ordinary topological K-theory on the left-hand side and with arbitrary unconditional completions). The techniques are nevertheless rather different from ours because we cannot make use of the corresponding results for C^*-algebras and crossed products.

[1] In [Tu00] it is said that such a space always exists and is unique (at least if everything is assumed to be σ-compact), the given reference [Tu99] shows this in the case of étale metrisable groupoids. We do not venture into the details but content ourselves with the assumption that $\underline{\mathrm{E}}\mathcal{G}$ exists and is unique.

8.1 Topological K-theory and the general Bost conjecture

8.1.1 Topological K-theory for Banach algebras and groupoids

Definition 8.1.1 (Topological K-theory). For every $\mathcal{G}$-Banach algebra B, define

$$\mathrm{K}^{\mathrm{top,ban}}\left(\mathcal{G},\ B\right) := \lim_{\rightarrow} \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathcal{C}_0(Y),\ B\right),$$

where Y runs through the closed proper $\mathcal{G}$-compact subspaces of $\underline{\mathrm{E}}\mathcal{G}$.

To make sense of this definition we have to clarify some technical details:

- If Y is a locally compact Hausdorff left $\mathcal{G}$-space (with anchor map ρ), then we would like to think of $\mathcal{C}_0(Y)$ as a $\mathcal{G}$-Banach space. A technical obstacle is that $\mathcal{C}_0(Y)$ (or rather $\mathbb{C}_Y$) is a field of Banach spaces over Y and not a field of Banach spaces over X. But $\mathcal{C}_0(Y)$ is of course a $\mathcal{C}_0(X)$-Banach space with the multiplication $\chi\chi' = (\chi\circ\rho)\chi'$ for all $\chi \in \mathcal{C}_0(X)$ and $\chi' \in \mathcal{C}_0(Y)$. The fibre of $\mathcal{C}_0(Y)$ over $x \in X$ can be identified with $\mathcal{C}_0(Y_x)$ where $Y_x = \rho^{-1}(\{x\})$. This way we get a u.s.c. field of Banach algebras over X that we call $\rho_*\left(\mathbb{C}_Y\right)$. There is also a canonical action of $\mathcal{G}$ on $\rho_*\left(\mathbb{C}_Y\right)$: Let $\gamma \in \mathcal{G}$. Then we get an isomorphism α_γ from $\mathcal{C}_0(Y_{s(\gamma)})$ to $\mathcal{C}_0(Y_{r(\gamma)})$ by defining $\alpha_\gamma(\chi)\ (y) = \chi\left(\gamma^{-1}y\right)$ for all $\chi \in \mathcal{C}_0(Y_{s(\gamma)})$ and $y \in Y_{r(\gamma)}$. If we write $\mathcal{C}_0(Y)$, regarding it as a $\mathcal{G}$-Banach algebra, then what we mean is $\rho_*(\mathbb{C}_Y)$.

  This is an example of a rather general pushforward construction which is needed for the definition of proper $\mathcal{G}$-Banach algebras, presented in Section 8.3. It is also a version for u.s.c. fields of Banach spaces of the simple construction presented in Section 2.7 for $\mathcal{C}_0(X)$-Banach spaces.

- We also have to show that $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathcal{C}_0(Y),\ B\right)$, for Y as above, forms a directed system. If Y and Y' are closed, proper, $\mathcal{G}$-compact subspaces of $\underline{\mathrm{E}}\mathcal{G}$ such that $Y \subseteq Y'$, then we would like to get a homomorphism between the $\mathrm{KK}^{\mathrm{ban}}$-groups. More generally, let Y and Y' be $\mathcal{G}$-proper locally compact $\mathcal{G}$-spaces (with anchor maps ρ and ρ') and let $f\colon Y \to Y'$ be a $\mathcal{G}$-equivariant continuous proper map. This induces a non-degenerate $\mathcal{C}_0(X)$-linear homomorphism $\tilde{f}\colon \mathcal{C}_0(Y') \to \mathcal{C}_0(Y)$. Because f is equivariant, the map $\tilde{f}$, thought of as a homomorphism from $\rho'_*(\mathbb{C}_{Y'})$ to $\rho_*(\mathbb{C}_Y)$, is $\mathcal{G}$-equivariant.

  From the functoriality of Banach KK-theory we get a map[2]

  $$\tilde{f}^*\colon\ \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathcal{C}_0(Y),\ B\right) \to \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathcal{C}_0(Y'),\ B\right).$$

  If $Y = Y'$ and $f = \mathrm{Id}$, we have $\tilde{\mathrm{Id}} = \mathrm{Id}_{\mathcal{C}_0(Y)}$ and therefore $\tilde{\mathrm{Id}}^* = \mathrm{Id}$. If Y'' is another proper $\mathcal{G}$-compact $\mathcal{G}$-space and $g\colon Y' \to Y''$ is proper and $\mathcal{G}$-equivariant, then $g\circ f$ is proper and $\mathcal{G}$-equivariant and $\widetilde{g\circ f} = \tilde{f}\circ\tilde{g}$. Now $\widetilde{g\circ f}^* = \tilde{g}^*\circ\tilde{f}^*$, so indeed, we have a directed system of abelian groups.

8.1.2 Functoriality for equivariant homomorphisms and Morita morphisms

Let B and C be $\mathcal{G}$-Banach algebras and let $\varphi\colon B \to C$ be a $\mathcal{G}$-equivariant homomorphism.

[2] Compare Proposition 1.2.6 of [Laf02].

If Y and Y' are proper $\mathcal{G}$-compact locally compact Hausdorff $\mathcal{G}$-spaces and $f: Y \to Y'$ is continuous, proper and $\mathcal{G}$-equivariant, then the following diagram commutes because $\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}$ is bifunctorial:[3]

$$\begin{tikzcd}
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathcal{C}_0(Y),\ B\right) \arrow[r, "\varphi_*"] \arrow[d, "\tilde{f}^*"'] & \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathcal{C}_0(Y),\ C\right) \arrow[d, "\tilde{f}^*"] \\
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathcal{C}_0(Y'),\ B\right) \arrow[r, "\varphi_*"] & \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathcal{C}_0(Y'),\ C\right)
\end{tikzcd}$$

Passing to the direct limit we get a group homomorphism

$$\varphi_*^{\mathrm{top}}\colon\ \mathrm{K}^{\mathrm{top,ban}}(\mathcal{G},\ B) \to \mathrm{K}^{\mathrm{top,ban}}(\mathcal{G},\ C).$$

The assignment $B \mapsto \mathrm{K}^{\mathrm{top,ban}}(\mathcal{G}, B)$ together with $\varphi \mapsto \varphi_*^{\mathrm{top}}$ is a covariant functor from the category of $\mathcal{G}$-Banach algebras and $\mathcal{G}$-equivariant homomorphisms into the category of abelian groups. The same construction works if B and C are non-degenerate $\mathcal{G}$-Banach algebras and $F \in \mathbb{M}^{\mathrm{ban}}_{\mathcal{G}}(B, C)$ is a $\mathcal{G}$-equivariant Morita cycle. In this case we get a group homomorphism

$$\cdot \otimes_B [F]\colon\ \mathrm{K}^{\mathrm{top,ban}}(\mathcal{G},\ B) \to \mathrm{K}^{\mathrm{top,ban}}(\mathcal{G},\ C).$$

The assignment $B \mapsto \mathrm{K}^{\mathrm{top,ban}}(\mathcal{G}, B)$ together with $[F] \mapsto \cdot \otimes_B [F]$ is a covariant functor from the category of $\mathcal{G}$-Banach algebras and $\mathcal{G}$-equivariant Morita morphisms into the category of abelian groups.

Corollary 8.1.2. *If B and C are equivariantly Morita equivalent $\mathcal{G}$-Banach algebras, then*

$$\mathrm{K}^{\mathrm{top,ban}}(\mathcal{G},\ B) \cong \mathrm{K}^{\mathrm{top,ban}}(\mathcal{G},\ C).$$

8.1.3 The Baum-Connes map in the Banach algebra context

Let $\mathcal{G}$ carry a Haar system and let $\mathcal{A}(\mathcal{G})$ be an unconditional completion of $\mathcal{C}_c(\mathcal{G})$.

Definition 8.1.3 (Bost map). Let B be a $\mathcal{G}$-Banach algebra. Define the homomorphism of abelian groups

$$\mu^B_{\mathcal{A}}\colon\ \mathrm{K}^{\mathrm{top,ban}}\left(\mathcal{G},\ B\right) \to \mathrm{K}_0\left(\mathcal{A}\left(\mathcal{G}, B\right)\right)$$

to be the direct limit of the group homomorphisms

$$\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathcal{C}_0(Y),\ B\right) \overset{j_{\mathcal{A}}}{\to} \mathrm{KK}^{\mathrm{ban}}\left(\mathcal{A}\left(\mathcal{G}, \mathcal{C}_0(Y)\right),\ \mathcal{A}\left(\mathcal{G}, B\right)\right) \overset{\Sigma(\cdot)\left(\lambda_{Y,\mathcal{G},\mathcal{A}}\right)}{\to} \mathrm{K}_0\left(\mathcal{A}\left(\mathcal{G}, B\right)\right)$$

where Y runs through all closed, $\mathcal{G}$-compact, proper subspaces of $\underline{\mathrm{E}}\mathcal{G}$.

Again, we discuss the details of this definition:

What is $\lambda_{Y,\mathcal{G},\mathcal{A}}$?[4] If Y is a $\mathcal{G}$-compact proper $\mathcal{G}$-space, then the element $\lambda_{Y,\mathcal{G},\mathcal{A}}$ of $\mathrm{K}_0\left(\mathcal{A}(\mathcal{G}, \mathcal{C}_0(Y))\right)$ (or rather of $\mathrm{K}_0\left(\mathcal{A}(\mathcal{G}, \rho_*\mathbb{C}_Y)\right)$) was defined in [Laf06], paragraph 1.5.2, as follows (with some technical changes): Consider the groupoid $\mathcal{G} \ltimes Y$. It is locally compact Hausdorff and proper and satisfies $(\mathcal{G} \ltimes Y)^{(0)} = Y$ and $Y/(\mathcal{G} \ltimes Y) \cong \mathcal{G}\backslash Y$, this space being compact. We can hence find a

[3] See Definition and Proposition 3.5.10.

[4] That we also use the letter λ for the Haar system on $\mathcal{G}$ does not lead to much notational inconvenience: instead of $\mathrm{d}\lambda^x(\gamma)$ we just write $\mathrm{d}\gamma$ in the integrals that appear in this chapter.

cut-off-function for $\mathcal{G} \ltimes Y$. For technical reasons, we identify $\mathcal{G} \ltimes Y$ with $\mathcal{G} \times_{r,\rho} Y$: The range and source maps are then given by $r_{\mathcal{G}\ltimes Y}(\gamma, y) = y$ and $s_{\mathcal{G}\ltimes Y}(\gamma, y) = \gamma^{-1}y$, and the product is given by $(\gamma_1, y_1) \cdot (\gamma_2, y_2) = (\gamma_1\gamma_2, y_1)$ for all $(\gamma_1, y_1), (\gamma_2, y_2) \in \mathcal{G} \times_{r,\rho} Y$ such that $\gamma_1^{-1}y_1 = y_2$. The Haar system on $\mathcal{G} \ltimes Y$ is the following (expressed as an integral): If $\chi \in \mathcal{C}_c(\mathcal{G} \ltimes Y)$, then $\int_{(\mathcal{G}\ltimes Y)^{y'}} \chi(\gamma, y)\,\mathrm{d}(\gamma, y) := \int_{\mathcal{G}^{\rho(y')}} \chi(\gamma, y')\,\mathrm{d}\gamma$ for all $y' \in Y$. A cut-off function for $\mathcal{G} \ltimes Y$ is a function from Y to $\mathbb{R}_{\geq 0}$ with compact support such that $\int_{\mathcal{G}^y} c(\gamma^{-1}y)\,\mathrm{d}\gamma = 1$ for all $y \in Y$.

Now consider the function $\gamma \mapsto \left(Y_{r_{\mathcal{G}}(\gamma)} \ni y \mapsto c^{1/2}(y)\, c^{1/2}(\gamma^{-1}y)\right)$ with $\gamma \in \mathcal{G}$. This is an idempotent element of $\Gamma_c\left(\mathcal{G},\ r_{\mathcal{G}}^*\rho_*\mathbb{C}_Y\right)$ (actually, we can think of it as an idempotent element of the algebra $\Gamma_c\left(\mathcal{G} \ltimes Y,\ r_{\mathcal{G}\ltimes Y}^*\mathbb{C}_Y\right) = \mathcal{C}_c\left(\mathcal{G} \ltimes Y\right)$). It therefore gives an idempotent element of $\mathcal{A}\left(\mathcal{G},\ \rho_*\mathbb{C}_Y\right)$, and the element of $\mathrm{K}_0\left(\mathcal{A}(\mathcal{G},\ \rho_*\mathbb{C}_Y)\right)$ that it determines is denoted by $\lambda_{Y,\mathcal{G},\mathcal{A}}$.

This definition is independent of the choice of the cut-off function c; actually, we could take any cut-off pair $(c^<, c^>)$ instead of $(c^{1/2}, c^{1/2})$ in the formula for the idempotent: if $(c^<, c^>)$ is a cut-off pair for $\mathcal{G}\ltimes Y$, then $\gamma \mapsto \left(Y_{r_{\mathcal{G}}(\gamma)} \ni y \mapsto c^>(y)\, c^<(\gamma^{-1}y)\right)$ defines an idempotent element of $\mathcal{A}\left(\mathcal{G},\ \rho_*\mathbb{C}_Y\right)$ which depends continuously on the cut-off pair. Using linear homotopies (and an additional correction factor) one can connect any two cut-off pairs for $\mathcal{G} \ltimes Y$ through a continuous path of cut-off pairs with respect to the inductive limit topology (here we use that fact that $\mathcal{G}\backslash Y$ is compact). Hence $\lambda_{Y,\mathcal{G},\mathcal{A}}$ does not depend on the cut-off pair (or the cut-off function).

What is $\Sigma(\cdot)\left(\lambda_{Y,\mathcal{G},\mathcal{A}}\right)$? The action Σ of $\mathrm{KK}^{\mathrm{ban}}$ on the K-theory was defined in[5] [Laf02]. In our case, Σ is a homomorphism from $\mathrm{KK}^{\mathrm{ban}}\left(\mathcal{A}(\mathcal{G}, \rho_*\mathbb{C}_Y),\ \mathcal{A}(\mathcal{G}, B)\right)$ to the group of homomorphisms from $\mathrm{K}_0\left(\mathcal{A}(\mathcal{G}, \rho_*\mathbb{C}_Y)\right)$ to $\mathrm{K}_0\left(\mathcal{A}(\mathcal{G}, B)\right)$. Evaluating at $\lambda_{Y,\mathcal{G},\mathcal{A}}$ gives a homomorphism from $\mathrm{KK}^{\mathrm{ban}}\left(\mathcal{A}(\mathcal{G}, \rho_*\mathbb{C}_Y),\ \mathcal{A}(\mathcal{G}, B)\right)$ to $\mathrm{K}_0\left(\mathcal{A}(\mathcal{G}, B)\right)$. Because $\lambda_{Y,\mathcal{G},\mathcal{A}}$ is given by an idempotent of $\mathcal{A}(\mathcal{G},\ \rho_*\mathbb{C}_Y)$ we can actually obtain a more concrete description of $\Sigma(\cdot)\left(\lambda_{Y,\mathcal{G},\mathcal{A}}\right)$: If (E, T) is a cycle in $\mathbb{E}^{\mathrm{ban}}\left(\mathcal{A}(\mathcal{G}, \rho_*\mathbb{C}_Y),\ \mathcal{A}(\mathcal{G}, B)\right)$ and p is a choice of an idempotent in $\mathcal{A}(\mathcal{G},\ \rho_*\mathbb{C}_Y)$ giving $\lambda_{Y,\mathcal{G},\mathcal{A}}$ such that p commutes with $\mathcal{A}(\mathcal{G}, T)$, then the cycle $\left(p\mathcal{A}(\mathcal{G}, E), T|_{p\mathcal{A}(\mathcal{G},E)}\right)$ (with the canonical left $\mathbb{C}$-action) gives the element $\Sigma([(E, T)])\left(\lambda_{Y,\mathcal{G},\mathcal{A}}\right) \in \mathrm{KK}^{\mathrm{ban}}\left(\mathbb{C},\ \mathcal{A}(\mathcal{G}, B)\right) \cong \mathrm{K}_0\left(\mathcal{A}(\mathcal{G}, B)\right)$, where $p\mathcal{A}(\mathcal{G}, E) = (\mathcal{A}(\mathcal{G}, E^<)p,\ p\mathcal{A}(\mathcal{G}, E^>))$.

Passing to the direct limit: To see that Definition 8.1.3 makes sense we check that the group homomorphisms are compatible with continuous equivariant proper maps between the subspaces, allowing us to take the limit. Let therefore Y and Y' be proper $\mathcal{G}$-compact $\mathcal{G}$-spaces and let $f\colon Y \to Y'$ be a proper $\mathcal{G}$-equivariant continuous map. Let $\tilde{f}\colon \mathcal{C}_0(Y') \to \mathcal{C}_0(Y)$ be the induced homomorphism of $\mathcal{G}$-Banach algebras. Then we have to show that the following diagram commutes

$$
\begin{tikzcd}
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathcal{C}_0(Y),\ B\right) \arrow[r, "j_{\mathcal{A}}"] \arrow[d, "\tilde{f}^*"'] & \mathrm{KK}^{\mathrm{ban}}\left(\mathcal{A}\left(\mathcal{G}, \mathcal{C}_0(Y)\right),\ \mathcal{A}\left(\mathcal{G}, B\right)\right) \arrow[dr, "\Sigma(\cdot)\left(\lambda_{Y,\mathcal{G},\mathcal{A}}\right)"] \arrow[d, "\mathcal{A}(\mathcal{G},\ \tilde{f})^*"'] & \\
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathcal{C}_0(Y'),\ B\right) \arrow[r, "j_{\mathcal{A}}"] & \mathrm{KK}^{\mathrm{ban}}\left(\mathcal{A}\left(\mathcal{G}, \mathcal{C}_0(Y')\right),\ \mathcal{A}\left(\mathcal{G}, B\right)\right) \arrow[r, "\Sigma(\cdot)\left(\lambda_{Y',\mathcal{G},\mathcal{A}}\right)"] & \mathrm{K}_0\left(\mathcal{A}\left(\mathcal{G}, B\right)\right)
\end{tikzcd}
\tag{8.1}
$$

That the left part of the diagram commutes follows from Theorem 5.2.25. The right part commutes because of Proposition 1.2.9 of [Laf02] and Proposition E.7.1.

Proposition 8.1.4. *The assignment $B \mapsto \mu^B_{\mathcal{A}}$ is a natural transformation from the functor $B \mapsto \mathrm{K}^{\mathrm{top,ban}}(\mathcal{G}, B)$ to the functor $B \mapsto \mathrm{K}_0\left(\mathcal{A}(\mathcal{G},\ B)\right)$ (where we can take as our source category the category of non-degenerate $\mathcal{G}$-Banach algebras with the Morita morphisms as morphisms).*

[5] See Proposition 1.2.9 of [Laf02] and the discussion thereafter.

Proof. Let B and C be non-degenerate $\mathcal{G}$-Banach algebras and let F be a Morita cycle from B to C (and let $[F]$ denote the corresponding Morita morphism). We have to show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{top,ban}}(\mathcal{G},\ B) \arrow[r, "\mu_{\mathcal{A}}^{B}"] \arrow[d, "\otimes_B[F]"'] & \mathrm{K}_0(\mathcal{A}(\mathcal{G},B)) \arrow[d, "\otimes_{\mathcal{A}(\mathcal{G},B)}[\mathcal{A}(\mathcal{G},F)]"] \\
\mathrm{K}^{\mathrm{top,ban}}(\mathcal{G},\ C) \arrow[r, "\mu_{\mathcal{A}}^{B}"] & \mathrm{K}_0(\mathcal{A}(\mathcal{G},C))
\end{tikzcd}$$

Most of the objects in this diagram are defined as direct limits, so we check the corresponding diagram before taking the limit. To this end let Y and Y' be proper $\mathcal{G}$-compact $\mathcal{G}$-spaces and let $f\colon Y \to Y'$ be a proper $\mathcal{G}$-equivariant continuous map. Then we have to take the 5-vertex diagram (8.1), once for B and once for C, and connect the two diagram by five morphisms coming from the tensor products with F and $\mathcal{A}(\mathcal{G},F)$. The resulting diagram has the shape of a prism with ten vertices, eight squares and two triangles. The two triangles and two of the squares commute because Diagram (8.1) is commutative (in the version for B and the version for C). One of the remaining squares is

$$\begin{tikzcd}
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(Y),B) \arrow[r, "j_{\mathcal{A}}"] \arrow[d, "\otimes_B[F]"'] & \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{A}(\mathcal{G},\mathcal{C}_0(Y)),\mathcal{A}(\mathcal{G},B)) \arrow[d, "\otimes_{\mathcal{A}(\mathcal{G},B)}[\mathcal{A}(\mathcal{G},F)]"] \\
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(Y),C) \arrow[r, "j_{\mathcal{A}}"] & \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{A}(\mathcal{G},\mathcal{C}_0(Y)),\mathcal{A}(\mathcal{G},C))
\end{tikzcd}$$

This an the corresponding square for Y' commute because the descent is compatible with Morita morphisms.[6] The square

$$\begin{tikzcd}
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{A}(\mathcal{G},\mathcal{C}_0(Y)),\mathcal{A}(\mathcal{G},B)) \arrow[r, "\Sigma(\cdot)(\lambda_{Y,\mathcal{G},\mathcal{A}})"] \arrow[d, "\otimes_{\mathcal{A}(\mathcal{G},B)}[\mathcal{A}(\mathcal{G},F)]"'] & \mathrm{K}_0(\mathcal{A}(\mathcal{G},B)) \arrow[d, "\otimes_{\mathcal{A}(\mathcal{G},B)}[\mathcal{A}(\mathcal{G},F)]"] \\
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{A}(\mathcal{G},\mathcal{C}_0(Y)),\mathcal{A}(\mathcal{G},C)) \arrow[r, "\Sigma(\cdot)(\lambda_{Y,\mathcal{G},\mathcal{A}})"] & \mathrm{K}_0(\mathcal{A}(\mathcal{G},C))
\end{tikzcd}$$

commutes because the action of $\mathrm{KK}^{\mathrm{ban}}$ on the K-theory is compatible with Morita morphisms (we only know this for ordinary homomorphisms[7] yet, but in our particular case the action on $\lambda_{Y,\mathcal{G},\mathcal{A}}$ is given by the pushforward along a homomorphism from $\mathbb{C}$ to $\mathcal{A}(\mathcal{G},\mathcal{C}_0(Y))$ in the first variable given by an idempotent of $\mathcal{A}(\mathcal{G},\mathcal{C}_0(Y))$, and this clearly commutes with the multiplication by a Morita morphism from the right). The same is true for the corresponding diagram for Y'. Similarly and just as in Subsection 8.1.2, the square

$$\begin{tikzcd}
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(Y),B) \arrow[r, "\tilde{f}^*"] \arrow[d, "\otimes_B[F]"'] & \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(Y'),B) \arrow[d, "\otimes_B[F]"] \\
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(Y),C) \arrow[r, "\tilde{f}^*"] & \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(Y'),C)
\end{tikzcd}$$

as well as the the corresponding square after the descent commute. □

[6] See Corollary 5.2.29.

[7] This is included in Proposition 1.2.9 of [Laf02].

8.1.4 The Bost map and varying unconditional completions

Let $\mathcal{G}$ carry a Haar system and let $\mathcal{A}(\mathcal{G})$ and $\mathcal{B}(\mathcal{G})$ be unconditional completions of $\mathcal{C}_c(\mathcal{G})$ such that $\|\chi\|_{\mathcal{B}} \leq \|\chi\|_{\mathcal{A}}$ for all $\chi \in \mathcal{C}_c(\mathcal{G})$.

Definition and Proposition 8.1.5. Let B be a $\mathcal{G}$-Banach algebra and let $\iota_{\mathcal{A}}$ and $\iota_{\mathcal{B}}$ be the canonical maps from $\Gamma_c(\mathcal{G},\ r^*B)$ to $\mathcal{A}(\mathcal{G}, B)$ and $\mathcal{B}(\mathcal{G}, B)$, respectively. Let $\psi\colon \mathcal{A}(\mathcal{G}, B) \to \mathcal{B}(\mathcal{G}, B)$ be the homomorphism of Banach algebras such that $\psi \circ \iota_{\mathcal{A}} = \iota_{\mathcal{B}}$. Then

$$\psi_*\colon \mathrm{K}_*\left(\mathcal{A}(\mathcal{G}, B)\right) \to \mathrm{K}_*\left(\mathcal{B}(\mathcal{G}, B)\right)$$

is a homomorphism making the following diagram commutative

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{top,ban}}(\mathcal{G},\ B) \arrow[r, "\mu_{\mathcal{A}}^{B}"] \arrow[dr, "\mu_{\mathcal{B}}^{B}"'] & \mathrm{K}_0\left(\mathcal{A}\left(\mathcal{G}, B\right)\right) \arrow[d, "\psi_*"] \\
 & \mathrm{K}_0\left(\mathcal{B}\left(\mathcal{G}, B\right)\right)
\end{tikzcd}$$

Proof. This follows from Proposition 1.4.8 in [Laf06], compare also Proposition 1.5.4 of the same article which is the above assertion for $\mathcal{G}$-C*-algebras. □

8.2 The Bost conjecture and proper groupoids

In this section let $\mathcal{G}$ be proper and equipped with a Haar system. Let $\mathcal{A}(\mathcal{G})$ be an unconditional completion of $\mathcal{C}_c(\mathcal{G})$.

Definition 8.2.1 (Hereditary subalgebra). Let B_0 be a subalgebra of a complex algebra B. Then B is called *hereditary* if $B_0\ B\ B_0 \subseteq B_0$.

The following lemma is a variant of Lemme 1.7.9 of [Laf02], inspired by a remark of Cuntz that his kk-theory is invariant under a similar relation.

Lemma 8.2.2. *Let B be a Banach algebra and A be a topological algebra (with separately continuous multiplication) and let $\varphi\colon A \to B$ be a continuous homomorphism such that $\varphi(A)$ is a dense hereditary subalgebra of B and such that the kernel of φ is nilpotent. Then $\varphi\colon \pi_0\left(\widetilde{A}^{-1}\right) \to \pi_0\left(\widetilde{B}^{-1}\right)$ is a bijection.*

Proof. Let $x \in A$ such that $1 + \varphi(a) \in \widetilde{B}^{-1}$. Let $1 + b$ be the inverse of $1 + \varphi(a)$ in $\widetilde{B}$. Then, as in the proof of Lemma 1.7.9 of [Laf02], $b = -\varphi(a) + \varphi(a)^2 + \varphi(a)b\varphi(a)$ belongs to $\varphi(A)$. Find $a' \in A$ such that $\varphi(a') = b$. Then $\widetilde{\varphi}\left((1 + a)(1 + a')\right) = (1 + \varphi(a))(1 + \varphi(a')) = 1 = \widetilde{\varphi}\left((1 + a')(1 + a)\right)$. This means that $(1+a)(1+a') = 1+n$ for some n in the kernel of φ. But such an element is always invertible, so $1 + a$ is right-invertible in $\widetilde{A}$. Similarly, $1 + a$ is left-invertible in $\widetilde{A}$, so it is invertible. This shows the surjectivity of φ on the level of π_0.

To show injectivity we remark that $\varphi[0,1]$ is a continuous homomorphism from $A[0,1]$ to $B[0,1]$ with dense hereditary image and nilpotent kernel; we can hence use the first part of the proof: Let $a_0, a_1 \in A$ such that $1 + b_0$ and $1 + b_1$ are in the same connected component of $\widetilde{B}$ where $b_i = \varphi(a_i)$ for $i = 0, 1$. Because $\widetilde{B}^{-1}$ is open in the Banach space $\widetilde{B}$ there is a path $\widetilde{\beta}$ in $\widetilde{B}^{-1}$ between b_0 and b_1. It is of the form $\widetilde{\beta} = \chi + \beta$ with $\chi \in \mathcal{C}[0,1]$ and $\beta \in \mathcal{C}([0,1], B)$. Because $\chi(t) = 0$ for all $t \in [0,1]$, we can invert χ, and $1 + \chi^{-1}\beta$ is also a path from $1 + b_0$ to $1 + b_1$ in $\widetilde{B}^{-1}$. Because the image $\varphi[0,1]$ is dense in $B[0,1]$, we can find an $\alpha \in A[0,1]$ such that $\chi^{-1}\beta$ is so close to $\varphi[0,1]\ (\alpha)$ that $1+\varphi[0,1]\ (\alpha)$ is invertible in $\widetilde{B[0,1]}$; we can even achieve this with $\varphi(\alpha(0)) = b_0 = \varphi(a_0)$ and $\varphi(\alpha(1)) = b_1 = \varphi(a_1)$. Now the first part of the proof shows that $1+\alpha$ is invertible in $\widetilde{A[0,1]}$, so it is a path from $1+\alpha(0)$ to $1+\alpha(1)$ in $\widetilde{A}^{-1}$. The difference n_0 of $1+\alpha(0)$ and $1+a_0$ is in the kernel of φ, so it is nilpotent. So tn_0 is also nilpotent for all $t \in [0,1]$. The map $t \mapsto 1+a_0+tn_0$ is hence a path in the invertible elements of $\widetilde{A}$ from $1+a_0$ to $1+\alpha(0)$. Similarly, there is a path from $1+\alpha(1)$ to $1+a_1$. Putting the three paths together we get a path from $1+a_0$ to $1+a_1$ in $\widetilde{A}^{-1}$. Hence $1+a_0$ and $1+a_1$ are in the same connected component. $\square$

The following lemma is an elaborate version of Lemme 1.7.10 of [Laf02]; there are two minor differences: The first is that we allow $\|\cdot\|_1$ and $\|\cdot\|_2$ to be semi-norms rather than norms (with the restriction that the kernel of the homomorphisms into the completions are nilpotent), and secondly, we do not ask the homomorphism ψ to be injective. The first generalisation is necessary because we want to apply the result to unconditional completions in the groupoid setting where semi-norms appear naturally, the second generalisation seems to be already necessary in the setting of [Laf02], because in the proof of Lemme 1.7.8 there is no explicit argument given why the homomorphism from $\mathcal{B}(G,B)$ to $\mathcal{A}(G,B)$ is injective (although I have the feeling that I just lack a trivial argument).

Lemma 8.2.3. *Let A be a topological algebra (with separately continuous multiplication). Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be continuous semi-norms on A such that the completion of A with respect to both norms is a Banach algebra. Let ι_1 be the canonical continuous homomorphism from A into its completion B_1 with respect to $\|\cdot\|_1$ and define ι_2 and B_2 analogously. Assume that $\|a\|_1 \geq \|a\|_2$ for all $a \in A$, and let $\psi\colon B_1 \to B_2$ the homomorphism of Banach algebras that we get from this inequality. Assume also that $\iota_i(A)$ is hereditary in B_i and that the kernel of ι_i is nilpotent for all $i \in \{1,2\}$. Then the map*

$$\psi_*\colon \mathrm{K}_*(B_1) \to \mathrm{K}_*(B_2)$$

is an isomorphism.

Proof. This is proved analogously to Lemme 1.7.10 of [Laf02], based on our Lemma 8.2.2. $\square$

Lemma 8.2.4. *Let B be a non-degenerate $\mathcal{G}$-Banach algebra and let $\mathcal{A}(\mathcal{G})$ be a* regular *unconditional completion of $\mathcal{C}_c(\mathcal{G})$. Let ι be the canonical map from $\Gamma_c(\mathcal{G},\ r^*B)$ to $\mathcal{A}(\mathcal{G},B)$. Since $\mathcal{G}$ is proper, $\iota\left(\Gamma_c(\mathcal{G},\ r^*B)\right)$ is a hereditary subalgebra of $\mathcal{A}(\mathcal{G},B)$ and the kernel N of ι satisfies $\Gamma_c(\mathcal{G},\ r^*B)\ N\ \Gamma_c(\mathcal{G},\ r^*B) = 0$; in particular, it is nilpotent with $N^3 = 0$.*

Proof. Let $\mathcal{A}(\mathcal{G})$ act on the equivariant pair $\mathcal{H}(\mathcal{G})$ of locally convex monotone completions of $\mathcal{C}_c(\mathcal{G})$. Let $\beta^<, \beta^> \in \Gamma_c\left(\mathcal{G},\ r^*B\right)$. Let $K_r := r\left(\operatorname{supp}\beta^<\right)$ and $K_s := s\left(\operatorname{supp}\beta^>\right)$. The two sets K_r and K_s are compact subsets of $\mathcal{G}^{(0)}$. Because $\mathcal{G}$ is proper, the set $K := \{\gamma \in \mathcal{G} :\ r(\gamma) \in K_r,\ s(\gamma) \in K_s\}$ is compact. For all $\beta \in \Gamma_c\left(\mathcal{G},\ r^*B\right)$, we have $\operatorname{supp}\left(\beta^< * \beta * \beta^>\right) \subseteq K$. Because $\mathcal{A}(\mathcal{G})$ acts on $\mathcal{H}(\mathcal{G})$, we also have (by 7.3.14 and 7.3.6)

$$\left\|\beta^< * \beta * \beta^>\right\|_\infty \leq \left\|\beta^<\right\|_{\mathcal{H}^<} \|\beta\|_{\mathcal{A}} \left\|\beta^>\right\|_{\mathcal{H}^>}.$$

It follows that $\left(\beta^< * \beta_n * \beta^>\right)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $\Gamma_K\left(\mathcal{G},\ r^*B\right)$ whenever $\left(\beta_n\right)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $\Gamma_c\left(\mathcal{G},\ r^*B\right)$ for the semi-norm $\|\cdot\|_{\mathcal{A}}$; in this case, $\left(\beta^< * \beta_n * \beta^>\right)_{n\in\mathbb{N}}$ converges to some element of $\Gamma_K\left(\mathcal{G},\ r^*B\right)$, and hence $\iota\left(\beta^< * \beta_n * \beta^>\right) = \iota(\beta^<)\iota(\beta_n)\iota(\beta^>)$ converges to some element in the image of ι if $n \to \infty$. Thus the image of ι is hereditary in $\mathcal{A}(\mathcal{G},B)$.

Now let $\beta \in \Gamma_c\left(\mathcal{G},\ r^*B\right)$ satisfy $\iota(\beta) = 0 \in \mathcal{A}(\mathcal{G},B)$. Let $\beta^<, \beta^>$ be arbitrary elements of $\Gamma_c\left(\mathcal{G},\ r^*B\right)$. Because $\mathcal{A}(\mathcal{G})$ acts on $\mathcal{H}(\mathcal{G})$, we have $\|\beta^< * \beta * \beta^>\|_\infty \leq \|\beta^<\|_{\mathcal{H}^<} \|\beta\|_{\mathcal{A}} \|\beta^>\|_{\mathcal{H}^>} = 0$, so $\beta^< * \beta * \beta^> = 0$. This shows that the kernel N of ι satisfies $\Gamma_c(\mathcal{G},\ r^*B)\ N\ \Gamma_c(\mathcal{G},\ r^*B) = 0$. $\square$

For proper $\mathcal{G}$ the K-theory of $\mathcal{A}(\mathcal{G}, B)$ does not depend on the particular (regular) completion $\mathcal{A}(\mathcal{G})$:

Proposition 8.2.5. *Let B be a non-degenerate $\mathcal{G}$-Banach algebra. Let $\mathcal{B}(\mathcal{G})$ be another unconditional completion of $\mathcal{C}_c(\mathcal{G})$ such that $\|\chi\|_{\mathcal{B}} \leq \|\chi\|_{\mathcal{A}}$ for all $\chi \in \mathcal{C}_c(\mathcal{G})$. Let $\psi\colon \mathcal{A}(\mathcal{G}, B) \to \mathcal{B}(\mathcal{G}, B)$ be canonical the homomorphism of Banach algebras introduced in 8.1.5. If $\mathcal{B}(\mathcal{G})$ is a regular unconditional completion of $\mathcal{C}_(\mathcal{G})$, then also $\mathcal{A}(\mathcal{G})$ is regular and*

$$\psi_*\colon \mathrm{K}_*\left(\mathcal{A}(\mathcal{G}, B)\right) \to \mathrm{K}_*\left(\mathcal{B}(\mathcal{G}, B)\right)$$

is an isomorphism.

Proof. This follows from Lemma 8.2.3 and Lemma 8.2.4. $\square$

Corollary 8.2.6. *Let $\mathcal{A}_1(\mathcal{G})$ and $\mathcal{A}_2(\mathcal{G})$ be regular unconditional completions of $\mathcal{C}_c(\mathcal{G})$. Let B be a non-degenerate $\mathcal{G}$-Banach algebra. Then $\mathrm{K}_*\left(\mathcal{A}_1(\mathcal{G}, B)\right)$ and $\mathrm{K}_*\left(\mathcal{A}_2(\mathcal{G}, B)\right)$ are canonically isomorphic.*

Proof. Let $\|\cdot\|_{\mathcal{B}}$ be an unconditional norm on $\mathcal{C}_c(\mathcal{G})$ such that $\|\chi\|_{\mathcal{A}_i} \leq \|\chi\|_{\mathcal{B}}$ for all $\chi \in \mathcal{C}_c(\mathcal{G})$ and all $i \in \{1, 2\}$, define, for example, $\|\chi\|_{\mathcal{B}} := \max\left\{\|\chi\|_{\mathcal{A}_1}, \|\chi\|_{\mathcal{A}_2}\right\}$ for all $\chi \in \mathcal{C}_c(\mathcal{G})$. By the preceding proposition it follows that $\mathrm{K}_*\left(\mathcal{B}(\mathcal{G}, B)\right) \cong \mathrm{K}_*\left(\mathcal{A}_i(\mathcal{G}, B)\right)$ for all $i \in \{1, 2\}$. The resulting isomorphism $\mathrm{K}_*\left(\mathcal{A}_1(\mathcal{G}, B)\right) \cong \mathrm{K}_*\left(\mathcal{A}_2(\mathcal{G}, B)\right)$ does not depend on the particular norm $\|\cdot\|_{\mathcal{B}}$, we could have taken any unconditional norm dominating $\|\cdot\|_{\mathcal{A}_1}$ and $\|\cdot\|_{\mathcal{A}_2}$. $\square$

Example 8.2.7. Let G be a locally compact Hausdorff group action properly on some locally compact Hausdorff space X. Then $\mathrm{L}^1(G \ltimes X)$ and $\mathrm{L}^1(G,\ \mathcal{C}_0(X))$ are two regular unconditional completions of $\mathcal{C}_c(G \ltimes X)$. Because $G \ltimes X$ is a proper groupoid, we have a canonical isomorphism

$$\mathrm{K}_0\left(\mathrm{L}^1(G,\ \mathcal{C}_0(X))\right) \cong \mathrm{K}_0\left(\mathrm{L}^1(G \ltimes X)\right).$$

Because the unconditional norm given by $\mathrm{L}^1(G,\ \mathcal{C}_0(X))$ dominates $\|\cdot\|_1$, the isomorphism in K-theory is given by the canonical homomorphism from $\mathrm{L}^1(G,\ \mathcal{C}_0(X))$ to $\mathrm{L}^1(G \ltimes X)$.

Lemma 8.2.8. *If $\mathcal{G}$ is proper, then $X = \mathcal{G}^{(0)}$ is a model for $\underline{\mathrm{E}}\mathcal{G}$. If $\mathcal{G}$ is proper and $X/\mathcal{G}$ is compact, then the canonical homomorphism*

$$\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathcal{C}_0(X),\ B\right) \to \mathrm{K}^{\mathrm{top,ban}}\left(\mathcal{G},\ B\right)$$

is an isomorphism for all $\mathcal{G}$-Banach algebras B.

Proposition 8.2.9. *Assume that $\mathcal{G}$ is proper and that $X/\mathcal{G}$ is compact. Then the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\mathcal{C}_0(X),\ B\right) \arrow[r, "J^B_{\mathcal{A}}"] \arrow[d, "\cong"'] & \mathrm{RKK}^{\mathrm{ban}}\left(\mathcal{C}_0\left(X/\mathcal{G}\right);\ \mathcal{C}_0\left(X/\mathcal{G}\right),\ \mathcal{A}(\mathcal{G}, B)\right) \arrow[d, "\cong"] \\
\mathrm{K}^{\mathrm{top,ban}}\left(\mathcal{G},\ B\right) \arrow[r, "\mu^B_{\mathcal{A}}"] & \mathrm{K}_0\left(\mathcal{A}(\mathcal{G}, B)\right)
\end{tikzcd}$$

The isomorphism on the right-hand side is the given by the embedding $\mathbb{C} \mapsto \mathcal{C}_0\left(X/\mathcal{G}\right)$ as constant functions (compare Corollary 2.8.2). Before we come to the proof of Proposition 8.2.9 we state an immediate corollary:

Corollary 8.2.10. *If $\mathcal{G}$ is as above and $\mathcal{A}(\mathcal{G})$ is regular and B is non-degenerate, then Theorem 7.1.10 says that there is a natural split $M^B_{\mathcal{A}}$ of $J^B_{\mathcal{A}}$. Hence also $\mu^B_{\mathcal{A}}$ has a natural split.*

Proof of Proposition 8.2.9. We show that the diagram already commutes (up to isomorphism) on the level of cycles. Let therefore (E,T) be in $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(\mathcal{C}_0(X),\ B)$ and assume that T is $\mathcal{G}$-equivariant (which can be done because $\mathcal{G}$ is proper, see Proposition 7.2.1). Choose a cut-off pair $c = (c^<, c^>)$ for $\mathcal{G}$. Applying the Bost map to (E,T) gives $\left(p\mathcal{A}(\mathcal{G},E), T|_{p\mathcal{A}(\mathcal{G},E)}\right)$ where p is the idempotent in $\mathcal{A}(\mathcal{G})$ given by $\gamma \mapsto c^>(r(\gamma))c^<(s(\gamma)) \ \in \ \Gamma_c(\mathcal{G})$ as discussed after Definition 8.1.3 (p commutes with $\mathcal{A}(\mathcal{G},T)$ because T is $\mathcal{G}$-equivariant).

On the other hand, $J^B_{\mathcal{A},\mathcal{A}^c}(E,T)$ can be realised as precisely the same cycle using the homomorphism $j^<_{E,c}$ and $j^>_{E,c}$ introduced in Definition 7.2.26, compare also Definition 7.2.29. □

8.3 The pushforward construction

The pushforward construction that we are going to present here in some detail is needed for a precise discussion of the notion of a proper $\mathcal{G}$-Banach algebra, where $\mathcal{G}$ is a locally compact Hausdorff groupoid. The underlying idea is very simple: If X and Y are locally compact Hausdorff spaces and $p\colon Y \to X$ is continuous, then we want to know how to transform a field of Banach spaces over Y into a field over X. One way is to assemble, for every $x \in X$, all the fibres over points $y \in Y$ that satisfy $p(y) = x$ and make a single fibre out of them.

In the first part of this section we introduce the pushforward construction in a non-equivariant setting. The groupoid $\mathcal{G}$ comes back into play in the second part of the section, and in the third subsection, we discuss the relations between the descent and the pushforward (in the case that $\mathcal{G}$ carries a Haar system). The non-equivariant construction can also be found in the book [FD88], Paragraph 14.9; it is formulated in the language of Banach bundles rather than in the language of u.s.c. fields of Banach spaces.

8.3.1 The pushforward for fields

Let X and Y be locally compact Hausdorff spaces and let $p\colon Y \to X$ be continuous.

Definition and Proposition 8.3.1. Let E be a u.s.c. field of Banach spaces over Y. For all $x \in X$, define[8]

$$p_*(E)_x := \Gamma_0\left(Y_x,\ E|_{Y_x}\right).$$

On this family of Banach spaces over X define a structure of a u.s.c. field of Banach spaces over X as follows: For all $\xi \in \Gamma_0(Y,E)$, define the selection $p_*(\xi)\colon x \mapsto \xi|_{Y_x}$ of $p_*(E)$. Then $\Gamma_0 := \{p_*(\xi) :\ \xi \in \Gamma_0(Y,E)\}$ satisfies conditions (C1) - (C3) of the definition of a u.s.c. field of Banach spaces and therefore defines a structure of a u.s.c. field of Banach spaces over X on $p_*(E)$. It has the property $\Gamma_0 = \Gamma_0\left(X,\ p_*E\right)$.

Proof. Let $x \in X$. Then Y_x is a closed subspace of Y, so we can apply Proposition E.5.2 which says that the map $\xi \mapsto \xi|_{Y_x}$ is a metric surjection from $\Gamma_0(Y,E)$ onto $\Gamma_0\left(Y_x,\ E|_{Y_x}\right)$. In particular, the set Γ_0 defined above is total. It clearly is a $\mathbb{C}$-linear subspace of the space of all selections of

[8]This definition makes sense if $x \in p(Y)$, and can and should be interpreted as $p_*(E)_x = 0$ if $x \notin p(Y)$.

$p_*(E)$. So we have checked (C1) and (C2). As for (C3), let $x_0 \in X$, $\varepsilon > 0$ and $\xi \in \Gamma_0(Y, E)$. Let $L := \{y \in Y : \|\xi(y)\| \geq \|p_*(\xi)(x_0)\| + \varepsilon\}$. Then L is a compact subset of Y because ξ is vanishing at infinity. Hence its image $K := p(L)$ is a compact subset of X. This set K does not contain x_0, so its complement $U := X \setminus K$ is an open neighbourhood of x_0 such that for $u \in U$ we have

$$\|p(\xi)(u)\| = \sup_{y \in Y_x} \|\xi(y)\| \leq \|p_*(\xi)(x_0)\| + \varepsilon,$$

where the supremum is assumed to be zero if taken over the empty set. Hence we have shown that $|p_*(\xi)|$ is upper semi-continuous.

It remains to show that $\Gamma_0 = \Gamma_0(X, p_*E)$. Let ξ be in $\Gamma_0(Y, E)$ and $\varepsilon > 0$. Find a compact subset L of Y such that $\|\xi(y)\| \leq \varepsilon$ whenever $y \in Y \setminus L$. Let $K := p(L)$. Then K is a compact subset such that $\|p_*(\xi)(x)\| = \sup_{y \in Y_x} \|\xi(y)\| \leq \varepsilon$ for all $x \in X \setminus K$. So $p_*(\xi)$ vanishes at infinity. This shows that $\xi \mapsto p_*\xi$ is an (isometric) map from $\Gamma_0(Y, E)$ to $\Gamma_0(X, p_*E)$. The image is total and invariant under multiplication with elements of $\mathcal{C}_c(X)$, so it is dense. Hence the image is all of $\Gamma_0(X, p_*E)$. □

Definition and Proposition 8.3.2. Let E and F be u.s.c. fields of Banach spaces over Y and let T be a bounded continuous field of linear maps from E to F. For all $x \in X$, define

$$p_*(T)_x : p_*(E)_x \to p_*(F)_y,\ \xi \mapsto [Y_x \ni y \mapsto T_y(\xi(y))].$$

Then p_*T is a continuous field of linear maps bounded by $\|T\|$.

Proof. If $\xi \in \Gamma_0(X, E)$, then $p_*T \circ p_*\xi = p_*(T \circ \xi) \in \Gamma_0(X, p_*F)$. So p_*T maps a total subset of $\Gamma(X, p_*E)$ into $\Gamma(X, p_*F)$ and is hence continuous. □

Definition 8.3.3. Let E_1, E_2 and F be u.s.c. fields of Banach spaces over Y and let $\mu\colon E_1 \times_Y E_2 \to F$ be a bounded continuous field of bilinear maps. For all $x \in X$, define

$$p_*(\mu)_x : p_*(E_1)_x \times p_*(E_2)_x \to p_*(F)_x,\ (\xi_1, \xi_2) \mapsto [Y_x \ni y \mapsto \mu_y(\xi_1(y),\ \xi_2(y))].$$

Then $p_*\mu$ is a continuous field of bilinear maps bounded by $\|\mu\|$. If μ is non-degenerate, then so is $p_*\mu$, and vice versa. This definition respects the associativity of bilinear maps.

Using these definitions one can define a u.s.c. field p_*A of Banach algebras over X if A is a u.s.c. field of Banach algebras over Y. Similar definitions can be made for Banach modules and pairs.

Lemma 8.3.4. *Let Z be another locally compact Hausdorff space and let $q\colon Z \to Y$ be continuous. Let E be a u.s.c. field of Banach spaces over Z. Then $(p \circ q)_*E \cong p_*q_*E$. This is also true for bounded continuous fields of linear and bilinear maps.*

Proof. Let $x \in X$. Write Z_x for $(p \circ q)^{-1}(\{x\}) \subseteq Z$. The fibre of $(p \circ q)_*E$ at x is $\Gamma_0(Z_x, E|_{Z_x})$. The fibre of p_*q_*E is $\Gamma_0(Y_x, (q_*E)|_{Y_x})$. If $\xi \in \Gamma_0(Z_x, E|_{Z_x})$, then $q_*\xi \in \Gamma_0(Y_x, q_*(E|_{Z_x}))$. Note that $(q_*E)|_{Y_x} = q_*(E|_{Z_x})$. So $\xi \mapsto q_*\xi$ defines an isometric isomorphism from $[(p \circ q)_*E]_x$ to $[p_*q_*E]_x$. Now $\Gamma_0(X, (p \circ q)_*E)$ and $\Gamma_0(X, p_*q_*E)$ both come from $\Gamma_0(Z, E)$, in the first case through $\xi \mapsto (p \circ q)_*\xi$, in the second case through $\xi \mapsto p_*q_*\xi$. Hence $(p \circ q)_*E$ and p_*q_*E are isomorphic. □

Proposition 8.3.5 (Pushforward and Pullback). *Let Y' be another locally compact Hausdorff space and let $p'\colon Y' \to X$ be continuous. Let $q\colon Y' \times_X Y \to Y$ and $q'\colon Y' \times_X Y \to Y'$ be the canonical "projections". Let E be a u.s.c. field of Banach spaces over X. Then $p'^*(p_*E) \cong q'_*(q^*E)$, i.e., the two ways of going from the upper right to the lower left corner in the following diagram yield the same result:*

$$\begin{array}{ccc} Y' \times_X Y & \xrightarrow{q} & Y \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{p'} & X \end{array}$$

Proof. Let $y' \in Y'$. Then the fibre of $Y' \times_X Y$ over y' is $\{(y', y) : y \in Y_{p'(y')}\}$ and hence it is canonically homeomorphic to $Y_{p'(y')}$; let $\varphi_{y'}\colon (Y' \times_X Y)_{y'} \to Y_{p'(y')}$, $(y', y) \mapsto y$ be the homoeomorphism.

The fibre $(p'^*(p_*E))_{y'}$ of $p'^*(p_*E)$ over y' is $(p_*E)_{p'(y')} = \Gamma_0\left(Y_{p'(y')},\ E|_{Y_{p'(y')}}\right)$. The fibre $(q'_*(q^*E))_{y'}$ of $q'_*(q^*E)$ over y' is $\Gamma_0\left((Y' \times_X Y)_{y'},\ (q^*E)|_{(Y' \times_X Y)_{y'}}\right)$. Now

$$\left(\varphi_{y'}^*(E|_{Y_{p'(y')}})\right)_{(y',y)} = E_{\varphi_{y'}(y',y)} = E_y = (q^*E)_{(y',y)}$$

for all $y \in Y_{p'(y')}$. If $\xi_{y'} \in \Gamma_0\left(Y_{p'(y')},\ E|_{Y_{p'(y')}}\right)$, then define the selection $\Phi_{y'}\left(\xi_{y'}\right)$ by

$$\Phi_{y'}\left(\xi_{y'}\right)\ (y', y') := \xi_{y'}(y)$$

for all $y \in Y_{p'(y')}$. This is an isometric linear map from $\Gamma_0(Y_{p'(y')},\ E|_{Y_{p'(y')}})$ to the space of selections $\Sigma_0((Y' \times_X Y)_{y'},\ (q^*E)|_{(Y' \times_X Y)_{y'}})$. If $\xi_{y'} \in \Gamma_0(Y_{p'(y')},\ E|_{Y_{p'(y')}})$, then there exists a $\xi \in \Gamma_0\left(Y,\ E\right)$ such that $\xi_{y'} = \xi|_{Y_{p'(y')}}$. Then $\xi \circ q$ is a section in $\Gamma\left(Y' \times_X Y,\ q^*E\right)$. Now $(\xi \circ q)|_{(Y' \times_X Y)_{y'}}(y', y) = \xi(q(y', y)) = \xi(y) = \Phi_{y'}(\xi_{y'})\ (y', y)$ for all $y \in Y_{p'(y')}$, so $\Phi_{y'}(\xi_{y'})$ is a section, so $\Phi_{y'}$ takes it values in $\Gamma_0((Y' \times_X Y)_{y'},\ (q^*E)|_{(Y' \times_X Y)_{y'}})$. The image is clearly total and invariant under the action of $\mathcal{C}_c\left((Y' \times_X Y)_{y'}\right)$, so it is dense and hence $\Phi_{y'}$ is (isometric and) surjective.

