Münster J. of Math. **10** (2017), 409–424
DOI 10.17879/70299609822
urn:nbn:de:hbz:6-70299611617

Münster Journal of Mathematics

Obstructions to lifting cocycles on groupoids and the associated C*-algebras

Marius Ionescu and Alex Kumjian

(Communicated by Siegfried Echterhoff)

Abstract. Given a short exact sequence of locally compact abelian groups $0 \to A \to B \to C \to 0$ and a continuous C-valued 1-cocycle ϕ on a locally compact Hausdorff groupoid Γ we construct a twist of Γ by A that is trivial if and only if ϕ lifts. The cocycle determines a strongly continuous action of $\widehat{C}$ into $\operatorname{Aut} C^*(\Gamma)$ and we prove that the C^*-algebra of the twist is isomorphic to the induced algebra of this action if Γ is amenable. We apply our results to a groupoid determined by a locally finite cover of a space X and a cocycle provided by a Čech 1-cocycle with coefficients in the sheaf of germs of continuous $\mathbb{T}$-valued functions. We prove that the C^*-algebra of the resulting twist is continuous trace and we compute its Dixmier–Douady invariant.

1. Introduction

It is a well-known fact that given a 1-cocycle $\phi : \Gamma \to \mathbb{T}$ on an étale groupoid Γ, there is an automorphism α of $C^*(\Gamma)$ such that $\alpha(f)(\gamma) = \phi(\gamma)f(\gamma)$ for all $f \in C_c(\Gamma)$ (see [19, Prop. II.5.1]). If ϕ can be lifted to an $\mathbb{R}$-valued 1-cocycle $\tilde{\phi}$, there is a strongly continuous 1-parameter group of automorphisms $\tilde{\alpha} : \mathbb{R} \to \operatorname{Aut} C^*(\Gamma)$ such that $\alpha = \tilde{\alpha}_1$. But in general there is a cohomological obstruction to lifting ϕ. There is a central groupoid extension Σ_ϕ of Γ by $\mathbb{Z}$, called a twist, which is trivial precisely when ϕ can be lifted. Our goal in this paper is to describe the structure of $C^*(\Sigma_\phi)$ in terms of $C^*(\Gamma)$ and the automorphism α.

Our results will be proven in a somewhat more general form. Given a locally compact Hausdorff groupoid Γ, a short exact sequence of locally compact abelian groups

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

and a continuous 1-cocycle $\phi : \Gamma \to C$ we construct a twist Σ_ϕ of Γ by A which is trivial if and only if the cocycle lifts. As above the cocycle ϕ determines

This work was partially supported by Simons Foundation Collaboration grants #209277 (MI) and #353626 (AK)..

a strongly continuous action $\alpha^\phi : \widehat{C} \to \operatorname{Aut} C^*(\Gamma)$. We prove that $C^*(\Sigma_\phi)$ is isomorphic to the induced algebra for this action if Γ is amenable (see Theorem 4.3). So in particular, there is an action $\gamma : \widehat{B} \to \operatorname{Aut} C^*(\Sigma_\phi)$ such that $C^*(\Sigma_\phi) \rtimes_\gamma \widehat{B}$ is strong Morita equivalent to $C^*(\Gamma) \rtimes_{\alpha^\phi} \widehat{C}$.

In Example 3.6 we consider a groupoid Γ determined by a locally finite open cover $\mathcal{U} := \{U_i\}_{i\in I}$ of a space X (see [17]) and take $A := \mathbb{Z}$, $B := \mathbb{R}$ and $C := \mathbb{T}$. The 1-cocycle ϕ arises from a Čech 1-cocycle $\lambda = \{\lambda_{ij}\}_{i,j\in I}$ with coefficients in $\mathcal{T}$, the sheaf of germs of continuous $\mathbb{T}$-valued functions. If λ satisfies a certain liftability hypothesis, the corresponding twist Σ_ϕ is then determined by a Čech 2-cocycle $\lambda^\star$ with coefficients in the constant sheaf with fiber $\mathbb{Z}$ (we denote the sheaf by $\mathbb{Z}$ for the sake of simplicity) that measures the obstruction to lifting λ to a Čech 1-cocycle with coefficients in the sheaf of germs of continuous $\mathbb{R}$-valued functions. Finally, we show in Example 4.6 that $C^*(\Sigma_\phi)$ is a continuous trace algebra and compute its Dixmier–Douady invariant using the cohomology class $[\lambda]$ which is identified with $[\lambda^\star]$ via the standard isomorphism of Čech cohomology groups $\check{H}^1(X,\mathcal{T}) \cong \check{H}^2(X,\mathbb{Z})$. This example was a key motivation for this project and it was suggested by results of [16] (see also [22]).

In this note we assume that all topological spaces and groupoids are second countable locally compact and Hausdorff and all groups are second countable abelian locally compact and Hausdorff. Hence all spaces are paracompact. Moreover, we assume that all groupoids are endowed with a Haar system.

2. Preliminaries

In this section we present first a characterization of the induced algebra from an action of a closed subgroup of a locally compact abelian group on a C^*-algebra. Our characterization is essential in our proof of the main result, Theorem 4.3. For a related characterization of the induced algebra see [6]. Second, we provide conditions that guarantee that the natural map j as defined in equation (1) from the C^*-algebra of a subgroupoid into the multiplier algebra of the C^*-algebra of the groupoid is faithful. Our results generalize well-known facts about group C^*-algebras.

2.1. **A characterization of the induced algebra.** Let H be a closed subgroup of a locally compact abelian group G, let D be a C^*-algebra and let $\alpha : H \to \operatorname{Aut} D$ be a strongly continuous action. If $f : G \to D$ is a continuous function such that $f(x-h) = \alpha_h(f(x))$ for all $h \in H$, $x \in G$ then $x + H \mapsto \|f(x)\|$ yields a well-defined continuous function. We define the *induced C^*-algebra* (see [22, §3.6]) by

$$\operatorname{ind}_H^G(D,\alpha) := \big\{ f : G \to D \text{ continuous} \mid f(x-h) = \alpha_h(f(x)),\, h \in H,\, x \in G \\ \text{and } x + H \mapsto \|f(x)\| \in C_0(H/G) \big\}.$$

There is a canonical translation action $\beta : G \to \operatorname{Aut}(\operatorname{ind}_H^G(D,\alpha))$ given by $\beta_g(f)(x) := f(x-g)$. By [16, Lem. 3.1] $D \rtimes_\alpha H$ is strong Morita equivalent to $\operatorname{ind}_H^G(D,\alpha) \rtimes_\beta G$.

Point evaluation at 0 yields a surjective homomorphism $\pi : \operatorname{ind}_H^G(D,\alpha) \to D$ given by $\pi(f) = f(0)$. There is a natural translation action

$$\tau : G \to \operatorname{Aut}(C_0(G/H))$$

(where $\tau_g(k)(x+H) = k(x-g+H)$) and a G-equivariant homomorphism

$$j : C_0(G/H) \to M(\operatorname{ind}_H^G(D,\alpha))$$

determined by pointwise multiplication. It is easy to check that these maps satisfy the following conditions for all $f \in \operatorname{ind}_H^G(A,\alpha)$:

(i) For all $k \in C_0(G/H)$, $\pi(j(k)f) = k(H)\pi(f)$;
(ii) for all $h \in H$, $\pi(\beta_h(f)) = \alpha_h(\pi(f))$;
(iii) if $\pi(\beta_g(f)) = 0$ for all $g \in G$, then $f = 0$;
(iv) and

$$\lim_{x+H\to\infty} \|f(x)\| = 0.$$

We show below that these conditions characterize the induced algebra. Note that $\operatorname{ind}_H^G(D,\alpha)$ may be identified with the section algebra of a continuous C^*-bundle over G/H with fibers isomorphic to D and factor maps π_{x+H}.

