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> Functoriality and Stratifications of Moduli Spaces of Global $\mathbb{G}$-Shtukas

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Functoriality and Stratifications of Moduli Spaces of Global $\mathbb{G}$-Shtukas

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Abstract

Moduli spaces of global $\mathbb{G}$-shtukas play a crucial role in the Langlands-program for function fields. We analyze their functoriality properties concerning a change of the curve and a change of the group scheme $\mathbb{G}$ under various aspects. In particular we prove two finiteness results that could lead to a formulation of an André-Oort conjecture for $\mathbb{G}$-shtukas. Furthermore we define five axioms concerning stratifications of the considered moduli spaces, which are analogous to the axioms defined by Rapoport and He for Shimura varieties. The proof of these axioms requires some of our previous functoriality results.

Zusammenfassung

Modulräume für globale $\mathbb{G}$-Shtukas spielen eine wichtige Rolle im Langlands-Programm für Funktionenkörper. Wir untersuchen ihre Funktorialitätseigenschaften bezüglich einem Wechsel der Kurve und einem Wechsel des Gruppenschemas $\mathbb{G}$ unter verschiedensten Aspekten. Insbesondere beweisen wir zwei Endlichkeitsresultate, die zu einer Formulierung einer André-Oort Vermutung für globale $\mathbb{G}$-Shtukas führen könnten. Des Weiteren definieren wir fünf Axiome bezüglich Stratifizierungen der betrachteten Modulräume, welche analog sind zu den Axiomen, die Rapoport und He für Schimura-Varietäten definiert haben.Für deren Beweise werden Teile unserer vorherigen Funktorialitätsresultate benötigt.

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Contents

1 Introduction

Global $\mathbb{G}$-shtukas are the function field analogue of abelian varieties. Their moduli spaces play a crucial role in the Langlands-programm for function fields. This thesis is concerned about functoriality properties of these moduli spaces in various aspects as well as their stratifications and their geometry. Let us give a more detailed overview.
We choose a smooth projective geometrically irreducible curve C over a finite field $\mathbb{F}_q$ with q elements. Let further $\mathbb{G}$ be a smooth affine group scheme over C and denote by σ the $\mathbb{F}_q$-Frobenius of a scheme S over $\mathbb{F}_q$. Then a global $\mathbb{G}$-shtuka $\underline{\mathcal{G}} = (\mathcal{G}, s_1, \ldots, s_n, \tau_{\mathcal{G}})$ over S consists of a $\mathbb{G}$-torsor $\mathcal{G}$ over $C_S := C \times_{\mathbb{F}_q} S$, n sections $s_i : S \to C$ called paws and an isomorphism $\tau_{\mathcal{G}} : \sigma^\star \mathcal{G}|_{C_S \setminus \cup_i \Gamma_{s_i}} \to \mathcal{G}|_{C_S \setminus \cup_i \Gamma_{s_i}}$ outside the union of the graphs Γ_{s_i}. The precise definition of all the notations used in the introduction and the thesis are given in the preliminaries in the second chapter. The stack whose S-valued points parametrize the global $\mathbb{G}$-shtukas over S with n paws is denoted by $\nabla_n \mathscr{H}^1(C, \mathbb{G})$. Once we fix n closed points $(v_1, \ldots, v_n) = \underline{v}$ in C to which we refer as characteristic places we can introduce boundedness conditions Z_v for all $v \in \underline{v}$ and H-level structures. Here a bound Z_v is roughly a L^+G_v invariant closed subscheme of the affine flag variety $\hat{\mathcal{F}l}_{\mathbb{G}_v}$ (see § 2.6 for a correct definition) and H is a open compact subgroup of $\mathbb{G}(\mathbb{A}^{\underline{v}})$, where $\mathbb{A}^{\underline{v}}$ denotes the ring of the adeles outside $\underline{v}$. Then we denote by $\nabla_n^{\hat{Z}_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G})$ the stack which parametrize $\mathbb{G}$-shtukas $\underline{\mathcal{G}}$ over S bounded by $(\hat{Z}_v)_{v \in \underline{v}}$ together with an H-level structure. At the beginning of the third chapter we define all the parameters $(C, \mathbb{G}, \underline{v}, Z_{\underline{v}}, H)$ as a shtuka datum. This definition comes with the natural question if an appropriate change of this shtuka datum induces a morphism of the corresponding moduli spaces and what properties it has.
In section 3.1 we define a morphism of shtuka data and clarify what an appropriate change of the shtuka datum should be. Actually a morphism from $(C, \mathbb{G}, \underline{v}, \hat{Z}_{\underline{v}}, H)$ to $(C', \mathbb{G}', \underline{w}, \hat{Z}'_w, H')$ is a pair (π, f), where $\pi : C \to C'$ is a finite morphism and f is a morphism of group schemes from the Weil restriction $\pi_\star \mathbb{G}$ to $\mathbb{G}$ such that $\underline{w}, \hat{Z}'_{\underline{w}}$ and H' satisfy certain conditions.
In the following sections we answer then the questions about the functoriality of $\nabla_n^{\hat{Z}_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G})$. More precisely we consider in section 3.2 firstly the case that we only change the curve C, which yields the following main result of this section.

Theorem 1.1 (cf. Theorem 3.14)**.** *Let $(C, \mathbb{G}, \underline{v}, \hat{Z}_{\underline{v}}, H)$ be a shtuka datum and $\pi : C \to C'$ a finite morphism of smooth geometrically irreducible curves over $\mathbb{F}_q$ with $w_i := \pi(v_i)$ and $\underline{w} := (w_1, \ldots, w_n)$. Then the morphism $(\pi, id_{\pi_\star \mathbb{G}}) : (C, \mathbb{G}, \underline{v}, \hat{Z}_{\underline{v}}, H) \to (C', \pi_\star \mathbb{G}, \underline{w}, \pi_\star \hat{Z}_{\underline{v}}, \pi_\star H)$ of shtuka data (see definition 3.9 and remark 3.10) induces a finite morphism of the moduli stacks*

$$\pi_\star : \nabla_n^{\hat{Z}_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G}) \to \nabla_n^{\pi_\star \hat{Z}_{\underline{v}}, \pi_\star H} \mathscr{H}^1(C', \pi_\star \mathbb{G}).$$

The construction of this morphism and the proof of the theorem relies on a lemma in section 3.1 that states an equivalence of categories between $\mathbb{G}$-torsors over C and $\pi_\star \mathbb{G}$ over C'. In this thesis this theorem will find an application in the proof of the non-emptiness of KR-strata.
The next section 3.3 addresses then the questions about functoriality in the case that we only change the group scheme $\mathbb{G}$. Whereas we construct a morphism

$$\nabla_n^{\hat{Z}_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G}) \to \nabla_n^{\hat{Z}'_{\underline{v}}, H'} \mathscr{H}^1(C, \mathbb{G}')$$

for all morphisms (id_C, f) of shtuka data, we need to make different assumptions to state different results on the properties of this morphism. Assuming that $f : \mathbb{G} \to \mathbb{G}'$ is generically an isomorphism we have the following first main result of this section.

Theorem 1.2 (cf. Theoremm 3.20). *Let* $(id_C, f) : (C, \mathbb{G}, \underline{v}, \hat{Z}_{\underline{v}}, H) \to (C, \mathbb{G}', \underline{v}, \hat{Z}'_{\underline{v}}, H)$ *be a morphism of shtuka data, where* $f : \mathbb{G} \to \mathbb{G}'$ *is an isomorphism over* $C \backslash \underline{w}$*. Then the morphism*

$$f_\star : \nabla_n^{\hat{Z}_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G}) \to \nabla_n^{\hat{Z}'_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G}'), \quad (\underline{\mathcal{G}}, \gamma H) \mapsto (f_\star \underline{\mathcal{G}}, \gamma H)$$

is schematic and quasi-projective. In the case that $\mathbb{G}$ *is a parahoric Bruhat-Tits group scheme this morphism is projective. For any morphism* $(\underline{\mathcal{G}}', \gamma' H) : S \to \nabla_n^{\hat{Z}'_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G}')$

$$\text{the fiber product } S \times_{\nabla_n^{\hat{Z}'_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G}')} \nabla_n^{\hat{Z}'_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G}) \quad \text{is given by a closed subscheme of}$$

$$S \times_{\mathbb{F}_q} \big((L^+_{w_1}(\mathcal{G}')/L^+\widetilde{\mathbb{G}_{w_1}}) \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} (L^+_{w_m}(\mathcal{G}')/L^+\widetilde{\mathbb{G}_{w_m}}) \big).$$

If $\hat{Z}_{\underline{v}}$ *arises as a base change of* $\hat{Z}'_{\underline{v}}$ *for all* $v \in \underline{v}$*, the morphism* $f_\star$ *is surjective.*

This result will again be important in the fifth chapter about axioms on the moduli space $\nabla_n^{\hat{Z}_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G})$. If we assume that $f : \mathbb{G} \to \mathbb{G}'$ is a closed immersion instead of a generic isomorphism we get the second result of this section.

Theorem 1.3 (cf. Theorem 3.23). *Let* $f : \mathbb{G} \to \mathbb{G}'$ *be a closed immersion of smooth affine group schemes over* C*. Then the induced morphism* $f_\star : \nabla_n \mathscr{H}^1(C, \mathbb{G}) \to \nabla_n \mathscr{H}^1(C, \mathbb{G}')$ *is unramified and schematic.*

Assuming additionally that $\mathbb{G}$ is a parahoric Bruhat-Tits group scheme, we prove as well:

Theorem 1.4 (cf. Theorem 3.26). *Let* $\mathbb{G}$ *be a parahoric Bruhat-Tits group scheme and* $f : \mathbb{G} \to \mathbb{G}'$ *be a closed immersion of smooth affine group schemes and* $\underline{v} = (v_1, \ldots, v_n)$ *be a set of closed points in* C*. Then the induced morphism*

$$f_\star : \nabla_n \mathscr{H}^1(C, \mathbb{G})^{\underline{v}} \to \nabla_n \mathscr{H}^1(C, \mathbb{G}')^{\underline{v}} \quad \text{is proper and in particular finite.}$$

Also the morphism in this theorem occurs again in the fifth chapter and is needed to proof the non-emptiness of KR-strata.

The second part of the thesis is concerned about stratifications of the special fiber of the moduli space $\nabla_n^{\hat{Z}_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G})$. One can ask a lot of interesting questions about these stratifications, what is their dimension, are they equi-dimensional, are they smooth, are they affine or quasi-affine, what is their relation, are they non-empty, ... A lot of work about these questions has been done for stratifications of the special fiber of Shimura varieties. In [HR17] Rapoport and He introduce five axioms on Shimura varieties concerning these stratifications. Once these axioms are proved one can conclude the definition and existence of these characteristic subsets as Newton-stratification, Kottwitz-Rapoport stratification and EKOR-stratification in a most general way. Also their natural index set and some relations follow then from these axioms.
We translate these axioms to the moduli space $\nabla_n^{\hat{Z}_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G})$. Whereas in the fourth chapter

we discuss the necessary preparations to do this, we formulate these axioms in first section of the fifth chapter and prove some of them in its second section. The first axiom is about a change of the parahoric subgroup at the characteristic places and requires that this induces a surjective and projective morphism between the corresponding moduli spaces. This axiom is used to reduce some statements to the case of Iwahori-level. The proof of this axiom is mostly done in theorem 3.19. The second axiom is about the existence of local models, which allows to define the KR-stratification on $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$. The existence of these local models follows from [AH16]. The third axiom is about the existence of the Newton stratification. Once the global-local functor is established, the proof is known from [HV11]. The fourth axiom gives a relation between the Newton- and KR-stratification by requiring the existence of some central leaves. The fifth axiom is a basic non-emptiness statement and states that for all groups with Iwahori-level at the characteristic places the minimal KR-stratum is non-empty. We will explain the different steps of the proof of this axiom and also why we can not finish the second step completely at the moment and how a better understanding of the connected components of $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$ would help us to do so. In the case of Shimura varieties this basic non-emptiness is proven by M. Kisin and C.-F. Yu for this axiom under certain conditions on the Shimura datum. Although the idea to construct firstly an object lying in the basic Newton-stratum is similar to the case of Shimura varieties, the construction of this element works differently. In the sixth chapter we will draw some conclusions of the axioms. In the seventh chapter we discuss Drinfeld's moduli space with Iwahori level structure as an example of $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$.

2 Preliminaries

Before we start with the functoriality of $\nabla_n\mathscr{H}^1(C,\mathbb{G})$, we introduce the basic objects and notations that we use in this thesis. Most of the notations introduced in this chapter, can also be found in [AH13] and [AH14].

Let q be a power of some prime number p. We start with a smooth projective geometrically irreducible curve C over the field $\mathbb{F}_q$ with q elements. We denote by $Q := \mathbb{F}_q(C)$ its function field. For a closed point $v \in C$ we denote by A_v the completion of the stalk $\mathcal{O}_{C,v}$ and by Q_v the fraction field of A_v. Furthermore we choose a uniformizer z_v in A_v, denote the residue field of v by $\mathbb{F}_v$ and set $deg\ v = [\mathbb{F}_v : \mathbb{F}_q]$.

Let $\mathbb{G}$ be a smooth affine group scheme over C and $\mathbf{G} := \mathbb{G} \times_C Q$ its generic fiber. Later, from chapter 4 on, we assume $\mathbb{G}$ to be a parahoric Bruhat-Tits group scheme, which we will define in § 2.17. We write $\mathbb{G}_v := \mathbb{G} \times_C Spec\ A_v$ and $\mathbf{G}_v := \mathbb{G} \times_C Spec\ Q_v = \mathbb{G}_v \times_{A_v} Spec\ Q_v$ for the appropriate base changes.

For an $\mathbb{F}_q$-scheme S we denote by $\sigma_S : S \to S$ the absolut $\mathbb{F}_q$-Frobenius, which acts as the q-power map on the structure sheaf. Further we define σ as the endomorphism $id_C \times \sigma_S$ of $C_S := C \times_{\mathbb{F}_q} S$. For a morphism $s : S \to C$ we denote as usual by $\Gamma_s : S \to C_S$ the graph of s, which is a closed immersion.

Let $\mathcal{X}$ be a site with a final object x and G a sheaf of groups on $\mathcal{X}$. Then a (right) G-torsor is a sheaf $\mathcal{G}$ on $\mathcal{X}$ with a right action of G on $\mathcal{G}$ such that $G \times \mathcal{G} \simeq \mathcal{G} \times \mathcal{G}$, $(g,h) \mapsto (hg,h)$ is an isomorphism and $\mathcal{G}(U) \neq \varnothing$ for some covering $U \to X$. When we speak about a torsor, we always mean a right torsor and if nothing else is mentioned we mean a sheaf on the big étale site of a scheme. For any scheme S we write $S_{\acute{E}t}$ for the big étale site of this scheme. We denote by $\mathscr{H}^1(C,\mathbb{G})$ the stack fibered over $(\mathbb{F}_q)_{\acute{E}t}$, whose fiber category $\mathscr{H}^1(C,\mathbb{G})(S)$ is given by the category of $\mathbb{G}$-torsors over C_S.

§ 2.1 Global $\mathbb{G}$-Shtukas:

Let S be an $\mathbb{F}_q$-scheme. A global $\mathbb{G}$-shtuka over S is a tuple $\underline{\mathcal{G}} = (\mathcal{G}, s_1, \ldots, s_n, \tau_{\mathcal{G}})$, where

- $\mathcal{G}$ is a $\mathbb{G}$-torsor over C_S,
- $s_1, \ldots, s_n$ are morphisms $S \to C$ and
- $\tau_{\mathcal{G}} : \sigma^\star \mathcal{G}|_{C_S \setminus (\bigcup_i \Gamma_{s_i})} \to \mathcal{G}|_{C_S \setminus (\bigcup_i \Gamma_{s_i})}$ is an isomorphism of the $\mathbb{G}$-torsors $\sigma^\star\mathcal{G}$ and $\mathcal{G}$ restricted to $C_S \setminus (\Gamma_{s_1} \cup \cdots \cup \Gamma_{s_n})$.

We take the notation $\nabla_n \mathscr{H}^1(C,\mathbb{G})$ from [AH14, Definition 2.12] for the stack fibered over $(\mathbb{F}_q)_{\acute{E}t}$ whose S-valued points for a scheme S are given by $\mathbb{G}$-shtukas $\underline{\mathcal{G}}$ over S. Morphisms from $(\mathcal{G}, s_1, \ldots, s_n, \tau)$ to $(\mathcal{G}', s_1', \ldots, s_n', \tau')$ in the fiber category $\nabla_n \mathscr{H}^1(C,\mathbb{G})(S)$ only exist if $s_i = s_i'$ and are given by morphisms $f : \mathcal{G} \to \mathcal{G}'$ of $\mathbb{G}$-torsors over C_S such that $f \circ \tau = \tau' \circ \sigma^\star f$. Given two $\mathbb{G}$-shtukas $\underline{\mathcal{G}}$ and $\underline{\mathcal{G}}'$ over S with $s_i = s_i'$ as before, we also define a quasi-isogeny from $\underline{\mathcal{G}}$ to $\underline{\mathcal{G}}'$ to be an isomorphism $f : \mathcal{G}|_{C_S \setminus D_S} \to \mathcal{G}'|_{C_S \setminus D_S}$ of $\mathbb{G}$-torsors satisfying $f \circ \tau = \tau' \circ \sigma^\star f$, where D is some effective divisor on C. The moduli space $\nabla_n \mathscr{H}^1(C,\mathbb{G})$ is an ind-algebraic stack that is ind-separated and locally of ind-finite type [AH13, Theorem 3.14].

§ 2.2 Loop Groups:

Let $\mathbb{F}$ be a finite field and $\mathbb{H}$ be a smooth affine group scheme over $\mathbb{D} := Spec\ \mathbb{F}[\![z]\!]$, with generic fiber $\mathbf{H} := \mathbb{H} \times_{\mathbb{D}} \dot{\mathbb{D}}$ where $\dot{\mathbb{D}} := Spec\ \mathbb{F}(\!(z)\!)$. We are mainly interested in the case that $\mathbb{D} \simeq Spec\ A_v$ and $\mathbb{H} = \mathbb{G}_v$ for some closed point $v \in C$.
We recall that the sheaf of groups $L^+\mathbb{H}$ on $\mathbb{F}_{\acute{E}t}$, whose R-valued points for an $\mathbb{F}$-algebra R are given by

$$L^+\mathbb{H}(R) := \mathbb{H}(R[\![z]\!]) := \mathbb{H}(\mathbb{D}_R) := Hom_{\mathbb{D}}(\mathbb{D}_R, \mathbb{H}) \qquad \text{with } \mathbb{D}_R := Spec\ R[\![z]\!],$$

is an infinite-dimensional affine group scheme over $\mathbb{F}$. It is called the group of positive loops associated with $\mathbb{H}$. The group of loops associated with $\mathbf{H}$ is the sheaf of groups $L\mathbf{H}$ on $\mathbb{F}_{\acute{E}t}$, whose R-valued points are defined by

$$L\mathbf{H}(R) := \mathbf{H}(R(\!(z)\!)) := \mathbf{H}(\dot{\mathbb{D}}_R) := Hom_{\dot{\mathbb{D}}}(\dot{\mathbb{D}}_R, \mathbf{H}) \ ,$$

where we write $R(\!(z)\!) = R[\![z]\!]\left[\frac{1}{z}\right]$ and $\dot{\mathbb{D}}_R = Spec\ R(\!(z)\!)$. The loop group $L\mathbf{H}$ is an ind-scheme of ind-finite type over $\mathbb{F}$.

§ 2.3 Torsors for Loop Groups:

We write $\mathscr{H}^1(\mathbb{F}, L^+\mathbb{H})$ for the stack fibered over $(\mathbb{F})_{\acute{E}t}$ whose fiber category $\mathscr{H}^1(\mathbb{F}, L^+\mathbb{H})(S)$ is the category of $L^+\mathbb{H}$-torsors over S. In the same way $\mathscr{H}^1(\mathbb{F}, L\mathbf{H})$ denotes the stack fibered over $(\mathbb{F})_{\acute{E}t}$ whose fiber category $\mathscr{H}^1(\mathbb{F}, L\mathbf{H})(S)$ is the category of $L\mathbf{H}$-torsors over S. There is a natural 1-morphism

$$L : \mathscr{H}^1(\mathbb{F}, L^+\mathbb{H}) \to \mathscr{H}^1(\mathbb{F}, L\mathbf{H}), \quad \mathcal{L}^+ \mapsto \mathcal{L} \tag{1}$$

induced by the inclusion of sheaves $L^+\mathbb{H} \subset L\mathbf{H}$.
We now consider also the z-adic completions of $\mathbb{D}$ and $\mathbb{H}$ and denote them by $\hat{\mathbb{D}} := Spf\ \mathbb{F}[\![z]\!]$ and $\hat{\mathbb{H}} := \mathbb{H} \times_{\mathbb{D}} \hat{\mathbb{D}}$. Later when we pass from global $\mathbb{G}$-shtukas to local $\mathbb{G}_v$-shtukas we often need to know that $L^+\mathbb{H}$-torsors are equivalent to formal $\hat{\mathbb{H}}$-torsors. So we recall that for an $\mathbb{F}$-scheme S a z-adic formal scheme $\mathcal{H}$ over $\hat{\mathbb{D}}_S := \hat{\mathbb{D}} \hat{\times}_{\mathbb{F}} S$ together with an action $\hat{\mathbb{H}} \hat{\times}_{\hat{\mathbb{D}}} \mathcal{H} \to \mathcal{H}$ of $\hat{\mathbb{H}}$ is called a formal $\hat{\mathbb{H}}$-torsor if there is an étale covering $S' \to S$ and an $\hat{\mathbb{H}}$-equivariant isomorphism $\mathcal{H} \hat{\times}_{\hat{\mathbb{D}}_S} \hat{\mathbb{D}}_{S'} \longrightarrow \hat{\mathbb{H}} \times_{\hat{\mathbb{D}}} \hat{\mathbb{D}}_{S'}$, where $\hat{\mathbb{H}}$ is acting on itself by right multiplication.
We denote by $\mathscr{H}^1(\hat{\mathbb{D}}, \hat{\mathbb{H}})$ the category fibered in groupoids over $(\mathbb{F})_{\acute{E}t}$ whose fiber category $\mathscr{H}^1(\hat{\mathbb{D}}, \hat{\mathbb{H}})(S)$ is the groupoid of formal $\hat{\mathbb{H}}$-torsors over S. We remark that Arasteh Rad and Hartl proved in [AH14, Proposition 2.4] that there is a natural isomorphism of stacks $\mathscr{H}^1(\hat{\mathbb{D}}, \hat{\mathbb{H}}) \longrightarrow \mathscr{H}^1(\mathbb{F}, L^+\mathbb{H})$. It sends a formal $\hat{\mathbb{H}}$-torsor $\mathcal{H}$ to the sheaf

$$\mathcal{L}^+ : S_{\acute{E}t} \to \mathbf{Sets}, \quad T \mapsto \mathrm{Hom}_{\hat{\mathbb{D}}_S}(\hat{\mathbb{D}}_T, \mathcal{H})$$

which becomes a $L^+\mathbb{H}$-torsor under the action of $L^+\mathbb{H}(T) = \mathrm{Hom}_{\hat{\mathbb{D}}}(\hat{\mathbb{D}}_T, \hat{\mathbb{H}})$.

§ 2.4 Local $\mathbb{H}$-Shtukas:

Let S be a $\mathbb{F}$-scheme and $\hat{\sigma}$ its absolut $\mathbb{F}$-Frobenius. If $\mathbb{F}$ equals $\mathbb{F}_q$ or $\mathbb{F}_v$ we will write σ and σ_v respectively, instead of $\hat{\sigma}$. With the previous notations a local $\mathbb{H}$-shtuka over S is a pair $(\mathcal{L}^+, \tau_{\mathcal{L}})$ where

- $\mathcal{L}^+$ is a $L^+\mathbb{H}$-torsor over S and
- $\tau_{\mathcal{L}} : \hat{\sigma}^\star \mathcal{L} \longrightarrow \mathcal{L}$ is an isomorphism of the associated loop group torsors from (1) in § 2.3.

A morphism from $\underline{\mathcal{L}} = (\mathcal{L}^+, \tau_{\mathcal{L}})$ to $\underline{\mathcal{L}'} = (\mathcal{L}^{+\prime}, \tau'_{\mathcal{L}'})$ of two local $\mathbb{H}$-shtukas over S is a morphism $f : \mathcal{L}^+ \to \mathcal{L}^{+\prime}$ of $L^+\mathbb{H}$-torsors over S satisfying $\tau_{\mathcal{L}} \circ f = \tau_{\mathcal{L}'} \circ \hat{\sigma}^\star f$. A quasi-isogeny from $\underline{\mathcal{L}} = (\mathcal{L}^+, \tau_{\mathcal{L}})$ to $\underline{\mathcal{L}'} = (\mathcal{L}^{+\prime}, \tau'_{\mathcal{L}'})$ is an isomorphism $f : \mathcal{L} \to \mathcal{L}'$ of the associated $L\mathbf{H}$-torsors satisfying $\tau_{\mathcal{L}} \circ f = \tau_{\mathcal{L}'} \circ \hat{\sigma}^\star f$.
A local $\mathbb{H}$-shtuka $(\mathcal{L}^+, \tau_{\mathcal{L}})$ is called étale if $\tau_{\mathcal{L}} : \hat{\sigma}^\star \mathcal{L} \to \mathcal{L}$ comes already from an isomorphism $\tau_{\mathcal{L}} : \hat{\sigma}^\star \mathcal{L}^+ \to \mathcal{L}^+$ of the $L^+\mathbb{H}$-torsors. We denote the category of local $\mathbb{H}$-shtukas over S by $Sht_{\mathbb{H}}(S)$ and the category of étale local $\mathbb{H}$-shtukas over S by $\acute{E}tSht_{\mathbb{H}}(S)$.
We recall the Corollary [AH14, Corollary 2.9] that states that if $\mathbb{H}$ has a connected special fiber, then any étale local shtuka over an separably closed field k is already isomorphic to $(L^+\mathbb{H}_k,\ 1 \cdot \hat{\sigma}^\star)$.
Let $\mathbb{F}[\![\zeta]\!]$ be the power series ring over $\mathbb{F}$ in a variable ζ. We denote by $\mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$ the category of schemes over $Spec\ \mathbb{F}[\![\zeta]\!]$ on which ζ is locally nilpotent in the structure sheaf. Therefore $\mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$ is the full subcategory of formel schemes over $Spf\ \mathbb{F}[\![\zeta]\!]$ consisting of ordinary schemes. We will define boundedness conditions only for local shtukas over a scheme S in $\mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$.

§ 2.5 The Affine Flag Variety:

Let $\mathbb{H}$ be a smooth affine group scheme over $Spec\ \mathbb{F}[\![z]\!]$ as before, then the affine flag variety $\mathcal{F}l_{\mathbb{H}}$ is defined as the quotient sheaf $L\mathbf{H}/L^+\mathbb{H}$ on $\mathbb{F}_{\acute{E}t}$, that is the sheaf associated to the pre-sheaf

$$T \mapsto L\mathbf{H}(T)/L^+\mathbb{H}(T).$$

By [PR08, Theorem 1.4] $\mathcal{F}l_{\mathbb{H}}$ is represented by an ind-scheme which is ind-quasi-projective and in particular ind-separated and of ind-finite type over $\mathbb{F}$. By [Ric16a, Theorem A] $\mathcal{F}l_{\mathbb{H}}$ is ind-projective if and only if $\mathbb{H}$ is a Bruhat-Tits group scheme over $\mathbb{F}[\![z]\!]$ in the sense of [BT84, Definition 5.2.6]. We also remark that $L^+\mathbb{H}$ acts from the left on $\mathcal{F}l_{\mathbb{H}}$.

§ 2.6 Bounds in $\hat{\mathcal{F}}l_{\mathbb{H}}$

We fix an algebraic closure $\mathbb{F}(\!(\zeta)\!)^{alg}$ of $\mathbb{F}(\!(\zeta)\!)$. For a finite extension R of discrete valuation rings $\mathbb{F}[\![\zeta]\!] \subset R \subset \mathbb{F}(\!(\zeta)\!)^{alg}$ with residue field κ_R we denote similar as before with $\mathcal{N}ilp_R$ the category of R-schemes on which ζ is locally nilpotent. Furthermore we set $\hat{\mathcal{F}}l_{\mathbb{H},R} \coloneqq \mathcal{F}l_{\mathbb{H}} \hat{\times}_{\mathbb{F}} Spf\ R$ as well as $\hat{\mathcal{F}}l_{\mathbb{H}} \coloneqq \mathcal{F}l_{\mathbb{H},\mathbb{F}[\![\zeta]\!]}$.
Now let R and R' be two such finite extensions of discrete valuation rings and let $\hat{Z}_R \subset \hat{\mathcal{F}}l_{\mathbb{H},R}$ and $\hat{Z}'_{R'} \subset \hat{\mathcal{F}}l_{\mathbb{H},R'}$ be two closed ind-subschemes. We call $\hat{Z}_R$ and $\hat{Z}'_{R'}$ equivalent if there is a finite extension $\widetilde{R}$ of discrete valuation rings as above containing R and R' such that $\hat{Z}_R \times_{Spf\ R} Spf\ \widetilde{R} = \hat{Z}'_{R'} \times_{Spf\ R'} Spf\ \widetilde{R}$ as closed ind-subschemes of $\hat{\mathcal{F}}l_{\mathbb{H},\widetilde{R}}$.
Now a bound is defined (compare [AH14, Definition 4.8] and [AH13, Definition 4.5]) as an equivalence class $\hat{Z} = [\hat{Z}_R]$ of closed ind-subschemes $\hat{Z}_R \subset \hat{\mathcal{F}}l_{\mathbb{H},R}$ satisfying

- firstly that all subschemes $\hat{Z}_R$ are stable under the left action of $L^+\mathbb{H}$ on $\hat{\mathcal{F}}l_{\mathbb{H},R}$ and
- secondly that all the special fibers $Z_R \coloneqq \hat{Z}_R \times_{Spf\ R} Spec\ \kappa_R$ are quasi-compact and connected subschemes of $\mathcal{F}l_{\mathbb{H}} \hat{\times}_{\mathbb{F}} \kappa_R$.

We remark that in [AH13] and [AH14] the definition of a bound does not require the special fibers to be connected. Actually we make this assumption, because it does not change the theory and simplifies the formulation of certain statements. In fact if $\hat{Z}$ is the disjoint union of two bounds $\hat{Z}_1 \coprod \hat{Z}_2$ then the moduli space $\nabla_n^{\hat{Z}}\mathscr{H}^1(C,\mathbb{G})$ that will be defined in paragraph § 2.11 is the disjoint union of the moduli spaces $\nabla_n^{\hat{Z}_1}\mathscr{H}^1(C,\mathbb{G})$ and $\nabla_n^{\hat{Z}_2}\mathscr{H}^1(C,\mathbb{G})$.

§ 2.7 The Reflex Ring:

For an equivalence class $\hat{Z} = [\hat{Z}_R]$ as above we set $G_{\hat{Z}} \coloneqq \{\gamma \in Aut_{\mathbb{F}[\![\zeta]\!]}(\mathbb{F}(\!(\zeta)\!)^{alg}) \mid \gamma(\hat{Z}) = \hat{Z}\}$. The ring $R_{\hat{Z}}$ is defined as the intersection of the fixed field of $G_{\hat{Z}}$ in $\mathbb{F}(\!(\zeta)\!)^{alg}$ with all the finite extensions R over which a representative $\hat{Z}_R$ of $\hat{Z}$ exists. In the case that $\hat{Z}$ is a bound, we call $R_{\hat{Z}}$ the reflex ring of $\hat{Z}$.
It is not always clear if there exists a representative of $\hat{Z}$ over $R_{\hat{Z}}$. We write $\kappa_{\hat{Z}}$ and κ_R for the residue fields of $R_{\hat{Z}}$ and R respectively. Then the special fiber $Z_R \coloneqq \hat{Z}_R \times_{Spf\ R} \kappa_R$ arises from a unique closed subscheme $Z \subset \mathcal{F}l_{\mathbb{H}} \times_{\mathbb{F}} \kappa_{\hat{Z}}$. This follows from Galois descent for closed ind-subschemes of $\mathcal{F}l_{\mathbb{H}}$, which is effective. The subscheme Z is called the special fiber of $\hat{Z}$.

§ 2.8 Boundedness of Local $\mathbb{H}$-Shtukas:

Let $\hat{Z}$ be a bound with reflex ring $R_{\hat{Z}}$. Furthermore let $\mathcal{L}^+$ and $\mathcal{L}^{+\prime}$ be two $L^+\mathbb{H}$-torsors over a scheme S in $\mathcal{N}ilp_{R_{\hat{Z}}}$ and $\delta : \mathcal{L} \to \mathcal{L}'$ an isomorphism of the associated $L\mathbf{H}$-torsors. We choose a covering $S' \to S$ such that there are trivializations $\alpha : \mathcal{L}^+ \xrightarrow{\sim} L^+\mathbb{H}_{S'}$ and $\alpha' : \mathcal{L}^{+\prime} \xrightarrow{\sim} L^+\mathbb{H}_{S'}$. Then the automorphism $\alpha' \circ \delta \circ \alpha^{-1} : L\mathbf{H}_{S'} \xrightarrow{\sim} L\mathbf{H}_{S'}$ defines a morphism $S' \to L\mathbf{H}$.
For any finite extension R of $R_{\hat{Z}}$ we have an induced morphism

$$S' \times_{Spf\ R_{\hat{Z}}} Spf\ R \longrightarrow L\mathbf{H} \hat{\times}_{\mathbb{F}} Spf\ R \longrightarrow \hat{\mathcal{F}l}_{\mathbb{H},R}. \tag{2}$$

Now δ is said to be bounded by $\hat{Z}$ if for all trivializations α and α' and all finite extensions R of $R_{\hat{Z}}$ with a representative $\hat{Z}_R$, this morphism (2) factors through $\hat{Z}_R$. By [AH14, Remark 4.9] δ is bounded if and only if this condition is satisfied for one trivialization and for one such extension R. By definition a local $\mathbb{H}$-shtuka $(\mathcal{L}^+, \tau_{\mathcal{L}})$ is bounded by $\hat{Z}$ if $\tau_{\mathcal{L}}$ is bounded by $\hat{Z}$.

§ 2.9 A Version of the Theorem of Beauville-Laszlo:

Let $v \in C$ be a closed point and set $C^v := C\backslash\{v\}$ as well as $C^v_S := C^v \times_{\mathbb{F}_q} S$. We define $\mathscr{H}^1_e(C^v, \mathbb{G})$ as the category fibered in groupoids over $(\mathbb{F}_q)_{\acute{E}t}$ whose fiber category $\mathscr{H}^1_e(C^v, \mathbb{G})(S)$ consists of those $\mathbb{G}$-torsors over C^v_S that can be extended to a $\mathbb{G}$-torsor over C_S. By restricting a $\mathbb{G}$-torsor $\mathcal{G}$ over C_S to C^v_S we get a morphism $\mathscr{H}^1(C, \mathbb{G}) \to \mathscr{H}^1_e(C^v, \mathbb{G})$. We introduce further the notation $\widetilde{\mathbb{G}_v} = Res_{A_v/\mathbb{F}_q[\![z_v]\!]}\mathbb{G}_v$ and $\widetilde{\mathbf{G}_v} := \widetilde{\mathbb{G}_v} \times_{\mathbb{F}_q[\![z_v]\!]} \mathbb{F}_q(\!(z_v)\!)$. For $\mathcal{G} \in \mathscr{H}^1(C, \mathbb{G})$ the base change $\mathcal{G}_v := \mathcal{G} \times_{C_S} (Spf\ A_v \times_{\mathbb{F}_q} S)$ defines a formal $\mathbb{G}_v$-torsor over $Spf\ A_v \hat{\times}_{\mathbb{F}_q} S$ and its Weil restriction $Res_{A_v/\mathbb{F}_q[\![z_v]\!]}\mathcal{G}_v$ defines a formal $\widetilde{\mathbb{G}_v}$-torsor over $Spf\ \mathbb{F}_q[\![z_v]\!] \hat{\times}_{\mathbb{F}_q} S$. Using the category equivalence in § 2.3 it corresponds to an object in $\mathscr{H}^1(\mathbb{F}_q, L^+\widetilde{\mathbb{G}_v})(S)$ that we denote by $L^+_v(\mathcal{G})$ which defines a functor

$$L^+_v : \mathscr{H}^1(C, \mathbb{G}) \longrightarrow \mathscr{H}^1(\mathbb{F}_q, L^+\widetilde{\mathbb{G}_v}), \quad \mathcal{G} \mapsto L^+_v(\mathcal{G}).$$

Furthermore we have the functor

$$L_v : \mathscr{H}^1_e(C^v, \mathbb{G}) \to \mathscr{H}^1(\mathbb{F}_q, L\widetilde{\mathbf{G}_v}), \quad \mathcal{G}\big|_{C^v_S} \mapsto L(L^+_v(\mathcal{G})) = L_v(\mathcal{G})$$

which is independent of the extension $\mathcal{G}$ of $\mathcal{G}\big|_{C^v_S}$. Now a version of the theorem of Beauville-Laszlo, that is proven in [AH14, Lemma 5.1], states that the following diagram is cartesian.

$$\begin{array}{ccc} \mathscr{H}^1(C, \mathbb{G}) & \longrightarrow & \mathscr{H}^1_e(C^v, \mathbb{G}) \\ \downarrow {\scriptstyle L^+_v} & & \downarrow {\scriptstyle L_v} \\ \mathscr{H}^1(\mathbb{F}_q, L^+\widetilde{\mathbb{G}}_v) & \longrightarrow & \mathscr{H}^1(\mathbb{F}_q, L\widetilde{\mathbf{G}}_v) \end{array}$$

§ 2.10 The Global-Local Functor:

Now we fix n closed points $\underline{v} = \{v_1, \ldots, v_n\}$ of C. Then define $A_{\underline{v}}$ as the completion of the local ring $\mathcal{O}_{C^n, \underline{v}}$ at the closed point $\underline{v}$ and furthermore: This

$$\nabla_n\mathscr{H}^1(C, \mathbb{G})^{\underline{v}} := \nabla_n\mathscr{H}^1(C, \mathbb{G}) \times_{C^n} Spf\ A_{\underline{v}}.$$

So an S-valued point of $\nabla_n\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}$ is given by a global $\mathbb{G}$-shtuka $\underline{\mathcal{G}} = (\mathcal{G}, s_1, \dots, s_n, \tau_{\mathcal{G}})$ such that $s_i : S \to C$ factors through $Spf\ A_{v_i}$. We now want to associate with such a global $\mathbb{G}$-shtuka a local $\mathbb{G}_{v_i}$-shtuka for all $v_i \in \underline{v}$. We write $\mathbb{D}_{v_i} := Spec\ A_{v_i}$ and $\hat{\mathbb{D}}_{v_i} := Spf\ A_{v_i}$ as well as $\hat{\mathbb{D}}_{v_i,S} := \mathbb{D}_{v_i}\hat{\times}_{\mathbb{F}_{v_i}} S$. Then we have:

$$\hat{\mathbb{D}}_{v_i}\hat{\times}_{\mathbb{F}_q} S = \coprod_{l\in\mathbb{Z}/deg\ v_i} V(\mathfrak{a}_{v_i,l}) = \coprod_{l\in\mathbb{Z}/deg\ v_i} \hat{\mathbb{D}}_{v_i,S}$$

where $\mathfrak{a}_{v_i,l} := \langle a\otimes 1 - 1\otimes s^\star(a)^{q^l} \mid a \in \mathbb{F}_{v_i}\rangle$ and $V(\mathfrak{a}_{v_i,l})$ is the closed subscheme given by this ideal. We remark that σ cyclically permutes these components and that the $\mathbb{F}_{v_i}$-Frobenius $\sigma^{deg\ v_i}$ leaves all these components $V(\mathfrak{a}_{v_i,l})$ stable. For $\underline{\mathcal{G}} \in \nabla_n\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}$ the base change

$$\mathcal{G}_{v_i} := \mathcal{G}\hat{\times}_{C_S}(Spf\ A_{v_i}\hat{\times}_{\mathbb{F}_q} S) = \coprod_{l\in\mathbb{Z}/deg\ v_i} \mathcal{G}\hat{\times}_{C_S} V(\mathfrak{a}_{v_i,l})$$

defines a formal $\hat{\mathbb{G}}_{v_i}$-torsor over $\coprod_{l\in\mathbb{Z}/deg\ v_i} \hat{\mathbb{D}}_{v_i,S}$ which is an object in $\mathscr{H}^1(\hat{\mathbb{D}}_{v_i}, \hat{\mathbb{G}}_{v_i})(\coprod_{l\in\mathbb{Z}/deg\ v_i} S)$. Each component $\mathcal{G}\times_{C_S} V(\mathfrak{a}_{v_i,l})$ defines a formal $\hat{\mathbb{G}}_{v_i}$-torsor. Similar to the notation in § 2.3 we denote by $\mathcal{L}^+_{i,0}$ the $L^+\mathbb{G}_{v_i}$-torsor associated by [AH14, Proposition 2.4] with the formal $\hat{\mathbb{G}}_{v_i}$-torsor $\mathcal{G}\times_{C_S} V(\mathfrak{a}_{v_i,0})$. Then $(\mathcal{L}^+_{i,0}, \tau^{deg\ v_i})$ is a local $\mathbb{G}_{v_i}$-shtuka, where $\tau^{deg\ v_i} : (\sigma^{deg\ v_i})^\star\mathcal{L}_{i,0} \xrightarrow{\sim} \mathcal{L}_{i,0}$ is the isomorphism of $L\mathbf{G}_{v_i}$-torsors induced by $\tau_{\mathcal{G}}$ (compare also [AH14, Lemma 5.1]). More precisely one should write $\tau^{deg\ v_i} = \tau\circ\sigma^\star\tau\circ\dots\circ(\sigma^{deg\ v_i-2})^\star\tau\circ(\sigma^{deg\ v_i-1})^\star\tau$.
This now defines the global-local functor:

$$\begin{aligned}\Gamma_{v_i} : \nabla_n\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}(S) &\longrightarrow Sht_{\mathbb{G}_{v_i}}(S)\\ \underline{\mathcal{G}} = (\mathcal{G}, s_1,\dots,s_n,\tau) &\longmapsto (\mathcal{L}^+_{i,0}\ ,\ \tau^{deg\ v_i}) = \Gamma_{v_i}(\underline{\mathcal{G}}).\end{aligned}$$

We remark that this functor transforms by [AH14, Definition 5.4] quasi-isogenies into quasi-isogenies. If $v\notin\underline{v}$ the component $V(\mathfrak{a}_{v,0})$ exists only if S is an $\mathbb{F}_v$-scheme.
If we do not restrict $\mathcal{G}\times_{C_S}(Spf\ A_{v_i}\times_{\mathbb{F}_q} S)$ to the component $V(\mathfrak{a}_{v_i,0})$ but consider its Weil restriction $L^+_{v_i}(\mathcal{G})$ we get in a similar way a local $\widetilde{\mathbb{G}_{v_i}}$-shtuka $(L^+_{v_i}(\mathcal{G}), \tau_{v_i})$ where $\tau_{v_i} := L_{v_i}(\tau_{\mathcal{G}}) : \sigma^\star L_{v_i}(\mathcal{G}) \to L_{v_i}(\mathcal{G})$. We denote this local $\widetilde{\mathbb{G}_{v_i}}$-shtuka by $L^+_{v_i}(\underline{\mathcal{G}}) = (L^+_{v_i}(\mathcal{G}), \tau_{v_i})$. We remark that $L^+_v(\mathcal{G})$ does not only exists for $v\in\underline{v}$ but also for other places $v\in C$. In the case that $v\notin\underline{v}$ the local shtuka $L^+_v(\underline{\mathcal{G}})$ is étale.

§ 2.11 Boundedness of Global $\mathbb{G}$-Shtukas:

Recall that we fixed n closed points $\underline{v} = (v_1,\dots,v_n)$ in C. If the group scheme $\mathbb{G}$ is fixed we write for each of these points $\mathcal{F}l_{v_i} := \mathcal{F}l_{\mathbb{G}_{v_i}}$ for the corresponding affine flag variety over $\mathbb{F}_{v_i}$ and $\hat{\mathcal{F}}l_{v_i,R} = \hat{\mathcal{F}}l_{\mathbb{G}_{v_i},R} = \mathcal{F}l_{v_i}\hat{\times}_{\mathbb{F}_{v_i}} Spf\ R$ for a finite extension $A_{v_i}\subset R$. In each of these affine flag varietes $\hat{\mathcal{F}}l_{v_i} = \hat{\mathcal{F}}l_{v_i,A_{v_i}}$ we choose a bound $\hat{Z}_{v_i} = [\hat{Z}_{v_i,R}]$ with reflex ring $R_{\hat{Z}_{v_i}}$ and we write $\hat{Z}_{\underline{v}}$ for the tuple $(\hat{Z}_{v_1},\dots,\hat{Z}_{v_n})$. Choosing a uniformizer π_{v_i} in $R_{\hat{Z}_{v_i}}$ and defining $\mathbb{F}_R$ as the compositum of all the residue fields $R_{\hat{Z}_{v_i}}/(\pi_{v_i})$, we set $R_{\hat{Z}_{\underline{v}}} := \mathbb{F}_R[\![\pi_{v_1}\dots,\pi_{v_n}]\!]$. In particular the morphism $Spf\ R_{\hat{Z}_{\underline{v}}} \to C^n$ factors through $Spf\ A_{\underline{v}}$. This means that every point $\underline{\mathcal{G}} = (\mathcal{G}, s_1,\dots,s_n,\tau_{\mathcal{G}})$ in $\nabla_n\mathscr{H}^1(C,\mathbb{G})\times_{C^n} Spf\ R_{\hat{Z}_{\underline{v}}}(S)$ defines also an S-valued point in $\nabla_n\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}$ so that we write $\Gamma_{v_i}(\underline{\mathcal{G}})$ for its associated local $\mathbb{G}_{v_i}$-shtuka over S. The fact that $S\in Nilp_{R_{\hat{Z}_{v_i}}}$ allows us to ask if

$\Gamma_{v_i}(\underline{\mathcal{G}})$ is bounded by $\hat{Z}_{v_i}$.

We define $\nabla_n^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G})$ to be the stack consisting of these bounded global $\mathbb{G}$-shtukas. That means the fiber category $\nabla_n^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G})(S)$ is the full subcategory of $\nabla_n\mathscr{H}^1(C,\mathbb{G})\times_{C^n} Spf\ R_{\hat{Z}_{\underline{v}}}(S)$ that consists of those global $\mathbb{G}$-shtukas $\underline{\mathcal{G}}$ over S that are bounded by $\hat{Z}_{\underline{v}}$. By [AH13, Remark 7.2] the moduli space $\nabla_n^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G})$ is a closed ind-substack of $\nabla_n\mathscr{H}^1(C,\mathbb{G})\times_{C^n} Spf\ R_{\hat{Z}_{\underline{v}}}$. Moreover we denote by $\nabla_n^{\hat{Z}_{\underline{v}}}\mathcal{H}^1(C,\mathbb{G})_{\mathbb{F}_R} := \nabla_n^{\hat{Z}_{\underline{v}}}\mathcal{H}^1(C,\mathbb{G})\times_{Spf R_{\hat{Z}_{\underline{v}}}}\mathbb{F}_R$ the special fiber of $\nabla_n^{\hat{Z}_{\underline{v}}}\mathcal{H}^1(C,\mathbb{G})$.

§ 2.12 D-Level Structures:

Let D be a proper closed subscheme of C and let $D_S := D\times_{\mathbb{F}_q} S$ for some $\mathbb{F}_q$-scheme S and $\mathcal{G}$ a $\mathbb{G}$-torsor on C_S. By [AH13, Definition 3.1] a D-level structure on $\mathcal{G}$ is a trivialization $\Psi : \mathcal{G}\times_{C_S} D_S \to \mathbb{G}\times_C D_S$ and $\mathscr{H}^1_D(C,\mathbb{G})$ denotes the stack fibered over $(\mathbb{F}_q)_{\acute{E}t}$ whose fiber category $\mathscr{H}^1_D(C,\mathbb{G})(S)$ consists of pairs $(\mathcal{G},\Psi)$ where $\mathcal{G}\in\mathscr{H}^1(C,\mathbb{G})(S)$ and Ψ is a D-level structure. A morphism from $(\mathcal{G},\Psi)$ to $(\mathcal{G}',\Psi')$ in this fiber category is given by an isomorphism $f:\mathcal{G}\to\mathcal{G}'$ of $\mathbb{G}$-torsors such that $\Psi = \Psi'\circ(f\times id_{D_S})$. The moduli stack of global $\mathbb{G}$-shtukas with D-level structure is denoted by $\nabla_n\mathscr{H}^1_D(C,\mathbb{G})$. Its fiber category over S is given by tuples $(\underline{\mathcal{G}},\Psi) = (\mathcal{G},s_1,\ldots,s_n,\tau_{\mathcal{G}},\Psi)$ where $\underline{\mathcal{G}}\in\nabla_n\mathscr{H}^1(C,\mathbb{G})\times_{C^n}(C\backslash D)^n(S)$ (i.e. $s_i : S\to C$ factors through $C\backslash D$) and Ψ is a S-level structure on $\mathcal{G}$ satisfying $\Psi\circ(\tau\times id_{D_S}) = \sigma^\star(\Psi)$. A morphism from $(\underline{\mathcal{G}},\Psi)$ to $(\underline{\mathcal{G}}',\Psi')$ in this fiber category is a morphism $f\in\nabla_n\mathscr{H}^1(C,\mathbb{G})(S)$ (in particular an isomorphism $f:\mathcal{G}\to\mathcal{G}'$ of $\mathbb{G}$-torsors) satisfying $\Psi = \Psi'\circ(f\times id_{D_S})$.

If $D=\varnothing$ we have $\nabla_n\mathscr{H}^1_D(C,\mathbb{G}) = \nabla_n\mathscr{H}^1(C,\mathbb{G})$. If $v_1,\ldots,v_n\notin D$ and $\hat{Z}_{\underline{v}}$ is a bound as before we use the intuitive notations $\nabla_n\mathscr{H}^1_D(C,\mathbb{G})^{\underline{v}}$ for the base change $\nabla_n\mathscr{H}^1_D(C,\mathbb{G})\times_{C^n} Spf\ A_{\underline{v}}$ and $\nabla_n^{\hat{Z}_{\underline{v}}}\mathscr{H}^1_D(C,\mathbb{G})$ for the stack of $\mathbb{G}$-shtukas $\underline{\mathcal{G}}$ in $\nabla_n^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G})$ with a D-level structure.

§ 2.13 Local Shtukas and Local GL_r-Shtukas:

The category of local GL_r-shtukas over an F_q-scheme S can be defined more explicitly. We briefly describe this here since it is useful for the definition of the Tate functors.

We denote by $\mathcal{O}_S[\![z]\!]$ the sheaf of $\mathcal{O}_S$-algebras on $S_{\acute{E}t}$ which associates with every S-scheme Y the ring $\mathcal{O}_S[\![z]\!](Y) := \Gamma(Y,\mathcal{O}_Y)[\![z]\!]$. Now every sheaf of $\mathcal{O}_S[\![z]\!]$-modules that is fqqc-locally free of rank r is by [HV11, Prop 2.3] already Zariski locally free of rank r. We call these locally free sheafes of $\mathcal{O}_S[\![z]\!]$-modules of rank r. For a commutative ring R we set $R(\!(z)\!) := R[\![z]\!]\left[\frac{1}{z}\right]$. This leads to the intuitive notation $\mathcal{O}_S(\!(z)\!)$ for the sheaf on $S_{\acute{E}t}$ associated to the pre-sheaf $Y\mapsto\Gamma(Y,\mathcal{O}_Y)(\!(z)\!)$. The absolut $\mathbb{F}$ Frobenius was denoted $\hat{\sigma}$ and we use the same notation for the endomorphism of $\mathcal{O}_S[\![z]\!]$ and $\mathcal{O}_S(\!(z)\!)$ that acts as $\hat{\sigma}$ on sections of $\mathcal{O}_S$ and as the identity on z. For a sheaf M of $\mathcal{O}_S[\![z]\!]$-modules we can consider the pullback $\hat{\sigma}^\star M := M\otimes_{\mathcal{O}_S[\![z]\!],\hat{\sigma}}\mathcal{O}_S[\![z]\!]$.
Now by [HV11, Definition 4.1] a local shtuka of rank r over S is a pair (M,τ_M) consisting of a locally free sheaf M of $\mathcal{O}_S[\![z]\!]$-modules of rank r and an isomorphism

$$\tau_M : \hat{\sigma}^\star M\otimes_{\mathcal{O}_S[\![z]\!]}\mathcal{O}_S(\!(z)\!) \to M\otimes_{\mathcal{O}_S[\![z]\!]}\mathcal{O}_S(\!(z)\!).$$

The local shtuka (M,τ_M) is called étale if τ_M arises from an isomorphism $\hat{\sigma}^\star M\to M$ of $\mathcal{O}_S[\![z]\!]$-modules. A morphism from (M,τ_M) to $(M',\tau_{M'})$ between two local shtukas over S is a morphism $f : M\to M'$ of $\mathcal{O}_S[\![z]\!]$-modules satisfying $\tau_{M'}\circ\hat{\sigma}^\star f = f\circ\tau_M$. A quasi-isogeny from (M,τ_M) to

$(M', \tau_{M'})$ is a morphism $f : M \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S(\!(z)\!) \to M' \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S(\!(z)\!)$ of $\mathcal{O}_S(\!(z)\!)$-modules satisfying $\tau_{M'} \circ \hat{\sigma}^\star f = f \circ \tau_M$. We denote the category of local shtukas over S by $Sht_{\mathbb{F}_q}(S)$ and the category of étale local shtukas over S by $\acute{E}tSht_{\mathbb{F}_q}(S)$.
Now there is a category equivalence between local GL_r-shtukas as defined in § 2.4 and the category of local shtukas of rank r over S with isomorphisms as the only morphisms. It is naturally induced by the category equivalence [HV11, Lemma 4.2] of $\mathscr{H}^1(\mathbb{F}, L^+\mathrm{GL}_r)(S)$ and the category of locally free sheaves of $\mathcal{O}_S[\![z]\!]$-modules of rank r with isomorphisms as morphisms.

§ 2.14 Tate Functors on Local $\mathbb{H}$-Shtukas:

Now let S be a connected $\mathbb{F}_q$-scheme with geometric base point $\bar{s} \in S$ and algebraic fundamental group $\pi_1(S, \bar{s})$. We denote by $\mathfrak{FMod}_{\mathbb{F}[\![z]\!][\pi_1(S,\bar{s})]}$ (resp. $\mathfrak{FMod}_{\mathbb{F}(\!(z)\!)[\pi_1(S,\bar{s})]}$) the category of finite and free $\mathbb{F}_q[\![z]\!]$-modules (resp. $\mathbb{F}(\!(z)\!)$ vector spaces) equipped with a continuous action of $\pi_1(S, \bar{s})$. Then the dual Tate functor $\check{T}$ on étale local shtukas is defined as

$$\check{T}_- : \acute{E}tSht_{\mathbb{F}_q}(S) \to \mathfrak{FMod}_{\mathbb{F}[\![z]\!][\pi_1(S,\bar{s})]} \qquad \underline{M} := (M, \tau_M) \mapsto \check{T}_{\underline{M}} := (M \otimes_{\mathcal{O}_S[\![z]\!]} \kappa(\bar{s})[\![z]\!])^{\tau_M}$$

where the superscript τ_M denotes the τ_M invariants. The rational dual Tate functor is defined by

$$\check{V}_- : \acute{E}tSht_{\mathbb{F}_q}(S) \to \mathfrak{FMod}_{\mathbb{F}(\!(z)\!)[\pi_1(S,\bar{s})]} \qquad \underline{M} := (M, \tau_M) \mapsto \check{V}_{\underline{M}} := \check{T}_{\underline{M}} \otimes_{\mathbb{F}[\![z]\!]} \mathbb{F}(\!(z)\!)$$

We also need Tate functors for local $\mathbb{H}$-shtukas. To define these we denote by $Rep_{\mathbb{F}[\![z]\!]}\mathbb{H}$ the category of representations $\rho : \mathbb{H} \to \mathrm{GL}(V)$ where V is a finite free $\mathbb{F}[\![z]\!]$-module and ρ a morphism of algebraic groups over $\mathbb{F}[\![z]\!]$. Any such ρ naturally induces, as described in [AH14, section 3, above Definition 3.5], a functor $\rho_\star : \acute{E}tSht_{\mathbb{H}}(S) \to \acute{E}tSht_{\mathbb{F}}(S)$ that is compatible with quasi-isogenies.
Let $Funct^\otimes(Rep_{\mathbb{F}[\![z]\!]}\mathbb{H},\ \mathfrak{FMod}_{\mathbb{F}[\![z]\!][\pi_1(S,\bar{s})]})$ and $Funct^\otimes(Rep_{\mathbb{F}(\!(z)\!)}\mathbb{H},\ \mathfrak{FMod}_{\mathbb{F}(\!(z)\!)[\pi_1(S,\bar{s})]})$ be the categories of the appropriate tensor functors whose morphisms are isomorphisms of functors. Now the dual Tate functor $\check{\mathcal{T}}_-$ and the rational dual Tate functor $\check{\mathcal{V}}_-$ are defined by

$$\check{\mathcal{T}}_- : \acute{E}tSht_{\mathbb{H}}(S) \longrightarrow Funct^\otimes(Rep_{\mathbb{F}[\![z]\!]}\mathbb{H},\ \mathfrak{FMod}_{\mathbb{F}[\![z]\!][\pi_1(S,\bar{s})]}) \qquad \underline{\mathcal{L}} \mapsto (\check{\mathcal{T}}_{\underline{\mathcal{L}}} : \rho \mapsto \check{T}_{\rho_\star \underline{\mathcal{L}}})$$
$$\check{\mathcal{V}}_- : \acute{E}tSht_{\mathbb{H}}(S) \longrightarrow Funct^\otimes(Rep_{\mathbb{F}(\!(z)\!)}\mathbb{H},\ \mathfrak{FMod}_{\mathbb{F}(\!(z)\!)[\pi_1(S,\bar{s})]}) \qquad \underline{\mathcal{L}} \mapsto (\check{\mathcal{V}}_{\underline{\mathcal{L}}} : \rho \mapsto \check{V}_{\rho_\star \underline{\mathcal{L}}}).$$

§ 2.15 Tate Functors on Global $\mathbb{G}$-Shtukas:

Now we assume that the tuple $\underline{v} = (v_1, \ldots, v_n)$ is given by n pairwise different places on C and set $\widetilde{C} = C \backslash \{v_1, \ldots, v_n\}$. We denote by $\mathbb{O}^{\underline{v}} := \prod_{v \in (C \backslash \underline{v})} A_v$ the integral adeles of C outside $\underline{v}$ and by $\mathbb{A}^{\underline{v}} := \mathbb{O}^{\underline{v}} \otimes_{\mathcal{O}_{\widetilde{C}}} Q = \prod'_{v \in (C \backslash \underline{v})} Q_v$ the adeles of C outside $\underline{v}$. Let $Rep_{\mathbb{O}^{\underline{v}}}\mathbb{G}$ be the category of representations $\rho : \mathbb{G} \times_C Spec\ \mathbb{O}^{\underline{v}} \to \mathrm{GL}_{\mathbb{O}^{\underline{v}}}(V)$ where V is a finite free $\mathbb{O}^{\underline{v}}$-module and ρ a morphism of group schemes over $\mathbb{O}^{\underline{v}}$. Let S be a connected scheme over $Spf\ A_{\underline{v}}$ with a fixed geometric base point $\bar{s}$. We denote by $\mathfrak{Mod}_{\mathbb{O}^{\underline{v}}[\pi_1(S,\bar{s})]}$ (resp. $\mathfrak{Mod}_{\mathbb{A}^{\underline{v}}[\pi_1(S,\bar{s})]}$) the category of $\mathbb{O}^{\underline{v}}$-modules (resp. $\mathbb{A}^{\underline{v}}$-modules) with a continuous $\pi_1(S, \bar{s})$ action. For a finite subscheme $D \subset C$ we set $D_{\bar{s}} = D \times_{\mathbb{F}_q} \bar{s}$ as well as $\mathcal{G}|_{D_{\bar{s}}} = \mathcal{G} \times_{C_S} D_{\bar{s}}$. Then the dual Tate functor $\check{\mathcal{T}}_-$ and the rational dual Tate functor $\check{\mathcal{V}}_{-}$ on global $\mathbb{G}$-shtukas are defined by

$$\begin{aligned}\check{\mathcal{T}}_{-}:\ \nabla_n\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}(S) &\longrightarrow Funct^{\otimes}(Rep_{\mathbb{O}^{\underline{v}}}\mathbb{G},\ \mathfrak{Mod}_{\mathbb{O}^{\underline{v}}[\pi_1(S,\bar{s})]})\\ \underline{\mathcal{G}} &\mapsto \left(\check{\mathcal{T}}_{\underline{\mathcal{G}}}:\rho\mapsto \varprojlim_{D\subset\widetilde{C}}(\rho_\star\underline{\mathcal{G}}|_{D_{\bar{s}}})^{\tau_{\mathcal{G}}}\right)\\ \check{\mathcal{V}}_{-}:\ \nabla_n\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}(S) &\longrightarrow Funct^{\otimes}(Rep_{\mathbb{A}^{\underline{v}}}\mathbb{G},\ \mathfrak{Mod}_{\mathbb{A}^{\underline{v}}[\pi_1(S,\bar{s})]})\\ \underline{\mathcal{G}} &\mapsto \left(\check{\mathcal{V}}_{\underline{\mathcal{G}}}:\rho\mapsto \varprojlim_{D\subset\widetilde{C}}(\rho_\star\underline{\mathcal{G}}|_{D_{\bar{s}}})^{\tau_{\mathcal{G}}}\otimes_{\mathbb{O}^{\underline{v}}}\mathbb{A}^{\underline{v}}\right).\end{aligned}$$

We remark that the functor $\check{\mathcal{V}}$ transforms by [AH13, section 6] quasi-isogenies into isomorphisms. Furthermore it is useful to know that there is a natural isomorphism $\varprojlim_{D\subset C}(\rho_\star\mathcal{G}|_{D_{\bar{s}}})^{\tau_{\mathcal{G}}}\simeq\prod_{v\in C\setminus\underline{v}}\check{\mathcal{T}}_{L_v^+(\underline{\mathcal{G}})}(\rho_v)$ writing $\rho=(\rho_v)_{v\in\widetilde{C}}$ with $\rho_v:=\rho\times id_{A_v}$. Here $L_v^+(\underline{\mathcal{G}})$ is the étale local $\widetilde{\mathbb{G}}_v$-shtuka and $\check{\mathcal{T}}_{L_v^+(\underline{\mathcal{G}})}(\rho_v):=\check{\mathcal{T}}_{L_v^+(\underline{\mathcal{G}})}(\widetilde{\rho_v})$ where $\widetilde{\rho_v}$ is the representation of $\widetilde{\mathbb{G}}_v$ induced from ρ_v by Weil restriction (see [AH14, remark 5.6]).

