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ABSTRACT

In this dissertation, we explore the automorphism groups of the ab-initio generic structures. We establish the simplicity of the automorphism groups of the ab-initio generic structures which are obtained from a pre-dimension function with rational coefficients.

First we prove that there are no non-trivial bounded automorphisms in the automorphism group of collapsed ab-initio generic structures. Using a result of Lascar, we deduce the simplicity of *a new strongly minimal set* which was constructed in Hrushovski [17]. Then we prove that the automorphism group of an uncollapsed ab-initio generic structure that fixes every dimension-zero set pointwise is boundedly simple. In order to obtain the result we use a modification by Evans of the machinery developed in Tent and Ziegler [39].

We also prove that the automorphism groups of the generalized n-gons constructed in Tent [36] are boundedly simple. The result is obtained by modifying the proofs of Chapter 2. From this, it follows that there are simple groups with a spherical BN-pair of rank 2 which are non-Moufang and hence not of algebraic origin.

Finally, we make some observations about the small index property in automorphism groups of ab-initio generic structures. We improve a result of Lascar and conclude that the automorphism group of the almost strongly minimal generalized n-gons constructed in Tent [36] have the “almost small index property.” We also present some questions that we feel are interesting in the subject.

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CONTENTS

1 INTRODUCTION

The main motivation for studying automorphism groups of countable structures comes from the classical theorem of Engeler, Ryll-Nadrzewski, Svenonius (see Thm. 2.3.2). In this theorem, with the same perspective as the Klein's *Erlanger Programm*, one can see a connection between a countable $\aleph_0$-categorical structure and its automorphism group; namely we obtain a full characterization of a countable $\aleph_0$-categorical structure in purely group-theoretical terms. Later on, model theorists and group theorists studying permutation groups became very interested in these implicit features of automorphism groups of structures (for more information see e.g. [25, 7, 14, 21]).

One standard question that one can study is toward the *simplicity structure* of the automorphism group of a countable structure. By the term "simplicity structure" of an automorphism group, we simply mean to determine its normal and maximal normal subgroups. One remarkable result in this direction is a theorem by Lascar in [24] (see Thm.2.3.11). Lascar, first introduces bounded automorphisms and then proves that if M is a countable saturated structure which is in the algebraic closure of a $\emptyset$-definable strongly minimal set, then its strong automorphism group modulo the bounded automorphism group is a simple group. The proof of this result involves certain tools from descriptive set theory; it is well-known that one can assign a pointwise convergence topology to the automorphism group of a countable structure and obtain a Polish group (see e.g., [7]). The Polish topology on the automorphism group allows us to have access to topological tools and enables us to ask more structural questions about the automorphism group.

Our main motivation to study automorphism groups was to understand the simplicity structure of the automorphism group of some structures that were built by model-theoretic methods.

One useful method in model theory to construct countable structures is via the Fraïssé-construction method. In this method, one builds a countable homogeneous structure (we call it the Fraïssé universal object) from a countable class of finite structures which has the "joint embedding property" and the "amalgamation property". A countable class of finite structures has the amalgamation property if for every elements A, B_1, B_2 of the class and embeddings $f_i : A \to B_i$ for $i = 1, 2$, there exists an element D in the class such that B_1 and B_2 embed in D and the diagram commutes. Fraïssé's original example was to think of the class of finite linear orders as a set of approximations to the ordering in the rational numbers. One nice feature of the Fraïssé-method, in constructing a countable structure, is that we start to build a structure from a countable class of finite structures (with the amalgamation property), which is called a Fraïssé class. This helps us to verify some basic properties of the universal object by understanding the elements of the class and the amalgamation property (see e.g. [14]). Many interesting objects have been constructed or reconstructed using this method. Examples include various kinds of universal graphs, the random graph and more recently the rational Urysohn space (see e.g., [6]).

The automorphism groups of countable homogeneous structures have been studied in recent years, and a lot of machinery has been developed in order to determine their simplicity structure of them. Macpherson and Tent in [26], proved that the automorphism group of the free-homogeneous structures (homogeneous structures with the free-amalgamation property), which is transitive and not equal to the full-symmetric group is a simple group. Later, Tent and Ziegler, introduced a combinatorial tool in [39] called stationary independence to verify the simplicity of the automorphism group of some structures. For instance, they proved that the isometry group of the Urysohn space is simple modulo the normal subgroup of bounded isometries.

Smooth classes are a modified version of Fraïssé classes with a stronger property between elements of the class which is called self-sufficiency or closedness and denoted by "$\leqslant$" (see Def. 2.2.1). Every Fraïssé class is a smooth class when we choose $\leqslant$ as $\subseteq$. Similar to the Fraïssé classes, the key point in a smooth class is the amalgamation property (see Def. 2.2.3). Having the amalgamation property for a smooth class $(\mathbf{K}, \leqslant)$ allows us to construct a countable universal object called the$(\mathbf{K}, \leqslant)$-generic structure (we sometimes denote it by $\leqslant$-generic when $\mathbf{K}$ is clear from the context). Again, the $(\mathbf{K}, \leqslant)$-generic structure is the Fraïssé universal object when we choose $\leqslant$ as $\subseteq$. This method has been generalized to uncountable classes and later on and, Shelah generalized it to arbitrary classes and introduced the notion of abstract elementary classes. Later Hrushovski modified Fraïssé's construction of a universal countable structure in order to obtain stable structures with particular properties. In particular, he constructed:

1. an $\aleph_0$-categorical stable pseudo-plane;
2. a strongly minimal set with a new geometry which does not interpret any infinite group;
3. the fusion of two strongly minimal sets in disjoint languages,

thus obtaining counter-examples to conjectures by Lachlan and Zilber. Zilber conjectured that every non-locally modular strongly minimal set interprets an algebraically closed field. In [17], Hrushovski constructs a strongly minimal set which is not locally modular and does not interpret a field. In this work, he introduced the key notion of assigning a dimension function to the finite structure and used it to define a smooth class with the amalgamation property. His method was taken up by many model theorists to construct a variety of strange objects.

Let $\mathbb{K}$ be the class of all finite $\mathfrak{L}$-structures for a finite relational language $\mathfrak{L}$. We assign a pre-dimension function to each element of the class $\mathbb{K}$ (see Def. 2.2.11). The "uncollapsed" ab-initio $\leqslant$-generic structure is the $\leqslant$-generic structure that we build from the class $\mathbb{K}_0$ of all finite $\mathfrak{L}$-structures with non-negative pre-dimension (for the precise definition see Def. 2.2.11). The self-sufficiency notion "$\leqslant$" is also defined using the pre-dimension function. It is known that the theory of an uncollapsed ab-initio $\leqslant$-generic structure is stable; more precisely the theory of an ab-initio $\leqslant$-generic structure obtained from a pre-dimension function with rational coefficients is $\aleph_0$-stable and the theory of an uncollapsed ab-initio $\leqslant$-generic structures obtained from a pre-dimension function with irrational coefficients is stable but not $\aleph_0$-stable (e.g., see [4, 42]).

In [17], Hrushovski, in order to obtain *a new strongly minimal set*, which we denote it by M_μ, restricts the class of non-negative pre-dimension structures to a subclass which is smooth and has the amalgamation property; using a finite-to-one function on the 0-minimally algebraic sets (see Def. 2.2.29). Then he proves that the $\leqslant$-generic structure obtained from this restricted class is a strongly minimal structure. His method of collapsing the Morley rank in a $\leqslant$-generic structure has been later used by many model-theorists. We refer to the $\leqslant$-generic structure of the restricted class $\mathbb{K}_0^\mu$ of $\mathbb{K}_0$ as the "collapsed" ab-initio $\leqslant$-generic structure.

In Chapter 2, first we present some general background in model theory. We focus on smooth classes and the ab-initio generic structures. Then we list some of their properties and some facts about them. Later, we give an overview of the automorphism groups of countable structures and mention Lascar's simplicity result concerning the almost strongly minimal structures.

In Chapter 3, we investigate the simplicity of the automorphism group of the collapsed and uncollapsed ab-initio generic structures. In the first section, following Lascar's approach, we investigate the key step of understanding the bounded automorphism group in the automorphism group of the collapsed ab-initio generic structures. We indicate that the collapsing method is not applicable to the ab-initio generic structures derived from pre-dimensions with irrational coefficients. Hence the pre-dimension function that we will consider has rational coefficients. Let $(\mathbb{K}_0^\mu, \leqslant)$ be the restricted class of $(\mathbb{K}_0, \leqslant)$ using a μ-function (see Def. 2.2.31). Suppose $(\mathbb{K}_0^\mu, \leqslant)$ has the amalgamation property and let M be the $(\mathbb{K}_0^\mu, \leqslant)$-generic structure. The main result in this section is the following theorem.

Theorem. *There is no non-trivial bounded automorphism in the automorphism group of the* $(\mathbb{K}_0^\mu, \leqslant)$*-generic structure* M*.*

We present the proof only for binary relational languages. The result can be modified for any finite relational languages with n-ary relations with $n \geq 2$. We indicate that in the above theorem, the coefficients of the pre-dimension function are important. For example in a binary relational language, when the coefficients of the pre-dimension function are equal, it is known that the pre-geometry (see Def. 2.1.23) that one obtains is locally modular. Locally-modular strongly minimal structures always have non-trivial bounded automorphisms (see Thm. 2.3.14). Hence there are collapsed ab-initio generic structures with bounded automorphisms. We will discuss these cases in the section as well.

The proof of the theorem above is based on the combinatorial behavior of the pre-dimension and the dimension functions in the $(\mathbb{K}_0^\mu, \leqslant)$-generic structure M. Note that, in general, the collapsed ab-initio $\leqslant$-generic structures are not necessarily strongly minimal, hence we can not apply Lascar's simplicity theorem (Thm. 2.3.11) directly. However, the collapsed $\leqslant$-generic structure constructed in [17], is strongly minimal. Hence, from Lascar's theorem (Thm. 2.3.11), we deduce the following.

Corollary. *The automorphism group of* M_μ *is a simple group.*

As we mentioned before, the collapsed ab-initio $\leqslant$-generic structures are not necessarily strongly minimal. Hence, we can not answer the simplicity structure of the collapsed ab-initio generic structures fully just by Lascar's theorem. However, we can apply the same method as described in Section 2 of Chapter 3, to answer the question in full generality.

In the second section of Chapter 3, we prove the following theorem.

Theorem. *The automorphism group of uncollapsed ab-initio generic structures with rational coefficients which fixes pointwise every dimension-zero set is boundedly simple.*

As we mentioned before, Tent and Ziegler introduced stationary independence in [39] in order to provide a combinatorial tool to prove the simplicity of certain countable structures. In the uncollpased ab-initio generic structures it was not possible to work in the same setting as they considered in [39]. To obtain the result, we follow a modification by Evans in [9] of the machinery developed in [39]. First, we indicate an observation about the ab-initio generic structures. Then, we define the notion of stationary independence relation for a countable class of subsets of a structure, as Evans does (see Def. 3.2.9). From the pre-dimension function we can define a dimension function $\mathrm{d}(-)$ (see Def. 2.2.18). Then, we prove that the independence relation $\mathop{\smile\!\!\!\!\!\!|}\,^{d}$, which is derived from the dimension function, is indeed a stationary independence on the countable class $\mathfrak{Y} := \{\mathrm{gcl}(A) : A \subset M, A \text{ is finite}\}$ where $\mathrm{gcl}(A) = \{x \in M : \mathrm{d}(xA) = \mathrm{d}(A)\}$ for $A \subseteq M$ (see Thm. 3.2.21). Then we modify the definition of bounded automorphisms to gcl-bounded automorphisms (see Def. 3.2.23). The main step in this section is to prove that every non gcl-bounded automorphism moves almost maximally for the $\mathop{\smile\!\!\!\!\!\!|}\,^{d}$-independence relation on the family $\mathfrak{Y}$ (see Lem. 3.2.25 and Cor. 3.2.28). The observation that we noted before will be used in this part. Then we prove that there is no non-trivial gcl-bounded automorphism in the automorphism group of the uncollapsed generic structure. This proof is a modification of the proof of the theorem which we already mentioned for the bounded automophisms in the automorphism group of collapsed ab-initio generic structures. Finally, using a theorem by Evans, we obtain the result that the automorphism group of uncollapsed generic structures which fixes pointwise every dimension-zero set is boundedly simple (see Cor. 3.2.32).

The initial and main motivation for this dissertation was to investigate the simplicity of the automorphism group of the very homogeneous generalized n-gons constructed in [36]. It was unknown whether there is a simple group with a spherical BN-pair of rank 2 which is not of algebraic origin. The automorphism group of the very generalized n-gons constructed in [36] have a BN-pair and they were good candidates to verify this question. Indeed, here, we show that the automorphism group of the very generalized n-gons are simple groups. In [2], Baldwin adapted Hrushovski's generic structure to construct an almost strongly minimal non-Desarguesian projective plane. Later in [8] Debonis and Nesin, generalized Baldwin's method to construct $2^{\aleph_0}$-many almost strongly minimal generalized n-gons which do not interpret a group for odd n. In [36] with a similar approach Tent constructed very homogeneous generalized n-gons for all n with some stronger properties. She constructed generalized n-gons for $n \geq 3$ such that the automorphism group acts transitively on the set of ordered ordinary $(n+1)$-gons. The transitivity result implies that the automorphism group has a BN-pair. As a result of using Hrushovski construction, no infinite group is interpretable in this class of constructions (see Lemma 2.2.35 and Lemma 2.1.42).

In Chapter 4, first we give a brief background of the question in the context of the theory of buildings and present some facts about generalized polygons and BN-pairs. Then, we prove that in both cases of the very homogeneous generalized n-gons constructed in [36], the automorphism group is a simple group. The proofs are similar to the proofs in Chapter 3. Although the combinatorial behavior in the generic structure is very similar in both cases, the of full class and the sub-class (with finite elements with non-negative pre-dimension), the proofs needed to be modified.

In the second section of Chapter 4 we prove the simplicity of the automorphism group of the $\aleph_0$-stable generalized n-gons constructed in [36]. To obtain the result, we follow the same approach as we did in the proof of the simplicity of the automorphism group of the uncollapsed generic structure. In the third section of Chapter 4 we prove the simplicity of the automorphism group of the almost strongly minimal generalized n-gons. As we mentioned before, to obtain a finite Morley rank generic structure by the Hrushovski method, one needs to consider a finite-to-one function μ for 0-minimally algebraic pairs in order to restrict the class of finite structures. We give a more straightforward proof of simplicity in this case which is more algebraic and which requires only little background in model theory. The main result that we need is that there is no non-trivial bounded automorphism in the automorphism group of the almost strongly minimal generalized n-gons. Then, following Lascar's approach and applying Theorem 2.3.11, we obtain the simplicity result. This result will be published in Journal of Algebra [11].

However, the conclusion that we get at the end is the following.

Corollary. *There are simple groups with a spherical BN-pair of rank 2 which are non-Moufang and hence not of algebraic origin.*

In the last chapter we mention some results about the small index property and also present some questions in the subject. As we mentioned before, the automorphism group of a countable structure with the pointwise convergence topology is a Polish group. One interesting aspect of the topological approach is towards some theorems that are called "reconstruction theorems". The main idea is to reconstruct the structure from the information that one can obtain from the group theoretical properties of the automorphism group (including the properties that one obtains from the Polish topology). One key notion that leads to a reconstruction theorem is the small index property. We present some observations about the small index property and then slightly improve Lascar's result about the countable saturated almost strongly minimal structures. Then we conclude that the almost strongly minimal very homogeneous generic structure constructed in [36] has "almost" small index property. Overall, we investigated a very special case of smooth classes in this dissertation, namely those that are obtained from a pre-dimension function. Finally, we ask some questions about further possible generalizations of the simplicity results that we obtained here.

2 PRELIMINARIES

2.1 BACKGROUND

2.1.1 *Notation and preliminaries*

We assume basic knowledge of model theory as we mainly follow [38, 27] and basic knowledge of set theory that can be found in [20, 23]. We will use standard notation of model theory that can be found in [38, 27].

Sets are denoted by capital letter mainly such as A, B, C, X, Y. Single elements mainly by $a, b, c, \cdots, x, y, z$. Models by M, N and types by $p, q, \cdots$. Moreover the set of natural numbers is $\mathbb{N} = \{0, 1, 2, \cdots\}$. For a set X, let $\mathcal{P}(X)$ denote the power set of X. Tuples are always finite.

Let $\mathfrak{L}$ be a countable possibly many-sorted language. T is always a complete theory in $\mathfrak{L}$. A set $X \subseteq M^n$ is *definable over* A (or A*-definable*) if $X = \{\bar{x} \in M^n : M \models \phi(\bar{x}, \bar{a})\}$ for some $\phi \in \mathfrak{L}$ and $\bar{a} \in A$. We denote $\phi \in \mathfrak{L}_A$ to indicate that ϕ is a formula in language $\mathfrak{L}$ with parameters from A. Let $\phi \in \mathfrak{L}_M$, then denote by $\phi(M)$ the set of all realization of ϕ in M. We say that X is *type-definable* over A if X is the intersection of an arbitrary family of A-definable set. If $M \models T$, $A \subseteq M$ and $\bar{a} \in M$ let $\operatorname{tp}(\bar{a}/A)$ be the *complete type* of $\bar{a}$ over A. A type $p(\bar{x})$ is *isolated* by a formula $\phi(\bar{x}) \in \mathfrak{L}_A$ if any realization of ϕ also realizes p. We denote by $S_n(M)$ the collection of all complete n-types.

We proceed to define $\mathfrak{L}^{eq}$ in the following way. Let EQ^M be the collection of all $\mathfrak{L}$-equivalence formulas $E(\bar{x}_1, \bar{x}_2)$ on M^n. For each formula $E(\bar{x}_1, \bar{x}_2) \in EQ^M$ there is a sort S_E in $\mathfrak{L}^{eq}$. In particular also a sort for $S_=$ in $\mathfrak{L}^{eq}$. For E be an $\mathfrak{L}$-equivalence relation as above, $\mathfrak{L}^{eq}$ also contains a new n-ary function symbol f_E whose domain sort is $(S_=)^n$ and whose image sort is S_E. Note that $\mathfrak{L}^{eq}$ is also countable (since $\mathfrak{L}$ is countable). Now enlarge M to an $\mathfrak{L}^{eq}$-structure; interpretation of the sort $S_=$ in M^{eq} will be the set M itself, and relations and functions symbols in $\mathfrak{L}^{eq}$ will be interpreted in the usual way in $\mathfrak{L}$. The interpretation of sort S_E in M^{eq} will be the set $\{\bar{a}/E : \bar{a} \in M^n\}$ and the interpretation of the function f_E will be the function which takes $\bar{a}$ to $\bar{a}/E$. More details about $\mathfrak{L}^{eq}$ can be found in [38, 27, 30].

Suppose M is an $\mathfrak{L}$-structure. Let $\mathrm{Aut}(M)$ be the set of all automorphisms of M. If $A \subseteq M$, one defines the subgroup of A-automorphisms of $\mathrm{Aut}(M)$ as follows:

$$\mathrm{Aut}_A(M) = \{f \in \mathrm{Aut}(M) : f(a) = a \text{ for all } a \in A\}.$$

Let $M \models T$ and $A \subseteq M$. The *definable closure* of A, denoted by $\mathrm{dcl}(A)$, is the set of all elements of M which are fixed under A-automorphisms of M. The *algebraic closure* of A, denoted by $\mathrm{acl}(A)$ is the set of all elements of M which have finite orbit under A-automorphisms of M. By $\mathrm{acl}^{eq}(-)$ and $\mathrm{dcl}^{eq}(-)$ we mean algebraic closure and definable closure, respectively, in M^{eq}.

Definition 2.1.1. Let T be a complete theory. We say T has *weak elimination of imaginaries* if for every $e \in M^{eq}$ there is $\bar{a} \in \mathrm{acl}^{eq}(e) \cap M$ such that $e \in \mathrm{dcl}^{eq}(\bar{a})$.

Definition 2.1.2. Let $\bar{a}, \bar{b} \in M$ be n-tuples in M^n. We say that $\bar{a}$ and $\bar{b}$ have the same *strong type* over A, denoted by $\mathrm{stp}(\bar{a}/A) = \mathrm{stp}(\bar{b}/A)$, if for every finite A-definable equivalence relation E on M^n, $E\left(\bar{a}, \bar{b}\right)$ holds.

We will use the following fact about strong types.

Lemma 2.1.3. *([30]) The following are equivalent:*

1. $\mathrm{stp}(a/A) = \mathrm{stp}(b/A)$.
2. $tp(a/\mathrm{acl}^{eq}(A)) = tp(b/\mathrm{acl}^{eq}(A))$.

Corollary 2.1.4. Suppose T admits weak elimination of imaginaries and $M \models T$. Let $A \subset M$ and $\bar{a} \in M$. Then $\mathrm{tp}(\bar{a}/\mathrm{acl}(A)) = \mathrm{stp}(\bar{a}/A)$.

Proof. By the definition of weak elimination of imaginaries if $x \in \mathrm{acl}^{eq}(A) \setminus \mathrm{acl}(A)$ then there is $\bar{c} \in M$ such that $x \in \mathrm{dcl}^{eq}(\bar{c})$ and $\bar{c} \in \mathrm{acl}^{eq}(x)$. Note that $\mathrm{acl}(A) = \mathrm{acl}^{eq}(A) \cap M$ so $\bar{c} \in \mathrm{acl}(A)$. Now the result follows from $x \in \mathrm{dcl}^{eq}(\bar{c})$. □

2.1.1.1 *Saturated Models*

Definition 2.1.5. Let κ be an infinite cardinal. We say $M \models T$ is *κ-saturated* if $p \in S_n^M(A)$ is realized in M for all $A \subseteq M$ with $|A| < \kappa$. We say that M is *saturated* if it is $|M|$-saturated.

Fact 2.1.6. *([27]) Countable saturated models are unique up to isomorphism.*

Fact 2.1.7. *([27]) Let M be saturated. Let $A \subset M$ with $|A| < |M|$ and $b \in M$. The following are equivalent:*

1. *b has finite orbit under automorphisms of M fixing A pointwise.*

2. *$tp^M(b/A)$ has finitely many realizations (then we say tha type of b over A is algebraic).*

There are many different rank notions in model theory in order to study the complexity of definable sets and types (e.g., see [32]). Here we are only concerned with Morley rank which is defined as follows.

Definition 2.1.8. Let M be an $\mathfrak{L}$-structure and $\phi(\bar{x})$ is an $\mathfrak{L}_M$-formula. We define inductively the *Morley rank of* ϕ, $RM^M(\phi(\bar{x}))$ in M as follows:

1. $RM^M(\phi(\bar{x})) \geq 0$ if and only if $\phi(M)$ is nonempty,
2. if α is a limit ordinal, then $RM^M(\phi(\bar{x})) \geq \alpha$ if and only if $RM^M(\phi(\bar{x})) \geq \beta$ for all $\beta < \alpha$,
3. for any ordinal α, $RM^M(\phi(\bar{x})) \geq \alpha + 1$ if and only if there are $\mathfrak{L}_M$-formulas $\psi_1(\bar{x}), \psi_2(\bar{x}), \cdots$ such that $\psi_1(M), \psi_2(M), \cdots$ is an infinite family of pairwise disjoint subsets of $\phi(M)$ and $RM^M(\psi_i) \geq \alpha$ for all $i \in \omega$.

If $RM^M(\phi) \geq \alpha$ but $RM^M(\psi_i) \ngeq \alpha + 1$, then $RM^M(\phi) = \alpha$. If $RM^M(\psi_i) \geq \alpha$ for all ordinal α, then $RM^M(\phi) = \infty$. If $RM^M(\phi) = \alpha$ then *Morley degree* of ϕ, denoted by $dM^M(\phi)$ is the greatest k such that there are mutually contradictory formulas $\phi_1(\bar{x}), \cdots, \phi_k(\bar{x})$ each implying $\phi(\bar{x})$ and each with Morley rank α. If $p \in S_n(A)$, then $RM^M(p) = \inf\{RM^M(\phi) : \phi \in p\}$. If $RM^M(p)$ is an ordinal, then
$dM^M(p) = \inf\{dM^M(\phi) : \phi \in p, RM^M(\phi) = RM^M(p)\}$.

Fact 2.1.9. *Suppose M and N are $\aleph_0$-saturated models of T and N is an elementary extension of M. If $\phi \in \mathfrak{L}_M$, then $RM^M(\phi) = RM^N(\phi)$.*

Fact 2.1.9 allows us to define Morley rank of ϕ in such way that it does not depend on which model contains the parameters occurring in ϕ.

Definition 2.1.10. A formula ϕ is strongly minimal if $RM(\phi) = dM(\phi) = 1$. We say that a theory T is strongly minimal if the formula $v = v$ is strongly minimal.

Example 2.1.11. Classical examples of strongly minimal theories are the theory of algebraically closed fields of some fixed characteristic and the theory of a vector space over some fixed skew field both when considered in suitable languages.

2.1.1.2 *Stable Theories*

Stable theories were the first dividing line in the classification of first order theories which was introduced by Shelah; he proved that unstable theories have the largest possible number of non-isomorphic models in each uncountable cardinal (see e.g. [35]).

Definition 2.1.12. Let T be a complete theory in a countable language, and let κ be an infinite cardinal. We say T is *κ-stable* if whenever $M \models T$ and $A \subseteq M$ and $|A| = \kappa$, then $\left|S_n^M(A)\right| = \kappa$ for all n. We say that T is *stable* if T is κ-stable for some infinite cardinal κ.

Fact 2.1.13. *([38, 27])*

- *T is κ-stable if and only if $\left|S_1^M(A)\right| \leq \kappa$ for all $|A| \leq \kappa$.*
- *If T is $\aleph_0$-stable, then T is κ-stable for all κ.*

Fact 2.1.14. *([30]) Let T be a stable theory. Then either*

1. *T is superstable; or*
2. *for any $\kappa \geq \aleph_0$, T is κ-stable if and only if $\kappa = \kappa^{\aleph_0}$.*

Fact 2.1.15. *Strongly minimal theories are $\aleph_0$-stable.*

One important notion in model theory that was introduced by Shelah is the notion of dividing and forking (see [35]).

Definition 2.1.16. A formula $\phi\left(\bar{x}, \bar{b}\right)$ *divides over* A if there is some $k < \omega$ and an infinite sequence $\left\langle \bar{b}_i : i < \omega \right\rangle$ of realizations of $tp\left(b/A\right)$ such that $\left\{\phi\left(\bar{x}, \bar{b}_i\right) : i < \omega\right\}$ is k-inconsistent. A partial type $p\left(\bar{x}\right)$ divides over A if there is a formula $\phi\left(\bar{x}\right)$ implied by $p\left(\bar{x}\right)$ such that $\phi\left(\bar{x}\right)$ divides over A. A partial type $p\left(\bar{x}\right)$ *forks over* A if there are formulas $\phi_1\left(\bar{x}\right), \cdots, \phi_n\left(\bar{x}\right)$ such that $p\left(\bar{x}\right) \models \phi_1\left(\bar{x}\right) \vee \cdots \vee \phi_n\left(\bar{x}\right)$ and each $\phi_i\left(\bar{x}\right)$ divides over A.

Now we can define the following notion of independence

Definition 2.1.17. A set A is *forking-independent from C over B*, denoted by $A \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}_B C$, if $tp\left(\bar{a}/B, C\right)$ does not fork over B for all $\bar{a} \in A$.

Theorem 2.1.18. *([27, 30])*

Suppose T is stable. Then the forking-independence in T has the following properties.

1. *Invariance: if $A \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}_B C$ and $\left(A', B', C'\right) \equiv \left(A, B, C\right)$, then $A' \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}_{B'} C'$.*

2. *Extension: for any a, A and $B \supseteq A$ there is a' such that $tp(a/A) = tp(a'/A)$ and $a' \mathop{\smile\!\!\!\!\!\!\!\;|}_A B$.*

3. *Symmetry: if $A \mathop{\smile\!\!\!\!\!\!\!\;|}_B C$ then $C \mathop{\smile\!\!\!\!\!\!\!\;|}_B A$.*

4. *Finite Character: $A \mathop{\smile\!\!\!\!\!\!\!\;|}_C B$ if and only if $A_0 \mathop{\smile\!\!\!\!\!\!\!\;|}_C B$ for all $A_0 \subset_{<\omega} A$.*

5. *Local Character: for B and every finite A_0 there is a subset $A \subset B$ such that $|A| \leq |T|$ and $A_0 \mathop{\smile\!\!\!\!\!\!\!\;|}_A B$.*

6. *Transitivity: suppose $D \subseteq C \subseteq B$. Then $B \mathop{\smile\!\!\!\!\!\!\!\;|}_D A$ if and only if $B \mathop{\smile\!\!\!\!\!\!\!\;|}_C A$ and $C \mathop{\smile\!\!\!\!\!\!\!\;|}_D A$.*

7. *Uniqueness: there is a unique non-forking extension over an algebraically closed set.*

Fact 2.1.19. *The properties 1-7 in Theorem 2.1.18 characterize stable theories and forking with an abstract treatment (see [12]).*

2.1.1.3 *Morley categoricity theorem*

Definition 2.1.20. Let κ be an infinite cardinal. A theory T is *κ-categorical* if any two models of T of cardinality κ are isomorphic.

The following theorem of Morley is often considered the most inspiring theorem in the modern model theory.

Theorem 2.1.21. *(Morley [28]) Let T be a countable theory. If T is λ-categorical for some uncountable cardinal λ then T is κ-categorical for all uncountable cardinal κ.*

Fact 2.1.22. *Strongly minimal theories are $\aleph_1$-categorical.*

2.1.2 *Pre-geometries*

The following notion of pre-geometries is motivated by combinatorial properties of the algebraic closure in strongly minimal theories. The same concept also appears in the theory of matroids (see e.g. [43]).

Definition 2.1.23. Let X be a set and $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ be an operator. We call (X, cl) a *pre-geometry* if the following properties hold:

1. $A \subset cl(A)$ and $cl(cl(A)) = cl(A)$ for all $A \subseteq X$;

2. $cl(A) \subseteq cl(B)$ for $A \subseteq B \subseteq X$;

3. (finite character) if $x \in cl(A)$ then $x \in cl(A_0)$ for some finite $A_0 \subseteq A$;

4. (exchange property) if $A \subset X$ and $x, y \in X$ such that $x \in cl(A \cup \{y\})$, then $x \in cl(A)$ or $y \in cl(A \cup \{x\})$.

The pre-geometry (X, cl) is said to be *geometry* if $cl(\emptyset) = \emptyset$ and $cl(\{x\}) = \{x\}$ for all $x \in X$. A subset $A \subseteq X$ is called *cl-closed* if $cl(A) = A$.

Definition 2.1.24.
1. Suppose (X, cl) is a pre-geometry, we say $A \subseteq X$ is *independent* if $a \notin cl(A \setminus \{a\})$ for all $a \in A$. If $Y \subseteq X$, we say that B is a *basis* for Y if $B \subseteq Y$ is independent and $Y \subseteq cl(B)$.
2. ([27] Lemma 8.1.3) Let (X, cl) be a pre-geometry and $Y \subseteq X$. If $B_1, B_2 \subseteq Y$ are basis for Y then $|B_1| = |B_2|$. We call $|B_i|$ the dimension of Y and denote it by $dim_{cl}(Y)$.
3. (Localization) Suppose (X, cl) is a pre-geometry. If $A \subseteq X$ we also consider the *localization* of $cl(-)$ at A which is defined as $cl_A(B) = cl(A \cup B)$ for all $B \subseteq X$.

