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Decomposition of simple Cuntz semigroups

2014

Mathematik

Decomposition of simple Cuntz semigroups

Inaugural-Dissertation
zur Erlangung des Doktorgrades
der Naturwissenschaften im Fachbereich
Mathematik und Informatik
der Mathematisch-Naturwissenschaftlichen Fakultät
der Westfälischen Wilhelms-Universität Münster

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- 2014 -

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Tag der mündlichen Prüfung:	29. Januar 2015
Tag der Promotion:	29. Januar 2015

Abstract

Let Cu be the category of positively ordered abelian monoids satisfying axioms ($\mathcal{O}1$) to ($\mathcal{O}6$), as given in [29]. It is well-known that any simple and stably finite semigroup S in the category Cu is the union of two subsemigroups: the subsemigroup of all compact elements, which we call $\mathsf{C}(S)$, and the subsemigroup of all noncompact elements (and 0), which we call $\mathsf{D}(S)$. We show that a large class of simple and stably finite semigroups S in Cu, including the Cuntz semigroups $\mathsf{Cu}(A)$ of every simple, separable, nonelementary, and stably finite C*-algebra A, as well as all the simple, separable, nonelementary, and weakly cancellative semigroups $S \in \mathsf{Cu}$, admit what we call a *predecessor map*, a faithful semigroup homomorphism $\gamma_S \colon \mathsf{C}(S) \to \mathsf{D}(S)$ that is characterised by the property that $\gamma_S(x) = \max\{y \in S \mid y < x\}$ for every nonzero $x \in \mathsf{C}(S)$. We call a semigroup with this property *decomposable*, and denote by Cu_{dec} the category of all simple and decomposable semigroups in Cu.
Next, we describe the categories C and D of which the semigroups $\mathsf{C}(S)$ and $\mathsf{D}(S)$ are objects, and introduce the notion of a *composition map* γ between simple semigroups $C \in \mathsf{C}$ and $D \in \mathsf{D}$. We denote by Cu_{com} the class of all triples (C, D, γ) where C is a simple semigroup in C, and D is a simple semigroup in D, and $\gamma \colon C \to D$ is a composition map between them. We show that for every semigroup S in Cu_{dec}, the decomposition $(\mathsf{C}(S), \mathsf{D}(S), \gamma_S)$ is in Cu_{com}, and conversely, that every triple (C, D, γ) in Cu_{com} can be composed into a semigroup S in Cu_{dec}. Moreover, we introduce a notion of morphism on Cu_{com}, turning it into a category, and show that composition and decomposition are functorial. We then show that the composition and decomposition functors implement a category equivalence between the category Cu_{dec} and the category Cu_{com}.
We then apply these results to the Elliott invariant, and show that for the class of simple, separable, unital, exact, nonelementary, and cancellative C*-algebras A, the extended Elliott invariant $\widetilde{\mathsf{Ell}}(\cdot)$ is equivalent to the invariant $\overline{\mathsf{Ell}}(A) = ((\mathsf{K}_0(A), \mathsf{K}_0(A)_+, [\mathbb{I}_A]), \mathsf{K}_1(A), \mathsf{D}(A), \gamma_A)$, where $\mathsf{D}(A) := \mathsf{D}(\mathsf{Cu}(A))$ and $\gamma_A := \gamma_{\mathsf{Cu}(A)}$. By equivalent, we mean that we exhibit a category equivalence $(\mathsf{E}, \mathsf{E}')$ between appropriate image categories for $\widetilde{\mathsf{Ell}}(\cdot)$ and $\overline{\mathsf{Ell}}(\cdot)$ with the property that $\mathsf{E}(\overline{\mathsf{Ell}}(A)) \cong \widetilde{\mathsf{Ell}}(A)$ and $\mathsf{E}'(\widetilde{\mathsf{Ell}}(A)) \cong \overline{\mathsf{Ell}}(A)$. Moreover, we show that $\mathsf{D}(A)$ and γ_A can be interpreted as preduals for the tracial data $\mathsf{T}_1(A)$, ρ_A (the simplex of tracial states and the pairing map) from the original Elliott invariant. Thus, we show that the extended Elliott invariant is, for this class of C*-algebras, a quite natural extension of the original invariant. We then finish the dissertation with some preliminary results concerning the question of whether, or for which classes of semigroups, the predecessor maps γ form a natural transformation from the functor $\mathsf{C}(\cdot)$ to the functor $\mathsf{D}(\cdot)$, a question that for the time being will remain mostly unanswered.

Acknowledgements

I would like to thank my doctoral advisor, Joachim Cuntz, for successfully guiding me along the often difficult path of writing this dissertation, and of course my friends and colleagues from the operator theory workgroup and the SFB, for making my time in Münster such an inspiring and unforgettable experience.
In particular, my gratitude belongs to Hannes Thiel and Dominic Enders for their part in many enlightening discussions and for taking an interest in my research. Hannes Thiel moreover deserves my thanks for reading an early draft of this dissertation and pointing out many opportunities for improvements.
Finally, I want to thank my mother and my brothers for never stopping to believe in me, and all my friends, especially Stephan Rave, Ulrike Liermann, Christian Volk, and Magdalena Kotalla, for their support, for lifting my spirits when it mattered, and for reminding me that there is more to life than mathematics.

Contents

1 Introduction

1.1 Introduction

If A is a simple and stably finite C*-algebra, then the Cuntz semigroup $\mathsf{W}(A)$ is known to be the disjoint union of two subsemigroups: the first one, $\mathsf{V}(A)$, consists of all equivalence classes of projections (and is isomorphic to the Murray-von Neumann semigroup), and the second one, $\mathsf{W}(A)_+$, consists of all remaining equivalence classes. It was shown by N. P. Brown, F. Perera, and A. S. Toms in [9] that for the special case of a simple, separable, unital, exact, finite, and $\mathcal{Z}$-stable C*-Algebra A (where $\mathcal{Z}$ is the Jiang-Su algebra introduced in [20]), the latter semigroup is isomorphic to a semigroup of lower semicontinuous, affine, positive functions on the trace simplex of A. Moreover, it follows from their results that the ordered abelian semigroup $\mathsf{W}(A)$ can be reconstructed from three parts: the ordered semigroup $\mathsf{V}(A)$, the ordered semigroup $\mathsf{W}(A)_+$, and a morphism γ_A of ordered semigroups that maps the former into the latter. This morphism, under the conditions imposed above, is the map that sends the equivalence class $[p] \in \mathsf{V}(A)$ of a projection to the continuous, positive, affine function $\gamma_A([p]) := (\widehat{p}\colon \tau \mapsto \tau(p))$ on the trace simplex that evaluates each trace at the projection p. Moreover, it follows that for $x \in \mathsf{V}(A)^\times$, the element $\gamma_A(x)$ is the maximum among all elements in $\mathsf{W}(A)$ that are strictly below x. We call such an element the *predecessor* of x in $\mathsf{W}(A)$.

In the same year, K. T. Coward, G. A. Elliott, and C. Ivanescu introduced the stabilised Cuntz semigroup $\mathsf{Cu}(A)$ of a C*-algebra A in [11], and showed that it belongs to a category Cu, the objects of which satisfy a number of axioms that need not hold in general for other ordered abelian semigroups. Moreover, they showed that $\mathsf{Cu}(A) \cong \mathsf{W}(A \otimes \mathcal{K})$. The semigroup $\mathsf{Cu}(A)$ is also called the Cuntz semigroup of A, and it has largely replaced the semigroup $\mathsf{W}(A)$ in all applications. The reason for that is mostly that $\mathsf{Cu}(A)$ is an object of Cu while $\mathsf{W}(A)$ is not, and the additional structure carried by $\mathsf{Cu}(A)$ often makes the stabilised variant easier to reason about. In particular, the aforementioned theorem of Brown, Perera, and Toms has an analogous formulation for $\mathsf{Cu}(A)$; this result was first published by N. P. Brown and A. S. Toms in [10] and has since appeared in a number of slightly different forms.

In 2009, during a problem session at the American Institute of Mathematics Research Conference Center in Palo Alto, J. Cuntz asked if the equivalence class of a projection p in a simple, separable, stably finite, and nonelementary C*-algebra A always has a predecessor in the semigroup $\mathsf{Cu}(A)$. The participants of that session proved that this question has a positive answer, and hence that predecessors exist under much more general conditions than previously known. That result was the starting point of this dissertation, in which we will analyse the properties of predecessors and of the map γ_A that sends compact elements to their predecessors.

The dissertation is organised as follows: Chapter 1 is the introduction that you are reading right now. In Chapter 2, we define the Cuntz semigroup $\mathsf{Cu}(A)$ of a C*-algebra A and cite the most important structure theorems thereof. In Chapter 3, we define the category Cu, define a large number of properties that objects of Cu may or may not satisfy, and cite the most important structure results for semigroups in Cu. In Chapter 4, we show how the axiomatic definition of Cu relates to the structure of the ordered abelian semigroup $\mathsf{Cu}(A)$ introduced in Chapter 2. In Chapter 5, we define the cone of functionals on a semigroup in Cu, and give a rough overview of quasitracial weights on a C*-algebra and the relationship between these weights and the functionals on $\mathsf{Cu}(A)$. We claim no originality for the results in Chapter 2 to Chapter 5: everything contained therein should already be widely known among experts in the field.

Our original research starts in Chapter 6, where we use the techniques that were employed to answer the problem of the Palo Alto session, as mentioned before, to show that a large class of semigroups in Cu satisfies a condition we call *decomposability*; this class includes the Cuntz semigroups of all simple, separable, nonelementary, and stably finite C*-algebras, as well as all semigroups in Cu that are simple, separable, nonelementary, and weakly cancellative. Decomposability by definition means that every nonzero compact element $x \in S$ has a nonzero predecessor $\gamma_S(x)$, and that the so-called *predecessor map* γ_S is additive. The rationale behind the name "decomposable" is not elaborated upon at this point; it will become clear in Chapter 8.

In Chapter 7, we gather the fundamental properties of simple and decomposable semigroups S in Cu. In particular, we will introduce the category C of all algebraically ordered abelian monoids and the category D of all semigroups in Cu that have no nonzero compact elements. We analyse the subsemigroup $\mathsf{C}(S)$ consisting of all compact elements of S, and the subsemigroup $\mathsf{D}(S)$ consisting of 0 and of the noncompact elements of S, and we show that $\mathsf{C}(\cdot)$ and $\mathsf{D}(\cdot)$ are functors from the category of simple and decomposable semigroups in Cu to the categories C, D respectively. We also introduce the *extended predecessor map* $\varepsilon_S\colon S \to \mathsf{D}(S)$, a map that is defined on all of S and naturally extends the predecessor map $\gamma_S\colon \mathsf{C}(S) \to \mathsf{D}(S)$, sharing many of the latter map's features.

Chapter 8 is undoubtedly the heart of our research. We introduce the notion of a *composition map* $\gamma\colon C \to D$ between simple semigroups $C \in \mathsf{C}$ and $D \in \mathsf{D}$, and show that the predecessor map $\gamma_S\colon \mathsf{C}(S) \to \mathsf{D}(S)$ defined earlier for simple and decomposable semigroups is of this type. Every simple and decomposable semigroup $S \in \mathsf{Cu}$ can be decomposed into the triple $(\mathsf{C}(S), \mathsf{D}(S), \gamma_S)$. The original semigroup S can be recovered from this data: we introduce a construction that associates to each triple (C, D, γ) – where the first component is a simple object of C, the second component is a simple object of D, and the third component is a composition map between them – a semigroup $C \sqcup_\gamma D$ in the category Cu, and we show that $S \cong \mathsf{C}(S) \sqcup_{\gamma_S} \mathsf{D}(S)$ for every simple and decomposable semigroup S in Cu. Indeed, we show that a simple semigroup S in Cu is decomposable if and only if it is of the form $C \sqcup_\gamma D$ for a *composable triple* (C, D, γ) as above, finally providing the justification for using the name "decomposable" in Chapter 6. Additionally, we introduce a notion of morphism that turns the class of composable triples (C, D, γ) into a category, and we show that the decomposition functor $\mathsf{Dec}\colon S \mapsto (\mathsf{C}(S), \mathsf{D}(S), \gamma_S)$ from the category Cu_{dec} of simple and decomposable semigroups into the category Cu_{com} of composable triples is an equivalence of categories, with its converse equivalence functor given by $\mathsf{Com}\colon (C, D, \gamma) \mapsto C \sqcup_\gamma D$. It is at this point that we are first confronted with the question of whether or not the predecessor map is a natural transformation from the functor $\mathsf{C}(\cdot)$ to the functor $\mathsf{D}(\cdot)$, a question which is still largely unanswered. We will return to that problem later, in Chapter 10.

In Chapter 9, we apply our composition and decomposition results to the most famous invariant in the classification theory of C*-algebras, the Elliott invariant. In particular, we cite the definition of the Elliott invariant Ell and the extended Elliott invariant $\widetilde{\mathsf{Ell}}$, and introduce our own invariant $\overline{\mathsf{Ell}}$ for the category of simple, separable, unital, exact, cancellative, and nonelementary C*-algebras. The new invariant $\overline{\mathsf{Ell}}$ consists of the ordered abelian group with order unit $\mathsf{K}_0(A)$, the abelian group $\mathsf{K}_1(A)$, the semigroup $\mathsf{D}(A)$, and the predecessor map γ_A. We are not trying to improve upon the Elliott invariant; the reason for this chapter is to provide a new picture of the extended Elliott invariant by exploring its connection to $\overline{\mathsf{Ell}}$. We show (unsurprisingly) that the original Elliott invariant Ell can be recovered functorially from $\overline{\mathsf{Ell}}$, and that $\overline{\mathsf{Ell}}$ and $\widetilde{\mathsf{Ell}}$ are equivalent under the restrictions mentioned above, i.e. for the category of simple, separable, unital, exact, cancellative, and nonelementary C*-algebras. We also show that the semigroup $\mathsf{D}(A)$ and the predecessor map γ_A can be interpreted as preduals of the trace simplex $\mathsf{T}_1(A)$ and the pairing map ρ_A, suggesting that the extended invariant $\widetilde{\mathsf{Ell}}$ is quite a natural generalisation of the invariant Ell.

In Chapter 10, we return to the question of whether or not the predecessor map is a natural transformation from $\mathsf{C}(\cdot)$ to $\mathsf{D}(\cdot)$, something that is plausible considering the results of Chapter 8. While a satisfactory answer to this question remains elusive, we do provide some preliminary results. We show that the predecessor map is indeed a natural transformation if we restrict to the class of simple and decomposable semigroups that are almost unperforated; moreover, we provide a condition that is equivalent to the naturality of the predecessor map for the class of simple and decomposable semigroups that are weakly cancellative. Finally, we show that naturality of the predecessor map holds under conditions that appear to be weaker than being almost unperforated, e.g. for semigroups that satisfy a strong cancellation property, and for semigroups with n-comparison for some $n \in \mathbb{N}$.

1.2 Notation

The notation used in this dissertation is fairly standard, and usually explained on first use. The symbol $\mathbb{N}$ denotes the set of natural numbers excluding zero, while the symbol $\mathbb{N}_0$ denotes the set of natural numbers including zero, i.e. $\mathbb{N} = \{1, 2, ...\}$ and $\mathbb{N}_0 = \{0, 1, 2, ...\}$. For a C*-algebra A, the symbol $\tilde{A}$ denotes the minimal unitisation of A; if A is already unital, then $\tilde{A} = A$. We use A_{min} to denote the Pedersen ideal of A, i.e. the minimal dense ideal of A. For any element $a \in A$, we use $\sigma(a)$ to denote the spectrum of a, and sometimes $\sigma_A(a)$ to avoid ambiguities. By $\mathcal{M}(A)$ we denote the multiplier algebra of A. If A and B are C*-algebras, then $A \otimes B$ shall always refer to the minimal tensor product of A and B (also known as the spatial tensor product of A and B).

We use M_n or $M_n(\mathbb{C})$ to denote the algebra of $n \times n$ matrices over the field of complex numbers, and $M_n(A)$ to denote the C*-algebra of $n \times n$ matrices over the C*-algebra A. For $n < m$, we will identify $M_n(A)$ with the upper left corner of $M_m(A)$. By $M_\infty(A)$ we mean the algebraic direct limit of the $M_n(A)$, i.e. the set of matrices with countably infinitely many rows and columns and with coefficients in A, of which all but finitely many vanish. As before, we will identify $M_n(A)$ with the upper left corner of $M_\infty(A)$.

The symbol $\mathcal{H}$ is used for Hilbert spaces. We use $\mathcal{K}(\mathcal{H})$ to refer to the algebra of compact operators on $\mathcal{H}$, and $\mathcal{B}(\mathcal{H})$ to refer to the algebra of bounded operators on $\mathcal{H}$. By $\mathcal{K}$ we shall always mean the algebra of compact operators on the Hilbert space $\ell^2(\mathbb{N})$. By $\mathcal{Z}$ we always mean the Jiang-Su algebra as introduced in [20]. By $\mathsf{QT}(A)$ we mean the set of all lower semicontinuous 2-quasitracial weights on the C*-algebra A, and by $\mathsf{QT}_1(A)$ the set of all normalised 2-quasitracial weights on A (the exact meaning of these terms will be explained in Chapter 5). Likewise, $\mathsf{T}(A)$ and $\mathsf{T}_1(A)$ denote the lower semicontinuous tracial weights and the normalised tracial weights on A, respectively.

If $X_1, ..., X_n$ are any sets, then $X_1 \sqcup ... \sqcup X_n$ refers to their disjoint union, i.e. to the set $\{(x, k) \mid k \in \{1, ..., n\} \text{ and } x \in X_k\}$. We will usually identify X_k with its copy $\{(x, k) \mid x \in X_k\}$ in the disjoint union via the obvious bijective map, unless there are sound reasons that require us to be more rigorous.

2 The Cuntz semigroup

2.1 Construction of the Cuntz semigroup

We shall now give an overview of the construction of the semigroups $\mathsf{V}(A)$ and $\mathsf{Cu}(A)$. Let $\mathcal{H}$ be the Hilbert space $\ell^2(\mathbb{N})$, and let $\mathcal{K} := \mathcal{K}(\mathcal{H})$ be the C*-algebra of compact operators on $\mathcal{H}$. The following lemma is elementary, and allows us to think of the elements of $A \otimes \mathcal{K}$ as certain infinite matrices with coefficients in A:

2.1.1 Lemma. *Let A be a C*-algebra, and let $a \in A \otimes \mathcal{K}$. Let $(e_{ij})_{i,j \in \mathbb{N}}$ be a complete system of matrix units for $\mathcal{K}$. Then there is a unique family $(a_{ij})_{i,j \in \mathbb{N}}$ of elements of A such that $a = \lim_n \sum_{i,j=1}^n a_{ij} \otimes e_{ij}$.*

Proof. We will only sketch the proof and leave out the calculations. Since the linear span of $\{x \otimes e_{ij} \mid x \in A \text{ and } i,j \in \mathbb{N}\}$ is dense in $A \otimes \mathcal{K}$, we can find a sequence $(x_n)_n$ such that $\lim_n \ x_n = a$ and $x_n = \sum_{i,j \in \mathbb{N}} a_{ij}^{(n)} \otimes e_{ij}$ with elements $a_{ij}^{(n)} \in A$ that vanish except for finitely many index pairs $(i,j) \in \mathbb{N}^2$. Let $E_n := \sum_{i=1}^n e_{ii}$, then $(\mathbb{I}_{\tilde{A}} \otimes E_n)_n$ is an approximate identity for $\tilde{A} \otimes \mathcal{K}$. Using this approximate identity, a mostly straightforward calculation shows that the sequences $(a_{ij}^{(n)})_n$ are all Cauchy, hence that $a_{ij} = \lim_n \ a_{ij}^{(n)}$ is well-defined for every index pair (i,j), and that $x = \lim_n \sum_{i,j=1}^n a_{ij} \otimes e_{ij}$ as expected. If $(b_{ij})_{i,j \in \mathbb{N}}$ is another sequence of elements in A with $x = \lim_n \sum_{i,j=1}^n b_{ij} \otimes e_{ij}$, then it follows that, for any approximate identity $(u_\nu)_\nu$ of A, we have $a_{ij} \otimes e_{ij} = \lim_\nu (u_\nu \otimes e_{ii}) x (e_\nu \otimes e_{jj})$ and $b_{ij} \otimes e_{ij} = \lim_\nu (u_\nu \otimes e_{ii}) x (e_\nu \otimes e_{jj})$, hence $a_{ij} \otimes e_{ij} = b_{ij} \otimes e_{ij}$ and therefore $a_{ij} = b_{ij}$ for every index pair (i,j). ■

In [25], J. Murray and J. von Neumann gave a construction that associates to A an abelian monoid $\mathsf{V}(A)$ as follows: let $\mathcal{P}(A \otimes \mathcal{K})$ be the set of projections in $A \otimes \mathcal{K}$. For any $p, q \in \mathcal{P}(A \otimes \mathcal{K})$, define relations $\precsim$ (subequivalence) and $\approx$ (equivalence) on $\mathcal{P}(A \otimes \mathcal{K})$ by

$$p \precsim q \quad :\Longleftrightarrow \quad \text{there is } v \in A \otimes \mathcal{K} \text{ such that } v^*v = p \text{ and } vv^* \leq q,$$
$$p \approx q \quad :\Longleftrightarrow \quad \text{there is } v \in A \otimes \mathcal{K} \text{ such that } v^*v = p \text{ and } vv^* = q.$$

It is easy to see that $\precsim$ is a preorder on $\mathcal{P}(A \otimes \mathcal{K})$, and hence that $\approx$ is an equivalence relation. Define $\mathsf{V}(A)$ as $\mathcal{P}(A \otimes \mathcal{K}) / \approx$. We can define a preorder on $\mathsf{V}(A)$ by $[p] \leq [q] \ :\Longleftrightarrow \ p \precsim q$. Note that this will not, in general, be an order relation, since $p \precsim q \precsim p$ does not always imply $p \approx q$ (more on that later). We want to define an addition on $\mathsf{V}(A)$ by $[p] + [q] := [\overline{p} + \overline{q}]$, where $\overline{p}, \overline{q}$ are any projections in $\mathcal{P}(A \otimes \mathcal{K})$ with $p \approx \overline{p}$, and $q \approx \overline{q}$, and $\overline{p} \perp \overline{q}$. We need to show that this addition is well-defined:

2.1.2 Lemma. *Let A be a C^*-algebra.*

(*i*) *For any two projections p, q in $A \otimes \mathcal{K}$, we can find projections $\overline{p}, \overline{q}$ in $A \otimes \mathcal{K}$ such that $p \approx \overline{p}$, and $q \approx \overline{q}$, and $\overline{p} \perp \overline{q}$.*

(*ii*) *For any projections p_1, p_2, q_1, q_2 in $A \otimes \mathcal{K}$ such that $p_1 \approx p_2$, and $q_1 \approx q_2$, and $p_1 \perp q_1$, and $p_2 \perp q_2$, we have that $p_1 + q_1 \approx p_2 + q_2$.*

Proof.

(*i*) Let $\mathcal{L}_1, \mathcal{L}_2$ be two orthogonal, infinite-dimensional subspaces of $\mathcal{H}$ with $\mathcal{H} = \mathcal{L}_1 \oplus \mathcal{L}_2$. We can find isometries $V, W \in \mathcal{B}(\mathcal{H})$ with $\operatorname{im}(VV^*) = \mathcal{L}_1$ and $\operatorname{im}(WW^*) = \mathcal{L}_2$. Let $\mathbb{I}$ be the unit of $\tilde{A}$; then $(\mathbb{I} \otimes V)$, $(\mathbb{I} \otimes W)$ act as multipliers on $A \otimes \mathcal{K}$. Let $v := (\mathbb{I} \otimes V)p \in A \otimes \mathcal{K}$ and $w := (\mathbb{I} \otimes W)q \in A \otimes \mathcal{K}$. Then $v^*v = p$ and $w^*w = q$, while $\overline{p} := vv^* \in A \otimes \mathcal{K}(\mathcal{L}_1)$ and $\overline{q} := ww^* \in A \otimes \mathcal{K}(\mathcal{L}_2)$. It follows that $p \approx \overline{p}$, and $q \approx \overline{q}$, and $\overline{p} \perp \overline{q}$.

(*ii*) Let $v, w \in \mathcal{K}$ be partial isometries such that $v^*v = p_1$, and $vv^* = p_2$, and $w^*w = q_1$, and $ww^* = q_2$. Then it follows that $v^*w = v^*p_2q_2w = 0$ and $w^*v = w^*q_2p_2v = 0$, and hence $(v+w)^*(v+w) = v^*v + w^*w = p_1 + q_1$. Likewise, we have $vw^* = vp_1q_1w^* = 0$ and $wv^* = wq_1p_1v^* = 0$, and hence $(v+w)(v+w)^* = vv^* + ww^* = p_2 + q_2$. Thus, we have shown that $p_1 + q_1 \approx p_2 + q_2$.

■

As long as we are only concerned with projections, it does not matter if $A \otimes \mathcal{K}$ is replaced by $M_\infty(A)$ in the above construction; the resulting semigroups will be isomorphic.

In [12], J. Cuntz associated to any C*-algebra A an ordered abelian group $\mathsf{K}_0^*(A)$. Implicit in this construction is the definition of an abelian monoid that has $\mathsf{K}_0^*(A)$ as its enveloping Grothendieck group; this monoid was later called $\mathsf{W}(A)$ by M. Rørdam. $\mathsf{W}(A)$ is defined similarly to $\mathsf{V}(A)$, but instead of containing only equivalence classes of projections, it contains equivalence classes of all positive elements in $M_\infty(A)$, and the equivalence and subequivalence relations on $M_\infty(A)_+$ have a more topological nature. In [11], K. T. Coward, G. A. Elliott, and C. Ivanescu associated yet another abelian monoid $\mathsf{Cu}(A)$ to the C*-algebra A, this one consisting of isomorphism classes of countably generated Hilbert (right) A modules. They proved that $\mathsf{Cu}(A)$ is always isomorphic to $\mathsf{W}(A \otimes \mathcal{K})$, but the Hilbert module picture of $\mathsf{Cu}(A)$ allowed them to show that the semigroups $\mathsf{Cu}(A)$ are objects of a special category Cu and satisfy a couple of very useful properties that need not always be satisfied by $\mathsf{W}(A)$. Both $\mathsf{W}(A)$ and $\mathsf{Cu}(A)$ are commonly referred to as *the Cuntz semigroup of A*; in this dissertation, the term *Cuntz semigroup of A* will always refer to $\mathsf{Cu}(A)$. We shall now give a construction of $\mathsf{Cu}(A)$ that follows Cuntz's original construction of $\mathsf{W}(A)$, the only difference being that $\mathsf{Cu}(A)$ is defined in terms of $(A \otimes \mathcal{K})_+$ instead of $M_\infty(A)_+$.

Let $(A \otimes \mathcal{K})_+$ be the set of positive elements in $A \otimes \mathcal{K}$. For any $a, b \in (A \otimes \mathcal{K})_+$, define relations $\precsim$ (subequivalence) and $\sim$ (equivalence) on $\mathcal{P}(A \otimes \mathcal{K})$ by

$$a \precsim b \quad :\Longleftrightarrow \quad \text{there is } (x_n)_n \text{ in } A \otimes \mathcal{K} \text{ such that } \lim_n x_n^* b x_n = a,$$
$$a \sim b \quad :\Longleftrightarrow \quad a \precsim b \text{ and } b \precsim a.$$

It is not hard to see that $\precsim$ is a preorder on $(A \otimes \mathcal{K})_+$, and hence that $\sim$ is an equivalence relation. Define $\mathsf{Cu}(A)$ as $(A \otimes \mathcal{K})_+ / \sim$. We can define a partial order on $\mathsf{Cu}(A)$ by $[a] \leq [b] \;:\Longleftrightarrow\; a \precsim b$. Note that this will, in fact, be an order relation, since $p \precsim q \precsim p$ always implies $p \sim q$ by definition. We want to define an addition on $\mathsf{Cu}(A)$ by $[a] + [b] := [\overline{a} + \overline{b}]$, where $\overline{a}, \overline{b}$ are any elements in $(A \otimes \mathcal{K})_+$ with $a \sim \overline{a}$, and $b \sim \overline{b}$, and $\overline{a} \perp \overline{b}$. The following lemma shows that this addition is well-defined:

2.1.3 Lemma. *Let A be a C^*-algebra.*

- (i) *For any two elements a, b in $(A \otimes \mathcal{K})_+$, we can find elements $\overline{a}, \overline{b}$ in $(A \otimes \mathcal{K})_+$ such that $a \sim \overline{a}$, and $b \sim \overline{b}$, and $\overline{a} \perp \overline{b}$.*
- (ii) *For any elements a_1, a_2, b_1, b_2 in $(A \otimes \mathcal{K})_+$ such that $a_1 \precsim a_2$, and $b_1 \precsim b_2$, and $a_2 \perp b_2$, we have $a_1 + b_1 \precsim a_2 + b_2$.*
- (iii) *For any elements a_1, a_2, b_1, b_2 in $(A \otimes \mathcal{K})_+$ such that $a_1 \sim a_2$, and $b_1 \sim b_2$, and $a_1 \perp b_1$, and $a_2 \perp b_2$, we have $a_1 + b_1 \sim a_2 + b_2$.*

Proof. The proof is similar to the proof of Lemma 2.1.2, but slightly more complicated. The statement in part (ii) was originally proven by J. Cuntz in [12], Proposition 1.1.

- (i) Let $\mathcal{L}_1, \mathcal{L}_2$ be two orthogonal, infinite-dimensional subspaces of $\mathcal{H}$ with $\mathcal{H} = \mathcal{L}_1 \oplus \mathcal{L}_2$. Let $(e_{ij})_{ij}$ be a complete system of matrix units for $\mathcal{K}(\mathcal{H})$, let $(f_{ij})_{ij}$ be a complete system of matrix units for $\mathcal{K}(\mathcal{L}_1)$, and let $(g_{ij})_{ij}$ be a complete system of matrix units for $\mathcal{K}(\mathcal{L}_2)$. Let $a = \lim_n \sum_{i,j=1}^n a_{ij} \otimes e_{ij}$ and $b = \lim_n \sum_{i,j=1}^n b_{ij} \otimes e_{ij}$ with a_{ij}, $b_{ij} \in A$ as in Lemma 2.1.1, and define $\overline{a} := \lim_n \sum_{i,j=1}^n a_{ij} \otimes f_{ij}$ and $\overline{b} := \lim_n \sum_{i,j=1}^n a_{ij} \otimes g_{ij}$. Then $\overline{a} \in (A \otimes \mathcal{K}(\mathcal{L}_1))_+$ and $\overline{b} \in (A \otimes \mathcal{K}(\mathcal{L}_2)))_+$, so we clearly have $\overline{a} \perp \overline{b}$. It remains to show that $a \sim \overline{a}$ and $b \sim \overline{b}$. We can pick a sequential approximate identity $(u_n)_n$ of $C^*(\{a_{ij}, b_{ij} \mid i, j \in \mathbb{N}\})$; then $a_{ij} = \lim_n u_n a_{ij} u_n$ and $b_{ij} = \lim_n u_n b_{ij} u_n$ for all $i, j \in \mathbb{N}$. Moreover, we can find a sequence of partial isometries $(v_n)_n \in \mathcal{K}(\mathcal{H})$ such that $v_n e_{ij} v_n^* = f_{ij}$ and $v_n^* f_{ij} v_n = e_{ij}$ whenever $i, j \in \{1, ..., n\}$, and such that $v_n e_{ij} v_n^* = 0$ and $v_n^* f_{ij} v_n = 0$ whenever $i > n$ or $j > n$. Let $x_n := u_n \otimes v_n$ for each $n \in \mathbb{N}$. Then $x_n a x_n^* = \sum_{i,j=1}^n u_n a_{ij} u_n \otimes f_{ij}$ and $x_n^* \overline{a} x_n = \sum_{i,j=1}^n u_n a_{ij} u_n \otimes e_{ij}$. Using the fact that $(u_n \otimes \sum_{k=1}^n e_{kk})_n$ and $(u_n \otimes \sum_{k=1}^n f_{kk})_n$ are approximate identities of $A \otimes \mathcal{K}$ and $A \otimes \mathcal{K}(\mathcal{L}_1)$, respectively, we conclude that $\overline{a} = \lim_n x_n a x_n^*$ and $a = \lim_n x_n^* \overline{a} x_n$. Analogously, we can find a sequence $(y_n)_n$ in $A \otimes \mathcal{K}$ such that $\lim_n y_n b y_n^* = \overline{b}$ and $\lim_n y_n^* \overline{b} y_n = b$. Thus, we have $a \sim \overline{a}$ and $b \sim \overline{b}$.

(*ii*) Fix $n \in \mathbb{N}$. Since $a_1 \precsim a_2$, we can find $z \in A \otimes \mathcal{K}$ with $\|z^* a_2 z - a_1\| \leq \frac{1}{2n}$; we may assume that $z \neq 0$. Find $m \in \mathbb{N}$ such that $\|a_2^{m+2/m} - a_2\| \leq \frac{1}{2n\|z\|^2}$, and let $x_n = a_2^{1/m} z$. Using the triangle inequality, we find that $\|x_n^* a_2 x_n - a_1\| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Thus, we have found a sequence $(x_n)_n$ in the right ideal $\overline{a_2(A \otimes \mathcal{K})}$ with $\lim_n x_n^* a_2 x_n = a_1$. Analogously, we can find a sequence $(y_n)_n$ in the right ideal $\overline{b_2(A \otimes \mathcal{K})}$ such that $\lim_n y_n^* b_2 y_n = b_1$. Since $a_2 \perp b_2$, it follows that $b_2 x_n = a_2 y_n = 0$ for all $n \in \mathbb{N}$, and hence $\lim_n (x_n + y_n)^*(a_2 + b_2)(x_n + y_n) = \lim_n (x_n^* a_2 x_n + y_n^* b_2 y_n) = a_1 + b_1$. Thus, we have $a_1 + b_1 \precsim a_2 + b_2$ as expected.

(*iii*) This follows immediately from part (*ii*).

■

Thus, we have made $\mathsf{Cu}(A)$ into an ordered abelian monoid. If p, q are projections in $\mathcal{P}(A \otimes \mathcal{K})$, then it can be shown (see [2], Lemma 2.18) that $p \precsim q$ in the sense of Murray and von Neumann if and only if $p \precsim q$ in the sense of Cuntz (we have therefore decided not to distinguish these relations in our notation). On the other hand, this implies that equivalence in the sense of Murray and von Neumann cannot, in general, be the same as equivalence in the sense of Cuntz. However, in this dissertation we will mostly be interested in C*-algebras that are simple and stably finite:

2.1.4 Definition. *A simple C*-algebra A is called* stably finite *if $A \otimes \mathcal{K}$ contains no infinite projections, i.e. if $p \approx q \leq p$ implies $p = q$ for all projections $p, q \in A \otimes \mathcal{K}$.*

We will see in Chapter 5 that a simple C*-algebra A is stably finite in the sense of this definition if and only if there is a faithful, semifinite, and lower semicontinuous quasitrace on A. This makes the above definition of stable finiteness appropriate for simple C*-algebras. If A is any C*-algebra such that $A \otimes \mathcal{K}$ contains no infinite projections, then it is indeed true for all projections $p, q \in A \otimes \mathcal{K}$ that $p \approx q$ if and only if $p \sim q$: we always have $p \approx q \implies p \sim q$ since subequivalence in the sense of Murray and von Neumann agrees with subequivalence in the sense of Cuntz. For the other direction, we need to show that $p \precsim q \precsim p$ implies $p \approx q$. By Lemma 2.1.2 (*i*), we may assume without loss of generality that $p \perp q$. We have $p \approx q' \leq q$ and $q \approx p' \leq p$. Let $p'' := p - p'$ and $q'' := q - q'$. Using Lemma 2.1.2 (*ii*), we find that

$$p' + (p'' + q'') = (p' + p'') + q'' = p + q'' \approx q' + q'' = q \approx p'.$$

It follows that $p' + (p'' + q'') \approx p' \leq p' + (p'' + q'')$; since $A \otimes \mathcal{K}$ contains no infinite projections, it follows that $p' = p' + (p'' + q'')$, which implies $p'' + q'' = 0$ and therefore $p'' = q'' = 0$. Thus, we have $q = q'$ and $p = p'$, which means that $p \approx q$, as we wanted to show. It follows that if $A \otimes \mathcal{K}$ contains no infinite projections, then the Murray-von Neumann semigroup $\mathsf{V}(A)$ can be understood as a subsemigroup of the Cuntz semigroup $\mathsf{Cu}(A)$ with the induced order. The remainder $\mathsf{Cu}(A) \setminus \mathsf{V}(A)$ is sometimes denoted by $\mathsf{Cu}(A)_+$.

2.2 Properties of the Cuntz semigroup

Let A be a C*-algebra, and let $a \in A_+$. For every $\varepsilon > 0$, let $\ell_\varepsilon\colon [0,\infty) \to [0,\infty)$ be the continuous function with $\ell_\varepsilon(t) = \max\{0,\ t-\varepsilon\}$. Then the element $\ell_\varepsilon(a) \in A_+$ is well-defined by functional calculus; we shall denote this element by $(a-\varepsilon)_+$. Moreover, let $A_a := \overline{aAa}$ denote the hereditary C*-subalgebra of A generated by a. We will now collect the most important structural properties of $\mathsf{Cu}(A)$.

2.2.1 Theorem. *Let A be any C*-algebra, and let $a, b \in (A \otimes \mathcal{K})_+$.*

- *(i) If $f, g \in \mathrm{C}_0(\sigma(a))_+$ such that $\{t \in \sigma(a) \mid f(t) > 0\} \subseteq \{t \in \sigma(a) \mid g(t) > 0\}$, then $[f(a)] \leq [g(a)]$.*
- *(ii) We have $[a] = [a^n]$ for every $n \in \mathbb{N}$.*
- *(iii) We have $[x^*x] = [xx^*]$ for every $x \in A \otimes \mathcal{K}$.*
- *(iv) If $a \leq b$, then $[a] \leq [b]$.*
- *(v) We have $[a+b] \leq [a]+[b]$. If $a \perp b$, then we have $[a+b] = [a]+[b]$.*
- *(vi) If $\|a-b\| < \varepsilon$, then there is a contraction $d \in A \otimes \mathcal{K}$ with $(a-\varepsilon)_+ = d^*bd$. In particular, this means that $[(a-\varepsilon)_+] \leq [b]$.*

Proof. Proofs for most of these statements can be found in [2]: for statement (i) see Proposition 2.5 of [2], for statements (ii) and (iii) see Corollary 2.6 of [2], for statement (iv) see Lemma 2.8 of [2], and for statement (vi) see Theorem 2.13 of [2] (as mentioned there, this result was originally obtained by E. Kirchberg and M. Rørdam in [22], Lemma 2.2). Statement (v) follows from Lemma 2.1.3 and our choice of definition for the addition in $\mathsf{Cu}(A)$. ∎

2.2.2 Lemma. *Let A be a C*-algebra, and let $a, b \in (A \otimes \mathcal{K})_+$. If $a \in (A \otimes \mathcal{K})_b$, then $a \precsim b$.*

Proof. Let $\varepsilon > 0$ be fixed; we want to find an element $x \in A \otimes \mathcal{K}$ such that $\|a - x^*bx\| < \varepsilon$. Since $(b^{1/m})_m$ is an approximate unit for $(A \otimes \mathcal{K})_b$, we can find a sufficiently large m such that $\left\|b^{1/m}ab^{1/m} - a\right\| < \frac{\varepsilon}{2}$. We have $b^{1/m}ab^{1/m} \sim a^{1/2}b^{2/m}a^{1/2} \precsim b^{2/m} \sim b$ by parts (i) and (iii) of the preceding theorem. Thus, we can find an $x \in A \otimes \mathcal{K}$ with $\left\|b^{1/m}ab^{1/m} - x^*bx\right\| < \frac{\varepsilon}{2}$. In total, we get $\|a - x^*bx\| < \varepsilon$ as expected. It follows that $a \precsim b$. ∎

2.2.3 Lemma. *Let A be a C*-algebra, and let $a, b \in (A \otimes \mathcal{K})_+$. If $a \precsim b$, then for every $\varepsilon > 0$ there is $x \in A \otimes \mathcal{K}$ such that $x^*x = (a-\varepsilon)_+$ and $xx^* \in (A \otimes \mathcal{K})_b$. In particular, there is a positive element $c \in (A \otimes \mathcal{K})_b$ such that $(a-\varepsilon)_+ \sim c$.*

Proof. We have $a \precsim b$, so for any given $\varepsilon > 0$ we can find some element $r \in A \otimes \mathcal{K}$ such that $\|r^*br - a\| < \varepsilon$. Using Theorem 2.2.1 (vi), we can find a contraction $d \in A \otimes \mathcal{K}$ such that $(a-\varepsilon)_+ = d^*r^*brd$. Let $x := b^{1/2}rd$. Then $x^*x = (a-\varepsilon)_+$, and $c := xx^* \in (A \otimes \mathcal{K})_b$ as required. That $(a-\varepsilon)_+ = x^*x \sim xx^* = c$ follows immediately from Theorem 2.2.1 (iii). ∎

The following proposition is often of enormous help:

2.2.4 Rørdam's proposition. *Let A be any C*-algebra, and let $a, b \in (A \otimes \mathcal{K})_+$. The following conditions are equivalent:*

(i) $a \precsim b$.

(ii) *For every $\varepsilon > 0$, we have $(a - \varepsilon)_+ \precsim b$.*

(iii) *For every $\varepsilon > 0$ there is $\delta > 0$ such that $(a - \varepsilon)_+ \precsim (b - \delta)_+$.*

Proof. The first result of this type was proven by M. Rørdam in [30], Proposition 2.4. In the form quoted above, it first appeared in [21], Proposition 2.6. A complete proof can be found in [2], Proposition 2.17. ∎

For example, the following result follows immediately from Rørdam's proposition (but it can be improved considerably, as we will show in Theorem 4.1.2 (*i*)):

2.2.5 Corollary. *Let A be a C*-algebra, let $a \in (A \otimes \mathcal{K})_+$, and let $(\varepsilon_n)_n$ be a sequence in $(0, \infty)$ that decreases towards 0. Then $([(a - \varepsilon_n)_+])_n$ is an increasing sequence in $\mathsf{Cu}(A)$ with supremum $[a]$.*

Proof. For any $0 < \delta_1 \leq \delta_2 < \infty$, we have $(a - \delta_2)_+ \leq (a - \delta_1)_+ \leq a$ by functional calculus, which implies $[(a - \delta_2)_+] \leq [(a - \delta_1)_+] \leq [a]$ by Theorem 2.2.1 (*iv*). It follows that $([(a - \varepsilon_n)_+])_n$ is an increasing sequence with $[a]$ as an upper bound. Let $b \in (A \otimes \mathcal{K})_+$ be such that $[b]$ is another upper bound for this sequence; we need to show that $[a] \leq [b]$. Fix any $\varepsilon > 0$. We can find a sufficiently large n with $\varepsilon_n \leq \varepsilon$, which implies $(a - \varepsilon)_+ \precsim (a - \varepsilon_n)_+ \precsim b$. But since $(a - \varepsilon)_+ \precsim b$ for every $\varepsilon > 0$, we get $a \precsim b$ from Rørdam's proposition, and hence $[a] \leq [b]$. ∎

3 The category Cu

An abelian semigroup M is called *positively ordered* if for all $a, b \in M$ we have $a \leq a + b$, and for all $a, b, c \in M$ we have $a \leq b \implies a + c \leq b + c$. Given such a semigroup M, we will denote by $M^\times$ the same semigroup with its zero element removed (if M does not have a zero element, then $M^\times := M$). If $X \subseteq M$, then an element $s \in M$ is called the supremum of X (denoted by $s = \sup X$) if s is an upper bound for X, and if for every other upper bound $t \in M$ of X, we have $s \leq t$. By antisymmetry of the order relation, it follows that the supremum of a subset is unique whenever it exists. We define a relation $\ll$ on M as follows:

$$x \ll y \quad :\Longleftrightarrow \quad \text{for every increasing sequence } (z_n)_n \text{ in } M \text{ that has a supremum } z \text{ in } M \text{ such that } y \leq z, \text{there is a natural number } n \text{ such that } x \leq z_n.$$

We say that x is *compactly contained* in y, or that x is *way below* y. This relation is most useful if every increasing sequence in M has a supremum in M, but it is well-defined for any partially ordered set M. Note that $x \ll y$ implies $x \leq y$, and that $w \leq x \ll y \leq z$ implies $w \ll z$. An element $x \in M$ is called *compact* if $x \ll x$, i.e. whenever $(x_n)_n$ is an increasing sequence with a supremum above x, then the terms x_n are eventually above x. In particular, every increasing sequence with supremum x is eventually constant with terms equal to x. Finally, a sequence $(x_n)_n$ in M is called *rapidly increasing* if $x_n \ll x_{n+1}$ for all n. If x is the supremum of of a rapidly increasing sequence $(x_n)_n$, then x is compact if and only if the sequence $(x_n)_n$ is eventually constant.

Using these notions, a particular category of positively ordered abelian monoids has been defined by Coward, Elliott, and Ivanescu in [11]. We give a slightly different definition, including the additional axioms ($\mathcal{O}5$) and ($\mathcal{O}6$) as given in [29]. Coward, Elliott, and Ivanescu showed that for all C*-algebras A and B, and every *-homomorphism $\varphi\colon A \to B$, the semigroup $\mathsf{Cu}(A)$ satisfies axioms ($\mathcal{O}1$) to ($\mathcal{O}4$), and the map $\mathsf{Cu}(\varphi)\colon \mathsf{Cu}(A) \to \mathsf{Cu}(B)$ satisfies axioms ($\mathcal{M}1$) to ($\mathcal{M}4$). It was shown by Rørdam and Winter in Lemma 7.1 of [34] that $\mathsf{Cu}(A)$ always satisfies axiom ($\mathcal{O}5$), and by Robert in Proposition 5.1.1 of [29] that $\mathsf{Cu}(A)$ always satisfies axiom ($\mathcal{O}6$).

The definition of the category Cu might best be understood as a work in progress, since additional axioms satisfied by every Cuntz semigroup $\mathsf{Cu}(A)$ are likely to be found. We will sometimes refer to semigroups of the form $\mathsf{Cu}(A)$ as *concrete* Cuntz semigroups, and to objects in the category Cu as *abstract* Cuntz semigroups. One should keep in mind, though, that the latter class is in all likelihood much larger than the former class.

3.1 Definition and basic properties of Cu

3.1.1 Definition. *The Category Cu is defined as follows.*
The objects of Cu are the positively ordered abelian monoids S that satisfy the following axioms:

($\mathcal{O}1$) *Every increasing sequence $(x_n)_n$ in S has a supremum in S.*
($\mathcal{O}2$) *Every $x \in S$ is the supremum of a rapidly increasing sequence in S.*
($\mathcal{O}3$) *If $x_1, x_2, y_1, y_2 \in S$ with $x_1 \ll y_1$ and $x_2 \ll y_2$, then $x_1 + x_2 \ll y_1 + y_2$.*
($\mathcal{O}4$) *If $(x_n)_n$ and $(y_n)_n$ are two increasing sequences in S, then* $\sup_n (x_n + y_n) = \sup_n x_n + \sup_n y_n$.
($\mathcal{O}5$) *S has* almost algebraic order: *if $x', x, y \in S$ with $x' \ll x \leq y$, then there is $z \in S$ such that $x' + z \leq y \leq x + z$.*
($\mathcal{O}6$) *S has* almost Riesz decomposition: *if $x', x, y, z \in S$ with $x' \ll x \leq y + z$, then there are elements $y_0 \leq x, y$ and $z_0 \leq x, z$ such that $x' \leq y_0 + z_0$.*

The morphisms of Cu are the additive maps $\alpha \colon S_1 \to S_2$ that satisfy the following axioms:

($\mathcal{M}1$) $\alpha(0) = 0$.
($\mathcal{M}2$) *If $x, y \in S_1$ and $x \leq y$, then $\alpha(x) \leq \alpha(y)$.*
($\mathcal{M}3$) *If $x, y \in S_1$ and $x \ll y$, then $\alpha(x) \ll \alpha(y)$.*
($\mathcal{M}4$) *If $(x_n)_n$ is an increasing sequence in S_1, then $\alpha(\sup_n x_n) = \sup_n \alpha(x_n)$.*

We prove the following two lemmas for the reader's convenience.

3.1.2 Lemma. *Let S be any semigroup in the category Cu, and let $x, y \in S$.*

(i) *There is $x' \in S$ with $x' \ll x$. If $0 < x$, we can moreover arrange that $0 < x'$.*
(ii) *If $x \ll y$, then there is $z \in S$ such that $x \ll z \ll y$.*
(iii) *If $x \ll y$ and y is noncompact, then there is $z \in S^\times$ such that $x + z \ll y$.*

Proof.

(i) Pick any rapidly increasing sequence $(x_n)_n$ with supremum x. We have $x_n \ll x_{n+1} \leq x$ and therefore $x_n \ll x$ for every n. Moreover, if $x > 0$, then $x_n > 0$ for sufficiently large n, since otherwise the supremum would be 0. Hence, the element $x' := x_n$ for sufficiently large n is as required.
(ii) Pick any rapidly increasing sequence $(y_n)_n$ with supremum y. Since $x \ll y$, we have $x \leq y_n$ for some n. But then $x \leq y_n \ll y_{n+1} \ll y_{n+2} \leq y$. Let $z := y_{n+1}$, and it follows that $x \ll z \ll y$ as required.

(*iii*) Using (*ii*), we can find some elements $v, w \in S$ such that $x \ll v \ll w \ll y$. By axiom ($\mathcal{O}5$), there is an element $u \in S$ such that $v + u \leq y \leq w + u$. Assume that u is zero. Then $y \leq w$, and therefore $y \leq w \ll y$, which implies $y \ll y$, so that y would be compact. Since y is noncompact, the assumption was wrong, and $u > 0$. Using (*i*), we can find an element $z \in S$ such that $0 < z \ll u$. Since $x \ll v$ and $z \ll u$, we have $x + z \ll v + u \leq y$ by axiom ($\mathcal{O}3$), and therefore $x + z \ll y$ for a nonzero $z \in S$.

■

3.1.3 Rørdam's proposition. *Let S be a semigroup in the category* Cu*, and let $x, y \in S$. The following conditions are equivalent:*

(*i*) $x \leq y$.

(*ii*) *For every $x' \ll x$, we have $x' \leq y$.*

(*iii*) *For every $x' \ll x$ there is some $y' \ll y$ such that $x' \leq y'$.*

Proof. This result is the analogue of Proposition 2.2.4 for semigroups in Cu.
(*i*) $\implies$ (*ii*): Clear, since the order $\leq$ is transitive, and $x' \ll x$ implies $x' \leq x$.
(*ii*) $\implies$ (*iii*): Let $x' \ll x$. By the previous lemma, we can find an element $x'' \in S$ such that $x' \ll x'' \ll x$. It follows from (*ii*) that $x'' \leq y$, and therefore $x' \ll x'' \leq y$, which implies $x' \ll y$. Pick any rapidly increasing sequence $(y_n)_n$ with supremum y. There is some n such that $x' \leq y_n$, and by construction we have $y_n \ll y_{n+1} \leq y$, which implies $y_n \ll y$. Hence, $y' := y_n$ satisfies $y' \ll y$ and $x' \leq y'$, as required.
(*iii*) $\implies$ (*i*): Pick any rapidly increasing sequence $(x_n)_n$ in S with supremum x. We have $x_n \ll x_{n+1} \leq x$ for all n, which implies $x_n \ll x$ for all n. By (*iii*), we can find elements $y_n \ll y$ with $x_n \leq y_n$ for every n. Since $y_n \ll y$ implies $y_n \leq y$, this means that $x_n \leq y$ for all n, so that y is an upper bound for the sequence $(x_n)_n$. Since x is the supremum of $(x_n)_n$, it follows that $x \leq y$. ■

We shall now introduce some interesting properties for semigroups S in the category Cu. We will see in the following section for which C*-algebras A the semigroup Cu(A) satisfies these.