§ 2.16 H-Level Structures:

Let H be an open compact subgroup of $\mathbb{G}(\mathbb{A}^{\underline{v}})$. In this paragraph we define H-level structures which is a generalization of the previous D-level structures. We denote by

$$\omega^{\circ}_{\mathbb{O}^{\underline{v}}}:Rep_{\mathbb{O}^{\underline{v}}}\mathbb{G}\longrightarrow\mathfrak{Mod}_{\mathbb{O}^{\underline{v}}}\ ,\qquad \omega^{\circ}_{\mathbb{A}^{\underline{v}}}:Rep_{\mathbb{A}^{\underline{v}}}\mathbb{G}\longrightarrow\mathfrak{Mod}_{\mathbb{A}^{\underline{v}}}$$

the forgetful functors and by $Isom^{\otimes}(\omega^{\circ}_{\mathbb{O}^{\underline{v}}},\check{\mathcal{T}}_{\underline{\mathcal{G}}})$ and $Isom^{\otimes}(\omega^{\circ}_{\mathbb{A}^{\underline{v}}},\check{\mathcal{V}}_{\underline{\mathcal{G}}})$ the sets of isomorphisms of tensor functors which are defined for every global $\mathbb{G}$-shtuka $\underline{\mathcal{G}}$ over S, where S is as before a scheme over $Spf\ A_{\underline{v}}$ with geometric base point $\bar{s}\in S$. By the definition of the Tate functor $\pi_1(S,\bar{s})$ acts on $\check{\mathcal{T}}_{\underline{\mathcal{G}}}$ and $\mathbb{G}(\mathbb{O}^{\underline{v}})$ (resp. $\mathbb{G}(\mathbb{A}^{\underline{v}})$) acts on $\omega^{\circ}_{\mathbb{O}^{\underline{v}}}$ (resp. $\omega^{\circ}_{\mathbb{A}^{\underline{v}}}$) since we have $\mathbb{G}(\mathbb{O}^{\underline{v}})=Aut^{\otimes}(\omega^{\circ}_{\mathbb{O}^{\underline{v}}})$ by the generalized tannakian formalism [Wed04, corollary 5.20]. This induces an action of $\mathbb{G}(\mathbb{O}^{\underline{v}})\times\pi_1(S,\bar{s})$ on $Isom^{\otimes}(\omega^{\circ}_{\mathbb{O}^{\underline{v}}},\check{\mathcal{T}}_{\underline{\mathcal{G}}})$ and of $\mathbb{G}(\mathbb{A}^{\underline{v}})\times\pi_1(S,\bar{s})$ on $Isom^{\otimes}(\omega^{\circ}_{\mathbb{A}^{\underline{v}}},\check{\mathcal{V}}_{\underline{\mathcal{G}}})$. Now by [AH14, Definition 6.3] a rational H-level structure $\bar{\gamma}$ on a global $\mathbb{G}$-shtuka $\underline{\mathcal{G}}$ in $\nabla_n\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}(S)$ is defined as a $\pi_1(S,\bar{s})$-invariant H-orbit $\bar{\gamma}=\gamma H$ in $Isom^{\otimes}(\omega^{\circ}_{\mathbb{A}^{\underline{v}}},\check{\mathcal{V}}_{\underline{\mathcal{G}}})$. We denote by $\nabla_n^H\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}$ the category fibered in groupoids over $(\mathbb{F}_q)_{\acute{E}t}$ with the following fiber categories. An object in $\nabla_n^H\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}(S)$ is a tuple $(\underline{\mathcal{G}},\gamma)$, where $\underline{\mathcal{G}}\in\nabla_n\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}(S)$ and $\bar{\gamma}$ is a H-level structure on $\underline{\mathcal{G}}$. A morphism from $(\underline{\mathcal{G}},\bar{\gamma})$ to $(\underline{\mathcal{G}}',\bar{\gamma}')$ over S is a quasi-isogeny $f:\underline{\mathcal{G}}\to\underline{\mathcal{G}}'$ that is an isomorphism at the characteristic places v_i and that satisfies $\check{\mathcal{V}}_f\circ\gamma H=\gamma' H$. (So $f:\underline{\mathcal{G}}|_{C_S\setminus T_S}\xrightarrow{\sim}\underline{\mathcal{G}}'|_{C_S\setminus T_S}$ for a finite subscheme $T\subset C$ with $v_1,\ldots,v_n\notin T$.)
Now let D be a finite subscheme $D\subset C$ with $v_1,\ldots,v_n\notin D$ and let H_D be the open and compact subgroup $ker(\mathbb{G}(\mathbb{O}^{\underline{v}})\to\mathbb{G}(\mathcal{O}_D))$ of $\mathbb{G}(\mathbb{A}^{\underline{v}})$. Then we remark hat by [AH13, Theorem 6.4] there is a canonical isomorphism of stacks

$$\nabla_n\mathscr{H}^1_D(C,\mathbb{G})^{\underline{v}}\xrightarrow{\sim}\nabla_n^{H_D}\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}\qquad(3)$$

Furthermore we note that for the conjugated group gHg^{-1} with $g\in\mathbf{G}(\mathbb{A}^{\underline{v}})$ there is by [AH13, Remark 6.6] a natural isomorphism $\nabla_n^H\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}\xrightarrow{\sim}\nabla_n^{gHg^{-1}}\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}$ sending $(\underline{\mathcal{G}},\gamma H)$ to $(\underline{\mathcal{G}},\gamma g^{-1}(gHg^{-1}))$.

In addition we remark that by [AH13, section 6] for a open compact subgroup $\widetilde{H} \subset \mathbb{G}(\mathbb{A}^{\underline{v}})$ contained in H we have a natural finite étale morphism

$$\nabla_n^{\widetilde{H}} \mathscr{H}^1(C, \mathbb{G})^{\underline{v}} \xrightarrow{\sim} \nabla_n^H \mathscr{H}^1(C, \mathbb{G})^{\underline{v}}, \quad (\underline{\mathcal{G}}, \gamma \widetilde{H}) \mapsto (\underline{\mathcal{G}}, \gamma H) \tag{4}$$

If we have additionally given a bound $\hat{Z}_v$ at all places $v \in \underline{v}$ we denote by $\nabla_n^{\hat{Z}_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G})$ the closed substack of $\nabla_n^H \mathscr{H}^1(C, \mathbb{G})^{\underline{v}} \times_{Spf\ A_{\underline{v}}} Spf\ R_{\hat{Z}_{\underline{v}}}$ that consists of those points $(\underline{\mathcal{G}}, \bar{\gamma})$ such that $\underline{\mathcal{G}}$ is bounded by $\hat{Z}_{\underline{v}}$.

§ 2.17 Parahoric Bruhat-Tits Group Schemes

From the fourth chapter forward we will assume $\mathbb{G}$ to be a parahoric Bruhat-Tits group scheme, where we call a smooth affine group scheme $\mathbb{G}$ over C a parahoric Bruhat-Tits group scheme, if

- all fibers are connected,
- the generic fiber $\mathbf{G}$ is a connected reductive group over Q and
- for all $v \in |C|$ the group $\mathbb{G}(A_v) \subset \mathbb{G}(Q_v) = \mathbf{G}(Q_v)$ is a parahoric subgroup in the sense of [BT84, Definition 5.2.6].

For each parahoric subgroup in $\mathbf{G}_v(Q_v)$ there is a unique smooth affine group scheme $\mathbb{H}$ over A_v with connected special fiber, with generic fiber equal to $\mathbf{G}_v$ and with $\mathbb{H}(A_v)$ equal to this parahoric subgroup. Since this group scheme is exactly given by $\mathbb{G}_v$ our definition of parahoric Bruhat-Tits group scheme coincides with the one in [AH13, Definition 3.11].
Now Bruhat-Tits group schemes can be constructed as follows. We start with a reductive group scheme $\mathbf{G}$ over the function field Q, which has a reductive model G over an open subscheme $C \smallsetminus \{w_1, \ldots, w_m\}$ of C. For each of the pairwise different closed points $w \in \underline{w} := \{w_1, \ldots, w_m\}$ we choose futhermore a parahoric subgroup $H_w \in \mathbb{G}(Q_w)$. Then H_w corresponds as explained above to a smooth affine group scheme $\mathbb{H}_w$ over A_w with generic fiber $\mathbf{G} \times_Q Q_w$. Consequently $\left(\coprod_{w \in \underline{w}} \mathbb{H}_w\right) \coprod G$ is a group scheme over $\left(\coprod_{w \in \underline{w}} SpecA_w\right) \coprod C \backslash \{w_1, \ldots, w_n\}$. Since $\left(\coprod_{w \in \underline{w}} SpecA_w\right) \coprod C \backslash \{w_1, \ldots, w_n\} \to C$ is an fpqc covering and the identification $\mathbb{H}_w \times_{A_w} Q_w = G \times_{C \smallsetminus \{\underline{w}\}} Q_w$ gives a descent datum for $\left(\coprod_{w \in \underline{w}} \mathbb{H}_w\right) \coprod G$, we can glue this group scheme using faithfully flat descent [BLR90, section 6.1, theorem 6] to a group scheme $\mathbb{G}$ over C. This group scheme is by [Gro65, Proposition 2.7.1] smooth and by [Gro67, Proposition 17.7.1] affine over C. Therefore $\mathbb{G}$ is by construction a parahoric Bruhat-Tits group scheme satisfying $\mathbb{G}_v = \mathbb{H}_v$ and $\mathbb{G} \times_C Q = \mathbf{Q}$.
Further we remark that if $\pi : \widetilde{C} \to C$ is a generically étale covering of C and $\mathbb{G}$ is a parahoric Bruhat-Tits group scheme over $\widetilde{C}$ then by [Hei10, Example (3) page 2] the Weil restriction $\pi_\star \mathbb{G}$ (see also lemma 3.2) of $\mathbb{G}$ along π is again a parahoric Bruhat-Tits group scheme. In addition we remark that parahoric Bruhat-Tits group schemes give an interesting class of smooth affine group schemes over C since moduli spaces of global $\mathbb{G}$-shtukas for such parahoric Bruhat-Tits group schemes $\mathbb{G}$ are used by Lafforgue to establish in [Laf12] and [Laf14] the Langlands-parametrization over the function field Q.

3 Functoriality of $\nabla_n^{\hat{Z}_v,H}\mathscr{H}^1(C,\mathbb{G})$

In this chapter we establish and analyze morphisms between moduli spaces of global $\mathbb{G}$-shtukas, which are functorial in changing the curve C and the group scheme $\mathbb{G}$. Apart from the interest of these morphisms in the study of $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$ in general, there are two other motivations. The first one is that the finiteness results in theorem 3.14 and theorem 3.26 supposedly enables us to formulate in some future work a André-Oort conjecture for moduli spaces of global $\mathbb{G}$-shtukas, as explained more detailed in remark 3.28. The second motivation is the study of stratifications of $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$ in the following chapters.
Now the third chapter is divided into three sections. In the first section we define a shtuka datum and morphisms of these. A shtuka datum contains all the necessary parameters to define a moduli space of $\mathbb{G}$-shtukas. Then a morphism is defined in such a way that it satisfies exactly the properties to induce a morphism of the corresponding moduli spaces. The fact that $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$ is indeed functorial in the shtuka datum is then seen in the following two sections. The second section discusses the case that we only change the curve C in the shtuka datum. The induced morphism is constructed and it is proven in theorem 3.14 proven to be finite. In the fifth chapter we use it again, when we sketch the proof of the fifth axiom.
A change $f:\mathbb{G}\to\mathbb{G}'$ of the group scheme $\mathbb{G}$ is analyzed in the third section. Before making any assumptions on f the morphism $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})\to\nabla_n^{\hat{Z}'_{\underline{v}},H'}\mathscr{H}^1(C,\mathbb{G}')$ is constructed in general. Then, assuming that f is generically an isomorphism, we prove in theorem 3.20 a projectivity and surjectivity result that is needed again for the first axiom in the fifth chapter. Afterwards we consider closed immersions of group schemes. In this situation we prove $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})\to\nabla_n^{\hat{Z}'_{\underline{v}},H'}\mathscr{H}^1(C,\mathbb{G}')$ to be unramified (theorem 3.23) and even finite if $\mathbb{G}$ is a parahoric Bruhat-Tits group scheme (theorem 3.26). This morphism is used as well for the fith axiom in the fifth chapter.

3.1 The Shtuka Datum

In this section, we define the category of Shtuka-data. While we can easily define the objects, we need some further explanations to define the morphisms.

Definition 3.1. *A Shtuka-datum is a tuple* $(C,\mathbb{G},\underline{v},Z_{\underline{v}},H)$ *where*

- C *is a smooth projective geometrically irreducible curve over* $\mathbb{F}_q$*,*
- $\mathbb{G}$ *is a smooth affine group scheme over* C*,*
- $\underline{v}=(v_1,\dots,v_n)$ *is a tuple of* n *closed points in* C *(not necessarily disjoint),*
- $\hat{Z}_{\underline{v}}$ *is a bound in the sense of § 2.6,*
- H *is an open compact subgroup of* $\mathbb{G}(\mathbb{A}^{\underline{v}})$*.*

Before we can define morphisms, we need the following lemmas. Let $\pi:X\to Y$ be a morphism of schemes. We recall that for any functor $F:\ (\mathbf{Sch}/X)^{op}\longrightarrow\mathbf{Set}$ the push forward $\pi_\star F:(\mathbf{Sch}/Y)^{op}\longrightarrow\mathbf{Set}$ with respect to π is defined by $(T\to Y)\longmapsto F(T\times_Y X)$. In the case that F

is a scheme (i.e. representable) and $\pi_\star F$ is also representable, we call $\pi_\star F$ the Weil restriction $\mathfrak{R}_{X/Y}(F)$ of F. The basic properties and some conditions for the existence of Weil restrictions are discussed and developed in [BLR90, Paragraph 7.6] and [CGP10]. We have the following lemma, where we call a morphism of schemes finite locally free, if it is finite, flat and of finite presentation.

Lemma 3.2. *Let $\pi : X \to Y$ be a surjective, finite locally free morphism of schemes, let $\mathbb{G}$ be a smooth affine group scheme over X and let $\mathcal{G}$ be a $\mathbb{G}$-torsor on the big étale site of X, then*

1. *$\pi_\star\mathbb{G}$ is a smooth affine group scheme over Y*
2. *$\pi_\star\mathcal{G}$ is a $\pi_\star\mathbb{G}$-torsor on the big étale site of Y.*

Proof: Since π is finite and faithfully flat we can apply theorem 4 in [BLR90, Paragraph 7.6] to see that the Weil restriction $\pi_\star\mathbb{G}$ exists indeed as a scheme. Let $U,V,W \in Fun((\mathbf{Sch}/X)^{op},\mathbf{Set})$ be arbitrary with natural transformations $f_1 : U \to W$ and $f_2 : V \to W$, then for $S \in (\mathbf{Sch}/Y)$ we have

$$\begin{aligned}\pi_\star(U\times_W V)(S) &= Hom_X(S\times_Y X, U\times_W V)\\ &= \{(f,g)\ |f\in Hom(S\times_Y X,U),\ g\in Hom(S\times_Y X,V),\ f_1\circ f = f_2\circ g\}\\ &= (\pi_\star U\times_{\pi_\star W}\pi_\star V)(S).\end{aligned}$$

This shows that $\pi_\star$ commutes with fiber products and it follows that $\pi_\star\mathbb{G}$ becomes a group scheme over Y.
Let $U \subset Y$ be an affine open. Then $\pi_\star(X\times_Y U) = U$ and the compatibility with the fiber product implies $\pi_\star(\mathbb{G}\times_X X\times_Y U) = \pi_\star\mathbb{G}\times_Y U$. Now $\pi_\star\mathbb{G}\times_Y U$ is affine because $\mathbb{G}$ is affine over X and π is finite. Since the Weil restriction of an affine scheme is by construction affine we conclude that $\pi_\star\mathbb{G}$ is affine over Y. Furthermore we know by [BLR90, Chapter 7.6, Proposition 5] that $\pi_\star\mathbb{G}$ is again of finite type and smooth over Y, which proves the first part.
Now let $\mathcal{G}$ be a $\mathbb{G}$-torsor over X. Since $\mathbb{G}$ is smooth and affine, $\mathcal{G}$ is represented by a smooth affine scheme over X, by faithfully flat descent, [Gro65, Proposition 2.7.1] and [Gro67, Proposition 17.7.1]. [BLR90, paragraph 7.6, Theorem 4] and [BLR90, paragraph 7.6, Proposition 5] tell us again, that $\pi_\star\mathcal{G}$ is a smooth scheme over Y. Using once more the compatibility of the fiber product with the Weil restriction, the action of $\mathbb{G}$ on $\mathcal{G}$ induces an action of $\pi_\star\mathbb{G}$ on $\pi_\star\mathcal{G}$ and additionally the isomorphism $\mathbb{G}\times_X\mathcal{G}\simeq\mathcal{G}\times_X\mathcal{G}$ yields an isomorphism $\pi_\star\mathbb{G}\times_Y\pi_\star\mathcal{G}\simeq\pi_\star\mathcal{G}\times_Y\pi_\star\mathcal{G}$. It remains to show that $\pi_\star\mathcal{G}$ has étale locally on Y a section to $\pi_\star\mathcal{G}$. Since $\pi_\star\mathcal{G}\to Y$ is smooth and surjective this is content of proposition [BLR90, paragraph 2.2, Prop. 14]. □

Now morphisms between $\mathbb{G}$-torsors are sent by $\pi_\star$ to morphisms of $\pi_\star\mathbb{G}$-torsors and in fact we have the following lemma.

Lemma 3.3. *Let $\pi : X \to Y$ be a surjective, finite locally free morphism and $\mathbb{G}$ a smooth affine group scheme over X. Then the functor*

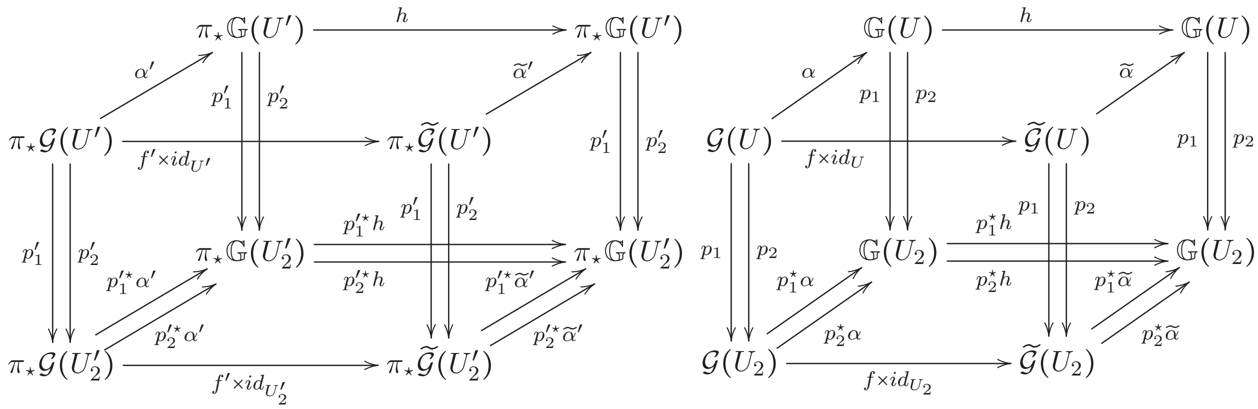
$$\pi_\star : \left\{\mathbb{G}\text{-}torsors\ on\ X\right\} \longrightarrow \left\{\pi_\star\mathbb{G}\text{-}torsors\ on\ Y\right\},$$
$$\mathcal{G}\longmapsto\pi_\star\mathcal{G}$$

induced by lemma 3.2, is an equivalence of categories. The inverse functor sends some $\pi_\star\mathbb{G}$-torsor $\widetilde{\mathcal{G}}$ to $\widetilde{\mathcal{G}} \times^{\pi^\star\pi_\star\mathbb{G}} \mathbb{G}$.

Proof: First we prove that $\pi_\star$ is fully faithful. So let $\mathcal{G}, \widetilde{\mathcal{G}}$ be two $\mathbb{G}$-torsors over X and $f' : \pi_\star\mathcal{G} \to \pi_\star\widetilde{\mathcal{G}}$ be a morphism of $\pi_\star\mathbb{G}$-torsors. We choose an étale covering $U' \to Y$ with $\pi_\star\mathcal{G}(U') \neq \varnothing \neq \pi_\star\widetilde{\mathcal{G}}(U')$. Now this implies automatically that $U := U' \times_Y X \to X$ is an étale covering with $\mathcal{G}(U) \neq \varnothing \neq \widetilde{\mathcal{G}}(U)$. We choose two sections $u \in \mathcal{G}(U) = \pi_\star\mathcal{G}(U')$ and $\widetilde{u} \in \widetilde{\mathcal{G}}(U) = \pi_\star\widetilde{\mathcal{G}}(U')$, which determine trivializations

$$\begin{array}{ll} \alpha' : \pi_\star\mathcal{G} \times_Y U' \to \pi_\star\mathbb{G} \times_Y U' & \qquad \widetilde{\alpha}' : \pi_\star\widetilde{\mathcal{G}} \times_Y U' \to \pi_\star\mathbb{G} \times_Y U' \\ \alpha : \mathcal{G} \times_X U \to \mathbb{G} \times_X U & \qquad \widetilde{\alpha} : \widetilde{\mathcal{G}} \times_X U \to \mathbb{G} \times_X U \end{array}$$

with $\alpha'^{-1}(1) = \alpha^{-1}(1) = u$ and $\widetilde{\alpha}'^{-1}(1) = \widetilde{\alpha}^{-1}(1) = \widetilde{u}$. Now we consider the following diagrams, where $U_2 := U \times_X U$, $U_2' := U' \times_Y U'$ with projections $p_1,\ p_2,\ p_1'$ and p_2' and $h := \widetilde{\alpha}' \circ (f' \times id_{U'}) \circ \alpha'^{-1}$. Note that since h is $\pi_\star\mathbb{G}$-equivariant h is determined by $h := h(1) \in \pi_\star\mathbb{G}(U') = \mathbb{G}(U)$. This same h defines then a morphism of $h : \mathbb{G} \times_X U \to \mathbb{G} \times_X U$ of $\mathbb{G}$-torsors on U and we set $f \times id_U := \widetilde{\alpha}^{-1} \circ h \circ \alpha : \mathcal{G} \times_X U \to \widetilde{\mathcal{G}} \times_X U$.



Now by definition of the Weil restriction we have equal sets at the corresponding vertices of the two cubes and the maps p_1', p_2' coincide with p_1 and p_2. Furthermore by definition of $\alpha', h, \widetilde{\alpha}'$ these maps coincide with the maps $\alpha, h, \widetilde{\alpha}$ in the right hand cube. The morphisms $p_1'^\star\alpha' : \pi_\star\mathcal{G} \times_Y U_2' \to \pi_\star\mathbb{G} \times_Y U_2'$ and $p_1^\star\alpha$ are uniquely determined by the preimage of $1 \in \pi_\star\mathbb{G}(U_2') = \mathbb{G}(U_2)$. But this preimage is in both cases given as $p_1^\star(u) = p_1(\alpha^{-1}(1)) = p_1'(\alpha'^{-1}(1)) \in \pi_\star\mathcal{G}(U_2') = \mathcal{G}(U_2)$. Hence the maps $p_1'^\star\alpha'$ and $p_1^\star\alpha$ coincide and equally $p_1'^\star h,\ p_1'^\star\widetilde{\alpha}',\ p_2'^\star\alpha',\ p_2'^\star h,\ p_2'^\star\widetilde{\alpha}'$ coincide with $p_1^\star h,\ p_1^\star\widetilde{\alpha},\ p_2^\star\alpha,\ p_2^\star h$ and $p_2^\star\widetilde{\alpha}$ respectively.

We further denote $g' := p_2'^\star\alpha' \circ p_1'^\star\alpha'^{-1}(1) \in \pi_\star\mathbb{G}(U_2')$ and $\widetilde{g}' := p_2'^\star\widetilde{\alpha}' \circ p_1'^\star\widetilde{\alpha}'^{-1}(1) \in \pi_\star\mathbb{G}(U_2')$. So that we have $g' = g := p_2^\star\alpha \circ p_1^\star\alpha^{-1}(1) \in \mathbb{G}(U_2)$ and $\widetilde{g}' = \widetilde{g} := p_2^\star\widetilde{\alpha} \circ p_1^\star\widetilde{\alpha}^{-1}(1) \in \mathbb{G}(U_2)$. With these notations we get the following bijections:

$$\begin{array}{ccc} Hom_{\mathbb{G}}(\mathcal{G}, \widetilde{\mathcal{G}}) & \xrightarrow[1:1]{f \mapsto h := (\widetilde{\alpha} \circ (f \times id_U) \circ \alpha^{-1})(1)} & \{h \in \mathbb{G}(U) \mid \widetilde{g} \circ p_1^\star h = p_2^\star h \circ g\} \\ \downarrow & & \| \\ Hom_{\pi_\star\mathbb{G}}(\pi_\star\mathcal{G}, \pi_\star\widetilde{\mathcal{G}}) & \xrightarrow[1:1]{f' \mapsto h' := (\widetilde{\alpha}' \circ (f' \times id_{U'}) \circ \alpha'^{-1})(1)} & \{h' \in \pi_\star G(U') \mid \widetilde{g}' \circ p_1'^\star h = p_2'^\star h \circ g'\} \end{array} .$$

Here the horizontal bijections are due to faithfully flat descent [BLR90, paragraph 6.1, Theorem 6] and the fact that the condition $\widetilde{g}\circ p_1^\star h = p_2^\star h\circ g$ is equivalent by definition of $g,\widetilde{g}$ to $p_1^\star(\widetilde{\alpha}^{-1}\circ h\circ\alpha) = p_2^\star(\widetilde{\alpha}^{-1}\circ h\circ\alpha)$ and for $\widetilde{g}'\circ p_1'^\star h' = p_2'^\star h'\circ g'$ respectively. The equality on the right follows from the identifications in the above cubes. To prove the fully faithfulness it remains to show that the bijective dashed arrow is given by $\pi_\star$. By definition of $\pi_\star f$ the following diagram commutes

$$\begin{array}{ccc} Hom(U',\pi_\star\mathcal{G}) & \xrightarrow{\pi_\star f} & Hom(U',\pi_\star\widetilde{\mathcal{G}}) \\ \| & & \| \\ Hom(U'\times_Y X,\mathcal{G}) & \xrightarrow{f} & Hom(U'\times_Y X,\pi_\star\widetilde{\mathcal{G}}) \end{array}$$

which shows that f and $\pi_\star f$ map to the same h on the right hand side.
It remains to show that $\pi_\star$ is essentially surjective. So let $\widetilde{\mathcal{G}}$ be a $\pi_\star\mathbb{G}$-torsor over Y and choose again an étale covering $U'\to Y$ and a trivialization $\alpha' : \widetilde{\mathcal{G}}\times_Y U' \xrightarrow{\sim} \pi_\star\mathbb{G}\times_Y U'$. Let $U_2' := U'\times_Y U'$ and $g' := p_2'^\star\alpha'\circ p_1'^\star\alpha'^{-1}(1) \in \pi_\star\mathbb{G}(U_2')$, where $p_2'^\star\alpha'\circ p_1'^\star\alpha'^{-1} : \pi_\star\mathbb{G}\times_Y U_2' \xrightarrow{\sim} \pi_\star\mathbb{G}\times_Y U_2'$. So the descent datum of $\widetilde{\mathcal{G}}$ is isomorphic to $(\pi_\star\mathbb{G}\times_Y U', g')$. Now $U := U'\times_Y X\to X$ is an ´étale covering and we set $\mathcal{G}_U := \mathbb{G}\times_X U$ as well as $U_2 := U\times_X U = U_2'\times_Y X$ with projections p_1,p_2. Let $g\in\mathbb{G}(U_2)$ be equal to g' using $\mathbb{G}(U_2) = \pi_\star\mathbb{G}(U_2')$. Then $(\mathcal{G}_U, g)$ is a descent datum that comes by [BLR90, beginning of paragraph 6.5 and paragraph 6.1, Theorem 6] from a $\mathbb{G}$-torsor $\mathcal{G}$ on X. Now it is clear that $(\pi_\star\mathcal{G}_U,\pi_\star g) = (\pi_\star\mathbb{G}\times_Y U', g')$. Therefore we have $\pi_\star\mathcal{G}\simeq\widetilde{\mathcal{G}}$ which proves that $\pi_\star$ is essentially surjective. We only need to prove that for every $\mathbb{G}$-torsor $\mathcal{G}$ on X the torsor $\pi^\star\pi_\star\mathcal{G}\times^{\pi^\star\pi_\star\mathbb{G}}\mathbb{G}$ is isomorphic to $\mathcal{G}$. With the same notation as above $\mathcal{G}$ is given by the $(\mathbb{G}\times_X U, g)$ and $\pi_\star\mathcal{G}$ is given by the descent datum $(\pi_\star\mathbb{G}\times_Y U', g')$. Restricting the latter torsor to X we get the descent datum $(\pi^\star(\pi_\star\mathbb{G}\times_Y U'), g\times id_X) = (\pi_\star\mathbb{G}\times_Y U'\times_Y X, g\times id_X)$. Using the adjunction $Hom_Y(\pi_\star\mathbb{G},\pi_\star\mathbb{G}) = Hom_X(\pi^\star\pi_\star\mathbb{G},\mathbb{G})$ we denote by $\varphi:\pi^\star\pi_\star\mathbb{G}\to\mathbb{G}$ the morphism corresponding to $id_{\pi_\star\mathbb{G}}$. Now applying the functor $\times^{\pi^\star\pi_\star\mathbb{G},\varphi}\mathbb{G}$ gives us the descent data $\big(\pi^\star(\pi_\star\mathbb{G}\times_Y U')\times^{\pi^\star\pi_\star\mathbb{G}}\mathbb{G}, (g'\times id_X, \mathbf{1}_\mathbb{G})\big)$. Since φ maps $(g'\times id_X)$ to g this descent data is isomorphic to $(\mathbb{G}\times_X U, g)$, which proves the lemma. □

Now let $\pi : C\to C'$ be a finite morphism from C to some other smooth projective geometrically irreducible curve C'. This morphism is then automatically faithfully flat [Har77, chapter II Prop. 6.8 and chapter III Prop. 9.7]. Let further $(C,\mathbb{G},\underline{v},Z_{\underline{v}},H)$ be a shtuka datum as in definition 3.1. Since $\mathbb{G}$ is a smooth affine group scheme over C, this allows us to apply lemma 3.2 and 3.3 in this situation.

Remark 3.4. We denote by $\mathscr{H}^1(C,\mathbb{G})$ the category fibered in groupoids, whose S-valued points for some $\mathbb{F}_q$-scheme S are given by isomorphy classes of $\mathbb{G}$-torsors over C_S. By lemma 3.3 $\pi_\star$ induces an isomorphism $\mathscr{H}^1(C,\mathbb{G})\longrightarrow\mathscr{H}^1(C',\pi_\star\mathbb{G})$.

Let $w_i = \pi(v_i)$ and $\underline{w} = (w_1,\ldots,w_n)$. Our next goal is to define the bound $\pi_\star Z_{\underline{v}} = (Z_{w_i})_i$ at the points w_i. We need the following lemma and general remark, where w is a closed point in C', A_w' is the completion of the local ring $\mathcal{O}_{C',w}$ and $\pi_w = \pi\times id_{Spec\ A_w'} : C\times_{C'} Spec\ A_w = \coprod_{v|w} Spec\ A_v \to Spec\ A_w'$.

Remark 3.5. In the next lemma, we need the following general fact about Weil restrictions. Let X,Y,S be schemes over some base scheme Z, $\pi : X\to Y$ a Z-morphism, M an X-scheme and $Y_S = Y \times_Z S$, $X_S = X \times_Z S$ and $M_S = M \times_Z S$ the appropriate base changes. Then we have $(\pi \times id_S)_\star(M \times_Z S) = \pi_\star M \times_Z S$. This is easily seen by the equation for $T \in \mathbf{Sch}/Y_S$:

$$\begin{aligned}(\pi_\star M \times_Z S)(T) &= (\pi_\star M \times_Y Y_S)(T) = Hom_Y(T, \pi_\star M) = Hom_X(T \times_Y X, M)\\ &= Hom_{X_S}(T \times_Y X, M_S) = (\pi \times id_S)_\star(M \times_Z S)(T).\end{aligned}$$

Lemma 3.6. *We have* $(\pi_w)_\star(\coprod_{v\in\pi^{-1}(w)} \mathbb{G}_v) = (\pi_\star \mathbb{G})_w$ *as a group scheme over* $SpecA'_w$.

Proof: This follows formally from remark 3.5 with $M = \mathbb{G}$, $X = C$, $Y = Z = C'$ and $S = Spec\ A_w$ since we have

$$(\pi_w)_\star(\coprod_{v|w} \mathbb{G}_v) = (\pi_w)_\star(\mathbb{G} \times_C \coprod_{v|w} Spec\ A_v) = (\pi_w)_\star(\mathbb{G} \times_{C'} Spec\ A_w) = \pi_\star \mathbb{G} \times_{C'} Spec\ A_w = (\pi_\star \mathbb{G})_w.$$

□

Corollary 3.7. *We have* $\prod_{v|w} L^+\widetilde{\mathbb{G}_v} = L^+\widetilde{(\pi_\star \mathbb{G})_w}$ *as group schemes over* $\mathbb{F}_q$.

Proof: Let R be a connected $\mathbb{F}_q$-algebra, then we have:

$$\begin{aligned}L^+\widetilde{(\pi_\star \mathbb{G})_w}(R) &= \widetilde{(\pi_\star \mathbb{G})_w}(R[\![z_w]\!]) = Hom_{Spec\ A_w}(Spec\ R[\![z_w]\!] \otimes_{\mathbb{F}_q} \mathbb{F}_w, (\pi_\star \mathbb{G})_w)\\ &= Hom_{Spec\ A_w}(Spec\ R\hat{\otimes}_{\mathbb{F}_q} A_w, (\pi_\star \mathbb{G})_w)\\ &= Hom_{Spec\ A_w}(Spf\ R\hat{\otimes}_{\mathbb{F}_q} A_w, (\pi_w)_\star(\coprod_{v|w} \mathbb{G}_v))\\ &= Hom_{\coprod_{v|w} Spec\ A_v}(Spec\ R\hat{\otimes}_{\mathbb{F}_q} A_w \otimes_{A_w} \prod_{v|w} A_v, \coprod_{v|w} \mathbb{G}_v)\\ &= \prod_{v|w} Hom_{Spec\ A_v}(Spec\ R[\![z_v]\!] \otimes_{\mathbb{F}_q} \mathbb{F}_v, \mathbb{G}_v) = \prod_{v|w} \widetilde{\mathbb{G}_v}(R[\![z_v]\!])\\ &= \prod_{v|w} L^+\widetilde{\mathbb{G}_v}(R).\end{aligned}$$

□

We have the following 2-cartesian diagrams:

$$\begin{CD} \prod_{v|w} \mathcal{F}l_{\widetilde{\mathbb{G}_v}} @>>> \mathbb{F}_q \\ @VVV @VVV \\ \prod_{v|w} \mathscr{H}^1(\mathbb{F}_q, L^+\widetilde{\mathbb{G}_v}) @>>> \prod_{v|w} \mathscr{H}^1(\mathbb{F}_q, L\widetilde{\mathbb{G}_v}) \end{CD} \qquad \begin{CD} \mathcal{F}l_{\widetilde{(\pi_\star \mathbb{G})_w}} @>>> \mathbb{F}_q \\ @VVV @VVV \\ \mathscr{H}^1(\mathbb{F}_q, L^+\widetilde{(\pi_\star \mathbb{G})_w}) @>>> \mathscr{H}^1(\mathbb{F}_q, L\widetilde{(\pi_\star \mathbb{G})_w}) \end{CD}$$

By corollary 3.7 the lower stacks in the diagrams are isomorphic, so that we get an isomorphism $\prod_{v|w} \mathcal{F}l_{\widetilde{\mathbb{G}_v}} \xrightarrow{\sim} \mathcal{F}l_{\widetilde{(\pi_\star \mathbb{G})_w}}$ and by the base change with the compositum $\mathbb{F}$ of the finite fields $\mathbb{F}_v$ for all $v|w$ we get an isomorphism $\prod_{v|w} \prod_{l\in\mathbb{Z}/deg\ v} \mathcal{F}l_{\mathbb{G}_v} \times_{\mathbb{F}_v} \mathbb{F} \simeq \prod_{l\in\mathbb{Z}/deg\ w} \mathcal{F}l_{(\pi_\star \mathbb{G})_w} \times_{\mathbb{F}_w} \mathbb{F}$. Since $\sigma^{deg\ w}$ invariant components are mapped to $\sigma^{deg\ w}$ invariant components, it restricts to an isomorphism

$$\prod_{v|w} \prod_{\substack{l\in\mathbb{Z}/deg\ v\\ deg\ w|l}} \mathcal{F}l_{\mathbb{G}_v} \times_{\mathbb{F}_v} \mathbb{F} \simeq \mathcal{F}l_{(\pi_\star \mathbb{G})_w} \times_{\mathbb{F}_w} \mathbb{F}.$$

Now let R be a DVR with $\mathbb{F}\subset R$ and such that there exists a representative $\hat{Z}_{v,R}$ of $\hat{Z}_v$ for all $v\in\underline{v}$. Consider the ind-closed subscheme

$$\prod_{v|w}\prod_{\substack{l\in\mathbb{Z}/deg\ v\\ deg\ w|l}}\hat{Z}_{v,l}\subset\prod_{v|w}\prod_{\substack{l\in\mathbb{Z}/deg\ v\\ deg\ w|l}}\mathcal{F}l_{\mathbb{G}_v}\times_{\mathbb{F}_v}Spf\ R$$

where $\hat{Z}_{v,l}$ is always the closed stratum $\mathcal{S}(1)\times_{\mathbb{F}_v}Spf\ R$ except for $v\in\underline{v}$ and $l=0$, where we set $\hat{Z}_{v_i,0}=\hat{Z}_{v_i,R}$. Here $\mathcal{S}(1)$ denotes the closed Schubert cell in $1\cdot L^+\mathbb{G}_v\in\mathcal{F}l_{\mathbb{G}_v}$. Via the previous isomorphism this defines an ind-closed subscheme in $\mathcal{F}l_{(\pi_\star\mathbb{G})_w}\times_{\mathbb{F}_w}Spf\ R$. This defines a bound $\hat{Z}_w$ in the sense of [AH14, Definition 4.8] in $\mathcal{F}l_{(\pi_\star\mathbb{G})_w}$ and we set $\pi_\star\hat{Z}_{\underline{v}}:=\hat{Z}_{\underline{w}}:=(\hat{Z}_{w_i})_i$.

Next we define $\pi_\star H$. We recall that H was an open compact subgroup of $\mathbb{G}(\mathbb{A}^{\underline{v}})$. Since $\underline{v}\subset\pi^{-1}(\underline{w})\subset|C|$ we have a quotient map of topological rings $\mathbb{A}^{\underline{v}}\longrightarrow\mathbb{A}^{\pi^{-1}(\underline{w})}$.
Since this map is open, it induces by [Con12, theorem 3.6] an open continuous group homomorphism $\mathbb{G}(\mathbb{A}^{\underline{v}})\longrightarrow\mathbb{G}(\mathbb{A}^{\pi^{-1}(\underline{w})})$. We have $\mathbb{A}^{\underline{w}}\times_{C'}C=\mathbb{A}^{\underline{w}}\times_{\eta'}\eta=\mathbb{A}^{\pi^{-1}(\underline{w})}$ where η and η' are the generic points of C and C'. This gives us with the definition of the Weil restriction $\pi_\star\mathbb{G}(\mathbb{A}^{\underline{w}})=\mathbb{G}(\mathbb{A}^{\pi^{-1}(\underline{w})})$, where both groups carry the same topology by [Con12, example 2.4]. Now the image of H under this morphism gives us an open compact subgroup in $\pi_\star\mathbb{G}(\mathbb{A}^{\underline{w}})$ that we denote by $\pi_\star H$.

Remark 3.8. We have seen in § 2.12 that there is the possibility to define level structures using finite closed subschemes D of C and in § 2.16 we have seen that D-level structures of a $\mathbb{G}$-shtuka correspond bijectively to H_D-level structures, where $H_D=ker(\mathbb{G}(\mathbb{O}^{\underline{v}})\to\mathbb{G}(\mathcal{O}_D))$. Now we can also consider the Weil restriction $\pi_\star D$ of D. It is a closed finite subset of C' consisting of the points $\{w\in|C'|\mid w\times_{C'}C\subset D\}$. And with a D-level structure of some $\mathbb{G}$-shtuka $\underline{\mathcal{G}}$ we could associate a $\pi_\star D$-level structure of the corresponding $\pi_\star\mathbb{G}$-shtuka $\pi_\star\underline{\mathcal{G}}$ (which will be defined in proposition 3.12). But compared to the associated $\pi_\star H$-level structure that we will define in theorem 3.14 we would lose some information at the points $D\backslash(\pi_\star D\times_{C'}C)$, which is seen in the following way. Since we have $\pi_\star D\times_{C'}C\subset D$ we have $H_D\subset H_{\pi_\star D\times_{C'}C}$ and hence $\pi_\star H_D\subset\pi_\star H_{\pi_\star D\times_{C'}C}$. Now

$$\begin{aligned}H_{\pi_\star D}&=ker\big(\pi_\star\mathbb{G}(\mathbb{O}^{\underline{w}})\to\pi_\star\mathbb{G}(\mathcal{O}_{\pi_\star D})\big)=ker\big(\mathbb{G}(\mathbb{O}^{\pi^{-1}\underline{w}})\to\mathbb{G}(\mathcal{O}_{\pi_\star D\times_{C'}C})\big)\\&=H_{\pi_\star D\times_{C'}C}\cap\mathbb{G}(\mathbb{A}^{\pi^{-1}(\underline{w})})=im\left(H_{\pi_\star D\times_{C'}C}\to\mathbb{G}(\mathbb{A}^{\pi^{-1}(\underline{w})})\right)=:\pi_\star H_{\pi_\star D\times_{C'}C}\supset\pi_\star H_D\end{aligned}$$

shows that $\pi_\star H_D$ is in general a finer level than $\pi_\star D$ (or equivalently $H_{\pi_\star D}$) and the previous equation shows that the information is lost exactly at the points $D\backslash(\pi_\star D\times_{C'}C)$.

All these previous explanations concerned the case that we change the curve in the shtuka datum but we can also change the group scheme in this datum. Let $f:\mathbb{G}\to\mathbb{G}'$ be any morphism of smooth affine group schemes over C and v a closed point in C. Firstly this induces a morphism $L^+\mathbb{G}_v\to L^+\mathbb{G}'_v$ of the positive loop groups as well as a morphism $L\mathbf{G}_v\to L\mathbf{G}'_v$ of the loop groups. Consequently we also get a morphism $\mathcal{F}l_{\mathbb{G}_v}\to\mathcal{F}l_{\mathbb{G}'_v}$ of the affine flag varieties. Secondly such a morphism induces a morphism $f_{\mathbb{A}^{\underline{v}}}:\mathbb{G}(\mathbb{A}^{\underline{v}})\to\mathbb{G}'(\mathbb{A}^{\underline{v}})$ of locally compact Hausdorf spaces by [Con12, Proposition 2.1]. Now we can define morphisms of shtuka data.

Definition 3.9. *A morphism between two shtuka data* $(C, \mathbb{G}, \underline{v}, \hat{Z}_{\underline{v}}, H)$ *and* $(C', \mathbb{G}', \underline{w}, \hat{Z}'_{\underline{w}}, H')$ *is a pair* (π, f) *such that:*

- $\pi : C \to C'$ *is a finite morphism with* $\pi(v_i) = w_i$
- $f : \pi_\star \mathbb{G} \to \mathbb{G}'$ *is a morphism of smooth affine group schemes over* C'
- *The morphism* $(\pi_\star Z_{\underline{v}})_R \to \prod_{w \in \underline{w}} \hat{\mathcal{F}l}_{(\pi_\star \mathbb{G})_w, R} \to \prod_{w \in \underline{w}} \hat{\mathcal{F}l}_{\mathbb{G}'_w, R}$ *factores through* $\hat{Z}'_{\underline{w},R}$, *where* R *is a DVR such that there exists representatives* $(\pi_\star \hat{Z}_{\underline{v}})_R$ *and* $\hat{Z}'_{\underline{w},R}$ *of the corresponding bounds*
- $f_{\mathbb{A}^{\underline{w}}}(\pi_\star H) \subset H'$

With this definition we have reached the goal of this section. In the next two sections we will prove that such a morphism induces a morphism of the corresponding moduli stacks and determine some of its properties. But before we give some remarks.

Remark 3.10.

- Let $\pi : C \to C'$ be a finite morphism and $\underline{w} = \pi(\underline{v})$. With the definition of $\pi_\star H$ and $\pi_\star Z_{\underline{v}}$ on page 18 it is clear that $(\pi, id_{\pi_\star \mathbb{G}}) : (C, \mathbb{G}, \underline{v}, Z_{\underline{v}}, H) \to (C', \pi_\star \mathbb{G}, \underline{w}, \pi_\star Z_{\underline{v}}, \pi_\star H)$ defines a morphism of shtuka data
- Every morphism (π, f) of shtuka data factorizes as $(id_C, f) \circ (\pi, id_{\pi_\star \mathbb{G}})$.
- If $f : \pi_\star \mathbb{G} \to \mathbb{G}'$ is an isomorphism in the generic fiber we have $\pi_\star \mathbb{G}(\mathbb{A}^{\underline{w}}) = \mathbb{G}'(\mathbb{A}^{\underline{w}})$ so that we can naturally choose $H = H'$.
- If $f : \pi_\star \mathbb{G} \to \mathbb{G}'$ is smooth in the generic fiber, then $f_{\mathbb{A}^{\underline{w}}} : \pi_\star \mathbb{G}(\mathbb{A}^{\underline{w}}) \to \mathbb{G}'(\mathbb{A}^{\underline{w}})$ is an open map by [Con12, Theorem 4.5] so that we can naturally choose $H' = f_{\mathbb{A}^w}(\pi_\star H)$
- If $f : \pi_\star \mathbb{G} \to \mathbb{G}'$ is proper in the generic fiber, then $f_{\mathbb{A}^{\underline{w}}} : \pi_\star \mathbb{G}(\mathbb{A}^{\underline{w}}) \to \mathbb{G}'(\mathbb{A}^{\underline{w}})$ is a topologically proper map by [Con12, Proposition 4.4] so that we can naturally choose $H = f^{-1}_{\mathbb{A}^w}(H')$

3.2 Changing the Coefficients

In this section we prove that a morphism of shtuka data (π, id), where we only change the curve, induces a finite morphism of the corresponding moduli stacks. We firstly prove this for the moduli stack $\nabla_n \mathscr{H}^1(C, \mathbb{G})$, where the characteristic sections are not fixed and no boundedness condition or level structures are imposed. For this purpose we need the following lemma:

Lemma 3.11. *Let* S *be an* $\mathbb{F}_q$*-scheme with* n *morphisms* $s_i : S \to C$ *for* $i = 1, \dots, n$*. Then the scheme theoretic image of* $\widetilde{C}_S := C_S \setminus \bigcup_i \Gamma_{s_i}$ *in* C_S *equals* C_S.

Proof: Since $D := \bigcup_i \Gamma_{s_i}$ is an effective Cartier-Divisor on C_S over S, we find an affine covering $(U_j)_{j \in J}$ of C_S with $U_j := Spec\ B_j$ such that D is the vanishing locus of an element $f_j \in B_j$ that can be written as $f_j = \frac{a_j}{b_j}$ with two regular elements $a_j, b_j \in B_j$ (see [GW10, after definition 11.24]). Now the ring homomorphism $B_j \to \Gamma(U_j \setminus D, \mathcal{O})$ is injective, which is seen as follows. An element $x \in B_j$ is send to 0 if and only if $f_j^m x = 0$ for some $m \in \mathbb{N}$. The latter condition implies

$a_j^m x = 0$ and since a_j is a non-zero divisor this means $x = 0$ so that *Spec* $B_j \backslash D$ is schematically dense in *Spec* B_j (compare also [Gro67, Lemma 20.2.9]). Now gluing all the U_j shows that for every affine open $V \subset C_S$ the ring homomorphism $\Gamma(V,\mathcal{O}) \to \Gamma(V\backslash D,\mathcal{O})$ is injective and we conclude that $\widetilde{C}_S$ is schematically dense in C_S (see [Gro67, p. 20.2.1]). □

Proposition 3.12. *Let $\pi : C \to C'$ be a finite morphism of smooth projective geometrically irreducible curves over $\mathbb{F}_q$ and let $\mathbb{G}$ be a smooth affine group scheme over C. This induces a finite morphism of the moduli stacks*

$$\pi_\star : \nabla_n\mathscr{H}^1(C,\mathbb{G}) \to \nabla_n\mathscr{H}^1(C',\pi_\star\mathbb{G}).$$

which factors through a closed immersion $\nabla_n\mathscr{H}^1(C,\mathbb{G}) \hookrightarrow \nabla_n\mathscr{H}^1(C',\pi_\star\mathbb{G}) \times_{C'^n} C^n$.

Proof: Let S be an $\mathbb{F}_q$-scheme and $(\mathcal{G}, s_1,\dots,s_n,\tau_{\mathcal{G}}) \in \nabla_n\mathscr{H}^1(C,\mathbb{G})(S)$. We describe its image $(\mathcal{G}', s_1',\dots,s_n',\tau_{\mathcal{G}'})$ to define the morphism. The torsor $\mathcal{G}'$ is given by $(\pi_S)_\star\mathcal{G}$ and the sections $s_i : S \to C$ are mapped to the composition $s_i' := \pi \circ s_i : S \to C'$. This implies $\pi_S(\bigcup_i \Gamma_{s_i}) \subset \bigcup_i \Gamma_{s_i'} \subset C_S'$. Let $\widetilde{C_S'} = C_S' \backslash (\cup_i \Gamma_{s_i'})$ and $\widetilde{C_S} = C_S\backslash(\cup_i\Gamma_{s_i})$. Then $U := C \times_{C'} \widetilde{C_S'} = C_S \times_{C_S'} \widetilde{C_S'}$ is open in $\widetilde{C_S}$. We denote by $\pi_U := \pi \times_{id_{C'}} id_{\widetilde{C_S'}} : U \to \widetilde{C_S'}$ and we have $(\pi_U)_\star(\mathbb{G}\times_C U) = \pi_\star\mathbb{G}\times_{C'}\widetilde{C_S'}$. Now we restrict $\tau_{\mathcal{G}} : \sigma^\star\mathcal{G}|_{\widetilde{C_S}} \to \mathcal{G}|_{\widetilde{C_S}}$ to U_S and apply lemma 3.3 to π_U. The category equivalence gives us the desired morphism $\tau_{\mathcal{G}'} : (\pi_U)_\star(\sigma^\star\mathcal{G}\times_{C_S'}\widetilde{C_S'}) = \sigma^\star\mathcal{G}'|_{\widetilde{C_S'}} \to (\pi_U)_\star(\mathcal{G}\times_{C_S'}\widetilde{C_S'}) = \mathcal{G}'|_{\widetilde{C_S'}}$. This defines a global $\pi_\star\mathbb{G}$-shtuka $(\mathcal{G}', s_1',\dots,s_n',\tau_{\mathcal{G}}')$ over S and therefore the morphism of the moduli stacks.

We now show that this morphism is representable and finite. Let S be again an arbitrary scheme over $\mathbb{F}_q$ and $\underline{\mathcal{G}}' : S \to \nabla_n\mathscr{H}^1(C',\pi_\star\mathbb{G})$ be given by $\underline{\mathcal{G}}' = (\mathcal{G}',\tau',s_1',\dots,s_n')$. Then $\nabla_n\mathscr{H}^1(C,\mathbb{G}) \times_{\nabla_n\mathscr{H}^1(C',\pi_\star\mathbb{G})} S$ is the category fibered in groupoids over $\mathbf{Sch}/\mathbb{F}_q$ whose fiber category over an $\mathbb{F}_q$-scheme T is given by

$$\left\{(\underline{\mathcal{G}}, g : T \to S, \beta) \;\middle|\; \underline{\mathcal{G}} = (\mathcal{G},\tau_{\mathcal{G}},s_1,\dots,s_n) \in \nabla_n\mathscr{H}^1(C,\mathbb{G})(T) \text{ and } \beta : g^\star\underline{\mathcal{G}}' \xrightarrow{\sim} \pi_\star\underline{\mathcal{G}}\right\}.$$

Using the n sections $s_1',\dots,s_n' : S \to C'$ and the morphism $\pi : C \to C'$ we set $\widetilde{S} := S \times_{C'^n} C^n$. Since $S \times_{\mathscr{H}^1(C',\pi_\star\mathbb{G})} \mathscr{H}^1(C,\mathbb{G}) = S$ by remark 3.4 we know that $Hom_{\mathbb{F}_q}(T,S)$ is in bijection with the tuples $\{g : T \to S, \mathcal{G} : T \to \mathscr{H}^1(C,\mathbb{G}), \alpha : g^\star\mathcal{G}' \xrightarrow{\sim} \pi_\star\mathcal{G}\}$. Consequently the $\mathbb{F}_q$-morphisms $T \to \widetilde{S}$ are in bijection with the tuples $(\mathcal{G}, s_1,\dots,s_n,g,\alpha)$, where $(\mathcal{G},g,\alpha) \in S(T)$ as before and $s_1,\dots,s_n : T \to C$ are morphisms making the following diagram commutative for all $i = 1,\dots,n$

$$\begin{array}{ccc} T & \xrightarrow{g} & S \\ {\scriptstyle s_i}\downarrow & & \downarrow{\scriptstyle s_i'} \\ C & \xrightarrow{\pi} & C' \end{array}.$$

We claim that we get a morphism $\nabla_n\mathscr{H}^1(C,\mathbb{G}) \times_{\nabla\mathscr{H}^1(C',\pi_\star\mathbb{G})} S \to \widetilde{S}$ that is injective on T-valued points (hence a monomorphism) and satisfies the valuative criterion for properness. Then this implies by [Gro66, Proposition 8.11.5] that $\nabla_n\mathscr{H}^1(C,\mathbb{G}) \times_{\nabla\mathscr{H}^1(C',\pi_\star\mathbb{G})} S$ is a closed subscheme of $\widetilde{S}$. So first of all a given object $(\mathcal{G},\tau_{\mathcal{G}},s_1,\dots,s_n,g,\alpha)$ in $(\nabla_n\mathscr{H}^1(C,\mathbb{G}) \times_{\nabla\mathscr{H}^1(C',\pi_\star\mathbb{G})} S)(T)$

is sent to $(\mathcal{G}, s_1, \dots, s_n, g, \alpha)$. Since $\alpha : g^\star \underline{\mathcal{G}'} \xrightarrow{\sim} \pi_\star \underline{\mathcal{G}}$ is not only an isomorphism of torsors, but also of $\pi_\star \mathbb{G}$-shtukas the n-sections $g \circ s'_i$ of $g^\star \underline{\mathcal{G}'}$ and $(s_i \circ \pi)$ of $\pi_\star \underline{\mathcal{G}}$ have to coincide, so that $(\mathcal{G}, s_1, \dots, s_n, g, \alpha)$ is a well defined object in $\widetilde{S}(T)$. This induces the morphism $\nabla_n \mathscr{H}^1(C, \mathbb{G}) \times_{\nabla \mathscr{H}^1(C', \pi_\star \mathbb{G})} S \to \widetilde{S}$. Further it was claimed, that this morphism is injective on T-valued points. So given two points $(\mathcal{G}, \tau_\mathcal{G}, s_1, \dots, s_n, g, \alpha)$ and $(\mathcal{G}, \widetilde{\tau}_\mathcal{G}, s_1, \dots, s_n, g, \alpha)$ we need to show that this implies $\tau_\mathcal{G} = \widetilde{\tau}_\mathcal{G}$. Since α is an isomorphism of $\pi_\star \mathbb{G}$-shtukas, we have $\alpha^{-1} \circ \pi_\star \tau_\mathcal{G} = g^\star \tau' \circ \sigma^\star \alpha^{-1} = \alpha^{-1} \circ \pi_\star \widetilde{\tau}_\mathcal{G} : \sigma^\star(\pi_\star \mathcal{G})|_{\widetilde{C'_T}} \xrightarrow{\sim} g^\star \mathcal{G}'|_{\widetilde{C'_T}}$, where we write again $\widetilde{C'_T} := C'_T \backslash \bigcup_i \Gamma_{s'_i}$. This implies $\pi_\star \tau_\mathcal{G} = \pi_\star \widetilde{\tau}_\mathcal{G}$ and using lemma 3.3 applied to $\pi \times id_{\widetilde{C'_T} \times_{C'} C}$ we see $\tau_\mathcal{G}\big|_{\widetilde{C'_T} \times_{C'} C} = \widetilde{\tau}_\mathcal{G}\big|_{\widetilde{C'_T} \times_{C'} C}$. We even need to know that $\tau_\mathcal{G} = \tau'_\mathcal{G}$. In the following diagram the restriction of $(\tau_\mathcal{G}, \tau'_\mathcal{G})$ to $\widetilde{C'_T} \times_{C'} C$ factors by the previous observation through the diagonal Δ.

$$\begin{array}{ccc}
\sigma^\star \mathcal{G}\big|_{\widetilde{C}_T} & \xrightarrow{(\tau_\mathcal{G}, \tau'_\mathcal{G})} & \mathcal{G}\big|_{\widetilde{C}_T} \times_{\widetilde{C}_T} \mathcal{G}\big|_{\widetilde{C}_T} \\
\uparrow & & \uparrow \Delta \\
\sigma^\star \mathcal{G}\big|_{\widetilde{C'_T} \times_{C'} C} & \longrightarrow & \mathcal{G}\big|_{\widetilde{C}_T}
\end{array} .$$

Since $\mathcal{G}$ is separated over C_T the diagonal is a closed immersion and $(\tau_\mathcal{G}, \tau'_\mathcal{G})$ factors already over $\widetilde{C}_T$ through the diagonal if the scheme theoretic image of $\sigma^\star \mathcal{G}\big|_{\widetilde{C'_T} \times_{C'} C}$ in $\sigma^\star \mathcal{G}\big|_{\widetilde{C}_T}$ equals $\sigma^\star \mathcal{G}\big|_{\widetilde{C}_T}$. Since taking the scheme theoretic image is stable under flat base change by [Gro66, Théoreme 11.10.5], this is the case if the scheme theoretic image of $\widetilde{C'_T} \times_{C'} C$ in $\widetilde{C}_T$ equals $\widetilde{C}_T$. By the same argument this follows if the scheme theoretic image of $\widetilde{C}'_T$ in C'_T equals C'_T. Now this is content of lemma 3.11 so that we can conclude as desired $\tau_\mathcal{G} = \tau'_\mathcal{G}$.

