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Continuous fields of C*-algebras and quasidiagonality

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Abstract. Let A be a separable and exact C*-algebra which is a continuous field of C*-algebras over a connected, compact metrizable space. If at least one of the fibers of A is quasidiagonal, then so is A. As an application we show that if G is an amenable group that is a central extension by a countable torsion-free group, then the C*-algebra of G is quasidiagonal.

1. Introduction

Quasidiagonal C*-algebras have now been studied for almost forty years. Quasidiagonality is a finite-dimensional approximation property. In fact, by a theorem of Voiculescu, quasidiagonal C*-algebras are precisely those C*-algebras which have good matrix models, in the sense that the relevant C*-algebraic structure can be seen in a matrix. Some important open problems in C*-algebras are connected in some way to quasidiagonality or notions around it. For example, the question of whether nuclearity and quasidiagonality are equivalent notions for reduced C*-algebras of countable discrete groups is still open an question.

In [12], Voiculescu proved that quasidiagonality is a homotopy invariant. His technique can be adapted to prove quasidiagonality for certain continuous fields of C*-algebras.

Theorem 1.1. *Let A be a separable exact* C*-*algebra. Suppose that A is a continuous field of* C*-*algebras over a compact, connected metrizable space X. If the fiber $A(x_0)$ of A is quasidiagonal for some $x_0 \in X$, then A is also quasidiagonal.*

Following the approach taken by Dădărlat in [4], we use this theorem to deduce the following result about the quasidiagonality of group C*-algebras:

Theorem 1.2. *Let $1 \to N \to G \to H \to 1$ be a central extension of amenable second countable locally compact groups. Assume that N is torsion-free and discrete. If $C^*(H)$ is quasidiagonal, then so is $C^*(G)$.*

2. Quasidiagonality and homotopy invariance

Definition 2.1. Let H be a Hilbert space. A subset $\mathcal{S} \subset \mathscr{L}(H)$ is called a *quasidiagonal set of operators* if there is an increasing net $(P_\alpha)_\alpha$ of finite-rank orthogonal projections on H, with $P_\alpha \to I$ strongly and such that $\lim_\alpha \|P_\alpha S - SP_\alpha\| = 0$ for every $S \in \mathcal{S}$. A representation (π, H) of a C*-algebra A is a called a *quasidiagonal representation* if $\pi(A)$ is a quasidiagonal set of operators. A C*-algebra is called *quasidiagonal* (QD) if it has a faithful quasidiagonal representation.

The main characterization of quasidiagonality is contained in the following theorem of Voiculescu (recall that a representation of a C*-algebra is called *essential* if its image does not contain any nonzero compact operator).

Theorem 2.2 ([12, Thm. 1])**.** *Let A be a separable* C**-algebra. The following statements are equivalent:*

(i) *A is QD.*
(ii) *Every faithful essential representation of A is quasidiagonal.*
(iii) *For every $\varepsilon > 0$, and every finite subset $\mathcal{F} \subset A$, there exist a representation (φ, H) of A and a finite-rank orthogonal projection $P \in \mathscr{L}(H)$ such that, for $a \in \mathcal{F}$,*
$$\|P\varphi(a)P\| \geq \|a\| - \varepsilon \quad \textit{and} \quad \|[P, \varphi(a)]\| < \varepsilon.$$
(iv) *There is a sequence $(\varphi_n)_n$ of contractive completely positive maps, $\varphi_n : A \to M_{k(n)}(\mathbb{C})$, such that, for $a, b \in A$,*
$$\lim_n \|\varphi_n(a)\| = \|a\| \quad \textit{and} \quad \lim_n \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0.$$

The following theorem is a confirmation of the topological nature of quasidiagonality:

Theorem 2.3 ([12, Thm. 5])**.** *If A homotopically dominates B, and A is QD, then B is also QD. In particular, quasidiagonality is an invariant of homotopy equivalence.*

Since the cone $CA := C_0([0, 1)) \otimes A$ of any C*-algebra A is homotopic equivalent to the zero C*-algebra (which is trivially a QD C*-algebra), the next corollary follows immediately from Theorem 2.3.