Theorem 2.2. *Let H be a closed subgroup of a locally compact abelian group G. Let D and E be C^*-algebras, $\alpha : H \to \operatorname{Aut} D$ and $\gamma : G \to \operatorname{Aut}(E)$ strongly continuous actions, $\rho : E \to D$ a surjective homomorphism, and $i : C_0(G/H) \to Z(M(E))$ a G-equivariant homomorphism. Suppose that*

(i) *for all $k \in C_0(G/H)$ and $e \in E$, $\rho(i(k)e) = k(H)\rho(e)$;*
(ii) *for all $h \in H$ and $e \in E$, $\rho(\gamma_h(e)) = \alpha_h(\rho(e))$;*
(iii) *if $\rho(\gamma_g(e)) = 0$ for all $g \in G$, then $e = 0$;*
(iv) *for all $e \in E$,*

$$\lim_{x+H\to\infty} \|\rho(\gamma_x(e))\| = 0.$$

Then there is a (unique) G-equivariant isomorphism $\Psi : E \to \operatorname{ind}_H^G(D,\alpha)$ such that $\pi \circ \Psi = \rho$ and $j(k)\Psi(e) = \Psi(i(k)e)$ for all $k \in C_0(G/H)$ and $e \in E$.

Proof. For $e \in E$, we define a function $\Psi(e) : G \to D$ by $\Psi(e)(x) := \rho(\gamma_{-x}(e))$. It is straight-forward to check that $\Psi(e)$ is continuous and that $\Psi(e) \in \operatorname{ind}_H^G(D,\alpha)$. Indeed we have

$$\Psi(e)(x-h) = \rho(\gamma_{h-x}(e)) = \rho(\gamma_h(\gamma_{-x}(e))) = \alpha_h(\rho(\gamma_{-x}(e))) = \alpha_h(\Psi(e)(x)).$$

It is also routine to check that the map thus defined, $\Psi : E \to \operatorname{ind}_H^G(D,\alpha)$, is an equivariant $*$-homomorphism. Injectivity follows from (iii) above. Let $k \in C_0(G/H)$ and $e \in E$. Then by (i) above and the G-equivariance of i it follows that, for all $x \in G$,

$$\begin{aligned}\Psi(i(k)e)(x) &= \rho(\gamma_{-x}(i(k)e)) = \rho(i(\tau_{-x}(k))\gamma_{-x}(e)) \\ &= \tau_{-x}(k)(H)\rho(\gamma_{-x}(e)) \\ &= k(x+H)\Psi(e)(x) = (j(k)\Psi(e))(x),\end{aligned}$$

and hence $j(k)\Psi(e) = \Psi(i(k)e)$. Thus Ψ is a map of $C_0(G/H)$-algebras. Since $\rho(e) = \Psi(e)(0)$ for all $e \in E$ we have $\pi \circ \Psi = \rho$.

Next we use [22, Prop. C.24] (see also [7, Prop. 14.1]), which states that a submodule of continuous sections of an upper semicontinuous C^*-bundle which is fiberwise dense must also be norm dense in the C^*-algebra of sections vanishing at infinity, to prove that $\Psi(E)$ is dense in $\operatorname{ind}_H^G(D, \alpha)$. This will prove that Ψ is surjective and therefore an isomorphism. We check hypotheses (a) and (b) of the proposition. Let $f \in \Psi(E)$ and let $k \in C_0(G/H)$, then $f = \Psi(e)$ for some $e \in E$ and so

$$j(k)f = j(k)\Psi(e) = \Psi(i(k)e) \in \Psi(E).$$

This proves condition (a). Now since each factor map is conjugate to $\pi \circ \beta_x$ for some $x \in G$, its restriction to $\Psi(E)$ is surjective. Thus (b) holds and the result follows. □

Remark 2.3. Let D_1 and D_2 be C^*-algebras. A homomorphism $\phi : D_1 \to M(D_2)$ is said to be *nondegenerate* if it extends uniquely to a unital map $\phi : M(D_1) \to M(D_2)$ or equivalently if $\{\phi(e_\lambda)\}_\lambda$ converges strictly to the unit of $M(D_2)$ for every approximate identity $\{e_\lambda\}_\lambda$ in D_1. Property (iv) above is equivalent to the requirement that $i : C_0(G/H) \to M(E)$ be nondegenerate.

Remark 2.4. Note that since $C_0(G/H) \cong C^*((G/H)\widehat{\ })$, the map i is determined by a strictly continuous homomorphism $u : (G/H)\widehat{\ } \to UM(E)$ where $UM(E)$ is the unitary group of the multiplier algebra $M(E)$. The nondegeneracy of i is equivalent to the requirement that $u_0 = 1$ and condition (i) above is satisfied if and only if $\rho(u_\chi e) = \rho(e)$ for all $\chi \in (G/H)\widehat{\ }$ and $e \in E$.

Remark 2.5. Our characterization of the induced algebra is provided to facilitate the proof of our main result, namely, that the C^*-algebra of a certain groupoid extension regarded as an obstruction to lifting a cocycle is an induced C^*-algebra. One could in principle use Echterhoff's more general characterization (see the main theorem of [6]), but this would have required the identification of the C^*-algebra of the quotient groupoid with a certain quotient of the putative induced algebra. The proof of this identification would use similar techniques with those in our proof and it would not necessarily lead to a simplification of our arguments.

2.6. **Multiplier algebras of groupoid C^*-algebras.** Given a C^*-algebra D let $M(D)$ denote its multiplier algebra. We view $M(D)$ as the C^*-algebra $\mathcal{L}(D_D)$ of adjointable maps on the Hilbert C^*-module D_D (see, for example, [18, §2.3] and [12]). Recall that D_D is a full Hilbert C^*-module via $d \cdot e = de$ and $\langle d, e\rangle_D = d^*e$.

Assume that Σ is a closed subgroupoid of a locally compact Hausdorff groupoid Γ such that $\Sigma^0 = \Gamma^0$. Assume that Σ is endowed with a Haar system $\beta = \{\beta^u\}_{u\in\Gamma^0}$ and that Γ is endowed with a Haar system $\lambda = \{\lambda^u\}_{u\in\Gamma^0}$. Then there is a $*$-homomorphism $j : C^*(\Sigma) \to M(C^*(\Gamma))$ defined for $a \in C_c(\Sigma)$ and

$f \in C_c(\Gamma)$ via

(1) $$(j(a)(f))(\gamma) = \int_\Sigma a(\eta) f(\eta^{-1}\gamma)\, d\beta^{r(\gamma)}(\eta)$$

such that $j(a)^* = j(a^*)$ (see [19, Prop. II.2.4]). It is known that if Σ and Γ are locally compact *groups* then the map j fails to be faithful in general (see, for example, [4]). However, if Σ is a clopen subgroup of Γ or Γ is an amenable group then the map j is a faithful *-homomorphism of $C^*(\Sigma)$ into $C^*(\Gamma)$ (see [21, Prop. 1.2], [4, Thm. 1.3], [3, Cor. 1.5]).

We prove next that the above mentioned results hold for groupoid C^*-algebras as well: the map j is faithful if Σ is a clopen subgroupoid of Γ or if Γ is an amenable groupoid.