Now let $\Phi := (\Phi_{y'})_{y' \in Y'}$. We show that Φ is an isomorphism of the u.s.c. fields $p'^*(p_*E)$ and $q'_*(q^*E)$ over Y'. We already know that it is a family of isometric isomorphisms between the fibres. Let $\xi \in \Gamma_0\left(Y, E\right)$. Then $p_*\xi \in \Gamma_0\left(X, p_*E\right)$. Moreover, $(p_*\xi) \circ p' \in \Gamma\left(Y',\ p'^*p_*E\right)$. If $\chi' \in \mathcal{C}_0\left(Y'\right)$, then $\chi'\left((p_*\xi) \circ p'\right) \in \Gamma_0\left(Y',\ p'^*p_*E\right)$. On the other hand, $\xi \circ q \in \Gamma\left(Y' \times_X Y,\ q^*E\right)$ and $(\chi' \circ q')(\xi \circ q) \in \Gamma_0\left(Y' \times_X Y,\ q^*E\right)$. This implies $q'_*\left((\chi' \circ q')(\xi \circ q)\right) \in \Gamma_0\left(Y',\ q'_*q^*E\right)$. We have

$$\begin{aligned} & \Phi_{y'}\left(\chi'\left((p_*\xi) \circ p'\right)(y')\right)\ (y', y) = \chi'(y')\Phi_{y'}\left((p_*\xi)(p'(y'))\right)\ (y', y) \\ = \ & \chi'(y')\Phi_{y'}\left(\xi|_{Y_{p'(y')}}\right)\ (y', y) = \chi'(y')\xi(y) = \left((\chi' \circ q')(\xi \circ q)\right)\ (y', y) \\ = \ & \left((\chi' \circ q')(\xi \circ q)\right)|_{(Y' \times_X Y)_{y'}}\ (y', y) \end{aligned}$$

for all $y' \in Y'$ and all $y \in Y_{p'(y')}$. So

$$\Phi \circ \left(\chi'\left((p_*\xi) \circ p'\right)\right) = q'_*\left((\chi' \circ q')(\xi \circ q)\right).$$

In both cases, the set of such sections is total and hence Φ is continuous in both directions. $\square$

8.3.2 The pushforward for equivariant fields

In this subsection let Y be a locally compact Hausdorff left $\mathcal{G}$-space with anchor map $\rho\colon Y \to X$. Let E be a $\mathcal{G} \ltimes Y$-Banach space. Then we define the structure of a $\mathcal{G}$-Banach space on $\rho_* E$ as follows:

Definition and Proposition 8.3.6. Let E be a $\mathcal{G} \ltimes Y$-Banach space with action α. Let $\gamma \in \mathcal{G}$ and $\xi_{s(\gamma)} \in (\rho_* E)_{s(\gamma)} = \Gamma_0\left(Y_{s(\gamma)},\ E|_{Y_{s(\gamma)}}\right)$. Define a section $\gamma\xi_{s(\gamma)} \in \Gamma_0\left(Y_{r(\gamma)},\ E|_{Y_{r(\gamma)}}\right) = (\rho_* E)_{r(\gamma)}$ by

$$\left(\gamma\xi_{s(\gamma)}\right)\ (y) := (\gamma, y)\left(\xi_{s(\gamma)}\left(\gamma^{-1}y\right)\right)$$

for all $y \in Y_{r(\gamma)}$. This defines an action of $\mathcal{G}$ on $\rho_* E$.

Proof. The action α of $\mathcal{G} \ltimes Y$ on E is an isomorphism from $s^*_{\mathcal{G}\ltimes Y} E$ and $r^*_{\mathcal{G}\ltimes Y} E$. Recall that we have identified $\mathcal{G} \ltimes Y$ with $\mathcal{G} \times_{r,\rho} Y$. Define $\pi_1\colon \mathcal{G} \ltimes Y \to \mathcal{G},\ (\gamma, y) \mapsto \gamma$. Then $\pi_{1,*}\alpha$ is an isomorphism from $\pi_{1,*}s^*_{\mathcal{G}\ltimes Y} E$ to $\pi_{1,*}r^*_{\mathcal{G}\ltimes Y} E$. Note that we have commutative squares

$$\begin{tikzcd} \mathcal{G} \ltimes Y \arrow[r, "r_{\mathcal{G}\ltimes Y}"] \arrow[d, "\pi_1"'] & Y \arrow[d, "\rho"] \\ \mathcal{G} \arrow[r, "r_{\mathcal{G}}"] & X \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G} \ltimes Y \arrow[r, "s_{\mathcal{G}\ltimes Y}"] \arrow[d, "\pi_1"'] & Y \arrow[d, "\rho"] \\ \mathcal{G} \arrow[r, "s_{\mathcal{G}}"] & X \end{tikzcd}$$

Applying Proposition 8.3.5 we have $\pi_{1,*}s^*_{\mathcal{G}\ltimes Y} E \cong s^*_{\mathcal{G}}\rho_* E$ and $\pi_{1,*}r^*_{\mathcal{G}\ltimes Y} E \cong r^*_{\mathcal{G}}\rho_* E$, and the resulting isomorphism from $s^*_{\mathcal{G}}\rho_* E$ to $r^*_{\mathcal{G}}\rho_* E$ is precisely the action of $\mathcal{G}$ on $\rho_* E$ defined above:

Let $\gamma \in \mathcal{G}$ and let $\xi_{s_{\mathcal{G}}(\gamma)} \in (\rho_* E)_{s_{\mathcal{G}}(\gamma)} = \Gamma_0(Y_{s_{\mathcal{G}}(\gamma)},\ E|_{Y_{s_{\mathcal{G}}(\gamma)}})$. Our identification $(\rho_* E)_{s_{\mathcal{G}}(\gamma)} = (\pi_{1,*}s^*_{\mathcal{G}\ltimes Y} E)_\gamma$ identifies $\xi_{s_{\mathcal{G}}(\gamma)}$ with $(\gamma, y) \mapsto \xi_{s_{\mathcal{G}}(\gamma)}(\gamma^{-1}y)$. Applying $(\pi_{1,*}\alpha)_\gamma$ to this section gives the section $(\gamma, y) \mapsto (\gamma, y)\xi_{s_{\mathcal{G}}(\gamma)}(\gamma^{-1}y)$ in $(\pi_{1,*}r^*_{\mathcal{G}\ltimes Y} E)_\gamma$. The identification $(\pi_{1,*}r^*_{\mathcal{G}\ltimes Y} E)_\gamma$ with $(r^*_{\mathcal{G}}\rho_* E)_\gamma = (\rho_* E)_{r_{\mathcal{G}}(\gamma)}$ gives the section $y \mapsto (\gamma, y)\xi_{s_{\mathcal{G}}(\gamma)}(\gamma^{-1}y)$.

We also check the algebraic properties of the action: Let $\gamma, \gamma' \in \mathcal{G}$ such that $s_{\mathcal{G}}(\gamma) = r_{\mathcal{G}}(\gamma')$. Let $\xi_{s_{\mathcal{G}}(\gamma')} \in (\rho_* E)_{s_{\mathcal{G}}(\gamma')} = \Gamma_0(Y_{s_{\mathcal{G}}(\gamma')},\ E|_{Y_{s_{\mathcal{G}}(\gamma')}})$. Then

$$\begin{aligned} \left((\gamma\gamma')\xi_{s_{\mathcal{G}}(\gamma')}\right)(y) &= (\gamma\gamma', y)\,\xi_{s_{\mathcal{G}}(\gamma')}\left((\gamma\gamma')^{-1}y\right) = (\gamma, y)\,(\gamma', \gamma^{-1}y)\,\xi_{s_{\mathcal{G}}(\gamma')}\left(\gamma'^{-1}\left(\gamma^{-1}y\right)\right) \\ &= (\gamma, y)\left[\left(\gamma'\xi_{s_{\mathcal{G}}(\gamma')}\right)(\gamma^{-1}y)\right] = \gamma\left(\gamma'\xi_{s_{\mathcal{G}}(\gamma')}\right)(y) \end{aligned}$$

for all $y \in Y_{s_{\mathcal{G}}(\gamma')}$, so $(\gamma\gamma')\xi_{s_{\mathcal{G}}(\gamma')} = \gamma(\gamma'\xi_{s_{\mathcal{G}}(\gamma')})$. Hence we have defined an action of $\mathcal{G}$ on $\rho_* E$. $\square$

Definition and Proposition 8.3.7. Let E and F be $\mathcal{G} \ltimes Y$-Banach spaces and let $\varphi\colon E \to F$ be a $\mathcal{G} \ltimes Y$-equivariant contractive continuous field of linear maps. Then $\rho_*\varphi$ is a $\mathcal{G}$-equivariant contractive continuous field of linear maps from $\rho_* E$ to $\rho_* F$.

Proof. Let $\gamma \in \mathcal{G}$ and $\xi_{s(\gamma)} \in (\rho_* E)_{s(\gamma)} = \Gamma_0\left(Y_{s(\gamma)},\ E|_{Y_{s(\gamma)}}\right)$. Then

$$\begin{aligned} &\left[\gamma(\rho_*\varphi)_{s(\gamma)}\xi_{s(\gamma)}\right](y) = (\gamma, y)\left((\rho_*\varphi)_{s(\gamma)}\xi_{s(\gamma)}\right)(\gamma^{-1}y) = (\gamma, y)\left(\varphi_{\gamma^{-1}y}\xi_{s(\gamma)}(\gamma^{-1}y)\right) \\ =\ &\varphi_y\left((\gamma, y)\xi_{s(\gamma)}(\gamma^{-1}y)\right) = \varphi_y\left((\gamma\xi_{s(\gamma)})(y)\right) = \left[(\rho_*\varphi)_{r(\gamma)}\left(\gamma\xi_{s(\gamma)}\right)\right](y) \end{aligned}$$

for all $y \in Y_{r(\gamma)}$, which means that $\rho_*\varphi$ is $\mathcal{G}$-equivariant. $\square$

Hence we have a functor from the $\mathcal{G} \ltimes Y$-Banach spaces to the $\mathcal{G}$-Banach spaces. The same type of calculation shows:

Definition and Proposition 8.3.8. Let E_1, E_2 and F be $\mathcal{G} \ltimes Y$-Banach spaces and let $\mu \colon E_1 \times_Y E_2 \to F$ be a $\mathcal{G} \ltimes Y$-equivariant contractive continuous field of bilinear maps. Then $\rho_*\mu$ is a $\mathcal{G}$-equivariant contractive continuous field of bilinear maps from $\rho_* E_1 \times_X \rho_* E_2$ to $\rho_* F$.

Lemma 8.3.9. *Let Z and Y be locally compact Hausdorff $\mathcal{G}$-spaces with anchor maps ρ^Z and ρ^Y, respectively. Let $q \colon Z \to Y$ be a continuous $\mathcal{G}$-equivariant map. Then Z is also a $\mathcal{G} \ltimes Y$-space: The anchor map is q and the continuous action is defined by $(\gamma, y)z := \gamma z$ for all $(\gamma, z) \in \mathcal{G} \ltimes Y$ and $y \in Y$ such that $s(\gamma, y) = \gamma^{-1} y = q(z)$. Note that $\rho^Y \circ q = \rho^Z$. If E is a $\mathcal{G} \ltimes Z$-Banach space, then $q_* E$ is a $\mathcal{G} \ltimes Y$-Banach space and*

$$\rho^Z_* E \cong \rho^Y_* q_* E.$$

This construction respects equivariant bounded continuous fields of linear and bilinear maps.

Proof. We just check that the isomorphism given by Lemma 8.3.4 is $\mathcal{G}$-equivariant: Let $\gamma \in \mathcal{G}$ and $\xi \in \Gamma_0(Z_{s(\gamma)},\ E|_{Z_{s(\gamma)}})$. Then

$$\left(\gamma \rho^Z_* \xi\right)(z) = (\gamma, z)\left[\xi(\gamma^{-1} z)\right]$$

for all $z \in Z_{r(\gamma)}$. On the other hand, we have, for all $y \in Y_x$,

$$\left(\gamma\left(\rho^Y_*(q_*\xi)\right)\right)(y) = (\gamma, y)\left[(q_*\xi)(\gamma^{-1} y))\right]$$

and thus for all $z \in Z_y$ (using $y = q(z)$):

$$\begin{aligned} & \left[\left(\gamma\left(\rho^Y_*(q_*\xi)\right)\right)(q(z))\right](z) = (\gamma, q(z))\left[(q_*\xi)(\gamma^{-1} q(z)))\right](z) \\ = \ & (\gamma, z)\left[\left((q_*\xi)(q(\gamma^{-1} z))\right)(\gamma^{-1} z)\right] = (\gamma, z)\left[\xi(\gamma^{-1} z)\right]. \end{aligned}$$

Hence the isomorphism is $\mathcal{G}$-equivariant. □

This construction respects the associativity of equivariant bilinear maps. Hence we can make the following definitions:

Definition and Proposition 8.3.10. Let B be a $\mathcal{G} \ltimes Y$-Banach algebra with multiplication μ. Then $\rho_* B$ together with $\rho_*\mu$ is a $\mathcal{G}$-Banach algebra. If $\varphi \colon B \to B'$ is a $\mathcal{G} \ltimes Y$-equivariant homomorphism between $\mathcal{G} \ltimes Y$-Banach algebras, then $\rho_*\varphi$ is a $\mathcal{G}$-equivariant homomorphism from $\rho_* B$ to $\rho_* B'$.

Definition and Proposition 8.3.11. Let B be a $\mathcal{G} \ltimes Y$-Banach algebra and let E be a right $\mathcal{G} \ltimes Y$-Banach B-module with multiplication μ_E. Then $\rho_* E$ together with $\rho_*\mu_E$ is a right $\mathcal{G}$-Banach $\rho_* B$-module. This construction respects equivariant homomorphisms of Banach modules. If F is another right $\mathcal{G} \ltimes Y$-Banach B-module and $T \in \mathrm{L}_B(E, F)$ is a B-linear bounded continuous field of linear maps, then $\rho_* T$ is in $\mathrm{L}_{\rho_*}(\rho_* E,\ \rho_* F)$ with $\|\rho_* T\| \leq \|T\|$.

Definition and Proposition 8.3.12. Let B be a $\mathcal{G} \ltimes Y$-Banach algebra and let $E = (E^<, E^>)$ be a $\mathcal{G} \ltimes Y$-Banach B-pair with bracket $\langle \cdot, \cdot \rangle_E$. Then $\rho_* E = (\rho_* E^<,\ \rho_* E^>)$ together with $\rho_* \langle \cdot, \cdot \rangle_E$ is a $\mathcal{G}$-Banach $\rho_* B$-pair. This construction respects equivariant concurrent homomorphisms of Banach pairs. If F is another right $\mathcal{G} \ltimes Y$-Banach B-pair and $T \in \mathrm{L}_B(E, F)$ is a B-linear bounded continuous field of operators, then $\rho_* T$ is in $\mathrm{L}_{\rho_*}(\rho_* E,\ \rho_* F)$ with $\|\rho_* T\| \leq \|T\|$.

Proposition 8.3.13. *Let B be a $\mathcal{G} \ltimes Y$-Banach algebra and let E and F be $\mathcal{G} \ltimes Y$-Banach B-pairs. Let $\xi^< \in \Gamma_0(X, E^<)$ and $\eta^> \in \Gamma_0(X, F^>)$. Then*

$$\rho_*\left(\left|\eta^>\right\rangle\left\langle\xi^<\right|\right) = \left|\rho_*(\eta^>)\right\rangle\left\langle\rho_*(\xi^<)\right| \in \mathrm{K}_{\rho_* B}\left(\rho_* E,\ \rho_* F\right).$$

It follows that $\rho_*\left(\mathrm{K}_B(E, F)\right) \subseteq \mathrm{K}_{\rho_* B}\left(\rho_* E,\ \rho_* F\right)$.

Proof. We check this formula only on the right-hand side: Let $x \in X$ and $\xi_x^> \in \rho_*(E^>)_x = \Gamma_0\left(Y_x,\ E^>|_{Y_x}\right)$. Then for all $y \in Y_x$:

$$\begin{aligned} & \left[\rho_*\left(|\eta^>\rangle\langle\xi^<|\right)_x^>\left(\xi_x^>\right)\right](y) = |\eta^>\rangle\langle\xi^<|_y^>\left(\xi_x^>(y)\right) = |\eta^>(y)\rangle\langle\xi^<(y)|^>(\xi_x^>(y)) \\ = \;& \eta^>(y)\ \langle\xi^<(y),\ \xi_x^>(y)\rangle = \rho_*(\eta^>)_x(y)\left\langle\rho_*(\xi^<),\ \xi^>\right\rangle(y) = \left[|\rho_*(\eta^>)\rangle\langle\rho_*(\xi^<)|^>(\xi_x^>)\right](y). \end{aligned}$$

□

Lemma 8.3.14. *Let B be a $\mathcal{G}\ltimes Y$-Banach algebra and let E and F be $\mathcal{G}\ltimes Y$-Banach B-pairs. Define $\pi_1\colon \mathcal{G}\ltimes Y \to \mathcal{G},\ (\gamma,y)\mapsto\gamma$. Then for all $T\in \mathrm{L}_B(E,F)$, we have*

$$\alpha^{\mathrm{L}(\rho_*E,\rho_*F)}s_{\mathcal{G}}^*(\rho_*T) = \pi_{1,*}\left(\alpha^{\mathrm{L}(E,F)}s_{\mathcal{G}\ltimes Y}^*T\right),$$

where we identify $r_{\mathcal{G}}^*\rho_*E$ and $\pi_{1,*}r_{\mathcal{G}\ltimes Y}^*E$ (and similar for F and B).

Proof. Let $\gamma\in\mathcal{G}$ and $\xi_{r(\gamma)}^> \in \rho_*E_{r_{\mathcal{G}}(\gamma)}^> = \Gamma_0\left(Y_{r_{\mathcal{G}}(\gamma)},\ E^>|_{Y_{r_{\mathcal{G}}(\gamma)}}\right)$. Then for all $y\in Y_{r_{\mathcal{G}}(\gamma)}$:

$$\begin{aligned} & \left[\alpha^{\mathrm{L}(\rho_*E,\rho_*F)}s_{\mathcal{G}}^*(\rho_*T)\right]_\gamma\left(\xi_{r(\gamma)}^>\right)\ (y) = \gamma\left[\left(s_{\mathcal{G}}^*(\rho_*T)\right)_\gamma\left(\gamma^{-1}\xi_{r(\gamma)}^>\right)\right]\ (y) \\ = \;& \gamma\left[(\rho_*T)_{s_{\mathcal{G}}(\gamma)}\left(\gamma^{-1}\xi_{r(\gamma)}^>\right)\right]\ (y) = (\gamma,y)\left(\left[(\rho_*T)_{s_{\mathcal{G}}(\gamma)}\left(\gamma^{-1}\xi_{r(\gamma)}^>\right)\right](\gamma^{-1}y)\right) \\ = \;& (\gamma,y)\left(T_{\gamma^{-1}y}\left[\left(\gamma^{-1}\xi_{r(\gamma)}^>\right)(\gamma^{-1}y)\right]\right) = (\gamma,y)\left(T_{\gamma^{-1}y}\left[(\gamma^{-1},\gamma^{-1}y)\left(\xi_{r(\gamma)}^>(\gamma\gamma^{-1}y)\right)\right]\right) \\ = \;& (\gamma,y)\left(T_{\gamma^{-1}y}\left[(\gamma,y)^{-1}\left(\xi_{r(\gamma)}^>(y)\right)\right]\right) = \left(\alpha^{\mathrm{L}(E,F)}(s_{\mathcal{G}\ltimes Y}^*T)\right)_{(\gamma,y)}(\xi_{r(\gamma)}^>(y)). \end{aligned}$$

A similar calculation holds for the left-hand side. □

Proposition 8.3.15. *Let A and B be $\mathcal{G}\ltimes Y$-Banach algebras. Let $(E,T)\in\mathbb{E}_{\mathcal{G}\ltimes Y}^{\mathrm{ban}}(A,\ B)$. Then $(\rho_*E,\ \rho_*T)$ is in $\mathbb{E}_{\mathcal{G}}^{\mathrm{ban}}(\rho_*A,\ \rho_*B)$.*

Proof. Surely E is a graded non-degenerate $\mathcal{G}$-Banach ρ_*A-ρ_*B-pair and ρ_*T is an odd continuous field of linear operators on ρ_*E. Now let $a\in\Gamma_0(X,\ \rho_*A)$. Then there is a $a'\in\Gamma_0(Y,\ A)$ such that $a=\rho_*a'$. Now

$$[a,\ \rho_*T] = \left[\rho_*a',\ \rho_*T\right] = \rho_*\left[a',T\right]\in \mathrm{K}_{\rho_*B}(\rho_*E).$$

Similarly,

$$a\left(\rho_*T^2-1\right) = \rho_*\left(a'(T^2-1)\right)\in\mathrm{K}_{\rho_*B}(\rho_*E).$$

Now let $\tilde a\in\Gamma_0\left(\mathcal{G},\ r_{\mathcal{G}}^*\rho_*A\right)$. As above, define $\pi_1\colon\mathcal{G}\ltimes Y\to\mathcal{G},\ (\gamma,y)\mapsto\gamma$. We identify $r_{\mathcal{G}}^*\rho_*A$ and $\pi_{1,*}r_{\mathcal{G}\ltimes Y}^*A$ (and do the same for B and E) and regard $\tilde a$ as an element of $\Gamma_0\left(\mathcal{G},\ \pi_{1,*}r_{\mathcal{G}\ltimes Y}^*A\right)$. We can then find an element $\tilde a'\in\Gamma_0\left(\mathcal{G}\ltimes Y,\ r_{\mathcal{G}\ltimes Y}^*A\right)$ such that $\pi_{1,*}\tilde a'=\tilde a$. Using Lemma 8.3.14 and suitable identifications we can now conclude

$$\begin{aligned} & \tilde a\left(\alpha^{\mathrm{L}(\rho_*E)}(s_{\mathcal{G}}^*\rho_*T) - r_{\mathcal{G}}^*\rho_*T\right) = \pi_{1,*}\tilde a'\left(\pi_{1,*}\alpha^{\mathrm{L}(E)}s_{\mathcal{G}\ltimes Y}^*(T) - \pi_{1,*}r_{\mathcal{G}\ltimes Y}^*(T)\right) \\ = \;& \pi_{1,*}\left(\tilde a'\left(\alpha^{\mathrm{L}(E)}s_{\mathcal{G}\ltimes Y}^* - r_{\mathcal{G}\ltimes Y}^*T\right)\right)\in\mathrm{K}_{\pi_{1,*}r_{\mathcal{G}\ltimes Y}^*B}\left(\pi_{1,*}r_{\mathcal{G}\ltimes Y}^*E\right) = \mathrm{K}_{r_{\mathcal{G}}^*\rho_*B}\left(r_{\mathcal{G}}^*\rho_*E\right). \end{aligned}$$

Proposition 4.7.5 tells us that we can define $\mathrm{KK}_{\mathcal{G}}^{\mathrm{ban}}$-cycles between $\mathcal{G}$-Banach algebras with locally compact Hausdorff $\mathcal{G}$ also using compact instead of locally compact operators, so we have shown that (ρ_*E,ρ_*T) is a $\mathrm{KK}_{\mathcal{G}}^{\mathrm{ban}}$-cycle. □

Lemma 8.3.16. *Let B be a $\mathcal{G} \ltimes Y$-Banach algebra. Then $\rho_*(B[0,1])$ is canonically isomorphic to $(\rho_* B)[0,1]$.*

Proof. Define $\pi_2^Y\colon [0,1] \times Y \to Y,\ (t,y) \mapsto y$ and $\pi_2^X\colon [0,1] \times X \to X,\ (t,x) \mapsto x$. Then a careful inspection of the definition of $B[0,1]$ shows that $B[0,1] = (\pi_2^Y)_*(\pi_2^Y)^* B$ and similarly $(\rho_* B)[0,1] = (\pi_2^X)_*(\pi_2^X)^*(\rho_* B)$. Now $\rho \circ \pi_2^Y = \pi_2^X \circ (\rho \times \mathrm{Id}_{[0,1]})$ implies that

$$\begin{aligned} \rho_*(B[0,1]) &= \rho_*(\pi_2^Y)_*(\pi_2^Y)^* B \cong (\rho \circ \pi_2^Y)_*(\pi_2^Y)^* B = (\pi_2^X \circ (\rho \times \mathrm{Id}_{[0,1]})_*(\pi_2^Y)^* B \\ &\cong (\pi_2^X)_*(\rho \times \mathrm{Id}_{[0,1]})_*(\pi_2^Y)^* B \cong (\pi_2^X)_*(\pi_2^X)^* \rho_* B = (\rho_* B)[0,1]. \end{aligned}$$

□

Lemma 8.3.17. *Let B be a $\mathcal{G} \ltimes Y$-Banach algebra. Let E be a left $\mathcal{G} \ltimes Y$-Banach B-module and F a right $\mathcal{G} \ltimes Y$-Banach B-module, one of them being non-degenerate. Then*

$$\rho_*(E) \otimes_{\rho_*(B)} \rho_*(F) \cong \rho_* \left(E \otimes_B F\right).$$

Proof. Let $\mu\colon E \times_Y F \to E \otimes_B F$ be the canonical field of bilinear maps. Then there is a canonical homomorphism from the left-hand to the right-hand side, namely $\rho_* \mu$. We check that it is a fibre-wise isomorphism:

Let $x \in X$. The fibre of the left-hand side over x is $\Gamma_0\left(Y_x,\ E|_{Y_x}\right) \otimes^{\mathcal{C}_0(Y_x)}_{\Gamma_0(Y_x, B|_{Y_x})} \Gamma_0\left(Y_x,\ F|_{Y_x}\right)$, the fibre of the right-hand side over x is $\Gamma_0\left(Y_x,\ (E \otimes_B F)|_{Y_x}\right)$. Both sides are $\mathcal{C}_0(Y_x)$-Banach spaces and the canonical map $(\rho_*\mu)_x$ is $\mathcal{C}_0(Y_x)$-linear and an isomorphism on the fibres. By Theorem A.2.15, which says that the $\mathcal{C}_0(Y_x)$-tensor product of locally convex $\mathcal{C}_0(Y_x)$-Banach spaces is locally $\mathcal{C}_0(Y_x)$-convex (plus the fact that quotients of locally convex $\mathcal{C}_0(Y_x)$-Banach spaces are locally convex), both sides are locally $\mathcal{C}_0(Y_x)$-convex and hence we have an isomorphism. □

By the two preceding lemmas and arguments that appeared several times in this thesis we can conclude:

Proposition 8.3.18. *Let A and B be $\mathcal{G} \ltimes Y$-Banach algebras. Then ρ_* gives a homomorphism*

$$\rho_*\colon\ \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G} \ltimes Y}\left(A, B\right) \to \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\rho_* A,\ \rho_* B\right).$$

8.3.3 The pushforward and the descent

Assume that $\mathcal{G}$ carries a Haar system.

Definition 8.3.19. Let E be a $\mathcal{G} \ltimes Y$-Banach space. For all $\xi \in \Gamma_c\left(\mathcal{G} \ltimes Y,\ r^*_{\mathcal{G} \ltimes Y} E\right)$, define $\hat{\iota}_E(\xi) \in \Gamma_c\left(\mathcal{G},\ r^*_{\mathcal{G}} \rho_* E\right)$ by

$$\hat{\iota}_E(\xi)\,(\gamma) := \left[Y_{r_{\mathcal{G}}(\gamma)} \ni y \mapsto \xi(\gamma, y)\right] \in \rho_* E_{r_{\mathcal{G}}(\gamma)}$$

for all $\gamma \in \mathcal{G}$.

Lemma 8.3.20. *If E is a $\mathcal{G} \ltimes Y$-Banach space, then $\hat{\iota}_E$ is continuous for the inductive limit topologies, injective, and has dense image.*

Proof. The map $\hat{\iota}_E$ is isometric for the sup-norm, so in particular, it is injective. This also shows that $\hat{\iota}_E$ is continuous for the inductive limit topologies. The image of $\hat{\iota}_E$ is dense because it is pointwise dense and invariant under multiplication with functions in $\mathcal{C}_c\left(\mathcal{G}\right)$. □

Lemma 8.3.21. *Let E_1, E_2 and F be $\mathcal{G} \ltimes Y$-Banach spaces and let $\mu\colon E_1 \times_Y E_2 \to F$ be a bounded equivariant continuous field of bilinear maps. Then $(\rho_*\mu)\left(\hat\iota_{E_1}(\xi_1),\ \hat\iota_{E_2}(\xi_2)\right) = \hat\iota_F\left(\mu(\xi_1, \xi_2)\right)$ for all $\xi_1 \in \Gamma_c\left(\mathcal{G} \ltimes Y,\ r^*_{\mathcal{G}\ltimes Y} E_1\right)$ and $\xi_2 \in \Gamma_c\left(\mathcal{G} \ltimes Y,\ r^*_{\mathcal{G}\ltimes Y} E_2\right)$; this could also be written as*

$$\hat\iota_{E_1}(\xi_1) \ * \ \hat\iota_{E_2}(\xi_2) = \hat\iota_F\left(\xi_1 * \xi_2\right).$$

Proof. We have

$$\begin{aligned}
\left[\hat\iota_F\left(\xi_1 * \xi_2\right)(\gamma)\right](y) &= (\xi_1 * \xi_2)(\gamma, y) = \int_{\mathcal{G}^{r(\gamma)}} \mu_y\left(\xi_1(\gamma', y),\ (\gamma', y)\xi_2\left((\gamma', y)^{-1}(\gamma, y)\right)\right) \mathrm{d}\gamma' \\
&= \int_{\mathcal{G}^{r(\gamma)}} \mu_y\left(\xi_1(\gamma', y),\ (\gamma', y)\xi_2\left(\gamma'^{-1}\gamma,\ \gamma'^{-1}y\right)\right) \mathrm{d}\gamma' \\
&= \int_{\mathcal{G}^{r(\gamma)}} \mu_y\left(\hat\iota_{E_1}(\xi_1)(\gamma')(y),\ \left[\gamma'\hat\iota_{E_2}(\xi_2)(\gamma'^{-1}\gamma)\right](y)\right) \mathrm{d}\gamma' \\
&= \left[\int_{\mathcal{G}^{r(\gamma)}} (\rho_*\mu)_{r(\gamma)}\left(\hat\iota_{E_1}(\xi_1)(\gamma'),\ \gamma'\hat\iota_{E_2}(\xi_2)(\gamma'^{-1}\gamma)\right) \mathrm{d}\gamma'\right](y) \\
&= \left[\left(\hat\iota_{E_1}(\xi_1) * \hat\iota_{E_2}(\xi_2)\right)(\gamma)\right](y)
\end{aligned}$$

for all $\gamma \in \mathcal{G}$ and $y \in Y_{r(\gamma)}$. □

Proposition 8.3.22. *For every $\mathcal{G} \ltimes Y$-Banach algebra B, the map $\hat\iota_B$ is a continuous injective homomorphism with dense image.*

Definition and Proposition 8.3.23. Let $\mathcal{H}(\mathcal{G})$ be a monotone completion of $\mathcal{C}_c(\mathcal{G})$ and let E be a $\mathcal{G} \ltimes Y$-Banach space. For all $\xi \in \Gamma_c\left(\mathcal{G} \ltimes Y,\ r^*_{\mathcal{G}\ltimes Y} E\right)$, define

$$\|\xi\|_{\mathcal{H}_Y} := \|\hat\iota_E(\xi)\|_{\mathcal{H}}.$$

This defines a semi-norm on $\Gamma_c\left(\mathcal{G} \ltimes Y,\ r^*_{\mathcal{G}\ltimes Y} E\right)$. If $B = \mathbb{C}_Y$, then $\Gamma_c\left(\mathcal{G} \ltimes Y,\ r^*_{\mathcal{G}\ltimes Y} \mathbb{C}_Y\right) = \mathcal{C}_c\left(\mathcal{G} \ltimes Y\right)$ and $\|\cdot\|_{\mathcal{H}_Y}$ is a monotone semi-norm on $\mathcal{C}_c\left(\mathcal{G} \ltimes Y\right)$. The map $\hat\iota_E$ extends to an isomorphism on the completions

$$\hat\iota_E\colon \mathcal{H}_Y\left(\mathcal{G} \ltimes Y,\ E\right) \cong \mathcal{H}\left(\mathcal{G},\ \rho_* E\right).$$

Proposition 8.3.24. *If $\mathcal{A}(\mathcal{G})$ is an unconditional completion of $\mathcal{C}_c\left(\mathcal{G}\right)$, then $\mathcal{A}_Y\left(\mathcal{G} \ltimes Y\right)$ is an unconditional completion of $\mathcal{C}_c\left(\mathcal{G} \ltimes Y\right)$. If B is a $\mathcal{G} \ltimes Y$-Banach algebra, then*

$$\hat\iota_B\colon \mathcal{A}_Y\left(\mathcal{G} \ltimes Y,\ B\right) \cong \mathcal{A}\left(\mathcal{G},\ \rho_* B\right)$$

as Banach algebras.

Proposition 8.3.25. *Let A and B be $\mathcal{G} \ltimes Y$-Banach algebras and let $\mathcal{A}(\mathcal{G})$ be an unconditional completion of $\mathcal{G}$. Then the following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}\ltimes Y}\left(A,\ B\right) \arrow[r, "j_{\mathcal{A}_Y}"] \arrow[d, "\rho_*"'] & \mathrm{KK}^{\mathrm{ban}}\left(\mathcal{A}_Y\left(\mathcal{G} \ltimes Y, A\right),\ \mathcal{A}_Y\left(\mathcal{G} \ltimes Y, B\right)\right) \arrow[d, "\cong"] \\
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\rho_* A,\ \rho_* B\right) \arrow[r, "j_{\mathcal{A}}"] & \mathrm{KK}^{\mathrm{ban}}\left(\mathcal{A}\left(\mathcal{G},\ \rho_* A\right),\ \mathcal{A}\left(\mathcal{G},\ \rho_* B\right)\right)
\end{tikzcd}$$

where the isomorphism on the right-hand side is given by the isomorphism of Proposition 8.3.24 in both variables.

Proof. This is true already on the level of cycles: Let $(E,T) \in \mathbb{E}^{\mathrm{ban}}_{\mathcal{G}\ltimes Y}(A,\ B)$. Then $\rho_*(E,T) = (\rho_* E,\ \rho_* T)$ by definition. The module $\mathcal{A}(\mathcal{G}, \rho_* E^>)$ is a completion of $\Gamma_c\left(\mathcal{G},\ r^*_{\mathcal{G}}\rho_* E^>\right)$ for the norm $\|\cdot\|_{\mathcal{A}}$. Using the continuous injective linear map $\hat\iota_{E^>}$ from $\Gamma_c\left(\mathcal{G}\ltimes Y,\ r^*_{\mathcal{G}\ltimes Y}E^>\right)$ to $\Gamma_c\left(\mathcal{G},\ r^*_{\mathcal{G}}\rho_* E^>\right)$ introduced in 8.3.19 we get, as in 8.3.23, a linear isometric isomorphism $\hat\iota_{E^>}\colon \mathcal{A}_Y(\mathcal{G}\ltimes Y,\ E^>) \to \mathcal{A}(\mathcal{G},\ \rho_* E^>)$; analogously, we get a linear isomorphism $\hat\iota_{E^<}\colon \mathcal{A}_Y(\mathcal{G}\ltimes Y,\ E^<) \to \mathcal{A}(\mathcal{G},\ \rho_* E^<)$. Together, this gives an isomorphism of Banach pairs $\hat\iota_E$ from $\mathcal{A}_Y(\mathcal{G}\ltimes Y,\ E)$ to $\mathcal{A}(\mathcal{G},\rho_* E)$ with coefficient maps $\hat\iota_A$ and $\hat\iota_B$ (the algebraic properties follow from Lemma 8.3.21). It is straightforward to show that this isomorphism is compatible with the grading and intertwines the operators, i.e., it is an isomorphism of cycles. $\square$

Proposition 8.3.26. *If $\mathcal{A}(\mathcal{G})$ is a regular unconditional completion of $\mathcal{C}_c(\mathcal{G})$, also $\mathcal{A}_Y(\mathcal{G}\ltimes Y)$ is regular.*

Proof. Let $\mathcal{A}(\mathcal{G})$ act on a $\mathcal{G}$-equivariant pair $\mathcal{H}(\mathcal{G}) = (\mathcal{H}^<(\mathcal{G}),\ \mathcal{H}^>(\mathcal{G}))$ of locally convex monotone completions of $\mathcal{C}_c(\mathcal{G})$. Then $\mathcal{A}_Y(\mathcal{G}\ltimes Y)$ acts on $\mathcal{H}_Y(\mathcal{G}\ltimes Y) = \left(\mathcal{H}^<_Y(\mathcal{G}\ltimes Y),\ \mathcal{H}^>_Y(\mathcal{G}\ltimes Y)\right)$. There is a canonical non-degenerate action of $\mathcal{C}_0(Y)$ on $\mathcal{H}_Y(\mathcal{G}\ltimes Y)$ making it a $\mathcal{C}_0(Y)$-Banach space; the trouble is that $\mathcal{H}_Y(\mathcal{G}\ltimes Y)$ needs not be locally $\mathcal{C}_0(Y)$-convex. But it is easy to see that the Gelfand transform $\mathfrak{G}(\mathcal{H}_Y)(\mathcal{G}\ltimes Y) := \left(\mathfrak{G}\left(\mathcal{H}^<_Y(\mathcal{G}\ltimes Y)\right),\ \mathfrak{G}\left(\mathcal{H}^>_Y(\mathcal{G}\ltimes Y)\right)\right)$ is a pair of locally convex monotone completions of $\mathcal{C}_c(\mathcal{G}\ltimes Y)$ on which $\mathcal{A}_Y(\mathcal{G}\ltimes Y)$ acts. We check that this pair is $\mathcal{G}\ltimes Y$-equivariant (we only consider the left-hand side, the right-hand side can be treated analogously).

Let $y \in Y$ and $\chi \in \mathcal{C}_c((\mathcal{G}\ltimes Y)_y)$ with $\chi \geq 0$. The semi-norm of χ as an element of the fibre of $\mathcal{H}^<_Y(\mathcal{G}\ltimes Y)$ over y is the infimum over the semi-norm of all extensions of χ to non-negative elements of $\mathcal{C}_c(\mathcal{G}\ltimes Y)$; these extensions are the same as all the non-negative extensions to $\mathcal{C}_c(\mathcal{G}\ltimes Y)$ of all extensions to $\mathcal{C}_c\left((\mathcal{G}\ltimes Y)^{\rho(y)}\right)$ of χ, where $(\mathcal{G}\ltimes Y)^{\rho(y)} = \{(\gamma', y') \in \mathcal{G}\ltimes Y : r(\gamma') = \rho(y) = \rho(y')\}$. So we can calculate the semi-norm of χ also as the infimum over all non-negative extensions $\tilde\chi$ of χ to $\mathcal{C}_c\left((\mathcal{G}\ltimes Y)^{\rho(y)}\right)$ of the semi-norm $\|\hat\iota_{\mathbb{C}_Y}\tilde\chi\|_{(\mathcal{H}^<)_{\rho(y)}}$.

Now let $(\gamma, y) \in \mathcal{G}\ltimes Y$. Note that $(\mathcal{G}\ltimes Y)^{s(\gamma,y)} = \{(\gamma', y') \in \mathcal{G}\ltimes Y : y' = \gamma^{-1}y\}$. Let $\chi \in \mathcal{C}_c\left((\mathcal{G}\ltimes Y)^{s(\gamma,y)}\right)$ with $\chi \geq 0$. We have to show that $\|\chi\|_{(\mathcal{H}^<_Y)_{\gamma^{-1}y}} = \|(\gamma,y)\cdot\chi\|_{(\mathcal{H}^<_Y)_y}$, where $(\gamma,y)\cdot\chi = \left[(\mathcal{G}\ltimes Y)^y \ni (\gamma', y') \mapsto \chi\left((\gamma,y)^{-1}(\gamma',y')\right) = \chi\left(\gamma^{-1}\gamma',\ \gamma^{-1}y\right)\right]$. By symmetry, it suffices to show that $\|\chi\| \geq \|(\gamma,y)\cdot\chi\|$.

Let $\tilde\chi \in \mathcal{C}_c\left((\mathcal{G}\ltimes Y)^{s(\gamma)}\right)$ be a non-negative extension of χ. Then

$$\gamma\tilde\chi := \left[(\mathcal{G}\ltimes Y)^{r(\gamma)} \ni (\gamma', y') \mapsto \tilde\chi\left(\gamma^{-1}\gamma',\ \gamma^{-1}y'\right)\right]$$

is a non-negative extension of $(\gamma,y)\chi$ to $(\mathcal{G}\ltimes Y)^{r(\gamma)}$. Hence $\|(\gamma,y)\chi\| \leq \|\gamma\tilde\chi\|_{\mathcal{H}^<_{r(\gamma)}} = \|\tilde\chi\|_{\mathcal{H}^<_{s(\gamma)}}$, where we have used that $\mathcal{H}^<(\mathcal{G})$ is $\mathcal{G}$-equivariant. The infimum over the right-hand side is $\|\chi\|$, so we have shown $\|\chi\| \geq \|(\gamma,y)\cdot\chi\|$. $\square$

8.4 Proper $\mathcal{G}$-Banach algebras

Definition 8.4.1 (Proper $\mathcal{G}$-Banach algebra). A $\mathcal{G}$-Banach algebra B is called *proper* if there is a proper locally compact Hausdorff $\mathcal{G}$-space Z (with anchor map ρ) and a $\mathcal{G}\ltimes Z$-Banach algebra $\hat B$ such that the $\mathcal{G}$-Banach algebra $\rho_*\hat B$ is isomorphic to B.

Proposition 8.4.2. *A $\mathcal{G}$-Banach algebra B is a proper if and only if there is a $\mathcal{G}\ltimes\underline{\mathrm{E}}\mathcal{G}$-Banach algebra $\hat B$ such that $\tilde\rho_*\tilde B$ is isomorphic to B, where $\tilde\rho$ denotes the anchor map of $\underline{\mathrm{E}}\mathcal{G}$. This means that we can assume without loss of generality that the space Z appearing in the above definition is equal to $\underline{\mathrm{E}}\mathcal{G}$.*

Proof. Let B be a proper $\mathcal{G}$-Banach algebra and let Z be a proper locally compact Hausdorff $\mathcal{G}$-space with anchor map ρ and let $\hat{B}$ be a $\mathcal{G} \ltimes Z$-Banach algebra such that $\rho_* \hat{B}$ is isomorphic to B. From the universal property of $\underline{\mathrm{E}}\mathcal{G}$ we can find a continuous $\mathcal{G}$-equivariant map q from Z to $\underline{\mathrm{E}}\mathcal{G}$. The equivariance of q means in particular that $\tilde{\rho} \circ q = \rho$. Now $B \cong \rho_* \hat{B} \cong \tilde{\rho}_* q_* \hat{B}$ and $q_* \hat{B}$ is a $\mathcal{G} \ltimes \underline{\mathrm{E}}\mathcal{G}$-Banach algebra. □

For the rest of this chapter, let $\mathcal{G}$ carry a Haar system.

The following proposition generalises Proposition 8.2.5, which discusses the case that $\mathcal{G}$ itself is proper. We are going to prove it by reducing it to this special case.

Proposition 8.4.3. *Let B be a proper non-degenerate $\mathcal{G}$-Banach algebra and let $\mathcal{A}(\mathcal{G})$ and $\mathcal{B}(\mathcal{G})$ be unconditional completions of $\mathcal{C}_c(\mathcal{G})$ such that $\|\chi\|_{\mathcal{A}} \geq \|\chi\|_{\mathcal{B}}$ for all $\chi \in \mathcal{C}_c(\mathcal{G})$. Let $\psi\colon \mathcal{A}(\mathcal{G}, B) \to \mathcal{B}(\mathcal{G}, B)$ be the canonical homomorphism of Banach algebras introduced in 8.1.5. If $\mathcal{B}(\mathcal{G})$ is a regular unconditional completion of $\mathcal{C}_(\mathcal{G})$, then*

$$\psi_*\colon\ \mathrm{K}_*\left(\mathcal{A}(\mathcal{G}, B)\right) \to \mathrm{K}_*\left(\mathcal{B}(\mathcal{G}, B)\right)$$

is an isomorphism making the following diagram commutative

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{top,ban}}(\mathcal{G},\ B) \arrow[r, "\mu^B_{\mathcal{A}}"] \arrow[dr, "\mu^B_{\mathcal{B}}"'] & \mathrm{K}_0\left(\mathcal{A}(\mathcal{G}, B)\right) \arrow[d, "\psi_*"] \\
 & \mathrm{K}_0\left(\mathcal{B}(\mathcal{G}, B)\right)
\end{tikzcd}$$

Proof. That the diagram is commutative was already stated in 8.1.5; it remains to show that ψ_* is an isomorphism.

Find a proper locally compact Hausdorff $\mathcal{G}$-space Z with anchor map ρ and a $\mathcal{G} \ltimes Z$-Banach algebra $\hat{B}$ such that $\rho_* \hat{B}$ is isomorphic to B. Then $\hat{B}$ is non-degenerate. Because $\mathcal{B}(\mathcal{G})$ is a regular unconditional completion of $\mathcal{C}_c(\mathcal{G})$, also $\mathcal{A}(\mathcal{G})$ is regular and $\mathcal{A}_Z(\mathcal{G} \ltimes Z)$ and $\mathcal{B}_Z(\mathcal{G} \ltimes Z)$ are regular unconditional completions of $\mathcal{C}_c(\mathcal{G} \ltimes Z)$ by Proposition 8.3.26. Moreover, $\|\chi\|_{\mathcal{A}_Z} \geq \|\chi\|_{\mathcal{B}_Z}$ for all $\chi \in \mathcal{C}_c(\mathcal{G} \ltimes Z)$, hence there is a canonical homomorphism $\psi^Z\colon \mathcal{A}_Z(\mathcal{G} \ltimes Z,\ \hat{B}) \to \mathcal{B}_Z(\mathcal{G} \ltimes Z,\ \hat{B})$. The following diagram commutes

$$\begin{tikzcd}
\mathcal{A}_Z\left(\mathcal{G} \ltimes Z,\ \hat{B}\right) \arrow[r, "\psi^Z"] \arrow[d, "\cong"'] & \mathcal{B}_Z\left(\mathcal{G} \ltimes Z,\ \hat{B}\right) \arrow[d, "\cong"] \\
\mathcal{A}(\mathcal{G}, B) \arrow[r, "\psi"] & \mathcal{B}(\mathcal{G}, B)
\end{tikzcd}$$

Hence also the following diagram commutes

$$\begin{tikzcd}
\mathrm{K}_0\left(\mathcal{A}_Z\left(\mathcal{G} \ltimes Z,\ \hat{B}\right)\right) \arrow[r, "\psi^Z_*"] \arrow[d, "\cong"'] & \mathrm{K}_0\left(\mathcal{B}_Z\left(\mathcal{G} \ltimes Z,\ \hat{B}\right)\right) \arrow[d, "\cong"] \\
\mathrm{K}_0\left(\mathcal{A}(\mathcal{G}, B)\right) \arrow[r, "\psi_*"] & \mathrm{K}_0\left(\mathcal{B}(\mathcal{G}, B)\right)
\end{tikzcd}$$

By Proposition 8.2.5, ψ^Z_* is an isomorphism, so ψ_* is an isomorphism as well. □

Theorem 8.4.4. *Let B be a non-degenerate proper $\mathcal{G}$-Banach algebra and let $\mathcal{A}(\mathcal{G})$ be a regular unconditional completion of $\mathcal{C}_c(\mathcal{G})$. Then the homomorphism*

$$\mu_{\mathcal{A}}^{B}\colon\ \mathrm{K}^{\mathrm{top,ban}}\left(\mathcal{G},\ B\right) \to \mathrm{K}_0\left(\mathcal{A}\left(\mathcal{G}, B\right)\right)$$

is split surjective. The split is natural in B.

This applies in particular to the regular unconditional completion $\mathrm{L}^1(\mathcal{G})$ and its symmetrised version $\mathrm{L}^1(\mathcal{G}) \cap \mathrm{L}^1(\mathcal{G})^*$.

Lemma 8.4.5. *Let B be a non-degenerate proper $\mathcal{G}$-Banach algebra such that there exists a proper $\mathcal{G}$-compact $\mathcal{G}$-space Z with anchor map ρ and a $\mathcal{G} \ltimes Z$-Banach algebra $\hat{B}$ such that $\rho_* \hat{B} \cong B$. Let $\mathcal{A}(\mathcal{G})$ be a regular unconditional completion of $\mathcal{C}_c(\mathcal{G})$. Then $\mu_{\mathcal{A}}^{B}$ is split surjective, the split being natural in B.*

Proof. Let Z, ρ and $\hat{B}$ as in the statement of the lemma. Because $\mathcal{G} \ltimes Z$ is proper and the quotient $(Z \ltimes \mathcal{G})\backslash Z$ is compact, we can apply Lemma 8.2.8 to get

$$\mathrm{K}^{\mathrm{top,ban}}\left(\mathcal{G} \ltimes Z,\ \hat{B}\right) = \mathrm{KK}^{\mathrm{ban}}_{\mathcal{G} \ltimes Z}\left(\mathcal{C}_0(Z),\ \hat{B}\right).$$

By Proposition 8.3.26, $\mathcal{A}_Z\left(\mathcal{G} \ltimes Z\right)$ is a regular unconditional completion of $\mathcal{C}_c\left(\mathcal{G} \ltimes Z\right)$ because $\mathcal{A}(\mathcal{G})$ is regular. So by Corollary 8.2.10, the homomorphism

$$\mu_{\mathcal{A}_Z}^{\hat{B}}\colon\ \mathrm{K}^{\mathrm{top,ban}}\left(\mathcal{G} \ltimes Z,\ \hat{B}\right) \to \mathrm{K}_0\left(\mathcal{A}_Z\left(\mathcal{G} \ltimes Z,\ \hat{B}\right)\right)$$

has a natural split. The diagram

$$\begin{tikzcd}
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G} \ltimes Z}\left(\mathbb{C}_Z,\ \hat{B}\right) \arrow[r, "j_{\mathcal{A}_Z}"] \arrow[d, "\rho_*"'] & \mathrm{KK}^{\mathrm{ban}}\left(\mathcal{A}_Z\left(\mathcal{G} \ltimes Z,\ \mathbb{C}_Z\right),\ \mathcal{A}_Z\left(\mathcal{G} \ltimes Z,\ \hat{B}\right)\right) \arrow[d, "\cong"] \\
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\rho_*\mathbb{C}_Z,\ B\right) \arrow[r, "j_{\mathcal{A}}"] & \mathrm{KK}^{\mathrm{ban}}\left(\mathcal{A}\left(\mathcal{G},\ \rho_*\mathbb{C}_Z\right),\ \mathcal{A}\left(\mathcal{G}, B\right)\right)
\end{tikzcd}$$

commutes by Proposition 8.3.25. Also the diagram

$$\begin{tikzcd}
\mathrm{KK}^{\mathrm{ban}}\left(\mathcal{A}_Z\left(\mathcal{G} \ltimes Z,\ \mathbb{C}_Z\right),\ \mathcal{A}_Z\left(\mathcal{G} \ltimes Z,\ \hat{B}\right)\right) \arrow[r, "\Sigma(\cdot)\left(\lambda_{Z,\mathcal{G}\ltimes Z,\mathcal{A}_Z}\right)"] \arrow[d, "\cong"'] & \mathrm{K}_0\left(\mathcal{A}_Z\left(\mathcal{G} \ltimes Z,\ \hat{B}\right)\right) \arrow[d, "\iota_{\hat{B},*}"] \\
\mathrm{KK}^{\mathrm{ban}}\left(\mathcal{A}\left(\mathcal{G},\ \rho_*\mathbb{C}_Z\right),\ \mathcal{A}\left(\mathcal{G}, B\right)\right) \arrow[r, "\Sigma(\cdot)\left(\lambda_{Z,\mathcal{G},\mathcal{A}}\right)"] & \mathrm{K}_0\left(\mathcal{A}\left(\mathcal{G}, B\right)\right)
\end{tikzcd}$$

is commutative, because $\left(\iota_{\hat{B}}\right)_* \lambda_{Z,\mathcal{G}\ltimes Z,\mathcal{A}_Z} = \lambda_{Z,\mathcal{G},\mathcal{A}}$ (this follows because the idempotents that define the two K-theory classes are identified under $\iota_{\hat{B}}$). Putting the two commuting squares together we get the following commutative diagram:

$$\begin{tikzcd}
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G} \ltimes Z}\left(\mathbb{C}_Z,\ \hat{B}\right) \arrow[r] \arrow[d, "\rho_*"'] & \mathrm{K}_0\left(\mathcal{A}_Z\left(\mathcal{G} \ltimes Z,\ \hat{B}\right)\right) \arrow[d, "\cong"] \arrow[l, bend right, dotted] \\
\mathrm{KK}^{\mathrm{ban}}_{\mathcal{G}}\left(\rho_*\mathbb{C}_Z,\ B\right) \arrow[r] & \mathrm{K}_0\left(\mathcal{A}\left(\mathcal{G}, B\right)\right) \arrow[l, bend right, dotted]
\end{tikzcd}$$

Because the top-arrow has a natural split (dashed arrow), also the bottom-arrow has a natural split (the other dashed arrow). But this means that $\mu_{\mathcal{A}}^{B}$ has a natural split:

$$\begin{tikzcd} \mathrm{KK}_{\mathcal{G}}^{\mathrm{ban}}\left(\rho_*\mathbb{C}_Z,\ B\right) \arrow[r] \arrow[d] & \mathrm{K}_0\left(\mathcal{A}\left(\mathcal{G},B\right)\right) \arrow[l, dashed, bend right] \\ \mathrm{K}^{\mathrm{top,ban}}\left(\mathcal{G},\ B\right) \arrow[ur, "\mu_{\mathcal{A}}^{B}"'] & \end{tikzcd}$$

□

Proof of Theorem 8.4.4. Let $\hat{B}$ be a $\mathcal{G} \ltimes \underline{\mathrm{E}}\mathcal{G}$-Banach algebra and let $\rho\colon \underline{\mathrm{E}}\mathcal{G} \to X = \mathcal{G}^{(0)}$ be the anchor map of the proper action of $\mathcal{G}$ on $\underline{\mathrm{E}}\mathcal{G}$; assume that $\rho_*\hat{B} \cong B$ as $\mathcal{G}$-Banach algebras. Then $\hat{B}$ is non-degenerate. For every open $\mathcal{G}$-invariant subspace U of $\underline{\mathrm{E}}\mathcal{G}$, define $\hat{B}_U$ to be the $\mathcal{G} \ltimes \underline{\mathrm{E}}\mathcal{G}$-Banach algebra with the following fibres: If $u \in U$, then the fibre over u is $\hat{B}_u$, if $y \in \underline{\mathrm{E}}\mathcal{G} \setminus U$, then the fibre over y is zero; the space $\Gamma(\underline{\mathrm{E}}\mathcal{G},\ \hat{B}_U)$ is defined to be the set of all elements of $\Gamma(\underline{\mathrm{E}}\mathcal{G},\ \hat{B})$ that vanish outside U. By definition, there is a $\mathcal{G} \ltimes \underline{\mathrm{E}}\mathcal{G}$-equivariant "injection" $\hat{\jmath}_U$ from $\hat{B}_U$ to $\hat{B}$. It descends to a $\mathcal{G}$-equivariant homomorphism $j_U := \rho_*\hat{\jmath}_U$ from $B_U := \rho_*\hat{B}_U$ to $B := \rho_*\hat{B}$. We can regard B_U as a subalgebra of B.