Remark 2.1.25. ([27] Lemma 8.1.3)

1. If (X, cl) is a pre-geometry, then (X, cl_A) is also a pre-geometry for all $A \subseteq X$.
2. If (X, cl) is a pre-geometry, then we can associate a natural geometry to it. Namely let $X_0 = X \setminus cl(\emptyset)$ and consider the equivalence relation $\sim$ on X_0 given by $a \sim b$ if and only if $cl(\{a\}) = cl(\{b\})$. Let $\hat{X}$ be $X_0/\sim$. Define $\hat{cl}(A/\sim) = \{b/\sim : b \in cl(A)\}$. Then $\left(\hat{X}, \hat{cl}\right)$ is a geometry.

The following definition helps us to classify some combinatorial aspects of pre-geometries in different structures.

Definition 2.1.26. Let (X, cl) be a pre-geometry.

- We say that (X, cl) is *locally-finite* if $|cl(A)| < \omega$ for all finite $A \subseteq X$.
- We say that (X, cl) is *trivial* (or *degenerated*) if $cl(A) = \bigcup_{a \in A} cl(\{a\})$ for all $A \subseteq X$.
- We say that (X, cl) is *modular* if for any finite-dimensional cl-closed sets $A, B \subseteq X$, $dim_{cl}(A \cup B) = dim_{cl}(A) + dim_{cl}(B) - dim_{cl}(A \cap B)$ holds.
- We say that (X, cl) is *locally-modular* if $\left(X, cl_{\{a\}}\right)$ is modular for some $a \in X$.

Remark 2.1.27. Trivial pre-geometries are modular.

The following lemma gives us a technical tool to work with in pre-geometries.

Lemma 2.1.28. *Let (X, cl) be a pre-geometry. Then the following are equivalent.*

1. *(X, cl) is modular.*

2. *If $A \subseteq X$ is closed and nonempty, $b \in X$ and $x \in cl(A, b)$ then there is $a \in A$ such that $x \in cl(a, b)$.*

3. *If $A, B \subseteq X$ is closed and nonempty and $x \in cl(A, B)$ then there is $a \in A$ and $b \in B$ such that $x \in cl(a, b)$.*

4. *If $A, B, C \subseteq M$ are cl-closed sets and $A \subseteq B$, then $cl(A \cup C) \cap B = cl(A \cup (B \cap C))$.*

Proof. See Lemma 8.1.13 in [27] for the proof of the equivalence of (1), (2) and (3). Here we first prove that (2) and (3) imply (4). Let $x \in cl(A \cup C) \cap B$, then, by (2), $x \in cl(a, C)$ for some $a \in A$ and since $A \subseteq B$, $x \in cl(a, B)$. Hence using the exchange property $x \in cl(a, B \cap C)$ and $x \in cl(A, B \cap C)$. Now for the other direction suppose that $x \in cl(A, B \cap C)$ then $x \in cl(a, d)$ for some $a \in A$ and $d \in B \cap C$. Since $A \subseteq B$, it follows that $x \in B$ and $x \in cl(A, C)$.

We claim that (4) implies (1). Suppose $A, B \subseteq X$ are cl-closed sets. We use induction on $dim_{cl}(A)$. If $dim_{cl}(A) = 0$ then we are done. Suppose $dim_{cl}(A) = dim_{cl}(A_0) + 1$ and by induction
$dim_{cl}(A_0 \cup B) = dim_{cl}(A_0) + dim_{cl}(B) - dim_{cl}(A_0 \cap B)$.

By (4) we know that $cl(A_0 \cup B) \cap A = cl(A_0 \cup (B \cap A))$. If $cl(A \cup B) = cl(A_0 \cup B)$ then we are done using the induction step. If $cl(A \cup B) \neq cl(A_0 \cup B)$, then $dim_{cl}(A \cup B) = dim_{cl}(A_0 \cup B) + 1$. It follows $dim_{cl}(A \cup B) = dim_{cl}(A_0 \cup B) + 1 = (dim_{cl}(A_0) + 1) + dim_{cl}(B) - dim_{cl}(A_0 \cap B)$. Using $cl(A_0 \cup B) \cap A = cl(A_0 \cup (B \cap A))$ and the exchange property it follows that $dim_{cl}(A_0 \cap B) = dim_{cl}(A \cap B)$. Hence $dim_{cl}(A \cup B) = dim_{cl}(A) + 1 + dim_{cl}(B) - dim_{cl}(A \cap B)$. □

Example 2.1.29. Here are some standard examples of pre-geometries.

1. Pure sets.

2. acl$(-)$ is a pre-geometry in strongly minimal structures.

   a) As we mentioned in Example 2.1.11, algebraically closed fields and vector spaces are classical examples of strongly minimal theories. In both examples algebraic closure and linear closure (linear span) are pre-geometries.

   b) Algebraically closed fields are the only known naturally arising example of non-locally modular ([27] 8.1.12) strongly minimal sets.

Theorem 2.1.30. *Suppose M is strongly minimal, non-trivial, locally modular, then there is an infinite group definable in M^{eq}.*

Zilber conjectured that every non-locally modular strongly minimal set interprets an algebraically closed field (this is called Zilber's trichotomy conjecture). This conjecture has been refuted by Hrushovski in [17]. He constructed a new strongly minimal set with a new geometry which does not interpret any infinite group (hence in particular no field). We will discuss this method in Subsection 2.2.1.1. Later, Hrushovski and Zilber introduced a new framework called Zarisiki geometries, in which the Zilber's trichotomy conjecture holds and which prevents Hrushovski's combinatorial objects to be constructed (see e.g., [18]).

Another combinatorial notion that helps us to understand the algebraic closure in a theory (or in a model) is the following notion of n-ampleness.

Definition 2.1.31. Let T be a complete stable theory. Then T is called n-ample if, possibly after adding parameters, there exist tuples $\bar{a}_0, \cdots, \bar{a}_n$ such that

1. $acl\left(\bar{a}_0, \cdots, \bar{a}_r\right) \cap acl\left(\bar{a}_0, \cdots, \bar{a}_{r-1}, \bar{a}_{r+1}\right) = acl\left(\bar{a}_0, \cdots, \bar{a}_{r-1}\right)$ where $0 \leq r \leq n-1$.

2. $\bar{a}_{r+1} \mathop{\smile\!\!\!\!\!\!\!\mid}_{\bar{a}_r} \bar{a}_0, \cdots, \bar{a}_{r-1}$ where $0 \leq r \leq n-1$.

3. $\bar{a} \not\mathop{\smile\!\!\!\!\!\!\!\mid} \bar{a}_0$.

Fact 2.1.32. *([31]) If T is n-ample then T is m-ample for all $m \leq n$.*

Example 2.1.33.

1. Fields are n-ample for all n (see e.g., [31]).

2. Recently Ould-Houcine and Tent in [29] showed that free groups are n-ample for all n. However, it is not known whether one can interpret a field inside a free group.

Definition 2.1.34. Let T be a complete theory.

- T is *CM-trivial* if T is not 2-ample.

- T is *one-based* if T is not 1-ample.

Remark 2.1.35. If a theory T is one-based then by Fact 2.1.32, T is CM-trivial.

Fact 2.1.36. *([27] Thm. 8.2.14) Suppose M is strongly minimal. Then $Th\left(M\right)$ is one-based if and only if "acl" is locally modular.*

Fact 2.1.37. *(Lem. 13 in [17]) The structure M_μ constructed in [17] is CM-trivial but not one-based.*

Definition 2.1.38. Suppose (X, cl) is a pre-geometry and $dim_{cl}(-)$ is the dimension function that is derived from $cl(-)$. Then, using $dim_{cl}(-)$, one can define the following closure operator $gcl(-)$ in X: suppose $A \subset X$ then

$gcl(A) := \{x \in X : dim_{cl}(\{x\} \cup A) = dim_{cl}(A)\}.$

For $A, B \subseteq X$, define $dim_{cl}(A/B) = dim_{cl}(A \cup B) - dim_{cl}(B)$. For simplicity we denote AB for $A \cup B$.

Remark 2.1.39. The $dim_{cl}(-)$ is integer valued and $dim_{cl}(A) \leq |A|$ for all A, hence $gcl(-)$ is a pre-geometry.

Proof. It is enough to check the exchange property. Suppose $a \in gcl(bA) \backslash gcl(A)$ then $dim_{cl}(a/Ab) = 0$ and $dim_{cl}(a/A) = 1$. Moreover, by our assumptions $dim_{cl}(b/A) > 0$. Then $dim_{cl}(bA) = dim_{cl}(aA)$. Hence $dim_{cl}(Aba) = dim_{cl}(Aa)$ and $dim_{cl}(b/Aa) = 0$. Hence $b \in gcl(Aa)$. □

Remark 2.1.40. Suppose $d(-)$ is a dimension function such that the dimension of single elements lie in $\{0, 1, e \in \mathbb{N} \backslash \{0, 1\}\}$. Then $gcl_d(-)$ is not necessarily a pre-geometry. To see this, it is enough to find elements a, b and subset $A \subset X$ such that $d(a/bA) = 0$, $d(a/A) = e$ and $d(b/A) = 1$.

Definition 2.1.41. Let (M, cl) be a pre-geometry and $n \in \mathbb{N}$. We call M to be *n-flat* if whenever $E = \{E_i : i \in \mathcal{P}(\{1, \cdots, n\})\}$ is a family of finite dimensional closed subsets of M and s ranges over the subsets of E, then $\sum_{s \in E} (-1)^{|s|} dim_{cl}(E_s) \leq 0$. We call M *flat* if it is n-flat for all $n \in \mathbb{N}$.

Lemma 2.1.42. *Let M be a saturated strongly minimal set whose geometry in 3-flat. Then M does not interpret an infinite group.*

Proof. The proof that we present here is the same proof as the proof of Lemma 14 in [17]. Suppose G is an infinite group in M of dimension g. Let a_1, a_2, a_3 be generic elements (elements with the maximal dimension) in G. Let $E_i = cl(a_j : j \neq i)$ for $i = 1, 2, 3, 4$ and let $E_4 = cl\left(a_1^{-1}a_2, a_1^{-1}a_3\right)$. Then $dim_{cl}(E_\emptyset) = 3g$ and $dim_{cl}(E_i) = 2g$, $dim_{cl}(E_{ij}) = g$. Moreover $E_1 \cap E_2 \cap E_3 = cl(\emptyset)$. Then 3-flatness implies that $g = 0$. □

2.2 SMOOTH CLASSES

SETTING Let $\mathfrak{L} = \{R_i(\bar{x}) : ln(\bar{x}) = r_i, i \leq n, \}$ be a finite relational language. Let A be an $\mathfrak{L}$-structure. We denote $R_i(A)$ for the set $\left\{\bar{a} : R_i^A(\bar{a})\right\}$. By notation $A \subseteq_{<\omega} X$ we mean that A is a finite subset of X. We denote AB for $A \cup B$.

There is a very useful method to construct a countable homogeneous structure from a countable class of finite structures, which is called Fraïssé construction method. It is known that if the empty set is included in the Fraïssé class (a countable class of finite structures with AP as it was described in the introduction) then there is a unique countable universal object which is called Fraïssé universal object (see e.g., Thm. 1.5 in [14]). Smooth classes are a modified version of Fraïssé classes with stronger notion of embedding between elements of the class.

Definition 2.2.1. A class $(\mathbf{K}, \leqslant)$ of finite $\mathfrak{L}$-structures, together with a relation $\leqslant$ on $\mathbf{K} \times \mathbf{K}$, is called *smooth* if:

1. $\leqslant$ is reflexive;
2. $\leqslant$ is transitive;
3. $B \leqslant C$ implies $B \subseteq C$;
4. if $A \leqslant C$, $B \subseteq C$ and $A \subset B$ then $A \leqslant B$.

When $A \leqslant B$, then we say that A is $\leqslant$-*closed* (or *self-sufficient*) in B. Moreover if M is an infinite $\mathfrak{L}$-structure, we define $A \leqslant M$ if $A \leqslant B$ for all finite subsets $A \subseteq B \subset M$.

The following definition is a modified version of Fraïssé's universal object for a Fraïssé class.

Definition 2.2.2. Let $(\mathbf{K}, \leqslant)$ be a smooth class. A countable structure M is called $(\mathbf{K}, \leqslant)$-*generic structure* if the following condition holds.

1. If $A' \leqslant M$ and $A' \leqslant B \in \mathbf{K}$ then there exists $B' \leqslant M$ such that $B \cong_{A'} B'$ (this property is called $\leqslant$-*richness*)
2. M is a union of $\langle A_i : i \in \omega \rangle$ such that $A_i \leqslant A_{i+1}$.
3. Suppose B and C are finite $\leqslant$-closed subset of M and let α be an isomorphism of B and C. Then α extends to an automorphism of M.

Definition 2.2.3. Suppose $(\mathbf{K}, \leqslant)$ is a smooth class and $A, B, C \in \mathbf{K}$ such that $A \leqslant B$ and $A \leqslant C$. The class $(\mathbf{K}, \leqslant)$ has the *amalgamation property* (AP) if there exists $D \in \mathbf{K}$ such that $B \leqslant D$ and $C \leqslant D$. The *free-amalgam* of B and C over A is the structure with domain $B \cup C$, whose only relations are those from B and C. We denote it by $B \otimes_A C$. For simplicity, we also denote $B \otimes C$ for $B \otimes_A C$ when $A = \emptyset$. A smooth class $(\mathbf{K}, \leqslant)$ has the *free-amalgamation property* if for $A, B, C \in \mathbf{K}$ if $A \leqslant B$ and $A \leqslant C$ implies $B \otimes_A C \in \mathbf{K}$. Moreover suppose

$A, B \in \mathbf{K}$. Then the class $(\mathbf{K}, \leqslant)$ has the *joint embedding property* (JEP) if there exists $C \in \mathbf{K}$ such that $A \leqslant C$ and $B \leqslant C$.

Theorem 2.2.4. (Fraïssé-Hrushovski*[22, 17]) Suppose a smooth class* $(\mathbf{K}, \leqslant)$ *has* AP *and* JEP*. Then there is a unique* $(\mathbf{K}, \leqslant)$*-generic structure.*

Remark 2.2.5. Note that if we consider $\leqslant$ as $\subseteq$, then $(\mathbf{K}, \leqslant)$ will be a Fraïssé class (see e.g. [21]) and the $(\mathbf{K}, \leqslant)$-generic structure is the Fraïssé's universal object.

Remark 2.2.6. Unlike Fraïssé's original construction, the theory of a $(\mathbb{K}, \leqslant)$-generic structure is not necessarily $\aleph_0$-categorical.

Definition 2.2.7. Suppose M is a $(\mathbf{K}, \leqslant)$-generic structure and A is a finite subset of M. Let $\mathrm{cl}(A)$ be the smallest $\leqslant$-closed subset B of M that includes A. We call $\mathrm{cl}(A)$, the $\leqslant$-*closure* of A. Note that it follows from Definition 2.2.1 part (4) that the $\leqslant$-closure of a finite set is well-defined and finite.

Definition 2.2.8. A pair (A, B) of finite subsets of M is called a *minimal pair* if $A \subset B$, $A \not\leqslant B$ and $A \leqslant B'$ for all $A \subseteq B' \subset B$.

Remark 2.2.9. We can extend the $\leqslant$-closedness notion to infinite subsets of M; define $X \leqslant M$ if for all minimal pairs (A, B), if $A \subset X$ then $B \subset X$. Similarly, we can extend the $\leqslant$-closure to an infinite subsets of M; for $Y \subset M$ define $\mathrm{cl}(Y) := \bigcap_{Y \subset Y' \leqslant M} Y'$.

Remark 2.2.10. Kueker and Laskowski in [22], give a characterization of when a generic structure is saturated (see Thm. 2.5 in and Cor. 2.6 [22]). They give examples when the generic structure is not saturated.

2.2.1 *Ab-initio structures*

In [17], Hrushovski introduced the key notion of assigning a dimension function to the finite structure and used it to defined a self-sufficiency "$\leqslant$". Recall that in our setting 2.2, the language $\mathfrak{L}$ is a finite relational language.

Definition 2.2.11. Let $\mathbb{K}$ be the class of all finite $\mathfrak{L}$-structures up to isomorphism. Let $\bar{w} = \langle w_i : i \leq n \rangle$ be a tuple of length n (where n is the number of relations in $\mathfrak{L}$) such that $w_i \in \mathbb{R}^{\geq 0}$ for $i \leq n$. Then we can define the following *pre-dimension* function associated with $\bar{w}$ for elements of $\mathbb{K}$:

$$\delta(A) = |A| - \sum_i w_i |R_i(A)|.$$

For $A, B \in \mathbb{K}$ define $\delta(A/B) = \delta(A \cup B) - \delta(B)$. We define the following *self-sufficiency* notion for $A, B \in \mathbb{K}$ and $A \subseteq B$:

$$A \leqslant_\delta B \text{ if } \delta\left(B'/A\right) \geq 0 \text{ for all } A \subseteq B' \subseteq B.$$

Then define $\mathbb{K}_{\delta,0}$ be a subclass of $\mathbb{K}$ such that $\mathbb{K}_{\delta,0} = \{A \in \mathbb{K} : \delta(A') \geq 0 \text{ for all } A' \subseteq A\}$.

Remark 2.2.12. If w_i's in the above definition are rational numbers then we can find the smallest integers κ and $\bar{\lambda}$ such $\frac{\lambda_i}{\kappa} = w_i$ for $i \leq n$. Then define

$$\delta'(A) = \kappa |A| - \sum_i \lambda_i |R_i(A)| .$$

It is easy to see that $\mathbb{K}_{\delta,0} = \mathbb{K}_{\delta',0}$ and $A \leqslant_\delta B$ if and only if $A \leqslant_{\delta'} B$.

NOTATION For simplicity, since in each subsection we fix a $\delta(-)$ function, we drop the subscripts δ and denote by $\mathbb{K}_0$ the class $\mathbb{K}_{\delta,0}$ and similarly denote by "$\leqslant$" the "$\leqslant_\delta$". When we intend to emphasize that the coefficients of the pre-dimension function $\delta(-)$ are κ and $\bar{\lambda}$ we denote it by $\delta_{\kappa,\bar{\lambda}}(-)$.

Now we can rephrase Definition 2.2.8 using the pre-dimension function as follows.

Definition 2.2.13. A pair (A, B) of finite subsets of M is called *minimal-pair* if $A \subset B$, $\delta(B/A) < 0$ and $\delta(B'/A) \geq 0$ for all $A \subseteq B' \subset B$.

Remark 2.2.14. Similar to Remark 2.2.9, using Definition 2.2.13, we can extend the notion of $\leqslant$-closedness to infinite subsets of M.

Theorem 2.2.15. *([4] Lemma 4.2) The class $(\mathbb{K}_0, \leqslant)$ is a smooth class and has the free-amalgamation property.*

Then the following holds.

Corollary 2.2.16. *There is up to isomorphism a unique countable $(\mathbb{K}_0, \leqslant)$-generic structure .*

Let $(\mathbb{K}_0, \leqslant)$ be a smooth class as is defined in Definition 2.2.11 using a pre-dimension function $\delta(-)$. In Theorem 2.2.15 and Corollary 2.2.16 we have seen that the class $(\mathbb{K}_0, \leqslant)$ is a smooth class with the amalgamation property and that the $\leqslant$-generic structure exists. We call the class $(\mathbb{K}_0, \leqslant)$ and the $(\mathbb{K}_0, \leqslant)$-generic structure the *uncollapsed ab-initio class* and the *uncollapsed ab-initio $\leqslant$-generic structure*, respectively.

Remark 2.2.17. The uncollapsed ab-initio $\leqslant$-generic structure obtained from a pre-dimension function with different rational coefficients is saturated (see Prop. 2.5 [42] or Thm. 2.28 in [4]). In [19], Ikeda give an example of ab-initio $\leqslant$-generic which is not saturated. Moreover, Baldwin and Shelah showed in [3] that the uncollapsed ab-initio $\leqslant$-generic structure obtained from a pre-dimension function with irrational coefficients is not saturated.

Definition 2.2.18. Let $\delta(-)$ be a pre-dimension. Let M be the $(\mathbb{K}_0, \leqslant)$-generic structure. Define the following *dimension* function:

$$\mathrm{d}_M^{\delta}(A) := \inf\left\{\delta(A') : A \subseteq A' \subset_{<\omega} M\right\}.$$

For simplicity we drop superscript and subscript when the context is clear. If the weights of δ (i.e. the w_i's in Def. 2.2.11) are rational numbers, then we can use "min" instead of "inf". If X is an infinite subset of M, then we define $\mathrm{d}(X) = \sup\{\mathrm{d}(X_0) : X_0 \subseteq_{<\omega} X\}$, where "sup" may also be replaced by "max" if the weights are rational numbers. Denote $\mathrm{d}(A/B)$ for $\mathrm{d}(AB) - \mathrm{d}(B)$ if $A, B \subset_{<\omega} M$ and for infinite B let $\mathrm{d}(A/B) := \inf\{\mathrm{d}(A/B') : B' \subset_{<\omega} B\}$. Again, we may use "min" in place of "inf" if the weights are rational numbers.

The following fact helps us to calculate $\mathrm{d}(-)$ of some finite set inside in the $\leqslant$-generic structure just by calculating pre-dimensions in a finite $\leqslant$-closed set that contains it.

Fact 2.2.19. *Let $A \subseteq B \leqslant M$. Then*
$\mathrm{d}(A) = min\{\delta(A') : A \subseteq A' \subseteq B\}$.

Proof. By the definition of a $\leqslant$-closed set, $\delta(C/B) \geq 0$ for all $C \supseteq B$. The result follows from Definition 2.2.18. □

Fact 2.2.20. *The dimension function $\mathrm{d}(-)$ satisfies the following properties:*

1. $\mathrm{d}(AB) + \mathrm{d}(A \cap B) \leq \mathrm{d}(A) + \mathrm{d}(B)$;
2. *if $A \subseteq B \subset M$, then $\mathrm{d}(A) \leq \mathrm{d}(B)$.*

Proof. For (1), without loss of generality suppose A and B are $\leqslant$-closed. Then $A \cap B$ is also $\leqslant$-closed and it is easy to see that $\delta(AB) \leqslant \delta(A) + \delta(B) - \delta(A \cap B)$. The result follows from Definition 2.2.18. (2) is also immediate from Definition 2.2.18. □

Similar to Definition 2.1.38, we can define the following closure operator in M,

Definition 2.2.21. Define a *closure-operator* derived from $\mathrm{d}(-)$ as follows: for a finite set A, let $\mathrm{gcl}(A) := \{x \in M : \mathrm{d}(x/A) = 0\}$. We borrowed the notation $\mathrm{gcl}(-)$ from [42] which stands for *geometric closure*.

Remark 2.2.22. It is easy to see that $\mathrm{gcl}(-)$ satisfies the following properties:

1. $A \subseteq \mathrm{gcl}(A)$, for $A \subset M$;
2. $\mathrm{gcl}(\mathrm{gcl}(A)) = \mathrm{gcl}(A)$;

3. $\mathrm{gcl}\,(A) \subseteq \mathrm{gcl}\,(B)$, for $A \subseteq B$.

Then the following holds.

Corollary 2.2.23. $\mathrm{gcl}\,(-)$ *is a closure operator.*

Fact 2.2.24. *([42]) For a finite set $X \subset M$;*

$$\mathrm{d}\,(\mathrm{gcl}\,(X)) = \mathrm{d}\,(X) = \mathrm{d}\,(\mathrm{cl}\,(X)) = \delta(\mathrm{cl}\,(X)).$$

Remark 2.2.25. By Remark 2.1.39, if $\mathrm{d}\,(\mathrm{A}) < |A|$ for all $A \subset M$ then the closure operator $\mathrm{gcl}\,(-)$ is a pre-geometry. If it is not the case then it can be shown that it won't not be a pre-geometry.

Remark 2.2.26. Having the $\leqslant$-generic structure, as we have seen in Definition 2.2.7, we can consider the $\leqslant$-closure. It is easy to show that in the ab-initio $\leqslant$-generic structures $\mathrm{cl}\,(-) \subseteq \mathrm{gcl}\,(-)$. Moreover, in the ab-initio case $\mathrm{cl}\,(-)$ has the finite closure property but $\mathrm{gcl}\,(-)$ is not necessarily finite.

Lemma 2.2.27. *([42]) Suppose M is the uncollapsed ab-initio $\leqslant$-generic structure then for $A \subseteq M$*

$$\mathrm{cl}\,(A) = acl\,(A)\,.$$

Proof. It is easy to see that for any finite set $X \subset M$, if an automorphism fixes X pointwise then it will fix $\mathrm{cl}\,(X)$ setwise. Let $a \in M$ is an element not in $\mathrm{cl}\,(X)$ using the $\leqslant$-richness and the free-amalgamation property one can find infinitely many copies of the same type of a over $\mathrm{cl}\,(X)$. Hence $a \notin \mathrm{acl}\,(X)$. □

Remark 2.2.28. ([42] Lemma 3.2) For all finite $A, A_1, A_2 \subset M$ and $B \subseteq M$:

1. $\mathrm{d}\,(A/B) \geq 0$;

2. $B \subseteq B' \subseteq M$ implies $\mathrm{d}\,(A/B) \geq \mathrm{d}\,(A/B')$;

3. $\mathrm{d}\,(A/B)) = \mathrm{d}\,(\mathrm{cl}\,(A)\,/B) = \mathrm{d}\,(A/\mathrm{cl}\,(B)) = \mathrm{d}\,(\mathrm{cl}\,(A)\,/\mathrm{cl}\,(B))$;

4. $\mathrm{d}\,(A_1A_2/B) = \mathrm{d}\,(A_1/A_2B) + \mathrm{d}\,(A_2/B)$.

2.2.1.1 *Collapsed ab-initio generic structures*

As we mentioned in the introduction, in order to obtain a strongly minimal structure Hrushovski restricts the uncollapsed ab-initio class to a smaller class, using a finite-to-one function over the following pairs of elements.

Definition 2.2.29. Let A and B be two disjoint finite sets. B is called 0-*algebraic* over A if $\delta(B/A) = 0$ and $\delta(B'/A) > 0$ for all proper subset $\emptyset \neq B' \subset B$. B is called 0-*minimally algebraic* over A if there is no proper subset A' of A such that B is 0-algebraic over A'. A pair (A, B) is called 0-*algebraic pair* if B is a 0-algebraic over A and we define a 0-*minimally algebraic pair* similarly.

Remark 2.2.30. ([17]) If B is 0-algebraic over A, then there is a unique A' subset of A such that B is 0-minimally algebraic over A'. Namely, $A' = \{a \in A : |R(a, B)| \geq 1\}$.

Definition 2.2.31. Consider $\Lambda \subset \mathbb{K}_0 \times \mathbb{K}_0$ such that $(A, B) \in \Lambda$ if B is 0-algebraically minimal over A and $A \neq \emptyset$. Define a function $\mu : \Lambda \to \mathbb{N}$ such that:

1. μ is finite-to-one;
2. $\mu(A, B) \geq \delta(A)$.

Let $\mathbb{K}_0^\mu \subset \mathbb{K}_0$ such that $A \in \mathbb{K}_0^\mu$ if for $A' \subset A$ and B' 0-algebraically minimal over A', the number of isomorphic copies of B' over A' in A is less or equal to $\mu(A', B')$.

Hrushovski, in order to construct M_μ considers a 3-ary relation with is symmetric. In section 3.1.3 we give more details about his construction. Then, he proves the following in his setting.

Theorem 2.2.32. *([17]) The class $(\mathbb{K}_0^\mu, \leqslant)$ has the amalgamation property and the theory of the $(\mathbb{K}_0^\mu, \leqslant)$-generic structure is strongly minimal.*

Theorem 2.2.33. *([17]) If the $(\mathbb{K}_0^\mu, \leqslant)$ has the amalgamation property, then we can construct the $(\mathbb{K}_0^\mu, \leqslant)$-generic structure. If the weights of the δ-function are rational numbers, then the $(\mathbb{K}_0^\mu, \leqslant)$-generic structure has finite Morley rank. Further, the Morley rank of the $(\mathbb{K}_0^\mu, \leqslant)$-generic structure is the coefficient κ of the pre-dimension function $\delta_{\kappa,\bar{\lambda}}$.*

Suppose that the class $(\mathbb{K}_0, \leqslant)$ is an uncollapsed ab-initio class and assume that μ is a finite-to-one function as defined in Definition 2.2.31. The restricted class $(\mathbb{K}_0^\mu, \leqslant)$ is called a *collapsed ab-initio class.* If $(\mathbb{K}_0^\mu, \leqslant)$ has AP then the $(\mathbb{K}_0^\mu, \leqslant)$-generic structure is called a *collapsed ab-initio $\leqslant$-generic structure*. As we will see in Theorem 2.2.33, the $\leqslant$-generic structure has finite Morley rank. This explains the terminology.

Remark 2.2.34. In Remark 2.2.17, we mentioned that the uncollapsed ab-initio $\leqslant$-generic structure obtained from a pre-dimension function with rational coefficients is saturated. The same result holds for the collapsed $\leqslant$-generic structure.

Note that "collapsing" with the μ function is just applicable for an ab-initio $\leqslant$-generic structure which is obtained from a pre-dimension function with rational coefficients.

In his original construction Hrushovski proves the following lemma. This implies that the $(\mathbb{K}_0^\mu, \leqslant)$-generic structure is not locally modular and no infinite group is interpretable.

Lemma 2.2.35. *([17] Lemma 13 and Lemma 15) Let M be the $(\mathbb{K}_0^\mu, \leqslant)$-generic structure. Then:*

1. *the geometry of M is flat and*
2. *$Th(M)$ is CM-trivial.*

Theorem 2.2.36. *Suppose $(\mathbb{K}_0^\mu, \leqslant)$ has the amalgamation property. Then in the $(\mathbb{K}_0^\mu, \leqslant)$-generic structure:*

$$\mathrm{gcl}(-) = acl\ (-)$$

holds.

Proof. Let $A \subseteq M$. First we want show that $\mathrm{gcl}(A) \subseteq \mathrm{acl}(A)$. Let $a \in \mathrm{gcl}(A) \setminus A$ and $A' = \mathrm{cl}(aA)$. It is clear that $A \leqslant A'$ and $\mathrm{d}(A'/A) = 0$. Then there is $B \subset A'$ such that $a \in B$ and B is 0-minimally algebraic over $A_0 \subset A'$. There are two possibilities for A_0: first assume $A_0 \subset A$. Then, since we bound the number of different copies of B over A_0 by $\mu(B, A_0)$, $a \in \mathrm{acl}(A)$. Else $A_0^c = A_0 \setminus A$ is nonempty but then one can see inductively that the elements of A_0^c are 0-minimally algebraic over some smaller sets. We conclude that $A_0^c \subset \mathrm{acl}(A)$. Hence $\mathrm{gcl}(A) \subseteq \mathrm{acl}(A)$.

