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Multiplier algebras of $C_0(X)$-algebras

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Abstract. If a C^*-algebra A is a $C_0(X)$-algebra then the multiplier algebra $M(A)$ is a $C(\beta X)$-algebra in a canonical way. In the case where A is σ-unital, we give necessary and sufficient conditions on A and X for $M(A)$ to be a continuous $C(\beta X)$-algebra.

Introduction

Let A be a C*-algebra which is a $C_0(X)$-algebra over a locally compact Hausdorff space X. Then the multiplier algebra $M(A)$ may be regarded in a natural way as a $C(\beta X)$-algebra over βX, the Stone-Čech compactification of X. The purpose of this paper is to characterize, for A σ-unital, when $M(A)$ is a continuous $C(\beta X)$-algebra. An elementary necessary condition is that the $C_0(X)$-algebra A should be continuous. The additional conditions for the characterization involve the interplay between the *base map* $\phi : \mathrm{Prim}(A) \to X$ (where $\mathrm{Prim}(A)$ is the primitive ideal space of A with the hull-kernel topology), the *structure map* $\mu : C_0(X) \to ZM(A)$ (where $ZM(A)$ is the center of $M(A)$), and the topology of X (Theorem 3.8). In the special case where A is separable and the base map ϕ is surjective it follows that $M(A)$ is a continuous $C(\beta X)$-algebra if and only if X is a disjoint union $X = U \cup D$ where $U := \{x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x\}$ is clopen and D is a discrete set (see Corollary 3.9). The maps ϕ and μ, and the ideals J_x of A, are described in the next section.

The structure of the paper is as follows. In the first section we collect some general results about $C_0(X)$-algebras. The second and third sections work gradually towards the main result. The fourth section gives applications to various classes of $C_0(X)$-algebras. For example, it is shown that if A is a stable, σ-unital C*-algebra with $\mathrm{Prim}(A)$ Hausdorff then $M(A)$ is a continuous $C(\beta X)$-algebra if and only if $X = \mathrm{Prim}(A)$ is basically disconnected.

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1. Preliminaries on $C_0(X)$-algebras

Let A be a C*-algebra and X a locally compact Hausdorff space. Then A is a $C_0(X)$-algebra if there is a *-homomorphism $\mu : C_0(X) \to ZM(A)$ such that $\mu(C_0(X))A$ is norm-dense in A.

The map μ is called the structure map. If X is compact then μ is necessarily unital, and in this case it is usual to speak of a "$C(X)$-algebra" rather than a "$C_0(X)$-algebra". An equivalent definition is that A is a $C_0(X)$-algebra if there is a continuous map $\phi : \mathrm{Prim}(A) \to X$ [31, Prop. C.5], [3, Prop. 4.1]. The map ϕ is called the base map.

The maps μ and ϕ uniquely determine each other as follows. Let $\theta_A : C^b(\mathrm{Prim}(A)) \to ZM(A)$ be the Dauns-Hofmann *-isomorphism. This has the property that

$$(\theta_A(f)a) + P = f(P)(a + P) \qquad (f \in C^b(\mathrm{Prim}(A)),\ a \in A,\ P \in \mathrm{Prim}(A))$$

or, equivalently, $\theta_A(f) - f(P)1 \in \tilde{P}$ (where $\tilde{P}$ is the ideal of $M(A)$ defined prior to Proposition 1.1 below). Then μ and ϕ are related by the equation $\mu(f) = \theta_A(f \circ \phi)$ for all $f \in C_0(X)$ [31, Prop. C.5]. Strictly, a $C_0(X)$-algebra is a triple (A, X, μ) (or (A, X, ϕ)), but we generally find it less cumbersome, and more in accord with common usage, to say that A is a $C_0(X)$-algebra with respect to μ or ϕ. Elementary examples show that it is not enough to state the space X and that one must specify μ or ϕ as well; and furthermore we shall see that the answer to our main question depends not only on X but also on μ and ϕ.

