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The Euler ring of the rotation group

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Abstract. The Euler rings for the closed subgroups of a compact Lie group are part of a Green functor on the universal induction category associated to this group. We use these structural data in order to obtain general information about the multiplicative structure: idempotent elements, units, restriction homomorphisms. We then apply these results to the group SO_3. Also some classical elementary geometry is interpreted in this algebraic context.

1. Introduction

The Euler ring $U(G)$ of a compact Lie group G was studied in [4] in the context of additive invariants and induction categories and in [10] for other reasons. An element of $U(G)$ is represented by a compact G-ENR (G-equivariant Euclidean neighborhood retract) X. The G-action on X induces for each subgroup H of G (always assumed to be closed, notation $H \leq G$) an action of the normalizer $N_G H$ on the H-fixed set X^H. We denote by $\chi(Y)$ the Euler characteristic of a space Y and let $\chi_H(X)$ be the Euler characteristic $\chi(N_G H \backslash X^H)$ of the orbit space $N_G H \backslash X^H$. Two G-ENRs X and Y define the same element of $U(G)$ if and only if for each H the equality $\chi_H(X) = \chi_H(Y)$ holds. Addition and multiplication in $U(G)$ are induced by disjoint union and Cartesian product, respectively. The value $\chi_H(X)$ only depends on the conjugacy class of H in G. Let $[X] \in U(G)$ denote the element represented by X. The assignment $\chi^G_H = \chi_H : U(G) \to \mathbb{Z}$, $[X] \mapsto \chi(N_G H \backslash X^H)$ is an additive homomorphism. We recollect a few general facts in 1.1, 1.2 and 1.3.

1.1. Additively, $U(G)$ is the free abelian group with basis the isomorphism classes of homogeneous spaces $[G/H]$ for closed subgroups H of G. Suppose an element of $U(G)$ has the expression $[X] = \sum_{(H)} a_H [G/H]$, $a_H \in \mathbb{Z}$, in terms of the basis elements; the sum is over conjugacy classes of isotropy groups of X. Apply the homomorphism χ_C to this equation. From the set of equations

$$\chi(N_G C \backslash X^C) = \textstyle\sum_H a_H \chi(N_G C \backslash (G/H^C))$$

the a_H can be determined recursively by downward induction. The integers a_H can also be computed as an Euler characteristic. Let $X_{(H)}$ denote the subspace of orbits isomorphic to G/H and let $G\backslash X^+_{(H)}$ be the one-point compactification of the orbit space $G\backslash X_{(H)}$. Then $a_H = \chi(G\backslash X^+_{(H)}) - 1$. One can also work with $NH\backslash X_H \cong G\backslash X_{(H)}$ where X_H is the subspace of points with isotropy group H; this has the advantage that X_H is a subset of X^H [4, IV.1.12]. For instance, if X_H is a union of k open intervals (as sometimes in our examples), then $NH\backslash X^+_H$ is the wedge of k circles and therefore $a_H = -k$. ♢

There is a canonical surjective ring homomorphism $\pi = \pi_G : U(G) \to A(G)$ onto the Burnside ring $A(G)$ of G. The kernel of π_G is the nilradical $N(G)$ of $U(G)$, see [4, p. 241]. The subgroup $N(G)$ is spanned by the $[G/H]$ for subgroups H which have infinite index in their normalizer N_GH; and $A(G)$ has a $\mathbb{Z}$-basis of isomorphism classes $[G/H]$ for subgroups H with finite Weyl group $W_GH = N_GH/H$. We let $\iota : A(G) \to U(G)$ denote the additive inclusion which is the identity on the canonical basis elements $[G/H]$. We call $b(x) = \iota\pi(x)$ the Burnside part of $x \in U(G)$. Thus $U(G)$ is additively the direct sum of the subgroups $N(G)$ and $\iota A(G)$. In order to simplify the notation, we also write H for the basis element $[G/H]$ if G is clear from the context. We also recall that elements in the Burnside ring are characterized by the Burnside marks $\varphi_H : [X] \mapsto \chi(X^H)$ for H with finite Weyl group. In contrast to the χ_H the φ_H are ring homomorphisms into $\mathbb{Z}$.

By the way, it will turn out that the nilpotent elements (which are not present in the Burnside ring) are quite useful. We assume the group theory and the geometry of the subgroups of SO_3 as known.

The numbers $\chi_K(G/L)$ can be computed by group theory.

1.2. The space $G/L^K = \{tL \mid t^{-1}Kt \leq L\}$ consists of a finite number of N_GK-orbits [4, p. 41] and $\chi_K(G/L)$ is the number of these orbits. Elements sL and tL are in the same N_GK-orbit if and only if the subgroups $s^{-1}Ks$ and $t^{-1}Kt$ are L-conjugate. The isotropy group of the N_GK-action at tL is $tN_L(t^{-1}Kt)t^{-1}$. The number of N_GK-orbits is equal to the number of L-conjugacy classes of subgroups $A \leq L$ which are G-conjugate to K.

For an L-space Y we can split the fixed point set $(G \times_L Y)^K$ according to the N_GK orbits of G/L^K. This yields the formula $\chi^G_K(G \times_L Y) = \sum_{(A)} \chi^L_A(Y)$ where the sum is taken over the L-conjugacy classes of groups A which are G-conjugate to K. ♢

1.3. For our applications we have to recall the conjugacy classes of closed subgroups of SO_3. We list them together with their normalizers.