Now we want to show that $\mathrm{acl}(A) \subseteq \mathrm{gcl}(A)$. If $e \notin \mathrm{gcl}(A)$ then $\mathrm{d}(e/A) > 0$. Let $E = \mathrm{cl}(eA)$ and $E' = E \cap \mathrm{gcl}(A)$. It is clear that $e \in E \setminus E'$ and e is not 0-minimally algebraic over any subset of E' and $\mathrm{gcl}(A)$. Then, by the amalgamation property, there are infinitely many isomorphic copies of e over E' and hence $e \notin \mathrm{acl}(A)$. □

2.2.1.2 *Weak elimination of imaginaries in the generic structures*

In his paper, Hrushovski mentions that flatness can be defined in structures with finite Morley rank and indicates that if a structure is flat then it *essentially* admits *elimination of imaginaries*. The following theorem and corollary are a precise version of what we will use in our proofs.

Theorem 2.2.37. *([4] Prop. 5.3) If T is stable and for every $M \models T$, any elementary extension of N of M, every n and every $\bar{a} \in N^n$, there is a unique minimal algebraically* *closed $B \subseteq N$ such that an automorphism α of M preserves $p = tp(\bar{a}/M)$ if and only if α fixes B setwise, then $T = Th(M)$ admits weak elimination of imaginaries.*

Corollary 2.2.38. *([4] Prop. 5.4) Let T be the theory of the an ab-initio generic model. Then for every $N \models T$ and every n and every $\bar{a} \in N^n$, there is a unique minimal $\leqslant$-closed $B \subset N$ such that an automorphism α of N preserves $p = tp(\bar{a}/N)$ if and only if α fixes B setwise. Hence, by Theorem 2.2.37, T admits weak elimination of imaginaries.*

2.3 THE AUTOMORPHISM GROUP OF COUNTABLE STRUCTURES

One natural object that has been studied in the the recent years is the automorphism group of a countable structure. Automorphism groups of structures can be studied from different perspectives (see e.g. [25]). One can study them in the terms of general group theory and permutation group theory. Also, since there is a natural topology on the automorphism group of a countable structure, one can study them in topological group theory terms. One direction of studies about the automorphism group of a countable structure is to understand how much information about the model theory of the structure one can obtain from knowing the group structure of the automorphism group. Many results have been shown in recent years and a lot of machinery has been developed in descriptive set theory and Ramsey theory in order to study such questions. Most of these results are about countable homogeneous structures (see e.g., [21]).

The following theorem was the first theorem studying the automorphism group of first-order structures. The interesting feature of this theorem is that it uses a purely group theoretical language. It is remarkable that no topological aspects of the structure is needed. In order to state the result we need the following definition.

Definition 2.3.1. A permutation group G of an infinite set Ω is called *oligomorphic* if for every natural number n, G has only finitely many orbits on Ω^n.

Theorem 2.3.2. *(Ryll-Nardzeswki, Engeler, Svenonius see e.g. [7, 25]) Let M be a countable infinite structure. Then the followings are equivalent:*

1. *a countable first order structure M is $\aleph_0$-categorical;*
2. $\mathrm{Aut}(M)$ *acts oligomorphically (see Def. 2.3.1) on M;*
3. *the number of n-types are finite for all $n \in \mathbb{N}$;*
4. *every n-type is isolated for for all $n \in \mathbb{N}$.*

Example 2.3.3. The Fraïssé universal objects are homogeneous and $\aleph_0$-categorical. They are a large class of typical examples that fit in the above setting.

2.3.1 *The automorphism group of a countable structure as a Polish group*

A separable completely metrizable topological space is called a *Polish space.* A topological group for which the topology is Polish is called a *Polish group*. In this section we mainly follow [7]. There is a natural topology defined on the symmetric group of a countable set Ω; namely the pointwise convergence topology. This topology makes $Sym(\Omega)$ into a topological group. Basis of the open sets are $\{g \in Sym(\Omega) : g(\bar{a}) = \bar{b}\}$ where $\bar{a}$ and $\bar{b}$ are tuples of distinct elements of the same length. In the similar way we can define a topology on the automorphism group of a countable $\mathfrak{L}$-structure M.

Corollary 2.3.4. *Automorphism group of first order countable structures are Polish.*

The following proposition is the main property that connects closed subgroups of $Sym(\Omega)$ and countable first order structures.

Proposition 2.3.5. *Let Ω be a countable set. A subgroup of G of $Sym(\Omega)$ is closed if and only if $G = Aut(M)$ for some first order structure M on Ω.*

2.3.2 *Lascar's simplicity result*

Suppose M is an $\mathfrak{L}$-structure. For $A \subseteq M$, the subgroup of the A-strong automorphisms is:

$$\mathrm{Autf}_A(M) = \{f \in \mathrm{Aut}_A(M) : \mathrm{stp}(\bar{m}/A) = \mathrm{stp}(f(\bar{m})/A) \text{ for all } \bar{m} \in M\}.$$

If A is the empty set, then $\mathrm{Autf}_\emptyset(M)$ becomes the group of all strong automorphisms of M, and we denote it simply by $\mathrm{Autf}(M)$. It is clear that if $X \subset M$ and X is algebraically closed in M^{eq}, then $\mathrm{Autf}_X(M) = \mathrm{Aut}_X(M)$.

Fact 2.3.6. *$Autf(M)$ is a normal subgroup of $Aut(M)$.*

Definition 2.3.7. An automorphism $\beta \in \mathrm{Aut}(M)$ is called *bounded* if there exists a finite set $A \subset M$ such that $\beta(m) \in \mathrm{acl}(mA)$ for all $m \in M$. In this case we say that β is bounded over A. Let $\mathrm{Bdd}(M)$ be the set of all bounded automorphisms of M.

Remark 2.3.8. Note that if the automorphism β is bounded over a finite set A, then it is bounded over all $B \supseteq A$. This simply follows from the definition of bounded automorphisms.

Remark 2.3.9. In general $\text{Bdd}(M)$ might not be a subgroup of $\text{Aut}(M)$. If $\text{acl}(-)$ has the exchange property, then it is a pre-geometry. Then $\text{Bdd}(M)$ is a normal subgroup of $\text{Aut}(M)$.

Example 2.3.10. There are non-trivial bounded automorphisms in locally-modular strongly minimal theories (e.g. infinite dimensional vector spaces). See Lemma 2.3.14.

To present Lascar's simplicity result, suppose that M is a countable structure in a countable language and there exists a strongly minimal formula $D(v)$, without parameters, such that M is in the algebraic closure of $D(M)$.

Theorem 2.3.11. *(Lascar [24]) The group* $\text{Autf}(M) / (\text{Bdd}(M) \cap \text{Autf}(M))$ *is simple.*

Remark 2.3.12. Suppose M is a strongly minimal structure. As we noted in Example 2.1.29, $\text{acl}(-)$ is a pre-geometry and there will be a notion of a dimension derived from $\text{acl}(-)$. Let Dim denotes the dimension in the strongly minimal structures. Suppose $\beta \in \text{Aut}(M)$ is a bounded automorphism then there exists $n \in \mathbb{N}$ such that for any set $X \subseteq_{<\omega} M$, $Dim(g(X)/X) \leq n$. This property is equivalent to Definition 2.3.7 in strongly minimal sets. Bounded automorphisms were originally defined in this way [24].

Remark 2.3.13. Lascar also proves that there are no non-trivial bounded automorphisms in the automorphism group of a saturated algebraically closed field of characteristic zero ([24] Thm. 15).

Lemma 2.3.14. *If M is locally modular, then there exist non-trivial automorphisms of M which are strong and bounded.*

Proof. This proof can be found in [24] and because of its frequent-use we recall it here. Without loss of generality we can assume M is modular; we can name non-algebraic points as the constants and add them to the language. Suppose B is a base for M. One chooses a finite subset B_0 of B with cardinality n. Let σ be a non-trivial permutation of B_0, and suppose $B_1 = B \backslash B_0$. It is known that there exists an $\alpha \in \text{Autf}(M)$ which it extends σ and fixes $\text{acl}(B_1)$. We claim that α is a bounded automorphism. Suppose X is a algebraically closed subset of M. Let $Y = \text{acl}(B_0 \cup X) \cap \text{acl}(B_1)$. Then the equivalent definition of modularity (see Definition 2.1.26), with $A := \text{acl}(B_0)$, $B := \text{acl}(B_0 \cup X)$ and $C := \text{acl}(B_1)$ gives us:

$$\text{acl}(B_0 \cup Y) = \text{acl}(B_0 \cup X) \cap \text{acl}(B_0 \cup B_1).$$

However $\text{acl}(B_0 \cup B_1) = M$, and thus $\text{acl}(B_0 \cup Y) = \text{acl}(B_0 \cup X)$. As α fixes B_0 (setwise) and Y (pointwise), $\text{acl}(B_0 \cup Y) = \text{acl}(B_0 \cup X) = \text{acl}(B_0 \cup \alpha(X))$ holds. This shows that $Dim(\alpha(X)/X) < Dim(B_0) = n$ which completes the proof. □

3 AUTOMORPHISM GROUPS OF AB-INITIO GENERIC STRUCTURES

In this chapter, we explore the simplicity structure of the automorphism group of the ab-initio generic structures in both collapsed and uncollapsed cases. In the first section, following Lascar's approach, we investigate the key step of understanding the bounded automorphism group in the automorphism group of the collapsed ab-initio generic structures. We conclude that the automorphism group of M_μ constructed in [17] is a simple group. In the second section, we prove that the automorphism groups of the uncollapsed ab-initio generic structures with rational coefficients which fix every dimension zero set pointwise are boundedly simple.

3.1 AUTOMORPHISM GROUP OF THE COLLAPSED AB-INITIO GENERIC STRUCTURES

In this section, we want to investigate the simplicity of the automorphism group of the collapsed ab-initio generic structures (recall Def. 2.2.1.1). This leads us to investigate the bounded automorphism group. In Theorem 2.3.11, we mentioned one simplicity result by Lascar for the automorphism group of the almost strongly minimal structures. Lascar, in order to obtain the result defines bounded automorphisms (see Def. 2.3.7) and then he proves that the bounded automorphism group intersected with the strong automorphism group is a maximal normal subgroup of the automorphism group. Following Lascar's approach, here first we investigate the bounded automorphisms and indeed we prove that there is no non-trivial bounded automorphisms in the automorphism group of the ab-initio collapsed $\leqslant$-generic structures (except some trivial cases see e.g. Sec. 3.1.2). Note that the collapsed ab-initio $\leqslant$-generic structures are not necessarily strongly minimal, hence we can not apply Theorem 2.3.11 directly. However, the $\leqslant$-generic structure constructed in [17], is indeed strongly minimal. Hence by Theorem 2.3.11, we deduce that the automorphism group of M_μ is a simple group. Finally, at the end of this section, we will investigate the question of simplicity in the different cases of the collapsed ab-initio generics.

When the coefficients of the predimention function δ in Definition 2.2.11 are irrational, then there will be no 0-algebraic pairs (see Def. 2.2.29). In this case, we are only able to approximate a 0-algebraic set over a finite set (for more information see [4]). Hence no collapsing, using a μ-function (see Def. 2.2.31), is possible. Throughout this section, we investigate the collapsed ab-initio generic structures with rational coefficients. Moreover, we mainly deal with binary relations and present the proofs for the binary case. The arguments can be modified for n-ary relations. In Subsection 3.1.3, in order to show that there is not much difference between binary and n-ary relations and also to show that the combinatorial structure of the proofs does not depend on the type of geometry (see Def. 2.1.23), we consider the collapsed case of 3-ary relation that has been studied in [17]. Recall that as we mentioned in Chapter 1, the structure in [17] is an example of a CM-trivial but not one-based strongly minimal set (hence non-locally modular) which does not interpret a group.

3.1.1 *Bounded automorphisms in the binary case*

We assume $\mathfrak{L}$ consists of a binary relation R which is irreflexive and symmetric. An $\mathfrak{L}$-structure can simply be considered as a graph without loops. The predimension function (see Def. 2.2.11) that we consider in this subsection for an $\mathfrak{L}$-structure A, is of the following form:

$$\delta(A) = \kappa|A| - \lambda|R(A)|,$$

where $\kappa, \lambda \in \mathbb{N}$, $\gcd(\kappa, \lambda) = 1$ and $\kappa > \lambda > 0$. For the case $\kappa \leq \lambda$, there will be non-trivial bounded automorphisms and we will discuss it in Subsection 3.1.2. Let the class $\mathbb{K}_0 = \{\text{finite } \mathfrak{L}\text{-structures } A : \delta(A') \geq 0 \text{ for all } A' \subseteq A\}$. Define the self-sufficiency "$\leqslant$" as it is defined in Definition 2.2.11. Let μ be a finite-to-one function over 0-minimally algebraic pairs as it is defined in Def. 2.2.31. Using the μ-function, we consider the restricted class $\mathbb{K}^{\mu} \subset \mathbb{K}$ (see Def. 2.2.11). Suppose $(\mathbb{K}_0^{\mu}, \leqslant)$ has the amalgamation property and let M denote the $(\mathbb{K}_0^{\mu}, \leqslant)$-generic structure. Here is the main theorem in this section

Theorem 3.1.1. *There is no non-trivial bounded automorphism in the automorphism group of the $(\mathbb{K}_0^{\mu}, \leqslant)$-generic structure M.*

In order to prove the theorem above, we first present some definitions and technical lemmas.

The following definition is a generalization of Definition 2.2.29.

Definition 3.1.2. Let A and B be two disjoint finite sets and $\nu \in \mathbb{N}$. B is called *ν-algebraic* over A if $\delta(B/A) = \nu$ and $\delta(B'/A) > \nu$ for all proper non-empty subsets $B' \subset B$. B is called *ν-minimally algebraic* over A, if there is no proper subset A' of A such that B is ν-algebraic over A'.

Lemma 3.1.3. *The following holds.*

1. $\delta(AB) \leq \delta(A) + \delta(B) - \delta(A \cap B) - \lambda \cdot |R(A \backslash B; B \backslash A)|$.
2. *Suppose* A, B *are disjoint then* $\delta(A/B) = \delta(A) - \lambda \cdot |R(A; B)|$.
3. *Suppose* $(A; B)$ *is a minimal pair (see Def. 2.2.8) then* $\delta(B/A') \leq \delta(B/A) < 0$ *for all* $A \subseteq A' \subset B$.
4. *Suppose* B *is a* ν*-algebraic set over* A*, then* B *is a connected graph.*

Proof. (1) and (2) follow from definition of $\delta(-)$. (3) follows immediately from Definition 2.2.8. (4) also follows from Definition 3.1.2 and cases (1) and (2). □

Lemma 3.1.4. *Suppose* A *is a finite* $\leqslant$*-closed set and* D *is* 0*-minimally algebraic over* $A_0 \subset A$*. If* $D \cap A \neq \emptyset$*, then* $D \subset A$*.*

Proof. Since D is 0-algebraic over A, $\delta(D'A) > \delta(DA) = \delta(A)$ for any proper non-empty subset D' of D. Then this implies

$$\delta(DA) - \delta(D'A) = \delta(D/D'A) < 0.$$

Let $D_0 := D \cap A$. By our assumption $D_0 \neq \emptyset$ and if $D_0 \neq D$, then $\delta((D \backslash D_0)/A) < 0$ which contradicts the fact that A is $\leqslant$-closed. Hence $D_0 = D$ and the result follows. □

Lemma 3.1.5. *Let* A *be a finite* $\leqslant$*-closed set and suppose* D_1, D_2 *are two distinct* ν*-minimally algebraic sets over* A *such that* AD_1 *or* AD_2 *is a* $\leqslant$*-closed set. Then* $D_1 \cap D_2 = \emptyset$*.*

Proof. From Lemma 3.1.3, for D_1, D_2 and A we have

$$\delta(D_1 D_2 / A) \leq \delta(D_1/A) + \delta(D_2/A) - \delta((D_1 \cap D_2)/A).$$

Without loss of generality suppose AD_1 is a $\leqslant$-closed set, then $\delta(D_1 D_2/A) \geq \delta(D_1/A)$. Then by ν-minimality, we deduce that $D_1 \cap D_2 = \emptyset$. □

Lemma 3.1.6. *Suppose* $A \subset M$ *is a finite set such that* $|A| \geq \lfloor \frac{\kappa}{\lambda} \rfloor$*. Then there are infinitely many isomorphism types of* 0*-algebraic sets over* A*.*

Proof. For any $i > 2$, let $C_i = \{c_1, \cdots, c_i\}$ be a cycle of length i (i.e. $C_i \models \bigwedge_{1 \leq j < i} R(c_j, c_{j+1}) \wedge R(c_i, c_1)$). It is easy to see that $\delta(C_i) = i \cdot (\kappa - \lambda)$. Let $A_0 \subseteq A$ such that $|A_0| = s$ where $s = \lfloor \frac{\kappa}{\lambda} \rfloor$ if $\lfloor \frac{\kappa}{\lambda} \rfloor \neq \frac{\kappa}{\lambda}$; otherwise $s = \lfloor \frac{\kappa}{\lambda} \rfloor - 1$. Enumerate $A_0 = \{a_1, \cdots, a_s\}$. Now let $i = m \cdot \lambda$ and consider $e := m \cdot (\kappa - \lambda)$ where $m \cdot \lambda > 2$. We claim that we can distribute e many edges between elements of C_i and A_0 such that $e = |R(C_i; A_0)|$ and $A_0 \leqslant C_i A_0$ and

$\delta(C_i/A_0) = 0$ and for all proper subsets $\emptyset \neq C \subset C_i$, we have $\delta(C/A_0) > 0$ (i.e. $A_0C_i \models \left[\bigwedge_{1\leq j\leq i}\left[\bigwedge_{j^*\leq l\leq (j+s_j)^*} R(c_j, a_l)\right]\right]$; where $\nu^* \equiv \nu \pmod{s}$ such that $\nu^* \in \{1, \cdots, s\}$ and $\left\lfloor \frac{\kappa-\lambda}{\lambda} \right\rfloor \leq s_j \leq \left\lfloor \frac{\kappa-\lambda}{\lambda} \right\rfloor + 1 \leq \left\lfloor \frac{\kappa}{\lambda} \right\rfloor$). It is clear that C_i and C_j are not isomorphic when $i \neq j$. Hence C_i's are infinitely many isomorphism types of 0-minimally algebraic sets over A_0 for $i = m \cdot \lambda > 2$. □

Corollary 3.1.7. *Let $A_0 \subset A$ and A be a finite $\leqslant$-closed set in M. Suppose $|A_0| \geq \left\lfloor \frac{\kappa}{\lambda} \right\rfloor$. Then there is a 0-minimally algebraic set $D \subset M$ over A_0 such that $D \cap A = \emptyset$ and $D \cup A \in \mathbb{K}_0^{\mu}$.*

Proof. Since by our assumption, A is a $\leqslant$-closed set, one can see $\mathrm{cl}(A_0) \subset A$. By Lemma 3.1.6, there are infinitely many isomorphism types of 0-minimally algebraic sets over A_0. Set A is finite and $\leqslant$-closed. By Lemma 3.1.5 and Lemma 3.1.4, 0-algebraic sets have empty intersection and if a 0-algebraic set over A_0 intersects with A, then it contained in A, so there is a 0-minimally algebraic set D over A, which is not contained in A and $D \cap A = \emptyset$ hence $D \cup A \in \mathbb{K}_0^{\mu}$. □

Note that for the class $\mathbb{K}_0$, Corollary 3.1.7 is immediate (follows from the free-amalgamation property see Thm. 2.2.15).

Now, we start proving that there is no non-trivial bounded automorphism in M. Here are some lemmas that we need in the proof.

Lemma 3.1.8. *Suppose $\beta \neq id_M$ is a bounded automorphism and A is a $\leqslant$-closed subset of M such that $\beta(m) \in acl(mA)$ for all $m \in M$. Let $b \in M$ such that $A \leqslant Ab \leqslant M$ and $\mathrm{d}(b/A) = \kappa$, then b is a fixed-point of β.*

Proof. Note that by Remark 3.2.24, we can assume $|A| \geq \kappa$. Suppose b is not a fixed-point. By Definition 2.3.7, $\beta(b) \in \mathrm{acl}(bA)$ and by Lemma 2.2.24

$$\mathrm{d}(\mathrm{acl}(\beta(b)\, bA)) = \delta(\mathrm{cl}(bA)) = \delta(bA) = \delta(A) + \kappa.$$

Let $B := \mathrm{cl}(Ab\beta(A)\beta(b))$. Since by our assumption β is a bounded automorphism, $\mathrm{d}(A\beta(A)) = \mathrm{d}(A)$ and $\delta(B) = \mathrm{d}(B) = \delta(A) + \kappa$. Let $B_0 := B \backslash A$. Since $\delta(B/A) = \delta(B_0/A) = \delta(B_0) - \lambda \cdot |R(B_0; A)|$ hence $\delta(B_0) - \lambda \cdot |R(B_0; A)| = \kappa$. Now let $C := B \cup D \in \mathbb{K}_0^{\mu}$ be an $\mathfrak{L}$-structure such that $D \cap B = \emptyset$, D is a 0-minimally algebraic set over Ab and $|R(b; D)| = 1$; existence of such a D is guaranteed by Corollary 3.1.7. Since $B \leqslant C$ holds, by $\leqslant$-richness of M we can find an isomorphic copy of C over B inside M which is $\leqslant$-closed. One can see that:

- $D \subset \mathrm{acl}(B)$;
- $D \nsubseteq \mathrm{cl}(A)$, $D \nsubseteq \mathrm{acl}(A)$ since $\mathrm{d}(D/A) = \delta(D/A) = \lambda$;

- $b \notin \mathrm{cl}(DA)$ and $\mathrm{d}(D/A) = \lambda$, so $\mathrm{d}(b/DA) = \kappa - \lambda$.

Since $\beta(b) \neq b$ holds, by Lemma 3.1.4 $\beta[D] \cap D = \emptyset$ and hence $\delta(B_0/D\beta[D]A) = \delta(B_0) - \lambda \cdot |R(B_0; D\beta[D]A)| = \delta(B_0) - \lambda \cdot |R(B_0; A)| - \lambda \cdot |R(B_0; D\beta[D])| \leq \kappa - 2\lambda$. But this is a contradiction since $\mathrm{d}(\beta(D)DA) = \mathrm{d}(DA) = \mathrm{d}(A) + \lambda$ and $\mathrm{d}(b/DA) = \mathrm{d}(b/\beta(D)DA) = \kappa - \lambda$. So b is a fixes point of β.

□

Lemma 3.1.9. *Let β and A be similar to the assumptions of Lemma 3.1.8. Suppose $c \in M$ such that $\mathrm{d}(c/A) \geq \lambda$. Then c is a fixed-point of β.*

Proof. There are two possibilities for $\mathrm{d}(c/A)$:

CASE 1: $\mathrm{d}(c/A) < \kappa$. Let $C := \mathrm{cl}(cA)$. Consider $D := C \cup \{b\} \in \mathbb{K}_0^\mu$ such that b is a new element and $R(c,b)$ holds in D. Then $\delta(A) < \delta(C) < \delta(D) = \delta(Cb)$. By the $\leqslant$-richness of M there is an isomorphic copy of D over C in M which is $\leqslant$-closed. Then $\mathrm{d}(D/A) = \mathrm{d}(bC/A) = \kappa - \lambda + \mathrm{d}(C/A) \geq \kappa$ and it follows that $\mathrm{d}(b/A) = \delta(b/A) = \kappa$. By Lemma 3.1.8 we already know that b is a fixed point of β. If c is not a fixed-point of β, then $\beta(c) \notin C$ and since $|R(b; c\beta(c)A)| \geq 2$ so $\mathrm{d}(b/C\beta(c)) \leq \delta(b/C\beta(c)) \leq \kappa - 2\lambda$ which contradicts the fact that $\mathrm{d}(b/C\beta(c)) = \mathrm{d}(b/C) = \kappa - \lambda$. So c is a fixed point of β.

CASE 2: $\mathrm{d}(c/A) = \kappa$. Then Ac is $\leqslant$-closed and the result follows from Lemma 3.1.8. □

Now we present the proof of Theorem 3.1.1.

Proof of Theorem. 3.1.1. Let β be a non-trivial bounded automorphism and let A be a $\leqslant$-closed set such that $\beta(m) \in \mathrm{acl}(mA)$ for all $m \in M$.

Note that by Lemma 3.1.9, every $b \in M$ with $\mathrm{d}(b/A) \geq \lambda$, is a fixed-point of β. First we prove that $\mathrm{acl}(A)$ is fixed pointwise. Suppose $d \in \mathrm{acl}(A)$. Let $D := \mathrm{cl}(dA)$ and $s := \lfloor \frac{\lambda}{\kappa - \lambda} \rfloor + 1$. Consider an $\mathfrak{L}$-structure $E := D \cup \{e_1, \cdots, e_s\}$ such that the e_i's are new elements and

$$E \models \bigwedge_{1 \leq j < s} R(e_j, e_{j+1}) \wedge R(d, e_1).$$

It is clear that $E \in \mathbb{K}_0^\mu$ and $D \leqslant E$. Moreover $\delta(e_1/D) = \delta(e_{j+1}/DE_j) = \kappa - \lambda$ where $E_j = \{e_1, \cdots, e_j\}$ for $0 \leq j \leq s$. By the $\leqslant$-richness of M, we can find an isomorphic copy of E inside M over D which is $\leqslant$-closed. Since $\mathrm{d}(e_s/D) = \mathrm{d}(e_s/A) \geq \lambda$, by Lemma 3.1.9, e_s is a fixed-point of β. Similar to the proof of Lemma 3.1.9, if $e_{s-1} \neq \beta(e_{s-1})$, then $\mathrm{d}(e_s/DE_{s-1}\beta(e_{s-1})) \leq \delta(e_s/DE_{s-1}\beta(e_{s-1})) < \delta(e_s/DE_{s-1})$ which is a contradiction. So e_{s-1} is a fixed point. Inductively we can prove that all elements of $E\backslash D$ are fixed pointwise by β. A similar argument proves that $\beta(d) = d$. So every $d \in \mathrm{acl}(A)$ is a fixed-point of β.

Now let $c \in M$ such that $\mathrm{d}(c/A) > 0$ and $\mathrm{d}(c/A) < \lambda$. Similar to the previous case let $D = \mathrm{cl}(cA)$. Consider $E = D \cup \{e_1, \cdots, e_s\}$ such that $E \models \bigwedge_{1 \leq j < s} R(e_j, e_{j+1}) \wedge R(d, e_1)$ where $s = \left\lfloor \frac{\lambda - \mathrm{d}(cA)}{\kappa - \lambda} \right\rfloor + 1$. With the same argument as above we can embed E over D such that $\delta(E) = \mathrm{d}(E)$ and $\mathrm{cl}(Ac) \leqslant E \leqslant M$. Then $\mathrm{d}(e_s/A) \geq \lambda$ and by Lemma 3.1.9, e_s is a fixed-point. Inductively, similar to the previous case, we can prove that every element of $E\backslash D$ and also c is a fixed point of β. □

3.1.2 *Simplicity in the binary case*

As we have seen in the previous Chapter, understanding the bounded automorphism group is a crucial part in understanding the simplicity structure of the automorphism group. Here, we establish the simplicity structure of the collapsed ab-initio structures using the results that we know about bounded automorphisms.

Recall that $\mathfrak{L} = \{R\}$ such that $R(-,-)$ is a irreflexive symmetric binary relation. In Subsection 3.1.1, we have seen that if $\kappa > \lambda$ then there is no non-trivial bounded automorphism (see Thm. 3.1.1). The $\leqslant$-generic structure in this case is not almost strongly minimal and we can not use Theorem 2.3.11 directly to answer the question of simplicity. However, in this case the automorphism group is indeed boundedly simple (see Def. 3.2.1) but to prove bounded simplicity we need different machinery that we will develop in the next section.

Here, we consider all possible cases of coefficients κ and λ (integer valued) for the pre-dimension function $\delta(-)$. Let $\mathbb{K}_0$ be the class that we defined in 2.2.11. Similar to Definition 2.2.31, let $\mathbb{K}_0^{\mu} \subseteq \mathbb{K}_0$ and assume $\left(\mathbb{K}_0^{\mu}, \leqslant\right)$ is a smooth class with AP.

Since by our assumptions, the coefficients κ and λ are natural numbers, hence the induced dimension $\mathrm{d}(-)$ is also integer-valued. As we have seen in Remark 2.1.40, if the $\mathrm{d}(a) \in \{0,1\}$ for all single elements a in M, then $\mathrm{acl}(-)$ is a pre-geometry and otherwise it is not.

Case 1: $\kappa = 1$, $\lambda = 1$. It is clear that $\emptyset \in \mathbb{K}_0^{\mu}$. The pre-dimension of any cycle is zero. Hence $\mathrm{acl}(\emptyset)$ contains all cycles of length n for all $n \in \mathbb{N}$ and there are no cycles in $M\backslash\mathrm{acl}(\emptyset)$. The set $M\backslash\mathrm{acl}(\emptyset)$ contains countably many disjoint connected countable trees with of finite valency; namely the valency in each vertex is bounded by the μ-function. The automorphisms that move only finitely many connected components are bounded automorphisms and hence

Bdd (M) is a not-trivial normal subgroup. From Remark 2.1.40, acl$(-)$ is a pre-geometry. Hence $1 \neq$Bdd $(M) \lhd$ Aut (M). Since the $\left(\mathbb{K}_0^\mu, \leqslant\right)$-generic structure is strongly minimal then by Theorem 2.3.11, Autf (M) / (Autf $(M) \cap$Bdd (M)) is a simple group.

Case 2: $\kappa = 1$, $\lambda > 1$. By Remark 2.1.40 acl$(-)$ is a pre-geometry. If $\lambda = 2$ then acl$(\emptyset)$ is an infinite set and consists of infinitely many disjoint sets of two elements which are connected by one edge. Moreover, since having any relation (i.e. edge) between two elements makes the pre-dimension zero, there will be no edges between points in $M\backslash$acl$(\emptyset)$. If $\lambda > 2$, then there will be no edges between any two points of the $\leqslant$-generic structure and acl$(\emptyset) = \emptyset$. Hence in both cases the geometries are disintegrated (see Def. 2.1.26 i.e. acl $(a_1, a_2) =$ acl $(a_1) \cup a_2 = a_1 \cup$ acl (a_2) for $a_1, a_2 \notin$ acl$(\emptyset)$). Similar to the previous case Bdd (M) is not trivial. Hence $1 \neq$Bdd $(M) \lhd$ Aut (M). Again similar to Case 1, since the generic model is strongly minimal then by Theorem 2.3.11, Autf (M) / (Autf $(M) \cap$Bdd (M)) is a simple group.