Next we claimed that the morphism satisfies the valuative criterion for properness. So let

$$\begin{array}{ccccc}
Spec\ K & \xrightarrow{(\mathcal{H}, \tau_\mathcal{H}, r_1, \dots, r_n, f, \beta)} & S \times_{\nabla_n \mathscr{H}^1(C', \pi_\star \mathbb{G})} \nabla_n \mathscr{H}^1(C, \mathbb{G}) & & \\
{\scriptstyle j}\downarrow & \nearrow & \downarrow & & \\
Spec\ R & \xrightarrow[(\mathcal{G}, s_1, \dots, s_n, g, \alpha)]{} & \widetilde{S} & \xrightarrow[finite]{} & S
\end{array}$$

be a commutative diagram, where R is a complete discrete valuation ring with fraction field K, maximal ideal $\mathfrak{m}$ and algebraically closed residue field $\kappa_R = R/\mathfrak{m}$. Note that R is a κ_R-algebra. We have to prove that there exists a unique dashed arrow making everything commutative. The commutativity of the square shows $\mathcal{H} = j^\star \mathcal{G}$, $s_i \circ j = r_i : Spec\ K \to C$, $f = g \circ j : Spec\ K \to S$ and $j^\star \alpha = \beta$. To define this dashed arrow we have to extend $(\mathcal{G}, s_1, \dots, s_n)$ to a $\mathbb{G}$-shtuka $(\mathcal{G}, \tau_\mathcal{G}, s_1, \dots, s_n)$ over R such that α extends to an isomorphism $\alpha : g^\star \underline{\mathcal{G}'} \to \pi_\star \underline{\mathcal{G}}$ and $j^\star \tau_\mathcal{G} = \tau_\mathcal{H}$. So we define this isomorphism $\tau_\mathcal{G} : \sigma^\star \mathcal{G}|_{\widetilde{C_R}} \to \mathcal{G}|_{\widetilde{C_R}}$. Since $\mathscr{H}^1(\widetilde{C'_R}, \pi_\star \mathbb{G})$ and $\mathscr{H}^1(\widetilde{C'_R} \times_{C'} C, \mathbb{G})$ are isomorphic, $\tau_\mathcal{G}|_{\widetilde{C'_R} \times_{C'} C}$ is defined by $\alpha \circ g^\star \tau_{\mathcal{G}'} \circ \sigma^\star \alpha^{-1}$. Furthermore we know that $\tau_\mathcal{G}$ is defined on the generic fiber $\widetilde{C_K} \subset \widetilde{C_R}$ by $\tau_\mathcal{G}|_{\widetilde{C_K}} = j^\star \tau_\mathcal{G} = \tau_\mathcal{H}$. So let $p \in \widetilde{C_R} \backslash (\widetilde{C'_R} \times_{C'} C \cup \widetilde{C_K})$, i.e. $p \in \big((\bigcup_i \Gamma_{s'_i} \times_{C'} C) \backslash \bigcup_i \Gamma_{s_i} \big) \bigcap C_{R/\mathfrak{m}}$. It remains to show, that $\tau_\mathcal{G}$ extends to p. Since p is closed we choose an open $V \subset \widetilde{C}_{R/\mathfrak{m}}$ with $V \bigcap (\bigcup_i \Gamma_{s'_i} \times_{C'} C) = p$ and set $\widetilde{V} = V \backslash p$. Then we consider the

2-cartesian diagram of stacks fibered over $\kappa_{R_{\acute{E}t}}$ (compare [AH14, Lemma 5.1]):

$$\begin{array}{ccc} \mathscr{H}^1(V,\mathbb{G}) & \longrightarrow & \mathscr{H}_e^1(\widetilde{V},\mathbb{G}) \\ \downarrow L_p^+ & & \downarrow L_p \\ \mathscr{H}^1(\kappa_R, L^+\mathbf{G}_p) & \longrightarrow & \mathscr{H}^1(\kappa_R, L\mathbb{G}_p) \end{array} .$$

Here $\mathscr{H}_e^1(\widetilde{V},\mathbb{G})(X)$ is the full subcategory of $\mathscr{H}^1(\widetilde{V},\mathbb{G})(X)$ consisting of those $\mathbb{G}$-torsors over $\widetilde{V}_X := \widetilde{V}\times_{\kappa_R} X$ that can be extended to a $\mathbb{G}$-torsor over V_X. Now $\sigma^\star\mathcal{G}|_{V_R}$ and $\mathcal{G}|_{V_R}$ define two R-valued points in $\mathscr{H}^1(V,\mathbb{G})$ and $\tau_{\mathcal{G}}|_{\widetilde{V}_R}$ is an isomorphism in $\mathscr{H}_e^1(\widetilde{V},\mathbb{G})(R)$ that is already defined. Since R has algebraically closed residue field, we can choose trivializations $\alpha_1 : L_p^+(\sigma^\star\mathcal{G}) \to L^+\mathbb{G}_p$ and $\alpha_2 : L_p^+(\mathcal{G}) \to L^+\mathbb{G}_p$ ([AH14, Proposition 2.4]). Then $\alpha_2^{-1}\circ\tau_{\mathcal{G}}\circ\alpha_1 : L\mathbf{G}_{p,R} \to L\mathbf{G}_{p,R}$ is an isomorphism in $\mathscr{H}^1(\kappa_R, L\mathbf{G})(R)$ given by an element $h \in L\mathbf{G}_p(R)$. We know by assumption that the pull back of h to K is given by an element $h_K \in L^+\mathbb{G}_p(K)$, since $\tau_{\mathcal{G}}$ is generically already an isomorphism over V. But since $L^+\mathbb{G}_p$ is closed in $L\mathbb{G}_p$ it follows that $h \in L^+\mathbb{G}_p(R)$. This implies that the isomorphism $\tau_{\mathcal{G}}|_{\widetilde{V}}$ comes from an isomorphism in $\mathscr{H}^1(V,\mathbb{G})(R)$. So $\tau_{\mathcal{G}}$ extends uniquely to p and the valuative criterion is proved. This proves that $\nabla_n\mathscr{H}^1(C,\mathbb{G})\times_{\nabla\mathscr{H}^1(C',\pi_\star\mathbb{G})}S$ is a closed subscheme of $\widetilde{S}$ and since $\widetilde{S}$ is finite over S it proves as well that $\nabla_n\mathscr{H}^1(C,\mathbb{G}) \to \nabla_n\mathscr{H}^1(C',\pi_\star\mathbb{G})$ is a finite morphism. □

The next goal is to prove that for any shtuka datum $(C,\mathbb{G},\underline{v},\hat{Z}_{\underline{v}},H)$ the morphism $\pi : C \to C'$ induces also a finite morphism $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) \to \nabla_n^{\pi_\star\hat{Z}_{\underline{v}},\pi_\star H}\mathscr{H}^1(C',\pi_\star\mathbb{G})$. For this we need the following lemma that concerns the boundedness condition. Given a global $\mathbb{G}$-shtuka $\underline{\mathcal{G}}$ in $\nabla_n\mathscr{H}^1(C,\mathbb{G})$ over S, we recall that we introduced in § 2.10 the global-local functor Γ_{v_i} that associates with it a local $\mathbb{G}_{v_i}$-shtuka $\Gamma_{v_i}(\underline{\mathcal{G}})$ over S. On the other hand we explained (compare also [AH14, Remark 5.6]) that base change with $Spf\ A_{v_i}\times_{\mathbb{F}_q} S \simeq \prod_{l\in\mathbb{Z}/deg\ v_i} V(\mathfrak{a}_{v_i,l})$ gives a local $\widetilde{\mathbb{G}_{v_i}}$-shtuka $L_{v_i}^+(\underline{\mathcal{G}})$ over S. Here $\widetilde{\mathbb{G}_{v_i}}$ denotes the Weil restriction $Res_{A_{v_i}/\mathbb{F}_q[\![z_{v_i}]\!]}\mathbb{G}_{v_i}$. Now let Z_{v_i} be a bound in $\hat{\mathcal{F}l}_{\mathbb{G}_{v_i}}$ and R an DVR over $A_{v_i} = \mathbb{F}_{v_i}[\![z_{v_i}]\!]$ with a representative $\hat{Z}_{v_i,R} \subset \hat{\mathcal{F}l}_{\mathbb{G}_{v_i},R}$. We have $\hat{\mathcal{F}l}_{\widetilde{\mathbb{G}_{v_i}}}\hat{\times}_{\mathbb{F}_q[\![z_{v_i}]\!]}Spf\ R = \prod_{\mathbb{Z}/deg\ v_i}\hat{\mathcal{F}l}_{\mathbb{G}_{v_i},R}$. Let $\mathcal{S}(1) := L_{v_i}^+\mathbb{G}_{v_i}\cdot 1 \subset \mathcal{F}l_{\mathbb{G}_{v_i}}$ be the closed Schubert variety and $\mathcal{S}(1)_R = \mathcal{S}(1)\times_{\mathbb{F}_q} Spf R$ then $\hat{Z}_{v_i,R}\times\mathcal{S}(1)_R\times\cdots\times\mathcal{S}(1)_R$ defines a bound in $\hat{\mathcal{F}l}_{\widetilde{\mathbb{G}_v}}$ that we also denote by $Z_{v_i}\times\mathcal{S}(1)\times\cdots\times\mathcal{S}(1)$. We have the following lemma.

Lemma 3.13. *Let $\underline{\mathcal{G}} \in \nabla_n\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}(S)$ as before. The local $\mathbb{G}_{v_i}$-shtuka $\Gamma_{v_i}(\underline{\mathcal{G}})$ is bounded by $\hat{Z}_{v_i}$ if and only if the local $\widetilde{\mathbb{G}_{v_i}}$-shtuka $L_v^+(\underline{\mathcal{G}})$ is bounded by $\hat{Z}_{v_i}\times\mathcal{S}(1)\times\cdots\times\mathcal{S}(1)$.*

Proof: We choose an étale covering S' of S that trivializes $L_{v_i}^+(\mathcal{G})$ as well as $\sigma^\star L_{v_i}^+(\mathcal{G})$. In particular S' trivializes also $\Gamma_{v_i}(\mathcal{G})$ and $\sigma^{d\star}\Gamma_{v_i}(\mathcal{G})$. We fix such trivializations and call them $\widetilde{\alpha} : L_{v_i}^+(\mathcal{G})_{S'} \longrightarrow L^+\widetilde{\mathbb{G}}_{v_i S'}$, $\widetilde{\alpha}' : \sigma^\star L_{v_i}^+(\mathcal{G})_{S'} \longrightarrow L^+\widetilde{\mathbb{G}}_{v_i S'}$, $\alpha : \Gamma_{v_i}(\mathcal{G})_{S'} \longrightarrow \mathbb{G}\times_C V(\mathfrak{a}_{v_i,0})$ and $\alpha' : \sigma^{d\star}\Gamma_{v_i}(\mathcal{G})_{S'} \longrightarrow \mathbb{G}\times_C V(\mathfrak{a}_{v_i,0})$. Denote by τ_j the Frobenius morphism $\tau_{\mathcal{G}}$ restricted to $V(\mathfrak{a}_{v_i,j})$ for $j = 0,\ldots,d-1$, where $d = deg\ v_i = [\mathbb{F}_{v_i} : \mathbb{F}_q]$. So the local shtuka $\Gamma_{v_i}(\underline{\mathcal{G}})$ is given by $(\mathcal{G}\times_C V(\mathfrak{a}_{v_i,0}), \tau_0\circ\sigma^\star\tau_1\circ\cdots\circ\sigma^{(d-1)\star}\tau_{d-1})$ and $\alpha\circ\tau_0\circ\sigma^\star\tau_1\circ\cdots\circ\sigma^{(d-1)\star}\tau_{d-1}\circ\alpha'^{-1} : L\mathbf{G}_{v_i,S'} \xrightarrow{\sim} L\mathbf{G}_{v_i,S'}$ computed in $\mathscr{H}^1(\mathbb{F}_{v_i}, L\mathbf{G}_{v_i})(S')$ defines a morphism $S' \to L\mathbf{G}_{v_i}$. Now $\Gamma_{v_i}(\underline{\mathcal{G}})$ is bounded by Z_{v_i} if and only if the morphism $S'\times_{R_{Z_{v_i}}} Spf\ R \to L\mathbf{G}_{v_i}\times_{\mathbb{F}_{v_i}} Spf\ R \to \hat{\mathcal{F}l}_{\mathbb{G}_{v_i},R}$ factors through

$Z_{v_i,R}$. Since τ is an isomorphism outside the graphs of s_i, $\tau_1,\ldots,\tau_{d-1}$ are isomorphisms. Hence $\sigma^\star\tau_1\circ\cdots\circ\sigma^{(d-1)\star}\tau_{d-1}\circ\alpha'^{-1}$ comes from some other trivialization $\beta'^{-1}:\mathbb{G}\times_C V(\mathfrak{a}_{v_i,0})\to\sigma^\star\Gamma_{v_i}(\mathcal{G})$. This shows that $\tau_0\circ\sigma^\star\tau_1\circ\cdots\circ\sigma^{(d-1)\star}\tau_{d-1}$ is bounded by Z_{v_i} if and only if τ_0 is bounded by Z_{v_i}. Now $\widetilde{\alpha}\circ\tau\circ(\widetilde{\alpha'})^{-1}:L\widetilde{\mathbf{G}_{v_i,S'}}\xrightarrow{\sim}L\widetilde{\mathbf{G}_{v_i,S'}}$ computed in $\mathscr{H}^1(\mathbb{F}_q,L\widetilde{\mathbf{G}}_{v_i})(S')$ defines in the same way a morphism $S'\to L\widetilde{\mathbf{G}_{v_i}}$ which induces a morphism $S'\times_{R_{Z_{v_i}}}Spf\ R\to\mathcal{F}l_{\widetilde{\mathbb{G}}_{v_i}}\times_{\mathbb{F}_q}Spf\ R=\prod_{l\in\mathbb{Z}/deg v_i}\mathcal{F}l_{\mathbb{G}_{v_i}}\times_{\mathbb{F}_{v_i}}Spf\ R$. Note that the morphism in the j-th component of $\prod\mathcal{F}l_{\mathbb{G}_{v_i}}\times_{\mathbb{F}_{v_i}}Spf\ R$ is exactly defined by $\widetilde{\alpha}\circ\tau_j\circ(\widetilde{\alpha}')^{-1}$. Since $\tau_1,\ldots,\tau_{d-1}$ are isomorphisms the morphism into the j-th component with $j\geqslant 1$ always factors through $\mathcal{S}(1)_R$. This implies that τ is bounded by $Z_{v_i}\times\mathcal{S}(1)\times\cdots\times\mathcal{S}(1)$ if and only if τ_0 is bounded by Z_{v_i}. □

Now we can prove:

Theorem 3.14. *Let $(C,\mathbb{G},\underline{v},\hat{Z}_{\underline{v}},H)$ be a shtuka datum and $\pi:C\to C'$ a finite morphism of smooth geometrically irreducible curves over $\mathbb{F}_q$ with $w_i:=\pi(v_i)$ and $\underline{w}:=(w_1,\ldots,w_n)$. Then the morphism $(\pi,id_{\pi_\star\mathbb{G}}):(C,\mathbb{G},\underline{v},\hat{Z}_{\underline{v}},H)\to(C',\pi_\star\mathbb{G},\underline{w},\pi_\star\hat{Z}_{\underline{v}},\pi_\star H)$ of shtuka data (see definition 3.9 and remark 3.10) induces a finite morphism of the moduli stacks*

$$\pi_\star:\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})\to\nabla_n^{\pi_\star\hat{Z}_{\underline{v}},\pi_\star H}\mathscr{H}^1(C',\pi_\star\mathbb{G}).$$

Proof: Let S be an $\mathbb{F}_q$-scheme and $(\mathcal{G},s_1,\ldots,s_n,\tau_{\mathcal{G}},\gamma)\in\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})(S)$. We describe again its image $(\mathcal{G}',s_1',\ldots,s_n',\tau_{\mathcal{G}'},\gamma')$ in $\nabla_n^{\hat{Z}_{\underline{w}},\pi_\star H}\mathscr{H}^1(C',\pi_\star\mathbb{G})(S)$ to define the morphism. The $\pi_\star\mathbb{G}$-torsor $\underline{\mathcal{G}}'=(\mathcal{G}',s_1',\ldots,s_n',\tau_{\mathcal{G}'})$ is already defined by the morphism in proposition 3.12, but we have to prove that it lies indeed in $\nabla_n^{\hat{Z}_{\underline{w}}}\mathscr{H}^1(C,\pi_\star\mathbb{G})(S)$. We will do this first and then define the $\pi_\star H$-level structure γ'. Since the section $s_i:S\to C$ is mapped to $s_i':=\pi\circ s_i$ and s_i is required to factor through $Spf\ A_{v_i}$, we easily see with $\pi(v_i)=w_i$ that s_i' factors through $Spf\ A_{w_i}$. Furthermore this shows that $C'_S\backslash\cup_i\Gamma_{s_i'}\supset C'^\star\times_{\mathbb{F}_q}S$ and $C_S\backslash\cup_i\Gamma_{s_i}\supset C^\star\times_{\mathbb{F}_q}S$ where we use the notation $C^\star=C\backslash\{v_1,\ldots,v_n\}$ and $C'^\star=C'\backslash\{w_1,\ldots w_n\}$. It remains to show that $\tau_{\mathcal{G}'}$ is bounded by $Z_{\underline{w}}$ to see $\underline{\mathcal{G}}'\in\nabla_n^{\hat{Z}_{\underline{w}}}\mathscr{H}^1(C',\pi_\star\mathbb{G})(S)$.

Now by assumption $\tau_{\mathcal{G}}$ is bounded by Z_{v_i}, which means by definition that the local shtukas $\Gamma_{v_i}(\underline{\mathcal{G}})$ are bounded by Z_{v_i} for all i. By lemma 3.13 this is equivalent to the fact that $L^+_{v_i}(\underline{\mathcal{G}})$ is bounded by $Z_{v_i}\times\mathcal{S}(1)\times\cdots\times\mathcal{S}(1)$. Now consider the following 2-cartesian diagram (compare § 2.9 and [AH14, Lemma 5.1]) where we set $U:=C'^\star\times_{C'}C$.

$$\begin{array}{ccc}\mathscr{H}^1(C,\mathbb{G}) & \longrightarrow & \mathscr{H}^1_e(U,\mathbb{G})\\ \downarrow{\scriptstyle\prod_v L^+_v} & & \downarrow{\scriptstyle\prod_v L_v}\\ \prod_{v\in\pi^{-1}(\underline{w})}\mathscr{H}^1(\mathbb{F}_q,L^+\widetilde{\mathbb{G}_v}) & \longrightarrow & \prod_{v\in\pi^{-1}(\underline{w})}\mathscr{H}^1(\mathbb{F}_q,L\widetilde{\mathbf{G}_v})\end{array}\qquad(5)$$

Here $\mathscr{H}^1_e(U,\mathbb{G})(S)$ is the full subcategory of $\mathscr{H}^1(U,\mathbb{G})(S)$ consisting of those $\mathbb{G}$-torsors over U_S that can be extended to a $\mathbb{G}$-torsor over C_S. Now the categories $\mathscr{H}^1(C,\mathbb{G})$ and $\mathscr{H}^1_e(U,\mathbb{G})$ are by lemma 3.3 equivalent to $\mathscr{H}^1(C',\pi_\star G)$ and $\mathscr{H}^1_e(C'^\star,\pi_\star\mathcal{G})$. Furthermore the categories

$$\prod_{v\in\pi^{-1}(\underline{w})}\mathscr{H}^1(\mathbb{F}_q,L^+\widetilde{\mathbb{G}_v})=\prod_{w\in\underline{w}}\mathscr{H}^1(\mathbb{F}_q,\prod_{v\in\pi^{-1}(w)}L^+\widetilde{\mathbb{G}_v})\quad\text{and}$$

$$\prod_{v\in\pi^{-1}(\underline{w})}\mathscr{H}^1(\mathbb{F}_q,L\widetilde{\mathbf{G}_v})=\prod_{w\in\underline{w}}\mathscr{H}^1(\mathbb{F}_q,\prod_{v\in\pi^{-1}(w)}L\widetilde{\mathbf{G}_v})$$

are by corollary 3.7 equivalent to $\prod_{w\in\underline{w}}\mathscr{H}^1(\mathbb{F}_q,L^+\widetilde{\pi_\star\mathbb{G}_w})$ and $\prod_{w\in\underline{w}}\mathscr{H}^1(\mathbb{F}_q,L\widetilde{\pi_\star\mathbf{G}_w})$. Therefore the whole diagram (5) is equivalent to the diagram

$$\begin{array}{ccc}
\mathscr{H}^1(C',\pi_\star\mathbb{G}) & \longrightarrow & \mathscr{H}_e^1(C'^\star,\pi_\star\mathbb{G}) \\
\Big\downarrow{\scriptstyle \prod_w L_w^+} & & \Big\downarrow{\scriptstyle \prod_w L_w} \\
\prod_{w\in\underline{w}}\mathscr{H}^1(\mathbb{F}_q,L^+\widetilde{\pi_\star\mathbb{G}_w}) & \longrightarrow & \prod_{w\in\underline{w}}\mathscr{H}^1(\mathbb{F}_q,L\widetilde{\pi_\star\mathbf{G}_w})\ .
\end{array}$$

Now we choose some covering S' over S that trivializes $L_v^+\mathcal{G},\sigma^\star L_v^+\mathcal{G}$ for all $v\in\pi^{-1}(\underline{w})$ and fix some trivializations $\alpha_v: L_v^+(\mathcal{G})_{S'}\to L^+\widetilde{\mathbb{G}}_{vS'}$, $\alpha'_v:\sigma^\star L_v^+(\mathcal{G})_{S'}\to L^+\widetilde{\mathbb{G}}_{vS'}$. Then $(\mathcal{G},\tau)$ defines a tuple $\prod_{w\in\underline{w}}\big(\prod_{v|w}L^+\widetilde{\mathbb{G}_{v,S'}},\ \prod_{v|w}\alpha_v\circ L_v(\tau|_U)\circ\alpha'^{-1}_v\big)$ and the equivalence of the diagrams shows that it corresponds to the tuple $\prod_{w\in\underline{w}}\big(L^+\widetilde{\pi_\star\mathbb{G}_{w,S'}},\ \alpha_w\circ L_w(\pi_\star\tau|_{C'^\star})\circ\alpha'^{-1}_w\big)$ defined in the same way by the shtuka $(\pi_\star\mathcal{G},\pi_\star\tau)$. Here α_w,α'_w are the trivializations corresponding to $\prod_{v|w}\alpha_v$ and $\prod_{v|w}\alpha'_v$. Now choose some finite extension $R\supset\mathbb{F}_q[\![\zeta]\!]$ such that there are representatives $Z_{v,R}$ for all $v\in\underline{v}$. Using the 2-cartesian diagram

$$\begin{array}{ccc}
\prod\limits_{v\in\pi^{-1}(\underline{w})}\mathcal{F}l_{\widetilde{\mathbb{G}_v}} & \longrightarrow & \mathbb{F}_q \\
\Big\downarrow & & \Big\downarrow \\
\prod_{v\in\pi^{-1}(\underline{w})}\mathscr{H}^1(\mathbb{F}_q,L^+\widetilde{\mathbb{G}_v}) & \longrightarrow & \prod_{v\in\pi^{-1}(\underline{w})}\mathscr{H}^1(\mathbb{F}_q,L\widetilde{\mathbf{G}_v})
\end{array}$$

the tuple $(\prod_{v|w}L^+\widetilde{\mathbb{G}_{v,S'}},\ \prod_{v|w}\alpha_v\circ L_v(\tau|_U)\circ\alpha'^{-1}_v)$ defines an $S'\times_{R_{\hat{\underline{Z}}}}Spf\ R$-valued point in $\prod_{v|w}\mathcal{F}l_{\widetilde{\mathbb{G}_v}}\times_{\mathbb{F}_q}Spf\ R=\prod_{v|w}\prod_{l\in\mathbb{Z}/deg\ v}\mathcal{F}l_{\mathbb{G}_v}\times_{\mathbb{F}_v}Spf\ R$. By lemma 3.13 the boundedness of $\underline{\mathcal{G}}$ at all the points $v\in(\underline{v}\cap\pi^{-1}(w))$ by Z_v is equivalent to the boundedness of $L_v^+(\underline{\mathcal{G}})$ by $\prod_{l\in\mathbb{Z}/deg\ v}Z_{v,l}$ with $Z_{v,0}=Z_v$ for all $v\in\underline{v}$ and $Z_{v,l}=\mathcal{S}(1)_R$ for all $v\notin\underline{v}$ and $l\neq 0$, which means by definition, that the above $S'\times_{R_{\hat{\underline{Z}}}}Spf\ R$ valued point factors through $\prod_{v|w}\prod_{l\in\mathbb{Z}/deg\ v}Z_{v,l}$. The tuple $(L^+\widetilde{\pi_\star\mathbb{G}_{w,S'}},S',\alpha_w\circ L_w(\pi_\star\tau|_{C^\star})\circ\alpha'^{-1}_w)$ defines in the same way a morphism $S'\times_{R_{\hat{\underline{Z}}}}Spf\ R\to\mathcal{F}l_{\widetilde{(\pi_\star\mathbb{G})_w}}\times_{\mathbb{F}_q}Spf\ R=\prod_{l\in\mathbb{Z}/deg\ w}\mathcal{F}l_{\pi_\star\mathbb{G}_w}\times_{\mathbb{F}_w}Spf\ R$. Composing with the isomorphism $\prod_{v|w}\mathcal{F}l_{\widetilde{\mathbb{G}_v}}\simeq\mathcal{F}l_{\widetilde{\pi_\star\mathbb{G}_w}}$, the above morphism factors also through $\prod_{v|w}\prod_{l\in\mathbb{Z}/deg\ v}Z_{v,l}$. With lemma 3.13 and the definition of $\pi_\star Z_{\underline{v}}$ on page 18 it follows, that $\pi_\star\underline{\mathcal{G}}$ is bounded by $\pi_\star Z_{\underline{v}}$.

Next we have to define the $\pi_\star H$-level structure γ'. We fix a geometric base point $\bar{s}\in S$ and we choose for all closed points $v\in C\backslash\underline{v}$ a trivialization $L_v^+\underline{\mathcal{G}}_{\bar{s}}\simeq(L^+\widetilde{\mathbb{G}}_{v,\kappa(\bar{s})},\tau=1)$, which exists by [AH14, Corollary 2.9]. This provides also trivializations

$$L_w^+(\pi_\star\mathcal{G}_{\bar{s}})=\prod_{v|w}L_v^+\mathcal{G}_{\bar{s}}\simeq\prod_{v|w}L^+\widetilde{\mathbb{G}}_{v,\kappa(\bar{s})}=L^+\widetilde{\pi_\star\mathbb{G}}_{w,\kappa(\bar{s})}.$$

We denote by $\underline{\mathcal{L}_v}$ and $\underline{\mathcal{L}_w}$ the shtukas $L_v^+\underline{\mathcal{G}}_{\bar{s}}$ and $L_w^+(\pi_\star\mathcal{G}_{\bar{s}})$. Now these trivializations induce isomorphisms

$$\beta:\omega^\circ_{\mathbb{O}^{\underline{v}}}=\prod_{v\in C\backslash\underline{v}}\mathcal{T}_{L^+\widetilde{\mathbb{G}_v}}\simeq\prod_{v\in C\backslash\underline{v}}\mathcal{T}_{\underline{\mathcal{L}_v}}=\mathcal{T}_{\mathcal{G}}$$

$$\pi_\star\beta:\omega^\circ_{\mathbb{O}^{\underline{w}}}=\prod_{w\in C'\backslash\underline{w}}\mathcal{T}_{L^+\widetilde{\pi_\star\mathbb{G}_w}}\simeq\prod_{w\in C'\backslash\underline{w}}\mathcal{T}_{\underline{\mathcal{L}_w}}=\mathcal{T}_{\pi_\star\mathcal{G}}.$$

We write $\omega^\circ_{\mathbb{A}^{\underline{v}}}:=\omega^\circ_{\mathbb{O}^{\underline{v}}}\otimes_{\mathbb{O}^{\underline{v}}}\mathbb{A}^{\underline{v}}$ and $\omega'^\circ_{\mathbb{A}^{\underline{w}}}:=w^\circ_{\mathbb{O}^{\underline{w}}}\otimes_{\mathbb{O}^{\underline{w}}}\mathbb{A}^{\underline{w}}$. Now $\beta^{-1}\circ\gamma\in Aut^\otimes(\omega^\circ_{\mathbb{A}^{\underline{v}}})$ is given by an element $g\in\mathbb{G}(\mathbb{A}^{\underline{v}})$ and the H-orbit of γ is $\beta\circ gH$. Now we can use the projection

$\mathbb{G}(\mathbb{A}^{\underline{v}}) \twoheadrightarrow \mathbb{G}(\mathbb{A}^{\pi^{-1}(\underline{w})}) = \pi_\star\mathbb{G}(\mathbb{A}^{\underline{w}})$ to define $\pi_\star g \in \pi_\star\mathbb{G}(\mathbb{A}^{\underline{w}})$ as the image of g. This corresponds to an element in $Aut(\omega'^{\circ}_{\mathbb{A}^{\underline{w}}})$. Therefore $\pi_\star\beta \circ \pi_\star g$ defines an element γ' and consequently an $\pi_\star H$-orbit in $Isom^{\otimes}(\omega'^{\circ}_{\mathbb{A}^{\underline{w}}}, \check{\mathcal{V}}_{\pi_\star\underline{\mathcal{G}}})$. This orbit is independent of the representative γ since $\pi_\star H$ was defined as the image of H under the above projection. Let $\rho \in \pi_1(S,\overline{s})$ since $\gamma H \subset Isom^{\otimes}(\omega^{\circ}_{\mathbb{A}^{\underline{v}}}, \check{\mathcal{V}}_{\underline{\mathcal{G}}})$ is $\pi_1(S,\overline{s})$ invariant, we know that there is $h \in H$ such that $\rho\gamma = \gamma h$. This defines a group homomorphism $\varphi : \pi_1(S,\overline{s}) \to H$ and we set $\overline{\varphi} : \pi_1(S,\overline{s}) \to H \to \pi_\star H$. Let $\rho \in \pi_\star(S,\overline{s})$ and $\gamma' \in Isom^{\otimes}(\omega'^{\circ}_{\mathbb{A}^{\underline{w}}}, \check{\mathcal{V}}_{\pi_\star\underline{\mathcal{G}}})$ be as above, then ρ operates by $\rho\gamma = \gamma'\overline{\varphi}(\rho)$ and in particular $\pi_1(S,\overline{s})\gamma' \subset \gamma'\pi_\star H$. This means that the orbit $\gamma'\pi_\star H$ is $\pi_1(S,\overline{s})$ invariant and defines a level structure γ' of $\underline{\mathcal{G}}'$.

After constructing this morphism, we now prove that it is representable by a scheme and finite. By proposition 3.12 it is clear that the morphism $\nabla_n^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G}) \to \nabla_n^{\hat{Z}_{\underline{w}}}\mathscr{H}^1(C',\pi_\star\mathbb{G})$ is finite. Now we find some finite subscheme $D \subset C$ such that $H_D := ker(\mathbb{G}(\mathbb{O}^{\underline{v}}) \to \mathbb{G}(\mathcal{O}_D))$ is a subgroup of finite index in H. Then we have by § 2.16 the following diagram:

$$\begin{array}{ccccccc}
\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) & \xleftarrow{\text{finite étale}} & \nabla_n^{\hat{Z}_{\underline{v}},H_D}\mathscr{H}^1(C,\mathbb{G}) & \xrightarrow{\sim} & \nabla_n^{\hat{Z}_{\underline{v}}}\mathscr{H}^1_D(C,\mathbb{G}) & \xrightarrow{\text{finite étale}} & \nabla_n^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \text{finite}} \\
\nabla_n^{\hat{Z}_{\underline{w}},\pi_\star H}\mathscr{H}^1(C',\pi_\star\mathbb{G}) & \xleftarrow{\text{finite étale}} & \nabla_n^{\hat{Z}_{\underline{w}},H_{\pi_\star D}}\mathscr{H}^1(C',\pi_\star\mathbb{G}) & \xrightarrow{\sim} & \nabla_n^{\hat{Z}_{\underline{w}}}\mathscr{H}^1_{\pi_\star D}(C',\pi_\star\mathbb{G}) & \xrightarrow{\text{finite étale}} & \nabla_n^{\hat{Z}_{\underline{w}}}\mathscr{H}^1(C',\pi_\star\mathbb{G})
\end{array}$$

where the horizontal arrows are finite (and even étale) by [AH13, section 6]. This implies firstly that the morphism $\nabla_n^{\hat{Z}_{\underline{v}},H_D}\mathscr{H}^1(C,\mathbb{G}) \to \nabla_n^{\hat{Z}_{\underline{w}},H_{\pi_\star D}}\mathscr{H}^1(C',\pi_\star\mathbb{G})$ is finite and consequently that $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) \to \nabla_n^{\hat{Z}_{\underline{w}},\pi_\star H}\mathscr{H}^1(C',\pi_\star\mathbb{G})$ is finite. □

3.3 Changing the Group $\mathbb{G}$

Now let $f : \mathbb{G} \to \mathbb{G}'$ be a morphism of smooth affine group schemes over C. In this section we explain how this induces a morphism between the moduli stacks of $\mathbb{G}$-shtukas and $\mathbb{G}'$-shtukas. Further we prove some of its properties, depending on f. First of all we recall, that given a sheaf M on $C_{\acute{E}t}$ with an action of $\mathbb{G}$, we can define the sheaf $M \times^{\mathbb{G}} \mathbb{G}'$ whose R-valued points are given by the set $\{(a,b) \mid a \in M(R), b \in \mathbb{G}'(R)\}/\sim$, where $(a,b) \sim (c,d)$ if and only if $(a,b) = (cg, f(g^{-1})d)$ for some $g \in \mathbb{G}(R)$. Actually this construction works for any sheaf of groups on any site. Now this construction is functorial for $\mathbb{G}$-equivariant morphisms $\varphi : M_1 \to M_2$ and commutes obviously with base change. We also write $f_\star M = M \times^{\mathbb{G}} \mathbb{G}'$ and note that if M is a $\mathbb{G}$-torsor then $f_\star M$ is a $\mathbb{G}'$-torsor. With these facts we see that for a given $\mathbb{G}$-shtuka $(\mathcal{G}, s_1, \ldots, s_n, \tau_{\mathcal{G}})$ over S, the tuple $(f_\star\mathcal{G}, s_1, \ldots, s_n, f_\star\tau_{\mathcal{G}})$ defines a $\mathbb{G}'$-shtuka over S. Therefore we get a morphism

$$\nabla_n\mathscr{H}^1(C,\mathbb{G}) \longrightarrow \nabla_n\mathscr{H}^1(C,\mathbb{G}') \qquad (\mathcal{G}, s_1, \ldots, s_n, \tau_{\mathcal{G}}) \mapsto (f_\star\mathcal{G}, s_1, \ldots, s_n, f_\star\tau_{\mathcal{G}}). \tag{6}$$

Now we want to show that this morphism also induces a morphism of these moduli stacks with additional H-level structure. So we fix n closed points $\underline{v} = (v_1, \ldots, v_n)$ in C and let $H \subset \mathbb{G}(\mathbb{A}^{\underline{v}})$ be an open and compact subgroup. Let further S be a connected $\mathbb{F}_q$-scheme with a geometric base point $\overline{s} \in S$ and $(\underline{\mathcal{G}}, \gamma) = (\mathcal{G}, s_1, \ldots, s_n, \tau_{\mathcal{G}}, \gamma)$ be a $\mathbb{G}$-shtuka over S with an H-level structure γH. We already mentioned that by [Con12, Proposition 2.1] $f : \mathbb{G} \to \mathbb{G}'$ induces a continuous

homomorphism $f_{\mathbb{A}^{\underline{v}}} : \mathbb{G}(\mathbb{A}^{\underline{v}}) \to \mathbb{G}'(\mathbb{A}^{\underline{v}})$ (see also above definition 3.9). For an open compact subgroup $H' \subset \mathbb{G}'(\mathbb{A}^{\underline{v}})$ satisfying $f_{\mathbb{A}^{\underline{v}}}(H) \subset H'$ we now construct an H'-level structure on the shtuka $f_\star\underline{\mathcal{G}} = (f_\star\mathcal{G}, s_1, \ldots, s_n, f_\star\tau_{\mathcal{G}})$.
We choose for every $v \in \widetilde{C} = C\backslash\underline{v}$ a trivialization $\alpha_v : L_v^+(\underline{\mathcal{G}}_{\bar{s}}) \xrightarrow{\sim} (L^+\widetilde{\mathbb{G}}_{v,\bar{s}}, 1\cdot\sigma^\star)$ which exists by [AH14, Proposition 2.9]. Since $f_\star$ commutes with base change this induces trivializations $f_\star\alpha_v : L_v^+((f_\star\underline{\mathcal{G}})_{\bar{s}}) \xrightarrow{\sim} (L^+\widetilde{\mathbb{G}'}_{v,\bar{s}}, 1\cdot\sigma^\star)$. We denote by $\omega^\circ_{\mathbb{O}^{\underline{v}}} : Rep_{\mathbb{O}^{\underline{v}}}\mathbb{G} \to \mathfrak{Mod}_{\mathbb{O}^{\underline{v}}[\pi_1(S,\bar{s})]}$ and $\omega'^\circ_{\mathbb{O}^{\underline{v}}} : Rep_{\mathbb{O}^{\underline{v}}}\mathbb{G}' \to \mathfrak{Mod}_{\mathbb{O}^{\underline{v}}[\pi_1(S,\bar{s})]}$ the forgetful functors and by $\underline{\mathcal{L}}_v$ and $\underline{\mathcal{L}'}_v$ the local shtukas $L_v^+(\underline{\mathcal{G}}_{\bar{s}})$ and $L_v^+((f_\star\underline{\mathcal{G}})_{\bar{s}})$. Then the previous trivializations provide isomorphisms of tensor functors

$$\beta : \omega^\circ_{\mathbb{O}^{\underline{v}}} = \prod_{v\in C\backslash\underline{v}} \mathcal{T}_{L^+\widetilde{\mathbb{G}_v}} \xrightarrow{\sim} \prod_{v\in C\backslash\underline{v}} \mathcal{T}_{\underline{\mathcal{L}_v}} = \mathcal{T}_{\underline{\mathcal{G}}}$$
$$f_\star\beta : \omega'^\circ_{\mathbb{O}^{\underline{v}}} = \prod_{v\in C\backslash\underline{v}} \mathcal{T}_{L^+\widetilde{\mathbb{G}'_v}} \xrightarrow{\sim} \prod_{v\in C\backslash\underline{v}} \mathcal{T}_{\underline{\mathcal{L}'_v}} = \mathcal{T}_{f_\star\underline{\mathcal{G}}}.$$

It follows that $\beta^{-1}\circ\gamma \in Aut^\otimes(\omega^\circ_{\mathbb{A}^{\underline{v}}})$ is given by an element $g \in \mathbb{G}(\mathbb{A}^{\underline{v}})$ and the H-orbit of γ is given by $\beta\circ gH$. Now we view the image $f(g)$ of g under the map $f_{\mathbb{A}^{\underline{v}}} : \mathbb{G}(\mathbb{A}^{\underline{v}}) \to \mathbb{G}'(\mathbb{A}^{\underline{v}})$ as an automorphism in $Aut^\otimes(\omega'^\circ_{\mathbb{O}^{\underline{v}}})$ and define $\gamma' := f_\star\gamma := f_\star\beta\circ f(g)$. Since $f_{\mathbb{A}^{\underline{v}}}(H) \subset H'$ the H' orbit of $f(g)$ is independent of the chosen representative γ in the orbit γH. Since $\pi_1(S,\bar{s})$ leaves γH invariant there is for all $\rho \in \pi_1(S,\bar{s})$ an $h \in H$ such that $\rho\cdot\gamma = \gamma\cdot h$. This defines a group homomorphism $\varphi : \pi_1(S,\bar{s}) \to H$ and we set $\varphi' : \pi_1(S,\bar{s}) \to H \xrightarrow{f_{\mathbb{A}^{\underline{v}}}|_H} H'$. Now $\rho \in \pi_1(S,\bar{s})$ operates on $\gamma' \in Isom(\omega', \check{\mathcal{V}}_{\underline{\mathcal{G}}})$ by $\rho\cdot\gamma' = \gamma'\cdot\bar{\varphi}'(\rho)$. In particular $\pi_1(S,\bar{s})\gamma' \subset \gamma'H'$ so that $\gamma'H'$ is $\pi_1(S,\bar{s})$ invariant and defines a H'-level structure on $f_\star\underline{\mathcal{G}}$. A morphism $(\underline{\mathcal{G}},\bar{\gamma})$ to $(\underline{\mathcal{F}},\bar{\eta})$ induces naturally a morphism $(f_\star\underline{\mathcal{G}},\bar{\gamma}') \to (f_\star\underline{\mathcal{F}},\bar{\eta}')$ so that we get a morphism of moduli stacks

$$\nabla_n^H\mathscr{H}^1(C,\mathbb{G}) \to \nabla_n^{H'}\mathscr{H}^1(C,\mathbb{G}') \quad (\underline{\mathcal{G}},\bar{\gamma}) \mapsto (f_\star\underline{\mathcal{G}},\bar{\gamma}'). \tag{7}$$

Next we show that this morphism behaves well with respect to boundedness conditions. We note that for all $v \in \underline{v}$ the morphism $f : \mathbb{G} \to \mathbb{G}'$ induces a morphism $L^+\mathbb{G}_{v_i} \to L^+\mathbb{G}'_v$ as well as a morphism $L\mathbb{G}_v \to L\mathbb{G}'_v$ and consequently also a morphism $\mathcal{F}l_{\mathbb{G}_v} \to \mathcal{F}l_{\mathbb{G}'_v}$.

Lemma 3.15. *Let $\hat{Z}_{\underline{v}}$ be a bound in $\prod_{i-1}^n \widehat{\mathcal{F}l}_{\mathbb{G}_{v_i}}$ and $\underline{\mathcal{G}}$ a $\mathbb{G}$-shtuka over S bounded by $\hat{Z}_{\underline{v}}$. Let further $\hat{Z}'_{\underline{v}}$ be a bound in $\prod_{i=1}^n \widehat{\mathcal{F}l}_{\mathbb{G}'_{v_i}}$ such that after choosing representatives over some DVR R the morphism $\hat{Z}_{\underline{v},R} \to \prod_{i=1}^n \widehat{\mathcal{F}l}_{\mathbb{G}'_{v_i}}$ factors through $\hat{Z}'_{\underline{v},R}$. Then $f_\star\underline{\mathcal{G}}$ is bounded by $\hat{Z}'_{\underline{v}}$.*

Proof: We have to prove that for $v \in \underline{v}$ the local shtuka $\Gamma_v(f_\star\underline{\mathcal{G}})$ is bounded by $\hat{Z}'_v$. We choose some covering $S' \to S$ with $S'/Spec\ R$ that trivializes $L_v^+\sigma^\star\mathcal{G}$ and $L_v^+\mathcal{G}$ at the same time and fix such trivializations, which we denote by $\alpha : L_v^+\sigma^\star\mathcal{G}_{S'} \to L^+\mathbb{G}_{v,S'}$ and $\alpha' : L_v^+\mathcal{G}_{S'} \to L^+\mathbb{G}_{v,S'}$. Then $f_\star\alpha$ and $f_\star\alpha'$ are trivializations of $L_v^+(f_\star\sigma^\star\mathcal{G})_{S'}$ and $L_v^+(f_\star\mathcal{G})_{S'}$. Now we have the automorphism $\alpha'\circ\tau^{deg\ v}\circ\alpha^{-1} : L\mathbf{G}_{v,S'} \xrightarrow{\sim} L\mathbf{G}_{v,S'}$ and we let $1_{S'} : S' \to L\mathbf{G}_{v,S'}$ be the unit morphism. The composition defines an S' valued point $1_{S'}\circ\alpha'\circ\tau^{deg\ v}\circ\alpha^{-1}$ in $L\mathbf{G}_{v,S'}$. The composition of this point with the morphism $L\mathbb{G}_{v,S'} \to L\mathbb{G}'_{v,S'}$ induced by f defines an S'-valued point in $L\mathbb{G}'_{v,S'}$. Since the diagram

$$\begin{array}{ccccc}
S' & \xrightarrow{1_{S'}} & L\mathbf{G}_{v,S'} & \xrightarrow{\alpha'\circ\tau^{deg\ v}\circ\alpha^{-1}} & L\mathbf{G}_{v,S'} \\
 & \searrow & \downarrow & & \downarrow \\
 & & L\mathbf{G}'_{v,S'} & \xrightarrow{f_\star(\alpha'\circ\tau^{deg\ v}\circ\alpha^{-1})} & L\mathbf{G}'_{v,S'}
\end{array}$$

commutes, this is exactly the S' valued point defined by

$$f_\star(\alpha')f_\star(\tau^{deg\ v})f_\star(\alpha^{-1}) = f_\star(\alpha'\tau^{deg\ v}\alpha^{-1}).$$

By assumption $\underline{\mathcal{G}}$ is bounded by $\hat{Z}_{\underline{v}}$ and consequently the morphism $1_{S'} \circ \alpha' \circ \tau^{deg\ v} \circ \alpha^{-1}$ factors after projection to $\widehat{\mathcal{F}l}_{\mathbb{G}_v,S'}$ through $\hat{Z}_{v,R}$ and maps then into $\hat{Z}'_{v,R}$. This means exactly that $\Gamma_v(f_\star\underline{\mathcal{G}})$ is bounded by $\hat{Z}'_v$, so that $f_\star\underline{\mathcal{G}}$ is bounded by $\hat{Z}'_{\underline{v}}$.

□

This lemma and the previous explanations show.

Corollary 3.16. *The morphism* $(id,f) : (C,\mathbb{G},\underline{v},\hat{Z}_{\underline{v}},H) \to (C,\mathbb{G}',\underline{v},\hat{Z}'_{\underline{v}},H')$ *of shtuka data induces a morphism*

$$f_\star : \quad \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) \longrightarrow \nabla_n^{\hat{Z}'_{\underline{v}},H'}\mathscr{H}^1(C,\mathbb{G}')$$
$$(\mathcal{G},s_1,\ldots,s_n,\tau_{\mathcal{G}},\gamma) \longmapsto (f_\star\mathcal{G},s_1,\ldots,s_n,f_\star\tau_{\mathcal{G}},f_\star\gamma).$$

Proof: Follows directly from lemma 3.15 and the morphism (7) on page 26. □

Now we are interested in some special classes of morphisms $f : \mathbb{G} \to \mathbb{G}'$.

Generic Isomorphisms of $\mathbb{G}$

First of all we want to consider morphisms $f : \mathbb{G} \to \mathbb{G}'$ which are generically an isomorphism, that means $f \times id_Q : \mathbf{G} \xrightarrow{\sim} \mathbf{G}'$. In this case $f : \mathbf{G} \to \mathbf{G}'$ is already an isomorphism over some open subscheme in C. So we fix such an $f : \mathbb{G} \to \mathbb{G}'$ and denote by U the maximal open subscheme in C such that $f \times id_U : \mathbb{G}_U \to \mathbb{G}'_U$ is an isomorphism and denote by $\underline{w} = (w_1,\ldots,w_m)$ the finite set of closed points in the complement $C\backslash U$.
Before we come to the moduli stacks of the global $\mathbb{G}$-shtukas, we prove a proposition that describes the morphism $\mathscr{H}^1(C,\mathbb{G}) \to \mathscr{H}^1(C,\mathbb{G}')$. For this proposition we need the following lemma.

Lemma 3.17. *Let* $\mathcal{L}'_+$ *be an* $L^+\widetilde{\mathbb{G}'_w}$ *torsor over an* $\mathbb{F}_q$*-scheme* S*. Then the quotient stack* $\left[\mathcal{L}'_+/L^+\widetilde{\mathbb{G}_w}\right]$ *is represented by a scheme* $\mathcal{L}'_+/L^+\widetilde{\mathbb{G}_w}$ *over* S *that is étale locally on* S *isomorphic to* $L^+\widetilde{\mathbb{G}'_w}/L^+\widetilde{\mathbb{G}_w}$*. In the case that* $\mathbb{G}_w$ *is parahoric* $\mathcal{L}'_+/L^+\widetilde{\mathbb{G}_w}$ *is projective.*

Proof: Let $\mathcal{L}' := \mathcal{L}'_+ \times^{L^+\widetilde{\mathbb{G}'_w}} L\widetilde{\mathbf{G}'_w}$ be the associated $L\widetilde{\mathbf{G}'_w}$-torsor of $\mathcal{L}'_+$. By [AH14, Theorem 4.4] the quotient stack $\left[\mathcal{L}'/L^+\widetilde{\mathbb{G}_w}\right]$ is represented by an ind-quasi-projective ind-scheme $\mathcal{L}'/L^+\widetilde{\mathbb{G}_w}$ over S. The closed morphism $\mathcal{L}'_+ \to \mathcal{L}'$ realizes $\left[\mathcal{L}'_+/L^+\widetilde{\mathbb{G}_w}\right]$ as a closed sub-sheaf of $\mathcal{L}'/L^+\widetilde{\mathbb{G}_w}$. Since $\mathcal{L}'_+$ is affine over S the quotient $\left[\mathcal{L}'_+/L^+\widetilde{\mathbb{G}_w}\right]$ is given by a closed subscheme in $\mathcal{L}'/L^+\widetilde{\mathbb{G}_w}$. It is clear that after passing to a covering $S' \to S$ that trivializes $\mathcal{L}'_+$, the scheme $\mathcal{L}'_+/L^+\widetilde{\mathbb{G}_w}$ becomes isomorphic to $L^+\widetilde{\mathbb{G}'_w}/L^+\widetilde{\mathbb{G}_w} \times_{\mathbb{F}_q} S'$. Since $\mathcal{L}'/L^+\widetilde{\mathbb{G}_w}$ is by [AH14, Theorem 4.4] ind-projective if $\mathbb{G}_w$ is parahoric, we see that the last statement about the projectivity of $\mathcal{L}'_+/L^+\widetilde{\mathbb{G}_w}$ follows. □

Now we can prove:

Proposition 3.18. *Let* $f : \mathbb{G} \to \mathbb{G}'$ *be a morphism of smooth affine group schemes over* C*, which is an isomorphism over* $C\backslash\underline{w}$*. Then the morphism*

$$f_\star : \ \mathscr{H}^1(C,\mathbb{G}) \to \mathscr{H}^1(C,\mathbb{G}'), \quad \mathcal{G} \mapsto f_\star\mathcal{G}$$

is schematic and quasi-projective. Étale locally it is relatively representable by the morphism

$$(L^+\widetilde{\mathbb{G}'_{w_1}}/L^+\widetilde{\mathbb{G}_{w_1}})\times_{\mathbb{F}_q}\cdots\times_{\mathbb{F}_q}(L^+\widetilde{\mathbb{G}'_{w_m}}/L^+\widetilde{\mathbb{G}_{w_m}})\longrightarrow\mathbb{F}_q.$$

That means that for any $\mathbb{F}_q$-morphism $S\to\mathscr{H}^1(C,\mathbb{G}')$ there is an étale covering $S'\to S$ such that the fiber product $S'\times_{\mathscr{H}^1(C,\mathbb{G}')}\mathscr{H}^1(C,\mathbb{G})$ is given by $S'\times_{\mathbb{F}_q}(\prod_{w\in\underline{w}}L^+\widetilde{\mathbb{G}'_w}/L^+\widetilde{\mathbb{G}_w})$, where the product is taken over $\mathbb{F}_q$. In the case that the fibers $\mathbb{G}_w$ for all $w\in\underline{w}$ are parahoric group schemes this morphism is projective.

Proof: Let $S\to\mathscr{H}^1(C,\mathbb{G}')$ be given by a $\mathbb{G}$-torsor $\mathcal{G}'$ over C_S. Let $g:T\to S$ be an S-scheme. Then a T-valued point of the fiber product $S\times_{\mathscr{H}^1(C,\mathbb{G}')}\mathscr{H}^1(C,\mathbb{G})$ is given by a tuple $(g,\mathcal{G},\alpha)$ where $\mathcal{G}\in\mathscr{H}^1(C,\mathbb{G})$ and $\alpha:f_\star\mathcal{G}\xrightarrow{\sim}g^\star\mathcal{G}'$. Using the theorem of Beauville-Laszlo from § 2.9 we write $\mathcal{G}=(\mathcal{G}|_{U_T},\prod_{w\in\underline{w}}\mathcal{L}_w,\varphi)$ with $U_T:=(C\backslash\underline{w})\times_{\mathbb{F}_q}T$, $\mathcal{L}_w\in\mathscr{H}^1(\mathbb{F}_q,L^+\widetilde{\mathbb{G}_w})(T)$ and $\varphi=(\varphi_w)_{w\in\underline{w}}:\prod_{w\in\underline{w}}L_w(\mathcal{G})\xrightarrow{\sim}\prod_{w\in\underline{w}}L(\mathcal{L}_w)$. In the same way we write $\mathcal{G}'=(\mathcal{G}'|_{U_S},\prod_{w\in\underline{w}}\mathcal{L}'_w,\psi)$. In particular $f_\star\mathcal{G}$ is given by $(f_\star(\mathcal{G}|_{U_T}),f_{w,\star}\mathcal{L}_w,f_\star\varphi)$ and the isomorphism α is determined by $\alpha_U:f_\star\mathcal{G}\to g^\star\mathcal{G}'$ and $\alpha_w:f_{w,\star}\mathcal{L}_w\to L^+_w(g^\star\mathcal{G}')$ satisfying

$$\begin{array}{ccc} L_w(f_\star(\mathcal{G}|_{U_T})) & \xrightarrow{f_\star\varphi} & L(f_{w,\star}(\mathcal{L}_w)) \\ \downarrow{\scriptstyle L_w(\alpha_U)} & & \downarrow{\scriptstyle L(\alpha_w)} \\ L_w(g^\star(\mathcal{G}'|_{U_T})) & \xrightarrow{g^\star\psi} & L(g_w^\star\mathcal{L}'_w) \end{array}.$$

Since $f|_U=id$ we have $f_\star(\mathcal{G}|_{U_T})=\mathcal{G}|_{U_T}$ and the point $(\mathcal{G}|_{U_T},\prod_{w\in\underline{w}}\mathcal{L}_w,\varphi)$ is equivalent to $(g^\star\mathcal{G}'|_{U_T},\prod_{w\in\underline{w}}\mathcal{L}_w,(\varphi_w\circ L_w(\alpha_U^{-1})))$ by the isomorphism $(\alpha_U^{-1},\prod id_{\mathcal{L}_w})$. This shows that the category of tuples $(\mathcal{G},\alpha)$ as above is equivalent to the category of tuples $(\mathcal{L}_w,\alpha_w)_{w\in\underline{\mathfrak{p}}w}$ where $\mathcal{L}_w\in\mathscr{H}^1(C,L^+\widetilde{\mathbb{G}_w})(T)$ and $\alpha_w:f_\star\mathcal{L}_w\xrightarrow{\sim}g^\star\mathcal{L}'_w$. Namely we associate with some arbitrary tuple $(\mathcal{L}_w,\alpha_w)_{w\in D}$ the tuple $((g^\star\mathcal{G}'|_{U_T},\mathcal{L}_w,\varphi),\beta)$ where $\beta|_U=id$ and $\beta_w=\alpha_w$ and φ is uniquely determined by the condition $f_\star\varphi=\psi\circ L(\alpha_w^{-1})$. This is unique because $f\times id_{Q_w}:\mathbf{G}_w\to\mathbf{G}'_w$ is an isomorphism.
Now we note that the isomorphisms $\alpha_w:f_\star\mathcal{L}_w\xrightarrow{\sim}g^\star\mathcal{L}'_w$ are in bijection with the $L^+\widetilde{\mathbb{G}_w}$ equivariant morphisms $\mathcal{L}_w\to g^\star\mathcal{L}'_w$. This shows that the tuples $(\mathcal{L}_w,\alpha_w)$ parametrize exactly the T-valued points of the quotient stack $\left[g^\star\mathcal{L}_w/L^+\widetilde{\mathbb{G}_w}\right]$ over $\mathbb{F}_q$. It follows with the lemma 3.17 that the fiber product $S\times_{\mathscr{H}^1(C,\mathbb{G}')}\mathscr{H}^1(C,\mathbb{G})$ is given by the scheme $g^\star\mathcal{L}_{w_1}/L^+\widetilde{\mathbb{G}_{w_1}}\times\cdots\times g^\star\mathcal{L}_{w_m}/L^+\widetilde{\mathbb{G}_{w_m}}$. In particular the morphism $f_\star$ is representable and the remaining statements follow directly from the previous lemma. □

Now let us turn to the moduli stacks of global $\mathbb{G}$-shtukas. The following results will be again of interest in later chapters. Let us firstly assume that $\underline{w}\subset\underline{v}$ and that all the closed points $\underline{w}$ are $\mathbb{F}_q$-rational. In particular the group homomorphism $f:\mathbb{G}\to\mathbb{G}'$ is an isomorphism outside the fixed characteristic places $v_1,\ldots,v_n$. Then we have the following theorem.

Proposition 3.19. *Let $f:\mathbb{G}\to\mathbb{G}'$ be a morphism of smooth affine group schemes over C, which is an isomorphism over $C\backslash\underline{w}$ with $\underline{w}\subset\underline{v}$ and $w_i\in C(\mathbb{F}_q)$ for all $w_i\in\underline{w}$. Let $H\subset\mathbb{G}(\mathbb{A}^{\underline{v}})=\mathbb{G}'(\mathbb{A}^{\underline{v}})$ be an open compact subgroup, let $\hat{Z}'_{v_i}$ be a bound in $\mathcal{F}l_{\mathbb{G}'_{v_i}}$ for all i and let $\hat{Z}_{v_i}$ be the base change of $\hat{Z}'_{v_i}$ under the map $\mathcal{F}l_{\mathbb{G}_{v_i}}\to\mathcal{F}l_{\mathbb{G}'_{v_i}}$. Then the morphism*

$$f_\star:\ \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})\to\nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}'),\quad(\underline{\mathcal{G}},\gamma H)\mapsto(f_\star\underline{\mathcal{G}},\gamma H)$$

is schematic and quasi-projective. Étale locally it is relatively representable by the morphism

$$(L^+\widetilde{\mathbb{G}'_{w_1}}/L^+\widetilde{\mathbb{G}_{w_1}}) \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} (L^+\widetilde{\mathbb{G}'_{w_m}}/L^+\widetilde{\mathbb{G}_{w_m}}) \longrightarrow \mathbb{F}_q.$$

That means that for any $\mathbb{F}_q$-scheme S there is an étale covering $S' \to S$ such that the fiber product $S' \times_{\nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})} \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C',\mathbb{G}')$ is given by $S' \times_{\mathbb{F}_q} (\prod_{w\in\underline{w}} L^+\mathbb{G}'_w/L^+\mathbb{G}_w)$, where the product is taken over $\mathbb{F}_q$. In particular $f_\star$ is a surjective morphism. In the case that $\mathbb{G}$ is a parahoric Bruhat-Tits group scheme this morphism is projective.

Proof: Since f is an isomorphism outside $\underline{w}$, for two open subgroups $\tilde{H} \subset H \subset \mathbb{G}(\mathbb{A}^{\underline{v}}) = \mathbb{G}'(\mathbb{A}^{\underline{v}})$ the diagram

$$\begin{array}{ccc} \nabla_n^{\hat{Z}_{\underline{v}},\widetilde{H}}\mathscr{H}^1(C,\mathbb{G}) & \longrightarrow & \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) \\ \downarrow f_\star & & \downarrow f_\star \\ \nabla_n^{\hat{Z}'_{\underline{v}},\widetilde{H}}\mathscr{H}^1(C,\mathbb{G}') & \longrightarrow & \nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}') \end{array}$$

is cartesian. In particular we can assume $H \subset \mathbb{G}(\mathbb{O}^{\underline{v}}) = \mathbb{G}'(\mathbb{O}^{\underline{v}})$, because otherwise we can prove the theorem for the compact open subgroup $\widetilde{H} := H \cap \mathbb{G}(\mathbb{O}^{\underline{v}})$. This implies the assertions of the theorem for the group H since the vertical arrows on the left and the right in the previous diagram are relatively represented by the same morphism. Now for each S-valued point $(\underline{\mathcal{G}}, \gamma H)$ in $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$ we find an isomorphic point $(\underline{\mathcal{G}}', \gamma' H)$ with $\gamma' \in Isom^{\otimes}(\omega^{\circ}_{\mathbb{O}^{\underline{v}}}, \check{\mathcal{T}}_{\underline{\mathcal{G}}'})$. This is due to the fact, that we can pull back global $\mathbb{G}$-shtukas along quasi-isogenies of local $\mathbb{G}_v$-shtukas [AH13, Theorem 5.2] and is explained in the proof of [AH13, Theorem 6.4]. We get a morphism $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) \to \mathscr{H}^1(C,\mathbb{G})$ sending $(\underline{\mathcal{G}}, \gamma H) = (\underline{\mathcal{G}}', \gamma' H)$ to $\mathcal{G}'$. This is the morphism $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) \to \nabla_n^{\hat{Z}_{\underline{v}},\mathbb{G}(\mathbb{O}^{\underline{v}})}\mathscr{H}^1(C,\mathbb{G})$ from (4) in § 2.16 composed with the morphism (3) with $D = \varnothing$ in § 2.16 and the natural morphism $\nabla_n\mathscr{H}^1(C,\mathbb{G}) \to \mathscr{H}^1(C,\mathbb{G})$. Now using proposition 3.18 it suffices to prove that $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$ is given by the fiber product

$$\mathcal{M} := \nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}') \times_{\mathscr{H}^1(C,\mathbb{G}')} \mathscr{H}^1(C,\mathbb{G}).$$

There is a natural morphism $p : \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) \to \mathcal{M}$ which sends an S-valued point $(\underline{\mathcal{G}}, \gamma H)$, where we can assume as before $\gamma \in Isom^{\otimes}(\omega^{\circ}_{\mathbb{O}^{\underline{v}}}, \check{\mathcal{T}}_{\underline{\mathcal{G}}})$, to $((f_\star\underline{\mathcal{G}}, \gamma H), \mathcal{G}, id_{f_\star\mathcal{G}})$, which is well defined by 3.16. We need to prove that this morphism induces an equivalence of the fibered categories. First we see that it is fully faithful. Let $(\underline{\mathcal{G}}_1, \gamma_1 H)$ and $(\underline{\mathcal{G}}_2, \gamma_2 H)$ be two S-valued points in $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$, where we assume again $\gamma_1, \gamma_2 \in Isom^{\otimes}(\omega^{\circ}_{\mathbb{O}^{\underline{v}}}, \check{\mathcal{T}}_{\underline{\mathcal{G}}})$. Let $g \in Hom((\underline{\mathcal{G}}_1, \gamma_1 H), (\underline{\mathcal{G}}_2, \gamma_2 H))$. Since $\check{\mathcal{V}}_g \circ \gamma_1 = \gamma_2 \mod H$ we see that $\check{\mathcal{V}}_g = \gamma_2 \circ h \circ \gamma_1^{-1}$ for some $h \in H$, which implies that $\check{\mathcal{V}}_g$ already comes from a tensor isomorphism in $\mathrm{Isom}^{\otimes}(\check{\mathcal{T}}_{\mathcal{G}_1}, \check{\mathcal{T}}_{\mathcal{G}_2})$. By [AH14, Proposition 3.6] it follows that g is not only a quasi-isogeny but also a morphism of the global $\mathbb{G}$-shtukas $\underline{\mathcal{G}}_1 \to \underline{\mathcal{G}}_2$. Therefore $Hom((\underline{\mathcal{G}}_1, \gamma_1 H), (\underline{\mathcal{G}}_2, \gamma_2 H))$ equals the morphisms of $\mathbb{G}$-torsors such that g is a morphism of the global $\mathbb{G}$-shtukas $\underline{\mathcal{G}}_1$ and $\underline{\mathcal{G}}_2$ compatible with the level structure. Since $f_\star$ is an isomorphism outside of $\underline{v}$ the latter condition is equivalent to the statement that $f_\star g$ is a morphism of $\mathbb{G}'$-shtukas compatible with the level structure. But this says exactly that

$$Hom((\underline{\mathcal{G}}_1, \gamma_1 H), (\underline{\mathcal{G}}_2, \gamma_2 H)) = Hom_{\mathcal{M}}(((f_\star\underline{\mathcal{G}}_1, \gamma_1 H), \mathcal{G}_1, id_{f_\star\underline{\mathcal{G}}_1}), ((f_\star\underline{\mathcal{G}}_2, \gamma_2 H), \mathcal{G}_2, id_{f_\star\underline{\mathcal{G}}_2})).$$

For the essential surjectivity let $((E,s_1,\dots,s_n,\tau_E,\gamma_E H),\mathcal{G},\psi)$ be an S-valued point in $\mathcal{M}$, with $\gamma_E \in Isom^{\otimes}(\omega^{\circ}_{\mathbb{O}^{\underline{v}}},\check{\mathcal{T}}_{\underline{E}})$ as before. This is isomorphic to

$$((f_\star\mathcal{G},s_1,\dots,s_n,\sigma^\star\psi\circ\tau_E\circ\psi^{-1},\check{\mathcal{T}}_\psi\circ\gamma_E H),\mathcal{G},id_{f_\star\mathcal{G}})$$

by $(\psi^{-1},id_{\mathcal{G}})$. We need to show that it comes from an element

$$(\underline{\mathcal{G}},\gamma_{\mathcal{G}}H)=(\mathcal{G},s_1,\dots,s_n,\tau_{\mathcal{G}},\gamma_{\mathcal{G}}H)\in\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}).$$

Here $s_1,\dots,s_n$ and $\mathcal{G}$ are already uniquely defined. Therefore we need to define the isomorphism $\tau_{\mathcal{G}}:\sigma^\star\mathcal{G}|_{C_S\backslash\underset{i}{\cup}\Gamma_{s_i}}\to\mathcal{G}|_{C_S\backslash\underset{i}{\cup}\Gamma_{s_i}}$. Since all the closed points $\underline{w}$ are $\mathbb{F}_q$-rational $(C\backslash\underline{w})_S$ is contained in $C_S\backslash\underset{i}{\cup}\Gamma_{s_i}$. Note that this is not the case if w_i splits, because in this case $w_i\times_{\mathbb{F}_q}S$ has $deg\ w_i$ components isomorphic to S and Γ_{s_i} surjects only to one of these. By assumption $f\times id_S:\mathbb{G}_S\to\mathbb{G}'_S$ is an isomorphism over $(C\backslash\underline{w})_S$. This together with the fact that $\tau_{\mathcal{G}}$ has to satisfy $f_\star\tau_{\mathcal{G}}=\sigma^\star\psi\circ\tau_E\circ\psi^{-1}$ an the inclusion $(C\backslash\underline{w})_S\subset C_S\backslash\underset{i}{\cup}\Gamma_{s_i}$ defines therefore a unique $\tau_{\mathcal{G}}:\sigma^\star\mathcal{G}|_{C_S\backslash\underset{i}{\cup}\Gamma_{s_i}}\to\mathcal{G}|_{C_S\backslash\underset{i}{\cup}\Gamma_{s_i}}$. Now $\underline{\mathcal{G}}$ is a global $\mathbb{G}$-shtuka with H-level structure $\gamma_{\mathcal{G}}:=\check{\mathcal{T}}_\psi\circ\gamma_E$ that is mapped to $((\underline{E},\gamma_E H),\mathcal{G},\psi)\in\mathcal{M}(S)$. It just remains to prove that $\underline{\mathcal{G}}$ is bounded by $\hat{Z}_{\underline{v}}$ to see that $(\underline{\mathcal{G}},\gamma_{\mathcal{G}})$ lies indeed in $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$. Let R be an extension of A_{v_i} with representatives $\hat{Z}_{v_i,R}$ and $\hat{Z}'_{v_i,R}$ of the bounds $\hat{Z}_{v_i}$ and $\hat{Z}'_{v_i}$. We choose an étale covering S' of S and trivializations $\alpha:L^+\mathbb{G}_{v_i,S'}\to\Gamma_{v_i}(\mathcal{G}_{S'})$ and $\alpha':L^+\mathbb{G}_{v_i,S'}\to\Gamma_{v_i}(\sigma^\star\mathcal{G}_{S'})$. Then $\alpha^{-1}\circ\tau_{\mathcal{G}}\circ\alpha'=(f_v)_\star(\alpha^{-1}\circ\tau_{\mathcal{G}}\circ\alpha'):L\mathbf{G}_{v_i,S'}\to L\mathbf{G}_{v_i,S'}$ defines an S'-valued point of $L\mathbf{G}_{v_i}$ and hence an induced morphism $S'\to\hat{\mathcal{F}}l_{\mathbb{G}_{v_i},R}\to\hat{\mathcal{F}}l_{\mathbb{G}'_{v_i},R}$. By assumption $\underline{E}$ and hence $f_\star\underline{\mathcal{G}}$ is bounded by $\hat{Z}'_{\underline{v}}$. This means that this morphism factors through $\hat{Z}'_{v_i,R}$ and since $\hat{Z}_{v_i}$ arises from base change it factors by the universal property of the fiber product also through $\hat{Z}_{v_i,R}$. This shows that $\underline{\mathcal{G}}$ is bounded by $\hat{Z}_{v_i}$ for all $v_i\in\underline{v}$. □

If $(id_C,f):(C,\mathbb{G},\underline{v},\hat{Z}_{\underline{v}},H)\to(C,\mathbb{G}',\underline{v},\hat{Z}'_{\underline{v}},H)$ is a morphism of shtuka data, where $\hat{Z}_{\underline{v}}$ does not arise as a base change of $\hat{Z}'_{\underline{v}}$ or if $f:\mathbb{G}\to\mathbb{G}'$ is an isomorphism outside $\underline{w}$ without any conditions relating $\underline{w}$ to the characteristic points $\underline{v}$ or their residue field, the morphism $f_\star:\ \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})\to\nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}'),\quad(\underline{\mathcal{G}},\gamma H)\mapsto(f_\star\underline{\mathcal{G}},\gamma H)$ is still representable, but in general not surjective anymore. More precisely, we have the following theorem.