The $\hat{B}_U$, where U runs through the open $\mathcal{G}$-invariant subsets of $\underline{\mathrm{E}}\mathcal{G}$ such that $\mathcal{G}\backslash U$ is relatively compact, form a directed system: If U and V are open $\mathcal{G}$-invariant and $\mathcal{G}$-relatively compact subsets of $\underline{\mathrm{E}}\mathcal{G}$ with $U \subseteq V$, then there is an obvious homomorphism $\hat{\jmath}_{U,V}\colon \hat{B}_U \to \hat{B}_V$ such that $\hat{\jmath}_U = \hat{\jmath}_V \circ \hat{\jmath}_{U,V}$. Also the B_U form a directed system, just take the $j_{U,V} := \rho_*\hat{\jmath}_{U,V}$ as connecting maps. We can regard B as the direct limit of the B_U. More importantly, the $\mathcal{A}\left(\mathcal{G}, B_U\right)$ form a directed system with connecting maps $\alpha_{U,V} := \mathcal{A}(\mathcal{G},\ j_{U,V})\colon \mathcal{A}(\mathcal{G}, B_U) \to \mathcal{A}(\mathcal{G}, B_V)$. The Banach algebra $\mathcal{A}(\mathcal{G}, B)$ is the direct limit of this system with embeddings $\alpha_U := \mathcal{A}(\mathcal{G},\ j_U)\colon \mathcal{A}(\mathcal{G}, B_U) \to \mathcal{A}(\mathcal{G}, B)$. Because the K-theory of Banach algebras is continuous, we get:

$$\mathrm{K}_*\left(\mathcal{A}(\mathcal{G}, B)\right) = \varinjlim \mathrm{K}_*\left(\mathcal{A}(\mathcal{G}, B_U)\right)$$

where U runs through the $\mathcal{G}$-invariant open subsets of $\underline{\mathrm{E}}\mathcal{G}$ such that $\mathcal{G}\backslash U$ is relatively compact.

Now let U be such a set. Find a closed set $Z \subseteq \underline{\mathrm{E}}\mathcal{G}$ such that $U \subseteq Z$ and $\mathcal{G}\backslash Z$ is compact. Define $\rho^Z := \rho|_Z$. Then $\hat{B}_U|_Z$ is a $\mathcal{G} \ltimes Z$-Banach algebra and $(\rho^Z)_*\hat{B}_U|_Z$ is isomorphic to B_U. So B_U satisfies the hypotheses of Lemma 8.4.5, so $\mu_{\mathcal{A}}^{B_U}\colon\ \mathrm{K}^{\mathrm{top,ban}}\left(\mathcal{G},\ B_U\right) \to \mathrm{K}_0\left(\mathcal{A}\left(\mathcal{G}, B_U\right)\right)$ is split surjective. Let σ_U denote the natural split constructed above. It is easy to see, using the naturality of the split, that $\sigma_V \circ (\alpha_{U,V})_* = (j_{U,V})_* \circ \sigma_U$. Define $\tau_U := (j_U)_* \circ \sigma_U\colon\ \mathrm{K}_0\left(\mathcal{A}\left(\mathcal{G}, B_U\right)\right) \to \mathrm{K}^{\mathrm{top,ban}}\left(\mathcal{G},\ B\right)$. Then $\tau_V = \tau_U \circ (\alpha_{U,V})_*$. The universal property of the direct limit shows that there exists a natural homomorphism $\tau\colon\ \mathrm{K}_0\left(\mathcal{A}\left(\mathcal{G}, B\right)\right) \to \mathrm{K}^{\mathrm{top,ban}}\left(\mathcal{G},\ B\right)$ such that $\tau \circ (\alpha_U)_* = \tau_U$ for all U.

Note that
$$\mu_{\mathcal{A}}^{B} \circ \tau_U = \mu_{\mathcal{A}}^{B} \circ (j_U)_* \circ \sigma_U = (\alpha_U)_* \circ \mu_{\mathcal{A}}^{B_U} \circ \sigma_U = (\alpha_U)_*$$
because σ_U is a split. Passing to the limit shows that $\mu_{\mathcal{A}}^{B} \circ \tau = \mathrm{Id}$, i.e., τ is a natural split. □

Remark 8.4.6 (The case of locally compact groups). Let $\mathcal{G} = G$ be a locally compact Hausdorff group. If Z is a proper G-space, then we can model an action of the groupoid $G \ltimes Z$ on a Banach algebra using G-$\mathcal{C}_0(Z)$-Banach algebras, as we have discussed in Chapter 4. More precisely, we can regard a $G \ltimes Z$-Banach algebra as a G-$\mathcal{C}_0(Z)$-Banach algebra which is locally $\mathcal{C}_0(Z)$-convex. In this situation we have the following corollary of the above theorem:
If B is a proper G-Banach algebra and $\mathcal{A}(G)$ is a regular unconditional completion of $\mathcal{C}_c(G)$, then

$$\mu_{\mathcal{A}}^{B}\colon\ \mathrm{K}^{\mathrm{top,ban}}\left(G, B\right) \to \mathrm{K}_0(\mathcal{A}(G, B))$$

is split surjective. In particular this is true for $\mathcal{A}(G) = \mathrm{L}^1(G)$.

Appendix A

Locally $\mathcal{C}_0(X)$-Convex $\mathcal{C}_0(X)$-Banach Spaces

Let X be a locally compact Hausdorff space.

A.1 Restriction and fibres

A.1.1 Restriction

Let V be a *closed* subspace of the locally compact Hausdorff space X. Let ι_V denote the inclusion map from V to X. Let $r_V = \iota_V^*$ denote the restriction map from $\mathcal{C}_0(X)$ onto $\mathcal{C}_0(V)$, being a homomorphism and a quotient map (with kernel $\mathcal{C}_0(X \setminus V)$). If $\mathcal{E}$ is a $\mathcal{C}_0(V)$-Banach space, then we can make it a $\mathcal{C}_0(X)$-Banach space by using r_V; the category of $\mathcal{C}_0(V)$-Banach spaces sits as a subcategory in the category of $\mathcal{C}_0(X)$-Banach spaces.

The restriction functor is a left inverse of this inclusion:

Two pictures of the restriction functor

Definition A.1.1 (Restriction (tensor product picture)). Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. Then we define the *restriction of $\mathcal{E}$ to V* to be the $\mathcal{C}_0(V)$-Banach space

$$\mathcal{E}|_V := \iota_V^*(\mathcal{E}) = \mathcal{E} \otimes^{\mathcal{C}_0(V)} \mathcal{C}_0(V).$$

If $\mathcal{F}$ is another $\mathcal{C}_0(X)$-Banach space and $T \in \mathrm{L}^{\mathcal{C}_0(X)}(\mathcal{E}, \mathcal{F})$, then we define

$$T|_V := \iota_V^*(T) = T \otimes 1 \colon \ \mathcal{E} \otimes^{\mathcal{C}_0(V)} \mathcal{C}_0(V) \to \mathcal{F} \otimes^{\mathcal{C}_0(V)} \mathcal{C}_0(V).$$

This defines a functor from the category of $\mathcal{C}_0(X)$-Banach spaces to the category of $\mathcal{C}_0(V)$-Banach spaces, linear and contractive on the morphisms sets and compatible with the tensor product.

There is an alternative and equivalent definition of the restriction $\mathcal{E}|_V$ which constructs it as a quotient of $\mathcal{E}$. To give this definition we first introduce some additional notation.

Definition A.1.2. Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. For every open subset $U \subseteq X$, define

$$\mathcal{E}_U := \mathcal{C}_0(U)\mathcal{E}.$$

Note that $\mathcal{E}_U$ is a (closed) $\mathcal{C}_0(X)$-Banach subspace of $\mathcal{E}$.

Definition A.1.3 (Restriction (quotient picture)). Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. Define the restriction of $\mathcal{E}$ to V to be

$$\mathcal{E}|_V := \mathcal{E}/\mathcal{E}_{X\setminus V}.$$

This space has a canonical $\mathcal{C}_0(X)$-action which induces a $\mathcal{C}_0(V)$-action such that $\mathcal{E}|_V$ is a $\mathcal{C}_0(V)$-Banach space.

For all $e \in \mathcal{E}$, we will denote by $e|_V$ the corresponding element of the restriction $\mathcal{E}|_V$. The canonical projection map from $\mathcal{E}$ onto $\mathcal{E}|_V$ will be denoted by $\pi_V^{\mathcal{E}}$ or just π_V, if the space $\mathcal{E}$ is understood. The map $\pi_V^{\mathcal{E}}$ is a homomorphism of Banach modules with coefficient map r_V.

Also in this picture there is a canonical way of turning the restriction of $\mathcal{C}_0(X)$-Banach spaces into a functor:

Definition and Lemma A.1.4. Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{C}_0(X)$-Banach spaces and $T \in \mathrm{L}^{\mathcal{C}_0(X)}(\mathcal{E}, \mathcal{F})$. Then there is a unique map $T|_V \colon \mathcal{E}|_V \to \mathcal{F}|_V$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "T"] \arrow[d, "\pi_V^{\mathcal{E}}"'] & \mathcal{F} \arrow[d, "\pi_V^{\mathcal{F}}"] \\
\mathcal{E}|_V \arrow[r, "T|_V"] & \mathcal{F}|_V
\end{tikzcd}$$

It is in $\mathrm{L}^{\mathcal{C}_0(X)}\left(\mathcal{E}|_V,\ \mathcal{F}|_V\right)$ and satisfies $\|T|_V\| \leq \|T\|$.

Proof. Note that if $U \subseteq X$ is open, then T maps $\mathcal{E}_U$ into $\mathcal{F}_U$. So in particular, T maps $\mathcal{E}_{X\setminus V}$ into $\mathcal{F}_{X\setminus V}$. Hence the map $T|_V$ exists and is unique. By linear algebra it is linear and $\mathcal{C}_0(X)$-linear, by Banach space theory it is continuous with $\|T|_V\| \leq \|T\|$. □

To be able to switch between the two pictures of the restriction we construct natural connecting maps. This can be done using some suitable universal properties. For the moment, write $\mathcal{E}|_V^{\mathrm{tp}}$ and $\mathcal{E}|_V^{\mathrm{q}}$ for the restriction of $\mathcal{E}$ in the tensor product and the quotient picture.

We already have a map from $\mathcal{E}$ onto $\mathcal{E}|_V^{\mathrm{q}}$, namely the quotient map $\pi_V^{\mathcal{E}}$. There is a corresponding map in the tensor product picture which is, maybe, a bit less obvious: We have $\mathcal{E} \cong \mathcal{E} \otimes^{\mathcal{C}_0(X)} \mathcal{C}_0(X)$ and the quotient map $r_V \colon \mathcal{C}_0(X) \to \mathcal{C}_0(V)$. Together this gives a map

$$\mathrm{Id} \otimes r_V \colon \mathcal{E} \cong \mathcal{E} \otimes^{\mathcal{C}_0(X)} \mathcal{C}_0(X) \to \mathcal{E} \otimes^{\mathcal{C}_0(X)} \mathcal{C}_0(V).$$

This map is surjective and a quotient map as r_V and Id are surjective and quotient maps.

Proposition A.1.5. *There are contractive $\mathcal{C}_0(V)$-linear maps from $\mathcal{E}|_V^{\mathrm{tp}}$ to $\mathcal{E}|_V^{\mathrm{q}}$ and vice versa such that the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "r_v \otimes \mathrm{Id}"] \arrow[d, "\pi_V^{\mathcal{E}}"'] & \mathcal{E}|_V^{\mathrm{tp}} \arrow[dl, shift left] \\
\mathcal{E}|_V^{\mathrm{q}} \arrow[ur, shift left] &
\end{tikzcd}$$

From this it follows that $\mathcal{E}|_V^{\mathrm{tp}}$ and $\mathcal{E}|_V^{\mathrm{q}}$ are isometrically isomorphic.

Proof. First we use Lemma E.6.6: The map $\pi_V^{\mathcal{E}}$ is a homomorphism with coefficient map r_V. The Banach $\mathcal{C}_0(V)$-module $\mathcal{E} \otimes^{\mathcal{C}_0(X)} \mathcal{C}_0(V) = \mathcal{E}|_V^{\mathrm{tp}}$ has the universal property for such homomorphisms, so there is a unique $\mathcal{C}_0(V)$-linear contractive map from $\mathcal{E}|_V^{\mathrm{tp}}$ to $\mathcal{E}|_V^{\mathrm{q}}$ such that the diagram commutes.

For the inverse map, note that $\mathrm{Id} \otimes r_V$ is $\mathbb{C}$-linear and contractive. If $e \in \mathcal{E}$ and $\varphi \in \mathcal{C}_0(X \setminus V)$, then φe is in the kernel of $r_V \otimes \mathrm{Id}$. By the definition of the quotient Banach space structure on $\mathcal{E}|_V^{\mathrm{q}}$ there is a unique contractive and linear map from $\mathcal{E}|_V^{\mathrm{q}}$ to $\mathcal{E}|_V^{\mathrm{tp}}$ such that the diagram commutes. □

Standard constructions and restriction

Lemma A.1.6. *Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. For all $e \in E$, we have*

$$\begin{aligned}
\|e|_V\| &= \inf\left\{\|\varphi e\| : \varphi \in \mathcal{C}_b(X),\ \varphi|_V \equiv 1,\ 0 \leq \varphi \leq 1\right\} \\
&= \inf\left\{\|\varphi e\|,\ \varphi \in \mathcal{C}_b(X);\ \exists U \subseteq X \textit{ open}:\ \varphi|_U \equiv 1,\ 0 \leq \varphi \leq 1, V \subseteq U, X \setminus U \textit{ comp.}\right\}.
\end{aligned}$$

Proof. Denote the three terms that are to be shown to be equal by A, B, and C. We now prove $A \leq B \leq C \leq A$.

$A \leq B$: Let $\varphi \in \mathcal{C}_b(X)$ such that $0 \leq \varphi \leq 1$ and $\varphi|_V \equiv 1$. Then $1 - \varphi \in \mathcal{C}_b(X)$ such that $0 \leq 1-\varphi \leq 1$ and $\varphi|_V \equiv 0$. By Cohen's Factorisation Theorem we can write $e = \chi f$ with $f \in \mathcal{E}$ and $\chi \in \mathcal{C}_0(X)$. Then $(1-\varphi)e = (1-\varphi)\chi f \in \mathcal{C}_0(X \setminus V)\mathcal{E}$, so $e|_V = \varphi e|_V$. In particular, $\|e|_V\| \leq \|\varphi e\|$. Taking the infimum we obtain $A \leq B$.

$B \leq C$: This is trivial.

$C \leq A$: Let $\varepsilon > 0$. Find a $k \in \mathbb{N}$ and $\varphi_1, \ldots, \varphi_k \in \mathcal{C}_0(X \setminus V)$ and $e_1, \ldots, e_k \in \mathcal{E}$ such that $\|e - \sum_{i=1}^k \varphi_i e_i\| \leq \|e|_V\| + \varepsilon/2$. For every $i \in \{1, \ldots, k\}$, we can find a compact subset K_i of $X \setminus V$ such that $|\varphi_i(x)|\,\|e_i\| \leq \frac{\varepsilon}{2k}$ for all $x \in X \setminus K_i$. Let $K := \bigcup_{i=1}^k K_i$. Then K is a compact subset of the open subset $X \setminus V$ so we can find a compact neighbourhood K' of K that is still contained in $X \setminus V$. Define $U := X \setminus K'$. Let φ be an element of $\mathcal{C}_b(X)$ such that $0 \leq \varphi \leq 1$, $\varphi|_U \equiv 1$ and $\varphi|_K \equiv 0$. Note that $\|\varphi\varphi_i\|\,\|e_i\| \leq \frac{\varepsilon}{2k}$. Now

$$\left\|\varphi e - \varphi \sum_{i=1}^k \varphi_i e_i\right\| \leq \left\|e - \sum_{i=1}^k \varphi_i e_i\right\| \leq \left\|e|_V\right\| + \varepsilon/2$$

and therefore

$$\|\varphi e\| \leq \left\|\varphi e - \varphi \sum_{i=1}^k \varphi_i e_i\right\| + \left\|\varphi \sum_{i=1}^k \varphi_i e_i\right\| \leq \|e|_V\| + \frac{\varepsilon}{2} + \sum_{i=1}^k \|\varphi\varphi_i\|\,\|e_i\| \leq \|e|_V\| + \varepsilon.$$

So the infimum C is less than or equal to $\|e|_V\| = A$. □

The preceding lemma shows that the function $V \mapsto \|e|_V\|$, defined on the set of closed subsets of X, is upper semi-continuous (in an appropriate sense).

Proposition A.1.7. *Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{C}_0(X)$-Banach spaces and $T \in \mathrm{L}^{\mathcal{C}_0(X)}(\mathcal{E}, \mathcal{F})$.*

1. *If T is isometric, then also $T|_V$ is isometric.*

2. *If T is surjective and a quotient map, then so is $T|_V$.*

3. *If T has dense image, then so has $T|_V$.*

4. *If T is an isometric isomorphism, then so is $T|_V$.*

Proof. 1. We have for all $e \in \mathcal{E}$:

$$\begin{aligned} \|T(e)|_V\| &= \inf\{\|\varphi T(e)\| : \varphi \in \mathcal{C}_b(X)\ \varphi|_V = 1,\ 0 \le \varphi \le 1\} \\ &= \inf\{\|T(\varphi e)\| : \varphi \in \mathcal{C}_b(X)\ \varphi|_V = 1,\ 0 \le \varphi \le 1\} \\ &\overset{T \text{ isom.}}{=} \inf\{\|\varphi e\| : \varphi \in \mathcal{C}_b(X)\ \varphi|_V = 1,\ 0 \le \varphi \le 1\} = \|e|_V\|. \end{aligned}$$

2. In the commuting square defining the operator $T|_V$ three arrows are quotient maps, hence so is the fourth.

3. From abstract non-sense we can deduce that reflectors (such as $\cdot|_V$) respect epimorphisms. But the epimorphisms in the categories of $\mathcal{C}_0(X)$- and $\mathcal{C}_0(V)$-Banach spaces are precisely the morphisms with dense image.

   For a direct argument, let $f \in F$ and $\varepsilon > 0$. We want to find $e \in E$ with $\|T|_V(e|_V) - f|_V\| \le \varepsilon$. Since T has dense image, we can find $e \in E$ such that $\|T(e) - f\| \le \varepsilon$. Now $T|_V(e|_V) = T(e)|_V$ and hence $\|T|_V(e|_V) - f|_V\| = \|(T(e) - f)|_V\| \le \|T(e) - f\| \le \varepsilon$.

4. This follows, for example, from 1. and 2. □

Because restriction is a special case of the pullback construction, we know that restriction commutes with the tensor product:

Proposition A.1.8. *Let $\mathcal{E}^1$ and $\mathcal{E}^2$ be $\mathcal{C}_0(X)$-Banach spaces. Then there is a natural isomorphism*

$$\left(\mathcal{E}^1 \otimes^{\mathcal{C}_0(X)} \mathcal{E}^2\right)|_V \cong (\mathcal{E}^1)|_V \otimes^{\mathcal{C}_0(V)} (\mathcal{E}^2)|_V$$

interchanging the canonical bilinear maps from $\mathcal{E}^1 \otimes \mathcal{E}^2$ into the two spaces.

Definition A.1.9. Let $\mathcal{E}^1$, $\mathcal{E}^2$ and $\mathcal{F}$ be $\mathcal{C}_0(X)$-Banach spaces and let $\mu\colon \mathcal{E}^1 \times \mathcal{E}^2 \to \mathcal{F}$ be $\mathcal{C}_0(X)$-bilinear and continuous. Define $\mu|_V := \iota_V^*(\mu)$.

$\mu|_V$ is the unique $\mathcal{C}_0(V)$-bilinear continuous map making the following diagram commutative:

$$\begin{tikzcd} \mathcal{E}^1 \times \mathcal{E}^2 \arrow[r, "\mu"] \arrow[d, "\pi_V^{\mathcal{E}^1} \times \pi_V^{\mathcal{E}^2}"'] & \mathcal{F} \arrow[d, "\pi_V^{\mathcal{F}}"] \\ \mathcal{E}^1|_V \times \mathcal{E}^2|_V \arrow[r, "\mu|_V"] & \mathcal{F}|_V \end{tikzcd}$$

Moreover, we have $\widehat{\mu|_V} = \hat{\mu}|_V$ if we identify $\left(\mathcal{E}^1 \otimes^{\mathcal{C}_0(X)} \mathcal{E}^2\right)|_V$ and $(\mathcal{E}^1)|_V \otimes^{\mathcal{C}_0(V)} (\mathcal{E}^2)|_V$.

A.1.2 Fibres

Definition A.1.10 (The fibres of a $\mathcal{C}_0(X)$-Banach space). Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. If $x \in X$, then define

$$\mathcal{E}_x := \mathcal{E}|_{\{x\}} = \mathcal{E} \otimes_{\mathrm{ev}_x} \mathbb{C}.$$

The space $\mathcal{E}_x$ is a Banach space called *the fibre of $\mathcal{E}$ in x*. For all $e \in \mathcal{E}$, we will denote by e_x the corresponding element of the fibre $\mathcal{E}_x$. The canonical projection map from $\mathcal{E}$ onto $\mathcal{E}_x$ will be denoted by $\pi_x^{\mathcal{E}}$ or just π_x, if the space $\mathcal{E}$ is understood.

Definition A.1.11. Let $x \in X$. Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{C}_0(X)$-Banach spaces and $T \in \mathrm{L}^{\mathcal{C}_0(X)}(\mathcal{E}, \mathcal{F})$. Write T_x for the pullback

$$T_x := T|_{\{x\}} = 1 \otimes_{\mathrm{ev}_x} T \in \mathrm{L}\left(\mathcal{E}_x,\ \mathcal{F}_x\right).$$

It is the unique map such that the following diagram commutes

$$\begin{tikzcd} \mathcal{E} \arrow[r, "T"] \arrow[d, "\pi_x^{\mathcal{E}}"'] & \mathcal{F} \arrow[d, "\pi_x^{\mathcal{F}}"] \\ \mathcal{E}_x \arrow[r, "T_x"] & \mathcal{F}_x \end{tikzcd}$$

and satisfies $\|T_x\| \leq \|T\|$.

In the same spirit we define $\mu_x := \mu|_{\{x\}}$ for $\mathcal{C}_0(X)$-bilinear continuous maps μ. As the fibre construction is a special case of the restriction, it follows that the fibre construction commutes with the tensor product, etc.

Example A.1.12. Let E be a Banach space. Then $\mathcal{E} := \mathcal{C}_0(X, E)$ is a $\mathcal{C}_0(X)$-Banach space and $\mathcal{E}_x \cong E$ for all $x \in X$. The same is true for $\mathcal{E}' := \mathcal{C}_0(X) \otimes^\pi E$.

Proof. To determine the fibres of $\mathcal{E}$ it is probably the easiest to use the quotient picture for the fibres. The space $\mathcal{E}_{X \setminus \{x\}}$ can be identified with $\mathcal{C}_0(X \setminus \{x\},\ E)$. Consider the evaluation map from $\mathcal{C}_0(X, E)$ to E which evaluates a function at x. Its kernel is $\mathcal{C}_0(X \setminus \{x\},\ E)$. From our knowledge about continuous fields of Banach spaces we can deduce that this evaluation map is a quotient map. So the fibre of $\mathcal{E}$ can indeed be identified with E.

To determine the fibres of $\mathcal{E}'$ one can use the tensor product picture. Note that $\mathbb{C} \otimes_{\mathrm{ev}_x} \mathcal{C}_0(X) \cong \mathbb{C}$ by Example 2.3.2. Now

$$\mathbb{C} \otimes_{\mathrm{ev}_x} (\mathcal{C}_0(X) \otimes_{\mathbb{C}} E) \cong (\mathbb{C} \otimes_{\mathrm{ev}_x} \mathcal{C}_0(X)) \otimes_{\mathbb{C}} E \cong \mathbb{C} \otimes E \cong E.$$

This can also be understood in the following way: If $p\colon X \to \{x\}$ denotes the constant map and $\iota_x\colon x \to X$ the inclusion, then $p \circ \iota_x = \mathrm{Id}_{\{x\}}$. Since $\mathcal{E}' = p^*(E)$ it follows that

$$\mathcal{E}'_x = \iota_x^*(\mathcal{E}') = \iota_x^*(p^*(E)) \cong (p \circ \iota_x)^*(E) \cong E.$$

□

A.2 Local $\mathcal{C}_0(X)$-convexity

A.2.1 Definition of local $\mathcal{C}_0(X)$-convexity

Definition A.2.1. Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. For all $e \in \mathcal{E}$, define

$$|e| : X \to [0, \infty[,\ x \mapsto \|e_x\|_{\mathcal{E}_x}.$$

As we have seen in 4.2.7 this function is upper semi-continuous and vanishes at infinity. Define

$$|||e||| := \|\ |e|\ \|_\infty = \|\mathfrak{g}_{\mathcal{E}}(e)\|.$$

Then $|||\cdot|||$ is a semi-norm on $\mathcal{E}$ such that $|||e||| \leq \|e\|$ for all $e \in \mathcal{E}$.

Definition A.2.2 (Locally $\mathcal{C}_0(X)$-convex). Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. Then $\mathcal{E}$ is called *locally $\mathcal{C}_0(X)$-convex* if $|||e||| = \|e\|$ for all $e \in \mathcal{E}$, i.e., if the Gelfand transformation is isometric.

Example A.2.3. Let $\mathcal{E}$ be a Banach space. Then $\mathcal{E}X$ is locally $\mathcal{C}_0(X)$-convex.

Example A.2.4. Let E be a u.s.c. field of Banach spaces over X. Then $\mathfrak{M}(E)$ is locally $\mathcal{C}_0(X)$-convex.

Proof. As we have seen in the proof of Theorem 4.3.1, the fibres of $\mathfrak{M}(E)$ are isometrically isomorphic to the fibres of E in a way that identifies ξ_x with $\xi(x)$ for all $\xi \in \mathfrak{M}(E) = \Gamma_0(X, E)$ and all $x \in X$. So $|||\xi||| = \sup_{x\in X} \|\xi_x\| = \sup_{x\in X} \|\xi(x)\| = \|\xi\|_{\mathfrak{M}(E)}$ for all $\xi \in \mathfrak{M}(E)$. So $\mathfrak{M}(E)$ is locally $\mathcal{C}_0(X)$-convex. □

Example A.2.5. The $\mathcal{C}[0,1]$-Banach space $\mathrm{L}^1[0,1]$ of Example 4.3.5 fails to be locally $\mathcal{C}[0,1]$-convex.

The name "locally $\mathcal{C}_0(X)$-convex" is motivated by the following proposition:

Proposition A.2.6. [1] *Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. Then the following are equivalent:*

1. *$\mathcal{E}$ is locally $\mathcal{C}_0(X)$-convex.*
2. $\forall \chi_1, \chi_2 \in \mathcal{C}_b(X), \chi_1, \chi_2 \geq 0, \chi_1 + \chi_2 = 1\, \forall e_1, e_2 \in \mathcal{E} :\ \|\chi_1 e_2 + \chi_2 e_2\| \leq \max\{\|e_1\|, \|e_2\|\}$.
3. $\forall \chi_1, \chi_2 \in \mathcal{C}_0(X), \chi_1, \chi_2 \geq 0, \chi_1 + \chi_2 \leq 1\, \forall e_1, e_2 \in \mathcal{E} :\ \|\chi_1 e_2 + \chi_2 e_2\| \leq \max\{\|e_1\|, \|e_2\|\}$.

Remark A.2.7. The locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces form a full subcategory of the category of all $\mathcal{C}_0(X)$-Banach spaces.

Proposition A.2.8. *Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{C}_0(X)$-Banach spaces and $T \in \mathrm{L}^{\mathcal{C}_0(X)}(\mathcal{E}, \mathcal{F})$ such that $\|T\| \leq 1$.*

1. *If for all $x \in X$ the operator $T_x\colon \mathcal{E}_x \to \mathcal{F}_x$ is isometric and $\mathcal{E}$ is locally $\mathcal{C}_0(X)$-convex, then T is isometric.*
2. *If for all $x \in X$ the operator $T_x\colon \mathcal{E}_x \to \mathcal{F}_x$ has dense image and $\mathcal{F}$ is locally $\mathcal{C}_0(X)$-convex, then T has dense image.*
3. *If for all $x \in X$ the operator $T_x\colon \mathcal{E}_x \to \mathcal{F}_x$ is surjective and a quotient map and $\mathcal{E}$ and $\mathcal{F}$ are locally $\mathcal{C}_0(X)$-convex, then T is surjective and a quotient map.*
4. *If for all $x \in X$ the operator $T_x\colon \mathcal{E}_x \to \mathcal{F}_x$ is an isometric isomorphism and both $\mathcal{E}$ and $\mathcal{F}$, are locally $\mathcal{C}_0(X)$-convex, then T is an isometric isomorphism.*

Proof. 1. Let $e \in \mathcal{E}$. Since $\mathcal{E}$ is locally $\mathcal{C}_0(X)$-convex, we have $\|e\| = |||e|||$. Now $|||e||| = |||T(e)||| \leq \|T(e)\| \leq \|e\|$, so we have equality throughout, and hence T is isometric.

2. The image of T is fibrewise dense. Since $\mathcal{F}$ is locally $\mathcal{C}_0(X)$-convex, we can conclude that the image of T, being a $\mathcal{C}_0(X)$-invariant subspace, is dense.

3. We use Lemma E.3.1. Let $f \in \mathcal{F}$ and $\varepsilon > 0$. For every $x \in X$, we pick some $e^x \in \mathcal{E}$ such that $\|(T(e^x) - f)_x\| \leq \varepsilon/2$, $\|e^x\| \leq \|f_x\|$ (this is possible since $T_x \circ \pi_x$ is a quotient map for all $x \in X$). Find a compact subset K of X such that $\|f_x\| \leq \varepsilon$ for all $x \subseteq X \setminus K$. Since for all $x \in X$ the function $|T(e^x) - f|$ is upper semi-continuous, the sets $U_x := \{y \in X : \|(T(e^x) - f)_y\| < \varepsilon$ are open (and contain x). So the set $\{U_x :\ x \in K\}$ forms an open cover of K. Let $S \subseteq K$ be a finite set such that $\{U_S :\ s \in S\}$ is a cover of K. Find a

[1] Compare [DG83], Theorem 2.5.

continuous partition of unity on K subordinate to this cover, i.e., a family $(\varphi_s)_{s\in S}$ of elements of $\mathcal{C}_0(X)$ such that $0 \le \varphi_s \le 1$, $\operatorname{supp}\varphi_s \subseteq U_s$ and $\sum_{s\in S}\varphi_s(k) = 1$ for all $k \in K$ as well as $\sum_{s\in S}\varphi_s \le 1$ on the whole of X. Define

$$e := \sum_{s\in S}\varphi_s e^s \in \mathcal{E}.$$

Since $\mathcal{E}$ is locally $\mathcal{C}_0(X)$-convex, we can conclude that $\|e\| \le \sup_{s\in S}\|f_s\| \le \|f\|$. Let $\psi := 1 - \sum_{s\in S}\varphi_s$. Note that $f = \sum_{s\in S}\varphi_s f + \psi f$.

Let $x \in X$. Let $s \in S$. If $x \in U^s$, then $\|T(e^s)_x - f_x\| \le \varepsilon$, so $\|T(\varphi_s e^s)_x - (\varphi_s f)_x\| \le \varphi_s(x)\varepsilon$. If $x \notin U_s$, then $\|T(\varphi_s e^s)_x - (\varphi_s f)_x\| = 0 \le \varphi_s(x)\varepsilon$. So

$$\left\|T(e)_x - \sum_{s\in S}\varphi_s f\right\| \le \sum_{s\in S}\varphi_s(x)\varepsilon \le \varepsilon.$$

On the other hand, $\|(\psi f)_x\| \le \varepsilon$, so

$$\left\|T(e)_x - f_x\right\| \le \left\|T(e)_x - \sum_{s\in S}\varphi_s f\right\| + \|(\psi f)_x\| \le 2\varepsilon.$$

This is true for all $x \in X$, so $|||T(e) - f||| \le 2\varepsilon$. Now $\mathcal{F}$ is locally $\mathcal{C}_0(X)$-convex, so $\|T(e) - f\| \le \varepsilon$. So T is surjective and a quotient map.

4. This follows from 1. and 2. (or 3.). □

Examples A.2.9. The following examples show that the hypotheses on the local $\mathcal{C}_0(X)$-convexity that appear in the preceding proposition cannot simply be dropped:

1. Let $X = [0,1]$ and $\mathcal{E}$ be $\mathrm{L}^1[0,1]$ as in Example 4.3.5. Let $\mathcal{F} := 0$ and $T := 0$. Then $T_x = 0$ for all $x \in [0,1]$. But also $\mathcal{E}_x = 0$ for all $x \in [0,1]$, so $0\colon \mathcal{E}_x \to 0$ is an isometric isomorphism. But T is not isometric. This shows that the condition that $\mathcal{E}$ is locally $\mathcal{C}_0(X)$-convex cannot be dropped in 1., 3. and 4.

2. Let $X := [0,1]$; $\mathcal{E} := 0$ and $\mathcal{F} := \mathrm{L}^1[0,1]$. Let T be the zero-map from 0 to $\mathcal{F}$. Then T_x is zero for all $x \in X$, but, again, this is an isometric isomorphism. However, the image of T is not dense in $\mathcal{F}$. This shows that the condition that $\mathcal{F}$ is locally $\mathcal{C}_0(X)$-convex cannot be dropped in 2., 3. and 4.

Corollary A.2.10. *Let E and F be u.s.c. fields of Banach spaces over X and let $(T_x)_{x\in X}$ be a bounded continuous field of morphisms from E to F.*

1. *If for all $x \in X$ the operator $T_x\colon E_x \to F_x$ is isometric, then $\mathfrak{M}(T)$ is isometric.*

2. *If for all $x \in X$ the operator T_x has dense image, then $\mathfrak{M}(T)$ has dense image.*

3. *If for all $x \in X$ the operator T_x is injective, then $\mathfrak{M}(T)$ is injective.*

4. *If for all $x \in X$ the operator T_x is surjective and a quotient map, then $\mathfrak{M}(T)$ is surjective and a quotient map.*

5. *If for all $x \in X$ the operator T_x is an isometric isomorphism, then $\mathfrak{M}(T)$ is an isometric isomorphism.*

Proof. The only one of these assertions that does not follow directly from Example A.2.4 and Proposition A.2.8 is 3.: Assume that all the T_x are injective and let ξ be in $\Gamma_0(X, E)$ such that $T \circ \xi = 0$. Let $x \in X$. Then $(T \circ \xi)(x) = T_x(\xi(x)) = 0$ and thus $\xi(x) = 0$. It follows that $\xi = 0$, so $\mathfrak{M}(T)$ is injective. □

A.2.2 The Gelfand transformation and local convexity

Proposition A.2.11. *Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. Then $\mathfrak{G}(\mathcal{E})$ is locally $\mathcal{C}_0(X)$-convex. The Gelfand transform $\mathfrak{g}_{\mathcal{E}}$ induces isometric isomorphisms of the fibres of $\mathcal{E}$ onto the fibres of $\mathfrak{G}(\mathcal{E})$.*

Proof. Since $\mathfrak{G}(\mathcal{E}) = \mathfrak{M}(\mathfrak{F}(\mathcal{E}))$, it follows from Example A.2.4 that $\mathfrak{G}(\mathcal{E})$ is locally $\mathcal{C}_0(X)$-convex. The fibres of $\mathfrak{G}(\mathcal{E})$ can be identified with the fibres of $\mathfrak{F}(\mathcal{E})$ which are the fibres of $\mathcal{E}$. The identification maps are induced by $\mathfrak{g}_{\mathcal{E}}$. □

So the Gelfand functor $\mathfrak{G}(\cdot)$ is a Banach functor from the category of $\mathcal{C}_0(X)$-Banach spaces to the full subcategory of locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces.

Proposition A.2.12. *The Gelfand functor has the following properties:*

1. *If $\mathcal{E}$ is a locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach space, then $\mathfrak{g}_{\mathcal{E}}$ is an isometric isomorphism from $\mathcal{E}$ to $\mathfrak{G}(\mathcal{E})$.*

2. *$\mathfrak{G}(\cdot)$ is a reflector: If $\mathcal{E}$ and $\mathcal{F}$ are $\mathcal{C}_0(X)$-Banach spaces with $\mathcal{F}$ locally $\mathcal{C}_0(X)$-convex and if $T \in \mathrm{L}^{\mathcal{C}_0(X)}(\mathcal{E}, \mathcal{F})$, then there is a unique $\hat{T} \in \mathrm{L}^{\mathcal{C}_0(X)}(\mathfrak{G}(\mathcal{E}), \mathcal{F})$ such that $T = \hat{T} \circ \mathfrak{g}_{\mathcal{E}}$. It satisfies $\|\hat{T}\| \leq \|T\|$.*

3. *The functor $\mathfrak{G}(\cdot)$ is naturally isomorphic to the functor $\mathfrak{G}(\mathfrak{G}(\cdot))$.*

4. *If $T \in \mathrm{L}^{\mathcal{C}_0(X)}(\mathcal{E}, \mathcal{F})$ is isometric (has dense image / is surjective and a quotient map), then so is (has / is) $\mathfrak{G}(T)$.*

Proof.
1. This follows directly from the definition of local $\mathcal{C}_0(X)$-convexity.

2. Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{C}_0(X)$-Banach spaces with $\mathcal{F}$ locally $\mathcal{C}_0(X)$-convex and let $T \in \mathrm{L}^{\mathcal{C}_0(X)}(\mathcal{E}, \mathcal{F})$. The homomorphism $\mathfrak{g}_{\mathcal{F}}$ is an isometric isomorphism and hence the operator $\hat{T} := \mathfrak{g}_{\mathcal{F}}^{-1} \circ \mathfrak{G}(T)$ is continuous with norm $\leq \|T\|$. Note that $\hat{T} \circ \mathfrak{g}_{\mathcal{E}} = \mathfrak{g}_{\mathcal{F}}^{-1} \circ \mathfrak{G}(T) \circ \mathfrak{g}_{\mathcal{E}} = T$. The operator $\hat{T}$ is unique with this property since the image of $\mathfrak{g}_{\mathcal{E}}$ is dense in $\mathfrak{G}(\mathcal{E})$.

3. This can, for example, be deduced from 1. and Proposition A.2.11.

4. Let $T \in \mathrm{L}^{\mathcal{C}_0(X)}(\mathcal{E}, \mathcal{F})$ be isometric. Then T_x is isometric for every $x \in X$ by Proposition 4.2.5. Hence $\mathfrak{G}(T)$ is isometric by Proposition A.2.8. Similarly one shows the statements for the maps with dense image [2] and the quotient maps. □

[2] This also follows since the reflector $\mathfrak{G}(\cdot)$ respects epimorphisms.

A.2.3 Standard constructions and local convexity

Proposition A.2.13. *The category of locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces is stable under the following constructions:*

1. *closed subspaces,*
2. *quotients,*
3. *finite products, and*
4. *finite fibre products.*

Proof. 1. Let $\mathcal{E}$ be a locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach space. If $\mathcal{E}_0$ is a closed $\mathcal{C}_0(X)$-invariant subspace, then the embedding is isometric in every fibre, so it is isometric for the semi-norms $|||\cdot|||$. Since $\mathcal{E}$ is $\mathcal{C}_0(X)$-convex, the norm on $\mathcal{E}$ coincides with the semi-norm, so the same holds on $\mathcal{E}_0$.

2. Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{C}_0(X)$-Banach spaces and let $T \in \mathrm{L}(\mathcal{E}, \mathcal{F})$ be $\mathcal{C}_0(X)$-linear, surjective and a quotient map. Let $f_1, f_2 \in \mathcal{F}$, $\varphi_1, \varphi_2 \in \mathcal{C}_0(X)$ such that $0 \leq \varphi_1, \varphi_2$ and $\varphi_1 + \varphi_2 \leq 1$. Let $\varepsilon > 0$. Find $e_1, e_2 \in \mathcal{E}$ such that $T(e_i) = f_i$ and $\|e_i\| \leq \|f_i\| + \varepsilon$ for $i = 1, 2$. Then

$$\begin{aligned} \|\varphi_1 f_1 + \varphi_2 f_2\| &= \|T(\varphi_1 e_1 + \varphi_2 e_2)\| \leq \|\varphi_1 e_1 + \varphi_2 e_2\| \\ &\leq \max\{\|e_1\|, \|e_2\|\} \leq \max\{\|f_1\|, \|f_2\|\} + \varepsilon. \end{aligned}$$

Since ε was arbitrary, it follows that $\|\varphi_1 f_1 + \varphi_2 f_2\| \leq \max\{\|f_1\|, \|f_2\|\}$. So $\mathcal{F}$ is locally $\mathcal{C}_0(X)$-convex.

3. Let $\mathcal{E}^1$ and $\mathcal{E}^2$ be $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces. Then the fibres of the product $\mathcal{E}^1 \times \mathcal{E}^2$ are (isometrically isomorphic to) the products of the fibres. If $(e^1, e^2) \in \mathcal{E}^1 \times \mathcal{E}^2$ and $x \in X$, then $\left\|(e^1, e^2)_x\right\| = \left\|(e^1_x, e^2_x)\right\| = \sup\{\left\|e^1_x\right\|, \left\|e^2_x\right\|\}$ and hence

$$\begin{aligned} \left\|(e^1, e^2)\right\| &= \sup\{\left\|e^1\right\|, \left\|e^2\right\|\} = \sup_{i\in\{1,2\}} \sup_{x\in X} \left\|e^i_x\right\| \\ &= \sup_{x\in X} \sup_{i\in\{1,2\}} \left\|e^i_x\right\| = \sup_{x\in X} \left\|(e^1, e^2)_x\right\| = |||(e^1, e^2)|||. \end{aligned}$$

4. This follows from 1. and 3. □

Sums of locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces need not be $\mathcal{C}_0(X)$-convex:

Example A.2.14. Let X have at least two points and let $\mathcal{E}^1 = \mathcal{E}^2 = \mathcal{C}_0(X)$. Then $\mathcal{E}^1 \oplus \mathcal{E}^2$ carries the norm $\|(e_1, e_2)\|_1 = \|e_1\| + \|e_2\| = \sup_{x\in X} |e_1(x)| + \sup_{x\in X} |e_2(x)|$ where $(e_1, e_2) \in \mathcal{E}^1 \oplus \mathcal{E}^2$. This is generally not the same as $\sup_{x\in X}(|e_1(x)| + |e_2(x)|)$ as the following example shows: Let $x_1, x_2 \in X$ be two distinct points. Find functions $e_1, e_2 \in \mathcal{C}_0(X)$ such that $e_i(x_i) = 1$, $0 \leq e_i \leq 1$ and $e_1 \cdot e_2 = 0$. Then $\|(e_1, e_2)\| = 2$ whereas the other norm is 1.

However, one can use the Gelfand-functor to find products in the category of locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces: Just apply the Gelfand functor to the ordinary product.

A.2.4 Tensor products

In this section we show that the $\mathcal{C}_0(X)$-tensor product of locally $\mathcal{C}_0(X)$-convex spaces is again locally $\mathcal{C}_0(X)$-convex.[3] When proving this, we have to be careful not to use the multiplicativity of the Gelfand functor, since we have deduced this multiplicativity from the multiplicativity of the functors $\mathfrak{F}(\cdot)$ and $\mathfrak{M}(\cdot)$. But in the proof of the multiplicativity of $\mathfrak{M}(\cdot)$, which is part of the proof of Proposition 4.1.2, we have already used the fact that we are going to prove now, so applying the multiplicativity of $\mathfrak{G}(\cdot)$ would result in a circular argument.

Theorem A.2.15. *Let $\mathcal{E}$ and $\mathcal{F}$ be locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces. Then their $\mathcal{C}_0(X)$-tensor product $\mathcal{E} \otimes^{\mathcal{C}_0(X)} \mathcal{F}$ is locally $\mathcal{C}_0(X)$-convex.*

The starting point of the proof is the following proposition:

Proposition A.2.16. *Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. Then the following are equivalent:*

1. *$\mathcal{E}$ is locally $\mathcal{C}_0(X)$-convex.*
2. $\forall e \in \mathcal{E}\ \forall \varphi_1, \varphi_2 \in \mathcal{C}_b(X): \ \varphi_1\varphi_2 = 0 \Rightarrow \|(\varphi_1 + \varphi_2)e\| = \max\{\|\varphi_1 e\|, \|\varphi_2 e\|\}$.
3. $\forall e \in \mathcal{E}\ \forall \varphi_1, \varphi_2 \in \mathcal{C}_0(X): \ \varphi_1\varphi_2 = 0 \Rightarrow \|(\varphi_1 + \varphi_2)e\| = \max\{\|\varphi_1 e\|, \|\varphi_2 e\|\}$.
4. $\forall e \in \mathcal{E}\ \forall \varphi_1, \varphi_2 \in \mathcal{C}_c(X): \ \varphi_1\varphi_2 = 0 \Rightarrow \|(\varphi_1 + \varphi_2)e\| = \max\{\|\varphi_1 e\|, \|\varphi_2 e\|\}$.

Proof. 1. $\Leftrightarrow$ 2.: This is part of proposition 7.14 of [Gie82].
The implications 2. $\Rightarrow$ 3. and 3. $\Rightarrow$ 4. are trivial.
4. $\Rightarrow$ 2.: Take a bounded approximate unit $(\chi_\lambda)_{\lambda\in\Lambda}$ of $\mathcal{C}_0(X)$ which is contained in $\mathcal{C}_c(X)$. Let $e \in E$ and $\varphi_1, \varphi_2 \in \mathcal{C}_b(X)$ such that $\varphi_1\varphi_2 = 0$. Then

$$\|(\varphi_1 + \varphi_2)e\| = \lim_\lambda \|(\chi_\lambda\varphi_1 + \chi_\lambda\varphi_2)e\| = \lim_\lambda \max\{\|\chi_\lambda\varphi_1 e\|, \|\chi_\lambda\varphi_2 e\|\} = \max\{\|\varphi_1 e\|, \|\varphi_2 e\|\}$$

since $(\chi_\lambda\varphi_1)(\chi_\lambda\varphi_2) = 0$ for every $\lambda \in \Lambda$ (allowing us to apply 4.). $\square$

For technical reasons, we want to refine this proposition a tiny bit. The condition $\varphi_1\varphi_2 = 0$ says that the sets $U_{\varphi_i} := \{x \in X : \ \varphi_i(x) \neq 0\}$, $i = 1, 2$, are disjoint. We can impose the slightly stronger condition that the supports, being the closures of these sets, do not intersect either. This is an easy consequence of the following trivial observation:

Lemma A.2.17. *Let φ be an element of $\mathcal{C}_0(X)$ and $\varepsilon > 0$. Let $U_\varphi := \{x \in X : \ \varphi(x) \neq 0\}$. Then there is a function $\varphi^\varepsilon \in \mathcal{C}_c(X)$ of compact support contained in U_φ such that $\|\varphi - \varphi^\varepsilon\| \leq \varepsilon$.*

From this follows:

Lemma A.2.18. *Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. Then $\mathcal{E}$ is locally $\mathcal{C}_0(X)$-convex if and only if*

4.' $\quad \forall e \in E\ \forall \varphi_1, \varphi_2 \in \mathcal{C}_c(X): \ \operatorname{supp}\varphi_1 \cap \operatorname{supp}\varphi_2 = \emptyset \Rightarrow \|(\varphi_1 + \varphi_2)e\| = \max\{\|\varphi_1 e\|, \|\varphi_2 e\|\}$.

[3] In [KR89b] the tensor product of locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces was defined to be $\mathfrak{G}(\cdot)$ of the $\mathcal{C}_0(X)$-tensor product. With the result presented here, applying the Gelfand functor is no longer necessary which simplifies some considerations in [KR89b].

Proof. It is clear that 4. $\Rightarrow$ 4.'. For the opposite direction, let $e \in \mathcal{E}$ and $\varphi_1, \varphi_2 \in \mathcal{C}_0(X)$ such that $\varphi_1\varphi_2 = 0$. Let $\varepsilon > 0$. Find functions φ_1^ε and φ_2^ε in $\mathcal{C}_c(X)$ such that the support of φ_i^ε is contained $U_{\varphi_i} := \{x \in X : \varphi_i(x) \neq 0\}$ and such that $\|\varphi_i - \varphi_i^\varepsilon\| \leq \varepsilon$. Note that the supports of these two functions are separated by the open sets U_{φ_i}. We can hence apply 4.' to get

$$\begin{aligned}
\|(\varphi_1 + \varphi_2)e\| &= \|((\varphi_1 - \varphi_1^\varepsilon) + \varphi_1^\varepsilon + (\varphi_2 - \varphi_2^\varepsilon) + \varphi_2^\varepsilon)e\| \\
&\leq \|(\varphi_1^\varepsilon + \varphi_2^\varepsilon)e\| + \|(\varphi_1 - \varphi_1^\varepsilon)e\| + \|(\varphi_2 - \varphi_2^\varepsilon)e\| \\
&\leq \|(\varphi_1^\varepsilon + \varphi_2^\varepsilon)e\| + 2\varepsilon\,\|e\| \\
&\overset{4.'}{=} \max\{\|\varphi_1^\varepsilon e\|, \|\varphi_2^\varepsilon e\|\} + 2\varepsilon\,\|e\| \\
&\leq \max\{\|\varphi_1 e\| + \varepsilon\,\|e\|, \|\varphi_2 e\| + \varepsilon\,\|e\|\} + 2\varepsilon\,\|e\| \\
&= \max\{\|\varphi_1 e\|, \|\varphi_2 e\|\} + 3\varepsilon\,\|e\|.
\end{aligned}$$

Since ε was arbitrary, we get the desired result. $\square$

Definition A.2.19. Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space and $e \in \mathcal{E}$. Then the support $\operatorname{supp} e$ of e is defined as

$$\operatorname{supp} e := X \setminus \{x \in X : \exists U \subseteq X, x \in U, U \text{open}\ \forall \varphi \in \mathcal{C}_0(U) : \varphi e = 0\}.$$

Define

$$\mathcal{E}_c := \{e \in \mathcal{E} : \operatorname{supp} e \text{ is compact}\}.$$

Lemma A.2.20. *Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space and $e \in \mathcal{E}$. If $\varphi \in \mathcal{C}_c(X)$ such that $\operatorname{supp}\varphi \cap \operatorname{supp} e = \emptyset$, then $\varphi e = 0$.*

Proof. Let K be the support of φ. For all $k \in K \subseteq X \setminus \operatorname{supp} e$, there is an open neighbourhood U_k of k such that $\psi e = 0$ for all $\psi \in \mathcal{C}_0(U_k)$. Now $\{U_k : k \in K\}$ is an open covering of K, so we can find a finite set $S \subseteq K$ such that $\{U_s : s \in S\}$ covers K. Find a continuous partition of unity $(\chi_s)_{s\in S}$ on K subordinate to $(U_s)_{s\in S}$ (with $\chi_s \in \mathcal{C}_c(X)$). Then $\chi_s\varphi$ is in $\mathcal{C}_0(U_s)$ so $\chi_s\varphi e = 0$. But $\sum_{s\in S}\chi_s\varphi = \varphi$, so $\varphi e = 0$. $\square$

Lemma A.2.21. *Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. If $e \in \mathcal{E}$ and $\varphi \in \mathcal{C}_0(X)$, then*

$$\operatorname{supp}(\varphi e) \subseteq \operatorname{supp}\varphi \cap \operatorname{supp} e.$$

Proof. Let $x \in X$ such that $x \notin (\operatorname{supp}\varphi \cap \operatorname{supp} e)$. If $x \notin \operatorname{supp}\varphi$, then $U := X \setminus \operatorname{supp}\varphi$ is a neighbourhood of x. Let $\psi \in \mathcal{C}_0(U)$. Then $\psi(\varphi e) = (\psi\varphi)e = 0e = 0$, so $x \notin \operatorname{supp}(\varphi e)$. If $x \notin \operatorname{supp} e$, then $U := X \setminus \operatorname{supp} e$ is a neighbourhood of x. Let $\psi \in \mathcal{C}_0(U)$. Then $\psi(\varphi e) = \varphi(\psi e) = \varphi 0 = 0$, so $x \notin \operatorname{supp}(\varphi e)$. $\square$

Lemma A.2.22. *Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space and $e \in \mathcal{E}$. Then $e \in \mathcal{E}_c$ if and only if there is an $\varphi \in \mathcal{C}_c(X)$ such that $\varphi e = e$. If $e \in \mathcal{E}_c$, then the φ can be chosen to be supported in any given compact neighbourhood of $\operatorname{supp} e$ and such that $0 \leq \varphi \leq 1$.*

Proof. If $\varphi e = e$ for some $\varphi \in \mathcal{C}_c(X)$ then this means $\operatorname{supp} e \subseteq \operatorname{supp}\varphi$, so the support of e is compact.