3.1.4 Definition. ***Simple:***

(*i*) *A subset $I \subseteq S$ is called an* ideal *of S if it is nonempty, order-hereditary, and closed under addition. An ideal I is called a* closed ideal *if, moreover, the supremum of any increasing sequence in I is itself an element of I.*

(*ii*) *The semigroup S is called* simple *if it contains no closed ideals besides $\{0\}$ and S.*

3.1.5 Definition. ***Separable:***

(*i*) *A subset $S_0 \subseteq S$ is called* dense *if, for every element $x \in S$, there is an increasing sequence $(x_n)_n$ in S_0 with $\sup_n x_n = x$.*

(*ii*) *The semigroup S is called* separable *if it contains a countable dense subset.*

3.1.6 Definition. ***Elementary:***

(*i*) *For any natural number $n \geq 1$, let $\mathcal{E}_n$ be the semigroup with elements $\{x_0, ..., x_n, x_\infty\}$ such that $x_0 < ... < x_n < x_\infty$ and $x_m + x_k := x_{m+k}$ if $m + k \leq n$, while $x_m + x_k = x_\infty$ if $m + k > n$. The elements of this semigroup will henceforth be referred to as $0, ..., n, \infty$. Moreover, let $\mathcal{E} := \mathbb{N}_0 \cup \{\infty\}$ with the obvious addition and order.*

(*ii*) *The semigroup S is called* elementary *if $S \cong \mathcal{E}$ or $S \cong \mathcal{E}_n$ for some n with $1 \leq n < \infty$.*

3.1.7 Remark. *This definition of the term* elementary *in the context of* Cu*-semigroups seems needlessly complicated at first. The reason why we want the semigroups $\mathcal{E}_n$ excluded from the class of nonelementary* Cu*-semigroups is that they and $\mathcal{E}$ are the only simple semigroups in* Cu *that do not satisfy the halving theorem (Theorem 3.1.25), nor any results that require it to be satisfied. Since the structure of the semigroups $\mathcal{E}_n$ is similar to that of $\mathcal{E}$, we have decided to widen the definition of elementary semigroups to include them. Please note that none of the semigroups $\mathcal{E}_n$ can arise as the Cuntz semigroup of a C*-algebra (we will prove this later, in Theorem 4.4.4).*

The following useful proposition provides an analogue of the Pedersen ideal in the context of semigroups in Cu.

3.1.8 Definition/Proposition. *For S in* Cu*, let $S_{min} := \{x \in S \mid \exists\, y \in S \colon x \ll y\}$. Then S_{min} is a (not necessarily closed) ideal in S, and it is the minimal dense ideal of S (meaning that S_{min} is dense in S, and moreover, that $S_{min} \subseteq J$ for every other dense ideal J of S).*

Proof. It follows immediately from the definition that S_{min} is hereditary and contains the zero element of S. Moreover, S_{min} is closed under addition by axiom ($\mathcal{O}$3), so S_{min} is an ideal of S. That S_{min} is dense in S is an immediate consequence of axiom ($\mathcal{O}$2). Let J be any dense ideal of S, and let $x \in S_{min}$. Pick an element $y \in S$ with $x \ll y$. Since J is dense in S, there is an increasing sequence $(y_n)_n$ in J with $\sup_n y_n = y$. Since $x \ll y$, there is an $n \in \mathbb{N}$ such that $x \leq y_n$. Since $y_n \in J$ and J is hereditary, it follows that $x \in J$. Since this is true for every $x \in S_{min}$, we have $S_{min} \subseteq J$. ∎

3.1.9 Definition. ***Stably finite:***

(*i*) *An element $x \in S$ is called* finite *if $x < x + y$ for every $0 \neq y \in S$, and infinite otherwise. Note that any nonzero S contains infinite elements, namely the elements $\infty \cdot z := \sup_n (n \cdot z)$ for nonzero $z \in S$.*

(*ii*) *A simple semigroup S in* Cu *is* stably finite *if every compact element of S is finite.*

3.1.10 Definition. ***Weakly cancellative:*** *The semigroup S is called* weakly cancellative *if $x + z \ll y + z$ implies $x \ll y$ for all elements $x, y, z \in S$.*

3.1.11 Definition. ***Almost divisible:*** *The semigroup S is called* almost divisible *if, for all $x \in S$ and all $n \in \mathbb{N}$, there is an element $y \in S$ such that $ny \leq x \leq (n + 1)y$.*

3.1.12 Definition. ***Almost unperforated:*** *The semigroup S is called* almost unperforated *if, for all $x, y \in S$ and all $n \in \mathbb{N}$, the inequality $(n+1)x \leq ny$ implies $x \leq y$.*

Next, we will prove some useful implications of the above properties.

3.1.13 Definition. ***Almost cofinal sequences:*** *Let $S \in$ Cu, and let B be any nonempty subset of S. An increasing sequence $(x_n)_n$ in B is called* almost cofinal *if, for every $y \in B$ and $y' \in S$ with $y' \ll y$, there is some $n \in \mathbb{N}$ such that $y' \leq x_n$.*

3.1.14 Proposition. ***Directed sets:*** *Let S be a semigroup in Cu.*

(i) If $B \subseteq S$ contains an almost cofinal sequence $(x_n)_n$, then B has a supremum and, moreover, we have $\sup B = \sup_n x_n$.

(ii) If $D \subseteq S$ is a countable directed subset, then D contains an almost cofinal sequence $(x_n)_n$. It follows that D has a supremum, and that $\sup D = \sup_n x_n$.

(iii) If $D \subseteq S$ is any directed subset and if S is separable, then D contains an almost cofinal sequence $(x_n)_n$. Therefore, D has a supremum, and $\sup D = \sup_n x_n$.

Proof.

(i) Let B be a nonempty subset, and let $(x_n)_n$ be an almost cofinal sequence in B. Let $s := \sup_n x_n$. If x is any element of B, then s dominates any element of S that is way below x, and therefore s dominates x by Rørdam's proposition (see Proposition 3.1.3); hence s is an upper bound for B. If $t \in S$ is another upper bound for B, then t dominates every x_n, since $x_n \in B$. Therefore, it dominates $s = \sup_n x_n$. Hence $s \leq t$, so that s is the least upper bound for B in S and therefore $\sup B = \sup_n x_n$.

(ii) Let $(y_n)_n$ be an enumeration of D. For $n = 1$, let $x_1 := y_1$. For $n > 1$, assume that elements $x_1 \leq ... \leq x_{n-1}$ of D have already been constructed in such a manner that $y_1, ..., y_i \leq x_i$ for $1 \leq i \leq n-1$. Since D is directed, we can find $x_n \in D$ such that $x_{n-1}, y_n \leq x_n$. It follows from this construction that $(x_n)_n$ is an almost cofinal sequence in D; in fact, every element of D is eventually dominated by the terms x_n.

(iii) Let S_0 be a countable and dense subset of S. Let $D_0 := \{y \in S_0 \mid \exists x \in D\colon\ y \leq x\}$. Pick any $x \in D$ and $x' \in S$ such that $x' \ll x$. Since S_0 is dense in S, we can find an increasing sequence $(y_n)_n \in S_0$ such that $\sup_n y_n = x$. Since $x' \ll x$, there is some $n \in \mathbb{N}$ such that $x' \leq y_n \leq x$, hence $y_n \in D_0$. Thus:

(a) For every $x, x' \in S$ with $x' \ll x \in D$, there is $y \in D_0$ with $x' \leq y$.

(b) For every $y \in D_0$, there is $x \in D$ such that $y \leq x$.

(c) The set D_0 is countable, since it is a subset of S_0.

We shall construct the sequence $(x_n)_n$ inductively. Let $(z_n)_n$ be an enumeration of D_0 (which exists by (c)). For $n = 1$, pick an element $x_1 \in D$ such that $z_1 \leq x_1$ (which is possible by (b)). For $n > 1$, assume that elements $x_1 \leq ... \leq x_{n-1}$ of D have already been constructed in such a way that $z_1, ..., z_i \leq x_i$ for $1 \leq i \leq n-1$. Pick an element

$v \in D$ with $z_n \leq v$ (which, again, is possible by (b)). Since D is directed, we can find an element $x_n \in D$ such that $x_{n-1}, v \leq x_n$. It follows that $x_n \in D$, that $x_{n-1} \leq x_n$ and that $z_1, ..., z_n \leq x_n$. This completes the construction. Now, $(x_n)_n$ is an increasing sequence in D that eventually dominates every element of D_0. In particular, if $x \in D$ and $x' \in S$ such that $x' \ll x$, then $(x_n)_n$ eventually dominates x' by (a). Therefore, $(x_n)_n$ is almost cofinal.

■

3.1.15 Definition/Proposition. ***Infinity:*** *If $S \in$ Cu is simple or separable, then it contains a maximal element, denoted by ∞. It follows that $\infty + x = \infty$ for every $x \in S$. In particular, ∞ is properly infinite if S is nonzero.*

Proof. If S is separable, then this follows immediately from Proposition 3.1.14, since S is closed under addition and therefore directed. If $S = \{0\}$, then 0 is obviously the maximal element of S. It remains to prove the statement for simple and nontrivial S. Pick any $0 \neq x \in S$ and let $S_x := \{y \in S \mid y \leq \infty \cdot x := \sup_n (n \cdot x)\}$. We need to show that S_x is a closed ideal (the closed ideal generated by x). By simplicity, it will then follow that $S = S_x$ and thus that $\infty \cdot x$ is a maximal element of S. It is clear from the definition that $0 \in S_x$, that S_x is hereditary, and that S_x is closed under suprema of increasing sequences, so we need only show that S_x is closed under addition. Let y, z be any two elements of S_x. By axiom $(\mathcal{O}2)$, we can chose rapidly increasing sequences $(y_n)_n$ and $(z_n)_n$ such that $\sup_n y_n = y$ and $\sup_n z_n = z$. Fix n, and note that $y_n \ll y_{n+1} \leq \infty \cdot x$, hence $y_n \ll \infty \cdot x$, so that we can find $N_1 \in \mathbb{N}$ with $y_n \leq N_1 \cdot x$. Likewise, we can find $N_2 \in \mathbb{N}$ with $z_n \leq N_2 \cdot x$. Hence, we have $y_n + z_n \leq (N_1 + N_2) \cdot x \leq \infty \cdot x$. It follows that $y_n + z_n \leq \infty \cdot x$ for every $n \in \mathbb{N}$, and therefore $y + z = \sup_n y_n + \sup_n z_n = \sup_n (y_n + z_n) \leq \infty \cdot x$ (we have used axiom $(\mathcal{O}4)$ here); thus $y + z \in S_x$ and S_x is closed under addition. For the remainder of the statement, let x be any element of S. It follows from S being positively ordered that $\infty \leq \infty + x$, and it follows from maximality of ∞ that $\infty + x \leq \infty$, hence $\infty + x = \infty$ for all $x \in S$. In particular, we have $\infty = \infty + \infty$. ■

3.1.16 Proposition. *If S is simple or separable, then the minimal dense ideal is given by $S_{min} := \{x \in S \mid x \ll \infty\}$.*

Proof. This follows from the definition of S_{min} and the preceding proposition. ■

3.1.17 Proposition. *Let $S \in$ Cu be simple and nonzero, and let $x \in S$. Then x is finite if and only if $x < \infty$.*

Proof. Obviously, the element ∞ is infinite since $\infty > 0$ and $\infty + \infty = \infty$. Conversely, if x is infinite, then we can find a nonzero element $y \in S$ such that $x = x + y$. Since S is simple, we have $\sup_n (n \cdot y) = \infty$. It follows that $x = \sup_n (x + (n \cdot y)) = x + \sup_n (n \cdot y) = x + \infty = \infty$. ■

3.1.18 Proposition. *Let $S \in \mathsf{Cu}$. The following conditions are equivalent whenever S is simple and nonzero:*

(*i*) *S is stably finite.*

(*ii*) *If $x, y \in S$ are compact and $x + y = x$, then $y = 0$.*

(*iii*) *All elements of S_{min} are finite.*

(*iv*) *The maximal element ∞ is noncompact.*

Proof.
(*i*) $\implies$ (*ii*): This is obvious, since x is compact and therefore finite.
(*ii*) $\implies$ (*iv*): If ∞ was compact, then the fact that $\infty + \infty = \infty$ would imply that $\infty = 0$, which is impossible since S is nonzero, and ∞ is the maximal element of S.
(*iv*) $\implies$ (*iii*): Assume that S_{min} contains an infinite element x. By the two preceding propositions, it would follow that $x = \infty$ and hence that $\infty \ll \infty$, contradicting the fact that ∞ is noncompact.
(*iii*) $\implies$ (*i*): If $x \in S$ is compact, then $x \ll x$, so we have $x \in S_{min}$, and hence x is finite. ■

3.1.19 Proposition. *If $S \in \mathsf{Cu}$ is simple and weakly cancellative, then S is stably finite.*

Proof. Let $x, y \in S$ with x compact. If $x + y = x$, then $x + y = x \ll x = x + 0$, and it follows from weak cancellation that $y \ll 0$, which is equivalent to $y = 0$. Hence, S is stably finite. ■

3.1.20 Definition. *For $S \in \mathsf{Cu}$, let $\mathsf{C}(S) := \{x \in S \mid x \text{ is compact}\}$ denote the* compact part *of S, and let $\mathsf{D}(S) := \{x \in S \mid x \text{ is noncompact or zero}\}$ denote the* noncompact part *of S, so that S can be decomposed as $S^\times = \mathsf{C}(S)^\times \sqcup \mathsf{D}(S)^\times$.*

3.1.21 Lemma. *Let $S \in \mathsf{Cu}$ be simple and stably finite. Then the following statements hold:*

(*i*) *$\mathsf{C}(S)$ is an algebraically ordered subsemigroup of S. In fact, for every $x \in \mathsf{C}(S)$ and $y \in S$ with $x \leq y$, there is some $z \in S$ such that $x + z = y$, and z is compact if y is.*

(*ii*) *$\mathsf{D}(S)$ is an absorbing subsemigroup of S, which means that $x + y \in \mathsf{D}(S)$ for any $x \in \mathsf{D}(S)^\times$ and any $y \in S$.*

Proof.

(*i*) If $x, y \in \mathsf{C}(S)$, then $x \ll x$ and $y \ll y$. By axiom ($\mathcal{O}3$), we have that $x + y \ll x + y$, hence $x + y \in \mathsf{C}(S)$. This means that $\mathsf{C}(S)$ is a subsemigroup of S. The rest of the statement follows from the fact that S is almost algebraically ordered: let $x \in \mathsf{C}(S)$ and $y \in S$ such that $x \leq y$. Since $x \ll x \leq y$, we can find $z \in S$ such that $x + z \leq y \leq x + z$, hence $x + z = y$. That z must be compact if y is compact will follow from (*ii*).

(*ii*) Let $x \in \mathsf{D}(S)^\times$ and $y \in S$. We can find rapidly increasing sequences $(x_n)_n$ and $(y_n)_n$ with supremum x and y, respectively. Since x is noncompact, the sequence $(x_n)_n$ cannot be eventually constant, so we may assume that it increases strictly and rapidly. Since S is almost algebraically ordered, we can find a sequence $(v_n)_n$ such that $x_n + v_n \leq x_{n+2} \leq x_{n+1} + v_n$. The latter inequality implies $v_n > 0$, since $(x_n)_n$ is strictly increasing. Let $z_n := x_{2n}$ and $w_n := v_{2n}$ for all n; then $(z_n)_n$ is still strictly and rapidly increasing with supremum x, and $(w_n)_n$ is a sequence of nonzero elements such that $z_n + w_n \leq z_{n+1}$ for all n. Moreover, by axioms ($\mathcal{O}3$) and ($\mathcal{O}4$), the sequence $(z_n + y_n)_n$ is still rapidly increasing with supremum $x + y$. We show that it is also strictly increasing: by construction we have $z_n + y_n \ll z_{n+1} + y_{n+1} \leq \infty$ and therefore $z_n + y_n \ll \infty$ for all n, and from Proposition 3.1.18 it follows that $z_n + y_n < (z_n + y_n) + w_n \leq z_{n+1} + y_{n+1}$ for all n. Since $x + y = \sup_n (z_n + y_n)$, and since $(z_n + y_n)_n$ is rapidly increasing and not eventually constant, the supremum $x + y$ cannot be compact. Hence, $x + y \in \mathsf{D}(S)$. ■

3.1.22 Lemma. *Let $S_1, S_2 \in$ Cu such that S_1 is simple, and S_2 is simple and stably finite. Let $\alpha\colon S_1 \to S_2$ be any Cu-morphism. Then $\alpha(\mathsf{C}(S_1)) \subseteq \mathsf{C}(S_2)$ and $\alpha(\mathsf{D}(S_1)) \subseteq \mathsf{D}(S_2)$. Let $\mathsf{C}(\alpha) := \alpha|_{\mathsf{C}(S_1)}$, and let $\mathsf{D}(\alpha) := \alpha|_{\mathsf{D}(S_1)}$. Then $\mathsf{C}(\cdot)$ is a functor from Cu to the category M of all positively ordered abelian monoids, with additive, zero-preserving, and order-preserving maps as morphisms. Moreover, $\mathsf{D}(\cdot)$ is a functor from the full subcategory of all simple and stably finite semigroups in Cu to the category M.*

Proof. We shall first consider the case where $x \in \mathsf{C}(S_1)$; this part requires no assumptions on S_1, S_2 other than $S_1, S_2 \in$ Cu. Since $x \ll x$ and α is a Cu-morphism, we have $\alpha(x) \ll \alpha(x)$ by axiom ($\mathcal{M}3$), so that $\alpha(x) \in \mathsf{C}(S_2)$. Next, consider the case where $x \in \mathsf{D}(S_1)^\times$. If $\alpha = 0$, then there is nothing to show, so we assume $\alpha \neq 0$. Since α preserves the zero element, addition, order, and suprema, the set $\ker(\alpha) := \{y \in S_1 \mid \alpha(y) = 0\}$ is a closed ideal. Since S_1 is simple and $\alpha \neq 0$, it follows that $\ker(\alpha) = \{0\}$. Let $(x_n)_n$ be any rapidly increasing sequence in S_1 with supremum x. This sequence cannot be eventually constant, since its supremum is noncompact; we can therefore assume that $x_n \ll x_{n+1}$ and $x_n < x_{n+1}$ for every n. Since S_1 is almost algebraically ordered, we can find elements $(y_n)_n$ in S such that $x_n + y_n \leq x_{n+2} \leq x_{n+1} + y_n$; the latter inequality implies $0 < y_n$, since $x_{n+1} < x_{n+2}$. As in the preceding lemma, Let $z_n := x_{2n}$ and $w_n := y_{2n}$; then $(z_n)_n$ is still a rapidly increasing sequence with supremum x, and $(w_n)_n$ is a sequence of nonzero elements such that $z_n + w_n \leq z_{n+1}$ for each $n \in \mathbb{N}$. Since $z_n \ll z_{n+1}$ and α is a Cu-morphism, we have $\alpha(z_n) \ll \alpha(z_{n+1}) \leq \infty$ for all n, and therefore $\alpha(z_n) \ll \infty$ for every n. Since $\ker(\alpha)$ is trivial, it follows that $\alpha(w_n) > 0$ for all n. By Proposition 3.1.18, it follows that $\alpha(z_n) < \alpha(z_n) + \alpha(w_n) = \alpha(z_n + w_n) \leq \alpha(z_{n+1})$, so that $(\alpha(z_n))_n$ is a strictly increasing sequence in S_2. Since α preserves suprema, we have that $\alpha(x) = \alpha(\sup_n z_n) = \sup_n \alpha(z_n)$. If $\alpha(x)$ were compact, then the rapidly increasing sequence $(\alpha(z_n))_n$ would be eventually constant, which we have just shown is not the case. Thus, $\alpha(x)$ is noncompact, i.e. $\alpha(x) \in \mathsf{D}(S_2)$. Finally, that $\mathsf{C}(\cdot)$ and $\mathsf{D}(\cdot)$ are functorial, i.e. that they preserve composition of morphisms and identity morphisms, follows immediately from their definition. ■

We end this section with some very useful theorems due to L. Robert. The first one (mentioned to be true in [29] without an explicit proof) is a strengthening of axiom ($\mathcal{O}6$):

3.1.23 Theorem. *Let $S \in$ Cu and let $x', x, y_1, ..., y_n \in S$ such that $x' \ll x \leq y_1 + ... + y_n$. Then for each $1 \leq i \leq n$ there is an element $z_i \leq y_i$, x such that $x' \leq z_1 + ... + z_n$.*

Proof. We prove the theorem by induction. For $n = 1$, we have $x' \ll x \leq y_1$, so letting $z_1 := x$ yields an element with $z_1 \leq y_1, x$ and $x' \leq z_1$. For $n > 1$, find an element $x'' \in S$ with $x' \ll x'' \ll x$, and let $w_1 := (y_1 + y_2)$ and $w_i := y_{i+1}$ for $2 \leq i \leq n-1$. Then we have $x'' \ll x \leq w_1 + ... + w_{n-1}$. By the induction hypothesis, we can find elements $v_i \leq x$, w_i for each $1 \leq i \leq n-1$ such that $x'' \leq v_1 + ... + v_{n-1}$. Since $x' \ll x'' \leq v_1 + ...v_{n-1}$, we can use axioms ($\mathcal{O}2$), ($\mathcal{O}3$), and ($\mathcal{O}4$) to find elements $v_i' \ll v_i$ such that $x' \leq v_1' + ... + v_{n-1}'$. Since $v_1' \ll v_1 \leq w_1 = y_1 + y_2$, we can use axiom ($\mathcal{O}6$) to find $z_1, z_2 \in S$ such that $z_1 \leq y_1$, v_1, and $z_2 \leq y_2$, v_1, and $v_1' \leq z_1 + z_2$. Since $v_1 \leq x$, we have $z_1 \leq y_1$, x, and $z_2 \leq y_2$, x, and $v_1' \leq z_1 + z_2$. Moreover, we have $v_i' \leq v_i \leq x, w_i$ and therefore $v_i' \leq x$, y_{i+1} for $2 \leq i \leq n-1$. Thus, if we let $z_i := v_{i-1}'$ for $3 \leq i \leq n$, then we have $z_i \leq x$, y_i for $1 \leq i \leq n$. Finally, by construction, we have $x' \leq v_1' + (v_2' + ... + v_{n-1}')$, and thus $x' \leq (z_1 + z_2) + (z_3 + ... + z_n) = z_1 + ... + z_n$. This concludes the proof. ■

A proof for the following statement can be found in [29] (it forms the first part of the proof of Proposition 5.2.1]):

3.1.24 Theorem. ***Downwards directedness:*** *If $S \in$ Cu is simple and $x, y \in S$ are nonzero elements, then there is a nonzero element $z \in S$ such that $z \leq x, y$.*

Proof. We can pick $x'', x' \in S$ such that $0 < x'' \ll x' \ll x$. Since S is simple, we know that $x \leq \infty \cdot y = \infty_S$, and therefore we have $x' \leq ny$ for sufficiently large n. It follows that $x'' \ll x' \leq ny$. By the above theorem, we can find elements $z_i \leq y$, x' for $1 \leq i \leq n$ such that $x'' \leq z_1 + ... + z_n$. Since x'' is nonzero, at least one of these z_i must be nonzero; let $z := z_i$ for this index i. Then $0 < z \leq y, x'$ and therefore $0 < z \leq y, x$. ■

The next theorem will be used extensively in the following chapters:

3.1.25 Halving theorem. *If $S \in$ Cu is simple and nonelementary, then for each $x \in S^\times$ there is some $y \in S^\times$ such that $2y \leq x$.*

Proof. This theorem was proven in [29], Proposition 5.2.1. Please note the remark following its proof, where Robert clarifies that the elementary semigroups $\mathcal{E}_n$ are examples of simple semigroups in Cu that do not satisfy the halving theorem. His proof shows that these semigroups and $\mathcal{E}$ itself are the only simple Cu-semigroups for which halving is not always possible. ■

4 $\mathsf{Cu}(A)$ as an object of Cu

4.1 Basic notions

In this section, we collect results that show for which C*-algebras A the semigroup $\mathsf{Cu}(A)$ has the properties defined earlier for semigroups in Cu. The following result is fundamental:

4.1.1 Theorem. *If A is a C*-algebra, then the Cuntz semigroup $\mathsf{Cu}(A)$ is an object of Cu. If A, B are C*-algebras and $\alpha\colon A \to B$ is a *-homomorphism, then the induced map $\mathsf{Cu}(\alpha)\colon \mathsf{Cu}(A) \to \mathsf{Cu}(B)$ with $\mathsf{Cu}(\alpha)([a]) = [(\alpha \otimes id_{\mathcal{K}})(a)]$ is a morphism in Cu. Cu is a covariant functor from the category of C*-algebras (with *-homomorphisms as arrows) to Cu.*

Proof. This was shown by Coward, Elliott, and Ivanescu in [11]. Note, however, that their definition of $\mathsf{Cu}(A)$ differs from ours – that both definitions are equivalent was shown in the Appendix (Section 6) of [11]. A more accessible treatment of these facts is available in Chapter 4 of [2]. Moreover, the definition of the category Cu in both [11] and [2] differs from ours in that it does not include the two additional axioms ($\mathcal{O}5$) and ($\mathcal{O}6$); these two axioms were proposed by L. Robert in [29]. That $\mathsf{Cu}(A)$ always satisfies axiom ($\mathcal{O}5$) was proven by M. Rørdam and W. Winter in [34], Lemma 7.1; that $\mathsf{Cu}(A)$ always satisfies axiom ($\mathcal{O}6$) was proven by L. Robert in [29], Proposition 5.1.1. ■

4.1.2 Theorem. *Let A be a C*-algebra.*

(i) *If $a \in (A \otimes \mathcal{K})_+$ and $0 < \varepsilon_1 < \varepsilon_2$, then $[(a - \varepsilon_2)_+] \ll [(a - \varepsilon_1)_+]$ in $\mathsf{Cu}(A)$. If $(\varepsilon_n)_n$ is a sequence in $(0, \infty)$ that decreases strictly towards 0, then $([(a - \varepsilon_n)_+])_n$ is rapidly increasing with supremum $[a]$ in $\mathsf{Cu}(A)$.*

(ii) *For $a, b \in (A \otimes \mathcal{K})_+$, we have $[a] \ll [b]$ if and only if $[a] \leq [(b - \varepsilon)_+]$ for some $\varepsilon > 0$.*

(iii) *If $p \in A \otimes \mathcal{K}$ is a projection, then $[p]$ is compact in $\mathsf{Cu}(A)$. If A is simple, then an element $x \in \mathsf{Cu}(A)$ is compact if and only if $x = [p]$ for a projection $p \in A \otimes \mathcal{K}$.*

Proof.

(i) The first claim follows from results by Coward, Elliott, and Ivanescu in [11]. A more accessible proof can be found in [2], Lemma 4.32 in combination with [2], Lemma 4.18. Note that these proofs use the Hilbert module picture of $\mathsf{Cu}(A)$. As mentioned above, this picture of $\mathsf{Cu}(A)$ is equivalent to $\mathsf{Cu}(A)$ as defined here; see e.g. [2], Lemma 4.31 and [2], Lemma 4.33, or the appendix (Section 6) of [11]. The second claim follows immediately from the above and Corollary 2.2.5.

(*ii*) If $[a] \ll [b]$, then $[a] \leq [(b-\frac{1}{n})_+]$ for sufficiently large n, since the sequence $([(b-\frac{1}{n})_+])_n$ is increasing with supremum $[b]$ by (*i*). Conversely, if $[a] \leq [(b-\varepsilon)_+]$ for some $\varepsilon > 0$, find a sequence $(\varepsilon_n)_n$ as in (*i*). Then $\varepsilon_n < \varepsilon$ for sufficienty large n, which implies $[a] \leq [(b-\varepsilon)_+] \ll [(b-\varepsilon_n)_+] \leq [b]$ by (*i*) and Theorem 2.2.1 (*iv*). Hence $[a] \ll [b]$.

(*iii*) If p is a projection, then we have $p \precsim (p-\varepsilon)_+$ for $0 < \varepsilon < 1$ by Theorem 2.2.1 (*i*). It follows from part (*ii*) that $[p] \ll [p]$, so $[p]$ is compact. If A is simple, then by [7], Theorem 5.8, for every compact $x \in \mathsf{Cu}(A)$ there is a projection $p \in A \otimes \mathcal{K}$ with $x = [p]$. ■

4.2 Ideal structure

If A is a C*-algebra, then it is well-known that the closed ideals of A form a lattice when ordered by set inclusion. Likewise, the closed ideals of $\mathsf{Cu}(A)$ as defined in Definition 3.1.4 are easily seen to form a lattice when ordered by set inclusion; it can then be shown that the ideal lattice of a C*-algebra A is isomorphic to the ideal lattice of its Cuntz semigroup $\mathsf{Cu}(A)$, and we can explicitly describe the isomorphism. This is a very well-known and elementary result, but we could not find an explicit proof in the literature. Since we will use this result several times, usually to show that $\mathsf{Cu}(A)$ is simple if A is simple and vice versa, we include our own proof here. Unless explicitly stated otherwise, by an ideal of a C*-algebra A we will always mean a closed, two-sided ideal of A, and by an ideal of $\mathsf{Cu}(A)$ we will always mean a closed ideal of $\mathsf{Cu}(A)$.

4.2.1 Definition. ***Ideal lattice:***

(*i*) *For a C*-algebra A, let $\mathsf{Lat}(A)$ be the lattice of all (closed) ideals of A, ordered by set inclusion.*

(*ii*) *For a semigroup $S \in \mathsf{Cu}$, let $\mathsf{Lat}(S)$ be the lattice of all (closed) ideals of S, ordered by set inclusion.*

We will need a few lemmas before we can prove the isomorphism result.

4.2.2 Lemma. *Let A be a C*-algebra, and let B be a sub-C*-algebra of A. Then there is a canonical embedding $B \otimes \mathcal{K} \hookrightarrow A \otimes \mathcal{K}$; we can therefore regard $B \otimes \mathcal{K}$ as a sub-C*-algebra of $A \otimes \mathcal{K}$. Moreover, if I is an ideal of A, then $I \otimes \mathcal{K}$ is an ideal of $A \otimes \mathcal{K}$.*

Proof. For C*-algebras A, B, and C with $B \subseteq A$, the minimal C*-norm on the algebraic tensor product $A \odot C$ restricts to the minimal C*-norm on the algebraic tensor product $B \odot C$, which implies that the canonical inclusion $B \odot C \hookrightarrow A \odot C$ uniquely extends to an injective *-homomorphism $B \otimes C \hookrightarrow A \otimes C$. This is shown, for example, in Proposition 3.6.1 of [8]. The statement above follows immediately from this. Moreover, it is obvious that $I \otimes \mathcal{K}$ is an ideal of $A \otimes \mathcal{K}$ whenever I is an ideal of A. ■

4.2.3 Lemma. *Let A be a C*-algebra, let I be an ideal of A, and let $\iota\colon I \hookrightarrow A$ be the inclusion map. Then the induced map $\mathsf{Cu}(\iota)\colon \mathsf{Cu}(I) \to \mathsf{Cu}(A)$ is an order-embedding, i.e. it is an order-isomorphism between $\mathsf{Cu}(I)$ and $im(\mathsf{Cu}(\iota))$. Moreover, $im(\mathsf{Cu}(\iota))$ is an ideal of $\mathsf{Cu}(A)$. We can therefore identify $\mathsf{Cu}(I)$ with the ideal $im(\mathsf{Cu}(\iota)) = \{[a] \in \mathsf{Cu}(A) \mid a \in (I \otimes \mathcal{K})_+\}$ of $\mathsf{Cu}(A)$.*

Proof. Note that for a C*-subalgebra $B \subseteq A$, we cannot in general regard $\mathsf{Cu}(B)$ as a subset of $\mathsf{Cu}(A)$, since the induced map $\mathsf{Cu}(\iota)\colon \mathsf{Cu}(B) \mapsto \mathsf{Cu}(A)$ need not be injective. What we need to show here is that for an ideal $I \subseteq A$, the induced map $\mathsf{Cu}(\iota)$ is indeed injective, and even an order-embedding; for this it is sufficient to show that $a \precsim_{A\otimes\mathcal{K}} b$ implies $a \precsim_{I\otimes\mathcal{K}} b$ for all $a, b \in (I \otimes \mathcal{K})_+$. Moreover, we need to show that the image of $\mathsf{Cu}(\iota)$ is a closed ideal of $\mathsf{Cu}(A)$; since $\mathsf{Cu}(\iota)$ is additive, 0-preserving, and sup-preserving, it suffices to show that $a \precsim_{A\otimes\mathcal{K}} b$ for $a \in (A \otimes \mathcal{K})_+$ and $b \in (I \otimes \mathcal{K})_+$ implies that $a \in I \otimes \mathcal{K}$. For the first claim, let $a, b \in (I \otimes \mathcal{K})_+$ such that $a \precsim_{A\otimes\mathcal{K}} b$. By Theorem 2.2.1 (*ii*), this implies $a \precsim_{A\otimes\mathcal{K}} b^3$, so we can find a sequence $(x_n)_n$ in $A \otimes \mathcal{K}$ with $a = \lim_n\, x_n^* b^3 x_n$. Let $y_n := bx_n$, then $(y_n)_n$ is a sequence in $I \otimes \mathcal{K}$ with $a = \lim_n\, y_n^* b y_n$, so we have $a \precsim_{I\otimes\mathcal{K}} b$. For the second claim, let $a \in (A \otimes \mathcal{K})_+$ and $b \in (I \otimes \mathcal{K})_+$ such that $a \precsim_{A\otimes\mathcal{K}} b$. Then we can find a sequence $(x_n)_n$ in $A \otimes \mathcal{K}$ with $a = \lim_n\, x_n^* b x_n$. Since $b \in I \otimes \mathcal{K}$ and $I \otimes \mathcal{K}$ is a closed two-sided ideal of $A \otimes \mathcal{K}$, it follows that $a \in I \otimes \mathcal{K}$. This concludes the proof. ∎

4.2.4 Proposition. *Let A be a C*-algebra. Then the map $\alpha\colon \mathsf{Lat}(A) \to \mathsf{Lat}(A \otimes \mathcal{K})$, given by $I \mapsto I \otimes \mathcal{K}$, is a lattice isomorphism. Moreover, if $(e_{ij})_{i,j\in\mathbb{N}}$ is a complete system of matrix units for $\mathcal{K}$, then the inverse isomorphism $\alpha^{-1}\colon \mathsf{Lat}(A \otimes \mathcal{K}) \to \mathsf{Lat}(A)$ is given by $J \mapsto \{a \in A \mid a \otimes e_{11} \in J\}$.*

Proof. It follows from Lemma 4.2.2 that the map $I \mapsto I \otimes \mathcal{K}$ maps $\mathsf{Lat}(A)$ to $\mathsf{Lat}(A \otimes \mathcal{K})$, and this map is obviously inclusion-preserving, so it is an order homomorphism. Conversely, it is obvious that the subset $\{a \in A \mid a \otimes e_{11} \in J\}$ is an ideal of A whenever J is an ideal of $A \otimes \mathcal{K}$. Hence, the map $\alpha'\colon J \mapsto \{a \in A \mid a \otimes e_{11} \in J\}$ maps $\mathsf{Lat}(A \otimes \mathcal{K})$ to $\mathsf{Lat}(A)$. This map is also inclusion-preserving, and therefore an order homomorphism. From Lemma 2.1.1 it follows easily that $\alpha' \circ \alpha = \mathrm{id}_{\mathsf{Lat}(A)}$. Let $J \in \mathsf{Lat}(A \otimes \mathcal{K})$. Using the properties of matrix units in combination with an approximate identity of A, it is straightforward to check that Lemma 2.1.1 implies $J = \alpha(\alpha'(J))$, and hence we have $\alpha \circ \alpha' = \mathrm{id}_{\mathsf{Lat}(A\otimes\mathcal{K})}$. In total, it follows that α is a lattice isomorphism from $\mathsf{Lat}(A)$ to $\mathsf{Lat}(A \otimes \mathcal{K})$, and that $\alpha' = \alpha^{-1}$. ∎

4.2.5 Proposition. *Let A be a C*-algebra. Then the map $\alpha\colon \mathsf{Lat}(A \otimes \mathcal{K}) \to \mathsf{Lat}(\mathsf{Cu}(A))$, given by $I \otimes \mathcal{K} \mapsto \mathsf{Cu}(I)$, is a lattice isomorphism. Moreover, the inverse isomorphism $\alpha^{-1}\colon \mathsf{Lat}(\mathsf{Cu}(A)) \to \mathsf{Lat}(A \otimes \mathcal{K})$ is given by $J \mapsto \{a \in A \otimes \mathcal{K} \mid [a^*a] \in J\}$.*

Proof. It follows from Lemma 4.2.3 that $\mathsf{Cu}(I) = \{[a^*a] \in \mathsf{Cu}(A) \mid a \in I \otimes \mathcal{K}\}$ is an ideal of $\mathsf{Cu}(A)$ whenever I is an ideal of A, and it then follows from the preceding proposition that the map $\alpha\colon I \otimes \mathcal{K} \mapsto \mathsf{Cu}(I)$ is a well-defined map from $\mathsf{Lat}(A \otimes \mathcal{K})$ to $\mathsf{Lat}(\mathsf{Cu}(A))$.

Since it is obviously inclusion-preserving, it is an order homomorphism. Conversely, the map $\alpha'\colon J \mapsto \{a \in A \otimes \mathcal{K} \mid [a^*a] \in J\}$ is clearly inclusion-preserving; in order to show that α' is an order homomorphism from $\mathsf{Lat}(\mathsf{Cu}(A))$ to $\mathsf{Lat}(A \otimes \mathcal{K})$, we need to show that $I := \alpha'(J)$ is an ideal of $A \otimes \mathcal{K}$ whenever J is an ideal of $\mathsf{Cu}(A)$. Clearly, we have $0 \in I$. By [3], Proposition II.3.1.9 (ii), we have $(a+b)^*(a+b) \leq 2(a^*a+b^*b)$, which implies (by Theorem 2.2.1 (i), (iv), and (v)) that $[(a+b)^*(a+b)] \leq [a^*a] + [b^*b]$ for all $a, b \in A \otimes \mathcal{K}$. Since J is hereditary and closed under addition, it follows that I is closed under addition. If $a \in I$ and $b \in A \otimes \mathcal{K}$, then it follows that $[(ab)^*(ab)] = [b^*(a^*a)b] \leq [a^*a] \in J$ and, using Theorem 2.2.1 (iii), that $[(ba)^*(ba)] = [(ba)(ba)^*] = [b(aa^*)b^*] \leq [aa^*] = [a^*a] \in J$. Since J is hereditary, it follows that $[(ab)^*(ab)], [(ba)^*(ba)] \in J$ and therefore $ab, ba \in I$. If $(a_n)_n$ is a sequence in I and $\lim_n a_n = a \in A \otimes \mathcal{K}$, fix any $k \in \mathbb{N}$. We can find $n \in \mathbb{N}$ such that $\|a_n^*a_n - a^*a\| < \frac{1}{k}$, so by Theorem 2.2.1 (vi) we have $[(a^*a - \frac{1}{k})_+] \leq [a_n^*a_n] \in J$. Since J is hereditary, it follows that $[(a^*a - \frac{1}{k})_+] \in J$ for all $k \in \mathbb{N}$, and since $[a^*a] = \sup_k [(a^*a - \frac{1}{k})_+]$ and J is closed under suprema, it follows that $[a^*a] \in J$ and therefore $a \in I$. Thus, we have shown that $I = \alpha'(J) \in \mathsf{Lat}(A \otimes \mathcal{K})$ whenever $J \in \mathsf{Lat}(\mathsf{Cu}(A))$, and it follows that α' is indeed an order homomorphism from $\mathsf{Lat}(Cu(A)) \to \mathsf{Lat}(A \otimes \mathcal{K})$. It remains to show that $\alpha' = \alpha^{-1}$.
Let $I \in \mathsf{Lat}(A)$, and let a be any element of $A \otimes \mathcal{K}$. If $a \in I \otimes \mathcal{K}$, then it follows that $[a^*a] \in \mathsf{Cu}(I) = \alpha(I \otimes \mathcal{K})$, which implies that $a \in \alpha'(\alpha(I \otimes \mathcal{K}))$. Hence, we have $I \otimes \mathcal{K} \subseteq \alpha'(\alpha(I \otimes \mathcal{K}))$. Conversely, if $a \in \alpha'(\alpha(I \otimes \mathcal{K}))$ is positive, then we have $[a^2] \in \alpha(I \otimes \mathcal{K})$, so there is a $b \in I \otimes \mathcal{K}$ with $[a^2] = [b^*b]$. By Theorem 2.2.1 (ii), we have $a \sim a^2 \sim b^*b$, so there is a sequence $(x_n)_n$ in $A \otimes \mathcal{K}$ with $a = \lim_n x_n^*(b^*b)x_n$. Since $b \in I \otimes \mathcal{K}$ and $I \otimes \mathcal{K}$ is a closed, two-sided ideal of $A \otimes \mathcal{K}$, it follows that $a \in I \otimes \mathcal{K}$. Thus, we have $\alpha'(\alpha(I \otimes \mathcal{K}))_+ \subseteq I \otimes \mathcal{K}$. Since C*-algebras are the linear span of their positive elements, it follows that $\alpha'(\alpha(I \otimes \mathcal{K})) \subseteq I \otimes \mathcal{K}$ and therefore $\alpha'(\alpha(I \otimes \mathcal{K})) = I \otimes \mathcal{K}$.
Now, let $J \in \mathsf{Lat}(\mathsf{Cu}(A))$, and let $x \in \mathsf{Cu}(A)$. Assume that $x \in J$, and pick $a \in (A \otimes \mathcal{K})_+$ with $x = [a]$. Then by Theorem 2.2.1 (ii) we have $x = [a^2] = [a^*a]$, which implies $a \in \alpha'(J)$, and hence we have $x = [a^*a] \in \alpha(\alpha'(J))$. Thus, we have $J \subseteq \alpha(\alpha'(J))$ as expected. Conversely, if $x \in \alpha(\alpha'(J))$, then we can pick $a \in \alpha'(J)$ such that $x = [a^*a]$. From $a \in \alpha'(J)$ it follows that $[a^*a] \in J$, hence $x \in J$. Thus, we have $\alpha(\alpha'(J)) \subseteq J$ and therefore $\alpha(\alpha'(J)) = J$. In total, we have shown that α is a lattice isomorphism from $\mathsf{Lat}(A \otimes \mathcal{K})$ to $\mathsf{Lat}(\mathsf{Cu}(A))$, and that $\alpha' = \alpha^{-1}$. ■

4.2.6 Proposition. *Let A be a C*-algebra. Then the map $\alpha\colon \mathsf{Lat}(A) \to \mathsf{Lat}(\mathsf{Cu}(A))$, given by $I \mapsto \mathsf{Cu}(I)$, is a lattice isomorphism. Moreover, if $(e_{ij})_{i,j \in \mathbb{N}}$ is a complete system of matrix units for $\mathcal{K}$, then the inverse isomorphism $\alpha^{-1}\colon \mathsf{Lat}(\mathsf{Cu}(A)) \to \mathsf{Lat}(A)$ is given by $J \mapsto \{a \in A \mid [a^*a \otimes e_{11}] \in J\}$.*

Proof. This follows immediately from the two preceding propositions by composing the respective isomorphisms. ■

4.2.7 Corollary. *Let A be a C*-algebra. Then A is simple if and only if $\mathrm{Cu}(A)$ is simple.*

Proof. This follows immediately from the preceding proposition. ■

4.3 Separability

Next, we take a look at separability of $\mathrm{Cu}(A)$.

4.3.1 Lemma. *Let A be a C*-algebra. For any dense subset X of $(A \otimes \mathcal{K})$, let X' denote the set $X' := \{(x^*x - \varepsilon)_+ \mid x \in X,\ \varepsilon \in \mathbb{Q}_+\}$. Then for every $z \in \mathrm{Cu}(A)$ there is a sequence $(x_n)_n$ in X' such that $([x_n])_n$ is rapidly increasing in $\mathrm{Cu}(A)$ and has supremum z.*

Proof. Let a be an element of $(A \otimes \mathcal{K})_+$ with $[a] = z$. Given any rational $\varepsilon > 0$, we can find an element $x \in X$ with $\|a - x^*x\| < \varepsilon$. It follows that $\|(a - \varepsilon)_+ - x^*x\| < 2\varepsilon$ and $\|a - (x^*x - 2\varepsilon)_+\| < 3\varepsilon$. Using Theorem 2.2.1 (vi), we get $[(a-3\varepsilon)_+] \leq [(x^*x-2\varepsilon)_+] \leq [(a-\varepsilon)_+]$. Note that $(x^*x - 2\varepsilon)_+ \in X'$. It now follows from Theorem 4.1.2 (i) that we can find a rapidly increasing sequence $(x_n)_n$ in X' with $\sup_n [x_n] = [a] = z$. ■

4.3.2 Theorem. *Let A be any C*-algebra.*

(i) *If A is separable, then $\mathrm{Cu}(A)$ is separable.*

(ii) *If $\mathrm{Cu}(A)$ is separable, then there is a separable sub-C*-algebra $B \subseteq A$ such that the map $\mathrm{Cu}(\iota)\colon \mathrm{Cu}(B) \to \mathrm{Cu}(A)$ induced by the inclusion map $\iota\colon B \to A$ is an isomorphism.*

Proof.

(i) If A is separable, then $A \otimes \mathcal{K}$ is separable, so we can find a countable dense subset X of $A \otimes \mathcal{K}$. Choose X' for X as in the preceding lemma. Then X' is dense in $\mathrm{Cu}(A)$, and X' is countable since X is countable. Hence, $\mathrm{Cu}(A)$ is separable.

(ii) Let $(e_{ij})_{ij}$ be a complete system of matrix units for $\mathcal{K}$, and let $Y = \{y_n \mid n \in \mathbb{N}\}$ be a countable subset of $(A \otimes \mathcal{K})_+$ such that $\{[y_n] \mid n \in \mathbb{N}\}$ is dense in $\mathrm{Cu}(A)$. We will use induction to construct an increasing sequence $(B_n)_n$ of separable sub-C*-algebras of A and an increasing sequence $(X_n)_n$ of countable dense subsets $X_n \subseteq B_n \otimes \mathcal{K}$.
For $n = 0$, use Lemma 2.1.1 to find coefficients $y_{ij}^{(n)} \in A$ with $y_n = \lim_k \sum_{i,j=1}^k y_{ij}^{(n)} \otimes e_{ij}$ for each n. Let B_0 be the sub-C*-algebra of A generated by $\{y_{ij}^{(n)} \mid i, j, n \in \mathbb{N}\}$. Then B_0 is separable, and $Y \subseteq (B_0 \otimes \mathcal{K})_+$. Let X_0 be any countable dense subset of $B_0 \otimes \mathcal{K}$.
For $n > 0$, let $L_n := \{(x, y) \in X'_{n-1} \times X'_{n-1} \mid [x] \leq_{\mathrm{Cu}(A)} [y]\}$ where X'_{n-1} is as in the preceding lemma. For each pair $(x, y) \in L_n$, pick an enumerated set $S(x, y) = \{z_n \mid n \in \mathbb{N}\}$ in $A \otimes \mathcal{K}$ such that $x = \lim_n z_n^* y z_n$. Find coefficients $z_{ij}^{(n)} \in A$ for each z_n, and let $W(x, y) := \{z_{ij}^{(n)} \mid i, j, n \in \mathbb{N}\}$. Let $W_n := \bigcup_{(x,y)\in L_n} W(x, y)$, then W_n is countable. Let B_n be the sub-C*-algebra of A generated by B_{n-1} and W_n. Then B_n is separable, and $B_{n-1} \subseteq B_n$, hence $B_{n-1} \otimes \mathcal{K} \subseteq B_n \otimes \mathcal{K}$. Moreover, $S(x, y) \subseteq B_n \otimes \mathcal{K}$ for every pair $(x, y) \in L_n$. Let X_n be any countable dense subset of $(B_n \otimes \mathcal{K})$ that includes X_{n-1}.

Next, let $B := \overline{\bigcup_n B_n}$, so that $B \otimes \mathcal{K} = \overline{\bigcup_n B_n \otimes \mathcal{K}}$, and let $X := \bigcup_n X_n$, so that $X' := \bigcup_n X'_n$. Since each X_n was dense in $B_n \otimes \mathcal{K}$, the set X is dense in $B \otimes \mathcal{K}$. By the preceding lemma, it follows that for each $x \in (B \otimes \mathcal{K})_+$, there is a sequence $(x_n)_n$ in X' such that in $\mathsf{Cu}(B)$, the sequence $([x_n])_n$ is rapidly increasing with supremum $[x]$. Let $\iota\colon B \to A$ be the inclusion map. Since $Y \subseteq (B_0 \otimes \mathcal{K})_+ \subseteq (B \otimes \mathcal{K})_+$, the map $\mathsf{Cu}(\iota)\colon \mathsf{Cu}(B) \to \mathsf{Cu}(A)$ has dense image. We want to show that $\mathsf{Cu}(\iota)$ is also an order-embedding, for which it suffices to show that if $x, y \in (B \otimes \mathcal{K})_+$ such that $[x] \leq_{\mathsf{Cu}(A)} [y]$, then we also have $[x] \leq_{\mathsf{Cu}(B)} [y]$. So let x, y be any pair in $(B \otimes \mathcal{K})_+$ with $[x] \leq_{\mathsf{Cu}(A)} [y]$. As we have shown, we can find sequences $(x_n)_n$ and $(y_n)_n$ in X' such that $([x_n])_n$ and $([y_n])_n$ are rapidly increasing sequences in $\mathsf{Cu}(B)$ with supremum $[x]$ and $[y]$, respectively. Since $\mathsf{Cu}(\iota)$ is preserves compact containment and suprema, the sequences $([x_n])_n$ and $([y_n])_n$ are also rapidly increasing in $\mathsf{Cu}(A)$ with respective suprema $[x]$ and $[y]$. Fix any $n \in \mathbb{N}$. Since $[x] \leq_{\mathsf{Cu}(A)} [y]$, we can use Rørdam's proposition (Proposition 3.1.3) to see that there is some $m \in \mathbb{N}$ such that $[x_n] \leq_{\mathsf{Cu}(A)} [y_m]$. Since $x_n, y_m \in X'$, we can find some $k \in \mathbb{N}$ such that $x_n, y_m \in X'_k$. But then $(x_n, y_m) \in L_{k+1}$. By construction, we then have $S(x_n, y_m) \subseteq B_{k+1} \otimes \mathcal{K} \subseteq B \otimes \mathcal{K}$, and there is a sequence $(z_l)_l$ in $S(x_n, y_m)$ such that $\lim_l z_l^* y_m z_l = x_n$. Thus, we also have $[x_n] \leq_{\mathsf{Cu}(B)} [y_m]$ and therefore $[x_n] \leq_{\mathsf{Cu}(B)} [y]$. Since this is true for every n, and since the supremum of $([x_n])_n$ in $\mathsf{Cu}(B)$ is $[x]$, it follows that $[x] \leq_{\mathsf{Cu}(B)} [y]$. This proves that $\mathsf{Cu}(\iota)$ is an order-embedding with dense image. Since $\mathsf{Cu}(B)$ is closed under suprema and $\mathsf{Cu}(\iota)$ is sup-preserving, it follows that $\mathsf{Cu}(\iota)$ is surjective and therefore an order-isomorphism. Since every order-isomorphism preserves suprema and compact containment, it follows that $\mathsf{Cu}(\iota)$ is a Cu-isomorphism.

■

4.4 Elementary semigroups

Before we can turn towards elementary C*-algebras, we need some preparation.

