Westfälische Wilhelms-Universität Münster
Institut für mathematische Logik und Grundlagenforschung
Fachbereich Mathematik und Informatik

An Ample Geometry of Finite Rank

Inaugural-Dissertation

zur Erlangung des Doktorgrades der Naturwissenschaften
Dr. rer. nat.

eingereicht von: Isabel Müller
im Jahr: 2017
Unter Betreuung von: Prof. Dr. Dr. Katrin Tent
Im Bereich: Modelltheorie

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Abstract

Modern model theory started after the proof of Morley's famous categoricity theorem, which led to the study of a fundamental class of first order theories: the class of strongly minimal theories. In an attempt to classify the geometry of these strongly minimal theories, Zil'ber had conjectured that they split into three different types: trivial geometries, geometries which are vector space like and those which are field like. Hrushovski later refuted this conjecture by introducing a construction that had been modified and used a lot ever since. His counterexample to Zil'ber's conjecture provided a structure, which was not one-based, so could not be of trivial or vector space type, but nevertheless it forbade a certain point-line-plane configuration, which is always present in infinite fields. Hrushovski called that property CM-triviality and later Pillay, with some corrections by Evans, defined a whole hierarchy of new geometries. There, on the first two levels one finds non-one-based and non-CM-trivial theories and on the very top theories which interpret infinite fields. Recently, Baudisch, Pizarro and Ziegler and independently Tent have provided examples proving that this ample hierarchy is strict. While their examples are omega-stable of infinite rank, it remained open for over fifteen years if one can find geometries of finite rank which are non-CM-trivial, but nevertheless do not interpret an infinite field.

In this thesis under the supervision of Katrin Tent, we will introduce an almost strongly minimal structure which is strictly non-CM-trivial, using a Hrushovski-like construction. We furthermore show that there are no infinite groups definable in our theory and that its automorphism group is simple.

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CONTENTS

CHAPTER 1

INTRODUCTION

Model theory - that is algebraic geometry without fields. This is what Hodges writes in the introduction to his textbook on model theory [Hod93]. And indeed, if we have a look at the most relevant developments within model theory, from the early stages until its very modern developments, we find much truth in his words. Several of the most fundamental notions in model theory have their origin in algebraic geometry and are abstractions of the notions appearing therein.

Originally, model theory means the study of models. In the beginning of the last century, logic consisted of two a priori distinct parts: the study of mathematical structures, which can be seen as the "model-side", and the formal study of the syntax, the "theory-side". The field of model theory connects these two aspects. By choosing an appropriate *language* in which a mathematical structure M should be considered, we can assign to it its theory $\mathrm{Th}(M)$, i.e. the set of all formal statements which are true in M. On the other hand, to any given theory T, we can assign the class of its models $\mathrm{Mod}(T)$. Thus, instead of saying that model theory is merely a branch of mathematical logic as commonly phrased, it can rather be seen as the bridge connecting core mathematics and logic, translating phenomena on the one side into abstract notions on the other and obtaining new structural phenomena by reinterpreting the abstract notions in further structures.

The main fact which motivates this interaction, grounds in Gödel's famous completeness theorem [Gö30]. Its appearance is considered the birth of model theory. Recall that a theory is called ***inconsistent***, if we can deduce some first-order statement and its negation from it. Otherwise, it is called ***consistent***.

Fact (Gödel's Completeness Theorem, 1930) *Any consistent set of first order sentences has a model.*

This theorem was fundamental for the development of model theory and a lot of work appeared as a consequence in the time after. A very important parallel between model theory and algebraic geometry results from famous theorems by Tarski and Chevalley. Tarski, working on the logic side, proved that the theory of algebraically closed fields ACF has elimination of quantifiers, i.e. any definable set is a boolean combination of varieties. Furthermore, Chevalley showed that the projection of a constructible set is again constructible. Thus, as a consequence we get the following fact:

Fact (The Constructibility Theorem) *In an algebraically closed field, the definable sets are exactly the constructible ones.*

This fact states that the sets of interest for an algebraic geometer coincide with the sets of interest for a model theorist. Thus, it became the starting point of a fruitful interaction between algebraic geometry and model theory.

For some decades after Gödel's completeness theorem, the area of model theory was flourishing and much progress was made. But around 1960, "... *a feeling of exhaustion started pervading the whole theory. Daniel Lascar describes the situation as "un temps d'arrêt, comme si la machinerie, prête à tourner, ne savait quelle direction prendre." At this point Michael Morley appeared in the scene, causing what can be called a second birth of Model Theory.*" (Casanovas [Cas00])

The theorem of Morley yet again underlines the close connection model theory has to the theory of algebraically closed fields. Note that an algebraically closed field is uniquely determined by the transcendence degree over its prime field, whence in some fixed characteristic, for any uncountable cardinal κ, there is exactly one algebraically closed field up to isomorphism of cardinality κ. Morley managed to transfer this fact into the abstract setting of model theory. We say that some theory is κ-categorical for some cardinal κ, if it has exactly one model of cardinality κ up to isomorphism.

Fact (Morley's Categoricity Theorem, 1965) *Let* T *be a theory in a countable language. If* T *is* κ*-categorical for some uncountable cardinal* κ*, then it is categorical for any uncountable cardinal.*

Within his proof, Morley introduced various tools that turned out essential for the further developments in model theory, such as the Stone space of types and his notion of rank known as the Morley rank. He found that a κ-categorical theory for some uncountable cardinal κ is always ω-stable and he was able to discover analogous invariants in the general setting to the transcendence degree in algebraically closed fields.
His considerations were developed further by Baldwin and Lachlan in [BL71], where they described the structure of models of these *uncountably categorical* theories. They discovered that any such theory is controlled by a strongly minimal set. As we remind in Section 2.2, strongly minimal sets carry the notion of dimension arising as the cardinality of a maximal independent set and thus yield the desired analog of transcendence degree in algebraically closed fields. Baldwin and Lachlan showed that the dimension arising therein determines the models of an uncountably categorical theory up to isomorphism. It was in this context, that strongly minimal theories entered the attention of model theorists and proved to be a rich object of research.
Building upon Morley's Categoricity Theorem, Shelah started his famous and ample classification program, aiming to separate the theories which allow a classification of their models from those who do not. His fundamental work culminated in the book *Classification Theory* [She78] and is the core literature for modern model theory. He introduced various dividing lines in order to classify theories - the most important of them being undoubtedly the class of stable theories. Stability is the most fundamental dividing line, as any unstable theory possesses the maximal possible number of models in any sufficiently large cardinal. There are several ways to define stable theories, one being through the presence of a well-behaved independence notion which we call *non-forking independence*. The study of the properties of stable theories dominated model theory for many years.
In the end of the 1970's, Zil'ber studied uncountably categorical theories and with that, strongly minimal sets. An important feature of these strongly minimal sets is that the algebraic closure satisfies the Steinitz exchange principle. As a consequence, the algebraic closure induces a pregeometry on strongly minimal sets, which comes along with a well-defined notion of dimension. In the three classical, well-known examples of strongly minimal sets - pure sets without any structure, infinite dimensional vector spaces over division rings and algebraically closed fields - this dimension induced by the algebraic closure coincides with the natural notions of dimension we know within the three examples: the cardinality in the pure set, the linear dimension in vector spaces

and transcendence degree in algebraically closed fields. Let us emphasize that the three arising geometries therein are essentially different:

- The geometry of the pure set is ***disintegrated***, i.e. the algebraic closure of a set is the union of the algebraic closures of its singletons.
- The geometry of infinite dimensional vector spaces is not disintegrated, but still ***modular***: for any sets A and B the linear dimension of their union coincides with the sum of the linear dimensions of A and of B, after substracting the linear dimension of the intersection of A and B.
- The geometry of an algebraically closed field is neither disintegrated, nor modular.

The famous Trichotomy Conjecture of Zil'ber now states that these three types of geometries already describe fully the landscape of geometries that can appear in strongly minimal theories.

Conjecture (Zil'ber's Trichotomy Conjecture, 1984) *Let* T *be a strongly minimal theory. Then the pregeometry induced by* acl *on the models of* T *falls into one of the following three classes.*

(1) It is "set-like", i.e. disintegrated.

(2) It is "vectorspace-like", i.e. it is not disintegrated, but modular for all sets A and B with $d(A \cap B) > 0$. We then say the pregeometry is **locally modular**.

(3) It is "field-like" in a very strong sense: there is an algebraically closed field interpretable in T.

We want to emphasize the strong consequences Zil'ber's Conjecture would have in the interaction between model theory and algebra: starting with an arbitrary strongly minimal theory whose geometry is not locally modular, an actual infinite field is interpretable in it and hence we can do algebraic geometry.
Even though Zil'ber's conjecture had been refuted by Hrushovski [Hru93] soon after, work around it continued and proved to be a rich source of research. This continuation of research around Zil'ber's Conjecture can be partitioned into two leading questions:

(1) What further conditions should be put in order to make the conjecture hold?

(2) How far is the conjecture from being true? Can we fill the gap between vectorspace-like geometries and algebraically closed fields?

A lot of progress was done in both directions. Considering the first question, it turns out that there is a very natural setting in which Zil'ber's Conjecture holds: the Zariski geometries. The establishment of Zil'ber's Conjecture in this setting had far reaching consequences and allowed Hrushovski to give model theoretic proofs to profound number theoretic conjectures.

Example (DCF and Mordell-Lang) Loosely speaking, the Mordell-Lang conjecture states that given an abelian variety, the intersection of a proper subvariety with a subgroup is a finite union of translates of subgroups. Although being allocated within the area of number theory, it was Hrushovski who proved this famous conjecture in [Hru96], using model theoretic tools. This work was a groundbreaking result, which increased the interest of core mathematics into model theory significantly.
In order to assess the Mordell-Lang conjecture from a model theoretic angle, Hrushovski used two different approaches, depending on the characteristic of the underlying field. In characteristic zero, he studied the fields in the language of rings enlarged by a symbol ∂ for a difference function, which is additive and satisfies the condition $\partial(xy) = x\partial(y) + y\partial(x)$. This leads to the theory of ***differential fields***. Its model companion is the theory of differentially closed fields ***DCF***. Now, Hrushovski proved that in a differentially closed field, any type of rank one is either one-based, which is the analog of local modularity for arbitrary simple theories, or it is non-orthogonal to the (algebraically closed) field of constants, i.e. the field of all elements such that $\partial(x) = 0$ (conf. [HS]). Thus, he established a version of Zil'ber's Conjecture in the framework of differentially closed fields. For the case of positive characteristic, Hrushovski once again managed to establish the appropriate version of Zil'ber's Trichotomy Conjecture, this time in the framework of ***separably closed fields***. This led him to give a proof of the famous Mordell-Lang Conjecture in arbitrary characteristic in [Hru96] resulting in a fruitful interaction between number theory and model theory.

Example (ACFA and Munin-Mumford) The Munin-Mumford conjecture, which also has its origins in the area of number theory, can be seen as an analog of the Mordell-Lang conjecture, where the subgroup considered in the statement is the group of torsion elements of the abelian variety. It was first proven by Raynaud in [Ray83]. In 2001 however, Hrushovski in [Hru01] gave a completely new and independent proof using model theory.

In order to consider this conjecture from a model theoretic point of view, Chatzidakis and Hrushovski [CH00] and later together with Peterzil [CHP02] studied the underlying field in the language of rings enriched by one symbol σ for an automorphism. This leads to the theory of ***difference fields***. Its model companion is ***ACFA***, the theory of existentially closed difference fields. Here again, Zil'ber's Conjecture was established, in [CH00] for characteristic zero and in [CHP02] for positive characteristic. Chatzidakis, Hrushovski and Peterzil proved that any type of rank 1 is either one-based or almost internal to some field, which is either the field $\mathrm{Fix}(\sigma)$ or the field $\mathrm{Fix}(\sigma^n \mathrm{Frob}_p^m)$, for positive characteristic, where Frob_p is the Frobenius automorphism, sending an element to its p-th power. This allowed Hrushovski to give an entire model theoretic proof of the Munin-Mumford Conjecture [Hru01].

In this thesis, we are concerned with the second question on how to continue the work around Zil'ber's conjecture, i.e. how to fill the gap between non-locally modular structures and those which interpret an infinite field. In order to show that his new strongly minimal set is indeed a counterexample to Zil'ber's Conjecture, Hrushovski had to prove that there is no infinite field interpretable in it. He did so, by sorting out a geometrical property of his theory, the property of being CM-trivial, which can not be fulfilled if there is an infinite field around. Some years later it was Pillay who noted that the notions of not being one-based and not being CM-trivial are the first two steps of a whole hierarchy of geometries that, if they exist, would each be essentially different and not interpret an infinite field (conf. [Pil00]). After some corrections due to Evans, they defined what is now known as the *ample hierarchy*.
Though the motivation for this new classification tool came from the framework of strongly minimal theories, the notion of ampleness makes sense for arbitrary stable structures. It can best be understood as a combinatorial measurement of how complex the notion of forking independence is. If the theory is not ample at all, like in the case of pure sets or infinite dimensional vector spaces, then forking can be completely described in terms of algebraic closure. By definition, a theory is 1-ample if and only if it is not one-based and it is 2-ample if and only if it is not CM-trivial. In the same paper [Pil00] where the definition of ampleness first appeared, Pillay also proved that an infinite field is n-ample for all n, whence their theory stands on the very top of this ample hierarchy. Note that the counterexample constructed by Hrushovski is located at the second step of the hierarchy: it is not one-based, whence it is 1-ample, but it is CM-trivial and thus not 2-ample. A natural question arises, if there can be counterexamples to Zil'ber's

conjecture found in higher steps of the ample hierarchy. It has already been asked by Pillay in [Pil00]. For almost fifteen years this question had remained open, even when dropping the restriction on the theory to be strongly minimal. Only in 2014 Baudisch, Martin-Pizarro and Ziegler [BMPZ14a] as well as Tent [Ten14] managed to provide ω-stable examples of infinite Morley rank for any step of the ample hierarchy, which finally proved that the hierarchy is indeed strict. Nevertheless, there was no hope of collapsing the obtained examples to examples of finite rank, as each of them has trivial forking, and a theory of finite Morley rank with trivial forking is necessarily one-based.

There always has been a point of view suggesting that the notion of ampleness is very close to the notion of projective spaces. As projective spaces always rely on an underlying field, they cannot immediately be used to construct strictly ample examples. Nevertheless, it was pointed out by Tent that there is a more general way of viewing these spaces: they all relate to a wider class of geometrical objects, so-called Tits buildings.

Generally, buildings are combinatorial geometries associated to algebraic groups and Lie groups, which encode their algebraic properties. The notion was introduced by Jacques Tits and there are two different approaches to these geometrical objects: an early definition from 1959 [Tit59] views buildings as simplicial complexes. The more modern version [Tit74] however, defines them as chamber systems with a distance function that has its values in an associated Coxeter group. Both definitions and their correspondence are discussed in [Tit81]. The theory of buildings has profound applications within the classification of algebraic groups and Lie groups. In 2008, Tits co-received the Abel prize for his theory of buildings, which therein was described as a "central unifying principle with an amazing range of applications...".

It seems that looking at ampleness in connection with the theory of buildings is the appropriate point of view. Tent in [Ten00b] already had constructed many new examples of strictly 1-ample, almost strongly minimal theories which hence contradict Zil'ber's Conjecture and which are all (spherical) buildings. Furthermore, the first examples to prove that the ample hierarchy is strict, constructed in [BMPZ14a] and [Ten14], are also given by (right-angled) buildings.

In the present thesis under the supervision of Katrin Tent, who is the leading expert in the model theory of buildings, we construct a new almost strongly minimal set, which is strictly 2-ample. Although the arising structure cannot be a building, it relays on the general geometric properties behind the (modern) definition of buildings: it is an incidence geometry. We hope that this construction can be generalized to obtain counterexamples to Zil'ber's Conjecture for every step of the ample hierarchy.

Structure of this thesis

This thesis aims to produce a new almost strongly minimal 2-ample set.

First, we are going to provide the reader with enough background to follow the arguments appearing in this thesis. In particular, in Chapter 2, we introduce the necessary notions and facts on model theory, Zil'ber's Conjecture, Hrushovski Constructions and incidence geometries.
Building upon the assumption that the reader has a basic knowledge of model theory, we introduce any further results and notions from model theory used later in Section 2.1. We also introduce the construction technique of strong Fraïssé limits, which serves as a base in order to understand Hrushovski Constructions discussed in Section 2.2. We then revisit Zil'ber's Conjecture and introduce the ample hierarchy as a consequence of the research around it. We conclude the preliminary chapter with an introduction to incidence geometries in Section 2.3. As our construction essentially builds on the notion of incidence geometries, we try to provide several examples in order to enlighten these objects. We also introduce the notion of buildings, which are not directly needed to understand the geometry we construct, but as all intuition and notations origin in the study of buildings, we nevertheless feel the need to include them. We furthermore argue that, although all previously constructed examples around the ample hierarchy are indeed buildings, there is no hope of finding buildings as counterexamples to Zil'ber's conjecture of higher ampleness.

The core of this thesis is covered in Chapters 3, 5 and 6, where we execute the Hrushovski method on a new class of tripartite graphs in order to obtain a new counterexample to Zil'ber's Trichotomy Conjecture. In Chapter 3 we already construct the ab initio counterpart of our new almost strongly minimal 2-ample geometry. We start by introducing the amalgamation class $\mathcal{C}_0^{\text{fin}}$ and the predimension function which determines strong extensions. A large part of this chapter is devoted to come around the problem that our predimension function is not submodular. After locating the scenarios where submodularity fails, we are able to execute the amalgamation process and obtain the following theorem:

Theorem 1 (conf. Theorem 3.6.3) *The class $\mathcal{C}_0^{\text{fin}}$ has the amalgamation property with respect to strong embeddings. Moreover, if $C_0 \leq C_1$ is a minimal extension and*

$C_0 \leq_{n-2} C_2$, then either $C_1 \otimes_{C_0} C_2 \in \mathcal{C}_0^{\mathrm{fin}}$ or there is an isomorphic copy of C_1 over C_0 in C_2.

The above theorem yields the existence of a strong limit $\mathcal{M}_0$ of the class $\mathcal{C}_0^{\mathrm{fin}}$, which we call the ab-initio structure. In Chapter 4, we investigate the structure $\mathcal{M}_0$ and its model theoretic properties. That chapter is left independent from the rest of this thesis, as our main interest lays in the geometric properties of the collapsed counterpart of $\mathcal{M}_0$. Nevertheless, the considerations in Chapter 4 may facilitate the understanding of what follows, as many observations are repeated in a similar way when it comes do study the properties of the almost strongly minimal geometry in Chapters 5 and 6. We end that chapter with the following theorem:

Theorem 2 (conf. Theorem 4.4.1) *The theory $\mathrm{Th}(\mathcal{M}_0)$ is an ω-stable theory of infinite Morley rank $\omega \cdot (3(n-1)-1)$. It is 2-ample, witnessed by any complete flag, but not 3-ample.*

In the following Chapter 5 we collapse the structure $\mathcal{M}_0$ obtained above to obtain an almost strongly minimal geometry. On this account, one has to choose an appropriate, "tame" subclass $\mathcal{C}_\mu \subseteq \mathcal{C}_0$ and to prove that this class again has the amalgamation property. This is done in Section 5.1. As a corollary, we obtain again a strong Fraïssé limit of this new class, which we denote by $\mathcal{M}_\mu$. This provides the new almost strongly minimal 2-ample geometry we were aiming for. We start our analysis of $\mathcal{M}_\mu$ by showing that it is indeed an incidence geometry of the desired type. Then, we prove that the ω-saturated models of its theory are exactly the models which are $\mathcal{C}_\mu^{fin}$-saturated. This provides the following theorem:

Theorem 3 *The class $\mathcal{C}_\mu^{fin}$ has the amalgamation property with respect to strong embeddings. Its strong Fraïssé limit $\mathcal{M}_\mu$ is an incidence geometry of type $\bullet \overset{n}{-} \bullet \overset{n}{-} \bullet$. Furthermore, a model of its theory is ω-saturated if and only if it is $\mathcal{C}_\mu^{fin}$-saturated, whence in particular the structure $\mathcal{M}_\mu$ is ω-saturated.*

These results can be found in Proposition 5.1.5, Lemma 5.2.2 and Lemma 5.3.5.

The next chapter, Chapter 6, is devoted to show that the theory T_μ of our new geometry $\mathcal{M}_\mu$ is actually almost strongly minimal and exactly 2-ample. After proving that it is almost strongly minimal, applying a coordinatisation method used by Tent in [Ten00b], we describe how forking looks in its theory. From there, we can calculate the exact

Morley rank of our new geometry and show that Morley rank coincides with U-rank. We also use the description of forking to show that any 2-ample tuple is in a certain way witnessed by a complete flag. This yields on the one hand that the theory is indeed 2-ample, but also implies that it cannot be 3-ample. All together we obtain the following theorem.

Theorem 4 *The theory* T_μ *is almost strongly minimal of Morley rank* $3(n-1)-1$*. It is* 2*-ample, witnessed by any complete flag, but not* 3*-ample. In particular, the induced theory on the strongly minimal set yields a new counterexample to Zil'ber's Trichotomy Conjecture of ampleness* 2*.*

Chapter 6 concludes the construction part of our new geometry. The remaining part of the thesis is devoted to study interesting questions that naturally come up around the existence of such a new geometry. The first of them, treated in Chapter 7, concerns the existence of interpretable groups in T_μ. That question is of particular interested in connection with another famous Conjecture - the ***Alebraicity Conjecture***, also known by their authors, the Cherlin-Zil'ber Conjecture.

Conjecture (Algebraicity Conjecture, [Che79][Zil77]) *Any infinite simple group interpretable in a theory of finite Morley rank is an algebraic group over an algebraically closed field, which itself is interpretable in the group structure.*

This conjecture, even though being almost forty years old, is still unsolved and a whole area of model theory has developed around the quest of answering it. Even more surprisingly, there still is not a uniform line of thought on whether this conjecture should be answered affirmatively or negatively. If the conjecture would fail, then a minimal counterexample would be given by what is called a bad group. Until recently people seemed to start believing in the existence of a bad group, but ever since Frecon proved in [Fre16] that there are no bad groups of Morley rank 3, doubts on their existence started increasing again.

If there were groups interpretable in our structure, they would be of very interesting nature, as they could not interpret an infinite field and thus they could not be algebraic groups. Nevertheless, it turns out that there are no infinite definable groups interpretable in our new geometry.

Theorem 5 (conf. Proposition 7.4.1) *There are no infinite groups interpretable in the theory* T_μ*.*

A second question which naturally appears, concerns the existence of bounded automorphisms. This notion goes back to Lascar, who proved in [Las92] that in strongly minimal theories the group of strong automorphisms, i.e. of all automorphisms fixing the algebraic closure of the empty set pointwise, is a simple group modulo the normal subgroup of strong, bounded automorphisms. This result later has been generalized by Macpherson and Tent in [MT11] and then by Tent and Ziegler in [TZ13]. As the Hrushovski construction method can be used to produce novel exotic structures, assuming we can show that there are no bounded automorphisms we are bent to construct new exotic simple groups along with it for free. In this spirit, Ghadernezhad and Tent showed that there are no bounded automorphisms for the almost strongly minimal n-gons constructed by Tent in [Ten00b] and thereby obtained the first examples of non-algebraic simple groups with a BN-pair. In chapter 8, we adapt their arguments to obtain the final theorem:

Theorem 6 (conf. Proposition 8.3.2) *There are no bounded automorphisms in $\mathcal{M}_\mu$, nor in $\mathcal{M}_0$. Furthermore, any automorphism of $\mathcal{M}_\mu$ is strong, whence the automorphism group of $\mathcal{M}_\mu$ is a simple group.*

We conclude the thesis with a small epilogue on further interesting questions around the ample hierarchy in Chapter 9.

CHAPTER 2

PRELIMINARIES

In the following chapter, we provide the reader with a necessary background in model theory, Hrushovski constructions and incidence geometries required to follow the arguments in this thesis.

2.1 Some Model Theory

We expect the reader to be familiar with the basic notions of model theory, such as complete first order theories, first order languages, definable sets and the compactness theorem. A detailed presentation of the important notions and results in model theory, including all proofs of the statements below, can be found in [TZ12]. We intend to provide all definitions and theorems used later and which exceed the material of a basic course in model theory. First we fix some notation.

Notation

We denote finite tuples by small letters $a, b, \ldots, x, y, \ldots$ and finite sets by capital leters $A, B, \ldots$. Arbitrary sets are named $X, Y, \ldots$. If $X \subseteq Y$ is such that $|Y \setminus X|$ is finite, we say that Y is ***finite over*** X, or that it is a finite extension of X. In this case, we may also denote the infinite set Y by a capital letter $A, B, \ldots$.
We denote languages by $\mathcal{L}$. Theories are always be assumed to be consistent and be named T and the letter M stands for a model. By $\mathbb{M}$ we denote the monster model,

i.e. a very large, saturated model of some given theory. If M is some $\mathcal{L}$-structure and $X \subseteq M$ some subset of M, we mean by (M, X) the structure in the language $\mathcal{L}(X) := \mathcal{L} \cup \{c_x \mid x \in X\}$ where each c_x is a new constant symbol with the interpretation $c_x^M = x$.

For some model M and some subsets $A \subseteq M$ and $X \subseteq M^k$ we say that X is ***A-definable*** in M, if $X = \varphi(M, a)$ is the set of all realizations of some formula with parameters a in A. We say that X is definable, if it is A-definable for some set A. Furthermore, if Y is such that $X \subseteq Y \subseteq M$, we say that X is ***relatively definable*** in Y, if there is some definable set $X' \subseteq M$ such that $X = X' \cap Y$.

We usually denote types by p and q if they are complete and use π if we want to emphasize that a type is partial. The quantifier free type of some tuple b over some set A is denoted by $\mathrm{tp}^{qf}(b/A)$. If we want to emphasize the model M in which the type is considered, we write $\mathrm{tp}^M(b/A)$.

By the natural numbers $\mathbb{N}$ we mean the set $\{0, 1, 2, \dots\}$. For an arbitrary set X we denote by $\mathcal{P}(X)$ its power set, i.e. the collection of all its subsets. We often use the abbreviation AB or Ab to express the union $A \cup B$ or $A \cup \{b\}$. We call an element b ***algebraic*** over A, if there is a formula $\varphi(x, a)$ with only finitely many realizations, where a is a tuple in A and such that $\varphi(b, a)$ holds. The collection of all elements algebraic over A is denoted by $\mathbf{acl}(\boldsymbol{A})$. We denote the group of all automorphisms of some structure M fixing all elements of some set X pointwise by $\mathrm{Aut}_X(M)$.

Types and Saturation

In this section we introduce basic facts and tools used in model theory. One of the most fundamental notions which is used throughout the entire thesis, is the notion of a type. Let T be a theory and $M \models \mathrm{T}$ some model of T. We call a set of formulas $p(x) := \{\varphi(x, a) \mid a \in A \subseteq M\}$ in free variables x a ***type over*** A, if it is consistent, i.e. finitely satisfiable in M. If we want to emphasize the length $|x| = k$ of the tuple x, we also say that $p(x)$ is a k-type. Some type $p(x)$ is called a ***complete type*** over A, if for any tuple $a \in A$ and any formula $\varphi(x, a)$ in $L(A)$ we have that either $p(x) \vdash \varphi(x, a)$ or $p(x) \vdash \neg\varphi(x, a)$. Otherwise, it is called a ***partial type***. We denote the set of all complete n-types over some set A by $S_n(A)$ and the set of all complete types over A by $S(A)$. If p is a type over the monster model $\mathbb{M}$ of some theory, we say that p is a ***global type***. For an arbitrary set A and some element $b \in M$ we call

$$\mathrm{tp}(b/A) := \{\varphi(x, a) \mid a \in A, M \models \varphi(b, a)\}$$

the type of b over A. We say that some type p is an ***algebraic type***, if it contains an algebraic formula.
One of the big advantages in model theory is that instead of working in one fixed structure exclusively, we can work in richer elementary extensions and thus avoid to deal with approximations. For example, in order to make a statement of arbitrary large natural numbers, we can switch to a richer elementary extension in which an actual element being larger than any number in $\mathbb{N}$ exists. These rich models are called saturated.

Definition 2.1.1 Let κ be an arbitrary cardinal and T some theory. We say that some model M of T is ***κ-saturated***, if for any set $A \subseteq M$ of cardinality strictly less than κ, any type in $S(A)$ is realized in M. This has been shown to be equivalent to ask that any 1-type over A is realized in M. We call M ***saturated***, if it is $|M|$-saturated.

Probably the first question a model theorist should ask herself when given a consistent theory, is, whether this theory is already complete. The following fact provides a very useful tool to decide on this question.

Definition 2.1.2 Let T be a theory. We say that two models M_1 and M_2 possess the ***back-and-forth property*** or that they are ***partially isomorphic***, if we can extend any partial isomorphism between finite sets. This means that for any two finite sets $A_i \subseteq M_i$ such that there is a partial isomorphism $f : A_1 \to A_2$ and any two elements $b_i \in M_i$, there exist elements $c_i \in M_i$ such that both $f \cup \{(b_1, c_2)\}$ and $f^{-1} \cup \{(c_1, b_2)\}$ are again partial isomorphisms.

Fact 2.1.3 *(1) A countable theory* T *is complete if and only if any two of its ω-saturated models possess the back-and-forth property.*

(2) For two models M_1 and M_2 of the same complete theory with subsets $X_i \subseteq M_i$, the structures (M_1, X_1) and (M_2, X_2) are partially isomorphic if and only if the type $\mathrm{tp}^{M_1}(X_1)$ *equals the type* $\mathrm{tp}^{M_2}(X_2)$*.*

(3) If M is an ω-saturated model of T*, then two tuples b_1 and b_2 have the same type over some set $X \subseteq M$ if and only if there is some automorphism of M which fixes X pointwise and sends b_1 to b_2. In particular, some tuple b is algebraic over X if and only if its orbit under* $\mathrm{Aut}_X(M)$ *is finite.*

As outlined in the introduction, this thesis is motivated by the problem of understanding strongly minimal theories. We now introduce this notion.

Definition 2.1.4 Let M be a model of some theory T and $\varphi(x)$ a formula in one variable in T, possibly with parameters.

(1) We say that $X := \varphi(M)$ is ***minimal***, if it is infinite and any relatively definable set in X is either finite or co-finite.

(2) We call $\varphi(x)$ ***strongly minimal*** in T, if it defines a minimal set in any model of the theory T. Furthermore, we call a type $p(x)$ strongly minimal, if it contains a strongly minimal formula.

(3) Finally, the theory T is called a ***strongly minimal theory*** if the formula $x \doteq x$ is a strongly minimal formula in T.

The new geometry we are going to construct, though not being exactly strongly minimal, is still close enough, i.e. it is almost strongly minimal.

Definition 2.1.5 A theory T is called ***almost strongly minimal***, if there exists some finite set $B \subseteq \mathbb{M}$ and a strongly minimal formula $\varphi(x, B)$ with parameters in B such that for any model M of T and $D := \varphi(M, B)$ we get $M \subseteq \mathrm{acl}(BD)$.

For an exposition on almost strongly minimal theories you may consult [Bal72]. The definition of an almost strongly minimal theory states that any of its models is determined by some strongly minimal formula. The next fact gives a criterion on how to determine whether or not a type is strongly minimal.

Fact 2.1.6 *Let p be some type over a set X. Then p is strongly minimal, if it is not algebraic and for any superset $X \subseteq Y$, there is a unique non-algebraic extension of p to Y.*

Next we introduce the notion of imaginaries and the associated theory T^{eq}. Let $E(x, y)$ be a $\emptyset$-definable k-ary equivalence relation in T. We call an equivalence class M^k/E an ***imaginary element*** of T. Often it is necessary, or at least more convenient, to not only include elements of the ***home sort*** M in our statements, but also the imaginary elements of T. In the best case, these elements are already coded within the home sort, whence we say that T eliminates them.

Definition 2.1.7 We say that a theory T ***eliminates imaginaries***, if for any imaginary a/E there exists some finite tuple $d \in \mathbb{M}$ such that an automorphism of $\mathbb{M}$ fixes d

pointwise, if and only it fixes the class a/E as a set. We also say that d and the imaginary a/E are ***interdefinable***. We say that T has ***weak elimination of imaginaries***, if there exists a tuple d such that any automorphism which fixes d pointwise also fixes the class a/E and d only has finitely many conjugates under any automorphism that fixes the class a/E.

To any theory T in some language $\mathcal{L}$ we can naturally adjoin a theory T^{eq}, which eliminates imaginaries (conf. [TZ12, Section 8.4]). Any model M of T is in one-to-one correspondence with some model M^{eq} of T^{eq}, which consists of the home sort M and for any $\emptyset$-definable equivalence relation E one sort for M/E. This structure is considered in an associated language $\mathcal{L}^{eq}$, which contains projections from the home sort to any of the other sorts. Now, for some set $X \subseteq M$ we denote by $\mathrm{acl}^{eq}(X)$ the algebraic closure of X in the corresponding structure M^{eq} in the T^{eq} sense.
In the present thesis, imaginaries only appear indirectly. In fact, we show that the theory we obtain, weakly eliminates imaginaries. This already follows from the fact that our theory is almost strongly minimal. Nevertheless, we give a direct proof of this statement. To this end, we need the definition of a canonical base.

Definition 2.1.8 Let p be a global type in some stable theory T. There exists some definably closed set **Cb(p)** in $\mathbb{M}^{eq}$ such that an automorphism of $\mathbb{M}$ fixes the type p if and only if it fixes the set $\mathrm{Cb}(p)$ pointwise. We call $\mathrm{Cb}(p)$ the ***canonical base*** of p.

We can conclude weak elimination of imaginaries for our theory from the following fact:

Fact 2.1.9 *Let* T *be an arbitrary theory. If any global type in* T *has a real canonical base, i.e. a canonical base in the home sort* $\mathbb{M}$*, then the theory weakly eliminates imaginaries.*

Ranks and Stability

In this section we introduce two of the most important dividing lines developed by Shelah in his Classification Theory [She78]: stable theories and their subclass of ω-stable theories. We present the two principle notions of rank appearing in these two classes, the Morley rank and the Lascar rank and collect some of their basic properties, of which we make use later on.
As we already mentioned in the introduction, the most important dividing line introduced by Shelah consists of the dividing line of stability. In a stable theory, there are relatively few types over models, whereas any unstable theory has the maximal possible

number of types and hence one cannot classify its models. The definition we want to give here though, does not use the number of types, but another characterization of stable theories: the presence of a well-behaved notion of independence.

Lemma and Definition 2.1.10 (conf. [TZ12], Theorem 8.5.10) *Let* T *be a complete theory. Assume there exists a ternary relation* $\downarrow^{\mathrm{T}}$ *between subsets of models of* T *such that the following conditions are satisfied:*

- *(Invariance) The relation* $\downarrow^{\mathrm{T}}$ *is invariant under automorphisms of* $\mathbb{M}$*;*
- *(Local Character) There exists some cardinal* κ *such that for all* $A \subseteq \mathbb{M}$ *finite and* $X \subset \mathbb{M}$ *arbitrary, there is some* $C \subseteq X$ *with* $|C| < \kappa$ *and such that* $A \downarrow^{\mathrm{T}}_C X$*;*
- *(Transitivity) If* $A \downarrow^{\mathrm{T}}_X Y$ *and* $A \downarrow^{\mathrm{T}}_{XY} Z$ *then* $A \downarrow^{\mathrm{T}}_X YZ$*;*
- *(Weak Monotonicity) If* $A \downarrow^{\mathrm{T}}_X YZ$*, then* $A \downarrow^{\mathrm{T}}_X Y$*.*
- *(Weak Boundedness) For all* $A \subset \mathbb{M}$ *finite and* $X \subseteq \mathbb{M}$ *arbitrary, there is some cardinal* κ *such that for all* $X \subseteq Y$*, there are at most* κ *many isomorphism types of* $A' \subseteq \mathbb{M}$ *over* Y *with* $A' \cong_X A$ *and* $A' \downarrow^{\mathrm{T}}_X Y$*.*
- *(Existence) For any* $A \subset \mathbb{M}$ *finite and* $X \subseteq Y \subseteq \mathbb{M}$ *arbitrary, there is some* A' *such that* $\mathrm{tp}(A/X) = \mathrm{tp}(A'/X)$ *and* $A' \downarrow^{\mathrm{T}}_X Y$*.*

Then T *is stable and* $\downarrow^{\mathrm{T}} = \downarrow$ *coincides with the notion of non-forking independence.*

We omit the definition of non-forking independence at this point, as we are going to work exclusively in ω-stable theories, where this relation is described by the Morley rank of types.

Definition 2.1.11 Let T be a complete theory and $\varphi(x)$ a formula in T, possibly with parameters. We define the ***Morley rank*** of $\varphi(x)$, denoted by **MR**, successively.

- $\mathrm{MR}(\varphi(x)) \geq 0$ if and only if $\varphi(x)$ is consistent with T;
- For some ordinal α we set $\mathrm{MR}(\varphi(x)) \geq \alpha + 1$ if and only if there exist an infinite pairwise inconsistent family of formulas $\{\psi_i(x, c_i) \mid i \in \mathbb{N} \text{ and } c_i \in \mathbb{M}\}$ with $\psi_i(x) \to \varphi(x)$ and $\mathrm{MR}(\psi_i(x)) \geq \alpha$.
- If α is some limit ordinal, then we set $\mathrm{MR}(\varphi(x)) \geq \alpha$ if and only if $\mathrm{MR}(\varphi(x)) \geq \beta$ for all $\beta < \alpha$.

Now we define

$$
\begin{aligned}
\mathrm{MR}(\varphi(x)) = &\quad -\infty &&\text{if and only if } \varphi(x) \text{ is inconsistent with T;}\\
\mathrm{MR}(\varphi(x)) = &\quad \infty &&\text{if } \mathrm{MR}(\varphi(x)) \geq \alpha \text{ for any ordinal } \alpha \text{ and}\\
\mathrm{MR}(\varphi(x)) = &\quad \alpha &&\text{if } \mathrm{MR}(\varphi(x)) \geq \alpha \text{ and } \mathrm{MR}(\varphi(x)) \not\geq \alpha+1.
\end{aligned}
$$

For some type $p(x)$ we set $\mathrm{MR}(p(x)) := \inf\{\mathrm{MR}(\varphi(x)) \mid \varphi(x) \in p(x)\}$. If $p(x) = \mathrm{tp}(a/X)$, we also write $\mathrm{MR}(a/X)$ instead of $\mathrm{MR}(\mathrm{tp}(a/X))$. Finally, for an arbitrary theory T we define $\mathrm{MR}(\mathrm{T}) := \mathrm{MR}(x \doteq x)$, where x here denotes a singleton. A theory T is called ***ω-stable***, if and only if its Morely rank is bounded, i.e. $\mathrm{MR}(\mathrm{T}) < \infty$.

With the notion of Morley rank at hand, we can define an independence relation in any ω-stable theory. Thereby, we say that some set X is independent from some set Z over Y if and only if for any finite $A \subseteq X$ we have $\mathrm{MR}(A/Y) = \mathrm{MR}(A/YZ)$. We denote this by $X \mathop{\smile\!\!\!\!\!\!|}_Y Z$. It is not hard to see that this notion of independence satisfies all the properties listed in Fact 2.1.10, whence any ω-stable theory is indeed stable and the above defined notion of independence coincides with the non-forking independence $\mathop{\smile\!\!\!\!\!\!|}$. Fact 2.1.10 states that there is a unique notion of independence within ω-stable theories. The same does not hold for ranks. Indeed, there is another notion of rank on types in ω-stable theories, which may be essentially different from the Morley rank - the Lascar rank.