If $K := \operatorname{supp} e$ is compact and L is a compact neighbourhood of K, then we can find a function $\varphi \in \mathcal{C}_c(X)$ such that $\varphi|_L = 1$ and $0 \leq \varphi \leq 1$. Let M be a compact set containing the support of φ and χ_M be a function in $\mathcal{C}_c(X)$ such that $\chi_M|_M = 1$ and $0 \leq \chi_M \leq 1$. Then $\chi_M\varphi = \varphi$ and hence $\chi_M\varphi e = \varphi e$. On the other hand we have $\operatorname{supp}(\chi_M - \varphi) \subseteq X \setminus K$ and hence $(\chi_M - \varphi)e = 0$, i.e., $\chi_M e = \varphi e$. If M gets larger and larger, then $\chi_M e$ approaches e, so $e = \varphi e$. $\square$

Lemma A.2.23. *Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{C}_0(X)$-Banach spaces and $e \in \mathcal{E}_c$, $f \in \mathcal{F}_c$ such that* $\operatorname{supp} e \cap \operatorname{supp} f = \emptyset$. *Then $e \otimes f = 0 \in \mathcal{E} \otimes_{\mathcal{C}_0(X)} \mathcal{F}$.*

Proof. Let K be a compact neighbourhood of $\operatorname{supp} e$ and let L be a compact neighbourhood of $\operatorname{supp} f$ such that $K \cap L = \emptyset$. Find functions φ and ψ in $\mathcal{C}_c(X)$ such that $\operatorname{supp} \varphi \subseteq K$ and $\varphi e = e$ and $\operatorname{supp} \psi \subseteq L$ and $\psi f = f$. Now $e \otimes f = (\varphi e) \otimes (\psi f) = e \otimes (\varphi \psi f) = e \otimes 0 = 0$. □

Lemma A.2.24. *Let $\mathcal{E}$ be a $\mathcal{C}_0(X)$-Banach space. Then $\mathcal{E}$ is locally $\mathcal{C}_0(X)$-convex if and only if*

$$5. \qquad \forall e_1, e_2 \in \mathcal{E}_c : \ \operatorname{supp} e_1 \cap \operatorname{supp} e_2 = \emptyset \Rightarrow \|e_1 + e_2\| = \max\{\|e_1\|, \|e_2\|\}.$$

Proof. Assume that 5. is satisfied. We show 4.'. Let $e \in E$ and $\varphi_1, \varphi_2 \in \mathcal{C}_c(X)$ such that $\operatorname{supp} \varphi_1 \cap \operatorname{supp} \varphi_2 = \emptyset$. Let $e_i := \varphi_i e$ for $i = 1, 2$. Then $\operatorname{supp} e_i \subseteq \operatorname{supp} \varphi_i$ so $\operatorname{supp} e_1 \cap \operatorname{supp} e_2 = \emptyset$. An application of 5. now gives 4.'.

Assume now that 4.' holds. Let $e_1, e_2 \in \mathcal{E}_c$ such that $\operatorname{supp} e_1 \cap \operatorname{supp} e_2 = \emptyset$. Find $\varphi_1, \varphi_2 \in \mathcal{C}_c(X)$ such that $\varphi_i e_i = e_i$ for $i = 1, 2$ and $\operatorname{supp} \varphi_1 \cap \operatorname{supp} \varphi_2 = \emptyset$. Define $e := e_1 + e_2$. Note that $\varphi_2 e_1 = 0 = \varphi_1 e_2$, so $\varphi_i e = e_i$. An application of 4.' now gives 5. □

Proof of Theorem A.2.15. We use Lemma A.2.24. Let $t_1, t_2 \in \left(\mathcal{E} \otimes^{\mathcal{C}_0(X)} \mathcal{F}\right)_c$ such that $\operatorname{supp} t_1 \cap \operatorname{supp} t_2 = \emptyset$. Without loss of generality we assume that both, t_1 and t_2, are non-zero. Let L_1, L_2 be compact neighbourhoods of $\operatorname{supp} t_1$ and $\operatorname{supp} t_2$, respectively, such that $L_1 \cap L_2 = \emptyset$.

Find functions φ_1 and φ_2 such that $\operatorname{supp} \varphi_i \subseteq L_i$, $0 \le \varphi_i \le 1$ and $\varphi_i t_i = t_i$, for $i = 1, 2$. Note that

$$\|t_i\| = \|\varphi_i(t_1 + t_2)\| \le \|\varphi_i\| \, \|t_1 + t_2\| = \|t_1 + t_2\|$$

for $i = 1, 2$, which shows $\|t_1 + t_2\| \ge \max\{\|t_1\|, \|t_2\|\}$.

The other inequality is the non-trivial one. Let $\varepsilon > 0$. Find sequences $(e_n^1)_{n \in \mathbb{N}}$ and $(e_n^2)_{n \in \mathbb{N}}$ in $\mathcal{E}$ and $(f_n^1)_{n \in \mathbb{N}}$ and $(f_n^2)_{n \in \mathbb{N}}$ in $\mathcal{F}$ such that

$$\text{(A.1)} \qquad t_i = \sum_{n \in \mathbb{N}} e_n^i \otimes f_n^i \quad \text{and} \quad \sum_{n \in \mathbb{N}} \left\|e_n^i\right\| \left\|f_n^i\right\| \le \|t_i\| + \varepsilon$$

for $i = 1, 2$.
Without loss of generality we can assume

$$\text{(A.2)} \qquad \forall i \in \{1, 2\} \, \forall n \in \mathbb{N} : \ \operatorname{supp} e_n^i, \operatorname{supp} f_n^i \subseteq L_i,$$

$$\text{(A.3)} \qquad \forall i \in \{1, 2\} \, \forall n \in \mathbb{N} : \ \left\|f_n^i\right\| = 1,$$

$$\text{(A.4)} \qquad \forall n \in \mathbb{N} : \ \left\|e_n^1\right\| \ge \left\|e_n^2\right\| \qquad \text{or} \qquad \forall n \in \mathbb{N} : \ \left\|e_n^1\right\| \le \left\|e_n^2\right\|$$

Before justifying these assumptions we show how to use them to finish the proof. Assume that the first part of (A.4) holds. From (A.2) it follows that

$$\forall n \in \mathbb{N} : \ e_n^1 \otimes f_n^2 = 0 = e_n^2 \otimes f_n^1,$$

and hence

$$\sum_{n \in \mathbb{N}} (e_n^1 + e_n^2) \otimes (f_n^1 + f_n^2) = \sum_{n \in \mathbb{N}} e_n^1 \otimes f_n^1 + \sum_{n \in \mathbb{N}} e_n^2 \otimes f_n^2 = t_1 + t_2.$$

Moreover, we have

$$\left\|e_n^1 + e_n^2\right\| \overset{(A.2)}{=} \max\left\{\left\|e_n^1\right\|, \left\|e_n^2\right\|\right\} \overset{(A.4)}{=} \left\|e_n^1\right\|$$

and

$$\left\|f_n^1 + f_n^2\right\| \overset{(A.2)}{=} \max\left\{\left\|f_n^1\right\|, \left\|f_n^2\right\|\right\} \overset{(A.3)}{=} 1 = \left\|f_n^1\right\|$$

for all $n \in \mathbb{N}$. It follows that

$$\|t_1 + t_2\| \leq \sum_{n\in\mathbb{N}} \left\|e_n^1 + e_n^2\right\| \left\|f_n^1 + f_n^2\right\| = \sum_{n\in\mathbb{N}} \left\|e_n^1\right\| \left\|f_n^1\right\| \leq \|t_1\| + \varepsilon \leq \max\left\{\|t_1\|, \|t_2\|\right\} + \varepsilon.$$

If the second part of (A.4) holds, then we arrive at the same inequality. Since we have shown this for all $\varepsilon > 0$, it follows that

$$\|t_1 + t_2\| \leq \max\left\{\|t_1\|, \|t_2\|\right\}.$$

Now we justify the assumptions (A.2)-(A.4), step by step.

1. For (A.2), consider the sequences $(\varphi_i e_n^i)_{n\in\mathbb{N}}$ and $(\varphi_i f_n^i)_{n\in\mathbb{N}}$ for $i = 1, 2$. They satisfy the conditions $\operatorname{supp} \varphi_i e_n^i \subseteq L_i$ and $\operatorname{supp} \varphi_i f_n^i \subseteq L_i$ for all $n \in \mathbb{N}$, $i = 1, 2$. Moreover,

$$\sum_{n\in\mathbb{N}} \varphi_i e_n^i \otimes \varphi_i f_n^i = \varphi_i^2 \sum_{n\in\mathbb{N}} e_n^i \otimes f_n^i = \varphi_i^2 t_i = t_i$$

   for $i = 1, 2$ because $\varphi_i t_i = t_i$. Additionally,

$$\sum_{n\in\mathbb{N}} \left\|\varphi_i e_n^i\right\| \left\|\varphi_i f_n^i\right\| \leq \sum_{n\in\mathbb{N}} \left\|e_n^i\right\| \left\|f_n^i\right\| \leq \|t_i\| + \varepsilon,$$

   so substituting e_n^i with $\varphi_i e_n^i$ and f_n^i with $\varphi_i f_n^i$ gives sequences which satisfy (A.1) as well as (A.2).

2. We show that we can assume (A.3). Let $i \in \{1, 2\}$. We can assume that $f_n^i \neq 0$ for all $n \in \mathbb{N}$: Because $t_i \neq 0$ by assumption, there has to exist an $f_0^i \in \mathcal{F}$ such that $\operatorname{supp} f_0^i \subseteq L_i$ and $f_0^i \neq 0$. If $n \in \mathbb{N}$ such that $f_n^i = 0$, then substitute e_n^i by zero and f_n^i by f_0^i.

   Now consider the sequences $(\left\|f_n^i\right\| e_n^i)_{n\in\mathbb{N}}$ and $(\frac{1}{\|f_n^i\|} f_n^i)_{n\in\mathbb{N}}$. If we take these sequences instead of $(e_n^i)_{n\in\mathbb{N}}$ and $(f_n^i)_{n\in\mathbb{N}}$, then (A.1), (A.2) and (A.3) are satisfied.

3. For (A.4), we have to work a little harder. First of all, without loss of generality we may assume $\sum_{n\in\mathbb{N}} \|e_n^1\| \geq \sum_{n\in\mathbb{N}} \|e_n^2\|$. We show that in this case we can assume $\forall n \in \mathbb{N} : \ \|e_n^1\| \geq \|e_n^2\|$.

   Note that we have the freedom to rearrange the sequences $(e_n^i, f_n^i)_{n\in\mathbb{N}}$ in any order we like and that we can, informally speaking, replace some entry (e_n^i, f_n^i) by the two entries $(\lambda e_n^i, f_n^i)$ and $((1 - \lambda)e_n^i, f_n^i)$ for any $\lambda \in [0, 1]$. Both moves will not affect the properties (A.1), (A.2) or (A.3). Our strategy is to take one entry of $(e_n^2)_{n\in\mathbb{N}}$ after the other and split it up into smaller entries which we can match with entries of $(e_n^1)_{n\in\mathbb{N}}$ of the same size. Since $\sum_{n\in\mathbb{N}} \left\|e_n^1\right\| \geq \sum_{n\in\mathbb{N}} \left\|e_n^2\right\|$ it will be possible to match all entries of the sequence $(e_n^2)_{n\in\mathbb{N}}$ with entries of the other sequence. There might still be some bits of $(e_n^1)_{n\in\mathbb{N}}$ which are left over, but these entries will be matched with zero entries.

   For technical reasons, we would like to assume that $(e_n^2)_{n\in\mathbb{N}}$ has infinitely many non-zero entries: Because $t_2 \neq 0$ we know that at least one entry is non-zero. Substitute this entry by infinitely many "copies with weight 2^{-n}", where n runs through the natural numbers.

To gain space, we want the sequences to be indexed over a larger set, for notational convenience, we take $\mathbb{Z}$. So define $e_k^1 := e_k^2 := 0 \in \mathcal{E}$ for all $k \in \{0, -1, -2, \ldots\}$ and choose arbitrary f_k^1 and f_k^2 in $\mathcal{F}$ with norm 1 such that $\operatorname{supp} f_k^i \subseteq L_i$. Then the double-sequences $(e_k^i)_{k\in\mathbb{Z}}$ and $(f_k^i)_{k\in\mathbb{Z}}$ satisfy the relations (A.1), (A.2) and (A.3) (with $\mathbb{Z}$ replacing $\mathbb{N}$).

Description of the inductive procedure: We are going to give an inductive definition of a sequence $\left({}_n e^1, {}_n f^1, {}_n e^2, {}_n f^2\right)_{n\in\mathbb{N}_0}$ of such four-tuples of double-sequences, starting with the four double-sequences $\left(e^1, f^1, e^2, f^2\right) =: \left({}_0 e^1, {}_0 f^1, {}_0 e^2, {}_0 f^2\right)$ we have just defined. In each step, an entry of the sequence corresponding to $(e_k^2)_{k\in\mathbb{N}}$ is set to zero and "moved to the negative part of the double-sequence". Also some (parts of) entries of the sequence corresponding to $(e_k^1)_{k\in\mathbb{N}}$ are moved to the negative part, to ensure that the negative part of the sequences is always "balanced" in the sense that

$$\forall n \in \mathbb{N}_0 \; \forall k \in \mathbb{Z}_{\leq 0} : \; \left\| {}_n e_k^1 \right\| = \left\| {}_n e_k^2 \right\| . \tag{A.5}$$

Also, the procedure is designed in a way ensuring that the relations (A.1), (A.2) and (A.3) remain true.

In the limit, all positive entries of the sequences corresponding to $(e_k^2)_{k\in\mathbb{N}}$ vanish and we are left with sequences which are "balanced" on the negative side. There might still some non-vanishing entries of the sequence corresponding to $(e_k^1)_{n\in\mathbb{N}}$, but the sequence corresponding to $(e_k^2)_{k\in\mathbb{N}}$ vanishes, the condition (A.4) holds. Also the other relations hold for the limit.

The inductive definition: Let $n \in \mathbb{N}$ and assume that we have already defined the quadruple $\left({}_{n-1}e^1, {}_{n-1}f^1, {}_{n-1}e^2, {}_{n-1}f^2\right)$, satisfying the relations (A.1), (A.2) and (A.3) as well as $\forall k \in \mathbb{Z}_{\leq 0} : \; \left\|{}_{n-1}e_k^1\right\| = \left\|{}_{n-1}e_k^2\right\|$ and $\sum_{k\in\mathbb{N}} \left\|{}_{n-1}e_k^1\right\| \geq \sum_{k\in\mathbb{N}} \left\|{}_{n-1}e_k^2\right\|$, and such that the set $\{k \in \mathbb{Z}_{\leq 0} : \; {}_{n-1}e_k^2 \neq 0\}$ is finite whereas $\{k \in \mathbb{N} : \; {}_{n-1}e_k^2 \neq 0\}$ is infinite.

Note that $\left\|{}_{n-1}e_n^2\right\| < \sum_{m\in\mathbb{N}} \left\|{}_{n-1}e_m^2\right\| \leq \sum_{m\in\mathbb{N}} \left\|{}_{n-1}e_m^1\right\|$ so we can find a $p \in \mathbb{N}$ such that $r := \sum_{m=1}^{p-1} \left\|{}_{n-1}e_m^1\right\| < \left\|{}_{n-1}e_n^2\right\|$ and $\sum_{m=1}^{p} \left\|{}_{n-1}e_m^1\right\| \geq \left\|{}_{n-1}e_n^2\right\|$. Find $N \in \mathbb{Z}_{\leq 0}$ such that ${}_{n-1}e_k^2 = 0$ for all $k < N$.

Define

$${}_n e_k^1 := \begin{cases} {}_{n-1}e_l^1 & \text{if } k = N - l \text{ for some } l \in \{1, \ldots, p-1\} \\ \frac{\left\|{}_{n-1}e_n^2\right\| - r}{{}_{n-1}e_p^1}\, {}_{n-1}e_p^1 & \text{if } k = N - p \\ 0 & \text{if } k \in \{1, \ldots, p-1\} \\ \frac{\left\|{}_{n-1}e_n^1\right\| - (\left\|{}_{n-1}e_n^2\right\| - r)}{\left\|{}_{n-1}e_p^1\right\|}\, {}_{n-1}e_p^1 = {}_{n-1}e_p^1 - {}_n e_{N-p}^1 & \text{if } k = p \\ {}_{n-1}e_k^1 & \text{else,} \end{cases}$$

$${}_n f_k^1 := \begin{cases} {}_{n-1}f_l^1 & \text{if } k = N - l \text{ for some } l \in \{1, \ldots, p\} \\ {}_{n-1}f_k^1 & \text{else,} \end{cases}$$

$${}_n e_k^2 := \begin{cases} \frac{\left\|{}_{n-1}e_l^1\right\|}{\left\|{}_{n-1}e_n^2\right\|}\, {}_{n-1}e_n^2 & \text{if } k = N - l \text{ for some } l \in \{1, \ldots, p-1\} \\ \frac{\left\|{}_{n-1}e_n^2\right\| - r}{\left\|{}_{n-1}e_n^2\right\|}\, {}_{n-1}e_n^2 & \text{if } k = N - p \\ 0 & \text{if } k = n \\ {}_{n-1}e_k^2 & \text{else,} \end{cases}$$

$$
{}_n f^2_k := \begin{cases} {}_{n-1}f^2_l & \text{if } k = N - l \text{ for some } l \in \{1, \dots, p\} \\ {}_{n-1}f^2_k & \text{else.} \end{cases}
$$

The resulting quadruple $\left({}_n e^1, {}_n f^1, {}_n e^2, {}_n f^2\right)$ has all the properties of the original quadruple $\left({}_{n-1}e^1, {}_{n-1}f^1, {}_{n-1}e^2, {}_{n-1}f^2\right)$ that are listed above, plus it satisfies ${}_n e^2_n = 0$.

Note that $\left\|{}_n e^1 - {}_{n-1}e^1\right\|_1 = 2\left\|e^2_n\right\| = \left\|{}_n e^2 - {}_{n-1}e^2\right\|_1$. Hence $({}_n e^1)_{n\in\mathbb{N}}$ and $({}_n e^2)_{n\in\mathbb{N}}$ converge in l^1. The sequences $({}_n f^1)_{n\in\mathbb{N}}$ and $({}_n f^2)_{n\in\mathbb{N}}$ converge pointwise and are uniformly bounded by 1. Let $\left({}_\infty e^1, {}_\infty f^1, {}_\infty e^2, {}_\infty f^2\right)$ denote the limit-quadruple. The recursively defined sequences $\left(({}_n e^1_k \otimes {}_n f^1_k)_{k\in\mathbb{Z}}\right)_{n\in\mathbb{N}}$ and $\left(({}_n e^2_k \otimes {}_n f^2_k)_{k\in\mathbb{Z}}\right)_{n\in\mathbb{N}}$ converge in l^1. Hence the limit-quadruple satisfies (A.1). The relations (A.2), and (A.3) are stable under pointwise convergence of the involved sequences, hence they remain true in the limit as they are true in each step of the induction. The negative part of the sequences are balanced in every step of the induction, and ${}_\infty e^2_k = 0$ for all $k \in \mathbb{N}$. Hence (A.4) is true in the limit. □

Corollary A.2.25. *Let E^1 and E^2 be Banach spaces. Then*

$$
E^1 X \otimes^{\mathcal{C}_0(X)} E^2 X \cong \left(E^1 \otimes E^2\right) X.
$$

Proof. Define

$$
\begin{aligned}
\Phi\colon E^1 X \otimes^{\mathcal{C}_0(X)} E^2 X &\to \left(E^1 \otimes E^2\right) X, \\
f_1 \otimes f_2 &\mapsto \left(x \mapsto f_1(x) \otimes f_2(x)\right).
\end{aligned}
$$

This map is $\mathcal{C}_0(X)$-linear and of norm ≤ 1. Let $x \in X$. If we identify the fibre at x on both sides with $E^1 \otimes E^2$, then Φ_x is simply the identity and hence an isometric isomorphism.

From Theorem A.2.15 and Proposition A.2.8, 4., we can deduce that Φ is an isometric isomorphism. □

Corollary A.2.26. *Let $\mathcal{E}^1$ and $\mathcal{E}^2$ be $\mathcal{C}_0(X)$-Banach spaces. Then*

$$
\mathfrak{G}\left(\mathcal{E}^1 \otimes^{\mathcal{C}_0(X)} \mathcal{E}^2\right) \cong \mathfrak{G}\left(\mathcal{E}^1\right) \otimes^{\mathcal{C}_0(X)} \mathfrak{G}\left(\mathcal{E}^2\right). \tag{A.6}
$$

Proof. A direct argument for this is that $\mathfrak{g}_{\mathcal{E}^1} \otimes \mathfrak{g}_{\mathcal{E}^2}$ is a contractive $\mathcal{C}_0(X)$-linear map from $\mathcal{E}^1 \otimes^{\mathcal{C}_0(X)} \mathcal{E}^2$ to $\mathfrak{G}\left(\mathcal{E}^1\right) \otimes^{\mathcal{C}_0(X)} \mathfrak{G}\left(\mathcal{E}^2\right)$. So it factors through $\mathfrak{G}\left(\mathcal{E}^1 \otimes^{\mathcal{C}_0(X)} \mathcal{E}^2\right)$. The resulting map is a fibrewise isometric isomorphism and, since both sides are locally $\mathcal{C}_0(X)$-convex, it follows that it is an isometric isomorphism. □

A.2.5 The Gelfand functor and multilinear maps

In much the same way in which we have defined the Gelfand transform of a continuous linear operator between $\mathcal{C}_0(X)$-Banach spaces we can define it for continuous multilinear maps. Of course, in light of formula (A.6) it would actually possible to use tensor products to treat the multilinear case as a special case of the linear case. But for the sake of greater clarity, we present a direct construction for multilinear maps here in a separate section:

Proposition A.2.27. *Let $\mathcal{E}^1$, $\mathcal{E}^2$ and $\mathcal{F}$ be $\mathcal{C}_0(X)$-Banach spaces. Let $\mu \in \mathrm{M}^{\mathcal{C}_0(X)}\left(\mathcal{E}^1, \mathcal{E}^2; \mathcal{F}\right)$. Then there is a unique bilinear and continuous map $\mathfrak{G}(\mu) \in \mathrm{M}^{\mathcal{C}_0(X)}\left(\mathfrak{G}\left(\mathcal{E}^1\right), \mathfrak{G}\left(\mathcal{E}^2\right); \mathfrak{G}(\mathcal{F})\right)$ such that the* *following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{E}^1 \times \mathcal{E}^2 \arrow[r, "\mu"] \arrow[d] & \mathcal{F} \arrow[d] \\
\mathfrak{G}(\mathcal{E}^1) \times \mathfrak{G}(\mathcal{E}^2) \arrow[r, dashed, "\mathfrak{G}(\mu)"] & \mathfrak{G}(\mathcal{F})
\end{tikzcd}$$

It satisfies $\|\mathfrak{G}(\mu)\| \leq \|\mu\|$.

Proof. Uniqueness is obvious. To show existence define $\mathfrak{G}(\mu)$ on the dense subspace $\mathfrak{g}_{\mathcal{E}^1}(\mathcal{E}^1) \times \mathfrak{g}_{\mathcal{E}^2}(\mathcal{E}^2)$. If $e_i \in \mathcal{E}^i$ for all $i \in \{1,2\}$, then define

$$\mathfrak{G}(\mu)\left(\mathfrak{g}_{\mathcal{E}^1}(e_1), \mathfrak{g}_{\mathcal{E}^2}(e_2)\right) := \mathfrak{g}_{\mathcal{F}}\left(\mu(e_1, e_2)\right).$$

Suppose that $e_1' \in \mathcal{E}^1$ and $e_2' \in \mathcal{E}^2$ such that $\mathfrak{g}_{\mathcal{E}^1}(e_1) = \mathfrak{g}_{\mathcal{E}^1}(e_1')$ and $\mathfrak{g}_{\mathcal{E}^2}(e_2) = \mathfrak{g}_{\mathcal{E}^2}(e_2')$. Then $(e_1)_x = (e_1')_x$ and $(e_2)_x = (e_2')_x$ for all $x \in X$ and hence

$$\left(\mu(e_1, e_2)\right)_x = \mu_x((e_1)_x, (e_2)_x) = \mu_x((e_1')_x, (e_2')_x) = \left(\mu(e_1', e_2')\right)_x$$

which shows that

$$\mathfrak{g}_{\mathcal{F}}\left(\mu(e_1, e_2)\right) = \mathfrak{g}_{\mathcal{F}}\left(\mu\left(e_1', e_2'\right)\right).$$

Hence $\mathfrak{G}(\mu)$ is well-defined on a dense subspace. To calculate its norm just consider

$$\begin{aligned}
\left\|\mathfrak{g}_{\mathcal{F}}\left(\mu(e_1, e_2)\right)\right\| &= \sup_{x\in X} \left\|\left(\mu(e_1, e_2)\right)_x\right\| \\
\leq \quad & \|\mu\| \sup_{x\in X} \|(e_1)_x\| \, \|(e_2)_x\| \leq \|\mu\| \, \|\mathfrak{g}_{\mathcal{E}^1}(e_1)\| \, \|\mathfrak{g}_{\mathcal{E}^2}(e_2)\|.
\end{aligned}$$

So the norm of $\mathfrak{G}(\mu)$ is $\leq \|\mu\|$, so it can be extended to $\mathfrak{G}(\mathcal{E}^1) \times \mathfrak{G}(\mathcal{E}^2)$ by continuity. □

Proposition A.2.28. *Let* $\mathcal{E}^1, \mathcal{E}^2, \mathcal{F}$ *be* $\mathcal{C}_0(X)$*-Banach spaces. Let* $\mu \in \mathrm{M}^{\mathcal{C}_0(X)}\left(\mathcal{E}^1, \mathcal{E}^2; \mathcal{F}\right)$*. Then under the identification (A.6) we have*

$$\mathfrak{G}(\hat{\mu}) = \widehat{\mathfrak{G}(\mu)}$$

where $\hat{\mu}$ *and* $\widehat{\mathfrak{G}(\mu)}$ *are the linearisations of* μ *and* $\mathfrak{G}(\mu)$*, respectively.*

Proof. Let $e_1 \in \mathcal{E}^1$ and $e_2 \in \mathcal{E}^2$. Then by definition

$$\mathfrak{G}(\mu)\left(\mathfrak{g}_{\mathcal{E}^1}(e_1), \mathfrak{g}_{\mathcal{E}^2}(e_2)\right) = \mathfrak{g}_{\mathcal{F}}\left(\mu(e_1, e_2)\right).$$

Hence

$$\widehat{\mathfrak{G}(\mu)}\left(\mathfrak{g}_{\mathcal{E}^1}(e_1) \otimes \mathfrak{g}_{\mathcal{E}^2}(e_2)\right) = \mathfrak{g}_{\mathcal{F}}\left(\mu(e_1, e_2)\right).$$

On the other hand

$$\hat{\mu}\left(e_1 \otimes e_2\right) = \mu\left(e_1, e_2\right).$$

So

$$\mathfrak{G}(\hat{\mu})\, \mathfrak{g}_{\mathcal{E}^1 \otimes \mathcal{E}^2}\left(e_1 \otimes e_2\right) = \mathfrak{g}_{\mathcal{F}}\left(\mu\left(e_1, e_2\right)\right).$$

Now the identification (A.6) identifies $\mathfrak{g}_{\mathcal{E}^1 \otimes \mathcal{E}^2}\left(e_1 \otimes e_2\right)$ with $\mathfrak{g}_{\mathcal{E}^1}(e_1) \otimes \mathfrak{g}_{\mathcal{E}^2}(e_2)$, so we are done. □

Appendix B

Continuous Fields of Measures

The notion of a (faithful) continuous field of measures is underlying the definition of a Haar system. This appendix is a systematic collection of facts concerning continuous fields of measures and integration of sections in u.s.c. fields of Banach spaces. The results presented here that just concern continuous fields of measures and do not involve fields of Banach spaces are to a large extend folklore or appear in a similar form in the literature, compare [MRW87], for example.

B.1 Sections of compact support

Let X be a locally compact Hausdorff space.

B.1.1 Selections of compact support and linear maps

Let E be a family of Banach spaces over X. We now topologise the space $\Sigma_c(X, E)$ of bounded selections with compact support turning it into a locally convex space:

If K is a compact subset of X, then we write $\Sigma_K(X, E)$ for the space of all (locally) bounded selections of E with support contained in K. For all compact $K \subseteq X$, the vector space $\Sigma_K(X, E)$ becomes a Banach space when equipped with the sup-norm which we denote by $\|\cdot\|_K$. If K and L are compact subsets of X such that $K \subseteq L$, then the inclusion of $\Sigma_K(X, E)$ into $\Sigma_L(X, E)$ is a linear and isometric map. Hence we have an inductive system of Banach spaces indexed over the compact subsets of X. Since these spaces exhaust $\Sigma_c(X, E)$, we can identify the inductive limit (as a vector space) with $\Sigma_c(X, E)$. The inductive limit topology on $\Sigma_c(X, E)$ is then defined to be the inductive topology in the category of locally convex vector spaces.

By definition, the inductive limit topology has the following universal property: The inclusion of $\Sigma_K(X, E)$ into $\Sigma_c(X, E)$ is continuous for all compact $K \subseteq X$ and if F is a locally convex space and $T\colon \Sigma_c(X, E) \to F$ is a linear map, then T is continuous if and only if it is continuous when restricted to all $\Sigma_K(X, E)$, where K runs through the compact subsets of X.

We can regard the map $E \mapsto \Sigma_c(X, E)$ as a functor: Let F be another family of Banach spaces over X and let $T\colon E \to F$ be a locally bounded family of linear maps. Then the map $\xi \mapsto T \circ \xi$ is a continuous linear map from $\Sigma_c(X, E)$ to $\Sigma_c(X, F)$ which we denote by $\Sigma_c(X, T)$. In this way $E \mapsto \Sigma_c(X, E)$ becomes a functor from the category of families of Banach spaces over X and locally bounded families of morphisms to the category of locally convex spaces and continuous linear maps.

B.1.2 Sections of compact support and linear maps

Let E be a u.s.c. field of Banach spaces over X.

For every compact subset $K \subseteq X$, let $\Gamma_K(X, E)$ be the set of sections of E with (compact) support contained in K, i.e., $\Gamma_K(X, E) = \Gamma_c(X, E) \cap \Sigma_K(X, E)$. We equip the space $\Gamma_K(X, E)$ with the sup-norm $\|\cdot\|_K$ inherited from $\Sigma_K(X, E)$. With this norm, $\Gamma_K(X, E)$ is a closed subspace of $\Sigma_K(X, E)$, so in particular it is a Banach space. If K and L are compact subsets of X with $K \subseteq L$, then the embedding from $\Gamma_K(X, E)$ into $\Gamma_L(X, E)$ is isometric. As above, define on $\Gamma_c(X, E)$ the *inductive limit topology* as the finest locally convex topology such that all the embeddings $\Gamma_K(X, E) \hookrightarrow \Gamma_c(X, E)$ are continuous.

Definition and Proposition B.1.1. Let X be a locally compact Hausdorff space and let E and F be u.s.c. fields of Banach spaces over X. Let T be a continuous field of linear maps from E to F. Then $\xi \mapsto T \circ \xi$ defines a continuous linear map $\Gamma_c(X, T)$ from $\Gamma_c(X, E)$ to $\Gamma_c(X, F)$.

Proof. Let $K \subseteq X$ be compact. Since T is continuous, it is locally bounded by definition, so it is also bounded on K by some constant $C \geq 0$. For all $k \in K$ and $\xi \in \Gamma_K(X, E)$, we have $\|T_k(\xi(k))\| \leq \|T_k\| \, \|\xi_k\| \leq C \, \|\xi(k)\|$, so $\xi \mapsto T \circ \xi$ is a continuous linear map to $\Gamma_K(X, F)$ when restricted to $\Gamma_K(X, E)$. So it is also continuous as a map from $\Gamma_c(X, E)$ to $\Gamma_c(X, F)$ by the universal property of the inductive limit topology. $\square$

Proposition B.1.2. *The assignment $E \mapsto \Gamma_c(X, E)$ and $T \mapsto \Gamma_c(X, T)$ defines a functor from the category of u.s.c. fields of Banach spaces over X to the category of locally convex vector spaces.*

Lemma B.1.3. *Let K be a compact subset of X. Suppose that Ξ is a subset of $\Gamma(X, E)$ such that the span of $\{\xi(k) : \xi \in \Xi\}$ is dense in E_k for all $k \in K$. For each compact $L \subseteq X$ let Ξ_L be the closure in $\Gamma_L(X, E)$ of the span of all $\chi\xi$, with $\chi \in \mathcal{C}_L(X)$ and $\xi \in \Xi$. If L contains an open neighbourhood of K then $\Gamma_K(X, E) \subseteq \Xi_L$.*

Proof. Let L be a compact neighbourhood of K, let $\eta \in \Gamma_K(X, E)$ and let $\varepsilon > 0$. For all $k \in K$, find a section $\xi_k \in \Xi$ such that $\|\eta(k) - \xi_k(k)\|_{E_k} \leq \varepsilon/2$. Find a neighbourhood U_k of k in L such that $\|\eta(x) - \xi_k(x)\|_{E_x} \leq \varepsilon$ for all $x \in U_k$. Find a finite subset S of K such that $\{U_s : s \in S\}$ is an open cover of K. Find a continuous partition of unity $(\chi_s)_{s \in S}$ relative to K, subordinate to this cover. Note that the support of χ_s is contained in $U_s \subseteq L$, so $\chi_s \in \mathcal{C}_L(X)$. Define $\eta' := \sum_{s \in S} \chi_s \xi_s$. Then $\|\eta - \eta'\| \leq \varepsilon$. $\square$

Remark B.1.4. The preceding lemma does not seem to work if $K = L$. In general, the norm function of the η appearing in the proof might be non-continuous but merely upper semi-continuous. So it might happen that the norm does not vanish on the boundary of K. On the other hand, the product $\chi_s \xi_s$ will always have vanishing norm on the boundary, what makes approximation really difficult.

Corollary B.1.5. *If $\Xi \subseteq \Gamma(X, E)$ is a total subset, then the span of $\mathcal{C}_c(X)\Xi$ is dense in $\Gamma_c(X, E)$.*

Proposition B.1.6. *Let E and F be u.s.c. fields of Banach spaces over X. Let T be a continuous field of linear maps from E to F. If T_x has dense image for all $x \in X$, then $\xi \mapsto T \circ \xi$ from $\Gamma_c(X, E)$ to $\Gamma_c(X, F)$ has dense image.*

Proof. Let Ξ be the set $\{T \circ \xi : \xi \in \Gamma_c(X, E)\}$. Then Ξ is a total subset of F by assumption (and since $\Gamma_c(X, E)$ is total in E). It is closed under multiplication with $\mathcal{C}_c(X)$ and linear combinations, so by the preceding corollary, Ξ is dense in $\Gamma_c(X, F)$. $\square$

B.1.3 Sections of compact support and bilinear maps

Analogously to B.1.1, one can prove the following proposition.

Proposition B.1.7. *Let E, F and G be u.s.c. fields of Banach spaces over X. Let μ be a continuous field of continuous bilinear maps from $E \times_X F$ to G. Then $(\xi, \eta) \mapsto \mu \circ (\xi, \eta)$ defines a separately continuous bilinear map $\Gamma_c(X, \mu)$ from $\Gamma_c(X, E) \times \Gamma_c(X, F)$ to $\Gamma_c(X, G)$.*

Similarly to B.1.6 one proves:

Proposition B.1.8. *Let E, F and G be u.s.c. fields of Banach spaces over X. Let μ be a continuous field of continuous bilinear maps from $E \times_X F$ to G. If μ is non-degenerate, i.e., if the span of $\mu_x(E_x, F_x)$ is dense in G_x, then $(\xi, \eta) \mapsto \mu \circ (\xi, \eta)$ is a non-degenerate map from $\Gamma_c(X, E) \times \Gamma_c(X, F)$ to $\Gamma_c(X, G)$, i.e., its image spans a dense subset.*

Conjecture B.1.9. If we give a suitable definition for the $\mathcal{C}_c(X)$-balanced projective (!) tensor product $\Gamma_c(X, E) \otimes_{\mathcal{C}_c(X)} \Gamma_c(X, F)$ of $\Gamma_c(X, E)$ and $\Gamma_c(X, F)$, then

$$\Gamma_c(X, E) \otimes_{\mathcal{C}_c(X)} \Gamma_c(X, F) \cong \Gamma_c(X,\ E \otimes_X F).$$

B.2 Continuous fields of measures

Let X and Y be topological spaces and let $p\colon Y \to X$ be a continuous map.

Definition B.2.1 ((Faithful) continuous fields of measures). A *continuous field of measures* on Y over X (with coefficient map p) is a family $(\mu_x)_{x \in X}$ of measures[1] on Y such that $\operatorname{supp} \mu_x \subseteq Y_x := p^{-1}(\{x\})$ and such that, for all $\varphi \in \mathcal{C}_c(Y)$,

$$\mu(\varphi)\colon X \to \mathbb{C},\ x \mapsto \int_{y \in Y_x} \varphi(y)\, \mathrm{d}\mu_x(y), \tag{B.1}$$

is an element of $\mathcal{C}_c(X)$. It is called *faithful* if $\operatorname{supp} \mu_x = Y_x$.

Proposition B.2.2. *The map $\varphi \mapsto \mu(\varphi)$ appearing in the preceding definition is a continuous linear map from $\mathcal{C}_c(Y)$ to $\mathcal{C}_c(X)$. It is $\mathcal{C}_c(X)$-linear and positive in the sense that it maps non-negative functions to non-negative functions.*

Proof. We show continuity. Let K be a compact subset of Y. Then $L := p(K) \subseteq X$ is compact as well. Find a function χ in $\mathcal{C}_c(Y)$ such that $0 \leq \chi \leq 1$ and $\chi|_K = 1$. Then $\mu(\chi) \in \mathcal{C}_c(X)$ with $\mu \geq 0$. Let $\varphi \in \mathcal{C}_c(Y)$ be a function with support in K. Then $\mu(\varphi)$ has support in L. For all $x \in L$, we have

$$\begin{aligned} |\mu(\varphi)(x)| &= \left| \int_{y \in Y_x} \varphi(y)\, \mathrm{d}\mu_x(y) \right| \leq \int_{y \in Y_x} |\varphi(y)|\, \mathrm{d}\mu_x(y) \\ &\leq \int_{y \in Y_x} \chi(y) \sup_{y' \in Y_x} \left|\varphi(y')\right| \mathrm{d}\mu_x(y) = \mu(\chi)(x) \sup_{y' \in Y_x} \left|\varphi(y')\right| \leq \|\mu(\chi)\|_L \|\varphi\|_K . \end{aligned}$$

It follows that

$$\|\mu(\varphi)\|_L \leq \|\mu(\chi)\|_L \|\varphi\|_K .$$

[1] Here we just consider positive measures. With a bit of extra work it is probably be possible to show most of what is said here also for signed measures.

To see that $\chi \mapsto \mu(\chi)$ is $\mathcal{C}_c(X)$-linear, let $\varphi \in \mathcal{C}_c(Y)$ and $\chi \in \mathcal{C}_c(X)$. Then $\chi\varphi$ is defined as the function $y \mapsto \chi(p(y))\varphi(y)$ and we obtain

$$\mu(\chi\varphi)(x) = \int_{y \in Y_x} \chi(x)\varphi(y)\,\mathrm{d}\mu_x(y) = \chi(x)\;\mu(\varphi)(x)$$

for all $x \in X$. $\square$

Proposition B.2.3. *Let M be a continuous and positive $\mathcal{C}_c(X)$-linear map from $\mathcal{C}_c(Y)$ to $\mathcal{C}_c(X)$. Then there is a unique continuous field of measures μ such that $M(\varphi) = \mu(\varphi)$ for all $\varphi \in \mathcal{C}_c(Y)$.*

Proof. Let $x \in X$. Define $\mu_x(\varphi) := M(\varphi)(x)$. This is / defines a measure on Y and it is obvious that this is our unique choice. We prove:

If $\varphi \in \mathcal{C}_c(Y)$ such that $\varphi|_{Y_x} = 0$, then $\mu_x = 0$, i.e., $\operatorname{supp} \mu_x \subseteq Y_x$.

To see this, let $\varphi \in \mathcal{C}_c(Y)$ such that $\varphi|_{Y_x} = 0$. Let $\varepsilon > 0$. Let L_ε denote the set $\{y \in Y : |\varphi| \geq \varepsilon\}$. Then L_ε is compact. Hence $K_\varepsilon := p(L_\varepsilon)$ is compact. Note that K_ε does not contain x, so there is a continuous function $\chi_\varepsilon \in \mathcal{C}_c(X)$ such that $0 \leq \chi_\varepsilon \leq 1$, $\chi_\varepsilon(x) = 1$, and $\chi_\varepsilon|_{K_\varepsilon} = 0$. Now $\chi_\varepsilon\varphi$ is a function on Y such that $\|\chi_\varepsilon\varphi\|_\infty \leq \varepsilon$. Note that the support of $\chi_\varepsilon\varphi$ is contained in the support of φ. By continuity of M we see that $M(\chi_\varepsilon\varphi)$ becomes arbitrarily small if $\varepsilon \to 0$. Then we also have

$$M(\varphi)(x) = \chi_\varepsilon(x)M(\varphi)(x) = M(\chi_\varepsilon\varphi)(x) \to 0,$$

and hence $M(\varphi)(x) = 0$. $\square$

Lemma B.2.4. *Let μ be a faithful continuous field of measures on the locally compact space Y over X with coefficient map p. Then p is open.*

Proof. Let $y \in Y$. Let V be a neighbourhood of y in Y. We can find a continuous function φ of compact support contained in V such that $\varphi \geq 0$ and $\varphi(y) > 0$. Then the restriction of φ to $Y_{p(y)}$ is also a non-negative continuous function with compact support, positive in y, so from the faithfulness of μ we get $\mu_{p(y)}(\varphi) > 0$. By continuity of μ the function $\mu(\varphi)$ is in $\mathcal{C}_c(X)$. It is a non-negative function and satisfies $p(y) > 0$. So it is positive on a neighbourhood U of $p(y)$. Since U is contained in the image under p of the set where φ is positive, we can deduce that the image of V is a neighbourhood of $p(y)$ in X, so p is open. $\square$

Lemma B.2.5. *Let X and Y be locally compact Hausdorff spaces and let $p\colon Y \to X$ be continuous, open and surjective. Then for every compact set $K \subseteq X$, there is a compact subset $L \subseteq Y$ such that $K = p(L)$. If V is an open subset of Y and $K \subseteq p(V)$ is compact, then L can be chosen to be a subset of V.*

Proof. Let $A := p^{-1}(K)$. Since p is continuous and K is compact, the set A is closed. Since p is surjective, we have $p(A) = K$. For every $a \in A$, choose a compact neighbourhood U_a of a in Y. Since p is open, the set $p(U_a)$ is a neighbourhood of $p(a)$ for every $a \in A$. So we can find a finite subset S of A such that $\{p(U_s) : s \in S\}$ is a cover of K. Now $L' := \bigcup_{s \in S} U_s$ is a compact set and $p(L') \supseteq K$. Define L to be $L' \cap A$, which is compact. Note that we have $p(L) = K$ by construction.

Applying this result to V and $p(V)$ instead of Y and X shows that we can take $L \subseteq V$ if $K \subseteq p(V)$ for open $V \subseteq Y$. $\square$

Lemma B.2.6 (Local cut-off functions). *Let X and Y be locally compact Hausdorff spaces and let $p\colon Y \to X$ be continuous, open and surjective. Let $(\mu_x)_{x\in X}$ be a faithful continuous field of measures on Y over X with coefficient map p. For all open $V \subseteq Y$ and all compact $K \subseteq p(V)$, there is a function $\chi \in \mathcal{C}_c(Y)$ such that $\operatorname{supp}\chi \subseteq V$, $\chi \geq 0$ and $\mu(\chi)(x) = \int_{y\in Y_x} \chi(y)\,\mathrm{d}\mu_x(y) = 1$ for all $x \in K$.*

Proof. Use the preceding lemma to find a compact subset $L \subseteq V$ such that $p(L) = K$. Find a function $\chi' \in \mathcal{C}_c(Y)$ such that $\operatorname{supp}\chi' \subseteq V$, $0 \leq \chi \leq 1$ and $\chi' \equiv 1$ on L. Then $\mu(\chi')$ is continuous and positive on K, so it is strictly positive there. Find a function $\delta \in \mathcal{C}_c(X)$ such that $\operatorname{supp}\delta \subseteq p(V)$, $\delta \geq 0$ and $\delta(x) = \frac{1}{\mu(\chi')(x)}$ for all $x \in K$. Now define

$$\chi(y) := \chi'(y)\delta(p(y))$$

for all $y \in Y$. Then χ is clearly continuous with support contained in the support of χ', which is, in turn, contained in V. Moreover, $\chi \geq 0$ and for all $x \in K$, we have

$$\mu(\chi)(x) = \int_{y\in Y_x} \chi(y)\,\mathrm{d}\mu_x(y) = \delta(x)\int_{y\in Y_x} \chi'(y)\,\mathrm{d}\mu_x(y) = \frac{1}{\mu(\chi')(x)}\mu(\chi')(x) = 1. \qquad \square$$

The following result is Lemma 2.13 of [MRW87] (the only change is that we skip the unnecessary condition that $\mu\colon \mathcal{C}_c(Y) \to \mathcal{C}_c(X)$ should be onto).

Lemma B.2.7. *Let $(\mu_x)_{x\in X}$ be a faithful continuous field of measures on Y over X with (open) coefficient map p. Then for all open $V \subseteq Y$ and for all $\psi \in \mathcal{C}_c(X)$ with $\operatorname{supp}\psi \subseteq p(V)$, there is a $\varphi \in \mathcal{C}_c(Y)$ with $\operatorname{supp}\varphi \subseteq V$ such that $\mu(\varphi) = \psi$. In particular, $\mu\colon \mathcal{C}_c(Y) \to \mathcal{C}_c(X)$ is onto. If $\psi \geq 0$, then we can choose $\varphi \geq 0$.*

Proof. By the preceding lemma, we can find a function $\chi \in \mathcal{C}_c(Y)$ such that $\operatorname{supp}\chi \subseteq V$, $\chi \geq 0$ and $\mu(\chi) \equiv 1$ on $K := \operatorname{supp}\psi$. Define

$$\varphi(y) := \chi(y)\psi(p(y))$$

for all $y \in Y$. Then $\varphi \in \mathcal{C}_c(Y)$, $\operatorname{supp}\varphi \subseteq V$ and $\varphi \geq 0$ if $\psi \geq 0$. For all $x \in K$ we have

$$\mu(\varphi)(x) = \mu(\chi)(x)\psi(x) = \psi(x).$$

For all $x \notin K$, this formula is also true since $\psi(x) = 0$. So $\mu(\varphi) = \psi$. $\square$

Definition and Proposition B.2.8 (Pullback of continuous fields of measures). Let μ be a continuous field of measures on Y over X with coefficient map p. Assume that X' is another locally compact Hausdorff space and that $q\colon X' \to X$ is continuous. Let $Y' := q^*(Y) := Y \times_X X'$ and $p' := q^*(p)\colon Y' \to X'$. In order to define a continuous field of measures $(\mu'_{x'})_{x'\in X'}$ (or $q^*(\mu)$) on Y' we define it on each fibre as an integral: For all $x' \in X'$ and all $\varphi \in \mathcal{C}_c(Y'_{x'})$, define

$$\mu'_{x'}(\varphi) := \int_{y\in Y_{q(x')}} \varphi(y, x')\,\mathrm{d}\mu_{q(x')}(y).$$

We claim that μ' is continuous.

Proof. We organise this proof so that its similarities to the proof of B.3.1 (see below) become obvious. There probably is a common basis for the two propositions.

If $\varphi \in \mathcal{C}_c(Y')$ is of the form $(\chi \otimes \psi)|_{Y'}$ with $\chi \in \mathcal{C}_c(Y)$ and $\psi \in \mathcal{C}_c(X')$, then

$$\mu'(\varphi)(x') = \int_{y \in Y_{q(x')}} \varphi(y, x') \,\mathrm{d}\mu_{q(x')}(y) = \int_{y \in Y_{q(x')}} \chi(y) \,\mathrm{d}\mu_{q(x')}(y)\psi(x') = \mu(\chi)(q(x'))\psi(x')$$

for all $x' \in X'$, so $\mu'(\varphi) = (\mu(\chi) \circ q)\psi \in \mathcal{C}_c(X')$. For general $\varphi \in \mathcal{C}_c(Y')$ we have to use some approximation argument: Let K be a compact subset of Y'. Find some $\chi_0 \in \mathcal{C}_c(Y)$ such that $0 \leq \chi_0 \leq 1$ and $\chi_0 \equiv 1$ on $\pi_1(K)$ (where $\pi_1 \colon Y \times_X X' \to Y$ is the projection to the first coordinate). Then $\mu(\chi_0)$ is in $\mathcal{C}_c(X)$ and therefore bounded by some $C \geq 0$. Let $\varphi \in \mathcal{C}_K(Y')$. Then

$$\begin{aligned} \left|\mu'(\varphi)(x')\right| &= \left|\mu'(\varphi(\chi_0 \circ \pi_1))(x')\right| \leq \int_{y \in Y_{q(x')}} \chi_0(y) \left|\varphi(y, x')\right| \,\mathrm{d}\mu_{q(x')}(y) \\ &\leq \|\varphi\|_\infty \,\mu(\chi_0)(q(x_0)) \leq C \,\|\varphi\|_\infty \end{aligned}$$

for all $x' \in X'$. Since the support of $\mu'(\varphi)$ is contained in $p'(K)$, which is compact, it follows that μ defines a continuous linear map with norm $\leq C$ from $\mathcal{C}_K(Y')$ to the space $\mathcal{S}_{p'(K)}(X)$ of bounded functions on X' with support in $p'(K)$. Note that $\mathcal{C}_{p'(K)}(X)$ is a closed subspace of this space.

Let L be a compact subset of Y' of the form $L = M \times N$ with $M \subseteq Y$ and $N \subseteq X'$ compact and such that L is a compact neighbourhood of K in Y'. Then every $\varphi \in \mathcal{C}_K(Y')$ can be approximated in the sup-norm by sums of elements of the form $(\chi \otimes \psi)|_{Y'}$ with $\chi \in \mathcal{C}_M(Y)$ and $\psi \in \mathcal{C}_N(X')$. So $\mu'(\varphi)$ can be approximated in the sup-norm by elements of $\mathcal{S}_{p'(L)}(X)$ which are continuous. But this means that $\mu'(\varphi)$ is continuous as well and hence it is in $\mathcal{C}_c(X')$. So μ' is continuous. □

Remark B.2.9. If the μ in the preceding proposition is faithful, then μ' is faithful as well.

Definition and Proposition B.2.10 (Restriction of a field of measures). Let μ be a continuous field of measures on Y over X with coefficient map p and let V be an open subset of Y. Define $U := p(V)$ and assume that U is open in X (which is automatic if p is open). Then $\mu|_V := (\mu_u|_V)_{u \in U}$ defines a continuous field of measures on V over U with coefficient map $p|_V$. If μ is faithful, then so is $\mu|_V$.

Proof. Since V is open in Y and U is open in X, we can embed $\mathcal{C}_c(V)$ into $\mathcal{C}_c(Y)$ and $\mathcal{C}_c(U)$ into $\mathcal{C}_c(X)$. The map $\mu \colon \mathcal{C}_c(Y) \to \mathcal{C}_c(X)$, restricted to $\mathcal{C}_c(V)$, gives a linear, continuous and positive map $\mu|_V \colon \mathcal{C}_c(V) \to \mathcal{C}_c(U)$. So $\mu|_V$ is a continuous field of measures.

Let μ now be faithful. Let $u \in U$ and $\chi \neq 0$ be a non-negative function on $p^{-1}(\{u\}) \cap V$ with compact support. Then this function can be extended by zero to a non-negative function $\tilde{\chi}$ of compact support on the whole of $p^{-1}(\{u\})$. Then $0 < \mu_u(\tilde{\chi}) = \mu_u|_V(\chi)$. So $\mu|_v$ is faithful. □

Definition B.2.11. One says that a continuous field of measures μ on Y over X has *compact support* if $\bigcup_{x \in X} \operatorname{supp} \mu_x$ is relatively compact in Y. The support of $(\mu_x)_{x \in X}$ is said to be *proper* if, for all compact $K \subseteq X$, the set $\bigcup_{x \in K} \operatorname{supp} \mu_x \subseteq Y$ is relatively compact.

Definition B.2.12. For all continuous fields of measures μ on Y over X and all $\chi \in \mathcal{C}(Y)$, $\chi \geq 0$, define

$$(\chi\mu)_x(\varphi) := \int_{y \in Y_x} \chi(y)\varphi(y) \,\mathrm{d}\mu_x(y)$$

for all $x \in X$ and $\varphi \in \mathcal{C}_c(Y)$. Note that this simply means $(\chi\mu)(\varphi) = \mu(\chi\varphi)$.

If $\chi \in \mathcal{C}(X), \chi \geq 0$, then $\chi \circ p \in \mathcal{C}(Y)$ and we define $\chi\mu := (\chi \circ p)\mu$. Note that $(\chi\mu)(\varphi) = \mu(\chi\varphi) = \chi(\mu(\varphi))$ for all $\varphi \in \mathcal{C}_c(Y)$.