4.4.1 Lemma. *Let A be a C*-algebra, and let $B \subseteq A$ be a hereditary sub-C*-algebra. If A is separable (respectively simple), then B is separable (respectively simple). If A is simple and stably finite, then B is simple and stably finite. If A is simple and nonelementary, then B is simple and nonelementary.*

Proof. That B is separable if A is separable follows from the standard proof that all subspaces of separable metric spaces are separable. Let A be simple. In order to show simplicity of B, let J be a nonzero ideal of B. Then $\overline{AJA}$ is a nonzero ideal of A, and therefore $\overline{AJA} = A$. Thus, for every $b \in B$ and $\varepsilon > 0$, we can find finitely many elements $x_1, ..., x_n, y_1, ..., y_n \in A$ and $j_1, ..., j_n \in J$ such that $\|b - \sum_{k=1}^n x_k j_k y_k\| < \frac{\varepsilon}{3}$. Let $(u_\nu)_\nu$ be an approximate unit of B; we can pick ν so large that $\|b - u_\nu b u_\nu\| < \frac{\varepsilon}{3}$ and $\|\sum_{k=1}^n x_k j_k y_k - \sum_{k=1}^n x_k (u_\nu j_k u_\nu) y_k\| < \frac{\varepsilon}{3}$. It follows that $\|\sum_{k=1}^n u_\nu(x_k j_k y_k)u_\nu - \sum_{k=1}^n u_\nu(x_k(u_\nu j_k u_\nu)y_k)u_\nu\| < \frac{\varepsilon}{3}$ since $\|u_\nu\| \leq 1$, and likewise that $\|u_\nu b u_\nu - \sum_{k=1}^n u_\nu(x_k j_k y_k)u_\nu\| < \frac{\varepsilon}{3}$. In total, we find that $\|b - \sum_{k=1}^n (u_\nu x_k u_\nu) j_k (u_\nu y_k u_\nu)\| < \varepsilon$. Since J is an ideal of B, and since for $k = 1, ..., n$ we have $(u_\nu x_k u_\nu), (u_\nu y_k u_\nu) \in BAB \subseteq B$, it follows that $(u_\nu x_k u_\nu) j_k (u_\nu y_k u_\nu) \in J$. Since J is closed, we can conclude that $B \subseteq J$ and therefore $B = J$. So B is simple. That B is stably finite if A is simple and stably finite follows immediately from Definition 2.1.4. Finally, let A be simple and nonelementary. Let $(\pi, \mathcal{H})$ be any irreducible representation of A. Assume that B is elementary; then we could find a minimal nonzero projection p of B. This would imply that $pAp = pBp = \mathbb{C}p$, and it would follow from Kadison's transitivity theorem ([24], Theorem 5.2.2) that $\pi(p)$ is a rank one projection. Since $(\pi, \mathcal{H})$ is irreducible, containing a nonzero compact operator in its image would (by [24], Theorem 2.4.9) imply that $\mathcal{K}(\mathcal{H}) \subseteq \pi(A)$. Since A is simple, this would imply $A \cong \mathcal{K}(\mathcal{H})$, contradicting the fact that A is nonelementary. We conclude that B must indeed be simple and nonelementary if A is simple and nonelementary. ■

4.4.2 Lemma. *Let A be a C*-algebra, let $(\pi, \mathcal{H})$ be an irreducible representation of A on a Hilbert space of dimension at least n, and let $P \in \mathcal{B}(\mathcal{H})$ be an orthogonal projection of finite rank n. Then there is a *-homomorphism $\alpha : \mathrm{C}_0((0,1], M_n) \to A$ such that the kernel of the compressed map $P(\pi \circ \alpha)P$ is equal to $\mathrm{C}_0((0,1), M_n)$.*

Proof. This was proven by E. Blanchard and E. Kirchberg in Section 2.3 of [6]. ■

We can proof a version of the halving theorem for semigroups $\mathsf{Cu}(A)$ now. Note that, compared to Theorem 3.1.25, the requirements have passed from $\mathsf{Cu}(A)$ to A.

4.4.3 Lemma. *Let A be a simple and nonelementary C*-algebra. Then for every nonzero $x \in \mathsf{Cu}(A)$ there is a nonzero $y \in \mathsf{Cu}(A)$ such that $2y \leq x$.*

Proof. The stabilisation $A \otimes \mathcal{K}$ is simple and nonelementary as well. Pick any nonzero $a \in (A \otimes \mathcal{K})_+$ such that $x = [a]$. Then $(A \otimes \mathcal{K})_a$ is simple and nonelementary by Lemma 4.4.1. Let $(\pi, \mathcal{H})$ be any irreducible representation of $(A \otimes \mathcal{K})_a$. The Hilbert space $\mathcal{H}$ cannot be of finite dimension, for then π would be an isomorphism between $(A \otimes \mathcal{K})_a$ and $\mathcal{B}(\mathcal{H}) = \mathcal{K}(\mathcal{H})$ since $(A \otimes \mathcal{K})_a$ is simple, contradicting the fact that $(A \otimes \mathcal{K})_a$ is nonelementary. Pick any rank-2-projection $P \in \mathcal{B}(\mathcal{H})$ and find a *-homomorphism $\alpha : \mathrm{C}_0((0,1], M_2) \to (A \otimes \mathcal{K})_a$ as in Lemma 4.4.2. Define $e, f \in \mathrm{C}_0((0,1], M_2)$ by letting $e(t) := \left(\begin{smallmatrix} t & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $f(t) := \left(\begin{smallmatrix} 0 & 0 \\ 0 & t \end{smallmatrix}\right)$. Then $e \sim f$ and $e \perp f$ in $\mathrm{C}_0((0,1], M_2)$, and therefore $[e + f] = [e] + [e]$. Let $y := [\alpha(e)]$; it follows from Lemma 4.4.2 that $\alpha(e) \neq 0$, hence $y > 0$. Since $\alpha(e + f) \in (A \otimes \mathcal{K})_a$, we find that $2y = [\alpha(e)] + [\alpha(e)] = [\alpha(e)] + [\alpha(f)] = [\alpha(e + f)] \leq [a] = x$. This completes the proof. ■

This result allows us to show that a C*-algebra A is elementary if and only if its Cuntz semigroup $\mathsf{Cu}(A)$ is elementary. Moreover, we can now show that none of the elementary semigroups $\mathcal{E}_n$ as given in Definition 3.1.6 can arise as the Cuntz semigroup of a C*-algebra:

4.4.4 Theorem. *Let A be any C*-algebra. The following conditions are equivalent:*

(i) *A is elementary (i.e. $A \cong \mathcal{K}(\mathcal{H})$ for a nonzero Hilbert space $\mathcal{H}$).*

(ii) *$\mathsf{Cu}(A)$ is elementary.*

(iii) *$\mathsf{Cu}(A) \cong \mathcal{E}$.*

Proof.

$(i) \implies (iii)$: If A is elementary, then $A \otimes \mathcal{K}$ is elementary too; we may therefore assume that A is stable, and indeed that $A = \mathcal{K}(\mathcal{H})$ for some infinite-dimensional Hilbert space $\mathcal{H}$. If $a \in \mathcal{K}(\mathcal{H})_+$ is of finite rank, then the spectral theorem for compact operators implies that a is a linear combination of finitely many mutually orthogonal projections of finite rank; it follows that a is Cuntz equivalent to a projection of finite rank. Since $\mathcal{K}(\mathcal{H})$ is stably finite, it follows from our remarks in Chapter 2 that two projections in $\mathcal{K}(\mathcal{H})$ are Cuntz equivalent if and only if they are Murray-von Neumann equivalent; it follows that two finite rank projections are Cuntz equivalent if and only if they have identical rank. Let $Y := \{a \in \mathcal{K}(\mathcal{H})_+ \mid a \text{ has finite rank}\}$, then it follows that $\{[a] \in \mathsf{Cu}(\mathcal{K}(\mathcal{H})) \mid a \in Y\} \cong \mathbb{N}_0$ as ordered abelian monoids. Let X be the dense ideal of $\mathcal{K}(\mathcal{H})$ consisting of all finite rank operators. It is easy to see that $Y = X'$ in the sense of Lemma 4.3.1, so we find that $\{[a] \in \mathsf{Cu}(\mathcal{K}(\mathcal{H})) \mid a \in Y\}$ is dense in $\mathsf{Cu}(\mathcal{K}(\mathcal{H}))$. But if this subset is both isomorphic to $\mathbb{N}_0$ and dense in $\mathsf{Cu}(\mathcal{K}(\mathcal{H}))$, then it follows that $\mathsf{Cu}(\mathcal{K}(\mathcal{H})) \cong \mathcal{E}$, and hence that $\mathsf{Cu}(A) \cong \mathcal{E}$.

$(iii) \implies (ii)$: This is true by definition.

$(ii) \implies (i)$: Since all elementary semigroups in Cu are simple, it follows from Corollary 4.2.7 that A is simple. Assume that A is nonelementary. Then by Lemma 4.4.3 we could find for every nonzero $x \in \mathsf{Cu}(A)$ a nonzero $y \in \mathsf{Cu}(A)$ with $2y \leq x$. But since $\mathsf{Cu}(A)$ is elementary, there is a minimal nonzero element $e \in \mathsf{Cu}(A)$, and this element e is finite in $\mathsf{Cu}(A)$. Clearly, a nonzero $y \in \mathsf{Cu}(A)$ with $2y \leq e$ cannot be found. We conclude that the assumption was wrong, and that A is indeed elementary. ∎

4.5 Finiteness

We remind the reader that by Definition 2.1.4, we call a simple C*-algebra A *stably finite* if $A \otimes \mathcal{K}$ contains no infinite projections (this applies to both unital and nonunital A). In the simple case, stable finiteness passes from A to $\mathsf{Cu}(A)$ and vice versa:

4.5.1 Theorem. *Let A be a simple C*-algebra. The following conditions are equivalent:*

(i) *A is stably finite.*

(ii) *$\mathsf{Cu}(A)$ is stably finite.*

Proof.

$(i) \implies (ii)$: Let $x \in \mathsf{Cu}(A)$ be compact. Assume that x is infinite. Since A is simple, the semigroup $\mathsf{Cu}(A)$ is simple, so it follows from Proposition 3.1.17 that $x = \infty_{\mathsf{Cu}(A)}$ and

therefore $x+x=x$. Since x is infinite, it is nonzero, and by Theorem 4.1.2 (*iii*) we can find a nonzero projection $p \in A\otimes\mathcal{K}$ with $x=[p]$ since $\mathsf{Cu}(A)$ is simple. By Lemma 2.1.2, we can find another nonzero projection $q \in A\otimes\mathcal{K}$ such that $p \approx q$ and $p \perp q$. We have shown in Chapter 2 that for simple and stably finite A, two projections in $A\otimes\mathcal{K}$ are Cuntz equivalent if and only if they are Murray-von Neumann equivalent. Since we have $[p+q]=x+x=x=[p]$, it follows that $p+q \approx p < p+q$, which contradicts the fact that A is stably finite. Hence, x is finite in $\mathsf{Cu}(A)$; it follows that $\mathsf{Cu}(A)$ is stably finite.
$(ii) \implies (i)$: Let p,q be projections in $A\otimes\mathcal{K}$ such that $p \approx q \leq p$. Let $r := p-q$, then r is a projection in $A\otimes\mathcal{K}$ with $q \perp r$ and $p=q+r$. Since Murray-von Neumann equivalence always implies Cuntz equivalence, it follows that $[q]=[p]=[q+r]=[q]+[r]$. Since q is a projection, the element $[q]$ is compact in $\mathsf{Cu}(A)$; since $\mathsf{Cu}(A)$ is simple and stably finite, it follows that $[q]$ is finite and therefore $[r]=0$. This implies $r=0$, and therefore $p=q$; it follows that A is stably finite. ■

Next, we take a closer look at compactness in $\mathsf{Cu}(A)$ if A is simple and stably finite. A very similar result was proven by N. P. Brown and A. Ciuperca in [7], Theorem 3.5.

4.5.2 Lemma. *Let A be a C^*-algebra and $a \in (A\otimes\mathcal{K})_+$. Consider the following conditions:*
- (i) *0 is not a cluster point of $\sigma(a)$.*
- (ii) *There is a projection $p \in C^*(a)$ and numbers $C,D>0$ such that $a \leq Cp$ and $p \leq Da$.*
- (iii) *There is a projection $p \in A\otimes\mathcal{K}$ such that $a \sim p$.*
- (iv) *$[a]$ is compact in $\mathsf{Cu}(A)$.*

Then $(i) \implies (ii) \implies (iii) \implies (iv)$. If A is simple and stably finite, then all four conditions are equivalent. In general, all conditions are equivalent if $[a]$ is finite in $\mathsf{Cu}(A)$.

Proof.
$(i) \implies (ii)$: There is nothing to show for the case $a=0$, so we may assume that $a>0$. Let $p \in C^*(a)$ be the spectral projection onto the set $X := \sigma(a)\setminus\{0\} \subseteq (0,\infty)$. Since 0 is not a cluster point of $\sigma(a)$, the set X is compact; let C be the maximum of X, and let D be the inverse of the minimum of X. Then $a \leq Cp$ and $p \leq Da$.
$(ii) \implies (iii)$: If p is as in (ii), then we have $a \precsim p$ and $p \precsim a$ by Theorem 2.2.1 (i) and (iv).
$(iii) \implies (iv)$: This was already shown in Theorem 4.1.2 (iii).
$(iv) \implies (i)$ in case that $[a]$ is finite in $\mathsf{Cu}(A)$: Assume that 0 is a cluster point of $\sigma(a)$. Since $([(a-\frac{1}{n})_+])_n$ is a (rapidly) increasing sequence with supremum $[a]$ (see Theorem 4.1.2 (i)), there is some $n \in \mathbb{N}$ such that $a \precsim (a-\frac{1}{n})_+$. By a straightforward functional calculus argument, we can find a nonzero element $b \in C^*(a)_+$ such that $b \perp (a-\frac{1}{n})_+$ and $b+(a-\frac{1}{n})_+ \precsim a$. But then it follows that $[a]+[b] \leq [(a-\frac{1}{n})_+]+[b] \leq [a] \leq [a]+[b]$, so we have $[a]+[b]=[a]$. Since $[b]>0$, this contradicts the fact that $[a]$ is finite. Hence the assumption was wrong, and 0 is not a cluster point of $\sigma(a)$.
$(iv) \implies (i)$ in case that A is simple and stably finite: by Theorem 4.5.1, the semigroup

$\mathsf{Cu}(A)$ is simple and stably finite, hence every compact element $[a]$ of $\mathsf{Cu}(A)$ is automatically finite. Since we have already shown that (*iv*) implies (*i*) if $[a]$ is finite, we are done. ■

4.6 Other properties and notational conventions

Finally, we look at weak cancellation, almost divisibility, and almost unperforation of $\mathsf{Cu}(A)$.

4.6.1 Theorem. *If A is a C*-algebra of stable rank one, then $\mathsf{Cu}(A)$ has weak cancellation.*

Proof. Note that by [3], Proposition V.3.1.17, the stabilisation $A \otimes \mathcal{K}$ has stable rank one whenever A does, so it follows from [34], Theorem 4.3, that the following property holds in the semigroup $\mathsf{Cu}(A) \cong \mathsf{W}(A \otimes \mathcal{K})$ whenever A is a C*-algebra of stable rank one: If $x, y \in \mathsf{Cu}(A)$, $c \in (A \otimes \mathcal{K})_+$, and $\varepsilon > 0$, then $x + [c] \leq y + [(c-\varepsilon)_+]$ implies $x \leq y$. It remains to show that this property implies weak cancellation (it is, in fact, equivalent to weak cancellation, but we will not need this). So let $x, y, z \in \mathsf{Cu}(A)$ such that $x + z \ll y + z$. We can find an element $c \in (A \otimes \mathcal{K})_+$ with $z = [c]$. Using axioms ($\mathcal{O}2$), ($\mathcal{O}3$), and ($\mathcal{O}4$), we can find elements $z', y' \in \mathsf{Cu}(A)$ such that $y' \ll y$, and $z' \ll z$, and $x + z \leq y' + z'$. Since $z' \ll z = [c]$, we can find some $\varepsilon > 0$ such that $z' \leq [(c-\varepsilon)_+]$ by Theorem 4.1.2 (*ii*). Thus, we have $x + [c] \leq y' + [(c-\varepsilon)_+]$, and it follows from the above property that $x \leq y'$. Since $y' \ll y$, we obtain $x \ll y$. ■

4.6.2 Theorem. *Let A be a $\mathcal{Z}$-stable C*-algebra (i.e. $A \cong A \otimes \mathcal{Z}$, where $\mathcal{Z}$ is the Jiang-Su algebra). Then the Cuntz semigroup $\mathsf{Cu}(A)$ is almost unperforated and almost divisible.*

Proof. Note that $\mathsf{Cu}(A) \cong \mathsf{W}(A \otimes \mathcal{K})$, and that for any $\mathcal{Z}$-stable C*-algebra A, the stabilisation $A \otimes \mathcal{K}$ is $\mathcal{Z}$-stable as well. Using these facts, it follows from [32], Theorem 4.5, that $\mathsf{Cu}(A)$ is almost unperforated; and it follows from [2], Theorem 5.35, that $\mathsf{Cu}(A)$ is almost divisible. ■

Lastly, we want to introduce some simplified notation:

4.6.3 Notation. *Let A be a C*-algebra. Then we will use $\mathsf{C}(A)$ and $\mathsf{D}(A)$ instead of $\mathsf{C}(\mathsf{Cu}(A))$ and $\mathsf{D}(\mathsf{Cu}(A))$ to denote the compact and noncompact parts of $\mathsf{Cu}(A)$.*

If $A \otimes \mathcal{K}$ contains no infinite projections, then the Murray-von Neumann semigroup $\mathsf{V}(A)$ can be regarded as a subsemigroup of $\mathsf{Cu}(A)$ by the remarks at the end of Chapter 2. If A is simple and stably finite, then it follows from Theorem 4.1.2 (*iii*) that $\mathsf{C}(A) = \mathsf{V}(A)$ and $\mathsf{D}(A) = \mathsf{Cu}(A)_+ \cup \{0\}$.

5 Quasitraces and functionals on Cuntz semigroups

5.1 Functionals

We begin this section by defining the notion of functional on a semigroup in Cu:

5.1.1 Definition. *Let S be a semigroup in Cu. A* functional *on S is a map $\lambda\colon S \to [0,\infty]$ that is additive, zero-preserving, order-preserving, and supremum-preserving. The set of all functionals on S is denoted by $\mathsf{F}(S)$.*

A functional is not required to preserve compact containment, and therefore a functional need not be a Cu-morphism from the semigroup S to the semigroup $[0,\infty]$. The set $\mathsf{F}(S)$ shall be equipped with the following topology:

5.1.2 Definition. *For a semigroup $S \in \mathsf{Cu}$, the set $\mathsf{F}(S)$ is made into a topological space by requiring that a net of functionals $(\lambda_\nu)_\nu$ converges towards a functional λ if and only if the inequalities $\limsup_\nu \lambda_\nu(x') \leq \lambda(x) \leq \liminf_\nu \lambda_\nu(x)$ hold for every pair $x' \ll x$ in S.*

This topology was introduced in [15], Section 4. It follows easily from the definition of the topology that pointwise addition and multiplication by elements of $(0,\infty)$ are jointly continuous operations. Extending the scalar multiplication continuously to the compact interval $[0,\infty]$ requires the following definition:

5.1.3 Definition. *Let S be a semigroup in Cu, and let $\lambda \in \mathsf{F}(S)$ be a functional. Let $\ker(\lambda)$ be the closed ideal consisting of the elements of S on which λ takes the value 0, and let $\mathit{fin}(\lambda)$ be the (not necessarily closed) additive and hereditary ideal consisting of the elements of S on which λ takes a finite value. Let $\overline{\mathit{fin}}(\lambda)$ be the closed ideal generated by $\mathit{fin}(\lambda)$. Then the functionals $0 \cdot \lambda$ and $\infty \cdot \lambda$ are defined as follows:*

$$(0 \cdot \lambda)(x) := \begin{cases} 0 & \text{if } x \in \overline{\mathit{fin}}(\lambda), \\ \infty & \text{if } x \notin \overline{\mathit{fin}}(\lambda). \end{cases}$$

$$(\infty \cdot \lambda)(x) := \begin{cases} 0 & \text{if } x \in \ker(\lambda), \\ \infty & \text{if } x \notin \ker(\lambda). \end{cases}$$

This being done, the set $\mathsf{F}(S)$ forms a topological cone in the sense of the following definition:

5.1.4 Definition. *An extended topological cone is an abelian monoid M equipped with a scalar multiplication $[0,\infty] \times M \to M$ and a topology such that addition and scalar multiplication are jointly continuous. The scalar multiplication is supposed to satisfy the following conditions:*

- $(c+d)x = cx + dx$ *for all* $c, d \in [0,\infty]$ *and all* $x \in M$,
- $c(x+y) = cx + cy$ *for all* $c \in [0,\infty]$ *and all* $x, y \in M$,
- $(cd)x = c(dx)$ *for all* $c, d \in [0,\infty]$ *and all* $x \in M$,
- $c0_M = 0_M$ *for all* $c \in [0,\infty]$,
- $1x = x$ *for all* $x \in M$.

This definition is based on [15], where these cones are called non-cancellative topological cones. We prefer to call them extended cones, alluding to the extended scalar multiplication. Note that the definition does not require $0x = 0_M$ to hold for $x \in M$. If S_1, S_2 are semigroups in Cu and if $\alpha\colon S_1 \to S_2$ is a Cu-morphism, then we can define a map $\mathsf{F}(\alpha)\colon \mathsf{F}(S_2) \to \mathsf{F}(S_1)$ by $\mathsf{F}(\alpha)(\lambda) := \lambda \circ \alpha$.

5.1.5 Theorem. *$\mathsf{F}(\cdot)$ is a contravariant functor from the category Cu to the category of extended compact Hausdorff cones with continuous linear maps as morphisms. (In this context, a map is* linear *if it is additive, homogeneous for elements of $[0,\infty]$, and zero-preserving.)*

Proof. This was shown in [15], Theorem 4.4 and [15], Theorem 4.8. ■

5.1.6 Definition. *Let S be a semigroup in Cu. If $u \in S$ is any element, then $\mathsf{F}_u(S)$ shall be the subset of $\mathsf{F}(S)$ consisting of all functionals λ on S with $\lambda(u) = 1$, equipped wit the relative topology.*

5.1.7 Lemma. *Let S be a semigroup in Cu, and let $u \in S$ be compact. Then the space $\mathsf{F}_u(S)$ is a compact and convex subset of the extended cone $\mathsf{F}(S)$.*

Proof. Since u is compact, we have $u \ll u$. It then follows from Definition 5.1.2 that for every net $(\lambda_\nu)_\nu$ in $\mathsf{F}_u(S)$ that converges towards $\lambda \in \mathsf{F}(S)$, we have $\lambda(u) = \lim_\nu \lambda_\nu(u) = 1$. Hence, $\mathsf{F}_u(S)$ is closed in the compact Hausdorff space $\mathsf{F}(S)$ and therefore compact. It is obvious that $\mathsf{F}_u(S)$ is convex. ■

Every semigroup $S \in \mathsf{Cu}$ has two special functionals:

5.1.8 Definition. *For any semigroup $S \in \mathsf{Cu}$, the functional on S that maps every $x \in S$ to zero shall be denoted by λ_0, and the functional on S that maps $0 \in S$ to zero but maps every $x \in S^\times$ to ∞ shall be denoted by λ_∞. A functional $\lambda \in \mathsf{F}(S)$ is called* trivial *if $\lambda = \lambda_0$ or $\lambda = \lambda_\infty$, and nontrivial otherwise.*

The usual meaning of the term "trivial functional" is a functional that takes only the values 0 and ∞. For simple semigroups, which are our main concern, our definition is equivalent to this usage, as we will now show:

5.1.9 Proposition. *Let $S \in \mathsf{Cu}$ be simple, let $x, y, z \in S$, and let $\lambda \in \mathsf{F}(S)$ be nontrivial.*

(i) *λ is faithful, i.e. $\lambda(x) > 0$ whenever $x > 0$.*

(ii) *λ is semifinite, i.e. $\lambda(x) < \infty$ whenever $x \ll \infty$.*

(iii) *If $x \ll y < z$, then $\lambda(x) < \lambda(z)$.*

Proof.

(i) We know that $\ker(\lambda) := \{x \in S \mid \lambda(x) = 0\}$ is a closed ideal of S. Since S is simple and $\lambda \neq \mu$, it follows that $\ker(\lambda) := \{0\}$, which proves the statement.

(ii) We know that $\overline{\mathrm{fin}}(\lambda) := \overline{\{x \in S \mid \lambda(x) < \infty\}}$ is a closed ideal in S. Since S is simple and $\lambda \neq \lambda_\infty$, we have $\overline{\mathrm{fin}}(\lambda) = S$, so the ideal $\mathrm{fin}(\lambda)$ is dense in S. Thus, there is a sequence $(y_n)_n$ in $\mathrm{fin}(\lambda)$ such that $\sup_n y_n = \infty$. Since $x \ll \infty$, we have $x \leq y_n$ for sufficiently large n, and therefore $x \in \mathrm{fin}(\lambda)$. This proves the statement.

(iii) Using axiom ($\mathcal{O}5$), we can find an element $w \in \mathsf{Cu}(A)$ such that $x+w \leq z \leq y+w$; since $y < z$, it follows that $w > 0$. We can find $w' \in S$ such that $0 < w' \ll w$, and therefore $0 < w' \ll \infty$. Using parts (i) and (ii), we find that $0 < \lambda(w') < \infty$. Since $x \ll y$, we have $x \ll \infty$ and therefore $\lambda(x) < \infty$ by (i). Since $x + w' \leq x + w \leq z$, it follows that $\lambda(x) + \lambda(w') \leq \lambda(z)$, and therefore $\lambda(x) < \lambda(z)$, which proves the statement.

■

We will now show that nontrivial functionals always exist for nonzero, simple, and stably finite semigroups in Cu, which is a Cu-analogue of [4], Theorem 1.2.

5.1.10 Theorem. *For simple and nonzero S in Cu, the following conditions are equivalent:*

(i) *S is stably finite.*

(ii) *$\mathsf{F}(S)$ contains a nontrivial functional.*

(iii) *$\mathsf{F}_u(S)$ is nonempty for every nonzero $u \in S_{min}$.*

Proof.

(iii) $\Longrightarrow$ (ii): S_{min} is dense in S, so S_{min} must be nonzero since S is nonzero. Pick any nonzero $u \in S_{min}$. Since (iii) holds, there is a functional $\lambda \in \mathsf{F}_u(S)$. Then $\lambda \in \mathsf{F}(S)$ and $\lambda(u) = 1$, so λ is a nontrivial functional in $\mathsf{F}(S)$.

(ii) $\Longrightarrow$ (i): Let λ be a nontrivial functional of S. Since the maximal element ∞ of S is properly infinite by Proposition 3.1.15, we have $\lambda(\infty) \in \{0, \infty\}$. Assume that S is not stably finite. Then ∞ is compact by Proposition 3.1.18, hence $\infty \ll \infty$. Moreover, we have $0 < \infty$ since S is nonzero. Since λ is nontrivial, it follows from Proposition 5.1.9 that $0 < \lambda(\infty) < \infty$, contradicting $\lambda(\infty) \in \{0, \infty\}$. It follows that S is indeed stably finite.

(i) $\Longrightarrow$ (iii): This is the interesting part. As before, S_{min} must be nonzero since S is nonzero; let u be any nonzero element of S_{min}. Note that S_{min} is a positively ordered abelian monoid; let $G := \mathsf{Gr}(S_{min})$ be the Grothendieck enveloping group of S_{min}, and let $g\colon S_{min} \to G$ be the universal semigroup homomorphism from the Grothendieck construction (see e.g. [33], pp. 35–38, for an overview of the Grothendieck construction). Then g is additive and zero-preserving, and $G = g(S_{min}) - g(S_{min})$. Let $G_+ := \{g(x) - g(y) \mid x, y \in S_{min}$ and $y \leq x$ in $S\}$. Then $0 \in G_+$ since g is zero-preserving, and G_+ is closed under addition since g is additive and S_{min} is positively ordered. Hence G_+ is a cone in G. Since the order on S_{min} is anti-symmetric, it follows that $G_+ \cap (-G_+) = \{0\}$, so (G, G_+) is an ordered abelian group. Since $g(M) \subseteq G_+$ and $G = g(M) - g(M)$, it follows that $G = G_+ - G_+$, so that (G, G_+) is a directed abelian group (i.e., the partially ordered abelian group (G, G_+) is upwards directed). The map $g\colon M \to G$ is order-preserving with regard to the order on G induced by the cone G_+. Thus, we can regard it as a homomorphism of ordered abelian monoids $g\colon M \to (G, G_+)$.

Since S is stably finite and u is nonzero, we have $x < x + u$ for every $x \in S_{min}$ by Proposition 3.1.18, which implies that $g(u) > 0$ in the Grothendieck envelope G. Since S is simple and u is nonzero, it follows that $\sup_n nu = \infty$. Since $x \ll \infty$ for every $x \in S_{min}$, we find that for every $x \in S_{min}$ there is some $n \in \mathbb{N}$ such that $x \leq nu$. It follows that $g(u) > 0$, and that for every $x \in G$ there is some $n \in \mathbb{N}$ such that $x \leq ng(u)$, i.e. $g(u)$ is a strong unit for the directed abelian group (G, G_+). By the Goodearl-Handelman extension result, there is a state h on $(G, G_+, g(u))$, i.e. an additive, zero-preserving, and order-preserving map $h : G \to \mathbb{R}$ with $h(g(u)) = 1$ (see [18], Corollary 3.3]). Let $f := h \circ g$, then $f\colon S_{min} \to [0, \infty)$ is a state on the positively ordered abelian monoid (S_{min}, u).

Next, define a map $\mu\colon S \to [0, \infty]$ by $\mu(x) := \sup_{x' \ll x} f(x)$. Note that $x' \ll x$ implies $x' \in S_{min}$, so μ is well-defined. We claim that $\mu \in \mathsf{F}(S)$, i.e. that μ is zero-preserving, order-preserving, sup-preserving, and additive (compare [15], Lemma 4.7, and [30], Proposition 4.1). It is obvious that $\mu(0) = 0$, so μ is zero-preserving. Let $x, y \in S$ with $x \leq y$. By Rørdam's proposition, there is for every $x' \ll x$ some $y' \ll y$ such that $x' \leq y'$, hence $f(x') \leq f(y')$, which implies that $\mu(x) \leq \mu(y)$. Thus, μ is order-preserving. Next, let $(x_n)_n$ be any increasing sequence in S with supremum $x \in S$. Since μ is order-preserving, we know that $\mu(x) \geq \sup_n \mu(x_n)$. For every $x' \ll x$, we can find some $x'' \in S$ with $x' \ll x'' \ll x$. There is then some $n \in \mathbb{N}$ such that $x'' \leq x_n$. Since $x' \ll x''$, we can once again use Rørdam's proposition to find some $x'_n \ll x_n$ with $x' \leq x'_n$, which implies $f(x') \leq f(x'_n) \leq \mu(x_n) \leq \sup_n \mu(x_n)$. But from $f(x') \leq \sup_n \mu(x_n)$ for every $x' \ll x$, it follows that $\mu(x) \leq \sup_n \mu(x_n)$. In total, we find that $\mu(x) = \sup_n \mu(x_n)$, so μ is sup-preserving. Finally, let x, y be any elements of S. If $x' \ll x$ and $y' \ll y$, then $x' + y' \ll x + y$ by axiom ($\mathcal{O}3$); it follows that $f(x') + f(y') = f(x' + y') \leq \mu(x + y)$, and therefore $\mu(x) + \mu(y) \leq \mu(x + y)$. Conversely, for every $z \ll x + y$ we can find elements $x' \ll x$ and $y' \ll y$ with $z \leq x' + y'$ by axioms ($\mathcal{O}2$), ($\mathcal{O}3$), and ($\mathcal{O}4$). It follows that $f(z) \leq f(x' + y') = f(x') + f(y') \leq \mu(x) + \mu(y)$. From $f(z) \leq \mu(x) + \mu(y)$ for all $z \ll x + y$, it then follows that $\mu(x + y) \leq \mu(x) + \mu(y)$. In total, we find that $\mu(x + y) = \mu(x) + \mu(y)$, so μ is additive. Hence, $\mu \in \mathsf{F}(S)$ as claimed.

Finally, note that $\mu(u) = \sup_{u' \ll u} f(u') \leq f(u) = 1$. If $0 < u' \ll u$, we have $u \leq nu'$ for sufficiently large n since S is simple, and u' is nonzero, and $u \ll \infty$. It follows that $1 = f(u) \leq f(nu') = nf(u')$, which implies $0 < \frac{1}{n} \leq f(u') \leq \mu(u)$. Let $\alpha := \mu(u)$, then

$0 < \alpha \leq 1$. Let $\lambda := \alpha^{-1}\mu$, then $\lambda \in \mathsf{F}(S)$ and $\lambda(u) = 1$, hence $\lambda \in \mathsf{F}_u(S)$. This completes the proof. ■

We remind the reader that a convex set is *regularly embedded* in a real vector space if it is contained in a hyperplane that does not contain the origin. The following definitions are equivalent to those in [17], Chapter 10:

5.1.11 Definition.

(i) *A convex subset S of a real vector space E is called a* simplex *if it is affinely isomorphic to a regularly embedded convex subset S' of another real vector space E' such that S' is the base of a lattice cone (i.e. the cone $\{\alpha x \mid \alpha \geq 0,\, x \in S'\}$ is a lattice when equipped with its algebraic order).*

(ii) *If E is a locally convex, Hausdorff real vector space and $K \subseteq E$ is a compact simplex, then K is called a* Choquet simplex.

We will also refer to a compact, convex subset of an extended topological cone (as in Definition 5.1.4) as a Choquet simplex if it is affinely homeomorphic to a regularly embedded Choquet simplex in a locally convex, Hausdorff real vector space.

5.1.12 Theorem. *Let S be a semigroup in* Cu*, and let $u \in S$ be a full compact element (where* full *means that the closed ideal of S generated by u is all of S). Then $\mathsf{F}_u(S)$ is a Choquet simplex.*

Proof. This follows from results by Leonel Robert in [29], although it is not explicitly stated there. Let $\mathsf{F}_{sf}(S)$ be the subset of $\mathsf{F}(S)$ consisting of all semifinite functionals on S. Since u is full in S, it is easy to see that $\mathsf{F}_u(S) \subseteq \mathsf{F}_{sf}(S)$. Let $C := \{\lambda|_{S_{min}} \mid \lambda \in \mathsf{F}_{sf}(S)\}$. Regard C as a cone in the real vector space of all real-valued functions on S_{min}, and let $E := C - C$ be the $\mathbb{R}$-linear subspace generated by C. Let $K := \{\lambda \in C \mid \lambda(u) = 1\}$, then K is a regularly embedded convex subset of E, and the cone spanned by K is C (this is also true, by convention, in the case that $C = \{0\}$ and $K = \emptyset$). Note that every element $\lambda \in C$ extends uniquely to a functional $\tilde{\lambda} \in \mathsf{F}_{sf}(S)$ by the formula $\tilde{\lambda}(x) := \sup_{x' \ll x} \lambda(x')$. This bijective correspondence yields a linear isomorphism between C and $\mathsf{F}_{sf}(S)$; the restriction to K yields an affine isomorphism between K and $\mathsf{F}_u(S)$.

Let V be the real vector space of all linear, continuous, and real-valued functions on $\mathsf{F}_{sf}(S)$. It was shown in the proof of [29], Proposition 3.2.3, that the relative topology of $\mathsf{F}_{sf}(S)$ with regard to $\mathsf{F}(S)$ is the weak topology $\sigma(\mathsf{F}_{sf}(S), V)$. For each $\varphi \in V$, we want to define a seminorm $\|\cdot\|_\varphi$ on E by $\|\lambda - \mu\|_\varphi := \left|\varphi(\tilde{\lambda}) - \varphi(\tilde{\mu})\right|$. This seminorm is well-defined, for if $\lambda - \mu = \lambda' - \mu'$, then we have $\lambda + \mu' = \lambda' + \mu \in C$, and by unique extension we get $\tilde{\lambda} + \tilde{\mu'} = \tilde{\lambda'} + \tilde{\mu}$, which implies $\varphi(\tilde{\lambda}) - \varphi(\tilde{\mu}) = \varphi(\tilde{\lambda'}) - \varphi(\tilde{\mu'})$. Equip E with the locally convex topology induced by the family of seminorms $\{\|\cdot\|_\varphi \mid \varphi \in V\}$, then the linear isomorphism $\lambda \mapsto \tilde{\lambda}$ from C to $\mathsf{F}_{sf}(S)$ becomes a linear homeomorphism, and its restriction to K becomes an affine homeomorphism between K and $\mathsf{F}_u(S)$. Since $\mathsf{F}(S)$, and hence $\mathsf{F}_{sf}(S)$, are Hausdorff spaces by Theorem 5.1.5, it follows easily that $E = C - C$ with the topology defined above is a locally convex Hausdorff real vector space. Since $\mathsf{F}_u(S)$ is compact in $\mathsf{F}(S)$, and hence in $\mathsf{F}_{sf}(S)$, by Lemma 5.1.7, it follows that K is a regularly embedded, compact, convex subset of E, and that the cone spanned by K is C. Since $\mathsf{F}_u(S)$ is affinely homeomorphic to K, it only remains to show that C is lattice-ordered. Since C is linearly isomorphic to $\mathsf{F}_{sf}(S)$, it suffices to show that $\mathsf{F}_{sf}(S)$ is lattice ordered. Since $\mathsf{F}_{sf}(S)$ is order-hereditary in $\mathsf{F}(S)$, and since any pair $\lambda_1, \lambda_2 \in \mathsf{F}_{sf}(S)$ has an upper bound in $\mathsf{F}_{sf}(S)$, namely the sum $\lambda_1 + \lambda_2$, it is in fact sufficient to show that $\mathsf{F}(S)$ is lattice-ordered, which was done by Robert in Theorem 4.1.2 of [29]. In total, it follows that K is a compact simplex in the locally convex, Hausdorff real vector space E, i.e. K is a Choquet simplex, and that $\mathsf{F}_u(S)$ is affinely homeomorphic to K, so $\mathsf{F}_u(S)$ is a Choquet simplex as well. ■

Given any element $x \in S$, let the function $\tilde{x}\colon \mathsf{F}(S) \to [0,\infty]$ be defined by $\tilde{x}(\lambda) := \lambda(x)$. The following theorem describes the continuity properties of the maps $\tilde{x}$:

5.1.13 Proposition. *Let S be a semigroup in Cu.*

(*i*) *For any $x \in S$, the function $\tilde{x}\colon \mathsf{F}(S) \to [0,\infty]$ is lower semicontinuous.*

(*ii*) *If $x \in S$ is compact, then the function $\tilde{x}\colon \mathsf{F}(S) \to [0,\infty]$ is continuous.*

Proof. Let $(\lambda_\nu)_\nu$ be a net in $\mathsf{F}(S)$ that converges towards $\lambda \in \mathsf{F}(S)$.

(*i*) By Definition 5.1.2, we know that $\tilde{x}(\lambda) = \lambda(x) \leq \liminf_\nu \lambda_\nu(x) = \liminf_\nu \tilde{x}(\lambda_\nu)$, so the function $\tilde{x}$ is lower semicontinuous.

(*ii*) If x is compact, then $x \ll x$, so we have $\tilde{x}(\lambda) = \lambda(x) \geq \limsup_\nu \lambda_\nu(x) = \limsup_\nu \tilde{x}(\lambda_\nu)$. From this and part (*i*), we get that $\tilde{x}(\lambda) = \lim_\nu \tilde{x}(\lambda_\nu)$, so the function $\tilde{x}$ is continuous. ■

We close this section by showing that a number of conditions are equivalent to S being almost unperforated.

5.1.14 Theorem. *For $S \in \mathsf{Cu}$, the following conditions are equivalent:*

(*i*) *S is almost unperforated.*

(*ii*) *If $x, y \in S$ such that $x \leq \infty \cdot y$ and $\lambda(x) < \lambda(y)$ for all $\lambda \in \mathsf{F}_y(S)$, then $x \leq y$.*

(*iii*) *If $x, y \in S$ and $\varepsilon > 0$ such that $\lambda(x) \leq (1-\varepsilon)\,\lambda(y)$ for all $\lambda \in \mathsf{F}(S)$, then $x \leq y$.*

Proof.

(*i*) $\iff$ (*ii*): This was shown in [15], Proposition 6.2.

(*ii*) $\implies$ (*iii*): This was shown by Leonel Robert in [28], Lemma 1, for the special case where $S = \mathsf{Cu}(A)$. We shall prove the general case. Let $x, y \in S$ and $\varepsilon > 0$ such that $\lambda(x) \leq (1-\varepsilon)\lambda(y)$ for all $\lambda \in \mathsf{F}(S)$. First, we note that we automatically have $x \leq \infty \cdot y$, i.e. we have $x \in S_y$ where S_y is the closed ideal of S generated by y. For if not, the map

$\sigma\colon S \to [0,\infty]$ that takes the value zero everywhere on S_y, and takes the value ∞ everywhere outside S_y, would be a functional on S with $\sigma(x) = \infty$ and $\sigma(y) = 0$, contradicting the assumption that $\sigma(x) \leq (1-\varepsilon)\sigma(y)$. Next, for every functional $\lambda \in \mathsf{F}_y(S)$, we have $\lambda(y) = 1$ and therefore $\lambda(x) \leq (1-\varepsilon)\lambda(y) < \lambda(y)$. Hence, it follows from (ii) that $x \leq y$. This shows that condition (iii) holds.

$(iii) \implies (i)$: Let $x, y \in S$ and $n \in \mathbb{N}$ such that $(n+1)x \leq ny$. Clearly, it follows that $\lambda(x) \leq \frac{n}{n+1}\lambda(y)$ for every functional $\lambda \in \mathsf{F}(S)$, so the requirements of (iii) are satisfied for $\varepsilon = \frac{1}{n+1}$. Hence, it follows from (iii) that $x \leq y$. This shows that condition (i) holds. ■

5.2 Traces and quasitraces

If A is a C*-algebra, then the functionals on $\mathsf{Cu}(A)$ are closely related to tracial and quasitracial weights, which are defined as follows:

5.2.1 Definition. *Let A be a C*-algebra.*

(i) *A* quasitracial weight *on A is a map $\tau\colon A_+ \to [0,\infty]$ with the following properties:*
- $\tau(0) = 0$ *and* $\tau(ca) = c\,\tau(a)$ *for all* $a \in A_+$ *and* $c \in (0,\infty)$,
- $\tau(a+b) = \tau(a) + \tau(b)$ *for all* $a, b \in A_+$ *with* $ab = ba$,
- $\tau(xx^*) = \tau(x^*x)$ *for all elements* $x \in A$.

(ii) *A quasitracial weight τ is* lower semicontinuous *if* $\tau(\lim_n a_n) \leq \liminf_n \tau(a_n)$ *for every convergent sequence* $(a_n)_n$ *in* A_+.

(iii) *A quasitracial weight τ is* finite *if* $\tau(a) < \infty$ *for all* $a \in A_+$, *and* semifinite *if the set* $\{a \in A_+ \mid \tau(a) < \infty\}$ *is dense in* A_+. *We call τ* faithful *if* $\tau(a) > 0$ *whenever* $a \neq 0$.

(iv) *A quasitracial weight τ is* bounded *if* $\|\tau\| := \sup\{\tau(a) \mid a \in A_+ \text{ and } \|a\| \leq 1\} < \infty$. *We call τ* normalised *if* $\|\tau\| = 1$.

(v) *A quasitracial weight τ on A is called a* 2-quasitracial weight *if it extends to a quasitracial weight $\tilde{\tau}$ on $M_2(A)$ such that* $\tilde{\tau}(a \otimes e) = \tau(a)$ *for any* $a \in A_+$ *and any minimal projection* $e \in M_2$.

(vi) *A quasitracial weight τ on A is called a* tracial weight *if τ is additive on all of A_+, i.e. if* $\tau(a+b) = \tau(a) + \tau(b)$ *for all* $a, b \in A_+$.

Some articles use quasitracial functionals instead of *finite* quasitracial weights:

5.2.2 Definition. *Let A be a C*-algebra.*

(i) *A* quasitracial functional *on A is a map $\tau\colon A \to \mathbb{C}$ with the following properties:*
- $\tau(ca) = c\,\tau(a)$ *for all* $a \in A$ *and* $c \in \mathbb{C}$,
- $\tau(a+b) = \tau(a) + \tau(b)$ *for all* $a, b \in A$ *with* $ab = ba$,
- $\tau(a+ib) = \tau(a) + i\tau(b)$ *for all self-adjoint elements* $a, b \in A$,
- $\tau(xx^*) = \tau(x^*x) \geq 0$ *for all elements* $x \in A$.

(*ii*) *A quasitracial functional* τ *is* faithful *if* $\tau(a) > 0$ *whenever* $a \geq 0$ *and* $a \neq 0$.

(*iii*) *A quasitracial functional* τ *is* bounded *if* $\|\tau\| := \sup\{\tau(a) \mid a \in A_+ \text{ and } \|a\| \leq 1\} < \infty$. *We call* τ normalised *if* $\|\tau\| = 1$. *(We will soon see that quasitracial functionals are always bounded, but this is not obvious since quasitracial functionals need not be linear.)*

(*iv*) *A quasitracial functional* τ *on* A *is called a* 2-quasitracial functional *if it extends to a quasitracial functional* $\tilde{\tau}$ *on* $M_2(A)$ *such that* $\tilde{\tau}(a \otimes e) = \tau(a)$ *for any* $a \in A$ *and any minimal projection* $e \in M_2(A)$.

(*v*) *A quasitracial functional* τ *on* A *is called a* tracial functional *if* τ *is additive on all of* A_+, *i.e. if* $\tau(a + b) = \tau(a) + \tau(b)$ *for all* $a, b \in A_+$.

5.2.3 Lemma. *We can identify the quasitracial functionals with the finite quasitracial weights:*

(*i*) *Every finite quasitracial weight extends uniquely to a quasitracial functional. Conversely, every quasitracial functional restricts to a finite quasitracial weight.*

(*ii*) *Moreover, a quasitracial functional is normalised, faithful, 2-quasitracial, or tracial if and only if the corresponding finite quasitracial weight has the respective property.*

Proof. These are all straightforward calculations. ∎

5.2.4 Theorem. *Let* A *be a C*-algebra.*

(*i*) *Every tracial weight on* A *is automatically 2-quasitracial.*

(*ii*) *For any quasitracial weight* τ *on* A, τ *is finite* $\iff$ τ *is bounded* $\iff$ τ *is continuous.*

(*iii*) *Every finite quasitracial weight satisfies* $|\tau(a) - \tau(b)| \leq \|\tau\| \, \|a - b\|$ *for all* $a, b \in A_+$.

Proof.

(*i*) Another straightforward calculation shows that for a tracial weight τ, the obvious extension $\tau \otimes tr \colon \left[\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right] \mapsto \tau(x_{11}) + \tau(x_{22})$ has all the required properties.

(*ii*) It was shown in [5], Corollary II.2.3, that every quasitracial functional on a C*-algebra is automatically bounded; it follows immediately that the same is then true for the corresponding finite quasitracial weights. Moreover, by [6], Remark 2.27 (v), every bounded quasitracial weight is continuous (this was previously shown in [5], Corollary II.2.5, for 2-quasitracial functionals and hence for finite 2-quasitracial weights). Obviously, every bounded quasitracial weight is finite. It remains to show that a continuous quasitracial weight τ is necessarily bounded. If it was unbounded, we could fine a sequence $(a_n)_n$ in A_+ such that $\|a_n\| \leq 1$ and $\tau(a_n) \geq n$. Then $b_n := \frac{1}{n} a_n$ would be an element in A_+ with $\|b_n\| \leq \frac{1}{n}$ and $\tau(b_n) \geq 1$. It follows that $(b_n)_n$ would converge in norm towards 0, but $(\tau(b_n))_n$ would not converge towards $\tau(0) = 0$, contradicting the fact that τ is continuous. Hence, a continuous quasitrace must necessarily be bounded.

(*iii*) This was shown in [6], Remark 2.27 (v).

∎

Hence, every finite tracial weight is automatically lower semicontinuous and 2-quasitracial. In particular, if τ is any tracial functional, then the corresponding quasitracial weight is automatically lower semicontinuous and 2-quasitracial. Next, we shall define some interesting sets of lower semicontinuous 2-quasitracial weights:

5.2.5 Definition. *Let A be any C*-algebra.*

(i) *The set of lower semicontinuous 2-quasitracial weights on A is denoted by $\mathsf{QT}(A)$.*

(ii) *The set of lower semicontinuous tracial weights on A is denoted by $\mathsf{T}(A)$.*

(iii) *The set of normalised 2-quasitracial weights on A is denoted by $\mathsf{QT}_1(A)$.*

(iv) *The set of normalised tracial weights on A is denoted by $\mathsf{T}_1(A)$.*

For the sake of brevity, we shall use the terms *trace* and *quasitrace* in the following sense:

5.2.6 Definition. *The term* quasitrace *shall refer to a 2-quasitracial weight. Likewise, the term* trace *shall refer to a tracial weight.*

Hence, the elements of $\mathsf{QT}(A)$ are the lower semicontinuous quasitraces, and the elements of $\mathsf{T}(A)$ are the lower semicontinuous traces. By the above identifications, the elements of $\mathsf{QT}_1(A)$ are the normalised quasitraces and correspond precisely to the normalised quasitracial functionals on A; the elements of $\mathsf{T}_1(A)$ are the normalised traces and correspond precisely to the normalised tracial functionals, i.e. to the *tracial states*, on A.

The elements of $\mathsf{QT}(A)$ have some agreeable properties:

5.2.7 Theorem. *Let A be a C*-algebra and let $\tau \in \mathsf{QT}(A)$.*

(i) *τ is 2-subadditive: for all $a, b \in A_+$, we have $\tau(a+b) \leq 2(\tau(a)+\tau(b))$.*

(ii) *τ is order-preserving: for all $a, b \in A_+$ with $a \leq b$, we have $\tau(a) \leq \tau(b)$.*

Proof.

(i) See [6], Proposition 2.24.

(ii) See [6], Remark 2.27 (iv).

■

5.2.8 Definition. *Let A be a C*-algebra and let $\tau \in \mathsf{QT}(A)$.*

(i) *Let $\ker(\tau)$ denote the linear span of the set $\{a \in A_+ \mid \tau(a) = 0\}$.*

(ii) *Let $\mathit{fin}(\tau)$ denote the linear span of the set $\{a \in A_+ \mid \tau(a) < \infty\}$.*

(iii) *Let $\overline{\mathit{fin}}(\tau)$ denote the closure of $\mathit{fin}(\tau)$.*

We remind the reader that an algebraic ideal I of a C*-algebra A is called *strongly invariant* if it satisfies $x^*x \in I \iff xx^* \in I$ for every $x \in A$ (see [3], section II.5.2.3).

5.2.9 Proposition. *Let A be a C*-algebra, and let $\tau \in \mathsf{QT}(A)$.*

(i) *fin*(τ) *is a self-adjoint, positively generated, strongly invariant, and hereditary algebraic ideal of A. Moreover, we have* $\mathit{fin}(\tau)_+ = \{a \in A_+ \mid \tau(a) < \infty\}$.

(ii) $\ker(\tau)$ *and* $\overline{\mathit{fin}}(\tau)$ *are closed ideals of A. Moreover,* $\ker(\tau)_+ = \{a \in A_+ \mid \tau(a) = 0\}$.

Proof.

(i) It follows from Theorem 5.2.7 that $\{a \in A_+ \mid \tau(a) < \infty\}$ is hereditary and closed under addition. By the definition of quasitracial weight, it is obvious that this set is closed under multiplication with scalars in $[0, \infty)$, and that for any $x \in A$, it contains x^*x if and only if it contains xx^*. By [3], Section II.5.2.2 and Section II.5.2.3, it follows that its linear span is an algebraic ideal with all the properties listed above.

(ii) For $\overline{\mathrm{fin}}(\tau)$, this follows immediately from (a). For $\ker(\tau)$, the same argument that we used for $\mathrm{fin}(\tau)$ in (a) will show that $\ker(\tau)$ is also an algebraic ideal of A, and that $\ker(\tau)_+ = \{a \in A_+ \mid \tau(a) = 0\}$. Since τ is lower semicontinuous, it follows that $\ker(\tau)_+$ is closed, hence $\ker(\tau)$ is closed as well.