Definition 2.1.12 Let T be an ω-stable theory and $p(x)$ some type over the set X. Recall the definition of $\mathop{\smile\!\!\!\!\!\!|}$ from above. We define the ***Lascar rank*** of p, denoted by $\mathrm{U}(p)$ as follows:

- $\mathrm{U}(p) \geq 0$ for any type p.
- $\mathrm{U}(p) \geq \alpha + 1$ if and only if there exists some forking extension of p of rank at least α, i.e. there exists some $Y \supseteq X$ and $A \models p(x)$ such that $A \not\mathop{\smile\!\!\!\!\!\!|}_X Y$ and $\mathrm{U}(A/Y) \geq \alpha$.
- If α is a limit ordinal, then $\mathrm{U}(p) \geq \alpha$ if and only if $\mathrm{U}(p) \geq \beta$ for all $\beta < \alpha$.

As for the Morley rank we set $\mathrm{U}(p) = \alpha$ if α is the largest ordinal with $\mathrm{U}(p) \geq \alpha$ and $\mathrm{U}(p) = \infty$, if there exists no such α.
If p is a global type, we set $\mathrm{U}(p) = \mathrm{U}(p_{|M})$ for some small model M. This notion is well-defined as types are stationary over models in stable theories.

Note that even though Morley rank and Lascar rank define the same notion of independence, they differ in general. In particular, there are theories of finite Lascar rank with unbounded Morley rank. We say that that a theory is ***superstable***, if $\mathrm{U}(T) < \infty$. By the definition of the two ranks, it is easy to see that for any type p we have

$$\mathrm{U}(p) \leq \mathrm{MR}(p).$$

We mentioned before that the geometry we aim to construct is ω-stable, whence the Morley rank is bounded. So why bother at all introducing the notion of a Lascar rank? The reason is that this rank possesses a key property, which is not given for the Morley rank and simplifies the analysis of ranks: it satisfies the so-called Lascar inequalities.

Fact 2.1.13 *Let* T *be an ω-stable theory. Then the Lascar rank satisfies the* **Lascar inequalities***, i.e. for all finite sets a and b and arbitrary set X we have*

$$\mathrm{U}(a/bX) + \mathrm{U}(b/X) \leq \mathrm{U}(ab/X) \leq \mathrm{U}(a/bX) \oplus \mathrm{U}(b/X).$$

Hereby the addition $+$ on the left side denotes the ordinary ordinal sum and the one on the right side $\oplus$ denotes the sum arising by adding the coefficients in the Cantor normal form of the ordinal.

Note that if T is a theory of finite Morley rank, then it also is of finite Lascar rank and the two notions of addition coincide, whence actually $\mathrm{U}(ab/X) = \mathrm{U}(a/bX) + \mathrm{U}(b/X)$. Even though as mentioned above, Morley rank often does not satisfy the Lascar inequalities, it obtains this desired property in almost strongly minimal theories. This follows from the following fact in [TZ12, Proposition 6.4.9].

Fact 2.1.14 *Let φ be a strongly minimal formula defined over C and a and b algebraic over $\varphi(\mathbb{M}) \cup C$. Then*

$$\mathrm{MR}(ab/C) = \mathrm{MR}(a/bC) + \mathrm{MR}(b/C).$$

This yields the following corollary.

Corollary 2.1.15 *If* T *is almost strongly minimal, then it is of finite Morley rank. Furthermore, the Morley rank is additive, i.e. for all tuples a and b and for any set X we have*

$$\mathrm{MR}(ab/X) = \mathrm{MR}(a/bX) + \mathrm{MR}(b/X).$$

Proof Note that the Morley rank of some theory does not change by adding parameters. Thus we may assume that there is a strongly minimal formula $\varphi(x)$ such that for any model M of T we have $M = \operatorname{acl}(\varphi(M))$. Now, it is easy to see that T is of finite Morley rank, as any element is in the algebraic closure of finitely many elements of Morley rank one and the number of elements needed is uniformly bounded.
Assume φ is defined over C. Then all tuples a and b are algebraic over $\varphi(\mathbb{M}) \cup C$. Note that we may take a, b and X to be independent from C, whence additivity directly follows from Fact 2.1.14. ■

We conclude this section on Ranks and Stability with a general remark on the extension of types.

Lemma 2.1.16 *If $\pi(x)$ is a partial type over some set X of Morley rank α and Y a superset of X, then there exists an extension of $\pi(x)$ to a complete type $q(x)$ over Y such that* $\mathrm{MR}(q) = \alpha$.

Strong Fraïssé Limits

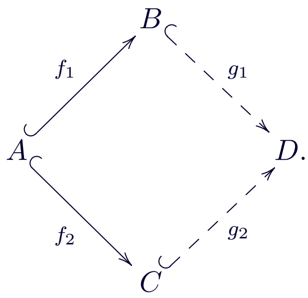
In this section we get to know one of the most important and universal construction methods of countable structures. It goes back to the article of Roland Fraïssé [Fra54], published in 1954, in which he describes how one can view the dense linear order of the rationals as some kind of limit structure of finite linear orderings. This construction cannot only be applied for these linear orders, but also for an ample amount of further classes, so called Fraïssé classes, which lead to a collection of new very exotic structures. A modification of these Fraïssé constructions has been used by Hrushovski in [Hru93] in order to refute Zil'ber's Trichotomy Conjecture. We now give an introduction into Fraïssé constructions. For a detailed exhibition see [TZ12, Section 4.4].

Definition 2.1.17 Assume $\mathcal{L}$ to be countable and $\mathcal{C}$ to be a class of finitely generated $\mathcal{L}$-structures which is countable up to isomorphism. Let furthermore $\mathcal{F}$ be a class of embeddings between $\mathcal{L}$-structures, which we call ***strong embeddings***. We call $\mathcal{C}$ a ***strong Fraïssé class***, if the following conditions are satisfied:

(HP) (***Strong Hereditary Property***) If B is a structure in $\mathcal{C}$ and A is a finitely generated $\mathcal{L}$-structure such that there exists a strong embedding $f : A \to B$, then $A \in \mathcal{C}$.

(JEP) (***Strong Joint Embedding Property***) For all structures B and C in $\mathcal{C}$, there is some structure $D \in \mathcal{C}$ and strong embeddings $f : B \to D$ and $g : C \to D$.

(AP) (***Strong Amalgamation Property***) For all structures A, B and C in $\mathcal{C}$ and strong embeddings $f_1 : A \to B$ and $f_2 : A \to C$, there exists some $D \in \mathcal{C}$ together with strong embeddings $g_1 : B \to D$ and $g_2 : C \to D$ such that the following diagram commutes:

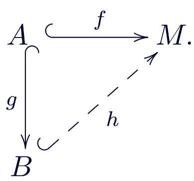


Let us furthermore say that some $\mathcal{L}$-structure A is ***strong*** in some structure B, if $A \subseteq B$ and the inclusion map is a strong embedding. In this case, we write $A \leq B$.
We see that these Fraïssé classes give rise to a limit structure, its so-called ***strong Fraïssé limit***. For a given $\mathcal{L}$-structure M we denote by $\text{age}(M) := \{A \subseteq M \mid A \text{ finitely generated}\}$ the set of all finitely generated substructures of M and call it the ***age*** of M. The Fraïssé limits we obtain by some amalgamation method, are characterized by the following condition.

Definition 2.1.18 Let $\mathcal{C}$ a class of finitely generated $\mathcal{L}$-structures and $\mathcal{F}$ be a class of strong embeddings between $\mathcal{L}$-structures. Some $\mathcal{L}$-structure M is called ***strongly $\mathcal{C}$-saturated***, if

(1) we have $\text{age}(M) = \mathcal{C}$ and

(2) for all $A \in \mathcal{C}$ with a strongly embedding $f : A \to M$ and $B \in \mathcal{C}$ together with a strong embedding $g : A \to B$, there exists a strong embedding $h : B \to M$ such that the following diagram commutes:



Let M be some $\mathcal{L}$-structure and $\mathcal{F}$ a class of strong embeddings between $\mathcal{L}$-structures. We call M ***ultrahomogeneous with respect to $\mathcal{F}$***, if any partial isomorphism between two substructures A and A' of M which are strongly embedded into M can be extended to an automorphism of M. In particular, if M is ultrahomogeneous with respect to $\mathcal{F}$, then two strongly embedded substructures A and A' of M have the same type, if and only if there is a partial isomorphism from A to A'.

Fact 2.1.19 *Let M be some countable $\mathcal{L}$-structure and $\mathcal{F}$ a class of strong embeddings between $\mathcal{L}$-structures. Then M is ultrahomogeneous with respect to $\mathcal{F}$ if and only if it is strongly $\mathcal{C}$-saturated. Furthermore, any two countable $\mathcal{C}$-saturated structures are isomorphic.*

The following theorem now establishes the relation between Fraïssé classes and $\mathcal{C}$-saturated countable structures.

Fact 2.1.20 *Let $\mathcal{C}$ be a countable class of finitely generated $\mathcal{L}$-structures and $\mathcal{F}$ a class of strong embeddings between $\mathcal{L}$-structures. There is a strongly $\mathcal{C}$-saturated countable structure M if and only if $\mathcal{C}$ is a strong Fraïssé class with respect to the strong embeddings in $\mathcal{F}$. Furthermore, the limit M is unique up to isomorphism and ultrahomogenous with respect to $\mathcal{F}$. We then call M the* **strong Fraïssé limit** *of $\mathcal{C}$.*

Indeed, we even get a slightly stronger result, not only guaranteeing that any finitely generated structure can be embedded into the limit, but also any countable one.

Fact 2.1.21 *For some $\mathcal{L}$ structure let $\mathcal{C}_\omega$ denote the class of all countable substructures of M. If $\mathcal{F}$ is a class of strong embeddings between $\mathcal{L}$-structures and M is ultrahomogenous with respect to $\mathcal{F}$, then for any finite subset A together with a strong embedding $f : A \to M$ and any $X \in \mathcal{C}_\omega$ with strong embedding $g : A \to X$, there is some strong embedding $h : X \to M$ such that $g \circ f = h$.*

2.2 Around Zil'ber's Conjecture

The Trichotomy Conjecture of Boris Zil'ber, although refuted soon after its appearance by Hrushovski, has influenced research in model theory during the last thirty years. In the following section, we want to motivate and state the conjecture, present its counterexample and have a look at the new classification tool - the ampleness of a structure - that arose from Hrushovski's new strongly minimal set.

Zil'ber's Conjecture

Inspired by Morleys Categoricity Theorem, there arose the question of classifying strongly minimal sets, which control uncountably categorical theories in an essential way. Recall that a theory is strongly minimal, if in all its models any definable set is either finite or co-finite. There are three classical, well-known examples of strongly minimal theories:

Example 2.2.1 The following provide strongly minimal theories:

(1) The infinite pure set without any further structure;

(2) An infinite dimensional vector space over some countable division ring;

(3) An algebraically closed field.

Within a strongly minimal theory, the algebraic closure satisfies the ***exchange principle***. This property has rich consequences, as it equips any strongly minimal theory with the structure of a pregeometry.

Definition 2.2.2 Let M be some set and cl a function from $\mathcal{P}(M) \to \mathcal{P}(M)$ which satisfies the following conditions for all $A \subseteq B \subseteq M$:

- (Monotonicity) We have $A \subseteq \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$
- (Idempotent) We have $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
- (Finite Character) It holds $\mathrm{cl}(A) = \bigcup_{A_0 \subseteq A \text{ finite}} \mathrm{cl}(A_0)$.
- (Exchange Principle) For all elements b and c we have that if $c \in \mathrm{acl}(Ab) \setminus A$, then $b \in \mathrm{acl}(Ac)$.

Then the pair (M, cl) is called a ***pregeometry***. If furthermore $\mathrm{cl}(\{x\}) = \{x\}$ for any singleton x, we call (M, cl) a ***geometry***.

The main consequence of the exchange property is that we can define a well behaved notion of dimension in any pregeometry M. For a finite set A, we let $d(A)$ denote the largest cardinality of an independent set, i.e. a set $A_0 \subseteq A$ with $a \notin \mathrm{cl}(A_0 \setminus \{a\})$ for any $a \in A_0$. In fact, there is a one to one correspondence between dimension functions and pregeometries.

Definition 2.2.3 A function $d : \mathcal{P}_{\text{fin}}(M) \to \mathbb{N}$ is called ***submodular***, if for all finite sets A and B we have

$$d(A \cup B) \leq d(A) + d(B) - d(A \cap B).$$

A monotone and submodular function d is called a ***dimension function***. If the inequality actually is an equality whenever $d(A \cap B) > 0$, we call the dimension function ***locally modular***.

Now, we already mentioned that we can obtain a dimension function from any pregeometry. For the examples given above, this dimension has a very natural interpretation: in a pure set, in coincides with the cardinality of a set, in a vector space it reflects the linear dimension and in algebraically closed fields, it is witnessed by the transcendence degree.
Reversely, from any dimension function d we can obtain a pregeometry by defining $b \in \text{cl}(X)$ if and only if there exists some finite $A \subseteq X$ such that $d(Ab) = d(A)$.
The Trichotomy Conjecture of Zil'ber aims to classify the pregeometries we obtain by acl on any strongly minimal theory.

Conjecture 2.2.4 (Zil'ber's Trichotomy Conjecture, 1984) *Let* T *be a strongly minimal theory. Then the pregeometry induced by* acl *on the models of* T *falls into one of the following three classes.*

(1) It is "set-like", i.e. for any set A we have $\text{acl}(A) = \bigcup_{a \in A} \text{acl}(a)$. We call such a geometry **disintegrated**.

(2) It is "vectorspace-like", i.e. it is not disintegrated and the assigned dimension function is locally modular.

(3) It is "field-like" in a very strong sense: there is an algebraically closed field interpretable in T.

Note the strong implication Zil'ber's Conjecture would have in the realm of the interaction between model theory and algebra: if we start with an arbitrary strongly minimal theory whose geometry is not locally modular, then an actual infinite field is interpretable in it and hence we can do algebraic geometry. Nevertheless, the conjecture had been refuted by Hrushovski [Hru93] soon after. We present his counterexample to Zil'ber's Conjecture in the next paragraph.

Before coming to that, we want to point out that even though the Trichotomy Conjecture had been refuted, work around it continued and proved to be a rich source of research. The continuation of Zil'ber's Conjecture can basically be partitioned into two streams:

(1) What further conditions should be put in order to make the conjecture hold?

(2) How far is the conjecture from being true? Can we fill the gap between vectorspace-like geometries and algebraically closed fields?

We gave some results on the first question in the introduction, whence we now turn towards the second question. We introduce Hrushovski's counter example and outline the new lines of classification obtained from it.

Hrushovski's Ab Initio

This section is devoted to introduce the counterexample constructed by Hrushovski in [Hru93] and outline in what sense it sits "between" the geometries named in Zil'ber's Conjecture. Finally, we see that there is a general description of the geometries that possess a geometry of essential different type than vector spaces and fields, which leads to the notion of the ample hierarchy.

Hrushovski's construction consists of two parts. He first builds a strong Fraïssé limit, where the notion of strongness is determined through a predimension function δ. This way, he obtains a theory which is ω-stable of infinite Morley rank and known as the ***ab initio*** structure. In order to decrease the rank and end up with a strongly minimal theory, he introduced a new method which is by now known as the Hrushovski ***collapse***, by only allowing a bounded number of copies of certain small extensions during the amalgamation process.
Recall that for a strong Fraïssé limit, we need a Fraïssé class which is to be amalgamated and a notion of strongness between its objects. The structures considered by Hrushovski are graphs A with a ternary relation $\mathcal{L} = \{\mathcal{K}\}$. Consider first the class $\mathcal{C}$ of all $\mathcal{L}$-structures for which $\mathcal{K}$ is irreflexive and symmetric, i.e. a collection of 3-element subsets. The ingenious idea now is that Hrushovski starts with a predimension function δ which in the strongly minimal collapsed structure yields the dimension function induced by acl, and defines the notion of strongness in dependence of this function. Recall that there is a one-to-one correspondence between a pregeometry and the dimension function. This way, he can determine the geometrical properties his structure is going

to have, from the very beginning.

We assume that all appearing graphs are contained in some common supergraph. By a subgraph we mean an induced subgraph and for two ternary graphs A and B we denote by AB the induced subgraph on $A \cup B$.

Definition 2.2.5 For a finite ternary graph A we set $\delta(A) = |A| - |\mathcal{K}(A)|$, where $\mathcal{K}(A)$ is the set of all relations in A. If B is another finite ternary graph, we set $\delta(B/A) = \delta(AB) - \delta(A)$. Finally, we say that some finite graph A is ***δ-strong*** in an arbitrary ternary graph X, write $\boldsymbol{A \leq_\delta X}$, if $\delta(B/A) \geq 0$ for any finite $B \subseteq X$.

Note that the function defined above is submodular. We now consider the subclass $\mathcal{C}_0$ of all structures A in $\mathcal{C}$ with $\delta(A) \geq 0$. It is easy to see that this class has the strong amalgamation property with respect to the notion of strongness defined above: the graph theoretic amalgam always provides a new example within the class $\mathcal{C}_0$. Thus, there exists a strong Fraïssé-limit $\mathcal{M}_0$, which we call Hrushovski's ab-initio structure.
Now recall that in order for δ to actually be a dimension function, besides being submodular it also has to be monotone. This is not yet the case.

Example 2.2.6 Consider some set $A = \{a_0, a_1, a_2\}$ consisting of three vertices without any relations and let B be the extension of A by one new vertex b together with two new relations $\mathcal{K}(a_0, a_1, b)$ and $\mathcal{K}(a_0, a_2, b)$. Then $3 = \delta(A) > \delta(B) = 2$.

On the other hand, recall that the δ-value of any finite substructure in $\mathcal{M}_0$ is non-negative. Thus, the value can only decrease finitely many times and any finite structure has a finite strong superstructure in $\mathcal{M}_0$. The submodularity of δ implies easily that the intersection of two strong sets is again strong, whence indeed for any finite set $A \subseteq \mathcal{M}_0$ there is a smallest strong superset of A in $\mathcal{M}_0$, which we call the ***closure*** of A and denote by $\mathbf{cl}(\boldsymbol{A})$. We now obtain a dimension function based on δ.

Fact 2.2.7 *Let $d : \mathcal{P}_{fin}(\mathcal{M}_0) \to \mathbb{N}$ be the function defined via $A \mapsto \delta(\mathrm{cl}(A))$. Then d is a dimension function and thus induces a pregeometry* Cl *on $\mathcal{M}_0$.*

As Hrushovski started with a δ function of his choice, so to prevent the Fraïssé limit to be locally modular, this is now witnessed through the dimension function d.

Example 2.2.8 Consider the sets in Example 2.2.6. As $B \in \mathcal{C}_0$, we can embed B strongly into $\mathcal{M}_0$. Clearly, the vertex b is in the closure Cl of A, as $B \subseteq \mathrm{Cl}(A)$. On the other hand, it is not in the closure of any of the vertices a_i, as each of them already provides a closed set in $\mathcal{M}_0$. Thus the geometry $(\mathcal{M}_0, \mathrm{Cl})$ is ***not disintegrated***.
Furthermore note that

$$d(\{a_0 b\} \cup \{a_1 b\}) = 2 < d(\{a_0 b\}) + d(\{a_1 b\}) - d(\{b\}) = 3,$$

whence d is ***not locally modular*** on $\mathcal{M}_0$.

Thus, we already obtained a geometry which does not fall into the first two cases. Before we consider the problem of interpretable fields, observe that we were aiming for a strongly minimal structure. It is easy to see that the present structure is ω-stable of infinite Morley rank. For a nice and detailed exposition on Hrushovski's new strongly minimal set, see also [Zie13].

The Collapse

In this paragraph we want to see how we can get from the infinite rank structure $\mathcal{M}_0$ to a strongly minimal counterpart, without changing the key geometric properties. Recall that we were aiming for a strongly minimal structure, in which the geometry is given by the algebraic closure. We observe that this is not yet the case in $\mathcal{M}_0$ as for a triple (a_0, a_1, b) with $\mathcal{K}(a_0, a_1, b)$, the vertex b is in the closure Cl of the set $\{a_0, a_1\}$, but not algebraic over it.
The main idea behind the collapse is to force the condition

$$d(b/A) = 0 \text{ if and only if } b \in \mathrm{acl}(A).$$

In order to do so, we have to be more careful during the amalgamation process and only allow a uniformly bounded number of copies of b over A, whenever $d(b/A) = 0$.

Definition 2.2.9 Let $A \leq B$ be a strong extension of structures in $\mathcal{C}_0$. We say that B is a 0-minimal extension of A, if $\delta(B/A) = 0$ and there is no proper strong subextension $A \leq B' \leq B$. We call B ***simple*** over A, or say that (A, B) is a ***simple pair***, if B is 0-minimal over A and not over any proper subset of A.

These simple extensions form the building blocks of any strong extensions of dimension 0. It hence suffices to uniformly bound the number of copies of simple extensions. Thus, we fix a function $\boldsymbol{\mu}$ from the set of all simple pairs in $\mathcal{M}_0$ into the natural numbers with the following properties:

(i) The value $\mu(A, B)$ only depends on the isomorphism type of the pair (A, B);

(ii) The value $\mu(A, B)$ is at least $\delta(A)$.

Denote furthermore for a simple pair (A, B), with A being contained in some structure D, by $\chi_D(A, B)$ the maximal number of disjoint copies of B over A in D. Now we define the subclass

$$\mathcal{C}_\mu := \{D \in \mathcal{C}_0 \mid \chi_D(A, B) \leq \mu(A, B) \text{ for all simple pairs } (A, B) \text{ with } A \subseteq D\}.$$

Then we get the following theorem which is proved in [Hru93].

Fact 2.2.10 *The class $\mathcal{C}_\mu$ has the amalgamation property with respect to strong embeddings. Its strong Fraïssé-limit $\mathcal{M}_\mu$ is strongly minimal and the pregeometry induced by* acl *coincides with the pregeometry induced by the dimension function d on $\mathcal{M}_\mu$. Furthermore, the geometry is not locally modular and there is no infinite field interpretable in $\mathcal{M}_\mu$.*

We do not give a complete proof of the above statement, but reason with some of its parts.

Proof (Sketch) We skip the proof of the class having the amalgamation property. Furthermore, the structure $\mathcal{M}_\mu$ turns out to be ω-saturated, which we also assume. Instead, we argue directly that algebraic closure and d-closure coincide. Assume $A \leq \mathcal{M}_\mu$ and there is some element $b \in \mathcal{M}_\mu$ in the d-closure of A, i.e. $d(b/A) = 0$. Then we can decompose the strong extension $A \leq \operatorname{cl}(Ab)$ into a chain of 0-minimal extensions. Clearly, by the property of the class, there are at most finitely many copies of each of these extensions, whence there also are only finitely many copies of b over A and b is algebraic over A. For the other direction, it follows from the amalgamation process that if $A \leq B$ is a minimal extension with $d(B/A) > 0$, then there are infinitely many copies of B over A in $\mathcal{M}_\mu$. Thus, for some element b we have $b \in \operatorname{acl}(A)$ if and only if $d(b/A) = 0$. Hence the two pregeometries coming from acl and from d coincide.
In order to show that $\mathcal{M}_\mu$ is strongly minimal, we have to show that over any set $A \leq \mathcal{M}_\mu$ there is exactly one non-algebraic 1-type. Note that for any singleton b we have $\delta(b/A) \leq 1$, whence also $d(b/A) \leq 1$. If $d(b/A) = 0$, then b is algebraic over A

by the considerations above. Otherwise $d(b/A) = \delta(b/A) = 1$ and the set $Ab \leq \mathcal{M}_\mu$ is strong in $\mathcal{M}_\mu$. We saw in Section 2.1 that the type of b over A is completely determined by the closure of Ab, whence this yields a unique non-algebraic type over A and $\mathcal{M}_\mu$ is strongly minimal.
There are two ways to show that there is no infinite field interpretable in $\mathcal{M}_\mu$. The first one is to show that the obtained geometry is ***flat***, a notion that is discussed in detail in Section 7.3. As a consequence, there is not even an infinite group interpretable in $\mathcal{M}_\mu$. Another argument uses that the geometry is CM-trivial. We discuss this line of thought in the following section. ∎

This concludes the overview of Hrushovski's example. The general method of constructing a strong Fraïssé limit with a notion of strongness given by some predimension function and of collapsing the structure by limiting the number of small extensions that we allow, has been used a lot in the following years to construct many interesting counterexamples to different questions. It is now known as a ***Hrushovski construction***.

The Ample Hierarchy

We already mentioned that the pregeometries appearing in Zil'ber's Trichotomy Conjecture are of essentially different type. Clearly, the three examples given in Example 2.2.1 fall exactly in the three corresponding types of geometries. While in disintegrated pregeometries any dependence between an element and a set already happens between the element and a singleton in the set, this is no longer the case in a vector space. Just note that for two base elements b_1 and b_2, its sum is algebraic over $\{b_1, b_2\}$, but not over any of the two vectors alone. Nevertheless, understanding independence is rather simple, as it is completely given by the algebraic closure: the dimension formula for vector spaces yields that any two sets are independent over the intersection of their algebraic closure. This property is known as being ***one-based***. In algebraically closed fields the independence is still harder to describe. It is easy to see, that the geometry in an algebraically closed field is not locally modular. For example, consider an algebraically closed field of characteristic 0 and choose an independent set of three elements, i.e. a set $\{a, b, c\}$ such that the transcendence degree of $\{a, b, c\}$ over $\mathbb{Q}$ is 3. Set furthermore $d := ac + b$. Then

$$\operatorname{acl}(a, b) \cap \operatorname{acl}(c, d) = \mathbb{Q}^{alg}(a, b) \cap \mathbb{Q}^{alg}(c, d) = \mathbb{Q}.$$

On the other hand, the tuple (a, b) is not independent from (c, d) over the empty set, as otherwise also $a, b \mathop{\smile\!\!\!\!\!\!\mid}_c d$ and thus $d \mathop{\smile\!\!\!\!\!\!\mid}_c d$, whence $d \in \operatorname{acl}(c)$, a contradiction.

We mentioned that the geometry constructed by Hrushovski is flat and hence cannot interpret an algebraically closed field, not even an infinite group. Besides this argument, Hrushovski pointed out another, rather combinatorial property that his structure possesses and which never occurs if there is an infinite field around. He called this property being CM-trivial. This property is a sharpening of the notion of one-basedness, which we briefly introduced above, and recall here for the matter of reference.

Definition 2.2.11 Let T be a stable theory. We call T ***one-based***, if for any sets A and B we have

$$A \mathop{\downarrow}_{\mathrm{acl}^{eq}(A)\cap\mathrm{acl}^{eq}(B)} B.$$

We say that T is ***CM-trivial***, if for all A, B and C with $A \downarrow_C B$ we have

$$A \mathop{\downarrow}_{\mathrm{acl}^{eq}(AB)\cap\mathrm{acl}^{eq}(C)} B.$$

As $A \downarrow_C B$ implies that $\mathrm{acl}^{eq}(A)\cap\mathrm{acl}^{eq}(B) \subseteq \mathrm{acl}^{eq}(AB)\cap\mathrm{acl}^{eq}(C)$ and also $\mathrm{acl}^{eq}(A)\cap \mathrm{acl}^{eq}(B) \subseteq \mathrm{acl}^{eq}(A)$, it immediately follows that any one-based theory also is CM-trivial. On the other hand, if T interprets an infinite field, it cannot be CM-trivial. This proves that there is no infinite field interpretable in the theory of Hrushovski's new strongly minimal set, whence his example is not a geometry of either of the types listed in Zil'ber's Trichotomy Conjecture.

Fact 2.2.12 (Proposition 3.2 in [Pil96b]) *If there is an infinite field interpretable in some stable theory* T*, then* T *is not CM-trivial.*

The above fact has first been proven by Pillay. In a following paper [Pil00], he observed that non-one-basedness and non-CM-triviality can be considered as the first two steps of an entire hierarchy of geometries of essentially different nature, which all refine the gap between vector spaces and algebraically closed fields. The following definition originates from Pillay [Pil00], with some modifications suggested by Evans.

Definition 2.2.13 Let T be a stable theory and $n \in \mathbb{N}$ arbitrary. We say that T is n-ample if, possibly after naming parameters, there are tuples $a_0, a_1 \ldots, a_n$ which satisfy the following properties:

(i) We have $\mathrm{acl}^{eq}(a_0)\cap\mathrm{acl}^{eq}(a_1) = \mathrm{acl}^{eq}(\emptyset)$;

(ii) For all $1 \leq i < n$ it holds

$$acl^{eq}(a_0, \ldots, a_{i-1} a_i) \cap acl^{eq}(a_0, \ldots, a_{i-1}, a_{i+1}) = acl^{eq}(a_0, \ldots, a_i);$$

(iii) For all $1 \leq i < n$ we have $a_0, \ldots, a_{i-1} \downarrow_{a_i} a_{i+1}$ and

(iv) It holds that $a_0 \not\downarrow a_n$.

One can understand the degree of ampleness of being a measure on how complicated the forking relation within the given theory is. We already observed that in vector spaces the independence is completely described by the algebraic closures. One can understand the ampleness of a theory in a way of characterizing, how far this statement is from being true.
The following remark is easy to see.

Remark 2.2.14 (1) The notions of ampleness form a hierarchy, i.e. any structure which is $n+1$ ample, is also n ample.

(2) A stable theory T is 1-ample, if and only if it is not one-based.

(3) A stable theory T is 2-ample, if and only if it is not CM-trivial.

The above remark implies in particular that infinite dimensional vector spaces over division rings are not 1-ample. Thus, they stand on the very bottom of the given hierarchy. It also states that algebraically closed fields are at least 2-ample. In fact, the theory of algebraically closed fields stands on the very top of the hierarchy.

Fact 2.2.15 (Proposition 3.13 in [Pil00]) *If there is an infinite field interpretable in some stable theory* T*, then* T *is* n*-ample for all* $n \in \mathbb{N}$.

Ever since the appearance of Hrushovski's new strongly minimal set, the question remained on whether there existed a counterexample to Zil'ber's conjecture, which is not CM-trivial. The introduction of the ample hierarchy now yields a vast range of possibilities of how these geometries could look like. Whenever one could construct a strongly minimal theory being strictly n-ample for some $n \geq 2$, one would have the desired counterexample. Nevertheless, this task turned out to be very challenging.
Anand Pillay discovered, that a stable theory is 1-ample if and only if there is a ***type-definable pseudoplane*** in it: consider a type-definable set $p(x, y)$, where we understand

the tuple x as coding points and the tuple y as coding lines. If any point is contained in infinitely many lines (i.e. $x \notin \mathrm{acl}(y)$) and any line contains infinitely many points and furthermore any two points intersect in only finitely many lines and any two lines in finitely many points, then we call $p(x, y)$ a type-definable pseudoplane. It is a ***free pseudoplane*** if any two lines (resp. points) intersect in a unique point (resp. line). Note that a free pseudoplane can be seen as the disjoint union of infinite-branching trees. It is 1-ample, but still CM-trivial.

The first to make progress in the direction of constructing a non-CM-trivial counterexample were Baudisch and Pillay in [BP00]. They produced a structure which is known as the ***free pseudospace*** and which can roughly be considered as gluing to free pseudoplanes together in a suitable way. Although this theory is not of finite Morley rank, it is at least ω-stable. Furthermore, it is shown to be not CM-trivial. The reason for both the free pseudoplane and the free pseudospace to not be of finite Morley rank is due to both structures having infinite diameter in the sense of graph distance.

It took some time for new examples to appear. Baudisch, Martin-Pizarro and Ziegler managed in [BMPZ14a] to generalize the construction of the free pseudospace to arbitrary ampleness and Katrin Tent at the same time understood the notion of ampleness to be a natural analog of right angled buildings and also provided examples in arbitrary ampleness in [Ten14]. They proved:

Fact 2.2.16 ([BMPZ14a] and [Ten14]) *The ample hierarchy is strict. For any n there is an ω-stable structure of infinite Morley rank, which is n-ample, but not $n+1$-ample.*

On the other hand, there was no hope of being able to collapse the structures obtained above into examples of finite Morley rank so that they could refute Zil'ber's Conjecture: all the examples given have trivial forking and hence any finite rank collapse would be one-based.

In this thesis under the supervision of Tent, we follow the line of thought of considering incidence geometries as the natural examples for ample structures and use a Hrushovski amalgamation method to obtain a counterexample to Zil'ber's Conjecture, which is 2-ample.

2.3 Incidence Geometries

Buildings are the geometrical objects, introduced by Jaques Tits in [Tit59] and later from another point of view in [Tit74], in order to study exceptional groups of Lie type. They are combinatorial geometries, which naturally appear along with algebraic groups and Lie groups and which reflect in a geometrical way the algebraic properties of these groups.

In this section we introduce all objects necessary to define incidence geometries and buildings, we view the known examples within the ample hierarchy as buildings and get to know a new counterexample to Zil'ber's Conjecture - The almost strongly minimal generalized n-gons constructed by Tent in [Ten00b]. The new 2-ample geometries, which to construct is the aim of this thesis, are roughly speaking two of these n-gons glued over each other in an appropriate way. Finally, we see that any building of finite Morley rank is either a generalized n-gon or interprets an algebraically closed field, whence the notion of a building is too strict to yield counter examples to Zil'ber's conjecture in higher ampleness.

Generalized N-Gons

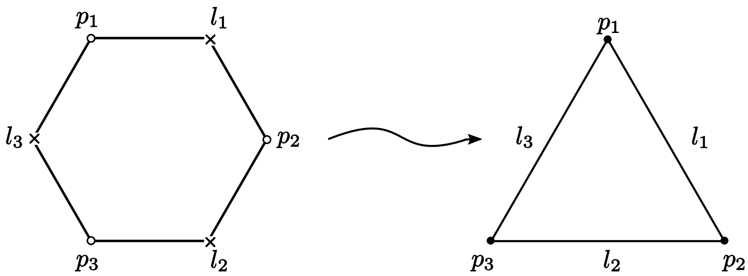
Recall that a ***k-partite graph*** is a graph $\Gamma = (V, \mathcal{K})$ such that the set of vertices V is partitioned into k-many disjoint sorts $P_1, \ldots, P_k$ and there are no edges between two vertices in the same sort. We consider all graphs equipped with the graph distance $dist$, where $dist(x, y)$ is the length of a shortest path in Γ from x to y, if such a path exists and ∞ otherwise. Now, the diameter of Γ is the supremum of $\{dist(x, y) \mid x, y \in \Gamma\}$ and the girth of Γ is the length of a smallest simple cycle in Γ.

Definition 2.3.1 Fix some $n \in \mathbb{N} \cup \{\infty\}$. A ***generalized n-gon*** is a bipartite graph Γ of diameter n and girth $2n$. Note that a simple $2n$-cycle is a generalized n-gon. This specific example is called an ***ordinary n-gon***. We say that a generalized n-gon is ***thick***, if any vertex is incident with at least three other vertices.

This definition might not come intuitively to all of the readers, considered that we already believe to know what an n-gon should be. We hope that the following example might help to see the motivation for the given terminology.

Example 2.3.2 Consider an ordinary triangle, i.e. a simple 6-cycle. We can partition its vertex set into two sorts, which we denote by points $\{p_1, p_2, p_3\}$ and lines $\{l_1, l_2, l_3\}$

such that the two neighbors of any vertex are in a different sort than the vertex itself. If we interpret an edge between some point and some line as saying that the point is contained in the line, then the picture below illustrates how the abstract ordinary triangle relates to the usual triangle that we already know.



Note that thick generalized n-gons consist of unions of ordinary n-gons and thus locally coincide with what we intuitively understand as being an n-gon. We hope that the reader finds the following examples helpful in order to get used to the notion of a generalized n-gon. When discussing n-gons, we from now on always consider the two sorts to stand for points and lines.

Examples 2.3.3

- A generalized 2-gon is a complete bipartite graph: as the distance between any point and line is always odd and has to be at most 2, we see that in fact, any point is connected to any line.
- A generalized 3-gon is a projective plane: the distance between two points or resp. two lines is even and at most 3, whence any two points (or any two lines) meet in a common line (or point resp.). Furthermore this line is unique, as there are no cycles of length 4.
- A generalized ∞-gon is a disjoint union of arbitrarily many trees such that there are at least two connected components if the graph is finite. This is immediate, as there are no simple cycles whatsoever. In particular, the free pseudoplane is a generalized ∞-gon.

These generalized n-gons are the building bricks for incidence geometries and buildings, which we want to define later in this section. But before we come to that, we want to introduce the almost strongly minimal generalized n-gons constructed by Tent in [Ten00b]. They are essential for the construction of the almost strongly minimal 2-ample geometries, which we are going to construct in the main part of this thesis and which arise by gluing the n-gons of Tent together in an appropriate way.

Almost Strongly Minimal Generalized N-gons

In the following we want to explain the construction of Tent from [Ten00b].
In order to obtain almost strongly minimal generalized n-gons, she uses a Hrushovski construction on a class of bipartite graphs, where we understand the two sorts as referring to points and lines. We always assume that the graphs we talk about are contained in some common supergraph. By a subgraph we mean an induced subgraph and for two graphs A and B we write AB for the induced subgraph on $A \cup B$. We denote the edges of some graph A by $\mathcal{K}(A)$.
Recall that for a Hrushovski construction, we need a predimension function on finite graphs, which determines strong embeddings and yields a dimension within the limit, and a class of graphs which possesses the amalgamation property with respect to these strong embeddings. For some finite graph A set

$$\delta_1(A) := (n-1)|A| - (n-2)|\mathcal{K}(A)|.$$

This clearly yields a submodular predimension function. For a graph A which is δ_1-strong in B, we write $A \leq_1 B$. Now consider the class $\mathcal{C}_1$ of finite bipartite graphs B satisfying the following properties:

(C1) There are no ordinary m-gons in B for any $m < 2n$.

(C2) If $A \subseteq B$ contains an ordinary m-gon for $m > n$, then $\delta_1(A) \geq 2n + 2$.

In [Ten00b] it is shown that the above class has the amalgamation property. As we are interested in constructing examples of finite Morley rank, we have to amalgamate more carefully so to bound the rank. We got to know this procedure as the collapse in Section 2.2. Recall that a simple extension $A \leq B$ is a strong extension with $\delta_1(B/A) = 0$ and such that there is no proper strong subextension and furthermore B is not a 0-minimal extension over any proper subset of A. We then also call (A, B) a simple pair.
Note that if within a graph A there are two vertices x and y such that $dist(x, y) = n+1$, then the extension of A by a new path of length $n-1$ between x and y is a valid extension in $\mathcal{C}_1$, which is simple over $\{x, y\}$. We call this extension a ***pure path extension*** of length $n-1$. The existence of these extensions ensures that the limit structure obtained by amalgamation is indeed of diameter n.

We now introduce a function μ from the set of all simple pairs (A, B) into the natural numbers with the following two properties:

(1) The value $\mu(A, B)$ does only depend on the isomorphism type of (A, B).

(2) If B is a pure path extension of two vertices in A, then $\mu(A, B) = 1$. Otherwise $\mu(A, B) \geq \max\{\delta(A), n\}$.