Case 3: $\kappa > 1$, $\kappa > \lambda \geq 1$. In this case, by Remark 2.1.40, acl$(-)$ is not a pre-geometry. Similar to Case (1), acl$(\emptyset)$ contains all $A \in \mathbb{K}_0^\mu$ such that $\delta(A) = 0$. For example in the case when $\kappa = 2$ and $\lambda = 1$, $M\backslash$acl$(\emptyset)$ contains all K_n (i.e. complete graph with n vertices) for $n = 1, 2, 3, 4$ but K_5 is not in the class $\mathbb{K}_0^\mu$. Note that for any $A \in \mathbb{K}_0^\mu$ and $a \in A$, one can consider $B := A \cup \{b\}$ such that b is a new element and $R(b, a)$ holds. It is easy to check that $A \leqslant B$ and $B \in \mathbb{K}_0^\mu$ and $\delta(B/A) > 0$ so by $\leqslant$-richness the valency of each vertex is infinite. The $\left(\mathbb{K}_0^\mu, \leqslant\right)$-generic structure M is connected. Let $I_a := \{b \in M : M \models R(a, b)\}$, then d (I_a) is infinite. Moreover for each $a \notin$ acl$(\emptyset)$, acl $(a) \backslash$acl$(\emptyset)$ is an infinite set. In this case we have the following result which we deduce its from the method that we will describe in next section: Aut$_{\text{acl}(\emptyset)}$ (M) is a non-tivial normal subgroup of Aut (M) and it is boundedly simple.

Case 4: $\kappa > 1$, $\lambda > \kappa > 1$. Similar to Case 2. Namely if $\left\lfloor \frac{\lambda}{\kappa} \right\rfloor > 2$ the $\leqslant$-generic is just a disjoint union of single points.

3.1.3 *The automorphism group of M_μ*

Note that, as we mentioned in Chapter 1, the construction in [17] is a an example of a CM-trivial strongly minimal set which does not interpret a group. Here, first we want to prove that there is no non-trivial bounded automorphism in the automorphism group of M_μ. Then since M_μ is almost strongly minimal, we can apply Theorem 3.1.1 and conclude that its automorphism group is a simple group (see Cor. 3.1.12).

Following [17], let the language $\mathfrak{L}$ consists of $R(-,-,-)$ which is a 3-ary symmetric irreflexive relation (i.e. if $\bar{a} = (a_1, a_2, a_3)$ and a_i's are distinct elements and $R(\bar{a})$ holds then $R(\bar{a}')$ holds for any permutations of elements of $\bar{a}$). The pre-dimension function that has been considered in [17] is

$\delta(A) = |A| - |R(A)|$.

Similar to the binary case, let the class $\mathbb{K}_0 = \{\text{finite } \mathfrak{L}\text{-structures } A : \delta(A') \geq 0 \text{ for all } A' \subseteq A\}$. Define the self-sufficiency "$\leqslant$" as it has been defined in Definition 2.2.11. To obtain a strongly minimal structure, Hrushovski considers a μ-function, finite-to-one, for 0-minimally algebraic pairs to restrict number of a 0-algebraic sets over a finite set. The smooth class that he considers is $\left(\mathbb{K}_0^{\mu}, \leqslant\right)$ and he proves that it has the amalgamation property ([17] Lemma 4). Hence the $\left(\mathbb{K}_0^{\mu}, \leqslant\right)$-generic structure exists and it is strongly minimal ([17] Corollary 8).

Here we modify the proof that there is no non-trivial bounded automorphism group in the automorphism group of this structure.

Similar to Lemma 3.1.6, first we need the following lemma

Lemma 3.1.10. *For a finite set $A \subset M$, there are infinitely many isomorphism types of 0-minimally algebraic sets over A.*

Theorem 3.1.11. *There are no non-trivial bounded automorphisms in the automorphism group of M_μ.*

Proof. Suppose β is a non-trivial bounded automorphism and A is a $\leqslant$-closed set such that $\beta(m) \in \operatorname{acl}(mA)$ for all $m \in M$. Let $a \in M$ be such that $\beta(a) \neq a$. Without loss of generality, we can assume that $a \in A$ (see Rem. 2.3.8). Consider $b_1, b_2 \notin \operatorname{acl}(A)$ such that $\operatorname{d}(b_1, b_2/A) = 2$. This implies that Ab_1b_2 is a $\leqslant$-closed set. Since β is bounded then $\beta(b_i) \in \operatorname{acl}(A, b_i)$ for $i = 1, 2$. Let $E := \operatorname{cl}(\beta(a), b_1, b_2, \beta(b_1), \beta(b_2), A)$ and let D be 0-minimally algebraic over $\{a, b_1, b_2\}$ such that $(D \cup \beta(D)) \cap E = \emptyset$; since E is a finite set, by Lemma 3.1.10 one can always find such an $\mathfrak{L}$structure D. From $\beta(a) \neq a$, we can conclude that $D \cap \beta(D) = \emptyset$ and no $R(x, y, z)$ holds in M such that $x \in D$, $y \in \beta(D)$ and $z \in E \cup D \cup \beta(D)$ (otherwise one can check that δ decreases since D and $\beta(D)$ are 0-minimally algebraic sets and it is impossible since E is $\leqslant$-closed). Let $d_1 \in D$. The set $ED\beta(D)$ is $\leqslant$-closed and then from Fact 2.2.19 it follows that

$$\operatorname{d}(d_1A) = \min\left\{\delta(A') : d_1A \subseteq A' \subseteq ED\beta(D)\right\}.$$

One can see that $\operatorname{d}(d_1A) = \delta(d_1A) = \delta(A) + 1$. Also it follows, by the same argument, that $\operatorname{d}(\beta(d_1)A) = \delta(A) + 1$ and $\operatorname{d}(d_1, \beta(d_1)A) = \delta(A) + 2$ (Fact 2.2.19 needs to be used) which contradicts with the definition of bounded automorphism. □

Now, Theorem 3.1.1 implies that $\mathrm{Autf}(M)$ has no non-trivial normal subgroup and then the following holds.

Corollary 3.1.12. *The strong automorphism group of M_μ is a simple group.*

3.2 AUTOMORPHISM GROUPS OF THE UNCOLLAPSED AB-INITIO GENERIC STRUCTURES

In this section, we prove that the automorphism group of the uncollapsed ab-initio generic structures with rational coefficients which fixes every dimension zero sets pointwise (i.e. $\mathrm{Aut}_{\mathrm{gcl}(\emptyset)}(M)$) is boundedly simple (see Def. 3.2.1) As we mentioned before, Tent and Ziegler in [39], introduced a combinatorial tool called stationary independence. This machinery is helping to prove the bounded simplicity of certain kind of homogeneous countable structures (see e.g. [6]). However, it was not possible to work in the same setting in the uncollpased ab-initio generic structures, as they have been considered in [39]. To obtain the result, we follow a modification by Evans in [9] of the machinery developed in [39].

First, we mention an observation about the ab-initio $\leqslant$-generic structures that we use in the proof of Lemma 3.2.27. Then we define *stationary independence* for a countable class of structures, as Evans does (see Def. 3.2.9). Afterwards, we introduce an independence relation $\mathop{\smile\!\!\!\!\!\mid}^{d}$ and prove that it is a stationary independence relation for the countable class $\mathfrak{Y} := \{\mathrm{gcl}(A) : A \subset M, A \text{ is finite}\}$ (see Thm. 3.2.21). We introduce gcl-bounded automorphisms. The main step in this section is to show that every non gcl-bounded automorphism moves *almost maximally* for $\mathop{\smile\!\!\!\!\!\mid}^{d}$-independence relation on the family $\mathfrak{Y}$ (see Lem. 3.2.27 and Cor. 3.2.28). Then we prove that there is no non-trivial gcl-bounded automorphism in this case by modifying the proof of Theorem 3.1.1 (see Thm. 4.2.24). We deduce the bounded simplicity of $\mathrm{Aut}_{\mathrm{gcl}(\emptyset)}(M)$ from Corollary 3.2.28, Theorem 4.2.24 and a theorem by Evans (see Thm. 3.2.30).

Here is the definition of bounded simplicity which was appeared in the statements.

Definition 3.2.1. Let G be a group and $m \in \mathbb{N} \setminus \{0\}$. Then G is *m-boundedly simple* if h is a product of $\leq m$ many conjugates of $g^{\pm 1}$ for all $1 \neq g, h \in G$. More precisely $G = \left(g^G \cup \left(g^{-1}\right)^G\right)^{\leq m}$. G is called *boundedly simple* if it is m-boundedly simple for some m.

Remark 3.2.2. Note that bounded simplicity implies simplicity but the converse is not necessarily true (e.g. infinite alternating groups). Moreover note that bounded simplicity is a first order property.

In this section similar to Section 3.1.1, we deal with $\mathfrak{L} = \{R(-,-)\}$ where R is binary irreflexive symmetric relation. The arguments can be modified for n-ary relations. Moreover for simplicity[1] we assume the following δ function on an $\mathfrak{L}$-structure A:

$$\delta(A) = \kappa|A| - |R(A)|$$

where $\kappa \in \mathbb{N}$ and $\kappa > 1$.

3.2.1 *An observation in the ab-initio generic structures*

In this subsection, two Lemmas 3.1.5 and 3.1.4 are used very often and their use is sometimes implicit.

Definition 3.2.3. Let A and B be two subsets of M such that $A \leqslant B$. Then

$$\mathrm{cl}^0_B(A) := \bigcup \{A' \subseteq B : A' \text{ is 0-algebraic over} A\} \cup A.$$

Note that we do not necessarily have $\mathrm{cl}^0_B\left(\mathrm{cl}^0_B(A)\right) = \mathrm{cl}^0_B(A)$.

Remark 3.2.4. If $A \leqslant B$, then $\mathrm{cl}^0_B(A) \setminus A$ is a disjoint union of 0-algebraic sets over A.

Proof. Follows from Definition 3.2.3 and Lemma 3.1.5. □

Lemma 3.2.5. *Let $A \leqslant B \leqslant M$. Then there exists a chain of $\leqslant$-closed subsets*

$$A = A_0 \leqslant A_1 \leqslant \cdots \leqslant A_n = B$$

where $\delta(A_i/A_{i-1}) = 0$ if i is odd and $A_i \setminus A_{i-1}$ is d_i-minimal over A_{i-1} if i is even. Further, $\sum_i d_i = \delta(A_n/A_0)$.

Proof. Let $A^1_1 := \mathrm{cl}^0_B(A_0)$ and define inductively $A^1_j := \mathrm{cl}^0_B(A_{j-1})$ and $A^{i+1}_j := \mathrm{cl}^0_B(A^i_j)$ for $i > 0$. Let $m \in \mathbb{N}$ be the smallest integer such that $A^{m+1}_1 = A^m_1$; it is clear that such an m exists; since A and B are finite subsets of M. Then define $A_1 := A^m_1$. Similarly for $j = 2k+1$, define $A_j := A^{m'}_{j-1}$ when m' is the smallest integer such that $A^{m'+1}_{j-1} = A^{m'}_{j-1}$ and A_{j-1} is defined. Suppose A_i is defined for $i = 2k+1$. Let $d_i := \min\{\mathrm{d}(a/A_i) : a \in B \setminus A_i\}$ and consider

$$C_{d_i} := \{\mathrm{cl}(aA_i) : \mathrm{d}(a/A_i) = d_i, a \in B \setminus A_i\}.$$

1 For a pre-dimension function $\delta_{\kappa,\lambda}(-)$ with $\kappa, \lambda \in \mathbb{N}$, $\gcd(\kappa, \lambda) = 1$ and $\kappa > \lambda > 0$, we need to modify Lemmas 3.2.7 and 3.2.8 in observation 3.2.1 and Lemmas 3.2.27 and 3.2.25. In this section, to present a clear proof, we skip the modifications for the general case. The modifications are similar to the case in Section 4.2.3 in Chapter 4.

Define $C'_{d_i} = \{\mathrm{cl}(aA_i) \in C_{d_1} : |\mathrm{cl}(aA_i)| = m_i\}$ where $m_i = \min\{|\mathrm{cl}(aA_i)| : \mathrm{cl}(aA_i) \in C_{d_i}\}$. Choose an element $a_{i+1} \in B \backslash A_i$ such that $\mathrm{cl}(a_{i+1}A_i) \in C'_{d_i}$ and define $A_{i+1} := \mathrm{cl}(a_{i+1}A_i)$. If $\delta(a_{i+1}/A_i) = \mathrm{d}(a_{i+1}/A_i)$ since A_{i+1} has a minimum size in C_{d_i}, we conclude that $A_{i+1} = a_{i+1}A_i$. If $\delta(a_{i+1}/A_i) > \mathrm{d}(a_{i+1}/A_i)$ then $a_{i+1}A_i \subsetneq A_{i+1}$ and again, since A_{i+1} has a minimum size in C_{d_i}, we conclude that $\delta(A'_i/A_{i-1}) > d_i$ for $A_{i-1} \subsetneq A'_i \subsetneq A_i$. So by definition, $A_{i+1} \backslash A_i$ is d_i-minimal over A_i. It is clear that $A_{i-1} \subsetneq A_i$ holds for $0 < i = 2k$. Since A and B are finite subsets so this process should stop and there is n $(\leq 2|B \backslash A| + 1)$ such that $A_n = B$. □

Remark 3.2.6. Let $A \leqslant B \leqslant M$ and consider a $\leqslant$-chain $\langle A_i : i \leqslant n \rangle$ obtained from Lemma 3.2.5 and assume the notation in its proof. For $i = 2k+1$ from Remark 3.2.4 it follows that $A_i^1 \backslash A_{i-1} = \bigcup_i E_i$ such that $E_i \cap E_j = \emptyset$ for $i \neq j$ and E_i's are 0-algebraic over A_{i-1}.

Lemma 3.2.7. *Let $A \leqslant B \leqslant M$. Then there exists $H \in \mathbb{K}_0$ such that $B \leqslant H$, $\delta(B/((H \backslash B)A)) = 0$ and $(H \backslash B)A \leqslant H$.*

Proof. By Lemma 3.2.5, there exists a chain of $\leqslant$-closed subsets

$$A = A_0 \leqslant A_1 \leqslant \cdots \leqslant A_n = B$$

where $\delta(A_i/A_{i-1}) = 0$ if i is odd and $A_i \backslash A_{i-1}$ is d_i-minimal over A_{i-1} if i is even. Let H' be a set of $(\sum_i d_i)$-many new elements enumerated as $\left\{h^i_j : 0 < j \leq d_i, i \text{ is even } \leq n\right\}$. For each even i, choose a_i to be an element in $A_i \backslash A_{i-1}$. Let H be a disjoint union of B and H' with additional edges: $H \models \bigwedge_{i,j} R\left(a_i, h^i_j\right)$. It is easy to see that $\delta(A_n/((H \backslash A_n)A_0)) = 0$ and $(H \backslash A_n)A_0 \leqslant H$. □

Lemma 3.2.8. *Let $A \leqslant B \leqslant M$. Let $H \leqslant M$ and $A = A_0 \leqslant \cdots \leqslant A_n = B$ be as given by Lemma 3.2.7. Suppose $H_1 \leqslant M$ such that $H_0 := H \cap H_1 = (H \backslash B)A$ and let $f \in Aut_A(M)$ be such that $f(H \backslash B) \cap (H \backslash B) = \emptyset$. Then $f(B \backslash A) \cap (B \backslash A) \subseteq A_1 \backslash A_0$. Moreover if we ask $A_1 = A_0$, then $f(B \backslash A) \cap (B \backslash A) = \emptyset$.*

Proof. Following the notation of the proof of Lemma 3.2.7, for even i let $a_i \in A_i \backslash A_{i-1}$ such that $|R(a_i, (H \backslash H_0))| = d(A_i/A_{i-1}) = d_i$. It is clear that $f(a_i) \neq a_i$ since by our assumption $f(h) \notin H_0 \backslash A$ for all $h \in H_0 \backslash A$; otherwise $\delta(A_i/H) < 0$. If the $\leqslant$-chain stops at A_1 (i.e. $n = 1$) then there is nothing to prove. Suppose $n \geq 2$, since $f(a_2) \neq a_2$ and $A_2 \backslash A_1$ is d_2-minimal over A_1, by Lemma 3.1.5, $f(A_2 \backslash A_1) \cap (A_2 \backslash A_1) = \emptyset$. Inductively, because of the construction of A_i's, we can prove that $f(A_i \backslash A_{i-1}) \cap (A_i \backslash A_{i-1}) = \emptyset$ for $i \geq 2$ and then $f(B \backslash A) \cap (B \backslash A) \subseteq A_1 \backslash A_0$. □

3.2.2 *Stationary independence on a countable family*

Let M be a countable structure and $\mathfrak{X}$ be a countable family of subsets of M. Suppose $\mathop{\smile\!\!\!\!\!\!|}^{\sharp}$ is ternary relation on $\mathfrak{X}^3$.

Definition 3.2.9. The ternary relation $\mathop{\smile\!\!\!\!\!\!|}^{\sharp}$ is a *stationary independence* relation for elements of $\mathfrak{X}$ if the following axioms are satisfied.

1. (Invariance) Suppose $A, B, C \in \mathfrak{X}$, $A \mathop{\smile\!\!\!\!\!\!|}^{\sharp}_{C} B$ is invariant under automorphisms of M.
2. (Monotonicity) Suppose $A, B, C, D \in \mathfrak{X}$. Then $A \mathop{\smile\!\!\!\!\!\!|}^{\sharp}_{B} CD$ implies $A \mathop{\smile\!\!\!\!\!\!|}^{\sharp}_{B} C$ and $A \mathop{\smile\!\!\!\!\!\!|}^{\sharp}_{BC} D$.
3. (Transitivity) Suppose $A, B, C, D \in \mathfrak{X}$. Then $A \mathop{\smile\!\!\!\!\!\!|}^{\sharp}_{B} C$ and $A \mathop{\smile\!\!\!\!\!\!|}^{\sharp}_{BC} D$ implies $A \mathop{\smile\!\!\!\!\!\!|}^{\sharp}_{B} D$.
4. (Symmetry) For $A, B, C \in \mathfrak{X}$, $A \mathop{\smile\!\!\!\!\!\!|}^{\sharp}_{C} B$ implies $B \mathop{\smile\!\!\!\!\!\!|}^{\sharp}_{C} A$.
5. (Existence) Suppose $A, B, C \in \mathfrak{X}$, then there is a $g \in \mathrm{Aut}_B(M)$ with $g(A) \mathop{\smile\!\!\!\!\!\!|}^{\sharp}_{B} C$.
6. (Stationarity) Suppose $A_1, A_2, B, C \in \mathfrak{X}$ with $B \subseteq A_i$ and $A_i \mathop{\smile\!\!\!\!\!\!|}^{\sharp}_{B} C$. Suppose $h : A_1 \to A_2$ is an isomorphism which is the identity on B and extends to an automorphism of M. Then there is some isomorphism k which contains $h \cup id_C$ and extends to an automorphism of M.

Definition 3.2.10. Suppose $\mathop{\smile\!\!\!\!\!\!|}^{\sharp}$ is an independence relation on the family $\mathfrak{X}$ and $g \in \mathrm{Aut}(M)$. Then g *moves almost maximally* if for all $B \in \mathfrak{X}$ and $a \in M$, there is a' in the Aut_B-orbit of a such that $a' \mathop{\smile\!\!\!\!\!\!|}^{\sharp}_{B} g(a')$.

The role of the stationary independence and Definition 3.2.10 can be seen in Theorem 3.2.30.

3.2.3 *Topology*

This chapter is based on Chapter 3 of Lascar's article [24]. Suppose M is a countable $\mathfrak{L}$-structure and $\mathfrak{Y}$ is a countable family of subset of M. Consider a countable family of partial isomorphisms $\mathfrak{J}$ which will satisfy conditions (1) - (7) as follows.

1. If $s \in \mathfrak{J}$, s is an partial isomorphism from X to Y, both X and Y belong $\mathfrak{Y}$.
2. $\mathfrak{J}$ contains all id_X, for $X \in \mathfrak{Y}$.

3. $\mathfrak{J}$ is closed under inverse functions: if $s : X \to Y$ belongs to $\mathfrak{J}$ then inverse function s^{-1} also belongs to $\mathfrak{J}$.

4. $\mathfrak{J}$ is closed under composition: if $s : X \to Y$ and $t : Y \to Z$ belong $\mathfrak{J}$. then $s \circ t$ also belongs to $\mathfrak{J}$.

5. If $s : X \to Y$ belongs to $\mathfrak{J}$ and $Z \in \mathfrak{Y}$ and $Z \leqslant X$, then $s \upharpoonright Z \in \mathfrak{J}$.

6. If $s : X \to Y$ belongs to $\mathfrak{J}$ and $Z \in \mathfrak{Y}$ and $X \leqslant Z$, then there exists $t \in \mathfrak{J}$ which is defined on Z and extends s.

7. If $s : X \to Y$ belongs to $\mathfrak{J}$ and $Z \in \mathfrak{Y}$ and $Y \leqslant Z$, then there exists $t \in \mathfrak{J}$ whose image contains Z and extends s.

Given a family the $\mathfrak{J}$ satisfying conditions (1) - (7), one defines

$$G(\mathfrak{J}) := \{f \in \mathrm{Aut}\,(M) : \text{ for all } X \in \mathfrak{Y}, f \upharpoonright X \in \mathfrak{J}\}\,.$$

Then it is easy to show that $G(\mathfrak{J})$ is a closed subgroup of $\mathrm{Aut}(M)$. If $s \in \mathfrak{J}$, one will denote $G_s(\mathfrak{J}) = \{f \in G(\mathfrak{J}) : f \text{ extends } s\}$. Then $G_s(\mathfrak{J})$ forms a basis of open sets for $G(\mathfrak{J})$, and the properties (6) and (7) ensure that they are not empty. For each $X \in \mathfrak{Y}$,

$$G_X(\mathfrak{J}) = \mathrm{Aut}_X(M) \cap G(\mathfrak{J})$$

and $G_X(\mathfrak{J})$ forms a neighborhood basis of identity such that all the open sets of the form $G_s(\mathfrak{J})$ are equal to $f.G_X(\mathfrak{J})$, where f is any element of $G_s(\mathfrak{J})$: thus $G(\mathfrak{J})$ is a Polish group.

3.2.4 *Our setting*

Let M be the ab-initio $\leqslant$-generic structure. Consider $\mathfrak{Y} := \{\mathrm{gcl}\,(A) : A \subset M, A \text{ is finite }\}$ and let the sub-base of neighborhoods of identity be

$$\{\mathrm{Aut}_X(M) : X \in \mathfrak{Y}\}\,.$$

In Lemma 3.2.19, we will show that the class $\mathfrak{Y}$ is not trivial and then similar to what we have seen in Section 3.2.3, define a basis of open sets: suppose s is an isomorphism $s : X \to Y$, and X and Y belong $\mathfrak{Y}$ define $O_s = \{f \in \mathrm{Aut}(M) : f \text{ extends } s\}$. Then the O_s's are open, and $\{O_s : s : X \to Y,\ X, Y \in \mathfrak{Y}\}$ form a basis of open sets.

The following two definitions of independence relations arise very naturally if there is a good notion of dimension $\mathrm{d}\,(-)$. This has been studied in the ab-initio generic structures (see e.g. [4, 42]).

Definition 3.2.11. For elements $A, B, C \in \mathfrak{Y}$, define $\mathop{\smile\hspace{-0.9em}|}^{d}$ as follows:

$$A \mathop{\smile\hspace{-0.9em}|}_{C}^{d} B \text{ if } \mathrm{d}(A/BC) = \mathrm{d}(A/C).$$

Remark 3.2.12. Note that we can extend the definition of $\mathop{\smile\hspace{-0.9em}|}^{d}$ to arbitrary subsets of M by specifying that $A \mathop{\smile\hspace{-0.9em}|}_{C}^{d} B$ if and only if $\mathrm{gcl}(A) \mathop{\smile\hspace{-0.9em}|}_{\mathrm{gcl}(C)}^{d} \mathrm{gcl}(B)$.

Definition 3.2.13. Suppose A, B and C are finite subsets of M. Define $A \mathop{\smile\hspace{-0.9em}|}_{B} C$ if $\mathrm{d}(A/B) = \mathrm{d}(A/BC)$ and $\mathrm{cl}(AB) \cap \mathrm{cl}(BC) = \mathrm{cl}(B)$.

For further properties of $\mathop{\smile\hspace{-0.9em}|}$ look at [4, 42]. It has been known that this independence relation coincides with the forking-independence (see Def. 2.1.17) in the uncollapsed ab-initio generic structures. This explains our notation $\mathop{\smile\hspace{-0.9em}|}$. Here we review some of properties of the $\mathop{\smile\hspace{-0.9em}|}$ independence which is defined as above.

Remark 3.2.14. ([42] Proposition 4.8) Let A and B be $\leqslant$-closed subsets of M, then the following are equivalent:

1. $A \mathop{\smile\hspace{-0.9em}|}_{A \cap B} B$,
2. $AB = A \otimes_{A \cap B} B$ and AB is $\leqslant$-closed,
3. $\mathrm{tp}(A/B)$ does not fork over $A \cap B$.

Remark 3.2.15. Note that $\mathop{\smile\hspace{-0.9em}|}$ is stronger that $\mathop{\smile\hspace{-0.9em}|}^{d}$; let A, B, C be finite subsets of M, then $A \mathop{\smile\hspace{-0.9em}|}_{C} B$ implies $\mathrm{gcl}(A) \mathop{\smile\hspace{-0.9em}|}_{\mathrm{gcl}(C)}^{d} \mathrm{gcl}(B)$ but the converse is not always true.

The following lemma was known as folklore. Since the precise statement was not stated anywhere we give the statement and the proof here.

Lemma 3.2.16. *Suppose $A, B, C \in \mathfrak{Y}$. Then the following are equivalent:*

1. $A \mathop{\smile\hspace{-0.9em}|}_{C}^{d} B$.
2. $\mathrm{gcl}(AC) \cup \mathrm{gcl}(BC) \leqslant M$ *and* $\mathrm{gcl}(AC) \cap \mathrm{gcl}(BC) = C$.

Proof. (2) to (1) follows from Definition 3.2.11 and Fact 2.2.24. For (1) to (2) let $C_0 \subset_{<\omega} C$ and $C_0 \subseteq A_0 \subset_{<\omega} \mathrm{gcl}(AC)$, $C_0 \subseteq B_0 \subset_{<\omega} \mathrm{gcl}(BC)$ be such that $\mathrm{d}(A_0) = \mathrm{d}(\mathrm{gcl}(AC))$, $\mathrm{d}(B_0) = \mathrm{d}(\mathrm{gcl}(BC))$ and $\mathrm{d}(C_0) = \mathrm{d}(C)$. We know that and $\mathrm{d}(ABC) = \mathrm{d}(\mathrm{gcl}(AC)\,\mathrm{gcl}(BC))$. Let $D \subset_{<\omega} \mathrm{gcl}(AC)\,\mathrm{gcl}(BC)$ such that $\mathrm{d}(D) = \mathrm{d}(ABC)$ and $A_0, B_0 \subseteq D$. Moreover $\delta(\mathrm{cl}(D)) = \mathrm{d}(D) \leq \delta(D)$. By definition $\delta(E) \geq \delta(\mathrm{cl}(D))$ for all $D \subseteq E \subset_{<\omega} M$. Let $E_1 = \mathrm{cl}(D) \cap \mathrm{gcl}(AC)$

and $E_2 = \mathrm{cl}(D) \cap \mathrm{gcl}(BC)$. It is clear that $A_0 \subset E_1$ and $B_0 \subset E_2$ so $\delta(E_1) \geq \delta(A_0)$ and $\delta(E_2) \geq \delta(B_0)$,

$$\delta(E_1E_2) \leq \delta(E_1) + \delta(E_2) - \delta(E_1 \cap E_2),$$

and since $D \subseteq E_1E_2$ so $\delta(E_1E_2) \geq \mathrm{d}(D)$. Thus E_1E_2 is $\leqslant$-closed. Suppose (E, F) is a minimal pair such that $E \subset \mathrm{gcl}(AC) \cup \mathrm{gcl}(BC)$. If $E \subset \mathrm{gcl}(AC)$ or $E \subset \mathrm{gcl}(BC)$, then it is clear that $F \subset \mathrm{gcl}(AC) \cup \mathrm{gcl}(BC)$. Suppose $E \subsetneq \mathrm{gcl}(AC)$ and $E \subsetneq \mathrm{gcl}(BC)$. Let $E_1' := E \cap \mathrm{gcl}(AC)$ and $E_2' := E \cap \mathrm{gcl}(BC)$. One can check $\mathrm{cl}(E_1'E_1) \subset \mathrm{gcl}(AC)$, $\mathrm{cl}(E_2'E_2) \subset \mathrm{gcl}(BC)$, and moreover $\mathrm{cl}(E_1'E_1)\,\mathrm{cl}(E_2'E_2)$ is $\leqslant$-closed. Hence $F \subset \mathrm{cl}(E_1'E_1)\,\mathrm{cl}(E_2'E_2) \subset \mathrm{gcl}(AC)\,\mathrm{gcl}(BC)$. For the second part $C \subseteq \mathrm{gcl}(AC)\,\mathrm{gcl}(BC)$ is clear. Let $z \in \mathrm{gcl}(AC) \cap \mathrm{gcl}(BC)$ and consider $Z_1 := \mathrm{cl}(zA_0)$ and $Z_2 := \mathrm{cl}(zB_0)$. It is clear that $\delta(Z_1/A_0) = \delta(Z_2/B_0) = 0$ then

$$\mathrm{d}(Z_1Z_2/C_0) = \delta(Z_1Z_2/C_0) \leq \delta(Z_1/C_0) + \delta(Z_2/C_0) - \delta(Z_1 \cap Z_2/C_0),$$

thus $\delta(z/C_0) = \delta(Z_1 \cap Z_2/C_0) = 0$. Hence $z \in \mathrm{gcl}(C_0) = C$ and we obtain the result. □

Lemma 3.2.17. *Let $A \in \mathfrak{Y}$ and $\bar{a}_1$ be a finite tuple in M. Then there is a finite subset A_0 of A, such that if $\bar{a}_2$ and $\bar{a}_1$ have the same type over A_0 for some $\bar{a}_2 \in M$, then they have the same type over A or $tp(\bar{a}_1/A_0)$ has a unique extension to A.*

Proof. Let $\bar{a}_1 \in M$ and consider A' be a finite $\leqslant$-closed subset of A such that $\mathrm{d}(A') = \mathrm{d}(A)$. Consider $A_0 := \mathrm{cl}(\bar{a}_1A') \cap A$. We claim thast if $\bar{a}_1$ and $\bar{a}_2$ have the same type over A_0 then for any finite subset A_1 of A, they have the same type over A_1. Since

$$\begin{aligned}\delta(\mathrm{cl}(\bar{a}_iA_0)) &\leq \delta(\mathrm{cl}(A_1A_0)\,\mathrm{cl}(\bar{a}_iA_0)) \\ &\leq \delta(\mathrm{cl}(A_1A_0)) + \delta(\mathrm{cl}(\bar{a}_iA_0)) - \delta(\mathrm{cl}(A_1A_0) \cap \mathrm{cl}(\bar{a}_iA_0))\end{aligned}$$

for $i = 1,2$ and $\delta(\mathrm{cl}(A_1A_0)) = \delta(\mathrm{cl}(A_1A_0) \cap \mathrm{cl}(\bar{a}_iA_0)) = \delta(A_0)$, then $\mathrm{cl}(A_1A_0\bar{a}_i) = \mathrm{cl}(A_1A_0)\,\mathrm{cl}(A_0\bar{a}_i)$. The way that A_0 is defined, we can see that $\mathrm{cl}(A_1A_0) \cap \mathrm{cl}(A_0\bar{a}_i) = A_0$ and $\delta(\mathrm{cl}(A_0\bar{a}_i)) = \delta(\mathrm{cl}(A_1A_0\bar{a}_i))$. Hence $\mathrm{cl}(A_0A_1) \downarrow_{A_0} \mathrm{cl}(A_0\bar{a}_i)$. Then by Remark 3.2.14, the set $\mathrm{cl}(A_1A_0\bar{a}_i) = \mathrm{cl}(A_1A_0) \otimes_{A_0} \mathrm{cl}(A_0\bar{a}_i)$. By the $\leqslant$-richness there is an automorphism $f \in \mathrm{Aut}_{\mathrm{cl}(A_1A_0)}(M)$ such that $f(\mathrm{cl}(A_1\bar{a}_1)) = \mathrm{cl}(A_1\bar{a}_2)$, thus $\bar{a}_1 \equiv_{\mathrm{cl}(A_1A_0)} \bar{a}_2$. So $\bar{a}_1$ and $\bar{a}_2$ have the same type for all finite $A_1 \subset A$ which includes A'. Hence the result follows. □

Corollary 3.2.18. *Suppose A is a finite $\leqslant$-closed subset of M and $\bar{a}_1, \bar{a}_2 \in M$. Assume $A_0 := \mathrm{cl}(\bar{a}_1A) \cap \mathrm{gcl}(A) = \mathrm{cl}(\bar{a}_2A) \cap \mathrm{gcl}(A)$, $\bar{a}_1\bar{a}_2 \cap \mathrm{gcl}(A) = \emptyset$ and $\bar{a}_1 \equiv_{A_0} \bar{a}_2$. Then $\bar{a}_1$ and $\bar{a}_2$ have the same type over $\mathrm{gcl}(A)$.*

Lemma 3.2.19. *Suppose $X \in \mathfrak{Y}$ and $a_1, a_2 \in M$ have the same type over $X \in \mathfrak{Y}$, then there is an automorphism $f \in \mathrm{Aut}_X(M)$ such that $f(a_1) = a_2$.*

Proof. Let X_0 be a finite $\leqslant$-closed subset of X such that $\mathrm{d}(X_0) = \mathrm{d}(X)$. Since $a_1, a_2 \in M$ have the same type over X so they have the same type over X_0. Let f_0 be a partial elementary isomorphism fixing X_0 such that $f_0(a_1) = a_2$. Using a back and forth construction, we build partial isomorphisms $f_0 \subset f_1 \subset f_2 \subset \ldots$ and then $f_\omega := \bigcup_{i<\omega} f_i$ will be the desired automorphism of M.