H	SO_3	O_2	SO_2	A_5	S_4	A_4	$D_m, m \geq 3$	D_2	$C_n, n \geq 2$	C_1
NH	SO_3	O_2	O_2	A_5	S_4	S_4	D_{2m}	A_4	O_2	SO_3

Here D_m is the dihedral group of order $2m$ and C_n is the cyclic group of order n. Moreover the alternating groups A_5 and A_4 are the icosahedral and the tetrahedral group and the symmetric group S_4 is the octahedral group. The subgroup $N(SO_3)$ is spanned by the C_n. Let $\nu : N(SO_2) \to N(SO_3)$ be the induction homomorphism which sends $[SO_2/C_n]$ to $[SO_3/C_n]$. ♢

Examples 1.4. (1) If we apply 1.1 to the trivial subgroup $C = 1$, we see that the sum $\sum a_H$ of the coefficients equals the Euler characteristic $\chi(G\backslash X)$ of the orbit space.

(2) In the case of the SO_2-space SO_3/L the orbit space $SO_2\backslash SO_3/L$ is a closed interval for infinite L and a 2-sphere for finite L. Therefore the sum of the coefficients is 1 or 2, respectively. Note that $SO_2\backslash SO_3$ is a 2-sphere with standard L-action.

(3) The orbit space $O_2\backslash SO_3/L$ has in each case Euler characteristic 1. It is a closed interval for infinite L, a 2-disk for finite $L \neq C_m$ and a projective plane for $L = C_m$. The sum of the coefficients is therefore in each case 1.

(4) If we apply 1.1 to a maximal isotropy type $H = C$ of X, then G/H^H consists of a single $N_G H$-orbit, and therefore $a_C = \chi(N_G C\backslash X^C)$.

(5) The SO_2-space $X = SO_3/C_m$ for the cyclic group C_m of order m has the isotropy groups C_m and C_1. For $m \geq 2$ the C_m-fixed set is O_2/C_m. From (1) and (4) we obtain $[X] = 2[SO_2/C_m] \in U(SO_2)$; this holds also for $m = 1$. ◊

2. Products and Green functors

A homomorphism $\rho : K \to L$ induces ring homomorphisms

$$U(\rho) : U(L) \to U(K), \qquad A(\rho) : A(L) \to A(K)$$

by viewing an L-space via ρ as K-space. In the case of an inclusion $\rho : K \subset L$ we call it the restriction res^L_K. There is also an additive induction homomorphism

$$\mathrm{ind}^L_K : U(K) \to U(L), \quad [X] \mapsto [L \times_K X].$$

Restriction and induction are the basic ingredients to make the Euler ring functor into a Mackey functor (Green functor). Here we use the setting of [4, p. 276].

The product $[G/K] \times [G/L]$ in $U(G)$ can be given another interpretation. There exists a canonical G-homeomorphism

$$G/K \times G/L \cong G \times_K \mathrm{res}^G_K G/L, \quad (uK, vL) \mapsto (uK, u^{-1}vL);$$

and $G/L = \mathrm{ind}^G_L(L/L)$. Thus, starting from the unit element $[L/L] \in U(L)$, we obtain the product as the composition

$$\text{(PROD)} \qquad [G/K] \times [G/L] = \mathrm{ind}^G_K \mathrm{res}^G_K [G/L] = \mathrm{ind}^G_K \mathrm{res}^G_K \mathrm{ind}^G_L [L/L].$$

The induction homomorphism has the simple property $\mathrm{ind}^G_K [K/K'] = [G/K']$. Hence the basic remaining problem is to express $\mathrm{res}^G_K [G/L]$ or $\mathrm{res}^G_K \mathrm{ind}^G_L [L/L]$ as a linear combination of the basis elements. The restriction data $\mathrm{res}^G_K [G/L]$ contain more information than the products. This already happens in the case of the orthogonal group O_2, as is shown by the next example; see also Table 3.

Example 2.1. The conjugacy classes of proper closed subgroups of O_2 are SO_2, D_m and C_m. The group D_m has a single conjugacy class of subgroups isomorphic to D_k if m/k is odd and two conjugacy classes if m/k is even. In the latter case we denote the conjugacy classes by D_k and $\mathrm{D}_k^\#$. We also understand that $\mathrm{D}_k \sim \mathrm{D}_k^\#$ if m/k is odd (to simplify notation). The data $\mathrm{res}_K \mathrm{O}_2/L$ are displayed in the next table

$K\backslash L$	O_2	SO_2	D_n	C_l
C_k	C_k	$2\mathrm{C}_k$	0	0
D_m	D_m	C_m	$(*)$	0
SO_2	SO_2	$2\mathrm{SO}_2$	C_n	$2\mathrm{C}_l$
O_2	O_2	SO_2	D_n	C_l

Table 1

with $(*) = \mathrm{D}_{(n,m)} + \mathrm{D}^\#_{(n,m)} - \mathrm{C}_{(n,m)}$ and (n,m) the greatest common divisor of n and m.

Most cases of the table are trivial. We explain the data $(*)$. The circle S^1 with the standard D_n-action yields in $U(\mathrm{D}_n)$ the element $\mathrm{D}_1 + \mathrm{D}_1^\# - \mathrm{C}_1$ as is directly seen from the orbit structure. The space $\mathrm{O}_2/\mathrm{D}_m \cong \mathrm{SO}_2/\mathrm{C}_m$ as D_n-space is S^1 with the action of $\mathrm{D}_{(n,m)}$ lifted to D_n.