Corollary 2.4. *For any separable* C**-algebra A, the cone CA of A is QD.*

Note that in Corollary 2.4, the C*-algebra A could be quite far from being quasidiagonal. We can regard CA as a continuous field of C*-algebras over $[0, 1]$, with fiber A at each $x \in [0, 1)$ and fiber 0 (a trivial quasidiagonal C*-algebra) at 1. Since quasidiagonality is a topological notion, this suggests the possibility of generalization. Indeed, it turns out that if A is exact and a continuous field of C*-algebras over a sufficiently nice space (a connected, compact metrizable space), and at least one fiber of A is quasidiagonal, then A is necessarily quasidiagonal. This is the result stated in Theorem 1.1.

3. Quasidiagonality of continuous fields of C*-algebras

We shall make use of the following known characterization of connectedness for compact metrizable spaces:

Proposition 3.1 ([8, Thm. 5.1]). *Let (X, d) be a compact metric space. The following statements are equivalent:*

(i) *X is connected.*
(ii) *For every $\varepsilon > 0$ and every pair of points $x, y \in X$ there exist $n \in \mathbb{N}$ and a finite sequence $x_0, x_1, x_2, \ldots, x_n$ of elements in X such that $x_0 = x$, $x_n = y$ and $d(x_{k-1}, x_k) < \varepsilon$ for each $1 \leq k \leq n$.*

Lemma 3.2. *Let H be a separable Hilbert space and $D \subset \mathscr{L}(H)$ a separable* C*-*subalgebra. Let $\mathcal{F} \subset D$ be a finite subset and $T \in \mathscr{L}(H)$ a finite-rank operator such that $0 \leq T \leq 1$ and $\|[T, a]\| < \delta$ for every $a \in \mathcal{F}$. If D is a quasidiagonal set of operators, then there is a finite-rank orthogonal projection $P \in \mathscr{L}(H)$ such that $T \leq P$ and $\|[P, a]\| < \delta$ for every $a \in \mathcal{F}$.*

Proof. Let K denote the range of T. By (the proof of) [3, Prop. 7.2.2], there is a finite-rank orthogonal projection P such that $\|[P, a]\| < \delta$ for every $a \in \mathcal{F}$ and $P\xi = \xi$ for every $\xi \in K$ (choose χ to be a basis for K in that proof). Note that $PT = T$. This implies that $TP = T$, and multiplying the inequality $0 \leq T \leq 1$ by P on both sides, it follows that $T \leq P$, as desired. □

Theorem 3.3. *Let A and D be* C*-*subalgebras of $\mathscr{L}(H)$, where H is a separable Hilbert space. Assume that A is separable, D is a quasidiagonal set of operators. Let X be a compact connected metrizable space and $x_0 \in X$. Put*

$$B = \{f \in C(X, A) : f(x_0) \in D\}.$$

Then B is QD.

Proof. By Theorem 2.2, it suffices to show that for each finite subset $\mathcal{F} \subset B$ and $\varepsilon > 0$, there exist a representation $\varphi : B \to \mathscr{L}(\mathcal{H}_\varphi)$ and a finite-rank orthogonal projection $P \in \mathscr{L}(\mathcal{H}_\varphi)$ such that, for $f \in \mathcal{F}$,

$$\|P\varphi(f)P\| \geq \|f\| - \varepsilon \quad \text{and} \quad \|[P, \varphi(f)]\| < \varepsilon.$$

Let d be a metric on X inducing the topology. Put $M := \max\{\|f\| : f \in \mathcal{F}\}$. By [1, p. 332], there is $\delta > 0$, such that for every pair of elements $Q, b \in \mathscr{L}(H)$ with $0 \leq Q \leq 1$ and $\|b\| \leq M$, the inequality $\|[Q, b]\| < 4\delta$ implies $\|[Q^{1/2}, b]\| < \varepsilon/10$. We may assume that $\delta < \varepsilon/10$. By uniform continuity, there is $r > 0$ such that if $d(x, y) < r$ then $\|f(x) - f(y)\| < \delta$ for all $f \in \mathcal{F}$. Write $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$. For each $1 \leq j \leq n$, there exists $x_{j,0} \in X$ such that $\|f_j(x_{j,0})\| = \|f_j\|$. By Proposition 3.1, we can find a positive integer m

such that for each $1 \leq j \leq n$ there is a finite subset $\{x_{j,0}, x_{j,1}, \dots, x_{j,m}\} \subset X$ such that $x_{j,m} = x_0$ and $d(x_{j,k-1}, x_{j,k}) < r$ for $1 \leq k \leq m$.