Lemma B.2.13. [2] *Let μ be a continuous field of measures on Y over X with compact support. For all $\varphi \in \mathcal{C}(Y)$, define $\mu(\varphi)$ as in (B.1). Then $\mu(\varphi)$ is an element of $\mathcal{C}_c(X)$.*

Proof. Let φ be in $\mathcal{C}(Y)$. Find a function $\chi \in \mathcal{C}_c(Y)$ such that $\chi \geq 0$ and $\chi(y) = 1$ for all $y \in \bigcup_{x\in X} \operatorname{supp} \mu_x$. Then $\chi\mu = \mu$ and $\mu(\varphi) = \mu(\chi\varphi) \in \mathcal{C}_c(X)$ because $\chi\varphi \in \mathcal{C}_c(Y)$. $\square$

Lemma B.2.14. *Let μ be a continuous field of measures on Y over X with proper support. For all $\varphi \in \mathcal{C}(Y)$, define $\mu(\varphi)$ as in (B.1). Then $\mu(\varphi)$ is an element of $\mathcal{C}(X)$.*

Proof. Let φ be in $\mathcal{C}(Y)$ and let $x \in X$. We check that $\mu(\varphi)$ is continuous in x. Find a compact neighbourhood K of x and a function $\chi \in \mathcal{C}_c(X)$ such that $\chi \geq 0$ and $\chi = 1$ on K. Then $\chi\mu$ has compact support and $(\chi\mu)(\varphi)(x') = \mu(\varphi)(x')$ for all $x' \in K$. Since $(\chi\mu)(\varphi)$ is continuous in x, so is $\mu(\varphi)$. $\square$

B.3 Continuous fields of measures and fields of Banach spaces

Let X and Y be locally compact Hausdorff spaces and let $p\colon Y \to X$ be continuous. Let E be a u.s.c. field of Banach spaces over X and let $(\mu_x)_{x\in X}$ be a continuous field of measures on Y over X with coefficient map p.

Definition and Proposition B.3.1. For every section $\xi \in \Gamma_c(Y,\, p^*E)$ with compact support and every $x \in X$, the function $Y_x \to E_x,\ y \mapsto \xi(y)$, is an element of $\mathcal{C}_c(Y_x, E_x)$ so we can define

$$\mu(\xi)(x) := \int_{y\in Y_x} \xi(y)\,\mathrm{d}\mu_x(y). \tag{B.2}$$

Then $\mu(\xi)$ is an element of $\Gamma_c(X, E)$ and the function $\xi \mapsto \mu(\xi)$ is a continuous linear map from $\Gamma_c(Y,\, p^*E)$ to $\Gamma_c(X, E)$. It is $\mathcal{C}(X)$-linear in the following way: If $\psi \in \mathcal{C}(X)$ and $\xi \in \Gamma_c(Y, p^*E)$, then we can define $(\psi\xi)(y) := \psi(p(y))\xi(y)$ for all $y \in Y$. This defines a $\mathcal{C}(X)$-action on $\Gamma_c(Y, p^*E)$. Then

$$\mu(\psi\xi) = \psi\mu(\xi).$$

Proof. First we have to check that our map is well-defined. For every element ξ of $\Gamma_c(Y,\, p^*E)$, the map $\mu(\xi)$ surely is a well-defined section of E with compact support. The question is whether it is continuous. Recall that if K' is a compact subset of X, then $\Sigma_{K'}(X, E)$ denotes the space of all bounded selections of E with support contained in K'.

If K is a compact subset of Y, then $\mu(\xi) \in \Sigma_{K'}(X, E)$ for all $\xi \in \Gamma_K(Y,\, p^*E)$ where $K' := p(K)$; actually, μ defines a continuous linear map from $\Gamma_K(Y,\, p^*E)$ to $\Sigma_{K'}(X, E)$. Indeed, find a function $\chi \in \Gamma_c(Y)$ such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on K. Then for all $\xi \in \Gamma_K(Y\,, p^*E)$ and all $x \in X$, we have

$$\begin{aligned} \left\|\mu(\xi)(x)\right\|_{E_x} &= \left\|\int_{y\in Y_x} \xi(y)\,\mathrm{d}\mu_x(y)\right\|_{E_x} = \left\|\int_{y\in Y_x} \chi(y)\xi(y)\,\mathrm{d}\mu_x(y)\right\|_{E_x} \\ &\leq \int_{y\in Y_x} \chi(y)\,\mathrm{d}\mu_x(y) \sup_{y\in Y_x} \left\|\xi(y)\right\|_{E_x} \leq \mu(\chi)(x)\left\|\xi\right\|_K . \end{aligned}$$

It follows that

$$\left\|\mu(\xi)\right\|_{K'} \leq \left\|\mu(\chi)\right\|_{K'} \left\|\xi\right\|_K .$$

[2] Compare [LG99], Lemma 3.1.

Note that $\|\mu(\chi)\|_{K'} < \infty$ since $\mu(\chi)$ is in $\mathcal{C}_{K'}(X, E)$ by the continuity of μ. As μ is obviously linear on $\Gamma_K(Y, p^*E)$, it is a continuous linear map.

Let ξ be in $\Gamma_c(Y,\, p^*E)$. Now we check that $\mu(\xi)$ is a section. Let K denote the support of ξ. Find a compact neighbourhood L of K. Then we can approximate ξ by sums of sections of the form $\chi \cdot \eta$, where $\chi \in \mathcal{C}_L(Y)$ and $\eta \in \Gamma(Y,\, p^*E)$ is such that there is an $\eta' \in \Gamma(X, E)$ with $\eta(y) = \eta'(p(y))$ for all $y \in Y$ (this follows from Lemma B.1.3 applied to the set Ξ of all such sections of p^*E). We show that $\mu(\chi\eta) \in \Gamma_{p(L)}(X, E)$, and, since this is a Banach space and μ is continuous, it follows that $\mu(\xi)$ is in the closed subspace $\Gamma_{p(L)}(X, E)$ of $\Sigma_{p(L)}(X, E)$ (and hence in $\Gamma_{p(K)}(X, E)$). Now

$$\mu(\chi\eta)(x) = \int_{y\in Y_x} \chi(y)\eta(y)\,\mathrm{d}\mu_x(y) = \int_{y\in Y_x} \chi(y)\eta'(x)\,\mathrm{d}\mu_x(y) = \int_{y\in Y_x} \chi(y)\,\mathrm{d}\mu_x(y)\ \cdot\ \eta'(x),$$

for all $x \in X$, or, in other words,

$$\mu(\chi\eta) = \mu(\chi)\eta' \in \Gamma_{p(L)}(X, E).$$

Together with the continuity result derived above we now know that μ is a continuous linear map from $\Gamma_K(Y,\, p^*E)$ to $\Gamma_{p(K)}(X, E)$ for all compact $K \subseteq Y$. So μ is a continuous linear map on all of $\Gamma_c(Y,\, p * E)$ with values in $\Gamma_c(X, E)$ by the universal property of the inductive limit topology.

To see $\mathcal{C}(X)$-linearity, let $\psi \in \mathcal{C}(X)$ and $\xi \in \Gamma_c(Y,\, p^*E)$. Then

$$\mu(\psi\xi)(x) = \int_{y\in Y_x} \psi(x)\xi(y)\,\mathrm{d}\mu_x(y) = \psi(x)\int_{y\in Y_x} \xi(y)\,\mathrm{d}\mu_x(y) = (\psi\mu(\xi))(x)$$

for all $x \in X$. □

Proposition B.3.2. *If p is surjective and μ is faithful, then $\xi \mapsto \mu(\xi)$ is surjective.*

Proof. Let $\eta \in \Gamma_c(X, E)$. Find a $\chi \in \mathcal{C}_c(X)$ such that $\chi \equiv 1$ on $\operatorname{supp}\eta$. By the surjectivity of the map $\mu\colon \mathcal{C}_c(Y) \to \mathcal{C}_c(X)$ we can find a $\chi' \in \mathcal{C}_c(Y)$ such that $\mu(\chi') = \chi$. Define $\xi(y) := \chi'(y)\eta(p(y))$ for all $y \in Y$. This surely is in $\Gamma_c(Y, p^*E)$ and we obtain

$$\begin{aligned}\mu(\xi)(x) &= \int_{y\in Y_x} \xi(y)\,\mathrm{d}\mu_x(y) = \int_{y\in Y_x} \chi'(y)\eta(x)\,\mathrm{d}\mu_x(y)\\ &= \int_{y\in Y_x} \chi'(y)\,\mathrm{d}\mu_x(y)\cdot\eta(x) = \mu(\chi')(x)\eta(x) = \chi(x)\eta(x)\end{aligned}$$

for all $x \in X$. Because $\chi \equiv 1$ on $\operatorname{supp}\eta$, it follows that $\mu(\xi) = \eta$. □

The following lemmas are proved as in the scalar case (Lemma B.2.13 and Lemma B.2.14).

Lemma B.3.3. [3] *Let the continuous field of measures μ on Y over X have compact support. For all $\xi \in \Gamma(Y,\, p^*E)$, define $\mu(\xi)$ as in (B.2). Then $\mu(\xi) \in \Gamma_c(X, E)$.*

Lemma B.3.4. *Let μ have proper support. For all $\xi \in \Gamma(Y, p^*E)$, define $\mu(\xi)$ as in (B.2). Then $\mu(\xi) \in \Gamma(X, E)$.*

[3] Compare [LG99], Lemma 3.1.

Appendix C

Some Details Concerning Chapter 5

C.1 Some proofs of results of Section 5.1

C.1.1 Proof of Lemma 5.1.3

The trick is to represent $\tilde{\mu}$ as a composition of continuous maps on the sections of suitably chosen fields of Banach spaces; continuity means here continuity for the uniform convergence on compact subsets if we are talking about the spaces of all sections and convergence in the inductive limit topology if we are talking about the sections with compact support. First, observe that the map $\xi_1 \mapsto \xi_1 \circ \pi_1$ is a continuous map from $\Gamma(\mathcal{G}, r^*E_1)$ to $\Gamma(\mathcal{G}^{(2)},\ \pi_1^* r^* E_1)$. Similarly, $\xi_2 \mapsto \xi_2 \circ \pi_2$ is continuous from $\Gamma(\mathcal{G}, r^*E_2)$ to $\Gamma(\mathcal{G}^{(2)},\ \pi_2^* r^* E_2)$, where π_2 is defined analogously to π_1. Now $r \circ \pi_2 = s \circ \pi_1$ by definition of $\mathcal{G}^{(2)}$, so $\pi_2^* r^* E_2 = \pi_1^* s^* E_2$. Since α is a continuous field of linear isomorphisms from s^*E_2 to r^*E_2, the pullback $\pi_1^*\alpha$ is a continuous field of linear isomorphisms from $\pi_1^* s^* E_2$ to $\pi_1^* r^* E_2$. This defines a continuous linear map (actually, a linear homeomorphism) from $\Gamma(\mathcal{G}^{(2)},\ \pi_1^* s^* E_2)$ to $\Gamma(\mathcal{G}^{(2)},\ \pi_1^* r^* E_2)$. Now there is the canonical map

$$\Gamma\left(\mathcal{G}^{(2)},\ \pi_1^* r^* E_1\right) \times \Gamma\left(\mathcal{G}^{(2)},\ \pi_1^* r^* E_2\right) \to \Gamma\left(\mathcal{G}^{(2)},\ \pi_1^* r^* E_1 \times_{\mathcal{G}^{(2)}} \pi_1^* r^* E_2\right)$$

mapping (η_1, η_2) to $(\gamma, \gamma') \mapsto (\eta_1(\gamma), \eta_2(\gamma'))$. Since this map defines the structure of a continuous field on the product field, it is continuous and takes total subsets to total subsets, a property shared also by the other maps we have used so far. Putting this together we have constructed a continuous linear map

$$(\xi_1, \xi_2) \mapsto \left[(\gamma, \gamma') \mapsto \left(\xi_1(\pi_1(\gamma, \gamma')),\ ((\pi_1^*\alpha) \circ \xi_2 \circ \pi_2)(\gamma, \gamma')\right) = (\xi_1(\gamma),\ \alpha_\gamma \xi_2(\gamma'))\right].$$

Note that this map takes the product of two total subsets to a total subset. Since μ is a continuous field of bilinear maps, we can pull it back to a continuous field of bilinear maps $\pi_1^* r^* \mu$ from $\pi_1^* r^* E_1 \times_{\mathcal{G}^{(2)}} \pi_1^* r^* E_2$ to $\pi_1^* r^* F$. Composing this map and the map defined above gives $\tilde{\mu}$, which is therefore continuous and well-defined.

Separate continuity on the sections of compact support follows from continuity for the uniform convergence on compact subsets and the (trivial) statement about the supports given in the lemma.

If μ is non-degenerate, then so is $\pi_1^* r^* \mu$, so it takes total subsets to total subsets. Hence the composition $\tilde{\mu}$ of maps that send (products of) total subsets to total subsets does the same.

Since the continuous sections of compact support form a total subset, it follows that the Ξ defined in the lemma is total. As a consequence, the span of $\mathcal{C}_c(\mathcal{G}^{(2)})\Xi$ is dense in $\Gamma_c(\mathcal{G}^{(2)}, \pi_1^* r^* F)$. Since the multiplication between $\mathcal{C}_c(\mathcal{G}^{(2)})$ and $\Gamma_c(\mathcal{G}^{(2)}, \pi_1^* r^* F)$ is (separately) continuous, it therefore suffices to find a subset Ψ of $\mathcal{C}_c(\mathcal{G}^{(2)})$ which generates a dense subspace and such that products of elements $\psi \in \Psi$ with $\xi \in \Xi$ are again in Ξ. Such a set is given by

$$\Psi := \{(\chi_1 \circ \pi_1) \cdot (\chi_2 \circ \pi_2) : \ \chi_1, \chi_2 \in \mathcal{C}_c(\mathcal{G})\}.$$

By the definition of $\tilde{\mu}$ it follows that for all $\chi_1, \chi_2 \in \mathcal{C}_c(\mathcal{G})$, $\xi_1 \in \Gamma_c(\mathcal{G}, r^* E_1)$ and $\xi_2 \in \Gamma_c(\mathcal{G}, r^* E_2)$:

$$\tilde{\mu}(\chi_1 \xi_1, \chi_2 \xi_2) = (\chi_1 \circ \pi_1) \cdot (\chi_2 \circ \pi_2) \cdot \tilde{\mu}(\xi_1, \xi_2).$$

What is left to show is that Ψ spans a dense subset of $\mathcal{C}_c(\mathcal{G}^{(2)})$. To see this note that the algebraic tensor product $\Phi := \mathcal{C}_c(\mathcal{G}) \otimes^{\mathrm{alg}} \mathcal{C}_c(\mathcal{G})$ spans a dense subset in $\mathcal{C}_c(\mathcal{G} \times \mathcal{G})$. Furthermore, the restriction map from $\mathcal{C}_c(\mathcal{G} \times \mathcal{G})$ to $\mathcal{C}_c(\mathcal{G}^{(2)})$ is continuous and surjective by Lemma E.4.1. The image of Φ under this restriction is the span of Ψ which therefore is dense.

C.1.2 Proof of Lemma 5.3.7

In this section we prove Lemma 5.3.7 which is a generalisation of 5.1.2. The proofs of these two results are completely analogous and the proof of the generalisation is only included to make it unnecessary for the reader to puzzle out the technical details (which took me some time).

We first state a lemma analogous to Lemma 5.1.3.

Lemma C.1.1. *Let $\xi_1 \in \Gamma(\mathcal{G}_V^W, r^* E_1)$ and $\xi_2 \in \Gamma(\mathcal{G}_U^V, r^* E_2)$ be sections (with arbitrary support). Then*

$$\tilde{\mu}(\xi_1, \xi_2)(\gamma, \gamma') = \mu_{r(\gamma)}\left(\xi_1(\gamma), \ \alpha_\gamma(\xi_2(\gamma'))\right)$$

is in $\Gamma(\mathcal{G}_V^W *_{s,r} \mathcal{G}_U^V, \ \pi_1^* r^* F)$, where $\pi_1 \colon \mathcal{G}_V^W *_{s,r} \mathcal{G}_U^V \to \mathcal{G}$ is the projection onto the first coordinate. If μ is non-degenerate, then $\tilde{\mu}$ is non-degenerate in two senses: Firstly, it sends the product of two total subsets to a total subset, and secondly, the set $\Xi := \{\tilde{\mu}(\xi_1, \xi_2) : \ \xi_1 \in \Gamma_c(\mathcal{G}_V^W, r^* E_1), \xi_2 \in \Gamma_c(\mathcal{G}_U^V, r^* E_2)\}$ spans a dense subset of $\Gamma_c(\mathcal{G}_V^W *_{s,r} \mathcal{G}_U^V, \pi_1^* r^* F)$.

The proof of this lemma is almost identical to the proof of Lemma 5.1.3. We just include it here because it was rather tedious to find the right places for all the Us, Vs and Ws, and if somebody needs this result, then it might save some work to find the proof spelled out in detail.

Proof. The first step is given by the maps $\xi_1 \mapsto \xi_1 \circ \pi_1$ and $\xi_2 \mapsto \xi_2 \circ \pi_2$, where the first map starts in $\Gamma(\mathcal{G}_V^W, r^* E_1)$ and ends in $\Gamma(\mathcal{G}_V^W *_{s,r} \mathcal{G}_U^V, \pi_1^* r^* E_1)$, the second starts in $\Gamma(\mathcal{G}_U^V, r^* E_2)$ and ends in $\Gamma(\mathcal{G}_V^W *_{s,r} \mathcal{G}_U^V, \pi_2^* r^* E_2)$. Here π_1 and π_2 are the projections on the first and second coordinate on the fibre product $\mathcal{G}_V^W *_{s,r} \mathcal{G}_U^V$. By definition, these maps are linear, continuous and map total sets to total sets.

As above, $r \circ \pi_2 = s \circ \pi_1$, so $\pi_2^* r^* E_2 = \pi_1^* s^* E_2$. Since α is an isometric isomorphism from $s^* E_2$ to $r^* E_2$, we know that $\pi_1^* \alpha$ is an isometric isomorphism from $\pi_1^* s^* E_2$ to $\pi_2^* r^* E_2$ (as fields over $\mathcal{G}_V^W *_{s,r} \mathcal{G}_U^V$). This defines a continuous linear isomorphism from $\Gamma(\mathcal{G}_V^W *_{s,r} \mathcal{G}_U^V, \pi_1^* s^* E_2)$ to $\Gamma(\mathcal{G}_V^W *_{s,r} \mathcal{G}_U^V, \pi_1^* r^* E_2)$. And as above, there is a canonical map

$$\Gamma\left(\mathcal{G}_V^W *_{s,r} \mathcal{G}_U^V, \ \pi_1^* r^* E_1\right) \times \Gamma\left(\mathcal{G}_V^W *_{s,r} \mathcal{G}_U^V, \ \pi_1^* r^* E_2\right) \to \Gamma\left(\mathcal{G}_V^W *_{s,r} \mathcal{G}_U^V, \ \pi_1^* r^* E_1 \times_{\mathcal{G}_V^W *_{s,r} \mathcal{G}_U^V} \pi_1^* r^* E_2\right)$$

mapping (η_1, η_2) to $(\gamma, \gamma') \mapsto (\eta_1(\gamma), \eta_2(\gamma'))$. Since this map defines the structure of a continuous field on the product field, it (is continuous and) takes total subsets to total subsets, a property shared also by the other maps we have used so far. Putting this together we have constructed a continuous linear map

$$(\xi_1,\xi_2)\mapsto\left[(\gamma,\gamma')\mapsto\left(\xi_1(\pi_1(\gamma,\gamma')),\ ((\pi_1^*\alpha)\circ\xi_2\circ\pi_2)(\gamma,\gamma')\right)=(\xi_1(\gamma),\ \alpha_\gamma\xi_2(\gamma'))\right].$$

Note that this map takes the product of two total subsets to a total subset.

Since μ is a continuous field of bilinear maps, we can pull it back to a continuous field of bilinear maps $\pi_1^*r^*\mu$ from $\pi_1^*r^*E_1\times_{\mathcal{G}_V^W *_{s,r}\mathcal{G}_U^V}\pi_1^*r^*E_2$ to $\pi_1^*r^*F$. Composing this map and the map defined above gives $\tilde{\mu}$, which is therefore (continuous and) well-defined.

If μ is non-degenerate, then so is $\pi_1^*r^*\mu$, so it takes total subsets to total subsets. Hence, the composition $\tilde{\mu}$ of maps that send (products of) total subsets to total subsets does the same.

Since the sections of compact support form a total subset, it follows that the Ξ defined above is total. As a consequence, the span of $\mathcal{C}_c(\mathcal{G}_V^W *_{s,r}\mathcal{G}_U^V)\Xi$ is dense in $\Gamma_c(\mathcal{G}_V^W *_{s,r}\mathcal{G}_U^V,\pi_1^*r^*F)$. Since the multiplication between $\mathcal{C}_c(\mathcal{G}_V^W *_{s,r}\mathcal{G}_U^V)$ and $\Gamma_c(\mathcal{G}_V^W *_{s,r}\mathcal{G}_U^V,\pi_1^*r^*F)$ is (separately) continuous, it therefore suffices to find a subset Ψ of $\mathcal{C}_c(\mathcal{G}_V^W *_{s,r}\mathcal{G}_U^V)$ which generates a dense subspace and such that products of elements $\psi\in\Psi$ with $\xi\in\Xi$ are again in Ξ. Such a set is given by

$$\Psi:=\{(\chi_1\circ\pi_1)\cdot(\chi_2\circ\pi_2):\ \chi_1\in\mathcal{C}_c(\mathcal{G}_V^W),\chi_2\in\mathcal{C}_c(\mathcal{G}_U^V)\}.$$

By the definition of $\tilde{\mu}$ it follows that for all $\chi_1\in\mathcal{C}_c(\mathcal{G}_V^W)$, $\chi_2\in\mathcal{C}_c(\mathcal{G}_U^V)$, $\xi_1\in\Gamma_c(\mathcal{G}_V^W,r^*E_1)$ and $\xi_2\in\Gamma_c(\mathcal{G}_U^V,r^*E_2)$:

$$\tilde{\mu}(\chi_1\xi_1,\chi_2\xi_2)=(\chi_1\circ\pi_1)\cdot(\chi_2\circ\pi_2)\cdot\tilde{\mu}(\xi_1,\xi_2).$$

What is left to show is that Ψ spans a dense subset of $\mathcal{C}_c(\mathcal{G}_V^W *_{s,r}\mathcal{G}_U^V)$. To see this note that the algebraic tensor product $\Phi:=\mathcal{C}_c(\mathcal{G}_V^W)\otimes^{\mathrm{alg}}\mathcal{C}_c(\mathcal{G}_U^V)$ spans a dense subset in $\mathcal{C}_c(\mathcal{G}_V^W\times\mathcal{G}_U^V)$. Furthermore, the restriction map from $\mathcal{C}_c(\mathcal{G}_V^W\times\mathcal{G}_U^W)$ to $\mathcal{C}_c(\mathcal{G}_V^W *_{s,r}\mathcal{G}_U^V)$ is continuous and surjective. The image of Φ under this restriction is the span of Ψ which therefore is dense. □

Now we can proceed with the Proof of 5.3.7. Again, this is just a variant of the proof of the special case $U=V=W=\mathcal{G}$ that has been discussed above.

First define the map $\Phi\colon\mathcal{G}_V^W *_{r,r}\mathcal{G}_U^W\to\mathcal{G}_V^W *_{s,r}\mathcal{G}_U^V$, $(\gamma,\gamma')\mapsto(\gamma,\gamma^{-1}\gamma')$. This is a homeomorphism. Let p_1 and p_2 denote the projections of $\mathcal{G}_V^W *_{r,r}\mathcal{G}_U^W$ onto the first and second component, respectively. Then $\pi_1\circ\Phi=p_1$, and we have $\Phi^*(\pi_1^*r^*F)=p_1^*r^*(F)=p_2^*r^*F$. The map Φ therefore induces an isomorphism

$$\Gamma_c\left(\mathcal{G}_V^W *_{s,r}\mathcal{G}_U^V,\ \pi_1^*r^*F\right)\to\Gamma_c\left(\mathcal{G}_V^W *_{r,r}\mathcal{G}_U^W,\ p_2^*r^*F\right)$$

which sends some η to $(\gamma,\gamma')\mapsto\eta(\gamma,\gamma^{-1}\gamma')$. In particular, it sends our $\tilde{\mu}(\xi_1,\xi_2)$ to

$$(\gamma,\gamma')\mapsto\mu_{r(\gamma)}\left(\xi_1(\gamma),\ \alpha_\gamma\left(\xi_2(\gamma^{-1}\gamma')\right)\right).$$

Note that this is the integrand in the convolution formula and a section of compact support.

Now we define a suitable continuous field of measures on $\mathcal{G}_V^W *_{r,r}\mathcal{G}_U^W$. Consider the map $p_2\colon\mathcal{G}_V^W *_{r,r}\mathcal{G}_U^W\to\mathcal{G}_U^W$. It is surjective since $r\colon\mathcal{G}_V^W\to W$ is surjective. Its fibres are of the form $p_2^{-1}(\{\gamma'\})=\{(\gamma,\gamma'):\ \gamma\in\mathcal{G}_V^W, r(\gamma)=r(\gamma')\}$ for each $\gamma'\in\mathcal{G}_U^W$. These fibres are homeomorphic to $\mathcal{G}_V^{r(\gamma')}\subseteq\mathcal{G}_V^W\subseteq\mathcal{G}$. If, for each $w\in W$, we restrict the measure λ^w on $\mathcal{G}$ to the open set $\mathcal{G}_V^W$, then we get a faithful continuous field $\lambda':=\lambda|_{\mathcal{G}_V^W}$ of measures on $\mathcal{G}_V^W$ over W with coefficient map r (see Proposition B.2.10). So we can put the measure $\lambda'^{r(\gamma')}$ on the fibre $p_2^{-1}(\{\gamma'\})$. Technically, we are forming the pullback $r^*\lambda'$ by r of the continuous field of measures λ' on $\mathcal{G}_V^W$ with coefficient map r (compare Definition B.2.8):

$$\begin{tikzcd}
\left(\mathcal{G}_V^W,\ \lambda'\right) \arrow[d, "r"] & & \left(\mathcal{G}_V^W *_{r,r} \mathcal{G}_U^W,\ r^*\lambda'\right) \arrow[ll, "p_1"'] \arrow[d, "p_2"] \\
W & & \mathcal{G}_U^W \arrow[ll, "r"]
\end{tikzcd}$$

By Proposition B.3.1 we can deduce that $r^*\lambda'$ maps $\Gamma_c\left(\mathcal{G}_V^W *_{r,r} \mathcal{G}_U^W,\ p_2^* r^* F\right)$ to $\Gamma_c\left(\mathcal{G}_U^W,\ r^*F\right)$, and this map is onto since λ' is faithful and so is $r^*\lambda'$.

The composition of the three maps $\tilde{\mu}$, the isomorphism induced by Φ, and $r^*\lambda'$ is our convolution product $(\xi_1, \xi_2) \mapsto \mu(\xi_1, \xi_2)$, which is therefore (well-defined, separately continuous and) non-degenerate if μ is non-degenerate.

C.2 The convolution with fields of compact operators

Let E and F be u.s.c. fields of Banach spaces over some topological space. Then we write $\mathrm{L}(E,F)_c$ for those continuous fields of linear maps that have compact support. In the same spirit we use the notation $\mathrm{L}_B(E,F)_c$ and $\mathrm{K}_B(E,F)_c$. Let $\mathcal{A}(\mathcal{G})$ be an unconditional completion of $\mathcal{C}_c(\mathcal{G})$.

C.2.1 The convolution with fields of linear maps

Definition and Proposition C.2.1. Let E and F be $\mathcal{G}$-Banach spaces. Let the field of operators $S = (S_\gamma)_{\gamma\in\mathcal{G}} \in \mathrm{L}(r^*E,^* F)_c$ have compact support. For all $\xi \in \Gamma_c(\mathcal{G},\ r^*E)$, define

$$(S * \xi)(\gamma) := \int_{\mathcal{G}^{r(\gamma)}} S_{\gamma'}\ \gamma'\xi(\gamma'^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')$$

and

$$(\xi * S)(\gamma) := \int_{\mathcal{G}^{r(\gamma)}} \xi(\gamma')\ \gamma' S_{\gamma'^{-1}\gamma}\,\mathrm{d}\lambda^{r(\gamma)}(\gamma') = \int_{\mathcal{G}^{r(\gamma)}} \gamma'\left[S_{\gamma'^{-1}\gamma}\left(\gamma'^{-1}\xi(\gamma')\right)\right]\,\mathrm{d}\lambda^{r(\gamma)}(\gamma')$$

for all $\gamma \in \mathcal{G}$. Then $S * \xi,\ \xi * S \in \Gamma_c(\mathcal{G},\ r^*F)$ and the maps $\xi \mapsto S * \xi$ and $\xi \mapsto \xi * S$ are linear, $\mathcal{C}_0(X/\mathcal{G})$-linear, and continuous with respect to the inductive limit topologies. We have

$$\|S * \xi\|_{\mathcal{A}(\mathcal{G},F)} \le \left\|\gamma \mapsto \|S_\gamma\|\right\|_{\mathcal{A}} \|\xi\|_{\mathcal{A}(\mathcal{G},E)}$$

and

$$\|\xi * S\|_{\mathcal{A}(\mathcal{G},F)} \le \|\xi\|_{\mathcal{A}(\mathcal{G},E)} \left\|\gamma \mapsto \|S_\gamma\|\right\|_{\mathcal{A}}.$$

In particular, $\xi \mapsto S * \xi$ and $\xi \mapsto \xi * S$ extend to linear and $\mathcal{C}_0(X/\mathcal{G})$-linear continuous maps from $\mathcal{A}(\mathcal{G}, E)$ to $\mathcal{A}(\mathcal{G}, F)$ (being also $\mathcal{C}_0(X)$-linear from the right and from the left, respectively).

Proof. Let us only consider $S * \xi$, the arguments for $\xi * S$ being very similar.

Let $\mathcal{G} *_{r,r} \mathcal{G}$ denote the space $\{(\gamma, \gamma') \in \mathcal{G}\times\mathcal{G} :\ r(\gamma) = r(\gamma')\}$ and let $\pi_i \colon \mathcal{G} *_{r,r} \mathcal{G} \to \mathcal{G}$ denote the projection onto the ith component. Then the map $(\gamma, \gamma') \mapsto \xi(\gamma'^{-1}\gamma)$ is in $\Gamma(\mathcal{G} *_{r,r} \mathcal{G},\ \pi_2^* s^* E)$. If we write α for the $\mathcal{G}$-action on E, then $\pi_2^*\alpha$ sends $\Gamma(\mathcal{G} *_{r,r} \mathcal{G},\ \pi_2^* s^* E)$ to $\Gamma(\mathcal{G} *_{r,r} \mathcal{G},\ \pi_2^* r^* E)$, so the map $(\gamma, \gamma') \mapsto \gamma'\xi(\gamma'^{-1}\gamma)$ is in $\Gamma(\mathcal{G} *_{r,r} \mathcal{G},\ \pi_2^* r^* E)$. Thirdly, the map $\pi_2^* S$ sends $\Gamma(\mathcal{G} *_{r,r} \mathcal{G},\ \pi_2^* r^* E)$ to

$\Gamma(\mathcal{G} *_{r,r} \mathcal{G},\ \pi_2^* r^* F)$, so $(\gamma, \gamma') \mapsto S_{\gamma'} \gamma' \xi(\gamma'^{-1}\gamma)$ is in $\Gamma(\mathcal{G} *_{r,r} \mathcal{G},\ \pi_2^* r^* F)$. More precisely, it is in $\Gamma_c(\mathcal{G} *_{r,r} \mathcal{G},\ \pi_2^* r^* F)$. The map which sends ξ to this element of $\Gamma_c(\mathcal{G} *_{r,r} \mathcal{G},\ \pi_2^* r^* F)$ is continuous for the inductive limit topology. Note that $r \circ \pi_2 = r \circ \pi_1$, so $\pi_2^* r^* F = \pi_1^* r^* F$.

Now the integral sends $\Gamma_c(\mathcal{G} *_{r,r} \mathcal{G},\ \pi_1^* r^* F)$ to $\Gamma_c(\mathcal{G},\ r^* F)$ and is continuous for the inductive limit topology, so the map $\xi \mapsto S * \xi$ is well-defined and continuous.

The proof of the inequalities for the norm is a variant of the proof of 5.2.6. Note that the map $\gamma \mapsto \|S_\gamma\|$ is in general neither continuous nor upper semi-continuous. However, it is locally bounded (by definition of a continuous field of operators) and has compact support. □

Proposition C.2.2. *Let B be a $\mathcal{G}$-Banach algebra and let E and F be right $\mathcal{G}$-Banach B-modules. Let S be in $\mathrm{L}_{r^*B}(r^*E, r^*F)_c$. Then the map $\xi \mapsto S * \xi$ from $\Gamma_c(\mathcal{G},\ r^*E)$ to $\Gamma_c(\mathcal{G},\ r^*F)$ is $\Gamma_c(\mathcal{G},\ r^*B)$-linear on the right. Hence the map $\xi \mapsto S * \xi$ from $\mathcal{A}(\mathcal{G},\ E)$ to $\mathcal{A}(\mathcal{G},\ F)$ is $\mathcal{A}(\mathcal{G},\ B)$-linear on the right. A similar statement is true for left modules and the map $\xi \mapsto \xi * S$.*

Proof. The assertion is proved just as the associativity of the convolution. □

Equipped with this knowledge, we can now analyse Definition 5.2.17: The equation $\langle \eta^<,\ S^> * \xi^> \rangle = \langle \eta^< * S^<,\ \xi^> \rangle$ that appears in the definition can again be proved similarly to the associativity of the convolution. This equation implies that the operator $\hat{S}$ that is defined in 5.2.17 is an element of $\mathrm{L}_{\mathcal{A}(\mathcal{G},B)}(\mathcal{A}(\mathcal{G}, E),\ \mathcal{A}(\mathcal{G}, F))$.

C.2.2 Fields of compact operators

Let B be a $\mathcal{G}$-Banach algebra and let E and F be $\mathcal{G}$-Banach B-pairs. For all $\eta^> \in \Gamma_c(\mathcal{G},\ r^*F^>)$ and all $\xi^< \in \Gamma_c(\mathcal{G},\ r^*E^<)$, define

$$\eta^> \bowtie \xi^< := \left(\int_{\mathcal{G}^{r(\gamma)}} \left| \eta^>(\gamma') \right\rangle \left\langle \gamma' \xi^<(\gamma'^{-1}\gamma) \right| \mathrm{d}\lambda^{r(\gamma)}(\gamma') \right)_{\gamma \in \mathcal{G}}.$$

Then $\eta^> \bowtie \xi^<$ is in $\mathrm{K}_{r^*B}(r^*E,\ r^*F)_c$ by 7.2.3, where the subscript c means that we are only considering those fields of operators which have compact support. Direct calculation yields

$$\widehat{\eta^> \bowtie \xi^<} = \left|\eta^>\right\rangle\left\langle\xi^<\right| \in \mathrm{K}_{\mathcal{A}(\mathcal{G},B)}(\mathcal{A}(\mathcal{G}, E),\ \mathcal{A}(\mathcal{G}, F)).$$

The fields of compact operators of the form $\eta^> \bowtie \xi^<$ span a dense subspace of $\mathrm{K}_{r^*B}(r^*E,\ r^*F)_c$ and the map $(\eta^>, \xi^<) \mapsto \eta^> \bowtie \xi^<$ is continuous for the inductive limit topology. On $\mathrm{L}_{r^*B}(r^*E,\ r^*F)_c$ we can define the semi-norm

$$\|S\|_{\mathcal{A}} := \left\| \gamma \mapsto \|S_\gamma\| \right\|_{\mathcal{A}}.$$

We have already seen that the map $S \mapsto \hat{S}$ is contractive for this norm. Because a dense subset of $\mathrm{K}_{r^*B}(r^*E,\ r^*F)_c$ (dense for the inductive limit topology and hence dense for the norm) is mapped to $\mathrm{K}_{\mathcal{A}(\mathcal{G},B)}(\mathcal{A}(\mathcal{G}, E),\ \mathcal{A}(\mathcal{G}, F))$, it follows that all of $\mathrm{K}_{r^*B}(r^*E,\ r^*F)_c$ is mapped into this closed subset. We can summarise this as follows:

Proposition C.2.3. *Let B be a $\mathcal{G}$-Banach algebra and let E and F be $\mathcal{G}$-Banach B-pairs. Let $\mathcal{A}(\mathcal{G})$ be an unconditional completion of $\mathcal{C}_c(\mathcal{G})$. If S is an element of $\mathrm{K}_{r^*B}(r^*E,\ r^*F)_c$, then $\hat{S}$ is compact, i.e., we have*

$$\hat{S} \in \mathrm{K}_{\mathcal{A}(\mathcal{G},B)}(\mathcal{A}(\mathcal{G}, E),\ \mathcal{A}(\mathcal{G}, F)).$$

C.2.3 The proof of Lemma 5.3.1

This Lemma can be proved by a careful revision of the argumentation in the previous paragraph: If $\mathcal{H}$ is an open subgroupoid of $\mathcal{G}$ and $\eta^>$ and $\xi^<$ have their support in $\mathcal{H}$, then also $\eta^> \bowtie \xi^<$ has its support in $\mathcal{H}$. Conversely, if $S \in \mathrm{K}_{r^*B}\left(r^*E, r^*F\right)_c$ has its support in $\mathcal{H}$, then we can choose summands of an approximation in the inductive limit topology to be of the form $\eta^> \bowtie \xi^<$ with $\eta^>$ and $\xi^<$ having their support in $\mathcal{H}$. This shows Lemma 5.3.1.

C.3 Some details concerning unconditional completions (Section 5.2)

C.3.1 The $\mathcal{A}(\mathcal{G})$-bimodule structure of $\mathcal{A}(\mathcal{G}, E)$

If E is a $\mathcal{G}$-Banach space, then E is a Banach $\mathbb{C}_X$-bimodule, where $\mathbb{C}_X$ denotes the constant field of Banach algebras over X with fibre $\mathbb{C}$, carrying the canonical $\mathcal{G}$-action. It follows that $\mathcal{A}(\mathcal{G}, E)$ is a $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G})$-bimodule. Because E is $\mathbb{C}_X$-non-degenerate, it follows that $\mathcal{A}(\mathcal{G}, E)$ is also non-degenerate, both as a left and a right $\mathcal{A}(\mathcal{G})$-Banach module.

If $T\colon E \to F$ is a $\mathcal{G}$-equivariant bounded continuous field of linear maps, then $\mathcal{A}(\mathcal{G}, T)$ is $\mathcal{A}(\mathcal{G})$-linear, both on the left and on the right. Similarly, if E_1, E_2 and F are $\mathcal{G}$-Banach spaces and $\mu\colon E_1 \times_X E_2 \to F$ is a $\mathcal{G}$-equivariant bounded continuous field of bilinear maps, then $\mathcal{A}(\mathcal{G}, \mu)$ is $\mathcal{A}(\mathcal{G})$-linear on the left in the first component, $\mathcal{A}(\mathcal{G})$-linear on the right in the second component and $\mathcal{A}(\mathcal{G})$-balanced.

The assignment $E \mapsto \mathcal{A}(\mathcal{G}, E)$ defines a functor from the category of $\mathcal{G}$-Banach spaces to the category of $\mathcal{C}_0(X/\mathcal{G})$-Banach $\mathcal{A}(\mathcal{G})$-bimodules.

If B is a $\mathcal{G}$-Banach algebra, then the multiplication on the $\mathcal{C}_0(X/\mathcal{G})$-Banach algebra $\mathcal{A}(\mathcal{G}, B)$ and the $\mathcal{C}_0(X/\mathcal{G})$-structure are compatible with the $\mathcal{A}(\mathcal{G})$-bimodule structure. Similar statements are true for $\mathcal{G}$-Banach modules and $\mathcal{G}$-Banach bimodules.

C.3.2 The descent, sums and tensor products

Proposition C.3.1. *Let E and F be $\mathcal{G}$-Banach spaces. Then there is a canonical bijective $\mathcal{C}_0(X/\mathcal{G})$-linear map*

$$\mathfrak{s}^{\mathcal{A}}_{E,F} := \mathfrak{s}\colon \mathcal{A}(\mathcal{G}, E) \oplus \mathcal{A}(\mathcal{G}, F) \to \mathcal{A}\left(\mathcal{G},\ E \oplus_X F\right)$$

such that $\|\mathfrak{s}\| \le 1$ and $\left\|\mathfrak{s}^{-1}\right\| \le 2$ and respecting the $\mathcal{A}(\mathcal{G})$-bimodule structures.

Proof. We define $\mathfrak{s} = \mathfrak{s}^{\mathcal{A}}_{E,F}$ on a dense subset: Let $\xi \in \Gamma_c\left(\mathcal{G},\ r^*E\right)$ and let $\eta \in \Gamma_c\left(\mathcal{G},\ r^*F\right)$. Then (ξ, η) can be regarded as an element of $\mathcal{A}(\mathcal{G}, E) \oplus \mathcal{A}(\mathcal{G}, F)$, whereas $\mathfrak{s}(\xi, \eta) := \gamma \mapsto (\xi(\gamma), \eta(\gamma))$ is an element of $\mathcal{A}\left(\mathcal{G},\ E \oplus_X F\right)$. We have

$$\begin{aligned} \|\mathfrak{s}(\xi,\eta)\|_{\mathcal{A}(\mathcal{G},E\oplus F)} &= \Big\|\ \gamma \mapsto \|\xi(\gamma)\| + \|\eta(\gamma)\|\ \Big\|_{\mathcal{A}} = \Big\|\ |\xi| + |\eta|\ \Big\|_{\mathcal{A}} \\ &\le \Big\|\ |\xi|\ \Big\|_{\mathcal{A}} + \Big\|\ |\eta|\ \Big\|_{\mathcal{A}} = \|(\xi,\eta)\|_{\mathcal{A}(\mathcal{G},E)\oplus\mathcal{A}(\mathcal{G},F)}\,. \end{aligned}$$

So $\mathfrak{s}$ is a $\mathcal{C}_0(X/\mathcal{G})$-linear map with norm $\|\mathfrak{s}\| \le 1$ on $\Gamma_c\left(\mathcal{G},\ r^*E\right) \oplus \Gamma_c\left(\mathcal{G},\ r^*F\right)$. Hence it extends to a $\mathcal{C}_0(X/\mathcal{G})$-linear map of norm less than or equal to 1 on the completion $\mathcal{A}(\mathcal{G}, E) \oplus \mathcal{A}(\mathcal{G}, F)$.

For the above ξ and η, we have $|\xi| \le |\mathfrak{s}(\xi, \eta)|$ and $|\eta| \le |\mathfrak{s}(\xi, \eta)|$. It follows from the properties of the unconditional norm that $\|\xi\|_{\mathcal{A}(\mathcal{G},E)} \le \|\mathfrak{s}(\xi,\eta)\|_{\mathcal{A}(\mathcal{G},E\oplus F)}$ and the same is true for the norm of η. This yields

$$\|(\xi,\eta)\|_{\mathcal{A}(\mathcal{G},E)\oplus\mathcal{A}(\mathcal{G},F)} = \|\xi\|_{\mathcal{A}(\mathcal{G},E)} + \|\eta\|_{\mathcal{A}(\mathcal{G},F)} \le 2\,\|\mathfrak{s}(\xi,\eta)\|_{\mathcal{A}(\mathcal{G},E\oplus F)}\,,$$

which shows that $\mathfrak{s}^{-1}$ is continuous with norm $\|\mathfrak{s}^{-1}\| \le 2$. $\square$

The preceding proposition remains true if E and F are not only $\mathcal{G}$-Banach spaces but $\mathcal{G}$-Banach B-modules over some $\mathcal{G}$-Banach algebra B. In this case, Ψ is also $\mathcal{A}(\mathcal{G}, B)$-linear.

Definition and Proposition C.3.2. Let E and F be $\mathcal{G}$-Banach spaces. Then there is a unique contractive linear map

$$\mathfrak{m}^{\mathcal{A}}_{E,F} \colon \mathcal{A}(\mathcal{G}, E) \otimes^{\mathcal{C}_0(X/\mathcal{G})}_{\mathcal{A}(\mathcal{G})} \mathcal{A}(\mathcal{G}, F) \to \mathcal{A}\left(\mathcal{G},\ E \otimes_X F\right)$$

such that

$$\left(\mathfrak{m}^{\mathcal{A}}_{E,F}\left(\xi \otimes \eta\right)\right)(\gamma') := \int_{\mathcal{G}^{r(\gamma')}} \xi(\gamma) \otimes \gamma\, \eta(\gamma^{-1}\gamma')\, \mathrm{d}\lambda^{r(\gamma')}(\gamma) \tag{C.1}$$

for all $\xi \in \Gamma_c(\mathcal{G}, r^*E)$, $\eta \in \Gamma_c(\mathcal{G}, r^*F)$ and $\gamma' \in \mathcal{G}$. The map $\mathfrak{m}^{\mathcal{A}}_{E,F}$ is $\mathcal{C}_0(X/\mathcal{G})$-linear, $\mathcal{A}(\mathcal{G})$-linear on the left and on the right and has dense image.

Proof. Note that $\otimes \colon E \times_X F \to E \otimes_X F$ is a $\mathcal{G}$-equivariant contractive continuous field of bilinear maps. It therefore gives a contractive $\mathcal{C}_0(X/\mathcal{G})$-linear and $\mathcal{A}(\mathcal{G})$-balanced map $\mathcal{A}(\mathcal{G}, \otimes) \colon \mathcal{A}(\mathcal{G}, E) \times \mathcal{A}(\mathcal{G}, F) \to \mathcal{A}\left(\mathcal{G}, E \otimes_X F\right)$. The linearisation of $\mathcal{A}(\mathcal{G}, \otimes)$ is the map $\mathfrak{m}^{\mathcal{A}}_{E,F}$. Because $\otimes$ is non-degenerate so is $\mathcal{A}(\mathcal{G}, \otimes)$. Hence also $\mathfrak{m}^{\mathcal{A}}_{E,F}$ is non-degenerate. $\square$

In exactly the same way one proves:

Definition and Proposition C.3.3. If E_B and ${}_BF$ are $\mathcal{G}$-Banach B-modules over some $\mathcal{G}$-Banach algebra B, then there is a unique contractive linear map

$$\mathfrak{m}^{\mathcal{A}}_{E,F} \colon \mathcal{A}(\mathcal{G}, E) \otimes^{\mathcal{C}_0(X/\mathcal{G})}_{\mathcal{A}(\mathcal{G},B)\oplus\mathcal{A}(\mathcal{G})} \mathcal{A}(\mathcal{G}, F) \to \mathcal{A}\left(\mathcal{G},\ E \otimes_B F\right)$$

such that

$$\left(\mathfrak{m}^{\mathcal{A}}_{E,F}\left(\xi \otimes \eta\right)\right)(\gamma') := \int_{\mathcal{G}^{r(\gamma')}} \xi(\gamma) \otimes \gamma\, \eta(\gamma^{-1}\gamma')\, \mathrm{d}\lambda^{r(\gamma')}(\gamma) \tag{C.2}$$

for all $\xi \in \Gamma_c(\mathcal{G}, r^*E)$, $\eta \in \Gamma_c(\mathcal{G}, r^*F)$ and $\gamma' \in \mathcal{G}$. The map $\mathfrak{m}^{\mathcal{A}}_{E,F}$ is $\mathcal{C}_0(X/\mathcal{G})$-linear, $\mathcal{A}(\mathcal{G})$-linear on the left and on the right and has dense image. If F is not only a left $\mathcal{G}$-Banach B-module but a $\mathcal{G}$-Banach B-C-bimodule, where C is another $\mathcal{G}$-Banach algebra, then $\mathfrak{m}^{\mathcal{A}}_{E,F}$ is $\mathcal{A}(\mathcal{G}, C)$-linear on the right (and similarly on the left-hand side).

Note that $\mathcal{A}(\mathcal{G}, E)$ and $\mathcal{A}(\mathcal{G}, F)$ are $\mathcal{A}(\mathcal{G}, B)$-non-degenerate if E and F are B-non-degenerate, respectively. It follows that the $\mathcal{A}(\mathcal{G}, B)$-balanced tensor product is automatically $\mathcal{A}(\mathcal{G})$-balanced and $\mathcal{C}_0(X/\mathcal{G})$-balanced if either E or F is B-non-degenerate, i.e.,

$$\mathcal{A}(\mathcal{G}, E) \otimes^{\mathcal{C}_0(X/\mathcal{G})}_{\mathcal{A}(\mathcal{G},B)\oplus\mathcal{A}(\mathcal{G})} \mathcal{A}(\mathcal{G}, F) = \mathcal{A}(\mathcal{G}, E) \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{A}(\mathcal{G}, F).$$

C.3.3 Some proofs concerning Section 5.2.8

Proof of Lemma 5.2.20. Let (E_1, T_1) and (E_2, T_2) be elements of $\mathbb{E}^{\mathrm{ban}}_{\mathcal{G}}(A, B)$, where A and B are $\mathcal{G}$-Banach algebras. Then there is a canonical $\mathcal{C}_0(X/\mathcal{G})$-linear homomorphism

$$\mathfrak{s} := \mathfrak{s}^{\mathcal{A}}_{E_1,E_2} \colon \mathcal{A}(\mathcal{G}, E_1) \oplus \mathcal{A}(\mathcal{G}, E_2) \to \mathcal{A}\left(\mathcal{G},\ E_1 \oplus_X E_2\right)$$

which is bijective such that the inverse maps on the left- and right-hand side both have norm ≤ 2 (compare Proposition C.3.1). Moreover, this homomorphism clearly respects the grading and the operators of the cycles, so it is a morphism of $\mathrm{KK}^{\mathrm{ban}}$-cycles. Such a morphism certainly induces a homotopy. $\square$

Proof of Lemma 5.2.21. On the one hand we have

$$\mathcal{A}\left(\mathcal{G},\psi\right)^{*}\left(E\right)=\mathcal{A}(\mathcal{G},E)\otimes_{\mathcal{A}(\mathcal{G},B)}^{\mathcal{C}_0(X/\mathcal{G})}\left(\mathcal{A}(\mathcal{G},C)\oplus\mathcal{C}_0(X/\mathcal{G})\right);$$

on the other hand

$$\mathcal{A}\left(\mathcal{G},\ \psi^{*}(E)\right)=\mathcal{A}\left(\mathcal{G},\ E\otimes_{B\oplus_X\mathbb{C}_X}\left(C\oplus_X\mathbb{C}_X\right)\right).$$

There is a canonical $\mathcal{C}_0(X/\mathcal{G})$-linear concurrent homomorphism

$$\mathfrak{m}^{\mathcal{A}}_{E,C\oplus_X\mathbb{C}_X}\colon\mathcal{A}(\mathcal{G},E)\otimes_{\mathcal{A}(\mathcal{G},B\oplus_X\mathbb{C}_X)}^{\mathcal{C}_0(X/\mathcal{G})}\mathcal{A}\left(\mathcal{G},\ C\oplus_X\mathbb{C}_X\right)\to\mathcal{A}\left(\mathcal{G},\ E\otimes_{B\oplus_X\mathbb{C}_X}\left(C\oplus_X\mathbb{C}_X\right)\right)$$

with coefficient maps $\mathrm{Id}_{\mathcal{A}(\mathcal{G},A)}$ and $\mathrm{Id}_{\mathcal{A}(\mathcal{G},C)}$. Note that non-degeneracy of the involved modules implies

$$\mathcal{A}(\mathcal{G},E)\otimes_{\mathcal{A}(\mathcal{G},B\oplus_X\mathbb{C}_X)}^{\mathcal{C}_0(X/\mathcal{G})}\mathcal{A}\left(\mathcal{G},\ C\oplus_X\mathbb{C}_X\right)=\mathcal{A}(\mathcal{G},E)\otimes_{\mathcal{A}(\mathcal{G},B)}\mathcal{A}\left(\mathcal{G},\ C\oplus_X\mathbb{C}_X\right).$$

There is a canonical homomorphism of $\mathcal{C}(X/\mathcal{G})$-Banach algebras

$$\mathfrak{s}^{\mathcal{A}}_{C,\mathbb{C}_X}\colon\mathcal{A}(\mathcal{G},C)\oplus\mathcal{A}(\mathcal{G})\to\mathcal{A}\left(\mathcal{G},\ C\oplus_X\mathbb{C}_X\right),$$

where the multiplication in the first algebra is defined as $(c,f)(c',f'):=(cc'+cf'+fc',ff')$ for all $c,c'\in\mathcal{A}(\mathcal{G},C)$ and $f,f'\in\mathcal{A}(\mathcal{G})$. It induces a canonical concurrent $\mathcal{C}_0(X/\mathcal{G})$-linear homomorphism

$$\mathcal{A}(\mathcal{G},E)\otimes_{\mathcal{A}(\mathcal{G},B)}\left(\mathcal{A}\left(\mathcal{G},C\right)\oplus\mathcal{A}(\mathcal{G})\right)\to\mathcal{A}(\mathcal{G},E)\otimes_{\mathcal{A}(\mathcal{G},B)}\mathcal{A}\left(\mathcal{G},\ C\oplus_X\mathbb{C}_X\right).$$

Now there are canonical concurrent $\mathcal{C}_0(X/\mathcal{G})$-linear homomorphisms

$$\mathcal{A}(\mathcal{G},E)\otimes_{\mathcal{A}(\mathcal{G},B)}\mathcal{A}\left(\mathcal{G},C\right)\to\mathcal{A}(\mathcal{G},E)\otimes_{\mathcal{A}(\mathcal{G},B)}\left(\mathcal{A}\left(\mathcal{G},C\right)\oplus\mathcal{A}(\mathcal{G})\right)$$

and

$$\mathcal{A}(\mathcal{G},E)\otimes_{\mathcal{A}(\mathcal{G},B)}\mathcal{A}\left(\mathcal{G},C\right)\to\mathcal{A}(\mathcal{G},E)\otimes_{\mathcal{A}(\mathcal{G},B)}\left(\mathcal{A}\left(\mathcal{G},C\right)\oplus\mathcal{C}_0(X/\mathcal{G})\right).$$

So we have connected $\mathcal{A}\left(\mathcal{G},\psi\right)^{*}\left(E\right)$ and $\mathcal{A}\left(\mathcal{G},\ \psi^{*}(E)\right)$ through a sequence of (inverses) of canonical $\mathcal{C}_0(X/\mathcal{G})$-linear concurrent homomorphisms having coefficient maps $\mathrm{Id}_{\mathcal{A}(\mathcal{G},A)}$ and $\mathrm{Id}_{\mathcal{A}(\mathcal{G},C)}$. Straightforward calculations show that these homomorphisms can be regarded as morphisms of $\mathrm{KK}^{\mathrm{ban}}$-cycles (if we take the canonical choices of operators on the above pairs). Moreover, all these morphisms give rise to homotopies. So $\mathcal{A}\left(\mathcal{G},\psi\right)_{*}\left(j_{\mathcal{A}}\left(E,T\right)\right)$ and $j_{\mathcal{A}}\left(\psi_{*}\left(E,T\right)\right)$ are homotopic. □

Proof of Lemma 5.2.22. We have to check that $\|\phi_B(\beta)\|\leq\|\beta\|$ for all $\beta\in\Gamma_c(\mathcal{G},\ r^{*}B[0,1])$. The first term is by definition

$$\|\beta\|=\left\|\ \gamma\mapsto\sup_{t\in[0,1]}\|\beta(\gamma)(t)\|\ \right\|_{\mathcal{A}}.$$

The second term is

$$\|\phi_B(\beta)\|=\sup_{t\in[0,1]}\left\|\ \gamma\mapsto\|\beta(\gamma)(t)\|\ \right\|_{\mathcal{A}}.$$

From the properties of the unconditional norm and the fact that for all $t\in[0,1]$ and all $\gamma\in\mathcal{G}$ we have

$$\|\beta(\gamma)(t)\|\leq\sup_{s\in[0,1]}\|\beta(\gamma)(s)\|$$

we can deduce that for all $t \in [0,1]$

$$\left\| \gamma \mapsto \|\beta(\gamma)(t)\| \right\|_{\mathcal{A}} \leq \|\beta\|,$$

so $\|\phi_B(\beta)\| \leq \|\beta\|$ as desired.