■

There are two trivial lower semicontinuous traces that exist on every C*-algebra:

5.2.10 Definition. *Let A be a C*-algebra. The symbol τ_0 shall refer to the lower semicontinuous trace on A with $\tau_0(a) = 0$ for every $a \in A_+$, and the symbol τ_∞ shall refer to the lower semicontinuous trace on A with $\tau_\infty(0) = 0$ and $\tau_\infty(a) = \infty$ for every nonzero $a \in A_+$. We call a quasitrace* trivial *if it is equal to τ_0 or τ_∞, and nontrivial otherwise.*

As for functionals, the usual meaning of the term “trivial quasitrace” is a quasitrace that takes only the values 0 and ∞. For simple C*-algebras, which are our main concern, our definition is equivalent to this usage, as we will show now:

5.2.11 Theorem. *Let A be a C*-algebra.*

(i) *If A is simple, then every nontrivial $\tau \in \mathsf{QT}(A)$ is semifinite and faithful.*

(ii) *If A is simple and unital, then every nontrivial $\tau \in \mathsf{QT}(A)$ is finite and faithful.*

(iii) *If A is exact, then every $\tau \in \mathsf{QT}(A)$ is a trace.*

Proof.

(i) By Proposition 5.2.9, we know that $\ker(\tau)$ and $\overline{\mathrm{fin}}(\tau)$ are closed ideals of A. Since A is simple and τ is nontrivial, it follows that $\ker(\tau) = \{0\}$ and that $\overline{\mathrm{fin}}(\tau)$ is all of A. Therefore, τ is faithful and semifinite.

(ii) We need only show that τ is finite. Again by Proposition 5.2.9, we know that $\mathrm{fin}(\tau)$ is an algebraic ideal of A with $\mathrm{fin}(\tau)_+ = \{a \in A_+ \mid \tau(a) < \infty\}$. Since simple, unital C*-algebras are algebraically simple and since $\tau \neq \tau_\infty$, it follows that $\mathrm{fin}(\tau)$ is equal to all of A, and this implies that τ takes finite values on all of A_+.

(*iii*) This was originally proven by U. Haagerup in [19] for quasitracial functionals on exact, unital C*-algebras; the manuscript has been widely known since 1991, and the article is finally about to be published. The generalisation to all lower semicontinuous 2-quasitracial weights on arbitrary exact C*-algebras was obtained by Blanchard and Kirchberg in [6], Remark 2.29 (*i*).

■

5.2.12 Corollary. *If A is a simple, unital, and exact C*-algebra, then every nontrivial lower semicontinuous quasitrace is of the form $\alpha\tau$ for a unique scalar $\alpha \in (0, \infty)$ and a unique normalised trace $\tau \in \mathsf{T}_1(A)$.*

Proof. This follows immediately from Theorem 5.2.11 (*ii*) and (*iii*). ■

This is why $\mathsf{T}_1(A)$ is the preferred invariant for the classification of simple, unital, and exact C*-algebras, while it is usually necessary to consider $\mathsf{T}(A)$ or $\mathsf{QT}(A)$ for more general classes of C*-algebras. Of course, we have yet to show that nontrivial quasitraces exist at all:

5.2.13 Theorem. *For a simple C*-algebra A, the following conditions are equivalent:*

(*i*) *A is stably finite and nonzero.*

(*ii*) *$\mathsf{QT}(A)$ contains a nontrivial quasitrace.*

Proof. This was shown by Blanchard and Kirchberg in [6], Remark 2.27 (*viii*), extending an earlier result by Blackadar and Cuntz in [4]. ■

One important property that all elements of $\mathsf{QT}(A)$ possess is the existence of a unique extension to the stabilisation of A:

5.2.14 Theorem. *Let A be a C*-algebra.*

(*i*) *Every $\tau \in \mathsf{QT}(A)$ extends uniquely to a lower semicontinuous quasitracial weight $\tilde{\tau}$ on $A \otimes \mathcal{K}$ such that $\tilde{\tau}(a \otimes e_{11}) = \tau(a)$ for every $a \in A_+$.*

(*ii*) *Conversely, for every lower semicontinuous quasitracial weight τ on $A \otimes \mathcal{K}$, the restriction of τ to $(A \otimes e_{11})_+$ induces an element of $\mathsf{QT}(A)$.*

(*iii*) *Moreover, the extension $\tilde{\tau}$ is tracial (or faithful, or semifinite) if and only if τ is tracial (or faithful, or semifinite).*

Proof.

(*i*) This follows from the results in [6], even though it is not explicitly stated there. By Remark 2.27 (*viii*) of [6], there is a unique extension of $\tau|_{\overline{\mathrm{fin}}(\tau)_+}$ to a semifinite lower semicontinuous quasitrace τ' on $\overline{\mathrm{fin}}(\tau) \otimes \mathcal{K}$. Let $\tilde{\tau}\colon (A \otimes \mathcal{K})_+ \to [0, \infty]$ be the function that agrees with τ' on $(\overline{\mathrm{fin}}(\tau) \otimes \mathcal{K})_+$, and takes the value ∞ everywhere else. It is easy to see that $\tilde{\tau}$, thus defined, is indeed a lower semicontinuous quasitracial weight on $A \otimes \mathcal{K}$ that extends τ. Let σ be another extension of τ to a lower semicontinuous quasitracial weight on $A \otimes \mathcal{K}$. Since τ, again by Remark 2.27 (*viii*) of [6], has a unique extension to a lower semicontinuous quasitrace on $M_n(A)$ for each $n \in \mathbb{N}$, it follows easily that $\tilde{\tau}$ and σ must agree on every element of $(A \odot \mathcal{F})_+$, where $\mathcal{F}$ shall denote the algebraic ideal of finite rank operators. Finally, if a is any element of $(A \otimes \mathcal{K})_+$, then $(a-\varepsilon)_+ \in (A \odot \mathcal{F})_+$ for every $\varepsilon > 0$, and thus we have $\tilde{\tau}(a) = \sup_{\varepsilon>0} \tilde{\tau}((a-\varepsilon)_+) = \sup_{\varepsilon>0} \sigma((a-\varepsilon)_+) = \sigma(a)$. In total, we find that $\tilde{\tau}$ is the unique extension of τ to a lower semicontinuous quasitracial weight on $A \otimes \mathcal{K}$.

(*ii*) This is perfectly obvious.

(*iii*) Yet another straightforward calculation shows that for a tracial weight τ on A, the unique extension $\tilde{\tau}$ on $A \otimes \mathcal{K}$ is the map $\tau \otimes tr \colon a \mapsto \sum_{i\in\mathbb{N}} \tau(a_{ii})$ – where the coefficients a_{ii} come from the identity $a = \lim_n \sum_{i,j=1}^n a_{ij} \otimes e_{ij}$ as in Lemma 2.1.1 – and that this map is additive on $(A \otimes \mathcal{K})_+$ since τ is additive on A_+. Conversely, if $\tilde{\tau}$ is additive on $(A \otimes \mathcal{K})_+$, then its restriction to $(A \otimes e_{11})_+$ is additive, so τ is additive on A_+. Moreover, it is clear from the construction of $\tilde{\tau}$ in part (*i*) that $\overline{\mathrm{fin}}(\tilde{\tau}) = \overline{\mathrm{fin}}(\tau) \otimes \mathcal{K}$, which means that $\tilde{\tau}$ is semifinite if and only if τ is semifinite. Since $\ker(\tilde{\tau})$ is a closed ideal of $A \otimes \mathcal{K}$, we can find a closed ideal J of A such that $J \otimes \mathcal{K} = \ker(\tilde{\tau})$ by Proposition 4.2.4. By the same proposition, we know that $J = \{x \in A \mid x \otimes e_{11} \in \ker(\tilde{\tau})\}$, which implies that $J_+ = \{x \in A_+ \mid x \in \ker(\tau)\}$, and therefore $J = \ker(\tau)$. It follows that $\ker(\tilde{\tau}) = \ker(\tau) \otimes \mathcal{K}$, and hence that $\tilde{\tau}$ is faithful if and only if τ is faithful.

■

Hence, we can identify the lower semicontinuous 2-quasitracial weights on A with the lower semicontinuous quasitracial weights on $A \otimes \mathcal{K}$. From here on, we will use the same symbol to denote both a quasitrace in $\mathsf{QT}(A)$, and its unique extension to a lower semicontinuous quasitrace on $A \otimes \mathcal{K}$. In [15], the authors define a topology on $\mathsf{QT}(A)$ as follows:

5.2.15 Definition. *For any C*-algebra A, the set* $\mathsf{QT}(A)$ *is made into a topological space by requiring that a net* $(\tau_\nu)_\nu$ *in* $\mathsf{QT}(A)$ *converges towards* $\tau \in \mathsf{QT}(A)$ *if and only if the inequality* $\limsup_\nu \tau_\nu((a-\varepsilon)_+) \leq \tau(a) \leq \liminf_\nu \tau_\nu(a)$ *holds for every* $a \in (A \otimes \mathcal{K})_+$ *and every* $\varepsilon > 0$.

Note that this topology is defined via the unique extensions of the quasitraces to $(A \otimes \mathcal{K})_+$. Once again, it is easy to see that addition and scalar multiplication with elements of $(0, \infty)$ are jointly continuous operations. We can extend the scalar multiplication continuously to the compact interval $[0, \infty]$ by defining $(0 \cdot \tau)$ to be the lower semicontinuous trace that vanishes on $\overline{\mathrm{fin}}(\tau)$ and that takes the value ∞ everywhere outside $\overline{\mathrm{fin}}(\tau)$, and by defining $(\infty \cdot \tau)$ to be the lower semicontinuous trace that vanishes on $\ker(\tau)$ and takes the value ∞ everywhere outside $\ker(\tau)$. With this extended scalar multiplication, we have the following:

5.2.16 Theorem. *Both* $\mathsf{QT}(A)$ *and* $\mathsf{T}(A)$ *are extended compact Hausdorff cones.*

Proof. This was shown in [15], Theorem 4.4 for $\mathsf{QT}(A)$, and in [15], Theorem 3.7 for $\mathsf{T}(A)$. ■

If $\alpha : A \to B$ is a *-homomorphism, then we can define a map $\mathsf{QT}(\alpha)\colon \mathsf{QT}(B) \to \mathsf{QT}(A)$ by $\mathsf{QT}(\alpha)(\tau) := \tau \circ \alpha$. We can define a map $\mathsf{T}(\alpha)\colon \mathsf{T}(B) \to \mathsf{T}(A)$ in the same way.

5.2.17 Theorem. *Both $\mathsf{QT}(\cdot)$ and $\mathsf{T}(\cdot)$ are contravariant functors from the category of C*-algebras with *-homomorphisms to the category of extended compact Hausdorff cones with continuous linear maps.*

Proof. The functoriality of $\mathsf{QT}(\cdot)$ and $\mathsf{T}(\cdot)$ is obvious; the rest follows immediately from Theorem 5.2.16. ■

The topology on $\mathsf{QT}(A)$ is closely related to the topology of pointwise convergence. Recall that for a C*-algebra A, we use A_{min} to denote the Pedersen ideal of A.

5.2.18 Theorem. *Let A be any C*-algebra.*

(i) *The topology of $\mathsf{QT}(A)$ is coarser than the topology of pointwise convergence on the positive elements of $(A \otimes \mathcal{K})_{min}$.*

(ii) *The relative topology on the subset of semifinite quasitraces in $\mathsf{QT}(A)$ is identical to the topology of pointwise convergence on the positive elements of $(A \otimes \mathcal{K})_{min}$.*

(iii) *The relative topology on the subset of semifinite traces in $\mathsf{T}(A)$ is identical to the topology of pointwise convergence on the positive elements of A_{min}.*

(iv) *The relative topology on the subset $\mathsf{T}_1(A)$ of normalised traces is identical to the topology of pointwise convergence on A_+.*

Proof.

(i) Let $\tau \in \mathsf{QT}(A)$, and let $(\tau_\nu)_\nu$ be a net in $\mathsf{QT}(A)$ such that $\lim_\nu \tau_\nu(a) = \tau(a)$ for every positive element a in $(A \otimes \mathcal{K})_{min}$. If b is any positive element in $A \otimes \mathcal{K}$, then $(b-\varepsilon)_+$ is in the Pedersen ideal for every $\varepsilon > 0$. Using Theorem 5.2.7 (ii), we find that $\limsup_\nu \tau_\nu((b-\varepsilon)_+) = \tau((b-\varepsilon)_+) \leq \tau(b)$. Moreover, for each $\varepsilon > 0$ we have $\tau((b-\varepsilon)_+) = \liminf_\nu \tau_\nu((b-\varepsilon)_+) \leq \liminf_\nu \tau_\nu(b)$, again using Theorem 5.2.7 (ii). Hence, we get $\tau(b) = \sup_{\varepsilon>0} \tau((b-\varepsilon)_+) \leq \liminf_\nu \tau_\nu(b)$ since τ is lower semicontinuous. Together, these inequalities imply that $\lim_\nu \tau_\nu = \tau$ in the sense of Definition 5.2.15.

(ii) This was proven in [15], Proposition 3.10, for the cone $\mathsf{T}(A)$. The proof for $\mathsf{QT}(A)$ is similar, but slightly more complicated. Let $(\tau_\iota)_\iota$ be a net of semifinite quasitraces in $\mathsf{QT}(A)$, and let τ be any semifinite quasitrace in $\mathsf{QT}(A)$. We need to show that if $(\tau_\iota)_\iota$ converges towards τ in the sense of Definition 5.2.15, then τ_ι converges towards τ pointwise on the positive elements of $(A \otimes \mathcal{K})_{min}$. Let a be any positive element of $(A \otimes \mathcal{K})_{min}$. Inspecting Definition 5.2.15, we find that we need only show that $s := \limsup_\iota \tau_\iota(a) \leq \tau(a)$. We can find a subnet $(\tau_{\iota_\alpha})_\alpha$ such that $\lim_\alpha \tau_{\iota_\alpha}(a) = s$. Since the space $\prod_{x \in (A\otimes\mathcal{K})_+} [0,\infty]$ is compact by Tychonoff's theorem, we may assume that $(\tau_{\iota_\alpha})_\alpha$ converges pointwise towards a function $\sigma\colon (A \otimes \mathcal{K})_+ \to [0,\infty]$ with $\sigma(a) = s$. Since every τ_{ι_α} is an order-preserving and 2-subadditive quasitracial weight, it is easy to see that σ is also an order-preserving and 2-subadditive quasitracial weight. In particular, it follows that $\mathrm{fin}(\sigma)$ is an algebraic ideal of $A\otimes\mathcal{K}$. If b is any positive element of the Pedersen ideal of $A\otimes\mathcal{K}$, then $\tau(b)<\infty$ since τ is semifinite. Moreover, for every $\varepsilon>0$ we find that $\sigma((b-\varepsilon)_+)=\limsup_\alpha \tau_{\iota_\alpha}((b-\varepsilon)_+)\leq \limsup_\iota \tau_\iota((b-\varepsilon)_+)\leq\tau(b)<\infty$ by Definition 5.2.15, since $(\tau_\iota)_\iota$ converges towards τ. It follows that $\overline{\mathrm{fin}}(\sigma)$ contains all positive elements of $(A\otimes\mathcal{K})_{min}$, hence $\overline{\mathrm{fin}}(\sigma)$ contains all of $(A\otimes\mathcal{K})$. But then $\mathrm{fin}(\sigma)$ is a dense ideal of $A\otimes\mathcal{K}$, which implies $(A\otimes\mathcal{K})_{min}\subseteq\mathrm{fin}(\sigma)$. We have $a\in(A\otimes\mathcal{K})_{min}$, so by [3], Theorem II.5.2.4 *(iii)*, the entire C*-algebra $C^*(a)$ is contained in $(A\otimes\mathcal{K})_{min}$. It follows that σ restricts to a finite quasitracial weight on $C^*(a)_+$. Since finite quasitracial weights are automatically continuous by Theorem 5.2.4 *(ii)*, we conclude that σ is continuous on $C^*(a)_+$. By the same calculation as above for b, we know that $\sigma((a-\varepsilon)_+)\leq\tau(a)$ for each $\varepsilon>0$, hence $\sigma(a)=\lim_{\varepsilon\to 0}\sigma((a-\varepsilon)_+)\leq\tau(a)$ by continuity. Thus, we have $\limsup_\iota\tau_\iota(a)=s=\sigma(a)\leq\tau(a)$, which proves the claim.

(iii) Let $\mathcal{F}$ be the algebraic ideal of $\mathcal{K}$ consisting of all finite rank operators. First, we will show that $(A\otimes\mathcal{K})_{min}=A_{min}\odot\mathcal{F}$. Let $(e_{ij})_{ij}$ be any complete system of matrix units for $\mathcal{K}$. It follows from [3], Theorem II.5.2.8, that $\{x\otimes e_{11}\mid x\in A_{min}\}$ is contained in $(A\otimes\mathcal{K})_{min}$. Let a be any positive element of A_{min}. By [3], Theorem II.5.2.4 *(iii)*, we have $C^*(a)\subseteq A_{min}$. For every index pair $i,j\in\mathbb{N}$, we then have $a\otimes e_{ij}=(a^{1/3}\otimes e_{i1})(a^{1/3}\otimes e_{11})(a^{1/3}\otimes e_{1j})\in(A\otimes\mathcal{K})_{min}$. Since A_{min} is the linear span of its positive elements, it follows that $\{x\otimes e_{ij}\mid x\in A_{min}\text{ and } i,j\in\mathbb{N}\}$ is contained in $(A\otimes\mathcal{K})_{min}$. Since every element $x\in A_{min}\odot\mathcal{F}$ is of the form $x=\sum_{ij=1}^n x_{ij}\otimes e_{ij}$ for some complete system of matrix units $(e_{ij})_{ij}$, some $n\in\mathbb{N}$, and some coefficients $a_{ij}\in A_{min}$, it follows that $A_{min}\odot\mathcal{F}$ is contained in $(A\otimes\mathcal{K})_{min}$. Finally, since $A_{min}\odot\mathcal{F}$ is a dense algebraic ideal of $A\otimes\mathcal{K}$, we conclude that $(A\otimes\mathcal{K})_{min}=A_{min}\odot\mathcal{F}$.

Let $(\tau_\iota)_\iota$ be a net of semifinite traces in $\mathsf{T}(A)$, and let τ be a semifinite trace in $\mathsf{T}(A)$. Using part *(ii)*, we need only show that if $(\tau_\iota)_\iota$ converges pointwise towards τ on the positive elements of A_{min}, then the unique extensions $\tilde{\tau}_\iota$ converge pointwise towards the unique extension $\tilde{\tau}$ on every positive element $a\in A_{min}\odot\mathcal{F}$. Let a be any positive element of $A_{min}\odot\mathcal{F}$; we can find a complete system of matrix units $(e_{ij})_{ij}$ for $\mathcal{K}$ and some $n\in\mathbb{N}$ such that $a=\sum_{i,j=1}^n a_{ij}\otimes e_{ij}$ with coefficients $a_{ij}\in A_{min}$. Since elements of $\mathsf{T}(A)$ extend to $(A\otimes\mathcal{K})_+$ in the obvious way, we find that $\tilde{\tau}_\iota(a)=\sum_{i=1}^n\tau_\iota(a_{ii})\to\sum_{i=1}^n\tau(a_{ii})=\tilde{\tau}(a)$. This proves the claim.

(iv) Let $\tau\in\mathsf{T}_1(A)$, and let $(\tau_\nu)_\nu$ be a net in $\mathsf{T}_1(A)$. Using *(iii)*, we need only show that if $\lim_\nu\tau_\nu(a)=\tau(a)$ for every positive $a\in A_{min}$, then $\lim_\nu\tau_\nu(a)=\tau(a)$ for every positive $a\in A$. Let a be any element in A_+, and let $\varepsilon>0$. Pick any $\delta>0$ with $\delta\leq\frac{\varepsilon}{3}$. Since $(a-\delta)_+$ is in the Pedersen ideal of A, we can find ν_0 such that $|\tau((a-\delta)_+)-\tau_\nu((a-\delta)_+)|\leq\frac{\varepsilon}{3}$ for all $\nu\geq\nu_0$. By Theorem 5.2.4 *(iii)*, we also have $|\tau(a)-\tau((a-\delta)_+)|\leq\delta\leq\frac{\varepsilon}{3}$ and $|\tau_\nu((a-\delta)_+)-\tau_\nu(a)|\leq\delta\leq\frac{\varepsilon}{3}$ for every ν. Ap-

plying the triangle inequality, we find that for every $\varepsilon > 0$ there is a ν_0 such that $|\tau(a) - \tau_\nu(a)| \leq \varepsilon$ for all $\nu \geq \nu_0$. Hence, $\lim_\nu \tau_\nu(a) = \tau(a)$ for every $a \in A_+$.

■

In particular, it follows that the space $\mathsf{T}_1(A)$ of normalised traces can be identified with the set of tracial states on A, equipped with the topology of pointwise convergence on A. We also get the following nice corollary for simple C*-algebras:

5.2.19 Corollary. *Let A be a simple C*-algebra.*

(i) *The topology of $\mathsf{QT}(A)$ is the topology of pointwise convergence on the positive elements of $(A \otimes \mathcal{K})_{min}$.*

(ii) *The relative topology on $\mathsf{T}(A)$ is the topology of pointwise convergence on the positive elements of A_{min}.*

(iii) *If A is unital, then the relative topology on $\mathsf{T}(A)$ is the topology of pointwise convergence on the positive elements of A.*

Proof.

(i) By Theorem 5.2.11 (i), we know that every quasitrace in $\mathsf{QT}(A) \setminus \{\tau_\infty\}$ is semifinite. By Theorem 5.2.18 (ii), it follows that the relative topology on $\mathsf{QT}(A) \setminus \{\tau_\infty\}$ is identical to the topology of pointwise convergence on the positive elements of $(A \otimes \mathcal{K})_{min}$. A straightforward calculation shows that this remains true when the trace τ_∞ is included.

(ii) As in part (i), it follows from Theorem 5.2.11 (i) and Theorem 5.2.18 (iii) that the relative topology on $\mathsf{T}(A) \setminus \{\tau_\infty\}$ is identical to the topology of pointwise convergence on the positive elements of A_{min}. The same straightforward calculation as before shows that this remains true when the trace τ_∞ is included.

(iii) If A is unital, then A_{min} is all of A, so the claim follows immediately from part (ii).

■

Finally, we shall describe the relationship between the lower semicontinuous quasitraces and the functionals on the Cuntz semigroup:

5.2.20 Theorem. *For every $\tau \in \mathsf{QT}(A)$, the map $\lambda_\tau \colon [a] \mapsto \lim_n \tau(a^{1/n})$ is well-defined and an element of $\mathsf{F}(\mathsf{Cu}(A))$. Moreover, the map $\mathsf{QT}(A) \to \mathsf{F}(\mathsf{Cu}(A))$ given by $\tau \mapsto \lambda_\tau$ is a homeomorphism. The inverse homeomorphism is given by $\lambda \mapsto \tau_\lambda$, where the quasitrace τ_λ is defined by $\tau_\lambda(a) := \int_0^\infty \lambda([(a-\varepsilon)_+])\, d\varepsilon$.*

Proof. This was shown in Proposition 4.2 of [15] and Theorem 4.4 of [15]. ■

It is easy to check that this homeomorphism is, moreover, an isomorphism of extended compact Hausdorff cones.

5.2.21 Theorem. *If A is a unital C*-algebra, then $\mathsf{T}_1(A)$ and $\mathsf{QT}_1(A)$ are Choquet simplices.*

Proof. See Definition 5.1.11 for the definition of a Choquet simplex. We prove the claim for $\mathsf{T}_1(A)$ first. Since $\mathsf{T}(A)$ is compact by Theorem 5.2.16, and $\mathsf{T}_1(A)$ is precisely the set of all $\tau \in \mathsf{T}(A)$ with $\tau(\mathbb{I}_A) = 1$, it follows easily from Definition 5.2.15 that $\mathsf{T}_1(A)$ is closed in $\mathsf{T}(A)$, and therefore that $\mathsf{T}_1(A)$ is compact. Let E be the real vector space of all real-valued functions on A_+, equipped with the topology of pointwise convergence. Then $\mathsf{T}_1(A)$ is a regularly embedded convex subset of E. By Theorem 5.2.18 (*iv*), the relative topology of $\mathsf{T}_1(A)$ with regard to E coincides with the relative topology of $\mathsf{T}_1(A)$ with regard to $\mathsf{QT}(A)$, so it follows that $\mathsf{T}_1(A)$ is a compact, convex, and regularly embedded subset of the locally convex, Hausdorff real vector space E. The cone spanned by $\mathsf{T}_1(A)$ in E is precisely the set of all finite traces on A; we need to show that this cone is a lattice when equipped with its algebraic order. By [15], Theorem 3.3, the larger cone $\mathsf{T}(A)$ is a lattice. Since the cone of finite traces is order-hereditary in $\mathsf{T}(A)$, and since any pair τ_1, τ_2 of finite tracial weights has a finite upper bound (namely, the sum $\tau_1 + \tau_2$), it follows that the cone of finite traces on A is itself a lattice. Thus, $\mathsf{T}_1(A)$ is a Choquet simplex. The claim for $\mathsf{QT}_1(A)$ follows easily from Theorem 5.2.20, since the linear homeomorphism between $\mathsf{F}(\mathsf{Cu}(A))$ and $\mathsf{QT}(A)$ restricts to an affine homeomorphism between $\mathsf{F}_{[\mathbb{I}_A]}(\mathsf{Cu}(A))$ and $\mathsf{QT}_1(A)$, and we have shown in Theorem 5.1.12 that $\mathsf{F}_{[\mathbb{I}_A]}(\mathsf{Cu}(A))$ is a Choquet simplex. ∎

It follows that, for any unital C*-algebra A, the induced homeomorphism between $\mathsf{QT}_1(A)$ and $\mathsf{F}_{[1]}(\mathsf{Cu}(A))$ is an isomorphism of Choquet simplices. If A is exact and unital, then the induced homeomorphism between $\mathsf{T}_1(A)$ (i.e. the tracial states on A with the topology of pointwise convergence) and $\mathsf{F}_{[1]}(\mathsf{Cu}(A))$ is an isomorphism of Choquet simplices.

5.3 Recovering the Cuntz semigroup

If A is a particularly nice C*-algebra, then the Cuntz semigroup can be fully recovered from standard invariants. Before we come to that result, a few more definitions are required.

5.3.1 Definition.

(*i*) *If C is an extended compact Hausdorff cone, let $\mathsf{Lsc}_{++}(C)$ be the set of lower semicontinuous, linear functions from C to the compact interval $[0,\infty]$ that take nonzero values at every nonzero element of C (where linear shall mean that a function is homogeneous with respect to scalars in $[0,\infty]$, and that it preserves addition and the zero element). This set is made into an ordered abelian semigroup by equipping it with pointwise addition and pointwise comparison.*

(*ii*) *If K is a Choquet simplex, let $\mathsf{LAff}_{++}(K)$ be the set of lower semicontinuous, affine functions from K to the half-open interval $(0,\infty]$. Again, this set is made into an ordered abelian semigroup by equipping it with pointwise addition and pointwise comparison.*

5.3.2 Proposition. *Let A be a simple and unital C*-algebra.*

(*i*) *Each element of $\mathsf{Lsc}_{++}(\mathsf{QT}(A))$ restricts to an element of $\mathsf{LAff}_{++}(\mathsf{QT}_1(A))$, and each element of $\mathsf{LAff}_{++}(\mathsf{QT}_1(A))$ extends uniquely to an element of $\mathsf{Lsc}_{++}(\mathsf{QT}(A))$. This correspondence induces an isomorphism of ordered abelian semigroups between $\mathsf{Lsc}_{++}(\mathsf{QT}(A))$ and $\mathsf{LAff}_{++}(\mathsf{QT}_1(A))$.*

(*ii*) *Each element of $\mathsf{Lsc}_{++}(\mathsf{T}(A))$ restricts to an element of $\mathsf{LAff}_{++}(\mathsf{T}_1(A))$, and each element of $\mathsf{LAff}_{++}(\mathsf{T}_1(A))$ extends uniquely to an element of $\mathsf{Lsc}_{++}(\mathsf{T}(A))$. This correspondence induces an isomorphism of ordered abelian semigroups between $\mathsf{Lsc}_{++}(\mathsf{T}(A))$ and $\mathsf{LAff}_{++}(\mathsf{T}_1(A))$.*

Proof. By Theorem 5.2.11 (*ii*), the simplex $\mathsf{QT}_1(A)$ is a base for the cone $\mathsf{QT}(A) \setminus \{\tau_\infty\}$. It is clear from this that for every element $f \in \mathsf{LAff}_{++}(\mathsf{QT}_1(A))$, the function $\tilde{f}$ with $\tilde{f}(\tau_0) = 0$, and $\tilde{f}(\tau_\infty) = \infty$, and $\tilde{f}(\alpha\tau) = \alpha f(\tau)$ for every $0 < \alpha < \infty$ and every $\tau \in \mathsf{QT}_1(A)$, is the unique extension of f to an element of $\mathsf{Lsc}_{++}(\mathsf{QT}(A))$. Conversely, it is obvious that every element of $\mathsf{Lsc}_{++}(\mathsf{QT}(A))$ restricts to an element of $\mathsf{LAff}_{++}(\mathsf{QT}_1(A))$, and that this correspondence respects pointwise addition and pointwise comparison. The proof for $\mathsf{T}(A)$ and $\mathsf{T}_1(A)$ proceeds in exactly the same way. ∎

In results concerning simple, unital, exact C*-algebras, it is preferable to use the simplex of tracial states, and consequently, to use LAff_{++}. In more general cases, it is often necessary to use the cones of lower semicontinuous quasitraces, and hence to use Lsc_{++}.

5.3.3 Proposition. *Let A be a C*-algebra and $x \in \mathsf{Cu}(A)$.*

(*i*) *The function $\hat{x}\colon \mathsf{QT}(A) \to [0, \infty]$, given by $\tau \mapsto \lambda_\tau(x)$, is lower semicontinuous.*

(*ii*) *If x is compact, then $\hat{x}$ is continuous.*

(*iii*) *If A is simple and x is nonzero, then $\hat{x} \in \mathsf{Lsc}_{++}(\mathsf{QT}(A))$.*

Proof.

(*i*) Since the map $\tau \mapsto \lambda_\tau$ is a homeomorphism, the statement that $\hat{x}$ is lower semicontinuous follows immediately from Proposition 5.1.13 (*i*).

(*ii*) If x is compact, than the continuity of $\hat{x}$ follows from Proposition 5.1.13 (*ii*).

(*iii*) Since every $\tau \in \mathsf{QT}(A) \setminus \{\tau_0\}$ is faithful, it follows that $\hat{x}$ takes a nonzero value on each nonzero quasitrace. That $\hat{x}$ is linear, homogeneous, and zero-preserving is evident.

∎

We are now ready to state the result that the Cuntz semigroup can, for sufficiently well-behaved C*-algebras, be recovered from $\mathsf{V}(A)$ and $\mathsf{T}(A)$:

5.3.4 Theorem. *Let A be a simple, separable, exact, stably finite C*-algebra with $\mathsf{Cu}(A)$ being almost unperforated and almost divisible. Then $\mathsf{Cu}(A) \cong \mathsf{V}(A) \sqcup \mathsf{Lsc}_{++}(\mathsf{T}(A))$, where compact elements of $\mathsf{Cu}(A)$ are identified with the respective elements of $\mathsf{V}(A)$, noncompact elements $x \in \mathsf{Cu}(A)$ are identified with the lower semicontinuous linear functions $\hat{x} \in \mathsf{Lsc}_{++}(\mathsf{T}(A))$, and the disjoint union $\mathsf{V}(A) \sqcup \mathsf{Lsc}_{++}(\mathsf{T}(A))$ is equipped with the following addition and order:*

- *When restricted to $\mathsf{V}(A)$, both addition and order agree with the usual addition and order on $\mathsf{V}(A)$.*
- *When restricted to $\mathsf{Lsc}_{++}(\mathsf{T}(A))$, both addition and order agree with the usual addition and order on $\mathsf{Lsc}_{++}(\mathsf{T}(A))$.*
- *For $x \in \mathsf{V}(A)$ and $f \in \mathsf{Lsc}_{++}(\mathsf{T}(A))$, addition is defined by $x + f := \hat{x} + f$ and $f + x := \hat{x} + f$, where the sums on the right-hand side are taken in $\mathsf{Lsc}_{++}(\mathsf{T}(A))$.*
- *For $x \in \mathsf{V}(A)$ and $f \in \mathsf{Lsc}_{++}(\mathsf{T}(A))$, the order is defined by $f \leq x$ iff $f(\tau) \leq \hat{x}(\tau)$ for all $\tau \in \mathsf{T}(A)$, and $x \leq f$ iff $\hat{x}(\tau) < f(\tau)$ for all $\tau \in \mathsf{T}(A) \setminus \{\tau_0, \tau_\infty\}$.*

Proof. This is the most complete formulation for the case of separable C*-algebras; similar results also hold in the nonseparable case (see the sources below). A complete proof of this theorem can be found in A. Tikuisis's doctoral thesis ([35], Theorem 2.2.5), but as mentioned there, the result was already well-known at the time. Indeed, many similar but slightly weaker results have been published earlier. The first one was a result by Brown, Perera, and Toms in [9], Corollary 5.8. However, since the creation of that article predates the introduction of $\mathsf{Cu}(A)$ in [11], it was formulated for $\mathsf{W}(A)$ instead; moreover, it only covered unital and $\mathcal{Z}$-stable C*-algebras. Another result, this time for $\mathsf{Cu}(A) = \mathsf{W}(A \otimes \mathcal{K})$ instead of $\mathsf{W}(A)$, was published by Brown and Toms in [10], Theorem 2.5; once again, the result only covered the unital and $\mathcal{Z}$-stable case. The generalisation to nonunital but $\mathcal{Z}$-stable C*-algebras was obtained by Elliott, Robert, and Santiago in [15], Corollary 6.8. Another generalisation to unital but non-$\mathcal{Z}$-stable C*-algebras was published by Ara, Perera, and Toms in [2], Theorem 5.27, but it required the additional condition of A having stable rank one. Tikuisis' doctoral thesis [35] appears to be the first publication that contains a proof of the result in the form stated above. ∎

5.3.5 Lemma. *Let C be an extended compact Hausdorff cone, and let $f, g \in \mathsf{Lsc}_{++}(C)$ such that f is continuous. Then the following conditions are equivalent:*

(i) *$f \ll g$ in $\mathsf{Lsc}_{++}(C)$.*

(ii) *$f(x) \leq g(x)$ for all $x \in C$, and $f(x) < g(x)$ whenever $0 < f(x) < \infty$.*

Proof.
(i) $\implies$ *(ii)*: Obviously, we have $f \leq g$ whenever $f \ll g$. Assume that $0 < f(x) = g(x) < \infty$ for some $x \in C$. For each $n \in \mathbb{N}$, let $g_n := \frac{n}{n+1}\, g$. Clearly, $(g_n)_n$ is an increasing sequence in $\mathsf{Lsc}_{++}(C)$ with $\sup_n g_n = g$. Since $f \ll g$, there is some n with $f \leq g_n$, and therefore $f(x) \leq g_n(x)$. But we have $0 < f(x) = g(x) < \infty$, and therefore $g_n(x) = \frac{n}{n+1}\, g(x) < f(x)$, a

contradiction. It follows that $f(x) < g(x)$ whenever $0 < f(x) < \infty$.
$(ii) \implies (i)$: Let $\mathrm{Set}(f) := \{x \in C \mid f(x) > 1\}$, and $\mathrm{Set}(g) := \{x \in C \mid g(x) > 1\}$. By [15], Proposition 5.1 (ii), it suffices to show that $\overline{\mathrm{Set}(f)} \subseteq \mathrm{Set}(g)$. Let $(x_\nu)_\nu$ be any net in $\mathrm{Set}(f)$ that converges towards x in C. Then $f(x_\nu) > 1$ for every ν, hence $f(x) \geq 1$ since f is continuous. If $f(x) < \infty$, then $g(x) > f(x) \geq 1$ by (ii). If $f(x) = \infty$, then $g(x) = \infty > 1$ since $f \leq g$ by (ii). Either way, we have $x \in \mathrm{Set}(g)$, which completes the proof. ∎

If A is as in Theorem 5.3.4, define a map $\gamma\colon \mathsf{V}(A)^\times \to \mathsf{Lsc}_{++}(\mathsf{T}(A))$ by letting $\gamma(x) := \hat{x}$. Then γ is additive and order-preserving, i.e. γ is a morphism of ordered abelian semigroups. Composing with the isomorphism from Theorem 5.3.4 and extending by $0 \mapsto 0$, we obtain a morphism of ordered abelian monoids $\gamma_A\colon \mathsf{C}(A) \to \mathsf{D}(A)$. Inspection of the order structure of $\mathsf{V}(A) \sqcup \mathsf{Lsc}_{++}(\mathsf{T}(A))$ shows that γ_A is characterised by the property that $\gamma_A(0) = 0$ and $\gamma_A(x) = \max\{y \in \mathsf{Cu}(A) \mid y < x\}$ for $x > 0$. We call an element $\gamma_A(x)$ like that the *predecessor* of the compact element x. It follows from the preceding lemma that the addition and the order structure of $\mathsf{Cu}(A)$ can be fully expressed in terms of the ordered semigroup structure of $\mathsf{C}(A)$ and $\mathsf{D}(A)$ and the morphism γ_A:

5.3.6 Corollary. *Let A be a simple, separable, exact, stably finite C*-algebra with $\mathsf{Cu}(A)$ being almost unperforated and almost divisible. Then the addition and order structure of $\mathsf{Cu}(A) = \mathsf{C}(A) \sqcup \mathsf{D}(A)^\times$ can be described as follows:*

- *When restricted to $\mathsf{C}(A)$, both addition and order agree with the usual addition and order on $\mathsf{C}(A)$.*
- *When restricted to $\mathsf{D}(A)$, both addition and order agree with the usual addition and order on $\mathsf{D}(A)$.*
- *For $x \in \mathsf{C}(A)$ and $y \in \mathsf{D}(A)$, addition is given by $x + y = y + x = \gamma_A(x) +_{\mathsf{D}(A)} y$.*
- *For $x \in \mathsf{C}(A)$ and $y \in \mathsf{D}(A)$, the order is given by $y \leq x$ if and only if $y \leq_{\mathsf{D}(A)} \gamma(x)$, and by $x \leq y$ if and only if $\gamma(x) \ll_{\mathsf{D}(A)} y$.*

Proof. Most of these points are trivial or follow immediately from Theorem 5.3.4. We need only show that for $x \in \mathsf{C}(A)$ and $y \in \mathsf{D}(A)$, we have $x \leq y$ if and only if $\gamma_A(x) \ll_{\mathsf{D}(A)} y$, since this is the only point where our description differs from the one in Theorem 5.3.4. Clearly, it suffices to show this for elements $x \in \mathsf{C}(A)^\times$ and $y \in \mathsf{D}(A)^\times$. Making use of the isomorphism between $\mathsf{Cu}(A)$ and $\mathsf{V}(A) \sqcup \mathsf{Lsc}_{++}(\mathsf{T}(A))$, we need only show that for $x \in \mathsf{V}(A)^\times$ and $f \in \mathsf{Lsc}_{++}(\mathsf{T}(A))$, we have $\hat{x} \ll_{\mathsf{Lsc}_{++}(\mathsf{T}(A))} f$ if and only if $\hat{x}(\tau) < f(\tau)$ for every $\tau \in \mathsf{T}(A) \setminus \{\tau_0, \tau_\infty\}$.
Since x is compact in $\mathsf{Cu}(A)$, it follows from Proposition 5.3.3 (ii) that $\hat{x}$ is continuous. Moreover, x is nonzero, so it follows from Proposition 5.1.9 (i) and (ii) that $0 < \lambda(x) < \infty$ for every nontrivial functional $\lambda \in \mathsf{F}(\mathsf{Cu}(A))$. In total. we find that $\hat{x}$ is continuous, that $0 < \hat{x}(\tau) < \infty$ for every $\tau \in \mathsf{T}(A) \setminus \{\tau_0, \tau_\infty\}$, that $\hat{x}(\tau_0) = f(\tau_0) = 0$, and that $\hat{x}(\tau_\infty) = f(\tau_\infty) = \infty$.

The statement that for all $x \in \mathsf{C}(A)^\times$ and for all $f \in \mathsf{Lsc}_{++}(\mathsf{T}(A))$ we have $\hat{x} \ll_{\mathsf{Lsc}_{++}(\mathsf{T}(A))} f$ if and only if $\hat{x}(\tau) < f(\tau)$ for every $\tau \in \mathsf{T}(A) \setminus \{\tau_0, \tau_\infty\}$ follows immediately from Lemma 5.3.5 now, so the proof is complete. ■

It is this result that we are going to generalise in the following chapters. To do this, we will show that predecessors of compact elements exist in $\mathsf{Cu}(A)$ for a much larger class of C*-algebras A than the one for which Theorem 5.3.4 holds. After analysing the properties of the predecessor map, we will then show that it allows us to fully recover the ordered semigroup $\mathsf{Cu}(A)$ from the compact part $\mathsf{C}(A)$, the noncompact part $\mathsf{D}(A)$, and the predecessor map γ_A, in a way analogous to Corollary 5.3.6. Moreover, we will show that this construction also works for a large class of abstract Cuntz semigroups S in Cu.

6 Predecessors of compact elements

6.1 Decomposability for semigroups in Cu

In the following chapters, we want to develop a way to decompose certain semigroups S in the category Cu into three components from which the semigroup S can be recovered functorially. The main requirement for this to work is that the following condition is satisfied:

6.1.1 Definition. *A semigroup S in Cu is called* decomposable *if there is an additive, zero-preserving, and faithful map $\gamma_S\colon \mathsf{C}(S) \to S$ such that $\gamma_S(x) = \max\{y \in S \mid y < x\}$ for every element $x \in \mathsf{C}(S)^\times$ (where faithful shall mean that $\gamma_S(x) > 0$ whenever $x > 0$). If such a map exists, it is necessarily unique; we then call $\gamma_S(x)$ the* predecessor *of the compact element x, and we call γ_S the* predecessor map *of S.*

Before we begin to prove decomposability for a large class of semigroups in Cu, we shall take a closer look at the implications of this definition.

6.1.2 Proposition. *Let S be a simple and decomposable semigroup in Cu. Then S is stably finite and nonelementary.*

Proof. Assume that S is elementary; then $S \cong \mathcal{E}$ or $S \cong \mathcal{E}_n$ for $1 \leq n < \infty$. Then S has a minimal nonzero element e which is compact. Since there are no nonzero elements below e, a predecessor $\gamma_S(e)$ with $0 < \gamma_S(e) < e$ cannot exist. This contradicts the decomposability of S, so S must indeed be nonelementary. Assume next that S is not stably finite. Then by Proposition 3.1.18, the infinity element ∞ of S is compact. Since γ_S is additive, we have $\gamma_S(\infty) = \gamma_S(\infty + \infty) = \gamma_S(\infty) + \gamma_S(\infty)$. Since S is simple, this is only possible if $\gamma_S(\infty) \in \{0, \infty\}$. So a predecessor $\gamma_S(\infty)$ with $0 < \gamma_S(\infty) < \infty$ cannot exist. Again, this contradicts the decomposability of S, so S must indeed be stably finite. ∎

In order to prove that a semigroup S in Cu is decomposable, it will be useful to have an equivalent formulation of decomposability. To achieve this, we will need the following relation:

6.1.3 Definition. *Let S be a semigroup in Cu and let $x, y \in S$. We shall write $x \lll y$ if there is an element $z \in S$ such that $x \ll z < y$.*

If S is a semigroup in Cu and x is any element of S, then it follows from axiom ($\mathcal{O}2$) and the definition of compact containment that the set $\{y \in S \mid y \ll x\}$ is directed (i.e. for any elements $y_1, y_2 \ll x$ there is an element $z \in S$ such that $y_1 \leq z$, and $y_2 \leq z$, and $z \ll x$).

Moreover, if $x, y, z \in S$ are such that $z \ll x + y$, then it follows from axioms ($\mathcal{O}2$), ($\mathcal{O}3$), and ($\mathcal{O}4$) as well as the definition of compact containment that we can find elements $x', y' \in S$ such that $x' \ll x$, and $y' \ll y$, and $z \leq x' + y'$. In general, neither of these two properties needs to hold for the relation $\lll$. We shall give a name to those semigroups S in Cu for which both conditions do hold:

6.1.4 Definition. *Let S be a semigroup in Cu.*

- *We say that S is* $\lll$-regular *if it satisfies the following conditions:*
  - (i) *For each $x \in S$, the set $\{y \in S \mid y \lll x\}$ is directed.*
  - (ii) *For all $x, y \in S^\times$ and $z \in S$ with $z \lll x + y$, there are elements $x', y' \in S$ such that $x' \lll x$, and $y' \lll y$, and $z \leq x' + y'$.*
- *We say that S is* weakly $\lll$-regular *if it satisfies the following weaker conditions:*
  - (i') *For each $x \in \mathsf{C}(S)^\times$, the set $\{y \in S \mid y \lll x\}$ is directed.*
  - (ii') *For all $x, y \in \mathsf{C}(S)^\times$ and $z \in S$ with $z \lll x + y$, there are elements $x', y' \in S$ such that $x' \lll x$, and $y' \lll y$, and $z \leq x' + y'$.*

We will require the following technical lemma:

6.1.5 Lemma. *Let S be a semigroup in Cu.*

(i) *If $x', x \in S$ with x noncompact, then $x' \lll x$ if and only if $x' \ll x$.*

(ii) *If $x', x \in S$ with $x' \lll x$, then there is an element $y \in S^\times$ such that $x' + y \ll x$.*

(iii) *If S is simple and $x, y \in S^\times$ such that x is compact and finite while y is noncompact, then for every $y' \ll y$ there is some $x' \lll x$ such that $x + y' \leq x' + y$.*

Proof.

(i) Obviously, $x' \lll x$ implies $x' \ll x$. For the other direction, let $x' \ll x$ and use Lemma 3.1.2 (ii) to find an element $y \in S$ with $x' \ll y \ll x$. If y was equal to x, then x would be compact, which it is not. Hence we have $y < x$, and therefore $x' \lll x$.

(ii) Since $x' \lll x$, there is some element $z \in S$ with $x' \ll z < x$. By Lemma 3.1.2 (ii), we can find $z' \in S$ such that $x' \ll z' \ll z < x$. Using the almost algebraic order of S, we can find an element $v \in S$ such that $z' + v \leq x \leq z + v$. Then $v > 0$, since otherwise we would have $x \leq z$, contradicting $z < x$. Using Lemma 3.1.2 (i), we can find another element $y \in S$ with $0 < y \ll v$. By axiom ($\mathcal{O}3$), we have $x' + y \ll z' + v$. Since we have $z' + v \leq x$ by our choice of v, it follows that $x' + y \ll x$ with $y > 0$.

(iii) This is trivial if $S = \mathcal{E}$, since y must then be the element ∞, and we can choose $x' := 0$. Moreover, S cannot be one of the elementary semigroups $\mathcal{E}_n$ for $1 \leq n < \infty$, since these semigroups do not contain any noncompact elements. We may therefore assume from here on that S is nonelementary. By the halving theorem (Theorem 3.1.25), we can find an element $h \in S^\times$ with $h + h \leq x$. If h were equal to x, then we would have $x + x = x$ for a finite and nonzero x, which cannot happen; hence we have $0 < h < x$. Pick elements $y'', y''' \in S$ such that $y' \ll y'' \ll y''' \ll y$. Using the almost algebraic order of S, we can find an element $e \in S$ such that $y'' + e \leq y \leq y''' + e$. If e was zero, then it would follow that $y \leq y''' \ll y$, contradicting the fact that y is noncompact; hence $e > 0$. Next, use downwards directedness of S (Theorem 3.1.24) to find an element $d \in S$ with $0 < d \leq e, h$. In particular, $d \leq e$ and $d < x$ (since $h < x$). By Lemma 3.1.2 (i), we can pick an element $d' \in S$ with $0 < d' \ll d$, and using the almost algebraic order of S, we can find an element $z \in S$ such that $d' + z \leq x \leq d + z$. If z was zero, then $x \leq d$ would follow, contradicting $d < x$. If z was equal to x, then we would have $d' + x \leq x$ for a finite x and a nonzero d', which cannot happen. Hence, we have $0 < z < x$. It follows that $x + y'' \leq z + d + y'' \leq z + e + y'' \leq z + y$. Since x is compact and $y' \ll y''$, we have $x + y' \ll x + y'' \leq y + z$, so by axioms $(\mathcal{O}2)$ and $(\mathcal{O}4)$ there is $x' \in S$ with $0 < x' \ll z$ and $x + y' \leq y + x'$. Since $z < x$, we have $x' \lll x$.

■

We are now able to prove an alternate characterisation of decomposability for simple and separable semigroups:

6.1.6 Theorem. *Let S be a simple and separable semigroup in* Cu. *Then the following three conditions are equivalent:*

(i) *S is decomposable.*

(ii) *S is stably finite and $\lll$-regular.*

(iii) *S is stably finite and weakly $\lll$-regular.*

Proof.
$(i) \implies (ii)$: First, we have already shown in Proposition 6.1.2 that S must be stably finite. Second, we need to prove that condition (i) of Definition 6.1.4 is satisfied. Let $x \in S$; we want to show that the set $\{y \mid y \lll x\}$ is directed. We need to consider three cases. If $x = 0$; then this set is empty and therefore directed. For the other cases, let $y_1, y_2 \in S$ with $y_1 \lll x$ and $y_2 \lll x$. If $x \in \mathsf{C}(S)^\times$, then we have $\gamma_S(x) = \max\{y \in S \mid y < x\}$, so it follows that $y_1 \ll \gamma_S(x)$ and $y_2 \ll \gamma_S(x)$. Using axiom $(\mathcal{O}2)$, we can find an element $z \in S$ such that $y_1 \leq z$, and $y_2 \leq z$, and $z \ll \gamma_S(x)$. Since $\gamma_S(x) < x$, it follows that $z \lll x$; this shows that $\{y \mid y \lll x\}$ is directed. Only the case where $x \in \mathsf{D}(S)^\times$ remains. We have $y_1, y_2 \lll x$, so in particular we have $y_1, y_2 \ll x$. Using axiom $(\mathcal{O}2)$, we can find elements $z, w \in S$ such that $y_1, y_2 \leq z \ll w \ll x$. We have $w < x$ since x is noncompact, and hence we have $y_1, y_2 \leq z \lll x$. Thus, the set $\{y \mid y \lll x\}$ is directed for every $x \in S$. In total, we have shown that condition (i) of Definition 6.1.4 is always satisfied.
Third, we need to prove that condition (ii) of Definition 6.1.4 is also satisfied. Let $x, y \in S^\times$ and $z \in S$ with $z \lll x + y$. Again, we have to consider three cases. If x, y are both in $\mathsf{C}(S)^\times$, then we have $z \ll \gamma_S(x+y) = \gamma_S(x) + \gamma_S(y)$. Using axioms $(\mathcal{O}2)$, $(\mathcal{O}3)$, and $(\mathcal{O}4)$, we can find elements $x' \ll \gamma_S(x)$ and $y' \ll \gamma_S(y)$ such that $z \leq x' + y'$. Since $\gamma_S(x) < x$ and $\gamma_S(y) < y$, it

follows that $x' \lll x$ and $y' \lll y$, which is what we wanted to show. Next, consider the case where x and y are both in $\mathsf{D}(S)^\times$. Since $z \ll x+y$, we can use axioms ($\mathcal{O}2$), ($\mathcal{O}3$), and ($\mathcal{O}4$) to find elements $x', y' \in S$ such that $x' \ll x$, and $y' \ll y$, and $z \leq x' + y'$. By Lemma 6.1.5 (i), it follows that $x' \lll x$ and $y' \lll y$ as required. Finally, consider the case where one of x and y is in $\mathsf{C}(S)^\times$ and the other is in $\mathsf{D}(S)^\times$. Without loss of generality, we will assume that x is compact while y is not. Since $z \ll x+y$ with x compact, we can find an element $y_0 \in S$ such that $y_0 \ll y$ and $z \ll x + y_0$ using axioms ($\mathcal{O}2$), ($\mathcal{O}3$), and ($\mathcal{O}4$). The compact element x is finite since S is stably finite, so using Lemma 6.1.5 (iii), we can now find an element $x_0 \in S$ such that $x_0 \lll x$ and $x + y_0 \leq x_0 + y$. It follows that $z \ll x_0 + y$; again using axioms ($\mathcal{O}2$), ($\mathcal{O}3$), and ($\mathcal{O}4$), we can now find elements $x', y' \in \mathsf{Cu}(A)$ such that $x' \ll x_0$, and $y' \ll y$, and $z \leq x' + y'$. Since $x' \ll x_0 \lll x$, we have $x' \lll x$. Since y is noncompact and $y' \ll y$, we have $y' \lll y$ by Lemma 6.1.5 (i). Hence, x' and y' are as required. In total, we have shown that condition (ii) of Definition 6.1.4 is always satisfied.