Choose a finite-rank orthogonal projection $P_0 \in \mathscr{L}(H)$ such that for every $1 \leq k \leq m$ we have

$$\|P_0 f(x_{j,k}) P_0\| \geq \|f(x_{j,k})\| - \varepsilon$$

for $f \in \mathcal{F}$ and $1 \leq j \leq n$. Let φ_j be the representation of B that is given by

$$\varphi_j(f) := f(x_{j,0}) \oplus f(x_{j,1}) \oplus \cdots \oplus f(x_{j,m}).$$

By the general facts on quasi-central approximate units, for each $1 \leq j \leq n$ there are positive finite-rank operators

$$P_0 \leq X_{j,0} \leq X_{j,1} \leq \cdots \leq X_{j,m} \leq 1$$

such that $X_{j,k+1} X_{j,k} = X_{j,k}$ for $0 \leq k \leq m-1$ and $\|[X_{j,k}, f(x_{j,k})]\| < \delta$ for $f \in \mathcal{F}$ and $0 \leq k \leq m$. In particular, $\|[X_{j,m}, f(x_0)]\| < \delta$ for $f \in \mathcal{F}$. By Lemma 3.2, we can choose each $X_{j,m}$ to be a finite-rank orthogonal projection. For each $1 \leq j \leq n$ define a bounded linear operator $T_j : \mathcal{H} \to \mathcal{H}^{m+1}$ by

$$T_j = X_{j,0}^{1/2} \oplus (X_{j,1} - X_{j,0})^{1/2} \oplus \cdots \oplus (X_{j,m} - X_{j,m-1})^{1/2}.$$

Then $T_j^* T_j = X_{j,m}$, and hence T_j is a partial isometry ($1 \leq j \leq n$). Moreover, for each $f \in \mathcal{F}$ and any $0 \leq k \leq m-1$ we have

$$\begin{aligned}\|[X_{j,k+1} - X_{j,k}, f(x_{j,k+1})]\| &\leq \|[X_{j,k+1}, f(x_{j,k+1})]\| + \|[X_{j,k}, f(x_{j,k})]\| \\ &\qquad + 2\|f(x_{j,k+1}) - f(x_{j,k})\| \\ &< 4\delta,\end{aligned}$$

hence,

$$\|[(X_{j,k+1} - X_{j,k})^{1/2}, f(x_{j,k+1})]\| < \varepsilon/10$$

for $0 \leq k \leq m-1$.

Fix $j \in \{1, 2, \dots, n\}$ and let $y_{j,0} := X_{j,0}^{1/2}$ and $y_{j,k+1} := (X_{j,k+1} - X_{j,k})^{1/2}$ for $0 \leq k \leq m-1$. Note that $P_j := T_j T_j^* = (y_{j,k} y_{j,l})_{k,l} \in \mathcal{B}(\mathcal{H}^{m+1})$ and that $y_{j,k} y_{j,l} = 0$ if $|k-l| > 1$, so that the projection P_j is tridiagonal. Furthermore, P_j almost commutes with $\varphi_j(\mathcal{F})$. Indeed, for any $f \in \mathcal{F}$ we have

$$\begin{aligned}\|[P_j, \varphi_j(f)]\| &\leq 3 \sup_{|k-l|\leq 1} \|f(x_{j,k+1}) y_{j,k} y_{j,l} - y_{j,k} y_{j,l} f(x_{j,l+1})\| \\ &\leq 6 \sup_{0 \leq k \leq m} \|[f(x_{j,k}), y_{j,k}]\| + 3 \sup_{|k-l|\leq 1} \|f(x_{j,k}) - f(x_{j,l})\| \\ &\leq 6\varepsilon/10 + 3\delta < \varepsilon.\end{aligned}$$

Put $P := P_1 \oplus P_2 \oplus \cdots \oplus P_n$ and $\varphi := \varphi_1 \oplus \varphi_2 \oplus \cdots \oplus \varphi_n$. Then for any $f \in \mathcal{F}$ we have

$$\|[P, \varphi(f)]\| = \sup_{1 \leq j \leq n} \|[P_j, \varphi_j(f)]\| < \varepsilon.$$