We now show how to construct the chain of partial isomorphisms. Fix $\langle B_i : i \in \omega \rangle$ to be a $\leqslant$-chain of finite $\leqslant$-closed subsets of M, sets such that $B_0 = X_0$, $B_1 = \mathrm{cl}(a_1 X_0)$, and $M = \bigcup_i B_i$. Let $E_0 := B_1 \cap X$, it is clear that $E_0 \leqslant B_1$. Let B_1' be an isomorphic copy of B_1 such that $B_1 \equiv_{E_0} B_1'$, $\mathrm{cl}(a_2 E_0) \subseteq B_1'$ and $E_0 \leqslant B_1' \leqslant M$ (actually $\mathrm{cl}(a_2 E_0) = B_1'$). Since $B_1' \cap X = B_1 \cap X = E_0$ and $\mathrm{tp}(B_1/E_0) = \mathrm{tp}(B_1'/E_0)$, we can extend f_0 to a partial isomorphism f_1 such that $f_1(B_1) = B_1'$ (so B_1 is in the domain i.e. forth) and f_1 fixes E_0 pointwise.

Let B_k be the smallest $\leqslant$-closed element in the $\leqslant$-chain such that $B_1' \leqslant B_k$. Let $E_1 := B_k \cap X$ then $E_1 \leqslant B_k$. Now find D_2 any isomorphic copy of B_k such that $B_1 \leqslant D_2 \leqslant M$, $D_2 \cap X = E_1$ and $D_2 \equiv_{E_1} B_k$ (by the free-amalgamation it is always possible). Since $\mathrm{tp}(D_2/E_1) = \mathrm{tp}(B_k/E_1)$ and $B_1 \subseteq D_2$, one can extend f_1 to a partial isomorphism f_2 such that $f_2(D_2) = B_k$ and f_2 fixes E_1 pointwise (so B_k is in the range i.e. back). Similarly this procedure can be continued and we can construct $f_0 \subset f_1 \subset f_2 \subset \ldots$ by back and forth fixing elements of X. Since $\bigcup B_i = \bigcup D_i = \bigcup B_i' = M$, f_ω is an automorphism of M. $\square$

Corollary 3.2.20. $\mathrm{Aut}_X(M)$ *is not trivial for all* $X \in \mathfrak{Y}$.

Corollary 3.2.20 proves that our base of neighborhoods of identity $\{\mathrm{Aut}_X(M) : X \in \mathfrak{Y}\}$ is not trivial. Now the next step will be the following.

Theorem 3.2.21. *The $\mathop{\smile\!\!\!\!\mid}^d$ is a stationary independence relation for elements of $\mathfrak{Y}$.*

Proof. (i, iv) Invariance under automorphisms of M and symmetry are trivial by the definition of $\mathop{\smile\!\!\!\!\mid}^d$.

(ii) For monotonicity, if $\mathrm{d}(A/B) = \mathrm{d}(A/BCD)$, then using Remark 2.2.28: $\mathrm{d}(A/B) \geq \mathrm{d}(A/BC) \geq \mathrm{d}(A/BCD) = \mathrm{d}(A/B)$. Thus we are done.

(iii) For transitivity, if $\mathrm{d}(A/BC) = \mathrm{d}(A/B)$ and $\mathrm{d}(A/BCD) = \mathrm{d}(A/BC)$ then $\mathrm{d}(A/B) = \mathrm{d}(A/BCD)$. Moreover, we know that $\mathrm{d}(A/B) \geq \mathrm{d}(A/BD) \geq \mathrm{d}(A/BCD)$. So this implies $\mathrm{d}(A/B) = \mathrm{d}(A/BD)$.

(v) Existence: suppose $B_0 \subset B$, $C_0 \subset C$ and $A_0 \subset A$ such that $\mathrm{d}(A_0) = \mathrm{d}(A)$, $\mathrm{d}(C_0) = C$ and $\mathrm{d}(B_0) = \mathrm{d}(B)$. Using the free-amalgamation (see Thm. 2.2.15) we can find A_0' an isomorphic copy of $\mathrm{cl}(A_0 B_0)$ over $B' := \mathrm{cl}(A_0 C_0 B_0) \cap B$ such

$A'_0 \mathop{\smile\!\!\!\!\!\!|}_{B'} C_0$ and $A'_0 \equiv_{B'} A_0$. Then by Lemma 3.2.19, there exists $g \in \mathrm{Aut}_B(M)$ such that $g(A_0) \subseteq A'_0$ and then by Remarks 3.2.15 and 3.2.12 $g(A) \mathop{\smile\!\!\!\!\!\!|}^{d}_{B} C$.

(vi) Stationarity: follows from Lemma 3.2.22. □

Lemma 3.2.22. *Suppose $A_1, A_2, B, C \in \mathfrak{X}$ with $B \subseteq A_i$ and $A_i \mathop{\smile\!\!\!\!\!\!|}_{B} C$. Suppose $h : A_1 \to A_2$ is an isomorphism which is the identity on B and extends to an automorphism of M. Then there is an automorphism of M which extends $h \cup id_C$.*

Proof. Note that $h \cup id_C$ is a partial isomorphism. We construct an automorphism $f \in \mathrm{Aut}(M)$ which extends $h \cup id_C$. By Lemma 3.2.16, $\mathrm{gcl}(A_i B) \cap \mathrm{gcl}(CB) = B$ for $i = 1, 2$. Let $B_0 \subset_{<\omega} B$ and $C_0 \subset_{<\omega} \mathrm{gcl}(CB) \backslash B$ such that $\delta(B_0) = \mathrm{d}(B_0) = \mathrm{d}(B)$ and $\delta(B_0 C_0) = \mathrm{d}(B_0 C_0) = \mathrm{d}(BC)$. Similarly let $A_1^0 \subset_{<\omega} A_1$ be such that $B_0 \subset A_1^0$ and $\delta(A_1^0) = \mathrm{d}(A_1^0) = \mathrm{d}(A_1)$. Let $A_2^0 := h(A_1^0)$ and by our assumptions A_1^0 and A_2^0 have the same type over B. Let $f_0 = id_{B_0 C_0}$ be a partial isomorphism. Using a back and forth construction, we build partial isomorphisms $f_0 \subset f_1 \subset f_2 \subset \dots$ so that $f := \bigcup_{i<\omega} f_i$ will be the desired automorphism of M.

Fix $\langle G_i : i \in \omega \rangle$ to be a $\leqslant$-chain of finite $\leqslant$-closed subsets of M where $G_0 = B_0 C_0$, $G_1 = \mathrm{cl}(C_0 A_1^0) = C_0 A_1^0$ and $M = \bigcup_i G_i$. Let $G'_1 = C_0 \cup h(A_1^0)$. Since $G'_1 \cap (BC) = G_1 \cap (BC) = B_0 C_0$ and $\mathrm{tp}(G_1 / B_0 C_0) = \mathrm{tp}(G'_1 / B_0 C_0)$, we can extend f_0 to a partial isomorphism f_1 such that $f_1(G_1) = G'_1$ (easy to check that f_1 is a partial isomorphism).

Let G_k be the smallest $\leqslant$-closed element in the $\leqslant$-chain such that $G'_1 \leqslant G_k$. Let $D_1 := G_k \cap (A_2 B)$ and $D_2 := G_k \cap (A_2 BC)$ then $D_1 \leqslant D_2 \leqslant G_k$ and $G'_1 \leqslant D_2$. Consider $D'_1 = h^{-1}(D_1) \leqslant D'_2 = D'_1 \cup (D_2 \backslash D_1) \leqslant G'_k = D'_2 \cup H'$ where H' over D'_2 is any isomorphic copy of $G_k \backslash D_2$ over D_2 such that $H' \cap (G_k \backslash D_2) = \emptyset$. [2] Extend f_1 to f'_2 such that $f'_2(D'_2) = D_2$ and f'_2 fixes $D_2 \backslash D_1$ pointwise (i.e. $f'_2 = h \restriction D'_1 \cup id_{D_2 \backslash D_1}$). Similarly extend f'_2 to f_2 such that $f_2(G_k \backslash D_2) = H'$. This procedure can be continued and we can construct $f_0 \subset f_1 \subset f_2 \subset \dots$ by back and forth fixing elements of X. Since $\bigcup G_i = \bigcup D_i = \bigcup G'_i = M$, f is an automorphism of M. □

3.2.5 *gcl-bounded automorphisms and moving almost maximally*

In Section 3.1.1, we showed that there is no non-trivial bounded automorphism for uncollapsed ab-initio generic structures with rational coefficients. The same

2 I would like to thank David Evans for pointing out that the existence of H' follows from the algebraic amalgamation lemma ([17]Lemma 3) which implies that there are finitely many zero algebraic sets over in $A_1 BC$. Then since we have the free-amalgamation property, existence of H' is guaranteed.

proof will work to prove that there is no non-trivial bounded automorphisms in this case. But the normal definition of a bounded automorphism is not good enough for our purpose. Roughly, in order to apply Theorem 3.2.30 we need to move 0-minimal algebraic sets almost maximally, which is not possible in this case (see Lemma 3.2.27).

Recall that for a finite set A we define $\operatorname{gcl}(A) := \{x \in M : \operatorname{d}(x/A) = 0\}$. Here is the modified definition of Definition 2.3.7.

Definition 3.2.23. An automorphism $\beta \in \operatorname{Aut}(M)$ is called gcl-*bounded* if there exists a finite set $A \subset M$ such that $\beta(m) \in \operatorname{gcl}(mA)$ for all $m \in M$.

Remark 3.2.24. Since by Remark 2.2.25 $\operatorname{cl}(A) \subseteq \operatorname{gcl}(A)$, in the definition of gcl-bounded automorphism, we may assume that the set A is $\leqslant$-closed and $|A| \geq \kappa$.

Lemma 3.2.25. *Suppose g is not a gcl-bounded automorphism. Then for every $n \in \mathbb{N}$, there is $X \subseteq_{<\omega} M$ such that $\operatorname{d}(g(X)/X) \geq n$.*

Proof. Suppose not. Let $n_0 \in \mathbb{N}$, be the smallest number such that $\operatorname{d}(g(Y)/Y) \leq n_0$ for all $Y \subset_{<\omega} M$. Let $Y_0 \subseteq_{<\omega} M$ be such that $\operatorname{d}(g(Y_0)/Y_0) = n_0$; we assume Y_0 is $\leqslant$-closed and $|Y_0| \geq \kappa$ (if $|Y_0| < \kappa$ replace Y_0 by $Y' \subset \operatorname{gcl}(Y_0)$ such that $|Y'| \geq \kappa$ and $Y_0 \subset Y'$). Let $Y_1 := \operatorname{cl}(g(Y_0)Y_0)$.

Let $b \in M$ be such that $\operatorname{d}(b/Y_1) = i$ for some $i > 0$.

By properties of $\operatorname{d}(-)$ (see Rem. 2.2.28) we know

$$\operatorname{d}(g(b)g(Y_0)/bY_0) = \operatorname{d}(g(b)/g(Y_0)Y_0b) + \operatorname{d}(g(Y_0)/Y_0b) \leq n_0.$$

Case (1) $\operatorname{d}(g(Y_0)/Y_0b) = \operatorname{d}(g(Y_0)/Y_0) = n_0$. Then the inequality above implies that $\operatorname{d}(g(b)/g(Y_0)Y_0b) = 0$ and hence $g(b) \in \operatorname{gcl}(g(Y_0)Y_0b) = \operatorname{gcl}(bY_1)$.

Case (2) $b \not\perp_{Y_0} g(Y_0)$ and let $Y_2 := \operatorname{cl}(bY_1g(Y_1))$. Note that $|Y_2| \geq \kappa$. Let $Y' \subset Y_2$ be such that $|Y'| = \kappa$, $b \in Y'$ and $Y' \setminus \{b\} \subset Y_0$. Enumerate Y' as $\{y_1, \cdots, y_\kappa\}$. Consider $Y_3 \in \mathbb{K}_0$ to be an $\mathfrak{L}$-structure such that $Y_3 := Y_2 \cup \{b_3\}$ with $Y_3 \models \bigwedge_i R(b_3, y_i)$. It is clear that $Y_2 \leqslant Y_3$ and $\delta(Y_3/Y_2) = 0$. By the $\leqslant$-richness and the free-amalgamation, we can find an isomorphic copy of Y_3 such that $b_3 \perp_{Y_0} g(Y_0)$. Then using Case (1), $g(b_3) \in \operatorname{gcl}(bY_1)$. Since $\operatorname{d}(b_3b/Y_1) = \operatorname{d}(b_3/bY_1) + \operatorname{d}(b/Y_1) = i$, one can see $\operatorname{d}(b/b_3Y_1) = \operatorname{d}(b_3/bY_1) = 0$, so $g(b) \in \operatorname{gcl}(g(b_3)g(Y_1))$. Then $g(b) \in \operatorname{gcl}(Y_2)$; note that from $\operatorname{d}(b_3/bY_1) = 0$ we can deduce $b_3 \in \operatorname{gcl}(bY_1)$.

As we saw from the cases above $g(b) \in \operatorname{gcl}(b \cup Y_1g(Y_1))$ for all elements $b \in M$, so g is gcl-bounded which is a contradiction. □

Corollary 3.2.26. *Suppose g is not a gcl-bounded automorphism. Let X be a finite $\leqslant$-closed set and $n \in \mathbb{N}$. Then there exists a $\leqslant$-closed set Z containing X such that $\operatorname{d}(g(Z)/Z) \geq n$.*

Proof. Let $\mathrm{d}(X) = n_0$. Note that $\mathrm{d}(Xg(X)) \leq 2n_0$. By Lemma 3.2.25, for every $n \in \mathbb{N}$, there is a finite set Y such that $\mathrm{d}(g(Y)/Y) \geq n$. Then by Remark 2.2.28,

$$\mathrm{d}(g(X)g(Y)/XY) = \mathrm{d}(g(X)/(g(Y)XY)) + \mathrm{d}(g(Y)/XY) \geq \mathrm{d}(g(Y)/XY).$$

By Remark 2.2.28 also follows that $\mathrm{d}(g(Y)/XY) \geq n - n_0$. Let $Z = \mathrm{cl}(XY)$ then $\mathrm{d}(g(Z)/Z) \geq n - n_0$ and since we can choose n and the corresponding Y large enough, we are done. □

Lemma 3.2.27. *Suppose g is not a gcl-bounded automorphism. Let $X, Y \in \mathbb{K}_0$, $X \leqslant M$ and $X \leqslant Y$. Moreover, assume $\delta(Y_0/X) > 0$ for all $X \subsetneq Y_0 \subseteq Y$. Then there exists $Y' \subset M$ such that $tp(Y'/X) = tp(Y/X)$ and $\mathrm{d}(Y'/X) = \mathrm{d}(Y'/Xg(Y'))$.*

Proof. Suppose $\delta(Y/X) = m$. We find $Y' \subset M$ in three steps.

Step 1 By our assumptions $m > 0$. Since g is not gcl-bounded, using Corollary 4.2.19, we can find a finite $\leqslant$-closed set $Z \subset M$ which contains $X_1 := \mathrm{cl}(X \cup g(X))$ such that $\mathrm{d}(g(Z)/Z) \geq m$. Specify $z'_1, \ldots, z'_m \in Z \backslash X_1$ such that $g(z'_i) \neq z'_i$ for all $0 < i \leq m$ and

$$\mathrm{d}\left(\left(\bigcup_{1 \leq i \leq m} z'_i\right)/X_1\right) \geq m.$$

Step 2 Using Lemma 3.2.5, there is a $\leqslant$-chain $X = X_0 \leqslant X_1 \ldots \leqslant X_n = Y$ such that $\delta(X_i/X_{i-1}) = 0$ if i is odd and $X_i \backslash X_{i-1}$ is d_i-algebraic over X_{i-1} if i is even, where $\sum_i d_i = m$. Note that by our assumptions $X_0 = X_1$. Then by Lemma 3.2.7, there is $H \in \mathbb{K}_0$ such that $X_n \leqslant H$, $\delta(H/((H \backslash Y)X_0)) = 0$ and $(H \backslash Y)X_0 \leqslant H$. Enumerate $H \backslash X_n = \left\{h^i_j : 1 \leq j \leq d_i \text{ and } i \text{ is even } \leq n\right\}$ lexicographically with respect to (i, j) as $\{h_i : 1 \leq i \leq m\}$ (in this case any enumeration works). Then consider the $\mathfrak{L}$-structure $A \in \mathbb{K}_0$ obtained from $H \otimes_X Z$ by identifying each h_i with z'_i for all $1 \leq i \leq m$. More precisely

1. the domain of A is $Y \cup Z$;
2. $A \models R(y_1, y_2)$ if and only if $Y \models R(y_1, y_2)$ for $y_1, y_2 \in Y$;
3. $A \models R(x_1, x_2)$ if and only if $Z \models R(x_1, x_2)$ for $x_1, x_2 \in Z$;
4. $A \models \bigwedge_i R(x_i, z'_i)$ if and only if $H \models \bigwedge_i R(x_i, h_i)$ for all $x_i \in Y$ and $z'_i \in Z \backslash X$.

Note that $\delta(A/Z) = 0$, $Y \leqslant A$ and $Z \leqslant A$.

Step 3 Let $Z_1 := \mathrm{cl}(Z \cup g(Z))$. Since $\mathbb{K}_0$ has the free-amalgamation property, then $A \otimes_Z Z_1 \in \mathbb{K}_0$. By $\leqslant$-richness of M, we can embed a $\leqslant$-closed copy of $A \otimes_Z Z_1$ over Z_1 in M , which we also denote by A, such that $g(A \backslash Z) \cap Z_1 = \emptyset$. Then by Remark 3.2.14, $A \mathop{\smile\!\!\!\!\!\!\mid}_Z Z_1$. Let $Y' \subset A$ be such that $\mathrm{tp}(Y'/X) = \mathrm{tp}(Y/X)$.

Claim. $Y' \downarrow_X g(X)$.

Proof. Both sets X and Y' are $\leqslant$-closed and Y' and $g(X)$ are free-amalgam over X and $\operatorname{cl}(Y'X_1) = Y'X_1$. □

Claim. $Y' \downarrow_{X \cup g(X)} g(Y')$.

Proof. Since $g(z_i) \neq z_i$ for the specified z_i's, one can see $g(Y') \neq Y'$. Moreover, by the Lemma 3.2.8, $g(Y') \cap Y' = \emptyset$. The sets Y' and $g(Y')$ are the free-amalgam over Z_1 and hence over X_1, if not it means $|R(Y' \backslash Z_1; g(Y') \backslash Z_1)| \geq 1$ which implies $\delta(Y'g(Y')/Z_1) < 0$ and this is impossible because of the fact that Z_1 is a $\leqslant$-closed set. Using the properties of step 2, it is easy to check that $\operatorname{cl}(Y'X_1) = Y'X_1$ and $\operatorname{cl}(g(Y')]X) = g(Y')X_1$. Then $Y' \downarrow_{X \cup g(X)} g(Y')$. □

The rest is done simply by transitivity of $\downarrow^d$ or more precisely since $Y' \downarrow_X g(X)$ and $Y' \downarrow_{X \cup g(X)} g(Y')$, $\operatorname{d}(Y'/X)) = \operatorname{d}(Y'/(Xg(X)))$ and $\operatorname{d}(Y'/(Xg(X))) = \operatorname{d}(Y'/(Xg(X)g(Y')))$ so $\operatorname{d}(Y'/X) = \operatorname{d}(Y'/(Xg(X)g(Y')))$ and then $\operatorname{d}(Y'/X) = \operatorname{d}(Y'/Xg(Y'))$ as we desired.

□

Corollary 3.2.28. *Every non gcl-bounded automorphism of M moves almost maximally for the $\downarrow^d$-independence relation on the family $\mathfrak{Y}$.*

Proof. Suppose g is not a gcl-bounded automorphism. Let $A \in \mathfrak{Y}$ and $a \in M$. Consider $A_0 \subset_{<\omega} A$ such that $\delta(A_0) = \operatorname{d}(A) = \operatorname{d}(A)$. Then either

Case (1) $a \in \operatorname{gcl}(A_0) = A$. Then automatically $a \downarrow_A^d g(a)$.

Case (2) $a \notin A$. Consider $A_1 := \operatorname{cl}(A_0 a) \cap A$. Now $A_1 \leqslant \operatorname{cl}(A_0 a)$ and since $a \notin A$, it follows $\delta(b/A_1) > 0$ for all $b \in \operatorname{cl}(A_0 a) \backslash A_1$. Then by Lemma 3.2.27 there is $A'_1 \models \operatorname{tp}(\operatorname{cl}(A_0 a)/A_1)$ such that $\operatorname{d}(A'_1/A_1 g(A'_1)) = \operatorname{d}(A'_1/A_1)$. Hence $\operatorname{gcl}(A'_1) \downarrow_A^d \operatorname{gcl}(g(A'_1))$. Let $a' \in A'_1$ such that $a' \models \operatorname{tp}(a/A_1)$ then by Lemma 3.2.18, $a' \models \operatorname{tp}(a/A)$ and $a' \downarrow_A^d g(a')$. □

3.2.5.1 *gcl-bounded automorphism*

Recall our notation for δ, that for an $\mathfrak{L}$-structure A it is of the following form:

$$\delta(A) = \kappa|A| - \lambda|R(A)|$$

where $\kappa, \lambda \in \mathbb{N}$ and $\kappa > \lambda > 0$. Note that, similar to the collapsed case when $\kappa \leq \lambda$ there are non-trivial gcl-bounded automorphisms. We will discuss it in Subsection 3.2.7. Let the class $\mathbb{K}_0 = \{\text{finite } \mathfrak{L}\text{-structures } A : \delta(A') \geq 0 \text{ for all } A' \subseteq A\}$ and let the self-sufficiency "$\leqslant$" be as it is defined in Definition 2.2.11. Then

Theorem 3.2.29. *There is no non-trivial gcl-bounded automorphism in the* $(\mathbb{K}_0, \leqslant)$*-generic structure* M*.*

Proof. The proof for Theorem 4.2.24 is the same as the proof for Theorem 3.1.1. The only difference in the proof is we need to substitute "acl $(-)$" with "gcl $(-)$" and "bounded automorphism" with "gcl-bounded automorphisms". Note that this phenomena is not accidental. Roughly, the intuition behind this is that in the collapsing structure gcl $(-)$ and acl$(-)$ coincide (see Thm. 2.2.36). □

3.2.6 *Bounded simplicity*

In order to answer the simplicity question in the uncollapsed generic structures, we need the following theorem of David Evans.

Theorem 3.2.30. *([9]) Suppose G fixes every element of* $\mathrm{gcl}(\emptyset)$*. Suppose* $g \in \mathrm{Aut}(M)$ *moves almost maximally. Then every element of G is product of a 16 conjugates of g.*

Lemma 3.2.31. $\mathrm{Aut}_{\mathrm{gcl}(\emptyset)}(M)$ *is a non-trivial normal subgroup of* $\mathrm{Aut}(M)$*.*

Proof. Any automorphism of M fixes $\mathrm{gcl}(\emptyset)$ setwise so $\mathrm{Aut}_{\mathrm{gcl}(\emptyset)}(M)$ is a normal subgroup. From Corollary 3.2.20 we also know that $\mathrm{Aut}_{\mathrm{gcl}(\emptyset)}(M)$ is not trivial. □

Corollary 3.2.32. *Let* M *be the uncollapsed ab-initio* $\leqslant$*-generic structure. Then* $\mathrm{Aut}_{\mathrm{gcl}(\emptyset)}(M)$ *is boundedly simple.*

Proof. By Theorem 3.2.21, $\downarrow^d$ is a stationary independence for the countable family $\mathfrak{Y}$. By Theorem 4.2.24 every non-trivial automorphism of M is not gcl-bounded. Corollary 3.2.28 implies that all automorphisms of M move almost maximally with respect to $\downarrow^d$. Now from Theorem 3.2.30, we conclude that $\mathrm{Aut}_{\mathrm{gcl}(\emptyset)}(M)$ is boundedly simple. □

3.2.7 *Further remarks*

Here we give a short proof that there is no non-trivial, bounded automorphism in the automorphism group of the uncollapsed $\leqslant$-generic structures which includes the case with irrational coefficients. It is known that the ab-initio $\leqslant$-generic structures that are derived from a pre-dimension function with irrational coefficients are stable but not $\aleph_0$-stable (see e.g. [4]). In [3], Baldwin and Shelah observed the connection between the theory of the ab-initio generic structures in this case and theory of random graphs with 'edge probability' n^w. Note that in this case, the algebraic closure and the geometric closure are the same; roughly because no 0-minimally algebraic pairs are possible.

Again, we assume $\mathfrak{L} = \{R(-,-)\}$ where R is binary irreflexive symmetric relation. Since we want to include irrational coefficients, we assume that δ for an $\mathfrak{L}$-structure A is of the the following form:

$$\delta(A) = |A| - w\,|R(A)|,$$

where $0 < w \leq 1$. The smooth class $(\mathbb{K}_0, \leqslant)$ is defined similarly to Definition 2.2.11. Let M be the $(\mathbb{K}_0, \leqslant)$-generic structure. Then

Theorem 3.2.33. *There is no non-trivial bounded automorphism in the uncollapsed ab-initio generic structures with coefficient $0 < w \leq 1$.*

Proof. Let β be a bounded automorphism and A a $\leqslant$-closed set such that $\beta(m) \in \mathrm{acl}(mA)$ for all $m \in M$. Recall that, in the uncollapsed case $\mathrm{acl}(A) = \mathrm{cl}(A)$ (see Lem. 2.2.27) and since A is a $\leqslant$-closed set then $\mathrm{acl}(A) = A$. Then from boundedness of β follows that A is fixed setwise. Similarly it is easy to see that all $\leqslant$-closed sets $B \supseteq A$ is also fixed setwise. Let $c \in M$ such that $\beta(c) \neq c$. Consider $C := \mathrm{cl}(cA)$ and let $D := C \cup \{d\}$ such that d is a new element and $R(d,b)$ holds in D. It is easy to see that $D \in \mathbb{K}_0$ and $C \leqslant D$. By the $\leqslant$-richness of M we can find an isomorphic copy D in M over C which is $\leqslant$-closed. But then, as we pointed out before, both D and C will be fixed setwise. Hence d is a fixed point. Then from $C \leqslant D$ we conclude that c is also a fixed-point. Hence $\beta = \mathrm{id}_M$. □

3.2.7.1 *Irrational coefficients*

Theorem 3.2.33, shows that there is no non-trivial bounded and gcl-bounded automorphisms in this case. But still we can not answer the question of simplicity. One missing part in the machinery that we developed in this section is that we can not show that every non-trivial automorphism moves almost maximally. Again, since 0-minimally algebraic pairs do not exists in this case hence the arguments, of Lemma 3.2.27 fail.

3.2.7.2 *Rational coefficients*

Again, since we study the $\leqslant$-generic structures in uncollapsed ab-initio case that are obtained from a pre-dimension function with rational coefficients, let the pre-dimension function over an $\mathfrak{L}$-structure A be of the following form:

$$\delta(A) = \kappa|A| - \lambda|R(A)|;$$

where $\kappa, \lambda \in \mathbb{N}$, $\gcd(\kappa, \lambda) = 1$. Let the class $\mathbb{K}_0 = \{\text{finite } \mathfrak{L}\text{-structures } A : \delta(A') \geq 0 \text{ for all } A' \subseteq A\}$.

CASE 1: $\kappa = 1$, $\lambda = 1$.

As we mentioned in Lemma 2.2.27, "cl$(-)$" and "acl$(-)$" coincide in the uncollapsed $\leqslant$-generic structures. It is easy to see that cl$(-)$ is not a pre-geometry. However, by Remark 2.1.40, gcl$(-)$ is a pre-geometry and it is disintegrated. Moreover, $\emptyset \in \mathbb{K}_0$ and $\text{acl}(\emptyset) = \emptyset$. However gcl$(\emptyset)$ contains countably many disjoint cycles of length n for all $n \in \mathbb{N}$, and finite structures that contain a cycle. There will be no cycles in $M \backslash \text{gcl}(\emptyset)$, therefore $M \backslash \text{gcl}(\emptyset)$ contains countably many disjoint connected countable trees of infinite valency at each vertex. Infinite valency at each vertex follows from the free-amalgamation property (see Thm. 2.2.15). By Theorem 3.2.33, in this case there will be no non-trivial bounded automorphisms. However, there will be gcl-bounded automorphisms; the gcl-bounded automorphisms are those which fix every element except finitely many connected components. Moreover the gcl-bounded automorphisms form a normal subgroup of Aut(M).

CASE 2: $\kappa = 1$, $\lambda > 1$.