For the second part of the lemma, let $t_0 \in [0,1]$. Let $\beta \in \Gamma_c(\mathcal{G},\, r^*B[0,1])$. The map ϕ_B sends β to $t \mapsto (\gamma \mapsto \beta(\gamma)(t))$. Now $\mathrm{ev}_{t_0}^{\mathcal{A}(\mathcal{G},B)}$ sends this function to $\gamma \mapsto \beta(\gamma)(t_0)$ in $\Gamma_c(\mathcal{G},\, r^*B) \subseteq \mathcal{A}(\mathcal{G},\, B)$. On the other hand, $\mathcal{A}(\mathcal{G},\, \mathrm{ev}_{t_0}^B)$ sends β to $\mathrm{ev}_{t_0}^B * \beta$, i.e., to $\gamma \mapsto (\mathrm{ev}_{t_0}^B)_{r(\gamma)}\beta(\gamma) = \beta(\gamma)(t_0)$. So $\mathrm{ev}_{t_0}^{\mathcal{A}(\mathcal{G},B)} \circ \phi_B$ and $\mathcal{A}(\mathcal{G},\, \mathrm{ev}_{t_0}^B)$ agree on a dense subset and are thus equal. □

Proof of Proposition 5.2.27. There is a canonical $\mathcal{C}_0(X/\mathcal{G})$-linear concurrent homomorphism with coefficient maps $\mathrm{Id}_{\mathcal{A}(\mathcal{G},A)}$ and $\mathrm{Id}_{\mathcal{A}(\mathcal{G},C)}$

$$\mathfrak{m} := \mathfrak{m}_{E,F}^{\mathcal{A}} \colon \mathcal{A}(\mathcal{G},E) \otimes_{\mathcal{A}(\mathcal{G},B)}^{\mathcal{C}_0(X/\mathcal{G})} \mathcal{A}(\mathcal{G},F) \to \mathcal{A}(\mathcal{G}, E \otimes_B F)$$

defined as in Equation (C.2). We show that $\mathfrak{m}$ induces a homotopy.

In a first step, assume that $S \in \mathrm{K}(r^*E)_c$ is a compact operator on r^*E with compact support. Let $\hat{S}$ be as in 5.2.17, i.e., let $\hat{S}$ denote the action of S on $\mathcal{A}(\mathcal{G},E)$ by convolution. Then Proposition 5.2.18 says that $\hat{S}$ is a compact operator on $\mathcal{A}(\mathcal{G},E)$. Because $\mathcal{A}(\mathcal{G},B)$ acts on $\mathcal{A}(\mathcal{G},F)$ by compact operators, it follows that $\hat{S} \otimes 1$ is a compact operator on $\mathcal{A}(\mathcal{G},E) \otimes_{\mathcal{A}(\mathcal{G},B)} \mathcal{A}(\mathcal{G},F)$ (see Proposition 1.3.7). On the other hand, Proposition 3.1.59 says that $S \otimes 1$ is a locally compact operator on $r^*E \otimes_{r^*B} r^*F = r^*(E \otimes_B F)$. It has compact support (the support is contained in the support of S). So $S \otimes 1$ is a compact operator with compact support. Hence $\widehat{S \otimes 1} \in \mathrm{K}_{\mathcal{A}(\mathcal{G},C)}(\mathcal{A}(\mathcal{G},\, E \otimes_B F))$. We show that the pair $(\hat{S} \otimes 1,\, \widehat{S \otimes 1})$ is in $\mathrm{K}(\mathfrak{m},\, \mathfrak{m})$, i.e., we show that $\mathfrak{m}$ intertwines the two operators and that we can approximate them simultaneously with finite rank operators.

Because the map $S \mapsto (\hat{S} \otimes 1,\, \widehat{S \otimes 1})$ is linear and contractive (if one takes the semi-norm $\|S\| = \|\gamma \mapsto \|S_\gamma\|\|_{\mathcal{A}}$ on $\mathrm{L}_{r^*B}(r^*E)_c$), it suffices to show this for $S = \xi^> \bowtie \xi^<$ with $\xi^< \in \Gamma_c(\mathcal{G},\, r^*E^<)$ and $\xi^> \in \Gamma_c(\mathcal{G},\, r^*E^>)$ (see Paragraph C.2.2 for the definition of $\bowtie$). Because $(\xi^>, \xi^<) \mapsto \xi^> \bowtie \xi^<$ is continuous it is sufficient to consider the case that $S = \xi^> \bowtie (\beta * \xi^<) = (\xi^> * \beta) \bowtie \xi^<$ with $\xi^< \in \Gamma_c(\mathcal{G}, r^*E^<)$ and $\xi^> \in \Gamma_c(\mathcal{G},\, r^*E^>)$ and $\beta \in \Gamma_c(\mathcal{G},\, r^*B)$. Let $\varepsilon > 0$.

The map $(\eta^>, \eta^<) \mapsto \eta^> \bowtie \eta^<$ is separately continuous and non-degenerate for the inductive limit topologies on $\Gamma_c(\mathcal{G},\, r^*F^>)$, $\Gamma_c(\mathcal{G},\, r^*F^<)$ and $\mathrm{K}_{r^*C}(r^*F)$. We can therefore find $n \in \mathbb{N}$, $\eta_1^<, \ldots, \eta_n^< \in \Gamma_c(\mathcal{G},\, r^*F^<)$ and $\eta_1^>, \ldots, \eta_n^> \in \Gamma_c(\mathcal{G},\, r^*F^>)$ such that

$$\|\xi^>\|_{\mathcal{A}}\ \|\xi^<\|_{\mathcal{A}} \left\| \gamma \mapsto \left\| \pi(\beta(\gamma)) - \sum_{i=1}^n \left(\eta_i^> \bowtie \eta_i^<\right)_\gamma \right\| \right\|_{\mathcal{A}} \leq \varepsilon.$$

Because the maps $\bowtie$, $\mathfrak{m}^>$ and the action of $S \otimes 1$ on $\mathcal{A}\left(\mathcal{G}, E \otimes_B F\right)^>$ are given by a convolution formula we use, for a moment, the symbol $*$ for all of them; a short calculation yields

$$\widehat{S \otimes 1}^> \left(\mathfrak{m}^>(\xi'^> \otimes \eta'^>)\right) = (S^> \otimes 1) * \xi'^> * \eta'^> = \xi^> * \beta * \xi^< * \xi'^> * \eta'^>$$

for all $\xi'^> \in \Gamma_c\left(\mathcal{G},\, r^*E^>\right)$ and $\eta'^> \in \Gamma_c\left(\mathcal{G},\, r^*F^>\right)$. Note that we have implicitly use some straightforward associativity laws. Now

$$\begin{aligned} &\left\| \xi^> * \beta * \xi^< * \xi'^> * \eta'^> - \xi^> * \left[\textstyle\sum_{i=1}^n \eta_i^> * \eta_i^<\right] * \xi^< * \xi'^> * \eta'^> \right\|_{\mathcal{A}} \\ \leq\ & \|\xi^>\|_{\mathcal{A}} \left\| (\gamma \mapsto \pi(\beta(\gamma))) - \textstyle\sum_{i=1}^n \eta_i^> * \eta_i^< \right\|_{\mathcal{A}} \|\xi^<\|_{\mathcal{A}} \|\xi'^> * \eta'^>\|_{\mathcal{A}} \leq \varepsilon \left\|\xi'^> * \eta'^>\right\|_{\mathcal{A}}. \end{aligned}$$

A similar formula is true for the left-hand side (= the bra-side). Using the density of the image of $\mathfrak{m}$ one can conclude that

$$\left\| \xi^{>} * \beta * \xi^{<} * \cdot - \xi^{>} * \left[\sum_{i=1}^{n} \eta_i^{>} * \eta_i^{<} \right] * \xi^{<} * \cdot \right\| \leq \varepsilon.$$

Now

$$\xi^{>} * \left[\sum_{i=1}^{n} \eta_i^{>} * \eta_i^{<} \right] * \xi^{<} * \cdot = \sum_{i=1}^{n} (\xi^{>} * \eta_i^{>}) * (\eta_i^{<} * \xi^{<}) * \cdot;$$

in other words, we can approximate $\widehat{S \otimes 1} = \xi^{>} * \beta * \xi^{<} * \cdot$ up to ε by $\sum_{i=1}^{n} \left| \mathfrak{m}^{>}(\xi^{>} \otimes \eta_i^{>}) \right\rangle \left\langle \mathfrak{m}^{<}(\eta_i^{<} \otimes \xi^{<}) \right|$.

In a similar manner one can show that we can approximate $\hat{S} \otimes 1$ by $\sum_{i=1}^{n} \left| \xi^{>} \otimes \eta_i^{>} \right\rangle \left\langle \eta_i^{<} \otimes \xi^{<} \right|$ up to ε. Hence $(\hat{S} \otimes 1,\ \widehat{S \otimes 1})$ is in $\mathrm{K}\,(\mathfrak{m}, \mathfrak{m})$.

Now we show that $\mathfrak{m}$ satisfies the conditions of Theorem 2.6.2, the sufficient condition for homotopy of $\mathrm{KK}^{\mathrm{ban}}$-cycles which will tell us that $\mathfrak{m}$ induces a homotopy. Let $a \in \mathcal{A}\,(\mathcal{G}, A)$. As in the proof of 5.2.19 we have $[a, \mathcal{A}(\mathcal{G}, T)] = \widehat{S}$ with $S = a * T - T * a \in \mathrm{K}_{r^*B}\,(r^*E)_c$ and similarly $[a, \mathcal{A}(\mathcal{G}, T \otimes 1)] = \widehat{S \otimes 1}$. It follows that

$$\begin{aligned}
[a,\ (\mathcal{A}(\mathcal{G}, T) \otimes 1,\ \mathcal{A}(\mathcal{G}, T \otimes 1))] &= ([a \otimes 1,\ \mathcal{A}(\mathcal{G}, T) \otimes 1]\,,\ [a,\ \mathcal{A}(\mathcal{G}, T \otimes 1)]) \\
&= ([a,\ \mathcal{A}(\mathcal{G}, T)] \otimes 1,\ [a,\ \mathcal{A}(\mathcal{G}, T \otimes 1)]) \\
&= \left(\widehat{S} \otimes 1,\ \widehat{S \otimes 1}\right) \in \mathrm{K}\,(\mathfrak{m}, \mathfrak{m})\,.
\end{aligned}$$

The second condition of Theorem 2.6.2 is checked analogously (and the third condition is void). $\square$

Appendix D

Some Details Concerning Chapter 6

D.1 Some proofs of results of Section 6.1

In this appendix we collect the proofs of most of the technical results of Section 6.1.

Proof of Proposition 6.1.9. Without loss of generality we may assume that Ω_1 is proper. We have to check whether the map

$$\mu\colon \mathcal{G} * (\Omega_1 * \Omega_2) \to (\Omega_1 * \Omega_2) \times (\Omega_1 * \Omega_2),\ (\gamma, \omega_1, \omega_2) \mapsto (\omega_1, \omega_2, \gamma\omega_1, \gamma\omega_2)$$

is proper. Define the map

$$\pi\colon (\Omega_1 * \Omega_2) \times (\Omega_1 * \Omega_2) \to \Omega_1 \times \Omega_1 \times \Omega_2,\ (\omega_1, \omega_2, \omega_1', \omega_2') \mapsto (\omega_1, \omega_1', \omega_2).$$

Since this map is continuous, it suffices to show that $\pi \circ \mu$ is proper. But

$$(\pi \circ \mu)(\gamma, \omega_1, \omega_2) = (\omega_1, \gamma\omega_1, \omega_2)$$

for all $(\gamma, \omega_1, \omega_2) \in \mathcal{G}*(\Omega_1*\Omega_2)$ which can be extended continuously to $(\mathcal{G}*\Omega_1)\times\Omega_2$. This extension is the product of a proper map and the identity on Ω_2 and hence proper, so $\pi \circ \mu$ is proper. $\square$

Proof of Proposition 6.1.12. Firstly, we show that Ω is free if and only if a map exists which has the properties of an inner product apart from continuity. Then we show that a free space Ω is proper if and only this "inner product" is continuous.

If Ω is free, then we define $\langle\omega, \omega'\rangle$ to be the unique $\gamma \in \mathcal{G}$ such that $\gamma\omega' = \omega$. Then by definition property 1, 4 and 5 hold. If $\gamma' \in \mathcal{G}$ such that $s(\gamma') = \rho(\omega)$, then $\gamma'\omega = (\gamma'\langle\omega, \omega'\rangle)\omega'$ so $\langle\gamma'\omega, \omega'\rangle$ has got to be equal to $\gamma'\langle\omega, \omega'\rangle$ by its defining property. Similarly, if $\gamma'' \in \mathcal{G}$ such that $s(\gamma'') = \rho(\omega')$, then $\omega = \langle\omega, \omega'\rangle\omega' = \left(\langle\omega, \omega'\rangle\gamma^{-1}\right)(\gamma\omega')$ so $\langle\omega, \gamma\omega'\rangle = \langle\omega, \omega'\rangle\gamma^{-1}$. So $\langle\cdot,\cdot\rangle$ has the properties 1-5.

Now let $\langle\cdot,\cdot\rangle$ be an inner product on Ω. Let $(\omega, \omega') \in \Omega \times_\sigma \Omega$. By definition of this fibre-product there is a $\gamma \in \mathcal{G}$ such that $\omega = \gamma\omega'$. We have to show that it is unique and we do this by showing that it is $\langle\omega, \omega'\rangle$. Because of 4 we have $\langle\omega', \omega'\rangle = \rho(\omega')$, and from 2 it follows that $\langle\omega, \omega'\rangle = \langle\gamma\omega', \omega'\rangle = \gamma\langle\omega', \omega'\rangle = \gamma\rho(\omega') = \gamma$. Hence Ω is free.

Now let Ω be a free $\mathcal{G}$-space. This implies that the continuous map

$$\mu\colon \mathcal{G} * \Omega \to \Omega \times_\sigma \Omega,\ (\gamma, \omega) \mapsto (\gamma\omega, \omega)$$

is a bijection. Its inverse map is given by

$$(\omega', \omega) \mapsto \left(\langle\omega', \omega\rangle\omega,\ \omega\right).$$

This shows that μ is a homeomorphism if and only if the inner product is continuous.

We now show that μ is a homeomorphism if and only if Ω is proper (Note that $\Omega \times_\sigma \Omega$ is closed in $\Omega \times \Omega$ since $\mathcal{G} \setminus \Omega$ is Hausdorff, so $\Omega \times_\sigma \Omega$ is locally compact Hausdorff).

If μ is a homeomorphism, then it is a proper if we consider it to have its values in the larger space $\Omega \times \Omega$. But this exactly means that Ω is proper. On the other hand, if Ω is proper then our μ is proper as well (with values in $\Omega \times_\sigma \Omega$). So by Lemma D.1.1 the map μ, being a continuous proper bijection between locally compact Hausdorff spaces, is a homeomorphism. $\square$

Lemma D.1.1. *Let Y and Z be locally compact Hausdorff spaces and let $f\colon Y \to Z$ be a continuous bijection. If f is proper, then f is a homeomorphism.*

Proof. Since Z is locally compact its topology is compactly generated. Let A be a closed subset of Y. Then we want to check that $f(A) \cap L$ is closed (or compact) for all compact subsets L of Z. Let L be such a compact set. Then $K := f^{-1}(L)$ is compact because f is proper. So $K \cap A$ is compact. As f is continuous we can deduce that $f(K \cap A)$ is compact, too. Now f is bijective and Z is Hausdorff, so $L \cap f(A) = f(K) \cap f(A) = f(K \cap A)$ is closed. It follows that $f(A)$ is closed and f is a homeomorphism. $\square$

Proof of Proposition 6.1.14. Note that the map $q\colon \Omega \times_{\mathcal{G}^0} \Omega \to \Omega^{-1} \times_{\mathcal{G}} \Omega$ is well-defined, continuous, surjective, and open. We check that the other maps are well-defined and continuous, too. Then we check that we have defined a locally compact groupoid in this way.

1. **The map $\epsilon_{\mathcal{H}}$:** Let $(\gamma, \omega) \in \mathcal{G} * \Omega$. Then $[\gamma\omega] = [\omega]$ by definition. On the other hand we have $[(\gamma\omega)^{-1}, \gamma\omega] = [\omega^{-1}\gamma^{-1}, \gamma\omega] = [\omega^{-1}, (\gamma^{-1}\gamma)\omega] = [\omega^{-1}, \omega]$. So $\epsilon_{\mathcal{H}}$ is a well-defined (and, by much the same argument, injective) map. The following square is commutative

$$\begin{tikzcd} \Omega \arrow[r] \arrow[d, "p"] & \Omega \times_{\mathcal{G}^{(0)}} \Omega \arrow[d, "q"] \\ \mathcal{G} \setminus \Omega \arrow[r, "\epsilon_{\mathcal{H}}"] & \mathcal{H} \end{tikzcd}$$

where the top arrow is the continuous map $\omega \mapsto (\omega, \omega)$ and p is the quotient map. By the definition of the quotient topology of $\mathcal{G} \setminus \Omega$ the map $\epsilon_{\mathcal{H}}$ is continuous as well.

2. **The maps $r_{\mathcal{H}}$ and $s_{\mathcal{H}}$:** Let $(\omega, \omega') \in \Omega \times_{\mathcal{G}^{(0)}} \Omega$ and $\gamma \in \mathcal{G}$ such that $\rho(\omega) = \rho(\omega') = s(\gamma)$. Then $[\omega^{-1}, \omega'] = [(\gamma\omega)^{-1}, \gamma\omega']$ and $[\omega] = [\gamma\omega]$ as well as $[\omega'] = [\gamma\omega']$. So $r_{\mathcal{H}}$ and $s_{\mathcal{H}}$ are well-defined. The following square is commutative:

$$\begin{tikzcd} \Omega \times_{\mathcal{G}^{(0)}} \Omega \arrow[r, "\pi_1"] \arrow[d, "q"] & \Omega \arrow[d, "p"] \\ \mathcal{H} \arrow[r, "r_{\mathcal{H}}"] & \mathcal{G} \setminus \Omega \end{tikzcd}$$

where π_1 is the continuous map that sends (ω, ω') to ω. By definition of the quotient topology on $\mathcal{H}$ the map $r_{\mathcal{H}}$ is continuous. The analogously constructed diagram for s is commutative so also $s_{\mathcal{H}}$ is continuous.

3. **The multiplication μ:** Define the map $\tilde{\mu}$ from Y to $\Omega \times_\rho \Omega$ to be

$$\tilde{\mu}((\omega_1, \omega_1'), (\omega_2, \omega_2')) := (\omega_1,\ \langle \omega_1', \omega_2 \rangle \omega_2')$$

for all $\omega_1', \omega_1, \omega_2', \omega_2 \in \Omega$ such that $\rho(\omega_1') = \rho(\omega_1)$, $\rho(\omega_2') = \rho(\omega_2)$ and $[\omega_1'] = \sigma(\omega_1') = \sigma(\omega_2) = [\omega_2]$. If $\gamma \in \mathcal{G}$ such that $s(\gamma) = \rho(\omega_1) = \rho(\omega_1')$, then

$$\tilde{\mu}(\gamma(\omega_1, \omega_1'), (\omega_2, \omega_2')) = \big(\gamma\omega_1, \langle\gamma\omega_1', \omega_2\rangle\omega_2'\big) = \big(\gamma\omega_1, \gamma\langle\omega_1', \omega_2\rangle\omega_2'\big) = \gamma\tilde{\mu}((\omega_1, \omega_1'), (\omega_2, \omega_2')).$$

So the multiplication μ is well-defined.

Since the fibre-product of open maps is open the canonical map from $Y := (\Omega \times_\rho \Omega) \times_\sigma (\Omega \times_\rho \Omega)$ to $\mathcal{H}^{(2)} = (\Omega^{-1} \times_{\mathcal{G}} \Omega) \times_\sigma (\Omega^{-1} \times_{\mathcal{G}} \Omega)$ is open and continuous. Since $\langle\cdot,\cdot\rangle$ is continuous we know that $\tilde{\mu}$ is continuous. By the definition of μ it is the map that makes the following square commutative:

$$\begin{array}{ccc} Y & \xrightarrow{\tilde{\mu}} & \Omega \times_\rho \Omega \\ \Big\downarrow{\scriptstyle q \times_\sigma q} & & \Big\downarrow{\scriptstyle q} \\ \mathcal{H}^{(2)} & \xrightarrow{\mu} & \mathcal{H} \end{array}$$

Since $q \times_\sigma q$ is open and surjective it follows that μ is continuous.

4. **The inversion ι:** Let $\tilde{\iota}$ denote the map $(\omega, \omega') \mapsto (\omega', \omega)$ from $\Omega \times_\rho \Omega$ onto itself. Then $\tilde{\iota}$ is continuous. If $(\omega, \omega') \in \Omega \times_\rho \Omega$ and $\gamma \in \mathcal{G}$ such that $\rho(\omega) = \rho(\omega') = s(\gamma)$ then $\tilde{\iota}(\gamma(\omega, \omega')) = \tilde{\iota}(\gamma\omega, \gamma\omega') = (\gamma\omega', \gamma\omega) = \gamma(\tilde{\iota}(\omega, \omega'))$. So the map $[\omega^{-1}, \omega'] \mapsto [\omega'^{-1}, \omega]$ is well-defined on $\Omega^{-1} \times_{\mathcal{G}} \Omega$. Since it makes the following diagram commutative it is continuous:

$$\begin{array}{ccc} \Omega \times_\rho \Omega & \xrightarrow{\tilde{\iota}} & \Omega \times_\rho \Omega \\ \Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle q} \\ \mathcal{H} & \xrightarrow{\iota} & \mathcal{H} \end{array}$$

Now we check the algebraic properties:

1. **Associativity:** Let $(\omega_1, \omega_1'), (\omega_2, \omega_2'), (\omega_3, \omega_3') \in \Omega \times_\rho \Omega$ such that $[\omega_1'] = [\omega_2]$ and $[\omega_2'] = [\omega_3]$. Now

$$\begin{aligned}
\big([\omega_1^{-1}, \omega_1']\ [(\omega_2)^{-1}, \omega_2']\big)\ [(\omega_3)^{-1}, \omega_3'] &= [\omega_1^{-1},\ \langle\omega_1', \omega_2\rangle\omega_2']\ [(\omega_3)^{-1}, \omega_3'] \\
&= \big[\omega_1^{-1},\ \big\langle\langle\omega_1', \omega_2\rangle\omega_2',\ \omega_3\big\rangle\,\omega_3'\big] \\
&= \big[\omega_1^{-1},\ \big\langle\omega_1', \omega_2\big\rangle\,\big\langle\omega_2',\ \omega_3\big\rangle\,\omega_3'\big] \\
&= \big[\omega_1^{-1},\ \omega_1'\big]\,\big[(\omega_2)^{-1},\ \big\langle\omega_2',\ \omega_3\big\rangle\,\omega_3'\big] \\
&= \big[\omega_1^{-1}, \omega_1'\big]\,\big(\big[(\omega_2)^{-1}, \omega_2'\big]\ \big[(\omega_3)^{-1}, \omega_3'\big]\big)\,.
\end{aligned}$$

2. **Units:** Let $(\omega, \omega') \in \Omega \times_\rho \Omega$. Then the elements $[\omega^{-1}, \omega]$ and $[\omega^{-1}, \omega']$ are composable and $[\omega^{-1}, \omega][\omega^{-1}, \omega'] = [\omega^{-1}, \langle\omega, \omega\rangle\omega'] = [\omega^{-1}, \omega']$. Similarly on the right-hand side.

3. **Inversion:** Let $(\omega, \omega') \in \Omega \times_\rho \Omega$. Then $[\omega^{-1}, \omega'][(\omega')^{-1}, \omega] = [\omega^{-1},\ \langle\omega', \omega'\rangle\omega] = [\omega^{-1}, \omega]$.

So $\mathcal{H}$ is a topological groupoid. It is locally compact Hausdorff since it is a quotient of a proper $\mathcal{G}$-space. The above diagrams show in particular that the quotient map from $\Omega \times_\rho \Omega$ to $\mathcal{H}$ is a strict morphism.

The range and source maps are open: As above, the following square is commutative:

$$\begin{tikzcd} \Omega \times_{\mathcal{G}^{(0)}} \Omega \arrow[r, "\pi_1"] \arrow[d, "q"] & \Omega \arrow[d, "p"] \\ \mathcal{H} \arrow[r, "r_{\mathcal{H}}"] & \mathcal{G} \setminus \Omega \end{tikzcd}$$

where π_1 is the map that sends (ω, ω') to ω. If ρ is open, then π_1 is open by lemma 3.4.5. Now q is surjective and continuous and $p \circ \pi_1$ is open, so $r_{\mathcal{H}}$ is open. Similarly for $s_{\mathcal{H}}$. □

Proof of Proposition 6.1.15. Note that the action is well-defined since we have $\langle \omega, \gamma\omega' \rangle (\gamma\omega'') = \langle \omega, \omega' \rangle \gamma^{-1}(\gamma\omega'') = \langle \omega, \omega' \rangle \omega'$ for all γ such that $\rho(\omega') = s(\gamma)$. Now we check that this map is indeed a continuous action on Ω:

1. **Compatibility with $s_{\mathcal{H}}$:** Let $\omega \in \Omega$ and $[(\omega')^{-1}, \omega''] \in \mathcal{H}$ such that $\sigma(\omega) = \sigma(\omega')$. Then $\sigma\left(\omega[(\omega')^{-1}, \omega'']\right) = \sigma\left(\langle \omega, \omega' \rangle \omega''\right) = \sigma(\omega'') = s_{\mathcal{H}}\left([(\omega')^{-1}, \omega'']\right)$.

2. **Units act trivially:** Let $\omega \in \Omega$. Then $\omega[\omega^{-1}, \omega] = \langle \omega, \omega \rangle \omega = \omega$, so $[\omega]$ acts identically on ω from the right.

3. **Associativity:** Let $\omega \in \Omega$, $[(\omega_1')^{-1}, \omega_1''], [(\omega_2')^{-1}, \omega_2''] \in \mathcal{H}$ such that $\sigma(\omega) = \sigma(\omega_1')$ and $\sigma(\omega_1'') = \sigma(\omega_2')$.

$$\left(\omega[(\omega_1')^{-1}, \omega_1'']\right)[(\omega_2')^{-1}, \omega_2''] = \left(\langle \omega, \omega_1' \rangle \omega_1''\right)[(\omega_2')^{-1}, \omega_2''] = \left\langle \langle \omega, \omega_1' \rangle \omega_1'',\ \omega_2' \right\rangle \omega_2''$$

and

$$\omega\left([(\omega_1')^{-1}, \omega_1''][(\omega_2')^{-1}, \omega_2'']\right) = \omega\left[(\omega_1')^{-1},\ \langle \omega_1'', \omega_2' \rangle \omega_2''\right] = \left\langle \omega, \omega_1' \right\rangle \left\langle \omega_1'',\ \omega_2' \right\rangle \omega_2''.$$

4. **Continuity:** σ is continuous by definition. The action μ of $\mathcal{H}$ on Ω is continuous because it makes the following diagram commutative:

$$\begin{tikzcd} \Omega \times_{\sigma} (\Omega \times_{\rho} \Omega) \arrow[r, "\tilde{\mu}"] \arrow[d, "\mathrm{Id} \times_{\sigma} q"] & \Omega \arrow[d, "\mathrm{Id}"] \\ \Omega * \mathcal{H} \arrow[r, "\mu"] & \Omega \end{tikzcd}$$

Here $\tilde{\mu}$ denotes the map that sends $(\omega, (\omega', \omega''))$ to $\langle \omega, \omega' \rangle \omega''$. This map is continuous and $\mathrm{Id} \times_{\sigma} q$ is surjective and open, so μ is continuous.

Now we check that ρ induces a continuous injection $\tilde{\rho}\colon \Omega/\mathcal{H} \to \mathcal{G}^{(0)}$: Let ω and ω' be in Ω such that $\rho(\omega) = \rho(\omega')$. Then $[\omega^{-1}, \omega'] \in \mathcal{H}$ and $\omega[\omega^{-1}, \omega'] = \omega'$, so ω and ω' are in the same $\mathcal{H}$-orbit of Ω. Vice versa, if $\omega, \omega' \in \Omega$ such that there is a $[(\omega'')^{-1}, \omega'] \in \mathcal{H}$ with $\omega[(\omega'')^{-1}, \omega'] = \langle \omega, \omega'' \rangle \omega' = \omega'$ then $\rho(\omega') = \rho\left(\langle \omega, \omega'' \rangle \omega'\right) = r(\langle \omega, \omega'' \rangle) = \rho(\omega)$, so we are done.

That the action of $\mathcal{H}$ on Ω is free and proper follows from the following lemma. □

Lemma D.1.2. *The map $(\omega, \omega') \mapsto [(\omega)^{-1}, \omega']$ is an inner product on the right $\mathcal{H}$-space Ω.*

Proof. First note that $\Omega \times_\rho \Omega$ is the same as $\Omega \times_{\tilde\rho} \Omega$ where $\tilde\rho$ denotes the canonical map from Ω to $\Omega/\mathcal{H}$. Now the map $(\omega, \omega') \mapsto [(\omega)^{-1}, \omega']$ is the map that we have called q earlier on, the quotient map. In particular, q is continuous. In order to show that q is an inner product we just check properties 2 and 4 of the definition of the inner product (and the first half of 1). This already implies the other conditions. Note that we have to reflect the formulae in the conditions because we are dealing with *right* spaces.

Let $(\omega, \omega') \in \Omega \times_\rho \Omega$. Then $s_{\mathcal{H}}\left([\omega^{-1}, \omega']\right) = \sigma(\omega')$ by definition (which shows property 1). If $(\omega'', \omega''') \in \Omega \times_\rho \Omega$ such that $\sigma(\omega') = \sigma(\omega'')$ then

$$\left[\omega^{-1},\ \omega'[(\omega'')^{-1}, \omega''']\right] = \left[\omega^{-1},\ \langle \omega', \omega''\rangle \omega'''\right] = \left[\omega^{-1}, \omega'\right]\left[(\omega'')^{-1}, \omega'''\right].$$

Hence we have property 2. Finally, if $\omega \in \Omega$, then $[\omega^{-1}, \omega] = \epsilon_{\mathcal{H}}(\omega)$ which shows 4. □

Proof of 6.1.19. 1. The map σ'' is well-defined since $\sigma'(\eta\omega') = \sigma'(\omega')$ for all $(\eta, \omega') \in \mathcal{H} * \Omega'$. The universal property of the quotient topology shows that σ'' is continuous because $(\omega, \omega') \mapsto \sigma'(\omega')$ is continuous from $\Omega \times_{\mathcal{H}^{(0)}} \Omega'$ to $\mathcal{K}^{(0)}$.

Let $\tilde\mu$ denote the map $((\omega, \omega'), \kappa) \to (\omega, \omega'\kappa)$ from $\Omega \times_{\mathcal{H}^{(0)}} \Omega' \times_{\mathcal{K}^{(0)}} \mathcal{K}$ to $\Omega \times_{\mathcal{H}^{(0)}} \Omega'$. It is continuous since the action of $\mathcal{K}$ on Ω' is. For all $\eta \in \mathcal{H}$ such that $\sigma(\omega) = \rho'(\omega') = s(\eta)$, we have

$$\eta \cdot \tilde\mu\left((\omega, \omega'), \kappa\right) = \eta(\omega, \omega'\kappa) = (\omega\eta^{-1},\ \eta(\omega'\kappa)) = (\omega\eta^{-1},\ (\eta\omega')\kappa) = \tilde\mu\left((\eta(\omega, \omega')), \kappa\right).$$

Hence the action μ of $\mathcal{K}$ on Ω'' is a well-defined map. It makes the following square commutative:

$$\begin{tikzcd}
\Omega \times_{\mathcal{H}^{(0)}} \Omega' \times_{\mathcal{K}^{(0)}} \mathcal{K} \arrow[r, "\tilde\mu"] \arrow[d, "q \times_{\mathcal{K}^{(0)}} \mathrm{Id}_{\mathcal{K}}"] & \Omega \times_{\mathcal{H}^{(0)}} \Omega' \arrow[d, "q"] \\
\Omega'' * \mathcal{K} \arrow[r, "\mu"] & \Omega''
\end{tikzcd}$$

where q denotes the canonical quotient map. Since $\tilde\mu$ is continuous and $q \times_{\mathcal{K}^{(0)}} \mathrm{Id}_{\mathcal{K}}$ is open and surjective we can deduce that μ is continuous.

2. Proceed as in 1. to see that our formulae indeed define a continuous $\mathcal{G}$-action on Ω''. It is trivially checked that we have defined a $\mathcal{G}$-$\mathcal{K}$-bimodule. □

Proof of 6.1.26. First we check that the fibres of $\rho''\colon \Omega'' \to \mathcal{G}^{(0)}$ are the orbits of the $\mathcal{K}$-action:

Let $[(\omega_1, \omega_1')]$, $[(\omega_2, \omega_2')]$ be elements of Ω'' such that $\rho(\omega_1) = \rho(\omega_2)$. Since ρ is a principal fibration with structure groupoid $\mathcal{H}$, we can find some $\eta \in \mathcal{H}$ such that $r(\eta) = \sigma(\omega_1)$ and $\omega_2 = \omega_1\eta$. Now $[(\omega_2, \omega_2')] = [(\omega_1\eta, \omega_2')] = [(\omega_1, \eta\omega_2')]$. Because $\rho'(\eta\omega_2') = r(\eta) = \sigma(\omega_1) = \rho'(\omega_1')$ and ρ' is a principal fibration with structure groupoid $\mathcal{K}$ we can find some $\kappa \in \mathcal{K}$ such that $r(\kappa) = \sigma'(\omega_2')$ and $\eta\omega_2'\kappa = \omega_1'$. Now this means

$$[(\omega_2, \omega_2')]\kappa = [(\omega_1, \eta\omega_2')]\kappa = [(\omega_1, \eta\omega_2'\kappa)] = [(\omega_1, \omega_1')].$$

So $[(\omega_1, \omega_1')]$ and $[(\omega_2, \omega_2')]$ are in the same $\mathcal{K}$-orbit. Since Ω'' is a $\mathcal{G}$-$\mathcal{K}$-bimodule the $\mathcal{K}$-orbits are thus exactly the fibres of ρ''.

To show that Ω'' is a free and proper $\mathcal{K}$-space we define a $\mathcal{K}$-values inner product $\langle\cdot,\cdot\rangle''$ on Ω'', using the $\mathcal{H}$-valued inner product $\langle\cdot,\cdot\rangle$ on Ω and the $\mathcal{K}$-valued inner product $\langle\cdot,\cdot\rangle'$ on Ω''. Note that we have just shown that $\Omega'' \times_{\Omega''/\mathcal{K}} \Omega''$ is equal to $\Omega'' \times_{\rho''} \Omega''$.

We define

$$\left\langle [(\omega_1, \omega_1')],\ [(\omega_2, \omega_2')] \right\rangle'' := \left\langle \omega_1',\ \langle \omega_1,\ \omega_2 \rangle\, \omega_2' \right\rangle'$$

for all $[(\omega_1, \omega_1')], [(\omega_2, \omega_2')] \in \Omega'' \times_{\rho''} \Omega''$. By standard arguments this is a well-defined and continuous map which clearly satisfies the axioms of an inner product.

Note that ρ'' is open because ρ and ρ' are open.

So Ω'' is a graph from $\mathcal{G}$ to $\mathcal{K}$. □

Proof of Proposition 6.1.27. The strict identity morphism are mapped to the (generalised) identity morphisms as we have seen above. Now let $f\colon \mathcal{G} \to \mathcal{H}$ and $f'\colon \mathcal{H} \to \mathcal{K}$ be strict morphisms. The product $\operatorname{Graph}(f) \times_{\mathcal{H}} \operatorname{Graph}(f')$ is given by

$$\Omega'' := \left(\mathcal{G}^{(0)} \times_{\mathcal{H}^{(0)}} \mathcal{H} \right) \times_{\mathcal{H}} \left(\mathcal{H}^{(0)} \times_{\mathcal{K}^{(0)}} \mathcal{K} \right).$$

Define a map λ from Ω'' to $\operatorname{Morph}(f' \circ f) = \mathcal{G}^{(0)} \times_{\mathcal{K}^{(0)}} \mathcal{K}$ by setting

$$\lambda\left([((g, \eta),\ (h, \kappa))] \right) := \left(g,\ f'(\eta)\kappa \right)$$

for all $g \in \mathcal{G}^{(0)}$, $\eta \in \mathcal{H}$, $h \in \mathcal{H}^{(0)}$, $\kappa \in \mathcal{K}$ such that $f(g) = r(\eta)$, $s(\eta) = h$ and $f'(h) = r(\kappa)$. That this map is well-defined can be shown as follows: If $\eta' \in \mathcal{H}$ such that $s(\eta) = s(\eta') = h$ then

$$\left(g, f'(\eta)\, \kappa \right) = \left(g, f'\left(\eta \eta'^{-1} \right) f'(\eta')\kappa \right)$$

so the right-hand side is "invariant under the action of $\mathcal{H}$" which we factor out on the left-hand side. This also shows that λ is continuous. By standard arguments the map λ respects the actions of $\mathcal{G}$ and $\mathcal{K}$. λ is (continuously) inverted by the map which sends (g, κ) to $[(g, f(g)),\ (f(g), \kappa)]$. □

Proof of Proposition 6.1.30. Consider the following diagram:

$$\begin{tikzcd}
\Omega \times_\rho \Omega \arrow[d, "q"'] \arrow[dr, "\langle \cdot, \cdot \rangle"] & \Omega * \mathcal{H} \arrow[l, "\mu", "\cong"'] \arrow[d, "\pi_2"] \\
\Omega^{-1} \times_{\mathcal{G}} \Omega \arrow[r, dashed, "\nu"] & \mathcal{H}
\end{tikzcd}$$

Here μ denotes the map which sends (ω, η) to $(\omega, \omega\eta)$. Since the action of $\mathcal{H}$ on Ω is free and proper, we can deduce that this map is a proper and continuous bijection from $\Omega * \mathcal{G}$ onto $\Omega \times_{\Omega/\mathcal{H}} \Omega$. Since ρ is a principal fibration, the latter space is equal to $\Omega \times_\rho \Omega$. By Lemma D.1.1, which we have already applied in almost the same situation, μ is a homeomorphism.

We write π_2 for the projection onto the second component. By Lemma 3.4.5 and because σ is open and surjective, π_2 is open and surjective.

By definition, the $\mathcal{H}$-valued inner product on the principal $\mathcal{H}$-space Ω is $\pi_2 \circ \mu^{-1}$. This map happens to be continuous, open and surjective. The map q is the quotient map (remember that $\Omega^{-1} \times_{\mathcal{G}} \Omega$ is constructed by factoring out the $\mathcal{G}$ action on $\Omega \times_\rho \Omega$; since this action is proper, $\Omega^{-1} \times_{\mathcal{G}} \Omega$ is locally compact Hausdorff). By definition, q is open and surjective.

We claim that μ factors to a map ν from $\Omega^{-1} \times_{\mathcal{G}} \Omega$ to $\mathcal{H}$ which is the desired isomorphism of groupoids and of $\mathcal{H}$-$\mathcal{H}$-bimodules. In particular, ν is a homeomorphism.

To check that ν is well-defined and injective it suffices to check that elements of $\Omega * \mathcal{H}$ which have the same image under $q \circ \mu$ are precisely those which have the same image under π_2. So let $(\omega_1, \eta_1), (\omega_2, \eta_2) \in \Omega * \mathcal{H}$. Now $\pi_2(\omega_1, \eta_1) = \pi_2(\omega_2, \eta_2)$ is equivalent to $\eta_1 = \eta_2$.

If this is the case, then $\mu(\omega_1, \eta_1) = (\omega_1, \omega_1\eta_1)$ and $\mu(\omega_2, \eta_2) = (\omega_2, \omega_2\eta_1)$. Now σ is a principal fibration and $\sigma(\omega_2\eta_1) = s(\eta_1) = \sigma(\omega_1\eta_1)$. So there is a $\gamma \in \mathcal{G}$ such that $\gamma\omega_1\eta_1 = \omega_2\eta_2$. This implies $\gamma\omega_1 = \omega_2$ and $\gamma(\omega_1, \omega_1\eta_1) = (\omega_2, \omega_2\eta_2)$. Hence $q(\mu(\omega_1, \eta_1)) = q(\mu(\omega_2, \eta_2))$.

On the other hand, assume that $q(\mu(\omega_1, \eta_1)) = q(\mu(\omega_2, \eta_2))$. This implies that there is a $\gamma \in \mathcal{G}$ such that $\gamma(\omega_1, \omega_1\eta_1) = (\omega_2, \omega_2\eta_2)$. This means $\gamma\omega_1 = \omega_2$ and $\gamma\omega_1\eta_1 = \omega_2\eta_2$. From this we have $\omega_2\eta_1 = \omega_2\eta_2$. Now the action of $\mathcal{H}$ is free, so $\eta_1 = \eta_2$.

It follows that ν is well-defined and injective. Since q is open and surjective, ν is continuous. Since π_2 and μ^{-1} are open and surjective and q is continuous and surjective, ν is open and surjective. So ν is a homeomorphism.

To have a better feeling for ν note that it maps a class $[\omega_1^{-1}, \omega_2]$ with $\rho(\omega_1) = \rho(\omega_2)$ to the unique $\eta \in \mathcal{H}$ such that $\omega_2 = \omega_1\eta$ (which happens to be $\langle\omega_1, \omega_2\rangle$).

To see that it even is a $\mathcal{H}$-$\mathcal{H}$-bimodule isomorphism let $(\omega_1, \omega_2) \in \Omega \times_\rho \Omega$ and $\eta_1, \eta_2 \in \mathcal{H}$ such that $\sigma(\omega_2) = r(\eta_2)$ and $\sigma(\omega_1) = s(\eta_1)$. Now, with $\eta_1\omega_1^{-1} = \left(\omega_1\eta_1^{-1}\right)^{-1}$,

$$\nu\left(\left[(\omega_1\eta_1^{-1})^{-1},\ \omega_2\eta_2\right]\right) = \left\langle\omega_1\eta_1^{-1},\ \omega_2\eta_2\right\rangle = \eta_1\left\langle\omega_1,\ \omega_2\right\rangle\eta_2 = \eta_1\nu\left(\left[(\omega_1)^{-1},\ \omega_2\right]\right)\eta_2.$$

Now we prove that ν is also a homomorphism of groupoids: Let $(\omega_1, \omega_2), (\omega_1', \omega_2') \in \Omega \times_\rho \Omega$ such that $\sigma(\omega_2) = \sigma(\omega_1')$. Then

$$\nu\left(\left[\omega_1^{-1},\ \omega_2\right]\left[\omega_1'^{-1},\ \omega_2'\right]\right) = \nu\left(\left[\omega_1^{-1},\ {}_{\mathcal{G}}\langle\omega_2, \omega_1'\rangle\ \omega_2'\right]\right) =: \eta.$$

Here ${}_{\mathcal{G}}\langle\omega_2, \omega_1'\rangle$ denotes the unique element $\gamma \in \mathcal{G}$ such that $\omega_2 = \gamma\omega_1'$. On the other hand, η is the unique element of $\mathcal{H}$ such that $\omega_1\eta = \gamma\omega_2'$. But

$$\omega_1\nu\left(\left[\omega_1^{-1},\ \omega_2\right]\right)\nu\left(\left[\omega_1'^{-1},\ \omega_2'\right]\right) = \omega_2\nu\left(\left[\omega_1'^{-1},\ \omega_2'\right]\right) = \gamma\omega_1'\nu\left(\left[\omega_1'^{-1},\ \omega_2'\right]\right) = \gamma\omega_2'.$$

By the uniqueness of η we get

$$\nu\left(\left[\omega_1^{-1},\ \omega_2\right]\left[\omega_1'^{-1},\ \omega_2'\right]\right) = \nu\left(\left[\omega_1^{-1},\ \omega_2\right]\right)\nu\left(\left[\omega_1'^{-1},\ \omega_2'\right]\right).$$

So ν is a homomorphism of groupoids. □

Proof of Proposition 6.1.32. Let Ω be a graph of a generalised isomorphism from $\mathcal{G}$ to $\mathcal{H}$ and let Ω' be a graph of the inverse isomorphism from $\mathcal{H}$ to $\mathcal{G}$. Let Ω have the anchor maps ρ and σ and Ω' have the anchor maps ρ' and σ'.

We first show that the left actions on Ω and Ω' are free and define an (algebraic) isomorphism between Ω' and Ω^{-1}. In a second step we show that this isomorphism also is a homeomorphism, implying that the left actions on Ω and Ω' are proper (because the right actions on Ω' and Ω are proper).

Because the morphisms $[\Omega]$ and $[\Omega']$ are inverses of each other we can find an isomorphism $\varphi_{\mathcal{G}}\colon {}_{\mathcal{G}}\Omega \times_{\mathcal{H}} \Omega'_{\mathcal{G}} \to {}_{\mathcal{G}}\mathcal{G}_{\mathcal{G}}$ of $\mathcal{G}$-$\mathcal{G}$-bimodules and an isomorphism $\varphi_{\mathcal{H}}\colon {}_{\mathcal{H}}\Omega' \times_{\mathcal{G}} \Omega_{\mathcal{H}} \to {}_{\mathcal{H}}\mathcal{H}_{\mathcal{H}}$ of $\mathcal{H}$-$\mathcal{H}$-bimodules. Define an "inner product"

$$\langle\cdot,\cdot\rangle_{\mathcal{G}}\colon \Omega \times_{\mathcal{H}^{(0)}} \Omega' \to \mathcal{G},\ (\omega, \omega') \mapsto \varphi_{\mathcal{G}}\left([\omega, \omega']\right)$$

where $[\omega, \omega']$ denotes the equivalence class of (ω, ω') in the quotient space $\Omega \times_{\mathcal{H}} \Omega'$ of $\Omega \times_{\mathcal{H}^{(0)}} \Omega'$ by the (diagonal) action of $\mathcal{H}$. The inner product if $\mathcal{G}$-linear in both components and $\mathcal{H}$-balanced in the sense that $\langle \omega\eta,\ \omega' \rangle_{\mathcal{G}} = \langle \omega,\ \eta\omega' \rangle_{\mathcal{G}}$. Similarly define a $\mathcal{H}$-bilinear $\mathcal{G}$-balanced inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}} \colon \Omega' \times_{\mathcal{G}^{(0)}} \Omega \to \mathcal{H}$.

As a first consequence, it is obvious that the maps $\sigma\colon \Omega \to \mathcal{H}^{(0)}$ and $\sigma'\colon \Omega' \to \mathcal{G}^{(0)}$ are surjective because $\varphi_{\mathcal{G}}$ and $\varphi_{\mathcal{H}}$ are surjective. Secondly, we can immediately deduce that the left actions on Ω and Ω' are free: Let $\omega \in \Omega$ and $\gamma \in \mathcal{G}$ such that $s(\gamma) = \rho(\omega)$ and $\gamma\omega = \omega$. Find some $\omega' \in \Omega'$ such that $\rho'(\omega') = \sigma(\omega)$. Then

$$\langle \omega, \omega' \rangle_{\mathcal{G}} = \langle \gamma\omega, \omega' \rangle_{\mathcal{G}} = \gamma \langle \omega, \omega' \rangle_{\mathcal{G}}$$

and hence $\gamma = \rho(\omega)$. Similarly, the left action of $\mathcal{H}$ on Ω' can be shown to be free.

The "inner product" $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ is "faithful":

1. If $\omega_1, \omega_2 \in \Omega$ and $\omega' \in \Omega'$ such that $\sigma(\omega_1) = \sigma(\omega_2) = \rho'(\omega')$ and $\langle \omega_1,\ \omega' \rangle_{\mathcal{G}} = \langle \omega_2,\ \omega' \rangle_{\mathcal{G}}$, then $\omega_1 = \omega_2$: This follows because by definition of the inner product and by the injectivity of $\varphi_{\mathcal{G}}$ we have $[\omega_1, \omega'] = [\omega_2, \omega']$, so there is an $\eta \in \mathcal{H}$ such that $(\omega_1\eta, \eta^{-1}\omega') = (\omega_2, \omega')$; because Ω' is a free left $\mathcal{H}$-space it follows that $\eta = \rho'(\omega')$ and hence $\omega_1 = \omega_2$.

2. Using the freeness of the right action on Ω one proves: If $\omega \in \Omega$ and $\omega_1', \omega_2' \in \Omega'$ such that $\sigma(\omega) = \rho'(\omega_1') = \rho'(\omega_2')$ and $\langle \omega,\ \omega_1' \rangle_{\mathcal{G}} = \langle \omega,\ \omega_2' \rangle_{\mathcal{G}}$, then $\omega_1' = \omega_2'$.

As a consequence, we have for all $\omega_1, \omega_2 \in \Omega$ and $\omega' \in \Omega'$ such that $\sigma(\omega_1) = \sigma(\omega_2) = \rho'(\omega')$:

$$\left\langle \omega_1,\ \omega' \right\rangle_{\mathcal{G}}^{-1} \omega_1 = \left\langle \omega_2,\ \omega' \right\rangle_{\mathcal{G}}^{-1} \omega_2,$$

because

$$\left\langle \left\langle \omega_1,\ \omega' \right\rangle_{\mathcal{G}}^{-1} \omega_1,\ \omega' \right\rangle = \left\langle \omega_1,\ \omega' \right\rangle_{\mathcal{G}}^{-1} \left\langle \omega_1,\ \omega' \right\rangle = \sigma'(\omega') = \left\langle \left\langle \omega_2,\ \omega' \right\rangle_{\mathcal{G}}^{-1} \omega_2,\ \omega' \right\rangle.$$

This implies that the canonical map from $\mathcal{G} \setminus \Omega$ to $\mathcal{H}^{(0)}$ is not only surjective, but also injective: If $\omega_1, \omega_2 \in \Omega$ with $\sigma(\omega_1) = \sigma(\omega_2)$, then we can find an $\omega' \in \Omega$ such that $\rho'(\omega') = \sigma(\omega_1) = \sigma(\omega_2)$. Then

$$\omega_2 = \left\langle \omega_2,\ \omega' \right\rangle_{\mathcal{G}} \left\langle \omega_1,\ \omega' \right\rangle_{\mathcal{G}}^{-1} \omega_1,$$

so ω_1 and ω_2 are in the same $\mathcal{G}$-orbit.

We now define a bijection from Ω' to Ω which is a bimodule homomorphism (if we regard Ω as Ω^{-1}): For all $\omega \in \Omega$ define

$$\Phi(\omega') = \left\langle \omega,\ \omega' \right\rangle_{\mathcal{G}}^{-1} \omega$$

where ω is an arbitrary element of Ω such that $\sigma(\omega) = \rho'(\omega')$. We have just seen that this definition is independent of the choice of ω. If $\gamma \in \mathcal{G}$ and $\eta \in H$ such that $s(\eta) = \rho'(\omega')$ and $r(\gamma) = \sigma'(\omega')$, then

$$\Phi\left(\eta\omega'\right) = \left\langle \omega,\ \eta\omega' \right\rangle_{\mathcal{G}}^{-1} \omega = \left\langle \omega\eta,\ \omega' \right\rangle_{\mathcal{G}}^{-1} \omega\eta\eta^{-1} = \Phi(\omega')\eta^{-1}$$

(with $\omega \in \Omega$ such that $\sigma(\omega) = r(\eta)$) and

$$\Phi\left(\omega'\gamma\right) = \left\langle \omega,\ \omega'\gamma \right\rangle_{\mathcal{G}}^{-1} \omega = \gamma^{-1} \left\langle \omega,\ \omega' \right\rangle_{\mathcal{G}}^{-1} \omega = \gamma^{-1}\Phi(\omega')$$

(with $\omega \in \Omega$ such that $\sigma(\omega) = \rho'(\omega')$).