A direct consequence of the table is now the multiplication table of $U(\mathrm{O}_2)$, by (PROD); and the restriction from $U(\mathrm{O}_2)$ to $U(\mathrm{SO}_2)$. ♢

We let φ_i for $i = 1,2,3,4$ denote the mark homomorphisms $[X] \mapsto \chi(X^H)$ for the exceptional groups $H = \mathrm{SO}_3, \mathrm{A}_5, \mathrm{S}_4, \mathrm{A}_4$, in that order and denote by $\varphi : U(\mathrm{SO}_3) \to \mathbb{Z}^4$ the ring homomorphism with components φ_i.

Proposition 2.2. *The ring homomorphisms*

$$\begin{aligned} (\pi, r_1): &\quad U(\mathrm{SO}_3) \to A(\mathrm{SO}_3) \times U(\mathrm{SO}_2) \\ (r_2, \varphi): &\quad U(\mathrm{SO}_3) \to U(\mathrm{O}_2) \times \mathbb{Z}^4 \\ (\pi, r): &\quad U(\mathrm{O}_2) \to A(\mathrm{O}_2) \times U(\mathrm{SO}_2) \end{aligned}$$

are injective. Here r, r_1, r_2 are restrictions. The cokernel of r_2 is free of rank 1.

Proof. An element in the kernel of π is a linear combination of the C_m. By the result $r(\mathrm{C}_m) = 2\mathrm{C}_m$ in 1.4 we see that such elements are detected by r. The analogous homomorphism $U(\mathrm{O}_2) \to A(\mathrm{O}_2) \times U(\mathrm{SO}_2)$ is also injective. Both results together imply the injectivity of (r_2, φ).

Let $\eta : U(\mathrm{O}_2) \to \mathbb{Z}$ be the surjective homomorphism which sends SO_2 to -1, the D_m to 1 and all other basis elements to zero. Then one verifies from the left column in Table 3 that the kernel of η is the image of r_2. □

The ring $U(\mathrm{SO}_2)$ is trivial. An additive basis consists of SO_2 and the cyclic groups C_m. All products of basis elements which do not involve the unit element are zero. The multiplication in $U(\mathrm{O}_2)$ is obtained from 2.1.

Corollary 2.3. *The product xy of elements x, y in $U(\mathrm{SO}_3)$ is obtained via the formula $xy = b(xy) + \frac{1}{2}\nu(r(x)r(y) - rb(xy))$.*

Proof. By construction, the element $xy - b(xy)$ is contained in the kernel of π and therefore a linear combination of the form $\sum_j a_j \mathrm{C}_j$. Then, by $r(\mathrm{C}_m) = 2\mathrm{C}_m$ in 2.1, $r(xy - b(xy)) = 2\sum_j a_j \mathrm{C}_j$ and hence $\frac{1}{2}\nu r(xy - b(xy)) = \sum_j a_j \mathrm{C}_j$. Now use that r is a ring homomorphism. □

Thus the multiplication table of the homogeneous spaces can be obtained in a simple manner from the Burnside ring and the right column in Table 2. The multiplication table of the $[\mathrm{SO}_3/H]$ for the Burnside ring $A(\mathrm{SO}_3)$ was determined by Schwänzl [12]. The result is stated in [3, p. 156]. Note that $r(x)r(y) = 0$ if x and y are finite groups. See also [6].

The restriction data are also a special case of the double coset formula (DCF), a main ingredient of a Mackey functor. We do not determine in this note the relevant DCFs, but later we use at least their existence.

The double coset formula has the following shape

(DCF) $$\mathrm{res}^G_K \mathrm{ind}^G_L = \textstyle\sum_\alpha a_\alpha \mathrm{ind}^K_{K\cap g^{-1}Lg} \circ c(g) \circ \mathrm{res}^L_{gKg^{-1}\cap L}$$

where $a_\alpha \in \mathbb{Z}$, the sum is taken over certain double cosets $KgL \subset G$ and $c(g)$ is induced by conjugation with g. In the case of a finite group the a_α equal 1 and the sum is over the double cosets [3, p. 164]. For the general case see [4, p. 280]. We mention without proof an example to show that the coefficients can be negative. Let us denote the summand in DCF (without its coefficient) by $[K, g, L]$, then the DCF in the case $K = L = \mathrm{O}_2$ reads

$$[\mathrm{O}_2, 1, \mathrm{O}_2] + [\mathrm{D}_2, g, \mathrm{D}_2] - [\mathrm{D}_1, 1, \mathrm{D}_1]$$

with an automorphism g of order 3 which permutes the 3 subgroups of order 2.

In this context there is also an interesting duality: we can interchange the roles of K and L. Then $\mathrm{res}^G_L \mathrm{ind}^G_K[K/K]$ is relevant. This is an instance of the fact that the induction category $\Omega(G)$ is self-dual, see [4, p. 274].

The multiplicative properties of the induction homomorphisms finally make the Euler ring functor into a Green functor. The basic property is the Frobenius reciprocity (= FR)

(FR) $$(\mathrm{ind}^G_H x) \cdot y = \mathrm{ind}^G_H (x \cdot \mathrm{res}^G_H y)$$

for $x \in U(H)$ and $y \in U(G)$.