Theorem 3.20. *Let $(id_C,f):(C,\mathbb{G},\underline{v},\hat{Z}_{\underline{v}},H)\to(C,\mathbb{G}',\underline{v},\hat{Z}'_{\underline{v}},H)$ be a morphism of shtuka data, where $f:\mathbb{G}\to\mathbb{G}'$ is an isomorphism over $C\backslash\underline{w}$. Then the morphism*

$$f_\star:\ \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})\to\nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}'),\quad(\underline{\mathcal{G}},\gamma H)\mapsto(f_\star\underline{\mathcal{G}},\gamma H)$$

is schematic and quasi-projective. In the case that $\mathbb{G}$ is a parahoric Bruhat-Tits group scheme this morphism is projective. For any morphism $(\underline{\mathcal{G}}',\gamma' H):S\to\nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}')$

the fiber product $S\times_{\nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}')}\nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$ *is given by a closed subscheme of*

$$S\times_{\mathbb{F}_q}\left((L^+_{w_1}(\mathcal{G}')/L^+\widetilde{\mathbb{G}_{w_1}})\times_{\mathbb{F}_q}\cdots\times_{\mathbb{F}_q}(L^+_{w_m}(\mathcal{G}')/L^+\widetilde{\mathbb{G}_{w_m}})\right).$$

If $\hat{Z}_{\underline{v}}$ arises as a base change of $\hat{Z}'_{\underline{v}}$ for all $v\in\underline{v}$, the morphism $f_\star$ is surjective.

Proof: In the case that $\hat{Z}_{v_i}$ does not arise by base change from $\hat{Z}_{v_i}$ the immersion $\hat{Z}_{v_i} \to \hat{\mathcal{F}l}_{\mathbb{G}_{v_i}}$ factors through the base change $\hat{Z}''_{v_i} := \hat{Z}'_{v_i} \times_{\hat{\mathcal{F}l}_{\mathbb{G}'_{v_i}}} \hat{\mathcal{F}l}_{\mathbb{G}_{v_i}}$. Since $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) \to \nabla_n^{\hat{Z}''_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$ is a closed substack we may therefore assume from the beginning that $\hat{Z}_{v_i}$ arises by base change from $\hat{Z}'_{v_i}$ for all $v_i \in \underline{v}$. Furthermore we can as in the previous theorem assume that $H \subset \mathbb{G}(\mathbb{O}^{\underline{v}})$. Let $S \to \nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}')$ be an S-valued point given by $(\underline{\mathcal{G}}',\gamma' H) = (\mathcal{G}', s'_1,\dots,s'_n,\tau_{\mathcal{G}'},\gamma' H)$, where we can assume as before that $\gamma' \in Isom^{\otimes}(\omega^{\circ}_{\mathbb{O}^{\underline{v}}}, \check{\mathcal{T}}_{\underline{\mathcal{G}}'})$. There is a natural morphism

$$S \times_{\nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}')} \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) \to S \times_{\mathscr{H}^1(C,\mathbb{G}')} \mathscr{H}^1(C,\mathbb{G})$$

sending an T-valued point $(g,\underline{\mathcal{G}},\gamma H,\psi)$ to $(g,\mathcal{G},\psi)$, where $g : T \to S$ is a morphism of schemes, $(\underline{\mathcal{G}},\gamma H)$ is a T-valued point in $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$ and $\psi : f_\star(\underline{\mathcal{G}},\gamma H) \xrightarrow{\sim} g^\star(\underline{\mathcal{G}}',\gamma' H)$ is an isomorphism of global $\mathbb{G}'$-shtukas. By proposition 3.18 it is now enough to show that this is a closed immersion.

Given a T-valued point $(g,\mathcal{G},\psi)$ in $S \times_{\mathscr{H}^1(C,\mathbb{G}')} \mathscr{H}^1(C,\mathbb{G})$, there can be at most one T-valued point $(g,(\underline{\mathcal{G}},\gamma H),\psi)$ in $S \times_{\nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}')} \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$ with $\underline{\mathcal{G}} = (\mathcal{G}, s_1,\dots,s_n,\tau_{\mathcal{G}})$ and $\gamma \in Isom^{\otimes}(\omega^{\circ}_{\mathbb{O}^{\underline{v}}}, \check{\mathcal{T}}_{\underline{\mathcal{G}}})$ mapping to $(g,(\underline{\mathcal{G}},\gamma H),\psi)$. This is because $\psi : f_\star(\underline{\mathcal{G}},\gamma H) \xrightarrow{\sim} g^\star(\underline{\mathcal{G}}',\gamma' H)$ is an isomorphism of global $\mathbb{G}$-shtukas. That means namely that $s_1,\dots,s_n$ are determined by $s'_1\circ g,\dots,s'_n\circ g$, that γH equals $\check{\mathcal{T}}_{\psi^{-1}}\circ g^\star\gamma' H$ and that there is at most one $\tau_{\mathcal{G}}$ since over the open subset $X := (C\backslash\underline{w})_S \bigcap (C_S\backslash\bigcup_i \Gamma_{s_i}) \subset C_S$ the isomorphism $\tau_{\mathcal{G}}$ is determined by $f_\star\tau_{\mathcal{G}} = \sigma^\star\psi\circ g^\star\tau_{\mathcal{G}'}\circ\psi$. Therefore we have to answer the question if the morphism $\tau_{\mathcal{G}}\big|_X : \sigma^\star\mathcal{G}\big|_X \to \mathcal{G}\big|_X$ can be extended to $C_S\backslash\bigcup_i \Gamma_{s_i}$. Note that if this is possible, then the global $\mathbb{G}$-shtuka $\underline{\mathcal{G}}$ is automatically bounded by $\hat{Z}_{\underline{v}}$ as we have seen at the end of the proof of the previous proposition 3.19.

Let $\mathbb{F}$ be the compositum of all $\mathbb{F}_{v_i}$ with $v_i \in \underline{v}$ and let $v_i^{(0)} \in C_{\mathbb{F}}$ be the closed point lying over v_i that equals the image of the characteristic morphism s_i. Then the definition $\widetilde{C_{\mathbb{F}}} := C_{\mathbb{F}}\backslash\big(\bigcup_i v_i^{(0)}\big)$ satisfies $\widetilde{C_{\mathbb{F}}} \times_{\mathbb{F}} S = C_S\backslash\left(\bigcup_i \Gamma_{s_i}\right)$ Let further

$$I = \left\{ w \in C_{\mathbb{F}} \;\middle|\; w|w_j \text{ for some } w_j \in \underline{w},\ w \neq v_i^{(0)} \text{ for all } v_i \in \underline{v} \right\}. \tag{8}$$

In other words that means that I is determined by $\bigcup_i \Gamma_{s_i} \subset (C_{\mathbb{F}}\backslash I) \times_{\mathbb{F}} S =: C_S^I$ and $((C_{\mathbb{F}}\backslash I) \times_{\mathbb{F}} S)\backslash\left(\bigcup_{v_i\in\underline{w}\cap\underline{v}} \Gamma_{s_i}\right) = (C\backslash\underline{w}) \times_{F_q} S$. The definition satisfies also the equation $(\widetilde{C_{\mathbb{F}}}\backslash I) \times_{\mathbb{F}} S = U$. Then by the theorem of Beauville-Laszlo from § 2.9 we have the following cartesian diagram

$$\begin{array}{ccc}
\mathscr{H}^1(\widetilde{C_{\mathbb{F}}},\mathbb{G}_{\mathbb{F}}) & \longrightarrow & \mathscr{H}^1_e(\widetilde{C_{\mathbb{F}}}\backslash I,\mathbb{G}_{\mathbb{F}}) \\
{\scriptstyle \prod_{w\in I} L_w^+}\big\downarrow & & \big\downarrow{\scriptstyle \prod_{w\in I} L_w} \\
\prod_{w\in I}\mathscr{H}^1(\mathbb{F},L^+\widetilde{\mathbb{G}_w}) & \longrightarrow & \prod_{w\in I}\mathscr{H}^1(\mathbb{F},L\widetilde{\mathbf{G}_w})
\end{array}$$

which means that $\sigma^\star\mathcal{G}|_{C_S\backslash\bigcup_i\Gamma_{s_i}}$ and $\mathcal{G}|_{C_S\backslash\bigcup_i\Gamma_{s_i}}$ are given by tuples $(\sigma^\star\mathcal{G}|_U, \prod_{w\in I} L_w^+(\sigma^\star\mathcal{G}), \prod_{w\in I} id_{L_w(\sigma^\star\mathcal{G})})$ and $(\mathcal{G}|_U, \prod_{v\in I} L_v^+(\mathcal{G}), \prod_{v\in I} id_{L_v(\mathcal{G})})$. The morphism $\tau_{\mathcal{G}}\big|_U : \sigma^\star\mathcal{G}|_U \to \mathcal{G}|_U$ determines isomorphisms $L_w(\tau_{\mathcal{G}}) : L_w(\sigma^\star\mathcal{G}) \to L_w(\mathcal{G})$ for all $w \in I$. The question if $\tau_{\mathcal{G}}$ can be extended to $C_S\backslash\bigcup_i \Gamma_{s_i}$ is then equivalent to the question if all isomorphisms $L_w(\tau_{\mathcal{G}})$ in $\mathscr{H}^1(\mathbb{F},\widetilde{L\mathbb{G}_w})$ already come from an isomorphism $L_w^+(\sigma^\star\mathcal{G})\to L_w^+(\mathcal{G})$ in $\mathscr{H}^1(\mathbb{F},L^+\widetilde{\mathbb{G}_w})$. Since $L^+\widetilde{\mathbb{G}_w}\subset\widetilde{L\mathbb{G}_w}$ is a quasi-compact closed subscheme, this is a closed condition on T which shows that

$$S\times_{\nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}')}\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})\to S\times_{\mathscr{H}^1(C,\mathbb{G}')}\mathscr{H}^1(C,\mathbb{G})$$

is a closed immersion. It rests to show that under our assumption on $\hat{Z}_{\underline{v}}$ the morphism $f_\star$ is surjective. This is not clear yet, since the closed subscheme

$$S\times_{\nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}')}\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})\hookrightarrow S\times_{\mathbb{F}_q}\Big((L_{w_1}^+(\mathcal{G}')/L^+\widetilde{\mathbb{G}_{w_1}})\times_{\mathbb{F}_q}\cdots\times_{\mathbb{F}_q}(L_{w_m}^+(\mathcal{G}')/L^+\widetilde{\mathbb{G}_{w_m}})\Big)$$

does not necessarily surject to S. For the proof of the surjectivity we show that for any algebraically closed field K and every global $\mathbb{G}'$-shtuka $\underline{\mathcal{G}'}=(\mathcal{G}',s_1,\dots,s_n,\tau_{\mathcal{G}'})$ in $\nabla_n^{\hat{Z}'_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G}')(K)$, there is a global $\mathbb{G}$-shtuka $\underline{\mathcal{G}}=(\mathcal{G},s_1,\dots,s_n,\tau_{\mathcal{G}})$ in $\nabla_n^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G})(K)$ with $f_\star\underline{\mathcal{G}}=\underline{\mathcal{G}'}$. By proposition 3.18 and the fact that K is algebraically closed the choice of a $\mathbb{G}$-torsor $\mathcal{G}$ over C_K with $f_\star\mathcal{G}=\mathcal{G}'$ corresponds to an element in $\prod_{w\in\underline{w}}(L^+\widetilde{\mathbb{G}'_w}/L^+\widetilde{\mathbb{G}_w})(K)$. Now let $\mathbb{F}'$ be the compositum of the fields $\mathbb{F}_w$ for all $w\in\underline{w}$. For a closed point $w\in\underline{w}\subset C$ there are exactly $deg\ w$ different closed points in $C_{\mathbb{F}}$ lying above w. We denote them by $w^{(0)},\dots,w^{(deg\ w-1)}$, where $w^{(0)}$ is a randomly chosen one and the others arise by applying successively σ on the residue field. If $w\in\underline{v}$ we choose $w^{(0)}$ as before to be the image of the characteristic morphism s_i. Now once again Beauville and Laszlo help us with the diagram

$$\begin{array}{ccc}\mathscr{H}^1(C_{\mathbb{F}'},\mathbb{G}'_{\mathbb{F}'}) & \longrightarrow & \mathscr{H}^1_e(V,\mathbb{G}'_{\mathbb{F}'})\\ \Big\downarrow{\scriptstyle\prod_{v\in J}L_v^+} & & \Big\downarrow{\scriptstyle\prod_{v\in J}L_v}\\ \prod_{v\in J}\mathscr{H}^1(\mathbb{F}',L^+\mathbb{G}'_v) & \longrightarrow & \prod_{v\in J}\mathscr{H}^1(\mathbb{F}',\mathbf{G}'_v)\end{array},$$

where $J=\{v\in C_{\mathbb{F}'}\mid v|w\ \text{ for some } w\in\underline{w}\}$ and $V:=C_{\mathbb{F}'}\backslash J$. It allows us to identify $\mathcal{G}'$ with the tuple $(\mathcal{G}'|_{V_K},\prod_{w\in\underline{w}}\prod_{i=1}^{deg\ w}L^+\mathbb{G}'_{w^{(i)}},(\epsilon_w^{(i)})_{w^{(i)}\in J})$ where $\epsilon_w^{(i)}:L_{w^{(i)}}(\mathcal{G}')\xrightarrow{\sim}L\mathbf{G}'_{w^{(i)}}$ already comes from an isomorphism of $L^+\mathbb{G}'_{w^{(i)}}$-torsors. Consequently $\sigma^\star\mathcal{G}'$ is identified with the tuple $(\sigma^\star\mathcal{G}'|_{V_K},\prod_{w\in\underline{w}}\prod_{i=1}^{deg\ w}L^+\mathbb{G}'_{w^{(i)}},(\sigma^\star\epsilon_w^{(i-1)})_{w^{(i)}\in J})$ with $\sigma^\star\epsilon_w^{(i-1)}:L_{w^{(i)}}(\sigma^\star\mathcal{G}')\xrightarrow{\sim}L\mathbf{G}'_{w^{(i)}}$ coming again from an isomorphism of $L^+\mathbb{G}'_{w^{(i)}}$-torsors. Note that the index i is computed in $\mathbb{Z}/deg\ w$ so that $-1=deg\ w-1$. We use again the intuitive notation $\tau'_{w^{(i)}}:=L_{w^{(i)}}(\tau_{\mathcal{G}'}):L_{w^{(i)}}(\sigma^\star\mathcal{G}')\to L_{w^{(i)}}(\mathcal{G}')$ and define for all $w^{(i)}\in J$ the element $c_w^{(i)}:=\epsilon_w^{(i)}\circ\tau'_{w^{(i)}}\circ\sigma^\star(\epsilon_w^{(i-1)})^{-1}$ in $L\mathbf{G}_{w^{(i)}}(K)$. The fact that $\tau_{\mathcal{G}'}$ is an isomorphism over $C_K\backslash\bigcup_{k=1}^n\Gamma_{s_k}$ implies that $c_w^{(i)}$ is an element in $L^+\mathbb{G}'_{w^{(i)}}(K)$ for all $w^{(i)}\in J_0:=J\backslash(\bigcup_{w\in\underline{w}\cap\underline{v}}w^{(0)})$. Equivalently we have $c_w^{(i)}\in L^+\mathbb{G}'_{w^{(i)}}(K)$ for all $w\in\underline{w}\cap\underline{v}$ and $i=1,\dots,deg\ w-1$ as well as for all $w\in\underline{w}\backslash(\underline{w}\cap\underline{v})$ and all $i=0,\dots,deg\ w-1$. We will now define the tuple $(b_w^{(i)})_{w^{(i)}\in J}\in\prod_{w^{(i)}\in J}L^+\mathbb{G}'_{w^{(i)}}/L^+\mathbb{G}_{w^{(i)}}(K)$ that will determine by proposition 3.18 the $\mathbb{G}$-torsor $\mathcal{G}$ over C_K mapping under $f_\star:\mathscr{H}^1(C,\mathbb{G})\to\mathscr{H}^1(C,\mathbb{G}')$ to $\mathcal{G}'$. If $w\in\underline{w}\cap\underline{v}$ we define

$$b_w^{(0)}:=1\text{ and }b_w^{(i)}:=\sigma^\star b_w^{(i-1)}\cdot(c_w^{(i)})^{-1}\in L^+\mathbb{G}'_{w^{(i)}}(K)\text{ for all }i=1,\dots,deg\ w-1$$

Now if $w \notin \underline{v}$ we can choose by [AH14, Corollary 2.9] an element $d_w^{(0)} \in L^+\mathbb{G}'_{w(0)}(K)$ with $d_w^{(0)} \cdot c_w^{(0)} \cdot \sigma^{deg\ w\star}(d_w^{(0)})^{-1} = 1$. Additionally we define $d_w^{(i)} := \sigma^\star d_w^{(i-1)} \cdot (c_w^{(i)})^{-1} \in L^+\mathbb{G}'_{w(i)}(K)$ for all $i = 1,\dots,deg\ w-1$. In particular $\widetilde{c}_w^{(0)} := \sigma^{(deg\ w-1)\star}(c_w^{(1)})^{-1} \circ \sigma^{(deg\ w-2)\star}(c_w^{(2)})^{-1} \circ \dots \circ \sigma^\star(c_w^{(deg\ w-1)})^{-1}$ satisfies the equation $\sigma^\star(d_w^{(deg\ w-1)})^{-1} = \sigma^{(deg\ w)\star}(d_w^{(0)})^{-1} \cdot \widetilde{c}_w^{(0)}$. Moreover we choose again by [AH14, Corollary 2.9] an element $\widetilde{d}_w^{(0)}$ in $L^+\mathbb{G}'_{w(0)}(K)$ with $\widetilde{d}_w^{(0)} \circ \widetilde{c}_w^{(0)} \circ \sigma^{deg\ w\star}(\widetilde{d}_w^{(0)})^{-1} = 1$ and use it to define

$$b_w^{(i)} := \sigma^{i\star}\widetilde{d}_w^{(0)} \cdot d_w^{(i)} \in L^+\mathbb{G}'_{w(i)}(K) \text{ for all } i = 0,\dots,deg\ w-1\ .$$

This choice results nicely into the equations

$$b_w^{(i)} \cdot c_w^{(i)} \cdot \sigma^\star(b_w^{(i-1)})^{-1} = \sigma^{i\star}\widetilde{d}_w^{(0)} \cdot \underbrace{d_w^{(i)} \cdot c_w^{(i)} \cdot \sigma^\star(d_w^{(i-1)})^{-1}}_{=1} \cdot \sigma^{i\star}(\widetilde{d}_w^{(0)})^{-1} = 1$$

for all $i = 1,\dots,deg\ w-1$ as well as

$$\begin{aligned} b_w^{(0)} \cdot c_w^{(0)} \cdot \sigma^\star(b_w^{(deg\ w-1)})^{-1} &= \widetilde{d}_w^{(0)} \cdot d_w^{(0)} \cdot c_w^{(0)} \cdot \underbrace{\sigma^\star(d_w^{(deg\ w-1)})^{-1}}_{=\sigma^{(deg\ w)\star}(d_w^{(0)})^{-1}\cdot\widetilde{c}_w^{(0)}} \cdot \sigma^{deg\ w\star}(\widetilde{d}_w^{(0)})^{-1} \\ &= \widetilde{d}_w^{(0)} \cdot \widetilde{c}_w^{(0)} \cdot \sigma^{deg\ w\star}(\widetilde{d}_w^{(0)})^{-1} = 1\ . \end{aligned}$$

Now the $\mathbb{G}$-torsor over C_K, determined by the choice of $(b_w^{(i)})_{w(i)\in J} \in \prod_{w\in\underline{w}} \prod_{i=0}^{deg\ w-1} L^+\mathbb{G}'_{w(i)}(K)$ and lying in the pre-image of $\mathcal{G}'$ under $f_\star : \mathscr{H}^1(C,\mathbb{G}) \to \mathscr{H}^1(C,\mathbb{G}')$, is given, as described in proposition 3.18, by

$$\mathcal{G} = \Big(\mathcal{G}'|_U, \prod_{w\in\underline{w}} \prod_{i=0}^{deg\ w-1} L^+\mathbb{G}_{w(i)}, (b_w^{(i)} \circ \epsilon_w^{(i)})_{w(i)\in J}\Big).$$

It lies indeed in the pre-image of $\mathcal{G}'$ since $f_\star\mathcal{G} = \Big(\mathcal{G}'|_U, \prod_{w\in\underline{w}} \prod_{i=0}^{deg\ w-1} L^+\mathbb{G}'_{w(i)}, (b_w^{(i)} \circ \epsilon_w^{(i)})_{w(i)\in J}\Big)$ is isomorphic to $\mathcal{G}'$ by $\Big(id_{\mathcal{G}'}\big|_U, \big((b_w^{(i)})^{-1}\big)_{w(i)\in J}\Big)$. Now we show that there is $\tau_{\mathcal{G}} : \sigma^\star\mathcal{G}|_{C_K\setminus \underset{k}{\cup}\Gamma_{s_k}} \to \mathcal{G}|_{C_K\setminus \underset{k}{\cup}\Gamma_{s_k}}$ with $f_\star\tau_{\mathcal{G}} = \tau_{\mathcal{G}'}$. We set $\tau_{\mathcal{G}}\big|_U := \tau_{\mathcal{G}'}$ and need to convince ourself that it extends to $C_K\setminus \underset{k}{\cup}\Gamma_{s_k}$. This is the case if and only if for all $w^{(i)} \in J$ the vertical right hand side morphism $b_w^{(i)} \circ \epsilon_w^{(i)} \circ L_{w(i)}(\tau_{\mathcal{G}}) \circ \sigma^\star(\epsilon_w^{(i-1)})^{-1} \circ \sigma^\star(b_w^{(i-1)})^{-1} \in L\mathbf{G}_{w(i)}(K)$ in the diagram

$$\begin{array}{ccccc} L_{w(i)}(\sigma^\star\mathcal{G}) & \xrightarrow{\sigma^\star\epsilon_w^{(i-1)}} & L\mathbf{G}_{w(i)} & \xrightarrow{\sigma^\star b_w^{(i-1)}} & L\mathbf{G}_{w(i)} \\ \Big\downarrow{\scriptstyle \tau_{w(i)}:=L_{w(i)}(\tau_{\mathcal{G}})} & & \Big\downarrow{\scriptstyle c_w^{(i)}} & & \Big\downarrow \\ L_{w(i)}(\mathcal{G}) & \xrightarrow[\epsilon_w^{(i)}]{} & L\mathbf{G}_{w(i)} & \xrightarrow[b_w^{(i)}]{} & L\mathbf{G}_{w(i)} \end{array}$$

is given by an element in $L^+\mathbb{G}_{w(i)}(K)$. By construction we have $b_w^{(i)} \circ c_w^{(i)} \circ \sigma^\star b_w^{(i-1)} = 1$ for all $w^{(i)} \in J$. This proves $f_\star\underline{\mathcal{G}} = \underline{\mathcal{G}}'$ and finally the theorem. □

Closed Subgroups of $\mathbb{G}'$

Secondly we take a closer look to the case that $f:\mathbb{G}\to\mathbb{G}'$ is a closed immersion of group schemes over C. We start with the following lemma that we mainly need for theorem 3.23.

Lemma 3.21. *Let $f:\mathbb{G}\to\mathbb{G}'$ be a closed immersion of smooth affine group schemes over C. Then the diagonal morphism $\Delta:\mathscr{H}^1(C,\mathbb{G})\to\mathscr{H}^1(C,\mathbb{G})\times_{\mathscr{H}^1(C,\mathbb{G}')}\mathscr{H}^1(C,\mathbb{G})$ of the induced morphism $f_\star:\mathscr{H}^1(C,\mathbb{G})\to\mathscr{H}^1(C,\mathbb{G}')$ is a monomorphism. The same is true for the diagonal morphism $\Delta:\nabla_n\mathscr{H}^1(C,\mathbb{G})\to\nabla_n\mathscr{H}^1(C,\mathbb{G})\times_{\nabla_n\mathscr{H}^1(C,\mathbb{G}')}\nabla_n\mathscr{H}^1(C,\mathbb{G})$ of the induced morphism $f_\star:\nabla_n\mathscr{H}^1(C,\mathbb{G})\to\nabla_n\mathscr{H}^1(C,\mathbb{G}')$.*

Proof: For the first diagonal morphism we have to prove that for any $\mathbb{F}_q$-scheme S the functor $\Delta_S:\mathscr{H}^1(C,\mathbb{G})(S)\to\mathscr{H}^1(C,\mathbb{G})\times_{\mathscr{H}^1(C,\mathbb{G}')}\mathscr{H}^1(C,\mathbb{G})(S)$ is fully faithful. Let $\mathcal{G}\in\mathscr{H}^1(C,\mathbb{G})(S)$, then this functor is cleary always faithful since $\varphi\in\operatorname{Aut}(\mathcal{G})$ is send to $(\varphi,\varphi)\in\operatorname{Aut}(\Delta(\mathcal{G}))$, where $\Delta(\mathcal{G})=(\mathcal{G},\mathcal{G},id_{f_\star\mathcal{G}})$. Note that it suffices to consider $\varphi\in\operatorname{Aut}(\mathcal{G})$ since all morphisms in $\mathscr{H}^1(C,\mathbb{G})$ are isomorphisms. To show that Δ_S is full, let $(\varphi,\psi)\in\operatorname{Aut}(\Delta(\mathcal{G}))$ which means by definition that

$$\begin{array}{ccc} f_\star\mathcal{G} & \xrightarrow{f_\star\varphi} & f_\star\mathcal{G} \\ {\scriptstyle id_{f_\star\mathcal{G}}}\downarrow & & \downarrow{\scriptstyle id_{f_\star\mathcal{G}}} \\ f_\star\mathcal{G} & \xrightarrow[f_\star\psi]{} & f_\star\mathcal{G} \end{array}$$

commutes. Therefore we have $f_\star\varphi=f_\star\psi$ and since $f:\mathbb{G}\to\mathbb{G}'$ is a closed immersion this implies $\varphi=\psi$ and hence that Δ_S is full.
More precisely, to see this, one chooses a covering $U\to C_S$ that trivializes $\mathcal{G}$ so that φ and ψ correspond to morphisms $\varphi,\psi:U\to\mathbb{G}$ satisfying the corresponding cocycle condition. The morphisms $f_\star\varphi$ and $f_\star\psi$ correspond to the compositions $U\xrightarrow{\varphi,\psi}\mathbb{G}\xrightarrow{f}\mathbb{G}'$ and the equality $f_\star\varphi=f_\star\psi$ means $f\circ\varphi=f\circ\psi$. Since f is a closed immersion this implies $\varphi=\psi$, which proves that the first diagonal morphism is a monomorphism. The proof for the second diagonal morphism $\Delta:\nabla_n\mathscr{H}^1(C,\mathbb{G})\to\nabla_n\mathscr{H}^1(C,\mathbb{G})\times_{\nabla_n\mathscr{H}^1(C,\mathbb{G}')}\nabla_n\mathscr{H}^1(C,\mathbb{G})$ works literally in the same way. □

Corollary 3.22. *The morphism $f_\star:\mathscr{H}^1(C,\mathbb{G})\to\mathscr{H}^1(C,\mathbb{G}')$ is representable by an algebraic space. In particular for every $\mathbb{F}_q$-morphism $\mathcal{G}':S\to\mathscr{H}^1(C,\mathbb{G}')$ and the natural projection $p_S:C_S\to S$, the Weil restriction $p_{S\star}(\mathcal{G}'/\mathbb{G}_S)$ is an algebraic space, that equals the fiber product $S\times_{\mathscr{H}^1(C,\mathbb{G}')}\mathscr{H}^1(C,\mathbb{G})$.*

Proof: Since the diagonal morphism in lemma 3.21 is a monomorphism it follows by [LMB00, Corollary 8.1.2] that $f_\star:\mathscr{H}^1(C,\mathbb{G})\to\mathscr{H}^1(C,\mathbb{G}')$ is representable by an algebraic space. By definition this means that the fiber product $S\times_{\mathscr{H}^1(C,\mathbb{G}')}\mathscr{H}^1(C,\mathbb{G})$ is an algebraic space and in particular given by a functor $(\mathbf{Sch}/S)^{op}\to\mathbf{Set}$. We show that this functor coincides with the Weil restriction functor $p_{S\star}(\mathcal{G}'/\mathbb{G}_S)$. By definition a T-valued point of this fiber product $S\times_{\mathscr{H}^1(C,\mathbb{G}')}\mathscr{H}^1(C,\mathbb{G})$ is given by a tuple $(g,\mathcal{G},\alpha)$ where $g:T\to S$ is a morphism of schemes, $\mathcal{G}$ is a $\mathbb{G}$-torsor over C_T and α is an isomorphism of $\mathbb{G}'$-torsors $f_\star\mathcal{G}\xrightarrow{\sim}g^\star\mathcal{G}'$. Since isomorphisms $f_\star\mathcal{G}\xrightarrow{\sim}g^\star\mathcal{G}'$ are in bijection with $\mathbb{G}$-equivariant morphisms $\mathcal{G}\to\mathcal{G}'$ the category of the tuples above is equivalent to the set of morphisms from C_T to the quotient $\mathcal{G}'/\mathbb{G}_S$.
Since $Hom_{C_S}(C_T,[\mathcal{G}'/\mathbb{G}_S])=Hom_S(T,p_{S,\star}([\mathcal{G}'/\mathbb{G}_S]))$ by definition of the Weil restriction, the

fiber product $S \times_{\mathscr{H}^1(C,\mathbb{G}')} \mathscr{H}^1(C,\mathbb{G})$ is given by $p_{S\star}(\mathcal{G}'/\mathbb{G}_S)$. □

Theorem 3.23. *Let $f : \mathbb{G} \to \mathbb{G}'$ be a closed immersion of smooth affine group schemes over C. Then the induced morphism $f_\star : \nabla_n\mathscr{H}^1(C,\mathbb{G}) \to \nabla_n\mathscr{H}^1(C,\mathbb{G}')$ is unramified and schematic.*

Proof: We first show that $f_\star$ is unramified and then conclude that it is representable by a scheme. Let B be any ring and $I \subset B$ an ideal with $I^2 = 0$ and $p : Spec\ \overline{B} := Spec\ B/I \to Spec\ B$ the natural projection arising in a diagram of the form

$$\begin{array}{ccc} Spec\ B/I & \xrightarrow{\quad g \quad} & \nabla_n\mathscr{H}^1(C,\mathbb{G}) \\ {\scriptstyle p}\big\downarrow & \overset{g_1}{\underset{g_2}{\nearrow}} & \big\downarrow{\scriptstyle f_\star} \\ Spec\ B & \xrightarrow[\quad g' \quad]{} & \nabla_n\mathscr{H}^1(C,\mathbb{G}') \end{array}\quad .$$

To prove that $f_\star : \nabla_n\mathscr{H}^1(C,\mathbb{G}) \to \nabla_n\mathscr{H}^1(C,\mathbb{G}')$ is unramified, we need to show that for any diagram of this kind there exists at most one dashed arrow making the diagram commutative, that means $g_1 = g_2$. This suffices since $\nabla_n\mathscr{H}^1(C,\mathbb{G})$ and $\nabla_n\mathscr{H}^1(C,\mathbb{G}')$ are locally of ind-finite type over the notherian scheme C^n.

The morphism $g : Spec\ \overline{B} \to \nabla_n\mathscr{H}^1(C,\mathbb{G})$ corresponds to a global $\mathbb{G}$-shtuka $\underline{\overline{\mathcal{G}}} := (\overline{\mathcal{G}}, \overline{s_1}, \dots, \overline{s_n}, \tau_{\overline{\mathcal{G}}})$ over $Spec\ \overline{B}$, where g_1 and g_2 correspond to global $\mathbb{G}$-shtukas $\underline{\mathcal{G}}_1 = (\mathcal{G}_1, s'_1, \dots, s'_n, \tau_{\mathcal{G}_1})$ and $\underline{\mathcal{G}}_2 = (\mathcal{G}_2, s''_1, \dots, s''_n, \tau_{\mathcal{G}_2})$ over $Spec\ B$. The commutativity of the upper triangle means that there are isomorphisms $\beta_1 : p^\star\underline{\mathcal{G}}_1 \xrightarrow{\sim} \underline{\overline{\mathcal{G}}}$ and $\beta_2 : p^\star\underline{\mathcal{G}}_2 \xrightarrow{\sim} \underline{\overline{\mathcal{G}}}$ of global $\mathbb{G}$-shtukas over $Spec\ \overline{B}$. Therefore we have to prove, that the isomorphism $\beta_2^{-1} \circ \beta_1$ arises already from an isomorphism $\underline{\mathcal{G}}_1 \to \underline{\mathcal{G}}_2$ of global $\mathbb{G}$-shtukas over $Spec\ B$. Furthermore we denote by $\underline{\mathcal{G}}' := (\mathcal{G}', s_1, \dots, s_n, \tau_{\mathcal{G}'})$ the global $\mathbb{G}'$-shtuka over $Spec\ B$ corresponding to $g' : Spec\ B \to \nabla_n\mathscr{H}^1(C,\mathbb{G}')$. The commutativity of the lower triangle gives us isomorphisms $\alpha_1 : f_\star\underline{\mathcal{G}}_1 \xrightarrow{\sim} \underline{\mathcal{G}}'$ and $\alpha_2 : f_\star\underline{\mathcal{G}}_2 \xrightarrow{\sim} \underline{\mathcal{G}}'$ of global $\mathbb{G}'$-shtukas over $Spec\ B$ satisfying $\gamma = p^\star\alpha_2 \circ f_\star\beta_2^{-1} = p^\star\alpha_1 \circ f_\star\beta_1^{-1}$ where $\gamma : f_\star\underline{\overline{\mathcal{G}}} \xrightarrow{\sim} p_\star\underline{\mathcal{G}}'$ is the isomorphism of global $\mathbb{G}'$-shtukas over $Spec\ \overline{B}$ coming from the commutativity of the square.

Now these isomorphisms imply directly that the paws s_i, s'_i and s''_i coincide for all i with $1 \leqslant i \leqslant n$. Although $f : \mathbb{G} \to \mathbb{G}'$ is a closed immersion it is by the following remark 3.24 a priori not so clear that the torsors $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic, but we now prove this as follows.

The $\mathbb{G}$-torsors $\mathcal{G}_1$ and $\mathcal{G}_2$ over C_B come with $\mathbb{G}$-equivariant maps to $\mathcal{G}'$ which are induced by $\alpha_1 : f_\star\mathcal{G}_1 \to \mathcal{G}'$ and $\alpha_2 : f_\star\mathcal{G}_2 \to \mathcal{G}'$. Therefore they define two C_B-valued points $h_1, h_2 : C_B \to \mathcal{G}'/\mathbb{G}_B$. In other words one can describe them as follows. Since

$$\begin{array}{ccc} Spec\ B & \xrightarrow{\mathcal{G}_1} & \mathscr{H}^1(C,\mathbb{G}) \\ & {\scriptstyle \mathcal{G}'}\searrow & \big\downarrow{\scriptstyle f_\star} \\ & & \mathscr{H}^1(C,\mathbb{G}') \end{array} \quad \text{and} \quad \begin{array}{ccc} Spec\ B & \xrightarrow{\mathcal{G}_2} & \mathscr{H}^1(C,\mathbb{G}) \\ & {\scriptstyle \mathcal{G}'}\searrow & \big\downarrow{\scriptstyle f_\star} \\ & & \mathscr{H}^1(C,\mathbb{G}') \end{array}$$

commute, that means $f_\star\mathcal{G}_1 \simeq \mathcal{G}' \simeq f_\star\mathcal{G}_2$, the $\mathbb{G}$-torsors $\mathcal{G}_1$ and $\mathcal{G}_2$ induce morphisms h_1, h_2 from $Spec\ B$ to the fiber product $Spec\ B \times_{\mathscr{H}^1(C,\mathbb{G}')} \mathscr{H}^1(C,\mathbb{G})$. In corollary 3.22 this fiber product was seen to be $p_{B\star}(\mathcal{G}'/\mathbb{G}_B)$ and by definition of the Weil restriction we have $Hom_{C_B}(C_B, (\mathcal{G}'/\mathbb{G}_B)) = Hom_B(Spec\ B, p_{B\star}(\mathcal{G}'/\mathbb{G}_B))$, so that h_1 and h_2 correspond consequently to morphisms $C_B \to \mathcal{G}'/\mathbb{G}_B$. First we show that they coincide on $\widetilde{C}_B := C_B \backslash \bigcup_i \Gamma_{s_i}$.

The $\mathbb{F}_q$-Frobenius induces a morphism $j : B/I \to B,\ \bar{b} \mapsto b^q$ which is well defined, because $I^2 = 0$ and in particular $I^q = 0$. We get the following commutative diagram:

$$\begin{array}{ccccc} Spec\ B & \xrightarrow{j} & Spec\ B/I & \xrightarrow{\overline{\mathcal{G}}} & \mathscr{H}^1(C,\mathbb{G}) \\ & \searrow^{\sigma_B} & \downarrow p & \nearrow_{\mathcal{G}_1}\nearrow_{\mathcal{G}_2} & \\ & & Spec\ B & & \end{array}$$

which implies

$$\sigma^\star\mathcal{G}_1 = j^\star p^\star \mathcal{G}_1 \xrightarrow[j^\star\beta_1]{\sim} j^\star\overline{\mathcal{G}} \xrightarrow[j^\star\beta_2^{-1}]{\sim} j^\star p^\star\mathcal{G}_2 = \sigma^\star\mathcal{G}_2.$$

By restricting this isomorphism to $\widetilde{C}_B$ and composing with $\tau_{\mathcal{G}_1}$ and $\tau_{\mathcal{G}_2}$ we get

$$\delta_0 : \mathcal{G}_1|_{\widetilde{C}_B} \xrightarrow[\tau_{\mathcal{G}_1}^{-1}]{\sim} \sigma^\star\mathcal{G}_1|_{\widetilde{C}_B} \xrightarrow[j^\star\beta_1]{\sim} j^\star\overline{\mathcal{G}}|_{\widetilde{C}_B} \xrightarrow[j^\star\beta_2^{-1}]{\sim} \sigma^\star\mathcal{G}_2|_{\widetilde{C}_B} \xrightarrow[\tau_{\mathcal{G}_2}]{\sim} \mathcal{G}_2|_{\widetilde{C}_B}\ ,$$

an isomorphism $\delta_0 := \tau_{\mathcal{G}_2} \circ j^\star\beta_2^{-1} \circ j^\star\beta_1 \circ \tau_{\mathcal{G}_1}^{-1}$ of $\mathbb{G}$-torsors over $\widetilde{C}_B$. It satisfies

$$\begin{aligned} \alpha_2|_{\widetilde{C}_B} \circ f_\star\delta_0 \circ \alpha_1^{-1}|_{\widetilde{C}_B} &= \underbrace{\alpha_2 \circ f_\star\tau_{\mathcal{G}_2}}_{=\tau_{\mathcal{G}'}\circ\sigma^\star\alpha_2} \circ f_\star j^\star\beta_2^{-1} \circ \underbrace{f_\star j^\star\beta_1 \circ \sigma^\star\alpha_1^{-1}}_{=j^\star(f_\star\beta_1\circ p^\star\alpha_1^{-1})} \circ \tau_{\mathcal{G}'}^{-1} \\ &= \tau_{\mathcal{G}'} \circ \sigma^\star\alpha_2 \circ j^\star f_\star\beta_2^{-1} \circ j^\star(f_\star\beta_2 \circ p^\star\alpha_2^{-1}) \circ \tau_{\mathcal{G}'}^{-1} = id_{\mathcal{G}'}|_{\widetilde{C}_B}. \end{aligned}$$

In other words δ_0 is an isomorphism from $(\mathcal{G}_1|_{\widetilde{C}_B}, \alpha_1|_{\widetilde{C}_B})$ to $(\mathcal{G}_2|_{\widetilde{C}_B}, \alpha_2|_{\widetilde{C}_B})$ of $\widetilde{C}_B$-valued points in $\mathcal{G}'/\mathbb{G}_B$. Therefore the restriction of $(h_1,h_2) : C_B \to \mathcal{G}'/\mathbb{G}_B \times_{C_B} \mathcal{G}'/\mathbb{G}_B$ to the open subscheme $\widetilde{C}_B$ in C_B factors through the diagonal in the following diagram

$$\begin{array}{ccc} C_B & \xrightarrow{(h_1,h_2)} & \mathcal{G}'/\mathbb{G}_B \times_{C_B} \mathcal{G}'/\mathbb{G}_B \\ \uparrow & \searrow & \uparrow \Delta \\ \widetilde{C}_B & \longrightarrow & \mathcal{G}'\backslash\mathbb{G}_B \end{array} \quad . \tag{9}$$

To see that $\mathcal{G}_1 \simeq \mathcal{G}_2$ over C_B we have to show that the morphism (h_1,h_2) factors through the diagonal Δ as well. Now since f is a closed immersion, the quotient $\mathcal{G}'/\mathbb{G}_B$ exists as a scheme by [Ana73, Theorem 4.C] and it is smooth and separated by [SGA70, $\mathrm{VI_B}$, Proposition 9.2(xii) and (x)]. In particular the diagonal Δ is a closed immersion. Therefore C_B factors through the diagonal if the scheme theoretic image of $\widetilde{C}_B$ in C_B equals C_B. This was proven in lemma 3.11. As a result of this, we conclude that δ_0 extends to an isomorphism $\delta : \mathcal{G}_1 \xrightarrow{\sim} \mathcal{G}_2$ of $\mathbb{G}$-torsor over C_B. The computation

$$\begin{aligned} \delta_0^{-1} \circ \tau_{\mathcal{G}_2} \circ \sigma^\star\delta_0 &= \tau_{\mathcal{G}_1} \circ j^\star\beta_1^{-1} \circ \underbrace{j^\star\beta_2 \circ \tau_{\mathcal{G}_2}^{-1} \circ \tau_{\mathcal{G}_2} \circ \sigma^\star\tau_{\mathcal{G}_2} \circ \sigma^\star j^\star\beta_2^{-1}}_{=j^\star\tau_{\overline{\mathcal{G}}}} \circ \sigma^\star j^\star\beta_1 \circ \sigma^\star\tau_{\mathcal{G}_1}^{-1} \\ &= \tau_{\mathcal{G}_1} \circ \underbrace{j^\star\beta_1^{-1} \circ j^\star\tau_{\overline{\mathcal{G}}} \circ \sigma^\star j^\star\beta_1}_{=\sigma^\star\tau_{\mathcal{G}_1}} \circ \sigma^\star\tau_{\mathcal{G}_1}^{-1} = \tau_{\mathcal{G}_1} \end{aligned}$$

shows that $\delta : \underline{\mathcal{G}}_1 \xrightarrow{\sim} \underline{\mathcal{G}}_2$ is an isomorphism of $\mathbb{G}$-shtukas over B, which finishes the proof that $f_\star : \nabla_n\mathscr{H}^1(C,\mathbb{G}) \to \nabla_n\mathscr{H}^1(C,\mathbb{G}')$ is an unramified morphism.

It rests to show that this morphism is schematic. We have proven in lemma 3.21 that the diagonal $\Delta_{f_\star}$ of $f_\star$ is a monomorphism, which implies together with [LMB00, Corollary 8.1.2] that the morphism $f_\star$ is representable by an algebraic space. It is clear that $f_\star$ is a separated morphism, since the moduli spaces of global $\mathbb{G}$-shtukas are separated. Furthermore we have proven that $f_\star$ is unramified and in particular locally quasi-finite [Gro67, Corollaire 17.4.3]. All together this allows us to apply [LMB00, Theorem A.2] which states that a separated, locally quasi-finite morphism of algebraic stacks that is representable by an algebraic space is already schematic. This finishes the proof of the theorem. □

Remark 3.24. Note that this is a particular property of the morphism of shtukas. Even if $f : \mathbb{G} \to \mathbb{G}'$ is a closed immersion, it is not true that $\mathscr{H}^1(C, \mathbb{G}) \to \mathscr{H}^1(C, \mathbb{G}')$ is an unramified morphism.

Corollary 3.25. *Let $(id_C, f) : (C, \mathbb{G}, \underline{v}, \hat{Z}_{\underline{v}}, H) \to (C, \mathbb{G}', \underline{v}, \hat{Z}'_{\underline{v}}, H')$ be a morphism of shtuka data, where $f : \mathbb{G} \to \mathbb{G}'$ is a closed immersion of smooth affine group schemes over C. Then the induced morphism*

$$f_\star : \nabla_n^{\hat{Z}_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G}) \to \nabla_n^{\hat{Z}'_{\underline{v}}, H'} \mathscr{H}^1(C, \mathbb{G}')$$

is unramified and schematic.

Proof: We first consider the induced morphism $f_\star : \nabla_n^{\hat{Z}_{\underline{v}}} \mathscr{H}^1(C, \mathbb{G}) \to \nabla_n^{\hat{Z}'_{\underline{v}}} \mathscr{H}^1(C, \mathbb{G}')$ of the moduli spaces of global $\mathbb{G}$-shtukas without level structures. We have the following commutative diagram

$$\begin{array}{ccc} \nabla_n^{\hat{Z}_{\underline{v}}} \mathscr{H}^1(C, \mathbb{G}) & \longrightarrow & \nabla_n \mathscr{H}^1(C, \mathbb{G})^{\underline{v}} \\ \downarrow {\scriptstyle f_\star} & & \downarrow {\scriptstyle f_\star} \\ \nabla_n^{\hat{Z}'_{\underline{v}}} \mathscr{H}^1(C, \mathbb{G}') & \longrightarrow & \nabla_n \mathscr{H}^1(C, \mathbb{G}')^{\underline{v}}. \end{array}$$

The vertical arrow on the right is an unramified morphism by theorem 3.23, where the horizontal arrows are closed immersions and in particular also unramified. As a consequence the vertical arrow on the left is unramified as well. To prove the statement for the morphism $f_\star$ of moduli spaces of global $\mathbb{G}$-shtukas with level H-structure, we choose similar to 3.14 some finite subscheme $D \subset C$ such that $H_D := ker(\mathbb{G}(\mathbb{O}^{\underline{v}}) \to \mathbb{G}(\mathcal{O}_D))$ and $H'_D := ker(\mathbb{G}'(\mathbb{O}^{\underline{v}}) \to \mathbb{G}'(\mathcal{O}_D))$ are subgroups of finite index in H (resp. H'). Then we have by § 2.16 the following commutative diagram

$$\begin{array}{ccccccc} \nabla_n^{\hat{Z}_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G}) & \xleftarrow[\text{étale}]{\text{finite}} & \nabla_n^{\hat{Z}_{\underline{v}}, H_D} \mathscr{H}^1(C, \mathbb{G}) & \xrightarrow{\sim} & \nabla_n^{\hat{Z}_{\underline{v}}} \mathscr{H}_D^1(C, \mathbb{G}) & \xrightarrow[\text{étale}]{\text{finite}} & \nabla_n^{\hat{Z}_{\underline{v}}} \mathscr{H}^1(C, \mathbb{G}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow {\scriptstyle unramified} \\ \nabla_n^{\hat{Z}_{\underline{v}}, H'} \mathscr{H}^1(C, \mathbb{G}') & \xleftarrow[\text{étale}]{\text{finite}} & \nabla_n^{\hat{Z}_{\underline{v}}, H'_D} \mathscr{H}^1(C, \mathbb{G}') & \xrightarrow{\sim} & \nabla_n^{\hat{Z}_{\underline{v}}} \mathscr{H}_D^1(C, \mathbb{G}') & \xrightarrow[\text{étale}]{\text{finite}} & \nabla_n^{\hat{Z}_{\underline{v}}} \mathscr{H}^1(C, \mathbb{G}') \end{array}$$

All the horizontal arrows are étale and in particular unramified. Furthermore we have seen that the vertical arrow on the right is unramified. As a result it follows that the morphism $\nabla_n^{\hat{Z}_{\underline{v}}} \mathscr{H}_D^1(C, \mathbb{G}) \to \nabla_n^{\hat{Z}_{\underline{v}}} \mathscr{H}_D^1(C, \mathbb{G}')$ is unramified [Gro67, Proposition 17.3.3 (v)] and finally that $f_\star : \nabla_n^{\hat{Z}_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G}) \to \nabla_n^{\hat{Z}'_{\underline{v}}, H'} \mathscr{H}^1(C, \mathbb{G}')$ is unramified [Gro67, Proposition 17.7.7]. It is clear that it is also schematic, which proves the corollary. □

Theorem 3.26. *Let $\mathbb{G}$ be a parahoric Bruhat-Tits group scheme and $f:\mathbb{G}\to\mathbb{G}'$ be a closed immersion of smooth affine group schemes and $\underline{v}=(v_1,\dots,v_n)$ be a set of closed points in C. Then the induced morphism*

$$f_\star:\nabla_n\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}\to\nabla_n\mathscr{H}^1(C,\mathbb{G}')^{\underline{v}}\quad \textit{is proper and in particular finite.}$$

Proof: We know by theorem 3.23 that this morphism is unramified and schematic and in particular locally quasi-finite. Moreover the morphism is quasi-compact. Since $\nabla_n\mathscr{H}^1(C,\mathbb{G})\to\mathscr{H}^1(C,\mathbb{G})$ is of ind-finite type, this follows from [AH13, Theorem 2.5] after choosing a representation $\rho:\mathbb{G}'\to\mathrm{GL}(\mathcal{V}_0)$ for some vector bundle $\mathcal{V}_0$ such that the quotient $GL(\mathcal{V}_0)/\mathbb{G}$ is quasi-affine (see [AH13, Proposition 2.2]). Therefore it suffices to prove that $f_\star$ satisfies the valuative criterion for properness to see that this morphism is proper and consequently also finite, due to the quasi-finiteness. Thus let R be a complete discrete valuation ring with uniformizer π such that its residue field $\kappa_R=R/\pi$ is algebraically closed and let $K=Frac(R)$ be the fraction field of R. Let us further denote by K^{alg} an algebraic closure of K and by R^{alg} the integral closure of R in K^{alg}. We need to prove that in every diagram of the form

$$\begin{array}{ccccc} \mathit{Spec}\ K^{alg} & \xrightarrow{i_K} & \mathit{Spec}\ K & \xrightarrow{g_1} & \nabla_n\mathscr{H}^1(C,\mathbb{G})^{\underline{v}} \\ \downarrow{\scriptstyle\widetilde{j}} & & \downarrow{\scriptstyle j} & \nearrow & \downarrow{\scriptstyle f_\star} \\ \mathit{Spec}\ R^{alg} & \xrightarrow[i_R]{} & \mathit{Spec}\ R & \xrightarrow[g_2]{} & \nabla\mathscr{H}^1(C,\mathbb{G}')^{\underline{v}} \end{array} \tag{10}$$

there exists a unique dashed arrow making the diagram commutative.
Here g_1,g_2,i_K,i_R,j and $\widetilde{j}$ are defined by the diagram. Choosing the closed embedding $\rho:\mathbb{G}'\hookrightarrow\mathrm{GL}(\mathcal{V}_0)$ it suffices, due to the separateness of the moduli spaces, to prove the valuative criterion for the composition $\rho_\star\circ f_\star:\nabla_n\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}\to\nabla_n\mathscr{H}^1(C,\mathrm{GL}(\mathcal{V}_0))^{\underline{v}}$. Therefore we may assume that $\mathbb{G}'$ equals $\mathrm{GL}(\mathcal{V}_0)$. We denote by $\underline{\mathcal{G}}=(\mathcal{G},s_1,\dots,s_n,\tau_{\mathcal{G}})$ the global $\mathbb{G}$-shtuka over K corresponding to g_1 and by $\underline{\mathcal{G}}'=(\mathcal{G}',s_1',\dots,s_n',\tau_{\mathcal{G}'})$ the global $\mathbb{G}'$-shtuka over R corresponding to g_2. Furthermore the commutativity of the square gives an isomorphism $\alpha:f_\star\underline{\mathcal{G}}\to j^\star\underline{\mathcal{G}}'$ of global $\mathbb{G}'$-shtukas over K.
Let $S:=\{v\in C\mid\mathbb{G}\times_C\mathbb{F}_v \text{ is not reductive }\}\cup\underline{v}$. Then $\mathbb{G}'/\mathbb{G}\times_C(C\backslash S)$ is by [Alp14, Theorem 9.4.1 and Corollary 9.7.7] an affine scheme over $C\backslash S$ and in particular $\mathcal{G}'/\mathbb{G}_R\times_{C_R}(C\backslash S)_R$ is an affine scheme over $(C\backslash S)_R$. Now the $\mathbb{G}$-torsor $\mathcal{G}|_{(C\backslash S)_R}$ with its $\mathbb{G}$ equivariant morphism $\mathcal{G}\to\mathcal{G}'$ induced by α defines an $(C\backslash S)_K$ valued point of the quotient $\mathcal{G}'/\mathbb{G}_R$.

$$\begin{array}{ccc} (C\backslash S)_K & \xrightarrow{s} & \mathcal{G}'/\mathbb{G}\times_{C_R}(C\backslash S)_R \\ & \searrow & \downarrow \\ & & (C\backslash S)_R \end{array} \tag{11}$$

Now the proof consists of several steps. In a first step we want to show that s factors through $(C\backslash S)_R$ which means that it gives a section $s_R:(C\backslash S)_R\to\mathcal{G}'/\mathbb{G}_R\times_{C_R}(C\backslash S)_R$ of the vertical morphism in diagram (11). This morphism s_R corresponds to a unique $\mathbb{G}$-torsor $\widetilde{\mathcal{E}}$ over $(C\backslash S)_R$ together with an isomorphism $\alpha_R:f_\star\widetilde{\mathcal{E}}\xrightarrow{\sim}\mathcal{G}'|_{(C\backslash S)_R}$ satisfying $j^\star\widetilde{\mathcal{E}}=\mathcal{G}|_{(C\backslash S)_K}$.
In the second step of the proof we then show that the base change of $\widetilde{\mathcal{E}}$ to R^{alg} extends uniquely to a $\mathbb{G}$-torsor over the whole relative curve $C_{R^{alg}}$. More precisely we show that there is a $\mathbb{G}$-torsor

$\mathcal{E}$ over $C_{R^{alg}}$ such that firstly the restriction $\mathcal{E}|_{(C\backslash S)_{R^{alg}}}$ is isomorphic to the $\mathbb{G}$-torsor $i_R^\star\widetilde{\mathcal{E}}$ over $(C\backslash S)_{R^{alg}}$ and secondly $f_\star\mathcal{E} \simeq i_R^\star\mathcal{G}'$ and $\widetilde{j}^\star\mathcal{E} \simeq i_K^\star\mathcal{G}$.
Then we show in the third step that this $\mathbb{G}$-torsor $\mathcal{E}$ over $C_{R^{alg}}$ gives rise to a unique $\mathbb{G}$-shtuka $(\mathcal{E}, r_1, \ldots, r_n, \tau_{\mathcal{E}})$ in $\nabla_n\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}(R^{alg})$ making the diagram (10) commutative. This will then finish the proof.