The definition of a $C_0(X)$-algebra was introduced by Kasparov [24] as the culmination of work by Fell [18], Tomiyama [30], Dauns and Hofmann [11], Lee [25], and others over the previous three decades. An account of the somewhat tangled history can be found in [31]. Other useful references are [15], [16], [10], [27], [17], [22], and [3].

For $x \in X$, let $J_x = \mu\{f \in C_0(X) \mid f(x) = 0\}A$, a norm-closed two-sided ideal of A by the Cohen factorization theorem (see [14, Thm. 16.1]). For $a \in A$ and $x \in X$, we often write $a_x = a + J_x \in A/J_x$. Then

$$(\mu(f)a)_x = f(x)a_x \qquad (a \in A,\ f \in C_0(X),\ x \in X).$$

This observation will be strengthened in Proposition 1.2 below. For $x \in X$ and $P \in \mathrm{Prim}(A)$, $J_x \subseteq P$ if and only if $\phi(P) = x$. Indeed, $J_x \subseteq P$ if and only if $f(\phi(P))(a+P) = (\theta_A(f\circ\phi)a)+P = 0$ for all $a \in A$ and all $f \in C_0(X)$ such that $f(x) = 0$. The latter holds if and only if $f(\phi(P)) = 0$ for all such f, that is, if and only if $\phi(P) = x$. It follows that $J_x = A$ if and only if $x \notin \mathrm{Im}(\phi)$. Note, too, that $\bigcap_{x\in X} J_x \subseteq \bigcap_{P\in\mathrm{Prim}(A)} P = \{0\}$.

For each $a \in A$, the norm function $x \to \|a_x\|$ $(x \in X)$ is upper semi-continuous [31, Prop. C.10]. The $C_0(X)$-algebra A is said to be *continuous* if, for all $a \in A$, the norm function $x \to \|a_x\|$ $(x \in X)$ is continuous. By Lee's theorem this happens if and only if the mapping $\phi : \mathrm{Prim}(A) \to X$ is open [31, Prop. C.10 and Thm. C.26]. In particular, if A is a continuous $C_0(X)$-algebra then $\mathrm{Im}(\phi)$ is open in X, and if $\mathrm{Prim}(A)$ is also compact then $\mathrm{Im}(\phi)$ is clopen in X.

Note that the question of whether A is a continuous $C_0(X)$-algebra depends crucially on the base map ϕ. For example, let A be a continuous $C_0(X)$-algebra where X has a nonisolated point x_0. Define a new map $\psi : \mathrm{Prim}(A) \to X$ by $\psi(P) = x_0$ ($P \in \mathrm{Prim}(A)$). Then (A, X, ψ) is a noncontinuous $C_0(X)$-algebra because ψ is not open. On the other hand, if A is a noncontinuous $C_0(X)$-algebra where X has an isolated point x_0 then, with ψ as above, (A, X, ψ) is a continuous $C_0(X)$-algebra.

Our next step is to show that if A is a $C_0(X)$-algebra with structure map μ then μ has a unique extension $\overline{\mu}$ such that $M(A)$ is a $C(\beta X)$-algebra with structure map $\overline{\mu}$. First, however, it is convenient to collect some elementary facts about the strict closure in $M(A)$ of an ideal J in A.

Let J be a proper, closed, two-sided ideal of a C^*-algebra A. The quotient map $q_J : A \to A/J$ has a canonical extension $\widetilde{q_J} : M(A) \to M(A/J)$ such that, for all $b \in M(A)$ and $a \in A$,

$$\widetilde{q_J}(b)(a + J) = ba + J \text{ and } (a + J)\widetilde{q_J}(b) = ab + J.$$

We define a proper, closed, two-sided ideal $\tilde{J}$ of $M(A)$ by

$$\tilde{J} = \ker \widetilde{q_J} = \{b \in M(A) \mid ba, ab \in J \text{ for all } a \in A\}.$$

Proposition 1.1. *Let J be a proper, closed, two-sided ideal of a C^*-algebra A. Then*