3. Idempotent elements

We use the formalism of Green functors to construct (almost) idempotent elements in Euler rings. We call a group G isolated if it is not the limit of proper subgroups. This is the case if and only if the component G_0 of the unit element is semisimple [4, IV.3.7]. Recall that the Burnside marks yield an embedding of $A(G)$ into the ring $C(G) = C(\Phi(G), \mathbb{Z})$ of continuous functions on the space $\Phi(G)$ of closed subgroups with finite Weyl group [4, IV.4.7]. A (nonconstant) function $\tau \in C(\Phi(G), \mathbb{Z})$ with values in $\{0, 1\}$ is a (nontrivial) idempotent in this ring. There exists a smallest positive integer $m(\tau)$ such that $m(\tau)\tau$ is contained in $A(G)$ [4, IV.6.5].

Suppose now that H is an isolated subgroup of G with finite Weyl group. Then we have the function $e(H,G)$ with value 1 at (H) and value zero elsewhere. Let $m(H,G)$ be the associated integer $m(\tau)$. There is also an induction process $\mathrm{ind}_H^G : C(H) \to C(G)$ and a restriction process $\mathrm{res}_H^G : C(G) \to C(H)$ compatible with the induction and restriction for the Burnside rings (see the proof of [4, IV.2.14]). The elements just defined satisfy

$$\mathrm{ind}_H^G e(H,H) = |W_G H| e(H,G), \quad \mathrm{res}_H^G e(H,G) = e(H,H).$$

We set $m(H,G)e(H,G) = t(H,G) \in A(G)$.

Lemma 3.1. *Let $m(H)$ denote the product of the prime-divisors of the commutator quotient $H/[H,H]$. Then the following relations hold*

$$m(H,G) = |W_G H| m(H), \qquad t(H,G)^2 = m(H,G)t(H,G),$$
$$\mathrm{ind}_H^G t(H,H) = t(H,G), \qquad \mathrm{res}_H^G t(H,G) = |W_G H| t(H,H).$$

Proof. Since $\mathrm{ind}_H^G t(H,H) = \mathrm{ind}_H^G m(H)e(H,H) = m(H)|W_G H|e(H,G)$, we see that $m(H,G)$ divides $m(H,H)|W_G H|$. Suppose $ne(H,G)$ is contained in $A(G)$. Let p be a prime divisor of the order of $H/[H,H]$; since H is isolated, this is a finite group. Let $W_p H$ denote a p-Sylow subgroup of WH and let $N_p H$ be its preimage in NH. By elementary p-group theory there exists a subgroup $V \subset H$ such that $|H/V| = p$ and V is normal in $N_p H$. We consider the congruence [4, IV.5.7] for $N_p H/V$. It is satisfied by the element $ne(H,G)$, and this congruence says that $n \equiv 0 \bmod |N_p H/V| = p|W_p H|$. Hence $m(H)|W_G|$ divides $m(H,G)$. The congruences also show that $m(H)e(H,H) \in A(H)$, since a $W_H C$-congruence for $A(H)$ gives a condition for the Burnside mark at H if and only $C \lhd H$ and H/C is cyclic. Compare also [9, Lemma 8] and [8, Prop. 3.2]. □

The restriction of $t_G = t(G,G)$ to each Burnside ring $A(H)$, H a proper subgroup of G, is zero by definition of t_G and [4, IV.5.8].

We now try to lift t_G to $U(G)$. For that purpose let us consider the localized situation $\pi[\frac{1}{m}] : U(G)[\frac{1}{m}] \to A(G)[\frac{1}{m}]$, i.e., we introduce $m = m(G)$ as a denominator. Since we are dealing with free abelian groups we can specify elements of $U(G)$ and $A(G)$ by their image in the localizations. The element $t'_G = t_G/m(G)$ is an idempotent element of the localized Burnside ring. It follows from [4, IV.1.14] that the kernel of $\pi[\frac{1}{m}]$ is the nilradical. By a general result in commutative algebra [1, II,§4.3], there exists a unique idempotent $T'_G \in U(G)[\frac{1}{m}]$ with image t'_G. Let $k \geq 1$ be minimal such that $T_G = m^k T'_G \in U(G)$. Then $T_G^2 = m^k T_G$ and $\pi(T_G) = m^{k-1} t_G$. Moreover T_G restricts to zero in each $U(H)$, $H \neq G$, since the restriction is nilpotent (being zero in $A(H)$) but also almost idempotent. We set $n(G) = m(G)^k$.

Proposition 3.2. *Let H be an isolated subgroup of G with finite Weyl group $W_G H$. Set $T(H,G) = \mathrm{ind}_H^G T_H$. Then $\mathrm{res}_H^G T(H,G) = |W_G H| T_G$ and*

$$T(H,G)^2 = |W_G H| n(H) T(H,G).$$

The restriction of $T(H,G)$ *to a proper subgroup of* H *is zero. If* H, K *are two such nonconjugate subgroups, then* $T(H,G)T(K,G) = 0$.

Proof. These assertions turn out to be a consequence of DCF and FR. We apply the DCF to $\mathrm{res}^G_K \mathrm{ind}^G_H T_H$. Since T_H restricts to zero for all proper subgroups only summands with $H \cap gKg^{-1} = H$ can yield a nonzero contribution. If K is not subconjugate to H each summand is zero. In the case that $K = H$ the summands with $gH \in W_G H$ remain, and each such summand yields the identity.