Now let $\mathcal{C}_1^\mu$ be the subclass of $\mathcal{C}_1$ consisting of all the structures $C \in \mathcal{C}_1$ which satisfy that for any simple pair (A, B) with $A \subseteq C$ there are at most $\mu(A, B)$-many disjoint copies of B over A in C. Tent shows that this class again has the amalgamation property:

Fact 2.3.4 (Theorem 4.4 in [Ten00b]) *The class $(\mathcal{C}_1^\mu, \leq_1)$ has the amalgamation property. Its strong Fraïssé limit Γ_n is a $\mathcal{C}_1^\mu$-saturated generalized n-gon.*

It is left to show that the limit structure Γ_n is almost strongly minimal, i.e. there exists a finite set B and an almost strongly minimal set D such that Γ_n is in the algebraic closure of BD. The strongly minimal set D is given by the set of all lines connected to one fixed point (so-called line pencils) or the set of all points contained in one fixed line (so-called point rows).

Fact 2.3.5 (Theorem 4.5 in [Ten00b]) *Let D be a point row or a line pencil. Then D is strongly minimal.*

For odd n it is known that any generalized n-gon is in the definable closure of a point-row and a finite set, due to certain definable bijections called projectivities. See [Ten00a] or [KTvM99] for a nice exhibition on these objects. For even n, Tent used a more involved geometric argument to prove the following fact:

Fact 2.3.6 (Theorem 4.6 in [Ten00b]) *The generalized n-gon Γ_n is almost strongly minimal. More precisely, if x_1, x_2 and x_3 are three vertices with*

$$dist(x_1, x_2) = dist(x_2, x_3) = n \text{ and } dist(x_1, x_3) \in \{n-1, n\},$$

depending on whether n is even or odd, and D is the set of all vertices connected to x_1, then $\Gamma_n \subseteq \mathrm{dcl}(D \cup \{x_1, x_2, x_3\})$.

Higher Rank Geometries and Buildings

We now introduce incidence geometries and buildings, which one can think of as higher rank analogies of generalized n-gons.
Fix some finite index set I. By an ***incidence geometry*** Γ over I, we only mean an $|I|$-partite graph. The cardinality of I is called the ***geometrical rank*** of Γ. A set of vertices $F \subseteq \Gamma$ is called a ***flag***, if all the vertices in F are pairwise incident. Note that this implies $|F| \leq |I|$. If equality holds, we call F a ***complete flag***. We also include the empty set in the definition of a flag. For a given flag F we call

$$\operatorname{Res}(F) := \{x \in \Gamma \mid F \cup \{x\} \text{ is a flag}\} \tag{2.1}$$

the ***residue*** of F. Note that the residue of any flag in the sorts $J \subseteq I$ is again an incidence geometry over $I \setminus J$. Now we have all the tools to define a geometry of a certain type.

Definition 2.3.7 Let I be a finite set and $M := (m_{ij})_{i,j \leq |I|}$ be a symmetric matrix with $m_{ii} = 1$ for all i, called a ***Coxeter matrix***. We call an incidence geometry Γ over I a ***geometry of type M***, if for any flag F on vertices in the sorts $I \setminus \{i, j\}$ for $i \neq j$, the bipartite graph $\operatorname{Res}(F)$ is a generalized m_{ij}-gon.

Often, the type of an incidence geometry is given by a ***Coxeter diagram***, rather than the Coxeter matrix M. The Coxeter diagram associated to the matrix M is a finite graph with vertices $s_1, \ldots, s_{|I|}$ and an edge between s_i and s_j for $i \neq j$ labeled m_{ij} whenever $m_{ij} \geq 3$. We intuitively understand that if s_i and s_j are not connected by an edge, then $m_{ij} = 2$.

Examples 2.3.8 (i) A geometry of type $\bullet\ \bullet$ is a complete bipartite graph.

(ii) A generalized n-gon is an incidence geometry of type $\bullet \overset{n}{-} \bullet$. The n-gons constructed by Tent in [Ten00b] are geometries of this type and yield examples which are 1-ample of finite Morley rank.

(iii) The free pseudoplane is an incidence geometry of type $\bullet \overset{\infty}{-} \bullet$. Recall that it is exactly 1-ample and ω-stable of infinite Morley rank.

(iv) A geometry of type $\bullet \overset{n}{-} \bullet \overset{n}{-} \bullet$ consists of three sorts, which we understand as points, lines and planes. The residue of each line is a complete bipartite graph. In the following chapters, we construct a geometry of this type, which is strictly 2-ample and of finite Morley rank.

(v) The free pseudo space as constructed by Baudisch and Pillay in [BP00] is a geometry of type $\bullet \overset{\infty}{-} \bullet \overset{\infty}{-} \bullet$. It is ω-stable of infinite Morley rank and strictly 2-ample.

(vi) The free k-pseudospaces as constructed in [BMPZ14a] and [Ten14] are geometries of type $\bullet \overset{\infty}{-} \bullet \overset{\infty}{-} \bullet \cdots \bullet \overset{\infty}{-} \bullet$ (k times). They provide examples of ω-stable theories of infinite rank, which are strictly k-ample.

We see in the above list of examples within the ample hierarchy, that it seems to be the natural guess to look for a geometry of type $\bullet \overset{n}{-} \bullet \cdots \bullet \overset{n}{-} \bullet$ (k times), when aiming to find a counterexample to Zil'ber's conjecture which is exactly k-ample.

Now we want to introduce the notion of a building. We can assign to each Coxeter diagram a unique group, its ***Coxeter group***. If the Coxeter diagram is over an index set I with values (m_{ij}) for $i, j \leq |I|$, then its associated Coxeter group is given through the presentation

$$W := \langle s_1, \ldots, s_{|I|} \mid (s_i s_j)^{m_{ij}} = 1 \rangle.$$

Note that it is in general not possible to recover the generating set $S := \{s_1, \ldots, s_n\}$ from the isomorphism type of W. Thus we often point out the generating set explicitly and call the pair (W, S) a ***Coxeter System***. Given a Coxeter system, we can assign to any element $w \in W$ its wordlength $l(w)$, i.e. the smallest number k such that w can be written as a word in k-many of the letters $\{s_1^{\pm 1}, \ldots, s_{|I|}^{\pm 1}\}$. We call a word ***reduced***, if there is no word of shorter length representing the same group element. Now we can finally give the definition of a building.

Definition 2.3.9 (conf. Definition 5.1 in [AB08]) Let (W, S) be a Coxeter system. A building Δ of type (W, S) is a pair (C, δ) consisting of a nonempty set C, whose elements are called chambers, together with a map $\delta : C \times C \to W$, such that for all $c, d \in C$ the following three conditions hold:

(B1) The value $\delta(c, d) = 1$ if and only if $c = d$.

(B2) If $\delta(c, d) = w$ and $c' \in C$ satisfies $\delta(c', c) = s \in S$, then $\delta(c', d) \in \{sw, w\}$. If in addition $l(sw) = l(w) + 1$, then $\delta(c', d) = sw$.

(B3) If $\delta(c, d) = w$, then for any $s \in S$, there is a chamber $c' \in C$ such that $\delta(c', c) = s$ and $\delta(c', d) = sw$.

This definition might appear rather combinatorial at first sight. In order to connect it to the definition of incidence geometries, we can think of the set C as the set of complete flags in an incidence geometry with Coxeter diagram on the vertices $S := \{s_1, \ldots, s_{|I|}\}$. The function δ measures the distance between two flags within the geometry, where we say for example that two complete flags F and F' are at distance s_i, if they coincide on all their vertices but the one in the i-th sort. We can illustrate this by writing $F \overset{s_i}{\mapsto} F'$, meaning that a change of some vertex in the i-th sort gets us from the complete flag F to the complete flag F'. That way, we can successively assign a distance value in the group (W, S) between any two complete flags. An incidence geometry now is a building, if the existing flag paths in it coincide with the group relations in W, i.e. any word assigned to a shortest path in Δ between two flags is a reduced word in W and there are no cycles in Δ which correspond to a nontrivial reduced word in W.
We call a building ***spherical***, if its associated Coxeter group W is finite. Furthermore, we say that a building is ***irreducible***, if its Coxeter diagram is connected. Moreover, a building is called a ***right angled*** building, if in the Coxeter Matrix all appearing entries are either 2, in which case the according generators commute, or infinite.
Note that thick generalized n-gons are exactly the spherical buildings of geometrical rank 2. We want to give the following example on how to view a generalized polygon as a building.

Example 2.3.10 Let Γ be a thick generalized n-gon. The associated Coxeter group is $W := \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^n = 1\rangle$.
For two flags $F = (a, b)$ and $F' = (a', b')$ in Γ, let k be the shortest distance between one vertex of F and one vertex of F'. We consider the example where the shortest distance is witnessed between the line b and the point a', whence k is odd. Then there is a unique ordinary $2n$-gon $(c_0 = a, c_1 = b, c_2, c_3, \ldots, c_k, c_{k+1} = a', c_{k+2} = b', \ldots, c_{2n-1}, c_{2n} = c_0)$ in Γ containing both flags. If $k < n - 1$, this yields a unique shortest flag-path

$$F = (a, b) \overset{s_1}{\mapsto} (c_2, b) \overset{s_2}{\mapsto} (c_2, c_3) \overset{s_1}{\mapsto} \ldots \overset{s_2}{\mapsto} (c_{k-1}, c_k) \overset{s_1}{\mapsto} (a', c_k) \overset{s_2}{\mapsto} (a', b') = F'$$

from F to F'. Thus we can assign as the distance between F and F' the value $\delta(F, F') = (s_1 s_2)^{\frac{k+1}{2}}$. If $k = n - 1$, then any shortest path between F and F' is of the form $(s_1 s_2)^{\frac{n}{2}}$ or $(s_2 s_1)^{\frac{n}{2}}$. Note that these two words represent the same group element in W as

$$(s_1 s_2)^{\frac{n}{2}} = \left((s_1 s_2)^{\frac{n}{2}}\right)^{-1} = (s_2 s_1)^{\frac{n}{2}}.$$

Note that all the examples given in 2.3.8 except our new geometry mentioned in (iv), are in fact buildings. Thus, in order to construct counterexamples to Zil'ber's Conjecture

in higher ampleness, one might want to construct buildings of finite Morley rank which are of type $\bullet \overset{n}{-} \bullet \ldots \overset{n}{-} \bullet$. Nevertheless, this turns out to be impossible.
This is due to the following observations: we can understand the graph distance as an indication for forking and dimension, whence for a building to be of finite Morley rank, it has to be of finite diameter. Otherwise there are vertices in the graph of arbitrary large finite distance that fork with any vertex on the path between them and the Morley rank of the theory it at least ω. It is known that a building is of finite diameter if and only if it is of spherical type, whence any possible example had to be spherical.
On the other hand, Kramer, Tent and van Maldeghem proved the following fact:

Fact 2.3.11 (Theorem 5.1 in [KTvM99]) *Let Δ be an infinite irreducible spherical building of geometrical rank at least three and of finite Morley rank. Then Δ is the building associated to a simple linear algebraic group over some algebraically closed field, which is definable in Δ.*

By the theorem of Pillay [Pil00], this implies that the theory of any infinite irreducible building of finite Morley rank is n-ample for all n and cannot be properly contained in the ample hierarchy.

Nevertheless, one should not be discouraged by this fact, as there is no structural reason why we should require the incidence geometry to be a building. Hence in the following chapters, we construct an incidence geometry of type $\bullet \overset{n}{-} \bullet \overset{n}{-} \bullet$ of finite Morley rank, which is 2-ample, but not 3-ample.

CHAPTER 3

THE CONSTRUCTION

3.1 Motivation and Notation

We saw in Section 2.2 that the natural examples of ω-stable geometries which are properly contained in the ample hierarchy, are given by right-angled buildings of type $\bullet \overset{\infty}{-} \bullet \overset{\infty}{-} \bullet \cdots \bullet \overset{\infty}{-} \bullet$. They are the natural higher rank generalizations of the free pseudoplane, which itself is a building of geometric rank 2 and type $\bullet \overset{\infty}{-} \bullet$. As is the latter, these right-angled buildings are of infinite Morley rank, which originates in the fact that all these incidence geometries have infinite diameter with respect to the graph distance.
In [Ten00b], Katrin Tent used a Hrushovski amalgamation method to construct generalized polygons of finite Morley rank. Recall that a generalized polygon is an incidence geometry of type $\bullet \overset{n}{-} \bullet$ for some $n \in \mathbb{N}$ and thus provides the natural finite diameter analog of the free pseudoplane. As in the pseudoplane, her examples are 1-ample and not 2-ample, witnessed by any complete flag.
In this dissertation under the supervision of Tent, we take these thoughts one step further and construct a geometry of geometric rank 3 of finite diameter and type $\bullet \overset{n}{-} \bullet \overset{n}{-} \bullet$ for arbitrary $n \geq 6$. Hereby, we can think of the three different types of vertices as corresponding to points $\mathcal{P}$, lines $\mathcal{L}$ and planes $\mathcal{E}$ respectively. Note that in such a geometry, the residue of any line is a generalized 2-gon, i.e. a complete bipartite graph on points and planes. This motivates that we distinguish between two different sets of edges: those that connect a line to either a point or plane, which we denote by $\mathcal{K}$

and those that connect points to planes, which we denote by $\mathcal{K}'$. Furthermore, we are interested in complete flags, which we denote by $\mathcal{F}$. If we want to refer only to the points (respectively edges et cetera) of some specific structure A, we also write $\mathcal{P}_A$ (respectively $\mathcal{K}_A$ et cetera). By $\mathcal{K}(A, B)$ and $\mathcal{K}'(A, B)$ respectively, we denote the number of edges connecting a vertex from A with some vertex from B and finally we use the notation $\mathcal{F}(A, B)$ (respectively $\mathcal{F}(A, B, C)$) for complete flags that contain at least one vertex from A and one vertex from B (respectively one from A, one from B and one from C).
We obtain the geometry through a Hrushovski construction using a predimension function δ based on the ideas introduced in Section 2.2. Once again, a complete flag are a witness for ampleness and we obtain a theory which is 2-ample, but not 3-ample. In Chapter 5, we use general methods to collapse the ab initio structure and thus obtain a strongly minimal geometry, which is 2-ample, but does not interpret an infinite field.
From now on for the rest of this thesis, we fix some $n \geq 6$. We assume that all graphs $A, B, \ldots$ that are considered in the following are contained in some common supergraph and we denote by AB the induced subgraph on the set $A \cup B$ therein. We frequently use suggestive expressions like "a line contains a point" or "two points intersect in a line" for the fact a point is connected to a line (or two points are connected to the same line). Usually, we use letters A, B etc. to denote finite structures, while X, Y and so on stand for infinite ones. We say that an extension $X \subseteq Y$ is finite or that Y is finite over X, if $Y \setminus X$ is finite. We may sometimes use letters $A, B, \ldots$ instead of Y in this case.

3.2 The Predimension Function

As outlined in Section 2.2, the main ingredient that sets Hrushovski's amalgamation method apart from general Fraïssé constructions, is the presence of some predimension function δ which assigns to any finite object in the amalgamation class some natural number and determines strong extensions.

Definition 3.2.1 Assume A to be a finite, tripartite graph. We set

$$\begin{aligned}\delta(A) := &(3(n-1)-1)|\mathcal{L}_A| + 2(n-1)(|\mathcal{P}_A| + |\mathcal{E}_A|)\\ &-(2(n-1)-1)|\mathcal{K}_A| - (n-1)(|\mathcal{K}'_A| - |\mathcal{F}_A|).\end{aligned}$$

As usual, for two finite tripartite graphs A and B we set

$$\delta(B/A) := \delta(AB) - \delta(A).$$

Furthermore, if X is an arbitrary set, we define

$$\boldsymbol{\delta(A/X)} := \min\{\delta(A/C) \mid C \subseteq X \text{ finite}\}.$$

We say that A is ***strong*** in B (write $\boldsymbol{A \leq B}$), if A is contained in B and we have $\delta(B'/A) \geq 0$ for all finite $B' \subseteq B$. Note that this is a priori not a first order property, whence we also use the following notations, which are first-order expressible:
We say that A is ***k-strong*** in B (write $\boldsymbol{A \leq_k B}$) for some natural number $k \geq 0$, if $A \leq B'$ for all $B' \subseteq B$ with $|B' \setminus A| \leq k$. If $\delta(l/A) \geq 0$ for all lines $l \in B$, we say that A is ***$\mathcal{L}$-strong*** in B and write $\boldsymbol{A \leq_{\mathcal{L}} B}$.

In the following, we want to motivate the above choice of the δ-function. First recall that within a geometry of type $\bullet \overset{n}{-} \bullet \overset{n}{-} \bullet$, the residue of any line is a complete bipartite graph, implying that whenever a point p is contained in some line l, which itself is contained in a plane e, then p must necessarily also be contained e. In that case, the vertices (p, l, e) form a complete flag and we call the edge between p and e ***induced***. Those induced edges do not provide any additional information and should not be counted, whence we oppose to the number of edges in $\mathcal{K}'$ the number of flags. If an edge is not induced, we also refer to it as an ***essential*** edge.
The next Lemma illustrates the inductive character of the above predimension function in relation to the predimension function δ_1 for the geometric rank 2 case, which was introduced in Section 2.3.

Lemma 3.2.2 *Let B be a tripartite graph and $x \in \mathcal{P} \cup \mathcal{E}$ a point or plane in B. Then for any $A \subseteq \mathrm{Res}^B(x)$ we have*

$$\delta(A/x) = \delta_1(A).$$

Proof Without loss of generalization we may assume that x is a point. Then

$$\begin{aligned}
\delta(A/x) &= \delta(A) - (2(n-1)-1)|\mathcal{K}(x,A)| - (n-1)(|\mathcal{K}'(x,A)| - |\mathcal{F}(x,A)|) \\
&= \delta(A) - (2(n-1)-1)|\mathcal{L}_A| - (n-1)(|\mathcal{E}_A| - |\mathcal{K}_A|) \\
&= (n-1)(|\mathcal{L}_A| + |\mathcal{E}_A|) - (n-2)|\mathcal{K}_A| \\
&= \delta_1(A).
\end{aligned}$$

3.3 The Amalgamation Class

The two main ingredients for a Hrushovski amalgamation are the class of structures we want to amalgamate and a predimension function which determines strong extensions. We already defined the predimension above, whence we now need to introduce the appropriate class of structures that have the amalgamation property and ensures that its Hrushovski limit is a geometry of type $\bullet \overset{n}{-} \bullet \overset{n}{-} \bullet$.

Definition 3.3.1 Let $\boldsymbol{\mathcal{C}_0}$ be the class of tripartite graphs X which satisfy the following conditions:

(C1) If the point p is contained in a line l and l is contained in some plane e, then p is contained in e.

(C2) There are no cycles of length less than $2n$ in the residue of a point or plane respectively.

(C3) If two different points intersect, i.e. there is a common vertex containing both the points, then they intersect either in a unique plane (and no line) or in a unique line and all the planes contained in that line. The dual holds for planes respectively.

(C4) Let $A \subseteq X$ be a finite substructure of X which satisfies $(C3)$.

  (a) If $|A| \geq 3$, then $\delta(A) \geq 3(n-1)+1$ and

  (b) if $x \in A$ is a point (or plane respectively) and $\mathrm{Res}^A(x)$ contains a cycle of length at least $2(n+1)$, then $\delta_1(\mathrm{Res}^A(x)) \geq 2(n+1)$.

Denote furthermore by $\boldsymbol{\mathcal{C}_0^{\mathrm{fin}}}$ the subclass of all finite structures in $\mathcal{C}_0$.

While conditions $(C1)-(C3)$ ensure that the limit structure is a geometry of the desired type, we use condition $(C4)(a)$ to guarantee that certain structures are always strongly embedded and $(C4)(b)$ to provide that the residue of any point or plane in the limit structure is a generalized n-gon as constructed by Tent in [Ten00b].

Remark 3.3.2 We remark that the class $\mathcal{C}_0$ is not closed under substructures due to condition $(C3)$. To see that consider the tripartite graph A consisting of two points, one line l and two planes such that any two vertices of different sorts are incident. Then $A \in \mathcal{C}_0$. The substructure $A' := A \setminus \{l\}$ though violates condition $(C3)$ and hence is no element of the class. Nevertheless, this being the only obstacle for substructures, the intersection of two structures in $\mathcal{C}_0$ is still a structure in $\mathcal{C}_0$.

3.4 Around Submodularity

In contrast to the Hrushovski amalgamations known so far, the predimension function given above lacks a major property: it is not submodular.

Example 3.4.1 Consider the tripartite graph consisting of one point p, one line l and two planes e, e' and set $A := \{l, e, e'\}$ and $B := \{p, e, e'\}$. Note that $A \leq AB$. Nevertheless, we calculate

$$\delta(B/A) = 1 > 0 = \delta(B/A \cap B),$$

contradicting submodularity.

The problem arising around submodularity is that, as outlined before, we have to count flags positively to oppose induced edges and thus, more edges from B to A do not immediately imply a decrease of the δ-value of B over A, as a single edge could induce several flags. In this section, we want to characterize the situations in which submodularity fails and so establish a slightly weaker version of it, which allows us to execute the amalgamation nevertheless.

First, we see that the only strong extensions by a line are those which extend a flag.

Example 3.4.2 (Extension by one line) Assume that $X \in \mathcal{C}_0$ and $B = X \cup \{l\} \in \mathcal{C}_0$ arises from X by adding a single new line $l \in \mathcal{L}$ and arbitrarily many edges from l into X. First note that, if the residue of the new line $\operatorname{Res}(l)$ does not only consist of vertices of one type (all planes or all points), then it is connected by condition (C1). Note further that if there were at least two points and at least two planes in $\operatorname{Res}(l) \subseteq X$, by (C3) they must share a common line other then l in X. But then, the extension would not be valid, as there would be a point-line cycle of length $4 < 2n$ in B in the residue of each of the planes. Thus, we may assume that there are i-many planes for $i = 0, 1$ (or i-many points respectively) and k-many points (or planes respectively) in $\operatorname{Res}(l)$. This yields

$$\delta(l/\operatorname{Res}(l)) = (3(n-1)-1) - (2(n-1)-1)(k+i) + ik(n-1),$$

which is non-negative if and only if $k \leq 1$. Thus, the only one-line extensions with non-negative delta are extensions of a partial flag (including the empty flag). As we use this a lot in the upcoming, we remark that

$$\begin{aligned} \delta(l/\operatorname{Res}(l)) &= 1 && \text{for } i = k = 1, \\ \delta(l/\operatorname{Res}(l)) &= n-1 && \text{for } i + k = 1 \text{ and} \\ \delta(l/\operatorname{Res}(l)) &= (n-1) - k(n-2) && \text{if } i = 1. \end{aligned}$$

Furthermore, whenever $X \in \mathcal{C}_0$ and B is an extension of X by a line extending a partial flag, then $B \in \mathcal{C}_0$.

Next we study strong one-point (respectively one-plane) extensions. As partial flags shall have infinite residues, we certainly want the extension of a partial flag by a point to be a strong extension. Furthermore, to obtain a structure of finite diameter in the limit, we want to ensure that any two planes meet in a common point (and vice versa). We see that these two types of extensions are indeed the only strong one-point extensions.

Lemma 3.4.3 *Let $y \in \mathcal{P} \cup \mathcal{E}$ be a point or plane and $X \in \mathcal{C}_0$. If $yX \in \mathcal{C}_0$ and $\delta(y/X) \geq 0$ then y is either connected to exactly one line in X and all the planes (respectively points) contained in that line, or it is connected to at most two planes (respectively points) and no line.*

Proof Without loss of generalization we may assume y to be a point. Consider some finite subset $A \subseteq X$ with $\emptyset \neq A \subseteq \operatorname{Res}(y)$. First note that

$$\delta(y/A) = 2(n-1) - (2(n-1)-1)|\mathcal{L}_A| - (n-1)(|\mathcal{E}_A| - |\mathcal{K}_A|). \tag{3.1}$$

Clearly, if there is no line in A, then $\delta(y/A) \geq 0$ if and only if $|\mathcal{E}_A| \leq 2$. Now assume that A contains at least one line. We want to show that there cannot be any further lines in A, if $\delta(y/A) \geq 0$.
By Remark 3.2.2 and condition $(C4)(a)$, we know

$$\delta_1(A) := (n-1)(|\mathcal{L}_A| + |\mathcal{E}_A|) - (n-2)|\mathcal{K}_A| \geq n-1. \tag{3.2}$$

If $\delta(y/A) \geq 0$, then by Equation (3.1), we have

$$n - 1 \geq (2(n-1)-1)|\mathcal{L}_A| + (n-1)|\mathcal{E}_A| - (n-1)|\mathcal{K}_A| - (n-1). \tag{3.3}$$

Multiplying (3.2) by $(n-1)$ and (3.3) by $(n-2)$, we get

$$\begin{aligned}(n-1)^2(|\mathcal{L}_A|+|\mathcal{E}_A|) &\geq (n-2)(2n-3)|\mathcal{L}_A|+(n-2)(n-1)|\mathcal{E}_A|-(n-1)(n-2)\\ \Rightarrow (n-1)|\mathcal{E}_A| &\geq (n^2-5n+5)|\mathcal{L}_A|-(n^2-3n+2).\end{aligned}$$

Equation (3.1) shows that we may consider all the planes in A to have degree at least 2, as otherwise we can take off those of less degree and still maintain the inequality. Thus we have $|\mathcal{K}_A| \geq 2|\mathcal{E}_A|$, and Equation (3.2) yields

$$\begin{aligned}(n-1)(|\mathcal{L}_A|+|\mathcal{E}_A|) &\geq (n-2)|\mathcal{K}_A|+(n-1)\\ &\geq 2(n-2)|\mathcal{E}_A|+(n-1)\\ \Rightarrow (n-1)|\mathcal{L}_A| &\geq (n-3)|\mathcal{E}_A|+(n-1).\end{aligned}$$

Putting the above pieces together, we get

$$\begin{aligned}(n-1)^2|\mathcal{L}_A| &\geq (n-3)(n-1)|\mathcal{E}_A|+(n-1)^2\\ &\geq (n-3)(n^2-5n+5)|\mathcal{L}_A|-(n-3)(n^2-3n+2)+(n-1)^2,\end{aligned}$$

which yields for $n \geq 6$ that

$$\frac{n^3-7n^2+13n-7}{n^3-9n^2+22n-16} \geq |\mathcal{L}_A|.$$

One can easily check that for $n \geq 6$, this is only possible if $|\mathcal{L}_A| \leq 4$. Thus, there are no cycles in A and moreover we may assume that A is a path consisting of $k+1$ lines, k planes and $2k$ edges. One calculates

$$\delta(y/A) = 2(n-1)-(2(n-1)-1)(k+1)-(n-1)(k-2k),$$

which is only non-negative if $k=0$ and there is exactly one line contained in A. Considering once more Equation 3.1, we see that if $\delta(y/A) \geq 0$ and A contains exactly one line, then any plane in A has to be connected to that line. ■

The following Lemma emphasizes the relation between the n-gons constructed by Tent [Ten00b] and the geometry presented here.

Lemma 3.4.4 *Assume $A \leq B$ and let x be a point or plane in A. Then* $\mathrm{Res}^A(x)$ *is δ_1-strong in* $\mathrm{Res}^B(x)$.

PROOF We may assume x to be a plane. Let $G \subseteq \mathrm{Res}^B(x)$ be a bipartite graph which contains $\mathrm{Res}^A(x)$ and set $\hat{G} := G \setminus \mathrm{Res}^A(x)$ and $\hat{A} := A \setminus \mathrm{Res}^A(x)$. Note that

$$\begin{aligned} 0 &\leq \delta(G/A) \\ &= \delta_1(G/G \cap A) - \underbrace{\Big((2(n-1)-1)|\mathcal{K}(\hat{G},\hat{A})| + (n-1)(|\mathcal{K}'(\hat{G},\hat{A})| - |\mathcal{F}(\hat{G},\hat{A})|\Big)}_{=:(*)}. \end{aligned}$$

We show that $(*) \geq 0$. Let $b \in \hat{G}$ be arbitrary and consider

$$(*)_b := (2(n-1)-1)|\mathcal{K}(b,\hat{A})| + (n-1)(|\mathcal{K}'(b,\hat{A})| - |\mathcal{F}(b,\hat{A})|.$$

If b is a line, then $(*)_b = 0$ by Example 3.4.2, as any vertex in A connected to b also has to be connected to x and thus is contained in $\mathrm{Res}^A(x)$, whence $(*)_b = 0$. If b is a point, then by Lemma 3.4.3 it is either connected to a unique plane different from x and $(*)_b = (n-1) > 0$ or again every edge between some vertex in A connected to b is in one to one correspondence with a complete flag, whence again $(*)_b = 0$. Thus, in particular $(*) \geq 0$ and

$$0 \leq \delta(G/A) \leq \delta_1(G/G \cap A) = \delta_1(G/A),$$

which proves the claim. ∎

Another important extension of $\mathcal{C}_0$-structures are path extensions within the residue of a point or plane. The next example gives a description of strong extensions of this kind.

Example 3.4.5 (Extension by a path in a residue) We show that strong path extensions within a residue have to be of length at least $n-1$ and any extension of length exactly $n-1$ is a ***pure path extension*** as defined below.
Consider a structure $A \in \mathcal{C}_0^{\mathrm{fin}}$. Let p be some point in A and x_0 and x_k be two vertices in the residue of p in A. Furthermore, let $B := \{x_1, x_2, \ldots, x_{k-1}\}$ consist of $k-1$ new vertices, such that $(x_0, x_1, \ldots, x_k)$ forms a path of length k in $\mathrm{Res}(p)$ with possibly more edges into A than the ones given within the path. Assume $AB \in \mathcal{C}_0$.
First we calculate that $\delta(B/p, x_0, x_k) = k-1-(n-2)$: note that $AB \in \mathcal{C}_0$ implies $\{p, x_0, x_1, \ldots, x_k\} \leq_1 A$. If now $A \leq AB$, we can apply Lemma 3.4.6 and obtain

$$0 \leq \delta(B/A) \leq \delta(B/p, x_0, x_k) = k-1-(n-2),$$

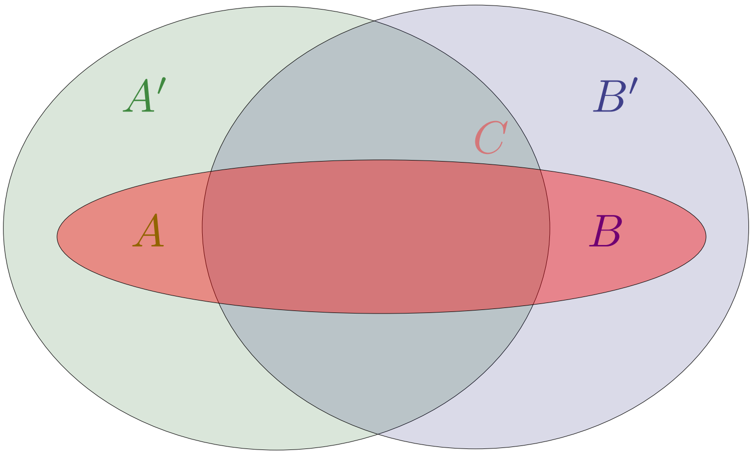
whence $k \geq n-1$. Furthermore, for $k = n-1$ the extension is a 0-extension and uniquely determined: there are no essential edges from some vertex of the path to $A \setminus \{p, x_0, x_k\}$. Thus, two strong path extensions of length $n-1$ from x_0 to x_{n-1} in $\mathrm{Res}(p)$ are isomorphic over A. We call this a ***pure path extension*** of length $n-1$.

With the knowledge about these important extensions, we are now ready to present a weaker form of submodularity which is satisfied in our setting and allows us to execute the amalgamation procedure.

Lemma 3.4.6 (Submodularity Lemma) *Consider two structures A' and B' in $\mathcal{C}_0^{\text{fin}}$ such that $A' \leq_1 A'B', B' \leq_3 A'B'$. Then, for any $C \subseteq A'B'$ with $A := C \cap A' \in \mathcal{C}_0^{\text{fin}}$ and $B := C \cap B' \in \mathcal{C}_0^{\text{fin}}$, we have*

$$\delta(AB) \leq \delta(A) + \delta(B) - \delta(A \cap B) - (n-2)K^{ess}(\hat{A}, \hat{B}),$$

where $K^{ess}(\hat{A}, \hat{B})$ denotes the number of essential edges connecting a vertex in $\hat{A} := A \setminus B$ to a vertex in $\hat{B} := B \setminus A$.



Proof Observe that

$$\begin{aligned}\delta(AB) &= \delta(A) + \delta(B) - \delta(A \cap B) \\ &\quad - \underbrace{(2(n-1)-1)K(\hat{A}, \hat{B}) - (n-1)(K'(\hat{A}, \hat{B}) - F(\hat{A}, \hat{B}))}_{=:(*)}.\end{aligned}$$

We want to show that $(*) \geq (n-2)K^{\text{ess}}(\hat{A}, \hat{B})$. Therefor, fix $b \in \hat{B}$ and consider the data $(*)_b$ in $(*)$ that involves b, i.e.

$$(*)_b := -(2(n-1)-1)K(b, \hat{A}) - (n-1)(K'(b, \hat{A}) - F(b, \hat{A})).$$

If there is no edge between b and some $a \in \hat{A}$, then $(*)_b = 0$. Hence we now always assume that there is an edge from b into $\hat{A}$.
First let b be a point. Consider the case that there is an edge from b to some line $l \in \hat{A}$. Note that there can exist at most one such edge and any plane in $\hat{A}$ which is connected to b also has to be connected to l, as $\delta(b/A') \geq 0$. Hence there is exactly one non-induced edge between $\hat{A}$ and $\hat{B}$ involving b. Now we count the flags that involve b.

F1 Consider the flags that involve b, l and some plane in $\hat{A}$. Then any of these flags is in one to one correspondence with an edge in $K'(b, \hat{A})$, whence it does not add anything to $(*)_b$.

F2 Consider the flags that involve b, l and some plane $e \in B$. As $B' \leq_3 A'B'$, we know that there can be at most one of these flags, as otherwise $\delta(l/B') < 0$. That flag adds at most $(n-1)$ to $(*)_b$.

F3 Consider flags that use b, some plane $e' \in \hat{A}$ (which has to be connected to l, as $\delta(b/A') \geq 0$) and some line $l' \in B$. Note that such a flag can only exist, if there is no flag in F2, as otherwise the path (e, l, e', l') would be a path extension of B' in the residue of b of length 3, contradicting $B' \leq_3 A'B'$. Furthermore, there can be at most one of these flags in F3, as any other such flag has to use a different plane $e'' \in \hat{A}$, which is again connected to l and as two different planes cannot intersect in two different lines, the second flag also has to use a different line $l'' \in B$. Then again, the path $(l'e'le''l'')$ is a path extension of B' in the residue of b of forbidden length.

This shows that $(n-1)(K'(b, \hat{A}) - F(b, \hat{A})) = -(n-1)(F2 + F3) \geq -(n-1)$, whence

$$(*)_b = (2(n-1) - 1) + (n-1)(K'(b, \hat{A}) - F(b, \hat{A})) \geq (n-2).$$

Now consider the case that there is an edge from b to some plane in $\hat{A}$ (and not to any line in $\hat{A}$). If this edge is part of a flag, then the flag is unique (containing that edge) and uses a line in B. In that case, the edge is a non-essential one and nothing is added to $(*)_b$. In addition to that there can be k edges between b and some planes in $\hat{A}$, which are not part of any flag, whence they correspond to essential edges between $\hat{A}$ and $\hat{B}$. Note that k can be arbitrarily large, as the edges between b and the planes in $\hat{A}$ might be induced through a line in $A' \setminus A$. Thus

$$(*)_b = (n-1)(K'(b, \hat{A}) - F(b, \hat{A})) = k(n-1) \geq k(n-2).$$

Clearly, the same argument is valid for b being a plane, only that we do not have to count those flags in $(F2)$ any more, which use point and plane in $\hat{B}$.
Now we consider the case that b is a line which is connected to a unique point $p \in \hat{A}$ (or plane respectively if there is no point). Note that any flag that involves p, b and some plane $e \in \hat{B}$ has already been counted in $(*)_e$. Thus, if there are no other edges from b to A, then there is exactly one non-induced edge between $\hat{A}$ and $\hat{B}$ involving b and the value added is $(2(n-1)-1) > (n-2)$. Otherwise, there is a plane $e \in A$ such that (p, b, e) is a complete flag. As $\delta(b/A') \geq 0$, there can be at most one such flag. Thus, the value added is $k(2(n-1)-1)-(n-1) \geq k(n-2)$, where $k=1$ if $e \in A \cap B$ (and there is one non-induced edge between $\hat{A}$ and $\hat{B}$ involving b) and $k=2$ if $e \in \hat{A}$ (and there are two non-induced edges between $\hat{A}$ and $\hat{B}$ involving b).
Finally, observe that the data above sums up to $(*)$. This concludes the proof. ∎

Remark 3.4.7 Note that in the conditions above it suffices to ask that $B' \leq_{\mathcal{L}} A'B'$ and there exists some $\tilde{B} \subseteq B'$ such that $B' \setminus A' \subseteq \tilde{B} \leq_1 \tilde{B} \cup (A' \setminus B')$, as the only time $B' \leq_1 A'B'$ is really needed, is in the very beginning of the proof, when we say that the point $a \in A \setminus B$ can be connected to at most one line in $B \setminus A$ and if it is connected to a such line l, then any plane in $B \setminus A$ which is connected to a, also is connected to l.

3.5 Dimension and Minimal Extensions

In the following section we want to study first properties of the dimension function d arising from δ and introduce specific minimal extensions which are used frequently afterwards.
For matter of reference we state the following properties of 0-extensions, which are direct consequences of the Submodularity Lemma.

Remark 3.5.1 Assume $A \subseteq B \subseteq C$ are $\mathcal{C}_0$-structures with $A \leq C$ and $\delta(B/A) = 0$. Then the following hold.

(i) If A is strong in C, then also AB is strong in C.

(ii) If $C' \leq C$ with $B \subseteq BC'$ and $B \cap C' = A$, then there is no non-induced edge between B and $C' \setminus A$. In particular the quantifier free type of B over C' is uniquely determined by its quantifier free type over A.

(iii) If B_1 and B_2 are two disjoint copies of B over A in C and $A \leq_{|AB|} C$, then there are no essential edges between B_1 and B_2. In particular $\delta(B_1/AB_2) = \delta(B_1/A)$.

Submodularity also yields that being strong is a transitive relation, preserved under intersections.

Lemma 3.5.2 *For all finite graphs A, B and C we have*

(1) If A and B are strong in C, then also their intersection $A \cap B$ is strong in C.

(2) If A is strong in B and B is strong in C, then A is also strong in C.

PROOF (1) Let A and B be strong in C. We first show that $A \cap B$ is strong in A. To this end, consider $A' \subseteq A$ such that $A \cap B \subseteq A'$. We may assume that $A' \leq_{\mathcal{L}} A$, whence $A' \leq_1 A'B$. As furthermore $B \leq A'B$, we can apply the Submodularity Lemma 3.4.6 and get

$$0 \leq \delta(A'/B) \leq \delta(A'/A' \cap B) = \delta(A'/A \cap B),$$

as desired. Clearly, the symmetric argument shows $A \cap B \leq B$.