$(ii) \implies (iii)$: This is perfectly obvious.

$(iii) \implies (i)$: Since S is separable and $\{y \in S \mid y \lll x\}$ is directed for every $x \in \mathsf{C}(S)^\times$, it follows from Proposition 3.1.14 that the set $\{y \in S \mid y \lll x\}$ contains an almost cofinal sequence, and therefore has a supremum, for every $x \in \mathsf{C}(S)^\times$. We can therefore define a map $\gamma_S \colon \mathsf{C}(S) \to S$ by letting $\gamma_S(x) := \sup\{y \in S \mid y \lll x\}$ for $x > 0$ and $\gamma_S(0) := 0$. We need to show that $\gamma_S(x) = \max\{z \in S \mid z < x\}$ for every $x \in \mathsf{C}(S)^\times$, that $\gamma_S(x) > 0$ for every $x \in \mathsf{C}(S)^\times$, and that $\gamma_S(x+y) = \gamma_S(x) + \gamma_S(y)$ for all $x, y \in \mathsf{C}(S)^\times$.

For the first claim, let $(z_n)_n$ be an almost cofinal sequence in $\{z \in S \mid z \lll x\}$; then it follows that $\gamma_S(x) = \sup_n z_n$. We have $z_n \lll x$ and therefore $z_n < x$ for all n; this implies that $\gamma_S(x) \leq x$. Moreover, for $\gamma_S(x)$ to be equal to x, the members of $(z_n)_n$ would have to be eventually equal to x, since x is compact. Since they are not, it follows that $\gamma_S(x) < x$. Now, let $z \in S$ be any element with $z < x$. For every $z' \ll z$, we have $z' \lll x$, and therefore $z' \leq \gamma_S(x)$. By Rørdam's proposition, this implies $z \leq \gamma_S(x)$. It follows that $\gamma_S(x) = \max\{z \in S \mid z < x\}$.

For the second claim, we first note that the semigroup S cannot be elementary, since none of the elementary semigroups satisfy condition (ii') of Definition 6.1.4 – to see this, let $z := 1$, $x := 1$, and $y := 1$; then $z \lll x + y$, but the only elements x', y' with $x' \lll x$ and $y' \lll y$ are $x' = y' = 0$, and these do not satisfy $z \leq x' + y'$. Thus, the semigroup S is simple and nonelementary. Since $x > 0$, we can use the halving theorem (Theorem 3.1.25) to find an element $h \in S$ with $h > 0$ and $h + h \leq x$. If h was equal to x, then it would follow that $x = x + x$, contradicting the fact that S is stably finite. Thus, we have $0 < h < x$. Since we already know that $\gamma_S(x) = \max\{z \in S \mid z < x\}$, it follows that $\gamma_S(x) \geq h > 0$.

For the third claim, observe that $\gamma_S(x+y) \leq \gamma_S(x) + \gamma_S(y)$ follows from condition (ii') of Definition 6.1.4. It remains to show that $\gamma_S(x) + \gamma_S(y) \leq \gamma_S(x+y)$. Let $(v_n)_n$ be an almost cofinal sequence in $\{z \in S \mid z \lll x\}$, and let $(w_n)_n$ be an almost cofinal sequence in $\{z \in S \mid z \lll y\}$. Then $\gamma_S(x) + \gamma_S(y) = \sup_n (v_n + w_n)$ by axiom ($\mathcal{O}4$). We have $v_n \lll x$

and $w_n \lll y$ for every n. Using Lemma 6.1.5 (*ii*), we can find a sequence $(r_n)_n$ in $S^\times$ such that $w_n + r_n \leq y$, and therefore $v_n + w_n + r_n \leq x + y$, for every n. Since $x + y$ is compact and S is stably finite, it follows that $v_n + w_n < x + y$ for every n. This implies that $\gamma_S(x) + \gamma_S(y) \leq x + y$; in fact, it implies that $\gamma_S(x) + \gamma_S(y) < x + y$, since for $\gamma_S(x) + \gamma_S(y)$ to be equal to the compact element $x + y$, the members of the increasing sequence $v_n + w_n$ would have to be eventually equal to $x + y$, which is not the case. Since we have already shown that $\gamma_S(x + y) = \max\{z \in S \mid z < x + y\}$, it follows that $\gamma_S(x) + \gamma_S(y) \leq \gamma_S(x + y)$, and therefore $\gamma_S(x + y) = \gamma_S(x) + \gamma_S(y)$, as we wanted to show. This completes the proof. ■

Our next goal is to show that two large classes of semigroups in Cu are decomposable. The first class consists of all semigroups $\mathsf{Cu}(A)$ where A is a simple, separable, nonelementary, and stably finite C*-algebra. The second class consists of all the semigroups $S \in \mathsf{Cu}$ that are simple, separable, nonelementary, and weakly cancellative. Neither of these two classes is more general than the other, and S being stably finite rather than weakly cancellative does not suffice for our proof (but it might yet turn out to be sufficient; the author has not been able to find a counterexample that shows it is not). We will treat the case of concrete semigroups $\mathsf{Cu}(A)$ first, and the case of abstract semigroups $S \in \mathsf{Cu}$ afterwards. As usual for our notation, we prefer to write γ_A instead of $\gamma_{\mathsf{Cu}(A)}$ when dealing with the Cuntz semigroup of a C*-algebra.

6.2 Predecessors in concrete Cuntz semigroups

The following theorem was the starting point of this dissertation; it is the result of a problem session at the American Institute of Mathematics Research Conference Center during a workshop on the Cuntz semigroup in 2009. The problem was proposed by J. Cuntz, and several participants of that session, most notably N. C. Phillips, were involved in finding the proof.

6.2.1 Theorem. *Let A be a simple and separable C*-algebra. For every compact and finite $x \in \mathsf{Cu}(A)^\times$, there is an element $\gamma_A(x) \in \mathsf{Cu}(A)$ such that $\gamma_A(x) = \max\{y \in \mathsf{Cu}(A) \mid y < x\}$. If A is nonelementary, then $\gamma_A(x)$ is noncompact (and therefore nonzero).*

Proof. If A is elementary, then $\mathsf{Cu}(A) = \mathcal{E}$, and the statement follows immediately; we may therefore assume that the C*-algebra A is simple, separable, and nonelementary. We may also assume, without loss of generality, that A is stable. Let $x \in \mathsf{Cu}(A)$ be compact and nonzero; then we can find a nonzero projection $p \in \mathcal{P}(A)$ such that $x = [p]$. The corner pAp is a simple, separable, and nonelementary C*-algebra by Lemma 4.4.1. Let φ be a pure state on pAp, and let $e \in (pAp)_+$ be a strictly positive element of the hereditary kernel $\mathrm{hk}(\varphi) := \{a \in pAp \mid \varphi(a^*a) = \varphi(aa^*) = 0\}$. Let $\gamma_A(x) := [e]$. Clearly, we have $\gamma_A(x) \leq x$. Let $z \in \mathsf{Cu}(A)$ be any element such that $z < x$. We shall show that $z \leq \gamma_A(x)$. Let $a \in A_+$

be such that $z = [a]$, and let $\varepsilon > 0$. Then we can find $b \in (pAp)_+$ such that $(a-\varepsilon)_+ \sim b$. The element b cannot be invertible in pAp, since otherwise it would be strictly positive and hence equivalent to p, and from $p \sim b \sim (a-\varepsilon)_+ \precsim a \precsim p$ it would follow that $a \sim p$, hence $z = x$, which contradicts $z < x$. But since b is noninvertible in pAp, it follows that $0 \in \sigma_{pAp}(b)$ and therefore $0 \in \sigma_{pAp}(b^2)$, so we can find a pure state ψ on pAp such that $\psi(b^2) = 0$. According to a theorem by Kishimoto, Ozawa, and Sakai ([23], Theorem 1.1), there is an approximately inner automorphism $\alpha \in \mathrm{Aut}(pAp)$ such that $\psi = \varphi \circ \alpha$. It follows that $\varphi(\alpha(b)^2) = 0$, hence $\alpha(b) \in \mathrm{hk}(\varphi) = \overline{eAe}$ and therefore $\alpha(b) \precsim e$. Since α is approximately inner, it follows that $(a-\varepsilon)_+ \sim b \sim \alpha(b) \precsim e$. Since this is true for every $\varepsilon > 0$, it follows from Rørdam's proposition (Proposition 2.2.4) that $a \precsim e$; hence $z \leq \gamma_A(x)$. Next, we want to show that 0 is a cluster point of the spectrum $\sigma(e)$. Let us assume that this is not so. Then we can find a projection $q \in C^*(e)$ and a scalar $C > 0$ such that $e \leq Cq$, which implies that q is also strictly positive in $\mathrm{hk}(\varphi)$, so $\mathrm{hk}(\varphi) = qAq$. Let $(\pi, \mathcal{H}, \xi)$ be the irreducible representation associated to the pure state φ. Since $(\pi, \mathcal{H})$ is irreducible, it follows from Kadison's transitivity theorem ([24], Theorem 5.2.2) that $\pi(q)$ is the projection onto the subspace $\{\xi\}^\perp$. But then $\pi(p-q)$ is a projection of rank one; since π is irreducible, we have $\mathcal{K}(\mathcal{H}) \subseteq \pi(pAp)$ by [24], Theorem 2.4.9. Since pAp is simple, it follows that $pAp \cong \mathcal{K}(\mathcal{H})$, which contradicts the fact that pAp is nonelementary. Hence, the assumption was wrong and 0 is indeed a cluster point of $\sigma(e)$. But since pAp is simple and $[e]$ is dominated by the finite element x, it is itself finite by Proposition 3.1.17, so it follows from Lemma 4.5.2 that $\gamma_A(x) = [e]$ is noncompact, and in particular that $\gamma_A(x) < x$. This completes the proof. ■

The element $\gamma_A(x)$ is the already mentioned *predecessor* of the compact element x. We have yet to show that the map γ_A as defined in the theorem above (and with $\gamma_A(0) := 0$) is additive. For this, we will need the following lemma, due to the author.

6.2.2 Lemma. *Let A be a simple, separable, and nonelementary C*-algebra, and let $p \in A \otimes \mathcal{K}$ be a finite, nonzero projection. Let $(\pi, \mathcal{H})$ be any irreducible representation of $p(A \otimes \mathcal{K})p$, and let $Q \in \mathcal{B}(\mathcal{H})$ be a projection of finite rank. The set*

$$her(\pi, \mathcal{H}, Q) := \{x \in p(A \otimes \mathcal{K})p \mid \pi(x^*x)Q = \pi(xx^*)Q = 0\}$$

is a hereditary sub-C-algebra of $p(A \otimes \mathcal{K})p$, and if a is any strictly positive element of $her(\pi, \mathcal{H}, Q)$, then $[a] = \gamma_A([p])$.*

Proof. It is easy to see that $\mathrm{her}(\pi, \mathcal{H}, Q)$ is a hereditary sub-C*-algebra. Let $\xi_1, ..., \xi_n$ be an orthonormal base for the subspace $Q\mathcal{H}$. Let φ be the pure state of the corner $p(A \otimes \mathcal{K})p$ associated to the cyclic representation $(\pi, \mathcal{H}, \xi_1)$. Let e be a strictly positive element of $\mathrm{hk}(\varphi)$; then we have $[e] = \gamma_A([p])$. By construction, $a \in \mathrm{hk}(\varphi)$ and therefore $a \precsim e$. It is sufficient to show that $(e-\varepsilon)_+ \precsim a$ for any $\varepsilon > 0$; it then follows from Rørdam's proposition that $e \precsim a$. Thus, fix any $\varepsilon > 0$. Since A is nonelementary, the element $[e] = \gamma_A([p])$ is noncompact;

since it is dominated by the finite element $[p]$, the element $[e]$ is finite itself by Proposition 3.1.17, so it follows from Lemma 4.5.2 that 0 is a cluster point of the spectrum $\sigma(e)$. This implies in turn that the operator $\pi((e-\varepsilon)_+)$ has an infinite-dimensional kernel. Pick any family $\eta_1, ..., \eta_n$ of orthogonal unit vectors in $\ker(\pi((e-\varepsilon)_+))$. Let $U \in \mathcal{U}(\mathcal{H})$ be any unitary operator that maps ξ_i to η_i for $1 \le i \le n$. By Kadison's transitivity theorem ([24], Theorem 5.2.2), we can find a unitary element $u \in \mathcal{U}(p(A \otimes \mathcal{K})p)$ such that $\pi(u)\xi_i = U\xi_i = \eta_i$ for $1 \le i \le n$. Let $d := (u^*(e-\varepsilon)_+ u)^{1/2}$, then $d \sim (e-\varepsilon)_+$. By construction, we have

$$\pi(d^2)Q \;=\; \pi(u)^*\pi((e-\varepsilon)_+)\pi(u)Q \;=\; \pi(u)^*\pi((e-\varepsilon)_+)UQ \;=\; 0,$$

so that $d \in \mathrm{her}(\pi, \mathcal{H}, Q)$ and therefore $d \precsim a$. It follows that $(e-\varepsilon)_+ \sim d \precsim a$, which completes the proof. ∎

Using this lemma, we can prove the additivity of the predecessor map:

6.2.3 Theorem. *Let A be a simple, separable, and nonelementary C*-algebra. If x, y are compact, nonzero elements of $\mathsf{Cu}(A)$ such that $x+y$ is finite, then $\gamma_A(x+y) = \gamma_A(x) + \gamma_A(y)$.*

Proof. We may assume, without loss of generality, that A is stable. Since $x+y$ is finite, so are x and y. We can find nonzero orthogonal projections $p, q \in A$ such that $x = [p]$ and $y = [q]$. Let $(\pi, \mathcal{H})$ be any irreducible representation of the hereditary sub-C*-algebra A_{p+q}. This (and every other) hereditary sub-C*-algebra is simple, separable, and nonelementary by Lemma 4.4.1. Moreover, A_{p+q} is finite and unital. Let $\mathcal{H}_p := \pi(p)\mathcal{H}$ and $\mathcal{H}_q := \pi(q)\mathcal{H}$. We shall identify the Hilbert space $\mathcal{H}$ with the direct sum $\mathcal{H}_p \oplus \mathcal{H}_q$, and the corner A_{p+q} with the corresponding algebra of 2×2 - matrices:

$$\begin{aligned}
\mathcal{H} &= \left\{ \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \;\middle|\; \xi_1 \in \mathcal{H}_p, \quad \xi_2 \in \mathcal{H}_q \right\}, \\
A_{p+q} &= \left\{ \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \;\middle|\; x_{11} \in pAp, \quad x_{21} \in qAp, \quad x_{12} \in pAq, \quad x_{22} \in qAq \right\}.
\end{aligned}$$

Let $(\pi_p, \mathcal{H}_p)$, $(\pi_q, \mathcal{H}_q)$ be the restrictions of $(\pi, \mathcal{H})$ to pAp, qAq respectively; then $(\pi_p, \mathcal{H}_p)$ and $(\pi_q, \mathcal{H}_q)$ are also irreducible representations. Pick unit vectors $\xi_p \in \mathcal{H}_p$ and $\xi_q \in \mathcal{H}_q$. Let φ_p be the pure state on pAp associated to the cyclic representation $(\pi_p, \mathcal{H}_p, \xi_p)$, and let φ_q be the pure state on qAq associated to the cyclic representation $(\pi_q, \mathcal{H}_q, \xi_q)$. Let a be a strictly positive element of $\mathrm{hk}(\varphi_p)$, and let b be a strictly positive element of $\mathrm{hk}(\varphi_q)$; note that a, b are orthogonal elements. Moreover, let $Q \in \mathcal{B}(\mathcal{H})$ be the projection onto the two-dimensional subspace spanned by ξ_p and ξ_q. Let c be a strictly positive element of $B := \mathrm{her}(\pi, \mathcal{H}, Q)$. Then $[a] = \gamma_A(x)$, $[b] = \gamma_A(y)$, and $[c] = \gamma_A(x+y)$ by Lemma 6.2.2. We need to show that $c \sim a+b$; it is sufficient for that to show that $B = A_{a+b}$. By definition, we know that $\pi_p(a)\xi_p = 0$ and $\pi_q(b)\xi_q = 0$; it follows immediately that $a+b \in \mathrm{her}(\pi, \mathcal{H}, Q) = B$, and therefore $A_{a+b} \subseteq B$. The proof of the other inclusion is a bit more involved.

Let x be any element of B_+. Note that we have $x \in B$ if and only if $(x)^{1/2} \in B$, since B is a C*-algebra, and so is closed under taking squares and square roots. Let $x = \begin{bmatrix} x_{11} & x_{12} \\ x_{12}^* & x_{22} \end{bmatrix}$. Then it follows from the definition of B that

$$\pi\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{12}^* & x_{22} \end{bmatrix}\right)\begin{bmatrix} \xi_p \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \pi\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{12}^* & x_{22} \end{bmatrix}\right)\begin{bmatrix} 0 \\ \xi_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

from which it subsequently follows that

$$\begin{array}{llll} (i) & \pi\left(\begin{bmatrix} x_{11} & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} \xi_p \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & (ii) & \pi\left(\begin{bmatrix} 0 & x_{12} \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 \\ \xi_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ (iii) & \pi\left(\begin{bmatrix} 0 & 0 \\ x_{12}^* & 0 \end{bmatrix}\right)\begin{bmatrix} \xi_p \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & (iv) & \pi\left(\begin{bmatrix} 0 & 0 \\ 0 & x_{22} \end{bmatrix}\right)\begin{bmatrix} 0 \\ \xi_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{array}$$

From (i) it follows immediately that $\pi_p(x_{11})\xi_p = 0$. Since x is positive, x_{11} is self-adjoint, and it follows that $x_{11} \in \mathrm{hk}(\varphi_p) = \overline{aAa}$. From (iv) it follows immediately that $\pi_q(x_{22})\xi_q = 0$. Again, since x_{22} is self-adjoint, we have $x_{22} \in \mathrm{hk}(\varphi_q) = \overline{bAb}$. From (ii), we get that

$$\pi\left(\begin{bmatrix} 0 & 0 \\ 0 & x_{12}^* x_{12} \end{bmatrix}\right)\begin{bmatrix} 0 \\ \xi_q \end{bmatrix} = \pi\left(\begin{bmatrix} 0 & 0 \\ x_{12}^* & 0 \end{bmatrix}\right)\pi\left(\begin{bmatrix} 0 & x_{12} \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 \\ \xi_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and therefore $\pi_q(x_{12}^* x_{12})\xi_q = 0$, which implies that $x_{12}^* x_{12} \in \mathrm{hk}(\varphi_q) = \overline{bAb}$. Since $\{b^{1/n}\}_n$ is an approximate identity for $\overline{bAb}$, it follows that

$$\begin{aligned} \left\| x_{12} - x_{12} b^{1/n} \right\|^2 &= \left\| \left(x_{12} - x_{12} b^{1/n} \right)^* \left(x_{12} - x_{12} b^{1/n} \right) \right\| \\ &= \left\| x_{12}^* x_{12} - x_{12}^* x_{12} b^{1/n} - b^{1/n} x_{12}^* x_{12} + b^{1/n} x_{12}^* x_{12} b^{1/n} \right\| \to 0. \end{aligned}$$

Thus, we find that $x_{12} = \lim_n x_{12} b^{1/n}$. Likewise, it follows from (iii) that $x_{12}^* = \lim_n x_{12}^* a^{1/n}$, and these identities imply that we have $x_{12} \in \overline{aAb}$. Hence we have shown that

$$x = \begin{bmatrix} x_{11} & x_{12} \\ x_{12}^* & x_{22} \end{bmatrix} \in \begin{bmatrix} \overline{aAa} & \overline{aAb} \\ \overline{bAa} & \overline{bAb} \end{bmatrix} = A_{a+b}.$$

Since this holds for every $x \in B_+$, and since the span of B_+ is all of B, we have $B \subseteq A_{a+b}$ and therefore $B = A_{a+b}$. It follows that $c \sim a + b$, and therefore $\gamma_A(x + y) = \gamma_A(x) + \gamma_A(y)$. ■

6.2.4 Corollary. *Let A be a simple, separable, nonelementary, and stably finite C*-algebra. Then $\mathsf{Cu}(A)$ is decomposable (and therefore $\lll$-regular).*

Proof. This follows immediately from Theorem 6.2.1 and Theorem 6.2.3. ■

6.3 Predecessors in abstract Cuntz semigroups

Our next goal is to prove decomposability for the abstract class of semigroups mentioned above. We require another useful lemma before we can proceed:

6.3.1 Lemma. *Let $S \in \mathsf{Cu}$ be simple and nonelementary, and let $x, y \in S$ be finite elements with $x \lll y$. Then there is a noncompact element $z \in S$ with $x \ll z < y$.*

Proof. Note that $x \lll y$ entails that y is nonzero. Moreover, it ensures that we can find an element $v \in S$ with $x \ll v < y$. Using Lemma 3.1.2 (ii), we can then find $v' \in S$ with $x \ll v' \ll v < y$. Using axiom ($\mathcal{O}$5), the almost algebraic order of S, we can find an approximate difference $d \in S$ such that $v' + d \leq y \leq v + d$. Since $v < y$, we necessarily have $d > 0$. Moreover, d is finite by Proposition 3.1.17, since it is dominated by the finite element y. Using the halving theorem (Theorem 3.1.25), we can find $s_0 \in S$ such that $0 < s_0$ and $s_0 + s_0 \leq d$. If $v' + s_0$ was equal to y, we would have $y + s_0 = v' + s_0 + s_0 \leq v' + d \leq y$, which cannot happen since y is finite and s_0 is nonzero. Thus, we have $v' + s_0 < y$. Using the halving theorem iteratively, we can construct a sequence $(s_n)_n$ in S such that $0 < s_n$ and $s_n + s_n \leq s_{n-1}$ for all $n \geq 1$. Let $z_n := v' + \sum_{i=1}^n s_i$. By construction, we have $z_{n+1} = z_n + s_{n+1} \leq v' + s_0$ for all $n \in \mathbb{N}$, so $(z_n)_n$ is an increasing sequence of elements above v' and below $v' + s_0$. In particular, since $v' + s_0 < y$, every z_n is finite by Proposition 3.1.17 since it is dominated by the finite element y. From this it follows that $z_n < z_n + s_{n+1} = z_{n+1}$ since s_{n+1} is always nonzero, so the sequence $(z_n)_n$ is strictly increasing. Let $z := \sup_n z_n$. Being the supremum of a strictly increasing sequence, the element z must be noncompact. By construction, we have $x \ll v' \leq z \leq v' + s_0 < y$, and therefore $x \ll z < y$ as required. ■

We now have all the tools we need to show that every simple, nonelementary, and weakly cancellative semigroup in Cu is decomposable:

6.3.2 Theorem. *Let S be a simple, separable, nonelementary, and weakly cancellative semigroup in Cu. Then S is weakly $\lll$-regular, and therefore decomposable.*

Proof. Note that S is stably finite by Proposition 3.1.19, since S is simple and weakly cancellative. First, we show that condition (i') of Definition 6.1.4 holds. Let $y_1, y_2 \in S$ and $x \in \mathsf{C}(S)^\times$ such that $y_1, y_2 \lll x$. Pick elements $v_1, v_2 \in S$ with $y_1 \ll v_1 < x$ and $y_2 \ll v_2 < x$. Using the almost algebraic order of S, we can find approximate differences $d_1, d_2 \in S$ such that $y_1 + d_1 \leq x \leq v_1 + d_1$ and $y_2 + d_2 \leq x \leq v_2 + d_2$. Note that $d_1, d_2 \leq x$. Moreover, the elements d_1 and d_2 must be nonzero, for otherwise we would have $x \leq v_1$ or $x \leq v_2$, contradicting the fact that $v_1, v_2 < x$. Using downwards directedness (Theorem 3.1.24), we can now find $d \in S$ with $0 < d \leq d_1, d_2$, and by Lemma 3.1.2 (i) we can find $d' \in S$ with $0 < d' \ll d$. In total, we have $0 < d' \ll d \leq d_1, d_2 \leq x$. Using the almost algebraic order of S again, we can find $z' \in S$ such that $z' + d' \leq x \leq z' + d$. We obviously have $z' \leq x$; we even have $z' < x$,

since otherwise we would have $x + d' \leq x$ for a finite x (since x is compact and S is weakly cancellative, hence stably finite by Proposition 3.1.19) and a nonzero d', which is impossible. Since S is weakly cancellative, it follows from $y_1 + d \leq y_1 + d_1 \leq x \ll x \leq z' + d$ that $y_1 \ll z'$; analogously, it follows that $y_2 \ll z'$. Using axiom $(\mathcal{O}2)$, we can find an element $z \in S$ with $y_1, y_2 \leq z \ll z'$. Since $z' < x$, we have $y_1, y_2 \leq z \lll x$, so z is as required. This shows that $\{y \in S \mid y \lll x\}$ is directed for every $x \in \mathsf{C}(S)^\times$, so we have shown that condition (i') of Definition 6.1.4 is satisfied.
Second, we show that condition (ii') of Definition 6.1.4 is satisfied too. Let $x, y \in \mathsf{C}(S)^\times$ and $z \in S$ such that $z \lll x + y$. By Lemma 6.1.5 (ii), we can find an element $w_0 \in S^\times$ with $z + w_0 \ll x + y$. Using the halving theorem (Theorem 3.1.25), we can find an element $w_1 \in S^\times$ with $w_1 + w_1 \leq w_0$. Since w_1 is dominated by $x + y$, which is compact and therefore finite (since S is weakly cancellative and therefore stably finite by Proposition 3.1.19), the element w_1 is itself finite by Proposition 3.1.17. Since $0 \ll 0 < w_1$, we can use Lemma 6.3.1 to find a noncompact $w \in S$ with $w \leq w_1$, hence $w + w \leq w_0$. It follows that the noncompact element w satisfies $z + w + w \ll x + y$. Pick any $w' \ll w$. Since x, y are compact and finite, we can use Lemma 6.1.5 (iii) to find an element $x' \in S$ with $x' \lll x$ and $x \leq x + w' \leq x' + w$. Likewise, we can find an element $y' \in S$ with $y' \lll y$ and $y \leq y + w' \leq y' + w$. It now follows that $z + (w + w) \ll x + y \leq x' + y' + (w + w)$; using weak cancellation, we find that $z \leq x' + y'$, so the elements x' and y' are as required. This shows that condition (ii') of Definition 6.1.4 is also satisfied. ■

6.3.3 Remark. *The simplicity requirement cannot be dropped from our decomposability results. To show this, let A be any simple, separable, unital, and nonelementary C^*-algebra with stable rank one. Then $A \oplus A$ is nonsimple, but still separable, unital, nonelementary, and with stable rank one; so $\mathsf{Cu}(A \oplus A)$ satisfies the requirements of both decomposition results, except for simplicity. We have $\mathsf{Cu}(A \oplus A) \cong \mathsf{Cu}(A) \oplus \mathsf{Cu}(A)$. Take a look at the compact element $[(\mathbb{I}_A, \mathbb{I}_A)]$, where $\mathbb{I}_A$ is the unit of A. Let $x := \gamma_A(\mathbb{I}_A)$. It is easy to see that $[(\mathbb{I}_A, x)] < [(\mathbb{I}_A, \mathbb{I}_A)]$ and $[(x, \mathbb{I}_A)] < [(\mathbb{I}_A, \mathbb{I}_A)]$ and that these elements cannot be identical (since $[\mathbb{I}_A]$ is compact, while x is noncompact). Moreover, it is evident that there is no element $z \in \mathsf{Cu}(A) \oplus \mathsf{Cu}(A)$ such that $[(\mathbb{I}_A, x)],\ [(x, \mathbb{I}_A)] \leq z < [(\mathbb{I}_A, \mathbb{I}_A)]$. Hence, the compact element $[(\mathbb{I}_A, \mathbb{I}_A)]$ has no predecessor in $\mathsf{Cu}(A) \oplus \mathsf{Cu}(A)$. It follows that $\mathsf{Cu}(A \oplus A)$ is not decomposable.*

7 Properties of the predecessor map

7.1 The predecessor map

We remind the reader of our definition of decomposability from the last chapter:

7.1.1 Definition. *Let S be a semigroup in the category Cu.*

(i) If the set $\{y \in S \mid y < x\}$ has a maximum for some compact element $x \in S$, then this maximum is called the predecessor *of x.*

(ii) If every $x \in \mathsf{C}(S)^\times$ has a predecessor, then the predecessor map $\gamma_S \colon \mathsf{C}(S) \to S$ *is defined by letting $\gamma_S(0) := 0$ and $\gamma_S(x) := \max\{y \in S \mid y < x\}$ for $x \in \mathsf{C}(S)^\times$.*

(iii) We say that S is decomposable *if every $x \in \mathsf{C}(S)^\times$ has a predecessor, if γ_S is additive, and if $\gamma_S(x) > 0$ for every $x > 0$.*

We will now analyse the properties of semigroups that are both simple and decomposable. The following result has already been proven:

7.1.2 Proposition. *Let S be a simple and decomposable semigroup in Cu. Then S is stably finite and nonelementary.*

Proof. This is Proposition 6.1.2 from the last chapter. ■

This allows us to characterise simple and decomposable semigroups under the additional assumption that they are separable:

7.1.3 Corollary. ***Characterisation (abstract case):*** *Let S be simple and separable. The following conditions are equivalent:*

(i) S is decomposable.

(ii) S is stably finite and $\lll$-regular.

Moreover, if these conditions are met, then S is nonelementary.

Proof. This follows immediately from Theorem 6.1.6 and Proposition 6.1.2. ■

7.1.4 Corollary. ***Characterisation (concrete case):*** *Let A be a simple and separable C*-algebra. The following conditions are equivalent:*

(i) $\mathsf{Cu}(A)$ is decomposable.

(ii) A is stably finite and nonelementary.

Proof. The implication $(ii) \implies (i)$ follows from Corollary 6.2.4. For the converse implication $(i) \implies (ii)$, note first that $\mathsf{Cu}(A)$ is simple by Corollary 4.2.7, and separable by Theorem 4.3.2. The preceding corollary now implies that $\mathsf{Cu}(A)$ is stably finite and nonelementary. It then follows from Theorem 4.5.1 that A is stably finite, and it follows from Theorem 4.4.4 that A is nonelementary. ■

We observe that the predecessor of a nonzero compact element can never be compact:

7.1.5 Corollary. ***Noncompactness:*** *Let S be simple and decomposable. Then $\gamma_S(x)$ is noncompact for every $x \in \mathsf{C}(S)^\times$. Hence, we will from here on regard the predecessor map as a map $\gamma_S \colon \mathsf{C}(S) \to \mathsf{D}(S)$.*

Proof. Let z be an element of $\mathsf{C}(S)^\times$, and assume that $x := \gamma_S(z)$ is compact. S is simple and, by Corollary 7.1.3, also stably finite and nonelementary. Since z and x are compact, they are finite elements of S. By the definition of predecessors, we have $x < z$, which implies $x \ll x < z$ and therefore $x \lll z$. By Lemma 6.3.1, there is a noncompact element $y \in S$ such that $x \ll y < z$. This implies $x < y < z$, which contradicts the fact that x is the maximum among all elements that are strictly below z. It follows that the assumption was wrong, and that $\gamma_S(z)$ must be noncompact. ■

The following observation will be of fundamental importance:

7.1.6 Theorem. ***Absorption (general case):*** *Let S be simple and decomposable. For any $x \in \mathsf{C}(S)$ and $y \in \mathsf{D}(S)^\times$, we have that $x + y = \gamma_S(x) + y$.*

Proof. S is simple and, by Corollary 7.1.3, also stably finite and nonelementary. We may assume that $x > 0$, since there is little to show otherwise. By the halving theorem, there is $z \in S^\times$ such that $2z \leq x$. If z were equal to x, then $2x = x$ for a compact and nonzero x, contradicting the fact that S is stably finite. Thus, we have $0 < z < x$. Let $y' \in S$ be any element with $y' \ll y$. Pick $r \in S$ such that $y' \ll r \ll y$. Using the almost algebraic order of S, we can find $s \in S$ such that $y' + s \leq y \leq r + s$. If s was zero, then $y \leq r \ll y$ and hence $y \ll y$, contradicting the noncompactness of y. Thus s is nonzero; by downwards directedness of S there is some $w \in S$ such that $0 < w \leq z, s$. In particular, we have $w < x$ since $z < x$. Pick any $w' \in S$ such that $0 < w' \ll w$. Using the almost algebraic order of S, we can find some $v \in S$ such that $w' + v \leq x \leq w + v$. If v were equal to x, then $w' + x \leq x$, which cannot happen since w' is nonzero, x is compact, and S is stably finite. Hence, $v < x$ and therefore $v \leq \gamma_S(x)$. It follows that

$$y' + x \leq y' + w + v \leq y' + s + v \leq y + v \leq y + \gamma_S(x).$$

Since this is true for every $y' \ll y$, it follows from axioms $(\mathcal{O}2)$ and $(\mathcal{O}4)$ that

$$\begin{aligned} x+y &= x+\sup\{y' \mid y' \ll y\} \\ &= \sup\{x+y' \mid y' \ll y\} \\ &\leq \gamma_S(x)+y \\ &\leq x+y. \end{aligned}$$

This concludes the proof. ∎

The following theorem is redundant, but its proof provides an interesting insight by showing how absorption works for the Cuntz semigroup of a C*-algebra. (This was the first proof of the absorption theorem that the author discovered; the more general statement above for semigroups in Cu was proven afterwards.) The proof is similar to the one of Lemma 6.2.2.

7.1.7 Theorem. ***Absorption (concrete case):*** *Let A be a simple, separable, stably finite, and nonelementary C*-algebra. Let $x \in C(A)$ and $y \in D(A)^\times$. Then $x+y=\gamma_A(x)+y$.*

Proof. We may assume, without loss of generality, that A is stable and that x is nonzero. We can find a nonzero projection $p \in A$ such that $x=[p]$, and we can find a nonzero element $a \in A_+$ such that $y=[a]$ and $a \perp p$. Let $(\pi,\mathcal{H})$ be an irreducible representation of A_{p+a}. Let $\mathcal{H}_p := \pi(p)\mathcal{H}$, and let $(\pi_p,\mathcal{H}_p)$ be the restriction of $(\pi,\mathcal{H})$ to the corner A_p. Then $(\pi_p,\mathcal{H}_p)$ is an irreducible representation of A_p. Pick any unit vector $\xi \in \mathcal{H}_p$, and let φ be the pure state on A_p associated to $(\pi_p,\mathcal{H}_p,\xi_p)$. Let e be a strictly positive element of the hereditary kernel $\mathrm{hk}(\varphi)$; then $\gamma_A(x)=[e]$ by the construction in Theorem 6.2.1. Let $Q \in \mathcal{B}(\mathcal{H})$ be the rank one projection onto the subspace spanned by ξ, then we have $A_{e+a}=\mathrm{her}(\pi,\mathcal{H},Q)$ by an argument analogous to the one in the proof of Theorem 6.2.3. We need to show that $p+a \precsim e+a$, which will imply $p+a \sim e+a$ (note that e is orthogonal to a, since p is orthogonal to a). Using Rørdam's proposition, it suffices to show that $((p+a)-\varepsilon)_+ \precsim e+a$ for every $\varepsilon>0$, so let $\varepsilon>0$ be fixed. Since y is noncompact, so is $x+y$. This means that 0 is a cluster point of $\sigma(a+p)$ by Lemma 4.5.2. But this implies that the operator $\pi(((p+a)-\varepsilon)_+)$ has an infinite-dimensional kernel. Pick any unit vector η in $\ker(\pi(((p+a)-\varepsilon)_+))$. Let $U \in \mathcal{U}(\mathcal{H})$ be any unitary operator that maps ξ to η. By Kadison's transitivity theorem ([24], Theorem 5.2.2), we can find a unitary element u in $(A_{p+a})^\sim$ (the minimal unitisation) such that $\tilde{\pi}(u)\xi=U\xi=\eta$, where $(\tilde{\pi},\mathcal{H})$ is the unique extension of $(\pi,\mathcal{H})$ to an irreducible representation of $(A_{p+a})^\sim$. Let $d := (u^*((p+a)-\varepsilon)_+u)^{1/2} \in A_{p+a}$, then $d \sim ((p+a)-\varepsilon)_+$ in $(A_{p+a})^\sim$, and thus in the ideal A_{p+a} of $(A_{p+a})^\sim$ by Lemma 4.2.3. By construction, we have

$$\pi(d^2)Q \;=\; \tilde{\pi}(u)^*\pi(((p+a)-\varepsilon)_+)\tilde{\pi}(u)Q \;=\; \tilde{\pi}(u)^*\pi(((p+a)-\varepsilon)_+)UQ \;=\; 0,$$

so that $d \in \mathrm{her}(\pi,\mathcal{H},Q)=A_{e+a}$, and thus $d \precsim (e+a)$. It follows that $((p+a)-\varepsilon)_+ \precsim (e+a)$, which completes the proof. ∎

7.1.8 Corollary. ***Functionals:*** *Let S be simple and decomposable. Then for every functional $\lambda \in \mathsf{F}(S)$ and every $x \in \mathsf{C}(S)$, we have $\lambda(\gamma_S(x)) = \lambda(x)$*

Proof. We may assume that $x \in \mathsf{C}(S)^\times$. If λ is λ_0 or λ_∞, then we have $\lambda(\gamma_S(x)) = \lambda(x)$ because both x and $\gamma_S(x)$ are nonzero. Thus, we need only consider nontrivial functionals. We have $0 < \gamma_S(x) < x$ and $x \ll x \leq \infty$; since S is simple, it follows from Proposition 5.1.9 that $0 < \lambda(\gamma_S(x)) \leq \lambda(x) < \infty$ for every nontrivial functional $\lambda \in \mathsf{F}(S)$. Since $\gamma_S(x) \in \mathsf{D}(S)^\times$, Theorem 7.1.6 implies that $x + \gamma_S(x) = \gamma_S(x) + \gamma_S(x)$. Thus, $\lambda(x) + \lambda(\gamma_S(x)) = 2\lambda(\gamma_S(x))$ and therefore $\lambda(x) = \lambda(\gamma_S(x))$. This concludes the proof. ■

If S is almost unperforated, then we get a characterisation of predecessors by functionals:

7.1.9 Theorem. ***Characterisation by functionals:*** *Let S be simple, decomposable, and almost unperforated. Let $x \in \mathsf{C}(S)^\times$. Then $\gamma_S(x)$ is fully characterised by the property that $\gamma_S(x) < x$ and that $\lambda(\gamma_S(x)) = \lambda(x)$ for every functional λ of S.*

Proof. We have already proven that $\gamma_S(x) < x$ and that $\lambda(\gamma_S(x)) = \lambda(x)$ for every functional λ. Let $v \in S$ be any element such that $v < x$ and $\lambda(v) = \lambda(x)$ for all λ. From $v < x$ it follows that $v \leq \gamma_S(x)$. Pick any $w \ll \gamma_S(x)$. Since $\gamma_S(x)$ is noncompact, we have $w \lll \gamma_S(x)$ by Lemma 6.1.5 (*i*) and therefore $\lambda(w) < \lambda(\gamma_S(x)) = \lambda(v)$ for every nontrivial functional by Proposition 5.1.9. Since S is almost unperforated, this implies $w \leq v$ by Theorem 5.1.14. Since this is true for every $w \ll \gamma_S(x)$, it follows from Rørdam's proposition that $\gamma_S(x) \leq v$, and thus $v = \gamma_S(x)$. ■

7.1.10 Corollary. ***Domination:*** *Let S be simple and decomposable, and let $x \in \mathsf{C}(S)$. Then $x < \gamma_S(x) + y$ for every $y \in S^\times$.*

Proof. We may assume that $x > 0$. Once again, note that S is stably finite. If y is noncompact, then $x < x + y = \gamma_S(x) + y$ by Theorem 7.1.6. If y is compact, then $\gamma_S(y)$, $\gamma_S(x)$ are noncompact by Corollary 7.1.5, so that $x < x + \gamma_S(y) = \gamma_S(x) + \gamma_S(y) = \gamma_S(x) + y$ by Theorem 7.1.6. Either way, it follows that $x < \gamma_S(x) + y$. ■

If S is weakly cancellative, we get another characterisation of predecessors:

7.1.11 Theorem. ***Characterisation by Domination:*** *Let S be simple, decomposable, and weakly cancellative, and let $x \in \mathsf{C}(S)^\times$.*

(i) If $z \in S$ is such that $x \leq z + y$ for every $y \in S^\times$, then $\gamma_S(x) \leq z$.

(ii) The predecessor $\gamma_S(x)$ is characterised by the property that $\gamma_S(x) < x < \gamma_S(x) + y$ for every $y \in S^\times$.

Proof.
(*i*): Pick any $v \ll \gamma_S(x)$. Find a $w \in S$ with $v \ll w \ll \gamma_S(x)$. Since S is almost algebraically ordered, there is $y \in S$ such that $v + y \leq \gamma_S(x) \leq w + y$. If y was zero, then

$\gamma_S(x) \leq w \ll \gamma_S(x)$, contradicting the fact that $\gamma_S(x)$ is noncompact by Corollary 7.1.5. Hence y must be nonzero. It follows that $v + y \leq \gamma_S(x) < x \ll x \leq z + y$. Now it follows from weak cancellation that $v \ll z$. Since this is true for every $v \ll \gamma_S(x)$, it follows from Rørdam's proposition that $\gamma_S(x) \leq z$.
(*ii*): We have $\gamma_S(x) < x$, and by Corollary 7.1.10 it is true that $x < \gamma_S(x) + y$ for every $y \in S^\times$. Let $z \in S$ be another element with this property. From (*i*) it follows that $\gamma_S(x) \leq z$, and $z < x$ implies that $z \leq \gamma_S(x)$, hence $z = \gamma_S(x)$. ■

7.1.12 Corollary. ***Monotonicity:*** *Let S be simple and decomposable. Then for $x, y \in \mathsf{C}(S)^\times$, we have $x < y$ if and only if $\gamma_S(x) \ll \gamma_S(y)$.*

Proof. If $x < y$, then $x \leq \gamma_S(y)$ and therefore $\gamma_S(x) < x \ll x \leq \gamma_S(y)$. For the other direction, let $x, y \in \mathsf{C}(S)^\times$ such that $\gamma_S(x) \ll \gamma_S(y)$. Since $\gamma_S(y)$ is noncompact, we can pick some $z \in S^\times$ such that $\gamma_S(x) + z \ll \gamma_S(y) < y$. By Corollary 7.1.10, it follows that $x < \gamma_S(x) + z$, and therefore $x < y$. ■

7.1.13 Remark. *In general, the predecessor map $\gamma_S \colon \mathsf{C}(S) \to \mathsf{D}(S)$ cannot be assumed to be injective, not even if $S = \mathsf{Cu}(A)$ – it is possible that several mutually incomparable elements of $\mathsf{C}(A)^\times$ all share the same predecessor. For example, if A is a simple, separable, unital, finite, exact, and $\mathcal{Z}$-stable C*-algebra, then by the results of Section 5.3, for two projections $p, q \in M_n(A)$ with $[p] \neq [q]$, we will still have $\gamma_A([p]) = \gamma_A([q])$ if $\tau(p) = \tau(q)$ for all tracial states $\tau \in \mathsf{T}_1(A)$. There are examples for this even among the class of simple, unital AF-algebras. Indeed, as mentioned in [26] (Section 6.3, right above the acknowledgements), such projections can always be found if the ordered abelian group $(\mathsf{K}_0(A), \mathsf{K}_0(A)_+, [\mathbb{I}_A])$ contains nonzero infinitesimal elements. It follows from Elliott's classification of AF-algebras (see [14]) that such AF-algebras exist.*

Next, we will show that the predecessor map γ_S has a natural extension to a map ε_S that is defined on the full semigroup S. The extended map ε_S shares many of the nice properties that γ_S possesses.

7.1.14 Definition/Proposition. ***The extended predecessor map:***
Let S be simple and decomposable. Then the set $\Gamma(x) := \{y \in S \mid y \lll x\}$ is directed and has a supremum for each $x \in S$. Therefore, we have a well-defined map $\varepsilon_S \colon S \to S$ with $\varepsilon_S(x) := \sup \Gamma(x)$. We call ε_S the extended predecessor map of S.

Proof. If $x = 0$, then $\Gamma(x) = \emptyset$; the empty set however is directed and has supremum 0. If x is noncompact, then we have $\Gamma(x) = \{y \in S \mid y \ll x\}$ by Lemma 6.1.5 (*i*); it follows easily from axiom $(\mathcal{O}2)$ that this set is directed, and from Rørdam's proposition that it has supremum x. If x is compact and nonzero, then we have $\Gamma(x) = \{y \in S \mid y \ll \gamma_S(x)\}$ since $\gamma_S(x)$ is the maximum among all elements strictly smaller than x; as before, it follows from axiom $(\mathcal{O}2)$ that this set is directed, and from Rørdam's proposition that it has supremum $\gamma_S(x)$. ■

7.1.15 Theorem. *Let S be simple and decomposable. The extended predecessor map has the following properties:*

(i) ε_S *agrees with* γ_S *on* $\mathsf{C}(S)$*.*

(ii) ε_S *is idempotent with image* $\mathsf{D}(S)$*.*

(iii) ε_S *is additive and zero-preserving.*

(iv) ε_S *is order-preserving and* sup*-preserving.*

(v) $\varepsilon_S(x) \ll \varepsilon_S(y)$ *implies* $x \ll y$ *for all* $x, y \in S$*.*

(vi) $\varepsilon_S(x) + y = x + y$ *for all* $x \in S$*,* $y \in \mathsf{D}(S)^\times$*.*

(vii) $\varepsilon_S(x) \leq x \leq \varepsilon_S(x) + y$ *for all* $x \in S$*,* $y \in S^\times$*.*

(viii) $\lambda \circ \varepsilon_S = \lambda$ *for every functional* λ *on* S*.*

Proof.

(i) We have already shown this in the previous proof.

(ii) We have already seen in Corollary 7.1.5 that $\varepsilon_S(x) = \gamma_S(x) \in \mathsf{D}(S)$ if x is compact and nonzero. We have shown in the previous proof that $\varepsilon_S(x) = x$ if x is zero or noncompact. Hence the image of ε_S is precisely $\mathsf{D}(S)$, and ε_S is the identity on $\mathsf{D}(S)$, which implies that ε_S is idempotent.

(iii) We have already shown in the previous proof that ε_S is zero-preserving. Let x, $y \in S$; we want to show that $\varepsilon_S(x+y) = \varepsilon_S(x) + \varepsilon_S(y)$. If $x = 0$ or $y = 0$, then the statement follows since $\varepsilon_S(0) = 0$. If x, $y \in \mathsf{C}(S)$, then the statement follows from the additivity of γ_S. If x, $y \in \mathsf{D}(S)$, then we have $x + y \in \mathsf{D}(S)$ by Lemma 3.1.21, and the statement follows from *(ii)*. Thus, we need only show that the statement holds for $x \in \mathsf{C}(S)^\times$ and $y \in \mathsf{D}(S)^\times$. In that case, we also have $x + y \in \mathsf{D}(S)^\times$ by Lemma 3.1.21, and it follows from *(i)*, *(ii)*, and Theorem 7.1.6 that $\varepsilon_S(x+y) = x + y = \gamma_S(x) + y = \varepsilon_S(x) + \varepsilon_S(y)$. Hence, ε_S is an additive map.

(iv) Let x, $y \in S$ and $x \leq y$. We want to show that $\varepsilon_S(x) \leq \varepsilon_S(y)$ holds. If $x = 0$, then the statement follows from $\varepsilon_S(x) = 0$. If $y = 0$, then $x = 0$ follows, and we have $\varepsilon_S(x) = 0 = \varepsilon_S(y)$. If x, $y \in \mathsf{D}(S)^\times$, then $\varepsilon_S(x) = x \leq y = \varepsilon_S(y)$. If x, $y \in \mathsf{C}(S)^\times$, then $\varepsilon_S(x) \leq \varepsilon_S(y)$ follows from Corollary 7.1.12. If $x \in \mathsf{C}(S)^\times$ and $x \in \mathsf{D}(S)^\times$, then $\varepsilon_S(x) < x \leq y = \varepsilon_S(y)$. Finally, if $x \in \mathsf{D}(S)^\times$ and $y \in \mathsf{C}(S)^\times$, then $x \leq y$ implies $x < y$, hence $\varepsilon_S(x) = x \leq \gamma_S(y) = \varepsilon_S(y)$. Thus, ε_S is order-preserving. Let $(x_n)_n$ be any increasing sequence in S. We want to show that ε_S preserves the supremum, i.e. that $\varepsilon_S(\sup_n x_n) = \sup_n \varepsilon_S(x_n)$. Let $x := \sup_n x_n$. If $(x_n)_n$ is eventually constant (e.g. if x is compact), then it follows immediately that $\varepsilon_S(\sup_n x_n) = \varepsilon_S(x) = \sup_n \varepsilon_S(x_n)$. Thus, we can assume that x is noncompact, and that $(x_n)_n$ is strictly increasing. It follows that $x_n \leq \varepsilon_S(x_{n+1}) \leq x_{n+1}$ for each n, but this immediately implies that $\varepsilon_S(\sup_n x_n) = \varepsilon_S(x) = x = \sup_n x_n = \sup_n \varepsilon_S(x_n)$. Thus, we have shown that ε_S is sup-preserving.

(v) Let $x, y \in S$ such that $\varepsilon_S(x) \ll \varepsilon_S(y)$. If $x \in \mathsf{D}(S)$, then $x = \varepsilon_S(x) \ll \varepsilon_S(y) \leq y$, hence $x \ll y$. If $x \in \mathsf{C}(S)^\times$, then $\varepsilon_S(x) \in \mathsf{D}(S)^\times$, hence $\varepsilon_S(y) \in \mathsf{D}(S)^\times$, so $\varepsilon_S(x) \ll \varepsilon_S(y)$ implies, in particular, that we can find some $z > 0$ with $\varepsilon_S(x) + z \ll \varepsilon_S(y)$ by Lemma 3.1.2 (*iii*). By Corollary 7.1.10, that means $x < \varepsilon_S(x) + z \ll \varepsilon_S(y) \leq y$. Hence, we always have $x \ll y$.

(vi) Let $x \in S$; we want to show that $x + y = \varepsilon_S(x) + y$ whenever $y \in \mathsf{D}(S)^\times$. This is evident when $x = 0$, and it follows from Theorem 7.1.6 if $x \in \mathsf{C}(S)^\times$. If $x \in \mathsf{D}(S)^\times$, then $\varepsilon_S(x) = x$, and there is nothing to show. Hence, ε_S satisfies absorption.

(vii) Let $x \in S$; we want to show that $\varepsilon_S(x) \leq x \leq \varepsilon_S(x) + y$ for every $y \in S^\times$. This is evident if $x = 0$, and it follows from Corollary 7.1.10 if $x \in \mathsf{C}(S)^\times$. If $x \in \mathsf{D}(S)$, then $\varepsilon_S(x) = x$, and therefore $\varepsilon_S(x) \leq x \leq \varepsilon_S(x) + y$ for every $y \in S^\times$. Hence, ε_S satisfies domination.

(viii) Let $\lambda\colon S \to [0, \infty]$ be any functional, and let $x \in S$. If $x \in \mathsf{D}(S)$, then $x = \varepsilon_S(x)$, and therefore $\lambda(x) = \lambda(\varepsilon_S(x))$. If $x \in \mathsf{C}(S)$, then the statement follows from Corollary 7.1.8. Hence, $\lambda = \lambda \circ \varepsilon_S$ for all functionals λ on S.

■

As with γ_A and as usual for our notation, we prefer to write ε_A instead of $\varepsilon_{\mathsf{Cu}(A)}$ for the extended predecessor map of the Cuntz semigroup of a C*-algebra.