We now apply this result and FR.

$$T(K,G) \cdot T(H,G) = \mathrm{ind}^G_K(T_K) \cdot \mathrm{ind}^G_H(T_H) = \mathrm{ind}^G_K(T_K \cdot \mathrm{res}^G_K \mathrm{ind}^G_H(T_H)).$$

The result is zero if K is not subconjugate to H and in the case $H = K$ equal to $|W_G H| n(H) T(H,G)$ by the previous result and the main property of T_H. □

4. The exceptional elements

Let us apply the general results of the preceding section to the group SO_3.

We also use the notation $(\mathrm{SO}_3, \mathrm{A}_5, \mathrm{S}_4, \mathrm{A}_4) = (H(1), H(2), H(3), H(4))$. We obtain the elements $T(H(j), G) = e_j \in U(\mathrm{SO}_3)$. By 3.1 and 3.2 they satisfy:

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_3^2 = 2e_3, \quad e_4^2 = 6e_4, \quad e_i e_j = 0$$

for $i \neq j$. We need another element $f_3 = \mathrm{ind}^{H(1)}_{H(3)} \eta_3$ where $\eta_3 \in U(\mathrm{S}_4)$ is the element with the Burnside marks $\eta_3(\mathrm{S}_4) = 1$, $\eta_3(\mathrm{A}_4) = 3$, and value zero for the remaining subgroups. From the definitions we verify the relation $2f_3 = e_3 + e_4$.

Proposition 4.1. *The elements* e_1, e_2, f_3, e_4 *are a* $\mathbb{Z}$*-basis of the kernel of the restriction* $r : U(\mathrm{SO}_3) \to U(\mathrm{O}_2)$.

Proof. Let $E \subset U(\mathrm{SO}_3)$ denote the span of the elements e_j and f_3. From the definition of E we see that E is contained in the kernel of r. Let $\varphi : U(\mathrm{SO}_3) \to \mathbb{Z}^4$ be the collection of the mark homomorphism for the $H(j)$ as in Proposition 2.2. By the congruences for the Burnside ring [3, p. 155], we see that the cokernel of φ has order 2 and the cokernel of φ restricted to the kernel of r has order 6. From the definition of the elements e_j and f_3 we also see that $\varphi|E$ is injective and $\varphi(E)$ a subgroup of index 6. Let x be contained in the kernel of r. We can find $e \in E$ such that $x - e$ is contained in the kernel of (r_2, φ). Hence $x - e = 0$, by Proposition 2.2. □

The elements in question have the following expansion in terms of the standard basis:

$$\begin{aligned}
e_1 &= \mathrm{SO}_3 - x - e_2 \\
e_2 &= \mathrm{A}_5 - \mathrm{A}_4 - \mathrm{D}_5 - \mathrm{D}_3 + \mathrm{C}_3 + 2\mathrm{C}_2 - \mathrm{C}_1 \\
f_3 &= \mathrm{S}_4 + \mathrm{A}_4 - \mathrm{D}_4 - \mathrm{D}_3 - \mathrm{C}_3 + \mathrm{C}_2 \\
e_4 &= 3\mathrm{A}_4 - \mathrm{D}_2 - 3\mathrm{C}_3 + \mathrm{C}_1
\end{aligned}$$

with $x = \mathrm{O}_2 + \mathrm{S}_4 - \mathrm{D}_4 - \mathrm{D}_3 + \mathrm{C}_2$. For the elements e_2, f_3, e_4 one computes the expansion in the Burnside ring of the groups $H(2), H(3), H(4)$ and then induces up to SO_3. A systematic method for the basis expansion (for finite groups) uses the Möbius function of the subgroup lattice; but since one has to reorganize in terms of conjugacy classes this amounts in practice to 1.1.

We describe some geometry related to these elements e_1 and e_2.

The space $P = SO_3/A_5$ is the Poincaré sphere. We consider it as an A_5-manifold. It has a single fixed point. When we excise an open invariant disk about the fixed point we obtain an A_5-manifold D (Poincaré disk), where each $H \neq A_5$ has a homology disk as fixed point set. Hence the Burnside mark is the function associated to the idempotent $1 - T_{H(2)}$ of $A(A_5)$. Moreover, one could imitate the plus-construction of Quillen and attach A_5-free 1- and 2-cells to D in order to kill the fundamental group and without changing the homology; then one obtains a 3-dimensional CW-space with empty or contractible fixed point sets. See [9] and [10] for such spaces in general. The basis expansion of $1 - T_{H(2)}$ can be determined by group theory. We can also use the geometry of D: the fixed point sets of the nontrivial cyclic groups are intervals, the other groups (except A_5) have a single fixed point; now one uses 1.1; in the case of the closed intervals one has to count the points with larger isotropy group on them in order to determine the reduced Euler characteristic of the one-point compactification. If we take the double $D \cup_{\partial D} D = S$ of the Poincaré disk we obtain a homology sphere. Its dimension function is not stably linear, i.e., not the difference of dimension functions of linear representations. In order to prove this, one can use [4, III.5.5].

There exists a finite SO_3-CW-complex X with empty fixed set X^H for $H = SO_3, A_5$ and with contractible fixed sets for all other groups. See [10] for the geometric significance of this Oliver disk. From these properties of the fixed sets we see that $[X]$ is an idempotent element of $U(SO_3)$. (Up to homotopy it is already idempotent on the space level.) From [11, p. 234] we see, that $[X]$ is the linear combination $x = O_2 + S_4 - D_4 - D_3 + C_2$ in terms of the basis elements. From the geometry we see that $1 - [X]$ is contained in the kernel of r_2. (In the Burnside ring it represents the element $1 - y$ in the notation of [3, p. 156].) The homotopy type of X is realized in [11] by a smooth action of SO_3 on an 8-dimensional disk. Again we can form the double $\Sigma = X \cup_{\partial X} X$ and obtain an 8-dimensional homotopy representation Σ. In order to determine its basis expansion in $U(SO_3)$ one must, by additivity in the Euler groups, determine ∂X.