Now, let $A \cap B \subseteq C' \subseteq C$ be a structure of minimal δ-value among those containing $A \cap B$. Note that $C' \in \mathcal{C}_0$ and strong in C. As furthermore $B \leq BC'$, the Submodularity Lemma yields that $0 \leq \delta(C'/B) \leq \delta(C'/B \cap C')$. Now, as we already have that $A \cap B$ is strong in B, we get

$$\delta(A \cap B) \leq \delta(C' \cap B) \leq \delta(C').$$

Thus we see that $\delta(C') = \delta(A \cap B)$, whence $A \cap B \leq C$.

(2) Let A' be the smallest $\mathcal{C}_0$-substructure of C that contains A and is strong in C. By (1), we know that A' is unique. Furthermore, again by (1) we know that $A' \cap B$ is strong in C and contains A, whence $A' = A' \cap B$ and thus $A' \subseteq B$. But $A \leq B$, whence $\delta(A') \geq \delta(A)$ and thus $A = A' \leq C$. ∎

We saw that the intersection of two closed sets is again closed. This yields that for any $X \in \mathcal{C}_0$ and any finite substructure $A \subseteq X$ there is a unique smallest closed set containing A. We call it the ***closure*** of A in X and denote it by $\mathbf{cl}^{\boldsymbol{X}}\boldsymbol{(A)}$. Furthermore, for finite A we call $\boldsymbol{d(A)}:= \delta(\mathrm{cl}^X(A))$ the ***dimension*** of A in X. If the context is clear, we omit the superscript X.
As usual, for an arbitrary subset $X \in \mathcal{C}_0$ and $A \in \mathcal{C}_0^{\mathrm{fin}}$ finite, we set

$$\boldsymbol{d(A/X)} \quad := \quad \min\{d(A/C) \mid C \subseteq X \text{ finite}\}.$$

We say that X is ***strong*** in Y (write $\boldsymbol{X \leq Y}$), if and only if $\delta(A/X) \geq 0$ for all $A \subset Y$ finite and we set $\mathbf{cl}(\boldsymbol{X}) := \cup\{\mathrm{cl}(C) \mid C \subseteq X \text{ finite}\}$.
The following lemma assures us, that these notions transfer smoothly from finite sets to infinite ones.

Lemma 3.5.3 *Let $X \subseteq Y$ be an arbitrary subset of $Y \in \mathcal{C}_0$ and $B \subset Y$ finite. Then, the following holds:*

(i) If $X = C$ is finite, then still $d(B/C) = \min\{d(B/C_0) \mid C_0 \subseteq C \text{ finite}\}$. In particular, for all finite D containing C we get that $d(B/D) \leq d(B/C)$.

(ii) The set X is strong in Y if and only if $\mathrm{cl}^Y(X) = X$.

Proof (i) Let $C_0 \subset C$ be an arbitrary finite subset of C and set $C_0' := \mathrm{cl}(C_0B) \cap \mathrm{cl}(C)$. Note that $\mathrm{cl}(C_0B) = \mathrm{cl}(C_0'B)$, whence

$$\begin{aligned} d(B/C_0') &= \delta(\mathrm{cl}(C_0'B)) - \delta(\mathrm{cl}(C_0')) \\ &= \delta(\mathrm{cl}(C_0B)) - \delta(\mathrm{cl}(C_0')) \\ &\leq \delta(\mathrm{cl}(C_0B)) - \delta(\mathrm{cl}(C_0)) \\ &= d(B/C_0). \end{aligned}$$

We also know that $\delta(\mathrm{cl}(CB)) \leq \delta(\mathrm{cl}(C_0'B) \cup \mathrm{cl}(C))$ whence submodularity for the sets $C_0' \leq \mathrm{cl}(C_0'B), \mathrm{cl}(C)$ yields

$$d(B/C) = \delta(\mathrm{cl}(CB)/\,\mathrm{cl}(C)) \leq \delta(\mathrm{cl}(C_0'B)/\,\mathrm{cl}(C_0')) = d(B/C_0') \leq d(B/C_0), \quad \blacksquare$$

as desired.

(ii) This is an implication of the following observation:

$$\begin{aligned} X \leq Y \quad &\Leftrightarrow \quad \delta(B/X) \geq 0 \text{ for all } B \subset Y \text{ finite} \\ &\Leftrightarrow \quad \delta(B/C) \geq 0 \text{ for all } B \subset Y, C \leq X \text{ finite} \\ &\Leftrightarrow \quad C \leq Y \text{ for all } C \leq X \text{ finite} \\ &\Leftrightarrow \quad X = \bigcup\{\mathrm{cl}(C) \mid C \subseteq X \text{ finite}\}. \end{aligned}$$

Next we want to introduce minimal extensions, which form the building blocks of arbitrary strong extensions.

Definition 3.5.4 (i) Let $A \leq B$ be a strong extension. We say that B is an ***i-minimal extension*** of A, if $\delta(B/A) = i$ and there is no proper subset $B' \subsetneq B$ with $A \subsetneq B'$ and $B' \leq B$. We say that B is a ***minimal extension*** of A, if it is an i-minimal extension for some i.

(ii) Assume that B is a 0-minimal extension of A. We call the pair (A, B) ***simple***, if B is not a 0-minimal extension of any proper subset of A. We then also may say that B is a ***simple extension*** of A.

Remark 3.5.5 Note that if the pair (A, B) is simple, then any $a \in A$ is connected to some $b \in B \setminus A$ by an essential edge.

On the way to understand minimal extensions, we first want to sort out few specific 0-minimal extensions, namely the only extensions which turn out to be in the definable closure of the base set in the limit structures we aim to obtain.

Definition 3.5.6 Let X be some structure in the class $\mathcal{C}_0$ and $X \leq Y$ with $Y \in \mathcal{C}_0$ an extension of X by either

(1) one point which is connected to exactly two planes in X;

(2) one plane which is connected to exactly two points in X or

(3) a pure path extension of length $n - 1$ in the residue of a point or plane in X as constructed in Example 3.4.5.

Then we call Y a ***rigid extension*** of X.

To motivate this notation, note that if Y is a rigid extension of X, then it is unique in the way that any structure in $\mathcal{C}_0$, which contains X strongly, contains at most one copy of Y over X. In fact, in both the structures we obtain in the following chapters of this thesis, the definable closure of any strong set X arises as a sequence of rigid extensions as defined above.

Next we want to construct specific simple extensions. They were first introduced in [GT14, Section 5] in order to study bounded automorphisms of the almost strongly minimal generalized n-gons constructed by Tent in [Ten00b]. Apart from the study of bounded automorphisms, we use these minimal extensions in order to determine the rank of the structure we aim to construct.

The following definition originates from [GT14, Definition 5.2] and was introduced for generalized n-gons. We adapt it slightly to fit our setting, where the same n-gons appear as residues of points or planes respectively.

Definition 3.5.7 Let x be some point or plane. We call a set $A_0 := \{x, s_0, \ldots, s_3\}$ a ***base configuration*** with respect to x, if the set $\{s_0, \ldots, s_3\}$ is contained in $\mathrm{Res}(x)$ and the following conditions are satisfied:

- The graph distance in $\mathrm{Res}(x)$ between s_i and s_{i+1} for $i < 3$, as well as the graph distance between s_0 and s_3 is n;
- The graph distance in $\mathrm{Res}(x)$ between s_i and s_{i+2} for $i = 0, 1$ is n or $n-1$, depending on whether n is even or odd.

Note that if n is even, then a base configuration with respect to some plane x consists of x together with either four points or four lines in $\mathrm{Res}(x)$ of pairwise distance n. If n is odd, then it contains x together with two points, say s_0 and s_2 and two lines s_1 and s_3 such that the distance of two vertices of different sorts is n and between vertices of the same sort it is $n-1$.

Next, we see that there are infinitely many 0-minimal extensions in $\mathcal{C}_0$ of any structure $A \in \mathcal{C}_0$ which contains a base configuration.

Fact 3.5.8 (Lemma 5.3 in [GT14]) *Let X be a structure in $\mathcal{C}_0$ containing a base configuration $A_0 = \{x, s_0, s_1, s_2, s_3\}$. Then for any $k \geq 2$ and any simple cycle $C_k = \{c_0, c_1, \ldots, c_{4k(n-2)} = c_0\}$ in the residue of x and of length $4k(n-2)$ with additional edges between s_i and $c_{(4l+i)(n-2)}$ for any $i \leq 3$ and $l < k$, the structure $\mathfrak{C}_k := X \cup C_k$ satisfies the following properties:*

(i) the structure $\mathfrak{C}_k$ is again in $\mathcal{C}_0$;

(ii) the pair $(A_0, \mathfrak{C}_k)$ is a simple pair and

(iii) the extension $X \leq \mathfrak{C}_k$ is 0-minimal, but not rigid.

The above fact is not hard to verify. Recall that the δ-value of a simple cycle of length $4k(n-2)$ in the residue of some point or plane x over x agrees with its δ_1-value, which is $4k(n-2)$. We furthermore added $4k$ new essential edges from the cycle into A_0 which

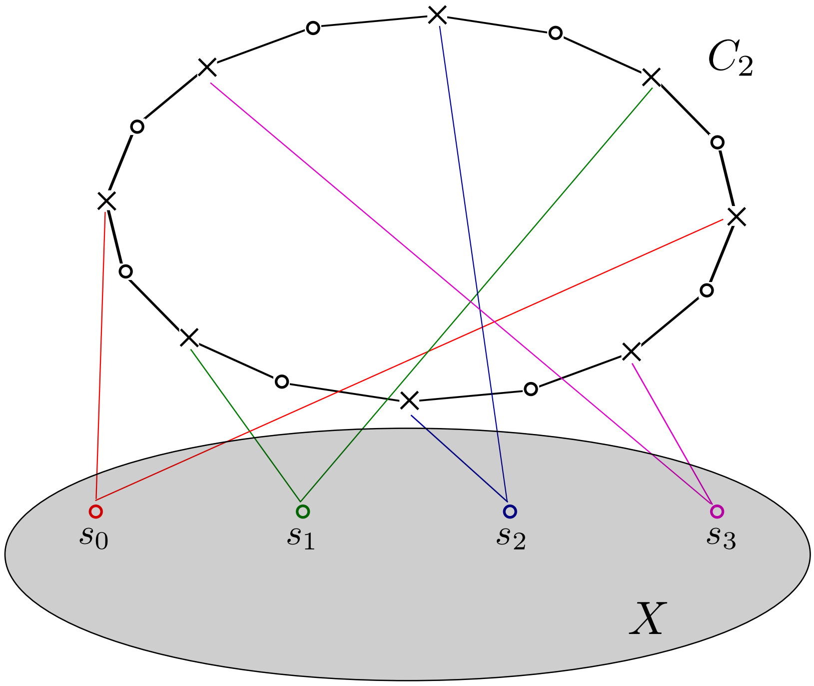


Figure 3.1: Minimal extension $\mathfrak{C}_k$ for $n = 4$ and $k = 2$

all produce a new flag, whence they add a value of $-4k(n-2)$ to the extension. Thus $X \leq \mathfrak{C}_k$ is indeed a 0-extension.
Another way to picture the extension is to think of it as a pure path extension between s_0 and s_1 of length n in the residue of x, followed by $4k-2$ many pure path extensions of length $n-1$ between different $c_{li(n-2)}$ and s_{i+1} in the residue of x and concluded by one pure path extension of length $n-2$ between $c_{4(k-1)(n-2)}$ and $c_{4k(n-2)} = c_0$. In this view, one can easily verify Fact 3.5.8. Confer also Figure 3.1.

As these extensions are used several times during the next chapters, we establish a notation for them. For an easy reference we fix the following definition:

Definition 3.5.9 Let X be some structure in $\mathcal{C}_0$ which contains a base configuration $A_0 := \{x, s_0, \ldots, s_3\}$ with respect to some point or plane $x \in X$. Fix some natural number $k \geq 2$. Then we denote by $\mathfrak{C}_k := X \dot\cup \{c_0, c_1, \ldots, c_{4k(n-2)} = c_0\}$ the extension of X by a simple cycle of length $4k(n-2)$ in the residue of x together with an additional edge between s_i and $c_{(4l+i)(n-2)}$ for any $i \leq 3$ and $l < k$. If we want to emphasize the base set A_0 over which the extension is simple, we may also write $\mathfrak{C}_k(A_0)$.

3.6 Amalgamation

In the following section we see that the class $\mathcal{C}_0$ has the amalgamation property with respect to strong embeddings. To this end, let us first define an appropriate notion of a free amalgam.

Definition 3.6.1 Let C_0 be a common substructure of C_1 and C_2. We define the ***free amalgam*** of C_1 and C_2 over C_0 (write $\boldsymbol{C_1 \otimes_{C_0} C_2}$) as the structure consisting of the disjoint union of C_1 and C_2 over C_0 and all relations given within C_1, C_2 and through condition $(C1)$ from the Definition 3.3.1 of the class.

Lemma 3.6.2 *Assume $C_0 \subseteq C_1$ and $C_0 \leq_k C_2$ for tripartite graphs C_i and $k \geq 1$. Then $C_1 \leq_k C_1 \otimes_{C_0} C_2$. Moreover, $\delta(D/C_1) = \delta(D/C_0)$ for any $D \subseteq C_2$ finite.*

Proof Consider $D \subseteq C_1 \otimes_{C_0} C_2$ finite and non-empty with $C_1 \cap D = \emptyset$. Set $\hat{C}_1 := C_1 \setminus C_0$. Then

$$\begin{aligned}\delta(D/C_1) &= \delta(D/C_0) - (2(n-1)-1)|\mathcal{K}(D, \hat{C}_1)| \\ &\quad -(n-1)(|\mathcal{K}'(D, \hat{C}_1)| - |\mathcal{F}(D, \hat{C}_1)|).\end{aligned}$$

As by the definition of the free amalgam there are no essential edges between D and $\hat{C}_1$, we get $|\mathcal{K}(D, \hat{C}_1)| = 0$. Furthermore, any edge in $\mathcal{K}'(D, \hat{C}_1)$ has to be induced through a line in C_0 and thus yields a flag in $\mathcal{F}(D, \hat{C}_1)$. Note that this flag is unique, as otherwise there was a vertex in D connected to two lines in C_0, contradicting $C_0 \leq_k C_2$. Thus $|\mathcal{K}'(D, \hat{C}_1)| - |\mathcal{F}(D, \hat{C}_1)| = 0$ and $\delta(D/C_1) = \delta(D/C_0)$, which is non-negative, if $|D| \leq k$. ∎

Theorem 3.6.3 *The class $\mathcal{C}_0^{\text{fin}}$ has the amalgamation property with respect to strong embeddings. Moreover, if $C_0 \leq C_1$ is a minimal extension and $C_0 \leq_{n-2} C_2$, then either $C_1 \otimes_{C_0} C_2 \in \mathcal{C}_0^{\text{fin}}$ or $C_0 \leq C_1$ is a rigid extension and there is an isomorphic copy of C_1 over C_0 in C_2.*

Proof Consider C_0, C_1 and $C_2 \in \mathcal{C}_0^{\text{fin}}$ such that $C_0 \leq C_1, C_2$ and denote $\hat{C}_i := C_i \setminus C_0$ for $i = 1, 2$. We may assume that the extension $C_0 \leq C_1$ is minimal, as otherwise we can decompose $C_0 \leq C_1$ into a finite chain of minimal extensions and amalgamate successively. Set $D := C_1 \otimes_{C_0} C_2$. If $D \in \mathcal{C}_0^{\text{fin}}$, then Lemma 3.6.2 yields that D already is the desired amalgam.

Now we consider the case $D \notin \mathcal{C}_0^{\text{fin}}$, i.e. at least one of the conditions (C1)-(C4) is violated. We show for each of the conditions that if it fails, then $\delta(C_1/C_0) = 0$ with $C_0 \leq C_1$ being a rigid extension and there is a strong isomorphic copy of C_1 over C_0 in C_2.

Case (C1): This condition is guaranteed to be true by definition of D.

Case (C2): Assume that there is a point-line cycle $(x_0, x_1, \ldots, x_{k-1}, x_k = x_0)$ of length less than $2n$ in the residue of some plane e in D. The cycle necessarily meets both $\hat{C}_1$ and $\hat{C}_2$. This immediately yields that $e \in C_0$, as every vertex of the cycle is connected to e and if e was for example in $\hat{C}_2$, there could be at most one vertex x_i contained in $\hat{C}_1$, which had to be a point, as there are no non-induced edges between $\hat{C}_1$ and $\hat{C}_2$, and which furthermore had to be connected to two different lines x_{i-1} and x_{i+1} in C_0, contradicting that the extension $C_0 \leq C_1$ is strong.
Hence, there exist path extensions of C_0 in both C_1 and C_2 in the residue of e, which are each of length at least $n-1$ by Example 3.4.5. As we assumed the length of the whole cycle to be less than $2n$, it follows that both path extensions are of length exactly $n-1$. Furthermore, they are 0-extensions, whence minimality of $C_0 \leq C_1$ implies that C_1 is a pure path extension of length $n-1$ of C_0, which is isomorphic to the according path in C_2 over C_0 by Example 3.4.5. Note that $C_0 \leq C_1$ thus is a rigid extension. Hence we can amalgamate those two structures over C_0 by identifying the amalgam with C_2 and embedding C_1 into C_2 over C_0. As $\delta(C_1/C_0) = 0$, we furthermore get that the embedding is strong.

Case (C3): Assume there are two points p_1 and p_2 in D that intersect. We check condition (C3) through a case distinction.
Case (1): First we assume that $p_i \in \hat{C}_i$ for $i = 1, 2$.

(1.1) Assume the p_i intersect in a line $l \in D$. As there are no essential edges between $\hat{C}_1$ and $\hat{C}_2$, the line has to be contained in C_0. Furthermore, because the extensions $C_0 < C_i$ are strong, the points cannot intersect in any other line nor in a plane which does not contain l, by Lemma 3.4.3. Thus condition (C3) is valid in the amalgam.

(1.2) Assume the p_i intersect in two different planes e_1 and e_2 and no line.

First consider the case where both $e_i \in C_0$. If $\mathrm{Res}^{C_0}(p_i)$ is exactly $\{e_1, e_2\}$ for both p_i, then $\delta(p_i/C_0) = 0$ and as $C_0 < C_1$ is a minimal extension, we know that C_1 consists exactly of $C_0 p_1$ and thus can be strongly embedded over C_0 into C_2 by sending p_1 to p_2. Furthermore, in this case $C_0 \leq C_1$ is a rigid extension. If one of the residues contains also a line l, then $l \in C_0$ has to be connected to the planes. But then, considering the other point, say p_2, we notice that in C_2 the planes e_1 and e_2 both intersect in l and in p_2, whence by condition (C3) the point p_2 also has to be contained in l, contradicting the assumptions.

Now let us assume that both e_i are in C_1, but not both in C_0, say $e_1 \in \hat{C}_1$. By definition of the amalgam, the edge between p_2 and e_1 has to be induced through some line $l \in C_0$. As the residue of p_2 in C_1 has to be connected, we conclude that also e_2 has to contain l, and thus in C_1 the planes e_1 and e_2 contain the line l and the point p_1, whence also p_1 has to be connected to l, which contradicts the assumptions.

Finally, assume the $e_i \in \hat{C}_i$. As all edges between $\hat{C}_1$ and $\hat{C}_2$ have to be induced, there exist lines l_1 and l_2 in C_0 such that (p_1, l_1, e_2) and (p_2, l_2, e_1) are complete flags. By the study of one-point extensions, we know that $\delta(p_1/C_2) = 1$ and $\delta(e_1/p_1 C_2) = -(n-2)$ if $l_1 \neq l_2$ and $\delta(e_1/p_1 C_2) = 1$, if $l_1 = l_2$. Because

$$0 \leq \delta(e_1 p_1/C_2) = \delta(e_1/p_1 C_2) + \delta(p_1/C_2),$$

the two lines have to coincide, whence once again p_1 and p_2 would intersect in one line, contradicting the assumptions.

Case (2): Now we consider the case that both points are in the same C_i, say in C_1.

(2.1) Assume the p_i intersect in a line l_1, which necessarily also lies in C_1. If they additionally intersect in a plane e that is not connected to l_1, then $e \in \hat{C}_2$ as condition $(C3)$ holds in C_1, and one of the points, say p_1, has to be contained in $\hat{C}_1$. As there are no non-induced edges between $\hat{C}_1$ and $\hat{C}_2$, there exists a line $l_2 \in C_0$ such that (p_1, l_2, e) is a complete flag. As above, the residue of e in C_1 has to be connected, whence l also contains p_2. But then, the points intersect in C_1

both in l_1 and l_2, whence $l_1 = l_2$ and the line contains the plane e, contradicting the assumptions.

(2.2) Assume the two points intersect in two different planes e_1 and e_2 and no line. Then either $e_i \in \hat{C}_i$ for $i = 1, 2$ and we are in the symmetric case of Case (1.2), or both planes lay in C_2. In that case, one of the points, say p_1, has to be contained in $\hat{C}_1$ and one of the planes, say e_2, in $\hat{C}_2$. As there are no non-induced edges between $\hat{C}_1$ and $\hat{C}_2$, there exists a line $l \in C_0$ such that (p_1, l, e_2) is a complete flag. As the residue of e_2 in C_1 is connected, also the point p_2 has to be contained in l, whence p_1 and p_2 intersect in l, contradicting the assumptions.

That proves that whenever condition (C3) is violated in D, then $C_0 \leq C_1$ is a rigid extension and there exists a strong isomorphic copy of C_1 over C_0 in C_2.

Case (C4)(a): Consider $D' \subseteq D$ with $|D'| \geq 3$. By possibly switching to its closure, we may assume that $D' \leq D$. By Lemma 3.6.2 we get $C_2 \leq D$ and $C_1 \leq_{n-2} D$. Assume $|D' \cap C_i| \geq 3$ for some $i = 1, 2$. If $|D' \cap C_2| \geq 3$, then by Lemma 3.5.2, also $D' \cap C_2 \leq D$ and $\delta(D') \geq \delta(D' \cap C_2) \geq 3(n-1) + 1$, as desired. Otherwise $|D' \cap C_2| \leq 2 < n - 2$, whence $C_1 \leq C_1 D'$ and we can repeat the argument.
Now assume $|D' \cap C_i| \leq 2$ for $i = 1, 2$, whence $|D'| \leq 4$. Note that any graph A on two vertices satisfies $\delta(A) \geq 3(n-1)$. One of the C_i, say C_1 contains two vertices. Now, an easy calculation shows that either $\delta(D'/D' \cap C_1) = 0$ and D' consists of two points (or planes respectively) in C_1, whence $\delta(D') = 4(n-1)$, or $\delta(D'/D \cap C_1) > 0$, whence

$$\delta(D') = \delta(D'/D' \cap C_1) + \delta(D' \cap C_1) \geq 1 + \delta(D' \cap C_1) \geq 3(n-1) + 1. \qquad \blacksquare$$

Case (C4)(b): Let $x \in D$ be a point or plane in D and $B \subsetneq \mathrm{Res}(x)$ containing a cycle of length at least $2(n+1)$. Denote by $B_i := B \cap C_i$. By Lemma 3.4.4, we know that B_0 is δ_1-strong in B_i. Note furthermore that $B = B_1 \otimes_{B_0} B_2$, whence condition $(C4)(b)$ follows from the corresponding result in [Ten00b].

Theorem 3.6.3 implies the existence of a $\mathcal{C}_0^{\text{fin}}$-saturated structure $\mathcal{M}_0$, the ***ab-initio structure*** corresponding to our given predimension and class. In the following chapter we study the properties of the ab-initio, independently from the construction of the collaps.
We want to conclude this section with a final remark on rigid structures and algebraic closure in $\mathcal{M}_0$.

Remark 3.6.4 Theorem 3.6.3 states that for any structures C_0, C_1 and C_2 in $\mathcal{C}_0^{\text{fin}}$, where $C_0 \leq C_2$ is a minimal extension and $C_0 \leq C_1$, we have that either $C_1 \otimes_{C_0} C_2$ is again a $\mathcal{C}_0^{\text{fin}}$-structure, or C_2 is a rigid extension of C_0. As the limit structure $\mathcal{M}_0$ is $\mathcal{C}_0^{\text{fin}}$-saturated, this implies that there are infinitely many copies of C_2 over C_0 for any minimal extension C_2 which is not rigid. Thus, for $A \leq \mathcal{M}_0$ and some tuple b, the following three conditions are equivalent:

(i) $b \in \text{acl}(A)$;

(ii) any decomposition of the strong extension $A \leq \text{cl}(Ab)$ into minimal extensions consists of a sequence of rigid extensions;

(iii) $b \in \text{dcl}(A)$.

CHAPTER 4

THE AB-INITIO STRUCTURE

Before collapsing the structure in the sense of Hrushovski, we want to study the properties of the geometry $\mathcal{M}_0$. These results will not be used in the following chapters and can thus be read independently from them. As a consequence, some of the proofs occurring below are very similar to their counterpart for the strongly minimal case. Several of the results obtained follow the very nice exposition by Martin Ziegler in [Zie13] of the original ab-initio structure by Hrushovski. A main difference, which needs a new approach, arises in the calculation of the rank of the theory.

4.1 Axiomatization

As it is standard in strong Fraïssé limits, we first note that the type of a finite tuple depends only on its strong closure.

Lemma 4.1.1 *Let M_1 and M_2 be $\mathcal{C}_0^{\text{fin}}$-saturated structures and $a_i \in M_i$ isomorphic finite tuples. Then a_1 and a_2 have the same type if and only if the isomorphism $a_1 \mapsto a_2$ extends to an automorphism between their closures $\mathrm{cl}^{M_1}(a_1) \mapsto \mathrm{cl}^{M_2}(a_2)$.*

Proof If a_1 and a_2 have the same type, they in particular have the same dimension in M_1 and M_2 respectively. As the closure of a_i can be characterized as the smallest A_i containing a_i and such that $\delta(A_i) = d(a_i)$, clearly the closure is determined by the type and a_1 and a_2 have isomorphic closures.

Now, assume that there is an isomorphism from $\mathrm{cl}^{M_1}(a_1)$ to $\mathrm{cl}^{M_2}(a_2)$ extending the map $a_1 \mapsto a_2$. We may assume that the M_i are ω-saturated. As furthermore both M_i are $\mathcal{C}_0^{\mathrm{fin}}$-saturated, the structures $(M_1, \mathrm{cl}(a_1))$ and $(M_2, \mathrm{cl}(a_2))$ and thus also (M_1, a_1) and (M_2, a_2) are partially isomorphic. Thus, Lemma 2.1.3 implies that a_1 and a_2 have the same type. ■

Below, we give an axiomatization of the theory $\mathrm{T}_0 := \mathrm{Th}(\mathcal{M}_0)$, based on the axiomatization in [MT17] introduced by Tent.

Proposition 4.1.2 *Let the theory T_0 consist of the axioms describing the following class of models M:*

(A1) Any model M is in $\mathcal{C}_0$.

(A2) Whenever $A \leq_k M$ for some $k \geq n-2$ and $A \leq B$ is a minimal strong extension of finite sets, then there exists some copy B' of B over A such that $B' \leq_{k-(n-2)} M$.

Then $\mathrm{T}_0 = \mathrm{Th}(\mathcal{M}_0)$. Furthermore, a model M of T_0 is ω-saturated if and only if it is $\mathcal{C}_0^{\mathrm{fin}}$-saturated.

Proof It is easy to see that the above is an infinite list of first order properties.
We first show that $\mathcal{M}_0 \models \mathrm{T}_0$. Clearly, we have that $\mathcal{M}_0$ is a $\mathcal{C}_0$-structure. Now consider some finite $A \leq_k \mathcal{M}_0$ with $k \geq n-2$. Then in particular, we have $A \leq_k \mathrm{cl}(A) \leq \mathcal{M}_0$. If $\mathrm{cl}(A) \otimes_A B$ is in $\mathcal{C}_0$, we can embed B strongly over $\mathrm{cl}(A)$ into $\mathcal{M}_0$. By Lemma 3.6.2, we further get that $B \leq_k \mathcal{M}_0$. If $\mathrm{cl}(A) \otimes_A B$ is not in $\mathcal{C}_0$, then by Theorem 3.6.3 there is a copy B' of B over A within $\mathrm{cl}(A)$ and B is either the extension of A by one point (respectively plane) which is connected to exactly two planes (respectively points) in A, or it is a pure path extension of A in the residue of some point or plane in A of length $n-1$. In both cases we have that $\delta(B'/A) = 0$ and $|B' \setminus A| \leq n-2$, whence for any $C \subseteq \mathcal{M}_0$ of size at most $k-(n-2)$, we get

$$\delta(C/B') = \delta(CB'/A) - \delta(B'/A) = \delta(CB'/A) \geq 0,$$

as $|CB' \setminus A| \leq k$ and $A \leq_k \mathcal{M}_0$. This proves that $B' \leq_{k-(n-2)} \mathcal{M}_0$, whence $\mathcal{M}_0$ is a model of T_0.

Now let $M \models \mathrm{T}_0$ be an arbitrary ω-saturated model of T_0. We show that M is $\mathcal{C}_0^{\mathrm{fin}}$-saturated, whence M is partially isomorphic to $\mathcal{M}_0$ and thus $\mathrm{Th}(\mathcal{M}_0) = \mathrm{Th}(M)$ by Lemma 2.1.3.

As $M \in \mathcal{C}_0$ and the empty set is strong in any $\mathcal{C}_0$-structure, it suffices to show that for any $A \leq M$ finite and any strong extension $A \leq B$ with $B \in \mathcal{C}_0^{\text{fin}}$ we can embed B strongly in M over A. Inductively, we may assume that the extension $A \leq B$ is minimal. Now let $\pi(x)$ be the set of formulas over A saying that any realization $B' \models \pi(x)$ is isomorphic to B over A and k-strong in M for any $k \in \mathbb{N}$. By the family of axioms $(A2)$, this is a type over A. As M is ω-saturated, there exists some realization $B' \models \pi(x)$ in M, which yields the desired strong embedding of B over A in M.
Thus M is $\mathcal{C}_0$-saturated and $\mathrm{T}_0 = \mathrm{Th}(\mathcal{M}_0)$. Furthermore, as any ω-saturated model of T_0 is partially isomorphic to $\mathcal{M}_0$, the ab-initio structure $\mathcal{M}_0$ is also ω-saturated. ∎

We gave a description of the theory T_0. In the following we study forking within T_0 and calculate the rank of the theory.

4.2 Forking

In this section we want to describe the forking relation in T_0. For the rest of this section, we fix some big saturated model $\mathbb{M} \models T_0$ and without further notice assume that all sets live within $\mathbb{M}$.
First, we note the following auxiliary Lemma.

Lemma 4.2.1 *Let A, X and Y be arbitrary sets such that A is finite over X with $d(A/X) = d(A/XY)$. Then $d(A \cap Y/X) = 0$.*

Proof Recall that the dimension function is monotone in the sense that for any A, Y and X we have $d(A/X) \geq d(A/XY)$. This yields

$$\begin{aligned} d(A/X) &= d(A/X(A \cap Y)) + d(A \cap Y/X) \\ &\geq d(A/XY) + d(A \cap Y/X) \\ &= d(A/X) + d(A \cap Y/X), \end{aligned}$$

whence $d(A \cap Y/X) = 0$ as desired. ∎

We now want to describe forking. Note that a pure description of forking by an independence witnessed through the dimension function is not possible, as by Remark 3.6.4, there are many non-algebraic types $\mathrm{tp}(B/A)$ with $d(B/A) = 0$. Nevertheless, we define a notion of independence in T_0 using the dimension function in a straight forward way and show that this notion indeed equals the non-forking independence.

Definition 4.2.2 For sets A, X and Y, where A is finite, we set $\boldsymbol{A} \mathop{\smile\!\!\!\!\!\!\,|}^{\boldsymbol{0}}_{\boldsymbol{X}} \boldsymbol{Y}$ if and only if $d(A/X) = d(A/XY)$ and $\mathrm{cl}(AX) \cap \mathrm{cl}(XY)$ is algebraic over X.

In other words, a finite set A is independent from Y over some set X, if it is d-independent and the extension $\mathrm{cl}(X) \leq \mathrm{cl}(AX) \cap \mathrm{cl}(XY)$ can be obtained by finitely many rigid extensions.
In order to show that the notion $\mathop{\smile\!\!\!\!\!\!\,|}^{0}$ coincides with non-forking independence, some prior observations are useful.

Remark 4.2.3 Let A be finite and X and Y arbitrary with $A \mathop{\smile\!\!\!\!\!\!\,|}_X Y$. Then by submodularity, we get

$$\begin{aligned} \delta(\mathrm{cl}(AX)/\,\mathrm{cl}(AX) \cap \mathrm{cl}(XY)) &\geq \delta(\mathrm{cl}(AX) \cup \mathrm{cl}(XY)/\,\mathrm{cl}(XY)) \\ &\geq \delta(\mathrm{cl}(AXY)/\,\mathrm{cl}(XY)) = d(A/XY) \\ &= d(A/X) = \delta(\mathrm{cl}(AX)/\,\mathrm{cl}(X)) \\ &\geq \delta(\mathrm{cl}(AX)/\,\mathrm{cl}(AX) \cap \mathrm{cl}(XY)). \end{aligned}$$

Thus, all inequalities are equalities and in particular $\mathrm{cl}(AX) \cup \mathrm{cl}(YX) \leq \mathbb{M}$.

Lemma 4.2.4 *Assume* $A \mathop{\smile\!\!\!\!\!\!\,|}^{0}_{X} Y$. *Then* $\mathrm{acl}(AX) \cap \mathrm{acl}(XY) \subseteq \mathrm{acl}(X)$.

Proof This follows by a study of minimal and rigid extensions. For an ease of notation, we may assume X, AX and XY to be closed. Let $b \in \mathrm{acl}(AX) \cap \mathrm{acl}(XY)$. Then in particular $d(b/AX) = d(b/XY) = 0$.
Clearly, if $b \in AX$, then there are no essential edges from $\mathrm{cl}(bX)$ to $Y \setminus X$, as otherwise $d(A/X) < d(A/XY)$. Thus, in that case $b \in \mathrm{acl}(X)$. Generally, note that Lemma 4.2.1 implies that $d(b/X) = 0$. In particular, there is a decomposition of $X \leq \mathrm{cl}(Xb)$ into a finite chain

$$X \leq XB_1 \leq \cdots \leq XB_k := \mathrm{cl}(Xb)$$

of 0-minimal extensions. Now, as $b \in \mathrm{acl}(AX)$, there also decomposition of $AX \leq \mathrm{cl}(AXb)$ into a finite chain

$$AX \leq AXD_1 \leq \cdots \leq AXD_l := \mathrm{cl}(AXb)$$

of rigid extensions. Set further $B_0 := D_0 := \emptyset$.

Claim: The sets $\{B_{i+1} \setminus B_i \mid i < k\}$ and $\{D_{i+1} \setminus D_i \mid i < l\}$ coincide.

Proof of the claim: Let i and j be minimal such that $B_i \cap D_j \neq \emptyset$. Such i and j have to exist, as $b \in B_k \cap D_l$. Then

$$AXB_{i-1}D_{j-1} \leq AXB_{i-1}D_j \text{ and } AXB_{i-1}D_{j-1} \leq AXB_iD_{j-1}$$

are both 0-minimal extensions of the same set and they intersect non-trivially, whence $B_i \setminus B_{i-1} = D_i \setminus D_{i-1}$. Now by submodularity, there are no essential edges from $B_i \setminus B_{i-1}$ to AD_{i-1}, as otherwise

$$\delta(B_i/AXD_{i-1}B_{i-1}) < \delta(B_i/XB_{i-1}) = 0.$$

Thus, we get that $X \leq X(D_i \setminus D_{i-1})$ is already a rigid extension and after resorting we may assume that $i = j = 1$. Inductively, we may now conclude that $\{B_{i+1} \setminus B_i \mid i < k\} = \{D_{i+1} \setminus D_i \mid i < l\}$, proving the claim.
The claim implies in particular that all the extensions $B_i \leq B_{i+1}$ are rigid, whence $b \in \operatorname{acl}(X)$. ■

Now, we want to show that the notion given in Definition 4.2.2 is indeed an independence relation in T_0 and hence has to coincide with the notion of non-forking independence, which we denote as usual by $\mathop{\smile\!\!\!\!\!\!|}\,$.

Lemma 4.2.5 *In T_0 the notion $\mathop{\smile\!\!\!\!\!\!|}^0$ satisfies all properties of stable forking as presented in [TZ12], i.e. it satisfies*

- *(Invariance) The relation $\mathop{\smile\!\!\!\!\!\!|}^0$ is invariant under automorphisms of $\mathbb{M}$;*
- *(Local Character) For all $A \subseteq \mathbb{M}$ finite and $X \subset \mathbb{M}$ arbitrary, there exists $C_0 \leq X$ finite such that $A \mathop{\smile\!\!\!\!\!\!|}^0_{C_0} X$;*
- *(Transitivity) If $A \mathop{\smile\!\!\!\!\!\!|}^0_X Y$ and $A \mathop{\smile\!\!\!\!\!\!|}^0_{XY} Z$ then $A \mathop{\smile\!\!\!\!\!\!|}^0_X YZ$;*
- *(Weak Monotonicity) If $A \mathop{\smile\!\!\!\!\!\!|}^0_X YZ$, then $A \mathop{\smile\!\!\!\!\!\!|}^0_X Y$.*
- *(Weak Boundedness) For all $A \subset \mathbb{M}$ finite and $X \subseteq Y \subset \mathbb{M}$ arbitrary, there are only finitely many isomorphism types of $A' \subseteq \mathbb{M}$ over Y with $A' \cong_X A$ and $A' \mathop{\smile\!\!\!\!\!\!|}^0_X Y$.*
- *(Existence) For any $A \subset \mathbb{M}$ finite and $X \subseteq Y \subseteq \mathbb{M}$ arbitrary, there is some A' such that $\operatorname{tp}(A/X) = \operatorname{tp}(A'/X)$ and $A' \mathop{\smile\!\!\!\!\!\!|}^0_X Y$.*

Proof We show the validity of the different properties of stable forking successively.

- (Invariance) As δ is clearly invariant under automorphisms, so is d. As furthermore closures are sent to closures, we get that $\mathop{\smile\!\!\!\!\!\mid}^0$ is invariant under automorphisms.

- (Local Character) For any $C \subseteq X$ finite, we know that $d(A/C)$ is a non-negative integer and thus can decrease only finitely many times. Therefore, we find some $C_0 \leq X$ with $d(A/C_0)$ being minimal and such that $\mathrm{cl}(AC_0) \cap \mathrm{cl}(X) = \mathrm{cl}(C_0)$. By Lemma 3.5.3.(i), the first property implies $d(A/C_0) = d(A/X)$ while the second property clearly yields $\mathrm{cl}(AC_0) \cap \mathrm{cl}(X) \subseteq \mathrm{acl}(C_0)$. Thus, we have $A \mathop{\smile\!\!\!\!\!\mid}^0_{C_0} X$.

- (Transitivity) The condition of transitivity on the dimension is clear, as the assumptions say that $d(A/X) = d(A/XY) = d(A/XYZ)$. Furthermore, Lemma 4.2.4 yields that

$$\mathrm{cl}(AX) \cap \mathrm{cl}(XYZ) \subseteq \mathrm{cl}(AX) \cap \mathrm{acl}(XY) \subseteq \mathrm{acl}(X),$$

  as desired.

- (Weak Monotonicity) Weak Monotonicity follows directly from the definition together with Lemma 3.5.3.(i), as

$$d(A/X) = d(A/XYZ) \leq d(A/XY) \leq d(A/X),$$

  whence $d(A/X) = d(A/XY)$ and the condition of the intersection of closures is immediate.