7.2 Simple and decomposable Cuntz semigroups

7.2.1 Proposition. *Let S be simple and decomposable. If $(x_n)_n$ is any increasing sequence in $D(S)$, then $\sup_n x_n \in D(S)$, and it follows immediately that $\sup_{n,S} x_n = \sup_{n,D(S)} x_n$. Moreover, if $x, y \in D(S)$, then $x \ll_S y$ if and only if $x \ll_{D(S)} y$.*

Proof. We know from Corollary 7.1.3 that S is nonelementary. If $(x_n)_n$ is eventually constant, then $\sup_n x_n \in \mathsf{D}(S)$, since all the terms x_n are. Conversely, if $(x_n)_n$ is not eventually constant, then $\sup_n x_n$ cannot be compact, so $\sup_n x_n \in \mathsf{D}(S)$ again. It follows directly that $\sup_{n,S} x_n = \sup_{n,\mathsf{D}(S)} x_n$. Let $x, y \in \mathsf{D}(S)$. First, assume that $x \ll_S y$. Let $(y_n)_n$ be any increasing sequence in $\mathsf{D}(S)$ with $y \leq \sup_{n,\mathsf{D}(S)} y_n$. Then $y \leq \sup_{n,S} y_n$, and since $x \ll_S y$, there is some n such that $x \leq y_n$. Since this is true for every increasing sequence $(y_n)_n$ in $\mathsf{D}(S)$, it follows that $x \ll_{\mathsf{D}(S)} y$. Next, assume that $x \ll_{\mathsf{D}(S)} y$. Let $(y_n)_n$ be any increasing sequence in S with $y \leq \sup_{n,S} y_n$. Let $z := \sup_n y_n$, then we have $\varepsilon_S(z) = \varepsilon_S(\sup_n y_n) = \sup_n \varepsilon_S(y_n)$ by Theorem 7.1.15. Since $y \leq z$, it follows that $y \leq \varepsilon_S(z)$: this is obvious for $z \in \mathsf{D}(S)$, and for $z \in \mathsf{C}(S)^\times$, we have $y < z$ and therefore $y \leq \varepsilon_S(z)$ since $y \in \mathsf{D}(S)$. Hence, we have $y \leq \sup_n \varepsilon_S(y_n)$. Since $(\varepsilon_S(y_n))_n$ is an increasing sequence in $\mathsf{D}(S)$ and $x \ll_{\mathsf{D}(S)} y$, it follows that $x \leq \varepsilon_S(y_n) \leq y_n$ for some n. Since this is true for every increasing $(y_n)_n$ in S, it follows that $x \ll_S y$. ■

7.2.2 Definition. *Let C denote the category with all algebraically ordered abelian monoids as objects, and with additive, order-preserving, and zero-preserving maps as morphisms. Let D denote the category with objects $\{S \in \mathsf{Cu} \mid S$ has no nonzero compact elements$\}$, and with Cu-maps as morphisms.*

7.2.3 Definition. *A semigroup $C \in \mathsf{C}$ is called* simple *if it has no ideals (closed or non-closed) apart from $\{0\}$ and S. Moreover, we call C* finite *if $x < x + y$ for every $x \in C$ and every $y \in C^{\times}$.*

7.2.4 Theorem. *Let Cu_{dec} be the category of simple and decomposable semigroups in Cu, with Cu-maps as morphisms. Then $\mathsf{C}(\cdot)$ is a functor from Cu_{dec} to C, and $\mathsf{D}(\cdot)$ is a functor from Cu_{dec} to D. Moreover, for every $S \in \mathsf{Cu}_{dec}$ the monoids $\mathsf{D}(S)$, $\mathsf{C}(S)$ are simple, $\mathsf{C}(S)$ is finite, and $\mathsf{D}(S)$ is stably finite.*

Proof. We show simplicity and finiteness of $\mathsf{C}(S)$ first. Let I be a proper ideal of $\mathsf{C}(S)$, and let x be an element of $\mathsf{C}(S) \setminus I$. Let J be the (not necessarily closed) ideal of S generated by I, and let $\overline{J}$ be the closed ideal of S generated by I. Clearly, x cannot be in J, for then it would be in I itself. But since x is compact, any increasing sequence in S with supremum x has to be eventually constant with members equal to x, so x cannot be the supremum of an increasing sequence in J either. It follows that x is not in the closed ideal $\overline{J}$; since S is simple, $\overline{J}$ must be the zero ideal of S, and hence I must be the zero ideal of $\mathsf{C}(S)$. So $\mathsf{C}(S)$ is simple. That $\mathsf{C}(S)$ is finite follows immediately from the fact that S is stably finite.

Next, we show that $\mathsf{D}(S)$ is simple. Let I be a proper ideal of $\mathsf{D}(S)$, and let x be an element of $\mathsf{D}(S) \setminus I$. Let J be the closed ideal of S generated by I. Again, if x was in J, then it would already be in I. It follows that $x \notin J$; since S is simple, it follows that $J = \{0\}$, and hence $I = \{0\}$. So $\mathsf{D}(S)$ is simple. That $\mathsf{D}(S)$ is stably finite will be trivial once we have shown that $\mathsf{D}(S)$ has no nonzero compact elements.

In light of Lemma 3.1.21 and Lemma 3.1.22, it only remains to show that $\mathsf{D}(S) \in \mathsf{Cu}$ whenever $S \in \mathsf{Cu}_{dec}$, and that $\mathsf{D}(\varphi)$ is a Cu-morphism whenever $\varphi\colon S_1 \to S_2$ is a Cu-morphism, and that $\mathsf{D}(S)$ contains no nonzero compact elements. Therefore, we show that $\mathsf{D}(S)$ satisfies axioms ($\mathcal{O}1$) to ($\mathcal{O}6$), and that $\mathsf{D}(\varphi)$ satisfies axioms ($\mathcal{M}1$) to ($\mathcal{M}4$):

- ($\mathcal{O}1$): By Proposition 7.2.1, we have $\sup_{n,S} x_n \in \mathsf{D}(S)$ for every increasing sequence $(x_n)_n$ in $\mathsf{D}(S)$, and $\sup_{n,\mathsf{D}(S)} x_n = \sup_{n,S} x_n$.
- ($\mathcal{O}2$): Every element $x \in \mathsf{D}(S)$ is the supremum of a rapidly increasing sequence $(x_n)_n$ in S. If this sequence contains infinitely many terms from $\mathsf{D}(S)$, then we can assume that all terms are from $\mathsf{D}(S)$; it then follows from Proposition 7.2.1 that x is the supremum of a rapidly increasing sequence in $\mathsf{D}(S)$. If $(x_n)_n$ contains only finitely many terms from $\mathsf{D}(S)$, then we can assume that all terms x_n are from $\mathsf{C}(S)^{\times}$. Moreover, for every n, there is some $m > n$ such that $x_n < x_m$, since otherwise the supremum x would be in $\mathsf{C}(S)^{\times}$. Hence, we can assume that $x_n < x_{n+1}$ for every $n \in \mathbb{N}$. It follows

that $\varepsilon_S(x_n) < x_n \ll x_n \leq \varepsilon_S(x_{n+1})$ for all $n \in \mathbb{N}$. Thus, the sequence $(\varepsilon_S(x_n))_n$ is again rapidly increasing. But by Theorem 7.1.15 and Proposition 7.2.1, we know that $(\varepsilon_S(x_n))_n$ is a sequence in $\mathsf{D}(S)$, and that $x = \varepsilon_S(x) = \varepsilon_S(\sup_{n,S} x_n) = \sup_{n,S} \varepsilon_S(x_n) = \sup_{n,\mathsf{D}(S)} \varepsilon_S(x_n)$, so x is again the supremum of a rapidly increasing sequence in $\mathsf{D}(S)$.

($\mathcal{O}3$): This is true because $\ll_S$ and $\ll_{\mathsf{D}(S)}$ agree on $\mathsf{D}(S)$.

($\mathcal{O}4$): This is true because $\sup_S$ and $\sup_{\mathsf{D}(S)}$ agree on $\mathsf{D}(S)$.

($\mathcal{O}5$): Let $x', x, y \in \mathsf{D}(S)$ such that $x' \ll_{\mathsf{D}(S)} x \leq y$. We want to show that there is a $z \in \mathsf{D}(S)$ such that $x' + z \leq y \leq x + z$. If $x = 0$, then $x' = 0$ follows, and $z := y$ is the element we are looking for. Assume that $x > 0$. By Proposition 7.2.1, we have $x' \ll_S x \leq y$, so we can find a $z \in S$ such that $x' + z \leq y \leq x + z$. If $z \in \mathsf{D}(S)$, then there is nothing else to show. If $z \in \mathsf{C}(S)^\times$, then $x' + \varepsilon_S(z) \leq x' + z \leq y \leq x + z = x + \varepsilon_S(z)$ by absorption, and $\varepsilon_S(z) \in \mathsf{D}(S)$ by Theorem 7.1.15.

($\mathcal{O}6$): Let $x, y, z, z' \in \mathsf{D}(S)$ such that $z' \ll_{\mathsf{D}(S)} z \leq x + y$. By Proposition 7.2.1 we have $z' \ll_S z \leq x + y$, so we can find $x_0, y_0 \in S$ such that $x_0 \leq x, z$, and $y_0 \leq y, z$, and $z' \leq x_0 + y_0$. If $x_0, y_0 \in \mathsf{D}(S)$, then there is nothing to show. If $x_0 \in \mathsf{C}(S)^\times$ and $y_0 \in \mathsf{C}(S)$, then we have $z' < x_0 + y_0$ since $x_0 + y_0$ is compact and nonzero. It follows that $z' \leq \varepsilon_S(x_0 + y_0) = \varepsilon_S(x_0) + \varepsilon_S(y_0)$; and that $\varepsilon_S(x_0) < x_0 \leq x, z$; and that $\varepsilon_S(y_0) \leq y_0 \leq y, z$; moreover, $\varepsilon_S(x_0)$, $\varepsilon_S(y_0) \in \mathsf{D}(S)$ by Theorem 7.1.15. The same argument works if $x_0 \in \mathsf{C}(S)$ and $y_0 \in \mathsf{C}(S)^\times$. If $x_0 \in \mathsf{C}(S)$ and $y_0 \in \mathsf{D}(S)^\times$, then by Theorem 7.1.15 we have $z' \leq x_0 + y_0 = \varepsilon_S(x_0) + y_0$ and $\varepsilon_S(x_0) \in \mathsf{D}(S)$; moreover, $\varepsilon_S(x_0) \leq x_0 \leq x, z$. The same argument works if $x_0 \in \mathsf{D}(S)^\times$ and $y_0 \in \mathsf{C}(S)$. The above covers all possible cases.

($\mathcal{M}1$): This is true because $\varphi(0) = 0$.

($\mathcal{M}2$): This is true because $\leq_{S_1}$ agrees with $\leq_{\mathsf{D}(S_1)}$ on $\mathsf{D}(S_1)$, and $\leq_{S_2}$ agrees with $\leq_{\mathsf{D}(S_2)}$ on $\mathsf{D}(S_2)$.

($\mathcal{M}3$): This is true because $\ll_{S_1}$ agrees with $\ll_{\mathsf{D}(S_1)}$ on $\mathsf{D}(S_1)$, and $\ll_{S_2}$ agrees with $\ll_{\mathsf{D}(S_2)}$ on $\mathsf{D}(S_2)$.

($\mathcal{M}4$): This is true because $\sup_{S_1}$ agrees with $\sup_{\mathsf{D}(S_1)}$ on $\mathsf{D}(S_1)$, and $\sup_{S_2}$ agrees with $\sup_{\mathsf{D}(S_2)}$ on $\mathsf{D}(S_2)$, and φ is sup-preserving.

Finally, let x be any element of $\mathsf{D}(S)$. We have already shown that x is the supremum of a rapidly increasing sequence $(x_n)_n$ in $\mathsf{D}(S)$. By Proposition 7.2.1, we know that $(x_n)_n$ is also rapidly increasing in S. If x is compact in $\mathsf{D}(S)$, then we have $x \leq x_n \leq x$ and therefore $x = x_n$ for sufficiently large n. But since $(x_n)_n$ is rapidly increasing in S, this implies that x is compact in S. Since $x \in \mathsf{D}(S)$, this can only happen for $x = 0$. Thus, $\mathsf{D}(S)$ contains no nonzero compact elements, and in particular $\mathsf{D}(S)$ is stably finite. ■

8 Composition and decomposition

8.1 Composition of semigroups

We will now proceed to prove a partial generalisation of the decomposition result from Section 5.3. This result will show that, for a simple, separable, nonelementary, and stably finite C*-algebra A, the Cuntz semigroup $\mathsf{Cu}(A)$ is a composite object built up from the compact and noncompact parts of the Cuntz semigroup by means of a morphism which maps the elements of the former into the latter. We first introduce a more general construction to compose two semigroups into one large semigroup:

8.1.1 Definition/Proposition. *Let $(M, +_M, \leq_M)$ and $(N, +_N, \leq_N)$ be any two positively ordered abelian semigroups. Let $\alpha\colon M \to N$ be a map that satisfies the following properties:*

- *The map α is additive and order-preserving.*
- *For all $x, y \in M^\times$, $\alpha(x) \ll_N \alpha(y)$ implies $x <_M y$.*
- *For all $x \in M$ and $y \in N^\times$, $\alpha(x) \ll_N y$ if and only if there is an element $z \in N^\times$ such that $\alpha(x) + z = y$.*

Define $(S, 0_S, +_S, \leq_S)$ as follows:

- *$S := M^\times \sqcup N^\times \sqcup \{0_S\}$.*
- *For $x, y \in M^\times$, $x +_S y := x +_M y$.*
- *For $x, y \in N^\times$, $x +_S y := x +_N y$.*
- *For $x \in M^\times$, $y \in N^\times$, $x +_S y := \alpha(x) +_N y =: y +_S x$.*
- *For $y \in S$, $0_S +_S y := y =: y +_S 0_S$.*
- *For $x, y \in M^\times$, $x \leq_S y :\iff x \leq_M y$.*
- *For $x, y \in N^\times$, $x \leq_S y :\iff x \leq_N y$.*
- *For $x \in M^\times$, $y \in N^\times$, $x \leq_S y :\iff \alpha(x) \ll_N y$.*
- *For $x \in M^\times$, $y \in N^\times$, $y \leq_S x :\iff y \leq_N \alpha(x)$.*
- *For $y \in S$, $0_S \leq_S y$ holds, and $y \leq_S 0_S$ holds if and only if $y = 0_S$.*

Then $(S, 0_S, +_S, \leq_S)$ is a positively ordered abelian monoid. If M has a zero element, we will identify M with $M^\times \cup \{0_S\}$. If N has a zero element, we will likewise identify N with $N^\times \cup \{0_S\}$. We call S the composition of M and N by α, and denote it by $M \sqcup_\alpha N$.

Proof. It is clear that 0_S is a neutral element for $+_S$, and that $+_S$ is commutative. It is also clear that $+_S$ is associative: in a multi-element sum, all the elements from $M^\times$ are replaced by their images under α, then everything is summed in N, where addition is associative.

Hence, $(S, 0_S, +_S)$ is a monoid. It is also clear that the relation $\leq_S$ is reflexive, and we can easily show that it is antisymmetric: let $x, y \in S$ such that $x \leq_S y \leq_S x$. If $x = 0$ or $y = 0$, then clearly $x = y = 0$. If $x, y \in M^\times$ or $x, y \in N^\times$, we also get $x = y$, since the respective orders on M, N are antisymmetric. Finally, if $x \in M^\times$ and $y \in N^\times$, then $x \leq y \leq x$ implies that $\alpha(x) \ll_N y \leq_N \alpha(x)$, so that $\alpha(x) \ll_N \alpha(x)$ and therefore $x <_M x$. Clearly, this case is impossible, so that $\leq_S$ is indeed antisymmetric. We show next that $\leq_S$ is transitive, and hence that $\leq_S$ is indeed a partial order on S. Let $x, y, z \in S$ such that $x \leq_S y \leq_S z$. There are eight different cases that need to be taken care of one by one:

- If any of the three elements is 0_S, then $x = 0_S$ and $x \leq_S z$ follows.
- If $x, y, z \in M^\times$ or $x, y, z \in N^\times$, then $x \leq z$ since the respective orders on M, N are transitive.
- If $x, y \in M^\times$ and $z \in N^\times$, then $x \leq_M y$ and $\alpha(y) \ll_N z$. Hence $\alpha(x) \leq_N \alpha(y) \ll_N z$, and therefore $\alpha(x) \ll_N z$ and $x \leq_S z$.
- If $x, z \in M^\times$ and $y \in N^\times$, then we have $\alpha(x) \ll_N y$ and $y \leq_N \alpha(z)$. It follows that $\alpha(x) \ll_N \alpha(z)$, and therefore $x <_M z$. Hence we have $x \leq_S z$.
- If $x \in N^\times$ and $y, z \in M^\times$, then $x \leq_N \alpha(y)$ and $y \leq_M z$. Hence $x \leq_N \alpha(y) \leq_N \alpha(z)$, and therefore $x \leq_S z$.
- If $x, y \in N^\times$ and $z \in M^\times$, then we have $x \leq_N y$ and $y \leq_N \alpha(z)$. It follows that $x \leq_N \alpha(z)$, and therefore $x \leq_S z$.
- If $x, z \in N^\times$ and $y \in M^\times$, then $x \leq_N \alpha(y)$ and $\alpha(y) \ll_N z$. It follows that $x \ll_N z$, hence $x \leq_N z$, and therefore $x \leq_S z$.
- If $x \in M^\times$ and $y, z \in N^\times$, then $\alpha(x) \ll_N y$ and $y \leq_N z$. It follows that $\alpha(x) \ll_N z$, and therefore $x \leq_S z$.

Next, we show that for all $x, y \in S$, we have $x \leq_S x + y$. If $y = 0_S$, then this is clear since $\leq_S$ was already shown to be reflexive. If $x = 0_S$, then it is also clear since 0_S is evidently the minimal element of S. If $x, y \in M^\times$ or $x, y \in N^\times$, then the statement is true since M and N are positively ordered. If $x \in M^\times$ and $y \in N^\times$, then $x +_S y = \alpha(x) +_N y \in N^\times$, and $\alpha(x) \ll_N \alpha(x) +_N y$. It follows that $\alpha(x) \ll_N x +_S y$ and therefore $x \leq_S x +_S y$. Finally, if $x \in N^\times$ and $y \in M^\times$, then $x \leq_N \alpha(y) +_N x = x +_S y$, and therefore $x \leq_S x +_S y$.

It remains to show that $\leq_S$ is translation invariant, i.e. that for any $x, y, z \in S$ with $x \leq_S y$, we also have $x +_S z \leq_S y +_S z$ (it follows from this that for $x_1, x_2, y_1, y_2 \in S$ with $x_1 \leq_S y_1$ and $x_2 \leq_S y_2$, we have $x_1 +_S x_2 \leq_S y_1 +_S y_2$). In order to prove translation invariance, there are once again eight different cases that need to be taken care of one by one:

- If $z = 0_S$, then the statement is true since 0_S is the neutral element. If $y = 0_S$, then $x = 0_S$ because 0_S is minimal in S, and since $z \leq_S z$, the statement is again true. For $x = 0_S$, the statement reduces to the form $z \leq_S y +_S z$, and we have already shown that this is always true.

- If $x, y, z \in M$ or $x, y, z \in N$, then the statement is true since M and N are positively ordered semigroups.
- If $x, y \in M^\times$ and $z \in N^\times$, then $x \leq_S y$ implies $x \leq_M y$, and therefore $\alpha(x) \leq_N \alpha(y)$. Since $x +_S z = \alpha(x) +_N z \leq_N \alpha(y) +_N z = y +_S z$ (because N is positively ordered), it follows that $x +_S z \leq_S y +_S z$.
- If $x, z \in M^\times$ and $y \in N^\times$, then $x \leq_S y$ implies $\alpha(x) \ll_N y$, so there is $r \in N^\times$ with $\alpha(x) +_N r = y$. We get $\alpha(x +_M z) \ll \alpha(x +_M z) +_N r = \alpha(x) +_N \alpha(z) +_N r = y +_N \alpha(z) = y +_S z$, which implies $x +_S z \ \leq_S \ y +_S z$.
- If $x \in N^\times$ and $y, z \in M^\times$, then $x \leq_S y$ implies $x \leq_N \alpha(y)$, and it follows that $x +_S z = \alpha(z) +_N x \leq_N \alpha(z) +_N \alpha(y) = \alpha(y +_M z) = \alpha(y +_S z)$ since N is positively ordered. Hence, $x +_S z \leq_S y +_S z$.
- If $x, y \in N^\times$ and $z \in M^\times$, then $x \leq_S y$ implies that $x \leq_N y$, and it follows from this that $x +_S z = x +_N \alpha(z) \leq_N y +_N \alpha(z) = y +_S z$ because N is positively ordered, so that $x +_S z \leq_S y +_S z$.
- If $x, z \in N^\times$ and $y \in M^\times$, then $x \leq_S y$ implies $x \leq_N \alpha(y)$, and it follows that $x +_S z = x +_N z \leq_N \alpha(y) +_N z = y +_S z$ because N is positively ordered. Thus, we get $x +_S z \leq_S y +_S z$.
- If $x \in M^\times$ and $y, z \in N^\times$, then $x \leq_S y$ implies $\alpha(x) \ll_N y$, which implies $\alpha(x) \leq_N y$, and hence $x +_S z = \alpha(x) +_N z \leq_N y +_N z = y +_S z$. It follows that $x +_S z \leq_S y +_S z$.

This proves that $(S, 0_S, +_S, \leq_S)$ is a positively ordered abelian monoid. ■

We know that any simple and decomposable semigroup $S \in \mathsf{Cu}$ can be decomposed into its components $\mathsf{C}(S) \in \mathsf{C}$, $\mathsf{D}(S) \in \mathsf{D}$, and $\gamma_S \colon \mathsf{C}(S) \to \mathsf{D}(S)$. We will now use the above construction to fully recover S from these components. More generally, we want to clarify, for simple semigroups C, D in the categories C and D respectively, what properties a map $\gamma \colon C \to D$ must satisfy so that the ordered semigroup structure of C and D can be extended to the disjoint union $S := C \sqcup_\gamma D$ in such a way that S becomes a simple and decomposable semigroup in Cu with γ as its predecessor map. This is achieved by the following definition:

8.1.2 Definition. ***Composition maps:*** *Let C, D be simple semigroups in C and D, respectively. A map $\gamma \colon C \to D$ is called a* composition map *if it satisfies the following conditions:*

($\mathcal{C}1$) *If $x \in C$, then $\gamma(x) = 0$ if and only if $x = 0$.*

($\mathcal{C}2$) *If $x, y \in C$, then $\gamma(x + y) = \gamma(x) + \gamma(y)$.*

($\mathcal{C}3$) *If $x, y \in C^\times$, then $\gamma(x) \ll \gamma(y)$ implies $x < y$.*

($\mathcal{C}4$) *If $x \in C$ and $y \in D^\times$, then $\gamma(x) \ll y$ if and only if there is an element $z \in D^\times$ such that $\gamma(x) + z = y$.*

($\mathcal{C}5$) *If $z \in C^\times$ and $x,\ y \in D^\times$ with $\gamma(z) \ll x + y$, then there are $x',\ y' \in D^\times$ such that $x' \ll x, \gamma(z)$, and $y' \ll y, \gamma(z)$, and $\gamma(z) \leq x' + y'$.*

Conditions ($\mathcal{C}1$) and ($\mathcal{C}2$) are obviously necessary if we want $C \sqcup_\gamma D$ to be a decomposable semigroup with γ as its predecessor map (compare Definition 7.1.1). Conditions ($\mathcal{C}2$), ($\mathcal{C}3$), and ($\mathcal{C}4$) are taken directly from Proposition 8.1.1. In that proposition, we moreover require the composition map to be order-preserving; this is automatically satisfied here since C is algebraically ordered, D is positively ordered, and γ is additive by condition ($\mathcal{C}2$). Condition ($\mathcal{C}4$) is moreover related to axiom ($\mathcal{O}5$), the almost algebraic order property, of semigroups in Cu, while condition ($\mathcal{C}5$) is related to axiom ($\mathcal{O}6$), the almost Riesz decomposition property, of semigroups in Cu. While D already satisfies ($\mathcal{O}5$) and ($\mathcal{O}6$) since it is an object of D, and therefore of Cu, we need conditions ($\mathcal{C}4$) and ($\mathcal{C}5$) to ensure that ($\mathcal{O}5$) and ($\mathcal{O}6$) will also be satisfied by $C \sqcup_\gamma D$. We will show in the next section (see Theorem 8.2.1) that whenever S is a simple and decomposable semigroup in Cu, then $\gamma_S \colon \mathsf{C}(S) \to \mathsf{D}(S)$ satisfies all the conditions ($\mathcal{C}1$) – ($\mathcal{C}5$); it follows that each of these conditions is necessary. It is interesting that no additional restrictions are necessary for C or D to make our composition argument work, apart from simplicity and the existence of a composition map between them.

8.1.3 Theorem. ***Composition:*** *Let C and D be simple semigroups in C and D, respectively, and let $\gamma \colon C \to D$ be a composition map. Then $S := C \sqcup_\gamma D$ is a simple and decomposable semigroup in Cu. Moreover, $\mathsf{C}(S) = C$, and $\mathsf{D}(S) = D$, and $\gamma_S = \gamma$ is the predecessor map.*

Proof. We already know from Proposition 8.1.1 that S is a positively ordered abelian monoid (note that γ is automatically order-preserving, as mentioned above). To show that $S \in \mathsf{Cu}$, we have no choice but to prove the axioms ($\mathcal{O}1$) to ($\mathcal{O}6$) one by one. The part of the proof where the remaining claims are shown to be true will be denoted by ($\mathcal{X}$). A number of intermediate results will also be proven there. We introduce the following notation for the rest of this proof: if $x \in S$, then $\varepsilon(x) := x$ in case $x \in D$, and $\varepsilon(x) := \gamma(x)$ in case $x \in C$. This means that we always have $\varepsilon(x) \in D$, and $\varepsilon(x) \leq_S x$, and $\varepsilon(x +_S y) = \varepsilon(x) +_D \varepsilon(y)$.

($\mathcal{O}1$): Let $(x_n)_n$ be an increasing sequence in S; we need to show that it has a supremum. This is trivially the case if the sequence is eventually constant, so we may assume that it is not. By passing to a subsequence, we may even assume without any loss of generality that $(x_n)_n$ increases strictly. We need to distinguish two cases. First, assume that the sequence contains infinitely many terms in D. Again, by passing to a subsequence, we may assume that all terms are in D. Using the fact that D is an object of D, we can define $x := \sup_{n,D} x_n$. Then $x \in D$ and $x_n \leq_D x$ for all n, from which it follows that $x_n \leq_S x$ for all n. Hence, x is an upper bound for $(x_n)_n$. Let $y \in S$ be another upper bound for $(x_n)_n$. If $y \in D$, then $x \leq_D y$ and therefore $x \leq_S y$, since x is the least upper bound in D. If $y \in C^\times$, then from $x_n \in D$ and $x_n \leq_S y$ it follows that $x_n \leq_D \gamma(y)$ for all n. Thus, $\gamma(y)$ is an upper bound for $(x_n)_n$ in D, and therefore $x \leq_D \gamma(y)$, which means precisely that $x \leq_S y$. Hence, $x = \sup_{n,S} x_n$ as expected. Second, assume that $(x_n)_n$ contains only finitely many terms in D. By passing to a subsequence, we may assume that $x_n \in C^\times$ for each n. Note that we may still assume that the sequence increases strictly, so that $x_n <_C x_{n+1}$ for each n. Since C is algebraically ordered, there are $z_n \in C^\times$ such that $x_n + z_n = x_{n+1}$. Then $\gamma(z_n) \in D^\times$ by ($\mathcal{C}1$), and since $\gamma(x_n) +_D \gamma(z_n) = \gamma(x_{n+1})$ by ($\mathcal{C}2$), we have $x_n \leq_S x_n +_S \gamma(z_n) = \gamma(x_n) +_D \gamma(z_n) = \gamma(x_{n+1})$, so we find that $\gamma(x_n) \leq_S x_n \leq_S \gamma(x_{n+1})$ for each n. Hence, $(\gamma(x_n))_n$ is another increasing sequence in S, and it has a supremum in S if and only if $(x_n)_n$ has a supremum in S, in which case both suprema will coincide. But since $(\gamma(x_n))_n$ is a sequence in D, we have already shown that it has a supremum in S, namely $\sup_{n,D} \gamma(x_n)$. Hence, we have $\sup_{n,S} x_n = \sup_{n,D} \gamma(x_n)$. This proves that every increasing sequence in S has a supremum in S.

($\mathcal{X}$): First, we note that we have already shown in the proof of ($\mathcal{O}1$) that $\sup_{n,S} x_n \in D$ for every sequence $(x_n)_n$ in S that is not eventually constant. Moreover, we have seen that $\sup_{n,S} x_n = \sup_{n,D} x_n$ if all the terms of such a sequence are in D, and that $\sup_{n,S} x_n = \sup_{n,D} \gamma(x_n)$ if all the terms of such a sequence are in C. In general, it follows that $\sup_{n,S} x_n = \sup_{n,D} \varepsilon(x_n) \in D$ unless all the terms x_n are eventually equal to some $x \in C^\times$, in which case we obviously have $\sup_{n,S} x_n = x \in C^\times$.

Second, we need to show that the elements of C are compact in S. Let $x \in C$, and let $(y_n)_n$ be any increasing sequence in S such that $x \leq y := \sup_{n,S} y_n$. We need to show that $x \leq_S y_n$ for sufficiently large n. This is trivial for $x = 0$, so we assume that $x \in C^\times$. Moreover, it is clearly the case if the sequence $(y_n)_n$ is eventually constant, so we may assume that it is not. By the above paragraph, we may then assume that all its terms are in $D^\times$. It follows from these assumptions that $y = \sup_{n,D} y_n$, and in particular that $y \in D^\times$ ($0 < y$ since $0 <_S x \leq_S y$). But then $x \leq_S y$ implies that $\gamma(x) \ll_D y$. Pick any $y' \in D^\times$ such that $\gamma(x) \ll_D y' \ll_D y$. Then $y' \leq_D y_n$, and therefore $\gamma(x) \ll_D y_n$, for sufficiently large n. But this means precisely that $x \leq_S y_n$ for sufficiently large n. Since this is true for every increasing sequence $(y_n)_n$ in S, we have shown that x is compact in S.

Third, we need to show that the elements of $D^\times$ are noncompact in S. Let $x \in D^\times$. We know that we can find a rapidly increasing sequence $(x_n)_n$ in $D^\times$ with $\sup_{n,D} x_n = x$, and therefore $x = \sup_{n,S} x_n$. Now, if x was compact in S, then $x \leq_S x_n \leq_S x$ for all sufficiently large n, so that $(x_n)_n$ would be eventually constant. But this is clearly not the case, since the sequence increases rapidly in $D^\times$, and D contains no nonzero compact elements. Hence, x is noncompact in S.

Fourth, we need to show that $\gamma(x) = \max\{y \in S \mid y <_S x\}$ for every $x \in C^\times$. Since $\gamma(x) \in D^\times$ and $\gamma(x) \leq_D \gamma(x)$, we see that $\gamma(x) \leq_S x$, and it follows that $\gamma(x) <_S x$ since x is compact and $\gamma(x)$ is not. Let $y \in S$ be another element such that $y <_S x$. If $y \in C$, then $y <_C x$. Since C is algebraically ordered, we can find $z \in C^\times$ such that $x = y +_C z$. It follows that $\gamma(x) = \gamma(y) +_D \gamma(z)$. Since $\gamma(z) \in D^\times$, it follows from axiom ($\mathcal{C}4$) that $\gamma(y) \ll_D \gamma(x)$, which means $y \leq_S \gamma(x)$. But if, on the other hand, we have $y \in D^\times$, then $y <_S x$ implies $y \leq_D \gamma(x)$ and therefore $y \leq_S \gamma(x)$. Either way, we have $y \leq_S \gamma(x)$, so that $\gamma(x)$ is indeed maximal among the elements of S that are strictly below x.

Fifth, we need to show that S contains no nontrivial closed ideals. Let $I \subseteq S$ be a nonzero closed ideal, and let $0 < x \in I$. If $x \in C^\times$, then $C \subseteq I$ since C is simple. Moreover, since $\gamma(x) \in D^\times$ and $\gamma(x) \leq_S x$, we have $\gamma(x) \in I$. Since D is simple, this means that $D \subseteq I$ and therefore $I = S$ (here we have silently used that suprema in D coincide with the respective suprema in S). If, on the other hand, $x \in D^\times$, then $D \subseteq I$ since D is simple. If y is any element of $C^\times$, then $\gamma(y) \in D^\times$ and therefore $\gamma(y) \in I$. But $\gamma(y) \ll_D \gamma(y) +_D \gamma(y) = \gamma(y +_C y)$ by axioms ($\mathcal{C}2$) and ($\mathcal{C}4$), so that $y \leq_S \gamma(y +_C y) = \gamma(y) +_D \gamma(y) = \gamma(y) +_S \gamma(y) \in I$, hence $y \in I$. Since C is simple, this means that $C \subseteq I$ and therefore $I = S$. Thus, S contains no nontrivial closed ideals.

Sixth, we want to show that for $x, y \in C$, we have $x \ll_S y$ if and only if $x \leq_C y$. If $x \ll_S y$, then $x \leq_S y$ follows immediately, and therefore $x \leq_C y$. Conversely, if $x \leq_C y$, then $x \leq_S y$. We have already shown that $x \ll_S x$, so it follows that $x \ll_S y$.

Seventh, we want to show that for $x, y \in D$, we have $x \ll_S y$ if and only if $x \ll_D y$. If $x \ll_S y$, pick any increasing sequence $(z_n)_n$ in D with $z := \sup_n z_n \geq_D y$. Then $(z_n)_n$ is also an increasing sequence in S, and $z = \sup_{n,S} z_n \geq_S y$. Since $x \ll_S y$, we have $x \leq_S z_n$ for some n, and therefore $x \leq_D z_n$ for some n. Since this is true for every increasing sequence $(z_n)_n$ in D, we have $x \ll_D y$. Conversely, if $x \ll_D y$, pick any increasing sequence $(z_n)_n$ in S with $z := \sup_{n,S} z_n \geq_S y$. We need to show that $x \leq_S z_n$ for sufficiently large n. If $(z_n)_n$ is eventually constant, then $y \leq_S z = z_n$ for some n, and from $x \ll_D y$ it follows that $x \leq_D y$ and therefore $x \leq_S y$, hence $x \leq_S z_n$. If $(z_n)_n$ is not eventually constant, then $z \in D$ and we may assume, without loss of generality, that $z_n \in D$ for every n, by passing to $(\varepsilon(z_n))_n$ if necessary. But then it follows that $z = \sup_{n,D} z_n \geq_D y$. Since $x \ll_D y$, we have $x \leq_D z_n$ and therefore $x \leq_S z_n$ for all sufficiently large n. We find that, whatever increasing sequence $(z_n)_n$ in S we pick, we always have $x \leq_S z_n$ eventually. Thus $x \ll_S y$.

($\mathcal{O}2$): Let x be any element of S. If $x \in C$, then it follows that x is compact in S. Let $x_n := x$ for all $n \in \mathbb{N}$. Then $(x_n)_n$ is a rapidly increasing sequence in S with supremum x. If $x \in D$, then we can find a rapidly increasing sequence $(x_n)_n$ in D with $\sup_{n,D} x_n = x$. It follows that this is also a rapidly increasing sequence in S, and that $\sup_{n,S} x_n = x$. Hence, every element in S is the supremum of a rapidly increasing sequence in S.

($\mathcal{O}3$): Let $x_1, x_2, y_1, y_2 \in S$ such that $x_1 \ll_S y_1$ and $x_2 \ll_S y_2$. We want to show that $x_1 +_S x_2 \ll_S y_1 +_S y_2$, and we need to take care of four cases. First, assume that $y_1, y_2 \in C^\times$. We have $x_1 \leq_S y_1$ and $x_2 \leq_S y_2$, and therefore $x_1 +_S x_2 \leq_S y_1 +_S y_2$. But $y_1 +_S y_2 \in C^\times$, and thus $y_1 +_S y_2$ is compact, so it follows that $x_1 +_S x_2 \ll_S y_1 +_S y_2$. Second, assume that $y_1 \in D^\times$ and $y_2 \in C$. Then $\varepsilon(x_1) \ll_S y_1$, which implies $\varepsilon(x_1) \ll_D y_1$. Therefore, we can find an element $u \in D^\times$ such that $\varepsilon(x_1) +_D u \ll_D y_1$. Next, pick an element $v \in D^\times$ such that $\varepsilon(x_1) +_D u \ll_D v \ll_D y_1$. Finally, we can find an element $w \in D^\times$ such that $\varepsilon(x_1) +_D u \ll_D v \leq v +_D w \ll_D y_1$. Observe that, since $x_2 \leq_S y_2$, we have $\varepsilon(x_2) \leq_S \gamma(y_2)$ and therefore $\varepsilon(x_2) \leq_D \gamma(y_2)$. Moreover, observe that $\gamma(y_2) \ll_D \gamma(y_2) +_D w$ by axiom ($\mathcal{C}4$). But then by axiom ($\mathcal{O}3$) in D we have

$$\begin{aligned}
x_1 +_S x_2 &\leq_S (x_1 +_S x_2) +_S u \\
&= (x_1 +_S u) +_S x_2 \\
&= (\varepsilon(x_1) +_D u) +_S x_2 \\
&= (\varepsilon(x_1) +_D u) +_D \varepsilon(x_2) \\
&\leq_D (\varepsilon(x_1) +_D u) +_D \gamma(y_2) \\
&\ll_D v +_D (\gamma(y_2) +_D w) \\
&= (v +_D w) +_D \gamma(y_2) \\
&\leq_D y_1 +_D \gamma(y_2) \\
&= y_1 +_S y_2,
\end{aligned}$$

which implies that $x_1 +_S x_2 \ll_S y_1 +_S y_2$ as required. Third, assume that $y_1 \in C$ and $y_2 \in D^\times$. This case works just like the one before. Fourth, assume that $y_1, y_2 \in D^\times$. Then $\varepsilon(x_1)$, $\varepsilon(x_2)$, y_1, $y_2 \in D$, and we have $\varepsilon(x_1) \ll_D y_1$, and $\varepsilon(x_2) \ll_D y_2$. Thus, by axiom ($\mathcal{O}3$) of D we have $\varepsilon(x_1) +_D \varepsilon(x_2) \ll_D y_1 +_D y_2$, hence $\varepsilon(x_1 +_S x_2) \ll_D y_1 +_D y_2$. If $x_1 +_S x_2 \in C$, this implies $\gamma(x_1 +_S x_2) \ll_D y_1 +_D y_2$ and therefore $x_1 +_S x_2 \leq_S y_1 +_D y_2$. Since $x_1 +_S x_2$ is compact in S (because it is in C), we get $x_1 +_S x_2 \ll_S y_1 +_S y_2$. On the other hand, if $x_1 +_S x_2 \in D$, then $\varepsilon(x_1 +_S x_2) \ll_D y_1 +_D y_2$ immediately implies $x_1 +_S x_2 \ll y_1 +_S y_2$. Since all the possible cases have now been taken care of, we have shown that S indeed satisfies axiom ($\mathcal{O}3$).

($\mathcal{O}4$): We prove this axiom in two steps. First, let $(x_n)_n$ be any increasing sequence in S, and let $y \in S$. We want to show that $\sup_{n,S} (x_n +_S y) = (\sup_{n,S} x_n) +_S y$. Let $x := \sup_{n,S} x_n$. If $(x_n)_n$ is eventually constant, then clearly $(\sup_{n,S} x_n) + y = x +_S y = \sup_{n,S} (x_n +_S y)$. If, on the other hand, $(x_n)_n$ is not eventually constant, then $x = \sup_{n,D} \varepsilon(x_n) \in D^\times$. We may assume that $(x_n)_n$ is strictly increasing, and that all terms x_n, $\varepsilon(x_n)$ are nonzero, and hence $\varepsilon(x_n) \in D^\times$. It follows that

$$\begin{aligned}
\varepsilon(x_n) +_S \varepsilon(y) &= \varepsilon(x_n) +_S y \\
&\leq_S x_n +_S y \\
&\leq_S \varepsilon(x_{n+1}) +_S y \\
&= \varepsilon(x_{n+1}) +_S \varepsilon(y),
\end{aligned}$$

which implies that $\sup_{n,S} (\varepsilon(x_n) +_S \varepsilon(y)) = \sup_{n,S} (x_n +_S y)$. But from this and from axiom ($\mathcal{O}4$) of D it follows that

$$\begin{aligned}
\sup_{n,S}(x_n +_S y) &= \sup_{n,S}(\varepsilon(x_n) +_S \varepsilon(y)) \\
&= \sup_{n,D}(\varepsilon(x_n) +_D \varepsilon(y)) \\
&= (\sup_{n,D} \varepsilon(x_n)) +_D \varepsilon(y) \\
&= (\sup_{n,S} x_n) +_S \varepsilon(y) \\
&= (\sup_{n,S} x_n) +_S y.
\end{aligned}$$

Second, let $(x_n)_n$ and $(y_n)_n$ be any two increasing sequences in S. We want to show that $\sup_{n,S}(x_n +_S y_n) = (\sup_{n,S} x_n) +_S (\sup_{n,S} y_n)$. If any one of these sequences is eventually constant, then this statement follows immediately from the previous paragraph. Thus, we may assume that neither sequence is eventually constant. As above, we may assume that both sequences are strictly increasing with nonzero terms. It follows that $x := \sup_{n,S} x_n = \sup_{n,D} \varepsilon(x_n) \in D^\times$ and $y := \sup_{n,S} x_n = \sup_{n,D} \varepsilon(y_n) \in D^\times$. Moreover, for every n we have $x_n <_S x_{n+1}$ and $y_n <_S y_{n+1}$, from which it follows that $x_n \leq_S \varepsilon(x_{n+1})$ and $y_n \leq_S \varepsilon(y_{n+1})$. These inequalities immediately imply $\varepsilon(x_n) + \varepsilon(y_n) \leq_S x_n +_S y_n \leq_S \varepsilon(x_{n+1}) +_S \varepsilon(y_{n+1})$. It follows now that indeed we have $\sup_{n,S}(x_n +_S y_n) = \sup_{n,S}(\varepsilon(x_n) +_S \varepsilon(y_n))$, but then by axiom ($\mathcal{O}4$) of D we have

$$\begin{aligned}
\sup_{n,S}(x_n +_S y_n) &= \sup_{n,S}(\varepsilon(x_n) +_S \varepsilon(y_n)) \\
&= \sup_{n,D}(\varepsilon(x_n) +_D \varepsilon(y_n)) \\
&= (\sup_{n,D} \varepsilon(x_n)) +_D (\sup_{n,D} \varepsilon(y_n)) \\
&= (\sup_{n,S} x_n) +_S (\sup_{n,S} y_n).
\end{aligned}$$

($\mathcal{O}5$): Let $x', x, y \in S$ such that $x' \ll_S x \leq_S y$. We need to find a $z \in S$ such that $x' +_S z \leq_S y \leq_S x +_S z$. We will distinguish between four cases. First, assume that $x, y \in C$. Then we can use the algebraic order on C to find $z \in C$ with $x +_S z = y$, from which it immediately follows that $x' +_S z \leq_S x +_S z = y \leq_S x + z$. Second, assume that $x \in C$ and $y \in D^\times$. Then it follows from $\gamma(x) \leq_S x \ll_S x \leq_S y$ and axiom ($\mathcal{C}4$) that we can find $z \in D^\times$ such that $\gamma(x) +_S z = y$, and therefore $x +_S z = y$. Again, this implies that $x' +_S z \leq_S x +_S z = y \leq_S x +_S z$. Third, assume that $x, y \in D^\times$. Let $(u_n)_n$ be a rapidly increasing sequence in $D^\times$ with $\sup_{n,D} u_n = x$. Then $(u_n)_n$ is also rapidly increasing in S with $\sup_{n,S} u_n = x$. Since $x' \ll_S x$, we can find $u := u_n \in D^\times$ such that $x' \leq_S u \ll_S x$. Since $u, x, y \in D$ and $u \ll_D x \leq_D y$, we can use axiom ($\mathcal{O}5$) of D to find a $z \in D$ such that $u +_D z \leq_D y \leq_D x +_D z$. It follows that $x' +_S z \leq_S u +_S z \leq_S y \leq_S x +_S z$. Fourth, assume that $x \in D^\times$ and $y \in C$. Then $x' \ll_S x <_S y$. It follows that $0 < y$ and therefore $\gamma(y) \in D^\times$. Moreover, we have $\varepsilon(x') \ll_S x \leq_S \gamma(y)$ and therefore $\varepsilon(x') \ll_D x \leq_D \gamma(y)$. Now, we can find a $u \in D^\times$

with $\varepsilon(x') +_D u \ll_D x$, which implies $x' +_S u = \varepsilon(x') +_D u \ll_D x \leq_D \gamma(y)$. We can find a $v \in D$ such that $x' +_S u +_S v \leq_S \gamma(y) \leq_S x +_S v$ using the previous case. It follows from ($\mathcal{C}4$) that $\gamma(y) \ll_D \gamma(y) +_D u \leq_D (x +_S v) +_D u = x +_S (v +_S u)$, which implies $y \leq_S x +_S (u +_S v)$. Let $z := u +_S v$, and we have $x' +_S z \leq_S \gamma(y) <_S y \leq_S x +_S z$. Since all possible cases are exhausted, we have shown that S satisfies axiom ($\mathcal{O}5$).

($\mathcal{O}6$): Let $z', z, x, y \in S$ such that $z' \ll_S z \leq_S x +_S y$. We need to find $x', y' \in S$ such that $x' \leq_S x, z,$ and $y' \leq_S y, z,$ and $z' \leq_S x' +_S y'$. We need to distinguish between three cases. First, assume that $z \in D^\times$. Then $\varepsilon(z') \ll_S z \leq_S \varepsilon(x) +_S \varepsilon(y)$. We can find $u \in D^\times$ such that $\varepsilon(z') \ll_D u \ll_D z$, which implies $z' \leq_S u \ll_D z$. But since we have u, z, $\varepsilon(x)$, $\varepsilon(y) \in D$ and $u \ll_D z \leq_D \varepsilon(x) +_D \varepsilon(y)$, we can use axiom ($\mathcal{O}6$) of D to find x', $y' \in D$ such that $x' \leq_D \varepsilon(x), z,$ and $y' \leq_D \varepsilon(y), z,$ and $u \leq_D x' +_D y'$. It follows immediately that $x' \leq_S \varepsilon(x), z,$ and $y' \leq_S \varepsilon(y), z,$ and $z' \leq_S u \leq_S x' +_S y'$. But then $x' \leq_S x, z$ and $y' \leq_S y, z$, so we are done.

Second, assume that $z' = z \in C$. We need to take care of some easy corner cases. If $z = 0$, then x', $y' := 0$ are as required. If $x = 0$, then $x' := 0$ and $y' := z$ are as required. If $y = 0$, then $x' := z$ and $y' := 0$ are as required. And if $z = x +_S y$, then $x' := x$ and $y' := y$ are as required. Thus, we may assume without loss of generality that $0 < x, y, z$ and that $z \ll_S z <_S x +_S y$. Then $z \leq_S \varepsilon(x) +_S \varepsilon(y)$, and $\gamma(z) <_S z \ll_S \varepsilon(x) +_S \varepsilon(y)$ since z is compact. It follows that $\gamma(z) \ll_S \varepsilon(x) +_S \varepsilon(y)$, which implies $\gamma(z) \ll_D \varepsilon(x) +_D \varepsilon(y)$. Since $z \in C^\times$ and $\varepsilon(x)$, $\varepsilon(y) \in D^\times$, we can use axiom ($\mathcal{C}5$) to find $x_0, y_0 \in D^\times$ such that $x_0 \ll_D \varepsilon(x), \gamma(z),$ and $y_0 \ll_D \varepsilon(y), \gamma(z),$ and $\gamma(z) \leq_S x_0 +_S y_0$. We can find $u \in D^\times$ with $x' := x_0 +_D u \ll_D \varepsilon(x), \gamma(z)$ and $y' := y_0 +_D u \ll_D \varepsilon(y), \gamma(z)$ using Lemma 3.1.2 (*iii*) and downwards directedness (Theorem 3.1.24). Then $x' \leq_S \varepsilon(x), \gamma(z)$, hence $x' \leq_S x, z,$ and analogously $y' \leq_S y, z$. Since $2u \in D^\times$, we find that $\gamma(z) \ll_D \gamma(z) +_D 2u$ by axiom ($\mathcal{C}4$), and thus $z \leq_S \gamma(z) +_S 2u$. But then we have

$$\begin{aligned} z &\leq_S \gamma(z) + 2u \\ &\leq_S (x_0 +_S y_0) +_S 2u \\ &= (x_0 +_S u) +_S (y_0 +_S u) \\ &= x' +_S y'. \end{aligned}$$

Thus, x' and y' again satisfy all the required conditions. Third, assume that $z' < z$ and $z \in C$. Then $z \ll_S z \leq_S x +_S y,$ so by the previous case we can find $x', y' \in S$ such that $x' \leq_S x, z,$ and $y' \leq_S y, z,$ and $z \leq_S x' +_S y'$. Since $z' <_S z \leq_S x' +_S y'$, these elements x', y' meet all the required conditions again. Since we have exhausted all possible cases, we have shown that S satisfies axiom ($\mathcal{O}6$). This completes the proof.

■

8.2 Decomposition of Cuntz semigroups

The following decomposition result is the perfect converse of the above:

8.2.1 Theorem. ***Decomposition:*** *Let S be any simple and decomposable semigroup in Cu. Then $\mathsf{C}(S)$ is a simple semigroup in C, and $\mathsf{D}(S)$ is a simple semigroup in D, and $\gamma_S \colon \mathsf{C}(S) \to \mathsf{D}(S)$ is a composition map. Moreover, we have $S \cong \mathsf{C}(S) \sqcup_{\gamma_S} \mathsf{D}(S)$, and this isomorphism can be chosen to be natural.*

Proof. This is obvious if $S = \{0\}$, so we may assume that S is nonzero. Since S is simple and decomposable, it follows that S is nonelementary and stably finite by Proposition 7.1.2. We have already shown in Theorem 7.2.4 that $\mathsf{C}(S) \in \mathsf{C}$, and that $\mathsf{D}(S) \in \mathsf{D}$, and that $\mathsf{C}(S)$, $\mathsf{D}(S)$ are simple and (stably) finite. Now, let us show that γ_S is indeed a composition map. It follows from the definition of decomposability that axioms $(\mathcal{C}1)$ and $(\mathcal{C}2)$ are satisfied, and it follows from Corollary 7.1.12 that axiom $(\mathcal{C}3)$ is satisfied as well. To show that $(\mathcal{C}4)$ is satisfied, let $x \in \mathsf{C}(S)$ and $y \in \mathsf{D}(S)^\times$. If we have $\gamma_S(x) \ll y$, then we can find an element $z_0 \in S^\times$ such that $\gamma_S(x)+z_0 \leq y$, since y is noncompact. It follows from Corollary 7.1.10 that $x < \gamma_S(x) + z_0$, and therefore $x < y$. By Lemma 3.1.21, this means that we can find $z \in S$ such that $x + z = y$. Since x is compact and y is noncompact, it follows that $z \in \mathsf{D}(S)^\times$. Lastly, it follows from Theorem 7.1.6 that $\gamma_S(x) + z = x + z = y$. Conversely, if there is an element $z \in \mathsf{D}(S)^\times$ such that $\gamma_S(x) + z = y$, then $x < \gamma_S(x) + z$ by Corollary 7.1.10, and therefore $x < y$. It follows that $\gamma_S(x) \leq x \ll x < y$, so that $\gamma_S(x) \ll y$ as required. Thus, axiom $(\mathcal{C}4)$ is satisfied.