(i) *$\tilde{J}$ is the strict closure of J in $M(A)$;*
(ii) *$\tilde{J} \cap A = J$;*
(iii) *if $P \in \mathrm{Prim}(A)$ then $\tilde{P}$ is primitive (and hence is the unique ideal in $\mathrm{Prim}(M(A))$ whose intersection with A is P);*
(iv) *$\tilde{J} = \bigcap\{\tilde{P} \mid P \in \mathrm{Prim}(A)$ and $P \supseteq J\}$ and for all $b \in M(A)$*

$$\|b + \tilde{J}\| = \sup\{\|b + \tilde{P}\| \mid P \in \mathrm{Prim}(A) \text{ and } P \supseteq J\};$$

(v) *$(A + \tilde{J})/\tilde{J}$ is an essential ideal in $M(A)/\tilde{J}$.*

Proof. Let (u_λ) be an approximate identity for A.

(i) Suppose that (b_α) is a net in J which is strictly convergent to some $b \in M(A)$. Let $a \in A$. Then ba is the norm-limit of $(b_\alpha a)$ and hence belongs to J. Similarly, $ab \in J$ and so $b \in \tilde{J}$.

Conversely, suppose that $b \in \tilde{J}$. Then $bu_\lambda \in J$ for all λ. For $a \in A$, we have

$$\|bu_\lambda a - ba\| \leq \|b\|\|u_\lambda a - a\| \to 0$$

and $\|(ab)u_\lambda - ab\| \to 0$, and so $bu_\lambda \to b$ strictly.

(ii) Let $b \in \tilde{J} \cap A$. Then $bu_\lambda \in J$ for all λ and so $b \in J$. Thus $\tilde{J} \cap A \subseteq J$ and the reverse inclusion is clear.

(iii) Let $\pi : A \to B(H)$ be an irreducible representation of A with kernel P and let $\tilde{\pi} : M(A) \to B(H)$ be the canonical extension to an irreducible representation of $M(A)$. It suffices to show that $\tilde{P} = \ker \tilde{\pi}$. Let $b \in \tilde{P}$ and

$a \in A$. Then $ba \in P$ and so $0 = \pi(ba) = \tilde{\pi}(b)\pi(a)$. Since π is nondegenerate, $\tilde{\pi}(b) = 0$.

Conversely, suppose that $b \in \ker \tilde{\pi}$ and $a \in A$. Then

$$\pi(ab) = \pi(a)\tilde{\pi}(b) = 0 = \pi(ba)$$

and so $ab, ba \in \ker \pi = P$. Hence $b \in \tilde{P}$.

(iv) Let $b \in M(A)$. Then $b \in \tilde{J}$ if and only if $ab, ba \in P$ for all $a \in A$ and all primitive ideals $P \supseteq J$. The latter holds if and only if $b \in \tilde{P}$ for all primitive ideals $P \supseteq J$. Since the canonical *-homomorphism of $M(A)/\tilde{J}_x$ into $\Pi_{P \supseteq J_x} M(A)/\tilde{P}$ is injective, it is isometric.

(v) Let $b \in M(A)$ and suppose that $b + \tilde{J}$ is in the annihilator of $(A+\tilde{J})/\tilde{J}$ in $M(A)/\tilde{J}$. For $a \in A$, we have $ba, ab \in \tilde{J} \cap A = J$ and so $b \in \tilde{J}$ as required. □

In the case when $J = A$ we may define $\tilde{J} = M(A)$ and then we still have that $\tilde{J}$ is the strict closure of J in $M(A)$ and that $\tilde{J} \cap A = A$.

Recall that if X is a locally compact Hausdorff space, with Stone-Čech compactification βX, then every $f \in C^b(X)$ has a unique extension $\overline{f} \in C(\beta X)$. If $f \in C_0(X)$ then $\overline{f}(y) = 0$ for all $y \in \beta X \setminus X$. In particular, it follows that if $x \in X$ and $y \in \beta X$ with $x \neq y$ then there exists $f \in C_0(X)$ such that $f(x) \neq \overline{f}(y)$. We shall use this fact in a moment.