- (Weak Boundedness) Recall that there is some $C \subseteq X$ finite with $A \mathop{\smile\!\!\!\!\!\mid}^0_C X$ and such that for any $A' \equiv_X A$ we have $A' \mathop{\smile\!\!\!\!\!\mid}^0_X Y$ if and only if $A' \mathop{\smile\!\!\!\!\!\mid}^0_C Y$.

  Claim: The type $\mathrm{tp}(A/C)$ is $\mathop{\smile\!\!\!\!\!\mid}^0$-stationary.

  We show that for any finite, closed set D containing C, there is a unique $\mathop{\smile\!\!\!\!\!\mid}^0$-independent extension of $\mathrm{tp}(A/C)$ to D. Consider two realizations A_1 and A_2 of $\mathrm{tp}(A/C)$ independent from D. Note that A_1 and A_2 also have the same type over $\mathrm{acl}(C) = \mathrm{dcl}(C)$, whence we may assume C to be algebraically closed in A_iD, i.e. $\mathrm{acl}(C) \cap A_iD = C$. By Remark 4.2.3 we get $\mathrm{cl}(A_iD) = \mathrm{cl}(A_iC) \cup D$, whence by submodularity there are no non-induced edges between $\mathrm{cl}(A_iC) \setminus C$ and $D \setminus C$. Thus $\mathrm{tp}^{qf}(\mathrm{cl}(A_1C)/D) = \mathrm{tp}^{qf}(\mathrm{cl}(A_2C)/D)$ and as $\mathrm{cl}(A_iC) \cup D$ is closed by Remark 4.2.3, we conclude by Lemma 4.1.1 that $\mathrm{tp}(A_1/D) = \mathrm{tp}(A_2/D)$, as desired.

  This proves that all types are stationary, whence in particular the forking notion is weakly bounded.

- (Existence) Consider an arbitrary type $\mathrm{tp}(A/X)$ for $X \leq \mathbb{M}$ small and fix some $X \subseteq Y$. Let $C \leq X$ be finite with $A \mathop{\smile\!\!\!\!\!\mid}^0_C X$. Recall that as $\mathbb{M}$ is ω-saturated, it is $\mathcal{C}_0^{\mathrm{fin}}$-saturated. Now, for any $D \leq Y$ finite such that $C \leq D$ is a minimal extension, Remark 3.6.4 yields that either $A \otimes_C D \in \mathcal{C}_0^{\mathrm{fin}}$ and there is a strong embedding of A over D such that $A \mathop{\smile\!\!\!\!\!\mid}^0_C D$, or D is a rigid extension of C, whence for any set A' we have $A' \mathop{\smile\!\!\!\!\!\mid}^0_C D$. Thus, the set of formulas $\pi(\bar{x}) := \mathrm{tp}(A/C) \cup \{\bar{x} \mathop{\smile\!\!\!\!\!\mid}^0_C D \mid D \subset Y \text{ finite}\}$ is consistent. By saturation of $\mathbb{M}$, there exists some realization A' of $\pi(x)$. By stationarity and the fact that $A' \mathop{\smile\!\!\!\!\!\mid}^0_C X$, we get that $\mathrm{tp}(A'/X) = \mathrm{tp}(A/X)$. By construction we also have $A' \mathop{\smile\!\!\!\!\!\mid}_C Y$, whence $A' \mathop{\smile\!\!\!\!\!\mid}_X Y$. ∎

4.3 Ranks

In this section we want to calculate the rank of types and show that T_0 is ω-stable of rank $\omega \cdot (3(n-1)-1)$.

Lemma 4.3.1 *The theory T_0 is ω-stable.*

Proof We show that there are only countably many 1-types over any countable set. First note that there are at most countably many 1-types over any finite strong set C. This follows from the fact that there are only finitely many pairwise different quantifier free types over C and for any realization b of some quantifier free type over C, the type $\mathrm{tp}(b/C)$ is uniquely determined by $\mathrm{cl}(Cb)$. As this closure is finite and for each $k \in \mathbb{N}$ there are at most finitely many closures of size k, there are at most countably many choices for the closure of Cb, whence there are only countably many diffierent types over C.

Now consider again types over some countable subset $X \subseteq \mathbb{M}$. Note that we may assume X to be closed in $\mathbb{M}$. Then, for any type $p(x) \in S_1(X)$ and any realization $b \models p(x)$, there is some finite set $C \leq X$ such that $b \mathop{\smile\!\!\!\!\!\mid}^0_C X$. By stationarity, any two elements which have the same type over C and are independent from X over C, also have the same type over X. As there are only countably many finite subsets of X and each type is determined by its type over a finite subset, of which there are at most countably many, we see that there are at most countably many types over X. Hence, the theory T_0 is ω-stable. ∎

In order to calculate the explicit Morley rank of T_0, we first show that any 0-extension is of finite Morley rank and conclude that T_0 is of rank at least ω. This part follows the exposition on Hrushovski new strongly minimal set in [Zie13]. As $\mathcal{M}_0$ is ω-saturated, it suffices to work in $\mathcal{M}_0$ in order to calculate ranks.

Lemma 4.3.2 *Let $A \leq \mathcal{M}_0$ be a strong finite subset of $\mathcal{M}_0$ and $B \leq \mathcal{M}_0$ a 0-minimal extension of A, which is not rigid. Then $\mathrm{tp}(B/A)$ is isolated and strongly minimal.*

Proof We show that for any strong C extending A, there is a unique non-algebraic type over C which contains $\mathrm{tp}(B/A)$. Then, Lemma 2.1.6 implies that $\mathrm{tp}(B/A)$ is strongly minimal. So let C be arbitrary, finite and $p(x)$ some type over C containing $\mathrm{tp}(B/A)$. Let B' be an arbitrary realization of $p(x)$. As B is a minimal extension of A and $A \leq B' \cap C \leq \mathcal{M}_0$, we get that either $B' \subseteq C$ or $B' \cap C = A$. In the first case $p(x)$ is clearly an algebraic type. For the second case, note that as $\delta(B/A) = 0$ and $C \leq \mathcal{M}_0$, Lemma 3.4.6 yields that actually $B'C \cong B' \otimes_A C$ and $\delta(B'/C) = 0$, whence also $B'C \leq \mathcal{M}_0$. As the extension $C \leq B'C$ is not rigid, we can amalgamate arbitrarily many copies of B' over C in $\mathcal{M}_0$ and by Lemma 4.1.1, all copies of B' realise $p(x)$, whence $p(x)$ is the unique non-algebraic extension of $tp(B/A)$ to C. It clearly is isolated, as

$$B' \models p(x) \text{ if and only if } tp^{qf}(B'/A) = tp^{qf}(B/A) \wedge B'C \cong B' \otimes_A C,$$

and the right hand side is easily seen to be expressible by a first order formula. ■

Recall that strongly minimal types are of Morley rank and U-rank 1. In order to lift the above Lemma to arbitrary 0-extensions, we need that a decomposition of any 0-extension into minimal ones is always unique up to permutation.

Lemma 4.3.3 *Let $A \leq B$ be a 0-extension and $A := B_0 \leq B_1 \leq \cdots \leq B_k =: B$ and $A := B'_0 \leq B'_1 \leq \cdots \leq B'_l =: B$ be two decompositions of $A \leq B$ into minimal extensions. Denote by $\hat{B}_{i+1}$ the graph $B_{i+1} \setminus B_i$ and similarly for $\hat{B}'_{i+1}$. Then $k = l$ and there is a permutation σ on $\{0, \dots, k\}$ such that $\hat{B}_{\sigma(i)} = \hat{B}'_i$.*

Proof We proceed by induction on k. If $k = 1$, then $A \leq B$ is a minimal extension, whence clearly $l = k = 1$ and $\hat{B}_1 = B \setminus A = \hat{B}'_1$. Now assume we proved the statement for all $k' < k$.

As the extensions $A \leq B_1$ and $A \leq B'_1$ are minimal, we get that either $B_1 \cap B'_1 = A$ or $B_1 = B'_1$. In the second case, the induction hypothesis concludes the proof. In the first case, note that as $\delta(B_1/A) = \delta(B'_1/A) = 0$ and $A \leq B$, we get that B_1 is also a minimal

strong extension of B'_1, whence we can repeat the argument for the minimal extensions $B'_1 \leq B'_1B_1$ and $B'_1 \leq B'_2$.
As $B_1 \subseteq B := \cup_{i \leq l} B'_i$, the intersection cannot always be empty. Assume that for $i_0 \leq k$ we have that $B_1 = B'_{i_0}$. Then the sequences $B_1 \leq B_2 \leq \cdots \leq B_k = B$ and $B_1 \leq B'_1 \leq \cdots \leq B'_{i_0-1} \leq B'_{i_0+1} \leq \cdots \leq B_l = B$ are two decompositions of the 0-extension $B_1 \leq B$ into minimal extensions of shorter length than k, whence by induction hypothesis $l-1 = k-1$ and there is a bijection σ' between $\{1, 2, \ldots k\}$ and $\{0, 1, \ldots, i_0-1, i_0+1, \ldots, l\}$ with $\hat{B}_{\sigma(i)} = \hat{B}'_i$. Now $\sigma := \sigma' \cup \{(0, i_0)\}$ yields the desired permutation of $\{0, \ldots, k\}$. ■

Corollary 4.3.4 *Let A be strong in $\mathcal{M}_0$ and B be some 0-extension of A. Then the type $\mathrm{tp}(B/A)$ has finite Morley and U-rank which coincide. Moreover, if $A := B_0 \leq B_1 \leq B_2 \cdots \leq B_k = B$ is the decomposition of $A \leq B$ into minimal extensions, where k' of them are non-rigid, then $\mathrm{MR}(\mathrm{tp}(B/A)) = \mathrm{U}(\mathrm{tp}(B/A)) = k'$.*

Proof By Lemma 4.3.3, we know that both k and k' are independent from the choice of the decomposition. Furthermore, Lemma 2.1.14 and Lemma 4.3.2 together imply that

$$\mathrm{MR}(\mathrm{tp}(B/A)) = \sum_{i=1}^{k} \mathrm{MR}(\mathrm{tp}(B_i/B_{i-1})) = \sum_{i=1}^{k} \mathrm{U}(\mathrm{tp}(B_i/B_{i-1})) = \mathrm{U}(\mathrm{tp}(B/A)).$$

If $B_{i-1} \leq B_i$ is a rigid extension, the corresponding type is clearly of rank zero. By Lemma 4.3.2 we further get that any other type has Morley rank and U-rank one, which concludes the proof. ■

The above lemma implies that T_0 is of U-rank at least ω and the rank of a type does not equal its dimension. Nevertheless, we see that the above observation of extensions of dimension zero having arbitrarily large finite Morley rank is the only obstruction to this.
The following Lemma is of auxiliary nature and it yields that for any $X \subseteq Y$ and any finite B with $d(B/X) > d(B/Y)$, we also have $\mathrm{U}(B/X) > \mathrm{U}(B/Y)$.

Lemma 4.3.5 *Let $X \leq XB \leq \mathbb{M}$ be a strong extension with $d(B/X) > 0$ and such that there exists some vertex $b \in B \setminus X$ connected to some vertex $x \in X$. Then, for any $k \in \mathbb{N}$ there exists some set Y_k containing X with the following properties:*

- $Y_k \leq \mathbb{M}$;
- $Y_k \cap XB = X$;

- *$d(B/Y_k) = d(B/X) - 1$ and*
- *there is a decomposition $Y_k := C_0 \leq C_1 \cdots \leq C_k \leq \mathrm{cl}(BY_k)$, such that $C_i \leq C_{i+1}$ are 0-minimal, non-rigid extensions for all $i < k$.*

Proof We may assume that x is a plane or a point, as otherwise, if b was only connected to some line $x \in X$, we could add a point and plane to X in the residue of x independent from B over X and follow the proof with the new set arising, while now b was connected through some edge induced via x to a plane or point in X.
Recall the notion of a base configuration from Definition 3.5.7. Let the set $A_0 := \{x, s_0, s_1, s_2, s_3\}$ be a base configuration with respect to x which contains at least one vertex, say s_0 of the same sort as b. As we can embed A_0 strongly over x independently from b into $\mathbb{M}$, without changing the assumptions on X, we may as well assume that X already strongly contains a base configuration A_0 with $b \mathop{\smile\!\!\!\!\!\!|}^0_x A_0$.
Now, we successively construct 0-minimal extensions over X. By Fact 3.5.8, we can embed a simple extension $\mathfrak{C}_2 := \{c_0, c_1, \ldots, c_{8(n-2)}\}$ of A_0 as in Definition 3.5.9 over BX. Let C'_1 be the union of X and the embedding of $\mathfrak{C}_2$. Note that now the set $A_1 := \{x, s_0, s_1, c_i, c_j\}$ for $i = 2(n-2)+1$ and $j = 7(n-2)-1$ forms again a base configuration independent from b, whence we can again embed a 0-minimal extension of BC'_1 into $\mathbb{M}$, which is isomorphic to $\mathfrak{C}_2$ over A_1. Let C'_2 denote the union of C'_1 together with this new extension (see Figure 4.1).
Continuing likewise, we construct a chain $X := C'_0 \leq C'_1 \leq C'_2 \leq \cdots \leq C'_k$ such that each C'_{i+1} is 0-minimal over C'_i, and simple over some set $A_i \subseteq C'_i$, which is not contained in C'_j for $j < i$.
As before, we denote by A_k the base configuration in C'_k independent from b over x. Let C'_{k+1} be the 0-minimal extension of BC_k arising by embedding a copy of $\mathfrak{C}_2$ over A_k as in the steps before and then replacing the edge (s_0, c_0) by the edge (b, c_0). As $b \mathop{\smile\!\!\!\!\!\!|}^0_x A_k$, this is a valid extension in the class $\mathcal{C}_0$, whence such a C'_{k+1} exists. Now let y be any vertex in C_{k+1} connected to s_1 and set $Y_k := Xy$ and $C_i := C'_i y$ for $i \leq k$. We claim that these sets satisfies our properties. It is not hard to see that Y_k is strong in $\mathbb{M}$ and clearly $Y_k \cap XB = X$. Furthermore $d(y/BX) = \delta(C'_{k+1}/BX) = 0$ and $d(y/X) = 1$, whence

$$d(B/X) = d(By/X) = d(B/Y_k) + d(y/X) = d(B/Y_k) + 1.$$

Thus, we have $d(B/Y_k) = d(X) - 1$, as desired. By construction, all the C_i are contained in $\mathrm{cl}(BY_k)$ and provide k-many 0-minimal extensions. ∎

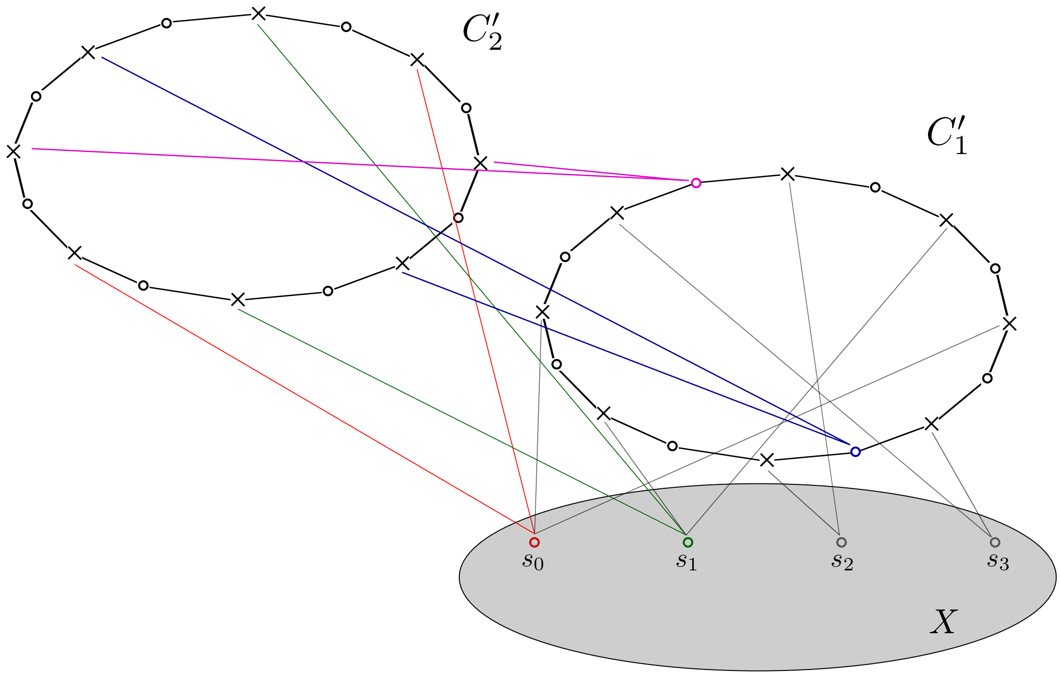


Figure 4.1: Minimal Extensions

In the following, we use these sets to inductively give a precise bound on the U-rank of types.

Proposition 4.3.6 *Let X be an arbitrary closed set in $\mathcal{M}_0$ and B a strong finite extension of X such that there is some edge between $B \setminus X$ and X. Furthermore, let k be maximal such that there exists a partition $X := B_0 \leq B_1 \leq \cdots \leq B_k \leq B_{k+1} \leq \cdots \leq B_K := BX$ into minimal extensions with $d(B_{i+1}/B_i) = 0$ for all $i \leq k$ and such that k' of them are non-rigid extensions. Then $\mathrm{U}(B/X) = \omega \cdot d(B/X) + k'$.*

Proof We prove the proposition by induction on $(d(B/X), k')$, ordered lexicographically. If $d(B/X) = 0$, the statement holds for arbitrary k' by Corollary 4.3.4. Now assume we have established the lemma for all (B', k) such that $d(B'/X) \leq m$ and $k' \in \mathbb{N}$ arbitrary. Consider B such that $d(B/X) = m+1$. We first show that for arbitrary l we get $\mathrm{U}(B/X) \geq \omega \cdot m + l$.
By Lemma 4.3.5, for any $l \in \mathbb{N}$ there exists some Y_l such that $d(B/Y_l) = m$ and the extension $Y_l \leq \mathrm{cl}(BY_l)$ can be decomposed into a chain of minimal extensions starting with l-many non-rigid 0-minimal extensions. Thus, by induction hypothesis we have

$\mathrm{U}(B/Y_l) \geq \omega{\cdot}m{+}l$. Furthermore, note that $B \not\perp_X Y_l$, as $d(B/Y_l) < d(B/X)$. This shows that $\mathrm{U}(B/X) \geq \omega{\cdot}m{+}l$ for all $l \in \mathbb{N}$, whence $\mathrm{U}(B/X) \geq \omega{\cdot}(m{+}1)$. Now assume that the extension $X \leq B$ can be decomposed as in the proposition. Then for each $i < k$ where $B_i \leq B_{i+1}$ is not a rigid extension, we have that $B \not\perp_{B_i} B_{i+1}$, as $\mathrm{cl}(B) \cap \mathrm{cl}(B_{i+1}) = B_{i+1}$ is not algebraic over B_i. Furthermore, we still have that $d(B/B_{i+1}) = m+1$, whence $\mathrm{U}(B/B_{i+1}) \geq \omega \cdot (m+1)$. This proves that $\mathrm{U}(B/X) \geq \omega \cdot (m+1) + k'$.
Now we want to establish the upper bound, using induction on k'. If

$$\mathrm{U}(B/X) \geq \omega \cdot (m+1) + k',$$

there exists some closed set Y containing X and such that $\mathrm{U}(B/Y) = \omega \cdot (m+1) + k'$. By induction hypothesis we know that $d(B/Y) = m+1$, whence $d(B \cap Y/X) = 0$. By Lemma 4.3.3, there is some decomposition

$$X := C_0 \leq \cdots \leq C_l = B \cap Y \leq C_{l+1} \cdots \leq C_k \leq C_{k+1} \leq \cdots \leq C_K := \mathrm{cl}(BY)$$

such that all the extensions $C_i \leq C_{i+1}$ for $i < l$ are 0-minimal and l' of them are not rigid and furthermore all the extensions $C_i \leq C_{i+1}$ for $l \leq i < k$ are 0-minimal and l'' of them are not rigid. Note further that $l' + l'' = k'$ and $B \perp_{B \cap Y} Y$. If $l' = 0$, then actually $B \cap Y \subseteq \mathrm{acl}(X)$ and $B \perp_X Y$, whence $\mathrm{U}(B/X) = \mathrm{U}(B/Y) = \omega \cdot (m+1) + k'$, as desired. Otherwise $0 \leq l'' < k'$ and Lascar inequalities together with the induction hypothesis yield that

$$\mathrm{U}(B/X) \leq \mathrm{U}(B/B \cap Y) \oplus \mathrm{U}(B \cap Y/X) = (\omega \cdot (m+1) + l'') \oplus l' = \omega \cdot (m+1) + k',$$

as desired. ■

Corollary 4.3.7 *Let $X \leq BX \leq \mathbb{M}$ be a finite extension of subsets of $\mathbb{M}$. Then $\mathrm{U}(B/X) = \omega \cdot d(B/X) + k'$, where k' is as in Proposition 4.3.6.*

Proof The only case left over is the case that there are no edges between B and X. Note that in this case k' equals zero. We may assume that X contains both a point and a plane. Note that for the U-ranks appearing as the rank of minimal extensions, the two notions of ordinal sums $+$ and $\oplus$ coincide, whence U is actually additive. Now, if B contains a point or plane b, then there is a vertex x of the same sort in X. As XB is strong, these two vertices intersect in a common plane resp. point c, which yields a rigid extension $XB \leq XBc$. Also, the maximal length of a chain of 0-minimal extensions in a decomposition of $X \leq Xc$ as well as of $Xc \leq XBc$ equals still zero. Furthermore, there is an edge between B and Xc as well as between c and X, whence by Proposition 4.3.6 we get

$$\begin{aligned} \mathrm{U}(B/X) &= \mathrm{U}(Bc/X) - \mathrm{U}(c/BX) \\ &= \mathrm{U}(Bc/X) = \mathrm{U}(B/cX) + \mathrm{U}(c/X) \\ &= \omega \cdot (d(B/cX) + d(c/X)) \\ &= \omega \cdot d(B/X), \end{aligned}$$

as desired.

If B only contains lines, we can consider an arbitrary plane c' in the residue of some line in B independent from X over B. Then

$$\begin{aligned} \mathrm{U}(B/X) &= \mathrm{U}(Bc'/X) - \mathrm{U}(c'/BX) \\ &= \omega \cdot (d(Bc'/X) - d(c'/BX)) \\ &= \omega \cdot d(B/X), \end{aligned}$$

as desired. ■

In the next Lemma, we use Corollary 4.3.7 in order to show that U-rank and Morley rank coincide.

Lemma 4.3.8 *Let X be an arbitrary closed set in $\mathbb{M}$ and B a finite strong extension of X. Then* $\mathrm{MR}(B/X) = \mathrm{U}(B/X)$.

Proof Let $X \leq B$ be given as in the Lemma. We proceed by induction over $\mathrm{U}(B/X)$, where the case $\mathrm{U}(B/X) < \omega$ is covered by Lemma 4.3.4. Now assume that for all X and B with $d(B/X) \leq m$ we have $\mathrm{MR}(B/X) = \mathrm{U}(B/X)$ and consider $X \leq B \leq \mathbb{M}$ with $d(B/X) = m+1$. Clearly, if $\mathrm{U}(B/X) = \omega \cdot (m+1)$ is a limit ordinal, it coincides with the Morley rank by induction hypothesis. Thus, assume now that $\mathrm{U}(B/X) = \omega \cdot (m+1) + k'$ for some $k' > 0$ and the two notions of rank coincide on all types of strictly smaller U-rank. We generally have that

$$\mathrm{MR}(B/X) \geq \mathrm{U}(B/X) = \omega \cdot d(B/X) + k'.$$

Consider some finite $C \subseteq X$ with $B \perp_C X$. If $\mathrm{MR}(B/X) > \mathrm{U}(B/X)$, then by the definition of Morley rank there exists some formula $\varphi(x, a)$ consistent with $\mathrm{tp}^{qf}(B/C)$ and such that $\mathrm{MR}(\mathrm{tp}^{qf}(B/C) \cup \{\varphi(x, a)\}) = \omega \cdot d(B/X) + k'$. By Lemma 2.1.16, we can extend this set to a complete type $q(x)$ over aC of Morley rank $\omega \cdot d(B/X) + k'$.

Now, pick an arbitrary realization B' of $q(x)$. If $B'C \not\leq \mathcal{M}_0$, then we would have $d(B'/C) < \delta(B'/C) = d(B/C)$, whence by induction hypothesis

$$\mathrm{MR}(q(x)) \leq \mathrm{MR}(B'/C) < \omega \cdot d(B/C),$$

a contradiction. Thus $B'C \leq \mathbb{M}$, whence in particular $B \equiv_C B'$ by Lemma 4.1.1. On the other hand, the rank conditions imply that $B' \not\downarrow_C a$, which provides $\mathrm{U}(B'/Ca) < \mathrm{U}(B'/C)$ and by induction hypothesis $\mathrm{MR}(B'/Ca) = \mathrm{U}(B'/Ca)$. Furthermore, by construction we have

$$\mathrm{MR}(B'/Ca) = \mathrm{MR}(q(x)) = \mathrm{U}(B/C) = \mathrm{U}(B'/C),$$

whence

$$\mathrm{MR}(B'/Ca) = \mathrm{U}(B'/Ca) < \mathrm{U}(B'/C) = \mathrm{MR}(B'/Ca),$$

a contradiction. ∎

Corollary 4.3.9 *The theory T_0 is ω-stable of Morley rank $\omega \cdot (3(n-1) - 1)$.*

Proof Let b be a line. Then b is closed, whence $\mathrm{MR}(b) = \omega \cdot (3(n-1) - 1)$ and $\mathrm{MR}(T_0) \geq \omega \cdot (3(n-1) - 1)$. On the other hand, let b and A be such that $\mathrm{MR}(b/A)$ is maximal. Then clearly b is a line and $d(b/A) = d(b) = 3(n-1) - 1$. Thus in particular Ab is strong, whence $A \leq Ab$ is the only decomposition of $\mathrm{cl}(Ab)$ into minimal extensions and $k = 0$. This concludes the proof. ∎

4.4 Ampleness

For the matter of completeness, we want to conclude the study of the ab-initio structure by observing its ampleness. As the proofs follow very much the same lines as in the collapsed case in Chapter 6.4, we do not give proofs here, but rather argue later, that the results for the collapsed structure transfer to the uncollapsed case. This way we obtain the following proposition.

Theorem 4.4.1 *The theory T_0 is an ω-stable theory of infinite Morley rank $\omega \cdot 3(n-1) - 1$, which is 2-ample, but not 3-ample. Any complete flag is a witness for 2-ampleness.*

CHAPTER 5

THE FINITE-RANK GEOMETRY

In the following chapter we collapse the structure $\mathcal{M}_0$ in order to obtain a geometry of the same type of finite rank. Afterwards, we study the properties of this geometry and see that forking and rank are given through the dimension function d and that the associated theory is almost strongly minimal and 2-ample, but not 3-ample.

5.1 The Collapse

As outlined in Section 2.2, we lower the rank of the structure $\mathcal{M}_0$ obtained in Chapter 3 by bounding the number of 0-minimal extensions that we allow during the amalgamation process.

Definition 5.1.1 We fix a ***μ-function*** from simple pairs (A, B) into the natural numbers satisfying the following three properties:

(1) The value $\mu(A, B)$ does only depend on the isomorphism type of (A, B);

(2) If $A \leq B$ is a rigid extension, then $\mu(A, B) = 1$;

(3) If $A \leq B$ is not a rigid extension, we have $\mu(A, B) \geq \delta(A)$.

Given a simple pair (A, B) and some structure $X \in \mathcal{C}_0$ with $A \subseteq X$, we denote by $\boldsymbol{\chi_X(A, B)}$ the maximal number of pairwise disjoint graphs B' in X, that are isomorphic to B over A. Note that $\chi_{\mathcal{M}_0}(A, B) \geq \mu(A, B)$ for all simple pairs (A, B) with $A \leq B$.

It is the goal of the collapse to construct a geometry $\mathcal{M}_\mu$ such that this inequality is an equality. In particular, there shall be only finitely many copy of 0-minimal extensions, and consequently of any extension of dimension 0.
Let now $\boldsymbol{\mathcal{C}_\mu}$ be the class of all the structures $X \in \mathcal{C}_0$ which satisfy $\chi_X(A, B) \leq \mu(A, B)$ for any simple pair (A, B) with $A \leq_{2|AB|} X$. Again we denote by $\boldsymbol{\mathcal{C}_\mu^{\mathbf{fin}}}$ the subclass of $\mathcal{C}_\mu$ consisting of all finite structures in $\mathcal{C}_\mu$.

Remark 5.1.2 In the original construction as explained in Section 2.2, the class $\mathcal{C}_\mu$ was defined using the slightly stronger condition that asks for *any* $A \subseteq X$ to satisfy $\chi_X(A, B) \leq \mu(A, B)$. For our setting, this is too restrictive. To see that, consider the structure $X = \{e_1, e_2, l, p_1, p_2\}$ consisting of two plains e_i, one line l and two points p_i, such that all points and planes are connected to the line l and thus to each other. Such a structure has to exists in our class $\mathcal{C}_\mu$, as residues of partial flags have to be infinite in the limit structure. Yet, the simple, rigid pair (A, B) with $A = \{e_1, e_2\}$ and $B := \{p\}$ such that p is connected to both the planes in A, violates the condition on the class, as $A \subseteq X$ and $\chi_X(A, B) = 2$ is larger than $\mu(A, B) = 1$.

Lemma 5.1.3 *Consider $C_0 \subseteq C_1, C_2$ with $C_i \in \mathcal{C}_\mu^{fin}$ such that C_1 is minimal over C_0 and $C_0 \leq_k C_2$ for $k = |C_1 \backslash C_0|$. Let (A, B) be a simple pair with $A \leq_{2|AB|} D := C_1 \otimes_{C_0} C_2$ and assume $\chi_D(A, B) > \mu(A, B)$. If either*

- *$A \subseteq C_2$ or*
- *$A \subseteq C_1$ and $|B| \leq |C_1|$,*

then $A \subseteq C_0$. Furthermore, the structure C_1 is a 0-minimal extension of C_0 and there exists a strong copy of C_1 over C_0 in C_2.

Proof We consider the case that $A \subseteq C_1$ and $|B| \leq |C_1|$. The proof for the case $A \subseteq C_2$ is essentially the same.
As $C_1 \in \mathcal{C}_\mu$, there is some copy B' of B over A, which intersects $\hat{C}_2$. By Lemma 3.6.2 we have $C_1 \leq_k D$ and as $|B' \setminus C_2| \leq k$, we have $C_1 \leq C_1 B'$. Furthermore, as $A \leq_{2|AB|} D$, also $AB_1' \leq D$ and the Submodularity Lemma yields $0 \leq \delta(B'/C_1) \leq \delta(B'/C_1 \cap AB')$. Thus

$$0 = \delta(B'/A) = \delta(B'/C_1 \cap AB') + \delta(B' \cap C_1/A),$$

which is only possible, if $B' \subseteq \hat{C}_2$, by the minimality of the extension $A \leq B$. Then $\delta(B'/C_1) = \delta(B'/A) = 0$ by Lemma 3.5.1(ii). As any vertex of A is connected to some vertex in B' by an essential edge in AB' and there are no essential edges between B' and $C_1 \setminus A$, we conclude that $A \subseteq C_0$ and $\delta(B/C_0) = \delta(B/A) = 0$. Now $A \subseteq C_0 \subseteq C_2$, so we can use a symmetric argument to obtain a copy B'' of B in $\hat{C}_1$, and minimality of the extension $C_0 \leq C_1$ gives $B'' = \hat{C}_1$. By Lemma 3.5.1(iii), the two copies B' and B'' are isomorphic over C_0, whence we can embed $C_1 = B'C_0$ into C_2 over C_0 and C_2 is a valid amalgam in $\mathcal{C}_\mu$. ■

Lemma 5.1.4 *Consider $C_0 \subseteq C_1, C_2$ with $C_i \in \mathcal{C}_\mu^{fin}$ such that C_1 is minimal over C_0 and $C_0 \leq_k C_2$ for $k = |C_1 \setminus C_0|$. Let (A, B) be a simple pair with $A \leq_{2|AB|} D := C_1 \otimes_{C_0} C_2$. If A is not fully contained in C_2 and $B_1, \ldots, B_l$ are disjoint copies of B in D which intersect $\hat{C}_1 := C_1 \setminus C_2$, then*

$$\delta\Big(\bigcup_{i=1}^{l} B_i/C_2A\Big) \leq -l.$$

Proof We first show that $\delta(B_i/C_2A) < 0$ for all i. Let $B_i' \subseteq B_i$ be the set consisting of $B_i \cap C_2$ together with all lines x in B_i that satisfy $\delta(x/C_2A) < 0$. Note that now

$$\begin{aligned} AB_i'C_2 &\leq_{\mathcal{L}} AB_iC_2, \\ (AB_i'C_2 \setminus AB_i) = C_2 \setminus AB_i &\subseteq C_2 \text{ with } C_2 \leq_1 C_2AB_i \text{ and} \\ AB_i &\leq_{|AB|} AB_iC_2, \end{aligned}$$

with $|AB| \geq 3$. Thus, by Remark 3.4.7 we can apply the Submodularity Lemma to $AB_i'C_2$ and AB_i, which yields

$$\delta(B_i/AC_2) \leq \delta(B_i/AB_i') - \mathcal{K}^{ess}(B_i \setminus AB_i', C_2 \setminus AB_i') \leq \delta(B_i/AB_i') \leq 0. \tag{5.1}$$

If B_i' is not empty, then $\delta(B_i/AB_i') < 0$ and we get the desired. If B_i' is empty, then $B_i \subseteq \hat{C}_1$, whence $|B| \leq k$. If A was contained in C_1, then Lemma 5.1.3 would imply that in fact $A \subseteq C_0$, which yields a contradiction. So A intersects both $\hat{C}_1$ and $\hat{C}_2$. By Remark 3.5.5, there is an edge between some $a \in A \cap \hat{C}_2$ and $b \in B_i \subseteq \hat{C}_1$, which is essential in AB_i. As there are no essential edges between $\hat{C}_1$ and $\hat{C}_2$, this edge has to be induced through some line in C_0, whence there is an essential edge between $B_i \setminus B_i'$ and $C_2 \setminus AB_i$ and the Inequality (5.1) is strict, proving the claim.
Lemma 3.5.1 now implies that for any $i \leq l$ we have

$$\delta(B_i/C_2AB_1 \ldots B_{i-1}) = \delta(B_i/AC_2)) < 0.$$

This yields

$$\begin{aligned}\delta(\bigcup_{i=1}^{l} B_i/C_2A) &= \delta(B_l/C_2AB_1\dots B_{l-1}) + \delta(\bigcup_{i=1}^{l-1} B_i/C_2A)\\ &\leq -1 + \delta(\bigcup_{i=1}^{l-1} B_i/C_2A) \leq \dots \leq -l,\end{aligned}$$

as desired. ■

Proposition 5.1.5 *The class $\mathcal{C}_\mu^{fin}$ has the amalgamation property with respect to strong embeddings. Moreover, if $C_0 \leq C_1$ is a minimal extension and $C_0 \leq_{|C_1 \setminus C_0|} C_2$, then either $C_1 \otimes_{C_0} C_2 \in \mathcal{C}_\mu$ or $\delta(C_1/C_0) = 0$ and there is an isomorphic copy of C_1 over C_0 in C_2.*

Proof Consider $C_0 \subseteq C_1, C_2 \in \mathcal{C}_\mu^{fin}$ such that $C_0 \leq C_1$ is a minimal extension and $C_0 \leq_{|C_1 \setminus C_0|} C_2$. Set $\hat{C}_1 := C_1 \setminus C_2$ and $\hat{C}_2 := C_2 \setminus C_1$. By Theorem 3.6.3, we may assume that $D := C_1 \otimes_{C_0} C_2 \in \mathcal{C}_0$. If $D \notin \mathcal{C}_\mu$, then there exists some simple pair (A, B) with $A \leq_{2|AB|} D$ and $\chi_D(A, B) > \mu(A, B)$. By Lemma 5.1.3, we may assume that A intersects $\hat{C}_1$. Let $B_1, \dots, B_k, B_{k+1}, \dots, B_{k+l}$ be disjoint copies of B over A in D, such that $B_i \subseteq C_2$ for $i \leq k$ and $B_i \cap \hat{C}_1 \neq \emptyset$ for $i = k+1, \dots, k+l$ and $k + l > \mu(A, B)$. Lemma 5.1.4 yields

$$\delta(\bigcup_{i=1}^{l} B_{k+i}/C_2A) \leq -l.$$

We now show that $k \leq 2$ and $\delta(A/C_2) \leq \delta(A) - k$.

Consider $a \in A \setminus C_2$. By Remark 3.5.5, there is an edge from a to each B_i for $i = 1, \dots, k$ which is non-induced in AB_i.

If the edges between a and the B_i for $i \leq k$ do become induced ones in AC_2, they have to be induced through a unique line $l \in C_2 \setminus AB_i$, as $\delta(a/C_2) \geq 0$. On the other hand, this line can induce only one edge from A to some B_j, as $\delta(l/AB_iB_j) \geq 0$. Thus, there is at most one copy of B in C_2.

If these edges do not become induced edges in AC_2, then there are at most two copies of B, as a is strong over C_2. Now submodularity yields for both cases that

$$\delta(A/C_2) \leq \delta(A/A \cap C_2) - k \leq \delta(A) - k,$$

as desired.

Finally, we have

$$0 \leq \delta(A \cup \bigcup_{i=k+1}^{k+l} B_i/C_2) \leq \delta(A/C_2) - l \leq \delta(A) - (k+l).$$

This implies $\chi_D(A, B) = k+l \leq \delta(A) \leq \mu(A, B)$, which yields the desired contradiction. ■

5.2 Geometrical Properties of $\mathcal{M}_\mu$

In the last section we saw that the class $\mathcal{C}_\mu^{fin}$ has the amalgamation property with respect to strong embeddings. Now, the results from Section 2.2 imply that there is a unique countable structure $\boldsymbol{\mathcal{M}_\mu}$, which is $\mathcal{C}_\mu^{fin}$-saturated. Denote by $\mathbf{T}_{\boldsymbol{\mu}} := \mathrm{Th}(\mathcal{M}_\mu)$ its theory. In this section we see that $\mathcal{M}_\mu$ is indeed, as planned, a geometry of type $\bullet \overset{n}{-} \bullet \overset{n}{-} \bullet$.

Lemma 5.2.1 *Any flag is strongly contained in $\mathcal{M}_\mu$ and every partial flag has an infinite residue.*

Proof Clearly, complete flags are strong in $\mathcal{M}_\mu$, due to condition $(C4)(a)$ of the class. Now consider a partial flag $\{xy\} \subseteq \mathcal{M}_\mu$ of rank 2.If xy was not strong in $\mathcal{M}_\mu$, then $\delta(\mathrm{cl}(\{xy\})) < 3(n-1)$. Consider the extension $\mathrm{cl}(\{xy\}) \leq F_{xy}^k(\mathrm{cl}(\{xy\})) := \mathrm{cl}(\{xy\}) \cup \{z_1, \ldots, z_k\}$, which arises from $\mathrm{cl}(\{xy\})$ by:

- adding k many new vertices $z_1, \ldots, z_k$, each of which completes the flag xy and
- adding edges from the z_i into $\mathrm{cl}(\{xy\})$, which are induced through xy by condition (C1) of the class.