Before proceeding further, we recall the definition of induced representations from closed subgroupoids following [9, §2] (see also [19, §II.2]). Let $\Sigma^\Gamma := \Gamma * \Gamma/\Sigma$ be the imprimitivity groupoid. Then $C_c(\Gamma)$ is a pre-$C_c(\Sigma^\Gamma)$-$C_c(\Sigma)$-imprimitivity bimodule with actions and inner products given by

$$\begin{aligned} F\cdot\phi(z) &= \int_\Gamma F\big([z,y]\big)\phi(y)\, d\lambda^{s(z)}(y),\\ \phi\cdot g(z) &= \int_\Sigma \phi(zh)g(h^{-1})\, d\beta^{s(z)}(h),\\ \langle\phi,\psi\rangle_*(h) &= \int_\Gamma \overline{\phi(y)}\psi(yh)\, d\lambda^{r(h)}(y),\\ {}_*\langle\phi,\psi\rangle\big([x,y]\big) &= \int_\Sigma \phi(xh)\overline{\psi(yh)}\, d\beta^{s(x)}(h). \end{aligned}$$

If L is a representation of $C^*(\Sigma)$ on $B(\mathcal{H}_L)$ then the induced representation $\operatorname{Ind}_\Sigma^\Gamma L$ acts on the completion $\mathcal{H}_{\operatorname{Ind} L}$ of $C_c(\Gamma)\odot\mathcal{H}_L$ with respect to the pre-inner product given on elementary tensors by

(2) $$(\phi\otimes h,\, \psi\otimes k) = (L(\langle\psi,\phi\rangle_*)h,\, k).$$

If $\phi\otimes_\Sigma h$ denotes the class of $\phi\otimes h$ in $\mathcal{H}_{\operatorname{Ind} L}$, then the induced representation is given by

(3) $$\operatorname{Ind}_\Sigma^\Gamma L(f)(\phi\otimes_\Sigma h) = f*\phi\otimes_\Sigma h$$

for $f \in C_c(\Gamma)$, where

$$f*\phi(\gamma) = \int_\Gamma f(\eta)\phi(\eta^{-1}\gamma)\, d\lambda^{r(\gamma)}(\eta).$$

In the following we suppress Σ from $\phi\otimes_\Sigma h$ to simplify slightly the notation.

2.6.1. *The clopen case.* Assume first that Σ is a clopen subgroupoid of Γ. Then the restriction of the Haar system $\lambda = \{\lambda^u\}_{u\in\Gamma^0}$ to Σ is a Haar system on Σ. We assume in the following that Σ is endowed with this Haar system, that is,

$\beta = \lambda|_\Sigma$. Then the map $i_\Sigma : C_c(\Sigma) \to C_c(\Gamma)$ defined via

$$i_\Sigma(f)(\gamma) = \begin{cases} f(\gamma) & \text{if } \gamma \in \Sigma, \\ 0 & \text{otherwise} \end{cases}$$

is a well-defined faithful $*$-homomorphism. We prove next that i_Σ extends to a faithful $*$-homomorphism of $C^*(\Sigma)$ into $C^*(\Gamma)$, generalizing [21, Prop. 1.2].

Proposition 2.7. *With notation as above, i_Σ extends to an embedding $i_\Sigma : C^*(\Sigma) \to C^*(\Gamma)$ and, therefore, $C^*(\Sigma)$ can be viewed as a subalgebra of $C^*(\Gamma)$.*

Proof. We need to show that $\|i_\Sigma(f)\|_{C^*(\Gamma)} = \|f\|_{C^*(\Sigma)}$ for all $f \in C_c(\Sigma)$. Let L be a representation of $C^*(\Gamma)$. Renault's disintegration theorem (see [20, Prop. 4.2] and also [13, §7]) implies that L is the integrated form of a unitary representation $(\mu, \Gamma^0 * \mathcal{H}, V)$ of Γ. Since Σ is a clopen subgroupoid of Γ, any unitary representation of Γ restricts to a unitary representation of Σ. Therefore $\|f\|_{C^*(\Sigma)} \geq \|i_\Sigma(f)\|_{C^*(\Gamma)}$ for all $f \in C_c(H)$.

For the converse inequality, let $(L, \mathcal{H}_L)$ be a representation of $C^*(\Sigma)$. Since Σ is a closed subgroupoid of Γ, the representation L can be induced to a representation $\operatorname{Ind}_\Sigma^\Gamma L$ of $C^*(\Gamma)$ as in (3). Let $\mathcal{H}_{\mathrm{res}}$ be the closed subspace of $\mathcal{H}_{\operatorname{Ind} L}$ obtained by completing $C_c(\Sigma) \odot \mathcal{H}_L$ via the inner product in (2). Then $U : \mathcal{H}_{\mathrm{res}} \to \mathcal{H}_L$ defined on elementary tensors via $U(\phi \otimes h) = L(\phi)h$ defines a unitary that intertwines L with a subrepresentation of $\operatorname{Ind}_\Sigma^\Gamma L$ restricted to $C^*(\Sigma)$. It follows that $\|f\|_{C^*(\Sigma)} \leq \|i_\Sigma(f)\|_{C^*(\Gamma)}$ for all $f \in C_c(\Sigma)$. Therefore $\|i_\Sigma(f)\|_{C^*(\Gamma)} = \|f\|_{C^*(\Sigma)}$ for all $f \in C_c(\Sigma)$ and one can view $C^*(\Sigma)$ as a subalgebra of $C^*(\Gamma)$. □

2.7.1. *The amenable case.* We assume now that (Σ, β) is a closed subgroupoid of (Γ, λ) and Γ is amenable in the sense of [2]. It follows that Σ is amenable as well (see [2, Prop. 5.1.1]).

Proposition 2.8. *Assume that (Σ, β) is a closed subgroupoid of an amenable groupoid (Γ, λ) such that $\Sigma^0 = \Gamma^0$. Then the map j defined in (1) is faithful.*

We recall first the definition of the left regular representation of $C^*(\Gamma)$. Let μ be a quasi-invariant measure on Γ^0 with full support. The *left regular representation* of $C^*(\Gamma)$ is the representation $L_\Gamma := \operatorname{Ind}_{\Gamma^0}^\Gamma \mu$ induced from μ. By using induction in stages (see [9, Thm. 4]), it follows that if Σ is a closed subgroupoid of Γ such that $\Sigma^0 = \Gamma^0$ then L_Γ is unitarily equivalent to $\operatorname{Ind}_\Sigma^\Gamma L_\Sigma$.

Proof of Proposition 2.8. The proof is virtually identical with the group case (see, for example, the proof of [4, Thm. 1.3]): by the amenability of Σ, any representation of $C^*(\Sigma)$ is weakly contained in the left regular representation of $C^*(\Sigma)$, which is itself contained in the restriction to $C^*(\Sigma)$ of the left regular representation of $C^*(\Gamma)$. □

3. Twists and short exact sequences

Let Γ be a locally compact Hausdorff groupoid and let G be a locally compact Hausdorff abelian group. The set of continuous 1-cocycles from Γ to G is defined via

$$\begin{aligned} Z_\Gamma(G) &= Z^1(\Gamma, G) \\ &:= \{\phi : \Gamma \to G \mid \phi(\gamma_1\gamma_2) = \phi(\gamma_1) + \phi(\gamma_2) \text{ for all } (\gamma_1, \gamma_2) \in \Gamma^2\}. \end{aligned}$$

Then $Z_\Gamma(G)$ is an abelian group and the map $G \mapsto Z_\Gamma(G)$ is a functor.

Definition 3.1. Let A be an abelian group and Γ a groupoid. A *twist* by A over Γ is a central groupoid extension

$$\Gamma^0 \times A \xrightarrow{j} \Sigma \xrightarrow{\pi} \Gamma,$$

where $\Sigma^0 = \Gamma^0$, j is injective, π is surjective, and $j(r(\sigma), a)\sigma = \sigma j(s(\sigma), a)$ for all $\sigma \in \Sigma$ and $a \in A$.

Example 3.2. The *semi-direct* product $\Gamma \times A$ of Γ and A is called the *trivial twist*. Recall from [11] that $(\gamma_1, a_1)(\gamma_2, a_2) = (\gamma_1\gamma_2, a_1 + a_2)$ provided that $s(\gamma_1) = r(\gamma_2)$, and $(\gamma, a)^{-1} = (\gamma^{-1}, -a)$. Then $\Gamma \times A$ is a twist by A via $j_0(u, a) = (u, a)$ for $(u, a) \in \Gamma^0 \times A$, and $\pi_0(\gamma, a) = \gamma$ for $(\gamma, a) \in \Gamma \times A$.