Similar to CASE (1), cl$(-)$ and acl$(-)$ coincide and cl$(-)$ is not a pre-geometry. Therefore acl$(-)$ is not a pre-geometry as well. If $\lambda = 2$ then gcl$(\emptyset)$ is an infinite set: the union of countably many infinite sets of two elements with an edge. Since having a relation between any two elements of the structure makes the predimension δ zero, so there will be no edges between elements of $M \backslash \text{gcl}(\emptyset)$. By Remark 2.1.40, gcl$(-)$ is a pre-geometry and it is disintegrated. If $\lambda > 2$, then there will be no more edges between points of $\leqslant$-generic structure and then $\text{acl}(\emptyset) = \emptyset$. In this case there will be no non-trivial bounded automorphisms by Theorem 3.2.33. But there are gcl-bounded automorphisms: the gcl-bounded automorphisms are those those which fix everything except finitely many elements. Moreover, the gcl-bounded automorphism group is a normal subgroup of Aut(M).

CASE 3: $\kappa > 1$, $\kappa > \lambda \geq 1$.

Similar to CASES (1) and (2), cl$(-)$ and acl$(-)$ coincide. cl$(-)$ is not a pre-geometry and hence acl$(-)$ is not a pre-geometry too. Moreover, by Remark 2.1.40, gcl$(-)$ is not a pre-geometry. In this case gcl$(\emptyset)$ contains all $A \in \mathbb{K}_0$ with $\delta(A) = 0$. Similar to CASE (2) of Section 3.1.2 (i.e. the collapsed $\leqslant$-generic structures) for $\kappa = 2$ and $\lambda = 1$, in $M \backslash \text{gcl}(\emptyset)$ one can find all K_n for $n = 1, 2, 3, 4$ but there will be no K_5. Let every $A \in \mathbb{K}_0$ we may pick $a \in A$. Let $B := A \cup \{b\}$

such that b is a new element and $R(b,a)$ holds. Then $A \leqslant B$ and $B \in \mathbb{K}_0$. This implies that the valency of each vertex is infinite. For any two distinct elements $a, b \in M$ let $A = \text{cl}(ab)$. Let $C = A \cup \{c\}$ be an $\mathfrak{L}$-structure such that c is a new element and $R(a,c)$ and $R(b,c)$ hold in C. It is clear that $A \leqslant C$ and $C \in \mathbb{K}_0$. This implies that the $\leqslant$-generic structure M is connected. Let a be an arbitrarily element of M, let $I_a := \{b \in M : M \models R(a,b)\}$. Similar argument also implies that $\text{d}(I_a)$ is infinite. Moreover for each $a \notin \text{gcl}(\emptyset)$, $\text{gcl}(a) \setminus \text{gcl}(\emptyset)$ is an infinite set.

CASE 4: $\kappa > 1$, $\lambda > \kappa > 1$.

If $\frac{\lambda}{\kappa} > 2$, then a relation between any two elements makes the pre-dimension δ negative. Hence the $\leqslant$-generic structure is a disjoint union of single points. Then the generic structure is just a set. In CASE (2) we already considered $\frac{\lambda}{\kappa} = 2$.

4 GENERALIZED n-GONS

4.1 BASICS OF THE THEORY OF BUILDINGS AND GENERALIZED n-GONS

4.1.1 *Introduction*

Bruhat–Tits buildings are combinatorial and geometrical structures which generalize certain aspects of projective spaces. They were initially introduced by Tits as a means to understand the structure of semisimple complex groups of Lie type and later used to study semisimple algebraic groups over an arbitrary field. Although the theory of semisimple algebraic groups was the initial motivation for the notion of a building, not all buildings arise from groups. In particular, projective planes and generalized quadrangles form two classes of graphs studied in incidence geometry which satisfy the axioms of a building but may not be connected with any group. This phenomenon turns out to be related to the low rank (more precisely rank two) of the corresponding *Coxeter system*.

In [40], Tits gave a complete classification of thick, irreducible spherical buildings of rank greater than three. The result shows that except for some exceptional cases, such buildings correspond to a simple algebraic group over some field. It is also shown that the buildings of that form come from groups with a BN-pair. In contrast to spherical buildings of rank greater 3, the buildings of rank less than 3 can not be classified so easily. Even in the finite case the projective planes are not classified: for example projective planes of order 11 are not fully known.

Groups with a BN-pair, introduced in [40], do naturally appear in (semi-)simple algebraic groups and Lie groups. It is known that the groups with a BN-pair of rank one are 2-transitive on a suitable set, and hence they fall into a wild class of structures for which the simplicity question might not be an adequate question (see Example. 4.1.5). To each group with a BN-pair one can associate a building. The buildings that one gets from the groups with a BN-pair of rank 2 are generalized n-gons. Our main concern are groups with a BN-pair of rank 2 which are more well-behaved. Standard examples of such groups are given by semi-simple and almost simple algebraic groups of relative rank 2 (see [40]). Here are some known facts about groups with a BN-pair:

1. finite groups with a BN-pair of rank at least two are simple algebraic groups;
2. infinite groups with a split BN-pair of rank at least two are *essentially* algebraic.

One natural question is how much can one say about the infinite groups with a non-split BN-pair. One can also reformulate the question as whether there are simple infinite groups with a BN-pair which is not algebraic? In [36], Tent modified Hrushovski's construction method from model theory to build a new class of generalized n-gons which the automorphism group is transitive on ordinary $(n+1)$-gons. She also modified the construction to obtain almost strongly minimal generalized n-gons with the same transitivity properties. These transitivity properties shows that the automorphism group of the generalized n-gons have a BN-pair; namely it was known that the automorphism group of a building of rank 2 has a BN-pair if and only if it acts transitively on ordered n-gons. In this chapter we show that the automorphism group of the almost strongly minimal generalized n-gons and the $\aleph_0$-stable generalized n-gons constructed in [36] are indeed simple groups. Hence this implies that there exists a simple non-algebraic group with a non-split BN-pair. For proving the simplicity in the case of the $\aleph_0$-stable n-gons, we modify our proof of the simplicity result of the uncollapsed generic structures (see Section sec:unmpp), and in the case of the almost strongly minimal generalized n-gons, we follow Lascar' approach [24].

4.1.2 *Basic definitions*

Definition 4.1.1. A *Coxeter group* of rank n is a group with the following presentation
$\left\langle r_1, \cdots, r_n : \left(r_i r_j\right)^{m_{ij}} = 1 \right\rangle$
where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. The pair (W, S) where W is a Coxeter group with generators $S = \{r_1, \cdots, r_n\}$ is called a *Coxeter system.* A Coxeter system (W, S) is called *spherical* if W is a finite group.

A part of the data defining a building Δ is a *Coxeter group* W which determines a highly symmetric simplicial complex $\Sigma = \Sigma(W, S)$, called the *Coxeter complex*. We do not give here a precise definition of a building, but roughly, a *building* Δ is glued together from multiple copies of Σ in a certain regular fashion. When W is a finite *Coxeter* group, the *Coxeter complex* is a topological sphere, and the corresponding building is said to be of the *spherical type*.

As we mentioned before, Tits [40] gave a complete classification of thick, irreducible, spherical buildings of rank greater than 3. The buildings of rank 2 are called *generalized n-gons* or *generalized polygons*. Here, we work with the following axiomatization of the generalized n-gons.

Definition 4.1.2. Let $\Gamma = (P, L, I)$ be a triple where P and L are two sets whose elements are called points and lines, respectively, and $I \subseteq (P \times L) \cup (L \times P)$ is a symmetric relation which is called the *incidence relation*. Γ can be considered as a bipartite graph where incidence relation can be considered as the edge relation between elements of P and L. We call Γ a *generalized n-gon* if:

1. the diameter of Γ is equal to n;
2. the girth of Γ (i.e. length of the shortest cycle) is equal to $2n$.

We call a generalized n-gon *thick* if every element is incident with at least three other elements. We denote by $d_\Gamma(x, y)$ the graph distance between x and y in Γ.

Definition/Notation 4.1.3. Suppose Γ is a generalized n-gon.

1. For $a \in \Gamma$, define $D(a) := \{b \in \Gamma : (a, b) \in I\}$. If a is a line then $D(a)$ is called a *point row* and if a is a point then $D(a)$ is called a *line pencil*.
2. A *path* in Γ is a sequence $(x_0, \cdots, x_k)$ of points and lines such that $(x_i, x_{i+1}) \in I$ for $0 \leq i \leq k-1$.
3. By an *ordinary n-gon* we mean a simple cycle of length $2n$. If we fix labels $x_0, x_1, \cdots, x_{2n} = x_0$ and a point x_0 for an ordinary n-gon, we call this tuple an *ordered* ordinary n-gon. Note that a projective plane is nothing but a generalized 3-gon.

4.1.2.1 *BN-pairs*

Definition 4.1.4. We say that a pair of subgroups B and N of a group G is a *BN-pair* if:

1. $\langle B, N \rangle = G$;
2. $T := N \cap B \lhd N$;
3. $W := N/T$ is a Coxeter system with generators S;
4. for all $s \in S$ and $w \in W$,
$$sBw \subseteq BswB \cup BwB.$$
5. for all $s \in S$,
$$sBs^{-1} \nsubseteq B.$$

The group W is called *Weyl* group associated to the BN-pair. Also (G, B, N, S) is called a *Tits System*. A BN-pair *splits* if there is a normal nilpotent subgroup U of B with $B = U.T$. Moreover, we call $|S|$ the *rank* of a BN-pair.

Example 4.1.5. (see [1]) Here are some standard examples of groups with a BN-pair.

1. Suppose G is any 2-transitive permutation group on a set X with more than 2 elements. Let B be the subgroup of G fixing a point x, and let N be the subgroup fixing or exchanging two points x and y. Then the subgroup T is the set of elements fixing both x and y, and W has order 2 with its nontrivial element represented by anything exchanging x and y.

2. Conversely, if G has a BN-pair of rank 1, then the action of G on the cosets of B is 2-transitive. Hence BN-pairs of rank 1 are more or less the same as 2-transitive actions on sets with more than 2 elements.

3. The typical example of an algebraic group with a BN-pair is: suppose G is the general linear group $GL_n(K)$ over a field K. Take B to be the upper triangular matrices, T to be the diagonal matrices, and N to be matrices with exactly one non-zero element in each row and column. There are $(n-1)$ generators w_i's, represented by the matrices obtained by swapping two adjacent rows of a diagonal matrix.

4. More generally, any group of Lie type has a BN-pair.

As we mentioned before, one can associate a building to each group with a BN-pair (see e.g. [40] Thm. 3.2.6). Here, we focus on groups with a BN-pair of rank 2. Note that the building that one gets from a group with a BN-pair of rank 2 is a generalized n-gon.

The following proposition allows us to go from a BN-pair of rank 2 to a transitive action on the n-gons of the associated generalized n-gon.

Proposition 4.1.6. *([40] Thm. 3.2.6 and Thm. 3.11) For a generalized n-gon, an automorphism group acts transitively on ordered ordinary n-gons if and only if the automorphism group has a BN-pair.*

4.1.2.2 *Moufang condition*

Definition 4.1.7. A generalized n-gon $\Gamma = (P, L, I)$ satisfies the *Moufang condition* if for every path $\gamma = (x_0, \cdots, x_n)$ of length n, the pointwise stabilizer G_γ of the set $D(x_1) \cup \cdots \cup D(x_n)$ acts transitively on the set of ordinary n-gons containing γ.

By the classification of Moufang polygons due to Tits and Weiss [41] any Moufang polygon arises from the standard BN-pair of an essentially simple algebraic group.

4.2 VERY HOMOGENEOUS GENERALIZED n-GONS

In [2], Baldwin adapted Hrushovski's generic construction to build an almost strongly minimal non-Desarguesian projective plane. Later, in [8] Debonis and Nesin generalized Baldwin's method to construct $2^{\aleph_0}$-many almost strongly minimal generalized n-gons which do not interpret a group. However, their method was only suitable to construct generalized n-gons for odd n. In [36] Tent used a similar approach to construct very homogeneous generalized n-gons for all n with some stronger properties. She constructed the generalized n-gons for $n \geq 3$ for which the automorphism group acts transitively on the set of ordered ordinary $(n+1)$-gons. As we mentioned in the introduction by Proposition 4.1.6, this transitivity result implies that the automorphism group has a BN-pair. By Lemma 2.2.35 and Lemma 2.1.42 no group is interpretable in these class of generalized n-gons .

Here, we prove that in the both cases of the very homogeneous generalized n-gons constructed in [36], the automorphism group is a simple group. In the previous chapter we investigated the automorphism group of the ab-initio generic construction (the full class). Although there is very similar combinatorial behavior in the generic structures arising from the full class and sub class, the proofs should be modified.

4.2.1 *Set-up*

Let the first-order language $\mathfrak{L}$ contain a binary irreflexive symmetric relation $R(-,-)$ and a predicate $P(-)$. An $\mathfrak{L}$-structure can be considered as a graph, bipartite with respect to P. The pre-dimension function $\delta(-)$ (see Def. 2.2.11) considered in [36] to construct generalized n-gons for such $\mathfrak{L}$-structures A, is of the following form:

$$\delta(A) = (n-1) \cdot |A| - (n-2) \cdot |R(A)| .$$

Let **K** be the class of all finite graphs A, bipartite with respect to P, satisfying the following two conditions:

1. A contains no $2m$-cycle for $m < n$;
2. if $B \subseteq A$ contains a $2m$-cycle for $m > n$, then $\delta(B) \geq 2n+2$.

Recall that, using the pre-dimension function δ, one can define a self-sufficiency "$\leqslant$" as follows (see Def. 2.2.11): for $A, B \in \mathbf{K}$ with $A \subseteq B$

$$A \leqslant B \text{ if and only if } \delta(B') \geq \delta(A)$$

for all $A \subseteq B' \subseteq B$.

Theorem 4.2.1. *([36] Thm. 3.15) The class $(\mathbf{K}, \leqslant)$ is a smooth class and it has the amalgamation property. Hence the $(\mathbf{K}, \leqslant)$-generic structure exists and it is $\aleph_0$-stable.*

To obtain an almost strongly minimal generalized n-gons, one can use the collapsing method. In [36], Tent considers the following restriction on the elements of the class.

Consider a μ-function, finite-to-one, from the set of pairs (A, B) such that B is 0-minimally algebraic over A to the natural numbers satisfying following assumptions.

1. If (A, B) and (A', B') have the same isomorphism type, then $\mu(A, B) = \mu(A', B')$.

2. If $A = \{a_0, a_{n-1}\}$ and $B = \{a_1, \cdots, a_{n-2}\}$ with AB having edges $R(a_i, a_{i+1})$ for $0 \leq i < n-1$, then $\mu(A, B) = 1$; otherwise $\mu(A, B) \geq \max\{\delta(A), n\}$.

Let $\mathbf{K}^\mu \subset \mathbf{K}$, be the class of those elements of $C \in \mathbf{K}$ with the following property: if B is a 0-minimally algebraic set over A and $A, B \subset C$, then the number of copies of B over A inside C is less than or equal to $\mu(A, B)$. Then

Theorem 4.2.2. *([36] Thm. 4.4 and Thm. 4.6) The class $(\mathbf{K}^\mu, \leqslant)$ is a smooth class and it has the samalgamation property. Hence* the *$(\mathbf{K}^\mu, \leqslant)$-generic structure exists and it is almost strongly minimal.*

Before starting to modify the proofs that were given in the previous chapter, we review some basic facts about the construction which will be used in the proofs very often.

4.2.2 *Remarks on the transitivity properties of the structures*

Let Γ be either case of the very homogeneous generalized n-gons constructed in [36].

Remark 4.2.3. The following holds

1. Suppose $A \subseteq B$, and B is $\leqslant$-closed in Γ and $b \in B$. Then

$$\mathrm{d}(bA) = \mathrm{d}_B(bA) = \min\{\delta(B') : bA \subseteq B' \subseteq B\}.$$

2. For every finite set $B \subset \Gamma$, $\mathrm{cl}(B) \subset \mathrm{acl}(B)$ and $\mathrm{d}(B) = \mathrm{d}(\mathrm{acl}(B))$.

3. Suppose $a, b \in \Gamma$ and $d_\Gamma(a,b) < n$. By the axioms for a generalized n-gon, there is a unique shortest path from a to b, which we denote it by $\gamma_{a,b}$.

4. Let $\gamma_m = (x_0, x_1 \cdots, x_m)$ be a path of length m in Γ. Then

$$\delta(\gamma_m) = (n-1)\cdot(m+1) - (n-2)\cdot m = n + m - 1.$$

   Note that, this implies $\delta(\gamma_m) \leqslant 2n = \delta(Z_{2n})$ for $m \leq n+1$ where Z_{2n} is a cycle of length $2n$. Hence $\gamma_m \leqslant Z_{2n}$.

5. By the $\leqslant$-richness of Γ and the axioms for a generalized n-gon,

   a) for γ_{n+1} there is a unique Z_{2n} such that $\gamma_{n+1} \leqslant Z_{2n}$;

   b) for γ_{n+3} there is a unique Z_{2n+2} such that $\gamma_{n+3} \leqslant Z_{2n+2}$.

Corollary 4.2.4. *Let γ'_m and γ_m be two distinct paths of length m in the structure Γ such that $m \leqslant n+1$. Then, there is an automorphism of Γ which maps γ'_m to γ_m.*

Proof. By Remark 4.2.3, both γ'_m and γ_m are isomorphic and $\leqslant$-closed. Then, by the $\leqslant$-genericity of Γ there exists an automorphism of Γ, which maps γ'_m to γ_m. □

Definition 4.2.5. Let $\{s_0, s_1, s_2, s_3\}$ be a set of distinct elements (points and lines) such that if n is even, $d_\Gamma(s_i, s_j) = n$ for $i \neq j$ and if n is odd, then $d_\Gamma(s_0, s_3) = d_\Gamma(s_i, s_{i+1}) = n$ for $i = 0,1,2$ and $d_\Gamma(s_0, s_2) = d_\Gamma(s_1, s_3) = n-1$. We say that $\{s_1, s_2, s_3, s_4\}$ forms a *base* configuration.

Remark 4.2.6. If $S := \{s_0, s_1, s_2, s_3\} \subset \Gamma$ is a base configuration, then $\mathrm{d}(S) = 4\cdot(n-1)$.

We present the following lemma for the class $(\mathbf{K}^\mu, \leqslant)$. Note that $\mathbf{K}^\mu \subset \mathbf{K}$. Thus, if we prove that some construction is in $\mathbf{K}^\mu$, then it is automatically in $\mathbf{K}$.

Lemma 4.2.7. *There are infinitely many isomorphic types of 0-minimally algebraic sets over a base configuration.*

Proof. For $l \geq 2$, let $C_l = \left\{c_1, \cdots, c_{4l(n-2)}\right\}$ be a simple $4l\cdot(n-2)$-cycle i.e. $C_l \models \left[\bigwedge_{1\leq i<4l(n-2)} R(c_i, c_{i+1})\right] \wedge R\left(c_{4l(n-2)}, c_1\right)$ with the following configuration over S:

$$SC_l \models \bigwedge_{0\leq i<4l} R\left(c_{1+i(n-2)}, s_{i^*}\right);$$

where $i^* \equiv i \ (mod\ 4)$ with $i^* = \{0,1,2,3\}$. Then $\delta(C_l/S) = (n-1)\cdot(4l\cdot(n-2)) - (n-2)\cdot(4l\cdot(n-2)+4l) = 0$.

Claim. $\delta(D'/S) > 0$ for any proper subsets D' of C_l.

Proof ofthe claim. If $D' = C_l \setminus \{c_j\}$, then $\delta(D'/S) > 0$. Consider the sets:

$$L_m^{i,j} := \left\{c_{1+i(n-2)+j}, \cdots, c_{1+i(n-2)+j+m}\right\},$$

$$T^i := \left\{c_{1+i(n-2)}, \cdots, c_{1+(i+1)(n-2)}\right\},$$

$$S^{i,j,k} := \left\{c_{1+i(n-2)-j}, \cdots, c_{1+i(n-2)}, \cdots, c_{1+i(n-2)+k}\right\},$$

where $1 \leq j, k \leq n-3$, $0 \leq i < 4l$ and $0 \leq m \leq n-j-2$. If $D' = C_l \backslash L_m^{i,j}$ for some i and j with $m < n-2$, then as the valency of each point is at most two, we find that

$$\delta(D'/S) = 0 - m \cdot (n-1) + (n-2) \cdot (m+1) = n - 2 + m.$$

The length of m-chains are less than $n-2$; hence $\delta(D'/S)$ is at least 2. In the general case every proper subset D' of C_l is a disjoint union of its connected components, so $\delta(D'/S)$ is the summation of the pre-dimension of its connected components over S. We show that each connected component has the pre-dimension greater than 0 over S which establishes the case. The simplest connected components are of the form $L_m^{i,j}$ or T^i or $S^{i,j,k}$ for which

$$\delta\left(L_m^{i,j}/S\right) = n + m - 2;$$

$$\delta\left(T^i/S\right) = (n-1) \cdot (n-1) - (n-2) \cdot n = 1;$$

$$\delta\left(S^{i,j,k}/S\right) = (n-1) \cdot (j+k+1) - (n-2) \cdot (j+k+1) = j + k + 1.$$

Any connected component can be partitioned into finitely many simple connected components which are connected by an edge (since we assume they are disjoint and that the combination is connected so there is an edge which connects the two connected components). It is enough to show that any disjoint combination of $L_m^{i,j}$, T^i and $S^{i,j,k}$ has pre-dimension greater than 0 over S. Because the configuration is symmetric, it is enough to check it for some fixed $i > 1$. One can see that any possible combination of $L_m^{i,-}$ and $S^{i,-,-}$ are again of the form $S^{i,-,-}$. Any combination of $L_m^{i,-}$ and T^{i+1} can be considered as a combination of $S^{i,-,-}$ and $S^{i+1,-,-}$. Moreover, T^i and T^{i+1} can never be connected by an edge. Also, the possible combination of $S^{i,-,-}$ and T^{i+1} can be considered as a partition of $S^{i,-,-}$ and $S^{i+1,-,-}$. Thus, it is enough to check the pre-dimension of the combination of $S^{i,-,-}$ and $S^{i+1,-,-}$ over S which is the following:

$$\delta\left(S^{i,j,-} \cup S^{i+1,-,k}/S\right) = (n-1)(j+k+n-1) - (n-2)(k+j+n) = k + j + 1.$$

□

From above, we see that the pre-dimension of each connected component over S is greater 0. Hence $\delta(D'/S) > 0$. This implies that the C_l's are 0-minimally algebraic sets over S for all $l > 1$. One can easily check that there is no cycle of length less than $2n$ in C_l for all $l > 1$ and that the properties of the class $\mathbf{K}^{\mu}$ are satisfied. Hence, each $C_l \cup S$ is an element of the class $\mathbf{K}^{\mu}$ for $l \geq 2$. Moreover, it is clear that C_l and $C_{l'}$ are not isomorphic when $l \neq l'$. Hence, there are infinitely many isomorphic types of 0-minimally algebraic sets over a base configuration. □

Corollary 4.2.8. *Suppose* $A := \{s_0, s_1, s_2, s_3\}$ *forms a base configuration and that it is* $\leqslant$*-closed in* Γ *where* Γ *is either the* $(\mathbf{K}, \leqslant)$*-generic structure or* the $(\mathbf{K}^{\mu}, \leqslant)$*-generic structure. Suppose* $\mathrm{d}(b/A) = n-1$ *and* $d_\Gamma(b, s_i) = n$ *for some* $i = 0, 1, 2, 3$. *Then, there exists infinitely many isomorphism types of* 0*-minimally algebraic sets over* $\{s_0, s_1, s_2, s_3, b\}$.

Proof. Since $\mathrm{d}(b/A) = n-1$, $d_\Gamma(b, a) \geq n-1$ for all $a \in A$. Suppose n is odd. By our assumption, there is at least one s_i with $d_\Gamma(b, s_i) = n$. From the axioms of a generalized n-gon, it follows that there is no cycle of odd length. Thus, there are two possibilities: either

1. $d_\Gamma(b, s_i) = n$ for $i = 0, 2$ and $\mathrm{d}(b, s_i) = n-1$ for $i = 1, 3$, or
2. $d_\Gamma(b, s_i) = n$ for $i = 1, 3$ and $\mathrm{d}(b, s_i) = n-1$ for $i = 0, 2$.

Without loss of generality suppose (1) holds. For $l \geq 2$, let C_l be the same structure as it was considered in Lemma 4.2.7, a cycle of length $4l \cdot (n-2)$. Moreover, assume C_l has the same configuration over $\{s_0, \cdots, s_3\}$, except that we replace $R(s_1, c_{n-1})$ with $R(s_1, b)$. Since the number of relations between C_l and $\{s_0, s_1, s_2, s_3, b\}$ is exactly the same number of relations between C_l and $\{s_0, s_1, s_2, s_3\}$ in Lemma 4.2.7, we find that $\delta(C_l / \{s_0, s_1, s_2, s_3, b\}) = 0$, and C_l is 0-minimally algebraic over $\{s_0, s_1, s_2, s_3, b\}$.

Suppose n is even. Again, by our assumption, there is at least one s_i with $d_\Gamma(b, s_i) = n$. Then, the only possibility is $\mathrm{d}(s_i, b) = n$ for all $i = 0, 1, 2, 3$. For the case that n is odd, let C_l be a cycle of length $4l \cdot (n-2)$ and suppose it has the same configuration as it is described in Lemma 4.2.7 over $\{s_0, \cdots, s_3\}$, except that this time we replace $R(s_0, c_1)$ with $R(s_0, b)$. One can see C_l is 0-minimally algebraic over $\{s_0, s_1, s_2, s_3, b\}$.

□

Corollary 4.2.9. *Suppose that* $\{s_0, s_1, s_2, s_3\} \subset A$ *forms a base configuration and* A *is* $\leqslant$*-closed in* Γ. *Let* $b \notin A$, $\mathrm{d}(b/A) = n-1$, *and* $d_\Gamma(b, s_i) = n$ *for some* $i = 0, 1, 2, 3$. *Then there is a set* D, 0*-minimally algebraic over* $\{s_0, s_1, s_2, s_3, b\}$, *such that* $D \cap \mathrm{cl}(Ab) = \emptyset$, $|R(D, b)| = 1$, *and* $D \cup \mathrm{cl}(Ab) \in \mathbf{K}^{\mu}$.

Proof. Follows from Corollary 4.2.8 and Lemma 4.2.10. □

Corollary 4.2.10. *Let $A := \{s_0, s_1, s_2, s_3\}$ be a set of distinct points or lines which form a base configuration. Suppose $A_1 \supseteq A$ with A_1 finite and $\leqslant$-closed in Γ. Then there is a set D, 0-minimally algebraic over A, such that $D \cap A_1 = \emptyset$ and $D \cup A_1 \in \mathbf{K}^\mu$.*

Proof. By Lemma 4.2.7, there are infinitely many isomorphism types of 0-minimally algebraic sets over A. The set A_1 is finite and $\leqslant$-closed. According to Lemma 3.1.5, 0-algebraic sets have empty intersection and by Lemma 3.1.4, any 0-algebraic set over A which intersects with A_1, is contained in A_1. So there is a 0-minimally algebraic set D over A, which is not contained in A_1 and $D \cup A_1$ satisfies the axioms of $\mathbf{K}^\mu$. Hence $D \cup A_1 \in \mathbf{K}^\mu$ and $D \cap A_1 = \emptyset$. □

4.2.3 $\aleph_0$-*Stable n-gons*

Here, we modify the proofs in Section 3.2 for the $\aleph_0$-stable generalized n-gons constructed in [36].

First, we present some remarks about the properties of $\aleph_0$-stable generalized n-gons. The class $(\mathbf{K}, \leqslant)$ in this section is the class that we defined in 4.2.1. Let Γ be the $(\mathbf{K}, \leqslant)$-generic structure.

Lemma 4.2.11.

1. *For $A, B \in \mathbf{K}$, the following holds,*
$$\delta(AB) = \delta(A) + \delta(B) - \delta(A \cap B) - (n-2) \cdot |R(A \backslash B; B \backslash A)|.$$
2. $\mathrm{cl}(A) \leqslant acl(A) \leqslant \mathrm{gcl}(A) \leqslant \Gamma$ *for* $A \subseteq \Gamma$.
3. $\mathrm{d}(acl(A)) = \mathrm{d}(A)$.
4. $acl(\mathrm{cl}(A)) = dcl(\mathrm{cl}(A))$.
5. *If A, B are $\leqslant$-closed and* $\mathrm{d}(AB) = \mathrm{d}(A) + \mathrm{d}(B)$, *then* $\mathrm{gcl}(A) \cap \mathrm{gcl}(B) = \emptyset$.
6. $acl(\emptyset) = \mathrm{gcl}(\emptyset) = \emptyset$.

Proof. (1) follows from Definition 2.2.11. (2) and (3) follow from Remark 2.2.25. For (4) note that $\mathrm{dcl}(\mathrm{cl}(A)) \subseteq \mathrm{acl}(\mathrm{cl}(A))$ always hold. It remains to show that $\mathrm{acl}(\mathrm{cl}(A)) \subseteq \mathrm{dcl}(\mathrm{cl}(A))$. Note that by (2), it follows that $\mathrm{acl}(\mathrm{cl}(A)) \subset \mathrm{gcl}(A)$. Let B be a 0-minimally algebraic set over $A_0 \subset \mathrm{dcl}(\mathrm{cl}(A))$. Then, the result follows from the fact that the axioms for a generalized n-gon guarantees either B is unique or there are infinitely many copies of B over A_0. (5) follows from (6). For (6) note that the pre-dimension of a single element is $(n-1)$ and every single element is $\leqslant$-closed. □

4.2.3.1 *Modification of the observation*

Here, we modify observation 3.2.1 in Chapter 3.

Lemma 4.2.12. *Suppose B is v-minimally algebraic over A and $AB \in \mathbf{K}$. Then, there exists $B_1 \leqslant B_2 \in \mathbf{K}$ such that $AB \leqslant B_1$ and $\delta\left(B_1/\left(B_2\backslash B_1\right)A\right) = 0$.*

Proof. Let $S_1 := \left\{s_0^1, \cdots, s_3^1\right\}$ and $S_2 := \left\{s_0^2, \cdots, s_3^2\right\}$ be two disjoint sets such that the elements of each set form a base configuration. Let $b \in B$ and $m := 8\left(n-2\right)$. Similar to Lemma 4.2.7, let $C_2 := \{b\} \cup \{b_i' : i < m\}$ be a cycle of length $8 \cdot (n-2)$ such that C_2 is 0-minimally algebraic over $S := S_1 \otimes S_2$ and $|R\left(b; S\right)| \leq 1$. It is easy to see that $\delta\left(C/b\right) \geq 0$ for all $C \subseteq C_2$ (e.g. $\delta\left(C_2/b\right) = 8 \cdot (n-2) - (n-1) \geq 0$). Hence $AB \leqslant AB \otimes_{\{b\}} C_2$. One can easily check the axioms and see that $AB \otimes_{\{b\}} C_2 \in \mathbf{K}$. Let $B_1 := AB \otimes_{\{b\}} C_2$. Let B_2 be a new structure on $B_1 \otimes S$ with additional edges such that $R\left(B_1; S\right) = R\left(C_2; S\right)$. Note that $|R\left(s; B_1\right)| = 1$ for $s \in S$. Hence $B_1 \leqslant B_2 \in \mathbf{K}$ and $\delta\left(B_1/\left(B_2\backslash B_1\right)A\right) = 0$. □

Lemma 4.2.13. *Let $A \leqslant B \leqslant \Gamma$. Then, there exists $H_1 \leqslant H_2 \in \mathbf{K}$ such that $B \leqslant H_1$, $\delta\left(H_1/\left(\left(H_2\backslash H_1\right)A\right)\right) = 0$ and $\left(H_2\backslash H_1\right)A \leqslant H_2$.*

Proof. By Lemma 3.2.5, there exists a chain of $\leqslant$-closed subsets

$$A = A_0 \leqslant A_1 \leqslant \cdots \leqslant A_n = B$$

where $\delta\left(A_i/A_{i-1}\right) = 0$ if i is odd and $A_i\backslash A_{i-1}$ is d_i-algebraic over A_{i-1} if i is even. For each even i, using Lemma 4.2.12, consider B_1^i and B_2^i such that

1. $B_1^i \leqslant B_2^i \in \mathbf{K}$ and $A_i \leqslant B_1^i$, and
2. $\delta\left(B_1^i/\left(B_2^i\backslash B_1^i\right)A_{i-1}\right) = 0$.