The elements e_1 and e_2 are idempotents of the ring $U = U(SO_3)$. As always one can use these idempotents to decompose the ring U,

$$U = e_1 U \oplus e_2 U \oplus (1 - e_1 - e_2)U.$$

The first two summands are isomorphic to the ring $\mathbb{Z}$ under the Burnside marks $\varphi_{H(j)} : e_j U \to \mathbb{Z}$.

5. The standard representation

We interpret classical geometry in terms of Euler rings. The table below is then relevant for Corollary 2.3.

Recall that a classical method to determine the conjugacy classes of finite subgroups of SO_3 uses the standard action on the unit sphere $S^2 \cong \mathrm{SO}_3/\mathrm{SO}_2$, see e.g. [13, Sec. 2.6]. A similar method works for arbitrary continuous orientable actions of finite groups on S^2, see [5, 12.5.8], and even for two-dimensional orientable homotopy representations. We recall the result and its consequences for our purposes. One also uses (1) in 1.4.

Proposition 5.1. *The standard representation of $L \neq \mathrm{SO}_3$ on S^2 has a singular set S^2_s with a finite number of (topologically) cyclic isotropy groups. Thus if $S^2_s = \bigcup_{i=1}^r L/L_i$, then $\mathrm{res}_L\, S^2 = \sum_i L_i - (r + \chi(S^2/L)) \in U(L)$. If we induce up to SO_3, then we obtain the values of the product $L \cdot \mathrm{SO}_2$ in $U(\mathrm{SO}_3)$. Since the induction from SO_2 to SO_3 is injective, we also obtain the values of the restriction to $U(\mathrm{SO}_2)$.* □

For the convenience of the reader we display the result. The only difference occurs for the group S_4, since this group has two conjugacy classes of elements of order 2. We denote by $C_2^{\#}$ the group of a transposition.

L	$\mathrm{res}_L\, \mathrm{SO}_3/\mathrm{SO}_2$	$\mathrm{res}_{\mathrm{SO}_2}\, \mathrm{SO}_3/L$
O_2	$\mathrm{SO}_2 + \mathrm{D}_1 - \mathrm{C}_1$	$\mathrm{SO}_2 + \mathrm{C}_2 - \mathrm{C}_1$
SO_2	$2\mathrm{SO}_2 - \mathrm{C}_1$	$2\mathrm{SO}_2 - \mathrm{C}_1$
D_m	$\mathrm{C}_m + \mathrm{D}_1 + \mathrm{D}_1^{\#} - \mathrm{C}_1$	$\mathrm{C}_m + 2\mathrm{C}_2 - \mathrm{C}_1$
C_m	$2\mathrm{C}_m$	$2\mathrm{C}_m$
A_5	$\mathrm{C}_5 + \mathrm{C}_3 + \mathrm{C}_2 - \mathrm{C}_1$	$\mathrm{C}_5 + \mathrm{C}_3 + \mathrm{C}_2 - \mathrm{C}_1$
S_4	$\mathrm{C}_4 + \mathrm{C}_3 + \mathrm{C}_2^{\#} - \mathrm{C}_1$	$\mathrm{C}_4 + \mathrm{C}_3 + \mathrm{C}_2 - \mathrm{C}_1$
A_4	$2\mathrm{C}_3 + \mathrm{C}_2 - \mathrm{C}_1$	$2\mathrm{C}_3 + \mathrm{C}_2 - \mathrm{C}_1$

TABLE 2

Suppose L is finite. The isotropy groups are cyclic. Each element $g \neq 1$ has two fixed points on S^2. All elements in the cyclic subgroup generated by g have the same fixed points. The isotropy group of a fixed point is therefore a maximal cyclic subgroup of L. The nontrivial isotropy groups of the action are therefore the maximal cyclic subgroups. If $C \leq L$ is such a group and a_C the coefficient of C, then $2 = \chi(S^2) = \chi((S^2)^C) = a_C|L/C^C| = a_C|W_L C|$. Hence $a_C = 1, 2$ and $|W_L C| = 2, 1$, respectively. From this information and the knowledge of the sum of the coefficients we obtain the data of the table.

6. RESTRICTION TO THE ORTHOGONAL GROUP

We now determine the restriction data for the projective plane $\mathrm{SO}_3/\mathrm{O}_2$. We need one more piece of notation. The group S_4 has two conjugacy classes of subgroups of order 2: the group $\mathrm{C}_2^{\#}$ generated by a transposition and C_2. The normalizer of $\mathrm{C}_2^{\#}$ is denoted $\mathrm{D}_2^{\#}$. The normalizer of C_2 is a D_4 and contains another conjugacy class D_2.