(Step 1) We can assume that $Spec\ A = C\backslash S$ is affine by enlarging S if necessary. Since we have seen that $\mathcal{G}'/\mathbb{G}_R \times_{C_R} (C\backslash S)_R$ is affine over $(C\backslash S)_R = Spec\ A_R := Spec\ (A \times_{\mathbb{F}_q} R)$ we can set $\mathcal{G}'/\mathbb{G}_R \times_{C_R} Spec\ A_R =: Spec\ B$ for some ring B. Therefore, to prove the assertion of the first step, namely that s in diagram (11) factors through $(C\backslash S)_R$ it is enough to show that the ring morphism $s^\star : B \to A \otimes_{\mathbb{F}_q} K =: A_K$ factors through A_R. We write $L := Frac(A_R)$ for the function field of C_R and $\mathcal{O} := (A_R)_{(\pi)} \subset L$ for the localisation of A_R at the prime ideal $(\pi) := ker(A_R \to A_{\kappa_R})$. The fact that A_R is normal due to the smoothness of C_R over R and the fact that the prime ideal $(\pi) \subset A_R$ corresponding to the generic point of $Spec\ A_{\kappa_R}$ is of height 1 in A_R, implies that $\mathcal{O}$ is a discrete valuation ring with uniformizer π. The normality of A_R allows us also by [Har77, chapter II, 6.3.A] to write $A_R = \bigcap_{\mathfrak{p}\subset A_R\ \mathfrak{p}\ of\ height\ 1} A_{R,\mathfrak{p}}$. For all prime ideals $\mathfrak{p} \subset A_R$ of height 1 we have either $\mathfrak{p} = (\pi)$ or $\pi \notin \mathfrak{p}$. In the second case $\mathfrak{p}$ comes from a closed point in $Spec\ A_K$ which means $A_{R,\mathfrak{p}} = A_{K,\mathfrak{p}}$. Since $A_K = \bigcap_{\mathfrak{q}\subset A_K max.ideal} A_{K,\mathfrak{q}}$ we conclude

$$A_R = \mathcal{O} \cap A_K \subset L.$$

Due to this equation it is enough to show that the composition $s_L : Spec\ L \xrightarrow{\eta} Spec\ A_K \xrightarrow{s|_{A_K}} Spec\ B$ of $s|_{A_K}$ with $\eta : Spec\ L \to Spec\ A_K$ factors through $Spec\ \mathcal{O}$.
The Frobenius pullback $(\sigma^\star\mathcal{G}_L, \sigma^\star\alpha_L)$ with $\sigma^\star\alpha_L : (f_\star\sigma^\star\mathcal{G}) \to \sigma^\star\mathcal{G}'_L$ gives an L-valued point of the quotient $\sigma^\star\mathcal{G}'/\mathbb{G}_R$. As before this quotient is affine over A_R and given exactly by $\sigma^\star\mathcal{G}'/\mathbb{G}_R \times_{C_R} Spec\ A_R = Spec(B \otimes_{A_R,\sigma} A_R) \to Spec\ A_R$, where the A_R-algebra structure of $B \otimes_{A_R,\sigma} A_R$ is given by multiplication in the second component. This means that the L-valued point $(\sigma^\star\mathcal{G}_L, \sigma^\star\alpha_L)$ is given by an A_R-morphism $Spec\ L \to Spec(B \otimes_{A_R,\sigma} A_R)$. In other words we can describe this morphism as follows. The Frobenius $\sigma := id_A \otimes \sigma_R : A_R \to A_R$ induces of course a morphism of the fraction field L which we denote again by $\sigma : L \to L$, $\frac{a}{b} \mapsto \frac{\sigma(a)}{\sigma(b)}$ for $a, b \in A_R$. It is not the absolut $\mathbb{F}_q$-Frobenius. Now the composition $\sigma \circ s_L^\star : B \to L$ is not an A_R-linear morphism, but it induces a unique A_R-linear morphism $\sigma^\star s_L^\star : B \otimes_{A_R,\sigma} A_R \to L$ making the following diagram commutative.

$$\begin{array}{ccccc} b & \qquad & B & \xrightarrow{s_L^\star} & L \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\ b \otimes 1 & & B \otimes_{A_R,\sigma} A_R & \xrightarrow{\sigma^\star s_L^\star} & L \end{array}$$

This morphism $\sigma^\star s_L^\star$ is the one coming from the tuple $(\sigma^\star\mathcal{G}_L, \sigma^\star\alpha_L)$.
The $\mathbb{G}'$-shtuka $\underline{\mathcal{G}'}$ is defined over R. In particular the restriction of $\tau_{\mathcal{G}'}$ to $Spec\ A_R$ is an isomorphism $\sigma^\star\mathcal{G}'|_{A_R} \to \mathcal{G}'|_{A_R}$ that induces an isomorphism $\overline{\tau_{\mathcal{G}'}}$ of A_R-algebras

$$\overline{\tau_{\mathcal{G}'}} : Spec\ (B \otimes_{A_R,\sigma} A_R) \to Spec\ B. \tag{12}$$

It sends a T-valued point $(\mathcal{E}_0,\delta)$ with $\delta : f_\star\mathcal{E}_0 \to \sigma^\star\mathcal{G}'$ to $(\mathcal{E}_0,\tau_{\mathcal{G}'}\circ\delta)$. We then would like to know, that the following diagram

$$\begin{array}{ccc} B & \xrightarrow{\overline{\tau^\star_{\mathcal{G}'}}} & B\otimes_{A_R,\sigma}A_R \\ & {\scriptstyle s^\star_L}\searrow \qquad \swarrow{\scriptstyle \sigma^\star s^\star_L} & \\ & L & \end{array} \tag{13}$$

of A_R-morphisms commutes, which can be seen as follows. By assumption (see diagram (10)) the diagram

$$\begin{array}{ccc} f_\star\sigma^\star\mathcal{G}_L & \xrightarrow{\sigma^\star\alpha_L} & \sigma^\star\mathcal{G}'_L \\ \downarrow{\scriptstyle f_\star\tau_{\mathcal{G}}} & & \downarrow{\scriptstyle \tau_{\mathcal{G}'}} \\ f_\star\mathcal{G}_L & \xrightarrow{\alpha_L} & \mathcal{G}'_L \end{array} \tag{14}$$

is a commutative diagram of isomorphisms of $\mathbb{G}'$-torsors over L. (Actually the whole diagram is already defined over A_K and the vertical arrow on the right is even defined over A_R.) Now $\sigma^\star s_L$ was corresponding to $(\sigma^\star\mathcal{G}_L,\sigma^\star\alpha_L)$, so that by the description of the morphism (12) the composition $\overline{\tau_{\mathcal{G}'}}\circ\sigma^\star s_L$ corresponds to the L-valued point $(\sigma^\star\mathcal{G}_L,\tau_{\mathcal{G}'}\circ\sigma^\star\alpha_L)$ of *Spec* B. This point in the fiber category $(\mathit{Spec}\ B)(L)$ is by $\tau_{\mathcal{G}}^{-1}$ isomorphic to $(\mathcal{G}_L,\tau_{\mathcal{G}'}\circ\sigma^\star\alpha_L\circ f_\star\tau_{\mathcal{G}}^{-1})$, which is by diagram (14) equal to $(\mathcal{G}_L,\alpha_L)$. Since $(\mathcal{G}_L,\alpha_L)$ is exactly the L-valued point s_L the commutativity of diagram (13) follows.

Now we choose a closed point $v\in C\backslash S$. Then we can consider the associated étale local $\widetilde{\mathbb{G}'_v}$-shtuka $L_v(\underline{\mathcal{G}}') = (L_v^+(\mathcal{G}'),\tau'_v := L_v(\tau_{\mathcal{G}'}))$ over R, which arises from the formal $\hat{\mathbb{G}}'_v$-torsor $\mathcal{G}'\times_{C_R} \mathit{Spf}\ A_{v,R}$ as described in § 2.10. Since R is strictly henselian the $L^+\widetilde{\mathbb{G}'_v}$-torsor $L_v^+(\mathcal{G}')$ is trivial so that we choose a trivialization $\beta : L_v^+(\mathcal{G}') \xrightarrow{\sim} L^+\widetilde{\mathbb{G}'_v}$. In particular the composition $\beta\circ\tau'_v\circ\sigma^\star\beta^{-1}$: $L^+\widetilde{\mathbb{G}'_v} \xrightarrow{\sim} L^+\widetilde{\mathbb{G}'_v}$ is given by an element $b\in L^+\widetilde{\mathbb{G}'_v}(R)$ so that $\beta : L_v(\underline{\mathcal{G}}') \xrightarrow{\sim} (L^+\widetilde{\mathbb{G}'_v},b)$.
We define $R_i := R/\pi^{q^i}$ and $b_i\in L^+\widetilde{\mathbb{G}'_v}(R_i)$ as the image of the projection of b under the map $L^+\widetilde{\mathbb{G}'_v}(R)\to L^+\widetilde{\mathbb{G}'_v}(R_i),\ b\mapsto b_i$.
Since $R_0=\kappa_R$ is algebraically closed, there exists by [AH14, Corollary 2.9] a $c_0\in L^+\widetilde{\mathbb{G}'_v}$ with $c_0 = b_0\cdot\sigma^\star c_0$. Note that $\sigma^\star c_0\in L_v^+(\widetilde{\mathbb{G}'_v})(R_1)$. We set inductively $c_i := b_i\cdot\sigma^\star c_{i-1}$ for $i\geqslant 1$ and $c := \lim\limits_{i\to\infty} c_i = \lim\limits_{k\to\infty} b\cdot\sigma^\star b\cdot\dots\cdot\sigma^{(k-1)\star}b\sigma^{k\star}c_0\in L_v^+(\widetilde{\mathbb{G}'_v})(R)$ which satisfies $c=b\cdot\sigma^\star c$. Replacing the trivialization β by $c^{-1}\circ\beta$ gives therefore an isomorphism of local $\widetilde{\mathbb{G}'_v}$-shtuka $c^{-1}\cdot\beta : L_v(\underline{\mathcal{G}}') \xrightarrow{\sim} (L_v^+\widetilde{\mathbb{G}'_v}, id)$ as becomes clear from the diagram

$$\begin{array}{ccccc} \sigma^\star L_v^+(\underline{\mathcal{G}}') & \xrightarrow{\sigma^\star\beta} & L_v^+\widetilde{\mathbb{G}'_v} & \xrightarrow{\sigma^\star c} & L_v^+\widetilde{\mathbb{G}'_v} \\ \downarrow{\scriptstyle \tau_v} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle id} \\ L_v^+(\underline{\mathcal{G}}') & \xrightarrow{\beta} & L_v^+\widetilde{\mathbb{G}'_v} & \xrightarrow{c^{-1}} & L_v^+\widetilde{\mathbb{G}'_v} \end{array}$$

Let $A_{v,R} := A_v\hat{\otimes}_{\mathbb{F}_v}R$ and $\Gamma(A,\mathbb{G}'/\mathbb{G})$ the ring of sections of $\mathbb{G}'/\mathbb{G}$ over *Spec* A. The trivializations $c^{-1}\circ\beta : L_v^+(\underline{\mathcal{G}}')\to L_v^+\widetilde{\mathbb{G}'_v}$ and $\sigma^\star(c^{-1}\circ\beta) : \sigma^\star L_v^+(\underline{\mathcal{G}}')\to L_v^+\widetilde{\mathbb{G}'_v}$ and the isomorphism τ_v induces after passing to the v-adic completion morphisms $\overline{c^{-1}\beta}$, $\overline{\sigma^\star(c^{-1}\beta)}$ and $\overline{\tau_v}$ as in the following diagram

$$\begin{array}{ccccc} \Gamma(A,\mathbb{G}'/\mathbb{G})\otimes_A A_{v,R} & \xrightarrow{\overline{c^{-1}\beta}} & B\otimes_{A_R} A_{v,R} & & \\ \Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle \overline{\tau_v^\star}} & \searrow^{s_L^{\star v}} & L^v := L\otimes_{A_R} A_{v,R} \\ \Gamma(A,\mathbb{G}'/\mathbb{G})\otimes_A A_{v,R} & \xrightarrow{\overline{\sigma^\star(c^{-1}\beta)}} & B\otimes_{A_{R,\sigma}} A_{v,R} & \nearrow_{\sigma^\star s_L^{\star v}} & \end{array} \tag{15}$$

The right hand side of the diagram arises as the v-adic completion of the diagram (13), where $s_L^{\star v}$ and $\sigma^\star s_L^{\star v}$ denote the induced morphism of the completion. Since $ord_\pi(\sigma(x)) = q\cdot \mathrm{ord}_\pi(x)$ for all $x\in L^v$ the diagram (15) implies

$$ord_\pi(s_L^{\star v}\circ\overline{c^{-1}\beta}(y)) = ord_\pi(\sigma^\star(s_L^{\star v}\circ\overline{c^{-1}\beta})(y)) = q\cdot ord_\pi(s_L^{\star v}\circ\overline{c^{-1}\beta}(y))$$

for $y\in\Gamma(A,\mathbb{G}'/\mathbb{G})$. This means that $ord_\pi(s_L^{\star v}\circ\overline{c^{-1}\beta}(y))$ equals 0 or ∞. In particular we have

$$s_L^{\star v}\circ\overline{c^{-1}\beta}:\Gamma(A,\mathbb{G}'/\mathbb{G})\otimes_A A_{v,R}\to\{x\in L^v\mid ord_\pi(x)\geqslant 0\}$$

which implies

$$\begin{array}{ccc} B\otimes_{A_R} A_{v,R} & \xrightarrow{s_L^{\star v}} & \{x\in L^v\mid ord_\pi(x)\geqslant 0\} \\ \Big\uparrow & & \\ B & \xrightarrow{s_L^{\star}} & \{x\in L^v\mid ord_\pi(x)\geqslant 0\}\cap L=\mathcal{O} \end{array}$$

This finishes the first step.

(Step 2) As we have described above, the proof of the first step gives us a $\mathbb{G}$-torsor $\widetilde{\mathcal{E}}$ over $(C\backslash S)_R$ with $j^\star\widetilde{\mathcal{E}} = \mathcal{G}|_{(C\backslash S)_K}$ and an isomorphism $\widetilde{\alpha_R}: f_\star\widetilde{\mathcal{E}}\xrightarrow{\sim}\mathcal{G}'|_{(C\backslash S)_R}$. We now show that $\widetilde{\mathcal{E}}\times_{(C\backslash S)_R}(C\backslash S)_{R^{alg}}$ extends to a $\mathbb{G}$-torsor $\mathcal{E}$ over $C_{R^{alg}}$ with $\mathcal{E}\times_{C_{R^{alg}}} C_{K^{alg}} = \mathcal{G}_{K^{alg}}$ and $\alpha_R: f_\star\mathcal{E}\xrightarrow{\sim}\mathcal{G}'_{R^{alg}}$.
Now the field $\widetilde{L} := Quot(A_{R^{alg}})$ has transcendence degree one over K^{alg} so that its cohomological dimension equals one by [Ser94, §2.3 Théoreme 1 and remark page 140]. Since $\mathbb{G}_L$ is reductive this implies by [BS68, subsection 8.6] that $\widetilde{\mathcal{E}}$ is trivial over $\widetilde{L}$. Therefore we can choose a finite extension K'/K and a trivialization $\gamma_{L'}:\widetilde{\mathcal{E}}\to\mathbb{G}_{L'}$, where $L' := Quot(A_{R'})$ and where R' is the integral closure in K'. We recall that we denoted by z_v a uniformizer of C at v, so that $A_v\hat{\otimes}R' = R'[\![z_v]\!]$ (do not confuse A_v with A) and L' is contained in $Quot(R'[\![z_v]\!])$. In particular the trivialization γ_L implies that the $\mathbb{G}$-torsor $\widetilde{\mathcal{E}}$ is trivial over $Quot(R'[\![z_v]\!])$. This fact allows us to apply [Ans18, 1) in Theorem 1.2] to see that $\widetilde{\mathcal{E}}_{Quot(R'[\![z_p v]\!])}$ extends to a $\mathbb{G}$-torsor $\widetilde{\mathcal{E}}_v$ over $R'[\![z_v]\!]$. (Note that [Ans18] use the notation $\mathcal{O}_E$ for our ring $\kappa_R[\![z_v]\!]$ and z for a uniformizer π' in our ring R'.) This corresponds by [AH14, Proposition 2.4] to a $L^+\widetilde{\mathbb{G}}_v$-torsor $L^+(\widetilde{\mathcal{E}}_v)$ over R' which becomes trivial after base change to the strictly henselian ring R^{alg}. We fix such a trivialization $\beta_v: L_v^+(\widetilde{\mathcal{E}}_{v\,R^{alg}})\xrightarrow{\sim} L^+\widetilde{\mathbb{G}}_{v\,R^{alg}}$ for all $v\in S$. They induce trivializations $L(\beta_v): L_v(\widetilde{\mathcal{E}}_{R^{alg}})\xrightarrow{\sim} L\widetilde{\mathbf{G}}_{v\,R^{alg}}$ and therefore isomorphisms

$$L_v\widetilde{\mathcal{E}}_{R^{alg}}\big/L^+\widetilde{\mathbb{G}}_{v,R^{alg}}\xrightarrow{\sim}\big(\mathcal{F}l_{\widetilde{\mathbb{G}_v}}\big)\times_{\mathbb{F}_q}R^{alg} =: \mathcal{F}l_{v\,R^{alg}}.$$

An T-valued point $(\mathcal{L}^+,\delta)$ with $\delta : \mathcal{L} \to L_v\widetilde{\mathcal{E}}_{R^{alg}}$ is send to $(\mathcal{L}^+,\beta_v\circ\delta)$. By the theorem of Beauville-Laszlo in § 2.9 we have the following cartesian diagram

$$\begin{array}{ccc} \mathscr{H}^1(C,\mathbb{G}) & \longrightarrow & \mathscr{H}^1_e(C\backslash S,\mathbb{G}) \\ \downarrow{\scriptstyle \prod_{v\in S} L_v^+} & & \downarrow{\scriptstyle \prod_{v\in S} L_v} \\ \prod_{v\in S}\mathscr{H}^1(\mathbb{F}_q, L^+\widetilde{\mathbb{G}_v}) & \xrightarrow{\;L\;} & \prod_{v\in S}\mathscr{H}^1(\mathbb{F}_q, L\widetilde{\mathbf{G}}_v) \end{array} \quad . \tag{16}$$

Due to this diagram the torsor $\mathcal{G}_{K^{alg}}$ corresponds to the tuple $\big(\mathcal{G}|_{(C\backslash S)_{K^{alg}}}, \prod_{v\in S} L_v^+(\mathcal{G}_{K^{alg}}), (\epsilon_v)_{v\in S}\big)$ with $\epsilon_v = id : L(L_v^+(\mathcal{G})) \to L_v(\mathcal{G}|_{(C\backslash S)_{K^{alg}}})$. Now for all $v\in S$ the tuple $(L_v^+(\mathcal{G}_{K^{alg}}),\ (\beta_v\times id_{K^{alg}})\circ\epsilon_v)$ gives an K^{alg}-valued point of the affine flag variety $\mathcal{F}l_{v\,R^{alg}}$. By assumption $\widetilde{\mathbb{G}_v}$ is parahoric so that $\mathcal{F}l_{v\,R^{alg}}$ is ind-projective over R^{alg} by [Ric16a, Theorem A]. As a consequence we can lift $(L_v^+(\mathcal{G}_{K^{alg}}),(\beta_v\times id_{K^{alg}})\circ\epsilon_v)$ to a unique R^{alg}-valued point $(\mathcal{E}_v,\beta_v\circ\delta_v)\in\mathcal{F}l_v(R^{alg})$ with $(\mathcal{E}_v\times_{R^{alg}}K^{alg},(\beta_v\circ\delta_v)\times id_{K^{alg}})\simeq(L_v^+(\mathcal{G}_{K^{alg}}),(\beta_v\times id_{K^{alg}})\circ\epsilon_v)$.
In particular the tuple $(\widetilde{\mathcal{E}}_{(C\backslash S)_{R^{alg}}},\prod_{v\in S}\mathcal{E}_v,\delta_v)$ defines a unique R^{alg}-valued point in $\mathscr{H}^1(C,\mathbb{G})$ given by a $\mathbb{G}$-torsor $\mathcal{E}$ over $C_{R^{alg}}$ with $\mathcal{E}\times_{C_{R^{alg}}}C_{K^{alg}}=\mathcal{G}_{K^{alg}}$. By diagram (16) with $\mathbb{G}$ replaced by $\mathbb{G}'$ we get an isomorphism $\alpha_{R^{alg}} : f_\star\mathcal{E}\xrightarrow{\sim}\mathcal{G}'_{R^{alg}}$. This finishes the second step.

(Step 3) We now have to show that the $\mathbb{G}$-torsor $\mathcal{E}$ over $C_{R^{alg}}$ is part of a global $\mathbb{G}$-shtuka $\underline{\mathcal{E}} = (\mathcal{E},r_1,\dots,r_n,\tau_{\mathcal{E}})$ defining the dashed arrow in diagram (10). The condition that $\alpha_{R^{alg}} : f_\star\underline{\mathcal{E}}\xrightarrow{\sim}i_R^\star\underline{\mathcal{G}'}$ needs to be an isomorphism of global $\mathbb{G}'$-shtukas defines r_i by $r_i = s_i'\circ i_R$ for all $1\leqslant i\leqslant n$. So we have to construct $\tau_{\mathcal{E}}$.
From the proof of the first step we get the commutative diagram

$$\begin{array}{ccccc} & & (C\backslash S)_R & & \\ & \swarrow{\scriptstyle \sigma^\star s_R} & & \searrow{\scriptstyle s_R} & \\ \sigma^\star\mathcal{G}'/\mathbb{G}_R\times_{C_R}(C\backslash S)_R & & \xrightarrow{\;\overline{\tau_{\mathcal{G}'}}\;} & & \mathcal{G}'/\mathbb{G}_R\times_{C_R}(C\backslash S)_R \end{array} \tag{17}$$

We defined $(\widetilde{\mathcal{E}},\widetilde{\alpha_R})$ with $\widetilde{\alpha_R} : f_\star\widetilde{\mathcal{E}}\to\mathcal{G}'$ to be the $(C\backslash S)_R$-valued point in $\mathcal{G}'/\mathbb{G}_R$ corresponding to s_R. Hence $(\sigma^\star\widetilde{\mathcal{E}},\sigma^\star\widetilde{\alpha_R})$ corresponds to $\sigma^\star s_R$ and the composition $\overline{\tau'_{\mathcal{G}}}\circ\sigma^\star s_R$ corresponds to $(\sigma^\star\widetilde{\mathcal{E}},\tau_{\mathcal{G}'}\circ\sigma^\star\widetilde{\alpha_R})$. The commutativity of (17) means that $(\sigma^\star\widetilde{\mathcal{E}},\tau_{\mathcal{G}'}\circ\sigma^\star\alpha_R)$ and $(\widetilde{\mathcal{E}},\widetilde{\alpha_R})$ are isomorphic as $(C\backslash S)_R$-valued points in $\mathcal{G}'/\mathbb{G}_R$. This gives us therefore an isomorphism $\tau_{\widetilde{\mathcal{E}}} : \sigma^\star\widetilde{\mathcal{E}}\to\widetilde{\mathcal{E}}$ of $\mathbb{G}$-torsors over $(C\backslash S)_R$ satisfying $\tau_{\mathcal{G}'}\circ\sigma^\star\widetilde{\alpha_R} = \widetilde{\alpha_R}\circ f_\star\tau_{\widetilde{\mathcal{E}}}$. This defines the isomorphism $\tau_{\mathcal{E}}$ restricted to $(C\backslash S)_{R^{alg}}$ by $\tau_{\mathcal{E}}|_{(C\backslash S)_{R^{alg}}} = \tau_{\widetilde{\mathcal{E}}}\times_R id_{R^{alg}}$ and we have to extend it to $C_{R^{alg}}\backslash\bigcup_i\Gamma_{r_i}$. We know additionally by $\mathcal{E}|_{C_{K^{alg}}}=\mathcal{G}$ and $\alpha : f_\star\underline{\mathcal{G}}\xrightarrow{\sim}j^\star\underline{\mathcal{G}}'_{K^{alg}}$ that $\tau_{\mathcal{E}}$ extends to $C_{K^{alg}}\backslash\bigcup_i\Gamma_{r_i}$. Therefore we only have to extend $\tau_{\mathcal{E}}$ at finitely many closed points of $C_{R^{alg}}\backslash\bigcup_i\Gamma_{r_i}$. This works similar as at the end of the proof of proposition 3.12. So for $p\in C_{R^{alg}}\backslash(\bigcup_i\Gamma_{r_i}\cup C_{K^{alg}})$ we choose an open neighboorhood $V\subset C_{\kappa_R}$ with $(V\times_{\kappa_R}R^{alg})\cap((\bigcup_i\Gamma_{r_i})\cup(C\backslash S)_{R^{alg}}) = p$. We write $\widetilde{V} := V\backslash p$ so that $\tau_{\mathcal{E}}$ is defined on $\widetilde{V}_{R^{alg}}$ and need to be extended to $V_{R^{alg}}$. Moreover the $\mathbb{G}$-torsors $\sigma^\star\mathcal{E}|_{V_{R^{alg}}}$ and $\mathcal{E}|_{V_{R^{alg}}}$ are two R^{alg}-valued points in $\mathscr{H}^1(V,\mathbb{G})(R^{alg})$ so that $\tau_{\mathcal{E}}|_{\widetilde{V}_{R^{alg}}}$ is an isomorphism in $\mathscr{H}^1_e(\widetilde{V},\mathbb{G})(R^{alg})$. Thanks again to Beauville and Laszlo (§ 2.9) the cartesian diagram

$$\begin{array}{ccc} \mathscr{H}^1(V,\mathbb{G}) & \longrightarrow & \mathscr{H}^1(\widetilde{V},\mathbb{G}) \\ \downarrow{\scriptstyle L_p^+} & & \downarrow{\scriptstyle L_p} \\ \mathscr{H}^1(\kappa_R, L^+\widetilde{\mathbb{G}_p}) & \longrightarrow & \mathscr{H}^1(\kappa_R, L\widetilde{\mathbf{G}_p}) \end{array}$$

makes it sufficient to show that the isomorphism $L_p(\tau_{\mathcal{E}}) : L_p(\sigma^\star\mathcal{E}) \to L_p(\mathcal{E})$ in $\mathscr{H}^1(\kappa_R, L\widetilde{\mathbf{G}_p})$ comes from an isomorphism in $\mathscr{H}^1(\kappa_R, L^+\widetilde{\mathbb{G}_p})$. After trivializing $L_p(\mathcal{E})$ the morphism $L_p(\tau_{\mathcal{E}})$ is given by an element $h \in L\widetilde{\mathbf{G}_p}(R^{alg})$. Since $\tau_{\mathcal{E}}$ is already defined on $V_{K^{alg}}$ the pullback of h to $h_K \in L\widetilde{\mathbf{G}_p}(K^{alg})$ is already given by an element in $L^+\widetilde{\mathbb{G}_p}(K^{alg})$. Since $L^+\widetilde{\mathbb{G}_p} \subset L\widetilde{\mathbf{G}_p}$ is a closed subgroup we conclude that h is already an element in $L^+\widetilde{\mathbb{G}_p}(R^{alg})$. This shows that $\tau_{\mathcal{E}}$ extends uniquely to $V_{R^{alg}}$ and hence to $C_{R^{alg}}\backslash\bigcup_i \Gamma_{r_i}$. Hereby we found the $\mathbb{G}$-shtuka $\underline{\mathcal{E}}$ over R^{alg} defining a unique dashed arrow in the diagram (10), which ends the proof of the theorem. □

Corollary 3.27. *Let $\mathbb{G}$ be a parahoric Bruhat-Tits group scheme and $(id_C, f) : (C, \mathbb{G}, \underline{v}, \hat{Z}_{\underline{v}}, H) \to (C, \mathbb{G}', \underline{v}, \hat{Z}'_{\underline{v}}, H')$ be a morphism of shtuka data, where $f : \mathbb{G} \to \mathbb{G}'$ is a closed immersion of smooth affine group schemes over C. Then the induced morphism*

$$f_\star : \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) \to \nabla_n^{\hat{Z}'_{\underline{v}},H'}\mathscr{H}^1(C,\mathbb{G}')$$

is finite.

Proof: The proof of this corollary works literally in the same way as the proof of corollary 3.25 with replacing unramified by finite. □

Remark 3.28. The results of this chapter can maybe used in some future work to formulate and prove some kind of André-Oort conjecture for global $\mathbb{G}$-shtukas. To formulate such a conjecture one needs the notion of special points and special subvarieties. In the case of Drinfeld modular curves an analogue of the André-Oort conjecture has been formulated and proved in [Bre05]. Later the notion of special subvarieties and the formulation of the André Oort conjecture was generalized in [Bre12] to the higher dimensional Drinfeld modular varieties. In the same paper this André-Oort conjecture was proven in some special cases. These results were extended in [Hub13]. To define Drinfeld modular varieties, one fixes a point $\infty \in C$ so that $C \smallsetminus \infty =: Spec\ A$ is affine and $\mathcal{M}_A^r$ contains Drinfeld A-modules of rank r. Now for certain finite extensions $A' \subset A$ coming from a morphism $C \to C'$ of curves, Breuer shows that there is a proper morphism $\mathcal{M}_A^r \to \mathcal{M}_{A'}^{r'}$ of moduli spaces and he uses the image of this morphism to define special subvarieties.

Now Drinfelds modular variety $\mathcal{M}_A^r$ can be embedded into $\nabla_2^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C, \mathrm{GL_r})$ for $n = 2$ and some specific choosen bound $\hat{Z}_{\underline{v}}$. The morphism $\mathcal{M}_A^r \to \mathcal{M}_{A'}^{r'}$ corresponds then to a morphism $\nabla_2^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C, \mathrm{GL_r}) \to \nabla_2^{\hat{Z}'_{\underline{w}}}\mathscr{H}^1(C', \mathrm{GL}_{r\cdot[C:C']})$ coming from a morphism of shtuka data $(C, \mathrm{GL}_r, \underline{v}, \hat{Z}_{\underline{v}}) \to (C', \mathrm{GL}_{r\cdot[C:C']}, \underline{w}, \hat{Z}_{\underline{w}})$. So extending the coefficients for Drinfeld modules generalizes to changing the curve for global $\mathbb{G}$-shtukas as in section 3.2, since we are not restricted to choose $n = 2$, $\mathbb{G} = \mathrm{GL}_r$ or some specific bound. Moreover we have seen that additionally to changing the curve, we can also change the group scheme $\mathbb{G}$ as in section 3.3. Although we do not know if this is precisely the correct definition it is conceivable to define a special subvariety of $\nabla_n^{\hat{Z}'_{\underline{w}},H'}\mathscr{H}^1(C',\mathbb{G}')$ to be the image of the morphism

$$f_\star \circ \pi_\star : \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) \to \nabla_n^{\hat{Z}'_{\underline{w}},H'}\mathscr{H}^1(C',\mathbb{G}')$$

arising from a morphism (π, f) of shtuka data, where $f : \pi_\star\mathbb{G} \hookrightarrow \mathbb{G}'$ is a closed immersion of (Bruhat-Tits) group schemes. Special points in $\nabla_n^{\hat{Z}'_{\underline{w}},H'}\mathscr{H}^1(C',\mathbb{G}')$ would then be defined to be those points which arise in the image of a morphism $\widetilde{f}_\star : \nabla_n^{\hat{Z}_{\underline{w}},H}\mathscr{H}^1(C',\mathbb{T}) \to \nabla_n^{\hat{Z}'_{\underline{w}},H'}\mathscr{H}^1(C',\mathbb{G}')^{\underline{v}}$, where $\widetilde{f} : \mathbb{T} \to \mathbb{G}'$ is a closed (Bruhat-Tits) group scheme that is generically a torus in $\mathbf{G}'$. Following this, an André-Oort conjecture for global $\mathbb{G}$-shtukas would then say that given a set S of special points, the Zariski closure of these points is a finite union of special subvarieties. Again, this is not a precise formulation but should give an impression of the flavor of a possible statement.

4 Stratifications

We now move our interest to the stratifications of the special fiber $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_R}$ of the moduli space of global $\mathbb{G}$-shtukas . From now on $(C,\mathbb{G},\underline{v},\hat{Z}_{\underline{v}},H)$ will be a shtuka datum where $\mathbb{G}$ is a parahoric Bruhat-Tits group scheme as defined in § 2.17. In the fifth chapter we will define five axioms on this moduli space. Their verification will then imply several statements on the Newton and Kottwitz-Rapoport stratification. In this chapter we do the necessary preparations to formulate these axioms. In particular we define in the first section a stratification map and explain how it determines a stratification of an algebraic stack. The second section is about the set $B(\mathbf{G}_v)$ of σ-conjugacy classes and ends with the definition of the Newton stratification. In the third section we introduce the local model for $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$. This results then in the definition of the Kottwitz-Rapoport stratification. In the fifth and the last section of this chapter we recall the definition of affine Deligne-Lusztig varieties and introduce the notion of σ-straight elements from [HN14, Section 1.3] and [HR17, Section 5.1]

4.1 Stratifications of Stacks

A stratification of a topological space X is defined to be a locally closed partition $X = \coprod_{i\in I} X_i$. It is said to have the strong stratification property if the closure of any stratum X_i is the union of other strata $\overline{X_i} = \coprod_{j\in J} X_j$ for $J \subset I$. Now this notion of a stratification generalizes in very much the same way to an algebraic stack $\mathcal{X}$.

Definition 4.1. *A stratification of an algebraic stack $\mathcal{X}$ is a family $(\mathcal{X}_i)_{i\in I}$ of locally closed reduced substacks $\mathcal{X}_i \subset \mathcal{X}$ such that the 1-morphism $\coprod_{i\in I} \mathcal{X}_i \to \mathcal{X}$ is representable and universally bijective.*

The locally closed substacks $\mathcal{X}_i$ are called strata and later we give them specific names depending on the stratification (Newton, Kottwitz-Rapoport, ...) we are talking about. Now we will prove that giving a stratification on $\mathcal{X}$ is the same as giving a stratification map on $\mathcal{X}$ in the following sense.

Definition 4.2. *Let $\mathcal{X}$ be an algebraic stack fibered over the category of schemes and I an index set, which we view as a groupoid. Then a Stratification map of $\mathcal{X}$ with index set I is given by a collection of functors $\varphi_k : \mathcal{X}(k) \to I$ for every algebraically closed field k with the following properties:*

1. *For every $f : Spec\ k \to Spec\ k'$ we have $\varphi_{k'} \circ \mathcal{X}(f) = \varphi_k$.*
2. *For every morphism of a scheme S to $\mathcal{X}$ the partition $S = \coprod_{i\in I} S_i$ defined by*

$$S_i = \left\{ s \in S \mid \varphi_{\kappa(s)^{alg}} \left(\kappa(s)^{alg} \to S \to \mathcal{X} \right) = i \right\}$$

is a stratification of S in the sense of definition 4.1. Here S_i is well defined by the first condition.

Lemma 4.3. *The stratifications of $\mathcal{X}$ are in bijection to the stratification maps on $\mathcal{X}$.*

Proof: It is not difficult to see that every stratification $(\mathcal{X}_i)_{i\in I}$ give rise to a stratification map. Namely let $x : Spec\ k \to \mathcal{X}$ be an object in $\mathcal{X}(k)$ with $k = k^{alg}$. By the universal bijectivity of $\prod_{i\in I} \mathcal{X}_i \to \mathcal{X}$ we know that $k \times_{\mathcal{X}} \prod_{i\in I} \mathcal{X}_i$ is a scheme consisting of one point that factors through exactly one stratum $\mathcal{X}_i$. We define $\varphi_k(x) = i$. This gives a collection of functors $\varphi_k : \mathcal{X}(k) \to I$ that are compatible with $\mathcal{X}(k \to k') : \mathcal{X}(k) \to (k')$. Let S be a scheme with $S \to \mathcal{X}$. Since $\mathcal{X}_i$ are locally closed the partition $(\mathcal{X}_i \times_{\mathcal{X}} S)_{i\in I}$ gives a stratification of S and the underlying topological space of $\mathcal{X}_i \times_{\mathcal{X}} S$ is exactly given by S_i from Definition 4.2. So (φ_k) is a stratification map. Now starting with some stratification map $(\varphi_k)_k$ we want to associate with it a stratification. We recall that the algebraic stack has an underlying topological space $|\mathcal{X}|$ as defined in [LMB00, Definition 5.1 and (5.5)]. The compatibility of (φ_k) guarantees that we have a map $\varphi : |\mathcal{X}| \to I$. We set $X_i := \varphi_{-1}(i)$. Now choosing an atlas $f : A \to \mathcal{X}$ induces a surjective continuous open map $|f| : |\mathcal{A}| \to |\mathcal{X}|$. Let $(A_i)_{i\in I}$ be the stratification of A from Definition 4.2. It is also given as $A_i = (\varphi \circ |f|)^{-1}(i) = |f|^{-1}(X_i)$. We conclude that X_i is locally closed as follows. Let U_i be an open in $|A|$ such that $A_i \subset U_i$ is closed. Then $|f|(U_i)$ is an open in $|\mathcal{X}|$ that contains X_i and we have $|f|(U_i)\backslash X_i = f(U_i\backslash A_i)$ by the surjectivity of $|f|$ and $A_i = |f|^{-1}(X_i)$. Since $|f|$ is an open map and $U_i\backslash A_i$ is open in U_i we conclude that X_i is closed in $|f|(U_i)$ and hence locally closed in $|\mathcal{X}|$. Now the open set $|f|(U_i)$ corresponds by [LMB00, (5.5)] to an open substack $\mathcal{X}_i^\circ$ of $\mathcal{X}$ with $|\mathcal{X}_i^\circ| = |f|(U_i)$. By [LMB00, Lemma 4.10 and Corollary 5.6.1] there exists then a unique reduced closed substack $\mathcal{X}_i \subset \mathcal{X}_I^\circ$ with $|\mathcal{X}_i| = X_i$. Now the locally closed substacks $\mathcal{X}_i$ define a stratification. Here the universal bijectivity follows again from the fact that $|S \times_{\mathcal{X}} \mathcal{X}_i| = S_i$. □

Remark 4.4. The proof shows that we also could define a stratification of $\mathcal{X}$ simply as a stratification of $|\mathcal{X}|$.

The next lemma tells us that in a given stratification of an algebraic stack $\mathcal{X}$ locally of finite type over k the strata are already defined by the closed points of $\mathcal{X}_i$. Of course it is not true, that any partition of closed points of $|\mathcal{X}|$ give rise to a stratification. We remark as well that the lemma fails if $\mathcal{X}$ is not locally of finite type. For example let $\mathfrak{m} = (x, y)$ in $k[x, y]$. Then $X = Spec\ k[x, y]_{\mathfrak{m}}$ has exactly one closed point $\mathfrak{m}$ and this point generalizes to several points x_i, which correspond to the curves intersecting with $\mathfrak{m}$. Now $\{\mathfrak{m}, x_i\}$ and $X\backslash\{\mathfrak{m}, x_i\}$ give stratifications that can not be distinguished by knowing a partition of the closed points (only $\mathfrak{m}$ in this case).

Lemma 4.5. *Let $\mathcal{X}$ be an algebraic stack locally of finite type over some field K and let $(\varphi_k)_k$ and $(\psi_k)_k$ be two stratification maps with*

$$\varphi_{K^{alg}} = \psi_{K^{alg}} : \mathcal{X}_{K^{alg}} \to I$$

Then the stratification maps are equal.

Proof: Assume $(\varphi_k)_k$ and $(\psi_k)_k$ are not equal, then the maps $\varphi : |\mathcal{X}| \to I$ and $\psi : |\mathcal{X}| \to I$ would be different as well. Now we choose an atlas $A \to \mathcal{X}$ and the surjectivity $|A| \to |\mathcal{X}|$ implies that the stratifications $(A_{\varphi,i})_{i\in I}$ and $(A_{\psi,i})_{i\in I}$ are different. The lemma follows therefore from the following claim:
Let S be a scheme locally of finite type over K and denote by C its closed points. By Hilberts Nullstellensatz C consists exactly of those points in S that have a finite residue field over K. Let S_1 and S_2 be two locally closed subsets of S with $S_1 \cap C = S_2 \cap C$. Then we claim $S_1 = S_2$.

The set $S_3 := S_1 \cap S_2 \subset S$ is another locally closed subset in S with $S_3 \cap C = S_1 \cap C$ and it suffices to prove $S_1 = S_3$ which implies analogously $S_3 = S_2$. Since S_3 is also locally closed in S_1 we can even assume without loss of generality $S_1 = S$. Now choose an open subset $U \in S$ such that S_3 is closed in U. Since S is locally of finite type the set C is dense in S and we have $S_3 = U \cap \overline{C} = U$. Then $C \subset U$ so that $S \setminus U$ is closed without containing a closed point which implies $S_3 = S$. □

4.2 Notations related to Weyl Groups

We denote by $\breve{Q}_v$ the completion of the maximal unramified extension of Q_v in an algebraic closure $\overline{Q_v}$ of Q_v and $\breve{A}_v$ its ring of integers. Using the identification $Q_v = \mathbb{F}_v(\!(z_v)\!)$ this means $\breve{Q}_v = \mathbb{F}_v^{alg}(\!(z_v)\!)$. We denote by $\Sigma_0 = Gal(\overline{Q_v}, \breve{Q}_v)$ the inertia group, by Σ the Galois group $Gal(\overline{Q_v}, Q_v)$ and by Σ_{nr} the Galois group $Gal(\breve{Q}_v, Q_v)$, which is generated by the $\mathbb{F}_v$-Frobenius σ_v.

Since $\mathbb{G}$ is a parahoric Bruhat-Tits group scheme, the group $\mathbf{G}_v$ is a connected reductive group over Q_v and $\mathbb{G}_v(A_v) \subset \mathbf{G}(Q_v)$ is a parahoric group in the sense of [BT84, Definition 5.2.6] or [HR08, Definition 1]. We denote by $\mathscr{B}_v = \mathscr{B}(\mathbf{G}_v, Q_v)$ the Bruhat-Tits building. Let $F \subset \mathscr{B}_v$ be the maximal facet that is fixed (point wise) by $\mathbb{G}(A_v)$ and choose an apartment $\mathscr{A}_v$ in $\mathscr{B}_v$ that contains F. We denote by A the maximal Q_v-split torus of $\mathbf{G}_v$ that corresponds to the appartement $\mathscr{A}_v$ so that $\mathscr{A}_v = \mathscr{A}(\mathbf{G}_v, A, Q_v)$. Let S_v be a maximal $\breve{Q}_v$-split torus of $\mathbf{G}_v$ defined over Q_v that contains A, this exists by [BT84, Corollaire 5.1.12].
Since $\mathbf{G}_v \times_{Q_v} \breve{Q}_v$ is quasi-split by [BS68, subsection 8.6] the centralizer $T_v := Z_{\mathbf{G}_v}(S_v)$ is a maximal torus (defined over Q_v) and we can choose a Borel subgroup $B_v \subset \mathbf{G}_v \times_{Q_v} \breve{Q}_v$ that contains $T_v \times_{Q_v} \breve{Q}_v$. Furthermore we denote by $N_v := N(T_v)$ the normalizer of T_v in $\mathbf{G}_v$. Let $\pi_1(\mathbf{G}_v)$ be the algebraic fundamental group of $\mathbf{G}_v$. It is the quotient of the the cocharacters $X_\star(T_v)$ by the coroot lattice. The action of Σ (resp. Σ_0) on $X_\star(T_v)$ induces an action on $\pi_1(\mathbf{G}_v)$ and we denote by $\pi_1(\mathbf{G}_v)_{\Sigma_0}$ (resp.$\pi_1(\mathbf{G}_v)_{\Sigma}$) the coinvariants under this action. Now we recall that Kottwitz [Kot97, §7] (see also [PR08, Section 2.a.2]) defines a surjective homomorphism

$$\kappa_{\mathbf{G_v}} : \mathbf{G}_v(\breve{Q}_v) \longrightarrow \pi_1(\mathbf{G}_v)_{\Sigma_0} \tag{18}$$

which is functorial in the group $\mathbf{G}_v$. We denote as usual by $T_v(\breve{Q}_v)_1$ the kernel of

$$\kappa_{T_v} : T_v(\breve{Q}_v) \to \pi_1(T_v)_{\Sigma_0} = X_\star(T_v)_{\Sigma_0}. \tag{19}$$

The [HR08, Lemma 5] implies that the group $T_v(\breve{Q}_v)_1$ equals also the intersection $\mathbb{G}(\breve{A}_v) \cap T_v(\breve{Q}_v)$. In the cited paper Rapoport and Haines work over a strictly henselian field. So let $\breve{\mathscr{B}}_v = \mathscr{B}(\mathbf{G}_v, \breve{Q}_v)$ be the building of $\mathbf{G}_v$ over $\breve{Q}_v$ and $\breve{\mathscr{A}}_v = \mathscr{A}(\mathbf{G}_v, S_v, \breve{Q}_v)$ be the appartement corresponding to the split torus S_v. Then we have a natural Σ_{nr} equivariant embedding $\mathscr{B}_v \hookrightarrow \breve{\mathscr{B}}_v$ and $\mathscr{A}_v \hookrightarrow \breve{\mathscr{A}}_v$ that identifies $\mathscr{B}_v$ (resp. $\mathscr{A}_v$) with the Σ_{nr} fixed points in $\breve{\mathscr{B}}_v$ (resp. $\breve{\mathscr{A}}_v$) [BT84, p. 5.1.20]. The factes of $\mathscr{A}_v$ correspond to Σ_{nr} invariant facets of $\breve{\mathscr{A}}_v$. so let $\breve{F}$ be the Σ_{nr} invariant facet in $\breve{\mathscr{A}}_v$ corresponding to F. The associated parahoric subgroup to $\breve{F}$ is $\mathbb{G}_v(\breve{A}_v)$ and the unique smooth affine group scheme over $\breve{A}_v$ with connected special fiber, generic fiber equal to $\mathbf{G}_v \times_{Q_v} \breve{Q}_v$ and the condition that the $\breve{A}_v$ valued points equal this parahoric group is equal to $\mathbb{G}_v \times_{A_v} \breve{A}_v$.

Definition 4.6 ([HR08, Definition 7]). *The (finite) Weyl group W_0 of $\mathbf{G}_v$ is defined as*

$$W_0 := N_v(\breve{Q}_v)/T_v(\breve{Q}_v)$$

The Iwahori-Weyl group $\widetilde{W}_v$ of $\mathbb{G}_v$ associated with S is defined as

$$\widetilde{W}_v = N_v(\breve{Q}_v)/T_v(\breve{Q}_v)_1$$

In the case that $S_v = T_v$, i.e. that $\mathbf{G}_v$ is split over $\breve{Q}_v$ this group $\widetilde{W}$ is often called extended Weyl group and sometimes also extended affine Weyl group. Together with the definition of $T_v(\breve{Q}_v)_1$ as the kernel of (19) we get an exact sequence

$$0 \to X_\star(T_v)_{\Sigma_0} \to \widetilde{W}_v \to W_0 \to 0. \tag{20}$$

For any parahoric subgroup $K \subset \mathbb{G}(\breve{Q}_v)$ that corresponds to a facet in $\breve{A}_v$ we set

$$W_K := (N_v(\breve{Q}_v) \cap K)/T_v(\breve{Q}_v)_1 \subset \widetilde{W}_v.$$

We fix a special vertex $\mathfrak{p} \subset \breve{A}_v$ and denote by $K_\mathfrak{p}$ the associated parahoric subgroup of $\mathbf{G}_v(\breve{Q}_v)$. Since the vertex is special the projection $\widetilde{W}_v \to W_0$ induces an isomorphism $W_{K_\mathfrak{p}} \xrightarrow{\sim} W_0$ (see [HR08, Proposition 13]). This gives a section in (20) and hence a presentation of $\widetilde{W}_v$ as semi-direct product

$$\widetilde{W}_v = X_\star(T_v)_{\Sigma_0} \rtimes W_0$$

We denote by $\mathbf{G}_{v,1}$ the kernel of $\kappa_{\mathbf{G}_v}$ and by $N_{v,1}$ the kernel of κ_{N_v}. We fix a base alcove $\mathfrak{a}_v$ in $\breve{\mathscr{A}}_v$ whose closure contains $\mathfrak{p}$. We denote by $\mathbb{S}$ the set of reflections at the walls of $\mathfrak{a}_v$ and by I_v the corresponding Iwahori subgroup. Then the quadrupel $(G_{v,1}, I_v, N_{v,1}, \mathbb{S})$ is a Tits system by [BT84, p. 5.2.12].

Definition 4.7. *The affine Weyl group of $\mathbf{G}_v$ is defined as*

$$W_{v,af} := N_{v,1}/(N_{v,1} \cap I_v)$$

It is a coxeter group with $\mathbb{S}$ as a system of generators and carries therefore the Bruhat order $\preccurlyeq$ and a length function l.

Now by [Ric16b, Lemma 1.2] we have $N_{v,1} \cap I_v = T_{v,1}$ and $N_v(\breve{Q}_v)/N_{v,1} \simeq \mathbf{G}_v(\breve{Q}_v)/\mathbf{G}_{v,1}$ which induces an exact sequence

$$1 \to W_{v,af} \to \widetilde{W}_v \xrightarrow{\kappa_{\mathbf{G}_v}} \pi_1(\mathbf{G}_v)_{\Sigma_0} \to 1 \tag{21}$$

Now $W_{v,af}$ acts simply transitiv on the set of alcoves in $\breve{\mathscr{A}}_v$ and the stabilizer $\Omega \subset \widetilde{W}_v$ of $\mathfrak{a}_v$ maps isomorphically to $\pi_1(\mathbf{G}_v)_{\Sigma_0}$. This gives a section of (21) so that we can write $\widetilde{W}_v = W_{v,af} \rtimes \pi_1(\mathbf{G}_v)_{\Sigma_0} = W_{v,af} \rtimes \Omega$. This semi-direct product is used to extend the Bruhat-order of $W_{v,af}$ to $\widetilde{W}_v$. Namely we set $(\alpha_1, \beta_1) \preccurlyeq (\alpha_2, \beta_2)$ if and only if $\beta_1 = \beta_2$ and $\alpha_1 \preccurlyeq \alpha_2$ in $W_{v,af}$. Also we extend the length function of $W_{v,af}$ to $\widetilde{W}_v$ by $l(\beta) = 0$ for all $\beta \in \Omega$.
In the following we denote by K_v the parahoric subgroup $\mathbb{G}_v(\breve{A}_v)$ in $\mathbf{G}_v(\breve{Q}_v)$. Then the Bruhat-order on $\widetilde{W}_v$ induces a partial order on the double coset space $W_{K_v}\backslash\widetilde{W}_v/W_{K_v}$. Namely for each double coset $\omega \in W_{K_v}\backslash\widetilde{W}_v/W_{K_v}$ there exists a unique representative $\widetilde{\omega} \in \widetilde{W}_v$ with $\widetilde{\omega} \preccurlyeq \widetilde{x}$ for all

$\widetilde{x} \in \omega$ [PRS13, page 51 above 4.2.7]. Using this representative we define $\omega_1 \preccurlyeq \omega_2$ if and only if $\widetilde{\omega}_1 \preccurlyeq \widetilde{\omega}_2$.

We recall from [PR08, Proposition 8.1] that we have $I_v \cdot N(\breve{Q}_v) \cdot I_v = \mathbf{G}_v(\breve{Q}_v)$ which induces a bijection $I_v \backslash \mathbf{G}_v(\breve{Q}_v)/I_v \simeq \widetilde{W}_v$. More generally we have

$$K_v \backslash \mathbf{G}_v(\breve{Q}_v)/K_v \simeq W_{K_v} \backslash \widetilde{W}_v / W_{K_v}. \tag{22}$$

This set enumerates the Schubert cells in the affine flag varierty $\mathcal{F}l_{v,\mathbb{F}_q^{alg}}$. We recall its definition:

Definition 4.8 ([PR08, Definition 8.3]). *Let $\omega \in W_{K_v} \backslash \widetilde{W}_v / W_{K_v}$, then the Schubert cell C_ω is the reduced subscheme $L^+\mathbb{G}_v \cdot n_\omega \cdot L^+\mathbb{G}_v / L^+\mathbb{G}_v \subset \mathcal{F}l_{v,\mathbb{F}_q^{alg}}$, where $n_\omega \in N_v(\breve{Q}_v)$ is a representative of ω. The Schubert variety $\mathcal{S}_\omega$ is the reduced scheme with underlying set the Zariski closure $\overline{C}_\omega$ of the Schubert cell C_ω. It is a projective variety over $\mathbb{F}_q^{alg}$.*

From [Ric13, Proposition 2.8] it follows that we have $\mathcal{S}_\omega \subset \mathcal{S}_{\omega'}$ if and only if $\omega \leqslant \omega'$. We would like to recall this proposition which describes these closure relations and also the dimension of the Schubert varieties using particular representatives in every coset of $W_{K_v} \backslash \widetilde{W}_v / W_{K_v}$.

By [Ric13, Lemma 1.6] there exists for all $\widetilde{\omega} \in \widetilde{W}_v$ a unique element ${}_{K_v}\omega^{K_v} \in \widetilde{W}_v$ with

$$l({}_{K_v}\omega^{K_v}) = \underset{\omega_1 \in W_{K_v}}{Max} \quad \underset{\omega_2 \in W_{K_v}}{Min} \; l(\omega_1 \widetilde{\omega} \omega_2) \tag{23}$$

As it is commonly done in the literatur we denote by ${}_{K_v}\widetilde{W}_v^{K_v}$ the set $\{{}_{K_v}\omega^{K_v} \mid \omega \in \widetilde{W}_v \}$. Then ${}_{K_v}\widetilde{W}_v^{K_v}$ maps bijectively to $W_{K_v} \backslash \widetilde{W}_v / W_{K_v}$ and we recall:

Proposition 4.9 ([Ric13, Proposition 2.8]). *Let $\omega \in {}_{K_v}\widetilde{W}_v^{K_v}$ then we have*

- $\mathcal{S}_\omega = \bigcup_{\substack{\omega' \in {}_{K_v}\widetilde{W}_v^{K_v} \\ \omega' \leqslant \omega}} \mathcal{C}_{\omega'}$
- $\dim \mathcal{S}_\omega = l(\omega)$

Now let $\hat{Z}_v$ be any bound in $\hat{\mathcal{F}}l_v$ as in § 2.6 and let Z_v be its special fiber and $\breve{Z}_v = Z_v \times_{\kappa_{\hat{Z}_v}} \mathbb{F}_q^{alg} \subset \mathcal{F}l_{v,\mathbb{F}_q^{alg}}$. By definition of the boundedness condition in § 2.6 the special fiber Z_v is a projective closed $L^+\mathbb{G}_v$ invariant subscheme of $\mathcal{F}l_{v,\kappa_{\hat{Z}_v}}$. The definition 4.8 of Schubert varieties shows that we can write $\breve{Z}_v^{red} = \bigcup_{\omega \in I} \mathcal{S}_\omega$ for some subset $I \subset W_{K_v} \backslash \widetilde{W}_v / W_{K_v}$. Since we used Schubert varieties here, the union is not disjoint. This leads to the following definition:

Definition 4.10. *We define the $\hat{Z}_v$-admissible subset of $W_{K_v} \backslash \widetilde{W}_v / W_{K_v}$ to be the set*

$$Adm(\hat{Z}_v) = \{\omega \in W_{K_v} \backslash \widetilde{W}_v / W_{K_v} \mid \mathcal{S}_\omega \subset \breve{Z}_v\} \subset W_{K_v} \backslash \widetilde{W}_v / W_{K_v}$$

The proposition 4.9 tells us directly that $\omega \in Adm(\hat{Z}_v)$ and $\omega' \preccurlyeq \omega$ implies $\omega' \in Adm(\hat{Z}_v)$ and that $\breve{Z}_v^{red} = \bigcup_{\omega \in Adm(\hat{Z}_v)} \mathcal{C}_\omega$ is a stratification of $\breve{Z}_v$.

In the setting of Shimura varieties the μ-admissible subset $Adm(\mu)$ corresponding to some dominant minuscule cocharacter μ is defined by $\left\{w \in \tilde{W} \mid w \leqslant t^{x(\mu)} \text{ for some } x \in W_0\right\}$ (compare [HR17]). This cocharacter μ corresponds in our setting to the bound $\hat{Z}_{\underline{v}}$. Now $Adm(\mu)$ is the natural index set of the Kottwitz-Rapoport strata in the special fiber of a Shimura variety. We will see in theorem 6.2 that our definition of $Adm(\hat{Z}_v)$ satisfy this property for the Kottwitz-Rapoport stratification of $\nabla_n^{\hat{Z}_{\underline{v}},H} \mathscr{H}^1(C, \mathbb{G})$ as well.

4.3 The Set $B(\mathbf{G}_v)$ and the Newton Stratification

In this section we introduce the necessary notations to define the Newton stratification. The idea of the Newton stratification is to define two geometric points of $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$ to lie in the same Newton stratum if their associated local-shtukas are quasi-isogenuous at a choosen subset of the characteristic places $\underline{v}$.
Let k be an algebraically closed field over $\mathbb{F}_v$ and $v \in \underline{v}$, then we define

$$B(\mathbf{G}_v) := \{b \in L\mathbf{G}_v(k)\}/\sim, \qquad \text{where } b_1 \sim b_2 \text{ if there exists } g \in L\mathbf{G}_v(k) \text{ with } b_1 = g^{-1}b_2\sigma_v{}^\star g$$

This set is independent of k by [RR96, Lemma 1.3]. The set of quasi-isogeny classes of local $\mathbb{G}_v$-shtukas over k is in bijection to this set $B(\mathbf{G}_v)$. It sends a local $\mathbb{G}_v$-shtuka $\underline{\mathcal{L}} = (L^+\mathbb{G}_v, b)$ with $b \in L\mathbf{G}_v(k)$ to $[b]$. Since k is algebraically closed every $L^+\mathbb{G}_v$-torsor is trivial over k and changing $\underline{\mathcal{L}}$ by a quasi-isogeny $g \in L\mathbf{G}_v$ would change b to $g^{-1}b\sigma_v{}^\star g$, so that the map is well defined and bijective. Furthermore we note that for a local shtuka $\underline{\mathcal{L}}$ over any field k' its quasi-isogeny class in $B(\mathbf{G}_v)$ does not depend on an algebraic closure $(k')^{alg}$. For such a local $\mathbb{G}_v$-shtuka we denote by $[\underline{\mathcal{L}}]$ the corresponding element in $B(\mathbf{G}_v)$.
The Kottwitz-map (18) on page 47 induces a map

$$\kappa_{\mathbf{G}_v} : B(\mathbf{G}_v) \to \pi_1(\mathbf{G}_v)_\Sigma$$

which we denote again by $\kappa_{\mathbf{G}_v}$. Let $X_\star(T_v)_{\Sigma_0} := X_\star(T_v)/\langle \gamma\alpha - \alpha | \gamma \in \Sigma_0,\ \alpha \in X_\star(T_v)\rangle$ be the coinvariants and $X_\star(T_v)^{\Sigma_0}$ the fixpoints of $X_\star(T_v)$ under the action of Σ_0. Note that

$$X_\star(T_v)^{\Sigma_0}_{\mathbb{Q}} := X_\star(T_v)^{\Sigma_0} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} X_\star(T_v)_{\Sigma_0,\mathbb{Q}},\ \alpha \mapsto \overline{\alpha} \tag{24}$$

is a bijection.
Let $(X_\star(T_v)^{\Sigma_0}_{\mathbb{Q}})^+ \subset X_\star(T_v)^{\Sigma_0}_{\mathbb{Q}}$ be the dominant elements with respect t o the chosen Borel subgroup B_v and let $\left((X_\star(T_v)^{\Sigma_0}_{\mathbb{Q}})^+\right)^{\langle\sigma_v\rangle}$ be the σ_v-invariants. Furthermore we denote as usual by

$$\nu_{\mathbf{G}_v} : B(\mathbf{G}_v) \to \left((X_\star(T_v)^{\Sigma_0}_{\mathbb{Q}})^+\right)^{\langle\sigma_v\rangle} \tag{25}$$

the Newton-map, compare [Kot97]. By [Kot97, p. 4.13] the product of the Newton- and Kottwitz-map

$$B(\mathbf{G}_v) \to \left((X_\star(T_v)^{\Sigma_0}_{\mathbb{Q}})^+\right)^{\langle\sigma_v\rangle} \times \pi_1(\mathbf{G}_v)_\Sigma \qquad [b] \mapsto (\nu_{\mathbf{G}_v}([b]),\ \kappa_{\mathbf{G}_v}([b])) \tag{26}$$

is injective and used to equip $B(\mathbf{G}_v)$ with a partial order as follows.
The choice of the Borel subgroup B_v determines a set of simple roots and the dominance order on $X_\star(T_v)^+_{\mathbb{Q}}$. By definition $\alpha \leqslant \alpha'$ in this dominance order if and only if $\alpha' - \alpha$ is a non-negative $\mathbb{Q}$-sum of the simple roots. Now this dominance order defines together with $\nu_{\mathbf{G}_v}$ and $\kappa_{\mathbf{G}_v}$ a partial order on $B(\mathbf{G}_v)$. Namely for $[b],[b'] \in B(\mathbf{G}_v)$ we set

$$[b] \leqslant [b'] \text{ if and only if } \nu_{\mathbf{G}_v}([b]) \leqslant \nu_{\mathbf{G}_v}([b']) \text{ and } \kappa_{\mathbf{G}_v}([b]) = \kappa_{\mathbf{G}_v}([b']).$$

We will also equip the product $\prod_{v\in\underline{v}} B(\mathbf{G}_v)$ with the partial ordering defined by $([b_v])_{v\in\underline{v}} \leqslant ([b'_v])_{v\in\underline{v}}$ if and only if $[b_v] \leqslant [b'_v]$ for all $v \in \underline{v}$. Now we recall the following proposition.

Proposition 4.11 (compare [RR96, Theorem 3.6], [HV11, Theorem 7.3])**.** *Let S be an $\mathbb{F}_v$-scheme and $\underline{\mathcal{L}}$ a local $\mathbb{G}_\nu$-shtuka over S and $b \in B(\mathbf{G}_v)$. Then the set $\{s \in S \,|[\underline{\mathcal{L}}_s] \leqslant b\}$ is a Zariski closed subset of S. We equip it with the reduced subscheme structure and denote this subscheme with $N_{\leqslant b}$. Furthermore $N_b := \{s \in S \mid [\underline{\mathcal{L}}_s] = b\}$ defines an open subset of $N_{\leqslant b}$ and hence a locally closed subscheme of S. N_b is called the Newton stratum associated with $\underline{\mathcal{L}}$ and b.*

Proof: Compare [RR96, Theorem 3.6] and [HV11, Theorem 7.3]. □

This proposition implies directly that we get the following stratification maps in the sense of definition 4.2 on $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathcal{H}^1(C,\mathbb{G})_{\mathbb{F}_R} := \nabla_n^{\hat{Z}_{\underline{v}},H}\mathcal{H}^1(C,\mathbb{G})\times_{SpfR_Z}\mathbb{F}_R$.

Definition 4.12 (Newton stratification)**.** *For every algebraically closed field k over $\mathbb{F}_q$ we define*

$$\begin{aligned}\delta_{\mathbb{G},v,k} : \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_R}(k) &\xrightarrow{\Gamma_\nu} Sht_{\mathbf{G}_v}(k) && \longrightarrow B(\mathbf{G}_v)\\ (\mathcal{G},s_1,\ldots,s_n,\tau,\gamma) &\longmapsto \Gamma_v(\underline{\mathcal{G}}) \simeq (L^+\mathbb{G}_v,b_v) && \mapsto [b_v]\end{aligned}$$

Furthermore we define

$$\begin{aligned}\delta_{\mathbb{G},k} := \left(\prod_{v\in\underline{v}}\delta_{\mathbb{G},v,k}\right) : \quad \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_R}(k) &\longrightarrow \prod_{v\in\underline{v}}B(\mathbf{G}_v)\\ (\underline{\mathcal{G}},\gamma) &\longmapsto \left(\delta_{\mathbb{G},v,k}(\underline{\mathcal{G}})\right)_{v\in\underline{v}}\end{aligned}$$

Then by proposition 4.11 $\delta_{\mathbb{G},v} = (\delta_{\mathbb{G},v,k})_k$ and $\delta_{\mathbb{G}} = (\delta_{\mathbb{G},k})_k$ are stratification maps in the sense of definition 4.2 and for $b_v \in B(\mathbf{G}_v)$ (resp. $\underline{b} := (b_v)_{v\in\underline{v}} \in \prod_{v\in\underline{v}}B(\mathbf{G}_v)$) we denote by $\mathcal{N}_{b_v}$ (resp. $\mathcal{N}_{\underline{b}}$) the associated locally closed substack of $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathcal{H}^1(C,\mathbb{G})_{\mathbb{F}_R}$ and call it the Newton stratum associated with b_v (resp. $\underline{b}$).