Proposition 1.2. *Let A be a $C_0(X)$-algebra with structure map μ. Then μ has a unique extension to a *-homomorphism $\overline{\mu} : C(\beta X) \to ZM(A)$ such that*

$$\overline{\mu}(\overline{f}) = \mu(f) \qquad (f \in C_0(X)). \tag{1}$$

Moreover $\overline{\mu}(1) = 1_{M(A)}$ and $(\overline{\mu}(f)a)_x = f(x)a_x$ for all $f \in C(\beta X)$, $a \in A$, $x \in X$.

Hence $M(A)$ is a $C(\beta X)$-algebra with structure map $\overline{\mu}$, and the corresponding base map $\overline{\phi} : \mathrm{Prim}(M(A)) \to \beta X$ satisfies $\overline{\phi}(\tilde{P}) = \phi(P)$ for all $P \in \mathrm{Prim}(A)$.

Proof. Let $\phi : \mathrm{Prim}(A) \to X$ be the base map such that $\mu(f) = \theta_A(f \circ \phi)$ for all $f \in C_0(X)$. Define $\overline{\mu}(f) = \theta_A(f \circ \phi)$ for all $f \in C(\beta X)$. Then (1) holds and also $\overline{\mu}(1) = \theta_A(1) = 1_{M(A)}$. Let $f \in C(\beta X)$, $a \in A$ and $x \in X$, and let $P \in \mathrm{Prim}(A)$ with $P \supseteq J_x$. Recall that $\phi(P) = x$. Then

$$(\overline{\mu}(f)a) + P = (f \circ \phi)(P)(a + P) = f(x)(a + P)$$

and so $\overline{\mu}(f)a - f(x)a \in \bigcap_{P \supseteq J_x} P = J_x$, as required.

Thus $M(A)$ is a $C(\beta X)$-algebra with structure map $\overline{\mu}$. Let $\overline{\phi}$ denote the corresponding base map. Let $P \in \mathrm{Prim}(A)$, $a \in A \setminus P$ and $f \in C_0(X)$. Then $\theta_A(f \circ \phi) + P = f(\phi(P))(a + P)$ and so

$$f(\phi(P)))(a + \tilde{P}) = \theta_A(f \circ \phi)a + \tilde{P} = \theta_{M(A)}(\overline{f} \circ \overline{\phi})a + \tilde{P} = \overline{f}(\overline{\phi}(\tilde{P}))(a + \tilde{P}).$$

So $(f(\phi(P)) - \overline{f}(\overline{\phi}(\tilde{P})))a \in \tilde{P} \cap A = P$. Hence $f(\phi(P)) = \overline{f}(\overline{\phi}(\tilde{P}))$. Since f was arbitrary, $\phi(P) = \overline{\phi}(\tilde{P})$ (by the remark immediately preceding this proposition). Thus $\overline{\phi}$ has the required property.

Finally, suppose that $\rho : C(\beta X) \to ZM(A)$ is a *-homomorphism such that $\rho(\overline{f}) = \mu(f)$ for all $f \in C_0(X)$. Then $\rho(C(\beta X))A$ is norm-dense in A and so the central projection $\rho(1)$ in $ZM(A)$ must be $1_{M(A)}$ because A is an essential ideal of $M(A)$. Hence $M(A)$ is a $C(\beta X)$-algebra with structure map ρ. Let $\sigma : \mathrm{Prim}(M(A)) \to \beta X$ be the corresponding base map such that

$$\rho(g) = \theta_{M(A)}(g \circ \sigma) \qquad (g \in C(\beta X)).$$

Then the same argument given for $\overline{\phi}$ applies to σ and so $\phi(P) = \sigma(\tilde{P})$ for all $P \in \mathrm{Prim}(A)$. Hence $\overline{\phi} = \sigma$, since $\{\tilde{P} \mid P \in \mathrm{Prim}(A)\}$ is dense in $\mathrm{Prim}(M(A))$. Thus $\rho = \overline{\mu}$. □

Thus in the language of Proposition 1.2, the main question of this paper is as follows. Suppose that A is a $C_0(X)$-algebra with base map ϕ. Under what circumstances is $\overline{\phi}$ open? Note that since the canonical embedding of $\mathrm{Prim}(A)$ in $\mathrm{Prim}(M(A))$ is an open map, the openness of ϕ is certainly a necessary condition for the openness of $\overline{\phi}$.