Moreover we ask $E_i \cap E_j = \emptyset$ for $i \neq j$ where $E_k := B_2^k\backslash A_k$. Let $H_1 := \bigcup_i B_1^i$ and $H_2 := \bigcup_i B_2^i$. It is easy to see that $\delta\left(H_1/\left(\left(H_2\backslash H_1\right)A_0\right)\right) = 0$ and $\left(H_2\backslash H_1\right)A_0 \leqslant H_2$.

□

Lemma 4.2.14. *Let $A \leqslant B \leqslant \Gamma$. Let $H_1 \leqslant H_2 \leqslant \Gamma$ and $A = A_0 \leqslant \cdots \leqslant A_n = B$ be as given by Lemma 4.2.13. Suppose $H \leqslant \Gamma$ such that $H_0 := H \cap H_2 = \left(H_2\backslash H_1\right)A$ and let $f \in \mathrm{Aut}_A\left(\Gamma\right)$ be such that $f\left(H_2\backslash H_1\right) \cap \left(H_2\backslash H_1\right) = \emptyset$. Then $f\left(B\backslash A\right) \cap \left(B\backslash A\right) \subseteq A_1\backslash A_0$. Moreover if we ask $A_1 = A_0$, then $f\left(B\backslash A\right) \cap \left(B\backslash A\right) = \emptyset$.*

Proof. This follows from $f\left(H_2\backslash H_1\right) \cap \left(H_2\backslash H_1\right) = \emptyset$ and Lemmas 4.2.13 and 4.2.14. □

4.2.3.2 *Stationary independence on a countable family*

Recall that for $A \subset \Gamma$, we define $\mathrm{gcl}(A) := \{x \in \Gamma : \mathrm{d}(x/A) = 0\}$. Similar to Section 3.2.4, let $\mathfrak{Y} := \{\mathrm{gcl}(A) : A \subset \Gamma, A \text{ is finite }\}$ and let the bases of neighborhoods of the identity be

$$\{\mathrm{Aut}_X(\Gamma) : X \in \mathfrak{Y}\}.$$

Define $\mathop{\downarrow}^{d}$ as in Definition 3.2.11 define and $\mathop{\downarrow}^{\circ}$ as was defined $\mathop{\downarrow}$ in Definition 3.2.13. Note that unlike the uncollapsed ab-initio generic structures $\mathop{\downarrow}^{\circ}$ is not the forking independence relation. Now, a theorem analogue to Theorem 3.2.21 holds in this setting.

Theorem 4.2.15. *The relation $\mathop{\downarrow}^{d}$ is a stationary independence relation for elements of $\mathfrak{Y}$.*

4.2.3.3 *gcl-bounded automorphisms*

Recall that we say an automorphism β is gcl-bounded if there exists a finite set $A \subset \Gamma$ such that $\beta(m) \in \mathrm{gcl}(mA)$ for all $m \in \Gamma$.

Remark 4.2.16. Suppose β is a gcl-bounded automorphism and $A \subset_{<\omega} \Gamma$ such that $\beta(m) \in \mathrm{gcl}(mA)$ for all $m \in \Gamma$. Then

1. we can assume A is $\leqslant$-closed since by Lemma 4.2.11, $\mathrm{cl}(A) \subseteq \mathrm{gcl}(A)$;
2. we can assume $\mathrm{d}(A) > 4 \cdot (n-2)$; since $\mathrm{gcl}(mA) \subseteq \mathrm{gcl}(mB)$ for $B \supseteq A$ and $m \in \Gamma$.

Hence, from the remark above we can make the following assumption for the set A.

ASSUMPTION If n is even we assume A contains $\{s_0, s_1, s_2, s_3\}$ and $\{s'_0, s'_1, s'_2, s'_3\}$ as subsets such that each of them form a base configuration and $P(s_0) \Leftrightarrow \neg P(s'_0)$ (i.e. four points and four lines). If n is odd, A contains $\{s_0, s_1, s_2, s_3\}$ as a subset such that the s_i's form a base configuration.

Lemma 4.2.17. *Suppose $A \leqslant \Gamma$. Let $a \in A$ and $f \in \mathrm{Aut}(\Gamma)$ be such that $f(a) \neq a$ and $f(a) \in A$. Let $l \in \mathbb{N}$ such that $0 < l \leq n-2$. Then $f(x) \neq x$ for all $x \in \Gamma \backslash A$ with $d_\Gamma(x, a) = \mathrm{d}(x/A) = l$.*

Proof. Let $x \in \Gamma \backslash A$ and $d_\Gamma(x, a) = \mathrm{d}(x/A) = l \leq n-2$. Let $\gamma_{x,a}$ and $\gamma_{f(x),f(a)}$ be the shortest path connecting x to a and $f(x)$ to $f(a)$, respectively. If $f(x) = x$, then $\delta(x/A) = 2l - (n-1) < l$. This contradicts our assumption that $\mathrm{d}(x/A) = l$. Hence $f(x) \neq x$. □

Lemma 4.2.18. *Suppose $A \leqslant \Gamma$. Let $a \in A$ and $f \in \mathrm{Aut}(\Gamma)$ such that $f(a) \neq a$ and $f(a) \in A$. If $l \in \{n-1, n\}$, then there exists $x \in \Gamma$ such that $d_\Gamma(x, a) = l$, $\mathrm{d}(x/A) = n-1$ and $f(x) \neq x$.*

Proof. Let $x_0 \in \Gamma \backslash A$ be such that $\mathrm{d}(x_0/A) = d_\Gamma(x_0, a) = n-2$. By Lemma 4.2.17 $f(x_0) \neq x_0$. Let $A_1 := \mathrm{cl}(x_0 f(x_0) A)$. Pick $x_1 \in \Gamma$ such that $\mathrm{d}(x_1/A_1) = d_\Gamma(x_1, x_0) = 1$. Then by Lemma 4.2.17 $f(x_1) \neq x_1$ holds. This establishes the case $l = n-1$.

Let $A_2 := \mathrm{cl}(A_1 x_1 f(x_1))$ and pick $x_2 \in \Gamma$ such that $\mathrm{d}(x_2/A_2) = d_\Gamma(x_2, x_1) = 1$. Again, by Lemma 4.2.17 $f(x_2) \neq x_2$ holds. It is easy to see that $\mathrm{d}(x_2/A) = n-1$. This establishes the case $l = n$. Note that for the case of $n > 3$, in the first step, we could pick $x_1 \in \Gamma$ with $\mathrm{d}(x_1/A) = d_\Gamma(x_1, x_0) = 2$. □

Corollary 4.2.19. *Suppose $g \in \mathrm{Aut}(\Gamma)$ and $g \neq id_\Gamma$. Let X be finite and $\leqslant$-closed, and let $k \in \mathbb{N}$. Then, there exists a $\leqslant$-closed set Z containing X such that Z contains disjoint sets $S_1, \cdots, S_k$ such that*

1. *S_i form a base configuration for each i;*
2. *$g(s) \neq s$ for all $s \in \bigcup_i S_i$;*
3. *$\mathrm{d}(\bigcup_i S_i) = \sum_i \mathrm{d}(S_i)$;*
4. *$S_j \subsetneq g(\bigcup_i S_i)$ for $j \leq k$.*

Proof. Let $z_1 \in \Gamma \backslash X$ such that $g(z_1) \neq z_1$; note that existence of such an element follows from Lemma 4.2.17. Let $Z_1 := \mathrm{cl}(z_1 g(z_1) X)$. By applying Lemma 4.2.18, we may pick $z_2 \in \Gamma \backslash Z_1$ such that $d_\Gamma(z_2, Z) = n$, $\mathrm{d}(z_2/Z) = n-1$ and $g(z_2) \neq z_2$. Let $Z_2 := \mathrm{cl}(z_2 g(z_2) Z_1)$. Similarly we pick $z_3 \in \Gamma \backslash Z_2$ and $z_4 \in \Gamma \backslash \mathrm{cl}(Z_2 z_3 g(z_3))$ such that z_3 and z_4 satisfy the distance conditions of the base configuration for $\{z_1, z_2, z_3, z_4\}$ with $g(z_3) \neq z_3$ and $g(z_4) \neq z_4$. Let $S_1 := \{z_1, z_2, z_3, z_4\}$. Inductively, using Lemma 4.2.18, we can continue the procedure to find k-many disjoint sets S_i with the base configuration as desired. □

Lemma 4.2.20. *Suppose g is not a gcl-bounded automorphism. Let $X, Y \in \mathbf{K}$ and $X \leqslant Y$. Moreover assume $\delta(Y_0/X) > 0$ for all $X \subsetneq Y_0 \subseteq Y$. Then, there exists $Y' \subset \Gamma$ such that $tp(Y'/X) = tp(Y/X)$ and $\mathrm{d}(Y'/X) = \mathrm{d}(Y'/Xg(Y'))$.*

Proof. We find $Y' \subset \Gamma$ in four steps.

Step 1 Using Lemma 3.2.5, there is a $\leqslant$-chain $X = X_0 \leqslant X_1 \ldots \leqslant X_n = Y$ such that $\delta(X_i/X_{i-1}) = 0$ if i is odd and $X_i \backslash X_{i-1}$ is d_i-algebraic over X_{i-1} if i is even, where $\sum_i d_i = m$. Note that $X_0 = X_1$ by our assumptions. By Lemma 4.2.13, there is $H_1 \leqslant H_2 \in \mathbf{K}$ such that $X_n \leqslant H_1$, $\delta(H_2/((H_2 \backslash H_1) X_0)) = 0$

and $(H_2 \backslash X_n) X_0 \leqslant H_2$. Suppose $|H_2 \backslash H_1| = m$ and enumerate $H_2 \backslash H_1$ as $\{h_i : 1 \leq i \leq m\}$. Note that from the way that H_2 and H_1 are constructed in Lemma 4.2.13, we have $\frac{m}{4} \in \mathbb{N}$.

Step 2 Since g is not gcl-bounded, using Corollary 4.2.19, we can find a finite $\leqslant$-closed set $Z \subset \Gamma$ which contains $X_1 := \mathrm{cl}(X \cup g(X))$ and Z contains disjoint sets $S_1, \cdots, S_{\frac{m}{4}}$ that they satisfy properties (1) to (4) of Corollary 4.2.19. Enumerate the elements of $\bigcup_i S_i$ as $\{z_1, \cdots, z_m\}$. Note that $g(z_i) \neq z_i$ for all $0 < i \leq m$.

Step 3 One can check that $H_2 \otimes_X Z \in \mathbf{K}$. Consider an $\mathfrak{L}$-structure $A \in \mathbf{K}$ obtained from $H_2 \otimes_X Z$ by identifying each h_i with z_i for all $1 \leq i \leq m$. More precisely the domain of A is $H_1 \cup Z$ and, further we ask the following;

1. $A \models R(y_1, y_2)$ if and only if $H_1 \models R(y_1, y_2)$ for $y_1, y_2 \in H_1$;
2. $A \models R(x_1, x_2)$ if and only if $Z \models R(x_1, x_2)$ for $x_1, x_2 \in Z$;
3. $A \models \bigwedge_i R(x_i, z_i)$ if and only if $H_2 \models \bigwedge_i R(x_i, h_i)$ for all $x_i \in H_1$ and $z_i \in Z \backslash X$.

Note that $\delta(A/Z) = 0$, $Y \leqslant A$ and $Z \leqslant A$.

Step 4 Let $Z_1 := \mathrm{cl}(Z \cup g(Z))$. One can check that $A \otimes_Z Z_1 \in \mathbf{K}$. By the $\leqslant$-richness of Γ, we can embed a $\leqslant$-closed copy of $A \otimes_Z Z_1$ over Z_1 in Γ, which we also denote by A, such that $g(A \backslash Z) \cap Z_1 = \emptyset$. Then $A \mathop{\smile\!\!\!\!\!\!\mid}^{\circ}_{Z} Z_1$ holds. Let $Y' \subset A$ be such that $\mathrm{tp}(Y'/X) = \mathrm{tp}(Y/X)$.

Claim. $Y' \mathop{\smile\!\!\!\!\!\!\mid}^{\circ}_{X} g(X)$.

Proof. Both X and Y' are $\leqslant$-closed subsets of Γ, further Y' and $g(X)$ are free-amalgam over X, and $\mathrm{cl}(Y'X_1) = Y'X_1$. □

Claim. $Y' \mathop{\smile\!\!\!\!\!\!\mid}^{\circ}_{X \cup g(X)} g(Y')$.

Proof. Since $g(z_i) \neq z_i$ holds for the specified z_i's, one can see $g(Y') \neq Y'$. Moreover, by the Lemma 4.2.14 $g(Y') \cap Y' = \emptyset$ holds. The sets Y' and $g(Y')$ are free-amalgam over Z_1. Hence they are free-amalgam over X_1; otherwise it implies $|R(Y' \backslash Z_1; g(Y') \backslash Z_1)| \geq 1$ and then $\delta(Y'g(Y')/Z_1) < 0$ which is impossible because of the fact that Z_1 is a $\leqslant$-closed set. Using the properties of the construction in step 2, it is easy to check that $\mathrm{cl}(Y'X_1) = Y'X_1$ and $\mathrm{cl}(g(Y')X_1) = g(Y')X_1$. Then $Y' \mathop{\smile\!\!\!\!\!\!\mid}^{\circ}_{X \cup g(X)} g(Y')$ follows. □

One can use the transitivity property of $\mathop{\smile\!\!\!\!\!\!\mid}^{d}$ to establish the result, but we prove the result without using that. Since $Y' \mathop{\smile\!\!\!\!\!\!\mid}^{\circ}_{X} g(X)$, and $Y' \mathop{\smile\!\!\!\!\!\!\mid}^{\circ}_{X \cup g(X)} g(Y')$, $\mathrm{d}(Y'/X)) = \mathrm{d}(Y'/(Xg(X)))$ and $\mathrm{d}(Y'/(Xg(X))) = \mathrm{d}(Y'/(Xg(X)g(Y')))$.

Thus $\mathrm{d}(Y'/X) = \mathrm{d}(Y'/(Xg(X)g(Y')))$, so $\mathrm{d}(Y'/X) = \mathrm{d}(Y'/Xg(Y'))$ as desired.

□

Now Lemma 4.2.20 implies the same corollary as Corollary 3.2.28.

Corollary 4.2.21. *Every non gcl-bounded automorphism of* Γ *moves almost maximally for the* $\downarrow^{d}$*-independence relation on the family* $\mathfrak{Y}$.

Proof. The proof is the same as for Corollary 3.2.28. □

4.2.3.4 *gcl-bounded automorphisms*

Similar to the previous chapter, we want to prove that there is no non-trivial gcl-bounded automorphism in the automorphism group of an $\aleph_0$-stable generalized n-gons. The proof that we present here is a modification of Theorem 3.1.1. Also we have to modify Lemmas 3.1.8 and Lemma 3.1.9. Note that exactly the same proof works for Theorem 4.2.28 which is given in Section 4.2.4.

Lemma 4.2.22. *Suppose* β *is a non-trivial bounded automorphism over the set* A*. If* $A \leqslant Ab \leqslant \Gamma$ *and* $\mathrm{d}(b/A) = n-1$*, then* b *is fixed by* β*.*

Proof. Note that from $\mathrm{d}(b/A) = n-1$ it follows that $b \notin \mathrm{gcl}(A)$. Suppose $b \neq \beta(b)$. Then by the definition of bounded automorphism $\beta(b) \in \mathrm{gcl}(bA)$ and

$$\mathrm{d}(b\beta(b)A) = \mathrm{d}(bA) = \delta(\mathrm{cl}(bA)) = \delta(bA) = \delta(A) + n - 1.$$

Let $B := \mathrm{cl}(Ab\beta(Ab))$. Since $\mathrm{d}(A\beta(A)) = \mathrm{d}(A)$, we have that $\mathrm{d}(B) = \delta(B) = \delta(A) + n - 1$. Let $B_0 := B \backslash A$. Then

$$\delta(B/A) = \delta(B_0/A) = \delta(B_0) - (n-2) \cdot |R(B_0; A)| = n - 1,$$

and hence

$$\delta(B_0) - (n-2) \cdot |R(B_0; A)| = n - 1.$$

Since $\mathrm{d}(b/A) = n-1$, $d_\Gamma(b,a) \geq n-1$ for all $a \in A$. By our assumption, A contains enough elements for our purpose; namely A contains s_i for $0 \leq i \leq 3$ which form a base configuration. Let $C := B \cup D$ such that D is 0-algebraic over B and D is 0-minimally algebraic over $\{s_0, s_1, s_2, s_3, b\}$. Moreover we ask that $|R(b, D)| = 1$ and $\beta(D) \cap B = \emptyset$.
Since B is finite and contains finitely many 0-minimally algebraic sets over $\{s_0, s_1, s_2, s_3, b\}$, by Corollary 4.2.7 there exist such D and $C \in \mathbf{K}$. Since $\beta(b) \neq b$ and $\beta(D)$ is 0-algebraic over B, $D \neq \beta(D)$. By Lemma 3.1.5, D and $\beta(D)$ are

disjoint sets. It is clear that $B \leqslant C$, and by the $\leqslant$-richness of Γ, we can find an isomorphic copy of C over B in Γ which is $\leqslant$-closed. One can see that $D \subset \mathrm{gcl}(B)$ since $\delta(D/B) = 0$. Moreover, from the fact that $DAb \leqslant \Gamma$ it follows that:

1. $D \nsubseteq \mathrm{cl}(A)$ and $D \nsubseteq \mathrm{gcl}(A)$ since $\mathrm{d}(D/A) = \delta(D/A) = n-2$;
2. $b \notin \mathrm{cl}(DA)$ and $b \notin \mathrm{gcl}(DA)$ since $\mathrm{d}(b/DA) = \delta(b/DA) = 1$.

By Lemma 3.1.4, we have $\beta(D) \cap D = \emptyset$, hence

$$\delta(B_0/D\beta(D)A) = \delta(B_0) - (n-2) \cdot |R(B_0, D\beta(D)A)|.$$

Now by our assumption $|R(B_0, D\beta(D))| = 2$, so $\delta(B_0/D\beta(D)A) \leq 0$. This means that $B_0 \subset \mathrm{gcl}(D\beta(D)A) = \mathrm{gcl}(DA)$. Thus $b \in \mathrm{gcl}(DA)$ and $\mathrm{d}(b/DA) = 0$, which contradicts (2). Hence b is a fixed point. □

Lemma 4.2.23. *Suppose $A \leqslant Ab \leqslant \Gamma$, $\mathrm{d}(b/A) = n-1$ and $e \in \Gamma$ such that $R(b,e)$ holds. Then e is fixed by β.*

Proof. Two cases could happen: either
CASE ONE: $\mathrm{d}(e/A) = n-1$. Then Ae is $\leqslant$-closed, and the result holds from Lemma 3.1.8.
CASE TWO: $\mathrm{d}(e/A) < n-1$. Let $E = \mathrm{cl}(eA)$. Then

$$\delta(A) < \delta(E) = \mathrm{d}(E) < \delta(Eb) = \mathrm{d}(bE),$$

so $b \notin \mathrm{gcl}(e,A)$. If $\beta(e) \neq e$, then $|R(b; e\beta(e)A)| = 2$ and $b \in \mathrm{gcl}(e\beta(e)A)$. It follows that $\mathrm{d}(b/\mathrm{cl}(e\beta(e)A)) \leq 0$ and it is a contradiction. So e is a fixed point of β. □

Theorem 4.2.24. *There is no non-trivial gcl-bounded automorphism in the $(\mathbf{K}, \leqslant)$-generic structure Γ.*

Proof. Let β be a non-trivial bounded automorphism and A be a $\leqslant$-closed subset of Γ such that $\beta(m) \in \mathrm{gcl}(mA)$ for all $m \in \Gamma$. Note that by Lemma 4.2.23, every $b \in \Gamma$ with $\mathrm{d}(b/A) = n-1$ is a fixed point of β. First we prove that $\mathrm{gcl}(A)$ is fixed pointwise. Consider $d \in \mathrm{gcl}(A)$ and $D := \mathrm{cl}(dA)$. Let $E \supset D$ be such that $E = D \cup \{e_1, \cdots, e_{n-1}\}$ with

$$E \models R(d, e_1) \wedge \left[\bigwedge_{1<i\leq n-1} R(e_{i-1}, e_i)\right].$$

Note that $|R(D; e)| = 1$ and $|R(D; \{e_i : 1 < i \leq n-1\})| = 0$. It is clear that $D \leqslant E$ and $\delta(e_1/D) = \delta\left(e_i/\left(\{e_j : 1 \leq j < i\} \cup D\right)\right) = 1$ for every $1 < i \leq n-1$. Using the $\leqslant$-richness of Γ, we can find an isomorphic copy of E in Γ over D which is $\leqslant$-closed. Since $\mathrm{d}(e_{n-1}/D) = \mathrm{d}(e_{n-1}/A) = n-1$, we find that e_{n-1} is a fixed point of β. Then by Lemma 4.2.23, e_{n-2} is also a fixed point.

Claim. All e_i's for $i = 1, \cdots, n-3$ are fixed points of β.

Proof of the claim. Suppose e_i is a fixed point with the minimum index $1 \leq i \leq n-2$ such that for all $j \geq i$, e_j is a fixed point. If $i = 1$, then we are done. Suppose $i > 1$ and e_{i-1} is not a fixed point then $|R(e_i; e_{i-1}\beta(e_{i-1}))| = 2$ and hence

$$\delta(e_i / e_{i-1}\beta(e_{i-1})A) \leq 0.$$

This implies that $e_i \in \mathrm{gcl}(e_{i-1}\beta(e_{i-1})A)$ and $\mathrm{d}(e_i/e_{i-1}A) \leq \delta(e_i/e_{i-1}\beta(e_{i-1})A) < 1$ which is a contradiction. Thus the claim follows. □

If $\beta(d) \neq d$, then $|R(e_1; D\beta(D))| = 2$ and $\delta(e_1/D\beta(D)) < 1$, which is a contradiction. This proves that every $d \in \mathrm{gcl}(A)$ is a fixed point of β.

Now let $c \in \Gamma$ such that $\mathrm{d}(c/A) > 0$ and $\mathrm{d}(c/A) = m \neq n-1$. Similar to the previous case, build a set $E = \mathrm{cl}(cA) \cup \{e_i : 0 < i \leq n-m-1\}$ such that

$$E \models R(c; e_1) \wedge \left[\bigwedge_{1 < i \leq n-m-1} R(e_{i-1}; e_i) \right],$$

$|R(\mathrm{cl}(cA); e_1)| = 1$, and $|R(cl(cA); \{e_i : 1 < i \leq n-m-1\})| = 0$. By the same argument as above, $E \in \mathbf{K}$, and we can find an isomorphic copy of E in Γ such that $\mathrm{cl}(Ac) \leqslant E \leqslant \Gamma$. Since $\mathrm{d}(e_{n-m-1}/A) = n-1$ then e_{n-m-1} is a fixed point. By Lemma 4.2.23, e_{n-m-2} is also fixed by β. Similar to the proof of the claim, we can show that c is a fixed point too.

□

4.2.3.5 *Bounded simplicity of the $\aleph_0$-stable n-gons*

Corollary 4.2.25. *Let Γ be the $\leqslant$-generic structure. Then* $\mathrm{Aut}(\Gamma)$ *is boundedly simple.*

Proof. First of all $\mathrm{Autf}(\Gamma) = \mathrm{Aut}(\Gamma)$ since $Th(\Gamma)$ admits weak elimination of imaginaries (see Cor. 2.1.4), $\mathrm{gcl}(\emptyset) = \emptyset$ and "$\mathrm{dcl}(-) = \mathrm{acl}(-)$" (by Lemma 4.2.11 part (4)). By Theorem 3.2.21, $\downarrow^d$ is a stationary independence relation for the countable family $\mathfrak{Y}$. By Theorem 4.2.24 every non-trivial automorphism of Γ is not gcl-bounded. Then, Corollary 4.2.21 implies that all automorphisms of Γ move almost maximally with respect to $\downarrow^d$. By Lemma 4.2.11, $\mathrm{gcl}(\emptyset) = \emptyset$. Finally from Theorem 3.2.30, we conclude that $\mathrm{Aut}(\Gamma)$ is boundedly simple. □

4.2.4 *The almost strongly minimal generalized n-gons*

In this section, we prove the simplicity of the automorphism group of the almost strongly minimal generalized n-gons constructed in [36]. Recall that to obtain a finite Morley rank generalized n-gon by the Hrushovski method one needs to consider a finite-to-one function for 0-minimally algebraic pairs in order to restrict the class of finite structures. Note that one can obtain bounded simplicity in the collapsed case by having the result that there is no non-trivial bounded automorphism in this case and modifying the previous result of simplicity of the automorphism group of the $\aleph_0$-stable generalized n-gon. However, here we give a more straightforward proof which is more algebraic and which requires only little background in model theory to understand. The main result that we need is that there is no non-trivial bounded automorphism in the automorphism group of the $(\mathbf{K}^{\mu}, \leqslant)$-generic structure. Then, following Lascar's approach and applying Theorem 2.3.11, we obtain the simplicity of the automorphism group of the $(\mathbf{K}^{\mu}, \leqslant)$-generic structures. This part is going to be published in Journal of Algebra [11].

Again, we denote by Γ the $(\mathbf{K}^{\mu}, \leqslant)$-generic structure. First, we present a few lemmas and remarks in this case. Recall that $\gamma_m = (x_0, x_1 \cdots, x_m)$ denotes a path of length m in Γ.

Lemma 4.2.26. *Let Γ be the $(\mathbf{K}^{\mu}, \leqslant)$-generic structure. Then $acl(\gamma_m) = \mathrm{cl}(\gamma_m)$ for $m \leq n$.*

Proof. From Remark 4.2.3

$$\delta(\gamma_m) \leq 2n - 1.$$

Hence, it is clear that $\mathrm{cl}(\gamma_m) = \gamma_m$. We want to show that there is no 0-algebraic set X over $\mathrm{cl}(\gamma_m)$. Suppose X is a set such that $X \cap \mathrm{cl}(\gamma_m) = \emptyset$ and $X \cup \mathrm{cl}(\gamma_m)$ contains a $2n$-cycle. Then by the axioms of the construction, since $2n$-cycles are $\leqslant$-closed, $\delta(X\mathrm{cl}(\gamma_m))$ is at least $2n$. Then, $\delta(\mathrm{cl}(\gamma_m)) < \delta(X\mathrm{cl}(\gamma_m))$ so X can not be a 0-algebraic set over $\mathrm{cl}(\gamma_m)$. Now we claim that if X is a 0-minimally algebraic over $\mathrm{cl}(\gamma_m)$, one can always find a $2n$-cycle in $X \cup \mathrm{cl}(\gamma_m)$ which is a contradiction. To see that, first observe that $|R(X; \mathrm{cl}(\gamma_m))| > 1$. Otherwise $\delta(X) - (n-2) \cdot |R(X; \mathrm{cl}(\gamma_m))| = \delta(X/\mathrm{cl}(\gamma_m)) = 0$ implies that $\delta(X) = n - 2$ which is impossible. Second, note that by Remark 3.1.3 part (4), X is connected and therefore from $|R(X; \mathrm{cl}(\gamma_m))| > 1$ it follows that X can not be 0-minimally algebraic over a single element. If X is 0-minimally algebraic over at least two distinct element of γ_m, $X \cup \mathrm{cl}(\gamma_m)$ contains a cycle of length at least $2n$, since γ_m is connected. This establishes the result. □

Remark 4.2.27. Suppose A is an ordinary n-gon (see Def. 4.1.3) and $B := A \cup \{b\}$ with $\delta(B/A) = 1$. Then, B is a $\leqslant$-closed set.

Next theorem, is the main step in proving the simplicity of the automorphism group of the almost strongly minimal generalized n-gons and it is similar to Theorem 3.1.1 and Theorem 4.2.24.

Theorem 4.2.28. *Let Γ be the finite Morley rank $(\mathbf{K}^{\mu}, \leqslant)$-generic generalized n-gon constructed in [36]. Then, there is no non-trivial bounded automorphism in* $\mathrm{Aut}(\Gamma)$*.*

Proof. The proof is exactly the same as the proof of Theorem 4.2.24, if one substitutes "gcl-bounded automorphisms" with "bounded automorphisms" and "gcl" with "acl". Note that these substitutions are not accidental. Roughly, the intuition behind this is that in the collapsing procedure $\mathrm{gcl}(-)$ and $\mathrm{acl}(-)$ coincide (see Thm. 2.2.36). □

It is known that any generalized n-gon is in the definable closure of a point row or a line pencil with finitely many elements (see e.g.[37]). Moreover a point row or a line pencil in the very homogeneous generalized n-gons are strongly minimal sets ([36] Thm. 4.6). Hence the following theorem holds.

Theorem 4.2.29. *([36] Thm. 4.6) Let Γ be the $(\mathbf{K}^{\mu}, \leqslant)$-generic generalized n-gon. Then Γ is almost strongly minimal.*

Here, for our purpose we show the following.

Theorem 4.2.30. *Let $k, l_1, l_2, l_3 \in \Gamma$ such that $d_\Gamma(k, l_2) = d_\Gamma(k, l_3) = n$ and (l_2, l_1, l_3) is a path of length 2. Then, every $m \in \Gamma$ is in the algebraic closure of $D(k)$ and l_1, l_2, l_3.*

Proof. Let $\gamma_{k,l_1} := (k = s_0, s_1, \cdots, s_{n-2}, s_{n-1} = l_1)$ be the shortest path between k and l_1. First we claim that all elements m with $d_\Gamma(m, l_2) = 1$ or $d_\Gamma(m, l_3) = 1$ are in the definable closure of $D(k)$ and l_1, l_2, l_3. Without loss of generality, suppose $d_\Gamma(m, l_2) = 1$ and $m \neq l_1$. Then $(k, s_1, \cdots, s_{n-2}, l_1, l_2, m)$ is a path of length $n+1$ and by the axioms of a generalized n-gon, there is a unique $(n-1)$-path $\gamma_{m,k}$ between m and k. Let $a \in D(k) \cap \gamma_{m,k}$. One can see that $(a, k, s_1, \cdots, s_{n-2}, l_1, l_2)$ is path of length $n+1$ and hence $m \in \mathrm{dcl}(a, l_1, l_2)$. This establishes the first case.