4.4 The Local Model and the Kottwitz-Rapoport Stratification

We recall the definition of the local model of $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$ from [AH16, §4.4]. Using this local model roof we can then define the Kottwitz-Rapoport stratification.
We denote by $\nabla_n^{\hat{Z}_{\underline{v}},H}\widetilde{\mathscr{H}^1(C,\mathbb{G})}$ the stack fibered over $(\mathbb{F}_q)_{\acute{E}t}$ whose S-valued points consists of tuples $\left(\underline{\mathcal{G}},\gamma,(\epsilon_v)_{v\in\underline{v}}\right)$, where $(\underline{\mathcal{G}},\gamma) \in \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})(S)$ and $\epsilon_v : L^+\mathbb{G}_{v,S} \xrightarrow{\sim} \Gamma_v(\sigma^\star\mathcal{G})$ is a trivialization of the $L^+\mathbb{G}_v$-torsor $\Gamma_v(\sigma^\star\mathcal{G})$ over S. The map

$$\nabla_n^{\hat{Z}_{\underline{v}},H}\widetilde{\mathscr{H}^1(C,\mathbb{G})} \longrightarrow \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) \quad \left(\underline{\mathcal{G}},\gamma,(\epsilon_v)_{v\in\underline{v}}\right) \mapsto \left(\underline{\mathcal{G}},\gamma\right) \tag{27}$$

is a $\prod_{v\in\underline{v}}L^+\mathbb{G}_v$-torsor. Furthermore we fix for all $v \in \underline{v}$ a finite extension R_v of $R_{\hat{Z}_v}$ with a representative $\hat{Z}_{v,R_v} \subset \hat{\mathcal{F}l}_{\mathbb{G}_v,R_v}$ of $\hat{Z}_v$. Choosing for all $v_i \in \underline{v}$ a uniformizer π'_{v_i} in R_{v_i} we write κ' for the compositum of all the residue fields $R_{v_i}/(\pi_{v_i})$ and define $R_{\underline{v}} = \kappa'[\![\pi'_{v_1},\ldots,\pi'_{v_n}]\!]$. Then we denote by $\nabla_n^{\hat{Z}_{\underline{v}},H}\widetilde{\mathscr{H}^1(C,\mathbb{G})}_{R_{\underline{v}}} := \nabla_n^{\hat{Z}_{\underline{v}},H}\widetilde{\mathscr{H}^1(C,\mathbb{G})}\times_{R_{\hat{Z}_{\underline{v}}}}R_{\underline{v}}$ the base change to the ring $R_{\underline{v}}$. There is a smooth morphism

$$\hat{\psi} : \nabla_n^{\hat{Z}_{\underline{v}},H}\widetilde{\mathscr{H}^1(C,\mathbb{G})}_{R_{\underline{v}}} \longrightarrow \prod_{v\in\underline{v}}\hat{Z}_{v,R_{\underline{v}}} \tag{28}$$

defined as follows (compare [AH16]). Let $(\underline{\mathcal{G}},\gamma,(\epsilon_v)_{v\in\underline{v}})$ with $\underline{\mathcal{G}} = (\mathcal{G},s_1,\ldots,s_n,\tau_{\mathcal{G}})$ be an S-valued point in $\nabla_n^{\hat{Z}_{\underline{v}},H}\widetilde{\mathscr{H}^1(C,\mathbb{G})}_{R_{\underline{v}}}(S)$ and choose a trivialization $\alpha_v : \Gamma_v(\mathcal{G})_{S'} \to L^+\mathbb{G}_{v,S'}$ over some étale covering $S' \to S$. The composition $\alpha_v \circ \Gamma_v(\tau_{\mathcal{G}}) \circ \epsilon_v : \boldsymbol{L}\mathbf{G}_{v,S'} \to \boldsymbol{L}\mathbf{G}_{v,S'}$ defines an S'-valued point in $\mathcal{F}l_{\mathbb{G}_v}$ which is independent of α_v and hence descends to an S-valued point in $\mathcal{F}l_{\mathbb{G}_v}$. The induced morphism $S \to \hat{\mathcal{F}}l_{\mathbb{G}_v,R_v}$ factors by the boundedness condition through $\hat{Z}_{v,R_v}$. This defines the map (28).

Definition 4.13. *The product $\prod_{v\in\underline{v}} \hat{Z}_{v,R_v}$ is called a local model for $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$.*

We recall the following theorem:

Theorem 4.14 ([AH16, Theorem 4.4.6])**.** *Consider the local model roof*

$$\begin{array}{ccc} & \nabla_n^{\hat{Z}_{\underline{v}},H}\widetilde{\mathscr{H}^1(C,\mathbb{G})}_{R_{\underline{v}}} & \\ \swarrow & & \searrow{\scriptstyle \hat{\psi}} \\ \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{R_{\underline{v}}} & & \prod\limits_{v\in\underline{v}} \hat{Z}_{v,R_{\underline{v}}} \end{array}$$

induced from the $\prod\limits_{v\in\underline{v}} L^+\mathbb{G}_v$-torsor in (27) *and the smooth morphism $\hat{\psi}$ in* (28)*. Let y be a geometric point of $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{R_{\underline{v}}}$. Then the $\prod\limits_{v\in\underline{v}} L^+\mathbb{G}_v$-torsor*

$$\nabla_n^{\hat{Z}_{\underline{v}},H}\widetilde{\mathscr{H}^1(C,\mathbb{G})} \to \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) \quad \big(\underline{\mathcal{G}},\gamma,(\epsilon_v)_{v\in\underline{v}}\big) \mapsto \big(\underline{\mathcal{G}},\gamma\big)$$

admits locally over an étale neighboorhood of y a section s such that the composition with $\hat{\psi}$ is étale.

As we have already explained in § 2.7 the special fiber $\hat{Z}_{v,R_v} \times_{Spf\ R_v} \kappa_{R_v}$ arises by Galois descent from a unique closed subscheme $Z_v \subset \mathcal{F}l_{\mathbb{G}_v} \times_{\mathbb{F}_v} \kappa_{\hat{Z}_v}$ which we called the special fiber of $\hat{Z}_v$. In particular the morphism $\hat{\psi} \times id_{\kappa'} : \nabla_n^{\hat{Z}_{\underline{v}},H}\widetilde{\mathscr{H}^1(C,\mathbb{G})}_{\kappa'} \longrightarrow \prod\limits_{v\in\underline{v}} Z_{v,\kappa'}$ induced by (28) arises from a morphism $\psi : \nabla_n^{\hat{Z}_{\underline{v}},H}\widetilde{\mathscr{H}^1(C,\mathbb{G})}_{\mathbb{F}_R} \longrightarrow \prod\limits_{v\in\underline{v}} Z_{v,\mathbb{F}_R}$ (that means $\hat{\psi} \times_{Spf\ R_{\underline{v}}} id_{\kappa'} = \psi \times_{\mathbb{F}_R} id_{\kappa'}$). This gives the local model roof in the special fiber:

$$\begin{array}{ccc} & \nabla_n^{\hat{Z}_{\underline{v}},H}\widetilde{\mathscr{H}^1(C,\mathbb{G})}_{\mathbb{F}_R} & \\ \swarrow & & \searrow{\scriptstyle \psi} \\ \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_R} & & \prod\limits_{v\in\underline{v}} Z_{v,\mathbb{F}_R} \end{array}. \tag{29}$$

Concerning the left $\prod_{v\in\underline{v}} L^+\mathbb{G}_v$ action on $\prod\limits_{v\in\underline{v}} Z_v$ the morphism ψ is $\prod\limits_{v\in\underline{v}} L^+\mathbb{G}_v$ equivariant. The diagram (29) induces by definition of the quotient stack a morphism

$$\lambda_{\mathbb{G}} : \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_R} \to \left[\prod_{v\in\underline{v}} L^+\mathbb{G}_v \Big\backslash \prod_{v\in\underline{v}} Z_{v,\mathbb{F}_R}\right].$$

This morphism completes together with the projection $\prod\limits_{v\in\underline{v}} Z_{v,\mathbb{F}_R} \to \left[\prod\limits_{v\in\underline{v}} L^+\mathbb{G}_v \Big\backslash \prod\limits_{v\in\underline{v}} Z_{v,\mathbb{F}_R}\right]$ the diagram (29) to a cartesian diagram. In particular $\lambda_{\mathbb{G}}$ is a smooth morphism.

Now a morphism $f : \mathbb{G} \to \mathbb{G}'$ of smooth affine group schemes that induces a morphism of shtuka data $(id, f) : (C, \mathbb{G}, \underline{v}, \hat{Z}_v, H) \to (C, \mathbb{G}', \underline{v}, \hat{Z}'_v)$ induces also a morphism of the local models. Later we will need the following lemma.

Lemma 4.15. *Let $(id, f) : (C, \mathbb{G}, \underline{v}, \hat{Z}_v, H) \to (C, \mathbb{G}', \underline{v}, \hat{Z}'_v)$ be a morphism of shtuka data such that $f : \mathbb{G} \to \mathbb{G}'$ is a morphism of parahoric Bruhat-Tits group schemes over C, such that $f|_{C \setminus \underline{v}}$ is an isomorphism and such that $\hat{Z}_{\underline{v}}$ arises from base change of $\hat{Z}'_{\underline{v}}$. Then the induced morphism*

$$p_{\mathbb{G},\mathbb{G}',v} : Z_v \to Z'_v$$

of the special fibers of the local models is proper and surjective. We denote the induced morphism

$$p_{\mathbb{G},\mathbb{G}'} : \left[\prod_{v\in\underline{v}} L^+\mathbb{G}_v \Big\backslash \prod_{v\in\underline{v}} Z_{v,\mathbb{F}_R}\right] \to \left[\prod_{v\in\underline{v}} L^+\mathbb{G}'_v \Big\backslash \prod_{v\in\underline{v}} Z'_{v,\mathbb{F}_R}\right] \qquad \text{by} \qquad p_{\mathbb{G},\mathbb{G}'}.$$

Proof: Since Z_v and Z'_v are both projective it is clear that the morphism is projective. The surjectivity follows by the surjectivity of $\mathcal{F}l_{\mathbb{G}_v} \to \mathcal{F}l_{\mathbb{G}'_v}$ and the condition $Z_v = Z'_v \times_{\mathcal{F}l'_v} \mathcal{F}l_v$. □

Now we use the constructed morphism $\lambda_{\mathbb{G}}$ to define the Kottwitz-Rapoport stratification of $\nabla_n^{\hat{Z}_n,H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_R}$. Before we do this we take a closer look to the stack $\left[\prod_{v\in\underline{v}} L^+\mathbb{G}_v \Big\backslash \prod_{v\in\underline{v}} Z_{v,\mathbb{F}_R}\right]$.

Lemma 4.16. *The topological space of the quotient stack $\left[\prod_{v\in\underline{v}} L^+\mathbb{G}_v \Big\backslash \prod_{v\in\underline{v}} \breve{Z}_v\right]$ is given by the set $\prod_{v\in\underline{v}} Adm(\hat{Z}_v)$ endowed with the product topology, where each set $Adm(\hat{Z}_v)$ carries the topology induced by the partial order of $W_{K_v}\backslash\widetilde{W}_v/W_{K_v}$. In particular each element $\underline{\omega} = (\omega_v)_{v\in\underline{v}}$ in $\prod_{v\in\underline{v}} Adm(\hat{Z}_v)$ (resp. $\omega_v \in Adm(\hat{Z}_v)$) defines a locally closed substack $\left[\prod_{v\in\underline{v}} L^+\mathbb{G}_v \Big\backslash \prod_{v\in\underline{v}} \breve{Z}_v\right]_{\underline{\omega}}$ (resp. $\left[\prod_{v\in\underline{v}} L^+\mathbb{G}_v \Big\backslash \prod_{v\in\underline{v}} \breve{Z}_v\right]_{\omega_v}$) of $\left[\prod_{v\in\underline{v}} L^+\mathbb{G}_v \Big\backslash \prod_{v\in\underline{v}} \breve{Z}_v\right]$, which defines a stratification on it.*

Proof: This follows by the stratification of $\breve{Z}_v = \bigcup_{\omega\in Adm(\hat{Z}_v)} C_\omega$ (see definition 4.10). An $\mathbb{F}_q^{alg}$-valued point of $\left[\prod_{v\in\underline{v}} L^+\mathbb{G}_v \Big\backslash \prod_{v\in\underline{v}} \breve{Z}_v\right]$ is namely given by an $\prod_{v\in\underline{v}} L^+\mathbb{G}_v$-torsor (which is trivial over $\mathbb{F}_q^{alg}$) and an equivariant map to $\prod_{v\in\underline{v}} \breve{Z}_{v,\mathbb{F}_R}$, which is given by an element in $\prod_{v\in\underline{v}} L^+\mathbb{G}_v \cdot n_{\omega_v} L^+\mathbb{G}_v \big/ L^+\mathbb{G}_v$ for some $\omega_v \in Adm(\hat{Z}_v)$ and some representatives n_{ω_v} of ω_v. The isomorphy classes of $\mathbb{F}_q^{alg}$-valued points are therefore given by $\prod_{v\in\underline{v}} Adm(\hat{Z}_v)$. The claim on the topology and locally closed substacks follows by the closure relations in proposition 4.9. □

Definition 4.17 (Kottwitz-Rapoport stratification)**.** *For every algebraically closed field k over $\mathbb{F}_q$ we define*

$$\lambda_{\mathbb{G},k} : \quad \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_q^{alg}}(k) \xrightarrow{\lambda_{\mathbb{G}}} \left\|\left[\prod_{v\in\underline{v}} L^+\mathbb{G}_v \Big\backslash \prod_{v\in\underline{v}} \breve{Z}_v\right]\right\| = \prod_{v\in\underline{v}} Adm(\hat{Z}_v)$$

Furthermore we define for $v \in \underline{v}$

$$\lambda_{\mathbb{G},v,k} : \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_q^{alg}}(k) \xrightarrow{\lambda_{\mathbb{G}}} \prod_{\tilde{v}\in\underline{v}} Adm(\hat{Z}_{\tilde{v}}) \longrightarrow Adm(\hat{Z}_v)$$

Then $\lambda_{\mathbb{G}} = (\lambda_{\mathbb{G},k})_k$ *and* $\lambda_{\mathbb{G},v} = (\lambda_{\mathbb{G},v,k})_k$ *are stratification maps in the sense of definition 4.2 and for* $\underline{\omega} = (\omega_v)_v \in \prod_{v\in\underline{v}} Adm(\hat{Z}_v)$ *(resp.* $\omega_v \in Adm(\hat{Z}_v)$*) we denote by* $KR_{\underline{\omega}}$ *(resp.* KR_{ω_v}*) the associated locally closed substack of* $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathcal{H}^1(C,\mathbb{G})_{\mathbb{F}_q^{alg}}$ *and call it the Kottwitz-Rapoport stratum associated with* $\underline{\omega}$ *(resp.* ω_v*). That means* $KR_{\underline{w}}(k) = \lambda_{\mathbb{G},k}^{-1}(k)$.

It is clear, that $KR_{\underline{\omega}}$ is given as the pullback of $\left[\prod_{v\in\underline{v}} L^+\mathbb{G}_v \Big\backslash \prod_{v\in\underline{v}} \breve{Z}_v\right]_{\underline{\omega}}$ under the map $\lambda_{\mathbb{G}}$.

4.5 σ-Straight Elements and Affine Deligne-Lusztig Varieties

As before we denote by σ_v the $\mathbb{F}_v$-Frobenius that generates $Gal(\breve{Q}_v/Q_v)$. Since N_v and $T_{v,1}$ are defined over Q_v the definition 4.6 of the Iwahori-Weyl group shows directly that we have a natural action of $\langle\sigma_v\rangle$ on $\widetilde{W}_v$. We call the group $\mathbf{G}_v$ residually split if this action of σ_v is trivial and we note as in [HZ16, beginning of section 7] that split implies residually split, whereas residually split implies quasi-split.
The above action gives us a semi-direct product $\widetilde{W}_v \rtimes \langle\sigma_v\rangle$ which allows us to write $(\omega\sigma_v)^n = \omega\sigma_v(\omega)\ldots\sigma_v^{n-1}(\omega)\sigma_v^n$, where $\omega \in \widetilde{W}_v$. We extend the length function on $\widetilde{W}_v$ to $\widetilde{W}_v \rtimes \langle\sigma_v\rangle$ by defining $l(\sigma_v) = 0$ and we recall the following definition.

Definition 4.18 (compare also [HN14, section 1.3] and [HR17, section 5.1])**.**
An element $\omega \in \widetilde{W}_v$ *is called* σ_v*-straight if* $l((\omega\sigma_v)^m) = ml(\omega)$ *for all* $m \in \mathbb{N}$*. Let* $B(\widetilde{W}_v) = \widetilde{W}_v/\sim$*, with* $\omega_1 \sim \omega_2$ *if and only if* $\omega_1 = g^{-1}\omega_2\sigma_v(g)$ *for some* $g \in \widetilde{W}_v$*, denote the* σ_v *conjugacy classes of* $\widetilde{W}_v$*. Then such a* σ_v*-conjugacy class* $[\omega] \in B(\widetilde{W}_v)$ *is called* σ_v*-straight if there is a representative* $g^{-1}\omega\sigma_v(g) \in [\omega]$ *which is* σ_v*-straight. We denote the set of* σ_v*-straight conjugacy classes by* $B(\widetilde{W}_v)_{\sigma_v-str}$*. If* $\langle\sigma_v\rangle$ *operates trivially on* $\widetilde{W}_v$*, we only speak about straight elements and straight conjugacy classes.*

Remark 4.19. In [He14, section 2.4] He defines $\omega \in \widetilde{W}_v$ to be σ_v-straight if and only if $l(\omega) = \langle\nu_{\mathbf{G}_v}(\omega), 2\rho\rangle$ with the remark that this definition coincides with the previous one by [HN14, Lemma 1.1]. Here we denote as usual by ρ the halfsum of all positive coroots.

We recall that we have $\widetilde{W}_v = X_\star(T)_{\Sigma_0} \rtimes W_0$ and that we denoted its projection to $\pi_1(\mathbf{G})_{\Sigma_0}$ in section 4.2 by $\kappa_{\mathbf{G}_v}$. Its projection to $\pi_1(\mathbf{G})_\Sigma$ is invariant under σ_v-conjugation and we denote the resulting map still by

$$\kappa_{\mathbf{G}_v} : B(\widetilde{W}_v) \to \pi_1(\mathbf{G})_\Sigma$$

Furthermore we referred in equation (25) in section 4.3 to the Newton-map $\nu_{\mathbf{G}_v}$. Viewing it as a map $\mathbf{G}_v(\breve{Q}_v) \to \left((X_\star(T_v)_{\mathbb{Q}}^{\Sigma_0})^+\right)^{\langle\sigma_v\rangle}$ which is invariant under σ_v-conjugacy, we can describe its restriction to $N_v(\breve{Q}_v)$ as follows (compare also [He14, section 1.7] and [HR17, section 5.1]). For $\dot{\omega} \in N_v(\breve{Q}_v)$ let ω be its image in $\widetilde{W}_v = N_v(\breve{Q}_v)/T_{v,1} = X_\star(T_v)_{\Sigma_0} \rtimes W_0$. Then there is a natural number n such that $(\omega\sigma_v)^n = \lambda \in X_\star(T_v)_{\Sigma_0}$ and $\sigma_v{}^n$ acts trivially on $\widetilde{W}_v$. In particular, using the bijection (24) in section 4.3, we get an element $\frac{\lambda}{n}$ in $(X_\star(T_v)_{\mathbb{Q}}^{\Sigma_0})^{\langle\sigma_v\rangle}$, which is independent of n. The unique dominant element in its W_0-orbit is then defined to be $\nu_{\mathbf{G}_v}(\dot{\omega}) \in \left(\left(X_\star(T_v)_{\mathbb{Q}}^{\Sigma_0}\right)^+\right)^{\langle\sigma_v\rangle}$.
Since the described map $N_v(\breve{Q}_v) \to \widetilde{W}_v \to \left(\left(X_\star(T_v)_{\mathbb{Q}}^{\Sigma_0}\right)^+\right)^{\langle\sigma_v\rangle}$ is invariant under σ_v it induces a

map

$$B(\widetilde{W}_v) \longrightarrow \left(\left(X_\star(T_v)_{\mathbb{Q}}^{\Sigma_0}\right)^+\right)^{\langle\sigma_v\rangle}, \qquad \text{still denoted by } \nu_{\mathbf{G}_v}.$$

Writing $\varrho_v : B(\widetilde{W}_v) \longrightarrow \left(\left(X_\star(T_v)_{\mathbb{Q}}^{\Sigma_0}\right)^+\right)^{\langle\sigma_v\rangle} \times \pi_1(\mathbf{G}_v)_\Sigma, \qquad [\omega] \mapsto (\nu_{\mathbf{G}_v}([\omega]), \kappa_{\mathbf{G}_v}([\omega]))$

we recall the following proposition. Note that in contrast to the map (26) in section 4.3 this map ϱ_v is not injective.

Theorem 4.20 ([He14, Theorem 3.5]). *Let $\omega \in \widetilde{W}_v$ be σ-straight, $\dot{\omega} \in N_v(\breve{Q}_v)$ a representative and I_v the chosen Iwahori subgroup. Then the set $I_v\omega I_v \subset N_v(\breve{Q}_v)$ is contained in a single σ-conjugacy class of $\mathbf{G}_v(\breve{Q}_v)$.*

Then He uses this theorem to prove:

Proposition 4.21 ([He14, Proposition 3.6]). *Let $[\omega], [\omega'] \in B(\widetilde{W}_v)$ such that $\omega \in \widetilde{W}_v$ and $\omega' \in \widetilde{W}_v$ are of minimal length in their σ_v-conjugacy classes $[\omega], [\omega']$. Choose furthermore two representatives $\dot{\omega}, \dot{\omega}' \in N_v(\breve{Q}_v)$ of ω and ω' respectively. Then $\dot{\omega}$ and $\dot{\omega}'$ are in the same σ_v-conjugacy class of $\mathbf{G}(\breve{Q}_v)$ if and only if $\varrho_v([\omega]) = \varrho([\omega'])$.*

In particular we have a well defined map:

$$B(\widetilde{W}_v) \to B(\mathbf{G}_v)$$

and the following theorem implies that it restricts to a bijection

$$\Psi : B(\widetilde{W}_v)_{\sigma-str} \longrightarrow B(\mathbb{G}_v) \tag{30}$$

Theorem 4.22 ([He14, Theorem 3.7]). *For any σ_v-straight σ_v-conjugacy class $x \in B(\widetilde{W}_v)_{\sigma_v-str}$ we choose a minimal length representativ $\omega_x \in \widetilde{W}_v$ with some lift $\dot{\omega}_x$ then:*

$$\mathbf{G}_v(\breve{Q}_v) = \coprod_{x \in B(\widetilde{W}_v)_{\sigma_v-str}} \mathbf{G}_v(\breve{Q}_v) \underset{\sigma_v}{\cdot} \dot{\omega}_x$$

where $g \underset{\sigma_v}{\cdot} \dot{\omega}_x := g \cdot \dot{\omega}_x \cdot \sigma_v^\star g^{-1}$.

Affine Deligne-Lusztig varieties

Let K be any σ_v-invariant parahoric subgroup of $\mathbb{G}_v(\breve{Q}_v)$, for example the group $K_v := \mathbb{G}_v(\breve{A}_v)$. Recall from equation (22) on page 49 that choosing a representative $\dot{\omega} \in N_v(\breve{Q}_v)$ of $\omega \in W_K\backslash\widetilde{W}_v/W_K$ gives

$$\mathbb{G}(\breve{Q}_v) = \coprod_{\omega \in W_K\backslash\widetilde{W}_v/W_K} K\omega K$$

Definition 4.23. *Let $b_v \in B(\mathbf{G}_v)$ and $\omega \in W_K\backslash\widetilde{W}_v/W_K$. Then the affine Deligne-Lusztig variety associated with b_v, ω, K is defined as*

$$X_{K,\omega}(b_v) = \left\{gK \in L\mathbf{G}_v(\mathbb{F}_q^{alg})/K \;\middle|\; g^{-1}b_v\sigma(g) \in K\omega K\right\}$$

For a subset $C \subset W_K\backslash\widetilde{W}_v/W_K$ we set

$$X_{K,C}(b_v) = \bigcup_{\omega \in C} X_{K,\omega}(b_v).$$

If K equals the chosen Iwahori group I_v, we obmit it from the notation and write $X_\omega(b_v)$ (resp. $X_C(b_v)$).

Note that $X_{K,\omega}(b)$ are the $\mathbb{F}_q^{alg}$-valued points of a locally closed ind-subscheme of the partial affine flag variety $\mathcal{F}l_K$, which is actually a scheme locally of finite type over $\mathbb{F}_q^{alg}$ by the theorem [AH14, Theorem 4.18]. If $C \subset W_K \backslash \widetilde{W}_v / W_K$ is closed under the Bruhat order (for example $C = Adm(\hat{Z}_v)$) then $X_{C,\omega}(b)$ are the $\mathbb{F}_q^{alg}$-valued points of a closed subvariety of the partial affine flag variety $\mathcal{F}l_K$. Its irreducible components are projective by [AH14, Corollary 4.26].

5 Axioms on the Moduli Space $\nabla_n^{\hat{Z}_v,H}\mathscr{H}^1(C,\mathbb{G})$

There is an analogous definition of the stratifications, which we introduced in the fourth chapter, for the special fiber of Shimura varieties. In general one can ask a lot of interesting questions about these stratifications: what is their dimension, are they equi-dimensional, are they smooth, are they affine or quasi-affine, what is their relation, do they have the strong stratification property, are they non-empty, ... A lot of work about these questions has been done in the case of Shimura varieties, which is spread out in the literature.
In [HR17] Rapoport and He define five axioms on Shimura varieties. Once these axioms are verified, one can conclude the definition and existence of these characteristic subsets as Newton stratification, Kottwitz-Rapoport stratification and EKOR-stratification in a most general setup. Moreover one can specify precisely their natural index set and draw some further conclusions.
In this chapter we firstly define and then verify five analogous axioms for the moduli space $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$ of global $\mathbb{G}$-shtukas. But before we start doing this, we give a comparison of our setting to the analogous case of Shimura varieties. Here the setting is given as follows (compare also [HR17, chapter 3]):

We start with a Shimura datum $(G,\{h\})$ and an open compact subgroup $\mathbf{K} = K^pK$ of $G(\mathbb{A}_f)$, where $K^p \subset G(\mathbb{A}_f^p)$ and where $K = K_p$ is a parahoric subgroup of $G(\mathbb{Q}_p)$ for a fixed prime number p. We set $G_{\mathbb{Q}_p} := G \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and denote by $\{\mu\}$ the conjugacy class of cocharacters of $G_{\mathbb{Q}_p}$ corresponding to $\{h\}$. The corresponding Shimura variety $Sh_{\mathbf{K}} = Sh(G,\{h\})_{\mathbf{K}}$ is a quasi-projective variety defined over the refelex field $\mathbf{E}$. Let E be the completion of $\mathbf{E}$ at a prime $\mathfrak{p}$ over p and $\mathcal{O}_E$ be its ring of integers with residue field κ_E. Furthermore let $\mathbf{S_K}$ be an integral model over $\mathcal{O}_E$ and $Sh_K = \mathbf{S_K} \times_{Spec\ \mathcal{O}_E} \kappa_E$ its special fiber. Then the comparison to the analogous setting of moduli spaces of global $\mathbb{G}$-shtukas is given by the following table:

$\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$ over $R_{\hat{Z}_{\underline{v}}}$	the integral model $\mathbf{S_K}$ over $\mathcal{O}_E$
$\mathbf{G} := \mathbb{G} \times_C Spec\ Q$ notice that $\mathbb{G}$ can vary in the fibers	$G_{\mathbb{Q}_p}$
fixed characteristic places $\underline{v} = (v_1, \ldots, v_2)$	fixed prime p
$H \subset \mathbb{G}(\mathbb{A}^{\underline{v}}) = \mathbf{G}(\mathbb{A}^{\underline{v}})$ an open compact subgroup	$K^p \subset \mathbf{G}(\mathbb{A}_f^p)$
$(\mathbb{G}(A_v))_{v\in\underline{v}} \subset (\mathbb{G}(Q_v))_{v\in\underline{v}} = (\mathbf{G}_Q(Q_v))_{v\in\underline{v}}$ this subgroup is uniquely determined by $\mathbb{G}$	$K \subset \mathbf{G}(\mathbb{Q}_p)$ a parahoric subgroup
morphism $f : \mathbb{G} \to \mathbb{G}'$ which is an isomorphism over $C\backslash\underline{v}$	changing the subgroup $K \subset K' \subset \mathbf{G}(\mathbb{Q}_p)$
the bound $\hat{Z}_{\underline{v}}$	the cocharacter $\{\mu\}$
$\prod_{v\in\underline{v}} Adm(\hat{Z}_{\underline{v}})$	$Adm(\{\mu\})$
global $\mathbb{G}$-shtukas	abelian varieties
local $\mathbb{G}_v$-shtukas	p-divisible groups

Table 1: Comparision of the settings

5.1 The Axioms

The first axiom concerns a change of the parahoric structure at the characteristic places.

Axiom 1. Let $(id_C, f) : (C, \mathbb{G}, \underline{v}, \hat{Z}_{\underline{v}}, H) \to (C, \mathbb{G}', \underline{v}, \hat{Z}'_{\underline{v}}, H)$ be a morphism of Shtuka data, such that $f : \mathbb{G} \to \mathbb{G}'$ is a morphism of parahoric Bruhat-Tits group schemes over C, such that $f|_{C\setminus\underline{v}}$ is an isomorphism and such that $\hat{Z}_{\underline{v}}$ arises from basechange of $\hat{Z}'_{\underline{v}}$. Then there is a natural morphism of stacks

$$\pi_{\mathbb{G},\mathbb{G}'} : \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) \to \nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}')$$

which is projective and surjective.

The second axiom concerns the local model of the moduli space $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$, which guarantees the existence of the Kottwitz-Rapoport stratification.

Axiom 2 (Existence of the Kottwitz-Rapoport stratification)**.** There is a smooth morphism of algebraic stacks

$$\lambda_{\mathbb{G}} : \nabla_n^{H,\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_R} \longrightarrow \left[\prod_{v\in\underline{v}} L^+\mathbb{G}_v \Big\backslash \prod_{v\in\underline{v}} Z_{v,\mathbb{F}_R}\right].$$

This morphism is compatible with a change of the group scheme $\mathbb{G}$ as in axiom 1 and lemma 4.15, that means $\lambda_{\mathbb{G}'} \circ \pi_{\mathbb{G},\mathbb{G}'} = p_{\mathbb{G},\mathbb{G}'} \circ \lambda_{\mathbb{G}}$.

The third axiom is about the existence of the Newton stratification and its closure relations.

Axiom 3 (Existence of the Newton stratification)**.** There is a stratification map (see definition 4.2)

$$\delta_{\mathbb{G}} : \nabla_n^{H,\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_R} \to \prod_{v\in\underline{v}} B(\mathbf{G}_v).$$

For $\underline{b} = (b_v)_{v\in\underline{v}} \in \prod_{v\in\underline{v}} B(\mathbf{G}_v)$ we denote by $\mathcal{N}_{\underline{b}}$ the associated locally closed substack and call it the Newton stratum associated with $\underline{b}$. For a scheme $S \to \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_R}$ we denote by $\mathcal{N}_{\underline{b},S}$ the pullback of $\mathcal{N}_{\underline{b}}$ to S. We require that the map $\delta_{\mathbf{G}_v}$ satisfies the following two conditions

(i) For every map $\pi_{\mathbb{G},\mathbb{G}'}$ as in Axiom 1 we have $\delta_{\mathbb{G}} = \delta_{\mathbb{G}'} \circ \pi_{\mathbb{G},\mathbb{G}'}$.

(ii) If there is a scheme $S \to \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_R}$ and $\underline{b}, \underline{b}' \in \prod_{v\in\underline{v}} B(\mathbf{G}_v)$ with $\mathcal{N}_{\underline{b},S} \cap \overline{\mathcal{N}_{\underline{b}',S}} \neq \varnothing$ then we have $b \leqslant b'$ in the partial ordered set $\prod_{v\in\underline{v}} B(\mathbf{G}_{v_i})$.

Note that we do not require this stratification to have the strong stratification property. As long as $\mathbb{G}$ is not hyperspecial this property may not be satisfied.
Our next axiom relates the two previous axioms about the Newton and the Kottwitz-Rapoport stratification. To formulate it, we denote by $L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v} \subset L^+\mathbb{G}_v(\mathbb{F}_q^{alg}) \times L^+\mathbb{G}_v(\mathbb{F}_q^{alg})$ the graph of the Frobenius. It operates on $L\mathbf{G}_v(\mathbb{F}_q^{alg})$ by

$$L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v} \times L\mathbf{G}_v(\mathbb{F}_q^{alg}) \to L\mathbf{G}_v(\mathbb{F}_q^{alg}) \quad ((h, \sigma_v(h)), g) \mapsto h^{-1} g \sigma_v(h)$$

so that the set of $L^+\mathbb{G}_v$–σ_v-conjugacy classes is given by $L\mathbf{G}_v(\mathbb{F}_q^{alg}) \Big/ L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v} = L\mathbf{G}_v(\mathbb{F}_q^{alg})/\sim$ with $g_1 \sim g_2$ if and only if there exists a $h \in L^+\mathbb{G}_v(\mathbb{F}_q^{alg})$ with $g_1 = h^{-1} g_2 \sigma_v(h)$. Note that we can similar denote by $L\mathbf{G}_v(\mathbb{F}_q^{alg})_{\sigma_v} \subset L\mathbf{G}_v(\mathbb{F}_q^{alg}) \times L\mathbf{G}_v(\mathbb{F}_q^{alg})$ the graph of the Frobenius on $L\mathbf{G}_v(\mathbb{F}_q^{alg})$, which allows us to write $B(\mathbf{G}_v) = L\mathbf{G}_v(\mathbb{F}_q^{alg}) \big/ L\mathbf{G}_v(\mathbb{F}_q^{alg})_{\sigma_v}$. The reader should be aware, that unlike for $B(\mathbf{G}_v)$ (see [RR96, Lemma 1.3]) this quotient changes if we replace $\mathbb{F}_q^{alg}$ by some other algebraically closed field. Now we have the two embeddings $L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v} \subset L^+\mathbb{G}_v(\mathbb{F}_q^{alg}) \times L^+\mathbb{G}_v(\mathbb{F}_q^{alg})$ and $L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v} \subset L\mathbf{G}_v(\mathbb{F}_q^{alg})_{\sigma_v}$ which induce the two projection maps

$$l_{\mathbb{G}} : \prod_{v \in \underline{v}} L\mathbf{G}_v(\mathbb{F}_q^{alg}) \big/ L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v} \longrightarrow \prod_{v \in \underline{v}} L^+\mathbb{G}_v(\mathbb{F}_q^{alg}) \big\backslash L\mathbf{G}_v(\mathbb{F}_q^{alg}) \big/ L^+\mathbb{G}_v(\mathbb{F}_q^{alg})$$

and

$$d_{\mathbb{G}} : \prod_{v \in \underline{v}} L\mathbf{G}_v(\mathbb{F}_q^{alg}) \big/ L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v} \longrightarrow \prod_{v \in \underline{v}} L\mathbf{G}_v(\mathbb{F}_q^{alg}) \big/ L\mathbf{G}_v(\mathbb{F}_q^{alg})_{\sigma_v} = \prod_{v \in \underline{v}} B(\mathbf{G}_v)$$

We already mentioned on page 49 in (22) that we can identify $L^+\mathbb{G}_v(\mathbb{F}_q^{alg}) \big\backslash L\mathbf{G}_v(\mathbb{F}_q^{alg}) \big/ L^+\mathbb{G}_v(\mathbb{F}_q^{alg})$ with the set $W_{\mathbb{G}_v} \big\backslash \widetilde{W}_v \big/ W_{\mathbb{G}_v}$. Note also that if $f : \mathbb{G} \to \mathbb{G}'$ is a morphism as in axiom 1, which implies $L\mathbf{G}_v = L\mathbf{G}'_v$, then the embedding $L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v} \subset L^+\mathbb{G}'_v(\mathbb{F}_q^{alg})_{\sigma_v}$ gives a projection $L\mathbf{G}_v(\mathbb{F}_q^{alg}) \big/ L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v} \to L\mathbf{G}'_v(\mathbb{F}_q^{alg}) \big/ L^+\mathbb{G}'_v(\mathbb{F}_q^{alg})_{\sigma_v}$ compatible with $d_{\mathbb{G}}$ and $l_{\mathbb{G}}$.
Now we can formulate the fourth axiom, which guarantees the existence of the so called central leaves.

Axiom 4 (Central leaves)**.**

(i) There exist a natural map

$$\Upsilon_{\mathbb{G}} : \nabla_n^{\hat{Z}_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G})(\mathbb{F}_q^{alg}) \longrightarrow \prod_{v \in \underline{v}} L\mathbf{G}_v(\mathbb{F}_q^{alg}) \big/ L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v}$$

such that the following diagram commutes

$$\begin{array}{ccc}
 & \overset{\lambda_{\mathbb{G}}}{\nearrow} & \prod_{v \in \underline{v}} L^+\mathbb{G}_v(\mathbb{F}_q^{alg}) \big\backslash L\mathbf{G}_v(\mathbb{F}_q^{alg}) \big/ L^+\mathbb{G}_v(\mathbb{F}_q^{alg}) \\
 & & \uparrow l_{\mathbb{G}} \\
\nabla_n^{\hat{Z}_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G})(\mathbb{F}_q^{alg}) & \xrightarrow{\Upsilon_{\mathbb{G}}} & \prod_{v \in \underline{v}} L\mathbf{G}_v(\mathbb{F}_q^{alg}) \big/ L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v} \\
 & & \downarrow d_{\mathbb{G}} \\
 & \underset{\delta_{\mathbb{G}}}{\searrow} & \prod_{v \in \underline{v}} B(\mathbf{G}_v)
\end{array}$$

where $\lambda_{\mathbb{G}}$ and $\delta_{\mathbb{G}}$ are the maps from the second and the third axiom. Furthermore we require that for every morphism $\pi_{\mathbb{G},\mathbb{G}'}$ as in the first axiom the diagram

$$\begin{array}{ccc}
\nabla_n^{\hat{Z}_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G})(\mathbb{F}_q^{alg}) & \xrightarrow{\Upsilon_{\mathbb{G}}} & \prod_{v \in \underline{v}} L\mathbf{G}_v(\mathbb{F}_q^{alg}) \big/ L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v} \\
\downarrow \pi_{\mathbb{G},\mathbb{G}'} & & \downarrow \\
\nabla_n^{\hat{Z}_{\underline{v}}, H} \mathscr{H}^1(C, \mathbb{G}')(\mathbb{F}_q^{alg}) & \xrightarrow{\Upsilon_{\mathbb{G}'}} & \prod_{v \in \underline{v}} L\mathbf{G}'_v(\mathbb{F}_q^{alg}) \big/ L^+\mathbb{G}'_v(\mathbb{F}_q^{alg})_{\sigma_v}
\end{array} \tag{31}$$

commutes as well.

(ii) $Im\Upsilon_{\mathbb{G}} = l_{\mathbb{G}}^{-1}(Im\ \lambda_{\mathbb{G}})$. Note that the inclusion $\subset$ already follows from the definition.

(iii) Let $f : \mathbb{G} \to \mathbb{G}'$ as in the first axiom and $y \in Im(\Upsilon_{\mathbb{G}})$. Denote by y' the image under the projection in the above diagram (31). Then we require that the restriction

$$\pi_{\mathbb{G},\mathbb{G}'}(\mathbb{F}_q^{alg})\Big|_{\Upsilon_{\mathbb{G}}^{-1}(y)} : \Upsilon_{\mathbb{G}}^{-1}(y) \longrightarrow \Upsilon_{\mathbb{G}'}^{-1}(y')$$

is surjective with finite fibers.

Remark 5.1. The fibers of Υ are called central leaves, they give a partition of closed points, but are not a stratification. Now for $\omega \in Im(\lambda_{\mathbb{G}})$ the set $l_{\mathbb{G}}^{-1}(\omega)$ is the potential index set of the central leaves, which lie in the KR_ω.

The fifth axiom is a weak non-emptiness statement for the Kottwitz-Rapoport stratification. We recall that if $\mathbb{G}_v = \mathbb{I}_v$ is an Iwahori group scheme of $\mathbf{G}_v$, we have $W_{\mathbb{G}_v} = \{id\}$. In particular $Adm(\hat{Z}_v)$ is a subset of $\widetilde{W}_v = W_{\mathbb{G}_v}\backslash\widetilde{W}_v/W_{\mathbb{G}_v}$. Now the connected components of $\mathcal{F}l_v$ are given by $\pi_1(\mathbf{G}_v)_{\Sigma_0} = \Omega$ (see [PR08]) and a schubert cell C_ω lies in the connected component corresponding to $\beta \in \Omega$ if and only if ω equals by (21) to $(\alpha,\beta) \in W_{v,af} \rtimes \Omega$ for some $\alpha \in W_{v,af}$. Since $Z_v \subset \mathcal{F}l_v$ is by definition of the boundednes conditions connected it follows by 4.9 there is a $\mu_v \in \pi_1(\mathbf{G}_v)_{\Sigma_0} = \Omega$ such that all elements $\omega \in Adm(\hat{Z}_v)$ are of the form $(\alpha,\mu_v) \in W_{v,af} \rtimes \Omega$. In particular there is a unique element $\tau_v \in Adm(\hat{Z}_v)$ of length 0.

Axiom 5 (basic non-emptiness)**.** Let $\mathbb{G}$ be a group scheme such that $\mathbb{G}_v$ is an Iwahori subgroup of $\mathbf{G}_v$ for all $v \in \underline{v}$. Let $\tau_v \in Adm(\hat{Z}_v)$ be the unique length 0 element and $\underline{\tau} = (\tau_v)_{v\in\underline{v}} \in \prod_{v\in\underline{v}} Adm(\hat{Z}_v)$.

- *(weak version)* Then $KR_{\underline{\tau}}$ is not empty.
- *(strong version)* The map $KR_{\underline{\tau}} \to \pi_0(\nabla_n^{Z_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_q^{alg}})$ is surjective.

Here we write $\pi_0(\nabla_n^{Z_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_q^{alg}})$ for the geometric connected components of the moduli space $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_q^{alg}}$. So in other words the strong version expresses that the Kottwitz-Rapoport stratum KR_τ intersects all the geometric connected components of $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_q^{alg}}$. In particular it implies the weak version. In analogy to Shimura varieties one should expect also the strong version of this axiom to be true. However, proving such a result requires to have a good understanding of the connected components of $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_q^{alg}}$, what we do not have at the moment.

5.2 Verification of the Axioms

In this section we will verify some of the axioms. For the proof of first axiom, all the work is done in the third chapter and we just give a reference to it. The second and the third axiom follow mostly from our explanations and citations in the fourth chapter. Therefore most of the work we do in this section concerns the fourth and fifth axiom. The proof of the fifth axiom is not totally complete, but we sketch an idea how to prove it.

(Axiom 1) The construction of the morphism is done in subsection 3.3. The fact that the morphism $\pi_{\mathbb{G},\mathbb{G}'}$ is projective and surjective under the desired assumptions this is proved in theorem 3.20. Here the morphism was called $f_\star$.

(Axiom 2) The smooth morphism $\lambda_{\mathbb{G}}$ was constructed in section 4.4 and it was explained afterwards that it induces indeed a stratification map (compare definition 4.17). To prove the compatibility with a change of the group, let $f : \mathbb{G} \to \mathbb{G}'$ be a morphism as in the first axiom and $(\mathcal{G}, s_1, \dots, s_n, \tau_{\mathcal{G}}, \gamma)$ an $k = k^{alg}$-valued point in $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$. Choosing a trivialization $\Gamma_v(\underline{\mathcal{G}}) \simeq (L^+\mathbb{G}_v, b_v)$ we have $\Gamma_v(\pi_{\mathbb{G},\mathbb{G}'}(\underline{\mathcal{G}})) \simeq (L^+\mathbb{G}'_v, b_v)$. Using the bijection in (22) from section 4.2, let ω_v and ω'_v be the projections of b_v in $W_{\mathbb{G}_v}\backslash\widetilde{W}_v/W_{\mathbb{G}_v}$ and $W_{\mathbb{G}'_v}\backslash\widetilde{W}_v/W_{\mathbb{G}'_v}$ respectively. This means $\lambda_{\mathbb{G},v,k}(\underline{\mathcal{G}}) = \omega_v$ and $\lambda_{\mathbb{G}',v,k}(\pi_{\mathbb{G},\mathbb{G}'}(\underline{\mathcal{G}})) = \omega'_v$ and shows the compatibility $\lambda_{\mathbb{G}'} \circ \pi_{\mathbb{G},\mathbb{G}'} = p_{\mathbb{G},\mathbb{G}'} \circ \lambda_{\mathbb{G}}$.

(Axiom 3) The map $\delta_{\mathbb{G}}$ was described in definition 4.12. Right after this definition it was also explained that $\delta_{\mathbb{G}}$ is indeed a stratification map in the sense of definition 4.2. For the first condition in the third axiom we need to show $\delta_{\mathbb{G}} = \delta_{\mathbb{G}'} \circ \pi_{\mathbb{G},\mathbb{G}'}$ for all morphisms $\pi_{\mathbb{G},\mathbb{G}'}$ as in the first axiom. So let $f : \mathbb{G} \to \mathbb{G}'$ be again a morphism as in the first axiom, let k be an algebraically closed field and $(\underline{\mathcal{G}}, \gamma) \in \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})(k)$. We choose a trivialization $\Gamma_v(\underline{\mathcal{G}}) \simeq (L^+\mathbb{G}_v, b_v)$ for all $v \in \underline{v}$ so that $\delta_{\mathbb{G}}(\underline{\mathcal{G}}, \gamma) = ([b_v])_{v\in\underline{v}} \in \prod_{v\in\underline{v}} B(\mathbf{G}_v)$. As before we have $\pi_{\mathbb{G},\mathbb{G}'}(\underline{\mathcal{G}}, \gamma) = (f_\star\underline{\mathcal{G}}, \gamma)$ which implies $\Gamma_v(\pi_{\mathbb{G},\mathbb{G}'}(\underline{\mathcal{G}})) \simeq (L^+\mathbb{G}'_v, b_v)$. This means $(\delta_{\mathbb{G}'} \circ \pi_{\mathbb{G},\mathbb{G}'})(\underline{\mathcal{G}}, \gamma) = ([b_v])_{v\in\underline{v}}$ and proves the first condition. The second condition in the third axiom follows also from proposition 4.11, because it tells us that $\coprod_{\underline{c}\leqslant\underline{b}'} \mathcal{N}_{\underline{c},S}$ is a closed subscheme of S containing $\mathcal{N}_{\underline{b}',S}$ and consequently also $\overline{\mathcal{N}_{\underline{b}',S}}$. This means $\coprod_{\underline{c}\leqslant\underline{b}'} \mathcal{N}_{\underline{c},S} \cap \mathcal{N}_{\underline{b},S} \neq \varnothing$ which implies $\underline{b} \leqslant \underline{b}'$.

(Axiom 4) To verify (i) we describe $\Upsilon_{\mathbb{G}}$ as follows. Actually the set $L\mathbf{G}_v(\mathbb{F}_q^{alg})/L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v}$ determines precisely the set of isomorphism classes of local $\mathbb{G}_v$-shtukas over $\mathbb{F}_q^{alg}$. Note that this is different from $B(\mathbf{G}_v)$ which determines the isogeny classes. Now the map $\Upsilon_{\mathbb{G}}$ sends a global $\mathbb{G}$-shtuka $\underline{\mathcal{G}} = (\mathcal{G}, s_1, \dots, s_n, \tau_{\mathcal{G}})$ to the elements corresponding to the isomorphism class of the associated local $\mathbb{G}_v$-shtukas:

$$\begin{aligned}\Upsilon_{\mathbb{G}} : \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) &\to \prod_{v\in\underline{v}} L\mathbf{G}_v(\mathbb{F}_q^{alg})/L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v} \\ (\underline{\mathcal{G}}) &\mapsto (b_v)_{v\in v} \qquad\qquad \text{with } \Gamma_v(\underline{\mathcal{G}}) \simeq (L^+\mathbb{G}_v, b_v).\end{aligned}$$

It is clear by construction that $\lambda_{\mathbb{G}}$ and $\delta_{\mathbb{G}} = l_{\mathbb{G}} \circ \Upsilon_{\mathbb{G}}$ and $\delta_{\mathbb{G}} = d_{\mathbb{G}} \circ \Upsilon_{\mathbb{G}}$. Let $f : \mathbb{G} \to \mathbb{G}'$ as in the first axiom, $\underline{\mathcal{G}} \in \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})(\mathbb{F}_q^{alg})$ and $\Gamma_v(\underline{\mathcal{G}}) \simeq (L^+\mathbb{G}, b_v)$. Then $\Gamma_v(f_\star\underline{\mathcal{G}}) \simeq (L^+\mathbb{G}'_v, b_v)$ which implies the commutativity of diagram (31).

We don't prove the (ii) condition, but we will see in lemma 6.5 that it suffices to check this condition in the Iwahori case.

Now the statement of (iii) can actually be concluded from a detailed study of the proof of theorem 3.20. The crucial point is that we had some freedom to choose the element $b_w^{(0)}$, but let us explain the proof. We firstly show the surjectivity. We choose $y' \in Im(\Upsilon_{\mathbb{G}'})$ and a pre-image

$y = ([y_v])$ in $\prod_{v\in\underline{v}} L\mathbf{G}_v(\mathbb{F}_q^{alg})/L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v}$ under the natural projection, where $y_v \in L\mathbf{G}_v(\mathbb{F}_q^{alg})$ denotes a representative. Now let $\underline{\mathcal{G}}' = (\mathcal{G}', s_1, \ldots, s_n, \tau_{\mathcal{G}'}) \in \nabla_n\mathscr{H}^1(C,\mathbb{G}')(\mathbb{F}_q^{alg})$ with $\Upsilon(\underline{\mathcal{G}}') = y'$ which means $\Gamma_v(\underline{\mathcal{G}}') \simeq (L^+\mathbb{G}'_v, y_v)$ for all $v \in \underline{v}$. Let $\mathbb{F}$ be the compositum of all finite fields $\mathbb{F}_v$ for $v \in \underline{v}$ and denote for all $v \in \underline{v}$ as in theorem 3.20 by $v^{(0)}, \ldots, v^{(deg\ v-1)}$ the closed points in $C_{\mathbb{F}}$ lying above v. Here $v^{(0)}$ is the image of the characteristic morphism and the other points arise from applying σ to the residue field. We set $U := (C\backslash\underline{v})_{\mathbb{F}}$ as well as $I := \{v^{(i)} \in C_{\mathbb{F}} \mid v^{(i)}|v\ \}$ so that following Beauville and Laszlo as in § 2.9, we can again write down the cartesian diagram

$$\begin{array}{ccc}
\mathscr{H}^1(C_{\mathbb{F}}, \mathbb{G}'_{\mathbb{F}}) & \longrightarrow & \mathscr{H}_e^1(U, \mathbb{G}'_{\mathbb{F}'}) \\
\Big\downarrow {\scriptstyle \prod_{v\in I} L_v^+} & & \Big\downarrow {\scriptstyle \prod_{v\in I} L_v} \\
\prod_{v\in I}\mathscr{H}^1(\mathbb{F}, L^+\mathbb{G}'_v) & \longrightarrow & \prod_{v\in I}\mathscr{H}^1(\mathbb{F}, \mathbf{G}'_v)
\end{array} \quad .$$

Using this we write as before $\mathcal{G}' = (\mathcal{G}'|_U, \prod_{v^{(i)}\in I} L^+\mathbb{G}_{v^{(i)}}, (\epsilon_v^{(i)})_{v^{(i)}\in I})$ where $\epsilon_v^{(i)} : L_{v^{(i)}}(\mathcal{G}') \to L\mathbf{G}'_{v^{(i)}}$ are trivializations coming from an isomorphism $L^+_{v^{(i)}}(\mathcal{G}') \to L^+\mathbb{G}'_{v^{(i)}}$ of $L^+\mathbb{G}_v$-torsors. Then for $v \in \underline{v}$ and $i = 0, \ldots, deg\ v-1$ the element $c_v^{(i)} \in L\mathbf{G}_{v^{(i)}}(\mathbb{F}_q^{alg})$ was defined as $c_v^{(i)} := \epsilon_v^{(i)} \circ L_{v^{(i)}}(\tau_{\mathcal{G}'}) \circ \sigma^\star(\epsilon_v^{(i-1)})^{-1}$. For $i \geqslant 1$ it is even an element in $L^+\mathbb{G}_{v^{(i)}}(\mathbb{F}_q^{alg})$ and by definition of the global local functor we have $\Gamma_v(\underline{\mathcal{G}}') \simeq (L^+\mathbb{G}', \widetilde{c}_v)$ with $\widetilde{c}_v := c_v^{(0)} \cdot \sigma^\star c_v^{(deg\ v-1)} \cdot \sigma^{2\star} c_v^{(deg\ v-2)} \cdot \ldots \cdot \sigma^{deg\ v-1\star} c_1$. In particular we find $b_v^{(0)} \in L^+\mathbb{G}'_v(\mathbb{F}_q^{alg})$ with $b_v^{(0)} \cdot \widetilde{c}_v \cdot \sigma^{deg\ v\star}(b_v^{(0)})^{-1} = y_v$. Using this element $b_v^{(0)}$ we set inductively $b_v^{(i)} := \sigma^\star b_v^{(i-1)} \cdot (c_v^{(i)})^{-1}$ for $i = 1, \ldots, deg\ v$ and all $v \in \underline{v}$. Then $(b_v^{(i)}) \in \prod_{v^{(i)}\in I} L^+\mathbb{G}'_{v^{(i)}}(\mathbb{F}_q^{alg})$ defines by 3.18 a $\mathbb{G}$-torsor $\mathcal{G} = (\mathcal{G}'|_U, \prod_{v^{(i)}\in I} L^+\mathbb{G}'_{v^{(i)}}, (b_v^{(i)} \circ \epsilon_v^{(i)})_{v^{(i)}\in I})$ and extends similar as in theorem 3.20 to a $\mathbb{G}$-shtuka $\underline{\mathcal{G}} = (\mathcal{G}, s_1, \ldots, s_n, \tau_{\mathcal{G}})$ with $\tau_{\mathcal{G}}|_U := \tau_{\mathcal{G}'}|_U$ and $f_\star\underline{\mathcal{G}} = \underline{\mathcal{G}}'$. That $\tau_{\mathcal{G}}|_U$ extends to $\tau_{\mathcal{G}}$ is seen from the equation $b_v^{(i)} \cdot c_v^{(i)} \cdot \sigma^\star(b_v^{(i-1)})^{-1} = 1$ for $i \geqslant 1$ coming from

$$\begin{array}{ccccc}
L_{v^{(i)}}(\sigma^\star\mathcal{G}) & \xrightarrow{\sigma^\star\epsilon_v^{(i-1)}} & L\mathbf{G}_{v^{(i)}} & \xrightarrow{\sigma^\star b_v^{(i-1)}} & L\mathbf{G}_{v^{(i)}} \\
{\scriptstyle L_{v^{(i)}}(\tau_{\mathcal{G}})}\Big\downarrow & & \Big\downarrow{\scriptstyle c_v^{(i)}} & & {\scriptstyle id}\Big\downarrow \\
L_{v^{(i)}}(\mathcal{G}) & \xrightarrow[\epsilon_v^{(i)}]{} & L\mathbf{G}_{v^{(i)}} & \xrightarrow[b_v^{(i)}]{} & L\mathbf{G}_{v^{(i)}}
\end{array} \quad .$$

This equation together with the equation

$$\begin{aligned}
b_v^{(0)} \cdot c_v^{(0)} \cdot \sigma^\star(b_v^{(deg\ v-1)})^{(-1)} &= b_v^{(0)} \cdot c_v^{(0)} \cdot \sigma^\star(\sigma^{deg\ v-1\star} b_v^{(0)} \cdot \sigma^{deg\ v-2\star}(c_v^{(1)})^{-1} \cdot \ldots \cdot (c_v^{(deg\ v-1)})^{-1})^{-1} \\
&= b_v^{(0)} \cdot c_v^{(0)} \cdot \sigma^\star c_v^{(deg\ v-1)} \cdot \ldots \cdot \sigma^{(deg\ v-1)\star} c_v^{(1)} \cdot \sigma^{deg\ v\star}(b_v^{(0)})^{-1} = b_v^{(0)} \cdot \widetilde{c}_v \cdot \sigma^{deg\ a\star}(b_v^{(0)})^{-1} = y_v
\end{aligned}$$

and the definition of the global local functor implies $\Gamma_v(\underline{\mathcal{G}}) \simeq (L^+\mathbb{G}, y_v)$. This shows $\Upsilon_{\mathbb{G}}(\underline{\mathcal{G}}) = y$ and hence the surjectivity of

$$\pi_{\mathbb{G},\mathbb{G}'}(\mathbb{F}_q^{alg})\Big|_{\Upsilon_{\mathbb{G}}^{-1}(y)} : \Upsilon_{\mathbb{G}}^{-1}(y) \longrightarrow \Upsilon_{\mathbb{G}'}^{-1}(y').$$

It rests to prove that the fibers of this map are finite. Now for any other tuple $(d_v^{(i)})_{v^{(i)}\in I}$ in $\prod_{v\in\underline{v}}\prod_{i=1}^{deg\ v} L^+\mathbb{G}_{v^{(i)}}$ defining a $\mathbb{G}$-torsor $\widetilde{\mathcal{G}} = (\mathcal{G}'|_U, \prod_{v^{(i)}\in I} L^+\mathbb{G}'_{v^{(i)}}, (d_v^{(i)} \circ \epsilon_v^{(i)})_{v^{(i)}\in I})$ which is part of a $\mathbb{G}$-shtuka $\underline{\widetilde{\mathcal{G}}}$ with $f_\star\underline{\widetilde{\mathcal{G}}} = \underline{\mathcal{G}}'$ and $\Upsilon_{\mathbb{G}}(\underline{\widetilde{\mathcal{G}}}) = y$, we have $d_v^{(i)} \cdot c_v^{(i)} \cdot \sigma^\star(d_v^{(i-1)})^{-1} \in L^+\mathbb{G}_{v^{(i)}}(\mathbb{F}_q^{alg})$ for $i \geqslant 1$

$$\text{and } d_v^{(0)} \cdot \widetilde{c}_v \cdot \sigma^{d\star}(d_v^{(0)})^{-1} = y_v \text{ in } L\mathbf{G}_v(\mathbb{F}_q^{alg})/L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v}.$$

By choosing some $a_v^{(0)} \in L^+\mathbb{G}_v(\mathbb{F}_q^{alg})$ and defining

$$a_v^{(i)} := \sigma^\star a_v^{(i-1)} \cdot \sigma^\star d_v^{(i-1)} \cdot (c_v^{(i)})^{-1} \cdot (d_v^{(i)})^{-1} \in L^+\mathbb{G}_{v^{(i)}}(\mathbb{F}_q^{alg})$$

we can replace $(d_v^{(i)})_{v^{(i)}\in I}$ by $(a_v^{(i)} \cdot d_v^{(i)})_{v^{(i)}\in I}$ which defines a $\mathbb{G}$-torsor $\underline{\widetilde{\mathcal{G}_2}}$ isomorphic to $\underline{\widetilde{\mathcal{G}}}$ with $f_\star \underline{\widetilde{\mathcal{G}_2}} = \underline{\mathcal{G}}'$ and $\Upsilon_{\mathbb{G}}(\underline{\widetilde{\mathcal{G}_2}}) = y$. Since by definition

$$a_v^{(i)} \cdot d_v^{(i)} \cdot c_v^{(i)} \cdot \sigma^\star (d_v^{(i-1)})^{-1} \cdot \sigma^\star (a_v^{(i-1)})^{-1} = 1 \text{ for } i \geqslant 1 \text{ and}$$
$$a_v^{(0)} \cdot d_v^{(0)} \cdot c_v^{(0)} \cdot \sigma^{deg\ v\star} (d_v^{(0)})^{-1} \cdot \sigma^{deg\ v\star} (a_v^{(0)})^{-1} = y_v \quad \text{in } L\mathbf{G}_v(\mathbb{F}_q^{alg}) \big/ L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v},$$

we can assume that the tuple $(d_v^{(i)})_{v^{(i)}\in I}$ satisfies $d_v^{(i)} \cdot c_v^{(i)} \cdot \sigma^\star (d_v^{(i-1)})^{-1} = 1$ for $i \geqslant 1$. Therefore the fibers of $\pi_{\mathbb{G},\mathbb{G}'}(\mathbb{F}_q^{alg})\big|_{\Upsilon_{\mathbb{G}}^{-1}(y)}$ are given by the product over $v \in \underline{v}$ of the sets

$$\{h_v \in L^+\mathbb{G}'_v(\mathbb{F}_q^{alg})/L^+\mathbb{G}_v(\mathbb{F}_q^{alg}) \mid h_v \cdot y_v \cdot \sigma_v^\star h^{-1} = y_v\} \tag{32}$$

and we prove that this set is finite. Since $\mathbb{G}_v$ is connected, we find by [AH14, Remark 4.15] an element $f \in L\mathbf{G}_v(\mathbb{F}_q^{alg})$ defining a quasi-isogeny $(L^+\mathbb{G}_v, y) \to (L^+\mathbb{G}_v, f \cdot y \cdot \sigma_v^\star f^{-1})$ such that $x := f^{-1} y_v \sigma_v f$ is decent. That means that there exists a positive integer $s > 0$ with $(x\sigma_v^\star)^s = \nu_x(z_v)\sigma_v^{\star s}$, where $\nu_x = \nu_{\mathbf{G}_v}(x)$ is the Newton point and z_v the uniformizer in A_v. Now for all elements h in the above set (32) the element hf lies by [AH14, Remark 4.16] in $L\mathbf{G}(\mathbb{F}'_v)$, where $\mathbb{F}'_v$ denotes the finite field with $[\mathbb{F}'_v : \mathbb{F}_v] = s$. Moreover hf lies in the image of the morphism

$$L^+\mathbb{G}'_v \longrightarrow \mathcal{F}l_v \qquad b \mapsto bf.$$

Since $\mathcal{F}l_v$ is ind-projective and since $L^+\mathbb{G}'_v$ is affine and hence quasi-compact, this morphism factors through a projective subscheme, [HV11, Lemma 5.4]. Therefore hf is an $\mathbb{F}'_v$-valued point of an projective scheme. The set of these points is finite. □

Actually the proof even shows a stronger result as formulated in the fourth axiom:

Lemma 5.2. *For every $y' \in Im(\Upsilon_{\mathbb{G}'})$ and every preimage y of y' under the projection*

$$\prod_{v\in\underline{v}} L\mathbf{G}_v(\mathbb{F}_q^{alg}) \big/ L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v} \to \prod_{v\in\underline{v}} L\mathbf{G}'_v(\mathbb{F}_q^{alg}) \big/ L^+\mathbb{G}'_v(\mathbb{F}_q^{alg})_{\sigma_v}\ , \text{ the map}$$
$$\pi_{\mathbb{G},\mathbb{G}'}(\mathbb{F}_q^{alg})\big|_{\Upsilon_{\mathbb{G}}^{-1}(y)} : \Upsilon_{\mathbb{G}}^{-1}(y) \longrightarrow \Upsilon_{\mathbb{G}'}^{-1}(y')\ , \text{ is finite surjective.}$$

In particular y is an element in $Im(\Upsilon_{\mathbb{G}})$.