L	$\mathrm{res}_{\mathrm{O}_2}\,\mathrm{SO}_3/L$	$\mathrm{res}_L\,\mathrm{SO}_3/\mathrm{O}_2$
O_2	$\mathrm{SO}_2 + \mathrm{D}_2 - \mathrm{D}_1$	$\mathrm{O}_2 + \mathrm{D}_2 - \mathrm{D}_1$
SO_2	$2\mathrm{SO}_2 + \mathrm{D}_1 - \mathrm{C}_1$	$\mathrm{SO}_2 + \mathrm{C}_2 - \mathrm{C}_1$
D_{2m+1}	$\mathrm{D}_{2m+1} - 2\mathrm{D}_1 + \mathrm{C}_2$	D_{2m+1}
D_{2m}	$\mathrm{D}_{2m} + 2\mathrm{D}_2 - 3\mathrm{D}_1 + \mathrm{C}_1$	$\mathrm{D}_{2m} + \mathrm{D}_2 + \mathrm{D}_2^\# - \mathrm{C}_2 - \mathrm{D}_1 - \mathrm{D}_1^\# + \mathrm{C}_1$
C_m	C_m	C_m
A_5	$\mathrm{D}_5 + \mathrm{D}_3 + \mathrm{D}_2 - 3\mathrm{D}_1 + \mathrm{C}_1$	$\mathrm{D}_5 + \mathrm{D}_3 + \mathrm{D}_2 - 3\mathrm{C}_2 + \mathrm{C}_1$
S_4	$\mathrm{D}_4 + \mathrm{D}_3 + \mathrm{D}_2 - 3\mathrm{D}_1 + \mathrm{C}_1$	$\mathrm{D}_4 + \mathrm{D}_3 + \mathrm{D}_2^\# - \mathrm{C}_2 - 2\mathrm{C}_2^\# + \mathrm{C}_1$
A_4	$2\mathrm{D}_3 + \mathrm{C}_3 - \mathrm{D}_1$	$\mathrm{D}_2 + \mathrm{C}_3 - \mathrm{C}_2$

TABLE 3

Proposition 6.1. *The standard representations in the Euler ring.*

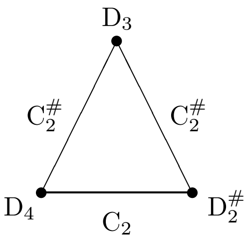
Proof. The table can be verified by different methods. Firstly, one could use group theory and 1.1.

Secondly, the left column is a consequence of known results. We can use the products of $\mathrm{SO}_3/\mathrm{O}_2$ with the homogeneous spaces in the Burnside ring [3, p. 281]. This gives the result modulo D_1 and the cyclic groups. Then one restricts further to SO_2, by using Proposition 5.1, and then one uses part (1) in 1.4.

We discuss the right column by using the geometry of the actions on the projective plane. The case of infinite L in the table is easily derived from the geometry.

Each element $g \neq 1$ has either a unique fixed point (order of g not 2) or a fixed point and a fixed S^1 (g of order 2); in each case the Euler characteristic is 1. The general formula $\chi(X/L) = \frac{1}{|L|}(\sum_{g\in L}\chi(X^g))$, see [3, p. 96], thus yields in our case that $\chi(X/L) = 1$. A closer study of the topology gives the result stated in part (3) of 1.4. Note that if an isotropy group of a fixed point is a dihedral group, then the orbit space is a manifold with boundary.

The most complicated case is perhaps the group S_4. The orbit space is a disk. The next figure indicates the subsets of a given orbit type.



In order to visualize the situation the reader may start with a standard cube in 3-space with vertices $(\gamma_1, \gamma_2, \gamma_3)$, $\gamma_j \in \{\pm 1\}$ and its symmetry group S_4.

The orbit space in the case A_5 has a similar shape. The vertices represent D_j for $j = 2, 3, 5$ and the edges C_2. The interior of the triangle corresponds in both cases to the free orbits.

In order to determine the coefficients of the groups one can use the reduced Euler characteristic 1.1. For example, in the S_4-case the set of points with isotropy group $C_2^{\#}$ consists of two open intervals; the one-point compactification is a wedge of two circles with reduced Euler characteristic -2. □

7. THE EULER RING OF THE UNITARY GROUP

We have the double covering $p : SU_2 \to SO_3$. It induces ring homomorphisms $A(p) = p^*$ and $U(p) = p^*$. Let $H^* = p^{-1}(H)$. Then $p^*[G/H] = [G^*/H^*]$. The homomorphism $U(p)$ is injective. The only basis elements of $U(SU_2)$ which are not contained in the image of $U(p)$ are the C_m with $m \equiv 1 \bmod 2$. It is easy to investigate their product behavior. By the way, $C_m^* = C_{2m}$. In $U(SU_2)$ we have

$$O_2^* \cdot C_n = C_n, \qquad SO_2^* \cdot C_n = 2\,C_n, \qquad SU_2 \cdot C_n = C_n.$$

The remaining products with C_n are zero. Thus the Euler ring of SU_2 is basically the same as the Euler ring of SO_3. The Euler ring $U(SU_2)$ was investigated by Hoffmann [7].

8. UNITS

We discuss units in Euler rings. Let R^* denote the group of units in the ring R. The unit element, represented by a point, of $U(G)$ will now be written as 1. Certain units can be obtained from representations (called linear units) and homotopy representations. Let $X * Y$ denote the join of two G-spaces. By additivity in the Euler groups we have $[X*Y] = [X]+[Y]-[X]\cdot[Y]$ and therefore the multiplicativity relation $(1-[X*Y]) = (1-[X])(1-[Y])$ holds. Thus we obtain a homomorphism of the Grothendieck monoid of isomorphism classes of finite G-complexes into the multiplicative monoid of $U(G)$. An element $u = 1-[X]$ is a unit in $U(G)$ if and only if for each H with finite Weyl group $\chi(X^H) \in \{0,2\}$. One method to see this is to consider the same element in the Burnside ring. Then its Burnside mark is a function with values ± 1 and therefore a unit of order two. Hence $u^2 = 1+n$ with a nilpotent element n. But elements of this form are units. The condition is satisfied if each fixed set X^H has the homology of a sphere.