Note that for all $\omega' \in \Omega'$, the element $\Phi(\omega')$ of Ω is unique with the property $\langle \Phi(\omega'),\ \omega' \rangle_{\mathcal{G}} = \sigma'(\omega')$. This could also have been our definition of Φ. We define an inverse homomorphism $\Psi\colon \Omega \to$

Ω' as follows: For all $\omega \in \Omega$, the element $\Psi(\omega)$ is the unique element of Ω' such that $\langle \omega,\ \Psi(\omega)\rangle_{\mathcal{G}} = \rho(\omega)$. Such an element exists, because we can find some $\omega' \in \Omega'$ such that $\sigma(\omega) = \rho'(\omega')$; then

$$\left\langle \omega,\ \omega' \langle \omega, \omega' \rangle_{\mathcal{G}}^{-1} \right\rangle_{\mathcal{G}} = \rho(\omega).$$

It is unique because the inner product is faithful. We show that Ψ is the inverse of Φ: If $\omega \in \Omega$, then

$$\langle \Phi(\Psi(\omega)),\ \Psi(\omega)\rangle_{\mathcal{G}} = \sigma'\left(\Psi(\omega)\right) = \langle \omega,\ \Psi(\omega)\rangle_{\mathcal{G}}\,;$$

if $\omega' \in \Omega'$, then

$$\left\langle \Phi(\omega'),\ \Psi(\Phi(\omega'))\right\rangle_{\mathcal{G}} = \rho\left(\Phi(\omega')\right) = \left\langle \Phi(\omega'),\ \omega' \right\rangle_{\mathcal{G}}.$$

What is left to show is that Φ and Ψ are continuous. For Ψ, this follows from the commutativity of the diagram

$$\begin{tikzcd}
\Omega \times_{\sigma,\rho'} \Omega' \arrow[r] \arrow[d] & \Omega' \times_{\rho',\rho'} \Omega' \arrow[d] \\
\Omega \arrow[r, "\Psi"] & \Omega'
\end{tikzcd}$$

where the top arrow is given by the continuous map $(\omega, \omega') \mapsto \left(\omega' \langle \omega, \omega' \rangle_{\mathcal{G}}^{-1}, \omega'\right)$ and the vertical arrows are the projections onto the first component (which are both continuous, surjective and open). The continuity of Φ is proved similarly. $\square$

Remark D.1.3. There is an interesting subtlety about the preceding proof (or rather about the Proposition that is proved): We have constructed an isomorphism Ψ from Ω^{-1} to Ω', and there is a canonical isomorphism from $\Omega \times_{\mathcal{H}} \Omega^{-1}$ to $\mathcal{G}$. In the proof, we have chosen an isomorphism $\varphi_{\mathcal{H}}$ from $\Omega \times_{\mathcal{H}} \Omega'$ to $\mathcal{G}$. The resulting diagram

$$\begin{tikzcd}
\Omega^{-1} \times_{\mathcal{G}} \Omega \arrow[rr] \arrow[dr] & & \Omega' \times_{\mathcal{G}} \Omega \arrow[dl] \\
& \mathcal{H} &
\end{tikzcd}$$

is not commutative in general (whereas the corresponding for $\mathcal{G}$ does commute). The reason is that we have used only the inner product coming from the isomorphism $\varphi_{\mathcal{G}}$ in the construction of Ψ. There still is some freedom to chose the isomorphism $\varphi_{\mathcal{H}}$: One could change it by some isomorphism of the $\mathcal{H}$-$\mathcal{H}$-bimodule $\mathcal{H}$ without changing anything in the proof. The above diagram can be shown to commute up to such an isomorphism. Such isomorphisms can exists: for example, if $\mathcal{H}$ is a group, then multiplication by any element in the center will give an isomorphism.

The same isomorphism also enters the following equality in the sense that it is only true if one corrects $\varphi_{\mathcal{H}}$ (and hence the $\mathcal{H}$-valued inner product) by the isomorphism:

$$\left\langle \omega_1,\ \omega' \right\rangle_{\mathcal{G}} \omega_2 = \omega_1 \left\langle \omega',\ \omega_2 \right\rangle_{\mathcal{H}}$$

for all $\omega_1, \omega_2 \in \Omega$ and $\omega' \in \Omega'$ such that $\sigma(\omega_1) = \rho'(\omega')$ and $\sigma'(\omega') = \rho(\omega_2)$.

D.2 Some proofs of results of Section 6.5

Proof of 6.5.1. We show that Γ satisfies the axioms (C1)-(C4):

(C1) First, Δ is a linear subspace of $\Gamma(Y, E)$. The map $\delta \mapsto p_!\delta$ is linear, so its image is a linear subspace.

(C2) To show (C2) we use the existence of a faithful continuous field of measures on Y over X; therefore we need the following lemma, which rephrases Lemma 3.2 of [LG99] in our context.[1]

Lemma D.2.1. *Let ν be a continuous field of measures on Y over X with proper[2] support. For all $\xi \in \Gamma(Y, E)$ and all $y \in Y$, define*

$$(\nu * \xi)(y) := \int_{z \in Y_{p(y)}} \alpha_{(y,z)}(\xi(z)) \, \mathrm{d}\nu_{p(z)}.$$

Then $\nu * \xi$ is an element of Δ.

To prove it we proceed as in [LG99]:

Proof. Let ξ be an element of $\Gamma(Y, E)$. Define $R \colon Y \times_X Y \to Y,\ (y, z) \mapsto y$ and $S \colon Y \times_X Y \to Y,\ (y, z) \mapsto z$. Define $p^*\nu$ to be a continuous field of measures on $Y \times_X Y$ over Y with coefficient map R which is given, for each $y \in Y$, by $(p^*\nu)_y = \nu_{p(y)}$ on $R^{-1}(\{y\}) \cong Y_{p(y)}$. This field is the pullback of ν in the diagram

$$\begin{tikzcd} (Y, \nu) \arrow[d, "p"'] & (Y \times_X Y, p^*\nu) \arrow[l, "S"'] \arrow[d, "R"] \\ X & Y \arrow[l, "p"'] \end{tikzcd}$$

If L is a compact subset of Y, then $p(L) = K$ is compact. Now

$$\bigcup_{l \in L} \operatorname{supp}(p^*\nu)_l = \bigcup_{l \in L} \left(\operatorname{supp} \nu_{p(l)} \times \{l\}\right) \subseteq \left(\bigcup_{k \in K} \operatorname{supp} \nu_k\right) \times L,$$

which is compact since the support of ν is proper. So the support of $p^*\nu$ is proper, too.

For all $(y, z) \in Y \times_X Y$, we have $\alpha_{(y,z)}(\xi(z)) = (\alpha \circ (\xi \circ S))(y, z)$, so $(y, z) \mapsto \alpha_{(y,z)}(\xi(z))$ belongs to $\Gamma(Y \times_X Y,\ R^*E)$. Now Lemma B.3.4 says that $\nu * \xi = \nu(\xi)$ is an elementr of $\Gamma(Y, E)$ because $p^*\nu$ has proper support. By construction, $\nu * \xi$ is in Δ. $\square$

Let us continue with the proof of (C2). By assumption, we can find a faithful continuous field μ of measures on Y over X. Let $y \in Y$. We show that for all $e \in E_y$ and $\varepsilon > 0$ there is a $\xi \in \Delta$ such that $\|\xi(y) - e\| \le \varepsilon$. This implies (C2). So let $e \in E_y$ and $\varepsilon > 0$. Find an arbitrary section $\xi' \in \Gamma(Y, E)$ such that $\xi'(y) = e$. The function $\zeta \colon z \mapsto \alpha_{(y,z)}\xi'(z)$ is in $\mathcal{C}\left(Y_{p(y)}, E_y\right)$. Find an open neighbourhood V of y in Y such that $\|\zeta(z) - \zeta(y)\| < \varepsilon$ for all $z \in V \cap Y_{p(y)}$. Now we

[1] Le Gall uses this lemma in conjunction with a strong result of E. Blanchard (see Proposition 3.13 of [Bla96]). However, this can only be done in case that $\mathcal{C}_0(Y)$ is separable. We wish to use the more general condition that we can find a faithful field of measures on Y, which is more natural in our setting.

[2] We only need compact support, but the proper case comes for free.

can find a local cut-off function in the sense of Lemma B.2.6, i.e., a function $\chi \in \mathcal{C}_c(Y)$ such that $\chi \geq 0$, $\operatorname{supp}\chi \subseteq V$, and such that $\mu(\chi)(p(y)) = \int_{z\in Y_{p(y)}} \chi(z)\,\mathrm{d}\mu_{p(y)}(z) = 1$.

Define $\nu := \chi\mu$. Then ν has compact support. By what we have just proved in Lemma D.2.1, $\xi := \nu * \xi'$ is contained in Δ. Now

$$\xi(y) = \left(\nu * \xi'\right)(y) = \int_{z\in Y_{p(y)}} \alpha_{(y,z)}(\xi'(z))\,\mathrm{d}\nu_{p(y)} = \int_{z\in Y_{p(y)}} \chi(z)\alpha_{(y,z)}(\xi'(z))\,\mathrm{d}\mu_{p(y)},$$

so that

$$\begin{aligned}
\|\xi(y) - e\| &= \left\|\int_{z\in Y_{p(y)}} \chi(z)\alpha_{(y,z)}(\xi'(z))\,\mathrm{d}\mu_{p(y)} - \int_{z\in Y_{p(y)}} \chi(z)e\,\mathrm{d}\mu_{p(y)}\right\| \\
&\leq \int_{z\in Y_{p(y)}\cap\operatorname{supp}\chi} \chi(z)\left\|\alpha_{(y,z)}(\xi'(z)) - e\right\|\,\mathrm{d}\mu_{p(y)} \\
&\leq \int_{z\in Y_{p(y)}\cap\operatorname{supp}\chi} \chi(z)\varepsilon\,\mathrm{d}\mu_{p(y)} = \varepsilon.
\end{aligned}$$

(C3) Let $\delta \in \Delta$, $x \in X$ and $\varepsilon > 0$. Find some $y \in Y$ with $p(y) = x$. Since δ is a section we can find an open neighbourhood V of y in Y such that $\|\delta(v)\| \leq \|\delta y\| + \varepsilon$ for all $v \in V$. Because p is open, the set $U := p(V)$ is an open neighbourhood of x in X. Let $u \in U$. Find a $v \in V$ such that $p(v) = u$. Then

$$\|p_!\delta(u)\| = \|\delta(v)\| \leq \|\delta(y)\| + \varepsilon = \|p_!\delta(x)\| + \varepsilon.$$

(C4) Let ζ be a selection of $p_!E$ such that for all $x \in X$ and all $\varepsilon > 0$ there is a neighbourhood U of x and a $\delta \in \Delta$ such that $\|p_!\delta(u) - \zeta(u)\| < \varepsilon$ for all $u \in U$. We show that there is a $\delta' \in \Delta$ such that $p_!\delta' = \zeta$.

For all $y \in Y$, define $\delta'(y) := \zeta(p(y))_y \in E_y$. Then δ' is a selection of E. By definition of $p_!E$, the selection δ' satisfies $\alpha_{(z,y)}\delta'(y) = \delta'(z)$ for all $(z,y) \in Y \times_X Y$. We have to show that δ' is in $\Gamma(Y,E)$. To this end, let $y \in Y$ and $\varepsilon > 0$. Find a neighbourhood U of $p(y)$ in X and a $\delta \in \Delta$ such that $\|p_!\delta(u) - \zeta(u)\| < \varepsilon$ for all $u \in U$. Let $V := p^{-1}(U)$, being a neighbourhood of y in Y. Then

$$\left\|\delta'(v) - \delta(v)\right\| = \|\zeta(p(v))_v - p_!\delta(p(v))_v\| = \|\zeta(p(v)) - p_!\delta(p(v))\| < \varepsilon$$

for all $v \in V$. So δ' is a section, and, by definition, $p_!\delta' = \zeta$. □

Proof of Proposition 6.5.3. We obviously have a functor which is isometric and linear on the morphism sets. Let E and F be $Y \times_X Y$-Banach spaces with action α and β, respectively. We have to compare $p_!\left(E \otimes_Y F\right)$ and $(p_!E) \otimes_X (p_!F)$. The fibre at $x \in X$ of the first X-Banach space consist of the families $(t_y)_{y\in Y_x}$ with $t_y \in E_y \otimes F_y$ and $(\alpha\otimes\beta)_{(z,y)}t_y = t_z$ for all $z, y \in Y_x$. The fibre at x of the second space is $(p_!E)_x \otimes (p_!F)_x$. We construct an isometric isomorphism from the second to the first space:

For all $e = (e_y)_{y\in Y_x} \in (p_!E)_x$ and $f = (f_y)_{y\in Y_x} \in (p_!F)_x$, define $\mu_x(e,f) := (e_y \otimes f_y)_{y\in Y_x}$. Then $\mu_x(e,f) \in p_!\left(E \otimes_Y F\right)_x$ because $(\mu_x(e,f))_y \in E_y \otimes F_y$ and

$$(\alpha\otimes\beta)_{(z,y)}(e_y \otimes f_y) = \left(\alpha_{(z,y)}e_y\right) \otimes \left(\beta_{(z,y)}f_y\right) = e_z \otimes f_z = \mu_x(e,f)_z$$

for all $(z, y) \in Y \times_X Y$. Moreover, μ_x is a contractive bilinear map. So it gives rise to a contractive linear map

$$\hat{\mu}_x \colon (p_! E)_x \otimes (p_! F)_x \to p_! (E \otimes_Y F)_x .$$

We show that, this way, we get a contractive continuous field of linear maps from $p_! E \otimes_X p_! F$ to $p_! (E \otimes_Y F)$. The sections of the form $x \mapsto (p_!\delta)(x) \otimes (p_!\delta')(x)$ with $\delta \in \Delta_E$ and $\delta' \in \Delta_F$ form a total subset in $\Gamma(X,\ p_! E \otimes_X p_! F)$. Now

$$\hat{\mu}_x \left((p_!\delta)(x) \otimes (p_!\delta')(x)\right) = \mu_x \left(\left(\delta(y)\right)_{y \in Y_x},\ \left(\delta'(y)\right)_{y \in Y_x}\right) = \left(\delta(y) \otimes \delta'(y)\right)_{y \in Y_x}$$

for all $x \in X$. Since $y \mapsto \delta(y) \otimes \delta'(y)$ is in $\Delta_{E \otimes_Y F}$, this shows that $\hat{\mu}$ is continuous.

Now we show that $\hat{\mu}$ is a continuous field of isometric isomorphisms. Let $x \in X$. Fix a $y \in Y_x$. Then

$$(p_! E)_x \otimes (p_! F)_x \cong E_y \otimes F_y \cong (p_! (E \otimes_Y F))_x ,$$

where the first isomorphism is given by componentwise evaluation at x and the second isomorphism (as a map from the right to the left) is given by (global) evaluation at x. The composition of the isomorphisms is $\hat{\mu}_x$.

A straightforward calculation shows that $\hat{\mu}$ is natural and respects the associativity of the tensor products. $\square$

Proof of Proposition 6.5.4. 1. Let E be a $Y \times_X Y$-Banach space.

- I^E **is an isomorphism:** Fibrewise, it is easy to see that I^E is an isometric isomorphism. The set of all $(p_!\delta) \circ p$, where $\delta \in \Delta_E$, is total in $p^* p_! E$. Let δ be an element of δ, then

$$I^E_y \left((p_!\delta)(p(y))\right) = I^E_y \left(\left(\delta(z)\right)_{y \in Y_{p(y)}}\right) = \delta(y).$$

  In other words, I^E identifies $(p_!\delta) \circ p$ and δ. In particular, I^E is a continuous field. It clearly is $Y \times_X Y$-equivariant.

- $E \mapsto I^E$ **is natural:** Let E and F be $Y \times_X Y$-Banach spaces and let T be a bounded equivariant isomorphism from E to F. Let $y \in Y$ and $(e_z)_{z \in Y_{p(y)}} \in (p^* p_! E)_y = (p_! E)_{p(y)}$. Then

$$I^F_y \left((p^* p_! T)_y(e)\right) = I^F_y \left((p_! T)_{p(y)}(e)\right) = I^F_y \left(\left(T_z e_z\right)_{z \in Y_{p(y)}}\right) = T_y e_y = T_y \left(I^E_y(e)\right).$$

  So $I^F \circ (p^* p_! T) = T \circ I^E$.

- $E \mapsto I^E$ **is multiplicative:** Let E and F be $Y \times_X Y$-Banach spaces. We have to check that the following diagram commutes:

$$\begin{tikzcd}
(p^* p_! E) \otimes_Y (p^* p_! F) \arrow[r, "="] \arrow[d, "I^E \otimes I^F"'] & p^* (p_! E \otimes_X p_! F) \arrow[r, "p^*(\hat{\mu})"] & p^* p_! (E \otimes_Y F) \arrow[d, "I^{E \otimes F}"] \\
E \otimes_Y F \arrow[rr, "="] & & E \otimes_Y F
\end{tikzcd}$$

  Let $y \in Y$ and $(e_z)_{z \in Y_{p(y)}} \in (p^* p_! E)_y = (p_! E)_{p(y)}$ and $(f_z)_{z \in Y_{p(y)}} \in (p^* p_! F)_y = (p_! F)_{p(y)}$. The map $\left(I^E \otimes I^F\right)_y = I^E_y \otimes I^F_y$ sends this to $e_y \otimes f_y \in (E \otimes_Y F)_y = E_y \otimes F_y$.

On the other hand, $p^*(\hat{\mu})$ sends it to $(e_z \otimes f_z)_{z \in Y_{p(y)}} \in (p_! E \otimes_X p_! F)_{p(y)}$. This is, in turn, mapped by $I_y^{E \otimes_Y F}$ to $e_y \otimes f_y$, so we are done.

2. Let E be a X-Banach space.

- J^E **is an isomorphism:** Fibrewise, this is clear. We just have to check that J^E is a continuous field. To this end, we determine Δ_{p^*E}. It is the set of sections $\delta \in \Gamma(Y, p^*E)$ such that $\delta(y) = \mathrm{Id}_{E_{p(y)}} \delta(y) = \alpha_{(z,y)}\delta(y) = \delta(z)$. Let δ be such a section. We have to check that $J^E \circ p_!\delta \in \Gamma(X, E)$. But $J_x^E p_!\delta(x) = \delta(y)$ for all $x \in X$ and $y \in Y$ such that $p(y) = x$. So $\delta = \left(J^E \circ p_!\delta\right) \circ p$. By Lemma D.2.2 the fact that $\delta \in \Gamma(Y, p^*E)$ implies $J^E \circ p_!\delta \in \Gamma(X, E)$. So J^E is continuous,
- $E \mapsto J^E$ **is natural:** Let E and F be X-Banach spaces and let $T \colon E \to F$ be a bounded continuous field of linear maps. Let $x \in X$ and $(e)_{y \in Y_x}$ be an element of $(p_! p^* E)_x$. Then

$$\begin{aligned} & J_x^F \left((p_! p^* T)_x (e)_{y \in Y_x}\right) = J_x^F \left(((p^*T)_y(e))_{y \in Y_x}\right) \\ = \quad & J_x^F \left((T_x(e))_{y \in Y_x}\right) = T_x(e) = T_x\left(J_x^E (e)_{y \in Y_x}\right). \end{aligned}$$

  So $J^F \circ (p_! p^* T) = T \circ J^E$.
- $E \mapsto J^E$ **is multiplicative:** Let E and F be X-Banach spaces. We have to check that the following diagram commutes:

$$\begin{tikzcd} (p_! p^* E) \otimes_X (p_! p^* F) \arrow[r, "\hat{\mu}"] \arrow[d, "J^E \otimes J^F"'] & p_! \left(p^* E \otimes_Y p^* F\right) \arrow[r, "="] & p_! p^* \left(E \otimes_X F\right) \arrow[d, "J^{E \otimes F}"] \\ E \otimes_X F \arrow[rr, "="] & & E \otimes_X F \end{tikzcd}$$

  Let $x \in X$ and $(e)_{y \in Y_x} \in (p_! p^* E)_x$ and $(f)_{y \in Y_x} \in (p_! p^* F)_x$. Then

$$\begin{aligned} & \left(J^E \otimes J^F\right)_x \left((e)_{y \in Y_x} \otimes (f)_{y \in Y_x}\right) = J_x^E (e)_{y \in Y_x} \otimes J_x^F (f)_{y \in Y_x} \\ = \quad & e \otimes f = J^{E \otimes_X F} (e \otimes f)_{x \in Y_x} = J^{E \otimes_X F}\left(\hat{\mu}\left((e)_{y \in Y_x} \otimes (f)_{y \in Y_x}\right)\right). \end{aligned}$$

  This means that the diagram is indeed commutative. □

Lemma D.2.2. *Let E be a continuous field of Banach spaces over X and let ξ be a selection of E (continuous or not). Then ξ is in $\Gamma(X, E)$ if and only if $\xi \circ p$ is in $\Gamma(Y, p^*E)$.*

Proof. If ξ is a section, then $\xi \circ p$ is a in $\Gamma(Y, p^*E)$ by the definition of the sections of p^*E.

Assume now that $\xi \circ p$ is a section of p^*E. Let $x \in X$ and $\varepsilon > 0$. Find a $y \in Y$ such that $p(y) = x$. Find a neighbourhood V of y in Y and a section ζ of E such that $\|\xi(p(v)) - \zeta(p(v))\| \leq \varepsilon$ for all $v \in V$. Let $U := p(V)$. Then U is an open neighbourhood of x in X. Let $u \in U$. Then we can find a $v \in V$ such that $p(v) = u$. Now

$$\|\xi(u) - \zeta(u)\| = \|\xi(p(v)) - \zeta(p(v))\| \leq \varepsilon.$$

Hence ξ is a section. □

Proof of Proposition 6.6.7. Define $\Omega := \mathrm{Graph}(f) = \mathcal{G}^{(0)} \times_{\mathcal{H}^{(0)}} \mathcal{H}$. Let (E,T) be a cycle in $\mathbb{E}^{\mathrm{ban}}_{\mathcal{H}}(A,B)$. We have a canonical concurrent homomorphism Φ from f^*E to Ω^*E from Proposition 6.6.3. We have to check that this is indeed an isomorphism of $\mathrm{KK}^{\mathrm{ban}}$-cycles. As Φ is already an isometric isomorphism and is surely compatible with the gradings, it is only left to check that it intertwines the operators. But here, we have to be a little bit more precise: The operator on Ω^*E is not uniquely defined, and it will suffice to find one "version of Ω^*T" which is compatible with Φ. Because Φ is an isomorphism, we can write down exactly what this means for Ω^*T; the result is, that the version we are looking for has to satisfy

$$\left(\Omega^*T^{>}\right)_g \left(e^{>}_{g,\eta}\right)_{f(g)=r(\eta)} = \left(\eta T^{>}_{f(g)}\eta^{-1}e^{>}_{g,\eta}\right)_{f(g)=r(\eta)}$$

for all $g \in \mathcal{G}^{(0)}$ and $(e^{>}_{g,\eta})_{f(g)=r(\eta)} \in (\Omega^*E^{>})_g$ (and similarly on the left-hand side). Define an operator $\tilde{T} \in \mathrm{L}_{\sigma^*B}(\sigma^*E)$ by setting $\tilde{T}^{>}_{(g,\eta)}(e^{>}_{(g,\eta)}) := \eta^{-1}T^{>}_{f(g)}\eta e^{>}_{(g,\eta)}$ for all $(g,\eta) \in \Omega$ and $e^{>}_{(g,\eta)} \in (\sigma^*E^{>})_{(g,\eta)} = E^{>}_{s(\eta)}$ (and analogously on the left-hand side). Using the notation of 3.4.24, this operator can be written as $\pi_2^*((\alpha^{\mathrm{L}(E)})^{-1}(r^*T))$, where π_2 denotes the canonical map from $\Omega = \mathcal{G}^{(0)} \times_{\mathcal{H}^{(0)}} \mathcal{H}$ to the second component $\mathcal{H}$. If we can show that $(f_\Omega^*E,\ \tilde{T})$ is homotopic in $\mathbb{E}^{\mathrm{ban}}_{\rho^*(\mathcal{G})}(f_\Omega^*A,\ f_\Omega^*B)$ to $(f_\Omega^*E,\ f_\Omega^*T)$, then we are done, because Φ intertwines f^*T and $\rho_!\tilde{T}$ (note that $\tilde{T}$ is $\Omega \times_\rho \Omega$-equivariant).

This homotopy can be constructed using Lemma 3.5.11: Let $\tilde{a} \in \Gamma(\Omega,\ f_\Omega^*A)$. We show that $\tilde{a}(f_\Omega^*T - \tilde{T})$ and $(f_\Omega^*T - \tilde{T})\tilde{a}$ are locally compact. For this, it suffices to consider the case that $\tilde{a}$ is of the form $a \circ \pi_2$ with $a \in \Gamma(\mathcal{H},\ s^*A)$. Note that $f_\Omega^*T = \sigma^*T = \pi_2^*s^*T$. Now

$$\begin{aligned}
(a\circ\pi_2)\left(f_\Omega^*T - \tilde{T}\right) &= (a\circ\pi_2)\left(\pi_2^*s^*T - \pi_2^*\left(\left(\alpha^{\mathrm{L}(E)}\right)^{-1}(r^*T)\right)\right)\\
&= \pi_2^*\left(a\left(s^*T - \left(\alpha^{\mathrm{L}(E)}\right)^{-1}(r^*T)\right)\right)\\
&= \pi_2^*\left(\left(\alpha^{\mathrm{L}(E)}\right)^{-1}\left(\alpha^A(a)\left(\alpha^{\mathrm{L}(E)}(s^*T) - r^*T\right)\right)\right).
\end{aligned}$$

Here α^A denotes the $\mathcal{G}$-action on A. The operator $\alpha^A(a)(\alpha^{\mathrm{L}(E)}(s^*T) - r^*T)$ is locally compact because (E,T) is a $\mathrm{KK}^{\mathrm{ban}}$-cycle. So $(\alpha^{\mathrm{L}(E)})^{-1}$ of this operator is locally compact by Proposition 3.4.25. By Proposition 3.3.22, the pullback by π_2 of the resulting operator is also locally compact. The same arguments show that $(f_\Omega^*T - \tilde{T})\tilde{a}$ is locally compact. $\square$

Appendix E

Some Remarks

E.1 A note concerning $\mathcal{C}_0(X)$-Banach algebras

There is an alternative definition of $\mathcal{C}_0(X)$-Banach algebras using structure homomorphisms which might be more familiar in the context of C*-algebras. In this appendix we would like to show how this definition is related to the definition in Section 2.2 (including a subtlety).

Definition E.1.1 (Structure homomorphism). Let B be a $\mathcal{C}_0(X)$-Banach algebra. Define a right action of $\mathcal{C}_0(X)$ on B by setting $b\varphi := \varphi b$ for all $b \in B$ and $\varphi \in \mathcal{C}_0(X)$. Define

$$\theta_B \colon \mathcal{C}_0(X) \to \mathrm{M}(B),\ \varphi \mapsto (b \mapsto b\varphi,\ b \mapsto \varphi b).$$

This map is a well-defined homomorphism of Banach algebras, called the *structure homomorphism* of the $\mathcal{C}_0(X)$-Banach algebra B.

Definition E.1.2 (Non-degenerate homomorphism). Let B and C be Banach algebras and $\varphi\colon B \to C$ a homomorphism. Then φ is called *non-degenerate* if $\varphi(B)C$ and $C\varphi(B)$ are dense in C.

The composition of non-degenerate homomorphisms is non-degenerate. The identity map on a Banach algebra is non-degenerate if and only the Banach algebra is non-degenerate. The non-degenerate Banach algebras together with the non-degenerate homomorphisms form a category.

Proposition E.1.3. *Let B be a Banach algebra and let θ_B be a non-degenerate homomorphism from $\mathcal{C}_0(X)$ to $\mathrm{M}(B)$ such that*

$$\forall b \in B\, \forall \varphi \in \mathcal{C}_0(X):\ \theta_B(\varphi)\, b \;=\; b\, \theta_B(\varphi).$$

Then B is a $\mathcal{C}_0(X)$-Banach algebra.

The condition given in the preceding proposition is in general not equivalent to the condition that the image of θ_B is contained in the centre of $\mathrm{M}(B)$. But if B is non-degenerate or has no annihilators, then they are equivalent by the following lemma.

Lemma E.1.4. *Let B be a non-degenerate Banach algebra and let $m \in \mathrm{M}(B)$. Then the following statements are equivalent:*

1. *$m \in \mathrm{Z}\,\mathrm{M}(B)$;*
2. *$\forall b \in B:\ mb = bm$.*

Proof. 1.⇒ 2.: Let $b, c \in B$. Then

$$\begin{aligned} m(bc) &= m(\psi_B(b)c) = (m\psi_B(b))c \overset{1.}{=} (\psi_B(b)m)c = \psi_B(b)(mc) \\ &= b(m\psi_B(c)) \overset{1.}{=} b(\psi_B(c)m) = (bc)m. \end{aligned}$$

By linearity and continuity of the maps $b \mapsto mb$ and $b \mapsto bm$ and by non-degeneracy of B we can conclude that $mb = bm$ for all $b \in B$.

2.⇒ 1.: Let $m' \in \mathrm{M}(B)$. In order to show $mm' = m'm$, let $b \in B$. Then

$$(mm')b = m(m'b) \overset{2.}{=} (m'b)m \overset{1.4.2}{=} m'(bm) \overset{2.}{=} m'(mb) = (m'm)b.$$

Also

$$b(mm') = (bm)m' \overset{2.}{=} (mb)m' \overset{1.4.2}{=} m(bm') \overset{2.}{=} (bm')m = b(m'm).$$

So $mm' = m'm$. □

The lemma also holds for Banach algebras without annihilators (instead of being non-degenerate). However, it does not hold in general, as the following example shows:

Example E.1.5. Let E be a Banach space. Let B be E equipped with the trivial multiplication. Then $\mathrm{M}(B)$ is $\mathrm{L}(E) \times \mathrm{L}(E)$. The centre of $\mathrm{M}(B)$ is $\mathrm{Z}\,\mathrm{L}(E) \times \mathrm{Z}\,\mathrm{L}(E)$. The set of elements of $\mathrm{M}(B)$ satisfying 2. is the diagonal of $\mathrm{L}(E) \times \mathrm{L}(E)$. If E has dimension $n \in \mathbb{N}$ then the centre has dimension $2n$ and the other set has dimension n^2. Obviously, already if $n = 1$, the two sets are not contained in one another.

E.2 A note concerning the local boundedness of fields of linear maps

A weaker form of the following result was mentioned in Section 3.1.

Proposition E.2.1. *Let X be a topological space. Let E and F be u.s.c. fields of Banach spaces over X and let T be a family of linear maps from E to F satisfying $T \circ \Gamma(X, E) \subseteq \Gamma(X, F)$. If X is completely regular and first countable, then T is locally bounded.*

Note that every metrisable space is completely regular and first countable, but the converse is false in general (a counterexample is the right half-open interval topology on $\mathbb{R}$; see Counterexamples in Topology [SS95], 51.).

Proof of Proposition E.2.1. (compare Proposition 1.1.9 of [Laf04]) Suppose that X is completely regular and first countable and that T is a field of morphisms. We show that if T is not locally bounded, then it cannot be a continuous. Suppose that $x \in X$ is such that $\sup_{u \in U} \|T_u\| = \infty$ for every neighbourhood U of x. Then, using the countable basis of neighbourhoods of x, we can find a sequence $(x_n)_{n \in \mathbb{N}}$ converging to x such that $\lim_{n \to \infty} \|T_{x_n}\| = \infty$. Without loss of generality we can assume that the members of this sequence are pairwise distinct and distinct from x. This means that we can find a sequence $(e_n)_{n \in \mathbb{N}}$ such that $e_n \in E_{x_n}$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \|e_n\|_{E_{x_n}} = 0$ and $\lim_{n \to \infty} \|T_{x_n}(e_n)\|_{F_{x_n}} = \infty$. By taking a subsequence of $(x_n)_{n \in \mathbb{N}}$ (and of $(e_n)_{n \in \mathbb{N}}$) we can even assume $\sum_{n \in \mathbb{N}} \|e_n\|_{E_{x_n}} < \infty$.

Let $n \in \mathbb{N}$. Then the subset $V_n := \{x_m : m \in \mathbb{N} \setminus \{n\}\} \cup \{x\}$ is compact in X. Since X is Hausdorff, the set V_n is closed. Since X is completely regular, we can find a function[1] $\varphi_n \in \mathcal{C}(X, \mathbb{C})$

[1] Here, it would probably be enough that X is regular instead of completely regular.

such that $0 \leq \varphi \leq 1$, $\varphi(x_n) = 1$ and $\varphi(v) = 0$ for all $v \in V_n$. And we can find a continuous section $\xi_n \in \Gamma_b(X, E)$ such that $\|\xi_n\|_\infty \leq 2\,\|e_n\|_{E_{x_n}}$ and $\xi_n(x_n) = e_n$. Now $\varphi_n\xi_n$ is an element of $\Gamma_b(X, E)$ such that $\|\varphi_n\xi_n\|_\infty \leq 2\,\|e_n\|_{E_{x_n}}$, $(\varphi_n\xi_n)(x_n) = e_n$, and $(\varphi_n\xi_n)(v) = 0$ for all $v \in V_n$.

Since $\sum_{n\in\mathbb{N}} \|e_n\|_{E_{x_n}} < \infty$ we can deduce that $(\varphi_n\xi_n)_{n\in\mathbb{N}}$ is absolutely summable in the Banach space $\Gamma_b(X, E)$, let $\xi := \sum_{n\in\mathbb{N}} \varphi_n\xi_n$ be the sum of this family. Then for each $n \in \mathbb{N}$, we have $\xi(x_n) = (\varphi_n\xi_n)(x_n) = e_n$. So what about $T \circ \xi$? For every $n \in \mathbb{N}$, we have $T_{x_n}(\xi(x_n)) = T(x_n)(e_n)$, so $\lim_{n\to\infty} \|T_{x_n}(\xi(x_n))\|_{E_{x_n}} = \infty$ by assumption. So $T \circ \xi$ is not locally bounded at x, hence it is not upper semi-continuous at x, hence $T \circ \xi$ is not contained in $\Gamma(X, F)$. It follows that T is not continuous. □

E.3 A lemma concerning quotient maps between Banach spaces

Lemma E.3.1. *Let X and Y be Banach spaces. If $T \in \mathrm{L}(X, Y)$ is a linear operator with norm ≤ 1 such that*

$$\forall y \in Y\ \forall \varepsilon > 0\ \exists x \in X :\ \|y - T(x)\| \leq \varepsilon\ \wedge\ \|x\| \leq \|y\|\,, \tag{E.1}$$

then T is surjective and a quotient map, i.e.,

$$\forall y \in Y\ \forall \varepsilon > 0\ \exists x \in X :\ T(x) = y\ \wedge\ \|x\| \leq \|y\| + \varepsilon.$$

Proof. Let $y \in Y$ and $\varepsilon > 0$. Define $y_0 := y$. Find an $x_0 \in X$ by property (E.1) such that $\|y_0 - T(x_0)\| \leq \varepsilon/2$ and $\|x_0\| \leq \|y_0\|$. For every $n \in \mathbb{N}_0$, define recursively $y_{n+1} := y_n - T(x_n)$ and find an element $x_{n+1} \in X$ such $\|y_{n+1} - T(x_{n+1})\| \leq 2^{-n-2}\varepsilon$ and $\|x_{n+1}\| \leq \|y_{n+1}\|$. By this choice it follows that

$$y_{n+1} = y_0 - \sum_{i=0}^{n} T(x_i)$$

for every $n \in \mathbb{N}_0$. Note that

$$\|x_{n+1}\| \leq \|y_{n+1}\| = \|y_n - T(x_n)\| \leq 2^{-n-1}\varepsilon$$

for every $n \in \mathbb{N}_0$ so we can deduce that $\sum_{i=0}^{\infty} x_i$ converges to some $x \in X$. But then

$$T(x) = \sum_{i=0}^{\infty} T(x_i) = y_0 = y$$

and

$$\|x\| = \Big\|\sum_{i=0}^{\infty} x_i\Big\| = \sum_{i=0}^{\infty} \|x_i\| \leq \sum_{i=0}^{\infty} \|y_i\| \leq \|y_0\| + \sum_{i=1}^{\infty} 2^{-i}\varepsilon = \|y_0\| + \varepsilon. \qquad \square$$

We can improve the above lemma by an ε:

Corollary E.3.2. *Let X and Y be Banach spaces. If $T \in \mathrm{L}(X, Y)$ is a linear operator with norm ≤ 1 such that*

$$\forall y \in Y\ \forall \varepsilon > 0\ \exists x \in X :\ \|y - T(x)\| \leq \varepsilon\ \wedge\ \|x\| \leq \|y\| + \varepsilon, \tag{E.2}$$

then T is surjective and a quotient map, i.e.,

$$\forall y \in Y\ \forall \varepsilon > 0\ \exists x \in X :\ T(x) = y\ \wedge\ \|x\| \leq \|y\| + \varepsilon.$$

Proof. We show that (E.1) follows from (E.2): Let $y \in Y$ and $\varepsilon > 0$. By (E.2) we can find an $x' \in X$ such that $\|y - T(x')\| \leq \varepsilon/2$ and $\|x'\| \leq \|y\| + \varepsilon/2$. Define $x := \frac{\|y\|x'}{\|y\|+\varepsilon/2}$. Then

$$\|x\| = \left\|x'\right\| \frac{\|y\|}{\|y\| + \varepsilon/2} \leq (\|y\| + \varepsilon/2) \frac{\|y\|}{\|y\| + \varepsilon/2} = \|y\|$$

and

$$\begin{aligned} \|y - T(x)\| &= \left\|y - T(x')\right\| + \left\|T(x') - T(x)\right\| \leq \varepsilon/2 + \left\|x - x'\right\| \\ &= \varepsilon/2 + \left\|x'\right\| \left|\frac{\|y\| - (\|y\| + \varepsilon/2)}{\|y\| + \varepsilon/2}\right| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

□

Corollary E.3.3. *If (E.2) is true for some dense subspace Y_0 of Y instead of Y, then it is true for all of Y and hence T is a metric surjection.*

Proof. Let $y \in Y$ and $\varepsilon > 0$. Find a $y_0 \in Y_0$ such that $\|y - y_0\| < \varepsilon/2$. Use (E.2) for y_0 to find an $x \in X$ such that $\|y_0 - T(x)\| < \varepsilon/2$ and $\|x\| \leq \|y_0\| + \varepsilon/2$. Then $\|y - T(x)\| < \varepsilon$ and $\|x\| \leq \|y\| + \varepsilon$. □

E.4 Some facts concerning $\mathcal{C}_0(X)$ and $\mathcal{C}_c(X)$

Let X be a locally compact Hausdorff space.

E.4.1 $\mathcal{C}_c(X)$ and subspaces

The following lemma is used in the proof of Lemma 5.1.3, see page 282.

Lemma E.4.1. *Let V be a closed subspace of X. Then the restriction map $G\colon \mathcal{C}_c(X) \to \mathcal{C}_c(V)$ which sends φ to $\varphi|_V$ is continuous and surjective.*

Proof. Continuity is obvious. To show surjectivity let $\psi \in \mathcal{C}_c(V)$. There are now two ways to proceed:

One can consider the Alexandroff compactification X^+. The function ψ is continuous on X^+ and vanishes on some neighbourhood of ∞. Since X^+ is compact, it is normal. We can hence apply Tietze's extension theorem to construct a continuous function on X^+ which agrees with ψ on the closed set $V \cup \{\infty\}$. Cut this function down by some function in $\mathcal{C}_c(X)$ which is 1 on the support of ψ to obtain an extension of ψ.

Alternatively, find a compact neighbourhood K of the support of ψ in X and a compact neighbourhood L of this set K. Since L is compact and therefore normal, we can find an extension φ of $\psi|_{L\cap V}$ to L of the same norm. Find a function $\chi \in \mathcal{C}_c(X)$ which is 1 on K and vanishes outside the interior of L. Then the product of χ and φ is a continuous extension of ψ to L which can be continuously extended by zero outside (the interior of) L. □

Remark E.4.2. Note that in the proof of the preceding lemma we have shown that a continuous function ψ on V with compact support can be extended to X preserving its sup-norm.

E.4.2 A short exact sequence

If $U \subseteq X$ is open then we embed $\mathcal{C}_0(U)$ into $\mathcal{C}_0(X)$ by continuation with zero.

Lemma E.4.3. *Let U_1, U_2 be open subsets of X. Define*

$$\Phi\colon \mathcal{C}_0(U_1 \cap U_2) \to \mathcal{C}_0(U_1) \times \mathcal{C}_0(U_2),\ f \mapsto (f, -f)$$

and

$$\Psi\colon \mathcal{C}_0(U_1) \times \mathcal{C}_0(U_2) \to \mathcal{C}_0(U_1 \cup U_2),\ (f_1, f_2) \mapsto f_1 + f_2.$$

Then the following sequence is an exact sequence of Banach spaces (with Φ isometric and Ψ a quotient map)

$$0 \to \mathcal{C}_0(U_1 \cap U_2) \xrightarrow{\Phi} \mathcal{C}_0(U_1) \times \mathcal{C}_0(U_2) \xrightarrow{\Psi} \mathcal{C}_0(U_1 \cup U_2) \to 0.$$

Proof. The map Φ is linear and isometric. Its image is clearly contained in the kernel of Ψ. Let (f_1, f_2) be in the kernel of Ψ, i.e., $f_1 + f_2 = 0$. On $X \setminus U_1$ the functions f_1 and $f_1 + f_2$ vanish, so f_2 has to vanish there as well. Analogously, f_1 has to vanish on $X \setminus U_2$. So both functions, f_1 and f_2, are supported in $U_1 \cap U_2$. Because we have $f_1 = -f_2$, it follows that $(f_1, f_2) = \Phi(f_1)$.

The only assertion that is left to show and that is not completely trivial is the fact that Ψ is surjective and a quotient map. We use Lemma E.3.1: Let $f \in \mathcal{C}_0(U_1 \cup U_2)$ and $\varepsilon > 0$. Then we can find a compact set $K \subseteq U_1 \cup U_2$ such that $|f|$ is less than $\varepsilon/2$ outside K. The sets U_1 and U_2 form an open cover of K, so we can find functions $\varphi_i \in \mathcal{C}_0(U_i)$ such that $0 \leq \varphi_i \leq 1$ and $\varphi_1(k)\varphi_2(k) = 1$ for all $k \in K$. Define $f_1 := \varphi_1 f$ and $f_2 := \varphi_2 f$. Then $f_i \in \mathcal{C}_0(U_i)$ and $(f_1 + f_2)(k) = (\varphi_1(k) + \varphi_2(k))f(k) = f(k)$ for all $k \in K$. If $x \in X \setminus K$, then $|(f_1 + f_2)(x)| = |\varphi_1(x)f(x) + \varphi_2(x)f(x)| \leq 2\,|f(x)| < \varepsilon$. So $\|f_1 + f_2 - f\| \leq \varepsilon$. On the other hand, we have $\|f_i\| \leq \|f\|$ and hence $\|(f_1, f_2)\| \leq \|f\|$. So Ψ is surjective and a quotient map by Lemma E.3.1. $\square$

E.4.3 Regularity conditions on X and $\mathcal{C}_0(X)$

Proposition E.4.4. *The following are equivalent for the locally compact Hausdorff space X:*

1. *X is σ-compact.*
2. *$\mathcal{C}_0(X)$ is σ-unital.*

Proof. 1. $\Rightarrow$ 2.: Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact subsets of X such that $K_n \subseteq K_{n+1}$ and $\bigcup_{n\in\mathbb{N}} U_n = X$. Define inductively an increasing sequence $(\chi_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_c(X)$ such that $0 \leq \chi_n \leq 1$ and $\chi \equiv 1$ on K_n for all $n \in \mathbb{N}$. Then $(\chi_n)_{n\in\mathbb{N}}$ is an approximate unit for $\mathcal{C}_0(X)$.

2. $\Rightarrow$ 1.: Let $(\chi_n)_{n\in\mathbb{N}}$ be an approximate unit for $\mathcal{C}_0(X)$. Let $K_n := \{x \in X : \chi_n(x) \geq 1/2\}$ for all $n \in \mathbb{N}$. Then $(K_n)_{n\in\mathbb{N}}$ is an increasing sequence of compact subsets of X. Let $x \in X$. Find a function f in $\mathcal{C}_0(X)$ such that $f(x) = 1$. Find an $n \in \mathbb{N}$ such that $\|\chi_n f - f\| < 1/2$. Then $|\chi_n(x) - 1| = |\chi_n(x)f(x) - f(x)| < 1/2$, so $\chi_n(x) > 1/2$. In particular, $x \in K_n$. $\square$

Proposition E.4.5. *The following are equivalent:*

1. *X is first countable.*
2. *X is metrisable and σ-compact.*
3. *The Alexandroff compactification X^+ of X is metrisable.*

4. *X is metrisable and separable.*

5. *$\mathcal{C}_0(X)$ is separable.*

Proof. The equivalence 1. $\Leftrightarrow$ 2. $\Leftrightarrow$ 3. is the corollary of Proposition in IX.2.10 of [Bou89]. If X is metrisable, then X is first countable if and only if it is separable (by Proposition 12 of IX.2.9 of the same book). This shows 4. $\Rightarrow$ 1., and, via the detour 2. $\Leftrightarrow$ 1., it shows 2. $\Rightarrow$ 4. Thus we have established the equivalence of the first four conditions.

1. $\Rightarrow$ 5.: Let X be first countable. Chose a countable base $(U_n)_{n\in\mathbb{N}}$ of its topology. Let M be the set of all pairs $(m,n) \in \mathbb{N}\times\mathbb{N}$ such that the closure of U_m is compact and lies in U_n. For all $(m,n)\in M$, find a function $\chi_{m,n}\in\mathcal{C}_c(X)$ such that $0\le\chi\le 1$, $\chi\equiv 1$ on U_m and $\chi\equiv 0$ outside U_n. We claim that the countable set $C := \{\chi_{m,n} :\ (m,n)\in M\}$ separates the points of X: If $x,y\in X$ with $x\neq y$, then we can find an element U_n in the base of the topology such that $x\in U_n$ and $y\notin U_n$. Furthermore, we can find a compact neighbourhood of x which lies in U_n. This compact neighbourhood must contain an element U_m of the base containing x. Now $(m,n)\in M$ and $\chi_{m,n}(x)=1$, whereas $\chi_{m,n}(y)=0$. Now the $\mathbb{Q}+\mathbb{Q}i$-linear algebra-span of C is a countable $*$-invariant subalgebra of $\mathcal{C}_0(X)$ separating the points of X, hence it is dense in $\mathcal{C}_0(X)$.

5. $\Rightarrow$ 1.-4.: Conversely, if $\mathcal{C}_0(X)$ is separable, the unit ball of $\mathcal{C}_0(X)^*$ is metrisable. This unit ball contains a homeomorphic image of X as a subspace, so X is metrisable, too. Moreover, $\mathcal{C}_0(X)$ is σ-unital, so X is σ-compact by the preceding proposition. Hence we have shown 5. $\Rightarrow$ 2.

Alternatively, let $(f_n)_{n\in\mathbb{N}}$ be a dense sequence of the unit ball of $\mathcal{C}_0(X)$. For all $n\in\mathbb{N}$, define $U_n := |f_n|^{-1}(]1/2,\ \infty[)$. We claim that $\{U_n :\ n\in\mathbb{N}\}$ is a basis of the topology of X: Let $x\in X$ and let U be an open neighbourhood of x. Find a function $\chi\in\mathcal{C}_c(X)$ such that $0\le\chi\le 1$, $\chi(x)=1$ and $\chi\equiv 0$ outside U. Find some $n\in\mathbb{N}$ such that $\|f_n-\chi\|<1/2$. Then $|f_n(y)|<1/2$ for all $y\in X\setminus U$, so $U_n\subseteq U$. On the other hand, $\chi(x)=1$, so $|f_n(x)|\ge|\chi(x)|-|\chi(x)-f_n(x)|>1/2$, hence $x\in U_n$. □

E.5 Restriction of u.s.c. fields onto closed subspaces

Let X be a topological space and let V be a closed subspace. Let E be a u.s.c. field of Banach spaces over X. If ι_V denotes the inclusion map from V to X, then the restriction $E|_V$ of E onto V is defined to be the pullback $\iota_V^*(E)$. By definition, the sections of $E|_V$ are the local closure of the restrictions of the sections of E. In Appendix E.5 we discuss under which circumstances all sections of $E|_V$ arise as restrictions, i.e., whether one can extend sections of $E|_V$ to sections of E.

Lemma E.5.1. *Let X be locally compact Hausdorff. Then for all $\xi\in\Gamma_c(V,\ E|_V)$ and all $\varepsilon>0$, there is an $\eta\in\Gamma_c(X,E)$ such that $\xi=\eta|_V$ and $\|\eta\|\le\|\xi\|+\varepsilon$.*

Proof. It is not hard to prove this lemma directly, but we prefer to reduce it to Proposition E.5.3 below.

Let $\xi\in\Gamma_c(V,E|_V)$ and $\varepsilon>0$. Let K be a compact neighbourhood of $\operatorname{supp}\xi$ in X (!). Find a function $\chi\in\mathcal{C}_c(X)$ such that $0\le\chi\le 1$, $\chi|_{\operatorname{supp}\xi}=1$ and $\chi=0$ outside the interior of K. Since K is compact, it is paracompact. The section $\xi|_{K\cap V}$ is bounded so we can find a section $\eta'\in\Gamma_b(K,E|_K)$ extending $\xi|_{K\cap V}$ and such that $\|\eta'\|\le\|\xi\|+\varepsilon$. Now $\eta:=\chi\eta'$ can be extended by zero to X and $\|\eta\|\le\|\eta'\|\le\|\xi\|+\varepsilon$. Since $\chi|_{\operatorname{supp}\xi}=1$ we have $\eta(v)=\eta'(v)=\xi(v)$ for all $v\in V\cap K$ and $\eta(v)=0=\xi(v)$ for all $v\in V\setminus K$. □

Proposition E.5.2. *Let X be locally compact Hausdorff. Then the map $\xi\mapsto\xi|_V$ from $\Gamma_0(X,E)$ to $\Gamma_0(V,\ E|_V)$ is a metric surjection.*

Proof. Let $\xi \in \Gamma_0(V, E|_V)$ and $\varepsilon > 0$. Find a $\xi' \in \Gamma_c(V, E|_V)$ such that $\|\xi - \xi'\| \leq \varepsilon/2$. Using Lemma E.5.1, find a section $\eta \in \Gamma_c(X, E)$ such that $\eta|_V = \xi'$ and $\|\eta\| \leq \|\xi'\| + \varepsilon/2 \leq \|\xi\| + \varepsilon$. Then $\|\eta|_V - \xi\| \leq \varepsilon/2$. So by Corollary E.3.2 the restriction map is a metric surjection. □

Proposition E.5.3. *Let X be paracompact. Then the map $\xi \mapsto \xi|_V$ from $\Gamma_b(X, E)$ to $\Gamma_b(V, E|_V)$ is a metric surjection. Moreover, every section of $E|_V$ can be extended to a section of E.*

Proof. Note that X is uniformisable and hence $\Gamma_b(X, E)$ is total for E and the restriction to V of sections in $\Gamma_b(X, E)$ is total for $E|_V$. Let $\xi \in \Gamma(V, E|_V)$ and $\varepsilon > 0$. For all $v \in V$, find a neighbourhood U_v of v in X and an element $\eta_v \in \Gamma_b(X, E)$ such that $\|\xi(u) - \eta_v(u)\| \leq \varepsilon$ for all $u \in U_v \cap V$ and such that $\|\eta_v\|_\infty \leq \|\xi(v)\| + \varepsilon$. We can find a locally finite refinement $(W_i)_{i \in I}$ of $\{U_v :\ v \in V\} \cup \{X \setminus V\}$. For every $i \in I$ such that W_i intersects V, there must be a $v \in V$ such that $W_i \subseteq U_v$; pick such a $v \in V$ and call it v_i. Define

$$\zeta_i := \begin{cases} \eta_{v_i}, & W_i \cap V \neq \emptyset \\ 0, & W_i \cap V = \emptyset \end{cases}$$

for all $i \in I$. Let $(\chi_i)_{i \in I}$ be a continuous partition of unity subordinate to $(W_i)_{i \in I}$. Define

$$\theta(x) := \sum_{i \in I} \chi_i(x) \zeta_i(x)$$

for all $x \in X$. It is not hard to see that θ is a section of E.

Let $v \in V$ and $J := \{i \in I :\ v \in W_i\}$. Then $\theta(v) = \sum_{j \in J} \chi_j(v) \zeta_j(v)$. If $j \in J$, then $v \in W_j \subseteq U_{v_j}$ and hence $\|\xi(v) - \zeta_j(v)\| = \left\|\xi(v) - \eta_{v_j}\right\| \leq \varepsilon$. It follows that $\|\xi(v) - \theta(v)\| \leq \varepsilon$.

This argument has two consequences:

1. If we start with a bounded ξ, then all ζ_i are bounded by $\|\xi\| + \varepsilon$, hence also θ is bounded by $\|\xi\| + \varepsilon$. Summarizing: For all $\xi \in \Gamma_b(V, E|_V)$ and all $\varepsilon > 0$, there is a $\theta \in \Gamma_b(X, E)$ such that $\|\theta\| \leq \|\xi\| + \varepsilon$ and $\|\xi - \theta|_V\| < \varepsilon$. An application of Corollary E.3.2 shows that the restriction map from $\Gamma_b(X, E)$ to $\Gamma_b(V, E|_V)$ is a metric surjection.

2. If we start with a general ξ, then we have constructed some $\theta \in \Gamma(X, E)$ such that $\|\xi - \theta|_V\| \leq \varepsilon$. In particular, $\xi - \theta|_V$ is bounded and we can find, by the first part of the argument, a $\theta' \in \Gamma_b(X, E)$ such that $\xi - \theta|_V = \theta'|_V$. So $\theta + \theta'$ is an extension of ξ to all of X. □

E.6 The pushout of B-induced Banach modules

Definition E.6.1 (Pushout, version II). Let B, C be Banach algebras and let E be a Banach B-module. If $\theta \colon B \to C$ is a morphism of Banach algebras, then define the pushout $\theta_\times(E)$ of E under θ to be the Banach C-module $E \otimes_\theta C$.

Definition E.6.2. Let B, C be Banach algebras, let $\theta \colon B \to C$ be a morphism of Banach algebras, and let E, E' be Banach B-modules. If $T \in \mathrm{L}_B(E, E')$, then define $\theta_\times(T) \in \mathrm{L}_C(E, E')$ by

$$\theta_\times(T)(e \otimes c) := T(e) \otimes c$$

for every $e \in E$, $c \in C$. In other words, we define $\theta_\times(T)$ to be $T \otimes \mathrm{Id}_C$.