Finally, if $f = f_j \in \mathcal{F}$, then using the fact that $X_{j,0} \geq P_0$, we obtain

$$\begin{aligned}\|P\varphi(f)P\| &= \sup_{1\leq k\leq n} \|P_k\varphi_k(f_j)P_k\| \\ &\geq \|P_j\varphi_j(f_j)P_j\| \\ &\geq \|P_0 f(x_{j,0})P_0\| \\ &\geq \|f(x_{j,0})\| - \varepsilon \\ &= \|f\| - \varepsilon.\end{aligned}$$

□

Proof of Theorem 1.1. By [2, Thm. A.1], there is a $C(X)$-linear $*$-monomorphism $A \hookrightarrow C(X, \mathcal{O}_2)$. Let $\mathcal{O}_2 \subset \mathscr{L}(H)$ be a faithful separable representation. Therefore A embeds in the C*-algebra

$$B = \{f \in C(X, \mathcal{O}_2) : f(x_0) \in A(x_0)\}.$$

Since by the previous theorem B is quasidiagonal, so is A. □

Remark 3.4. The hypothesis of connectedness cannot be removed. An easy example of this would be to consider a direct sum $A_1 \oplus A_2$, where A_1 is quasidiagonal and A_2 is not. This is a continuous field over $\{1, 2\}$ with fiber $A(x) \cong A_x$, $x \in \{1, 2\}$.

4. Applications

As a corollary of Theorem 1.1, we have the following result about the quasidiagonality of crossed products by $\mathbb{Z}$.

Theorem 4.1. *Let A be a separable, exact* C*-*algebra. Suppose that A is a continuous field of* C*-*algebras over a compact, connected metrizable space X. If the fiber $A(x_0)$ of A is a simple unital $A\mathbb{T}$-algebra for some $x_0 \in X$, then $A \rtimes_\alpha \mathbb{Z}$ is quasidiagonal for every $C(X)$-linear automorphism $\alpha \in \mathrm{Aut}(A)$.*

Proof. By [13, Cor. 8.6], $A \rtimes_\alpha \mathbb{Z}$ is a continuous field of C*-algebras over X with fiber at $x \in X$ isomorphic to $A(x) \rtimes_{\alpha^x} \mathbb{Z}$, for some automorphism $\alpha^x \in \mathrm{Aut}(A(x))$. By [7, Cor. 6.8] (see also [6, Thm. 2]) the fiber $A(x_0) \rtimes_{\alpha^{x_0}} \mathbb{Z}$ is AF-embeddable, hence quasidiagonal. The conclusion now follows from Theorem 1.1. □

By varying the hypothesis on the quasidiagonal fiber in the preceding theorem, or by varying the group, several other results about the quasidiagonality of crossed products could be deduced similarly.

In 1987 Jonathan Rosenberg proved the following result.

Theorem 4.2 ([11, Thm. A1])**.** *Let G be a countable discrete group. If $C^*_r(G)$ is quasidiagonal, then G is amenable.*

It is an open problem to determine whether the converse is true, that is, whether group C*-algebras of countable amenable groups are necessarily quasidiagonal. Not much is known about the class of countable (amenable) groups that satisfy the converse of Rosenberg's result. It is known, for example, that this class contains all amenable maximally almost periodic groups. Theorem 1.1 allows us to deduce that this class is closed under the formation of central extensions by torsion-free groups.

Proof of Theorem 1.2. Since N is a torsion-free, countable discrete abelian group, $\widehat{N}$ is a compact, connected metrizable space. By [5, Thm. 1.2] (see also [10, Lem. 6.3]), $C^*(G)$ is a continuous field of C*-algebras over $\widehat{N}$. Moreover, the fiber over the trivial character of N is isomorphic to $C^*(H)$, which is quasidiagonal by hypothesis. The conclusion now follows from Theorem 1.1. □

Note added in proof. It was recently established by N. Ozawa, M. Rørdam and Y. Sato in [9], that the class of groups that satisfy the converse of Rosenberg's theorem contains all elementary amenable groups.

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