Following [11, Def. 2.5], we say that two twists by A are properly isomorphic if there is a twist morphism between them which preserves the inclusion of $\Gamma^0 \times A$. The following lemma is a generalization of [11, Prop. 2.2].

Lemma 3.3. *A twist Σ by A is properly isomorphic to a trivial twist if and only if there is a groupoid homomorphism $\tau : \Gamma \to \Sigma$ such that $\pi\tau = \mathrm{id}_\Gamma$.*

Proof. If $\tilde{\tau} : \Gamma \times A \to \Sigma$ is a twist isomorphism then we can define $\tau : \Gamma \to \Sigma$ via $\tau(\gamma) = \tilde{\tau}(\gamma, 0_A)$, where 0_A is the identity of A. It is easy to check that τ is a groupoid homomorphism and that $\pi\tau = \mathrm{id}_\Gamma$.

Assume now that there is a groupoid homomorphism $\tau : \Gamma \to \Sigma$ such that $\pi\tau = \mathrm{id}_\Gamma$. Then we can define $\tilde{\tau} : \Gamma \times A \to \Sigma$ via $\tilde{\tau}(\gamma, a) = \tau(\gamma)j(s(\gamma), a)$. We check next that $\tilde{\tau}$ is a twist isomorphism. Let $(u, a) \in \Gamma^0 \times A$. Then

$$\begin{aligned} \tilde{\tau}(j_0(u, a)) &= \tilde{\tau}(u, a) = \tau(u)j(u, a) \\ &= \tau(u)\tau(u)j(u, a) = \tau(u)j(u, a)\tau(u) = j(u, a). \end{aligned}$$

If $(\gamma, a) \in \Gamma \times A$ then

$$\pi(\tilde{\tau}(\gamma, a)) = \pi(\tau(\gamma)j(s(\gamma), a)) = \gamma\pi(j(s(\gamma), a)) = \gamma = \pi_0(\gamma, a). \qquad \square$$

Following [11], we write $T_\Gamma(A)$ for the collection of proper isomorphism classes of twists by A and we write $[\Sigma] \in T_\Gamma(A)$. We endow $T_\Gamma(A)$ with the operation $[\Sigma] + [\Sigma'] := [\nabla^A_*(\Sigma *_\Gamma \Sigma')]$, where $\nabla^A \in \mathrm{Hom}_\Gamma(A \oplus A, A)$ is defined via $\nabla^A(a, a') = a + a'$ (see [11, Prop. 2.6]). Then $T_\Gamma(A)$ is an abelian group with neutral element $[\Gamma \times A]$. It can be shown that $A \mapsto T_\Gamma(A)$ is a half-exact functor.

Suppose that B and C are locally compact abelian groups. If $p : B \to C$ is a homomorphism and $\phi \in Z_\Gamma(C)$, then the *obstruction twist* determined by ϕ is defined via

$$\Sigma_\phi = \big\{(\gamma, b) \in \Gamma \times B \mid \phi(\gamma) = p(b)\big\}. \tag{4}$$

We establish that Σ_ϕ is indeed a twist in the following proposition and show that it has a Haar system in the next section (see equation (5) in Section 4).

If Γ is an étale groupoid, it has a basis consisting of open bisections, that is, open subsets to which the restrictions of both the range and source maps are injective.

Proposition 3.4. *Assume that $p : B \to C$ is a surjective homomorphism of locally compact abelian groups and let $\phi \in Z_\Gamma(C)$. Then Σ_ϕ is a closed subgroupoid of $\Gamma \times B$ and it is a twist of Γ by $A := \ker p$. Σ_ϕ is properly isometric to the trivial twist if and only if there is $\tilde{\phi} \in Z_\Gamma(B)$ such that $\phi = p_*\tilde{\phi}$. The projection $\pi_1 : \Sigma_\phi \to \Gamma$ is a surjective groupoid morphism. If Γ is étale and A is discrete then Σ_ϕ is étale and π_1 is a local homeomorphism.*

Proof. Note that Σ_ϕ is a closed subgroupoid of $\Gamma \times B$ because ϕ and p are continuous maps. Since p is surjective, it follows that Σ_ϕ is a twist by A via $j(u, a) = (u, a)$ and $\pi(\gamma, b) = \gamma$. We make the obvious identification

$$\Sigma_\phi^0 = \Gamma^0 \times \{0_B\} \simeq \Gamma^0,$$

where 0_B is the unit of B. If Σ_ϕ is properly isometric to the trivial twist then Lemma 3.3 implies that there is a groupoid homomorphism $\tau : \Gamma \to \Sigma_\phi$ such that $\pi_1\tau = \mathrm{id}_\Gamma$. Therefore there is $\tilde{\phi} \in Z_\Gamma(B)$ such that $\tau(\gamma) = (\gamma, \tilde{\phi}(\gamma))$ and, hence, $\phi = p_*\tilde{\phi}$. Conversely, if $\phi = p_*\tilde{\phi}$ for some $\tilde{\phi} \in Z_\Gamma(B)$, then define $\tau : \Gamma \to \Sigma_\phi$ via $\tau(\gamma) = (\gamma, \tilde{\phi}(\gamma))$. It follows that $\pi_1\tau = \mathrm{id}_\Gamma$ and Lemma 3.3 implies that Σ_ϕ is properly isometric to the trivial twist.

It is easy to check that π_1 is a groupoid morphism. Moreover, π_1 is surjective since p is surjective.

Now suppose that Γ is étale and A is discrete. Since $\pi_1 : \Sigma_\phi \to \Gamma$ is open, in order to prove that π_1 is a local homeomorphism it suffices to show that it is locally injective. Let $(\gamma, b) \in \Sigma_\phi$. Then there are an open bisection U in Γ such that $\gamma \in U$ and a neighborhood V_0 of 0_B such that $V_0 \cap \ker p = \{0_B\}$. Let V be an open neighborhood of b such that $V - V \subseteq V_0$. Then $W := (U \times V) \cap \Sigma_\phi$ is an open neighborhood of (γ, b) in Σ_ϕ. Given $(\gamma_1, b_1), (\gamma_2, b_2) \in W$ such that $\pi_1(\gamma_1, b_1) = \pi_1(\gamma_2, b_2)$, then $\gamma_1 = \gamma_2$ and

$$p(b_1 - b_2) = p(b_1) - p(b_2) = \phi(\gamma_1) - \phi(\gamma_2) = 0.$$

Hence

$$b_1 - b_2 \in (V - V) \cap \ker p \subset V_0 \cap \ker p = \{0_B\}$$

and so $b_1 = b_2$. Thus the restriction of π_1 to W is injective and therefore a local homeomorphism. It follows easily that Σ_ϕ is étale. $\square$

The following result generalizes an initial segment of the exact sequence given in [11, Prop. 3.3].

Proposition 3.5. *Given a short exact sequence of locally compact abelian groups*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0,$$

there is an exact sequence

$$0 \to Z_\Gamma(A) \xrightarrow{i_*} Z_\Gamma(B) \xrightarrow{p_*} Z_\Gamma(C) \xrightarrow{\delta} T_\Gamma(A),$$

where $\delta(\phi) := [\Sigma_\phi]$ *for* $\phi \in Z_\Gamma(C)$.