This is a valid, strong extension of $\mathrm{cl}(\{xy\})$ in $\mathcal{C}_\mu$. As remarked, the completion $F_{xy}^1(\mathrm{cl}(\{xy\}))$ of xy to a flag xyz is a strong extension of $\mathrm{cl}(\{xy\})$, whence we can embed z strongly over $\mathrm{cl}(\{xy\})$ in $\mathcal{M}_\mu$. This yields $\delta(\mathrm{cl}(\{xy\})z) \leq 3(n-1)$, contradicting the condition $(C4)(a)$ of the class. Now it easily follows that also flags of rank 1, i.e. single vertices, are strong in $\mathcal{M}_\mu$.
Furthermore, for any partial flag $\{xy\}$ and any $k \in \mathbb{N}$ we can embed $F_{xy}^k(xy)$ strongly over $\{xy\}$, whence in $\mathcal{M}_\mu$ the residue of xy is infinite. Thus, all partial flags of rank 2 have infinite residue. Observe, that we can complete any vertex to a complete flag, whence flags of rank 1 also have infinite residues. ■

Lemma 5.2.2 *The structure $\mathcal{M}_\mu$ is an incidence geometry of type $\bullet \overset{n}{-} \bullet \overset{n}{-} \bullet$.*

PROOF We first show that the residue of each point (respectively plane) p in $\mathcal{M}_\mu$ is a generalized n-gon. Clearly, there is no ordinary m-gon in $\mathrm{Res}(p)$ for $m < 2n$, by condition (C2). Consider $x, y \in \mathrm{Res}(p)$ arbitrary. We have to show that there is an ordinary $2n$-gon in $\mathrm{Res}(p)$ containing x and y. Let $d \in \mathbb{N} \cup \infty$ be the distance of x and y in $\mathrm{cl}(\{pxy\}) \cap \mathrm{Res}(p)$. We show that, if x and y are not yet contained in an ordinary $2n$-gon in $\mathrm{cl}(\{pxy\})$, then a pure path extension of $\mathrm{cl}(\{pxy\})$ in $\mathrm{Res}(p)$ between x and y of length $k := \max\{2n - d, n\}$ (respectively $k := \max\{2n - d, n - 1\}$ according to the sort of x and y) is a valid extension of $\mathrm{cl}(\{pxy\})$ satisfying all conditions of $\mathcal{C}_\mu$. By Example 3.4.5, it is furthermore a strong extension, whence it can be strongly embedded over $\mathrm{cl}(\{pxy\})$ into $\mathcal{M}_\mu$. This yields (possibly after repeating the procedure once more) an ordinary $2n$-gon in $\mathrm{Res}(p)$ containing x and y.

Denote by $C_1 := \mathrm{cl}(\{pxy\}) \cup \{z_1, \ldots, z_{k-1}\}$ the extension of $\mathrm{cl}(\{pxy\})$ by a pure path in $\mathrm{Res}(p)$ between x and y of length k. Clearly, the structure C_1 satisfies conditions (C1) to (C3). Also, condition (C4) can be easily verified. It is left to show that C_1 satisfies the μ-condition. To this end, assume (A, B) to be a minimal pair with $A \leq C_1$ and at least one copy B' of B over A intersects $\hat{C}_1$.

First consider the case that $A \not\subseteq \mathrm{cl}(\{pxy\})$ and take $a \in A \setminus \mathrm{cl}(\{pxy\})$. As any vertex in A has an essential edge to some $b \in B$, we see that if $a \in \{z_2, \ldots, z_{k-2}\}$, there can be at most two disjoint copies of B over A, whence $\chi_{C_1}(A, B) \leq \mu(A, B)$, as desired. If $a = z_1$ (or $a = z_{k-1}$ respectively) is a plane, then there are edges to z_2 and those points in $\mathrm{cl}(\{pxy\})$, that are connected to the line x. If the non-induced edge from $a = z_1$ into B uses a line in B, there can be at most two disjoint copies of B over A. If it uses a point in $\mathrm{cl}(\{pxy\})$, the line x cannot be contained in AB, as we required the edges to be non-induced in AB. But then again, there can be at most one copy of B over A, because if there were two disjoint copies B_1 and B_2 using an edge from a to some point in $\mathrm{cl}(\{pxy\})$ induced through x, we would have $\delta(x/AB_1B_2) < 0$, contradicting the fact $AB_1B_2 \leq C_1$.

Hence we can assume that $A \subseteq \mathrm{cl}(\{pxy\})$. Now if $B' \not\subseteq \hat{C}_1 := C_1 \setminus \mathrm{cl}(\{pxy\})$, then

$$0 > \delta(B'/(AB') \cap \mathrm{cl}(\{pxy\})) \geq 0,$$

a contradiction. Thus $B' \subseteq \hat{C}_1$. As before we get $0 = \delta(B'/A) = \delta(B'/\mathrm{cl}(\{pxy\})) \geq 0$, which yields that B' is a pure path extension of length $n - 1$ and $A = \{p, x, y\}$. Thus, the path $\{z_1, \ldots, z_{k-1}\}$ is the only copy of B' over A in C_1. This concludes that C_1 satisfies all conditions of $\mathcal{C}_\mu$ and thus $\mathrm{Res}(p)$ is a generalized n-gon.

Clearly condition (C1) of the class ensures that the residue of any line is a generalized 2-gon. ■

In the beginning of this chapter, we outlined that the main obstacle for the known ample geometries to be of finite rank, is their infinite diameter. The following easy Lemma states that in $\mathcal{M}_\mu$, the diameter is finite. In particular, any two points (respectively planes) lay in a common residue.

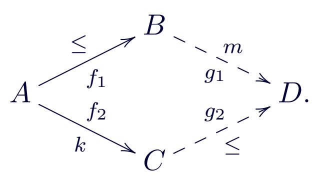
Lemma 5.2.3 *Any two points either intersect in a unique line (together with exactly all the planes that contain the line) or in a unique plane. The dual version holds for the intersection of two planes.*

PROOF If two points intersect in $\mathcal{M}_\mu$, then in the required way. This is assured by condition (C3) of our class. Furthermore, it is easy to check that if two points would not intersect, then the extension of their strong closure by one plane which is only connected to the two points is a valid strong extension in $\mathcal{C}_\mu$ and thus can be strongly embedded in $\mathcal{M}_\mu$. ■

5.3 Saturation

In this Section we see that the geometry $\mathcal{M}_\mu$ is ω-saturated. In [Her95] Herwig gives a general criterion for an ab-initio structure to be ω-saturated. Although the general criterion is not hit by our class, we can adapt the proof to our setting.
We fix the following notation: if $f : A \to B$ is an embedding of A into B such that $f(A) \leq_k B$, then we write $\boldsymbol{f : A \to_k B}$. The following diamond lemma is essential.

Lemma 5.3.1 *Consider $m \in \mathbb{N}$ arbitrary and $A, B \in \mathcal{C}_\mu$, together with a strong embedding $f_1 : A \to B$. Then there exists some $k := k(A, B, m) \in \mathbb{N}$ such that for any $C \in \mathcal{C}_\mu$ with $f_2 : A \to_k C$, there is some $D \in \mathcal{C}_\mu$ together with embeddings $g_1 : B \to_m D$ and $g_2 : C \overset{\leq}{\to} D$ such that the following diagram commutes:*



Proof Assume $A \leq B$ given and decompose the extension into a chain $A = A_0 \leq A_1 \leq \cdots \leq A_l := B$ of minimal extensions. We prove the claim by induction on the length l of the chain.

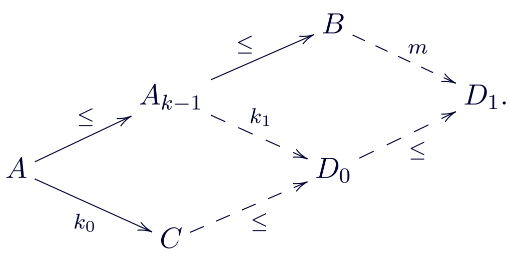
If $A \leq B$ already is a minimal extension, then set $k := k(A, B, m) := m + |B \setminus A|$. Now consider $A \rightarrow_k C$ arbitrary. If $D := B \otimes_A C \in \mathcal{C}_\mu$, then Lemma 3.6.2 yields that D is the desired structure.

Now assume $B \otimes_A C \notin \mathcal{C}_\mu$. By Lemma 5.1.5 we know that the extension $A \leq B$ is 0-minimal and we can embed B as B' over A into C. Set $D := C$. Then clearly $C \leq D$. We have to show that $B' \leq_m C$. Take $C' \subseteq C$ with $B' \subseteq C'$ and $|C' \setminus B'| \leq m$. Then $|C' \setminus A| \leq m + |B' \setminus A| = k$. Hence,

$$\delta(C'/B') = \delta(C'/A) - \delta(B'/A) = \delta(C'/A) \geq 0.$$

Thus, the $\mathcal{C}_\mu$-structure $D = C$ is as desired.

Now consider $l > 1$. By induction hypothesis, for $k_1 := k(A_{k-1}, B, m)$ there is some $k_0 := k(A, A_{k-1}, k_1)$ such that for all $C \in \mathcal{C}_\mu$ with $A \rightarrow_{k_0} C$ there exists some $D_0 \in \mathcal{C}_\mu$ such that $A_{k-1} \rightarrow_{k_1} D_0$ and $C \overset{\leq}{\dashrightarrow} D_0$. Now, as $A_{k-1} \leq B$ is minimal, we can again apply the induction hypothesis and obtain some $D_1 \in \mathcal{C}_\mu$ with $B \rightarrow_m D_1$ and $D_0 \overset{\leq}{\dashrightarrow} D_1$:



Note that $C \leq D_0 \leq D_1$ implies $C \leq D_1$, whence for $k := k(A, B, m) := k_0$ we have that whenever $A \rightarrow_k C$, then there is some $D := D_1 \in \mathcal{C}_\mu$ such that $B \rightarrow_m D$ and $C \overset{\leq}{\dashrightarrow} D$. ■

Definition 5.3.2 Let k be a natural number and $A \subseteq B$ with $B \in \mathcal{C}_\mu$. We write $\boldsymbol{A \lesssim_k B}$, if $A \leq B'$ for any B' with $A \subseteq B' \leq_1 B$ and $|B' \setminus A| \leq k$.

Remark 5.3.3 For $M \models \mathrm{T}_\mu$ and $A \subseteq M$ arbitrary finite, we have that $A \leq M$ if and only if $A \lesssim_k M$ for all $k \in \mathbb{N}$: assume the second condition holds and consider $B \subseteq M$ arbitrary finite containing A. Then the closure $\mathrm{cl}^M(B)$ of B is strong in M and $0 \leq \delta(\mathrm{cl}(B)/A) \leq \delta(B/A)$, as $A \lesssim_k M$ for $k = |\,\mathrm{cl}(B) \setminus A|$.

Lemma 5.3.4 *Assume* $A \lesssim_k B$ *and* $B \leq X$*. Then* $A \lesssim_k X$*.*

Proof Let $A \subseteq D \leq_1 X$ be arbitrary with $|D \setminus A| \leq k$. Then $B \leq BD$ and $D \leq_1 BD$, whence submodularity yields

$$0 \leq \delta(D/B) \leq \delta(D/D \cap B),$$

and thus

$$\delta(D/A) = \delta(D/D \cap B) + \delta(D \cap B/A) \geq 0,$$

as desired. ∎

Lemma 5.3.5 *A model of* T_μ *is* ω*-saturated if and only if it is* $\mathcal{C}_\mu$*-saturated. In particular, the structure* $\mathcal{M}_\mu$ *is* ω*-saturated.*

Proof Consider an arbitrary model M of T_μ. We first show that for all $A \leq B$ in $\mathcal{C}_\mu$ and $m \in \mathbb{N}$, if $A \leq_k M$ for $k = k(A, B, m)$ as in Lemma 5.3.1, then there exists a copy B' of B over A in M such that $B' \lesssim_m M$.
Note that for A, B and m fixed, the above is a first order sentence, whence it suffices to prove the claim for $M = \mathcal{M}_\mu$. Now, if $A \leq_k \mathcal{M}_\mu$, then also $A \leq_k \mathrm{cl}^{\mathcal{M}_\mu}(A)$. Note that $\mathrm{cl}^{\mathcal{M}_\mu}(A) \in \mathcal{C}_\mu$, whence by Lemma 5.3.1, there exists some $D \in \mathcal{C}_\mu$ such that $\mathrm{cl}^{\mathcal{M}_\mu}(A) \leq D$ and $B \leq_m D$. By $\mathcal{C}_\mu$-saturation of $\mathcal{M}_\mu$, we can embed D strongly into $\mathcal{M}_\mu$ over $\mathrm{cl}^{\mathcal{M}_\mu}(A)$. Denote the thereby arising copy of B in $\mathcal{M}_\mu$ by B'. As $B \leq_m D$, Lemma 5.3.4 yields $B' \lesssim_m M$, as desired.
Now, let $N \models \mathrm{T}_\mu$ be an arbitrary ω-saturated and $M \models \mathrm{T}_\mu$ an arbitrary $\mathcal{C}_\mu$-saturated model of T_μ. We show that M and N possess the back and forth-property for finite strong sets.
Let $A' \leq N$ and $A \leq M$ be arbitrary isomorphic finite structures with $f : A' \cong A$. First consider $b' \in N$. There is a closed set $B' := \mathrm{cl}^N(Ab)$, as δ is non-negative. As $\mathcal{C}_\mu$ is an elementary class and $N \models \mathrm{T}_\mu$, note that $B' \in \mathcal{C}_\mu$. Thus, we can embed B' as B strongly over A in M, extending f to B' as desired.
Now consider $b \in M$ and $B := \mathrm{cl}^M(Ab)$. Let $\pi(x)$ be the pre-image under f of $\mathrm{tp}^{qf}(B/A) \cup \{Ax \lesssim_k M \mid k \in \mathbb{N}\}$. We need to show that $\pi(x)$ is consistent, as by ω-saturation of N and Remark 5.3.3, this yields a strong embedding of B over A'. As $A' \leq N$, we saw above that for any k we can embed B as B' over A' such that $B' \lesssim_k N$, which proves that $\pi(x)$ is consistent. ∎

CHAPTER 6

RANK AND AMPLENESS

It is the goal of this chapter to study the first order properties of T_μ. In particular, we show that T_μ is almost strongly minimal of finite Morley rank $3(n-1)-1$ and 2-ample, but not 3-ample.

6.1 Coordinatisation

In this section we want to prove that T_μ is almost strongly minimal.
First we establish the fact, that the type of a subset within a model of T_μ is uniquely determined by the closure of the set.

Lemma 6.1.1 *Let $M_1, M_2 \models \mathrm{T}_\mu$ be two arbitrary models and $a_i \in M_i$ finite tuples of the same length. Then the following are equivalent:*

(i) $\mathrm{tp}^{M_1}(\mathrm{cl}^{M_1}(a_1)) = \mathrm{tp}^{M_2}(\mathrm{cl}^{M_2}(a_2))$.

(ii) $\mathrm{tp}^{M_1}(a_1) = \mathrm{tp}^{M_2}(a_2)$;

(iii) The map $f : a_1 \to a_2$ extends to an isomorphism from $\mathrm{cl}^{M_1}(a_1)$ to $\mathrm{cl}^{M_2}(a_2)$;

Proof As closures do not change in elementary extensions, we may assume the M_i to be ω-saturated. Clearly, (i) implies (ii).

(ii) $\Rightarrow$ *(iii)* As $\mathrm{tp}^{M_1}(a_1) = \mathrm{tp}^{M_2}(a_2)$, the set of formulas $f(\mathrm{tp}(\mathrm{cl}^{M_1}(a_1)/a_1))$ forms a type over a_2 and is hence realized in M_2. It is easy to see that its (unique) realization has to coincide with $\mathrm{cl}^{M_2}(a_2)$, whence f extends as desired.
(iii) $\Rightarrow$ *(i)* The proof of Lemma 5.3.5 yields that any two ω-saturated models possess the back-and-forth property, starting with two arbitrary closed finite subsets. Hence, if $\mathrm{cl}^{M_1}(a_1)$ and $\mathrm{cl}^{M_2}(a_2)$ are isomorphic, they clearly have the same type. ∎

One main consequence of the collapse is that algebraic sets are characterized through the dimension function.

Lemma 6.1.2 *Let M be a model of T_μ and $B \subset M$ a finite subset of M. Some tuple $a \in M$ is algebraic over B if and only if $d(a/B) = 0$.*

Proof We may assume that M is ω-saturated. Note that, as

$$d(a/B) = \delta(\mathrm{cl}(Ba)) - \delta(\mathrm{cl}(B)) = d(a/\,\mathrm{cl}(B)),$$

we may consider $B \leq M$.
First, assume that $d(a/B) = 0$. Then $\delta(\mathrm{cl}(aB)/B) = 0$. We decompose $B \leq \mathrm{cl}(aB)$ into a finite chain of 0-minimal extensions $B := B_0 \leq B_1 \leq \cdots \leq B_k = \mathrm{cl}(aB)$. Then for any $i < k$ there is some $B_i' \subseteq B_i$ such that (B_i', B_{i+1}) is a simple pair. Note, that any two different copies of B_{i+1} over B_i must be disjoint, by minimality of the extension. Now, if there were more than $\mu(B_i', B_{i+1})$ disjoint copies of B_{i+1} over B_i, we could realize their type in $\mathcal{M}_\mu$, contradicting the assumptions on μ. Hence, we get that $B_{i+1} \subseteq \mathrm{acl}(B_i)$ for any i, and thus $a \in \mathrm{acl}(B)$.
Now we see that $a \in \mathrm{acl}(B)$ implies $d(a/B) = 0$. Assume $d(a/B) > 0$ and let again $B = B_0 \leq B_1 \leq \cdots \leq B_k = \mathrm{cl}(aB)$ be a decomposition of $B \leq \mathrm{cl}(aB)$ into minimal extensions. Pick $i \leq k$ minimal such that $\delta(B_i/B_{i-1}) > 0$. By the first part of the proof we know that $B_{i-1} \subseteq \mathrm{acl}(B)$. Furthermore, Lemma 5.1.5 yields that for any $l \in \mathbb{N}$ the structure

$$B_i^l := (\ldots((B_i \otimes_{B_{i-1}} B_i) \otimes_{B_{i-1}} B_i)\ldots) \otimes_{B_{i-1}} B_i,$$

consisting of l copies of B_i freely amalgamated over B_{i-1} is a structure in $\mathcal{C}_\mu$ and $B_{i-1} \leq B_i^l$. As M is $\mathcal{C}_0^{\mathrm{fin}}$-saturated and $B_{i-1} \leq B_i^l \leq M$, we can embed ${B_i}^l$ strongly into M over B_{i-1}, providing l disjoint strong copies of B_i over B_{i-1}. By Lemma 6.1.1, all these copies have the same type as B_i over B_{i-1}, whence $B_i \not\subseteq \mathrm{acl}(B_{i-1})$ and hence $a \notin \mathrm{acl}(B)$, as desired. ∎

Remark 6.1.3 We just saw that the algebraic closure of some subset of a model of T_μ consists of all extensions of dimension zero over that set. This yields a huge difference to the ab-initio case, where the algebraic closure coincides with the definable closure and arises as the union of finite chains of rigid extensions. Note that here, we still have the same description of the definable closure, as for any 0-minimal extension $A \leq B$ which is not rigid, there are exactly $\mu(A, B) \geq \delta(A) > 1$ many copies of B over A in any model of T_μ.

The following example illustrates, that an arbitrary extension of dimension 1 does not necessarily yield a strongly minimal type.

Example 6.1.4 Consider an extension $A \leq B_0 \leq B_1$, where $A \leq B_0$ is 0-algebraic, but not rigid and the extension $B_0 \leq B_1$ is 1-minimal with $\mathrm{cl}(A \cup (B_1 \setminus B_0)) = B_1$. Let b be an arbitrary vertex in $B_1 \setminus B_0$. For another copy B_1' of B_1 over A such that $B_1 \cap B_1' = A$, we denote the thereby arising copy of b by b'. Then the type $\mathrm{tp}(b/A)$ is not strongly minimal, as there are two different extensions to a non-algebraic type over B_0, the one given by b and the one given by b'.
More concretely, let A be a base configuration and B_0 be a simple extension of A as constructed in 6.3.3. Furthermore, let B_1 be the extension of B_0 by one plane connected to some line in $B_0 \setminus A$. We can embed this structure strongly into $\mathcal{M}_\mu$ and get the desired counterexample.

Lemma 6.1.5 *Let M be any model of T_μ and $A \subseteq M$. If for some $b \in M$ we have $d(b/A) = 1$ and $A \leq \mathrm{cl}(Ab)$ is a minimal extension, then $\mathrm{tp}(b/A)$ is strongly minimal.*

Proof We may assume that M is saturated. The type $\mathrm{tp}(b/A)$ is strongly minimal if and only if it has a unique non-algebraic extension to any set C containing A. We may further assume that both A and C are strong in M. Consider an arbitrary realization b' of $\mathrm{tp}(b/A)$ and set $B' := \mathrm{cl}(Ab')$. By minimality, we either have that $d(b'/C) = 0$, whence b' would be algebraic over C by Lemma 6.1.2, or $d(b'/C) = 1$ and $B' \cap C = A$. Now submodularity yields

$$1 = d(b'/C) \leq \delta(B'/C) \leq \delta(B'/A) = 1,$$

whence $\delta(B'/C) = \delta(B'/A)$ and all edges from B' to C have to be induced via A. Thus, the quantifier free type of B' over C is uniquely determined by $\mathrm{tp}^{qf}(B'/A)$ and $d(b'/C) = 1$. As $B'C$ is necessarily strong in M, by Lemma 6.1.1 we get that $\mathrm{tp}(b'/C)$ is the unique non-algebraic extension of $\mathrm{tp}(b/A)$ to C. ■

Corollary 6.1.6 *Let M be any model of T_μ and $a, b \in M$ such that (a,b) forms a partial flag. Denote by*

$$D_{(a,b)} := \mathrm{Res}(a,b)$$

the set of vertices which extend (a,b) to a complete flag. Then $D_{(a,b)}$ is a strongly minimal set.

In [Ten00b] Tent constructs generalized n-gons which are almost strongly minimal, also using a Hrushovski construction. As for a given plane e (respectively point p), our delta function restricted to $\mathrm{Res}(e)$ coincides with the function used in [Ten00b], we obtain the following fact:

Fact 6.1.7 ([Ten00b], Theorem 4.6) *Let $x \in \mathcal{M}_\mu$ be an arbitrary point or plane in $\mathcal{M}_\mu$. Let $x_0, x_1, x_2 \in \mathrm{Res}(x)$ such that the distance between each two of the x_i in $\mathrm{Res}(x)$ is maximal possible. Let further $D := \mathrm{Res}^{\mathcal{M}_\mu}(xx_0)$ be the (strongly minimal) residue of the flag (x, x_0). Then $\mathrm{Res}^{\mathcal{M}_\mu}(x) \subseteq \mathrm{dcl}(x_0, x_1, x_2, D)$.*

This is called the ***coordinatization*** of the n-gon $\mathrm{Res}^{\mathcal{M}_\mu}(x)$. We use that fact to provide a coordinatization for the whole structure $\mathcal{M}_\mu$.

Theorem 6.1.8 (Coordinatization) *The theory T_μ is almost strongly minimal: there is a strongly minimal set $D \subseteq \mathcal{M}_\mu$ together with a finite set $B \subseteq \mathcal{M}_\mu$ such that any element of $\mathcal{M}_\mu$ is definable over BD.*

PROOF We fix a finite parameter set

$$B_0 = (p_1, e_1, p_2, e_2, p_2, e_3),$$

consisting of a 6-cycle of points p_i and planes e_i in $\mathcal{M}_\mu$ and pick $x_0 \in \mathrm{Res}(p_1)$ at maximal distance from e_1 and e_3 and $x_1 \in \mathrm{Res}(e_1)$ at maximal distance from p_1 and p_2. Note that $B := B_0 \cup \{x_0, x_1\}$ is a $\mathcal{C}_\mu$-structure and hence can be strongly embedded into $\mathcal{M}_\mu$, whence from now on we assume $B \leq \mathcal{M}_\mu$. By Corollary 6.1.6, the residue $D := D_{(p_1,e_1)}$ of the partial flag (p_1, e_1) is a strongly minimal set. We show that $\mathcal{M}_\mu \subseteq \mathrm{dcl}(BD)$.

Claim 1: The residues of p_1 and e_1 are contained in $\mathrm{dcl}(BD)$.
This follows immediately from Fact 6.1.7.

Claim 2: The residues of e_2 and p_2 are contained in $\mathrm{dcl}(BD)$.

It suffices to show that any point in $\mathrm{Res}(e_2)$ is contained in $\mathrm{dcl}(BD)$, as every line is uniquely determined by any two points in that line. Thus, consider $p \in \mathrm{Res}(e_2)$ arbitrary. See Figure 6.1. If the points p and p_1 are contained in a common line $l_0 \in \mathrm{Res}(p_1) \subseteq \mathrm{dcl}(BD)$, which is necessarily unique, then p is the unique point contained in e_2 and l_0, as l_0 is not contained in e_2.
If p and p_1 do not intersect in a common line, then there is some plane $e_4 \in \mathrm{Res}(p_1)$ which contains the two points. Now, either e_4 and e_2 intersect exactly in p, whence $p \in \mathrm{dcl}(BD)$, or they intersect in some line l_1 which is uniquely determined by e_4 and e_2 and thus in $\mathrm{dcl}(BD)$. Now consider another line $l_2 \in \mathrm{Res}(e_2)$ connected to p. If there is a plane in $\mathrm{Res}(p_1)$ connected to l_2, then $l_2 \in \mathrm{dcl}(BD)$, whence also $p \in \mathrm{dcl}(BD)$, as it is uniquely determined by l_1 and l_2. Otherwise consider a new point $p_4 \in \mathrm{Res}(e_2)$ connected to l_2. If p_4 and p_1 intersect in a line, then as above, $p_4 \in \mathrm{dcl}(BD)$, whence also $p \in \mathrm{dcl}(BD)$, as it is uniquely determined by p_4 and l_1. If p_4 and p_1 intersect in a plane e_5, then either e_5 and e_2 intersect only in p_4 and $p_4 \in \mathrm{dcl}(BD)$, or they intersect in a unique line $l_3 \in \mathrm{dcl}(BD)$. Then p lays on the unique path of length 4 between l_3 and l_1 in $\mathrm{Res}(e_2)$, whence $p \in \mathrm{dcl}(BD)$. Hence $\mathrm{Res}(e_2) \subseteq \mathrm{dcl}(BD)$, as desired.

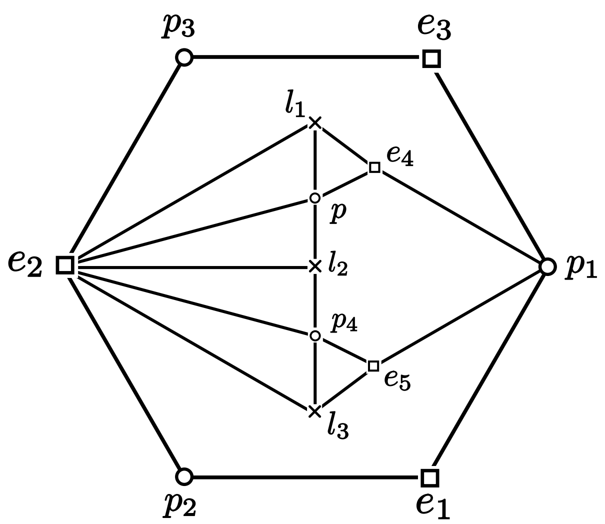


Figure 6.1: Possible Situation in Claim 2

A symmetric argument shows that also $\mathrm{Res}(p_2) \subseteq \mathrm{dcl}(BD)$.

Claim 3: If $e_7 \in \mathrm{Res}(p_1)$ and $p_8 \in \mathrm{Res}(e_1)$, then the residues of e_7 and p_8 are contained in $\mathrm{dcl}(BD)$.
We show the statement for $\mathrm{Res}(e_7)$, the argument for p_8 is exactly the same. As before, it suffices to show that any point in $\mathrm{Res}(e_7)$ is contained in $\mathrm{dcl}(BD)$. Assume p to be an arbitrary point in $\mathrm{Res}(e_7)$. Once more, the points p and p_2 either intersect in a unique plane e or a unique line l, contained in $\mathrm{Res}(p_2) \subseteq \mathrm{dcl}(BD)$. Exactly as in Claim 2, substituting e_2 and p_1 by e_7 and p_2, one can see that $p \in \mathrm{dcl}(BD)$.

Claim 4: The residues of p_2 and e_3 are contained in $\mathrm{dcl}(BD)$.
We show that all planes in $\mathrm{Res}(p_2)$ are contained in $\mathrm{dcl}(BD)$. Let e be an arbitrary plane in $\mathrm{Res}(p_2)$. Then e and e_7 intersect in a unique line l or a unique point p in $\mathrm{Res}(e_7) \subseteq \mathrm{dcl}(BD)$. Exactly as before we show that e is contained in $\mathrm{dcl}(BD)$.

Claim 5: Any vertex of $\mathcal{M}_\mu$ is contained in $\mathrm{dcl}(BD)$.
It suffices to show that an arbitrary point p is contained in $\mathrm{dcl}(BD)$. Clearly, for any point $p \in \mathrm{Res}(e_i)$ and for any plane e in $\mathrm{Res}(p_i)$ for $i = 0, \ldots, 5$ we have that $\mathrm{Res}(p)$ and $\mathrm{Res}(e)$ are contained in $\mathrm{dcl}(BD)$ (the proof of *Claim 3* applies). Hence, if the point p intersects with any of the p_i for $i = 0, 2, 4$ in a unique plane, it already is contained in $\mathrm{dcl}(BD)$. On the other hand, if p intersects with each p_i in a unique line l_i, then we obtain a substructure that contradicts the fact that B is strongly embedded in $\mathcal{M}_\mu$: if $l_i = l_j$ for some $i \neq j$, then the extension of B by $l = l_i$ is an extension of negative delta. If all the l_i are distinct, then the extension of B by the l_i and p is an extension of negative delta. Hence, any point p has to intersect in a unique plane with one of the p_i and is thus definable over BD. ■

Corollary 6.1.9 *The theory* T_μ *is almost strongly minimal. In particular, it is* $\aleph_1$*-categorical and* ω*-stable of finite Morley rank. Furthermore, the Morley rank is additive, i.e. for any tuples* a *and* b *and any set* X *we have*

$$\mathrm{MR}(ab/X) = \mathrm{MR}(a/bX) + \mathrm{MR}(b/X).$$

In the following chapter we want to describe forking in T_μ and show that it is given through the dimension function d. Based on these considerations, we show that the Morley rank of a finite set coincides with the dimension of it given by d.

6.2 Description of Forking

First, we want to relate the model theoretic independence of non-forking with the independence given by the dimension function d. Throughout this section, we work in a very saturated model $\mathbb{M} \models \mathrm{T}_\mu$.

Definition 6.2.1 We define the following relation of ***d-independence***: let X, Y and Z be subsets of some model $M \models \mathrm{T}_\mu$. Then we set

$$\mathbf{X} \mathop{\downarrow}^{\mathbf{d}}_{\mathbf{Y}} \mathbf{Z} \Leftrightarrow d(A/Y) = d(A/YZ) \text{ for all } A \subseteq X \text{ finite.}$$

Lemma 6.2.2 *If for two strong subsets $A_1, A_2 \in \mathcal{M}_\mu$ the induced subgraph on their union is not strong in $\mathcal{M}_\mu$, then in the strong closure* $\mathrm{cl}(A_1A_2)$ *there is a path from some vertex of A_1 to some vertex of A_2 that does not enter or use an edge induced through some line in* $\mathrm{acl}(A_1 \cap A_2)$.

Proof Set $C := \mathrm{acl}(A_1 \cap A_2) \cap \mathrm{cl}(A_1A_2)$. Note that C is a strong superset of $A_1 \cap A_2$ and contained in $\mathrm{acl}(A_1 \cap A_2)$, whence $\delta(C/A_1 \cap A_2) = 0$. Furthermore, the intersection $A_1 \cap A_2$ is strong, whence $\delta(A_1 \cap A_2) \leq \delta(C \cap A_i)$ for $i = 1, 2$ and thus submodularity yields

$$0 \leq \delta(C/A_i) \leq \delta(C) - \delta(C \cap A_i) \leq \delta(C) - \delta(A_1 \cap A_2) = 0,$$

whence $\delta(C/A_i) = 0$. If every path from A_1 to A_2 in $\mathrm{cl}(A_1A_2)$ enters or is induced by $\mathrm{acl}(A_1 \cap A_2)$ and thus by C eventually, then we can partition the closure of A_1A_2 into $\mathrm{cl}(A_1A_2) = B_1 \cup B_2$ in such a way that $A_iC \subseteq B_i$ and there are no non-induced edges between $B_1 \setminus C$ and $B_2 \setminus C$. Then for any $B' \subseteq \mathrm{cl}(A_1A_2)$ containing A_2B_1, we have

$$\begin{aligned}
\delta(B'/B_1A_2) &= \delta(B'/B_1) - \delta(A_2/B_1) \\
&= \delta(B' \cap B_2/C) - \delta(A_2/C) \\
&= \delta(B' \cap B_2/A_2) - \delta(C/A_2) \\
&= \delta(B' \cap B_2/A_2) \geq 0,
\end{aligned}$$

as $A_2 \leq \mathcal{M}_\mu$. Hence, the subgraph B_1A_2 is strong in $\mathrm{cl}(A_1A_2)$ and therefore also strong in $\mathcal{M}_\mu$, which is only possible if $\mathrm{cl}(A_1A_2) = B_1A_2$. A symmetric arguments gives that actually $\mathrm{cl}(A_1A_2) = A_1A_2 \leq \mathcal{M}_\mu$, contradicting the assumptions. ■

The next lemma yields a more concrete description of the d-independence in models of T_μ.

Lemma 6.2.3 *Let A be finite and X and Y arbitrary subsets of $\mathbb{M}$. Then the following two conditions are equivalent:*

(i) The set A is d-independent from Y over X (i.e. $A \mathop{\smile\!\!\!\!\!\mid}^{d}_{X} Y$).

(ii) The structure $X' := \mathrm{cl}(AX) \cap \mathrm{cl}(XY)$ is algebraic over X and

$$\mathrm{cl}(AXY) \cong \mathrm{cl}(AX) \otimes_{X'} \mathrm{cl}(XY).$$

In particular, $\mathrm{cl}(AX)\,\mathrm{cl}(XY) = \mathrm{cl}(AXY)$ is closed in $\mathbb{M}$.

Proof We may assume X to be closed. Set as in the Lemma $X' := \mathrm{cl}(AX) \cap \mathrm{cl}(XY)$. We first show that (i) implies (ii).
Assume $A \mathop{\smile\!\!\!\!\!\mid}^{d}_{X} Y$. By Lemma 3.5.3 we have

$$d(A/X) \leq d(A/X') \leq d(A/XY) = d(A/X), \tag{6.1}$$

whence

$$\begin{aligned} d(A/X) &= d(A/X') = d(AX'/X) - d(X'/X) \\ &= d(X'/AX) + d(A/X) - d(X'/X) \\ &= d(A/X) - d(X'/X). \end{aligned}$$

Thus $d(X'/X) = 0$ and by Lemma 6.1.2 we have $X' \subseteq \mathrm{acl}(X)$. Note that there are no essential edges between $\mathrm{cl}(AX) \setminus X'$ and $\mathrm{cl}(XY) \setminus X'$, as otherwise the inequality in (6.1) was strict, yielding a contradiction. Furthermore

$$d(A/XY) \leq \delta(\mathrm{cl}(AX)/\,\mathrm{cl}(XY)) \leq \delta(\mathrm{cl}(AX)/X') = d(A/X') = d(A/XY),$$

whence the structure $\mathrm{cl}(AX) \cup \mathrm{cl}(XY) = \mathrm{cl}(AXY)$ is strong in $\mathbb{M}$.
Now we prove that (ii) implies (i).
Assume that $X' \subseteq \mathrm{acl}(X)$ and $\mathrm{cl}(AXY) = \mathrm{cl}(AX) \cup \mathrm{cl}(XY)$, but $A \not\mathop{\smile\!\!\!\!\!\mid}_{X} Y$. We show that there are non-induced edges between $\mathrm{cl}(AX) \setminus X'$ and $\mathrm{cl}(XY) \setminus X'$. First note that X and X' are strong and as $X \subseteq X' \subseteq \mathrm{acl}(X)$, we have $\delta(X'/X) = 0$. Now we calculate

$$\begin{aligned} \delta(\mathrm{cl}(AX)/X') &= d(A/X') \\ &> d(A/XY) \\ &= \delta(\mathrm{cl}(AXY)/\,\mathrm{cl}(XY)) \\ &= \delta(\mathrm{cl}(AX)/\,\mathrm{cl}(XY)), \end{aligned}$$

whence $\delta(\mathrm{cl}(AX)/X') > \delta(\mathrm{cl}(AX)/\mathrm{cl}(XY))$. With the help of Lemma 3.4.6, this proves that there are non-induced edges between $\mathrm{cl}(AX) \setminus X'$ and $\mathrm{cl}(XY) \setminus X'$. ■

Our next goal is to show that the notion of d-independence coincides with the notion of forking independence. As we already saw that T_μ is ω-stable, we can use the classification of forking in stable theories, see e.g. [TZ12], in order to show that the two independence notions coincide.

Lemma 6.2.4 *The relation $\mathop{\smile\!\!\!\!\!\!|}^d$ coincides with model theoretic non-forking, i.e. it satisfies the properties of non-forking in stable theories:*

- *(Invariance) The relation $\mathop{\smile\!\!\!\!\!\!|}^d$ is invariant under automorphisms of $\mathbb{M}$;*
- *(Local Character) For all $A \subseteq \mathbb{M}$ finite and $X \subset \mathbb{M}$ arbitrary, there exists $C_0 \leq X$ finite such that $A \mathop{\smile\!\!\!\!\!\!|}^d_{C_0} X$;*
- *(Transitivity) If $A \mathop{\smile\!\!\!\!\!\!|}^d_X Y$ and $A \mathop{\smile\!\!\!\!\!\!|}^d_{XY} Z$ then $A \mathop{\smile\!\!\!\!\!\!|}^d_X YZ$;*
- *(Weak Monotonicity) If $A \mathop{\smile\!\!\!\!\!\!|}^d_X YZ$, then $A \mathop{\smile\!\!\!\!\!\!|}^d_X Y$.*
- *(Weak Boundedness) For all $A \subset \mathbb{M}$ finite and $X \subseteq Y \subset \mathbb{M}$ arbitrary, there are only finitely many isomorphism types of $A' \subseteq \mathbb{M}$ over Y with $A' \cong_X A$ and $A' \mathop{\smile\!\!\!\!\!\!|}_X Y$.*
- *(Existence) For any $A \subset \mathbb{M}$ finite and $X \subseteq Y \subseteq \mathbb{M}$ arbitrary, there is some A' such that $\mathrm{tp}(A/X) = \mathrm{tp}(A'/X)$ and $A' \mathop{\smile\!\!\!\!\!\!|}^d_X Y$.*

Proof We check the above listed properties.