Proof: Follows from the previous discussion. □

(Axiom 5) We explain the idea of the proof of the weak version of this axiom and remark the step that is missing. The strategy is as follows. We construct explicitly a $\mathbb{G}$-shtuka $\underline{\mathcal{G}}$ over $\mathbb{F}_q^{alg}$ in $\nabla_n\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}$ that lies in the basic locus corresponding to the unique basic element $([b_v])_{v\in\underline{v}}$ with $\kappa_{\mathbf{G}_v}([b_v]) = \mu_v$. Here μ_v was the element in $\pi_1(\mathbf{G}_v)_\Sigma$ corresponding to the connected component of $\mathcal{F}l_v$ containing Z_v. That means that $\underline{\mathcal{G}}$ should satisfy $\delta_{\mathbb{G}}(\underline{\mathcal{G}}) = ([b_v])$. A priori $\underline{\mathcal{G}}$ lies only in $\nabla_n\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}$ and not in $\nabla_n^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G})$ since we can a priori not say if $\underline{\mathcal{G}}$ is bounded

by $\hat{Z}_{\underline{v}}$. Then we will use a known result about Deligne-Lusztig varieties to transform our global $\mathbb{G}$-shtuka $\underline{\mathcal{G}}$ to a global $\mathbb{G}$-shtuka $\underline{\mathcal{G}}'$ over $\mathbb{F}_q^{alg}$ that lies in the KR-strata KR_{τ}, where τ_v was the unique length zero element in $Adm(\hat{Z}_v)$ and $\underline{\tau} = (\tau_v)_{v\in\underline{v}}$. This global $\mathbb{G}$-shtuka $\underline{\mathcal{G}}'$ will then be bounded by $\hat{Z}_{\underline{v}}$. The explicit construction of $\underline{\mathcal{G}}$ takes place in three steps. In the first step we define two smooth affine group schemes $\mathbb{T}$ and $\widetilde{\mathbb{T}}$ over C, whose generic fiber is a maximal torus of $\mathbf{G}$ with some well choosen properties. The group scheme $\mathbb{T}$ will be the Weil restriction of $\mathbb{G}_{m,\widetilde{C}}$ for some morphism of curves $\widetilde{\pi}: \widetilde{C} \to C$. In the second step, we need then to construct a global $\mathbb{G}_m$-shtuka $\underline{\mathcal{L}}$ in $\nabla_m\mathscr{H}^1(\widetilde{C},\mathbb{G}_m)$. Then we will apply the morphism $\widetilde{\pi}_\star$ from 3.14 to $\underline{\mathcal{L}}$ to get a global $\mathbb{T}$ and then a global $\widetilde{\mathbb{T}}$-shtuka. Then we will apply the morphism in theorem 3.20. In the third step we will then explain how we transform $\underline{\mathcal{G}}$ to $\underline{\mathcal{G}}'$. So far about the strategy, now let us start with the first step and construct explicitly this global $\mathbb{G}$-shtuka $\underline{\mathcal{G}}$ in $\nabla_n\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}(\mathbb{F}_q^{alg})$ with $\delta_{\mathbb{G}}(\underline{\mathcal{G}}) = ([b_v])_{v\in\underline{v}}$.

(Step 1) To begin with the first step we fix for all $v \in \underline{v}$ a maximal Q_v-torus T_v' in $\mathbf{G}_v$ such that T_v' modulo the center $Z(\mathbf{G}_v)$ is anisotropic. The existence of such a torus follows from the following theorem.

Theorem 5.3 (compare [PR94, Theorem 6.21 page 326]).
Let G be a reductive group over a non-Archimedian local field K, then there exists always a maximal torus T in G, which is modulo the center of G an anisotropic K-torus.

Proof: Actually the theorem in [PR94] differs in two points from this one. Firstly the cited book only considers local fields of characteristic 0. But knowing that all maximal tori of $\mathbf{G}$ are split and conjugated over an separabel closure K^{sep} (see [Con14, Corollary B.3.6]) the same proof works in positive characteristic. Secondly the theorem is only formulated for semisimple groups and states that there is a maximal K-torus, which is anisotropic over K.
Now once we know this, we write Z for the center and G^{ad} for the adjoint group of G so that we can consider the short exact sequence

$$0 \to Z \to G \to G^{ad} \to 0$$

and choose a maximal K-torus T' in the semisimple group G^{ad} which is anisotropic. Since $G \to G^{ad}$ is an epimorphism of connected algebraic groups, there exists by [Con14, Corollary 3.3.5] (see also [Hum75, Section 21.3 Coroallary C]) a maximal torus T in G such that the image of T equals T'. Since the image of T is exaclty T/Z and since $T' = T/Z$ was anisotropic, we see that T satisfies the desired properties. □

We recall that we say that an algebraic variety X over Q satisfies the weak approximation property if the embedding $X(Q) \to \prod_{v\in S} X(Q_v)$ is dense, where $X(Q_v)$ carries the topology induced by the v-adic on Q_v (see [Con12]) and S is any finite subset of $|C|$. This concept of weak approximation exists also for number fields K and there is the following useful corollary of the [PR94, Proposition 7.3 page 402] that states that an irreducible smooth K-rational variety satisfies the weak approximation property.

Corollary 5.4 ([PR94, Section 7.1, Corollary 3, page 405]).
Let G be a reductive algebraic group over a number field K, and let $\mathcal{T}$ be the variety of its maximal

tori. Then $\mathcal{T}$ has weak approximation. In particular, if S is a finite subset of places of K and, for each $v \in S$, $T(v)$ is a given maximal K_v-torus in G, then there exists a maximal K-torus T of G which, for any v in S, is conjugated to $T(v)$ via an element of $G(K_v)$.

Although this corollary is only formulated for characteristic 0 it holds as well for our group $\mathbf{G}$ over Q. The main ingrediant in the proof of the corollary is the rationality of the scheme of maximal tori. So let $\mathcal{T}$ be the Q-scheme of maximal tori in $\mathbf{G}$. Choosing some maximal torus in $\mathbf{G}$ and writing N for its normalizer in $\mathbf{G}$, the scheme $\mathcal{T}$ is represented by $\mathbf{G}/N$ and since $\mathbf{G}$ is smooth, $\mathcal{T}$ is in particular a smooth homogenous space for $\mathbf{G}$ (see for example [Con14, Theorem 3.2.6]). The Q-rationality of $\mathcal{T}$ (including the positive characteristic case) is proven in [BS68, Theorem 7.9], so that $\mathcal{T}$ satisfies the weak approximation property.
Now let $x_v \in \mathcal{T}(Q_v)$ be the point corresponding to the torus T'_v with normalizer N_v in $\mathbf{G}_v$ and let

$$\Phi_v : \mathbf{G}_v \to \mathcal{T}_v := \mathcal{T} \times_Q Q_v \simeq \mathbf{G}_v/N_v, \qquad g \mapsto g \cdot x_v.$$

Since the differential map $T_v\Phi_v : T_vG_v \to T_v\mathcal{T}_v$ is surjective, we can apply the theorem in [Ser92, chapter 3, §10, 2)]. The second part of this theorem tells us that every point in the set

$$V_v := \{g \cdot x_v \mid g \in \mathbf{G}_v(Q_v)\} \subset \mathcal{T}(Q_v)$$

has an open neighborhood that lies in the image of the map $\mathbf{G}(Q_v) \to \mathcal{T}(Q_v)$ (i.e. completely in V_v). In particular V_v is open in $\mathcal{T}(Q_v)$. Now using the weak approximation property we fix a maximal Q-torus $T \subset \mathbf{G}$ (corresponding to some $x \in \mathcal{T}(Q)$) such that $T_v := T \times_Q Q_v$ is $\mathbf{G}_v(Q_v)$-conjugated to the choosen torus T'_v (i.e. T_v lies in V_v) for all $v \in \underline{v}$. In particular T_v modulo the center of $\mathbf{G}_v$ is an anisotropic torus. Let L be the splitting field of the maximal torus T. It is a finite separable extension of Q. By [PR94, Proposition 2.2, page 55] T arises as a quotient of $\mathbf{T} := \prod_{i=1}^{l} Res_{L/Q}\mathbb{G}_{m,L}$, that means that we have an short exact sequence

$$0 \to F \to \mathbf{T} \to T \to 0$$

of Q-tori which split over L. Now the field extension L/Q corresponds to a curve C' over C and we set $\widetilde{C} := \coprod_{i=1}^{l} C'$. We denote by $\widetilde{\pi} : \widetilde{C} \to C$ and $\pi : C' \to C$ the natural morphisms to C. Further we set $\mathbb{T} := Res_{\widetilde{C}/C}\mathbb{G}_{m,\widetilde{C}} = \widetilde{\pi}_\star\mathbb{G}_{m,\widetilde{C}}$ for the Weil restriction of $\mathbb{G}_{m,\widetilde{C}}$ to C. Since $\widetilde{\pi}$ is generically étale it is by § 2.17 a parahoric group scheme over C. Note that we always used bold letters for the generic fiber of a group scheme over C. This concides with the previous notation, since we have indeed $\mathbb{T} \times_C Spec\ Q = \mathbf{T}$ and since we do not consider an integral model over C of the Q-torus T, there is no confusion about this torus. Although we have the generic morphism $\mathbf{T} \to T \to \mathbf{G}$ we do not know if this extends to a morphism $\mathbb{T} \to \mathbb{G}$. But we will now define a smooth affine group scheme $\widetilde{\mathbb{T}}$ over C with two C-morphisms

$$\begin{array}{ccc} \widetilde{\mathbb{T}} & \overset{f}{\dashrightarrow} & \mathbb{G} \\ {\scriptstyle u}\downarrow & & \\ \mathbb{T} & & \end{array}$$

and the property that the generic fiber of $\widetilde{\mathbb{T}}$ coincides with $\mathbf{T}$. It is clear that there is a Zariski open set $U \subset C$ such that $\mathbf{T} \to \mathbf{G}$ extends to $\mathbb{T}_U \to \mathbb{G}_U$. Therefore we only need to modify $\mathbb{T}$ at finitely many places $C \backslash U$ to get the group scheme $\widetilde{\mathbb{T}}$.
Let $w \in C\backslash U$ and z_w an uniformizer in A_w. Since $\mathbb{G}(A_w)$ is an open compact subgroup in $\mathbf{G}(Q_w)$ we can fix a natural number n such that the A_w-group scheme $\widetilde{\mathbb{T}}_w$ whose R-valued points are given by

$$\widetilde{\mathbb{T}}_w(R) = \left\{x \in \mathbb{T}(R) \mid x \in ker\Big(\mathbb{T}_w(R) \to \mathbb{T}(R/z_w^n)\Big) \right\}$$

maps into $\mathbb{G}_w$. Here $\widetilde{\mathbb{T}}_w$ is indeed a subgroup scheme of $\mathbb{T}_w$ (although not parahoric any more) since it can be written as $ker\Big(\mathbb{T}_w \to p_\star(\mathbb{T}_w \times_{Spec\ A_w} Spec\ A_w/z_w^n)\Big)$ where $p : Spec\ A_w/z_w^n \to Spec\ A_w$ is the natural projection so that $\mathbb{T}_w \to p_\star(\mathbb{T}_w \times_{Spec\ A_w} Spec\ A_w/z_w^n)$ arises naturally from the adjunction of the Weil restriction. Once we fix such a group scheme $\widetilde{\mathbb{T}}_w$ for all $w \in C\backslash U$ we can use faithfully flat descent [BLR90, section 6.1, theorem 6] for the group scheme $\big(\coprod_{w\in C\backslash U} \mathbb{T}_w\big) \coprod \mathbb{T}_U$ over $\big(\coprod_{w\in C\backslash U} Spec\ A_w\big) \coprod U$ to glue these to a group scheme $\widetilde{\mathbb{T}}$ over C satisfying, which is of finite type by[Gro65, Proposition 2.7.1], smooth by [Gro67, Proposition 17.7.1] and satisfies the desired properties. We denote by u the morphism $\widetilde{\mathbb{T}} \to \mathbb{T}$ and by f the morphism $\widetilde{\mathbb{T}} \to \mathbb{G}$.

(Step 2) Now we write $\underline{w} = (w_1, \ldots, w_m)$ for the set of all closed points in $\widetilde{C}$ which lie in the preimage of $\underline{v}$ under $\widetilde{\pi} : \widetilde{C} \to C$. Moreover let $\underline{\mathcal{L}} := (\mathcal{L}, r_1, \ldots, r_m, \tau_{\mathcal{L}})$ be any global $\mathbb{G}_m$-shtuka in $\nabla_m\mathscr{H}^1(\widetilde{C}, \mathbb{G}_m)^{\underline{w}}(\mathbb{F}_q^{alg})$. Then we have the morphism

$$\widetilde{\pi}_\star :\ \nabla_m\mathscr{H}^1(\widetilde{C}, \mathbb{G}_m)^{\underline{w}} \longrightarrow \nabla_n\mathscr{H}(C, \mathbb{T})^{\underline{v}}$$

which is a product of morphisms as in proposition 3.12 from section 3.2. Under this morphism $\underline{\mathcal{L}}$ is send to a global $\mathbb{T}$-shtuka $\widetilde{\pi}_\star\underline{\mathcal{L}}$ in $\nabla_n\mathscr{H}^1(C, \mathbb{T})^{\underline{v}}(\mathbb{F}_q^{alg})$. Due to the fact that $u : \widetilde{\mathbb{T}} \to \mathbb{T}$ is generically an isomorphism, the induced morphism

$$u_\star :\ \nabla_n\mathscr{H}^1(C, \widetilde{\mathbb{T}})^{\underline{v}} \longrightarrow \nabla_n\mathscr{H}^1(C, \mathbb{T})^{\underline{v}}$$

is surjective by theorem 3.20. This is good for us, because it means that $\widetilde{\pi}_\star\underline{\mathcal{L}}$ is coming from a global $\widetilde{\mathbb{T}}$-shtuka $\underline{\widetilde{\mathcal{L}}}$ in $\nabla_n\mathscr{H}^1(C, \widetilde{\mathbb{T}})^{\underline{v}}(\mathbb{F}_q^{alg})$. Afterwards we apply the morphism

$$f_\star : \nabla_n\mathscr{H}^1(C, \widetilde{\mathbb{T}})^{\underline{v}} \longrightarrow \nabla_n\mathscr{H}^1(C, \mathbb{G})^{\underline{v}}$$

from theorem 3.26 to $\underline{\widetilde{\mathcal{L}}}$ and call its image $\underline{\mathcal{G}}$. In particular for all $v \in \underline{v}$ the local $\mathbb{G}_v$-shtuka $\Gamma_v(\underline{\mathcal{G}})$ arise from the local $\widetilde{\mathbb{T}}_v$-shtuka $\Gamma_v(\underline{\widetilde{\mathcal{L}}})$ by applying $f_{v\star}$. Over $\mathbb{F}_q^{alg}$ the local $\widetilde{\mathbb{T}}_v$-shtuka $\Gamma_v(\underline{\mathcal{G}})$ is isomorphic to $(L^+\widetilde{\mathbb{T}_v}, b_v)$ for some $b_v \in L\mathbf{T}_v(\mathbb{F}_q)$. Since $\mathbf{T}_v$ is anisotropic modulo its center, the Newton point of b_v, wich is a morphism form the pro-algebraic multiplicative group $\mathbb{D}$ (see [Kot85]) to $\mathbf{T}_v$, factors through the center of $\mathbf{T}_v$. This means by definition that b_v is a basic element in $B(\mathbf{G}_v)$.
Note that we didn't work with any boundedness conditions yet. For sure there exists appropriate bounds $\hat{Z}_v'$ in $\mathcal{F}l_{\mathbb{G}_v}$ such that $\underline{\mathcal{G}}$ is even an element in $\nabla_n^{\hat{Z}_v'}\mathscr{H}^1(C, \mathbb{G})^{\underline{v}}$. If we denote by τ_v' the unique length zero element in $Adm(\hat{Z}_v')$ and $\tau_{\underline{v}}' := (\tau_v')_{v\in\underline{v}}$, then the third step will show that $\underline{\mathcal{G}}$ can be modified to a global $\mathbb{G}$-shtuka $\underline{\mathcal{G}}'$ that lies in the minimal Kottwitz-Rapoport stratum

$KR_{\tau'_{\underline{v}}}$ of $\nabla^{\hat{Z}'_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G})^{\underline{v}}$. So if we are lucky then $\hat{Z}'_{\underline{v}}$ equals $\hat{Z}_v$ and we are done with the proof. But in general we startet with an arbitrary bound $\hat{Z}_{\underline{v}}$. Therefore the goal of the second step is to construct for any bound $\hat{Z}_{\underline{v}}$ a global $\mathbb{G}_m$-shtuka $\underline{\mathcal{L}}$ over $\mathbb{F}_q^{alg}$ in $\nabla_m\mathscr{H}^1(\widetilde{C},\mathbb{G}_m)$ such that the global $\mathbb{G}$-shtuka resulting as in the explanation above lies in $\nabla_n^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G})(\mathbb{F}_q^{alg})$. Actually the third step shows that it suffices that $\underline{\mathcal{G}}$ is bounded by some larger bound lying in the same connected component in $\mathcal{F}l_{\mathbb{G}_v}$ as $\hat{Z}_v$. Or in other words that the Kottwitz point of b_v equals to μ_v, where $\Gamma_v(\underline{\mathcal{G}}) \simeq (L^+\mathbb{G}, b_v)$ and μ_v is the element in $\pi_1(\mathbf{G}_v)_\Sigma$ corresponding to the connected component of $\hat{Z}_v$.

Now unfortunately we can not construct this $\underline{\mathcal{L}}$ at the moment, but let us explain the idea how the construction should work. There is a condition on the bound $\hat{Z}_{\underline{v}}$ or rather the elements $\mu_v \in \pi_1(\mathbf{G}_v)_{\Sigma_0}$ that has to be satisfied to guarantee that the space $\nabla_n^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G})$ is non-empty. Let us look at the example GL_r and let $\underline{\mathcal{G}}$ be a $\mathrm{GL_r}$-shtuka. In that case $\mathcal{G}$ and $\sigma^\star\mathcal{G}$ correspond to vector bundles of the same degree. Since the number of zeros and poles of $\tau_{\mathcal{G}}$, counted with multiplicity, add to zero, we see that the sum $\sum_{v\in\underline{v}}\mu_v$ should equal to 0 in $\pi_1(\mathbf{G})_{\Sigma_0} = \mathbb{Z}$. Now it is not really clear to us how to formulate the right condition for a general group $\mathbb{G}$. This question is maybe also related to the question of the connected components of $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$.
Once we know this condition, it should give us in some way a divisor D on $\widetilde{C}$ that defines the desired global $\mathbb{G}_m$-shtuka $\underline{\mathcal{L}}$ in the following way:
First we note that the Picard variety $Pic^\circ_{\widetilde{C}/\mathbb{F}_q}$ is an abelian variety with a surjective endomorphism

$$1 - Frob_q: \quad Pic^\circ_{\widetilde{C}/\mathbb{F}_q}(\mathbb{F}_q^{alg}) \longrightarrow Pic^\circ_{\widetilde{C}/\mathbb{F}_q}(\mathbb{F}_q^{alg})$$
$$\mathcal{L} \longmapsto \mathcal{L}\otimes\sigma^\star\mathcal{L}^{-1}$$

Therefore we find for every line bundle $\mathcal{O}_{\widetilde{C}_{\mathbb{F}_q^{alg}}}(D)$ a line bundle $\mathcal{L}$ with $\mathcal{O}_{\widetilde{C}_{\mathbb{F}_q^{alg}}}(D) \simeq \mathcal{L}\otimes\sigma^\star\mathcal{L}^{-1}$ or equivalently $\mathcal{L}$ with $\sigma^\star\mathcal{L}\otimes\mathcal{O}_{\widetilde{C}_{\mathbb{F}_q^{alg}}}(D) \simeq \mathcal{L}$. The chosen divisor D should have support in $\underline{w}$ so that $\sigma^\star\mathcal{L}\otimes\mathcal{O}_{\widetilde{C}_{\mathbb{F}_q^{alg}}}(D)\big|_{\widetilde{C}_{\mathbb{F}_q^{alg}}} = \sigma^\star\mathcal{L}\big|_{\widetilde{C}_{\mathbb{F}_q^{alg}}\setminus\cup_i\Gamma_{r_i}}$. Then we can define

$$\tau_{\mathcal{L}}: \sigma^\star\mathcal{L}\big|_{\widetilde{C}_{\mathbb{F}_q^{alg}}\setminus\cup_i\Gamma_{r_i}} \to \mathcal{L}\big|_{\widetilde{C}_{\mathbb{F}_q^{alg}}\setminus\cup_i\Gamma_{r_i}}$$

to be the above isomorphism, which gives us a global $\mathbb{G}_m$-shtuka. Moreover the chosen divisor D should also guarantee that the Kottwitz point of $\kappa_{\mathbf{G}_v}(b_v)$, arising from the afterwards constructed $\underline{\mathcal{G}}$, equals μ_v, so that $\underline{\mathcal{L}}$ gives the desired global $\mathbb{G}_m$-shtuka in $\nabla_m\mathscr{H}^1(\widetilde{C},\mathbb{G}_m)^{\underline{w}}$. Note that it is not clear that this is possible under the restriction that D has degree zero on every component of $\widetilde{C}$.

(Step 3) We have constructed a global $\mathbb{G}$-shtuka $\underline{\mathcal{G}}$ over $\mathbb{F}_q^{alg}$ in $\nabla_n\mathscr{H}^1(C,\mathbb{G})$ lying in $\mathcal{N}_{\underline{b}}$, where $\underline{b} = (b_v)_{v\in\underline{v}}$ and b_v is the chosen basic element from the beginning. We now show that it can be modified to a global $\mathbb{G}$-shtuka $\underline{\mathcal{G}'}$ which lies in $KR_{\underline{\tau}}$, where $\underline{\tau} = (\tau_v)_{v\in\underline{v}}$ with τ_v the length zero element in $Adm(\hat{Z}_v)$.
For this purpose we need to know that the affine Deligne-Lusztig variety $X_{\tau_v}(b_v)$ is non-empty for all $v\in\underline{v}$. This can be seen as follows: The length 0 element τ_v is by definition σ_v-straight and

defines a σ_v-conjugacy class $[\tau_v] \in B(\widetilde{W}_v)$. We recall that we introduced the bijective map (30)

$$\Psi : B(\widetilde{W}_v)_{\sigma_v - str} \longrightarrow B(\mathbf{G}_v).$$

We can equip $B(\widetilde{W}_v)_{\sigma_v - str}$ with a partial ordering by defining $[\omega] \leqslant [\omega']$ if and only if this is the case for some σ_v-straight representatives in $\widetilde{W}_v$. Then by [He16, Theorem B] the map Ψ respects this order on $B(\widetilde{W}_v)_{\sigma_v - str}$ and the order of $B(\mathbf{G}_v)$. In particular $[\tau_v]$ is send to b_v, since the basic element b_v is minimal as well. Knowing that τ_v is σ_v-straight we can choose a representative $\dot{\tau}_v$ in $N_v(\breve{Q}_v)$, apply [He14, Theorem 4.5] and conclude

$$I_v \cdot \dot{\tau}_v \cdot I_v \subset [\dot{\tau}_v] = [b_v] = \{g \cdot \dot{\tau}_v \cdot \sigma_v^\star g^{-1}\} \in B(\mathbf{G}_v).$$

This means in particular $X_{\tau_v}(b_v) \neq \varnothing$. Following this we can choose an element $g_v \in X_{\tau_v}(b_v)$ for all $v \in \underline{v}$. This defines (up to isomorphism) a quasi-isogeny

$$g_v : (L^+\mathbb{G}_v, \tau_v) \longrightarrow (L^+\mathbb{G}_v, b_v) \qquad \text{for all } v \in \underline{v}.$$

We recall that we have $\Gamma_v(\underline{\mathcal{G}}) \simeq (L^+\mathbb{G}_v, b_v)$ by construction, so that we can use [AH14, Proposition 5.7] to pull back $\underline{\mathcal{G}}$ along the quasi-isogenies g_v to a global $\mathbb{G}$-shtuka $\underline{\mathcal{G}}'$ satisfying $\Gamma_v(\underline{\mathcal{G}}') \simeq (L^+\mathbb{G}_v, \tau_v)$. In particular $\underline{\mathcal{G}}'$ is bounded by $\hat{Z}_{\underline{v}}$ and lies in $KR_{\tau_{\underline{v}}}$.
Apart from the missing argument in step 2 this proves the weak version of the fifth axiom.

6 Consequences of the Axioms

In this chapter we collect some consequences of the axioms on $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$.

6.1 Kottwitz-Rapoport Stratification

The second axiom guarantees that we can define the Kottwitz-Rapoport stratification definition as in definition 4.17. By this definition it is clear that for all $\underline{\omega} \in \prod_{v\in\underline{v}} Adm(\hat{Z}_v)$ we have

$$\overline{KR_{\underline{\omega}}} = \bigcup_{\underline{\omega}'\leq\underline{\omega}} KR_{\underline{\omega}'}$$

so that the Kottwitz-Rapoport stratification satisfies the strong stratification property. Using theorem 4.14 from the fourth chapter and the known results about affine flag varieties we can prove the following proposition about the dimension and smoothness of KR-strata. As before we denote by K_v the parahoric subgroup $\mathbb{G}_v(\breve{A}_v)$ in $\mathbf{G}_v(\breve{Q}_v)$. The question when a KR-Strata is non-empty will be answered in theorem 6.2.

Proposition 6.1 (compare [HR17, remark 3.4])**.** *Let $\underline{\omega} = (\omega_v)_{v\in\underline{v}}$ with $\omega_v \in W_{K_v}\backslash\widetilde{W}_v/W_{K_v}$. If the Kottwitz-Rapoport stratum KR_ω is non-empty, then it is smooth of dimension $\sum_{v\in v} l({}_{K_v}\omega_v^{K_v})$, where ${}_{K_v}\omega^{K_v}$ is the element in ${}_{K_v}W_v^{K_v}$ corresponding to $\underline{\omega}$ by* (23)*.*

Proof: The smoothness follows from the fact that $\lambda_{\mathbb{G}}$ is smooth by the second axiom. Now let $y \in KR_{\underline{\omega}}$. Then by theorem 4.14 there is an étale neighboorhood U of y with a section to $\nabla_n^{\hat{Z}_{\underline{v}},H}\widetilde{\mathscr{H}^1(C,\mathbb{G})}$ such that the composition with $\hat{\psi}$ is étale. By definition of $KR_{\underline{\omega}}$ the element y lands in the product of the Schubert cells $\prod_{v\in\underline{v}} C_v =: C$. Then $C \times_{\mathcal{F}l_v} (KR_{\underline{\omega}} \cap U) \to C$ is étale and since C has dimension $\sum_{v\in\underline{v}} l({}_{K_v}\omega_v^{K_v})$ by proposition 4.9 the proposition follows. □

By the definition of the Kottwitz-Rapoport stratification, the image of $\lambda_{\mathbb{G}}$ is contained in $\prod_{v\in\underline{v}}^n Adm(\hat{Z}_v)$. Using the weak version of the fifth axiom we now prove that for every shtuka datum $(C,\mathbb{G},\underline{v},\hat{Z}_{\underline{v}},H)$ and every $\underline{\omega} \in \prod_{v\in\underline{v}} Adm(\hat{Z}_v)$ the Kottwitz-Rapoport stratum $KR_{\underline{\omega}}$ is non-empty. In other words:

Theorem 6.2 (compare [HR17, theorem 4.1])**.** *Assume that the weak version of the fifth axiom holds true. Then we have*

$$\lambda_{\mathbb{G}}(\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_q^{alg}}) = \prod_{v\in\underline{v}} Adm(\hat{Z}_v).$$

Proof: We firstly prove the theorem in the case that $(C,\mathbb{G},\underline{v},\hat{Z}_{\underline{v}},H)$ is a shtuka datum, where $\mathbb{G}_v$ is an Iwahori group scheme for all characteristic points $v \in \underline{v}$. By the second axiom $\lambda_{\mathbb{G}}$ is smooth and hence open. Therefore the image $Im(\lambda_{\mathbb{G}})$ is an open substack of $\left[\prod_{v\in\underline{v}} L^+\mathbb{G}_v \Big\backslash \prod_{v\in\underline{v}} \breve{Z}_v\right]$. Let M be the preimage of $Im(\lambda_{\mathbb{G}})$ under the projection map $\prod_{v\in\underline{v}} \breve{Z}_v \to \left[\prod_{v\in\underline{v}} L^+\mathbb{G}_v \Big\backslash \prod_{v\in\underline{v}} \breve{Z}_v\right]$ defined as the cartesian product

$$\begin{array}{ccc} M & \longrightarrow & \prod_{v\in\underline{v}} \breve{Z}_v \\ \downarrow & & \downarrow \\ Im(\lambda_{\mathbb{G}}) & \longrightarrow & \left[\prod_{v\in\underline{v}} L^+\mathbb{G}_v \backslash \prod_{v\in\underline{v}} \breve{Z}_v\right]. \end{array}$$

It is an open subscheme of $\prod_{v\in\underline{v}} \breve{Z}_v$ invariant under the action of $\prod_{v\in\underline{v}} L^+\mathbb{G}_v$. Now $\left|\prod_{v\in\underline{v}} \breve{Z}_v\right|$ is a disjoint union of finitely many orbits $\mathcal{O}_{\underline{\omega}}$ with $\underline{\omega}$ in $\prod_{v\in\underline{v}} Adm(\hat{Z}_v)$ by the second axiom and definition 4.10. Assume that one of these orbits $\mathcal{O}_{\underline{\omega}}$ does not lie in M. Since M is open this implies that also the closure $\overline{\mathcal{O}_{\underline{\omega}}}$ does not lie in M. Since the element $\underline{\tau} = (\tau_v)$ from the fifth axiom is the unique minimal element in $\prod_{v\in\underline{v}} Adm(\hat{Z}_v)$ we conclude from the closure relations of the Kottwitz-Rapoport stratification that $\mathcal{O}_{\underline{\tau}} \subset \overline{\mathcal{O}_{\underline{\omega}}}$ and consequently $\mathcal{O}_{\tau} \cap M \neq \emptyset$. Now this implies that $\underline{\tau}$ does not lie in the image of $\lambda_{\mathbb{G}}$ which is a contradiction to the weak version of the fifth axiom hence $\lambda_{\mathbb{G}}(\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_q^{alg}}) = \prod_{v\in\underline{v}} Adm(\hat{Z}_v)$.

Now let $(C,\mathbb{G},\underline{v},\hat{Z}'_{\underline{v}},H)$ be a general shtuka datum. Choosing for all $v \in \underline{v}$ an Iwahori subgroup in $\mathbb{G}'_v$, we find a morphism $f : \mathbb{G} \to \mathbb{G}'$ of group schemes as in the first axiom such that $\mathbb{G}_v$ is the chosen Iwahori subgroup for all $v \in \underline{v}$. Defining $\hat{Z}_{\underline{v}}$ by base change from $\hat{Z}'_{\underline{v}}$ as in the first axiom we get by the second axiom the following commutative diagram

$$\begin{array}{ccc} \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_q^{alg}} & \xrightarrow{\lambda_{\mathbb{G}}} & \prod_{v\in\underline{v}} Adm(\hat{Z}_v) \\ \downarrow{\scriptstyle \pi_{\mathbb{G},\mathbb{G}'}} & & \downarrow \\ \nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}')_{\mathbb{F}_q^{alg}} & \xrightarrow{\lambda'_{\mathbb{G}}} & \prod_{v\in\underline{v}} Adm(\hat{Z}'_v) \end{array}$$

The vertical arrow on the right hand side is surjective by construction and we just proved that $\lambda_{\mathbb{G}}$ is surjective. This implies the subjectivity of $\lambda_{\mathbb{G}'}$ and finishes the proof. □

Remark 6.3. If we know the strong version of the fifth axiom to be true, this theorem would even imply $\lambda_{\mathbb{G}}(X_{\mathbb{G}}) = \prod_{v\in\underline{v}} Adm(\hat{Z}_v)$ for every geometric connected component $X_{\mathbb{G}}$ in $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})_{\mathbb{F}_q^{alg}}$.

6.2 Newton Stratification

We have the following results that connects certain Kottwitz-Rapoport strata with Newton strata in the Iwahori case.

Proposition 6.4. *Let $(C,\mathbb{G},\underline{v},\hat{Z}_{\underline{v}},H)$ be a shtuka datum such that $\mathbb{G}_v$ is a Iwahori subgroup for all $v \in \underline{v}$ and let $\underline{\omega} = (\omega_v)_{v\in\underline{v}} \in \prod_{v\in\underline{v}} Adm(\hat{Z}_v) \subset \prod_{v\in\underline{v}} \widetilde{W}_v$ such that all ω_v are σ_v-straight elements in $\widetilde{W}_v$. Denote by $[\omega_v]$ their σ_v-conjugacy class in $B(\widetilde{W}_v)_{\sigma_v-str}$ and by $b_v := \Psi([\omega_v])$ their image in $B(\mathbf{G}_v)$ under the map Ψ in (30). Then we have $KR_{\underline{\omega}} \subset \mathcal{N}_{\underline{b}}$, where $\underline{b} := (b_v)_{v\in\underline{v}}$.*

Proof: The preimage of $\underline{\omega}$ under the map $l_{\mathbb{G}}$ in the fourth axiom is given by

$$L^+\mathbb{G}_v(\mathbb{F}_q^{alg})\dot{\omega}_v L^+\mathbb{G}_v(\mathbb{F}_q^{alg}) / L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v}$$

for some representative $\dot{\omega}_v \in N_v(\breve{Q}_v)$. By proposition [He14, Proposition 4.5] we know that $L^+\mathbb{G}_v(\mathbb{F}_q^{alg})\dot{\omega}_v L^+\mathbb{G}_v(\mathbb{F}_q^{alg})$ is containted in the single σ_v-conjugacy class $b_v := [\dot{\omega}_v] \in B(\mathbf{G}_v)$. Therefore we have

$$d_{\mathbb{G}}\left(\prod_{v\in\underline{v}} L^+\mathbb{G}_v(\mathbb{F}_q^{alg})\dot{\omega}_v L^+\mathbb{G}_v(\mathbb{F}_q^{alg})\big/ L^+\mathbb{G}_v(\mathbb{F}_q^{alg})_{\sigma_v}\right) = ([\dot{\omega}_v])_{v\in\underline{v}} = (b_v)_{v\in\underline{v}} =: \underline{b} \in \prod_{v\in\underline{v}} B(\mathbf{G}_v).$$

In particular we can for all $\underline{\mathcal{G}} \in \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})(\mathbb{F}_q^{alg})$ with $\lambda_{\mathbb{G}}(\underline{\mathcal{G}}) = \underline{\omega}$ conclude by the commutativity of the diagram in the fourth axiom $\delta_{\mathbb{G}}(\underline{\mathcal{G}}) = d_{\mathbb{G}} \circ \Upsilon_{\mathbb{G}}(\underline{\mathcal{G}}) = (b_v)_{v\in\underline{v}}$. Hence we have $KR_{\underline{\omega}}(\mathbb{F}_q^{alg}) \subset \mathcal{N}_{\underline{b}}(\mathbb{F}_q^{alg})$ and by lemma 4.5 this is enough to conclude $KR_{\underline{\omega}} \subset \mathcal{N}_{\underline{b}}$. □

6.3 Central Leaves

The central leaves of $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$ are defined as the substacks whose closed points equal the fibers of $\Upsilon_{\mathbb{G}}$. Concerning the image of $\Upsilon_{\mathbb{G}}$ we start with the following lemma, which tells us in combination with theorem 6.2 that it suffices to check condition (ii) in the fourth axiom only for an Iwahori subgroup.

Lemma 6.5 (compare [HR17, Lemma 3.1]). *Let $f : \mathbb{G} \to \mathbb{G}'$ induce a morphism of shtuka data as in the first axiom. In particular $\hat{Z}_{\underline{v}}$ arises as a base change of $\hat{Z}'_{\underline{v}}$. Then we have:*

$$Im(\Upsilon_{\mathbb{G}}) = l_{\mathbb{G}}^{-1}\left(\prod_{v\in\underline{v}} Adm(\hat{Z}_v)\right) \quad \text{if and only if} \quad Im(\Upsilon_{\mathbb{G}'}) = l_{\mathbb{G}'}^{-1}\left(\prod_{v\in\underline{v}} Adm(\hat{Z}'_v)\right)$$

Proof: Let us prove the first direction and assume $Im(\Upsilon_{\mathbb{G}}) = l_{\mathbb{G}}^{-1}\left(\prod_{v\in\underline{v}} Adm(\hat{Z}_v)\right)$ which means that $\Upsilon_{\mathbb{G}}$ in the following diagram

$$\begin{array}{ccc} \nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}) & \xrightarrow{\Upsilon_{\mathbb{G}}} & l_{\mathbb{G}}^{-1}\left(\prod_{v\in\underline{v}} Adm(\hat{Z}_v)\right) \\ \downarrow & & \downarrow \\ \nabla_n^{\hat{Z}'_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G}') & \xrightarrow[\Upsilon_{\mathbb{G}'}]{} & l_{\mathbb{G}'}^{-1}\left(\prod_{v\in\underline{v}} Adm(\hat{Z}'_v)\right) \end{array}$$

is surjective. Now by definition of $\lambda_{\mathbb{G}'}$ we know $Im(\lambda_{\mathbb{G}'}) \subset \prod_{v\in\underline{v}} Adm(\hat{Z}_v)$ and due to $\lambda_{\mathbb{G}} = \Upsilon_{\mathbb{G}'} \circ l_{\mathbb{G}'}$ we know $Im(\Upsilon_{\mathbb{G}'}) \subset l_{\mathbb{G}'}^{-1}\left(\prod_{v\in\underline{v}} Adm(\hat{Z}'_v)\right)$. Therefore to prove $Im(\Upsilon_{\mathbb{G}'}) = l_{\mathbb{G}'}^{-1}\left(\prod_{v\in\underline{v}} Adm(\hat{Z}_v)\right)$ it suffices to prove that the vertical morphism on the right hand side in the above diagram is surjective. This follows directly from the definitions, because $\hat{Z}_{\underline{v}}$ arises from base change of $\hat{Z}'_{\underline{v}}$. On the other hand, let us assume that $\Upsilon_{\mathbb{G}'}$ in the above diagram is surjective. Then we take an element $y \in l_{\mathbb{G}}^{-1}\left(\prod_{v\in\underline{v}} Adm(\hat{Z}_v)\right)$ and denote by $y' \in l_{\mathbb{G}'}^{-1}\left(\prod_{v\in\underline{v}} Adm(\hat{Z}'_v)\right)$ its image under the vertical map on the right hand side. In particular $\Upsilon_{\mathbb{G}'}^{-1}(y') \neq \varnothing$. Moreover lemma 5.2 states that $\pi_{\mathbb{G},\mathbb{G}'}(\mathbb{F}_q^{alg})\big|_{\Upsilon_{\mathbb{G}}^{-1}(y)} : \Upsilon_{\mathbb{G}}^{-1}(y) \longrightarrow \Upsilon_{\mathbb{G}'}^{-1}(y')$ is finite surjective. Consequently $\Upsilon_{\mathbb{G}}^{-1}(y) \neq \varnothing$ which implies that $\Upsilon_{\mathbb{G}}$ is surjective and hence the lemma. □

7 Drinfeld's Moduli Space with Iwahori Level

In this chapter we discuss Drinfeld´s moduli space as an example of the moduli stack $\nabla_n^{\hat{Z}_v}\mathscr{H}^1(C,\mathbb{G})$, to see that our results also apply to this more classical object. To connect Drinfeld's moduli space to some $\nabla_n^{\hat{Z}_v}\mathscr{H}^1(C,\mathbb{G})$ we will choose $\mathbb{G}$ to be a group scheme over C such that $\mathbb{G}_v$ is an Iwahori group scheme for some fixed point $v \in C$ and such that $\mathbb{G}|_{C\backslash\{v\}}$ equals $\mathrm{GL}_r \times_{\mathbb{F}_q} C\backslash\{v\}$. We will begin the first section with a precise definition for more general group schemes of this kind, where we also allow other parahoric group schemes at several points. This is followed by a useful description of the torsors for these group schemes in terms of elementary modifications of vector bundles.

7.1 Torsors for Parahoric Bruhat-Tits Group Gchemes with Generic Fiber GL_r

We take some smooth projective geometrically irreducible curve over $\mathbb{F}_q$ and fix n closed points $c_1,\dots,c_n \in C$. The diagonal torus in GL_r determines a standard appartement in the Bruhat-Tits building of GL_r over Q_{c_i} and the Borel subgroup of upper triangular matrices determines a base alcove τ in this appartement (see appendix page 85). We define the standard Iwahori subgroup $\mathcal{I}$ as the stabelizer of τ. It has also another description. Namely if we define $\Lambda_j^{(i)} \subset Q_{c_i}^r$ for $j = 0,\dots,r$ as the free A_{c_i}-submodule with basis $z_i^{-1}e_1,\dots,z_i{}^{-1}e_j, e_{j+1},\dots,e_r$, where z_i is some uniformizer in A_{c_i}, we have $\mathcal{I} = \bigcap_j Stab_{\mathrm{GL}_n(Q_{c_i})}\Lambda_j^{(i)}$. Now we choose for each of the points c_i a parahoric subgroup $\mathcal{P}_i$ in $\mathrm{GL}_r(A_{c_i})$ that contains the standard Iwahori subgroup. Such a parahoric subgroup corresponds to a facet in $\overline{\tau}$ and is determined by its type $T_i \subset \mathbb{S}$ with $s_0 \notin T_i$, where $(W_{aff},\mathbb{S})$ is the affine Weyl group with the Coxeter generating system $\mathbb{S}$ as described in the appendix on page 85. We will explain in remark 7.5 why it is not really a restriction to allow only the parahoric subgroups with $s_0 \notin T_i$ and containing $\mathcal{I}$. We have $\mathcal{P}_i = \bigcap_{k,s_k\notin T_i} Stab_{\mathrm{GL}_r(Q_{c_i})}\Lambda_k^{(i)}$. The construction of Bruhat and Tits (see [BT84, page 356] and [Tit79, subsection 3.4.1]) gives us a unique smooth affine group scheme $\mathbb{P}_i$ over $Spec\ A_{c_i}$ such that the generic fiber $\mathbb{P}_{i,Q_{c_i}} = \mathrm{GL}_{r,Q_{c_i}}$, such that $\mathbb{P}_i(A_{c_i}) = \mathcal{P}_i \subset \mathrm{GL}_r(Q_{c_i})$ and such that the special fiber is connected. We can describe this group scheme on R-valued points for every A_{c_i}-algebra R quite explicitely as follows (compare also [Hai05, section 3.2]). For simplicity we will write the complete flag of the type T_i with $l_i := \#(T_i\backslash\mathbb{S}) - 1$ now as $\{0 = k_0^{(i)} \leqslant \dots \leqslant k_{l_i}^{(i)}\}$ by identifying $\{0,\dots,r-1\}$ with $\{s_0,\dots,s_{r-1}\} = \mathbb{S}$.
Then consider the lattice chain of $n+1$ lattices

$$L_\bullet^{(i)} := L_0^{(i)} \to \dots \to L_r^{(i)}$$

where $L_0^{(i)} = \dots = L_{k_1^{(i)}-1}^{(i)} = \Lambda_0^{(i)}, \qquad L_{k_1^{(i)}}^{(i)} = \dots = L_{k_2^{(i)}-1}^{(i)} = \Lambda_{k_1^{(i)}}^{(i)}, \qquad \dots$
$L_{k_{l_i}^{(i)}}^{(i)} = \dots = L_{r-1}^{(i)} = \Lambda_{k_{l_i}^{(i)}}^{(i)}$ and $L_r^{(i)} = z_i{}^{-1}\Lambda_0^{(i)}$. By defining $L_{\bullet,R}^{(i)} := L_\bullet^{(i)} \otimes_{A_{c_i}} R$ we can describe the R-valued points of $\mathbb{P}_i$ as an r-tuple $(g_0,\dots,g_{r-1}) \in Aut_R(L_{0,R}^{(i)}) \times \dots \times Aut_R(L_{r-1,R}^{(i)})$ such that the

diagram

$$\begin{array}{ccccccc} L^{(i)}_{0,R} & \longrightarrow & \dots & \longrightarrow & L^{(i)}_{r-1,R} & \longrightarrow & L^{(i)}_{r,R} \\ \downarrow g_0 & & & & \downarrow g_{r-1} & & \downarrow g_0 \\ L^{(i)}_{0,R} & \longrightarrow & \dots & \longrightarrow & L^{(i)}_{r-1,R} & \longrightarrow & L^{(i)}_{r,R} \end{array}$$

commutes. Of course an R-valued point is in fact given by a l_i-tuple, but the redundant lattices in the chain are advantegous when we describe the global group scheme over C. The description gives us in this way morphisms $\rho_k : \mathbb{P}_i \to \mathrm{GL}_{r,A_{c_i}}, \quad (g_0,\dots,g_{r-1}) \mapsto g_k$.

Remark 7.1. There is exactly one parabolic subgroup P_i in GL_r of type T_i containing the Borel subgroup of upper triangular matrices . Now $\mathcal{P}_i$ equals the preimage of $P_i(A_{c_i}/(z_i))$ under the projection $\mathrm{GL}_r(A_{c_i}) \to \mathrm{GL}_r(A_{c_i}/(z_i))$. We can define the functor:

$$\{\text{flat } A_{c_i}\text{-algebras}\} \to \mathbf{Groups} \qquad R \mapsto \left\{ g \in \mathrm{GL}_r(R) \mid g \mod z_i \in P_i(R/(z_i)) \right\}$$

By [Yu15] this functor is representable by $\mathbb{P}_i$. One should be aware that this description of the functor of points for $\mathbb{P}_i$ is only true for flat A_{c_i}-algebras.

Now we have the group scheme $(\coprod_i \mathbb{P}_i) \coprod \mathrm{GL}_r \times_{\mathbb{F}_q} C\backslash\{c_1,\dots,c_n\} \to (\coprod_i \mathit{Spec}\, A_{c_i}) \coprod C\backslash\{c_1,\dots,c_n\}$. As described in § 2.17 we use faithfully flat descent along the map $(\coprod_i \mathit{Spec}\, A_{c_i}) \coprod C\backslash\{c_1,\dots,c_n\} \to C$ with the descent datum $\varphi = id$ to glue it to a group scheme $\mathbb{G} \to C$. Like in the local case we would like to have a description of this group scheme as an automorphism group.

We will define a category **pVec** fibered over $C_{\acute{E}t}$ of certain chains of vector bundles. $\mathbb{G}$ will then be the automorphism group of one of its objects. Let D be the divisor $\sum_{i=1}^n c_i$ on C. For $f : S \to C$ let $\mathcal{Q}$ be some locally free $\mathcal{O}_{f^\star D}$-module on the divisor $f^\star D$. Given a map $\mathcal{V}_0 \xrightarrow{\alpha} \mathcal{V}_1$ of two vector bundles over S, we recall that we call $\mathcal{V}_0$ an elementary modification of $\mathcal{V}_1$ by $\mathcal{Q}$ if there is a quotient map $\mathcal{V}_1 \xrightarrow{\varphi} \mathcal{Q} \to 0$ such that $V_0 \xrightarrow{\alpha} \mathcal{V}_1$ is the kernel of φ.

We now define the category **pVec** as the category fibered over $C_{\acute{E}t}$ whose fiber category $\mathbf{pVec}_S$ for some scheme S with $f : S \to C$ has the objects

$$\left\{ (\mathcal{V}_\bullet, \alpha_\bullet) = \mathcal{V}_0 \xrightarrow{\alpha_0} \mathcal{V}_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{r-1}} \mathcal{V}_r = \mathcal{V}_0(f^\star D) \right\}$$

where $\mathcal{V}_j$ are vector bundles over S and α_j are elementary modifications by some locally free sheaf $\mathcal{Q}_j$ on $S \times_C D$, where $\mathcal{Q}_j|_{c_i \times_C S}$ is locally free of rank $d_j^{(i)} := dim\ L_{j+1}^{(i)}/L_j^{(i)}$. The category **pVec** therefore depends on C, on r and the dimensions $d_j^{(i)}$, but we omit this in the notation. A morphism from $(\mathcal{V}_\bullet, \alpha_\bullet)$ to $(\mathcal{V}'_\bullet, \alpha'_\bullet)$ in $\mathbf{pVec}_S$ is given by a tuple $(g_0,\dots,g_{r-1})$ where $g_j : \mathcal{V}_j \to \mathcal{V}'_j$ is an isomorphism of vector bundles such that $g_{i+1} \circ \alpha_i = \alpha'_i \circ g_i$. There is the following "standard" object $(V_\bullet, a_\bullet)$ in $\mathbf{pVec}_C$. Let D_{jl} for $j = 1,\dots,r-1$ and $l = 1,\dots,r$ be the divisor on C defined by $D_{jl} = \sum_{k,\ s_m \notin T_k\ for\ some\ l \leqslant m \leqslant j} c_k$. In particular we have $D_{jl} = \varnothing$ if $j < l$. Then we define

$$V_\bullet := V_0 \xrightarrow{a_0} \dots \xrightarrow{a_{r-1}} V_r$$

as the chain of vector bundles with $V_0 = \bigoplus_{l=1}^r \mathcal{O}_C \cdot e_l$ and $V_j = \bigoplus_{l=1}^r \mathcal{O}_C(D_{jl}) e_l$ for $j = 1,\dots,r-1$ and $V_r = V_0(D)$. Here the a_j are given by the natural inclusion of the sheafs and a_j is an elementary

modification by the locally free sheaf on D associated to the module $\bigoplus_i L^{(i)}_{j+1}/L^{(i)}_j$. Using this standard object we define the group scheme $\widetilde{\mathbb{G}} \to C$ by $\widetilde{\mathbb{G}}(R) = Aut(V_\bullet \otimes_C Spec\ R)$ for every $Spec\ R \to C$. The construction of $V_\bullet$ shows that the pullback of $V_\bullet$ to $Spec\ A_{c_i}$ gives exactly the chain of modules $L^{(i)}_\bullet$ and the pullback to $C\backslash\{c_1,\dots,c_n\}$ gives the trivial modules $\Gamma(\mathcal{O}_C, C\backslash\{c_1,\dots,c_n\})^r$ with $a_i|_{C\backslash\{c_1,\dots,c_n\}} = id$. In particular this shows $\widetilde{\mathbb{G}} \times_C Spec\ A_{c_i} = \mathbb{P}_i$ and $\widetilde{\mathbb{G}} \times_C (C\backslash\{c_1,\dots,c_n\}) = \mathrm{GL}_r \times_{F_q} (C\backslash\{c_1,\dots,c_n\})$. So $\widetilde{\mathbb{G}}$ restricted to $(\coprod_i Spec\ A_{c_i}) \coprod (C\backslash\{c_1,\dots,c_n\})$ is isomorphic to $\mathbb{G}$ with the same descent datum which shows $\widetilde{\mathbb{G}} \simeq \mathbb{G}$.
The following lemma is the crucial step to see afterwards that the category of $\mathbb{G}$-torsors over C_S is equivalent to the fiber category $\mathbf{pVec}_{C_S}$. Actually **pVec** is not only a fibered category over $C_{\acute{E}t}$ but also a stack, since objects can be constructed locally by gluing.
Therefore the following lemma tells us exactly that **pVec** is a gerb in the sense of definition [Gir71, Def 2.1.1].

Lemma 7.2. *Let S be a scheme over C. Then any two objects $(\mathcal{V}_\bullet, \alpha_\bullet)$ and $(\mathcal{V}'_\bullet, \alpha'_\bullet)$ in $\mathbf{pVec}_S$ are Zariski locally isomorphic. That means there is a covering $U \to S$ such that $(\mathcal{V}_\bullet, \alpha_\bullet) \times_S U \simeq (\mathcal{V}'_\bullet, \alpha'_\bullet) \times_S U$.*

Proof: It is clearly enough to show that each object in **pVec** is locally isomorphic to $(V_\bullet, a_\bullet) \times_C S$. So let $(\mathcal{V}_\bullet, \alpha_\bullet)$ be any object over $S \to C$. Since the question is local we can assume that S is affine and consider the problem only for $S \times_C (C\backslash\{c_2,\dots,c_n\}) =: SpecA$. The argumentation over the other opens of the form $C\backslash\{\bigcup_{i\neq j} c_i\}$ is analogous. Let $C\backslash\{c_2,\dots,c_n\} =: SpecR$ and let $\mathfrak{m}_1 \subset R$ be the maximal ideal corresponding to c_1. Now $(\mathcal{V}_\bullet, \alpha_\bullet)$ corresponds to a chain of locally free A-modules $M_0 \xrightarrow{\alpha_0} M_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{r-1}} M_r$ such that $\alpha_{r-1} \circ \dots \circ \alpha_1$ equals the inclusion $M_0 = \mathfrak{m}_1 M_r \hookrightarrow M_r$ and M_{j+1}/M_j is a locally free $A/\mathfrak{m}_1 A$ module of rank $dim_{\mathbb{F}_{c_1}} L^{(1)}_{j+1}/L^{(1)}_j$. Let P_1 be the parabolic subgroup of GL_r as in remark 7.1. It is the stabilizer of the flag $0 \subset L^{(1)}_1/L^{(1)}_0 \subset \dots \subset L^{(1)}_r/L^{(1)}_0$. Let $\overline{M_j} := M_j/M_0$ then $0 \hookrightarrow \overline{M}_1 \dots \hookrightarrow \overline{M_r}$ gives us an $A/\mathfrak{m}_1$-valued point in the partial flag variety $\mathrm{GL}_r/P_1 =: Flag$. This morphism has Zariski locally a section (compare [Spr98, Theorem 8.5.2]). So after passing to a Zariski covering of $Spec\ A$ the $A/\mathfrak{m}_1$-valued point of $Flag$ comes from an element $g \in \mathrm{GL}_r(A/\mathfrak{m}_1)$ and we can identify $0 \to \overline{M}_1 \to \dots \to \overline{M}_r$ with $0 \to g\big(V_1/V_0 \otimes_{\mathbb{F}_{c_i}} A/\mathfrak{m}_1\big) \to \dots \to g\big(V_r/V_0 \otimes_{\mathbb{F}_{c_i}} A/\mathfrak{m}_1\big)$, where $V_j/V_0 \otimes_{\mathbb{F}_{c_1}} A/\mathfrak{m}_1 = L_j/L_0 \otimes_{\mathbb{F}_{c_1}} A/\mathfrak{m}_1 = \bigoplus_{i=1}^{dim_{\mathbb{F}_{c_1}} V_j/V_0} A/\mathfrak{m}_1 \cdot e_i$. We define $(\overline{v_1},\dots,\overline{v_r}) := (ge_1,\dots,ge_r)$ as a basis of $\overline{M}_r$ and choose lifts $v_1,\dots,v_r \in M_r$ with $v_{k^{(1)}_{l-1}+1},\dots,v_{k^{(1)}_l} \in M_{k_l}$ for $l = 1,\dots,l_1 = \#(\mathbb{S}\backslash T_1) - 1$. Now the Nakayama lemma [Eis95, corollary 4.7] applied to $M_{k^{(1)}_l} \big/ A \cdot v_{k^{(1)}_{l-1}+1} + \dots + A \cdot v_{k^{(1)}_l}$ shows that there is an $x \in \mathfrak{m}_1 A$ such that $v_{k^{(1)}_{l-1}+1},\dots,v_{k^{(1)}_l}$ is a basis for $M_{k^{(1)}_l}$ on $Spec\ A\big[\frac{1}{1-x}\big] =: U$. This means that $V_0 \times_C U \to \dots \to V_r \times_C U$ is isomorphic to $\mathcal{V}_0|_U \to \dots \to \mathcal{V}_r|_U$. Now $Spec\ A \times_C c_1 = Spec\ A/\mathfrak{m}_1 A \subset U$ and on $Spec\ A\backslash Spec\ A/\mathfrak{m}_1 A$ the object $(\mathcal{V}_\bullet, \alpha_\bullet)$ is clearly locally isomorphic to $(V_\bullet, a_\bullet)$ since all the α_j are isomorphisms. This proves the lemma. □

Once we know that **pVec** is a gerb and that $\mathbb{G}$ is the automorphism group of $(V_\bullet, a_\bullet)$ it follows by [Gir71, Corollaire 2.2.6] that **pVec** is equivalent to the gerb of $\mathbb{G}$-torsors on $C_{\acute{E}t}$, which means in particular that for any $\mathbb{F}_q$-scheme S the fiber category $\mathbf{pVec_S}$ is equivalent to the category $\mathscr{H}^1(C,\mathbb{G})(S)$ of $\mathbb{G}$-torsors over C_S. We describe this equivalence more explicitely, so let S be a $\mathbb{F}_q$-scheme.

Proposition 7.3. *We have a category equivalence*

$$\Phi : \mathbf{pVec}_{C_S} \to \mathscr{H}^1(C,\mathbb{G})(S)$$

Proof: Let $(\mathcal{V}_\bullet,\alpha_\bullet)$ in $\mathbf{pVec}_{C_S}$ then the functor Φ sends $(\mathcal{V}_\bullet,\alpha_\bullet)$ to $Isom\Big((\mathcal{V}_\bullet,\alpha_\bullet),(V_\bullet,a_\bullet)\otimes_C C_S\Big)$, where the latter sheaf becomes an $Aut(V_\bullet,a_\bullet)=\mathbb{G}$ torsor by composition on the right with an element in $\mathbb{G}$.
Now let $U\to C_S$ be a covering with an isomorphism $\gamma:(\mathcal{V}_\bullet,\alpha_\bullet)\times_{C_S}U\to(V_\bullet,a_\bullet)\times_C U$ as in 7.2 and set $U'':=U\times_{C_S}U$. Then $(\mathcal{V}_\bullet,\alpha_\bullet)$ is isomorphic to the object coming from the descent datum $\Big((V_\bullet\times_C U,a_\bullet),\varphi\Big)$ with $\varphi:=p_2^\star\gamma\circ p_1^\star\gamma^{-1}\in Aut(V_\bullet,a_\bullet)(U'')$. Now the same γ induces an isomorphism

$$Isom\Big((\mathcal{V}_\bullet,\alpha_\bullet)\times_{C_S}U,(V_\bullet,a_\bullet)\times_C U\Big)\to Isom\Big((V_\bullet,a_\bullet)\times_C U,(V_\bullet,a_\bullet)\times_C U\Big)=\mathbb{G}\times_C U.$$

Hence $Isom\Big((\mathcal{V}_\bullet,\alpha_\bullet),(V_\bullet,a_\bullet)\Big)$ is isomorphic to the torsor coming from the descent datum $(\mathbb{G}\times_C U,\varphi)$ and it follows that Φ is essentially surjectiv. Namelly if $\mathcal{G}$ is any $\mathbb{G}$-torsor in $\mathscr{H}^1(C,\mathbb{G})(S)$ isomorphic to the one coming from the descent datum $(\mathbb{G}\times_C V,\psi)$ for some covering $V\to C_S$ and $\psi\in\mathbb{G}(U'')$, then let $(\mathcal{V}_\bullet,\alpha_\bullet)$ be the object in $\mathbf{pVec}_{\mathbf{C_S}}$ coming from the descent datum $\Big((V_\bullet,a_\bullet)\times_C V,\psi\Big)$. This means $\Psi(\mathcal{V}_\bullet,\alpha_\bullet)\simeq\mathcal{G}$. Since the automorphisms of the descent data are the same, the proposition follows. □

Remark 7.4. Actually the proof shows more generally that we also have a category equivalence from $\mathbf{pVec}_U$ to the category of $\mathbb{G}$-torsors over U for all schemes U over C.

Remark 7.5. When we defined the group scheme $\mathbb{G}$, we have choosen a parahoric subgroup for every point c_i and made some assumptions on these subgroups ($s_0\notin T_i$ and $\mathcal{I}\subset\mathcal{P}_i$). We would like to explain why this does not cause any loss of generality. If we choose for every point c_i some arbitrary parahoric $\widetilde{\mathcal{P}}$ then the associated parahoric group scheme over $Spec\ A_{c_i}$ can be realized as the automorphisms of some other periodic lattice chain $\widetilde{L^{(i)}}$ in $Q_{c_i}^r$.
We can use these lattice chains to glue them to a chain of vector bundles $(\widetilde{V_\bullet},\widetilde{a_\bullet})$ over C with $(\widetilde{V_\bullet},\widetilde{a_\bullet})\times_C Spec\ A_{c_i}=\widetilde{L_\bullet^{(i)}}$ and $(\widetilde{V_\bullet},a_\bullet)\times_C C\backslash\{c_1,\dots,c_n\}=(\mathcal{O}^r_{C\backslash\{c_1\dots,c_n\}},id_\bullet)$. Then we can as before define a group scheme $\widetilde{\mathbb{G}}$ over C by $\widetilde{\mathbb{G}}(S)=Aut((\widetilde{V_\bullet},\widetilde{a_\bullet})\times_C S)$. Since we can always find a $g\in\mathrm{GL}_r(Q_{c_i})$ that transforms $\widetilde{L_\bullet^{(i)}}$ to some lattice chain $L_\bullet^{(i)}$ of the form described at the beginning of this section, it follows that $(\widetilde{V_\bullet},\widetilde{a_\bullet})$ is locally isomorphic to $(V_\bullet,a_\bullet)$ which implies that $\widetilde{\mathbb{G}}$ is locally, but not necesarrily globally, isomorphic to $\mathbb{G}$. Now the functor

$$\mathbf{pVec}_{C_S}\to\mathscr{H}^1(C,\widetilde{\mathbb{G}})(S)\qquad(\mathcal{V}_\bullet,\alpha_\bullet)\mapsto Isom((\mathcal{V}_\bullet,\alpha_\bullet),(\widetilde{V_\bullet},\widetilde{a_\bullet}))$$

gives a category equivalence as well. Nevertheless one should be aware that a $\mathbb{G}$-torsor $\mathcal{G}$ is in general not a $\widetilde{\mathbb{G}}$-torsor, since $\widetilde{\mathbb{G}}$ does not act in general on $\mathcal{G}$.
Above we chose a $g\in\mathrm{GL}_r(Q_{c_i})$ with $g\widetilde{L_\bullet^{(i)}}=L_\bullet^{(i)}$ which implies $g\widetilde{\mathcal{P}}_i g^{-1}=\mathcal{P}_i$. Parahoric subgroups of the same type are always conjugate, but since the operation of Ω does not fix the type there are further parahoric subgroups that are conjugate. In the case of GL_r we can find for every parahoric subgroup $\mathcal{P}_i$ with type $\widetilde{T}_i$ a conjugated parahoric subgroup $\mathcal{P}$ of type T satisfying the condition $s_0\notin T$, since the operation of Ω translates the type. This parahoric subgroup $\mathcal{P}$ can then be conjugated to the unique parahoric subgroup $\mathcal{P}_i$ of the same type $T_i=T$ containing the standard Iwahori subgroup $\mathcal{I}$.