Proof. Since Z_Γ is left exact, we only need to show that $\operatorname{Im} p_* = \ker \delta$. We have that $\phi \in \ker \delta$ if and only if Σ_ϕ is properly isometric to the trivial twist. By Lemma 3.4, this happens if and only if $\phi = p_* \tilde{\phi}$ for some $\tilde{\phi} \in Z_\Gamma(B)$. □

In the following example we consider the case $A = \mathbb{Z}$, $B = \mathbb{R}$ and $C = \mathbb{T}$. In a certain case where the groupoid Γ is equivalent to a space X, we indicate how the construction of the groupoid Σ_ϕ is related to the boundary map of Čech cohomology $\check{H}^1(X, \mathcal{T}) \to \check{H}^2(X, \mathbb{Z})$, where $\check{H}^1(X, \mathcal{T})$ is the first Čech cohomology of X with coefficients in $\mathcal{T}$, the sheaf of germs of continuous $\mathbb{T}$-valued functions, and $\check{H}^2(X, \mathbb{Z})$ is the second Čech cohomology of X with coefficients in the constant sheaf with fiber $\mathbb{Z}$ (see [18, §4.1] for background on Čech cohomology). This example is inspired by results in [16].

Example 3.6. Let X be a Hausdorff second countable locally compact space; thus X is also paracompact. Let $\mathcal{U} := \{U_i\}_{i \in I}$ be a locally finite open cover of X. Set $U_{ij} := U_i \cap U_j$ for $i, j \in I$ and similarly $U_{ijk} := U_i \cap U_j \cap U_k$ for $i, j, k \in I$. We now construct an étale groupoid associated to $\mathcal{U}$ (see [17, §1, Rem. 3] and [8, Ex. III.1.0]). Let

$$\Gamma_{\mathcal{U}} := \{(x, i, j) \mid x \in U_{ij}\} \quad \text{and} \quad \Gamma^0_{\mathcal{U}} := \{(x, i, i) \mid x \in U_i\};$$

note that $\Gamma^0_{\mathcal{U}}$ may be identified with the disjoint union $\bigsqcup_{i \in I} U_i$ and that $\Gamma_{\mathcal{U}}$ may be identified with the disjoint union $\bigsqcup_{i,j \in I} U_{ij}$. It is routine to check that with the topology obtained from this identification and with the following structure maps $\Gamma_{\mathcal{U}}$ is an étale groupoid:

$$\begin{aligned} s(x, i, j) &= (x, j, j), \\ r(x, i, j) &= (x, i, i), \\ (x, i, j)^{-1} &= (x, j, i), \\ (x, i, j)(x, j, k) &= (x, i, k). \end{aligned}$$

Note that $\Gamma_{\mathcal{U}}$ may also be viewed as the natural groupoid associated to the local homeomorphism $\bigsqcup_{i \in I} U_i \to X$ (see [10]). Next suppose that we are given a Čech 1-cocycle $\lambda = \{\lambda_{ij}\}_{i,j \in I}$ with coefficients in $\mathcal{T}$. So $\lambda_{ij} : U_{ij} \to \mathbb{T}$ is continuous for all $i, j \in I$ and $\lambda_{ik}(x) = \lambda_{ij}(x)\lambda_{jk}(x)$ for all $i, j, k \in I$ and $x \in U_{ijk}$. It is routine to check that the map $\phi : \Gamma_{\mathcal{U}} \to \mathbb{T}$ given by $\phi((x, i, j)) = \lambda_{ij}(x)$ is a continuous groupoid 1-cocycle.

Now suppose that each λ_{ij} lifts, that is, for each $i, j \in I$ there is a continuous function $\tilde{\lambda}_{ij} : U_{ij} \to \mathbb{R}$ such that $\lambda_{ij} = e \circ \tilde{\lambda}_{ij}$ where $e(t) = e^{2\pi\sqrt{-1}t}$ for $t \in \mathbb{R}$.

We may assume that $\tilde{\lambda}_{ii} = 0$ for all $i \in I$. We observe that for $i, j, k \in I$ the formula

$$(\lambda^\star)_{ijk}(x) := \tilde{\lambda}_{ij}(x) + \tilde{\lambda}_{jk}(x) - \tilde{\lambda}_{ik}(x)$$

for $x \in U_{ijk}$ defines a continuous integer-valued function. A routine computation shows that, for all $i, j, k, \ell \in I$ and $x \in U_{ijk\ell}$,

$$(\lambda^\star)_{jk\ell}(x) - (\lambda^\star)_{ik\ell}(x) + (\lambda^\star)_{ij\ell}(x) - (\lambda^\star)_{ijk}(x) = 0,$$

that is, $(\lambda^\star)_{ijk}$ gives a Čech 2-cocycle with values in $\mathbb{Z}$ (i.e. the constant sheaf with fiber $\mathbb{Z}$). It is normalized in the sense that $(\lambda^\star)_{ijk} = 0$ if $j = i$ or $j = k$. As in [17] we may construct a groupoid 2-cocycle $\phi^\star$ by the formula

$$\phi^\star((x, i, j), (x, j, k)) = (\lambda^\star)_{ijk}(x)$$

for all $x \in U_{ijk}$. We obtain a twist Σ by $\mathbb{Z}$ over $\Gamma_{\mathcal{U}}$ determined by $\phi^\star$ (see [19, Prop. I.1.14]). Indeed, let

$$\Sigma := \big\{(n, (x, i, j)) \mid n \in \mathbb{Z},\, x \in U_{ij}\big\}.$$

We identify $\Gamma^0_{\mathcal{U}}$ with Σ^0 via the map $(x, i, i) \mapsto (0, (x, i, i))$. The range and source maps factor through those of $\Gamma_{\mathcal{U}}$. The remaining structure maps are given by

$$\begin{aligned} (m, (x, i, j))(n, (x, j, k)) &:= \big(m + n + (\lambda^\star)_{ijk}(x), (x, i, k)\big), \\ (n, (x, i, j))^{-1} &:= \big(-n - (\lambda^\star)_{jij}(x), (x, j, i)\big). \end{aligned}$$

We claim that $\Sigma \cong \Sigma_\phi$. Define $\xi : \Sigma \to \Gamma_{\mathcal{U}} \times \mathbb{R}$ by

$$\xi((n, (x, i, j))) = \big((x, i, j), \tilde{\lambda}_{ij}(x) + n\big).$$

We first note that ξ induces an identification of the unit spaces. Given a pair of composable elements $(m, (x, i, j))$, $(n, (x, j, k)) \in \Sigma$ we have

$$\begin{aligned} \xi\big((m, (x, i, j))(n, (x, j, k))\big) &= \xi\big((m + n + (\lambda^\star)_{ijk}(x), (x, i, k))\big) \\ &= \big((x, i, k), \tilde{\lambda}_{ik}(x) + m + n + (\lambda^\star)_{ijk}(x)\big) \\ &= \big((x, i, k), m + n + \tilde{\lambda}_{ij}(x) + \tilde{\lambda}_{jk}(x)\big) \\ &= \big((x, i, j), \tilde{\lambda}_{ij}(x) + m\big)\big((x, j, k), \tilde{\lambda}_{jk}(x) + n\big) \\ &= \xi((m, (x, i, j)))\xi((n, (x, j, k))). \end{aligned}$$

Hence, ξ is a groupoid homomorphism. Moreover, ξ is injective and its image coincides with Σ_ϕ. It follows that the isomorphism class of Σ does not depend on the specific choice of the $\tilde{\lambda}_{ij}$.

Remark 3.7. With notation as in the above example any (normalized) Čech 2-cocycle with values in the constant sheaf with fiber $\mathbb{Z}$ gives rise to a groupoid as above which is isomorphic to a pullback. Let $\mathcal{R}$ be the sheaf of germs of continuous $\mathbb{R}$-valued functions on X. Since $\mathcal{R}$ is a fine sheaf, we have $H^n(X, \mathcal{R}) = 0$ for $n \geq 1$. Moreover, since $0 \to \mathbb{Z} \to \mathcal{R} \to \mathcal{T} \to 0$ is exact, the connecting map of the long exact sequence of cohomology $\partial^n : \check{H}^n(X, \mathcal{T}) \to \check{H}^{n+1}(X, \mathbb{Z})$ is an isomorphism for $n > 0$ (see [18, Thm. 4.37]). In Example 3.6 we have $[\lambda^\star] = \partial^1([\lambda])$.