Finally, we turn to axiom $(\mathcal{C}5)$, which is obviously connected to axiom $(\mathcal{O}6)$ of Cu. Let $z \in \mathsf{C}(S)^\times$ and $x,\ y \in \mathsf{D}(S)^\times$ with $\gamma_S(z) \ll x + y$. Since $x + y$ is noncompact, there is $s \in \mathsf{D}(S)^\times$ with $\gamma_S(z) + s = x + y$ by $(\mathcal{C}4)$. Again by Corollary 7.1.10, we have $z < \gamma_S(z) + s$, and hence $z \ll z < x + y$. By axiom $(\mathcal{O}6)$, we can find elements $x_0,\ y_0 \in S$ such that $x_0 \leq z, x,$ and $y_0 \leq z, y,$ and $z \leq x_0 + y_0$. At this point, we need to take care of some corner cases. First, observe that $z > 0$ and therefore $\gamma_S(z) > 0$. Second, if $x_0 = 0$, then $z \leq y_0 \leq z$, hence $y_0 = z$. Since both $\gamma_S(z)$ and x are nonzero, we can find $0 < r \leq x,\ \gamma_S(z)$ by downwards directedness. By Corollary 7.1.10, it follows that $z < r+\gamma_S(z)$, hence $\gamma_S(z) \ll r+\gamma_S(z)$. That means we can find $0 < x' \ll r$ and $0 < y' \ll \gamma_S(z)$ with $\gamma_S(z) \leq x' + y'$. Since $\gamma_S(z)$ is noncompact, we may assume that $x',\ y' \in \mathsf{D}(S)^\times$ by passing to predecessors and using Theorem 7.1.6 if necessary. By construction, we have $x' \ll r \leq x, \gamma_S(z)$ and $y' \ll \gamma_S(z) < z = y_0 \leq y$, hence $x' \ll x,\ \gamma_S(z)$ and $y' \ll y,\ \gamma_S(z)$. Thus, x' and y' meet the required conditions. Analogously, we can take care of the case that $y_0 = 0$, and may therefore assume that $x_0,\ y_0 > 0$. Third, if $z = x_0 + y_0$, then x_0 and y_0 are both compact, hence $\gamma_S(z) = \gamma_S(x_0) + \gamma_S(y_0)$. Let $x' := \gamma_S(x_0)$ and $y' := \gamma_S(y_0)$. We have $\gamma_S(z) = x' + y',$ and $x', y' \in \mathsf{D}(S)^\times$ since $x_0,\ y_0 > 0$. We also have $x' \ll x_0 \leq x$ and $y' \ll y_0 \leq y$ since x_0 and y_0 are compact. Moreover, since S is stably finite, and $0 < x_0, y_0$, and $x_0 + y_0 = z$, it follows that $x_0, y_0 < z$ and therefore

$x_0, y_0 \leq \gamma_S(z)$. But this implies that $x' = \gamma_S(x_0) < x_0 \ll x_0 \leq \gamma_S(z)$, and likewise that $y' = \gamma_S(y_0) < y_0 \ll y_0 \leq \gamma_S(z)$. Again, x' and y' meet all the required conditions.

From now on, we may therefore assume that the elements x_0, y_0 are both nonzero, and that $z < x_0 + y_0$. It follows that $z \leq \varepsilon_S(x_0 + y_0) = \varepsilon_S(x_0) + \varepsilon_S(y_0)$ and therefore $\gamma_S(z) \ll \varepsilon_S(x_0) + \varepsilon_S(y_0)$. Since $\gamma_S(z)$, $\varepsilon_S(x_0)$, $\varepsilon_S(y_0) \in \mathsf{D}(S)^\times$ and $\mathsf{D}(S)$ is an object in D, we can find x', $y' \in \mathsf{D}(S)^\times$ such that $x' \ll \varepsilon_S(x_0)$, $y' \ll \varepsilon_S(y_0)$, and $\gamma_S(z) \leq x' + y'$ (we have silently used Proposition 7.2.1 here). Since $x_0 \leq x, z$, we get $\varepsilon_S(x_0) \leq \varepsilon_S(x), \varepsilon_S(z) \leq x, \gamma_S(z)$. Likewise, from $y_0 \leq y, z$ it follows that $\varepsilon_S(y_0) \leq \varepsilon_S(y), \varepsilon_S(z) \leq y, \gamma_S(z)$. Therefore, we have $x' \ll x, \gamma_S(z)$ and $y' \ll y, \gamma_S(z)$. This shows that γ_S satisfies axiom ($\mathcal{C}5$), and hence that γ_S is indeed a composition map.

It remains to show that $S \cong \mathsf{C}(S) \sqcup_{\gamma_S} \mathsf{D}(S)$; we have to compare S to the definition of $\mathsf{C}(S) \sqcup_{\gamma_S} \mathsf{D}(S)$ in Definition 8.1.1. Evidently, $S = \mathsf{C}(S)^\times \sqcup \mathsf{D}(S)^\times \sqcup \{0\}$ is satisfied. Let $x \in \mathsf{C}(S)^\times$ and $y \in \mathsf{D}(S)^\times$. It follows from Theorem 7.1.6 that $x + y = \gamma_S(x) + y$. Moreover, we know that $y \leq x$ if and only if $y < x$, which happens if and only if $y \leq \gamma_S(x)$. Next, we want to show that $x \leq y$ if and only if $\gamma_S(x) \ll_{\mathsf{D}(S)} y$. If $x \leq y$, then $\gamma_S(x) < x \ll_S y$, hence $\gamma_S(x) \ll_S y$ and therefore $\gamma_S(x) \ll_{\mathsf{D}(S)} y$ by Proposition 7.2.1. Conversely, assume that $\gamma_S(x) \ll_{\mathsf{D}(S)} y$. Then we can find some $z \in \mathsf{D}(S)^\times$ such that $\gamma_S(x) + z \leq y$. By Corollary 7.1.10, it follows that $x < \gamma_S(x) + z \leq y$, and thus $x \leq y$. All the other conditions are easily seen to be satisfied as well. It is obvious that the family $(\alpha_S)_S$ of isomorphisms $\alpha_S \colon S \to \mathsf{C}(S)^\times \sqcup \mathsf{D}(S)^\times \sqcup \{0\}$ can be chosen to be natural. Thus, the theorem is proven. ∎

8.2.2 Corollary. *Let A be a simple, separable, nonelementary, and stably finite C^*-algebra. Then $\mathsf{Cu}(A) \cong \mathsf{C}(A) \sqcup_{\gamma_A} \mathsf{D}(A)$. Moreover, the isomorphism can be chosen to be natural.*

Proof. This follows immediately from the preceding Theorem and the fact that $\mathsf{Cu}(A)$ is decomposable by Corollary 6.2.4. ∎

The following result is a nice extension of Corollary 7.1.4:

8.2.3 Corollary. *Let A be any simple and separable C^*-algebra. The following conditions are equivalent:*

(i) *A is stably finite and nonelementary.*

(ii) *$\mathsf{Cu}(A)$ is decomposable.*

(iii) *$\mathsf{Cu}(A) \cong C \sqcup_\gamma D$ for a simple semigroup $C \in \mathsf{C}$, and a simple semigroup $D \in \mathsf{D}$, and a composition map $\gamma \colon C \to D$ between them.*

Proof. The equivalence of (i) and (ii) was already shown in Corollary 7.1.4. The implication (ii) $\implies$ (iii) follows from Theorem 8.2.1, since $\mathsf{Cu}(A)$ is simple. For the implication (iii) $\implies$ (ii), it follows from Theorem 8.1.3 that $C \sqcup_\gamma D$ is decomposable, and hence that $\mathsf{Cu}(A)$ is decomposable. ∎

8.2.4 Remark. *The above results show that the Cuntz semigroup of a simple, separable, stably finite, and nonelementary C*-algebra A is a composite object, built up from the compact part of the Cuntz semigroup, the noncompact part of the Cuntz semigroup, and the predecessor map. This partially generalises the decomposition result from the end of Chapter 5, where additionally the noncompact part of the Cuntz semigroup could be recovered from the cone of traces $\mathsf{T}(A)$. We will see in Chapter 9 that the pair $\mathsf{T}(A)$, ρ_A is dual, in a certain sense, to the pair $\mathsf{D}(A)$, γ_A, so they are still closely related.*

8.3 Morphisms, functoriality, and category equivalence

The beautiful symmetry of composition and decomposition raises the question of whether both operations form a category equivalence. We will show next that this is indeed the case.

8.3.1 Definition. *Let S_1 and S_2 be simple and decomposable semigroups in Cu. Moreover, let $\varphi\colon \mathsf{C}(S_1) \to \mathsf{C}(S_2)$ be a C-morphism and $\psi\colon \mathsf{D}(S_1) \to \mathsf{D}(S_2)$ a D-morphism. Then the map $\varphi \sqcup \psi\colon\ S_1 \to S_2$ is defined as follows:*

$$(\varphi \sqcup \psi)(x) := \begin{cases} \varphi(x) & \text{if } x \in \mathsf{C}(S_1), \\ \psi(x) & \text{if } x \in \mathsf{D}(S_1). \end{cases}$$

We say that the pair (φ, ψ) is almost natural *if the following conditions hold:*

(i) $\psi(\gamma_{S_1}(x)) \leq \gamma_{S_2}(\varphi(x))$ *for all* $x \in \mathsf{C}(S_1)$.

(ii) $\gamma_{S_2}(\varphi(x)) + \psi(y) = \psi(\gamma_{S_1}(x)) + \psi(y)$ *for all* $x \in \mathsf{C}(S_1)$, $y \in \mathsf{D}(S_1)^\times$.

8.3.2 Remark. *A bit of explanation is in order here. We call a pair (φ, ψ) as above* natural *if $\psi(\gamma_{S_1}(x)) = \gamma_{S_2}(\varphi(x))$ for all $x \in \mathsf{C}(S_1)$. One might expect that the class of Cu-morphisms between simple and decomposable semigroups $S_1, S_2 \in \mathsf{Cu}$ corresponds bijectively to the class of natural pairs between the triples $(\mathsf{C}(S_1), \mathsf{D}(S_1), \gamma_{S_1})$ and $(\mathsf{C}(S_2), \mathsf{D}(S_2), \gamma_{S_2})$. This would mean that the composition maps implement a natural transformation (in the category of ordered abelian monoids) from the functor $\mathsf{C}(\cdot)$ to the functor $\mathsf{D}(\cdot)$ when both functors are restricted to the category of simple and decomposable semigroups in Cu. It is an interesting question for which subcategories of simple and decomposable semigroups in Cu the conditions of naturality and almost naturality are equivalent; some preliminary results pertaining to this problem will be provided in Chapter 10. For now, we will proceed to show that the weaker condition of almost naturality, as given above, provides us with the correct notion of morphism.*

8.3.3 Theorem. ***Morphisms:*** *Let S_1, $S_2 \in Cu$ be simple and decomposable.*

(*i*) *If $\chi\colon S_1 \to S_2$ is a Cu-morphism, then the map $C(\chi)\colon C(S_1) \to C(S_2)$ is a C-morphism, the map $D(\chi)\colon D(S_1) \to D(S_2)$ is a D-morphism, and the pair $(C(\chi), D(\chi))$ is almost natural.*

(*ii*) *If $\varphi\colon C(S_1) \to C(S_2)$ is a C-morphism and $\psi\colon D(S_1) \to D(S_2)$ is a D-morphism such that the pair (φ, ψ) is almost natural, then the map $\varphi \sqcup \psi$ is a Cu-morphism.*

Proof.

(*i*) Most of this was already shown in Theorem 7.2.4. We need only show that the pair $(\mathsf{C}(\chi), \mathsf{D}(\chi))$ is almost natural. This is trivial if $\chi = 0$, so we may assume $\chi \neq 0$. Since S_1 is simple, this means that $\mathsf{C}(\chi)(\mathsf{C}(S_1)^\times) \subseteq \mathsf{C}(S_2)^\times$ and $\mathsf{D}(\chi)(\mathsf{D}(S_1)^\times) \subseteq \mathsf{D}(S_2)^\times$. We show condition (*i*) of Definition 8.3.1 first. Let x be any element of $\mathsf{C}(S_1)$. There is nothing to show for $x = 0$, so we may assume $x > 0$. Then $0 < \gamma_{S_1}(x) < x$ and therefore $0 < \chi(\gamma_{S_1}(x)) \leq \chi(x)$. Since $\chi(x)$ is compact and $\chi(\gamma_{S_1}(x))$ is noncompact, we have $\chi(\gamma_{S_1}(x)) < \chi(x)$, and it follows that $\chi(\gamma_{S_1}(x)) \leq \gamma_{S_2}(\chi(x))$. But this means precisely that $\mathsf{D}(\chi)(\gamma_{S_1}(x)) \leq \gamma_{S_2}(\mathsf{C}(\chi)(x))$. Next, we show condition (*ii*) of Definition 8.3.1. Let $x \in \mathsf{C}(S_1)$ and $y \in \mathsf{D}(S_1)^\times$. It follows from Theorem 7.1.6 that

$$\begin{aligned} \gamma_{S_2}(\chi(x)) + \chi(y) &= \chi(x) + \chi(y) &&= \chi(x+y) \\ &= \chi(\gamma_{S_1}(x) + y) &&= \chi(\gamma_{S_1}(x)) + \chi(y), \end{aligned}$$

but this means precisely that $\gamma_{S_2}(\mathsf{C}(\chi)(x)) + \mathsf{D}(\chi)(y) = \mathsf{D}(\chi)(\gamma_{S_1}(x)) + \mathsf{D}(\chi)(y)$.

(*ii*) If $\mathsf{C}(S_1) = \{0\}$, then $\varphi \sqcup \psi = \psi$, and $\varphi \sqcup \psi$ is a Cu-morphism. We may therefore assume that $\mathsf{C}(S_1)$ is nonzero, which implies that $\mathsf{D}(S_1)$ is also nonzero. By simplicity of S_1, it follows that $\varphi = 0$ or $\varphi(\mathsf{C}(S_1)^\times) \subseteq \mathsf{C}(S_2)^\times$ and, likewise, that $\psi = 0$ or $\psi(\mathsf{D}(S_1)^\times) \subseteq \mathsf{D}(S_2)^\times$. If $\psi = 0$, then we have $\gamma_{S_2} \circ \varphi = 0$ by condition (*ii*) of Definition 8.3.1, which implies $\varphi = 0$, since any nonzero element in $\mathsf{C}(S_2)$ has a nonzero predecessor by the definition of decomposability. Conversely, assume that $\varphi = 0$. Let $x \in \mathsf{C}(S_1)$ be nonzero. Pick any $y \in \mathsf{D}(S_1)^\times$, and pick any rapidly increasing sequence $(y_n)_n$ in $\mathsf{D}(S_1)$ with supremum y. Since $\sup_n y_n$ is noncompact, this sequence cannot be eventually constant; thus we may assume that it increases strictly. For each n, we have $y_n < y_{n+1} \ll y_{n+2} \leq \infty = \infty \cdot x$, so we can find for each n a number $m_n \in \mathbb{N}$ with $y_n < y_{n+1} \leq m_n \cdot x$, and therefore $y_n \leq \gamma_{S_1}(m_n \cdot x)$. It follows that $\psi(y_n) \leq \psi(\gamma_{S_1}(m_n \cdot x)) \leq \gamma_{S_2}(\varphi(m_n \cdot x)) = 0$ for each n by condition (*i*) of Definition 8.3.1. Since ψ is sup-preserving, it follows that $\psi(y) = 0$. Since this is true for every $y \in \mathsf{D}(S_1)^\times$, it follows that $\psi = 0$. Thus, we find that $\varphi = 0$ if and only if $\psi = 0$, and in that case $\varphi \sqcup \psi = 0$ is certainly a Cu-morphism. Hence, we may assume from now on that $\varphi(\mathsf{C}(S_1)^\times) \subseteq \mathsf{C}(S_2)^\times$, and that $\psi(\mathsf{D}(S_1)^\times) \subseteq \mathsf{D}(S_2)^\times$.

It is obvious that $\varphi \sqcup \psi$ is zero-preserving. Let $x \in \mathsf{C}(S)^\times$ and $y \in \mathsf{D}(S)^\times$. Then it follows from Theorem 7.1.6 and Definition 8.3.1 (*ii*) that

$$\begin{aligned}(\varphi \sqcup \psi)(x+y) &= \psi(x+y) \\ &= \psi(\gamma_{S_1}(x)+y) \\ &= \psi(\gamma_{S_1}(x))+\psi(y) \\ &= \gamma_{S_2}(\varphi(x))+\psi(y) \\ &= \varphi(x)+\psi(y) \\ &= (\varphi \sqcup \psi)(x)+(\varphi \sqcup \psi)(y).\end{aligned}$$

Thus, we have shown that $\varphi \sqcup \psi$ is additive.

Next, we show that $\varphi \sqcup \psi$ is order-preserving. Let $x \in \mathsf{C}(S_1)^\times$ and $y \in \mathsf{D}(S_1)^\times$. If $x \leq y$, then there is $z \in \mathsf{D}(S_1)^\times$ with $\gamma_{S_1}(x)+z = x+z = y$. Since ψ is additive and order-preserving, we get $\psi(\gamma_{S_1}(x))+\psi(z) = \psi(y)$, and therefore $\gamma_{S_2}(\varphi(x))+\psi(z) = \psi(y)$ by Definition 8.3.1 (*ii*). Since $z > 0$ implies $\psi(z) > 0$, we have $\varphi(x) \leq \gamma_{S_2}(\varphi(x))+\psi(z)$ by Corollary 7.1.10, and thus $(\varphi \sqcup \psi)(x) = \varphi(x) \leq \psi(y) = (\varphi \sqcup \psi)(y)$. If $y \leq x$, then $y \leq \gamma_{S_1}(x)$, hence $(\varphi \sqcup \psi)(y) = \psi(y) \leq \psi(\gamma_{S_1}(x)) \leq \gamma_{S_2}(\varphi(x)) \leq \varphi(x) = (\varphi \sqcup \psi)(x)$ by Definition 8.3.1 (*i*). Thus, $(\varphi \sqcup \psi)$ is order-preserving.

Next, we show that $(\varphi \sqcup \psi)$ is $\ll$-preserving. If $x \in \mathsf{C}(S_1)$ and $y \in S_1$, then $x \ll y$ is equivalent to $x \leq y$, which we have already shown implies $(\varphi \sqcup \psi)(x) \leq (\varphi \sqcup \psi)(y)$; since $(\varphi \sqcup \psi)(x) = \varphi(x) \in \mathsf{C}(S_1)$ is compact, this implies $(\varphi \sqcup \psi)(x) \ll (\varphi \sqcup \psi)(y)$. Analogously, if $x \in S_1$ and $y \in \mathsf{C}(S_1)$, then $x \ll y$ is equivalent to $x \leq y$, hence $(\varphi \sqcup \psi)(x) \leq (\varphi \sqcup \psi)(y)$ and therefore $(\varphi \sqcup \psi)(x) \ll (\varphi \sqcup \psi)(y)$ since $(\varphi \sqcup \psi)(y) = \varphi(y)$ is compact. Finally, if x, $y \in \mathsf{D}(S_1)$ such that $x \ll y$, then $(\varphi \sqcup \psi)(x) = \psi(x) \ll \psi(y) = (\varphi \sqcup \psi)(y)$ since ψ is $\ll$-preserving by Definition 7.2.2. Therefore, we have shown that $(\varphi \sqcup \psi)$ is also $\ll$-preserving.

Lastly, let $(x_n)_n$ be any increasing sequence in S_1. If $x := \sup_n x_n$ is compact, then the sequence is eventually constant. It follows immediately that

$$(\varphi \sqcup \psi)(\sup_n x_n) = (\varphi \sqcup \psi)(x) = \varphi(x) = \sup_n \varphi(x_n) = \sup_n (\varphi \sqcup \psi)(x_n).$$

If, on the other hand, $x := \sup_n x_n$ is noncompact, then we may assume that the sequence $(x_n)_n$ increases strictly (otherwise, an argument analogous to the one above can be applied). This implies $\varepsilon_S(x_n) \leq x_n \leq \varepsilon_S(x_{n+1})$ for every n. Since $(\varepsilon_S(x_n))_n$ is an increasing sequence in $\mathsf{D}(S_1)$, we have $x = \sup_n x_n = \sup_n \varepsilon_S(x_n)$. It follows that

$$\begin{aligned}\psi(\varepsilon_S(x_n)) &= (\varphi \sqcup \psi)(\varepsilon_S(x_n)) \\ &\leq (\varphi \sqcup \psi)(x_n)\end{aligned}$$

$$\leq (\varphi \sqcup \psi)(\varepsilon_S(x_{n+1}))$$
$$= \psi(\varepsilon_S(x_{n+1})),$$

which implies that $\sup_n (\varphi \sqcup \psi)(x_n) = \sup_n \psi(\varepsilon_S(x_n))$. But since ψ is sup-preserving by Definition 7.2.2 and $\sup_n x_n$ lies in $\mathsf{D}(S_1)$, it follows that

$$(\varphi \sqcup \psi)(\sup_n x_n) = \psi(\sup_n x_n) = \psi(\sup_n \varepsilon_S(x_n))$$
$$= \sup_n \psi(\varepsilon_S(x_n)) = \sup_n (\varphi \sqcup \psi)(x_n).$$

Thus, we have shown that $(\varphi \sqcup \psi)$ is sup-preserving. This completes the proof that $(\varphi \sqcup \psi)$ is a Cu-morphism.

■

8.3.4 Definition. *Let Cu_{dec} be the category of all simple and decomposable semigroups $S \in \mathsf{Cu}$, with Cu-maps as morphisms. Let Cu_{com} be the category of all composable triples (C, D, γ) where C is a simple semigroup in C, and D is a simple semigroup in D, and $\gamma\colon C \to D$ is a composition map (as defined in Definition 8.1.2), with almost natural pairs (φ, ψ) (as defined in Definition 8.3.1) as morphisms.*
We define two functors $\mathsf{Dec}\colon \mathsf{Cu}_{dec} \to \mathsf{Cu}_{com}$ and $\mathsf{Com}\colon \mathsf{Cu}_{com} \to \mathsf{Cu}_{dec}$ as follows: for all $S \in \mathsf{Cu}_{dec}$, we let $\mathsf{Dec}(S) := (\mathsf{C}(S), \mathsf{D}(S), \gamma_S)$, and for all Cu-morphisms $\chi\colon S_1 \to S_2$, we let $\mathsf{Dec}(\chi) := (\mathsf{C}(\chi), \mathsf{D}(\chi))$. Moreover, for all $(C, D, \gamma) \in \mathsf{Cu}_{com}$, we let $\mathsf{Com}(C, D, \gamma) := C \sqcup_\gamma D$, and for all almost natural pairs $(\varphi, \psi)\colon (C, D, \gamma) \to (C', D', \gamma')$, we let $\mathsf{Com}(\varphi, \psi) := \varphi \sqcup \psi$. It is easy to see that both Dec and Com preserve composition and identity morphisms, and are therefore functorial.

8.3.5 Theorem. *The composition and decomposition functors implement an equivalence between the categories Cu_{dec} and Cu_{com}.*

Proof. We remind the reader that, for any triple $(C, D, \gamma) \in \mathsf{Cu}_{com}$, the underlying set of the semigroup $C \sqcup_\gamma D$ is the disjoint union $C^\times \sqcup D^\times \sqcup \{0\}$. Up until now, we have intentionally been somewhat sloppy with our treatment of disjoint unions $X \sqcup Y$, by silently identifying the sets X and Y with their copies in $X \sqcup Y$. Due to the category-theoretic nature of this proof, we shall have to be more rigorous and spell out the canonical bijections between X, Y and their copies in $X \sqcup Y$ explicitly. This should not cause any undue confusion to the careful reader.

First, we need to show that the identity functor on Cu_{dec} is naturally isomorphic to the functor $\mathsf{Com} \circ \mathsf{Dec}$. For a semigroup $S \in \mathsf{Cu}_{dec}$, we have $\mathsf{Dec}(S) = (\mathsf{C}(S), \mathsf{D}(S), \gamma_S)$, so the underlying set of the semigroup $\mathsf{Com}(\mathsf{Dec}(S)) = \mathsf{C}(S) \sqcup_{\gamma_S} \mathsf{D}(S)$ is the disjoint union $\mathsf{C}(S)^\times \sqcup \mathsf{D}(S)^\times \sqcup \{0\}$. Let c_S be the canonical bijection that identifies $\mathsf{C}(S)^\times$ with its copy in $\mathsf{C}(S)^\times \sqcup \mathsf{D}(S)^\times \sqcup \{0\}$. Likewise, let d_S be the canonical bijection that identifies $\mathsf{D}(S)^\times$ with its copy in $\mathsf{C}(S)^\times \sqcup \mathsf{D}(S)^\times \sqcup \{0\}$. Let $\alpha_S := c_S \cup d_S \cup (0 \mapsto 0)$. By Theorem 8.2.1, we know that the map $\alpha_S \colon S \to \operatorname{im}(c_S) \cup \operatorname{im}(d_S) \cup \{0\}$ is an isomorphism in Cu_{dec} between S and $\mathsf{C}(S) \sqcup_\gamma \mathsf{D}(S)$, and therefore an isomorphism in Cu_{dec} between S and $\mathsf{Com}(\mathsf{Dec}(S))$. Let $S_1,\ S_2$ be any semigroups in Cu_{dec}, and let $\varphi : S_1 \to S_2$ be a morphism in Cu_{dec}. By Definition 8.3.4, we have $\mathsf{Dec}(\varphi) = (\varphi|_{\mathsf{C}(S_1)},\ \varphi|_{\mathsf{D}(S_1)})$, and it follows from Definition 8.3.4 that $\mathsf{Com}(\mathsf{Dec}((\varphi)) = (c_2 \circ \varphi|_{\mathsf{C}(S)} \circ c_1^{-1}) \cup (d_2 \circ \varphi|_{\mathsf{D}(S)} \circ d_1^{-1}) \cup (0 \mapsto 0)$. It is easy to see now that the diagram

$$\begin{array}{ccc} S_1 & \xrightarrow{\quad\varphi\quad} & S_2 \\ \downarrow{\scriptstyle \alpha_{S_1}} & & \downarrow{\scriptstyle \alpha_{S_2}} \\ \mathsf{Com}(\mathsf{Dec}(S_1)) & \xrightarrow[\mathsf{Com}(\mathsf{Dec}(\varphi))]{} & \mathsf{Com}(\mathsf{Dec}(S_2)) \end{array}$$

is commutative. Thus, the family $(\alpha_S)_{S \in \mathsf{Cu}_{dec}}$ is a natural isomorphism between the identity functor of Cu_{dec} and the endofunctor $\mathsf{Com} \circ \mathsf{Dec}$ of Cu_{dec}.

Second, we need to show that the identity functor on Cu_{com} is naturally isomorphic to the functor $\mathsf{Dec} \circ \mathsf{Com}$. For any triple $T = (C, D, \gamma)$ in Cu_{com}, we have $\mathsf{Com}(T) = C \sqcup_\gamma D$ by Definition 8.3.4, so the underlying set of $\mathsf{Com}(T)$ is the disjoint union $C^\times \sqcup D^\times \sqcup \{0\}$. Let c_T be the canonical bijection that identifies $C^\times$ with its copy in $C \sqcup_\gamma D$, and let d_T be the canonical bijection that identifies $D^\times$ with its copy in $C \sqcup_\gamma D$. By Definition 8.3.4, we then have $\mathsf{Dec}(\mathsf{Com}(T)) = (\operatorname{im}(c_T) \cup \{0\},\ \ \operatorname{im}(d_T) \cup \{0\},\ \ (d_T \circ \gamma \circ c_T^{-1}) \cup (0 \mapsto 0))$. Let $\beta_T := (c_T \cup (0 \mapsto 0),\ d_T \cup (0 \mapsto 0))$; obviously β_T is an isomorphism in Cu_{com} between the triples T and $\mathsf{Dec}(\mathsf{Com}(T))$. Let T_1, T_2 be objects in Cu_{com}, and let $(\varphi, \psi) \colon T_1 \to T_2$ be a morphism in Cu_{com}. By Definition 8.3.4, $\mathsf{Com}((\varphi, \psi)) = (c_{T_2} \circ \varphi \circ c_{T_1}^{-1}) \cup (d_{T_2} \circ \psi \circ d_{T_1}^{-1}) \cup (0 \mapsto 0)$, and $\mathsf{Dec}(\mathsf{Com}((\varphi, \psi)))$ is the pair $((c_{T_2} \circ \varphi \circ c_{T_1}^{-1}) \cup (0 \mapsto 0),\ (d_{T_2} \circ \psi \circ d_{T_1}^{-1}) \cup (0 \mapsto 0))$. It is easy to see now that the diagram

$$\begin{array}{ccc} T_1 & \xrightarrow{\quad(\varphi,\psi)\quad} & T_2 \\ \downarrow{\scriptstyle \beta_{T_1}} & & \downarrow{\scriptstyle \beta_{T_2}} \\ \mathsf{Dec}(\mathsf{Com}(T_1)) & \xrightarrow[\mathsf{Dec}(\mathsf{Com}((\varphi,\psi)))]{} & \mathsf{Dec}(\mathsf{Com}(T_2)) \end{array}$$

is commutative. Thus, the family $(\beta_T)_{T \in \mathsf{Cu}_{com}}$ is a natural isomorphism between the identity functor of Cu_{com} and the endofunctor $\mathsf{Dec} \circ \mathsf{Com}$ of Cu_{com}. In total, we have shown that the functors Dec and Com form an equivalence between the categories Cu_{dec} and Cu_{com}. ■

8.4 Construction of semigroups in Cu

In this section, we want to provide a method to construct a simple and decomposable semigroup in Cu from certain triples $((C,u),K,r)$, where (C,u) is a simple, algebraically ordered, abelian monoid with order unit, K is a metrisable Choquet simplex, and $r\colon K\to\mathcal{S}(C,u)$ is an affine and continuous map (here, $\mathcal{S}(C,u)$ denotes the set of states $\sigma\colon C\to\mathbb{R}_+$ on (C,u), equipped with the topology of pointwise convergence). We need some preparation first.

8.4.1 Lemma. *Let K be a nonempty Choquet simplex, and let $f,g\in \mathsf{LAff}_{++}(K)$ such that f is continuous and finite. The following conditions are equivalent:*

(*i*) *$f\ll g$ in $\mathsf{LAff}_{++}(K)$.*

(*ii*) *$f(x)<g(x)$ for every $x\in K$.*

Proof.
(*i*) $\implies$ (*ii*): Let $g_n:=(1-\frac{1}{n})g$ for each $n\in\mathbb{N}$. Clearly, $(g_n)_n$ is an increasing sequence in $\mathsf{LAff}_{++}(K)$ with supremum g. Since $f\ll g$ in $\mathsf{LAff}_{++}(K)$, we have $f\leq g_n$ for some n. Pick any $x\in K$. If $g(x)<\infty$, then $f(x)\leq g_n(x)<g(x)$. If $g(x)=\infty$, then $f(x)<g(x)$ since f is finite. Hence, we have $f(x)<g(x)$ for every $x\in K$.
(*ii*) $\implies$ (*i*): Let $(h_n)_n$ be any increasing sequence in $\mathsf{LAff}_{++}(K)$ with $g\leq\sup_n h_n$. We need to show that $f\leq h_n$ for some n. First, fix any $x\in K$. Since $f(x)<g(x)$, we can find some $\varepsilon(x)>0$ such that $f(x)+\varepsilon(x)\leq g(x)$. Let $V(x):=f^{-1}[(-\infty,\ f(x)+\frac{1}{2}\varepsilon(x))]$, and let $W_n(x):=h_n^{-1}[(f(x)+\frac{1}{2}\varepsilon(x),\ \infty]]$. Then $V(x)$ is open, and $x\in V_x$. $W_n(x)$ is open as well (since h_n is lower semicontinuous), and since $f(x)+\varepsilon(x)\leq g(x)\leq\sup_n h_n(x)$, we have $x\in W_n(x)$ for sufficiently large n. Let $N(x)\in\mathbb{N}$ be so large that $x\in W_{N(x)}(x)$. Let $U(x):=V(x)\cap W_{N(x)}(x)$. Then $U(x)$ is an open neighbourhood of x, and we have $f(y)<f(x)+\frac{1}{2}\varepsilon(x)<h_{N(x)}(y)$, and therefore $f(y)\leq h_{N(x)}(y)$, for every $y\in U(x)$. Since K is compact, we can find finitely many elements $x_1,...,x_k\in K$ with $K\subseteq U(x_1)\cup...\cup U(x_k)$. Let $n:=\max\{N(x_1),...,N(x_k)\}$. By construction, we have $f(y)\leq h_n(y)$ for every $y\in K$. We conclude that $f\ll g$ as expected. ■

The following theorem establishes a functorial relationship between the category of nonempty, metrisable Choquet simplices (with continuous, affine maps as arrows) and the category D:

8.4.2 Theorem.

(*i*) *If K is a nonempty metrisable Choquet simplex, then $S:=\mathsf{LAff}_{++}(K)\cup\{0\}$ is a semigroup in Cu. Moreover, S is simple and contains no nonzero compact elements. Hence, S is a simple semigroup in the category D.*

(*ii*) *If $\alpha\colon K'\to K$ is a continuous, affine map between nonempty, metrisable Choquet simplices K' and K, then the induced map $\alpha^*\colon f\mapsto f\circ\alpha$ is a D-morphism between the semigroups $\mathsf{LAff}_{++}(K)\cup\{0\}$ and $\mathsf{LAff}_{++}(K')\cup\{0\}$.*

Proof.

(i) We will prove the axioms ($\mathcal{O}1$) to ($\mathcal{O}6$) one by one. Afterwards, we will show that S contains no nonzero compact elements (this part is denoted by ($\mathcal{D}$)). It then follows easily that S is simple (which, for semigroups in D or Cu, means not having any non-trivial *closed* ideals): since any nonzero element $f \in S$ is a function in $\mathsf{LAff}_{++}(K)$, and hence takes only nonzero values on K, the element $\infty \cdot f = \sup_n (n \cdot f)$ is the function that takes the value ∞ everywhere on K. Since the closed ideal of S generated by f is precisely the set of all $g \in S$ with $g \leq \infty \cdot f$, it follows that the closed ideal generated by any nonzero $f \in S$ is all of S, and hence that S is a simple semigroup in D.

($\mathcal{O}1$): Every pointwise supremum of an increasing sequence of affine and lower semi-continuous functions is easily seen to be affine and lower semicontinuous. Thus, every increasing sequence in $S = \mathsf{LAff}_{++}(K) \cup \{0\}$ has a supremum in S.

($\mathcal{O}2$): Since it is obvious that 0 is compact, and hence the supremum of the rapidly increasing sequence with all terms equal to 0, we need only consider the case that $f \in \mathsf{LAff}_{++}(K)$. We will show that f is the supremum of a rapidly increasing sequence $(f_n)_n$ in $\mathsf{LAff}_{++}(K)$, and moreover, that the terms f_n can be chosen to be continuous and finite. We will essentially follow the proof of A. Tikuisis in [35], Subsection 2.2.2. First, we note that the lower semicontinuous function f takes a minimal value on the compact space K. Hence, there is some $\varepsilon > 0$ with $f(y) > \varepsilon$ for each $y \in K$. Second, we note that $f(x) = \sup \{g(x) \,|\, g : K \to \mathbb{R}$ affine and continuous, with $g \ll f\}$, by Proposition 11.8 of [17]. Strictly speaking, the theorem only covers the case where f takes values in $(0, \infty)$, but the same proof also works if f takes values in $(0, \infty]$. Note that the symbol $\ll$ in the theorem means that $g(y) < f(y)$ for every $y \in K$. Since the constant function ε is an element of this set, we may assume that all the functions g in this set take values in $(0, \infty)$; it then follows from the preceding lemma that our usage of the symbol $\ll$ coincides with the usage of $\ll$ in the theorem. Hence, we have $f(x) = \sup \{g(x) \,|\, g : K \to (0, \infty)$ affine and continuous, with $g \ll f\}$. Third, it now follows from [13] that we can find an increasing net $(h_\alpha)_\alpha$ of affine and continuous functions $h_\alpha \colon K \to (0, \infty)$ with $g = \sup_\alpha h_\alpha$. Fourth, as shown by Tikuisis in Lemma 2.2.6 of [35], we can find an increasing sequence $(h_{\alpha_n})_n$ with $g = \sup_n h_{\alpha_n}$ since K is metrisable. Fifth, let $g_n := (1 - \frac{1}{n})h_{\alpha_n}$ for each $n \in \mathbb{N}$. Then $(g_n)_n$ is an increasing sequence of continuous and finite functions in $\mathsf{LAff}_{++}(K)$ with $\sup_n g_n = g$. For each $n \in \mathbb{N}$ and $y \in K$, we have $(1 - \frac{1}{n})h_{\alpha_n}(y) \leq (1 - \frac{1}{n})h_{\alpha_{n+1}}(y) < (1 - \frac{1}{n+1})h_{\alpha_{n+1}}(y)$, and therefore $g_n(y) < g_{n+1}(y)$. It follows from the preceding lemma that $(g_n)_n$ is, indeed, rapidly increasing.

($\mathcal{O}3$): Let $f_1, f_2, g_1, g_2 \in S$ with $f_1 \ll g_1$ and $f_2 \ll g_2$. If any of these elements is zero, then the claim is easily seen to be true; we may therefore assume that all elements are functions in $\mathsf{LAff}_{++}(K)$. Using ($\mathcal{O}2$), we can find continuous and finite functions h_1, h_2 in $\mathsf{LAff}_{++}(K)$ such that $f_1 \leq h_1 \ll g_1$ and $f_2 \leq h_2 \ll g_2$. By the preceding lemma, we have $h_1(y) < g_1(y)$ and $h_2(y) < h_2(y)$ for every $y \in K$, and therefore $f_1(y)+f_2(y) \leq h_1(y)+h_2(y) < g_1(y)+g_2(y)$ for every $y \in K$. Since h_1+h_2 is continuous and finite, it follows again by the preceding lemma that $f_1 + f_2 \leq h_1 + h_2 \ll g_1 + g_2$, and therefore $f_1 + f_2 \ll g_1 + g_2$.

($\mathcal{O}4$): Since the order on $\mathsf{LAff}_{++}(K) \cup \{0\}$ is pointwise comparison, it follows immediately that $\sup_n (f_n + g_n) = (\sup_n f_n) + (\sup_n g_n)$ for every pair of increasing sequences $(f_n)_n$, $(g_n)_n$ in the semigroup S.

($\mathcal{O}5$): Let $f' \ll f \leq g$. If $f' = 0$, then $h := g$ is an element in S with $f'+h \leq g \leq f+h$. We may therefore assume that f' is nonzero, and hence that all three elements are in $\mathsf{LAff}_{++}(K)$. Using ($\mathcal{O}2$), we can find a continuous and finite element $v \in \mathsf{LAff}_{++}(K)$ such that $f' \ll v \ll f \leq g$. Since $v \ll g$, we have $v(y) < g(y)$ for all $y \in K$ by the preceding lemma. It follows that $h := g - v \in \mathsf{LAff}_{++}(K)$ such that $v + h = g$. Since $f' \leq v \leq f$, we have $f' + h \leq v + h = g$ and $g = v + h \leq f + h$, and therefore $f' + h \leq g \leq f + h$ as expected.

($\mathcal{O}6$): Let $f' \ll f \leq g_1 + g_2$. We exclude some corner cases first. If $f = 0$, then $f' = 0$, and the elements $h_1, h_2 := 0$ satisfy $h_1 \leq g_1, f$; and $h_2 \leq g_2, f$; and $f' \leq h_1 + h_2$. We may therefore assume that $f \neq 0$. If $f \neq 0$ and $f' = 0$, we can find by ($\mathcal{O}2$) an element $f'' \neq 0$ with $f' \ll f'' \ll f$, hence we may as well assume that $f' \neq 0$. Since $f', f \neq 0$, at least one of g_1, g_2 must be nonzero. If $g_1 = 0$, then $f' \ll f \leq g_2$. Let $h_1 := 0$ and $h_2 := f$, then $h_1 \leq g_1, f$; and $h_2 \leq g_2, f$; and $f' \leq h_1 + h_2$. An analogous argument works if $g_2 = 0$. We may therefore assume that f', f, g_1, g_2 are all in $\mathsf{LAff}_{++}(K)$. Using axioms ($\mathcal{O}2$) – ($\mathcal{O}4$), we can then find continuous and finite functions $v, g_1', g_2' \in \mathsf{LAff}_{++}(K)$ such that $g_1' \ll g_1$, and $g_2' \ll g_2$, and $f' \ll v \ll f$, and $v \leq g_1' + g_2'$. By [13], the semigroup consisting of all non-negative, continuous, and affine function $K \to [0, \infty)$ has the Riesz decomposition property. Since v, g_1, and g_2 are elements of this semigroup, we can find non-negative, continuous, and finite functions $w_1, w_2 \colon K \to [0, \infty)$ such that $w_1 \leq g_1'$, and $w_2 \leq g_2'$, and $v = w_1 + w_2$. By the preceding lemma, we know that $g_1' < g_1$, and $g_2' < g_2$, and $v < f$ pointwise on K. Since the lower semicontinuous functions $g_1 - g_1'$, $g_2 - g_2'$, $f - v$ all take a nonzero minimal value on the compact space K, we can find some $\varepsilon > 0$ such that $g_1' + \varepsilon \leq g_1$, and $g_2' + \varepsilon \leq g_2$, and $v + \varepsilon \leq f$. Let $h_1 := w_1 + \varepsilon$ and $h_2 := w_2 + \varepsilon$, then $h_1, h_2 \in \mathsf{LAff}_{++}(K)$. Since $w_i \leq g_i', v$, we have $h_i \leq g_i' + \varepsilon, v + \varepsilon$, and therefore $h_i \leq g_i, f$ for $i \in \{1, 2\}$. Moreover, we have $f' \leq v = w_1 + w_2 \leq h_1 + h_2$. Thus, the elements h_1, h_2 are as required.

($\mathcal{D}$): Assume that $f \ll f$ for some $f \in \mathsf{LAff}_{++}(K)$. By ($\mathcal{O}2$), we can find a continuous and finite function $g \in \mathsf{LAff}_{++}(K)$ such that $f \ll g \ll f$. By the preceding lemma, this implies that $f(y) \leq g(y) < f(y)$ for every $y \in K$, which is absurd. Thus, we conclude that S has no nonzero compact elements.

(*ii*) It is easy to see that the map α^* preserves addition and the zero element, and moreover, that it preserves the order and suprema. Hence, we need only show that α^* preserves compact containment. Let f, g be any elements of $\mathsf{LAff}_{++}(K)$ such that $f \ll g$; we need to show that $\alpha^*(f) \ll \alpha^*(g)$ (note that this claim is obvious for $f = 0$ or $g = 0$, so we do not consider those cases). It follows from the proof of axiom $(\mathcal{O}2)$ above that we can find a finite and continuous $h \in \mathsf{LAff}_{++}(K)$ such that $f \leq h \ll g$. By Lemma 8.4.1, it follows that $f(y) \leq h(y) < g(y)$ for every $y \in K$, and hence that $f(\alpha(y')) \leq h(\alpha(y')) < g(\alpha(y'))$ for every $y' \in K'$. Since $h \circ \alpha$ is a finite and continuous function in $\mathsf{LAff}_{++}(K')$, it follows again by Lemma 8.4.1 that $f \circ \alpha \leq h \circ \alpha \ll g \circ \alpha$ in $\mathsf{LAff}_{++}(K')$, and therefore $\alpha^*(f) \ll \alpha^*(g)$ in $\mathsf{LAff}_{++}(K')$. This shows that α^* preserves compact containment, and hence that it is a D-morphism.

■

8.4.3 Theorem. *Let (C, u) be a simple, algebraically ordered, abelian monoid with order unit, let K be a nonempty metrisable Choquet simplex, and let $r\colon K \to \mathcal{S}(C, u)$ be an affine and continuous map such that for $x_1, x_2 \in C$, we have $x_1 < x_2$ whenever $r(y)(x_1) < r(y)(x_2)$ for every element $y \in K$. Then the evaluation map $\gamma_r\colon x \mapsto (y \to r(y)(x))$ is a composition map from the semigroup $C \in \mathsf{C}$ to the semigroup $\mathsf{LAff}_{++}(K) \cup \{0\} \in \mathsf{D}$. It follows that $S := C \sqcup_{\gamma_r} (\mathsf{LAff}_{++}(K) \cup \{0\})$ is a simple and decomposable semigroup in Cu such that $\mathsf{C}(S) = C$, and $\mathsf{D}(S)^\times = \mathsf{LAff}_{++}(K)$, and $\gamma_S = \gamma_r$.*

Proof. Since r is affine and continuous, the function $\gamma_r(x)$ is affine and continuous. Moreover, the function $\gamma_r(x)$ takes only finite values. Clearly, we have $\gamma_r(0) = 0$. Since C is simple and $r(y)$ is a state, hence nonzero, for every $y \in K$, we have $\gamma_r(x)(y) = r(y)(x) > 0$ for every nonzero $x \in C$ and every $y \in K$. It follows that $\gamma_r(x)$ is indeed an element of $\mathsf{LAff}_{++}(K) \cup \{0\}$ for every $x \in C$. We will now prove the axioms $(\mathcal{C}1)$ to $(\mathcal{C}5)$, thus showing that γ_r is a composition map. The remaining claims will then follow from Theorem 8.1.3.
$(\mathcal{C}1)$: We have already shown that $\gamma_r(x) = 0$ if and only if $x = 0$.
$(\mathcal{C}2)$: We have $\gamma_r(x_1 + x_2)(y) = r(y)(x_1 + x_2) = r(y)(x_1) + r(y)(x_2) = \gamma_r(x_1)(y) + \gamma_r(x_2)(y)$ for all $x_1, x_2 \in C$ and for all $y \in K$. It follows that γ_r is an additive map.
$(\mathcal{C}3)$: Let x_1, x_2 be any elements of $C^\times$ such that $\gamma_r(x_1) \ll \gamma_r(x_2)$. By $(\mathcal{C}1)$, the elements $\gamma_r(x_1)$ and $\gamma_r(x_2)$ are nonzero, and therefore functions in $\mathsf{LAff}_{++}(K)$. As mentioned above, both functions are also continuous and finite. It follows from Lemma 8.4.1 that we have $\gamma_r(x_1)(y) < \gamma_r(x_2)(y)$ for all $y \in K$, and therefore $r(y)(x_1) < r(y)(x_2)$ for all $y \in K$. But this implies that $x_1 < x_2$.
$(\mathcal{C}4)$: Let $x \in C$ and $f \in \mathsf{LAff}_{++}(K)$ such that $\gamma_r(x) \ll f$. If $x = 0$, then $\gamma_r(x) + f = f$, and there is nothing else to show. If $x \neq 0$, then we have $\gamma_r(x)(y) < f(y)$ for every $y \in K$ by Lemma 8.4.1, since $\gamma_r(x)$ is nonzero, continuous, and finite. Let $g := f - \gamma_r(x)$, then $g \in \mathsf{LAff}_{++}(K)$ and $\gamma_r(x) + g = f$. Conversely, suppose that $x \in C$ and $f, g \in \mathsf{LAff}_{++}(K)$ such that $\gamma_r(x) + g = f$. If $x = 0$, then clearly we have $\gamma_r(x) = 0 \ll f$. If $x \neq 0$, then $\gamma_r(x)$

is nonzero, continuous, and finite, and it follows from $\gamma_r(x) + g = f$ that $\gamma_r(x)(y) < f(y)$ for all $y \in K$. By Lemma 8.4.1, this implies $\gamma_r(x) \ll f$.
$(\mathcal{C}5)$: Let $x \in C^\times$ and $f, g \in \mathsf{LAff}_{++}(K)$ such that $\gamma_r(x) \ll f + g$. By the proof of Theorem 8.4.2, we can find continuous and finite function $f_0, g_0 \in \mathsf{LAff}_{++}(K)$ such that $f_0 \ll f$, and $g_0 \ll g$, and $\gamma_r(x) \leq f_0 + g_0$. By Lemma 8.4.1, we have $f_0(y) < f(y)$ and $g_0(y) < g(y)$ for every $y \in K$. Thus, the functions $f - f_0$, $g - g_0$, and $\gamma_r(x)$ are all in $\mathsf{LAff}_{++}(K)$. Since lower semicontinuous functions take a minimal value on the compact space K, it follows that we can find some $\varepsilon > 0$ such that $f_0 + \varepsilon \leq f$, and $g_0 + \varepsilon \leq g$, and $\varepsilon < \gamma_r(x)$ pointwise on K. Let $h := \gamma_r(x) - \varepsilon$, then h is still affine, continuous, non-negative, and finite, and satisfies $0 \leq h \leq f_0 + g_0$. By [13], the semigroup of all affine, continuous, and non-negative functions $K \to [0, \infty)$ has the Riesz decomposition property. Since h, f_0, g_0 are elements of this semigroup, we can find such affine, continuous, and non-negative functions $f_1 \leq f_0$ and $g_1 \leq g_0$ with $h = f_1 + g_1$. Let $f' := f_1 + \frac{1}{2}\varepsilon$, and $g' := g_1 + \frac{1}{2}\varepsilon$, then $f', g' \in \mathsf{LAff}_{++}(K)$ are continuous, finite, and nonzero everywhere, and satisfy $\gamma_r(x) = f' + g'$. It follows from Lemma 8.4.1 that $f' \ll \gamma_r(x)$ and $g' \ll \gamma_r(x)$. Moreover, we have $f' < f_0 + \varepsilon$ and $g' < g_0 + \varepsilon$, and therefore $f' < f$ and $g' < g$ pointwise on K. By Lemma 8.4.1, we have $f' \ll f$ and $g' \ll g$. In total, we have $f' \ll f, \gamma_r(x)$; and $g' \ll g, \gamma_r(x)$; and $\gamma_r(x) = f' + g'$. Thus, the elements f', g' satisfy all requirements. This completes the proof. ■

9 Relationship to the Elliott invariant

We begin this section with a brief introduction to the Elliott invariant.

9.1.1 Definition. *Let A be any category. A* strong classification functor *for A is a pair (E, B) where B is a category, and $\mathsf{E}\colon \mathsf{A} \to \mathsf{B}$ is a functor with the following property: whenever X, Y are objects in A and $\beta\colon \mathsf{E}(X) \to \mathsf{E}(Y)$ is an isomorphism in B, then β lifts to an isomorphism α in A, i.e. there is an isomorphism $\alpha\colon X \to Y$ in A such that $\mathsf{E}(\alpha) = \beta$.*

The *Elliott invariant* $\mathsf{Ell}(\cdot)$ was introduced by George A. Elliott, who conjectured that $\mathsf{Ell}(\cdot)$ classifies (in the sense of strong classification defined above) the category of simple, separable, unital, nuclear C*-algebras, with unital *-homomorphisms as arrows. This conjecture is known as the *Elliott conjecture*. There are known counterexamples to the Elliott conjecture as stated above: there are simple, separable, unital, and nuclear C*-algebras A and B that are nonisomorphic, but satisfy $\mathsf{Ell}(A) \cong \mathsf{Ell}(B)$. The first such counterexample was constructed by M. Rørdam in [31]; the strongest counterexample so far was obtained by A. S. Toms in [37]. Nonetheless, the Elliott conjecture is known to be true for many important classes of C*-algebras (an overview can be found in [16]); it was shown by A. S. Toms and W. Winter in [38] that virtually all classes of C*-algebras for which the Elliott conjecture is known to be true consist exclusively of $\mathcal{Z}$-stable C*-algebras. Conversely, all known counterexamples to the Elliott conjecture involve C*-algebras that are not $\mathcal{Z}$-stable. It is therefore entirely possible that the Elliott invariant will turn out to classify the entire class of simple, separable, unital, nuclear, and $\mathcal{Z}$-stable C*-algebras.