In the next step we claim that all elements m with $d_\Gamma(m, k) = 2$ are in the algebraic closure of $D(k)$ and l_1, l_2, l_3. There are two possibilities in this case: either $d_\Gamma(m, s_1) = 1$ or $d_\Gamma(m, s_1) = 3$. Pick $e_2 \in D(l_2) \setminus \{l_1\}$. First assume $d_\Gamma(m, s_1) = 3$ and let $\gamma_{m,k} = (m, a, k)$ be the shortest path between m and k. Now $(m, a, k, s_1, \cdots, s_{n-2}, l_1, l_2, e_2)$ is a $(n+3)$-path, and by Remark 4.2.3, there is a unique $(n-1)$-path $\gamma_{m,e_2} := (m, x_0, \cdots, x_{n-3}, e_2)$ between m and e_2. Moreover $(x_0, m, a, k, s_1, \cdots, s_{n-2}, l_1, l_3)$ is a $(n+3)$-path, and again by Remark 4.2.3, there is a unique $(n-1)$-path $\gamma_{x_0,l_3} := (x_0, y_0, \cdots, y_{n-3}, l_3)$ between x_0 and l_3. It is easy to check that $d_\Gamma(x_0, e_2) = d_\Gamma(x_0, y_{n-3}) = n-2$ and $d_\Gamma(x_0, a) = 2$. Then $B := \{x_0, \cdots, x_{n-3}, y_0, \cdots y_{n-4}, m\}$ is 0-minimally algebraic over $A := \{e_2, y_{n-3}, a\}$. By the axioms of the class $\mathbf{K}^{\mu}$ the number of isomorphic copies of B over A is bounded by $\mu(A, B)$. Hence B is in the algebraic closure of A. By the previous case we know e_2 and y_{n-3} are in the definable closure of $D(k)$ and l_1, l_2, l_3. Hence B is in the algebraic closure of $D(k)$ and l_1, l_2, l_3. Now suppose $d_\Gamma(m, s_1) = 1$. Then $(m, s_1, \cdots, s_{n-2}, l_1, l_2, e_2)$ is a $(n+1)$-path. Let $\gamma_{m,e_2} = (m, z_0, \cdots, z_{n-3}, e_2)$ be the shortest path between m and e_2. Moreover $(z_0, m, s_1, \cdots, s_{n-2}, l_1, l_3)$ is a path of length $n+1$. Let $\gamma_{z_0,l_3} := (z_0, u_0, \cdots, u_{n-3}, l_3)$ be the shortest path between z_0 and l_3. Similar to the previous case one can see that $\{z_0, \cdots, z_{n-3}, u_0, \cdots, u_{n-4}, m\}$ is 0-minimally algebraic over $\{e_2, u_{n-3}, s_1\}$. Hence this establishes the case.

From the two previous cases by induction on the distance of an elements to l_2, l_3 and then k we show that every element is in the algebraic closure of $D(k)$ and l_1, l_2, l_3. Without loss of generality suppose $m \in \Gamma$ with $d_\Gamma(m, l_2) = 2$ and $m \neq s_{n-2}$. Then either that $d_\Gamma(m, l_1) = 1$ or $d_\Gamma(m, l_1) = 3$. Pick $a \in D(k) \setminus \{z_0\}$. First suppose $d_\Gamma(m, l_1) = 1$. It follows that $(a, k, s_1, \cdots, s_{n-2}, l_1, m)$ is a path of length $n+1$. Again by Remark 4.2.3, there is a unique shortest path $\gamma_{a,m}$ between a, m. Let $e \in D(a) \cap \gamma_{a,m}$. Then m is uniquely determined by e and l_1. It is clear that $d_\Gamma(e, k) = 2$ and from the previous cases we know that e is in the algebraic closure of $D(k)$ and l_1, l_2, l_3. Hence m is in the algebraic closure of $D(k)$ and l_1, l_2, l_3. Suppose $d_\Gamma(m, l_1) = 3$ and let $\gamma_{l_1,m} := (l_1, l_2, x, m)$. It follows that $(a, k, s_1, \cdots, s_{n-2}, l_1, l_2, x, m)$ is a path of length $n+3$. Again by Remark 4.2.3, there is a unique shortest path $\gamma_{a,m}$ between a, m. Let $e \in D(a) \cap \gamma_{a,m}$. Then m is uniquely determined by e and x. This establishes the case. Similarly we can continue by induction on the distance to k, l_2 and l_3, and prove that all elements are in the algebraic closure of $D(k)$ and l_1, l_2, l_3. □

Lemma 4.2.31. *Let* $k, l_1, l_2, l_3 \in \Gamma$ *be such that they satisfy the assumptions of Theorem* 4.2.30. *We denote by* $\gamma_{k,l_1} := (k = s_0, s_1, \cdots, s_{n-2}, s_{n-1} = l_1)$ *the shortest path between* k *and* l_1 *in* Γ. *Then,* $\mathrm{cl}(kl_1l_2l_3) = \mathrm{dcl}(kl_1l_2l_3) = \mathrm{acl}(kl_1l_2l_3) = S$ *where* $S = \{s_0, s_1, \cdots, s_{n-1}, s_n = l_2, s_{n+1} = l_3\}$.

Proof. It is easy to see that $\delta(S) = 2n$ and $\delta(S') \leq 2n-1$ for all proper subsets of $S' \subset S$. By Lemma 4.2.26 follows that there is no 0-minimally algebraic over S' for all proper subsets of $S' \subset S$. Then, it is enough to check that there is no 0-minimally algebraic set over S. Suppose B is a 0-minimally algebraic set over S. By Remark 3.1.3, B is connected. Similar to the arguments of Lemma 4.2.26, since B is 0-minimally algebraic over S, there is $\Delta_1 \subset B \cup S$ a $2n$-cycle that contains s_0 and s_n. Similarly let $\Delta_2 \subset B \cup S$ be a $2n$-cycle that contains s_0 and s_{n+1}. There are two possibilities: either $|\Delta_1 \cap \Delta_2| = n$ or $|\Delta_1 \cap \Delta_2| = n-1$. If $|\Delta_1 \cap \Delta_2| = n$, then one can see that $\delta(\Delta_1 \cap \Delta_2 / S) = 1$ and moreover $\Delta_1 \cup \Delta_2$ is a $\leqslant$-closed set which implies that $\delta(BS) \geq 2n+1$. If $|\Delta_1 \cap \Delta_2| = n-1$, then one can find a $(2n+2)$-cycle in $\Delta_1 \cup \Delta_2$ which by the axioms is $\leqslant$-closed and hence $\delta(BS) \geq 2n+2$. So no 0-minimally algebraic set over S is possible. □

Expand the language $\mathfrak{L}$ to $\mathfrak{L}_S = \mathfrak{L} \cup \mathcal{S}$ by adding constants $\mathcal{S} = \{s_0, \cdots, s_{n+1}\}$ with $s_0, s_{n-1}, s_n, s_{n+1}$ satisfying the assumptions of Theorem 4.2.30. By adding these constants to the language $\mathfrak{L}$, the structure Γ in the new language is in the algebraic closure of a strongly minimal $\emptyset$-definable set. By Remark 4.2.16 adding finite number of constants to $\mathfrak{L}$ does not change the fact of that there is no non-trivial bounded automorphisms; more precisely, a bounded automorphism over a finite set in the extended language $\mathfrak{L}_S$ is also a bounded automorphism in $\mathfrak{L}$. Hence, Γ satisfies the assumptions of Theorem 2.3.11 in the expanded language $\mathfrak{L}_s$.

Moreover, it is clear that $\mathrm{Aut}^{\mathfrak{L}_S}(\Gamma)$ is a closed subgroup of $\mathrm{Autf}^{\mathfrak{L}}(\Gamma)$ (in the pointwise convergence topology). By Theorem 2.2.37, $Th(\Gamma)$ admits weak elimination of imaginaries. Then by Corollary 2.1.4, $\mathrm{Autf}_S(\Gamma) = \mathrm{Aut}_{\mathrm{acl}(s)}(\Gamma)$. By Lemma 4.2.31, $\mathrm{acl}(S) = S$, and by Theorem 2.3.11, we conclude that $\mathrm{Aut}^{\mathfrak{L}_S}(\Gamma)$, which isomorphic to $\mathrm{Aut}_S^{\mathfrak{L}}(\Gamma)$, is a simple group. Now we want to deduce the simplicity of $\mathrm{Aut}(\Gamma)$ from the simplicity of $\mathrm{Aut}_S(\Gamma)$.

We fix the following notation: $S_0 = \emptyset$ and $S_i := \cup_{0 \le i} \{s_i\}$ for $1 \le i \le n+1$.

Lemma 4.2.32. $\mathrm{Aut}_{S_i}(\Gamma)$ *is a maximal subgroup of* $\mathrm{Aut}_{S_{i-1}}(\Gamma)$ *for* $1 \le i \le n+1$.

Proof. We are using the homogeneity properties of Γ, in order to prove this lemma. Let

$$\Omega = \{x \in \Gamma : x \models \mathrm{tp}(s_i/S_{i-1})\},$$

and let $g \neq id_\Gamma$ be an arbitrary automorphism in $\mathrm{Aut}_{S_{i-1}}(\Gamma) \setminus \mathrm{Aut}_{S_i}(\Gamma)$. We claim that $\langle g, \mathrm{Aut}_{S_i}(\Gamma)\rangle = \mathrm{Aut}_{S_{i-1}}(\Gamma)$. Suppose $b \in \Omega$ such that $g(s_i) = b$ and let $h \in \mathrm{Aut}_{S_{i-1}}(\Gamma) \setminus \mathrm{Aut}_{S_i}(\Gamma)$. Let $a = h(s_i)$. Note that $\delta(S_i) = (n-1)+i$ for $1 \le i \le n+1$. By the similar arguments to Lemma 4.2.31, S_i, aS_i, and bS_i are $\leqslant$-closed. Consider $c \in \Omega$ be such that c is d-independent from a, b and s_i over S_{i-1}; note that it is always possible to find such c since $\mathrm{d}(s_i/S_{i-1}) = 1$. From this it follows that cS_i is also $\leqslant$-closed. Now let $h_1, h_2 \in \mathrm{Aut}_{S_i}(\Gamma)$ such that $h_1(a) = h_2(b) = c$. It is easy to check that $g^{-1} \circ h_2^{-1} \circ h_1 \circ h$ fixes s_i, so $g^{-1} \circ h_2^{-1} \circ h_1 \circ h \in \mathrm{Aut}_{S_i}(\Gamma)$. This establishes the result. □

Theorem 4.2.33. Let Γ be the almost strongly minimal generalized n-gon constructed in [36]. Then $\mathrm{Aut}(\Gamma)$ *is a simple group.*

Proof. We prove this theorem inductively. It is clear that

$$\mathrm{Aut}_{S_{n+1}}(\Gamma) \le \mathrm{Aut}_{S_n}(\Gamma) \le \cdots \le \mathrm{Aut}_{S_1}(\Gamma) \le \mathrm{Aut}(\Gamma).$$

For the first step, we already know that $\mathrm{Aut}_{S_{n+1}}(\Gamma)$ is a simple group. Now assume $\mathrm{Aut}_{S_i}(\Gamma)$ is a simple group and we want to conclude that $\mathrm{Aut}_{S_{i-1}}(\Gamma)$ is also a simple group. Let $N \neq 1$ be a normal subgroup of $\mathrm{Aut}_{S_{i-1}}(\Gamma)$. Then, either $\mathrm{Aut}_{S_i}(\Gamma) \leq N$ or $N \cap \mathrm{Aut}_{S_i}(\Gamma) = 1$. By Lemma 4.2.32, $\mathrm{Aut}_{S_i}(\Gamma)$ is a maximal subgroup of $\mathrm{Aut}_{S_{i-1}}(\Gamma)$. Moreover, it is clear that $\mathrm{Aut}_{S_i}(\Gamma)$ is not normal in $\mathrm{Aut}_{S_{i-1}}(\Gamma)$. To establish the result, it is enough to prove that $N \cap \mathrm{Aut}_{S_i}(\Gamma) \neq 1$.

Claim. Suppose $N \cap \mathrm{Aut}_{S_i}(\Gamma) = 1$. Then N acts regularly on the realizations of the type $\mathrm{tp}(s_i/S_{i-1})$.

Proof. Let $a, b \models \mathrm{tp}(s_i/S_{i-1})$ and $a \neq b$ and let $g_1, g_2 \in N$ be such that $g_1 \neq g_2$ and $g_1(a) = g_2(a) = b$. Then $1 \neq g_1 g_2^{-1} \in N$ and $g_1 g_2^{-1}$ fixes b. We know that both S_i and bS_{i-1} are $\leqslant$-closed and isomorphic. Now let $h \in \mathrm{Aut}_{S_{i-1}}(\Gamma)$ such that $h(s_i) = b$. Then $1 \neq h^{-1} g_1 g_2^{-1} h \in \mathrm{Aut}_{S_i}(\Gamma) \cap N$ holds which is a contradiction. □

Now we show that in our case N can not act regularly. Let k_1, k_2 be two distinct elements of N and let $a := k_1(s_i)$ and $b := k_2(s_i)$. Let c be a realization of $\mathrm{tp}(s_i/S_{i-1})$ which is d-independent from a, b and s_i over S_i. Consider $h_1, h_2 \in \mathrm{Aut}_{S_i}(\Gamma)$ such that $h_1(k_1(s_i)) = h_2(k_2(s_i)) = c$. Now $h_1 k_1^{-1} h_1^{-1}(c) = h_2 k_2^{-1} h_2^{-1}(c) = s_i$ and $h_1 k_1^{-1} h_1^{-1} \neq h_2 k_2^{-1} h_2^{-1}$. Since N is a normal subgroup, $h_1 k_1^{-1} h_1^{-1}, h_2 k_2^{-1} h_2^{-1} \in N$, but this contradicts the regularity. Hence $N \cap \mathrm{Aut}_{S_i}(\Gamma) \neq 1$, so $N = \mathrm{Aut}_{S_i}(\Gamma)$. This implies that $\mathrm{Aut}_{S_{i-1}}(\Gamma)$ is a simple group. Inductively we conclude $\mathrm{Aut}(\Gamma)$ is simple.

□

Corollary 4.2.34. *There are simple groups with a spherical BN-pair of rank 2 which are non-split and hence not of algebraic origin.*

Proof. Let Γ be the almost strongly minimal generalized n-gon constructed in [36]. By Theorem 4.2.33, $\mathrm{Aut}(\Gamma)$ is a simple group. As we have mentioned before, $\mathrm{Aut}(\Gamma)$ is acting transitively on ordered n-gons and hence it has a BN-pair. That these BN-pairs are not of algebraic origin follows from the fact that the classification of Moufang polygons by Tits and Weiss [41] implies that in the algebraic case no point stabilizer Aut_x acts 6-transitively on $D(x)$. See [11] for more explanation. □

5 MISCELLANEOUS

5.1 SMALL INDEX PROPERTY

As we mentioned before, the automorphism group of a countable structure with the pointwise convergence topology is a Polish group. In this section, we present some observations about the small index property and then present an improvement of Lascar's result about countable saturated almost strongly minimal structures. Finally we conclude that the automorphism group of the almost strongly minimal generalized n-gons constructed in [36] has the "almost small index property". Here we mainly follow [5].

Here are some facts about Polish groups.

Fact 5.1.1. *Suppose H is a subgroup of G and G is a Polish group. Then*

1. *The closure $\bar{H}$ of H is a subgroup of G and $\bar{H}$ is normal in G if and only if H is.*
2. *If H is open, then H is also closed.*
3. *If H is closed and of finite index, then H is open.*
4. *If $K \leq H$ for some open K, then H is open.*

Moreover we have the following fact.

Fact 5.1.2. *Suppose G is a Polish group, then every open subgroup H has small index (i.e. $|G : H| < 2^{\aleph_0}$).*

The above fact motivates the following definition.

Definition 5.1.3. A group G has the *small index property* (SIP) if every group of small index (i.e. index less than $2^{\aleph_0}$) is open.

Example 5.1.4. Here are some structures such that the automorphism group has the small index property:

1. An infinite set without structure.
2. Then countable dense linear ordering $(\mathbb{Q}, <)$.
3. A vector space of dimension ω over a finite or countable division ring.

4. Countable $\aleph_0$-stable $\aleph_0$-categorical structures.

5. The random graph.

We prove the following by assuming the Continuum Hypothesis (CH). Note that CH implies that small index means $\leq \aleph_0$.

Theorem 5.1.5. *(with CH) Suppose G is a Polish group and H is a closed subgroup of G with a small index. Then if H has SIP in the induced topology, then G has SIP.*

Theorem 5.1.6. *Suppose that M is a countable saturated structure and it is almost strongly minimal. Suppose G is a subgroup of a countable index in* $\mathrm{Aut}(M)$. *Then there exists a finite set A in M such that* $\mathrm{Autf}_A(M) \subseteq G$.

This implies the following.

Corollary 5.1.7. *The automorphism group of the almost strongly minimal n-gon constructed in [36] has "almost SIP".*

By the term "almost SIP", we mean the same kind of result as Lascar's theorem 5.1.13 and Theorem 5.1.6. Namely for a subgroup H of a small index in $\mathrm{Aut}(M)$, there is a finite set $A \subset M$ such that $\mathrm{Autf}_A(M) \subseteq H$.

We use the following fact quite often.

Fact 5.1.8. *Suppose H, K are subgroups of G. Then $|H : H \cap K| \leq |G : K|$.*

Fact 5.1.9.

1. *A subgroup H of a Polish group is Polish (in the induced topology) if and only if H is G_δ if and only if H is closed.*

2. *If H is a closed subgroup of G, then index of H in G is either $\leq \aleph_0$ or $2^{\aleph_0}$.*

3. *For a closed subgroup H of G, $|G : H| \leq \aleph_0$ if and only if H is open.*

Proof. The facts above can be found in any standard book of descriptive set theory. For example see page 5 of [5]. □

Remark 5.1.10. The automorphism group of every countable structure is a closed subgroup of the symmetric group of the underlying set. Conversely, one can associate a first order structure to every closed subgroup of the symmetric group of a countable set such that the automorphism group of the structure is exactly the group that we started with.

Fact 5.1.11. *Suppose G is a Polish group and H is a closed subgroup of small index in G. Then if G has SIP then H has SIP, too (with the induced topology from G).*

Proof. It is clear that H is a Polish group (this follows from Fact. 5.1.9). Let K be a subgroup of countable index in H. Then K has small index in G. Since G has SIP, K is open in G. Then $H \cap K = K$ is open in H with respect to the induced topology. Hence H has SIP. □

Proof of Theorem. 5.1.5. Suppose K is a subgroup of small index in G. By Fact 5.1.8, the index $|H : K \cap H|$ is also small. Since H has SIP, one can see $K \cap H$ is open in H. By Fact 5.1.9 part (4), H is open in G. Hence $K \cap H$ is open in G. Then from Fact 5.1.1 part (4), it follows that K is open in G. So the result follows. □

The following corollary is an easy consequence of Theorem 5.1.5 but irrelevant to our topic.

Corollary 5.1.12. *Any small index closed subgroup of the automorphism group of the random graph has SIP. In other words, any closed subgroup without SIP has index $2^{\aleph_0}$.*

Here is Lascar's result of "almost SIP" in [24].

Theorem 5.1.13. *(Lascar [24]) Suppose M is a countable saturated structure and assume $D(v)$ is a strongly minimal formula (without parameters) such that M is in the algebraic closure of $D(M)$. Suppose G is a subgroup of a countable index in* $\mathrm{Aut}(M)$. *Then there exists a finite set A in M such that* $\mathrm{Autf}_A(M) \subseteq G$.

The proof that Lascar gives in [24] can be modified for the case that we prove here, but the following proof is simple and it is an application of Corollary 5.1.12.

Proof of Theorem. 5.1.6. We claim that a similar result holds when we allow $D(v)$ to have parameters. Suppose $D(v)$ have parameters $\{s_1, \cdots, s_n\}$. Without loss of generality we assume that $D(v)$ has one parameter $\{s\}$. Let $\mathfrak{L}^s = \mathfrak{L} \cup \{s\}$. Let M^s be the structure M in $\mathfrak{L}^s$. Since M is saturated, M^s is also saturated and M^s is in the algebraic closure of $D(v)$ which is without parameter in $\mathfrak{L}^s$.

Let G be a group of countable index in $\mathrm{Aut}^{\mathfrak{L}}(M)$. One can see that $\left|\mathrm{Aut}^{\mathfrak{L}}(M) : \mathrm{Aut}^{\mathfrak{L}}_s(M)\right| \leq \aleph_0$ holds. Then by Fact 5.1.8, $|\mathrm{Aut}_s(M) : G \cap \mathrm{Aut}_s(M)| \leq \aleph_0$. By Theorem 5.1.13, $\mathrm{Aut}^{\mathfrak{L}^s}(M^s)$ has "almost SIP". Hence there exists a finite set $A \subset M^s$ such that $\mathrm{Autf}^{\mathfrak{L}^s}_A(M^s) \subset G^s \cap \mathrm{Aut}^{\mathfrak{L}^s}(M^s)$. Using that $\mathrm{Aut}^{\mathfrak{L}}_s(M)$ is homeomorphic to $\mathrm{Aut}^{\mathfrak{L}^s}(M^s)$ we conclude that that $\mathrm{Autf}_{\{s\} \cup A} \subset \mathrm{Autf}_A(M) \cap \mathrm{Aut}_s(M)$ and hence $\mathrm{Autf}_{\{s\} \cup A} \subseteq G$. So the "almost SIP" holds. □

5.1.0.1 *Extension property*

In [16], Hrushovski, proved that the class of all finite graphs has the extension property. His result has been generalized by Herwig in [13] to a broader class of structures. We can deduce the small index property of the automorphism group of a saturated structure when there is an ample generic automorphism in the automorphism group (for more details, see [25, 21]). The extension property for different structures have been used to prove that the ample generic automorphisms exist (see e.g.,[15]).

It has been asked if the uncollapsed/collapsed ab-initio class has EP. We give a negative answer to this. Let the language $\mathfrak{L}$ consists of a binary relation R which is irreflexive and symmetric. Consider the following pre-dimension function for an $\mathfrak{L}$-structure A, is of the following form:

$$\delta(A) = 2|A| - |R(A)|.$$

The extension property can be defined just in the terms of a class of structures and one can easily check that $\mathbb{K} \in \{\mathbb{K}_0, \mathbb{K}_0^{\mu}\}$ does not have EP. However, the following definition seems to be the right notion of the extension property in this setting, we later prove that $\mathbb{K} \in \{\mathbb{K}_0, \mathbb{K}_0^{\mu}\}$ with this refinement still does not hold EP.

Definition 5.1.14. Let $(\mathbf{K}, \leqslant)$ be a smooth class with AP. Let M be the $(\mathbf{K}, \leqslant)$-generic structure. The class $(\mathbf{K}, \leqslant)$ has the *extension property* (EP) if for all $A \in \mathbf{K}$ and $f_1, \cdots, f_n$ arbitrary partial isomorphisms between $\leqslant$-closed subsets of A, there is $B \in \mathbf{K}$ and $\theta_1, \cdots, \theta_n \in \operatorname{Aut}(B)$ such that $A \leqslant B$ and θ_i's are extending the f_i's.

Here we claim that $\mathbb{K} \in \{\mathbb{K}_0, \mathbb{K}_0^{\mu}\}$ do not have EP.
Let $B := \{b_1, b_2, b_3, p_1, p_2\}$ such that

$$B \models \bigwedge_{1 \leq i \neq j \leq 3} R(b_i, b_j) \wedge \bigwedge_{1 \leq i \leq 3} R(b_i, p_1) \wedge R(b_1, p_2) \wedge R(b_2, p_2).$$

From the structure of B, $\delta(B) = 2$ and $\delta(p_1) = \delta(p_2) = 2$. Then $p_i \leqslant B$ for $i = 1, 2$. Suppose f is a partial isomorphism which sends p_1 to p_2. We claim that for any $B \subset B'$, and any extension f' of f to an automorphism of B', we have $B \not\leqslant B'$. Then either

- Case (1) f' is fixing b_1, b_2, b_3. Then $R(p_1, b_3)$ implies $R(f(p_1) = p_2, b_3)$ which is a contradiction; since B is $\leqslant$-closed and a new edge between elements of B decreases the pre-dimension.

- Case (2) f' is fixing b_i, b_j for some $1 \leq i \neq j \leq 3$. Enumerate the set of b_i's as $\{b_{i_1}, b_{i_2}, b_{i_3}\}$ such that the element b_{i_1} be the one which is not fixed. From the construction, we know that $|R(b_{i_1}; p_1 b_{i_2} b_{i_3})| = 3$ holds. Then $|R(f'(b_{i_1}); p_2 b_{i_2} b_{i_3})| = 3$ which implies that $\delta(f'(b_{i_1})/B) < 0$. But this contradicts the $\leqslant$-closedness of B.

- Case (3) f' is fixing only one of b_1, b_2, b_3. Similar to Case (2) assume b_{i_1} and b_{i_2} are not a fixed point. We know that $\delta(b_{i_1} b_{i_2}) = 3$ and $|R(b_{i_1} b_{i_2}; p_1 b_{i_3})| = 4$. Then $\delta(f'(b_{i_1}) f'(b_{i_2})) = 3$ and $|R(f'(b_{i_1}) f'(b_{i_2}); p_2 b_{i_3})| = 4$. Note that if $f'(b_{i_1}) f'(b_{i_2}) \subset B$, then we are in case (1). If $|f'(b_{i_1}) f'(b_{i_2}) \cap B| = 1$ then we are in case (2). If $|f'(b_{i_1}) f'(b_{i_2}) \cap B| = \emptyset$ then it is easy to see that $\delta(f'(b_{i_1}) f'(b_{i_2})/B) < 0$. Again, this contradicts the $\leqslant$-closedness of B.

- Case (4) f' is not fixing any of the b_1, b_2, b_3 and the orbit of p_1 under f' is finite. Suppose $n \in \mathbb{N}$ such that $f'^n(p_1) = p_1$ and write $p_{i+1} = f'^i(p_1)$ and $b_j^{i+1} = f'^i(b_j)$ for $1 \leq i < n$. Then $\delta\left(\bigcup_{i,j} b_j^i p_i\right) \leq 1$. This contradicts the fact that $\delta(p_1) = 2$ and p_1 is $\leqslant$-closed.

- Case (5) f' is not fixing any of the b_1, b_2, b_3 and the orbit of p_1 under f' is infinite. Then we never end up with a finite set.

Hence EP fails. We have chosen the coefficients $\kappa = 2$ and $\lambda = 1$ for simplicity. One can build similar constructions and use the same arguments for and $\kappa > \lambda > 0$ and deduce the following.

Corollary 5.1.15. *The ab-initio classes in the binary relational language in both, the collapsed and the uncollapsed cases arising from a pre-dimension function $\delta_{\kappa,\lambda}$ with coefficients $\kappa > \lambda > 0$ do not have EP.*

Remark 5.1.16. From Corollary above we see that even with the refinement of the definition the ab-initio classes do not have EP. This suggests that we might choose a different class of to see EP.

Moreover, we claim that the "almost SIP" is the right way of looking at automorphism groups of the ab-initio generic structures. Since the algebraic closure of finite sets are finite in the uncollapsed generic structures, hence, with the pointwise convergence topology, may be "SIP" is not very likely.

Question A: One reasonable question in this case is to see if the ab-initio generic structures in the collapsed and uncollapsed case have "almost SIP". Almost SIP in the uncollapsed case would mean the following: suppose G is a subgroup of countable index in $\mathrm{Aut}(M)$. Then there exists a finite set A in M such that $\mathrm{Aut}_{\mathrm{gcl}(A)}(M) \subseteq G$.

5.2 FURTHER QUESTIONS

Remark 5.2.1. We mentioned before that the notion of the pre-dimension and dimension appear in the theory of Matroids. Here we especially mean to consider infinite Matroids. One can use many of the techniques that we mentioned and determine the automorphism group of some specific Matroids. However it might be reasonable to answer the question of simplicity in the more general setting of Matroids.

QUESTION B. What can we say about the simplicity of an ab-initio generic structure that is obtained from a pre-dimension function with irrational coefficients? As we have mentioned before this case has been studied in [3]. From Theorem 3.2.33, we know that there are no non-trivial bounded automorphisms in its automorphism group. One reasonable guess would be that the strong automorphism group of these structures are simple. However our machinery here is not adequate to answer this question.

QUESTION C. Except the trivial cases, as we have seen in Theorem 3.1.11 $\leqslant$-generic structures does not have any non-trivial bounded automorphism. Moreover, the Hrushoski's original example is CM-trivial without any non-trivial bounded automorphism. We can ask if this is coincidental? Namely, are there strongly minimal CM-trivial but not one-based structures with non-trivial bounded automorphisms? Another variant of the question would be: is the bounded automorphism group in the higher ampleness hierarchy trivial? However, we know that non-trivial bounded automorphisms exists in the algebraically closed fields of characteristic p: namely the Frobenius automorphisms are bounded. Moreover, it is not known if there are strongly minimal sets which are $(n-1)$-ample but not n-ample for $n > 2$. These two questions are phrased in the strongly minimal sets and one can ask the same questions in a broader setting.

Let $(\mathbf{K}, \leqslant)$ be a smooth class with AP and let M be the $(\mathbf{K}, \leqslant)$-generic structure. The following definition helps us to define a property called *algebraic closure property* (AC) for a smooth class. In this dissertation, we only dealt with smooth classes with AC.

Definition 5.2.2. Let M be an $\mathfrak{L}$-structure $A \subset M$ and $A \subseteq B \in \mathbf{K}$. Then

1. by a copy of B over A in M we mean the image of an embedding of B over A into M.

2. $\chi_M(B; A)$ is the number of distinct copies of B over A in M.

Definition 5.2.3. We say $(\mathbf{K}, \leqslant)$ has the *algebraic closure* property (AC) if there is a function $\eta : \omega \times \omega \to \omega$ such that for any $A \leqslant B$ and $A \subset M$, we have $\chi_M(B; A) \leq \eta(|A|, |B|)$.

Remark 5.2.4. It is clear that $(\mathbf{K}, \leqslant)$ has AC if and only if $\mathrm{cl}(A) \subseteq \mathrm{acl}(A)$ for any $A \subset M$.

QUESTION D. As we have mentioned before, all the cases that we considered in Chapter 3 and 4 have AC. One can define a different self-sufficiency notion from a pre-dimension function with AC (see e.g., [10]). One reasonable question would be to verify the bounded automorphism group in this case. However smooth classes without AC are also interesting to consider, for example the smooth classes in [33, 34]. Their generic structure mainly lay in a broader framework, namely in simple theories. Some examples of these classes are obtained by using the pre-dimension function but with a different closedness relation. One could ask if we can generalize the methods here and answer the simplicity question in that setting, too. Of course with the same approach one would ask about the bounded automorphisms and it remains open to verify them.

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