Remark 3.8. Note that every Čech 2-cocycle with values in an arbitrary sheaf of abelian groups is cohomologous to a normalized cocycle $a = \{a_{ijk}\}_{ijk}$ (that is, $a_{ijk} = 0$ if $j = i$ or $j = k$). Indeed given a Čech 2-cocycle $c = \{c_{ijk}\}_{ijk}$ a routine calculation shows that $c_{iij}(x) = c_{iii}(x) = c_{jii}(x)$ for all $i, j \in I$ and $x \in U_{ij}$. For $i, j \in I$, define a 1-cochain λ by $\lambda_{ij} = 0$ if $i \neq j$ and $\lambda_{ii}(x) = c_{iii}(x)$ for $x \in U_{ii}$. Then $a := c - d^1\lambda$ is a normalized 2-cocycle cohomologous to c. (Recall that $(d^1\lambda)_{ijk}(x) := \lambda_{jk}(x) - \lambda_{ik}(x) + \lambda_{ij}(x)$ for $x \in U_{ijk}$.)

4. The structure of the C^*-algebra $C^*(\Sigma_\phi)$

Let Γ be a locally compact Hausdorff groupoid endowed with Haar system $\{\lambda^u\}_{u\in\Gamma^0}$ and let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of locally compact abelian groups. In the sequel, we identify A with its image $i(A)$ in B. Let $\phi \in Z_\Gamma(C)$ and let Σ_ϕ be the obstruction twist as in (4). We prove below that

$$C^*(\Sigma_\phi) \cong \operatorname{ind}_{\widehat{C}}^{\widehat{B}} C^*(\Gamma)$$

if Γ is amenable (see Theorem 4.3). We begin by describing the action of $\widehat{C}$ on $C^*(\Gamma)$. The following lemma follows immediately from [19, Prop. II.5.1 (iii)].

Lemma 4.1. *Assume that C is a locally compact abelian group. Given $\phi \in Z_\Gamma(C)$, the map*

$$\alpha_t^\phi(f)(\gamma) = \langle t, \phi(\gamma)\rangle f(\gamma),$$

for $f \in C_c(\Gamma)$, $t \in \widehat{C}$, and $\gamma \in \Gamma$, defines a strongly continuous action $\alpha^\phi : \widehat{C} \to \operatorname{Aut} C^*(\Gamma)$.

We define next a Haar system on Σ_ϕ. Choose Haar measures μ_A, μ_B, and μ_C on A, B, and, respectively C such that

$$\int_B f(b)\, d\mu_B(b) = \int_C \int_A f(b+a)\, d\mu_A(a) d\mu_C(p(b)).$$

One can always make such a choice since A, B, and C are locally compact abelian groups (see, e.g., [5, p. 79]). Let $\pi_1 : \Sigma_\phi \to \Gamma$ be the projection map (see Lemma 3.4). Note that $\pi_1^{-1}(\gamma) = \{\gamma\} \times A_\gamma$ where $A_\gamma := \{b \in B \mid \phi(\gamma) = p(b)\}$ is a coset in B, since $A_\gamma = b_\gamma + A$ for some $b_\gamma \in B$ with $\phi(\gamma) = p(b_\gamma)$. We write μ_γ for the measure defined on A_γ via

$$\int_{A_\gamma} f(\gamma, b)\, d\mu_\gamma(b) := \int_A f(\gamma, b_\gamma + a)\, d\mu_A(a).$$

The measure μ_γ is independent of the choice of b_γ because μ_A is a Haar measure on A. We define now a Haar system $\{\nu^u\}_{u\in\Gamma^0}$ on Σ_ϕ via

$$\int_{\Sigma_\phi^u} f(\gamma, b)\, d\nu^u(\gamma, b) = \int_{\Gamma^u} \int_{A_\gamma} f(\gamma, b)\, d\mu_\gamma(b) d\lambda^u(\gamma), \tag{5}$$

for all $u \in \Gamma^0$. It is easy to check that (5) defines a Haar system on Σ_ϕ using the fact that $\{\lambda^u\}$ is a Haar system on Γ and μ_A is a Haar measure on A.

The main goal of this section is to prove that $C^*(\Sigma_\phi)$ is isomorphic to the induced algebra $\operatorname{ind}_{\widehat{C}}^{\widehat{B}}(C^*(\Gamma), \alpha^\phi)$ using Theorem 2.2. We begin by defining a map

$$\rho : C_c(\Sigma_\phi) \to C_c(\Gamma)$$

via

$$\rho(f)(\gamma) = \int_{A_\gamma} f(\gamma, b)\, d\mu_\gamma(b). \tag{6}$$

Lemma 4.2. *With notation as above, the map ρ defined in* (6) *extends to a surjective $*$-homomorphism $\rho : C^*(\Sigma_\phi) \to C^*(\Gamma)$ which factors through the map $\iota : C^*(\Sigma_\phi) \to M(C^*(\Gamma \times B))$ given in Section 2.6. Moreover, if either Γ is amenable or Σ_ϕ is a clopen subset of $B \times \Gamma$, then ι is injective.*

Proof. We prove first that ρ is a $*$-homomorphism. Let f and g in $C_c(\Sigma_\phi)$ and let $\gamma \in \Gamma$. Then, using the fact that μ_A is a Haar measure on A, we obtain

$$\begin{aligned}
\rho(f * g)(\gamma) &= \int_{A_\gamma} (f * g)(\gamma, b)\, d\mu_\gamma(b) \\
&= \int_{A_\gamma} \int_{\Gamma^{r(\gamma)}} \int_{A_\alpha} f(\alpha, b') g(\alpha^{-1}\gamma, -b' + b)\, d\mu_\alpha(b') d\lambda^{r(\gamma)}(\alpha) d\mu_\gamma(b) \\
&= \int_{\Gamma^{r(\gamma)}} \int_{A_\alpha} f(\alpha, b') \int_{A_\gamma} g(\alpha^{-1}\gamma, -b' + b)\, d\mu_\gamma(b) d\mu_\alpha(b') d\lambda^{r(\gamma)}(\alpha) \\
&= \int_{\Gamma^{r(\gamma)}} \rho(f)(\alpha) \rho(g)(\alpha^{-1}\gamma)\, d\lambda^{r(\gamma)}(\alpha) \\
&= \big(\rho(f) * \rho(g)\big)(\gamma).
\end{aligned}$$

It is easy to show that $\rho(f^*) = \rho(f)^*$ for all $f \in C_c(\Sigma_\phi)$.

Recall from Section 2.6 (see [19, Prop. II.2.4]) that there is a bounded $*$-homomorphism from $C^*(\Sigma_\phi)$ into $M(C^*(\Gamma \times B))$. Since $C^*(\Gamma \times B)$ is $*$-isomorphic to $C^*(\Gamma) \otimes C^*(B)$ and $C^*(B)$ is isomorphic to $C_0(\widehat{B})$, it follows that $M(C^*(\Gamma \times B))$ is $*$-isomorphic to $C_b(\widehat{B}, M(C^*(\Gamma)))$ where $M(C^*(\Gamma))$ is given the strict topology (see [1, Cor. 3.4]). Then ρ is just the composition of the above bounded $*$-homomorphisms with evaluation at 0. Hence ρ extends to a bounded $*$-homomorphism $\rho : C^*(\Sigma_\phi) \to M(C^*(\Gamma))$. However, since $\rho(C_c(\Sigma_\phi)) \subseteq C_c(\Gamma)$, $C_c(\Sigma_\phi)$ is dense in $C^*(\Sigma_\phi)$, and $C_c(\Gamma)$ is dense in $C^*(\Gamma)$, it follows that ρ is a bounded $*$-homomorphism from $C^*(\Sigma_\phi)$ into $C^*(\Gamma)$.