The map $\theta_\times$ defines a functor from the category of Banach B-modules to the category of Banach C-modules, linear and contractive on the morphism sets.

Definition E.6.3. Let B, C be Banach algebras and let E_B, F_C be Banach modules. Assume that Θ_θ is a homomorphism from E_B to F_C with coefficient map $\theta\colon B \to C$. Then define a C-linear and contractive map

$$\widehat{\Theta}\colon \theta_\times(E) \to F,\; e \otimes c \mapsto \Theta(e)c.$$

Proposition E.6.4 (The pushout of a B-induced Banach module). *Let B, C be Banach algebras, let $\theta\colon B \to C$ be a homomorphism and let E be a B-induced Banach module. Then*

$$\theta_*(E) \cong \theta_\times(E).$$

More precisely, the θ_* and $\theta_\times$ are naturally isometrically equivalent Banach functors from the category of B-induced Banach B-modules to the category of Banach C-modules.

Proof. Define the contractive natural homomorphism

$$\eta_E\colon E \otimes_\theta C \to E \otimes_{\widetilde{\theta}} \widetilde{C} = \theta_*(E),\; e \otimes c \mapsto e \otimes c.$$

We first show that it is injective. Let $t \in E \otimes_\theta C$ lie in the kernel of η_E. Then we can write $t = \sum_{k\in\mathbb{N}} e_k \otimes c_k$ with $\sum_{k\in\mathbb{N}} \|e_k\|\,\|c_k\| \leq \infty$.

We are going to show the following: For every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ and $f_1, \dots, f_N \in E$, $b_1, \dots, b_N \in B$, and $d_1, \dots, d_N \in C$ such that

$$\left\| \sum_{k\in\mathbb{N}} e_k \otimes c_k - \sum_{n=1}^{N} \left(f_n b_n \otimes d_n - f_n \otimes \theta(b_n) d_n\right) \right\|_{E\otimes_\pi C} \leq \varepsilon.$$

Let $\varepsilon > 0$. We know that this t is zero as an element of $E \otimes_{\widetilde{\theta}} \widetilde{C}$, so we can find an $N \in \mathbb{N}$ and $f_1, \dots, f_N \in E$, $b_1, \dots, b_N \in B$, $d_1, \dots, d_N \in C$ and $\lambda_1, \dots, \lambda_N, \mu_1, \dots, \mu_N \in \mathbb{C}$ such that

$$\left\| \sum_{k\in\mathbb{N}} e_k \otimes c_k - \sum_{n=1}^{N} \left(f_n(b_n + \lambda_n) \otimes (d_n + \mu_n) - f_n \otimes (\theta(b_n) + \lambda_n)(d_n + \mu_n)\right) \right\|_{E\otimes_\pi \widetilde{C}} \leq \varepsilon/3.$$

Sorting out what this norm is we first note that all the terms disappear in which there is a λ, so we obtain

$$\left\| \sum_{k\in\mathbb{N}} e_k \otimes c_k - \sum_{n=1}^{N} \left(f_n b_n \otimes d_n - f_n \otimes \theta(b_n) d_n\right) + \sum_{n=1}^{N} f_n b_n \otimes \mu_n - \sum_{n=1}^{N} f_n \otimes \theta(b_n)\mu_n \right\|_{E\otimes_\pi \widetilde{C}} \leq \varepsilon/3.$$

Let $\rho\colon \widetilde{C} \to \widetilde{C},\; c + \lambda \mapsto \lambda$. This is a $\mathbb{C}$-linear projection of norm 1 by the definition of $\widetilde{C}$. Consider the map $1 \otimes \rho\colon E \otimes_\pi \widetilde{C} \to E \otimes_\pi \widetilde{C}$. It is also a projection of norm ≤ 1. Using this projection one immediately sees that the canonical map from $E \otimes_\pi C$ into $E \otimes_\pi \widetilde{C}$ is an isometry. If we apply this projection to the expression in the norm we get

$$\left\| \sum_{n=1}^{N} f_n b_n \otimes \mu_n \right\|_{E\otimes_\pi \widetilde{C}} \leq \varepsilon/3.$$

It remains to show that $\left\|\sum_{n=1}^{N} f_n \otimes \theta(b_n)\mu_n\right\| \leq \varepsilon/3$, but this can be achieved using the above estimate on $\left\|\sum_{n=1}^{N} f_n b_n \otimes \mu_n\right\|$:

$$\left\|\sum_{n=1}^{N} f_n(\mu_n b_n)\right\|_E = \left\|\sum_{n=1}^{N} f_n(\mu_n b_n) \otimes 1\right\|_{E\otimes_\pi \widetilde{C}} = \left\|\sum_{n=1}^{N} f_n b_n \otimes \mu_n\right\|_{E\otimes_\pi \widetilde{C}} \leq \varepsilon/3.$$

Because[2] E is B-induced it follows that

$$\left\|\sum_{n=1}^{N} f_n \otimes_B \mu_n\theta(b_n)\right\|_{E\otimes_B\widetilde{C}} \leq \|1\otimes\theta\| \left\|\sum_{n=1}^{N} f_n \otimes_B \mu_n b_n\right\|_{E\otimes_B B} \leq \left\|\sum_{n=1}^{N} f_n(\mu_n b_n)\right\|_E \leq \varepsilon/3.$$

Hence we have shown injectivity.

We now show that η_E is surjective and a quotient map: Let $t \in E\otimes_{\widetilde{\theta}}\widetilde{C}$. Let $\varepsilon > 0$. Find sequences $(e_n)_{n\in\mathbb{N}}$ in E, $(c_n)_{n\in\mathbb{N}}$ in C and $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ such that $\sum_{n\in\mathbb{N}} \|e_n\| \|c_n+\lambda_n\| \leq \|t\| + \varepsilon/2$ and $t = \sum_{n\in\mathbb{N}} e_n \otimes (c_n+\lambda_n)$. For every $n\in\mathbb{N}$, find sequences $(e_n^k)_{k\in\mathbb{N}}$ in E and $(b_n^k)_{k\in\mathbb{N}}$ in C such that $\sum_{k\in\mathbb{N}} \left\|e_n^k\right\| \left\|b_n^k\right\| \|c_n+\lambda_n\| \leq \|e_n\|\|c_n+\lambda_n\| + 2^{-n-1}\varepsilon$ and $e_n = \sum_{k\in\mathbb{N}} e_n^k b_n^k$ (this is possible because E is B-induced). Then

$$\sum_{k,n\in\mathbb{N}} e_n^k \otimes b_n^k(c_n+\lambda_n) = \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} e_n^k b_n^k \otimes (c_n+\lambda_n) = \sum_{n\in\mathbb{N}} e_n \otimes (c_n+\lambda_n) = t$$

and

$$\begin{aligned}\sum_{k,n\in\mathbb{N}} \left\|e_n^k\right\| \left\|b_n^k(c_n+\lambda_n)\right\| &\leq \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} \left\|e_n^k\right\| \left\|b_n^k\right\| \|c_n+\lambda_n\| \\ &\leq \sum_{n\in\mathbb{N}} \|e_n\|\|c_n+\lambda_n\| + 2^{-n-1}\varepsilon = \|t\| + \varepsilon.\end{aligned}$$

□

Definition E.6.5. Let B, C be Banach algebras and let $\theta\colon B\to C$ be a morphism. Let E be a B-induced Banach B-module. Let $\mu_E\colon E\otimes_B B\to E$ be the continuous B-linear map given through $\mu_E(e\otimes b) = eb$. Note that μ_E is an isometric isomorphism. Define

$$\theta_E^\times := (\mathrm{Id}_E \otimes\theta)\circ\mu_E^{-1} : \ E \to \theta_\times(E).$$

Note that $\theta_E^\times$ is a contractive homomorphism with coefficient map θ.

Lemma E.6.6 (Universal property of $\left(\theta_\times(E), \theta_E^\times\right)$). *Let B, C be Banach algebras and let E_B, F_C be Banach modules. Let Θ_θ be a homomorphism with coefficient map θ from E to F. Then $\widehat{\Theta}$, defined as above, is the unique continuous C-linear map from $\theta_\times(E)$ to F such that*

$$\Theta_\theta = \widehat{\Theta}_{\mathrm{Id}_C} \circ \left(\theta_E^\times\right)_\theta.$$

Proof. Let $e\in E$ and $b\in B$. Then

$$\widehat{\Theta}\left(\theta_E^\times(eb)\right) = \widehat{\Theta}\left((\mathrm{Id}_E\otimes\theta)(e\otimes b)\right) = \widehat{\Theta}\left(e\otimes\theta(b)\right) = \Theta(e)\theta(b) = \Theta(eb).$$

Since E is non-degenerate, this proves the above equality.

We still have to show uniqueness. Let $\Gamma \in \mathrm{L}_C\left(\theta_\times(E), F\right)$ such that $\Theta = \Gamma\circ\theta_E^\times$. Then

$$\Gamma\circ(\mathrm{Id}_E\otimes\theta) = \Gamma\circ\theta_E^\times\circ\mu = \Theta\circ\mu = \widehat{\Theta}\circ(\mathrm{Id}_E\otimes\theta).$$

Because E is non-degenerate, we can deduce that $\Gamma = \widehat{\Theta}$. □

[2] We use that $E\otimes_B B\to E$ is isometric.

E.7 Cut-off pairs for actions of groupoids

Let $\mathcal{G}$ be a locally compact Hausdorff groupoid carrying a Haar system λ and having unit space X. Let Y be a locally compact Hausdorff left $\mathcal{G}$-space with anchor map ρ. A *cut-off pair* for the $\mathcal{G}$-action on Y (or just a cut-off pair for Y if $\mathcal{G}$ is understood) is a pair $(c^<, c^>)$ of elements of $\mathcal{C}_b(Y)$ which form a cut-off pair[3] for the groupoid $\mathcal{G} \ltimes Y$, i.e., such that $c^<, c^> \geq 0$, such that $\operatorname{supp} c^< \cap \mathcal{G}K$ is compact for all compact subsets $K \subseteq Y$, such that the same property holds for $c^>$ and such that

$$\int_{\mathcal{G}^{\rho(y)}} c^< \left(\gamma^{-1}y\right)\; c^> \left(\gamma^{-1}y\right)\, \mathrm{d}\lambda^{\rho(y)}(\gamma) = 1$$

for all $y \in Y$. Such cut-off pairs exist if the quotient space $\mathcal{G} \setminus Y$ is σ-compact.[4] For the rest of this section, let $\mathcal{G} \setminus Y$ even be compact (i.e., let Y be $\mathcal{G}$-compact). Let $c = (c^<, c^>)$ be a cut-off for Y. Then we define

$$\tilde{p}_c \colon \mathcal{G} \ltimes Y \to \mathbb{R},\ (\gamma, y) \mapsto c^>(y) \cdot c^<(\gamma^{-1}y).$$

This is an element of $\mathcal{C}_c\left(\mathcal{G} \ltimes Y\right)$. Define

$$(p_c(\gamma))(y) := \tilde{p}_c(\gamma, y) = c^>(y) \cdot c^<(\gamma^{-1}y)$$

for all $(\gamma, y) \in Y \ltimes Y$, so that $p_c(\gamma) \in \mathcal{C}_c\left(Y_{r(\gamma)}\right)$ for all $\gamma \in \mathcal{G}$. Then p_c is an element of $\Gamma_c\left(\mathcal{G}, r^*\rho_*\mathbb{C}_Y\right)$, where the push-forward $\rho_*\mathbb{C}_Y$ is defined as in Section 8.3 (it could also be written $\mathcal{C}_0(Y)$, regarded as a $\mathcal{G}$-Banach space). We have[5] $p_c = \tilde{\iota}_{\mathbb{C}_Y}\left(\tilde{p}_c\right)$ and

$$\tilde{p}_c^2 = \tilde{p}_c \quad \text{and} \quad p_c^2 = p_c.$$

We just check the first equality, the second is then a consequence of the fact that $\tilde{\iota}_{\mathbb{C}_Y}$ is a homomorphism (see Proposition 8.3.22). We have

$$\begin{aligned}
(\tilde{p}_c)^2(\gamma, y) &= \int_{\mathcal{G}^{\rho(y)}} \tilde{p}_c\left(\gamma', y\right)\; \tilde{p}_c\left((\gamma', y)^{-1}(\gamma, y)\right)\, \mathrm{d}\lambda^{\rho(y)}(\gamma') \\
&= \int_{\mathcal{G}^{\rho(y)}} c^>(y) c^<\left(\gamma'^{-1}y\right) c^>\left((\gamma'^{-1}y\right) c^<\left(\left(\gamma'^{-1}\gamma\right)^{-1}\gamma'^{-1}y)\right)\, \mathrm{d}\lambda^{\rho(y)}(\gamma') \\
&= c^>(y)c^<(\gamma^{-1}y) \int_{\mathcal{G}^{\rho(y)}} c^<\left(\gamma'^{-1}y\right)\; c^>\left(\gamma'^{-1}y\right)\, \mathrm{d}\lambda^{\rho(y)}(\gamma') = \tilde{p}_c(\gamma, y)
\end{aligned}$$

for all $(\gamma, y) \in \mathcal{G} \ltimes Y$. If $\mathcal{A}(\mathcal{G})$ is an unconditional completion of $\mathcal{C}_c\left(\mathcal{G}\right)$, then p_c defines an idempotent in $\mathcal{A}\left(\mathcal{G},\ \rho_*\mathbb{C}_Y\right)$ which we also call p_c. This idempotent determines a class $\lambda_{Y,\mathcal{G},\mathcal{A}}$ in $\mathrm{K}_0\left(\mathcal{A}\left(\mathcal{G},\ \rho_*\mathbb{C}_Y\right)\right)$. It is not hard to see that this class does not depend on the choice of the cut-off pair.

Now let Y' be another locally compact Hausdorff $\mathcal{G}$-compact proper $\mathcal{G}$-space (with anchor map ρ') and let $f \colon Y \to Y'$ be a proper $\mathcal{G}$-equivariant continuous map. Write $\tilde{f}$ for the homomorphism of $\mathcal{G}$-Banach algebras from $\rho'_*\mathbb{C}_{Y'}$ to $\rho_*\mathbb{C}_Y$ induced by f.

Proposition E.7.1. *We have*

$$\lambda_{Y',\mathcal{G},\mathcal{A}} = \mathcal{A}\left(\mathcal{G}, \tilde{f}\right)_*\left(\lambda_{Y,\mathcal{G},\mathcal{A}}\right).$$

[3] See Definition 7.1.6.

[4] See the discussion after Definition 7.1.6 for a way to construct cut-off pairs from cut-off functions; a cut-off function exists according to [Tu04].

[5] See 8.3.19 for a definition of $\tilde{\iota}$.

Proof. Let $c' = (c'^{<}, c'^{>})$ be a cut-off pair for Y'. Then $c' \circ f := (c'^{<} \circ f,\ c'^{>} \circ f)$ is a cut-off pair for Y: $c'^{<} \circ f$ and $c'^{>} \circ f$ are obviously non-negative, continuous and of compact support (f is proper!). For all $y \in Y$, we have

$$\int_{\mathcal{G}^{\rho(y)}} c'^{<} \left(f(\gamma^{-1}y)\right)\ c'^{>} \left(f(\gamma^{-1}y)\right)\ \mathrm{d}\lambda^{\rho(y)}(\gamma) = \int_{\mathcal{G}^{\rho(y)}} c'^{<} \left(\gamma^{-1}f(y)\right)\ c'^{>} \left(\gamma^{-1}f(y)\right)\ \mathrm{d}\lambda^{\rho(y)}(\gamma) = 1,$$

where we have used the equivariance of f and the fact that c' is a cut-off pair. We also have

$$\tilde{p}_{c'} \circ (\mathrm{Id}_{\mathcal{G}} \ltimes f) = \tilde{p}_{c' \circ f}$$

as can be shown as follows: Let $(\gamma, y) \in \mathcal{G} \ltimes Y$. Then $(\gamma, f(y)) \in \mathcal{G} \ltimes Y'$ and

$$\tilde{p}_{c'}\left(\gamma, f(y)\right) = c'^{>}(f(y))c'^{<}(\gamma^{-1}f(y)) = (c'^{>} \circ f)(y)\ (c'^{<} \circ f)(\gamma^{-1}y) = \tilde{p}_{c' \circ f}\left(\gamma, y\right).$$

It follows that $\tilde{f} \circ p_{c'} = p_{c \circ f}$ and finally $\lambda_{Y', \mathcal{G}, \mathcal{A}} = \mathcal{A}\left(\mathcal{G}, \tilde{f}\right)_{*} (\lambda_{Y, \mathcal{G}, \mathcal{A}})$. $\square$

E.8 Monotone completions and operators given by kernels

Let X and Y be locally compact Hausdorff spaces and let $q\colon Y \to X$ be a continuous map. For all $x \in X$, write Y_x for $q^{-1}(\{x\}) \subseteq Y$. For $\chi \in \mathcal{C}(X)$ and $\varphi \in \mathcal{C}_c(Y)$ define $\varphi\chi := \varphi \cdot (\chi \circ q) \in \mathcal{C}_c(Y)$. In this way, $\mathcal{C}_c(Y)$ is a $\mathcal{C}(X)$-module and also a non-degenerate $\mathcal{C}_c(X)$-module.

Let $\mathcal{H}(Y)$ be a monotone completion of $\mathcal{C}_c(Y)$ (for the definition of a monotone semi-norm see 3.2.1). The monotone completion $\mathcal{H}(Y)$ is a $\mathcal{C}_0(X)$-Banach space. The semi-norm $\|\cdot\|_{\mathcal{H}}$ is called locally $\mathcal{C}_0(X)$-convex (or simply locally convex) if $\mathcal{H}(Y)$ is locally $\mathcal{C}_0(X)$-convex.

Examples E.8.1. 1. One of the simplest examples for a monotone semi-norm on $\mathcal{C}_c(Y)$ is the sup-norm $\|\chi\|_\infty = \sup_{y\in Y} |\chi(y)|$; in this case, $\mathcal{H}(Y)$ is just $\mathcal{C}_0(Y)$ as a (locally convex $\mathcal{C}_0(X)$-Banach space).

2. Let $\mu = (\mu_x)_{x\in X}$ be a continuous field of measures on Y over X and let $p \in [1, \infty[$. Then we define the p-semi-norm $\|\cdot\|_p$ on $\mathcal{C}_c(Y)$ (with respect to μ) by

$$\|\chi\|_p := \sup_{x\in X} \left(\int_{Y_x} |\chi(y)|^p \, \mathrm{d}\mu_x(y) \right)^{\frac{1}{p}}$$

for all $\chi \in \mathcal{C}_c(Y)$. The completion of $\mathcal{C}_c(Y)$ for this semi-norm is denoted by $\mathrm{L}^p(Y, \mu)$ or simply by $\mathrm{L}^p(Y)$ if μ is understood. Note that $\mathrm{L}^p(Y)$ is a locally convex $\mathcal{C}_0(X)$-Banach space.

E.8.1 Monotone completions and fields of Banach spaces

Let E be a u.s.c. field of Banach spaces over X.

The following definition should be compared to Definition 3.2.4 which covers the special case that $X = Y$ and $q = \mathrm{Id}_X$.

Definition E.8.2 ($\mathcal{H}(Y, E)$). We define the following semi-norm on $\Gamma_c(Y, q^*E)$:

$$\|\xi\|_{\mathcal{H}} := \left\| y \mapsto \|\xi(y)\|_{E_{q(y)}} \right\|_{\mathcal{H}}.$$

The Hausdorff completion of $\Gamma_c(Y, q^*E)$ with respect to this semi-norm will be denoted by $\mathcal{H}(Y, E)$ (and usually not by $\mathcal{H}(Y, q^*E)$). The Banach space $\mathcal{H}(Y, E)$ carries a canonical action of $\mathcal{C}_0(X)$ such that it is a $\mathcal{C}_0(X)$-Banach space.

The canonical map from $\Gamma_c(Y, q^*E)$ to $\mathcal{H}(Y, E)$ is continuous if we take the inductive limit topology on $\Gamma_c(Y, q^*E)$ and the norm topology on $\mathcal{H}(Y, E)$. It follows that if Ξ is dense in $\Gamma_c(Y, q^*E)$ for the inductive limit topology, then its canonical image in $\mathcal{H}(Y, E)$ is dense for the semi-norm topology.

Definition E.8.3. Let $x \in X$. For all $\varphi \in \mathcal{C}_c(Y_x)_{\geq 0}$, define $\|\varphi\|_{\mathcal{H}(Y_x)}$ to be the value of the semi-norm of the extension of φ to Y by 0. This defines a monotone semi-norm on $\mathcal{C}_c(Y_x)$. The completion of $\mathcal{C}_c(Y_x, E_x)$ with respect to this semi-norm will be denoted by $\mathcal{H}(Y_x, E)$. The restriction map yields a linear map of norm ≤ 1 from $\mathcal{H}(Y, E)$ to $\mathcal{H}(Y_x, E)$.

Example E.8.4. If $p \in [1, \infty[$, if μ is some continuous field of measures on Y over X, if $\mathcal{H}(Y) = \mathrm{L}^p(Y, \mu)$ and $x \in X$, then the semi-norm $\|\cdot\|_{\mathrm{L}^p(Y_x)}$ on $\mathcal{C}_c(Y_x)$ is simply given by

$$\|\chi\|_{\mathrm{L}^p(Y_x)} = \left(\int_{Y_x} |\chi(y)|^p \, \mathrm{d}\mu_x(y) \right)^{\frac{1}{p}}$$

for all $\chi \in \mathcal{C}_c(Y_x)$.

Proposition E.8.5. *If q is open, then there is an isometric isomorphism*

$$\mathcal{H}(Y,E)_x \cong \mathcal{H}(Y_x,E).$$

Proof. Let $P_x\colon \mathcal{H}(Y,E) \to \mathcal{H}(Y_x,E)$ denote the linear map induced by the restriction map and let $\pi_x\colon \mathcal{H}(Y,E) \to \mathcal{H}(Y,E)_x$ denote the quotient map.

First we show that the kernel of π_x, which is $\mathcal{C}_0(X\setminus\{x\})\mathcal{H}(Y,E)$ by definition, is contained in the kernel of P_x: Let $\varphi \in \mathcal{C}_0(X\setminus\{x\})$ and $\xi \in \Gamma_c(Y,\ q^*E)$. Then $(\varphi\xi)(y) = \varphi(q(y))\xi(y) = 0$ for all $y \in Y_x$. So $P_x(\varphi\xi) = 0$. By continuity, this is also true for all $\xi \in \mathcal{H}(Y,E)$. This means in particular that we get a continuous linear map Φ_x from $\mathcal{H}(Y,E)_x$ to $\mathcal{H}(Y_x,E)$ of norm ≤ 1.

Now we show that $\|P_x(\xi)\|_{\mathcal{H}(Y_x,E)} = \|\pi_x(\xi)\|_{\mathcal{H}(Y,E)_x}$ for all $\xi \in \Gamma_c(Y,\ q^*E)$. This will show that Φ_x is isometric on a dense subset, so it is isometric throughout $\mathcal{H}(Y,E)_x$. Since P_x has dense image it follows that Φ_x has dense image and thus we are done.

The inequality $\leq$ is already known, we have to show $\geq$. Let $\xi \in \Gamma_c(Y,\ q^*E)$ and $\varepsilon > 0$. Find a function $\varphi \in \mathcal{C}_c(Y)_+$ such that $\|\xi(y)\| \leq \varphi(y)$ for all $y \in Y_x$ and $\|\varphi\|_{\mathcal{H}(Y)} \leq \|\xi\|_{\mathcal{H}(Y_x,E)} + \varepsilon$.

Let K be a compact neighbourhood in Y of the support of φ and the support of ξ. Let $\varepsilon' > 0$. We are going to show that we can find a function $\psi \in \mathcal{C}_c(Y)_+$ and a function $\chi \in \mathcal{C}_c(X)_+$ such that $\|\xi(y)\| \leq \psi(y)$ for all $y \in Y$, $\operatorname{supp}\chi\psi \subseteq K$ and $\|\chi\psi - \varphi\|_\infty \leq \varepsilon'$. Using the fact that the "inclusion" $\mathcal{C}_K(Y,q^*E) \to \mathcal{H}(Y,E)$ is continuous, we can choose ε' so small that $\|\psi\chi - \varphi\|_{\mathcal{H}(Y)} \leq 2\varepsilon$ and hence $\|\chi\psi\|_{\mathcal{H}(Y)} \leq \|\xi\|_{\mathcal{H}(Y_x,E)} + \varepsilon$. But $\|\pi_x(\xi)\|_{\mathcal{H}(Y,E)_x} \leq \|\chi\psi\|_{\mathcal{H}(Y)}$, so we are done. Note that by the monotony of the semi-norm it suffices to find χ and ψ such that $\chi(q(y))\psi(y) \leq \varphi(y) + \varepsilon'$ for all $y \in Y$ (instead of $\|\chi\psi - \varphi\|_\infty \leq \varepsilon'$).

For all $y \in Y_x$, we can find a function $\psi_y \in \mathcal{C}_c(Y)_+$ such that $|\xi| \leq \psi_y$ and $\psi(y) \leq \varphi(y) + \varepsilon'$. Using a compactness argument and the continuity of φ we can thus find a function $\psi \in \mathcal{C}_c(Y)_+$ such that $|\xi| \leq \psi$ and $\psi(y) \leq \varphi(y) + 2\varepsilon'$ for all $y \in Y_x$. By multiplying ψ with a function in $\mathcal{C}_c(Y)$ between 0 and 1 which is 1 on the support of ξ and vanishes outside K, we can assume without loss of generality that the support of ψ is contained in K.

Both functions, φ and ψ, are continuous, so $y \mapsto \max\{\psi(y) - \varphi(y), 0\}$ is continuous. By Lemma E.8.6 the function $s\colon x' \mapsto \sup_{y\in Y_{x'}} |\psi(y) - \varphi(y)|$ is continuous. Note that $s(x) \leq 2\varepsilon'$. Find a function $\chi \in \mathcal{C}_c(X)$ such that $0 \leq \chi \leq 1$, $\chi(x) = 1$ and $\chi(x')s(x') \leq 3\varepsilon'$ for all $x' \in X$. It follows that $\chi(q(y))\psi(y) \leq \varphi(y) + 3\varepsilon'$ for all $y \in Y$. □

Lemma E.8.6. *Let $f \in \mathcal{C}_c(Y)_+$ and $q\colon Y \to X$ be open and surjective. Then the map $s\colon X \to \mathbb{R}$, $x \mapsto \sup_{y\in Y_x} f(y)$, is continuous.*

Proof. Let $x_0 \in X$. We show that s is lower and upper semi-continuous in x_0.

Let $\varepsilon > 0$. The set $\{f(y) :\ y \in Y_{x_0}\}$ is compact, so there is an $y_0 \in Y_{x_0}$ such that $y_0 = \sup_{y\in Y_{x_0}} f(y)$. Let V be a neighbourhood of y_0 in Y such that $f(v) > f(y_0) - \varepsilon$ for all $v \in V$. Then $U := q(V)$ is an open neighbourhood of x_0 in X. For all $u \in U$, we have $s(u) > s(x_0) - \varepsilon$, so s is lower semi-continuous in x_0.

Let $\varepsilon > 0$. For all compact neighbourhoods K of x_0 in X, define

$$A_K := \{y \in Y :\ f(y) \geq s(y_0) + \varepsilon,\ q(y) \in K\}.$$

These sets are closed and contained in the compact support of f. The intersection $\bigcap_K A_K$ is the set $\{y \in Y_{x_0} :\ f(y) \geq s(y_0) + \varepsilon\}$ which is empty. So the intersection of a finite number of A_K has to be empty. It follows, that there is a compact neighbourhood K of x_0 in X such that A_K is empty. So $s(x) \leq s(x_0) + \varepsilon$ for all $x \in K$. In other words: s is upper semi-continuous. □

E.8.2 Monotone completions, modules and pairs

Definition E.8.7 (The right $\Gamma_0(X,B)$-module structure). Let $\|\cdot\|_{\mathcal{H}}$ denote a monotone semi-norm on $\mathcal{C}_c(Y)$. Let B be a u.s.c. field of Banach algebras over X and let E be a right Banach B-module. Define

$$(\xi\beta)(y) := \xi(y)\,\beta(q(y))$$

for all $\xi \in \Gamma_c(Y,\ q^*E)$, $\beta \in \Gamma(X,B)$ and $y \in Y$. This defines an action of $\Gamma(X,B)$ — and hence of $\Gamma_0(X,B)$) — on $\Gamma_c(Y,\ q^*(E))$ which is compatible with the action of $\mathcal{C}(X)$ (and of $\mathcal{C}_0(X)$). The action of $\Gamma_0(X,B)$ satisfies

$$\|\xi\beta\|_{\mathcal{H}} \leq \|\xi\|_{\mathcal{H}}\,\|\beta\|_\infty$$

for all $\xi \in \Gamma_c(Y,\ q^*E)$ and $\beta \in \Gamma_0(X,B)$; it therefore lifts to an action of $\Gamma_0(X,B)$ on $\mathcal{H}(Y,E)$. If E is non-degenerate, then so is $\mathcal{H}(Y,E)$.

A similar definition can be made for left Banach B-modules.

Definition E.8.8 ((Locally convex) pair of monotone completions). Let $(\mu_x)_{x\in X}$ be a continuous field of measures on Y over X. A *(locally convex) pair of monotone completions* with respect to μ is a pair $\mathcal{H}(Y) = (\mathcal{H}^<(Y),\ \mathcal{H}^>(Y))$ such that $\mathcal{H}^<(Y)$ and $\mathcal{H}^>(Y)$ are (locally convex) monotone completions of $\mathcal{C}_c(Y)$ and such that the bilinear map

$$\langle\cdot,\cdot\rangle_{\mathcal{C}_c(X)}\colon \mathcal{C}_c(Y)\times\mathcal{C}_c(Y)\to\mathcal{C}_c(X),\ (\varphi^<,\varphi^>)\mapsto\left(x\mapsto\int_{Y_x}\varphi^<(y)\varphi^>(y)\,\mathrm{d}\mu_x(y)\right),$$

satisfies

$$\left\|\langle\varphi^<,\ \varphi^>\rangle_{\mathcal{C}_c(X)}\right\|_\infty \leq \left\|\varphi^<\right\|_{\mathcal{H}^<}\left\|\varphi^>\right\|_{\mathcal{H}^>}$$

for all $\varphi^<,\varphi^> \in \mathcal{C}_c(Y)$. Note that in this case the map $\langle\cdot,\cdot\rangle_{\mathcal{C}_c(X)}$ can be extended to a continuous bilinear map $\langle\cdot,\cdot\rangle_{\mathcal{C}_0(X)}\colon \mathcal{H}^<(Y)\times\mathcal{H}^>(Y)\to\mathcal{C}_0(X)$ which is $\mathcal{C}_0(X)$-bilinear.

Examples E.8.9. Let μ be a continuous field of measures on Y over X.

1. The pairs $\left(\mathrm{L}^1(Y),\ \mathcal{C}_0(Y)\right)$ and $\left(\mathcal{C}_0(Y),\ \mathrm{L}^1(Y)\right)$ are certainly locally convex pairs of monotone completions.

2. If $p,p' \in]1,\infty[$ such that $\frac{1}{p}+\frac{1}{p'}=1$, then $\left(\mathrm{L}^{p'}(Y),\ \mathrm{L}^p(Y)\right)$ is a locally convex pair of monotone completions.

Definition and Proposition E.8.10 (The pair $\mathcal{H}(Y,E)$). Let $(\mu_x)_{x\in X}$ be a continuous field of measures on Y over X. Let $\mathcal{H}(Y)$ be a pair of monotone completions with respect to μ. Let B be a u.s.c. field of Banach algebras over X and let E be a Banach B-pair. Then the pair $\mathcal{H}(Y,E) := (\mathcal{H}^<(Y,E^<),\ \mathcal{H}^>(Y,E^>))$ is a $\mathcal{C}_0(X)$-Banach $\Gamma_0(X,B)$-pair if we equip it with the bracket

$$\begin{aligned}\langle\cdot,\cdot\rangle_{\Gamma_c(X,B)}\colon \Gamma_c(Y,\ q^*E^<)\times\Gamma_c(Y,\ q^*E^>) &\to \Gamma_c(X,B),\\ (\xi^<,\xi^>) &\mapsto \left(x\mapsto\int_{Y_x}\left\langle\xi^<(y),\ \xi^>(y)\right\rangle_{E_{q(y)}}\mathrm{d}\mu_x(y)\right),\end{aligned}$$

which extends to a bracket on $\mathcal{H}^<(Y,E^<)\times\mathcal{H}^>(Y,E^>)$ which is $\mathcal{C}_0(X)$-bilinear and $\mathcal{C}_0(X,B)$-bilinear.

Proof. We have to check that the bracket $\langle\cdot,\cdot\rangle_{\Gamma_c(X,B)}$ satisfies

$$\left\|\langle\xi^<,\ \xi^>\rangle_{\Gamma_c(X,B)}\right\|_\infty \leq \left\|\xi^<\right\|_{\mathcal{H}^<}\left\|\xi^>\right\|_{\mathcal{H}^>}$$

for all $\xi^< \in \Gamma_c(Y,\ q^*E^<)$ and $\xi^> \in \Gamma_c(Y,\ q^*E^>)$. Let $x \in X$. Then

$$\begin{aligned}\left\|\int_{Y_x}\left\langle\xi^<(y),\ \xi^>(y)\right\rangle_{E_{q(y)}}\,\mathrm{d}\mu_x(y)\right\| &\leq \int_{Y_x}\left\|\xi^<(y)\right\|\left\|\xi^>(y)\right\|\,\mathrm{d}\mu_x(y)\\ &\leq \int_{Y_x}\varphi^<\varphi^>\,\mathrm{d}\mu_x(y) \leq \left\|\varphi^<\right\|_{\mathcal{H}^<}\left\|\varphi^>\right\|_{\mathcal{H}^>}\end{aligned}$$

for all $\varphi^<, \varphi^> \in \mathcal{C}_c(Y)$ such that $|\xi^<| \leq \varphi^<$ and $|\xi^>| \leq \varphi^>$. By taking the infimum on the right-hand side we obtain

$$\left\|\int_{Y_x}\left\langle\xi^<(y),\ \xi^>(y)\right\rangle_{E_{q(y)}}\,\mathrm{d}\mu_x(y)\right\| \leq \left\|\xi^<\right\|_{\mathcal{H}^<}\left\|\xi^>\right\|_{\mathcal{H}^>}. \qquad \square$$

E.8.3 Operators given by kernels

Let Y' be another locally compact Hausdorff space and let $q'\colon Y' \to X$ be continuous. Let $(\mu'_x)_{x\in X}$ be a continuous field of measures on Y' over X. Let $Y \times_X Y'$ be the fibre product of Y and Y' over X and $Q\colon Y' \times_X Y \to X,\ (y', y) \mapsto q(y) = q'(y')$.

Definition E.8.11 (The tensor product of monotone semi-norms). Let $\mathcal{H}(Y)$ be a monotone completion of $\mathcal{C}_c(Y)$ and $\mathcal{H}'(Y')$ a monotone completion of $\mathcal{C}_c(Y')$. For all $\zeta \in \mathcal{C}_c(Y' \times_X Y)$, define

$$\|\zeta\|_{\mathcal{H}'\otimes\mathcal{H}} := \inf\left\{\sum_{i=1}^n\left\|\chi'_i\right\|_{\mathcal{H}'}\left\|\chi_i\right\|_{\mathcal{H}} :\quad \chi_i \in \mathcal{C}_c(Y),\ \chi'_i \in \mathcal{C}_c(Y'), \left|\zeta(y', y)\right| \leq \sum_{i=1}^n \chi'_i(y')\chi_i(y)\right\}.$$

The semin-norm $\|\cdot\|_{\mathcal{H}'\otimes\mathcal{H}}$ on $\mathcal{C}_c(Y' \times_X Y)$ is monotone; the completion is called $\mathcal{H}'\otimes\mathcal{H}\ (Y'\times_X Y)$.

Lemma E.8.12. *Let B be a u.s.c. field of Banach algebras over X and let E, E' be Banach B-pairs. Let $k = \left(k_{(y',y)}\right)_{(y',y)\in Y'\times_X Y} \in \mathrm{L}_{Q^*B}\left(Q^*E,\ Q^*E'\right)$ be a continuous field of operators with compact support. Define an operator T_k from $\mathcal{H}(Y,E)$ to $\mathcal{H}'(Y',E')$ by*

$$T_k^>\left(\xi^>\right)(y') := \int_{Y_{q'(y')}} k^>_{(y',y)}\xi^>(y)\,\mathrm{d}\mu_{q'(y')}(y), \quad y' \in Y',$$

and

$$T_k^<\left(\xi'^<\right)(y) := \int_{Y'_{q(y)}} k^<_{(y',y)}\xi'^<(y')\,\mathrm{d}\mu_{q(y)}(y'), \quad y \in Y,$$

for all $\xi^> \in \Gamma_c\left(Y,\ q^*E^>\right)$, $\xi'^< \in \Gamma_c\left(Y',\ q'^*E'^<\right)$.

This operator is continuous and satisfies

$$\|T_k\| \leq \|k\|_{\mathcal{H}'^>\otimes\mathcal{H}^<}.$$

If k is compact, then T_k is compact.

Proof. Assume that $\left\|k_{(y',y)}\right\| \leq \sum_{i=1}^{n} \chi_i'(y')\chi_i(y)$ for all $(y',y) \in Y' \times_X Y$ with $\chi_i \in \mathcal{C}_c(Y)$ and $\chi_i' \in \mathcal{C}_c(Y')$. This implies that

$$\begin{aligned}\left\|T_k^{>}\left(\xi^{>}\right)(y')\right\| &\leq \int_{Y_{q'(y')}} \left\|k_{(y',y)}^{>}\right\| \left\|\xi^{>}(y)\right\| \,\mathrm{d}\mu_{q'(y')}(y) \\ &\leq \int_{Y_{q'(y')}} \sum_{i=1}^{n} \chi_i'(y')\chi_i(y) \left\|\xi^{>}(y)\right\| \,\mathrm{d}\mu_{q'(y')}(y) \\ &\leq \sum_{i=1}^{n} \chi_i'(y') \left\|\chi_i\right\|_{\mathcal{H}^{<}} \left\|\xi^{>}\right\|_{\mathcal{H}^{>}}\end{aligned}$$

for all $y' \in Y'$. By the monotony of the semi-norm $\left\|\cdot\right\|_{\mathcal{H}'^{>}}$ this yields

$$\left\|T_k^{>}\left(\xi^{>}\right)\right\|_{\mathcal{H}'^{>}} \leq \sum_{i=1}^{n} \left\|\chi_i'\right\|_{\mathcal{H}'^{>}} \left\|\chi_i\right\|_{\mathcal{H}^{<}} \left\|\xi^{>}\right\|_{\mathcal{H}^{>}},$$

so $T_k^{>}$ is continuous with norm $\leq \sum_{i=1}^{n} \left\|\chi_i'\right\|_{\mathcal{H}'^{>}} \left\|\chi_i\right\|_{\mathcal{H}^{<}}$. Taking the infimum yields

$$\left\|T_k^{>}\right\| \leq \left\|k\right\|_{\mathcal{H}'^{>} \otimes \mathcal{H}^{<}}.$$

On the left-hand side we have for all $\xi'^{<} \in \Gamma_c\left(Y',\ q'^{*}E'^{<}\right)$:

$$\begin{aligned}\left\|T_k^{<}\left(\xi'^{<}\right)(y)\right\| &\leq \int_{Y'_{q(y)}} \left\|k_{(y',y)}^{<}\right\| \left\|\xi'^{<}(y')\right\| \,\mathrm{d}\mu'_{q(y)}(y') \\ &\leq \int_{Y'_{q(y)}} \sum_{i=1}^{n} \chi_i'(y')\chi_i(y) \left\|\xi'^{<}(y')\right\| \,\mathrm{d}\mu'_{q(y)}(y') \\ &\leq \sum_{i=1}^{n} \chi_i(y) \left\|\chi_i'\right\|_{\mathcal{H}'^{>}} \left\|\xi'^{<}\right\|_{\mathcal{H}'^{<}}\end{aligned}$$

for all $y \in Y$, and hence

$$\left\|T_k^{<}\left(\xi'^{<}\right)\right\|_{\mathcal{H}^{<}} \leq \sum_{i=1}^{n} \left\|\chi_i'\right\|_{\mathcal{H}'^{>}} \left\|\chi_i\right\|_{\mathcal{H}^{<}} \left\|\xi'^{<}\right\|_{\mathcal{H}'^{<}}.$$

As above, this shows that $T_k^{>}$ is continuous with norm $\leq \sum_{i=1}^{n} \left\|\chi_i'\right\|_{\mathcal{H}'^{>}} \left\|\chi_i\right\|_{\mathcal{H}^{<}}$, implying

$$\left\|T_k^{<}\right\| \leq \left\|k\right\|_{\mathcal{H}'^{>} \otimes \mathcal{H}^{<}}.$$

Together, we get $\left\|T\right\| \leq \left\|k\right\|_{\mathcal{H}'^{>} \otimes \mathcal{H}^{<}}$.

If $\xi'^{<} \in \Gamma_c\left(Y',\ q'^{*}E'^{<}\right)$ and $\xi^{>} \in \Gamma_c\left(Y,\ q^{*}E^{>}\right)$, then

$$\begin{aligned}
\left\langle \xi'^{<},\ T_k^{>}\xi^{>}\right\rangle_x &= \int_{Y'_x} \left\langle \xi'^{<}(y'),\ (T_k^{>}\xi^{>})(y')\right\rangle \mathrm{d}\mu'_x(y') \\
&= \int_{Y'_x} \left\langle \xi'^{<}(y'),\ \left(\int_{Y_{q'(y')}} k^{>}_{(y',y)}\xi^{>}(y)\,\mathrm{d}\mu_x(y)\right)\right\rangle \mathrm{d}\mu'_x(y') \\
&= \int_{Y'_x}\int_{Y_x} \left\langle \xi'^{<}(y'),\ k^{>}_{(y',y)}\xi^{>}(y)\right\rangle \mathrm{d}\mu_x(y)\,\mathrm{d}\mu'_x(y') \\
&= \int_{Y_x}\int_{Y'_x} \left\langle k^{<}_{(y',y)}\xi'^{<}(y'),\ \xi^{>}(y)\right\rangle \mathrm{d}\mu'_x(y')\,\mathrm{d}\mu_x(y) \\
&= \int_{Y_x} \left\langle \int_{Y'_x} k^{<}_{(y',y)}\xi'^{<}(y')\,\mathrm{d}\mu'_x(y'),\ \xi^{>}(y)\right\rangle \mathrm{d}\mu_x(y) \\
&= \int_{Y_x} \left\langle \left(T_k^{<}\xi'^{<}\right)(y),\ \xi^{>}(y)\right\rangle \mathrm{d}\mu_x(y) = \left\langle T_k^{<}\xi'^{<},\ \xi^{>}\right\rangle_x
\end{aligned}$$

for all $x \in X$. Since $T_k^{>}$ and $T_k^{<}$ are clearly $\Gamma_0(X,B)$- and $\mathcal{C}_0(X)$-linear, the pair T_k is an element of $\mathrm{L}^{\mathcal{C}_0(X)}_{\Gamma_0(X,B)}\left(\mathcal{H}(Y,E),\ \mathcal{H}'(Y',E')\right)$.

Note that the map $k \mapsto T_k$ is, in particular, continuous for the inductive limit topology on the space of elements of $\mathrm{L}_{Q^*B}\left(Q^*E,\ Q^*E'\right)$ with compact support.

Assume now that k is compact, i.e., assume that k is an element of $\mathrm{K}^{\mathbf{loc}}_{Q^*B}(Q^*E, Q^*E')$ with compact support. We first show T_k is compact if k is of a very simple form; we then move on to the general situation step by step.

1. If $\eta^{<} \in \Gamma_c\left(Y,\ q^*E^{<}\right)$ and $\eta'^{>} \in \Gamma_c\left(Y',\ q'^*E'^{>}\right)$, then the operator $\left|\eta'^{>}\right\rangle\left\langle\eta^{<}\right|^{>}$, as a map from $\Gamma_c\left(Y,\ q^*E^{>}\right) \subseteq \mathcal{H}^{>}(Y,E^{>})$ to $\Gamma_c\left(Y',\ q'^*E'^{>}\right) \subseteq \mathcal{H}'^{>}(Y',E'^{>})$, is given by

$$\begin{aligned}
\left|\eta'^{>}\right\rangle\left\langle\eta^{<}\right|^{>}(\xi^{>})(y') &= \eta'^{>}(y')\langle\eta^{<},\ \xi^{>}\rangle_{q'(y')} = \eta'^{>}(y')\int_{Y_{q'(y')}} \langle\eta^{<}(y),\ \xi^{>}(y)\rangle\,\mathrm{d}\mu_{q'(y')}(y) \\
&= \int_{Y_{q'(y')}} \left|\eta'^{>}(y')\right\rangle\left\langle\eta^{<}(y)\right|^{>}(\xi^{>}(y))\,\mathrm{d}\mu_{q'(y')}(y)
\end{aligned}$$

for all $\xi^{>} \in \Gamma_c\left(Y,\ q^*E^{>}\right)$. A similar expression can be derived for $\left|\eta'^{>}\right\rangle\left\langle\eta^{<}\right|^{<}$, showing that $\left|\eta'^{>}\right\rangle\left\langle\eta^{<}\right|$ is given by the kernel $k_{(y',y)} = \left|\eta'^{>}(y')\right\rangle\left\langle\eta^{<}(y)\right|$. Conversely, if k is such a kernel, then T_k is compact. The same holds for linear combinations of such kernels.

2. Let $\eta^{<} \in \Gamma_c\left(Y' \times_X Y,\ Q^*E^{<}\right)$ and $\eta'^{>} \in \Gamma_c\left(Y' \times_X Y,\ Q^*E'^{>}\right)$. Assume that $k_{(y',y)} := \left|\eta'^{>}(y',y)\right\rangle\left\langle\eta^{<}(y',y)\right|$ for all $(y',y) \in Y' \times_X Y$.

   (a) If $\eta^{<}$ is of the form $\eta^{<}(y',y) = \chi'(y')\tilde{\eta}^{<}(y)$ with $\chi' \in \mathcal{C}_c(Y')$ and $\tilde{\eta}^{<} \in \Gamma_c(Y,\ q^*E^{<})$ and if the section $\eta'^{>}$ is of the form $\eta'^{>}(y',y) = \tilde{\eta}'^{>}(y')\chi(y)$ with $\chi \in \mathcal{C}_c(Y)$ and $\tilde{\eta}'^{>} \in \Gamma_c(Y',\ q'^*E'^{>})$, then

$$\begin{aligned}
k_{(y',y)} &= \left|\eta'^{>}(y',y)\right\rangle\left\langle\eta^{<}(y',y)\right| = \left|\tilde{\eta}'^{>}(y')\,\chi(y)\right\rangle\left\langle\chi'(y')\,\tilde{\eta}^{<}(y)\right| \\
&= \left|\chi'(y')\,\tilde{\eta}'^{>}(y')\right\rangle\left\langle\chi(y)\,\tilde{\eta}^{<}(y)\right|,
\end{aligned}$$

   so we are back in case 1.

(b) Approximate the section $\eta'^{>}$ in the inductive limit topology by sections which are of the form $(y', y) \mapsto \sum_{i=1}^{n} \eta_i'^{>}(y')\chi_i(y)$ with $\eta_i'^{>} \in \Gamma_c\left(Y',\ q'^{*}E'^{>}\right)$ and $\chi_i \in \mathcal{C}_c(Y)$. Do the same for $\eta^{<}$. Then the resulting kernel approximates k in the inductive limit topology. Hence T_k is compact also in this case.

3. Now consider a general k. Since it is locally compact, we can approximate it locally by operators which are sums of those considered in 2. By using continuous partitions of unity we can approximate k by such operators in the inductive limit topology. Hence T_k is compact. □

E.8.4 The pullback of monotone completions

Assume that $q\colon Y \to X$ is continuous and open. Let Y' be another locally compact Hausdorff space and let $q'\colon Y' \to X$ be another continuous and open map. Write $Y \times_X Y'$ for the fibre product of Y and Y' over the maps q and q', and let $\pi\colon Y \times_X Y' \to Y$ and $\pi'\colon Y \times_X Y' \to Y'$ be the canonical projections. Let $Q\colon Y \times_X Y' \to X,\ (y, y') \mapsto q(y) = q'(y')$.

Let $\mathcal{H}(Y)$ be a monotone completion of $\mathcal{C}_c(Y)$. We are now going to define a monotone completion $q'^{*}\mathcal{H}(Y \times_X Y')$ of $\mathcal{C}_c(Y \times_X Y')$ such that

$$q'^{*}\mathfrak{F}\left(\mathcal{H}(Y, E)\right) \cong \mathfrak{F}\left(q'^{*}\mathcal{H}\ \left(Y \times_X Y',\ q'^{*}E\right)\right)$$

for all u.s.c. fields of Banach spaces E over Y. We need such a construction in order to properly define groupoid actions on monotone completions.

Definition E.8.13 (The completion $q'^{*}\mathcal{H}(Y \times_X Y')$). On $\mathcal{C}_c(Y \times_X Y')$ define the monotone semi-norm

$$\|\chi\|_{q'^{*}\mathcal{H}} := \sup_{y' \in Y'} \left\| Y_{q'(x')} \ni y \mapsto \chi(y, y') \right\|_{\mathcal{H}\left(Y_{q'(x')}\right)}$$

for all $\chi \in \mathcal{C}_c(Y \times_X Y')$.

To see that $\|\chi\|_{q'^{*}\mathcal{H}} < \infty$, let $K := \pi\left(\operatorname{supp}\chi\right) \subseteq Y$. Find a function $\delta \in \mathcal{C}_c\left(Y\right)$ such that $0 \le \delta \le 1$ and $\chi|_K = 1$. For all $y' \in Y'$, we have

$$\left\| Y_{q'(x')} \ni y \mapsto \chi(y, y') \right\|_{\mathcal{H}\left(Y_{q'(x')}\right)} \le \left\| Y_{q'(x')} \ni y \mapsto \delta(y) \left\|\chi\right\|_{\infty} \right\|_{\mathcal{H}\left(Y_{q'(x')}\right)} \le \|\chi\|_{\infty} \|\delta\|_{\mathcal{H}}.$$

Definition and Proposition E.8.14. Let E be a u.s.c. field of Banach spaces over X. For all $\xi \in \Gamma_c\left(Y \times_X Y',\ Q^{*}E\right)$, define

$$\Phi(\xi)\ (y') := \iota_{E,q'(y')}\left(Y_{q'(y')} \ni y \mapsto \xi(y, y')\right) \in \mathcal{H}\left(Y, E\right)_{q'(y')}$$

for all $y' \in Y'$, where $\iota_{E,x}$ denotes the canonical map from $\Gamma_c\left(Y_x,\ (q^{*}E)|_{Y_x}\right)$ to $\mathcal{H}\left(Y, E\right)_x \cong \mathcal{H}\left(Y_x, E\right)$ for all $x \in X$. Then $\Phi(\xi)$ is in $\Gamma_c\left(Y',\ q'^{*}\mathfrak{F}\left(\mathcal{H}(Y, E)\right)\right)$ and $\left\|\Phi(\xi)\right\|_{\infty} = \|\xi\|_{q'^{*}\mathcal{H}}$.

Because the image of Φ is dense, it follows that we can realise $\Gamma_0\left(Y',\ q'^{*}\mathfrak{F}\left(\mathcal{H}(Y, E)\right)\right)$ as the completion of $\Gamma_c\left(Y \times_X Y',\ Q^{*}E\right)$ for the semi-norm $\|\cdot\|_{q'^{*}\mathcal{H}}$, in other words:

$$q'^{*}\mathfrak{F}\left(\mathcal{H}(Y, E)\right) \cong \mathfrak{F}\left(q'^{*}\mathcal{H}\ \left(Y \times_X Y',\ q'^{*}E\right)\right)$$

as u.s.c. fields of Banach spaces over Y'.

Proof. The map $\Phi(\xi)$ surely is a bounded *selection* of compact support, and almost by definition we have $\|\Phi(\xi)\|_\infty = \|\xi\|_{q'^*\mathcal{H}}$. Moreover, $\xi \mapsto \Phi(\xi)$ is linear and if the support of ξ is contained in L, then the support of $\Phi(\xi)$ is contained in $q'(L)$. It hence suffices to check that $\Phi(\xi)$ is a *section* for ξ taken from a dense subset of $\Gamma_c\left(Y \times_X Y',\ Q^*E\right)$. If $\xi_0 \in \Gamma_c\left(Y,\ q^*E\right)$ and $\delta' \in \mathcal{C}_c\left(Y'\right)$, then $\xi\colon (y, y') \mapsto \delta'(y')\xi_0(y)$ is in $\Gamma_c\left(Y \times_X Y',\ Q^*E\right)$ and the linear span of such sections is dense. If ξ is of this form, then

$$\Phi(\xi)\left(y'\right) = \iota_{E,q'(y')}\left(Y_{q'(y')} \ni y \mapsto \delta'(y')\xi_0(y)\right) = \delta'(y')\iota_{E,q'(y')}\left(\xi_0|_{Y_{q'(y')}}\right) = \delta'(y')\left(\iota_E(\xi_0)\right)_{q'(y')}$$

for all $y' \in Y'$, so $\Phi(\xi) = (\delta \circ q') \cdot \left(\mathfrak{g}_{\mathcal{H}(Y,E)}\left(\iota_E(\xi_0)\right) \circ Q\right)$, where ι_E is the canonical map from $\Gamma_c\left(Y,\ q^*E\right)$ to $\mathcal{H}(Y, E)$. In particular, because $\mathfrak{g}_{\mathcal{H}(Y,E)}\left(\iota_E(\xi_0)\right)$ is in $\Gamma\left(X,\ \mathfrak{F}\left(\mathcal{H}(Y, E)\right)\right)$, we can conclude that $\Phi(\xi)$ is a section. $\square$

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Lebenslauf

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