- (Invariance) As δ is clearly invariant under automorphisms, so is d and thus also $\mathop{\smile\!\!\!\!\!\!|}^d$.
- (Local Character) For any $C \subseteq X$ finite, we know that $d(A/C)$ is a non-negative integer and thus can decrease only finitely many times. Therefore, we find some $C_0 \leq X$ with $d(A/C_0)$ being minimal, whence by Lemma 3.5.3.(i) $A \mathop{\smile\!\!\!\!\!\!|}_{C_0} X$.
- (Transitivity) Transitivity is clear, as the assumptions say that $d(A/X) = d(A/XY) = d(A/XYZ)$.

- (Weak Monotonicity) Weak Monotonicity follows directly from the definition together with Lemma 3.5.3.(i), as

$$d(A/X) = d(A/XYZ) \leq d(A/XY) \leq d(A/X),$$

whence $d(A/X) = d(A/XY)$.

- (Weak Boundedness) Recall that there is some $C \subseteq X$ finite with $A \mathop{\smile\!\!\!\!\!\mid}^d_C X$ and that for any $A' \equiv_X A$ we have $A' \mathop{\smile\!\!\!\!\!\mid}^d_X Y$ if and only if $A' \mathop{\smile\!\!\!\!\!\mid}^d_C Y$. We consider the set $\mathrm{cl}(AC) \cap \mathrm{acl}(C)$, which is algebraic over C, say by some formula $\varphi(x, C)$. Denote by $D := \mathrm{cl}(\varphi(\mathbb{M}, C))$ the finite closure of realizations of φ. Then, for any $A' \equiv_X A$ we get that $\mathrm{cl}(A'C) \cap \mathrm{acl}(C) \subseteq D$, by Lemma 6.1.1.

Claim 1: There are only finitely many d-independent extensions of $\mathrm{tp}(A/C)$ to D.

Note first that generally $A \mathop{\smile\!\!\!\!\!\mid}^d_C \mathrm{acl}(C)$, as for $D' \subset \mathrm{acl}(C)$ finite we have $d(D'/C) = 0$, whence

$$d(A/CD') = d(AD'/C) - d(D'/C) = d(D'/AC) + d(A/C) = d(A/C).$$

Thus, any extension of $\mathrm{tp}(A/C)$ to D is d-independent. Clearly, there are only finitely many pairwise different extensions of $\mathrm{tp}(A/C)$ to a quantifier free type over D. Consider A_1 and A_2 to be two realizations of p with $\mathrm{tp}^{qf}(A_1/D) = \mathrm{tp}^{qf}(A_2/D)$. As Lemma 6.1.1 yields that $\mathrm{tp}(A_1/C) = \mathrm{tp}(A_2/C)$ if and only if $\mathrm{tp}(\mathrm{cl}(A_1C)/C) = \mathrm{tp}(\mathrm{cl}(A_2C)/C)$, we may assume that $A_iC \leq \mathbb{M}$. Now, Lemma 6.2.3 implies

$$\mathrm{cl}(A_1D) = \mathrm{cl}(A_1C) \cup \mathrm{cl}(CD) = A_1D \cong A_2D = \mathrm{cl}(A_2D).$$

Thus, again using Lemma 6.1.1, we see that any two realizations of p which have the same quantifier free type over D, have the same full type over D. This proves Claim 1.

Claim 2: For every realization $A' \models p$ of p, the type $\mathrm{tp}(A'/D)$ is d-stationary.

We show that for any finite, closed set D' containing D, there is a unique d-independent extension of $\mathrm{tp}(A'/D)$ to D'. Consider A'_1, A'_2 with $\mathrm{tp}(A'_1/D) = \mathrm{tp}(A'_2/D) = \mathrm{tp}(A'/D)$ and $A'_i \mathop{\smile\!\!\!\!\!\mid}^d_D D'$. Note that by transitivity we also have $A'_i \mathop{\smile\!\!\!\!\!\mid}^d_C D'$. Thus, Lemma 6.2.3 yields that there are no non-induced edges in $\mathrm{cl}(A'_iD')$ between $A'_iC \setminus (A'_iC \cap D')$ and $D' \setminus (A'_iC \cap D')$. Note that $A'_iC \cap D' \subseteq$

$A'_iC \cap \mathrm{acl}(C) \subseteq D$ by our assumptions. Hence in particular there are no non-induced edges between $A'_iC \backslash D$ and $D' \backslash D$, whence $\mathrm{tp}^{qf}(A'_1/D') = \mathrm{tp}^{qf}(A'_2/D')$ and as $A'_iD' = A'_iC \cup D' = \mathrm{cl}(A'_iD')$ is closed, we conclude as above that $\mathrm{tp}(A'_1/D') = \mathrm{tp}(A'_2/D')$, as desired.

This proves that there are only finitely many global d-independent extensions of p to $\mathbb{M}$.

- (Existence) Consider an arbitrary type $\mathrm{tp}(A/X)$ and let C and D be as in the proof of weak boundedness. Assume as before that $AC \leq \mathbb{M}$. Recall that $p := \mathrm{tp}(A/D)$ is d-stationary and $A \mathop{\smile\!\!\!\!\!\!|}^d_D X$.

  Now for any $D' \leq Y$ finite such that $D \leq D'$ is a minimal extension, Proposition 5.1.5 yields that either $A \otimes_D D' \in \mathcal{C}_\mu$ and there is a strong embedding of A over D' such that $A \mathop{\smile\!\!\!\!\!\!|}^d_D D'$, or D' is a 0-minimal extension of D, whence for any set A' we have $A' \mathop{\smile\!\!\!\!\!\!|}^d_D D'$. Thus, the set of formulas $\pi(\bar{x}) := \mathrm{tp}(A/D) \cup \{\bar{x} \mathop{\smile\!\!\!\!\!\!|}^d_D D' \mid D' \subset Y \text{ finite}\}$ is consistent. By saturation of $\mathbb{M}$, there exists some realization A' of $\pi(x)$. By stationarity and the fact that $A' \mathop{\smile\!\!\!\!\!\!|}^d_D X$, we get that $\mathrm{tp}(A'/X) = \mathrm{tp}(A/X)$. By construction we also have $A' \mathop{\smile\!\!\!\!\!\!|}_D Y$, whence $A' \mathop{\smile\!\!\!\!\!\!|}_X Y$. ■

Remark 6.2.5 Note the following consequence from the proof of weak monotonicity: for any finite set A and any set X, there exists a finite subset $D \subseteq X$ such that $A \mathop{\smile\!\!\!\!\!\!|}_D X$ and $\mathrm{tp}(A/D)$ is stationary.

6.3 The Rank

In this section we want to prove that the Morley rank coincides with the dimension on finite sets. There is one direction, which we can see directly.

Lemma 6.3.1 *Let $M \models \mathrm{T}_\mu$ be a saturated model of T_μ. Then, for any finite set $B \subset M$ and $X \subseteq M$ arbitrary, we have $\mathrm{MR}(B/X) \leq d(B/X)$.*

Proof We prove this by induction on $d(B/X)$. For $d(B/X) = 0$, Lemma 6.1.2 yields the desired statement. Now assume we have shown the inequality for all sets B and X with $d(B/X) \leq k$ and consider the case that $d(B/X) = k + 1$. Let $C \subseteq X$ be finite with $B \mathop{\smile\!\!\!\!\!\!|}_C X$. We may assume that $BC \leq M$. If the Morley rank of B over C was strictly bigger than $k + 1$, there existed some tuple $a \in M$ and some formula

$\varphi(x, y)$ such that $\varphi(x, a)$ is consistent with the quantifier free type of B over C and $\mathrm{MR}(\mathrm{tp}^{qf}(B/C) \cup \{\varphi(x, a)\}) = k+1$. We can extend $\mathrm{tp}^{qf}(B/C) \cup \{\varphi(x, a)\}$ to a complete type $p(x)$ of Morley rank $k+1$. Consider an arbitrary realization $B' \models p(x)$. If $B'C \not\leq M$, then $d(B'/C) < d(B/C)$, whence by induction hypothesis

$$k + 1 = \mathrm{MR}(B'/Ca) \leq \mathrm{MR}(B'/C) \leq d(B'/C) < d(B/C) = k + 1,$$

a contradiction. Otherwise, the structure $B'C$ is strong in M, whence $B \equiv_C B'$ and $\mathrm{MR}(B'/C) = \mathrm{MR}(B/C) > k + 1 = \mathrm{MR}(B'/Ca)$ and consequently $B' \not\perp_C a$. This yields $d(B'/Ca) \leq k$, whence

$$k + 1 = \mathrm{MR}(B'/Ca) \leq d(B'/Ca) \leq k,$$

yet again a contradiction. ■

The rest of this section is devoted to showing the counterpoint, i.e. that $MR \geq d$. Recall from Lemma 2.1.15 that the Morley rank in almost strongly minimal theories is additive. We show that $d(b/A) = \mathrm{MR}(b/A)$ through a case distinction. First we use induction to establish the claim for any vertex b which is connected to some point or plane in the base set A. From there we deduce the general statement taking into account the fact that $\mathcal{M}_\mu$ is of bounded diameter. We first establish an easy Corollary from Lemma 6.1.5, which serves as the induction base.

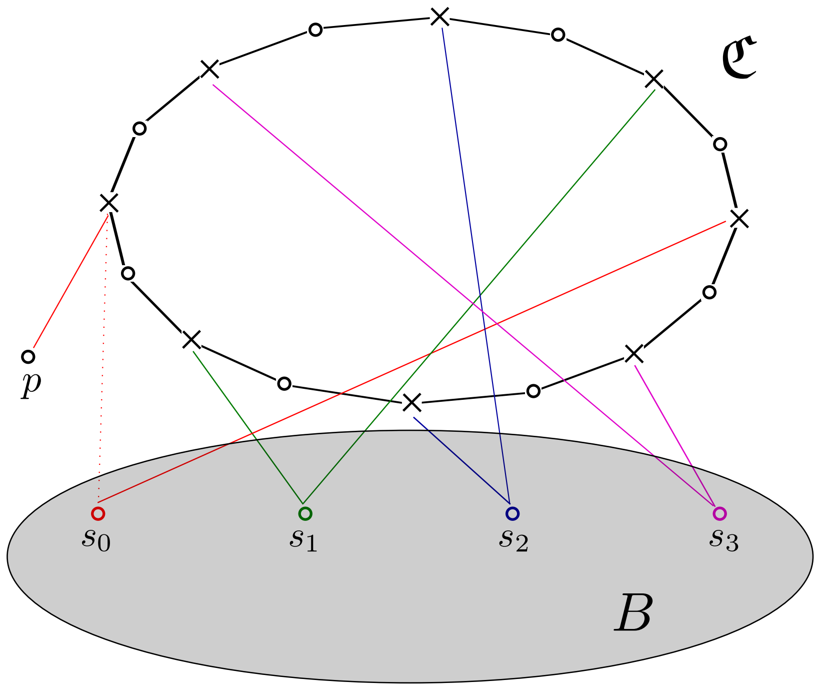
Lemma 6.3.2 *Let X be an arbitrary set and b some vertex in $\mathbb{M}$ such that $d(b/X) = 1$. Then $\mathrm{MR}(b/X) = d(b/X) = 1$.*

Proof Let $B := \mathrm{cl}(bX)$. We may assume that $X \leq XB \leq \mathbb{M}$ is closed in $\mathbb{M}$. We decompose the extension $X \leq B$ into a finite chain of minimal extensions $X := B_0 \leq B_1 \leq \cdots \leq B_k =: B$. By the additivity of d we have for exactly one minimal extension, say $B_{i_0} \leq B_{i_0+1}$ that $d(B_{i_0+1}/B_{i_0}) = 1$ and for any other extension we get that $d(B_{i+1}/B_i) = 0$. Now, Lemma 6.1.2 and Lemma 6.1.5 imply together with the additivity of Morley rank that $\mathrm{MR}(B/X) = \mathrm{MR}(b/X) = 1$. ■

In the following we want to show that for all non-zero extensions $X \leq B$ there is some set Y containing X such that $d(B/Y) = d(B/X) - 1$. This is what is needed in order to show that Morley rank and dimension coincide. The following Lemma is the key tool behind that proof and is originally stated for almost strongly minimal n-gons in [GT14] as constructed by Tent in [Ten00b]. Recall that the same n-gons appear in the residues of any point or plane in models of T_μ.

Lemma 6.3.3 *Let $B \leq \mathcal{M}_\mu$ be a finite set, which contains a base configuration $A_0 \subseteq B$ with respect to some plane $e \in B$ and let p a point in* Res(e) *with $p \downarrow_e B$. Then there are infinitely many pairwise disjoint* 0*-algebraic extensions $\mathfrak{C}$ of Bp in* Res(e) *such that $(A_0p, \mathfrak{C})$ is a simple pair with $|\mathcal{K}(p, \mathfrak{C} \setminus B)| = 1$.*

Proof Let $A_0 := \{e, s_0, \ldots, s_3\}$ be the base configuration in B with respect to some plane $e \in B$ and p a point in Res(e) with $p \downarrow_e B$. By Fact 3.5.8, there are arbitrarily many simple extensions over A_0 in $\mathcal{M}_\mu$, whence in particular, there is some simple extension $\mathfrak{C}$ of A_0 outside of B.



If $p \in$ Res(e) with $p \downarrow_e B$, we can transform the extension $A_0 \leq \mathfrak{C}$ into a simple extension $A_0p \leq \mathfrak{C}p$ by removing one edge from one of the points in $\{s_0, \ldots, s_3\}$ to $\mathfrak{C}$ and adding an edge from the same vertex in $\mathfrak{C}$ to the point p. By the given independence, this is a valid extension and it clearly is still simple over A_0p. Furthermore, we have $|\mathcal{K}(p, \mathfrak{C} \setminus B)| = 1$. ∎

Lemma 6.3.4 *If X is an arbitrary set and b some vertex in $\mathbb{M}$, which is connected to a vertex $x \in X$. Then $d(b/X) =$ MR(b/X).*

Proof We execute the proof for the case that b is a point in the residue of some plane $x \in X$. The cases for b being a line or plane are proved in exactly the same way. Without loss we may assume that $X \leq \mathbb{M}$.

Our plan is to establish the above statement by induction on $d(b/X)$. The induction base for all finite X and all b with $d(b/X) \leq 1$ is already settled by Lemmata 6.1.2 and 6.3.2. Assume now that for any set X and any vertex $b \in \mathbb{M}$ which is connected to a vertex $x \in X$ and such that $d(b/X) \leq k$, we already know that $d(b/X) = \mathrm{MR}(b/X)$. Now consider extension $X \leq \mathrm{cl}(Xb)$ for some vertex b in the residue of a plane x in X with $d(b/X) = k+1$. We want use additivity of Morley rank and dimension function in order to calculate $\mathrm{MR}(b/X)$ successively. Then, the induction hypothesis yields the desired statement.

Possibly by considering a strong extension $Y \leq \mathbb{M}$ of X with $b \mathop{\smile\!\!\!\!\!\!\mid}_X Y$ and such that Y contains a base configuration $A_0 \leq Y$ with respect to x, we may assume that such a configuration already exists in X. By Fact 3.5.8, we can embed a 0-algebraic extension $\mathfrak{C}$ into $\mathbb{M}$ over $\mathrm{cl}(bX)$ in such a way that $\mathfrak{C} \subset \mathrm{Res}(x)$ is simple over bA_0 and there is exactly one edge from $\mathfrak{C}$ to b.

Let $c \in \mathfrak{C}$ be one vertex in $\mathfrak{C}$ which is connected to x and one other vertex in A_0. Then $\delta(c/X) = 1$. We claim that actually $Xc \leq \mathbb{M}$. Otherwise $d(c/X) = 0$, whence $b \mathop{\smile\!\!\!\!\!\!\mid}_X c$. By Lemma 6.2.3, we get that $\mathrm{cl}(Xc) \cap \mathrm{cl}(Xb)$ is algebraic over X, whence $\mathrm{cl}(Xc)$ does not contain b. As there are no non-induced edges between $\mathfrak{C}$ and $\mathrm{cl}(bX) \setminus bX$, one sees that $\mathrm{cl}(Xc) \subseteq \mathfrak{C}$. This now easily yields a contradiction, as $\mathfrak{C}$ is a simple extension over $A_0 b$. Thus we have $Xc \leq \mathbb{M}$.

Now we can use that both d and MR are additive and the induction hypothesis yields for $d(c/X) = 1 \leq k$ and $d(b/Xc) = k$ that

$$\begin{aligned} d(b/X) &= d(b/Xc) + d(c/X) \\ &= \mathrm{MR}(b/Xc) + \mathrm{MR}(c/X) \\ &= \mathrm{MR}(b/X). \end{aligned}$$

Thus we have $d(b/X) = \mathrm{MR}(b/X)$, as desired. ∎

Now, as any two points intersect in a common plane, we deduce the general case for points and planes from the above lemma.

Lemma 6.3.5 *If X is an arbitrary finite set and b is some point or plane in $\mathcal{M}_\mu$, then $d(b/X) = \mathrm{MR}(b/X)$.*

Proof Again, we stick to the case where b is a point. If b is connected to some plane in X, we are done. Otherwise, let p be another point in $\mathbb{M}$ with $p \downarrow Xb$. Then p and b intersect in some plane e. Note that $d(e/Xbp) = 0$. Furthermore we have

$$\begin{aligned} d(b/X) &= d(b/Xp) \\ &= d(be/Xp) - d(e/Xbp) \\ &= d(b/Xep) + d(e/Xp) \\ &\overset{\text{Lemma 6.3.4}}{=} \mathrm{MR}(b/Xep) + \mathrm{MR}(e/Xp) \\ &= \mathrm{MR}(be/Xp) \\ &= \mathrm{MR}(b/X). \end{aligned}$$

This proves the lemma. ■

Finally, we want to establish Lemma 6.3.5 for lines.

Lemma 6.3.6 *If X is an arbitrary finite set and b is any vertex in $\mathcal{M}_\mu$, then $d(b/X) = \mathrm{MR}(b/X)$.*

Proof We are only left to check the case where b is some line in $\mathcal{M}_\mu$. Again, if b is already connected to some point or plane in X, we are done. Otherwise let b be a line and p an arbitrary point contained in b. Then

$$\begin{aligned} d(b/X) &= d(bp/X) - d(p/Xb) \\ &= d(b/Xp) + d(p/X) - d(p/Xb) \\ &= \mathrm{MR}(b/Xp) + \mathrm{MR}(p/X) - \mathrm{MR}(p/Xb) \\ &= \mathrm{MR}(b/X), \end{aligned}$$

as desired. ■

Corollary 6.3.7 *The theory T_μ is of Morley rank $3(n-1)-1$. Furthermore, the Morley rank of any type $\mathrm{tp}(B/X)$ equals its dimension $d(B/X)$.*

Proof This easily follows by a decomposition of $X \leq BX$ into minimal extensions under the use of additivity of both Morley rank and the dimension function. ■

6.4 Ampleness

So far we have seen that the theory T_μ is almost strongly minimal and forking is given by d-independence. In this last section, we prove that the theory is 2-ample, but not 3-ample. A witness for 2-ampleness is given by a complete flag.
One problem to overcome when discussing ampleness are imaginaries. In our case, it follows from Lemma 6.1.8 together with the fact that $\mathrm{acl}(B)$ is infinite for B as in the Coordinatisation Lemma, that the theory T_μ admits weak elimination of imaginaries. Nevertheless, we want to include a direct proof of this fact using Fact 2.1.9.

Lemma 6.4.1 *The theory T_μ has weak elimination of imaginaries.*

Proof We show that every global type has a real canonical base. Let $\mathfrak{p}(x)$ be a global type which does not fork over some model M and let $p(x) := \mathfrak{p}_{|M}(x)$ be its restriction to M. We may assume that for any realization $A \models p$ we have $AM \leq \mathbb{M}$. Let B' be the set of all the vertices b in M for which there is a vertex $a \in A \setminus M$ with an edge between a and b that is not induced through any other vertex in M. Set furthermore $B := B' \cup (A \cap M)$. We claim that B is a canonical base for $\mathfrak{p}(x)$.

Let $f \in \mathrm{Aut}(\mathbb{M})$ be an automorphism which fixes $\mathfrak{p}(x)$. If it does not fix B pointwise, then clearly there is some $b \in B'$ that is moved by f. By definition of B', there is an $a \in A \setminus M$ which is connected to b and hence also to $f(b)$ by a non-induced edge. Note further that b and $f(b)$ have the same type, whence $a \in \mathrm{acl}(b, f(b))$. Now, we get that $f(b)$ can not lay in M, as models are algebraically closed and it can neither lay in $\mathbb{M} \setminus M$, as d is additive and $d(a/M) > 0 = d(a/Mf(b))$, contradicting that $\mathfrak{p}$ is the non-forking extension of p, preserved under f. Thus, f fixes B pointwise.

Now assume $f \in \mathrm{Aut}(\mathbb{M})$ fixes B pointwise. By Lemma 6.1.1, we know that f fixes $\mathfrak{p}(x)_{|\,\mathrm{cl}(B)}$. We want to show that f also fixes $\mathfrak{p}(x)$. First note that for any realization A of $p(x)_{|M}$ we have $A \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{cl}(B)} M$. For any $\mathrm{cl}(B) \subseteq C \leq M$ there are no non-induced edges between $\mathrm{cl}(AB) \setminus \mathrm{cl}(B)$ and $C \setminus \mathrm{cl}(B)$. Furthermore $\mathrm{cl}(AC) \subseteq \mathrm{cl}(AM) = AM$, whence AC is closed, as otherwise for $D \subseteq M$ with $\mathrm{cl}(AC) = AD$ we get a contradiction via

$$\delta(AC) > \delta(AD) = \delta(A/D) + \delta(D) = \delta(A/C) + \delta(D) \geq \delta(AC).$$

Now we easily see that f fixes the quantifier free part of $p(x)$, and as the quantifier free type of strong sets determines the full type, we get that $f(p) = p$. As $\mathfrak{p}(x)$ is stationary

over M and $p(x)$ has a unique non-forking extension from $\mathrm{cl}(B) \subseteq M$ to M, we conclude that f fixes $\mathfrak{p}(x)$.
This shows that any global type has a real canonical base, whence T_μ has weak elimination of imaginaries. ∎

Indeed, the Lemma above is the best we can do. If T_μ had full elimination of imaginaries, then for any finite set A there was some real tuple a such that an automorphism fixes A setwise, if and only if it fixes a pointwise. The following example yields that this is not the case.

Example 6.4.2 Consider two arbitrary lines x_1 and x_2 such that $A := \{x_1, x_2\}$ is strongly embedded in $\mathcal{M}_\mu$. If A had a real canonical parameter a, then clearly $a \in \mathrm{dcl}(A)$. On the other hand we saw that the definable closure of any set consists of rigid extensions of its strong closure. Note that there are no rigid extensions of $\mathrm{cl}(A) = A = \mathrm{dcl}(A)$, whence $a \subseteq A$. Furthermore, by homogeneity there is an automorphism of $\mathcal{M}_\mu$ switching x_1 and x_2 and hence fixing A setwise, but not fixing any point in A. This yields that $a = \emptyset$. Now, clearly there are automorphisms of $\mathcal{M}_\mu$ which do not fix A, a contradiction.

Now, we are ready to prove that complete flags are witnesses for 2-ampleness.

Theorem 6.4.3 *The theory T_μ is 2-ample and any complete flag is a witness for that.*

Proof Assume $\{abc\}$ to be a complete flag with b being the line.

(i) As for any nonempty set $B \subseteq \mathbb{M}$ and for any proper subset $A \subseteq \{abc\}$ we have $\delta(B/A) > 0$ by condition (C4) of the class, Lemma 6.1.2 implies that all partial flags are algebraically closed. In particular $\mathrm{acl}^{eq}(a) \cap \mathrm{acl}^{eq}(b) = \mathrm{acl}^{eq}(\emptyset)$ and $\mathrm{acl}^{eq}(ab) \cap \mathrm{acl}^{eq}(ac) = \mathrm{acl}^{eq}(a)$.

(ii) We have $a \mathop{\smile\!\!\!\!\!\mid}_b c$, as flags are strongly embedded, whence $d(a/b) = \delta(a/b) = \delta(a/bc) = d(a/bc)$ and

(iii) Finally $a \not\mathop{\smile\!\!\!\!\!\mid} c$, as $d(a) = \delta(a) > \delta(a/c) = d(a/c)$.

This proves that any flag abc is a witness for 2-ampleness. ∎

Lemma 6.4.4 *If a triple of strong finite sets ABC witnesses ampleness over an arbitrary parameter set X, then there exist vertices $a \in \mathrm{cl}(AX) \setminus \mathrm{acl}(BX)$, $b \in \mathrm{acl}(BX) \setminus \mathrm{acl}(X)$ and $c \in \mathrm{cl}(BCX) \setminus \mathrm{acl}(BX)$ such that $\{abc\}$ is a complete flag with b being the line.*

Proof We choose $D \subseteq X$ finite in a way that

$$AC \mathop{\downarrow}_{D} X,\ A \mathop{\downarrow}_{BD} C \text{ and } A \mathop{\not\downarrow}_{D} C.$$

First consider the case where $\mathrm{cl}(AD) \cup \mathrm{cl}(CD) = \mathrm{cl}(ACD)$ is already closed. As $A \not\downarrow_D C$, Lemma 6.2.3 implies that there is some non-induced edge between $a \in \mathrm{cl}(AD) \setminus \mathrm{acl}(D)$ and $c \in \mathrm{cl}(CD) \setminus \mathrm{acl}(D)$. Note that neither a nor c are contained in $\mathrm{acl}(X)$, as $AC \downarrow_D X$. If a was in $\mathrm{acl}(BX)$, then $a \in \mathrm{acl}(AX) \cap \mathrm{acl}(BX) = \mathrm{acl}(X)$, a contradiction. Similarly, if $c \in \mathrm{acl}(BX)$, then

$$c \in \mathrm{acl}(ABX) \cap \mathrm{acl}(ACX) \cap \mathrm{acl}(BX) = \mathrm{acl}(AX) \cap \mathrm{acl}(BX) = \mathrm{acl}(X),$$

yet again a contradiction. On the other hand, the independence $A \downarrow_{BD} C$ yields that the edge between a and c has to be induced through some line $b \in \mathrm{acl}(BD) \setminus \mathrm{acl}(X)$. This finishes the proof for the case $\mathrm{cl}(ACD) = \mathrm{cl}(AD) \cup \mathrm{cl}(CD)$.

Now assume that $\mathrm{cl}(AD) \cup \mathrm{cl}(CD)$ is not strong. By Lemma 6.2.2 there is a path from $\mathrm{cl}(AD)$ to $\mathrm{cl}(CD)$ in $\mathrm{cl}(ACD)$ outside of $\mathrm{acl}(\mathrm{cl}(AD) \cap \mathrm{cl}(CD))$, hence in particular outside of $\mathrm{acl}(D)$ and, as $AC \downarrow_D X$, even outside of $\mathrm{acl}(X)$. Note that the path eventually has to leave $\mathrm{acl}(AX)$, as otherwise the last vertex would be contained in $\mathrm{acl}(AX) \cap \mathrm{acl}(CX) \subseteq \mathrm{acl}(AX) \cap \mathrm{acl}(BX) = \mathrm{acl}(X)$, a contradiction. Hence, let $a \in \mathrm{cl}(ACD)$ be the last vertex of that path that is still contained in $\mathrm{acl}(AX)$ and $c \in \mathrm{cl}(ACD) \setminus \mathrm{acl}(AX)$ its neighbor. Note that

$$a, c \in \mathrm{cl}(ACD) \subseteq \mathrm{cl}(ABCD) = \mathrm{cl}(ABD) \cup \mathrm{cl}(BCD),$$

as $A \downarrow_{BD} C$. If a was in $\mathrm{acl}(BCX)$, then by the same independence, $a \in \mathrm{acl}(BX)$ and thus $a \in \mathrm{acl}(X)$, a contradiction. Thus, the vertex $a \in \mathrm{cl}(ABD) \setminus \mathrm{acl}(BD)$. Similarly, if c was in $\mathrm{acl}(ABX)$, then $c \in \mathrm{acl}(ABX) \cap \mathrm{acl}(ACX) = \mathrm{acl}(AX)$, a contradiction. Hence, $c \in \mathrm{cl}(BCD) \setminus \mathrm{acl}(BD)$. On the other hand, there are no non-induced edges between $\mathrm{cl}(ABD) \setminus \mathrm{acl}(BD)$ and $\mathrm{cl}(BCD) \setminus \mathrm{acl}(BD)$, whence the edge between a and c has to be induced through some line $b \in \mathrm{acl}(BD) \setminus \mathrm{acl}(X)$. This proves the lemma. ■

Theorem 6.4.5 *The theory* T_μ *is not* 3*-ample.*

Proof As T_μ has weak elimination of imaginaries, it suffices to show that there are no real sets A, B, C and D together with an arbitrary set X such that (A, B, C, D) is 3-ample over X.

Aiming for a contradiction, we assume A, B, C, D and X to be given as above. Note that (A, C, D) is 2-ample over X. With Lemma 6.4.4 we know that there is a complete flag (a, c, d) with $a \in \mathrm{cl}(AX) \setminus \mathrm{acl}(CX)$, the line $c \in \mathrm{acl}(CX) \setminus \mathrm{acl}(X)$ and $d \in \mathrm{cl}(DX) \setminus \mathrm{acl}(CX)$. Assume $C' \supseteq C$ closed and finite such that $C'X \subseteq \mathrm{acl}(CX)$ and $c \in \mathrm{cl}(C'X)$. Then, we have $A \mathop{\smile\!\!\!\!\!\!|}_{BX} C'$ and by Lemma 6.2.3 one of the vertices a or c has to be contained in $\mathrm{acl}(BX)$. This now yields the desired contradiction, as

$$\mathrm{acl}(AX) \cap \mathrm{acl}(BX) = \mathrm{acl}(CX) \cap \mathrm{acl}(BX) = \mathrm{acl}(X),$$

and neither a nor c is contained in $\mathrm{acl}(X)$. ■

6.5 A new Counterexample to the Trichotomy Conjecture

In this last section, we see that the induced theory T_D on the strongly minimal set D with fixed parameter set B as in 6.1.8, is still 2-ample and does not interpret an infinite field. As a corollary, we obtain the following result:

Theorem 6.5.1 *There is a new 2-ample counterexample to the Trichotomy Conjecture of Zil'ber given by the strongly minimal theory T_D which is 2-ample and does not interpret an infinite field.*

Pillay proved in [Pil00, Proposition 3.8] that a theory of finite Lascar rank is n-ample if and only if all its types of rank 1 are. Here, we say that a type $p(x)$ is n-ample, if there is an n-ample tuple $a_0, \ldots, a_n$ such that $a_n \models p(x)$. It was pointed out in [PW13] that Pillay's proof implies that any type internal to a family of types which are non-n-ample, is itself non-n-ample.

Definition 6.5.2 Let $\Sigma(x)$ be $\emptyset$-invariant family of partial types and $\pi(x)$ a partial type over some parameter set A. We say that $\pi(x)$ is ***$\Sigma(x)$-internal***, if for any $\alpha \models \pi(x)$, there is some set $B \mathop{\smile\!\!\!\!\!\!|}_{A} \alpha$ and realizations $\beta_0, \ldots, \beta_k$ of types in $\Sigma(x)$ based on B such that $\alpha \in \mathrm{dcl}(A\beta_0 \ldots \beta_k)$.

Note that ampleness of some theory is invariant under fixing parameters. Thus, from now on we fix the parameter set B from Proposition 6.1.8, whence the strongly minimal set D becomes $\emptyset$-definable. We denote by $\mathbf{T}_D$ the theory induced by T_μ on D, i.e. the theory of D together with the structure given by all intersections of $\emptyset$-definable subsets of M^m with D^m.
The main tool we need, is the following:

Lemma 6.5.3 *For any type $p(x)$ in T_μ of rank 1, there is some type $q(x)$ in T_D of the same rank, such that $q(x)$ is $p(x)$-internal. Furthermore, for any type $q(x)$ in T_D there is some type $p(x)$ in T_μ of the same rank such that $p(x)$ is $q(x)$-internal.*

Proof Let $p(x)$ be a type over some parameter set A in T_μ and $\alpha \models p(x)$ arbitrary. Let $q(x)$ be the partial type saying $x \in D$. Then clearly, the rank of $q(x)$ is 1. We have seen in Proposition 6.1.8 that T_μ is almost strongly minimal over D in a strict sense, i.e. $(\mathcal{M}_\mu, B) = \mathrm{dcl}(D)$. Thus, there are finitely many realizations $d_1, \ldots, d_k$ of $q(x)$, such that $\alpha \in \mathrm{dcl}(d_1, \ldots, d_k)$, whence $p(x)$ is internal to $q(x)$.
On the other hand, if $q(x)$ is an arbitrary type in T_D, then it can be viewed as a type $p(x)$ in the sense of T_μ. Clearly, then $p(x)$ is internal to $q(x)$. ∎

Lemma 6.5.4 *The induced theory T_D is 2-ample.*

Proof Suppose not. Then, there is some type $q(x)$ over some set $A \subseteq D$ in T_D of rank 1, such that $q(x)$ is not 2-ample. By Lemma 6.5.3, there is some type $p(x)$ also of rank 1, which is internal to $q(x)$. By the remark above, this implies that also $p(x)$ is not 2-ample, contradicting the fact that T_μ is 2-ample. ∎

Lemma 6.5.5 *There is no infinite field interpretable in the theory T_D.*

Proof Any field interpretable in T_D is also interpretable in T_μ. As T_μ is not 3-ample, no such infinite field can exist. ∎

This concludes that the theory T_D is indeed a strictly 2-ample strongly minimal theory contradicting Zil'bers Trichotomy Conjecture.

CHAPTER 7

GROUPS IN PREGEOMETRIES

It is of general interest to model theorists to study the interpretations of abstract model theoretic notions in algebraic structures and see if they reflect known or new algebraic phenomena. In this sense, model theory possesses a fruitful back and forth relation with core mathematics, as on the one hand many of its notions are motivated by algebraic phenomena and on the other hand new objects can be discovered by studying the meaning model theoretic notions have in concrete mathematical structures.
One interesting question relates to the notion of non-forking independence, which always exists in stable theories. In many of the classical mathematical structures, this combinatorial notion, which relies on definable objects exclusively, has a very natural interpretation. It coincides with algebraic independence in algebraically closed fields, with linear independence in vector spaces and in the theory of the free group, which was added rather surprisingly to the picture of stable theories by Zlil Sela [Sel14], it gives back the fundamental notion of JSJ-decompositions. This las fact has been proven by Perin and Sklinos in [PS16].

Another key question that is studied intensively concerns interpretable groups and their relation to the algebraic structure of the models of the underlying theory. In this chapter, we want to study the existence of groups interpretable in the theory T_μ. Recall that T_μ has weak elimination of imaginaries, whence instead of interpretable groups, we may just consider definable, or more generally, type-definable ones. We start by recalling the notion of a type-definable group.

Definition 7.0.1 For a given theory T, we say that $(G, \cdot)$ is a ***type-definable group*** in T, if G is a type-definable set in T and $\cdot$ is a relatively definable group operation on G. It is an ***interpretable group***, if it is type-definable in T^{eq}.

The motivation for studying groups definable in a first order theory comes from the strong connections the structure of a definable groups often has to the structure of the underlying theory. In many cases it has been shown that the category of definable groups in some theory T is exactly the category one would expect from a purely algebraic point of view. As a starting point it was shown that any group definable in an algebraically closed field is definably isomorphic to an algebraic group. This question had been posed by Poizat and was first answered by van den Dries for characteristic zero and then extended by Hrushovski for the general case. A presentation of the proof can be found in [Poi01, Theorem 4.13]. Later, Anand Pillay proved in [Pil88] that any group definable in an o-minimal expansion of the real field can be equipped with a smooth manifold structure so that the group operations are smooth, and hence is a Lie group. He furthermore showed in the same paper that any infinite field definable in an *o*-minimal structure is either real closed or algebraically closed.
The probably most important open question in the model theory of groups is expressed in the ***Algebraicity Conjecture*** of Cherlin and Zil'ber. It is closely related to the Trichotomy Conjecture of Zil'ber and asks for a classification of groups of finite Morley rank.

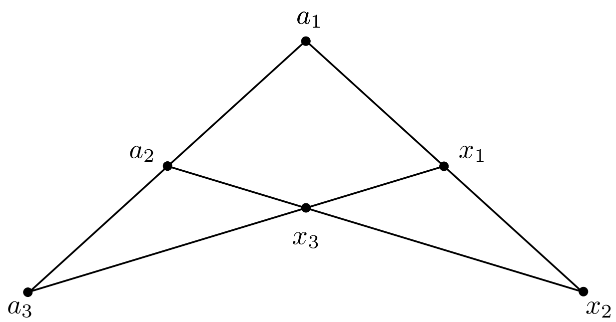
Conjecture 7.0.2 (Algebraicity Conjecture, [Che79][Zil77]) *Any infinite simple group interpretable in a theory of finite Morley rank is an algebraic group over an algebraically closed field, which itself is interpretable in the group structure.*

Although a lot of work and progress circulates around the study of this conjecture, an answer remains yet to be given. Note that for groups definable in a strongly minimal theory, the Algebraicity Conjecture would follow from an affirmation of Zil'ber's Trichotomy conjecture. This is due to the fact by Hrushovski and Pillay in [HP87], which states that one-based groups are abelian-by-finite. Clearly, any simple, abelian-by-finite group is finite, whence the only infinite simple groups that could appear, would be those in geometries where an infinite field is definable.

On the other hand, if the Algebraicity Conjecture would fail, then a minimal counterexample would be given by what is called a ***bad group*** - a simple group of finite Morley rank which is non-solvable and such that all of its proper definable subgroups are nilpotent. In [Pil96b] Pillay showed that if such a group existed, then its forking complexity would be at least 2-ample. As the example introduced in this thesis is the first structure of finite Morley rank not interpreting an infinite field and being 2-ample, it is of special interest to ask if there are any groups definable in it. Nevertheless, we see in the sequel that there are no infinite groups interpretable in our geometry, which is mainly a consequence of its Hrushovski construction nature.

7.1 The Group Configuration Theorem

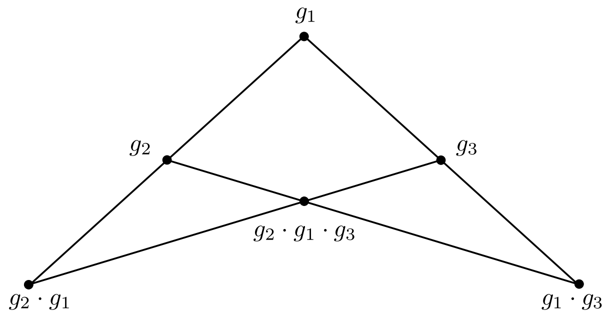
One of the cornerstones concerning the existence of groups in stable theories is the group configuration theorem, which first appeared in Hrushovski's PhD thesis [Hru86]. It states that we can deduce the existence of an infinite, type-definable group in some theory T from the presence of a certain diagram, the so-called ***group configuration***.
By a group configuration over some set D, we mean a set $\{a_1, a_2, a_3, x_1, x_2, x_3\}$ of possibly infinite tuples a_i and x_i which can be aligned in a ***group configuration diagram***



such that the following properties are satisfied:

- On each line in the diagram, any tuple is algebraic over the other two tuples in the same line together with D;
- Any three non-collinear tuples are independent over D.