To prove that ρ is surjective let $\mathfrak{b}$ be a *Bruhat approximate cross-section* for B over A. Recall (see, for example, [18, Prop. C.1]) that $\mathfrak{b} : B \to [0, \infty)$ is a continuous function such that $\operatorname{supp} \mathfrak{b} \cap (K + A)$ is compact for every compact set K in B, and such that $\int_A \mathfrak{b}(b+a) d\mu_A(a) = 1$ for all $b \in B$. Given $g \in C_c(\Gamma)$ define $f \in C_c(\Sigma_\phi)$ via $f(\gamma, b) = g(\gamma)\mathfrak{b}(b)$. It follows immediately that $\rho(f) = g$ and, hence, ρ is surjective.

The final assertion follows from Proposition 2.7 if Σ_ϕ is a clopen subset of $\Gamma \times B$. If Γ is amenable, the assertion follows from Proposition 2.8 and the fact that $\Gamma \times B$ is amenable (see [2, Prop. 5.1.2]). □

We are now ready to state and prove the main result of the paper.

Theorem 4.3. *Let* Γ *be a locally compact Hausdorff amenable groupoid endowed with Haar system* $\{\lambda^u\}_{u\in\Gamma^0}$ *and let*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of locally compact abelian groups. Let $\phi \in Z_\Gamma(C)$ *and let* Σ_ϕ *be the obstruction twist as in* (4) *endowed with the Haar system defined in* (5)*. Then there are a surjective* $*$*-homomorphism* $\rho : C^*(\Sigma_\phi) \to C^*(\Gamma)$*, a strongly continuous action* $\gamma : \widehat{B} \to \mathrm{Aut}(C^*(\Sigma_\phi))$ *and a* $\widehat{B}$*-equivariant homomorphism* $i : C_0(\widehat{B}/\widehat{C}) \to Z(M(C^*(\Sigma_\phi)))$ *such that the four conditions of Theorem 2.2 are satisfied. Therefore there is a unique* $\widehat{B}$*-equivariant isomorphism*

$$\Psi : C^*(\Sigma_\phi) \to \mathrm{ind}_{\widehat{C}}^{\widehat{B}}\, C^*(\Gamma)$$

defined via

$$\Psi(f)(t) = \gamma_{-t}(f)$$

for $f \in C_c(\Sigma_\phi)$ *and* $t \in \widehat{B}$*.*

Proof. The existence of the map $\rho : C^*(\Sigma_\phi) \to C^*(\Gamma)$ is proved in Lemma 4.2. The strongly continuous action $\gamma : \widehat{B} \to \mathrm{Aut}(C^*(\Sigma_\phi))$ is given by

$$\gamma_t(f)(\sigma, b) = \langle t, b\rangle f(\sigma, b).$$

In order to define the map $i : C_0(\widehat{B}/\widehat{C}) \to Z(M(C^*(\Sigma_\phi)))$ we identify $\widehat{B}/\widehat{C}$ with $\widehat{A}$ and we also identify $C_0(\widehat{A})$ with $C^*(A)$ via the Fourier transform. Recall that we view the multiplier algebra $M(C^*(\Sigma_\phi))$ as the C^*-algebra of the adjointable operators on the Hilbert C^*-module $C^*(\Sigma_\phi)_{C^*(\Sigma_\phi)}$. Then we define $i : C^*(A) \to Z(M(C^*(\Sigma_\phi)))$ via

$$(i(h)f)(\sigma) = \int_A h(a) f((r(\sigma), -a)\sigma)\, d\mu_A(a) \tag{7}$$

for all $\sigma \in \Sigma_\phi$, $h \in C_c(A)$ and $f \in C_c(\Sigma_\phi)$. A straight-forward but tedious computation shows that $i(h)$ is an adjointable operator on $C^*(\Sigma_\phi)_{C^*(\Sigma_\phi)}$: let f and g be in $C_c(A)$ and let $\sigma = (\gamma, b) \in \Sigma_\phi$. Then

$$\begin{aligned}
\langle i(h)f,\, g\rangle(\gamma, b) &= (i(h)f)^* * g(\gamma, b) \\
&= \int_{\Sigma_\phi^{r(\gamma)}} (i(h)f)^*(\eta, b') g(\eta^{-1}\gamma, -b' + b)\, d\nu^{r(\gamma)}(\eta, b') \\
&= \int_{\Gamma^{r(\gamma)}} \int_{A_\gamma} \overline{i(h)f(\eta^{-1}, -b')} g(\eta^{-1}\gamma, -b' + b)\, d\mu_\gamma(b') d\lambda^{r(\gamma)}(\eta)
\end{aligned}$$

which, using (7), interchanging the integrals on A, using the fact that μ_A is a Haar measure on A, and interchanging back the integrals, equals

$$\begin{aligned}
&\int_{\Gamma^{r(\gamma)}} \int_{A_\gamma} \overline{f(\eta^{-1}, -b')} (i(h^*)g)(\eta^{-1}\gamma, -b' + b)\ d\mu_\gamma(b') d\lambda^{r(\gamma)}(\eta) \\
&\quad = \langle f, i(h^*)(g)\rangle(\gamma, b).
\end{aligned}$$

Hence $i(h)$ is an adjointable operator on $C^*(\Sigma_\phi)_{C^*(\Sigma_\phi)}$. Moreover, $i(h)$ belongs to $Z(M(C^*(\Sigma_\phi)))$ since Σ_ϕ is a twist. Note that as in Remark 2.4, i is determined by the map $u : A \to UM(C^*(\Sigma_\phi))$ given as the composition

$$A \to M(C^*(A) \otimes C_0(\Gamma^0)) \to M(C^*(\Sigma_\phi)).$$

Then condition (i) of Theorem 2.2 follows from the fact that $\rho(u_a f) = \rho(f)$ for all $a \in A$ and $f \in C_c(\Sigma_\phi)$. The nondegeneracy of i (see Remark 2.3) follows from the fact that $u_0 = 1_{M(C^*(\Sigma_\phi))}$. Hence condition (iv) of Theorem 2.2 holds. Condition (ii) follows by a straight-forward computation. Since $B \times \Gamma$ is amenable, it follows by Lemma 4.2 that

$$\iota : C^*(\Sigma_\phi) \to M(C^*(\Gamma \times B)) \cong C_b(\widehat{B}, M(C^*(\Gamma)))$$

is injective. Note that ι is $\widehat{B}$-equivariant with respect to natural actions. Hence, if $\rho(\gamma_\chi(f)) = 0$ for all $\chi \in \widehat{B}$, we have $f = 0$ and so condition (iii) holds. □

Remark 4.4. As noted in the proof of the above theorem (see Remark 2.4) one can replace conditions (i) and (iv) of Theorem 2.2 by the following two conditions on the corresponding map $u : A \to UM(C^*(\Sigma_\phi))$: condition (i) may be replaced by the requirement that $\rho(u_a f) = \rho(f)$ for all $a \in A$ and $f \in C^*(\Sigma_\phi)$ and condition (iv) may be replaced by the requirement that $u_0 = 1_{M(C^*(\Sigma_\phi))}$.

Remark 4.5. Theorem 4.3 also holds if one replaces the requirement that Γ be amenable by the requirement that Σ_ϕ be a clopen subset of $\Gamma \times B$ (see Lemma 4.2).

Example 4.6. With notation as in Example 3.6, we first observe that $C^*(\Gamma_{\mathcal{U}})$ is a continuous trace algebra with $\operatorname{Prim} C^*(\Gamma_{\mathcal{U}}) = X$ and trivial Dixmier–Douady invariant $\delta(C^*(\Gamma)) = 0$, that is, it is strong Morita equivalent to $C_0(X)$ (see [10]). Next we let $\alpha : \mathbb{Z} \to \operatorname{Aut} C^*(\Gamma_{\mathcal{U}})$ be the action determined by the $\mathbb{T}$-valued groupoid 1-cocycle ϕ defined by the Čech 1-cocycle λ_{ij} (see [19, Prop. II.5.1]); so

$$(\alpha_n f)(x, i, j) := \lambda_{ij}(x)^n f(x, i, j)$$

for all $f \in C_c(\Gamma_{\mathcal{U}})$, $(x, i, j) \in \Gamma_{\mathcal{U}}$ and $n \in \mathbb{Z}$. It is straight-forward to see that α fixes every ideal in $C^*(\Gamma_{\mathcal{U}})$.