Let us denote by $R(G;\mathbb{R})$ the representation ring of real representations of G. We are not interested in its multiplicative structure. Additively it is the free abelian group with basis the irreducible real representations. If V is a representation with unit sphere $S(V)$, then $1-[S(V)] \in U(G)^*$. Since $S(V \oplus W) = S(V) * S(W)$ we obtain, by the remark above, a homomorphism $R(G;\mathbb{R}) \to U(G)^*$ which assigns to the representation V the element $1-[S(V)]$. If V is a one-dimensional representation with kernel H, then $[S(V)] = [G/H]$ and $1-[S(V)]$ has order two. Let $\overline{R}(G)$ denote the quotient of $R(G;\mathbb{R})$ by the even multiples of the one-dimensional representations. We obtain an induced homomorphism $\Delta_G : \overline{R}(G) \to U(G)^*$.

Suppose $x, y \in U(G)$ satisfy $x^2 = x$ and $y^2 = 2y$. Then $1-2x$ and $1-y$ are units of order two. From Proposition 4.1 we conclude that

$$E^* = \{1 - 2a_1e_1 - 2a_2e_2 - a_3e_3 \mid a_j \in \{0,1\}\}$$

is a subgroup of $U(\mathrm{SO}_3)^*$. Idempotents $x \in U(G)$ in general are represented by spaces X with empty or contractible fixed point sets [10]. See also [4, IV.7] for idempotents in the Burnside ring.

Proposition 8.1. *The homorphism Δ_G is an isomorphism for SO_2 and O_2. For $G = \mathrm{SO}_3$ the map Δ_G is split injective with complementary group E^*.*

Proof. We list the irreducible real representations, see e.g. [2, II.5].

SO_2. The trivial representation ε and the 2-dimensional representations U_j, $j \geq 1$ on $\mathbb{R}^2$ where $A \in \mathrm{SO}_2$ acts as $x \mapsto A^j x$.

O_2. The trivial representation ε and the determinant representation η. The 2-dimensional representations V_j, $j \geq 1$ with restriction U_j to SO_2.

SO_3. For each $j \geq 0$ a representation W_j of dimension $2j+1$. The restriction to SO_2 is $\varepsilon + U_1 + \ldots + U_j$.

SO_2. From $1 - [S(U_j)] = 1 - \mathrm{C}_j$ we see that $\Delta(\sum k_j U_j) = 1 - \sum k_j C_j$, and this implies the claim about $U(\mathrm{SO}_2)^*$.

O_2. By multiplying if necessary with $-1 = \Delta(1-\varepsilon)$ and $1 - \mathrm{SO}_2 = \Delta(1-\eta)$ we can assume that a unit has the form $x = 1 + \sum d_k \mathrm{D}_k + \sum c_k \mathrm{C}_k$. Suppose its restriction to $U(\mathrm{SO}_2)$ has the form $1 + \sum k_j \mathrm{C}_j$. We multiply x with $y = \Delta(\sum k_j V_j)$ and obtain a unit with trivial restriction to SO_2. Suppose $xy = 1 + \sum m_k \mathrm{D}_k + \sum n_k \mathrm{C}_k$. Then $m_k + 2n_k = 0$ for each k. Let k be maximal with $m_k \neq 0$ and take the D_k Burnside mark $1 + 2m_k = 1 - 4n_k$. This value cannot be ± 1. Hence xy is the trivial unit. This implies the claim about $U(\mathrm{O}_2)^*$.

SO_3. The composition $\mathrm{res} \circ \Delta : \overline{R}(\mathrm{SO}_3) \to U(\mathrm{SO}_3)^* \to U(\mathrm{SO}_2)^*$ is an isomorphism, since the restriction $\overline{R}(\mathrm{SO}_3) \to \overline{R}(\mathrm{SO}_2)$ is an isomorphism. If we multiply a unit in $U(\mathrm{SO}_3)^*$ with a suitable element in the image of Δ, we can assume that the SO_2-restriction is trivial. The units in $U(\mathrm{O}_2)$ which restrict to the trivial unit have the form $a\mathrm{O}_2 + b\mathrm{SO}_2 + \sum d_k \mathrm{D}_k + \sum c_k \mathrm{C}_k$ with $d_k + 2c_k = 0$ and $(a,b) = (1,0)$, $= (-1,0)$. In the first case the unit in $U(\mathrm{O}_2)$ is trivial. In the second case we have, by Proposition 2.2, $1 = b = \sum d_k$. Again we take the D_k Burnside mark for a maximal k with $d_k \neq 0$ and conclude that $-1 + 2d_k = \pm 1$. Since this is impossible, we conclude that a unit of the assumed form is not contained in the restriction from SO_3. We finally have to consider the case that a unit x has trivial restriction to $U(\mathrm{O}_2)$. Then $1 - x$ is contained in the kernel of r_2 and therefore, by Proposition 4.1, a linear combination of e_1, e_2, f_3, e_4. We consider the Burnside marks for $H(j)$, $j = 1,2,3,4$ and conclude that $x \in E^*$. □

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