Note that whenever there is some group G type-definable over D in some theory T, then we can find a group configuration. Just consider three generically independent elements g_1, g_2 and g_3 in G over D. Then the set $\{g_1, g_2, g_1 \cdot g_2, g_3, g_1 \cdot g_3, g_2 \cdot g_1 \cdot g_3\}$ provides the following group configuration:



The fundamental theorem of Hrushovski now provides a counterpart to this phenomenon. He showed that in the reverse, whenever we find a group configuration in a stable theory, then there actually is some type-definable group present.

Fact 7.1.1 (Hrushovski, [Hru86]) *If some stable structure contains a group configuration $\{a_1, a_2, a_3, x_1, x_2, x_3\}$ over D, then possibly after a base change there is some type-definable group G which acts definably, faithfully and transitively on some type-definable set equivalent to* $\mathrm{tp}(x_1)$.

In the view of the Algebraicity Conjecture, it is a particularly interesting question to ask whether or not there are groups definable in a theory of finite rank. Excluding the existence of a group configuration within his new strongly minimal set, Hrushovski proved that there are no type-definable groups present. Even though our geometry differs in certain aspects from the one obtained by Hrushovski in [Hru93], we see that his arguments for the non-existence of interpretable groups can be transferred rather immediately.

7.2 Trivial Forking

We have seen that the geometry $\mathcal{M}_\mu$ is an almost strongly minimal structure, which is 2-ample, but not 3-ample. It has been shown by Pillay [Pil00] that any stable theory which interprets an infinite field is n-ample for all n, whence we immediately know that there is no infinite field interpretable in T_μ. One way to show that no group can definably appear in a theory, is to show that the theory has trivial forking. In this section we see that, even though the geometry is not trivial, there is no infinite group interpretable in T_μ.

We recall the notion of a stable theory having trivial forking.

Definition 7.2.1 Let T be stable. We say that T has ***trivial forking***, if for any set D coming from an arbitrary model of T and for any three tuples of elements a, b and c which are pairwise independent over D, we have that $\{a, b, c\}$ is independent over D as a set.

For an overview of trivial forking see for example [Goo91]. Note that, if T is a theory with trivial forking, then so is T^{eq}. Thus, there is no infinite group interpretable in a theory with trivial forking, as otherwise for generically independent elements a and b of the group, we would have that $\{a, b, a \cdot b\}$ is pairwise independent, but clearly not independent, as $a \cdot b \in \mathrm{dcl}(a, b)$.
An easy example shows that the theory T_μ does not have trivial forking.

Example 7.2.2 Consider three points a, b and c together with a plane e which contains all the three points. This is a structure in $\mathcal{C}_\mu$ and hence can be strongly embedded into $\mathcal{M}_\mu$. By the characterization of forking, one easily checks that $\{a, b, c\}$ is pairwise independent and any two-element set is closed. Nevertheless, it is not independent as a set, as the plane e is contained in the algebraic closure of ab and is clearly not independent from c over the empty set.

Note that the above example also shows that the theory of the ab-initio T_0 does not have trivial forking. In the finite rank case, this fact already follows from a more general correlation between the triviality of forking and the ampleness of the underlying theory. We want to include that fact coming from [Goo91, Proposition 9], as it provides an interesting relation between triviality and ampleness in the case of finite rank.

Fact 7.2.3 *Let* T *be a theory of finite rank. If* T *has trivial forking, then it is one-based. Furthermore, its forking is totally trivial, i.e. whenever* $a \mathop{\smile\!\!\!\!\!\!|}_D b$ *and* $a \mathop{\smile\!\!\!\!\!\!|}_D c$*, then* $a \mathop{\smile\!\!\!\!\!\!|}_D bc$*.*

7.3 Flat Geometries

In the realm of Hrushovski constructions, or more generally, whenever there is a predimension present, there is another canonical method of excluding the existence of interpretable infinite groups, introduced by Hrushovski in [Hru93], which consists in showing that the induced geometry is flat.

Definition 7.3.1 A geometry with respect to the dimension function d is called ***flat***, if for any finite family $\{E_i \mid i \in I\}$ of finite-dimensional closed subsets we have

$$\sum_{J \subseteq I} (-1)^{|J|} d(E_J) \leq 0,$$

where E_J is defined as $\cap_{i \in J} E_i$ for $J \neq \emptyset$ and $E_\emptyset := \cup_{i \in I} E_i$.

Lemma 7.3.2 (conf. Lemma 14 in [Hru93]) *If the geometry (T, d) is flat, then there are no infinite groups type-definable in it.*

Proof Assume on the contrary that the geometry given by d is flat and nevertheless there is a group G type-definable in it of Morley rank g. We saw in Section 7.1, that we can obtain a group configuration $(a_1, a_2, a_3, x_1, x_2, x_3)$ from it. For $i \in \{1, 2, 3, 4\}$ let E_i be the strong closure of one of the for lines in the group configuration. By the given dependences within the configuration, we easily see that

$$d(E_\emptyset) = 3g; d(E_i) = 2g; d(E_{ij}) = g \text{ and } d(E_{ijk}) = 0$$

for any pairwise distinct $i, j, k \in \{1, 2, 3, 4\}$. Now flatness yields that

$$\begin{aligned} 0 &\geq \sum_{J \subseteq I} (-1)^{|J|} d(E_J) \\ &= 3g - 4(2g) + 6g \\ &= g. \end{aligned}$$

Thus $g = 0$ and as it coincides with the Morley rank of G, we see that G is a finite group. ■

Note that restricted to families of size two, the flatness condition coincides with the submodularity condition on the dimension function. While as discussed in Section 3.4 the predimension function δ we use in our new almost strongly minimal geometries is not submodular, its associated dimension function d is. This follows from the fact that submodularity holds for closed sets, which are the object of interest for the dimension function d. Nevertheless, the present geometry given by the dimension function is not flat. This in particular shows that the notion of flatness is a strict generalization of the notion of submodularity, i.e. any flat geometry is submodular, but the converse is not true.

Lemma 7.3.3 *The geometry of T_μ is not flat.*

Proof For some flag (a, b, c), consider the sets $E_1 := \{ab\}, E_2 := \{ac\}$ and $E_3 := \{bc\}$. Then with the notation as introduced above we get

$$\begin{aligned}\sum_{s\subseteq\{1,2,3\}} (-1)^{|s|} d(E_s) &= 3(n-1) + 1 - 3\cdot 3(n-1) + 2\cdot 2(n-1) + 3(n-1) - 1 \\ &= n - 1 > 0,\end{aligned}$$

contradicting the definition of flatness. ■

As in the case of triviality, there is yet again a more general principle contradicting the flatness of T_μ, by using an interesting connection between the flatness of the geometry and its ampleness. The following Lemma was mentioned in [Hru93].

Lemma 7.3.4 *If (T, d) is a combinatorial geometry of finite Morley rank, which is flat, then it is CM-trivial.*

Proof Assume not. Then there is some 2-ample tuple (a, b, c) over some set $D = \mathrm{acl}(a) \cap \mathrm{acl}(b) \cap \mathrm{acl}(c)$ of finite rank. Consider $E_1 := \mathrm{cl}(ab), E_2 := \mathrm{cl}(ac)$ and $E_3 := \mathrm{cl}(bc)$. Note that $d(abc) = d(ab) + d(bc) - d(b)$ and $d(c/D) > d(c/a)$ by the definition of independence. Then

$$\begin{aligned}\sum_{J\subseteq\{1,2,3\}} (-1)^{|J|} d(E_J) &= d(abc) - d(ab) - d(ac) - d(bc) + d(a) + d(b) + d(c) - d(D) \\ &= -d(ac) + d(a) + d(c) - d(D) \\ &= d(c/D) - d(c/a) > 0,\end{aligned}$$

contradicting flatness. ■

7.4 Groups in $\mathcal{M}_\mu$

We saw that so far, the known methods of showing that there is no infinite group definable in some theory, cannot be applied to our geometry. Nevertheless, there is no type-definable group. Indeed, the sets considered in [Hru93] arising from a group configuration yield a contradiction even in our case. The difference is that we have to prove the contradiction directly, rather than being able to merely deduce it from flatness. This suggests that the notion of flatness might be too strong in the question around definable groups in combinatorial geometries. It would be an interesting question to find a reformulation of this property which covers both the new strongly minimal set and the geometry introduced in this work.

We now give the proof that there is no group configuration in models of T_μ.

Proposition 7.4.1 *There is no infinite group interpretable in* T_μ.

Proof Let G be a group, interpretable in $\mathcal{M}_\mu$ over some set $D' \subseteq \mathcal{M}_\mu^{eq}$ and consider generically independent elements $a', b', x' \in G$ and set $c' := b' \cdot a', y' := a' \cdot x'$ and $z' := c' \cdot x' = b' \cdot y'$. We show that G is finite.
As T_μ admits weak elimination of imaginaries, there exist real finite tuples a, b, c, x, y, and z and a finite strong set $D \leq \mathcal{M}_\mu$ such that any primed set is interalgebraic with the corresponding unprimed set. In particular, the sets $A := \mathrm{cl}(aD), B := \mathrm{cl}(bD), C := \mathrm{cl}(cD), X := \mathrm{cl}(xD), Y := \mathrm{cl}(yD)$ and $Z := \mathrm{cl}(zD)$ form a group configuration over D. As in Lemma 7.3.2, we set $I := \{1, 2, 3, 4\}$ and let the E_i be the four different closures of collinear sets in the group configuration. Then as above, we see that

$$\sum_{s \subseteq I} (-1)^{|s|} d(E_s) = d(A/D).$$

On the other hand, we can calculate the sum as follows:

- Vertices outside of D: any vertex which is not in D is counted with positive sign exactly once in $E_\emptyset$ and once amongst the E_s for $|s| = 2$ and with negative sign twice amongst the E_s for $|s| = 1$. This sums up to zero.

- Vertices in D: the vertices in D appear everywhere, namely for the positive occurrences once in $E_\emptyset$, six times in the E_s for $|s| = 2$ and once in E_I and for the negative occurrences four times in the E_s where $|s| = 1$ and four times for the case where $|s| = 3$. This again sums up to zero.

- Edges: the edges within one of the sets A, B, C, X, Y or Z are exactly counted as above and sum up to zero. Edges between two sets which are aligned in the corresponding group configuration appear exactly once in $E_\emptyset$ and once in E_s for some s with $|s| = 1$, which again sums up to zero. Edges between two non-aligned sets are only counted in $E_\emptyset$. Notice that any two sets are independent over D, whence any such edge has to connect a point to a plane, induced via some line in D.

- Flags: again, any flag using vertices from aligned sets sum up to zero. If a flag uses at least two vertices from two non-aligned sets, then as above the independence of the chosen sets over D imply, that the line of the flag is contained in D. Those flags are counted exactly once in $E_\emptyset$.

As D is a strong set, we get that for any edge between two non-aligned sets, there is exactly one flag containing that edge, with the line in D. Thus these edges and flags again sum up to zero.
Those two calculations together yield $d(A/D) = 0$, whence A is algebraic over D. On the other hand, the set A is also interalgebraic over D with a generic element in G, whence G is finite. ■

CHAPTER 8

BOUNDED AUTOMORPHISMS

We want to conclude the study of our new 2-ample geometry by investigating its automorphism group. We can view any automorphism group of some structure as a topological group by equipping it with the topology of pointwise convergence. In the case where the structure is countable, its automorphism group is a Polish group, i.e. a separable, completely metrizable group. When we deal with countable structures which possess some homogeneity properties, the groups arising as their automorphism groups are interesting both as permutation groups and as topological groups. The methods of Fraïssé and Hrushovski amalgamation provide and abundance of new interesting groups as automorphism groups of their homogeneous limits. In this chapter we prove that the automorphism group of our new 2-ample geometry $\operatorname{Aut}(\mathcal{M}_\mu)$ is a simple group.

8.1 Lascar's Result

By a famous result of Lascar in [Las92], we know that if M is an almost strongly minimal set over some finite parameter set B, then the group of strong automorphisms of (M, B) is simple modulo the normal subgroup of its bounded automorphisms. Recall that an automorphism is called ***strong*** if it fixes any element of $\operatorname{acl}^{eq}(\emptyset)$. It is easy to check that the set of all strong automorphisms of some structure M forms a normal subgroup which we denote by $\operatorname{Aut}_f(M)$. For a stable theory T, we get that the quotient $\operatorname{Aut}(M)/\operatorname{Aut}_f(M)$ is an invariant of T for sufficiently saturated models of T. This

quotient is called the ***Galois Group*** of T. Once again, this terminology has its origin in algebraic geometry, as the strong automorphisms of the algebraically closed field $\mathbb{C}$ are exactly those fixing $\mathbb{Q}^{alg}$, whence the quotient $\operatorname{Aut}(\mathbb{C})/\operatorname{Aut}_f(\mathbb{C})$ is isomorphic to the absolute Galois group of $\mathbb{Q}$.
In general, not much is known about the Galois group of a stable theory. On the other hand, especially in the almost strongly minimal context, there are powerful tools to study the normal subgroup of strong automorphisms, or more precisely, its quotient by the subgroup of so-called bounded strong automorphisms.
Next we want to give the definition of a bounded automorphism.

Definition 8.1.1 Let M be some structure and $\operatorname{Aut}(M)$ its automorphism group. Then we call an automorphism $\sigma \in \operatorname{Aut}(M)$ ***bounded***, if there exists some finite set D such that $\sigma(x) \in \operatorname{acl}(Dx)$ for any element $x \in M$. We denote the set of all bounded automorphisms by $\mathbf{Bdd}(\boldsymbol{M})$. Furthermore, we denote the set of all bounded strong automorphisms by $\mathbf{Bdd}_{\boldsymbol{f}}(\boldsymbol{M})$.

The definition of a bounded automorphism originally given by Lascar in [Las92] differs from the one given above. It states that some automorphism σ is bounded, if there exists some natural number n such that for any $X \subseteq M$ we have $\dim(\sigma(X)/X) \leq n$. The next Lemma states that the two notions coincide for strongly minimal theories.

Lemma 8.1.2 *If M is strongly minimal with dimension function* dim*, then an automorphism $\sigma \in \operatorname{Aut}(M)$ is bounded if and only if there exists some natural number n such that for any $X \subseteq M$ we have $\dim(\sigma(X)/X) \leq n$.*

Proof Clearly, if σ is a bounded automorphism, then the conditions in the Lemma are satisfied with respect to the natural number $n := \dim(A)$. Conversely, consider an arbitrary automorphism σ such that $\dim(\sigma(X)/X))$ is bounded by some natural number. Let X be of finite dimension such that $\dim(\sigma(X)/X)$ is maximal. Note that we can assume X to be finite. Then for any element b we have

$$\dim(\sigma(Xb)/Xb)) = \dim(\sigma(b)/X\sigma(X)b) + \dim(\sigma(X)/Xb) \leq \dim(\sigma(X)/X),$$

whence either $\sigma(b) \in \operatorname{acl}(X\sigma(X)b)$ or $\sigma(X) \not\perp_X b$ and then strong minimality implies that $b \in \operatorname{acl}(X\sigma(X))$. Thus, if we set $D := X\sigma(X)\sigma^2(X)$, then for any b we get $\sigma(b) \in \operatorname{acl}(Db)$. This yields that σ is a bounded automorphism with respect to D. ∎

Remark 8.1.3 Note that the group of bounded automorphisms forms a normal subgroup. One easily checks for example that for some bounded automorphism σ and any finite dimensional X we have

$$\begin{aligned}\dim(\sigma^{-1}(X)/X) &= \dim(\sigma^{-1}(X)X) - \dim(X)\\ &= \dim(X\sigma(X)) - \dim(X)\\ &= \dim(\sigma(X)/X),\end{aligned}$$

by invariance of dim under automorphisms. Now it is immediate from submodularity that $\mathrm{Bdd}(M)$ forms a group. To see that it is normal, just note that for some bounded automorphism σ with respect to the natural number n and an arbitrary automorphism τ we have

$$\dim(\tau^{-1}\sigma\tau(X)/X) = \dim(\sigma(\tau(X))/\tau(X)) \leq n$$

for any set X.

We saw that $\mathrm{Bdd}_f(M) \trianglelefteq \mathrm{Aut}_f(M) \trianglelefteq \mathrm{Aut}(M)$. Now, the result of Lascar states that in the case that M is almost strongly minimal, there are no proper normal subgroups contained between $\mathrm{Bdd}_f(M)$ and $\mathrm{Aut}_f(M)$.

Fact 8.1.4 (Lascar, [Las92]) *Let M be an almost strongly minimal structure over some finite parameter set B. Then the group $\mathrm{Aut}_f(M,B)/\mathrm{Bdd}_f(M,B)$ is a simple group.*

Let us exhibit what this result means for the three main examples of strongly minimal structures. This is also well explained in [EGT16]. If M is just a countable set, then any automorphism is strong, whence Aut_f is the whole symmetric group of M. Furthermore, the bounded automorphisms consist exactly of those having finite support. Thus, Lascar's result easily implies that the normal subgroup structure of $\mathrm{Sym}(M)$ consists of the chain

$$\{1\} \trianglelefteq \{\sigma \text{ of finite support and even order }\} \trianglelefteq \{\sigma \text{ of finite support }\} \trianglelefteq \mathrm{Sym}(M).$$

This result had first been proven by Schreier and Ulam in [SU33].
For the case where M is a countably infinite dimensional vector space over some countable division ring F, again any automorphism is strong and $\mathrm{Aut}_f(M) = \mathrm{GL}(\aleph_0, F)$. Furthermore, the bounded automorphisms are exactly those with co-finite dimensional

Eigenspaces. The fact that $\mathrm{GL}(\aleph_0, F)/\operatorname{Bdd}(M)$ is a simple group, had been first proven by Rosenberg in [Ros58].

If M is an algebraically closed field, then we already mentioned that the strong automorphisms are exactly those fixing the algebraic closure of the prime field. If the field is of characteristic 0, then there is no non-trivial bounded automorphism (conf. [Las92]), whence the group of strong automorphisms is a simple group. In a private correspondence with Lascar in 1991, Ziegler generalized this fact to fields in arbitrary characteristic, showing that the only bounded automorphisms are powers of the Frobenius map. A lot of work has been done around the study of bounded automorphisms in fields with enlarged structure. So did for example Konnerth show in [Kon02] that there are no non-trivial bounded automorphisms of differentially closed fields in characteristic 0 and Blossier, Hardouin and Martin-Pizarro in [BHMP16] give a full characterization of the bounded automorphisms in fields considered in the ring language enlarged by symbols for automorphisms, for various cases. Furthermore, the proof of Lascar has been used in [GT14] to construct new simple groups with a BN-pair, which do not arise from algebraic groups. The existence of such groups had been unknown until then.

In his proof of Fact 8.1.4, Lascar shows that if σ is an arbitrary automorphism which is not bounded, then any strong automorphism is contained in the subgroup generated by the conjugates of σ by strong automorphisms. His proof uses the fact that we can view the automorphism group of a countable structure as a Polish group and can thus apply Baires Categoricity Theorem. Recall that in strongly minimal theories T, the algebraic closure possesses the exchange property and thus induces a pregeometry on T which gives rise to a dimension function that reflects the independence given in all stable theories.

Later on Macpherson and Tent in [MT11] and then Tent and Ziegler in [TZ13] further developed and extended the result of Lascar. In [MT11] it was shown that whenever M arises by free amalgamation and its automorphism group is a proper subgroup of $\operatorname{Sym}(M)$ which acts transitively on M, then $\operatorname{Aut}(M)$ is a simple group. Here, the independence given by the algebraic closure in the strongly minimal setting, is replaced by an independence notion which relates to free amalgamation. In [TZ13], a broader framework is introduced, to which Lascar's result transfers. The authors introduce the notion of a *stationary independence relation*, which extends the independence given by free amalgamation. They furthermore replace the notion of an unbounded automorphism by the notion of *moving almost maximally*, which coincides with boundedness in the strongly minimal case. Their main result is that whenever M is a countable structure which carries a notion of stationary independence, then its automorphism group is simple modulo the normal subgroup of automorphisms not moving almost maximally. As a consequence they show that the isometry group of the Urysohn space, which is the universal Polish group, is simple modulo the isometries with bounded displacement.

8.2 Bounded Automorphisms in Generalized N-Gons

Recently, the result of Lascar has been further generalized by Evans, Ghadernezhad and Tent in [EGT16], where they are interested in Hrushovski limits. These always possess a dimension function, which again gives rise to a notion of closure cl^d. Loosely speaken, they prove that Lascar's theorem is still valid, if one exchanges the algebraic closure in the definition of a bounded automorphism, with the closure coming from the dimension function. As in the strongly minimal case these two closures coincide, this extends Lascar's result directly to a broader context. In particular they show that the automorphism groups of Hrushovski's ab initio generic structures as constructed in [Hru93] and [Hru88] are simple modulo the group of automorphisms which fix $\mathrm{cl}^d(\emptyset)$ pointwise.

Note that in order to deduce the simplicity of an automorphism group, applying a Lascar-like theorem is just one out of two steps. A second, equally important step consists in the study of bounded automorphisms. For Hrushovski's ab-initio and strongly minimal structures, this has been done by Ghadernezhad in [Gha13]. Furthermore, Ghadernezhad and Tent showed in [GT14] that there are no bounded automorphisms of the almost strongly minimal n-gons as constructed by Tent in [Ten00b]. In order to show that our new almost strongly minimal 2-ample geometries has a simple automorphism group, we can transfer their proof into our setting. Thus, we now want to give a brief overview of their proof from [GT14].

From now on let Γ_n denote the generalized n-gon as constructed in [Ten00b] and reviewed in Section 2.3 and let σ be an arbitrary bounded automorphism of Γ_n with respect to some finite set $D \leq \Gamma_n$. We may assume that D contains a base configuration as already introduced in Definition 3.5.7. We want to show that σ is trivial. As a first step in [GT14], the authors prove that σ fixes any vertex b such that $d_1(b/D\sigma^{-1}(D)) = n-1$ and from there they deduce that σ fixes the entire n-gon. In the proof of this first step, they use the following fact:

Fact 8.2.1 (Corollary 5.4 in [GT14]) *Let A_0 be a base set if n is odd and the union of two base sets of different type if n is even. Let $b \in \Gamma_n$ be such that $d_1(A_0 b) = d(A_0) + n - 1$. Then for any finite strong set B containing $A_0 b$, there is a set C not contained in B which is 0-minimal over B and such that there is exactly one edge from C to b.*

Now we want to summarize the main idea of the proof. As σ is bounded with respect to D, we may assume that $\sigma(A_0) \subseteq D$. Note first that $D \leq Db \leq \Gamma_n$ and set $B := \mathrm{cl}(Db\sigma(b))$. Consider a set C as in Fact 8.2.1 such that both C and $\sigma(C)$ are disjoint from B. Then clearly $d_1(b/CD) = \delta_1(b/CD) = 1$.
Assume $\sigma(b) \neq b$. As $b \notin \sigma^{-1}(D)$, also $\sigma(b) \notin D$. Thus, there are at least two edges from $B \setminus D$ to $C\sigma(C)$, one edge involving b and another one involving $\sigma(b)$. Note that $DC\sigma(C)$ is closed. This follows as $\mathrm{cl}(DC\sigma(C)) \subseteq BC\sigma(C)$ and it cannot contain b or $\sigma(b)$. On the other hand, the set D is closed and there are no edges from $C\sigma(C)$ to B other than those to $Db\sigma(b)$.
Thus we calculate

$$\begin{aligned} 0 &\leq \delta_1(B/DC\sigma(C)) \\ &= \delta_1(B/D) - (n-2)\mathcal{K}(B \setminus D, C\sigma(C)) \\ &\leq d_1(b\sigma(b)/D) - 2(n-2) \\ &= d_1(b/D) - 2(n-2) \\ &= n - 1 - 2(n-2) < 0, \end{aligned}$$

yielding the desired contradiction.
This concludes that all "generic" points over $D\sigma^{-1}(D)$ are fixed by σ and it is not hard to see that thus the whole n-gon Γ_n is fixed, whence σ is trivial and there are no non-trivial bounded automorphisms of Γ_n.

Recall from Section 2.3 that Γ_n is almost strongly minimal with respect to the base set A_0 as parameter set. Together with Lascar's result, the considerations above imply that $\mathrm{Aut}_{A_0}(\Gamma_n)$ is a simple group. The conclusion for the whole automorphism group $\mathrm{Aut}(\Gamma_n)$ now follows from group theoretic arguments, see [GT14, Proposition 6.5].

8.3 Bounded Automorphisms in the new 2-Ample Geometry

In this section we see that there are no non-trivial bounded automorphisms of the structures $\mathcal{M}_\mu$ and $\mathcal{M}_0$ respectively. We adapt the proof of [GT14] for the almost strongly minimal generalized n-gons constructed by Tent in [Ten00b] outlined above to our setting of incidence geometries. Thus, we first show that any bounded automorphism fixes any plane and we can then deduce that they have to fix the entire structure $\mathcal{M}_\mu$. The basic ideas and definitions for this part are coming from [GT14]. Using an embedding of $\mathcal{M}_\mu$ into $\mathcal{M}_0$, we finally deduce the same statement for $\mathcal{M}_0$. Afterwards, we use the results from [EGT16] to conclude that the automorphism group of $\mathcal{M}_\mu$ is simple.

Recall that by Lemma 6.3.3, for any finite strong set $B \leq \mathcal{M}_\mu$, which contains a base configuration $A_0 \subseteq B$ with respect to some plane $e \in B$ and any point p in $\operatorname{Res}(e)$ with $p \mathop{\smile\!\!\!\!\!\mid}_e B$, there are infinitely many pairwise disjoint 0-algebraic extensions C of Bp in $\operatorname{Res}(e)$ such that (A_0p, C) is a simple pair with $|\mathcal{K}(p, C \setminus B)| = 1$.
The next Lemma refers to Lemma 6.1 in [GT14].

Lemma 8.3.1 *Let σ be a bounded automorphism of $\mathcal{M}_\mu$ with respect to some finite set $D \leq \mathcal{M}_\mu$ and $e \in D$ some plane in D. Then σ fixes any point $p \in \operatorname{Res}(e)$ with $p \mathop{\smile\!\!\!\!\!\mid}_e D\sigma^{-1}(D)$.*

Proof Let e and p be as in the statement of the lemma. We may assume that D contains some base configuration $A_0 \leq D$ with respect to e. By boundedness of σ we may further assume that $\sigma(eA_0) \subseteq D$, without harming the independence $p \mathop{\smile\!\!\!\!\!\mid}_e D$. Aiming for a contradiction, assume that p is not fixed under σ and set $B := \operatorname{cl}(Dp\sigma(p))$. As $p \mathop{\smile\!\!\!\!\!\mid}_e D$, we have $d(pD) = \delta(D) + n - 1$. Further, the boundedness of σ implies $d(B) = d(Dp)$, whence

$$\begin{aligned}\delta(B/D) &= d(B/D)\\ &= d(p/D)\\ &= n-1\\ &= \delta(\hat{B}) - \underbrace{\Big((2(n-1)-1)|\mathcal{K}(\hat{B}, D)| + (n-1)(|\mathcal{K}'(\hat{B}, D)| - |\mathcal{F}(\hat{B}, D)|\Big)}_{=:(*)},\end{aligned}$$

with $\hat{B} := B \setminus D$.
By Lemma 6.3.3 there is some 0-extension C of B disjoint from B which is simple over A_0p and such that there is exactly one edge from p to C. We may further assume that

$\sigma(C)$ is also disjoint from B, by choosing C large enough. Recall that the extension C is simple over $A_0p \subseteq Dp$, whence $\sigma(C)$ is simple over $\sigma(A_0p)$. On the other hand, as $p \mathop{\smile\!\!\!\!\mid}_e \sigma^{-1}(D)$ and $\sigma(p) \neq p$, we get that $\sigma(p) \notin Dp$. Thus σ cannot fix C as a set and by minimality we get that $C \cap \sigma(C) = \emptyset$.

Now we have $DC \leq DCp \leq \mathcal{M}_\mu$, whence $d(p/DC) = \delta(p/DC) = 1$. Thus, the point p is not algebraic over DC. On the other hand, note that the only edges from $\hat{B}$ to $C\sigma(C)$ which are not induced through some vertex in D are one edge from the point b to some line in C in $\mathrm{Res}(e)$ and another such edge from $\sigma(b)$ into $\sigma(C)$. Note that as above, the set $DC\sigma(C)$ is closed. Thus we get

$$\begin{aligned} 0 &\leq \delta(B/DC\sigma(C)) \\ &= \delta(\hat{B}) - (*) - 2(n-2) \\ &= n - 1 - 2(n-2) \\ &< 0, \end{aligned}$$

a contradiction. ■

Proposition 8.3.2 *There are no non-trivial bounded automorphisms of $\mathcal{M}_\mu$.*

Proof Let σ be an arbitrary bounded automorphism of $\mathcal{M}_\mu$ with respect to some closed set D. We first show that σ fixes any plane in $\mathcal{M}_\mu$. So consider an arbitrary plane e in $\mathcal{M}_\mu$. Recall that σ is also bounded with respect to $\mathrm{cl}(eD)$. Now pick two points p_1 and p_2 in the residue of e such that

$$p_1 \mathop{\smile\!\!\!\!\mid}_e D\sigma^{-1}(D) \text{ and } p_2 \mathop{\smile\!\!\!\!\mid}_e p_1 D\sigma^{-1}(D).$$

By Lemma 8.3.1, we know that σ fixes both p_1 and p_2. By the given independence, these two point intersect in a unique plane, which is exactly e. Thus σ fixes e. Now it is easy to see that any automorphism of $\mathcal{M}_\mu$ which fixes all the planes has to fix the whole structure and hence is trivial. ■

From the above result we can directly deduce that there neither exist non-trivial bounded automorphisms of the uncollapsed structure $\mathcal{M}_0$.

Proposition 8.3.3 *Let $\mathcal{M}_0$ be the ab-initio structure obtained in section 3. Then the group of bounded automorphisms of $\mathcal{M}_0$ is trivial.*

Proof We already saw that there are no non-trivial bounded automorphisms of any structure $\mathcal{M}_\mu$. We show that if there was some nontrivial bounded automorphism of $\mathcal{M}_0$, then also of $\mathcal{M}_\mu$. So let σ be a bounded automorphism of $\mathcal{M}_0$ with respect to some finite closed set $D \subseteq \mathcal{M}_0$. We may assume that σ acts non-trivially on D. Note that we can choose the function μ in such a way that $D \in \mathcal{C}_\mu$. By Lemma 2.1.21, we can embed $\mathcal{M}_\mu$ inside $\mathcal{M}_0$ in such a way that $D \leq \mathcal{M}_\mu \leq \mathcal{M}_0$, where we denote the arising copy again by $\mathcal{M}_\mu$.
Now, recall from Remark 3.6.4 that the algebraic closure in the sense of $\mathcal{M}_0$ of any set $X \subseteq \mathcal{M}_0$ is exactly the closure of X under rigid extensions, which coincides with its definable closure in $\mathcal{M}_\mu$. Thus, the embedding $\mathcal{M}_\mu$ is already algebraically closed in $\mathcal{M}_0$ and in particular, for any subset $A \subseteq \mathcal{M}_\mu$ we have that $\mathrm{acl}^{\mathcal{M}_0}(A) \subseteq \mathcal{M}_\mu$. This directly implies that σ fixes $\mathcal{M}_\mu$ setwise, as for any $a \in \mathcal{M}_\mu$ we have that $\sigma(a) \in \mathrm{acl}^{\mathcal{M}_0}(Da) \subseteq \mathcal{M}_\mu$. As furthermore $\mathrm{acl}^{\mathcal{M}_0}(A) \subseteq \mathrm{acl}^{\mathcal{M}_\mu}(A)$ for any set $A \subseteq \mathcal{M}_\mu$, we obtain a bounded automorphism $\sigma_{|\mathcal{M}_\mu}$ of $\mathcal{M}_\mu$, contradicting Proposition 8.3.2. ■

8.4 New Simple Groups

We now use the results from [EGT16, Section 3] in order to show that the automorphism group $\mathrm{Aut}(\mathcal{M}_\mu)$ is a simple group. To this end, we need to introduce some further definitions. First of all, we define the following family of sets:

$$\mathcal{X} := \{\mathrm{acl}(A) \mid A \subseteq \mathcal{M}_\mu \text{ finite }\}.$$

We first want to show that the relation of non-forking independence is a stationary independence relation as defined in [EGT16, Definition 2.2]

Remark 8.4.1 The relation of non-forking independence $\mathop{\smile\!\!\!\!\!\mid}$ in $\mathcal{M}_\mu$ is a ***stationary independence relation compatible with*** **acl**, i.e. it satisfies the following list of properties for all A, B, C and D in $\mathcal{X}$ and finite tuples a and b:

(1) (Compatibility) We have $a \mathop{\smile\!\!\!\!\!\mid}_b C$ if and only if $a \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}(b)} C$. Furthermore $a \mathop{\smile\!\!\!\!\!\mid}_B C$ if and only if $e \mathop{\smile\!\!\!\!\!\mid}_B C$ for all $e \in \mathrm{acl}(a, B)$ if and only if $\mathrm{acl}(a, B) \mathop{\smile\!\!\!\!\!\mid}_B C$.

(2) (Invariance) If σ is an automorphism of the structure, then $A \mathop{\smile\!\!\!\!\!\mid}_B C$ if and only if $\sigma(A) \mathop{\smile\!\!\!\!\!\mid}_{\sigma(B)} \sigma(C)$.

(3) (Monotonicity) If $A \mathop{\smile\!\!\!\!\!\mid}_B CD$, then $A \mathop{\smile\!\!\!\!\!\mid}_B C$ and $A_{BC} D$.

(4) (Transitivity) If $A \downfree_B C$ and $A \downfree BCD$, then also $A \downfree_B CD$.

(5) (Symmetry) If $A \downfree_B C$, then $C \downfree_B A$.

(6) (Existence) There is some automorphism σ fixing B such that $\sigma(A) \downfree_B C$.

(7) (Stationarity) Suppose A_1, A_2, B and C are in $\mathcal{X}$ with $A_1 \equiv_B A_2$ and $A_i \downfree_B C$. Then $A_1 \equiv_{BC} A_2$.

Proof The axioms follow clearly from the characterization of stable forking. ■

We remark that Transitivity is already implied by the other axioms.
Now, we conclude that the automorphism groups of $\mathcal{M}_\mu$ and $\mathcal{M}_0$ are simple, using the following theorem from [EGT16].

Fact 8.4.2 ([EGT16], Theorem 3.2) *Suppose $\downfree$ is a stationary independence relation compatible with* acl *and $B \in \mathcal{X}$ is such that there is an $\mathrm{Aut}_B(\mathcal{M}_\mu)$-invariant set D, where the elements of $D \setminus B$ have rank* 1 *over B and $\mathrm{acl}(D, B) = \mathcal{M}_\mu$. Suppose $\sigma \in \mathrm{Aut}(\mathcal{M}_\mu / \mathrm{acl}(\emptyset))$ is an unbounded automorphism of $\mathcal{M}_\mu$. Then every element of $\mathrm{Aut}(\mathcal{M}_\mu / \mathrm{acl}(\emptyset))$ is a product of* 96 *conjugates of $\sigma^{\pm 1}$.*

Corollary 8.4.3 *The automorphism group of $\mathcal{M}_\mu$ is simple.*

Proof We already saw in Remark 8.4.1 that $\downfree$ defines a stationary independence relation compatible with acl. Recall also that $\mathrm{acl}(\emptyset) = \emptyset$. Now, choose B and D as in the proof of Theorem 6.1.8. By substituting B with its algebraic closure, we obtain the conditions of Fact 8.4.2. Thus, there is no proper normal subgroup of $\mathrm{Aut}(\mathcal{M}_\mu)$ which contains an unbounded automorphism. On the other hand, we have seen in 8.3.2 that there are no bounded automorphisms of $\mathcal{M}_\mu$, whence conclusively, there are no non-trivial normal subgroups in $\mathrm{Aut}(\mathcal{M}_\mu)$ and the group is simple. ■

Remark 8.4.4 The notion of an unbounded automorphism used in Theorem 3.2. of [EGT16] differs from the one given in Definition 8.1.1. Nevertheless, as $\mathcal{M}_\mu$ is almost strongly minimal, it is what is called *monodimensional* in [EGT16]. Thus by Proposition 3.11 of the same paper, the two notions of unbounded automorphisms coincide in our context.

CHAPTER 9

EPILOGUE

In this thesis we have seen that there are counter examples to the Trichotomy Conjecture of Zil'ber of higher ampleness. This gives a partial answer to the conjecture of Pillay from [Pil00], which had been open for over fifteen years. Moreover, we expect the full conjecture to be answered affirmatively in the near future. By the inductive character of the construction presented in this thesis, which lifts the ideas from Tent in [Ten00b] to a higher dimension, there is hope that the methods can be further generalized to produce higher ample examples.

Conjecture 9.0.1 *There are almost strongly minimal incidence geometries of geometrical rank $k+1$ and of type $\bullet \overset{n}{-} \bullet \overset{n}{-} \bullet \cdots \bullet \overset{n}{-} \bullet$ which are k-ample, but not $k+1$-ample.*

We have already mentioned that the non-abelian free group was added to the picture of stable theories by Sela. Its theory turns out to provide a very natural counterexample to many phenomena stable groups usually show. For example, it is the first known group which is n-ample for all n ([OHT12] and [Skl15]) in which there is no infinite field definable ([BS15]). We can transfer this question to the strongly minimal case.

Question 9.0.2 Is there a strongly minimal theory, which is n-ample for all n, but does still not interpret an infinite field?

Another aspect in which the free group falls out of the picture of other stable groups is equationality. Equationality is an analogue of noetherianity for arbitrary first order theories and it implies stability. Although equational theories form a proper subclass of stable theories, a natural example that witnesses that had long been missing. The very first, and for a long time only, stable non-equational theory had been constructed by Hrushovski and Srour in [HS89] and is a colored version of the free pseudospace constructed by Baudisch and Pillay in [BP00], which we mentioned in 2.3.8. Much later, Sela showed in [Sel12] that the theory of the free group is non-equational. This result has then been generalized to arbitrary free products of stable groups (excluding $\mathbb{Z}_2 * \mathbb{Z}_2$) in [MS17].

In [BMPZ14b], Baudisch, Pizarro and Ziegler show that the ω-stable buildings which fill the ample hierarchy, are indeed equational. It would be interesting to answer this question for the present theory. Note that there is no stable non-equational theory known, which is of finite rank. Thus the following question should be answered:

Question 9.0.3 Is the theory T_μ equational?

Although the ample hierarchy now has been proven to be strict, there are few things known about structures which are at least 2-ample and their properties. Note for example that groups, which are not 1-ample are almost abelian and those of finite Morley rank which are not 2-ample, are almost nilpotent. There might still a lot to be learned about the structural consequences, which the bound on the ampleness has on a stable structure. Moreover, Pillay has proved in [Pil96a, Section 4.1] that a stable theory is 1-ample if and only if there is a type-definable pseudoplane in it. This raises the following question:

Question 9.0.4 Is there a good notion of a type-definable "k-pseudospace", such that some stable theory T is k-ample if and only if there is a type-definable "k-pseudospace" in it?

We hope these questions to be motivating to see that the study of the ample hierarchy provides an ample amount of open interesting questions, and its answers may help us significantly to understand the geometrical properties of stable first order theories.

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