For each $i \in I$ the ideal J_i in $C^*(\Gamma_{\mathcal{U}})$ corresponding to $U_i \subset X$ may be identified with $C^*((\Gamma_{\mathcal{U}})_{V_i})$ where $V_i := \{(x, j, j) \mid x \in U_{ij}\} \subset \Gamma^0_{\mathcal{U}}$ (hence $V_i \cong \bigsqcup_{j \in I} U_{ij}$). Note that $(\Gamma_{\mathcal{U}})_{V_i} = \{(x, j, k) \mid x \in U_{ijk}\}$; let $w_i : (\Gamma_{\mathcal{U}})_{V_i} \to \mathbb{R}$ be defined by

$$w_i(x, j, k) := \begin{cases} \lambda_{ji}(x) & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Then w_i may be identified with a multiplicative unitary for the ideal $C^*((\Gamma_{\mathcal{U}})_{V_i})$ and $\alpha_1|_{J_i} = \operatorname{Ad} w_i$. Hence, α is locally unitary. A quick calculation shows that $\lambda_{ij} w_i(x, k, k) = w_j(x, k, k)$ for all $x \in U_{ijk}$. Therefore the Phillips–Raeburn obstruction $\eta(\alpha)$ is the class $[\lambda] \in \check{H}^1(X, \mathcal{T})$, the first Čech cohomology of X

with coefficients in $\mathcal{T}$, the sheaf of continuous $\mathbb{T}$-valued functions on X (see [14, §2.10]). Using the usual isomorphism $\check{H}^1(X,\mathcal{T}) \cong \check{H}^2(X,\mathbb{Z})$, we identify $\eta(\alpha)$ with $[\lambda^\star] \in \check{H}^2(X,\mathbb{Z})$. Note that the element $\eta(\alpha)$ may also be identified with the class of $\operatorname{Prim} C^*(\Gamma_{\mathcal{U}}) \rtimes_\alpha \mathbb{Z}$ regarded as a circle bundle under the dual action of $\mathbb{T}$ (see [15, p. 224]).

By Theorem 4.3, we have

$$C^*(\Sigma_\phi) \cong \operatorname{ind}_{\mathbb{Z}}^{\mathbb{R}}(C^*(\Gamma),\alpha).$$

Next we observe that $C^*(\Sigma_\phi)$ is a continuous trace algebra (see the remark preceding [16, Lem. 3.3]) and that $\operatorname{Prim} C^*(\Sigma_\phi) \cong \mathbb{T} \times X$ (see the proof of [16, Prop. 3.4]). Since $\delta(C^*(\Gamma)) = 0$, it follows by [16, Cor. 3.5] that

$$\delta(C^*(\Sigma_\phi)) = z \times \eta(\alpha) = z \times [\lambda^\star] \in \check{H}^3(\mathbb{T} \times X,\mathbb{Z}),$$

where z is the standard generator of $\check{H}^1(\mathbb{T},\mathbb{Z})$.

Acknowledgments. The authors would like to thank Alcides Buss and Dana P. Williams for useful conversations and suggestions that led to a significant shortening of our original proof of Proposition 2.8. The second author would like to thank the first author and his colleagues at the Naval Academy for their hospitality and support during a recent visit.

References

[1] C. A. Akemann, G. K. Pedersen and J. Tomiyama, Multipliers of C^*-algebras, J. Functional Analysis **13** (1973), 277–301. MR0470685

[2] C. Anantharaman-Delaroche and J. Renault, *Amenable groupoids*, Monographies de L'Enseignement Mathématique, 36, L'Enseignement Mathématique, Geneva, 2000. MR1799683

[3] M. E. B. Bekka, A. T. Lau, and G. Schlichting, On invariant subalgebras of the Fourier-Stieltjes algebra of a locally compact group, Math. Ann. **294** (1992), no. 3, 513–522. MR1188135

[4] M. E. B. Bekka and A. Valette, Lattices in semi-simple Lie groups, and multipliers of group C^*-algebras, Astérisque No. 232 (1995), 67–79. MR1372525

[5] A. Deitmar and S. Echterhoff, *Principles of harmonic analysis*, 2nd ed., Universitext, Springer, Cham, 2014. MR3289059

[6] S. Echterhoff, On induced covariant systems, Proc. Amer. Math. Soc. **108** (1990), no. 3, 703–706. MR0994776

[7] J. M. G. Fell and R. S. Doran, *Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles. Vol. 1*, Pure Appl. Math., 125, Academic Press, Boston, MA, 1988. MR0936628

[8] A. Haefliger, Differential cohomology, in *Differential topology* (Varenna, 1976), 19–70, Liguori, Naples, 1979. MR0660658

[9] M. Ionescu and D. P. Williams, Irreducible representations of groupoid C^*-algebras, Proc. Amer. Math. Soc. **137** (2009), no. 4, 1323–1332. MR2465655

[10] A. Kumjian, Preliminary algebras arising from local homeomorphisms, Math. Scand. **52** (1983), no. 2, 269–278. MR0702958

[11] A. Kumjian, On equivariant sheaf cohomology and elementary C^*-bundles, J. Operator Theory **20** (1988), no. 2, 207–240. MR1004121

[12] E. C. Lance, *Hilbert C^*-modules*, London Math. Soc. Lecture Note Ser., 210, Cambridge University Press, Cambridge, 1995. MR1325694

[13] P. S. Muhly and D. P. Williams, *Renault's equivalence theorem for groupoid crossed products*, New York Journal of Mathematics. NYJM Monographs, 3, State University of New York, University at Albany, Albany, NY, 2008. MR2547343

[14] J. Phillips and I. Raeburn, Automorphisms of C^*-algebras and second Čech cohomology, Indiana Univ. Math. J. **29** (1980), no. 6, 799–822. MR0589649

[15] J. Phillips and I. Raeburn, Crossed products by locally unitary automorphism groups and principal bundles, J. Operator Theory **11** (1984), no. 2, 215–241. MR0749160

[16] I. Raeburn and J. Rosenberg, Crossed products of continuous-trace C^*-algebras by smooth actions, Trans. Amer. Math. Soc. **305** (1988), no. 1, 1–45. MR0920145

[17] I. Raeburn and J. L. Taylor, Continuous trace C^*-algebras with given Dixmier–Douady class, J. Austral. Math. Soc. Ser. A **38** (1985), no. 3, 394–407. MR0779202

[18] I. Raeburn and D. P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, Math. Surveys Monogr., 60, American Mathematical Society, Providence, RI, 1998. MR1634408

[19] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Math., 793, Springer, Berlin, 1980. MR0584266

[20] J. Renault, Représentation des produits croisés d'algèbres de groupoïdes, J. Operator Theory **18** (1987), no. 1, 67–97. MR0912813

[21] M. A. Rieffel, Induced representations of C^*-algebras, Advances in Math. **13** (1974), 176–257. MR0353003

[22] D. P. Williams, *Crossed products of C^*-algebras*, Math. Surveys Monogr., 134, American Mathematical Society, Providence, RI, 2007. MR2288954

Received December 12, 2016; accepted June 6, 2017

Marius Ionescu
Department of Mathematics,
United States Naval Academy,
Annapolis, MD, USA
E-mail: `ionescu@usna.edu`

Alex Kumjian
Department of Mathematics,
University of Nevada,
Reno, NV, USA
E-mail: `alex@unr.edu`