9.1.2 Definition. *The category Ell has as objects all tuples (G, H, K, r), where G is a simple preordered abelian group with a strong unit, H is an abelian group, K is a Choquet simplex, and $r\colon K \to \mathcal{S}(G)$ is an affine and continuous map (here, $\mathcal{S}(G)$ denotes the set of all states on G, with the topology of pointwise convergence). The morphisms from the object (G, H, K, r) to the object (G', H', K', r') are the tuples $(\alpha_1, \alpha_2, \alpha_3)$, where $\alpha_1\colon G \to G'$ is a unital (i.e. order unit preserving) and positive (i.e. order-preserving) homomorphism of preordered abelian groups with strong unit, $\alpha_2\colon H \to H'$ is a group homomorphism, and $\alpha_3\colon K' \to K$ is an affine and continuous map, such that the following diagram is commutative:*

$$\begin{array}{ccc} K' & \xrightarrow{\alpha_3} & K \\ {\scriptstyle r'}\big\downarrow & & \big\downarrow{\scriptstyle r} \\ \mathcal{S}(G') & \xrightarrow[\alpha_1^*]{} & \mathcal{S}(G) \end{array} \quad .$$

The map $\alpha_1^*\colon \mathcal{S}(G') \to \mathcal{S}(G)$ *that appears in this diagram is the dual map of* α_1*, which sends the state* $\sigma \in \mathcal{S}(G')$ *to the state* $\sigma \circ \alpha_1 \in \mathcal{S}(G)$*.*
The Elliott invariant*, denoted by* $\mathsf{Ell}(\cdot)$*, is the functor from the category of simple, separable, unital, exact C*-algebras (with unital *-homomorphisms as morphisms) to the Elliott category that is defined on objects by* $\mathsf{Ell}(A) := ((\mathsf{K}_0(A), \mathsf{K}_0(A)_+, [\mathbb{I}_A]), \mathsf{K}_1(A), \mathsf{T}_1(A), \rho_A)$*. Here, the* pairing map $\rho_A\colon \mathsf{T}_1(A) \to \mathcal{S}(\mathsf{K}_0(A), \mathsf{K}_0(A)_+, [\mathbb{I}_A])$ *is given by* $\rho_A(\tau)([p]-[q]) := \tau(p) - \tau(q)$*. The* trace simplex $\mathsf{T}_1(A)$ *is the Choquet simplex of tracial states as defined in Chapter 5. On morphisms* $\varphi\colon A \to B$*, the functor is defined by* $\mathsf{Ell}(\varphi) := (\mathsf{K}_0(\varphi), \mathsf{K}_1(\varphi), \mathsf{T}_1(\varphi))$*, where* $\mathsf{T}_1(\varphi)\colon \mathsf{T}_1(B) \to \mathsf{T}_1(A)$ *is given by* $\mathsf{T}_1(\varphi)(\tau) := \tau \circ \varphi$ *for* $\tau \in \mathsf{T}_1(B)$*.*

Next, we want to introduce the extended Elliott invariant (see [27]), which is usually defined by appending the Cuntz semigroup $\mathsf{Cu}(A)$ (or, previously, $\mathsf{W}(A)$) to the Elliott invariant $\mathsf{Ell}(A)$. All known counterexamples to the Elliott conjecture consist of pairs of C*-algebras that have isomorphic Elliott invariants, but nonisomorphic Cuntz semigroups (as mentioned in [16], Section 5.2). Moreover, the counterexample provided by A. S. Toms in [37] consists of two C*-algebras that are indistinguishable by virtually all known invariants with the notable exception of the Cuntz semigroup. Thus, an invariant consisting of the original Elliott invariant and the Cuntz semigroup is the obvious candidate for an invariant that might classify the entire class of simple, separable, unital, and nuclear C*-algebras, as envisaged by the Elliott conjecture. The appropriate notion of morphism on the image category of the extended Elliott invariant is, however, somewhat complicated to define and use (again, see [27]). Since we are exclusively interested in C*-algebras A with a cancellative Murray-von Neumann semigroup $\mathsf{V}(A)$, and since for C*-algebras with this condition an equivalent version of the extended Elliott invariant is much easier to define and use, we will restrict throughout this chapter to C*-algebras A with cancellative $\mathsf{V}(A)$ and use the alternative picture of the extended Elliott invariant exclusively. We will moreover restrict to nonelementary C*-algebras. In so doing, we ensure that the Cuntz semigroup $\mathsf{Cu}(A)$ is simple and decomposable, so we can restrict the image category of the invariant accordingly.

9.1.3 Definition. *A C*-algebra A is called* cancellative *if the Murray-von Neumann semigroup* $\mathsf{V}(A)$ *is cancellative, i.e. if* $x+z \leq y+z$ *implies* $x \leq y$ *for all elements* $x, y, z \in \mathsf{V}(A)$*. Note that every C*-algebra of stable rank one is cancellative, but the converse does not hold in general, see [36]. Moreover, every simple and cancellative C*-algebra is stably finite (in the sense of Definition 2.1.4). In particular, it follows that* $\mathsf{V}(A)$*,* $\mathsf{C}(A)$*, and* $\mathsf{K}_0(A)_+$ *are naturally isomorphic as algebraically ordered abelian monoids for the category of simple, unital, and cancellative C*-algebras A.*

9.1.4 Definition. *The category* $\widetilde{\mathsf{Ell}}$ *has as objects all tuples* (H, S, u)*, where H is an abelian group, S is a simple and decomposable semigroup in* Cu *such that* $\mathsf{C}(S)$ *is cancellative,*

and $u \in S^\times$ is a compact element. The morphisms from the object (H, S, u) to the object (H', S', u') are pairs (α_1, α_2) where $\alpha_1 \colon H \to H'$ is a group homomorphism and $\alpha_2 \colon S \to S'$ is a Cu*-morphism such that $\alpha_2(u) = u'$.*
The extended Elliott invariant*, denoted by $\widetilde{\mathsf{Ell}}(\cdot)$, is the functor from the category of simple, separable, unital, nonelementary, and cancellative C*-algebras (with unital *-homomorphisms as arrows) to $\widetilde{\mathsf{Ell}}$, defined on objects by $\widetilde{\mathsf{Ell}}(A) := (\mathsf{K}_1(A), \mathsf{Cu}(A), [\mathbb{I}_A])$, where $\mathsf{K}_1(A)$ is the K_1-group of A, $\mathsf{Cu}(A)$ is the Cuntz semigroup of A, and $[\mathbb{I}_A] \in \mathsf{Cu}(A)$ is the equivalence class of the unit element of A. On morphisms, the functor is defined in the natural way as $\widetilde{\mathsf{Ell}}(\varphi) := (\mathsf{K}_1(\varphi), \mathsf{Cu}(\varphi))$.*

The following two theorems show the extent to which $\mathsf{Ell}(\cdot)$ and $\widetilde{\mathsf{Ell}}(\cdot)$ are related. Results very similar to these were first obtained by F. Perera and A. Toms in [27] (with two main differences to this version: their article makes use of the semigroup $\mathsf{W}(A)$ instead of $\mathsf{Cu}(A)$, and it uses the general picture of the extended Elliott invariant for not-necessarily cancellative C*-algebras).

9.1.5 Theorem. *On the category of simple, separable, unital, exact, nonelementary, and cancellative C*-algebras A, the Elliott invariant can be recovered functorially from the extended Elliott invariant, i.e. there is a functor $\mathsf{G} \colon \widetilde{\mathsf{Ell}} \to \mathsf{Ell}$ such that $\mathsf{G}(\widetilde{\mathsf{Ell}}(A))$ is naturally isomorphic to $\mathsf{Ell}(A)$ for every C*-algebra A with the properties above.*

Proof. To begin, let $\mathsf{Gr}(\cdot)$ denote the Grothendieck enveloping group (see e.g. [33], pp. 35–38, for an overview), which we will regard as a functor from the category of algebraically ordered, cancellative, abelian monoids with order unit (with zero-preserving, unit-preserving, additive maps as morphisms) to the category of directed abelian groups with order unit (with order-preserving, unit-preserving, additive maps as morphisms); the positive cone of the enveloping group is meant to be the image of the universal morphism from the original monoid to its enveloping group.
The functor $\mathsf{G}(\cdot)$ sends the object (H, S, u) to the object $(\mathsf{Gr}(\mathsf{C}(S), u), H, \mathsf{F}_u(S), r_S)$, where r_S is the affine map from $\mathsf{F}_u(S)$ to $\mathcal{S}(\mathsf{Gr}(\mathsf{C}(S), u))$ – the state space of $\mathsf{Gr}(\mathsf{C}(S), u)$ – given by $r_S(\lambda) := \mathsf{Gr}(\lambda|_{\mathsf{C}(S)})$. Note that $\mathsf{F}_u(S)$ is a Choquet simplex by Theorem 5.1.12. On morphisms, the functor $\mathsf{G}(\cdot)$ sends the pair $(\alpha_1, \alpha_2) \colon (H, S, u) \to (H', S', u')$ to the morphism $(\mathsf{Gr}(\mathsf{C}(\alpha_2)), \alpha_1, \alpha_2^*)$, where the map $\alpha_2^* \colon \mathsf{F}_{u'}(S') \to \mathsf{F}_u(S)$ is given by $\alpha_2^*(\lambda) := \lambda \circ \alpha_2$. A straightforward calculation shows that $r_S \circ \alpha_2^* = \mathsf{Gr}(\mathsf{C}(\alpha_2))^* \circ r_{S'}$, so the commutation requirement for morphisms in the category Ell is met. Since $\mathsf{C}(\mathsf{Cu}(A)) = \mathsf{C}(A)$ is naturally isomorphic to $\mathsf{K}_0(A)_+$ under the restrictions we imposed on A, it follows that $\mathsf{Gr}(\mathsf{C}(A), [\mathbb{I}_A])$ is naturally isomorphic to $(\mathsf{K}_0(A), \mathsf{K}_0(A)_+, [\mathbb{I}_A])$. Moreover, we have shown in Chapter 5 that $\mathsf{T}_1(A) \cong \mathsf{F}_{[\mathbb{I}_A]}(\mathsf{Cu}(A))$; this isomorphism is natural as well. Using these isomorphisms, it follows that for any unital *-homomorphism $\varphi \colon A \to B$, the following diagram is commutative:

$$\begin{array}{ccc} \mathsf{F}_{[\mathbb{I}_B]}(\mathsf{Cu}(B)) & \xrightarrow{r_{\mathsf{Cu}(A)}} & \mathcal{S}(\mathsf{Gr}(\mathsf{C}(A),[\mathbb{I}_A])) \\ \cong\downarrow & & \downarrow\cong \\ \mathsf{T}_1(B) & \xrightarrow[\rho_A]{} & \mathcal{S}(\mathsf{K}_0(A),\mathsf{K}_0(A)_+,[\mathbb{I}_A]) \end{array} .$$

It follows from all of this that $\mathsf{G}(\widetilde{\mathsf{Ell}}(A))$ is indeed naturally isomorphic to $\mathsf{Ell}(A)$. ■

Let Ell^* be the full subcategory of Ell consisting of all tuples (G,H,K,r) with the additional requirements that (G,G_+) is partially ordered (not just preordered), that K is a nonempty and metrisable Choquet simplex, and that for all $x,y\in G_+$, we have $x<y$ whenever $r(z)(x)<r(z)(y)$ for all $z\in K$. If A is a simple, separable, unital, and exact C*-algebra, then $\mathsf{Ell}(A)$ is in Ell^* if and only if A is stably finite and $A\otimes\mathcal{K}$ has *strict comparison of projections*, i.e. for any projections p,q in $A\otimes\mathcal{K}$, we have $p\precsim q$ whenever $\tau(p)<\tau(q)$ for every tracial state τ of A (note that the Choquet simplex $\mathsf{T}_1(A)$ is equipped with the topology of pointwise convergence on A, so its topology is metrisable for all separable A). Every simple, separable, unital, exact, stably finite, and $\mathcal{Z}$-stable C*-algebra has a weakly unperforated Cuntz semigroup by Theorem 4.6.2, hence $A\otimes\mathcal{K}$ has strict comparison of projections (as a consequence of Theorem 5.1.14). It follows that $\mathsf{Ell}(A)$ is in Ell^* for such C*-algebras A. Moreover, such C*-algebras are automatically nonelementary, and by [32], Theorem 6.7, they have stable rank one, which implies that they are cancellative.

9.1.6 Theorem. *On the category of simple, separable, unital, exact, stably finite, and $\mathcal{Z}$-stable C*-algebras A, the extended Elliott invariant can be recovered functorially from the Elliott invariant, i.e. there is a functor $\mathsf{G}^*\colon \mathsf{Ell}^*\to\widetilde{\mathsf{Ell}}$ such that $\mathsf{G}^*(\mathsf{Ell}(A))$ is naturally isomorphic to $\widetilde{\mathsf{Ell}}(A)$ for every C*-algebra A with the properties above.*

Proof. Define the functor by $\mathsf{G}^*\colon((G,G_+,u),H,K,r)\mapsto(H,G_+\sqcup_{\gamma_r}(\mathsf{LAff}_{++}(K)\cup\{0\}),u)$ on objects, where $\gamma_r\colon G_+\to\mathsf{LAff}_{++}(K)\cup\{0\}$ is the evaluation map $\gamma_r(x)(y):=r(y)(x)$. We have shown in Theorem 8.4.3 that γ_r is a composition map, and that $G_+\sqcup_{\gamma_r}(\mathsf{LAff}_{++}(K)\cup\{0\})$ is a simple and decomposable semigroup in Cu. Note that the compact part of this semigroup is G_+, which is clearly cancellative (since it is a cone in the group G).
If $(\alpha_1,\alpha_2,\alpha_3)\colon(G,H,K,r)\to(G',H',K',r')$ is a morphism in Ell, then $r\circ\alpha_3=\alpha_1^*\circ r'$, which implies $\alpha_3^*\circ\gamma_r=\gamma_{r'}\circ\alpha_1|_{G_+}$, where $\alpha_3^*\colon\mathsf{LAff}_{++}(K)\cup\{0\}\to\mathsf{LAff}_{++}(K')\cup\{0\}$ is the map with $\alpha_3^*(f):=f\circ\alpha_3$. We have shown in Theorem 8.4.2 that α_3^* is a D-morphism. It follows that the pair $(\alpha_1|_{G_+},\alpha_3^*)$ is natural with regard to the composition maps γ_r, $\gamma_{r'}$. We can therefore define the functor on morphisms by $\mathsf{G}\colon(\alpha_1,\alpha_2,\alpha_3)\mapsto(\alpha_2,\ \alpha_1|_{G_+}\sqcup\alpha_3^*)$. It is easy to check that G, thus defined, is indeed functorial. If A is a simple, separable, unital, exact, stably finite, and $\mathcal{Z}$-stable C*-algebra, then $\mathsf{G}^*(\mathsf{Ell}(A))=(\mathsf{K}_1(A),\ \mathsf{K}_0(A)_+\sqcup_{\gamma_{\rho_A}}(\mathsf{LAff}_{++}(\mathsf{T}_1(A))\cup\{0\}),[\mathbb{I}_A])$. Since A is cancellative, $\mathsf{K}_0(A)_+$ is naturally isomorphic to $\mathsf{C}(A)$. It follows from Theorem 5.3.4 and Proposition 5.3.2 that $\mathsf{LAff}_{++}(\mathsf{T}_1(A))\cup\{0\}$ is isomorphic to $\mathsf{D}(A)$, and this isomor-

phism is easily seen to be natural, too. Moreover, we have shown in Section 5.3 that under these isomorphisms, the evaluation map $\gamma_{\rho_A}\colon x \mapsto \hat{x}$ corresponds to the predecessor map $\gamma_A\colon \mathsf{C}(A) \to \mathsf{D}(A)$. Finally, we have shown in Corollary 8.2.2 that $\mathsf{C}(A) \sqcup_{\gamma_A} \mathsf{D}(A)$ is naturally isomorphic to $\mathsf{Cu}(A)$. It follows that $\mathsf{G}(\mathsf{Ell}(A))$ is naturally isomorphic to $\widetilde{\mathsf{Ell}}(A)$. ∎

We will now introduce a third invariant, based on the decomposition results of Chapter 8, and investigate its relationship to the extended Elliott invariant $\widetilde{\mathsf{Ell}}$:

9.1.7 Definition. *The category $\overline{\mathsf{Ell}}$ has as objects all tuples (G, H, D, c), where G is a simple directed abelian group with a strong unit, H is an abelian group, D is a simple semigroup in D, and $c\colon G_+ \to D$ is a composition map (in the sense of Definition 8.1.2). The morphisms from the object (G, H, D, c) to the object (G', H', D', c') are the tuples $(\alpha_1, \alpha_2, \alpha_3)$, where $\alpha_1\colon G \to G'$ is a unital (i.e. order unit preserving) and positive (i.e. order-preserving) homomorphism of ordered abelian groups with strong unit, $\alpha_2\colon H \to H'$ is a group homomorphism, and $\alpha_3\colon D \to D'$ is a D-morphism, such that $(\alpha_1|_{G_+}, \alpha_3)$ is almost natural (with respect to c, c') in the sense of Definition 8.3.1.*
The invariant $\overline{\mathsf{Ell}}(\cdot)$ is the functor from the category of simple, separable, unital, nonelementary, and cancellative C-algebras (with unital *-homomorphisms as arrows) to the category $\overline{\mathsf{Ell}}$, defined on objects by $\overline{\mathsf{Ell}}(A) := ((\mathsf{K}_0(A),\ \mathsf{K}_0(A)_+,\ [\mathbb{I}_A]),\ \mathsf{K}_1(A),\ \mathsf{D}(A),\ \gamma_A)$, where the triple $(\mathsf{K}_0(A),\ \mathsf{K}_0(A)_+,\ [\mathbb{I}_A])$ is the ordered K_0-group with order unit, $\mathsf{K}_1(A)$ is the K_1-group, $\mathsf{D}(A)$ is the noncompact part of the Cuntz semigroup, and $\gamma_A\colon \mathsf{K}_0(A)_+ \to \mathsf{D}(A)$ is the predecessor map of $\mathsf{Cu}(A)$ (we have used the identification of $\mathsf{C}(A)$ with the positive cone $\mathsf{K}_0(A)_+$ here). On morphisms $\varphi\colon A \to B$, the functor is defined as $\overline{\mathsf{Ell}}(\varphi) := (\mathsf{K}_0(\varphi),\ \mathsf{K}_1(\varphi),\ \mathsf{D}(\varphi))$.*

The result we want to contribute is the following theorem, showing that the invariants $\overline{\mathsf{Ell}}(\cdot)$ and $\widetilde{\mathsf{Ell}}(\cdot)$ are equivalent on a large category of C*-algebras:

9.1.8 Theorem. *On the category of simple, separable, unital, nonelementary, and cancellative C*-algebras, the invariants $\overline{\mathsf{Ell}}(\cdot)$ and $\widetilde{\mathsf{Ell}}(\cdot)$ are equivalent, i.e. there is a category equivalence $\mathsf{E}\colon \overline{\mathsf{Ell}} \to \widetilde{\mathsf{Ell}}$ such that $\mathsf{E}(\overline{\mathsf{Ell}}(A))$ is naturally isomorphic to $\widetilde{\mathsf{Ell}}(A)$.*

Proof. The equivalence $\mathsf{E}\colon \overline{\mathsf{Ell}} \to \widetilde{\mathsf{Ell}}$ is defined by sending the object $((G,\ G_+,\ u),\ H,\ D,\ c)$ to the object $(H,\ G_+ \sqcup_c D,\ u)$, and by sending the morphism $(\alpha_1,\ \alpha_2,\ \alpha_3)$ to the morphism $(\alpha_2,\ \alpha_1|_{G_+} \sqcup\ \alpha_3)$. Note that G_+ is always cancellative as a subset of a group, so the semigroup $G_+ \sqcup_c D$ satisfies the requirement that its compact part be cancellative. The reverse equivalence $\mathsf{E}'\colon \widetilde{\mathsf{Ell}} \to \overline{\mathsf{Ell}}$ is defined on objects by sending the tuple (H, S, u) to the tuple $(\mathsf{Gr}(\mathsf{C}(S),\ u),\ H,\ \mathsf{D}(S),\ \gamma_S)$, and on morphisms by sending the pair $(\alpha_1,\ \alpha_2)$ to the triple $(\mathsf{Gr}(\mathsf{C}(\alpha_2)),\ \alpha_1,\ \mathsf{D}(\alpha_2))$, where $\mathsf{Gr}(\cdot)$ is again the Grothendieck enveloping group. It follows easily from the construction of the Grothendieck enveloping group that the Grothendieck functor $\mathsf{Gr}(\cdot)$ from the category of algebraically ordered, cancellative, abelian monoids with order unit (with zero-preserving, unit-preserving, additive maps as morphisms) to the category of directed abelian groups with order unit (with order-preserving, unit-preserving, additive maps as morphisms) is a category equivalence, with an inverse functor $\mathsf{Gr}'(\cdot)$ defined on objects by $\mathsf{Gr}' \colon (G, G_+, u) \mapsto (G_+, u)$ and on morphisms by $\mathsf{Gr}' \colon \varphi \mapsto \varphi|_{G_+}$. We also know from Theorem 8.3.5 that the pair of functors $\mathsf{Com}(\cdot), \mathsf{Dec}(\cdot)$ implements a category equivalence between Cu_{com} and Cu_{dec}. Apart from moving around some components, adding or removing bracket pairs, and preserving additional components of our tuples, the functor $\mathsf{E}(\cdot)$ is essentially the composition of the functors $\mathsf{Com}(\cdot)$ and $\mathsf{Gr}'(\cdot)$, while the functor $\mathsf{E}'(\cdot)$ is essentially the composition of the functors $\mathsf{Gr}(\cdot)$ and $\mathsf{Dec}(\cdot)$. Clearly, then, the pair $\mathsf{E}(\cdot)$, $\mathsf{E}'(\cdot)$ is itself a category equivalence from the category $\overline{\mathsf{Ell}}$ to the category $\widetilde{\mathsf{Ell}}$. Finally, it follows easily from Corollary 8.2.2 that $\mathsf{E}(\overline{\mathsf{Ell}}(A))$ is naturally isomorphic to $\widetilde{\mathsf{Ell}}(A)$ on the category of simple, separable, unital, nonelementary, and cancellative C*-algebras. ■

Next, we want to analyse the relationship between $\mathsf{D}(A)$ and $\mathsf{T}_1(A)$. It will turn out that we can identify $\mathsf{T}_1(A)$ with the dual of $\mathsf{D}(A)$, i.e. with the Choquet simplex of all suitably normalised functionals on $\mathsf{D}(A)$. Moreover, under this identification, the pairing map ρ_A will turn out to correspond to the dual map of the predecessor map.

9.1.9 Theorem. *Let A be any simple, separable, unital, exact, nonelementary, and stably finite C*-algebra. Then the trace simplex $\mathsf{T}_1(A)$ is isomorphic (as a Choquet simplex) to the simplex $\mathsf{F}_{\gamma_A([\mathbb{I}_A])}(\mathsf{D}(A))$. Moreover, this isomorphism is given by $\tau \mapsto \lambda_\tau|_{\mathsf{D}(A)}$.*

Proof. We have shown in Chapter 5 that the map $\tau \mapsto \lambda_\tau$ is an isomorphism of Choquet simplices (i.e. an affine homeomorphism) between the trace simplex $\mathsf{T}_1(A)$ and the simplex $\mathsf{F}_{[\mathbb{I}_A]}(\mathsf{Cu}(A))$. Thus, we only need to show that the map $\alpha \colon \lambda \mapsto \lambda|_{\mathsf{D}(A)}$ is an affine homeomorphism between $\mathsf{F}_{[\mathbb{I}_A]}(\mathsf{Cu}(A))$ and $\mathsf{F}_{\gamma_A([\mathbb{I}_A])}(\mathsf{D}(A))$. It follows from Theorem 7.1.15 (*viii*) that this map is bijective, and that its inverse is given by the map $\alpha^{-1} \colon \lambda \mapsto \lambda \circ \varepsilon_A$ from $\mathsf{F}_{\gamma_A([\mathbb{I}_A])}(\mathsf{D}(A))$ to $\mathsf{F}_{[\mathbb{I}_A]}(\mathsf{Cu}(A))$. Both maps are clearly affine. Moreover, from the definition of the topology on $\mathsf{F}(\mathsf{Cu}(A))$ and $\mathsf{F}(\mathsf{D}(A))$ (see Definition 5.1.2) and the fact that $\ll_{\mathsf{Cu}(A)}$ and $\sup_{\mathsf{Cu}(A)}$ agree with $\ll_{\mathsf{D}(A)}$ and $\sup_{\mathsf{D}(A)}$ on $\mathsf{D}(A)$ (see Proposition 7.2.1) it is immediately evident that α is continuous. It remains to show the continuity of α^{-1}. Let $(\lambda_\nu)_\nu$ be a net of functionals in $\mathsf{F}_{\gamma_A([\mathbb{I}_A])}(\mathsf{D}(A))$ that converges in $\mathsf{F}_{\gamma_A([\mathbb{I}_A])}(\mathsf{D}(A))$ towards a functional λ. We need to show that $(\lambda_\nu \circ \varepsilon_A)_\nu$ converges towards $\lambda \circ \varepsilon_A$ in $\mathsf{F}_{[\mathbb{I}_A]}(\mathsf{Cu}(A))$, i.e. that for every pair $x' \ll x$ in $\mathsf{Cu}(A)$ we have $\limsup_\nu \lambda_\nu(\varepsilon_A(x')) \leq \lambda(\varepsilon_A(x)) \leq \liminf_\nu \lambda_\nu(\varepsilon_A(x))$. The inequality on the right is obviously satisfied, since $\varepsilon_A(x) \in \mathsf{D}(A)$ and the λ_ν converge towards λ as functionals on $\mathsf{D}(A)$. The inequality on the left is satisfied if $x \in \mathsf{D}(A)$: for every $x' \ll x$ we can find an element $y \in \mathsf{D}(A)$ with $x' \ll y \ll x$ by Proposition 7.2.1 and the fact that $\mathsf{D}(A)$ satisfies axiom $(\mathcal{O}2)$; we then have $\varepsilon_A(x') \leq \varepsilon_A(y) \ll \varepsilon_A(x)$ and therefore $\limsup_\nu \lambda_\nu(\varepsilon_A(x')) \leq \limsup_\nu \lambda_\nu(\varepsilon_A(y)) \leq \lambda(\varepsilon_A(x))$. We still need to show that the inequality $\limsup_\nu \lambda_\nu(\varepsilon_A(x')) \leq \lambda(\varepsilon_A(x))$ holds for all $x \in \mathsf{C}(A)^\times$ and all $x' \ll x$; it suffices to show that $\limsup_\nu \lambda_\nu(\varepsilon_A(x)) \leq \lambda(\varepsilon_A(x))$ for every $x \in \mathsf{C}(A)^\times$. This will require a bit of work. First, we show that we can construct a sequence $(z_n)_n$ in $\mathsf{D}(A)^\times$ such that $\lambda(z_n) \leq \frac{1}{n}$

for all n. Use the halving theorem to find some $v_n \in \mathsf{Cu}(A)^\times$ with $nv_n \le \gamma_A([\mathbb{I}_A])$. Let $z_n := \varepsilon_A(v_n)$, then $z_n \in \mathsf{D}(A)^\times$ and $nz_n \le \gamma_A([\mathbb{I}_A])$. It follows that $\lambda(z_n) \le \frac{1}{n}$. Next, pick $z_n' \in \mathsf{D}(A)^\times$ with $z_n' \ll z_n$ for each n. Then $\varepsilon_A(x + z_n') = x + z_n' \ll x + z_n = \varepsilon_A(x + z_n)$ since both $x + z_n'$ and $x + z_n$ are noncompact. Since $(\lambda_\nu)_\nu$ converges towards λ in $\mathsf{F}(\mathsf{D}(A))$, it follows that $\limsup_\nu \lambda_\nu(\varepsilon_A(x)) \le \limsup_\nu \lambda_\nu(\varepsilon_A(x + z_n')) \le \lambda(\varepsilon_A(x + z_n)) = \lambda(\varepsilon_A(x) + z_n)$ for all n, and therefore $\limsup_\nu \lambda_\nu(\varepsilon_A(x)) \le \lambda(\varepsilon_A(x)) + \frac{1}{n}$ for all n. But this clearly implies $\limsup_\nu \lambda_\nu(\varepsilon_A(x)) \le \lambda(\varepsilon_A(x))$, so we are done. ∎

9.1.10 Theorem. *Let A be any simple, separable, unital, exact, nonelementary, and cancellative C*-algebra, and let $\gamma_A^*\colon \mathsf{F}_{\gamma_A([\mathbb{I}_A])}(\mathsf{D}(A)) \to \mathcal{S}(\mathsf{K}_0(A)_+, [\mathbb{I}_A])$ be the dual of the predecessor map, i.e. $\gamma_A^*(\lambda) := \lambda \circ \gamma_A$. Moreover, let $\iota\colon \mathcal{S}(\mathsf{K}_0(A)_+, [\mathbb{I}_A]) \to \mathcal{S}(\mathsf{K}_0(A), \mathsf{K}_0(A)_+, [\mathbb{I}_A])$ be the natural isomorphism that is defined by $\iota(\sigma)([p] - [q]) := \sigma([p]) - \sigma([q])$, and let $\beta\colon \mathsf{T}_1(A) \to \mathsf{F}_{\gamma_A([\mathbb{I}_A])}(\mathsf{D}(A))$ be the isomorphism $\tau \mapsto \lambda_\tau|_{\mathsf{D}(A)}$ mentioned above. Then the following diagram is commutative:*

$$\begin{array}{ccc} \mathsf{F}_{\gamma_A([\mathbb{I}_A])}(\mathsf{D}(A)) & \xrightarrow{\quad\gamma_A^*\quad} & \mathcal{S}(\mathsf{K}_0(A)_+, [\mathbb{I}_A]) \\ {\scriptstyle\beta^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle\iota} \\ \mathsf{T}_1(A) & \xrightarrow[\quad\rho_A\quad]{} & \mathcal{S}(\mathsf{K}_0(A), \mathsf{K}_0(A)_+, [\mathbb{I}_A]) \end{array} .$$

Proof. This is a simple calculation: let $\lambda \in \mathsf{F}_{\gamma_A([\mathbb{I}_A])}(\mathsf{D}(A))$, and let $[p] - [q] \in \mathsf{K}_0(A)$. Let $\tau := \beta^{-1}(\lambda)$, so that $\lambda_\tau \in \mathsf{F}_{[\mathbb{I}_A]}(\mathsf{Cu}(A))$ with $\lambda_\tau|_{\mathsf{D}(A)} = \lambda$. Then by Corollary 7.1.8, we have

$$\begin{aligned} \iota(\gamma_A^*(\lambda))([p] - [q]) &= \gamma_A^*(\lambda)([p]) - \gamma_A^*(\lambda)([q]) \\ &= \lambda(\gamma_A([p])) - \lambda(\gamma_A([q])) \\ &= \lambda_\tau(\gamma_A([p])) - \lambda_\tau(\gamma_A([q])) \\ &= \lambda_\tau([p]) - \lambda_\tau([q]) \\ &= \tau(p) - \tau(q) \\ &= \rho_A(\tau)([p] - [q]) \\ &= \rho_A(\beta^{-1}(\lambda))([p] - [q]). \end{aligned}$$

This proves that the diagram above is commutative. ∎

Thus we have, for sufficiently nice C*-algebras, presented a picture of the extended Elliott invariant that looks very similar to the original Elliott invariant; the difference being that the tracial components $\mathsf{T}_1(A)$ and ρ_A have been replaced by appropriate preduals, namely by $\mathsf{D}(A)$ and γ_A. This would serve as a strong indication that the extended Elliott invariant is, indeed, a quite natural extension of the original Elliott invariant.

10 The naturality problem

Let A, B be simple, separable, nonelementary, and stably finite C*-algebras, let $\chi\colon A \to B$ be a *-homomorphism, and let $p \in \mathcal{P}(A\otimes\mathcal{K})$ be a projection. We are trying to answer the following question: under what circumstances is it true that $(\gamma_B \circ \mathsf{C}(\chi))([p]) = (\mathsf{D}(\chi)\circ\gamma_A)([p])$? Put another way, is there a (suitably large) full subcategory of simple, separable, nonelementary, and stably finite C*-algebras such that the predecessor maps γ form a natural transformation (in the category of ordered abelian semigroups) from the functor $\mathsf{C}(\cdot)$ to the functor $\mathsf{D}(\cdot)$? We have already seen in Theorem 8.3.3 that the pair $(\mathsf{C}(\chi), \mathsf{D}(\chi))$ is almost natural, and this result immediately makes it plausible that full naturality might be satisfied for well-behaved C*-algebras. We begin with the following observation:

10.1.1 Proposition. *Let S_1, S_2 be simple and decomposable semigroups in* Cu*, and let $\chi\colon S_1 \to S_2$ be a* Cu*-morphism. If S_2 is almost unperforated, then $\gamma_{S_2} \circ \mathsf{C}(\chi) = \mathsf{D}(\chi) \circ \gamma_{S_1}$, i.e. the pair $(\mathsf{C}(\chi), \mathsf{D}(\chi))$ is natural with regard to γ_{S_1}, γ_{S_2}.*

Proof. Let $x \in \mathsf{C}(S_1)$. If x is zero, or if χ is the zero morphism, then clearly we have $(\gamma_{S_2}\circ\mathsf{C}(\chi))(x) = (\mathsf{D}(\chi)\circ\gamma_{S_1})(x)$. We may therefore assume that $x \neq 0$, and that χ is faithful. Let $y := (\mathsf{D}(\chi) \circ \gamma_{S_1})(x) = \chi(\gamma_{S_1}(x))$. We have $\gamma_{S_1}(x) < x$ and therefore $y \leq \chi(x)$; since $y \in \mathsf{D}(S_2)^\times$ and $\chi(x) \in \mathsf{C}(S_2)^\times$, we even have $y < \chi(x)$. Let λ be any functional on S_2, then $\lambda\circ\chi$ is a functional on S_1. By Corollary 7.1.8, we have $(\lambda\circ\chi)(\gamma_{S_1}(x)) = (\lambda\circ\chi)(x)$; it follows that $\lambda(y) = \lambda(\chi(x))$ for every $\lambda \in \mathsf{F}(S_2)$. By Theorem 7.1.9, this implies $y = \gamma_{S_2}(\chi(x))$. But this means precisely that $\gamma_{S_2}(\chi(x)) = \chi(\gamma_{S_1}(x))$, hence $(\gamma_{S_2} \circ \mathsf{C}(\chi))(x) = (\mathsf{D}(\chi) \circ \gamma_{S_1})(x)$. Since this is true for every $x \in \mathsf{C}(S_1)$, it follows that $\gamma_{S_2} \circ \mathsf{C}(\chi) = \mathsf{D}(\chi) \circ \gamma_{S_1}$. ■

Next, we consider a condition that is seemingly weaker then being almost unperforated:

10.1.2 Definition. *We call a semigroup S in* Cu strongly cancellative *if $x+z \leq y+z$ implies $x \leq y$ whenever $x, y, z \in \mathsf{D}(S)_{min}$.*

Since every nontrivial semigroup S in Cu contains infinite elements, it is impossible to require full cancellation for S, and requiring cancellation for S_{min} is the best we can hope for. But even for S_{min}, full cancellation may be impossible if compact elements are involved. As mentioned in Remark 7.1.13, even among the most well-behaved class of C*-algebras, it is possible to encounter examples A such that $\mathsf{Cu}(A)$ contains two incomparable compact elements x and y with $\gamma_A(x) = \gamma_A(y)$. If z is any noncompact element of $\mathsf{Cu}(A)$, then it follows from Theorem 7.1.6 that $x + z = \gamma_A(x) + z = \gamma_A(y) + z = y + z$, even though neither

$x \leq y$ nor $y \leq x$. If S is strongly cancellative as in the definition above, and $x, y, z \in S_{min}$ are such that $x+z \leq y+z$, then $\varepsilon_S(x)+\varepsilon_S(z) \leq \varepsilon_S(y)+\varepsilon_S(z)$, and therefore $\varepsilon_S(x) \leq \varepsilon_S(y)$. this appears to be the strongest form of cancellation that stands a chance of being satisfied by interesting classes of C*-algebras. Indeed, we will now show that strong cancellation is a consequence of almost unperforatedness for simple semigroups:

10.1.3 Proposition. *Let S be a simple and almost unperforated semigroup in Cu. Then S is strongly cancellative.*

Proof. Let $x, y, z \in \mathsf{D}(S)_{min}$ such that $x+z \leq y+z$. Pick any $x' \ll x$ in S, then x' is automatically in S_{min} as well. By Lemma 3.1.2 (*iii*), there is $v \in S^\times$ such that $x'+v \leq x$. Let λ be any nontrivial functional on S, then λ is faithful and semifinite by Proposition 5.1.9. It follows that we have $\lambda(x')+\lambda(z) < \lambda(x')+\lambda(v)+\lambda(z) \leq \lambda(y)+\lambda(z)$, and therefore $\lambda(x') < \lambda(y)$ for every nontrivial functional on S. Since S is almost unperforated, it follows from Theorem 5.1.14 that $x' \leq y$. Since this is true for every $x' \ll x$, it follows from Rørdams proposition that $x \leq y$. ■

10.1.4 Proposition. *Let S_1, S_2 be simple and decomposable semigroups in Cu, and let $\chi\colon S_1 \to S_2$ be a Cu-morphism. If S_2 is strongly cancellative, then $\gamma_{S_2} \circ \mathsf{C}(\chi) = \mathsf{D}(\chi) \circ \gamma_{S_1}$, i.e. the pair $(\mathsf{C}(\chi), \mathsf{D}(\chi))$ is natural with regard to γ_{S_1}, γ_{S_2}.*

Proof. As before, we may assume that χ is faithful. Let $x \in \mathsf{C}(S_1)^\times$. Let $y_1 := (\mathsf{D}(\chi) \circ \gamma_{S_1})(x)$, then $y_1 = \chi(\gamma_{S_1}(x)) \ll \chi(x)$ and therefore $y_1 \in \mathsf{D}(S_2)_{min}$. Let $y_2 := (\gamma_{S_2} \circ \mathsf{C}(\chi))(x)$, then $y_2 = \gamma_{S_2}(\chi(x)) \ll \chi(x)$ and therefore $y_2 \in \mathsf{D}(S_2)_{min}$ as well. Using Theorem 7.1.6 in S_1 and in S_2 yields

$$
\begin{aligned}
y_1 + y_1 &= \chi(\gamma_{S_1}(x) + \gamma_{S_1}(x)) = \chi(\gamma_{S_1}(x) + x) = y_1 + \chi(x), \\
y_1 + y_2 &= \chi(\gamma_{S_1}(x)) + \gamma_{S_2}(\chi(x)) = \chi(\gamma_{S_1}(x)) + \chi(x) = y_1 + \chi(x).
\end{aligned}
$$

Thus, we have $y_1 + y_1 = y_1 + y_2$. Since S_2 is strongly cancellative, it follows that $y_1 = y_2$, and therefore $(\gamma_{S_2} \circ \mathsf{C}(\chi))(x) = (\mathsf{D}(\chi) \circ \gamma_{S_1})(x)$. Since this is true for every $x \in \mathsf{C}(S_1)^\times$ (and, obviously, also for $x = 0$), we have $\gamma_{S_2} \circ \mathsf{C}(\chi) = \mathsf{D}(\chi) \circ \gamma_{S_1}$. ■

10.1.5 Remark. *There is a weaker notion than strong cancellation that would suffice for the proof of the preceding proposition: in [1], p. 126 (right above Lemma 5.3), the authors call an abelian monoid* strongly separative *if, for all $a, b \in M$, the equality $2a = a + b$ implies $a = b$. Clearly, the above proof shows that if S_1, S_2 are simple and decomposable semigroups in Cu with $\mathsf{D}(S_2)_{min}$ being strongly separative, then for every Cu-morphism $\chi\colon S_1 \to S_2$ we have $\gamma_{S_2} \circ \mathsf{C}(\chi) = \mathsf{D}(\chi) \circ \gamma_{S_1}$.*

Next, we take a closer look at semigroups satisfying only weak cancellation. Whether or not all pairs $(\mathsf{C}(\chi), \mathsf{D}(\chi))$ coming from Cu-morphisms $\chi\colon S_1 \to S_2$ between simple, decomposable, and weakly cancellative semigroups in Cu are natural is still an open question. The question remains open even if we further restrict to Cuntz semigroups of simple, separable, and nonelementary C*-algebras of stable rank one, and to Cu-morphisms that are induced by *-homomorphisms.

10.1.6 Lemma. *Let S_1, S_2 be simple and decomposable semigroups in Cu, and let $\chi\colon S_1 \to S_2$ be a nonzero Cu-morphism. If S_2 is weakly cancellative, then one of the following conditions holds:*

(i) $\mathsf{D}(\chi)(\gamma_{S_1}(x)) = \gamma_{S_2}(\mathsf{C}(\chi)(x))$ *for every* $x \in \mathsf{C}(S_1)$.

(ii) $\mathsf{D}(\chi)(\gamma_{S_1}(x)) < \gamma_{S_2}(\mathsf{C}(\chi)(x))$ *for every* $x \in \mathsf{C}(S_1)^\times$.

Proof. Assume that $\mathsf{D}(\chi)(\gamma_{S_1}(x)) = \gamma_{S_2}(\mathsf{C}(\chi)(x))$ is true for some $x \in \mathsf{C}(S_1)^\times$. Let $y \in \mathsf{C}(S)^\times$ be any other nonzero compact element. Since $\chi \neq 0$, we have $\mathsf{D}(\chi)(\gamma_{S_1}(y)) > 0$ and $\gamma_{S_2}(\mathsf{C}(\chi)(y)) > 0$. It now follows from Theorem 7.1.6 and almost naturality (see Definition 8.3.1) of $(\mathsf{C}(\chi), \mathsf{D}(\chi))$ that

$$\begin{aligned}
\gamma_{S_2}(\mathsf{C}(\chi)(y)) + \mathsf{C}(\chi)(x) &= \gamma_{S_2}(\mathsf{C}(\chi)(y)) + \gamma_{S_2}(\mathsf{C}(\chi)(x)) \\
&= \gamma_{S_2}(\mathsf{C}(\chi)(y)) + \mathsf{D}(\chi)(\gamma_{S_1}(x)) \\
&= \mathsf{D}(\chi)(\gamma_{S_1}(y)) + \mathsf{D}(\chi)(\gamma_{S_1}(x)) \\
&= \mathsf{D}(\chi)(\gamma_{S_1}(y)) + \gamma_{S_2}(\mathsf{C}(\chi)(x)) \\
&= \mathsf{D}(\chi)(\gamma_{S_1}(y)) + \mathsf{C}(\chi)(x).
\end{aligned}$$

But since S_2 is weakly cancellative, compact elements can be cancelled from sums in S_2. Since $\mathsf{C}(\chi)(x)$ is a compact element in S_2, it follows that we have $\mathsf{D}(\chi)(\gamma_{S_1}(y)) = \gamma_{S_2}(\mathsf{C}(\chi)(y))$. Hence, the naturality condition is either true for every compact element in S_2, or it is false for every nonzero compact element in S_2. This concludes the proof. ■

10.1.7 Lemma. *Let S_1, S_2 be simple and decomposable semigroups in Cu, and let $\chi\colon S_1 \to S_2$ be a nonzero Cu-morphism. If S_2 is weakly cancellative and $\mathsf{C}(S_1) \neq \{0\}$, then the following conditions are equivalent:*

(i) The naturality condition $\mathsf{D}(\chi) \circ \gamma_{S_1} = \gamma_{S_2} \circ \mathsf{C}(\chi)$ *holds.*

(ii) For every $x \in S_2^\times$ *there is some* $y \in S_1^\times$ *such that* $\chi(y) \leq x$.

(iii) For every $x \in \mathsf{D}(S_2)^\times$ *there is some* $y \in \mathsf{D}(S_1)^\times$ *such that* $\mathsf{D}(\chi)(y) \leq x$.

Proof.
$(i) \implies (ii)$: Let $x \in S_2^\times$, and let $z \in \mathsf{C}(S_1)^\times$. Since $\chi \neq 0$, we know that $\mathsf{C}(\chi)(z) > 0$. We need to find an element $y \in S_1^\times$ such that $\chi(y) \leq x$. By Corollary 7.1.10, we have $\mathsf{C}(\chi)(z) \ll \gamma_{S_2}(\mathsf{C}(\chi)(z)) + x$. Pick elements $u, v \in S_2^\times$ such that $u \ll \gamma_{S_2}(\mathsf{C}(\chi)(z))$, $v \ll x$, and $\mathsf{C}(\chi)(z) \leq u + v$. By (i), we know that $\gamma_{S_2}(\mathsf{C}(\chi)(z)) = \mathsf{D}(\chi)(\gamma_{S_1}(z))$. Let $(w_n)_n$ be a rapidly increasing sequence in $\mathsf{D}(S_1)^\times$ with supremum $\gamma_{S_1}(z)$. Since χ is a Cu-morphism, we know

that $\sup_n \chi(w_n) = \chi(\gamma_{S_1}(z))$, and hence $\sup_n \mathsf{D}(\chi)(w_n) = \mathsf{D}(\chi)(\gamma_{S_1}(z)) = \gamma_{S_2}(\mathsf{C}(\chi)(z))$. Hence, we can find an element $w := w_n \in \mathsf{D}(S_1)^\times$ such that $w \ll \gamma_{S_1}(z)$ and $u \leq \mathsf{D}(\chi)(w)$. Now, we can pick $y \in \mathsf{D}(S_1)^\times$ such that $w + y \ll \gamma_{S_1}(z)$ by Lemma 3.1.2 (*iii*). Then

$$\begin{aligned} u + \chi(y) &= u + \mathsf{D}(\chi)(y) \\ &\leq \mathsf{D}(\chi)(w) + \mathsf{D}(\chi)(y) \\ &= \mathsf{D}(\chi)(w + y) \\ &\ll \mathsf{D}(\chi)(\gamma_{S_1}(z)) \\ &= \gamma_{S_2}(\mathsf{C}(\chi)(z)) \\ &\leq \mathsf{C}(\chi)(z) \\ &\leq u + v. \end{aligned}$$

Since S_2 is weakly cancellative, it follows that $\chi(y) \ll v$, and therefore $\chi(y) \leq x$.
(*ii*) $\implies$ (*iii*) : This is obvious, since for any $y_0 \in S_1^\times$ there is $y \in \mathsf{D}(S_1)^\times$ with $y \leq y_0$, namely $y := \varepsilon_{S_1}(y_0)$.
(*iii*) $\implies$ (*i*) : Let $x \in \mathsf{C}(S_1)^\times$. By almost naturality of the pair $(\mathsf{C}(\chi), \mathsf{D}(\chi))$, we already know that $\mathsf{D}(\chi)(\gamma_{S_1}(x)) \leq \gamma_{S_2}(\mathsf{C}(\chi)(x))$, from which it follows that $\mathsf{D}(\chi)(\gamma_{S_1}(x)) < \mathsf{C}(\chi)(x)$. Using Theorem 7.1.11, we need only show that $\mathsf{C}(\chi)(x) < \mathsf{D}(\chi)(\gamma_{S_1}(x)) + y$ for every $y \in S_2^\times$. Pick any $y \in S_2^\times$. Then $z := \varepsilon_{S_2}(y) \in \mathsf{D}(S_2)^\times$ and $z \leq y$. Using (*iii*), we can find an element $w \in \mathsf{D}(S_1)^\times$ such that $\mathsf{D}(\chi)(w) = \chi(w) \leq z \leq y$. By Corollary 7.1.10, we have $x \leq \gamma_{S_1}(x) + w$, and therefore $\mathsf{C}(\chi)(x) = \chi(x) \leq \chi(\gamma_{S_1}(x)) + \chi(w) = \mathsf{D}(\chi)(\gamma_{S_1}(x)) + \mathsf{D}(\chi)(w)$, hence $\mathsf{C}(\chi(x)) \leq \mathsf{D}(\chi)(\gamma_{S_1}(x)) + \mathsf{D}(\chi)(w) \leq \mathsf{D}(\chi)(\gamma_{S_1}(x)) + y$. Since the term on the left side is compact, while the term on the right side is noncompact, we get $\mathsf{C}(\chi)(x) < \mathsf{D}(\chi)(\gamma_{S_1}(x)) + y$. ∎

The property of n-comparison for semigroups S in Cu is described by L. Robert in [28], Definition 2. We will use a variant of this definition; Robert has shown in [28], Lemma 1 that this variant is equivalent to n-comparison if $S = \mathsf{Cu}(A)$:

10.1.8 Definition. *Let A be a C^*-algebra, and let $n \in \mathbb{N}_0$. We say that $\mathsf{Cu}(A)$ has n-comparison if the following condition holds: Whenever $x, y_0, ..., y_n \in \mathsf{Cu}(A)$ and $\varepsilon > 0$ are such that $\lambda(x) \leq (1 - \varepsilon)\lambda(y_i)$ for $0 \leq i \leq n$ and for all $\lambda \in \mathsf{F}(\mathsf{Cu}(A))$, then $x \leq y_0 + ... + y_n$.*

By Theorem 5.1.14, $\mathsf{Cu}(A)$ has 0-comparison if and only if $\mathsf{Cu}(A)$ is almost unperforated.

10.1.9 Theorem. *Let A, B be simple, separable, nonelementary, and stably finite C^*-algebras, and let $\alpha\colon A \to B$ be any *-homomorphism. If B has stable rank one, and $\mathsf{Cu}(B)$ has n-comparison for some $n \in \mathbb{N}_0$, then the condition $\mathsf{D}(\alpha) \circ \gamma_A = \gamma_B \circ \mathsf{C}(\alpha)$ is satisfied.*

Proof. Note that $\mathsf{Cu}(B)$ is weakly cancellative by Theorem 4.6.1, since B has stable rank one. Once again, we may assume that α is injective, and hence that $\mathsf{Cu}(\alpha)$ is faithful. If

A is stably projectionless, then the condition $\mathsf{D}(\alpha) \circ \gamma_A = \gamma_B \circ \mathsf{C}(\alpha)$ is trivially satisfied, so we may additionally assume that $\mathsf{Cu}(A)$ contains a nonzero compact element v. Then all the requirements of Lemma 10.1.7 are satisfied; we will show that condition (ii) of that lemma is met. Let x be any nonzero element of $\mathsf{Cu}(B)$. By the halving theorem, we can find an element $z \in \mathsf{Cu}(B)^\times$ such that $(n+1)\, z \leq x$. Let $u := \mathsf{Cu}(\alpha)(v) \in \mathsf{C}(B)^\times$. The evaluation map $\tilde{z} \colon \mathsf{F}_u(\mathsf{Cu}(B)) \to [0, \infty]$, defined by $\tilde{z}(\lambda) := \lambda(z)$, is lower semicontinuous by Proposition 5.1.13. Since $\mathsf{F}_u(\mathsf{Cu}(B))$ is compact by Lemma 5.1.7, $\tilde{z}$ attains a minimum, and by Proposition 5.1.9 (i) this minimum must be nonzero. Using the halving theorem, we can find a sequence $(v_m)_m$ in $\mathsf{Cu}(A)^\times$ such that $mv_m \leq v$ for every m. Let $u_m := \mathsf{Cu}(\alpha)(v_m)$, then $mu_m \leq u$ for each m, and therefore $\tilde{u}_m(\lambda) \leq \frac{1}{m}$ for every $\lambda \in \mathsf{F}_u(\mathsf{Cu}(B))$. Let $0 < \varepsilon < 1$; it follows that for sufficiently large m, we have $\tilde{u}_m \leq (1-\varepsilon)\, \tilde{z}$, and therefore $\lambda(u_m) \leq (1-\varepsilon)\, \lambda(z)$ for every $\lambda \in \mathsf{F}_u(\mathsf{Cu}(B))$. The element u is nonzero because v is nonzero and $\mathsf{Cu}(\alpha)$ is faithful; it is also compact, which implies $u \ll \infty$. It follows from Proposition 5.1.9 (i) and (ii) that every nontrivial functional in $\mathsf{F}(\mathsf{Cu}(B))$ is of the form $C\lambda$ for $C \in (0, \infty)$ and $\lambda \in \mathsf{F}_u(\mathsf{Cu}(B))$. But this implies that $\lambda(u_m) \leq (1-\varepsilon)\, \lambda(z)$ holds for every $\lambda \in \mathsf{F}(\mathsf{Cu}(B))$. Since $\mathsf{Cu}(B)$ has n-comparison, it follows that $u_m \leq (n+1)\, z \leq x$. Let $y := v_m$, then we have $y \in \mathsf{Cu}(A)^\times$ and $\mathsf{Cu}(\alpha)(y) = u_m$, which means that $\mathsf{Cu}(\alpha)(y) \leq x$. Thus, condition (ii) of Lemma 10.1.7 is satisfied. ∎

11 Bibliography

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