7.2 Drinfeld's Moduli Space with Iwahori Level Structure

In this section we will first of all fix a particular Shtuka data $(C, \mathbb{G}, \underline{v}, \hat{Z}_{\underline{v}}, H)$ in the sense of definition 3.1 that defines a moduli space $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$ of $\mathbb{G}$-shtukas. We give a definition of Drinfelds moduli space with Iwahori level structure and prove that this is isomorphic to a closed substack of $\nabla_n^{\hat{Z}_{\underline{v}},H}\mathscr{H}^1(C,\mathbb{G})$.
We fix a curve C with the two closed points $v_1 = \infty$ and v_2 with residue fields $\mathbb{F}_{v_1}$ and $\mathbb{F}_{v_2}$ lying in the opens $U_1 = Spec\ A$ and U_2 of C. Let $\mathbb{I}_{v_2}$ be the Iwahori group scheme defined over $Spec\ A_{v_2}$, that corresponds to the standard Iwahori group of GL_r. Its R-valued points $\mathbb{I}_{v_2}(R)$ for an A_{v_2}-algebra R are given as the automorphisms of the lattice chain $\Lambda_{\bullet,R}$ as described at the beginning of chapter 7.1. Then we use faithfully flat descent along the map $U_1 \coprod Spec\ A_{v_2} \to C$ to glue the group scheme $\mathrm{GL}_{r,U_1} \coprod \mathbb{I}_{v_2} \to U_1 \coprod Spec\ A_{v_2}$ to a Bruhat-Tits group scheme over C. As explained in the chapter 7.1 it is also the automorphism group of some chain of vector bundles.
Let Q be the function field of C and let $G := \mathbb{G} \times_C Spec\ Q = \mathrm{GL}_r$ be the generic fiber of $\mathbb{G}$. For the completions of the stalks at the points v_1 and v_2 we have $A_{v_1} = \mathbb{F}_{v_1}[\![z_1]\!]$ and $A_{v_2} = \mathbb{F}_{v_2}[\![z_2]\!]$ and its quotient fields are $Q_{v_1} = \mathbb{F}_{v_1}(\!(z_1)\!)$ and $Q_{v_2} = \mathbb{F}_{v_2}(\!(z_2)\!)$ respectively. By construction the base changes of $\mathbb{G}$ to these rings are given as

$$\mathbb{G}_{v_1} := \mathbb{G} \times_{\mathbb{P}^1_{\mathbb{F}_q}} Spec\ A_{v_1} = \mathrm{GL}_{r,A_{v_1}} \qquad \mathbf{G}_{v_1} = \mathbb{G} \times_C Spec\ Q_{v_1} = GL_{r,Q_{v_1}}$$

$$\mathbb{G}_{v_2} := \mathbb{G} \times_{\mathbb{P}^1_{\mathbb{F}_q}} Spec\ A_{v_2} = \mathbb{I}_{v_2} \qquad \mathbf{G}_{v_2} = \mathbb{G} \times_C Spec\ Q_{v_2} = GL_{r,Q_{v_2}}$$

Now we define the bounds $\hat{Z}_1$ and $\hat{Z}_2$ as the scheme theoretic closure of the orbits

$$\hat{Z}_{v_1} = (L^+\mathbb{G}_{v_1} \times_{\mathbb{F}_{v_1}} \mathbb{F}_{v_1}[\![\zeta_1]\!]) \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & (z_1-\zeta_1)^{-1} \end{pmatrix} (L^+\mathbb{G}_{v_1} \times_{\mathbb{F}_{v_1}} \mathbb{F}_{v_1}[\![\zeta_1]\!]) \Big/ (L^+\mathbb{G}_{v_1} \times_{\mathbb{F}_{v_1}} \mathbb{F}_{v_1}[\![\zeta_1]\!]) \subset \hat{\mathcal{F}l}_{\mathbb{G}_{v_1}}$$

and

$$\hat{Z}_{v_2} = \bigcup_{s \in W_0} \left(L^+\mathbb{G}_{v_2} \times_{\mathbb{F}_{v_2}} \mathbb{F}_{v_2}[\![\zeta_2]\!]\right) s \begin{pmatrix} z_2-\zeta_2 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} s^{-1} \left(L^+\mathbb{G}_{v_2} \times_{\mathbb{F}_{v_2}} \mathbb{F}_{v_2}[\![\zeta_2]\!]\right) \Big/ \left(L^+\mathbb{G}_{v_2} \times_{\mathbb{F}_{v_2}} \mathbb{F}_{v_2}[\![\zeta_2]\!]\right) \subset \hat{\mathcal{F}l}_{\mathbb{G}_{v_2}}$$

where s denotes here the permutation matrix corresponding to $s \in W_0$. This definition has also the advantage that $\hat{Z}_1$ and $\hat{Z}_2$ are reduced and irreducible and in particular flat [Har77, chapter III proposition 9.7] so that $\nabla_2^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G})$ is flat. The reflex rings of these bounds are A_{v_1} and A_{v_2}. Now $(C, \mathbb{G}, \underline{v}, \hat{Z}_{\underline{v}})$ defines the moduli stack $\nabla_2^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G})$ over $Spf\ A_{\underline{v}} := Spf\ \kappa[\![z_1,z_2]\!]$, where κ is the compositum of $\mathbb{F}_{v_1}$ and $\mathbb{F}_{v_2}$. There is the covering $\nabla_2^{\hat{Z}_{\underline{v}}}\widetilde{\mathscr{H}^1(C,}\mathbb{G})$ of this moduli stack whose S-valued points are given by

$$\nabla_2^{\hat{Z}_{\underline{v}}}\widetilde{\mathscr{H}^1(C,}\mathbb{G})(S) = \left\{ (\mathcal{G}, s_1, s_2, \tau_{\mathcal{G}}, \epsilon_1, \epsilon_2) \;\middle|\; \begin{array}{l} \underline{\mathcal{G}} \in \nabla_2^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G})(S) \\ \text{and } \ \epsilon_i : \Gamma_{v_i}(\sigma^\star\mathcal{G}) \to L^+\mathbb{G}_{v_i,S} \\ \text{is a trivialization} \end{array} \right\}$$

Actually the map $\nabla_2^{\hat{Z}_{\underline{v}}}\widetilde{\mathscr{H}^1(C,\mathbb{G})} \longrightarrow \nabla_2^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G})$, $(\underline{\mathcal{G}},\epsilon_1,\epsilon_2) \mapsto \underline{\mathcal{G}}$ is an $L^+\mathbb{G}_{v_1} \times L^+\mathbb{G}_{v_2}$-torsor. Furthermore for a point $(\underline{\mathcal{G}},\epsilon_1,\epsilon_2) \in \nabla_2^{\hat{Z}_{\underline{v}}}\mathscr{H}^1\widetilde{(C,\mathbb{G})}(S)$ and a trivialization $\alpha_i : \Gamma_{v_i}(\mathcal{G}) \to L^+\mathbb{G}_{v_i,S}$ the composition

$$L(\alpha_i) \circ \Gamma_{v_i}(\tau_{\mathcal{G}}) \circ L(\epsilon_i^{-1}) : L\mathbf{G}_{v_i,S} \to L\mathbf{G}_{v_i,S}$$

corresponds to a morphism $S \to L\mathbf{G}_{v_i}$ and induces therefore an S-valued point in $\mathcal{F}l_{v_i}$ that is independent of α_i. In particular it gives a morphism $S \to \hat{\mathcal{F}}l_{v_i}$ which factors by the boundedness condition through $\hat{Z}_{v_i}$. This defines a morphism $\nabla_2^{\hat{Z}_{\underline{v}}}\widetilde{\mathscr{H}^1(C,\mathbb{G})} \to \hat{Z}_{v_1}\hat{\times}_{\mathbb{F}_q}\hat{Z}_{v_2}$ that forms the local model roof

$$\begin{array}{ccc} & \nabla_2^{\hat{Z}_{\underline{v}}}\widetilde{\mathscr{H}^1(C,\mathbb{G})} & \\ \swarrow & & \searrow \\ \nabla_2^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G}) & & \hat{Z}_{v_1}\hat{\times}_{\mathbb{F}_q}\hat{Z}_{v_2} \end{array} .$$

Let κ be the residue field of $Spf\ A_{\underline{v}}$. Since we are interested in the stratifications of the special fiber $\nabla_2^{\hat{Z}_{\underline{v}}}\mathscr{H}^1(C,\mathbb{G})\times_\kappa$ its worth to describe the special fibers $Z_{v_i} := \hat{Z}_{v_i} \times_{Spf\ A_{v_i}} \kappa_{v_i}$ in terms of Schubert varieties.

By the Bruhat decomposition $X_\star(T) = L^+\mathbb{G}_{v_1}\backslash L\mathbf{G}_{v_1}/L^+\mathbb{G}_{v_1}$ and since $\mu_1 = (0,\dots,0,-1)$ is a minimal element in $X^+(T)$ the special fiber Z_{v_1} is given by the Schubert variety $\mathcal{S}(z^{\mu_1})$.

For $\mathbb{G}_{v_2}$ we have the Cartan decomposition $\widetilde{W} = X_\star(T) \rtimes W_0 = L^+\mathbb{G}_{v_2}\backslash L\mathbf{G}_{v_2}/L^+\mathbb{G}_{v_2}$. We recall that we can identify $\widetilde{W}$ in this case with a subgroup of $L\mathbf{G}_{v_2}(\mathbb{F}_q)$ by sending $z^\mu \cdot s \in \widetilde{W}$ with $\mu = (u_1,\dots,u_r)$ to $\begin{pmatrix} z_{v_2}^{u_1} & & 0 \\ & \ddots & \\ 0 & & z_{v_2}^{u_r} \end{pmatrix} \cdot s$ where $s \in W_0$ corresponds to some permutation matrix.

Furthermore we can write $\widetilde{W} = W_{aff} \rtimes \Omega$ where Ω is the stabilizer of the base alcove and in this case given by $\Omega = \mathbb{Z}\cdot\beta$ with $\beta = z^{(0,\dots,0,1)}s_{r-1},\dots,s_1$ (see appendix A). In particular β is a length 0 element that corresponds to the matrix $\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ z_{v_2} & & & 0 \end{pmatrix}$ and $s_1\dots s_{r-1}\beta$ corresponds to $\begin{pmatrix} z_{v_2} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & 1 \end{pmatrix}$. Recall that $s_0 = z^{(-1,0,\dots,0,1)}s_1s_2,\dots,s_{r-1}\dots,s_2s_1$ corresponds to the matrix $\begin{pmatrix} 0 & & & z_{v_2}^{-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ z_{v_2} & & & & 0 \end{pmatrix}$ so that one verifies $z^{(-1,0,\dots,0,\underbrace{1}_{i},0\dots)} = s_i\dots s_{r-1}s_0s_1\dots s_{i-2}s_{r-1}\dots s_1$ for $i = 2,\dots,r$.

Multiplying this element from the left to $s_1\dots s_{r-1}\beta = z^{(1,0,\dots,0)}$ gives the identities

$$s_1\dots s_{r-1}\beta = z^{(1,0,\dots,0)}, \quad s_2\dots s_{r-1}s_1\beta = z^{(0,1,0,\dots)}, \ \dots\ , s_0\dots s_{r-2}\beta = z^{(0,\dots,0,1)} \tag{33}$$

The generic fiber Z_{v_2} is therefore the union of the Schubert varieties $\bigcup_{s\in W_0}\mathcal{S}(z^{(s(\mu_2))})$ with $\mu_2 = (1,0,\dots,0)$ and it contains the $L^+\mathbb{G}_{v_2}$ orbits of all the elements $\omega \in \widetilde{W}$ with $\omega \leqslant z^{s(\mu_2)}$ for some $s \in W_0$. This is by definition the admissible subset

$$Adm(\mu_2) := \{\omega \in \widetilde{W} \mid \exists s \in W_0 \text{ with } \omega \leqslant z^{s(\mu_2)}\},$$

that can also be described as follows.

Lemma 7.6. *The set $Adm(\mu_2)$ corresponds bijectively to the set*

$$\left\{ (a_{ij}) = A \in \mathrm{GL}_r(\mathbb{F}_q(z)) \;\middle|\; \begin{array}{l} A \text{ is monomial with exactly one entry equal to } z \\ \text{and r-1 entries equal to 1 satisfying} \\ (a_{ij}) \neq 1 \text{ for } i > j \text{ and } (a_{ij}) \neq z \text{ for } i < j \end{array} \right\}$$

Proof: Let A be as in the set described above and $a_{kl} = z$ with $k > l$. This implies $a_{ii} = 1$ for all $i > k$ and all $i < l$. Denote by $(w_1, \dots, w_m)$ the set $\{j \in \mathbb{N} \mid l < j < k \text{ and } a_{jj} = 1\}$. Now A in the above set is uniquely determined by these conditions ($a_{kl} = z$ and $a_{w_i w_i} = 1$). The element $s_k \dots s_{r-1} s_0 s_1 \dots s_{l-2}\beta$ (if $k = r$ let $s_k \dots s_{r-1} = id$ and if $l = 1$ let $s_0 s_1 \dots s_{l-2} = id$) corresponds to the monomial matrix with $a_{kl} = z$ and $a_{ii} = 1$ if and only if $i > k$ or $i < l$. It follows $\omega = s_{w_1} \dots s_{w_m} s_k \dots s_{r-1} s_0 s_1 \dots s_{l-2}\beta$ corresponds to A. Now $s_{w_1} \dots s_{w_m} s_k \dots s_{r-1} s_0 s_1 \dots s_{l-2}$ is obviously a subword of one of the elements in (33) so that ω lies in $Adm(\mu_2)$. Furthermore it is clear that all subwords of $s(\mu_2)$ with $s \in W_0$ and therefore all elements in $Adm(\mu_2)$ arises from a matrix A as above. □

Let us further denote with b_0 the basic Newton polygon $(\frac{1}{r}, \dots, \frac{1}{r})$ in $B(\mathbf{G}_{v_1})$. Let $\mathcal{N}_{b_0}$ be the Newton stratum corresponding to b_0. That means $\mathcal{N}_{b_0}(S)$ corresponds of those points $\underline{\mathcal{G}} \in \nabla_2^{\hat{Z}_v}\mathscr{H}^1(C, \mathbb{G})_\kappa$ such that for all geometric points $\bar{s} \in S$ the local shtuka $\Gamma_{v_1}(\underline{\mathcal{G}})_{\bar{s}}$ is isomorpic to $(L^+\mathbb{G}_{v_1,\kappa(\bar{s})}, b_0)$.

The Drinfeld moduli space

We just recall the definition of the Drinfeld moduli varieties and then define the Drinfeld moduli space with Iwahori level structure. Let $d \in \mathbb{N}$ and $U_1 = Spec\ A$. The Drinfeld moduli space $\mathcal{D}^r$ (without level structure) is defined as the category fibered over $(\mathbb{F}_q)_{\acute{E}t}$ whose fiber category for some scheme S is given by

$$\mathcal{D}^r(S) = \left\{ (E, \varphi, \gamma) \;\middle|\; \begin{array}{l} \text{where } \gamma : S \to Spec\ A \text{ and } (E, \varphi) \text{ is a} \\ \text{Drinfeld A-module of rank } r \text{ over } (S, \gamma) \end{array} \right\}.$$

We refer to the appendix B for the definition of Drinfeld modules. Morphisms from (E, φ, γ) to (E', φ', γ') in this fiber category only exist if $\gamma = \gamma'$ and are given by isomorphisms of Drinfeld A-modules. $\mathcal{D}^r$ is a Deligne-Mumford stack of finite type over $\mathbb{F}_q$. There is a map $\mathcal{D}^r \to Spec\ A$ sending (E, φ, γ) to γ and we can consider the base change $\mathcal{D}^r \times_{Spec\ A} Spf\ A_{v_2} =: \mathcal{D}^r_{A_{v_2}}$ as well as its special fiber $\mathcal{D}^r \times_{Spec\ A} Spec\ \mathbb{F}_{v_2} =: \mathcal{D}^r_{v_2}$. The fibercategory $\mathcal{D}^r_{A_{v_2}}(S)$ (resp. $\mathcal{D}^r_{v_2}(S)$) of this stack consists only of Drinfeld A-modules whose characteristic γ factors through $Spf A_{v_2}$ (resp. $Spec\ \mathbb{F}_{v_2}$). Now we define the Drinfeld moduli space with Iwahori level at v_2 $\mathcal{D}^r_{I,A_{v_2}}$ as the stack fibered over $(\mathbb{F}_q)_{\acute{E}t}$ whose fiber category $\mathcal{D}^r_{I,A_{v_2}}(S)$ has the objects

$$\{\underline{E}_0 \xleftarrow{\alpha_1} \underline{E}_1 \xleftarrow{\alpha_2} \dots \xleftarrow{\alpha_r} \underline{E}_r\}$$

where $\underline{E}_i := (E_i, \varphi_i)$ are Drinfeld A-modules in $\mathcal{D}^r_{A_{v_2}}(S)$ and $\alpha_i : \underline{E}_i \to \underline{E}_{i-1}$ are isogenies of order q, (i.e. $ker(\alpha_i)$ is a finite group scheme of order $\#\mathbb{F}_{v_i}$) and the composition $\alpha_r \circ \dots \circ \alpha_1 :$ $underlineE_r \to \underline{E}_1$ has kernel $\underline{E}_r[v_2]$.

A morphism in the fiber category from $(\underline{E}_0 \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_r} \underline{E}_r)$ to $(\underline{E}'_0 \xleftarrow{\alpha'_1} \dots \xleftarrow{\alpha'_r} \underline{E}'_r)$ is given by a tuple $(\delta_0, \dots, \delta_r)$ where $\delta_i : E_i \to E'_i$ is an isogeny with $\alpha'_i \circ \delta_i = \delta_{i-1} \circ \alpha_i$ for $i = 1, \dots, r$. We denote by $\mathcal{D}^r_{I,v_2}$ the special fiber $\mathcal{D}^r_{A_{v_2}} \times_{Spf\ A_{v_2}} \mathbb{F}_{v_2}$. Now we have the following remark.

Remark 7.7. Using the category (anti) equivalence [Har17, Theorem 3.5] of Drinfeld A-modules over R and effective Anderson A-motives (M, τ_M) of dimension 1 over R with the condition that M is finitely generated as $R\{\tau\}$ module, we see that the category $\mathcal{D}^r_{I,A_{v_2}}$ is equivalent to the category whose objects are given as

$$\{\underline{M}_1 \xrightarrow{\beta_1} \dots \xrightarrow{\beta_r} \underline{M}_r\}$$

where $\underline{M_1}, \dots, \underline{M_r}$ are pure Anderson A-Motives of rank r and dimension 1 over $(S, \gamma : S \to Spf\ A_{v_2})$ and $\beta_1, \dots, \beta_r$ are isogenies of degree 1.

Proposition 7.8. *We have a faithful essentially surjective functor* $\Psi : \mathcal{N}_{b_0} \to \mathcal{D}^r_{I,v_2}$.

Proof: Let $S = Spec\ R$ be a scheme over $\mathbb{F}_q$ and $\underline{\mathcal{G}} = (\mathcal{G}, s_1, s_2, \tau) \in \mathcal{N}_{b_0}(S)$. The equivalence in 7.3 maps $\mathcal{G}$ and $\sigma^\star\mathcal{G}$ to a chain of vector bundles $\mathcal{V}_0 \xrightarrow{\alpha_1} \dots \mathcal{V}_{r-1} \xrightarrow{\alpha_r} \mathcal{V}_r$ and $\sigma^\star\mathcal{V}_0 \xrightarrow{\alpha_1} \dots \sigma^\star\mathcal{V}_{r-1} \xrightarrow{\sigma^\star\alpha_r} \sigma^\star\mathcal{V}_r$, where all the arrows are elementary modifications by $\mathcal{O}_{S\times v_2}$. Furthermore $\tau : \sigma^\star\mathcal{G}|_{C_S\setminus(\Gamma_{s_1}\cup\Gamma_{s_2})} \to \mathcal{G}|_{C_S\setminus(\Gamma_{s_1}\cup\Gamma_{s_2})}$ induces by the same equivalence and remark 7.4 a tuple $(\tau_0, \dots, \tau_r)$, where $\tau_i : \sigma^\star\mathcal{V}_i|_{C_S\setminus(\Gamma_{s_1}\cup\Gamma_{s_2})} \to \mathcal{V}_i|_{C_S\setminus(\Gamma_{s_1}\cup\Gamma_{s_2})}$ satisfying $\tau_i \circ \alpha_i = \sigma^\star\alpha_i \circ \tau_{i-1}$ for $i = 1, \dots, r$. Let M_i be the locally free A_R-module of rank r corresponding to $\mathcal{V}_i|_{C_S\setminus\Gamma_{s_1}}$ where $C_S\setminus\Gamma_{s_1} = Spec\ A_R$. Then $\underline{M}_i = (M_i, \tau_{M_i})$ is an Anderson A-motive and we get a chain $(\underline{M}_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_r} \underline{M}_r)$ of isogenies of degree 1 of Anderson A-motives that are isomorphisms outside Γ_{s_2}.

We want to prove that all the $\underline{M}_i$ are effective of dimension 1 and pure. To prove the dimension and the effectivity it suffices to prove it for the local shtuka $\underline{M}_i \otimes_{A_R} R\llbracket z_2\rrbracket$ associated to $\underline{M}_i$. Now we can choose a covering $Spec\ R' \to Spec\ R$ and a trivialisation $\beta' : L^+_{v_2}(\mathcal{G}) \times_R R' \xrightarrow{\sim} L^+\mathbb{G}_{v_2,R'}$ and $\beta : \sigma^\star L^+_{v_2}(\mathcal{G}) \times_R R' \xrightarrow{\sim} L^+\mathbb{G}_{v_2,R'}$. This corresponds by remark 7.4 to trivialisations $\beta'_i : M_i \otimes_{A_R} R'\llbracket z_2\rrbracket \to R'\llbracket z_2\rrbracket^r$ and $\beta_i : \sigma^\star M_i \otimes_{A_R} R'\llbracket z_2\rrbracket \to R'\llbracket z_2\rrbracket^r$ such that

$$\begin{array}{ccccccc}
R'\llbracket z_2\rrbracket^r & \xrightarrow{\begin{pmatrix} z_2 & & \\ & 1 & \\ & & \ddots \end{pmatrix}} & R'\llbracket z_2\rrbracket^r & \xrightarrow{\begin{pmatrix} 1 & & & \\ & z_2 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}} & \dots & \xrightarrow{\begin{pmatrix} z_2 & & \\ & 1 & \\ & & \ddots \end{pmatrix}} & R'\llbracket z_2\rrbracket \\
\downarrow{\scriptstyle \beta_0^{-1}} & & \downarrow{\scriptstyle \beta_1^{-1}} & & & & \downarrow{\scriptstyle \beta_r^{-1}} \\
\sigma^\star M_0 \otimes_{A_R} R'\llbracket z_2\rrbracket & \xrightarrow{\sigma^\star\alpha_1\times id} & \sigma^\star M_1 \otimes_{A_R} R'\llbracket z_2\rrbracket & \xrightarrow{\sigma^\star\alpha_2\times id} & \dots & \xrightarrow{\sigma^\star\alpha_r\times id} & \sigma^\star M_r \otimes_{A_R} R'\llbracket z_2\rrbracket \\
\downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_1} & & & & \downarrow{\scriptstyle \tau_r} \\
M_0 \otimes_{A_R} R'\llbracket z_2\rrbracket & \xrightarrow{\alpha_1\times id} & M_1 \otimes_{A_R} R'\llbracket z_2\rrbracket & \xrightarrow{\alpha_2\times id} & \dots & \xrightarrow{\alpha_r\times id} & M_r \otimes_{A_R} R'\llbracket z_2\rrbracket \\
\downarrow{\scriptstyle \beta'_0} & & \downarrow{\scriptstyle \beta'_1} & & & & \downarrow{\scriptstyle \beta'_r} \\
R'\llbracket z_2\rrbracket^r & \xrightarrow{\begin{pmatrix} z_2 & & \\ & 1 & \\ & & \ddots \end{pmatrix}} & R'\llbracket z_2\rrbracket^r & \xrightarrow{\begin{pmatrix} 1 & & & \\ & z_2 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}} & \dots & \xrightarrow{\begin{pmatrix} z_2 & & \\ & 1 & \\ & & \ddots \end{pmatrix}} & R'\llbracket z_2\rrbracket
\end{array}$$

commutes. Since $\beta' \circ \tau \circ \beta^{-1}$ factors through Z_{v_2} we can choose the trivializations β' and β in such a way, that $\beta' \circ \tau \circ \beta^{-1}$ is given by a matrix T as in lemma 7.6. Since $\beta'_0 \circ \tau_0 \circ \beta_0^{-1}$ arises as $(\rho_0)_\star(\beta' \circ \tau \circ \beta^{-1})$ where $\rho_0 : \mathbb{I}_{v_2} \to \mathrm{GL}_{\mathrm{r},\mathrm{A}_{\mathrm{v}_2}}$ was the standard representation, it follows that also $\beta'_0 \circ \tau_0 \circ \beta_0^{-1}$ is given by T. Since $T \in \mathrm{Mat}_r(R\llbracket z_2\rrbracket)$ and $\beta'_0, \beta_0 \in \mathbb{I}_{v_2}(R'\llbracket z_2\rrbracket) \subset \mathrm{GL}_r(R'\llbracket z_2\rrbracket)$ this implies $\tau_0 \times id_{R'} \in \mathrm{Mat}_r(R'\llbracket z_2\rrbracket)$. In addition $\mathrm{coker}(\tau_0 \times id_{R'})$ is locally free of rank 1 over R' and annihilated by

(z_2) since coker$T$ has this property. The fact that R' is faithfully flat over R implies that the same is true for τ_0 which means that $\underline{M}_0$ is an effective Anderson A-motive of dimension 1. We have $\beta_i' \circ \tau_i \circ \beta_i^{-1} = diag(\underbrace{z_2, \ldots, z_2}_{i}, 0, \ldots, 0)\underbrace{\beta_0 \circ \tau_0 \circ \beta^{-1} + 0}_{T} diag(\underbrace{z_2^{-1}, \ldots, z_2^{-1}}_{i}, 0, \ldots, 0)$ and by the conditions on T this gives again a monomial matrix lying in $\mathrm{Mat}_r(R[\![z_2]\!])$ with determinant t. Similar as above it follows that $\underline{M}_i = (M_i, \tau_i)$ is an effective Anderson motive of dimension 1. Since $\underline{\mathcal{G}}$ lies in the Newton stratum $\mathcal{N}_{b_0}$ the local shtuka $\Gamma_{v_1}(\underline{\mathcal{G}})_{\overline{s}}$ is for every geometric point $\overline{s} \in Spec\ R$ isomorphic to the iso-shtuka $\left(\kappa_{\overline{s}}[\![\frac{1}{t}]\!], \begin{pmatrix} 0 & & z_2^{-1} \\ 1 & & \\ & \ddots & \\ & 1 & 0 \end{pmatrix}\right)$ Since $\Gamma_{v_1}(\underline{\mathcal{G}})$ equals the local shtuka of M_i at v_1 for all i this means by definition that $\underline{M}_i$ is pure. By proposition **??** in the appendix $\underline{M}_i$ is pure if and only if M_i is finitely generated as $R\{\tau\}$ module. Therefore the chain of Anderson motives $(\underline{M}_0 \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_r} \underline{M}_r)$ defines by remark 7.7 an object in $\mathcal{D}^r_{I,v_2}(S)$ and it is clear, gives a functor $\Psi : \mathcal{N}_{b_0} \to \mathcal{D}^r_{I,v_2}$.

Now let $f_1, f_2 : \underline{\mathcal{G}} \to \underline{\mathcal{G}}'$ be two morphisms in $\mathcal{N}_{b_0}$ with $\Psi(f_1) = \Psi(f_2)$. The construction of the functor and the equivalence 7.3 that this implies $f_1|_{C_S \setminus \Gamma_{s_1}} = f_2|_{C_S \setminus \Gamma_{s_1}}$. As before we deduce with lemma 3.11 $f_1 = f_2$, which means that Ψ is faithful.

To see that the functor is essentially surjective, let $(\underline{M}_0 \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_r} \underline{M}_r) \in \mathcal{D}^r_{I,\mathbb{F}_{v_2}}(S)$. One has to extend the locally free module M_0 to a locally free module $\mathcal{V}_0$ of rank r over C_S. This determines then extensions $\mathcal{V}_i$ of M_i satisfying the condition that α_i extends to $\mathcal{V}_i$ and that α_i is an isomorphism outside of Γ_{s_2}. Using proposition 7.3 this corresponds to an $\mathbb{G}$-torsor $\mathcal{G}$ with an Frobenius morphism $\tau : \sigma^\star \mathcal{G}|_{C_S \setminus (\Gamma_{s_1} \cup \Gamma_{s_2})} \to \mathcal{G}|_{C_S \setminus (\Gamma_{s_1} \cup \Gamma_{s_2})}$. The matrices T in lemma 7.6 are the only ones that corresponds to an element $\omega \in \widetilde{W}$ and that satisfies the condition that $det(T) \in z_2(\mathbb{F}_{v_2}[z_2])^\star$ and that T conjugated by $diag(z_2, \ldots, z_2, 0, \ldots, 0)$ liies again in $Mat_r(R[\![z_2]\!])$. This implies that τ is bounded by Z_{v_2}. Since all the $\underline{M}_i$ are pure $\underline{\mathcal{G}} = (\mathcal{G}, s_1, s_2, \tau)$ is an element in $\mathcal{N}_{b_0}(S)$ with $\Psi(\underline{\mathcal{G}}) = (\underline{M}_0 \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_r} \underline{M}_r)$. □

Remark 7.9. If one choose the extension of M_0 for all points in $\mathcal{D}^r_{I,\mathbb{F}_{v_2}}$ in a compatible way one gets a fully faithful functor $\mathcal{D}^r_{I,\mathbb{F}_{v_2}} \to \mathcal{N}_{b_0}$.

A GL_r over Local Fields

For the discussion about Dinfeld's moduli space with Iwahori level structure a good knowledge of the combinatorics of GL_n over local fields is advantageous. We discuss the notions of root datum, dominant coroots, positiv coroots, coroot basis, (affine, extended) Weyl group, fundamental group, Bruhat order, the standard appartement and the Bruhat-Tits building.
As a scheme $G = Gl_n$ is given by $Spec\ A$, with $A := \mathbb{Z}[X_{ij}, Y]_{1\leqslant i,j\leqslant n} \big/ (Y\det(X_{ij}) - 1)$. The maps

$$
\begin{array}{rll}
m : G \times G \to G & m^\star : A \to A \otimes A & X_{ij} \mapsto \sum_{l=1}^{n} X_{il} \otimes X_{lj} \\
e : Spec\ \mathbb{Z} \to G & e^\star : A \to \mathbb{Z} & X_{ij} \mapsto \delta_{ij} \\
i : G \to G & i^\star : A \to A & X_{ij} \mapsto (-1)^{i+j} Y \cdot (j, i - minor)
\end{array}
$$

make $Spec\ A$ into a group scheme.
We choose the subgroup of upper triangular matrices as a Borel subgroup B. For the maximal Torus T we chose the diagonal matrices. Now the character group $X^\star(T) := Hom(T, \mathbb{G}_m)$ is isomorphic to $\mathbb{Z}^n$. Let $\alpha \in Hom(T, \mathbb{G}_m)$ given by $\alpha^\star : \mathbb{Z}[z, z^{-1}] \to A,\ z \mapsto \prod_{i=1}^{n} x_{ii}^{r_i}$ with $r_i \in \mathbb{Z}$. Then this α is mapped to $(r_1, \ldots, r_n)$. We denote by χ_i the corresponding basis in $X^\star(T)$.
The Cocharactergroup $X_\star(T) := Hom(\mathbb{G}_m, T)$ is also isomorphic to $\mathbb{Z}^n$.

$$\mathbb{Z}^n \to X_\star(T) \quad (r_1, \ldots, r_n) \mapsto \gamma \text{ with } x_{ii} \mapsto z^{r_i}$$

We denote by $e_i \in X_\star(T)$ the image of the standard basis vectors in $\mathbb{Z}^n$. (Note that in [**bruhat gorups**] the negative of this basis is choosen.) Note that there is a natural pairing $X^\star(T) \times X_\star(T) \longrightarrow \mathbb{Z} = End(\mathbb{G}_m)$. Now we will consider Gl_n over some base field K. At the moment this can be any field, later we will require K to be a non-archimedean local field.

The finite Weyl group and its longest element: Let $N := N_{G(K)}(T(K))$ be the normalizer of T in G. It is given by the subgroup of the general permutation matrices, i.e. monomoial matrices with entries in K. We define $W_0 := N/T(K)$. Therefore W_0 is given by the permutation matrices with entries in $\{0,1\}$ and we have a natural isomorphism $W_0 \simeq S_n$. S_n is a Coxetergroup of type A_{n-1}, the set of generators is given by $s_i := \sigma_{i,i+1}$ for $i = 1, \ldots, n-1$. W_0 has a longest element ω_0 which is given by $(n, n-1, \ldots, 1)$. One possible representation is $s_1 s_2 \ldots s_{n-1} s_1 \ldots s_{n-2} \ldots s_1 s_2 s_1$. The length of this element is $\frac{n(n-1)}{2}$. The length equals always the cardinality of a positive system (see [Hum75, section 1.8]).
Now W_0 acts on $X_\star(T)$ as well as on $X^\star(T)$. Namely let for $\omega \in W_0$ $c_\omega : T \to T,\ t \mapsto \omega t \omega^{-1}$ be the well defined conjugation, then we have

$$W_0 \times X^\star(T) \to X^\star(T) : (\omega, \chi) \mapsto \chi \circ c_{\omega^{-1}} \qquad W_0 \times X_\star(T) \to X_\star(T) : (\omega, \lambda) \mapsto c_\omega \circ \lambda$$

Using the identifications $W_0 \simeq S_n$, $X^\star(T) \simeq \mathbb{Z}^n$ and $X_\star(T) \simeq \mathbb{Z}^n$ this operation is given by

$$S_n \times \mathbb{Z}^n \to \mathbb{Z}^n\ (\sigma, (x_1, \ldots, x_n)) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \qquad S_n \times \mathbb{Z}^n \to \mathbb{Z}^n\ (\sigma, (\lambda_1, \ldots, \lambda_n)) \mapsto (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)})$$

The adjoint representation: The Lie-algebra is $\mathfrak{gl}_n = T_e G = Der_k(A, k) = \langle \partial_{ij} \rangle$. Since $T_e G$ is also given as the kernel of $ker\Big(G\big(k(\epsilon)/(\epsilon^2)\big) \to G\big(k\big)\Big)$, we identify $\mathfrak{gl}_n$ with $M_n(k)$, where

the Lie bracket is given by $[X,Y] = XY - YX$. For all $g \in G(k)$ we note the conjugation by $c_g : G \to G, h \mapsto ghg^{-1}$, this corresponds to a morphism $c_g^\star : A \to A$.
Now G operates on $\mathfrak{gl}_n$ by $G \times \mathfrak{gl}_n \quad (g,\chi) \mapsto c_g^\star \circ \chi \in Der_k(A,k)$. Identifying $\mathfrak{gl}_n$ with $M_n(k)$, this operation is given by $G \times M_n \to M_n \quad (g,M) \mapsto gMg^{-1}$.
This gives a n^2-dimensional representaion of G which is called adjoint representation. Now we are interested in the operation of T on $\mathfrak{gl}_n$

$$T \times \mathfrak{gl}_n \to \mathfrak{gl}_n \qquad \left(x = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}, (a_{ij})\right) \mapsto x(a_{ij})x^{-1} = \begin{pmatrix} \frac{x_1}{x_1}a_{11} & \dots & \frac{x_1}{x_n}a_{1n} \\ \vdots & \frac{x_i}{x_j}a_{ij} & \vdots \\ \frac{x_n}{x_1}a_{n1} & \dots & \frac{x_n}{x_n}a_{nn} \end{pmatrix}$$

We denote by χ_i the character $(0,\dots,1,\dots,0)$, so that T operates on the one dimensional subvectorspace $\langle \partial_{ij} \rangle$ by the character $\alpha_{ij} := \chi_i - \chi_j$. In particular T operates trivially on the one dimensional subvector spaces ∂_{ii} for every i.

The root datum: We would like to describe the root datum $(X, \Phi, \check{X}, \check{\Phi})$ of G. Like always we have $X = X^\star(T)$ and $\check{X} = X_\star(T)$. The adjoint representation shows us $\Phi = \{\chi_i - \chi_j \mid i \neq j\}$ and with the identification $X^\star(T) \simeq \mathbb{Z}^n$ we have $\Phi = \{\chi_i - \chi_j | \ i \neq j\}$.
It rests to specify $\check{\Phi}$. For $\alpha = \chi_i - \chi_j \in \Phi$ we have $T_\alpha := ker(\alpha)^\circ = \left\{\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \mid x_i = x_j\right\}$ and $G_\alpha := C_G(T_\alpha) = \begin{pmatrix} \star & & & \\ & \star & \star & \\ & \star & \star & \\ & & & \star \end{pmatrix}$ (with two stars in the i-th and j-th colum/row). Let $N_\alpha(T) = \{g \in G_\alpha \ |gT = Tg\} = \begin{pmatrix} \star & & & \\ & 0 & \star & \\ & \star & 0 & \\ & & & \star \end{pmatrix} \cup T$. Then there is exactly one element $id \neq s_\alpha \in W_0(G_\alpha, T) = N_\alpha(T)/T \subset W_0$. This element is represented by $n_\alpha := \begin{pmatrix} 1 & & & & \\ & 0 & & 1 & \\ & & 1 & & \\ & 1 & & 0 & \\ & & & & 1 \end{pmatrix} \in C_G(T_\alpha)$. Now the coroot $\check{\alpha}$ is defined to be the unique element in $X_\star(T)$ such that

$$s_\alpha(x) = x - \langle x, \check{\alpha} \rangle \alpha \qquad \forall x \in X^\star(T).$$

Now if $x = (x_1,\dots,x_n) \in X_\star(T)$ and $\alpha = \chi_i - \chi_j$ we have

$$s_\alpha(x) = (x_1, \dots, \underbrace{x_j}_{i}, \dots, \underbrace{x_i}_{j} \dots, x_n) = x - \langle x, e_i - e_j \rangle (e_i - e_j)$$

Consequently we have $\check{\alpha} = e_i - e_j$ and we set $\check{\Phi} = \{\mu \in X_\star(T) \ |\mu = \check{\alpha} \ \text{ for some } \alpha \in \Phi\}$. This defines the root datum.

Positive roots: For each $\alpha \in \Phi$ the root group U_α is the unique subgroup of G, that is isompor-phic to the additive group $\mathbb{G}_a$ and satisfies $tU_\alpha(x)t^{-1} = U_\alpha(\alpha(t)x)$ for all x in some ring R and $t \in T(R)$, here we write $U_\alpha(x)$ for the elment in $U_\alpha(R) = R$ corresponding to x. For $\alpha = \chi_i - \chi_j \in \Phi$ the subgroup U_α is given by $\begin{pmatrix} 1 & & & & \\ & 1 & & \star & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$, where the star is at position (i,j). Now a root is called positiv if $U_\alpha \subset B \cap G_\alpha$. Therefore the set of positive roots is given by $\Phi^+ = \{\chi_i - \chi_j \ |i < j\}$. This is a positive system in the sense of [Spr98, (7.46)]. Note that it depends on a choice of the Borel subgroup. The bijection between Φ and $\check{\Phi}$ gives us also a system of positive coroots given by $\{e_i - e_j \mid i < j\}$.

The algebraic fundamental group of G: We define Q to be $\mathbb{Z}$-submodul of $X^\star(T)$ generated by Φ and analogously $\check{Q}$ as the $\mathbb{Z}$-submodul of $X_\star(T)$ generated by $\check{\Phi}$. We have $\check{Q} = \{(\mu_1, \dots, \mu_n) | \sum_i \mu_i = 0\}$. Then the algebraic fundamental group is given by

$$\pi_1(G) := X_\star(T) \diagup \check{Q} \simeq \mathbb{Z} \qquad (\mu_1, \dots, \mu_n) \mapsto \sum_i \mu_i$$

Simple roots and simple coroots: We set furthermore:

$$V := X^\star(T) \otimes_{\mathbb{Z}} \mathbb{R} \quad \check{V} := X_\star(T) \otimes_{\mathbb{Z}} \mathbb{R} \quad V_\Phi := Q \otimes_{\mathbb{Z}} \mathbb{R} \quad \check{V}_{\check{\Phi}} := \check{Q} \otimes_{\mathbb{Z}} \mathbb{R}$$

Now a subset $D \subset \Phi$ is called basis of the root system if D is a vectorspace basis of V_Φ and if we can write all $\alpha \in \Phi$ as $\alpha = \sum_{\beta \in D} n_\beta \beta$, such that n_β are integers with the same sign. Now every positive system contains exactly one basis and conversely each basis of the root system is contained in exactly one positive system. The root basis conatined in our positive system Φ^+ is given by the set $D = \{\chi_i - \chi_{i+1} | \ i = 1, \dots, n\}$. The elements $\alpha \in D$ (respectively $\check{\alpha} \in \check{D} := \{\check{\alpha} \mid \alpha \in D\}$) are called simple roots (resp. simple coroots).

The largest positive (co)root Now the set of positive (co)roots is partially ordered by $\alpha \geqslant \beta$ if and only if $\alpha - \beta$ is a non-negative linear combination of simple (co)roots. There is a largest root which is given by $(1, 0, \dots, 0, -1)$. Similary there is a largest positive coroot given by $(1, 0, \dots, 0, -1)$.

Dominant characters and dominant cocharacters: A character $x \in X^\star(T)$ is called dominant if $\langle x, \check{\beta} \rangle \geqslant 0$ for all $\beta \in D$. It is clear that $(x_1, \dots, x_n)$ is dominant if and only if $x_1 \geqslant \dots \geqslant x_n$. Analogously we define a cocharacter $\mu \in X_\star(T)$ to be dominant if $\langle \beta, \mu \rangle \geqslant 0$ for all $\beta \in D$, which means that a cocharacter $\mu = (\mu_1, \dots, \mu_n) \in X_\star(T)$ is dominant if and only if $\mu_1 \geqslant \dots \geqslant \mu_n$.

Fundamental weights and the halfsum of positive coroots: For each $\beta \in D$ let $\omega_\beta \in \check{V}_\Phi$ be the linear form defined by $\omega_\beta(\alpha) = \langle \alpha, \omega_\beta \rangle = \delta_{\alpha,\beta}$ for all $\alpha \in D$. So the fundamental weight attached to the simple root $\beta = e_i - e_{i+1}$ is given by $(\underbrace{\tfrac{1}{2}, \dots, \tfrac{1}{2}}_{i}, \underbrace{\tfrac{-1}{2}, \dots, \tfrac{-1}{2}}_{n-i})$. We denote by $\check{\rho} := \frac{1}{2} \sum_{\alpha \in \Phi^+} \check{\alpha}$ the halfsum of all positive coroots. There is a theorem [Bou68, Chapitre 6, §1, proposition 29] telling us that we have $\check{\rho} = \sum_{\alpha \in D} \omega_\alpha$. Hence we have $\check{\rho} = \frac{1}{2}(n-1, n-3, n-5, \dots, +3-n, 1-n)$.

The center of G: The center of G is equal to the kernel of the adjoint representation and this is easily seen to be the intersection $\bigcap_{\alpha \in \Phi} ker(\alpha)$. Therefore the center of G is seen to be $\mathbb{G}_m$ diagonally embedded in G.

The derived group of G: It is equal to Sl_n.

The affine (co-)roots
The affine roots are by definition given as $\Phi_{aff} := \Phi \times \mathbb{Z} \subset X^\star(T) \times \mathbb{Z}$. Here we define $\Phi^+_{aff} := \Phi^+ \times \{0\} \cup \Phi \times \mathbb{Z}_{>0}$ and $\Phi^-_{aff} := \Phi^- \times \{0\} \cup \Phi \times \mathbb{Z}_{<0}$ as the positive and negative affine roots. Note

that every affine root $a = (\alpha, k) \in \Phi_{aff}$ defines an affine function $a : \check{V} \to \mathbb{R}, x \mapsto \langle \alpha, x \rangle - k$. These affine functions make our definition of affine roots equivalent to the one in [Tit79, Section 1.6] and [Lan96, Defintion 7.1]. We call α the vector part of a.

From now on let K be a non archimedean local field. Let π be a uniformizer in K and ω_K a valuation with $\omega_K(\pi) = 1$.

The extended Weyl group: The centralizer $Z(T)$ of the maximal Torus T is T itself. Then the morphism $\nu : T(K) \to \check{V} \simeq \mathbb{R}^n$ defined by $\langle \nu(t), \chi \rangle = -\omega_K(\chi(t))$ for all $t \in T(K)$ and $\chi \in X^\star(T)$ is given by $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto \begin{pmatrix} -\omega_K(t_1) \\ \cdots \\ -\omega_K(t_n) \end{pmatrix}$. We denote the kernel of ν by Z_b, which is given by $\begin{pmatrix} \mathcal{O}_K^\star & & \\ & \ddots & \\ & & \mathcal{O}_K^\star \end{pmatrix}$. We define $\Lambda := T(K)/Z_b \simeq \mathbb{Z}^n$. We use the identification $X_\star(T) \xrightarrow{\sim} \Lambda, \quad \mu \mapsto \mu(\pi) \cdot Z_b$. This results also in the identification $\mathbb{Z}^r \to \Lambda, \quad (r_1, \dots, r_n) \mapsto \begin{pmatrix} \pi^{r_1} \\ \cdots \\ \pi^{r_n} \end{pmatrix}$.
The extended Weylgroup is defined by $\widetilde{W} = N(K)/Z_b$. It projects to the finite Weyl group $W_0 = N(K)/Z(K)$ and we have always a section $W_0 \to \widetilde{W}$, since we can identify W_0 with $N_{G(\mathcal{O}_K)}T(\mathcal{O}_K)/T(\mathcal{O}_K)$. Using the short exact sequence $0 \to \Lambda \to \widetilde{W} \to W_0 \to 0$ a version of the splitting lemma implies that $\widetilde{W}$ is a semidirect product $\Lambda \rtimes_\varphi W_0$, where $\varphi : W_0 \to Aut(\Lambda)$ is given by conjugation. In the case of Gl_n, W_0 embedds in $N(K)$ as the permutation matrices, so $\widetilde{W} = \mathbb{Z}^n \rtimes S_n$. We write the elements in $\widetilde{W}$ as $t^\mu \omega$ with $\mu \in \Lambda = X_\star(T)$ and $\omega \in W_0$.

The standard appartement attached to T: Note that there is the extended Bruhat-Tits building and the reduced Bruhat-Tits building. Whereas [Tit79] uses the extended building, [BT84] and [Lan96] deal with the reduced building. We follow [Tit79] for the construction (but compare also Landvogt).
Furthermore we set $\check{V}_0 := \{v \in \check{V} | \langle \alpha, v \rangle = 0 \ \forall \alpha \in \Phi\} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \cdot \mathbb{R}$. Now the affine space of the appartement in the extended building is $\check{V}$. The affine space of the reduced appartement is given by $\check{V}_\Phi = \check{V}/\check{V}_0$.

Now Λ acts on $\check{V}$ by $(\lambda \cdot v) = v + \nu(\lambda)$ and W_0 acts by conjugation on T and hence on $\check{V}$. This gives an action of $\widetilde{W}$ on $\check{V}$ defined by $\widetilde{W} \times \check{V} \to \check{V}, \quad (t^\mu \omega, v) \mapsto \omega(v) + \nu(\mu)$, where we abbreviate $\nu(t^\mu)$ by $\nu(\mu)$ (compare [Lan96, 1.5 and 1.6]). This action induces of course an action of $N(K)$ on $\check{V}$. With the identification $\Lambda \simeq X_\star(T) \simeq \mathbb{Z}^n$ we have $\nu(\mu) = -\mu$. We remark that there is a choice of this action as one could also define $(\lambda \cdot v) = v - \nu(\lambda)$. With this action the choice of the base alcove would result in a different standard Iwahori subgroup.
So far we have an affine space with an action of $N(K)$. Landvogt calls it the empty appartement, since there is missing the structure of a polysimplex.

The half appartements and walls: For some affine root $(\alpha, k) \in \Phi_{aff}$ the halfappartement $A_{\alpha,k}$ is defined to be the subset $\{v \in \check{V} \quad | \langle \alpha, v \rangle - k \geqslant 0\}$. We call $\partial A_{\alpha,k}$ a wall of the appartement. In the case of Gl_n the walls $\partial A_{\alpha,k}$ are hyperplanes defined by the equation $v_i - v_j - k = 0$. We denote by $s_{\alpha,k}$ the reflection at the wall $\partial A_{\alpha,k}$ for every affine root. We have $s_{\alpha,0}(x) = x - \langle \alpha, x \rangle \check{\alpha}$. Note that $v \in \partial A_{\alpha,k}$ implies $v + (1, \dots, 1)\mathbb{R} \in \partial A_{\alpha,k}$.

We illustrate this in the case of Gl_2 and Gl_3.

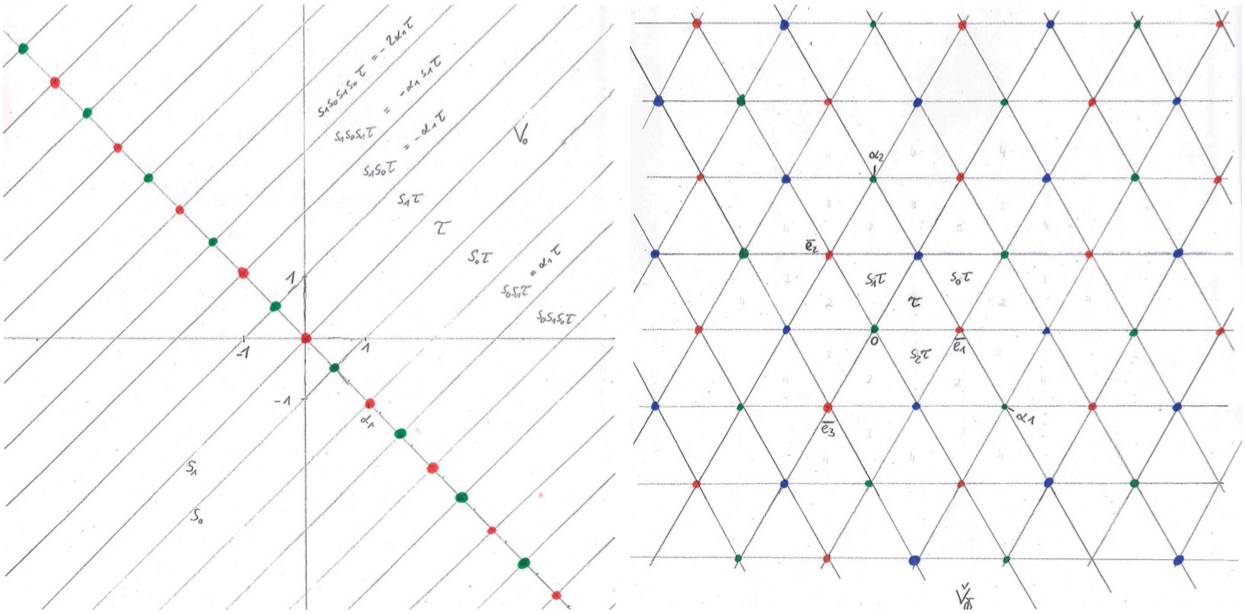


Figure 1: Standard appartement of Gl_2 and Gl_3.

In the figure for GL_2 the only simple root is marked by $\alpha_1 = (1,-1)$. The lines show the walls in the appartement. Let $\alpha_1 = (1,-1,0)$ and $\alpha_2 = (0,1,-1)$ be the simple roots for GL_3. The figure for GL_3 only shows the plane $\check{V}_\Phi$ in $\check{V}$ that is spanned by $0, \alpha_1, \alpha_2$. So for $\check{V}$ there is the additional axis $\check{V}_0 = (1,1,1) \cdot \mathbb{R}$ coming vertically out of the paper. We mark the projections of the standard basis in $\check{V}_\Phi$ by $\bar{e}_1, \bar{e}_2$ and $\bar{e}_3$. The lines show the intersection of the walls $\partial A_{\alpha,k}$ with $\check{V}_\Phi$.

The facets and the alcoves: For $x, y \in \check{V}$, we set $x \sim y \quad :\Leftrightarrow \{(\alpha,k) \in \Phi \,|x \in A_{\alpha,k}\} = \{(\alpha,k) \in \Phi \,|y \in A_{\alpha,k}\}$. The equivalence classes of this relation are the facets in the appartement. The facets of maximal dimension are called alcoves. The alcoves are also determined as the connected components of $\check{V} \setminus \bigcup_{(\alpha,k)\in\Phi_{aff}} \partial A_{\alpha,k}$. There is exactly one alcove which lies in the B positive Weyl chamber $\bigcup_{\alpha\in\Phi^+} A_{\alpha,0}$ and whose closure contains the origin. We call it the base alcove and denote it by τ. For GL_n it is explicitely given as $\tau = \{v \in \check{V} \mid 0 < \langle\alpha, v\rangle < 1 \ \forall \alpha \in \Phi^+\} = \{v = (v_1, \dots, v_n) \in \check{V} \mid 0 < v_i - v_j < 1 \ \text{ for all } 1 \leqslant i < j \leqslant n\}$. It depends on the choice of a Borel, since the Borel subgroup determines a set of positive roots.
(Note that the negative choice of the basis e_i of $X_\star(T)$ affects that the coefficients v_i are ascending instead of descending.)

The affine Weyl group: The affine Weylgroup W_{aff} is the subgroup of $\widetilde{W}$ generated by the reflections on the walls that are adjacent to τ. If we denote by $\alpha_i = (0, \dots, 1, -1, \dots, 0)$ the i-th simple root $(1 \leqslant i \leqslant n-1)$ and by s_i the reflection s_{α_i} at the wall $\partial A_{\alpha_i,0}$, then W_{aff} is generated by $W_0 =< s_1, \dots, s_{n-1} >$ and the element $s_0 := t^{(1,0,\dots,0,-1)} s_1 s_2 \dots s_{n-1} \dots s_2 s_1$. Here s_0 is the reflection at the additional wall $\partial A_{\alpha,1}$ adjacent to τ, where $\alpha = \sum_{i=1}^{n-1} \alpha_i = (1, 0 \dots, -1)$ was the

largest positive coroot. This is easily seen since $\partial A_{\alpha,1}$ equals $\{v = (v_1, \dots, v_n) \in \check{V} \mid v_1 - v_n = 1\}$ which shows that the relfection s_0 fixes $\partial A_{\alpha,1}$.
We note that $(W_{aff}, \mathbb{S})$ with $\mathbb{S} := \{s_0, \dots, s_{n-1}\}$ is a Coxeter group. One checks also that $\{s_0, \dots, s_{n-1}\}$ generates exactly $\check{Q} \rtimes W_0 = W_{aff}$.

The type of a facet: There is a bijection of $\mathcal{P}(\mathbb{S}) \backslash \mathbb{S}$ to the set of facets contained in $\bar{\tau}$, where $\mathcal{P}(\mathbb{S})$ denotes the power set of $\mathbb{S}$. A subset $T \subset \mathbb{S}$ is send to the facet $F_T := \{a \in \bar{\tau} \mid \{s \in \mathbb{S} \mid a \in L_s\} = T\}$, where L_s is the wall consisting of the fixpoints of s. Now for every facet F in the appartment, there is exactly one element $\omega \in W_{aff}$ such that $\omega \cdot F$ is a facet in $\bar{\tau}$. So $\omega \cdot F = F_T$ for some subset $T \subset \mathbb{S}$ and we call T the type of the facet. The facets of type $\varnothing$ are the alcoves. In particular W_{aff} acts simply transitiv on the set of alcoves, so that $\omega \mapsto \omega\tau$ is a bijection of W_{aff} to the set of alcoves. In the figure for GL_2 we marked the facets of type $\{s_0\}$ with green and the facets of type $\{s_1\}$ with red. In the figure for GL_3 the facets of type $\{s_0, s_1\}$ (resp. $\{s_0, s_2\}$, $\{s_1, s_2\}$) are the blue (resp. red, green) points.

The group Ω: The group Ω is defined as the stabilizer of τ (not pointwise). Since W_{aff} operates simply transitiv on alcoves and Ω is the stabilizer of τ we see $\widetilde{W} = W_{aff} \rtimes \Omega$, where Ω operates by conjugation on W_{aff}. The exact sequence $0 \to W_{aff} \to \widetilde{W} \to \Omega \to 0$ shows $\Omega \simeq \pi_1(G)$. Note that in contrast to W_{aff} the type of the facets in the appartement is not invariant under the operation of Ω.
To be explicit in the case of GL_n, we set $\beta = t^{(0,\dots,0,1)} s_{n-1} \dots s_2 s_1$, so that $\beta \cdot (v_1, \dots, v_n) = (v_2, v_3 \dots, v_n, v_1 - 1)$ for $(v_1, \dots, v_n) \in \check{V}$ which shows that β stabilizes $\tau = \{v = (v_1, \dots, v_n) \in \check{V} \mid 0 < v_i - v_j < 1 \text{ for all } 1 \leqslant i < j \leqslant n\}$. Since $\Omega \simeq \pi_1(\mathrm{GL}_n)$ is free of rank one, this implies already $\Omega = \mathbb{Z} \cdot \beta$, because if there were some other element $\widetilde{\beta} = t^{(a_1,\dots,a_n)}\omega$ with $m \cdot \widetilde{\beta} = \beta$ and $m > 1$ this would imply $m \cdot \sum_{i=1}^n a_i = 1$ which is not possible.

B Drinfeld A-Modules and Anderson A-Motives

We recall the definition of a Drinfeld A-module and Anderson A-motive from [Har17]. Let C be a smooth projective geometrically irreducible curve over $\mathbb{F}_q$. We fix a closed point $\infty \in C$ and set $A = \Gamma(C \smallsetminus \infty, \mathcal{O}_C))$. Let R be a ring with an ring homomorphism $\gamma : A \to R$. We denote with $\sigma := id_A \otimes Frob_{q,R}$ the endomorphism of $A_R := A \times_{\mathbb{F}_q} R$ with $(a \otimes b) \mapsto (a \otimes b^q)$ for $a \in A$ and $b \in R$. Furthermore we denote by $\mathcal{J}$ the ideal $ker(\gamma \otimes id_R : A_R \to R) = (a \otimes 1 - 1 \otimes \gamma(a) : a \in A)$ and recall

Definition B.1 ([Har17, Definition 3.7]).
A Drinfeld A-module of rank $r \in \mathbb{N}$ over R is a pair $\underline{E} = (E, \varphi)$ consisting of a smooth affine group scheme E over Spec R of relative dimension 1 and a ring homomorphism $\varphi : A \to End_{R-groups}(E), a \mapsto \varphi a$ satisfying the following conditions:

1. *Zariski-locally on Spec R there is an isomorphism $\alpha : E \xrightarrow{\sim} \mathbb{G}_{a,R}$ of $\mathbb{F}_q$-module schemes such that*

2. *the coefficients of* $\Phi_a := \alpha \circ \varphi_a \alpha^{-1} = \sum_{i \geqslant 0} b_i(a)\tau^i \in End_{R-groups, \mathbb{F}_q-lin}(\mathbb{G}_{a,R}) = R\{\tau\}$ *satisfy* $b_0(a) = \gamma(a), b_{r(a)} \in R^\star$ *and* $b_i(a)$ *is nilpotent for all* $i > r(a) := -[\mathbb{F}_\infty : \mathbb{F}_q] ord_\infty(a)$.

If $b_i(a) = 0$ *for all* $i > r(a)$ *we say that* $\underline{E}$ *is in standard form.*

A morphism between two Drinfeld A-modules (E, φ) and (E', φ') is a morphism of group schemes $f : E \to E'$ such that $f \circ \varphi_a = \varphi'_a \circ f \forall a \in A$. An isogeny from (E, φ) to (E', φ') is a morphism that is finite and surjective. As described in [Har17] every Drinfeld A-module is isomorphic to one in standard form. We note that Drinfeld A-modules generalize to Anderson A-modules (see [Har17, Theorem 3.9]). We also recall the Definition of an effective Anderson A-motive. For an A_R-module M we set $\sigma^\star M := M \otimes_{A_R, \sigma} A_R = M \otimes_{R, Frob_{q,R}} R$.

Definition B.2 ([Har17, Definition 1.1]). *An effective A-motive of rank r over an A-ring* (R, γ) *is a pair* $\underline{M} = (M, \tau_M)$ *consisting of a locally free* A_R*-module* M *of rank r and an* A_R*-homomorphism* $\tau_M : \sigma^\star M \to M$ *whose cokernel is annihilated by* $\mathcal{J}^n$ *for some positive integer* n*. We say that* $\underline{M}$ *has dimension* d *if* coker τ_M *is a locally free* R*-module of rank* d *and annihilated by* $\mathcal{J}^d$*. We write* $rk\underline{M} = r$ *and* $dim\underline{M} = d$ *for the rank and the dimension of* $\underline{M}$*.*
A morphism $f : (M, \tau_M) \to (N, \tau_N)$ *between effective A-motives is an* A_R*-homomorphism* $f : M \to N$ *which satisfies* $f \circ \tau_M = \tau_N \circ \sigma^\star f$*.*

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