Proposition 1.2 has a useful corollary.

Corollary 1.3. *Let A be a $C_0(X)$-algebra with structure map μ and base map ϕ. The following are equivalent.*

(i) the *-homomorphism $\mu : C_0(X) \to ZM(A)$ is injective;
(ii) the mapping $\phi : \mathrm{Prim}(A) \to X$ has dense range;
(iii) the mapping $\overline{\phi} : \mathrm{Prim}(M(A)) \to \beta X$ is surjective;
(iv) the *-homomorphism $\overline{\mu} : C(\beta X) \to ZM(A)$ is injective.

Proof. (i) $\Longrightarrow$ (ii). If $\mathrm{Im}(\phi)$ is not dense in X, there exists a nonzero $f \in C_0(X)$ such that $f \circ \phi = 0$. Then $\mu(f) = \theta_A(f \circ \phi) = 0$.

(ii) $\Longrightarrow$ (iii). Since $\{\tilde{P} \mid P \in \mathrm{Prim}(A)\}$ is dense in the compact space $\mathrm{Prim}(M(A))$, $\mathrm{Im}(\overline{\phi})$ is the closure of $\mathrm{Im}(\phi)$ in βX. So if $\mathrm{Im}(\phi)$ is dense in X then $\overline{\phi}$ is surjective.

(iii) $\Longrightarrow$ (iv). Suppose that (iii) holds and that $\overline{\mu}(g) = 0$ for some $g \in C(\beta X)$. Then $\theta_{M(A)}(g \circ \overline{\phi}) = \overline{\mu}(g) = 0$ and so $g \circ \overline{\phi} = 0$ since $\theta_{M(A)}$ is injective. Since $\overline{\phi}$ is surjective, $g = 0$.

(iv) $\Longrightarrow$ (i). Suppose that (iv) holds and that $\mu(f) = 0$ for some $f \in C_0(X)$. Then $\overline{\mu}(\overline{f}) = \mu(f) = 0$. Hence $\overline{f} = 0$ and so $f = 0$. □

Definition. Let A be a $C_0(X)$-algebra with structure map μ and let $\overline{\mu} : C(\beta X) \to ZM(A)$ be as in Proposition 1.2. For $x \in \beta X$, we define

$$H_x = \overline{\mu}\{f \in C(\beta X) \mid f(x) = 0\}M(A),$$

a closed two-sided ideal of $M(A)$.

Note that H_x is defined in relation to $(M(A), \beta X, \overline{\mu})$ in the same way that J_x (for $x \in X$) is defined in relation to (A, X, μ). It follows, in particular, that for $Q \in \mathrm{Prim}(M(A))$: $Q \supseteq H_x$ if and only if $\overline{\phi}(Q) = x$. Also, for each $b \in M(A)$, the function $x \to \|b + H_x\|$ $(x \in \beta X)$ is upper semi-continuous.

Note, too, that if $x \in \beta X$, $f \in C(\beta X)$ and $g := f - f(x)1$ then $\overline{\mu}(g)1 \in H_x$ and hence $\overline{\mu}(f) + H_x = f(x)(1 + H_x)$.

Proposition 1.4. *Let A be a $C_0(X)$-algebra with structure map μ.*

(i) For all $x \in X$, $J_x = \overline{\mu}\{f \in C(\beta X) \mid f(x) = 0\}A$.

(ii) For all $x \in X$, $J_x \subseteq H_x \subseteq \tilde{J}_x$ and $J_x = H_x \cap A$.

(iii) For all $b \in M(A)$,

$$\|b\| = \sup\{\|b + \tilde{J}_x\| \mid x \in X\} = \sup\{\|b + H_x\| \mid x \in X\}.$$

Proof. (i) Let $f \in C(\beta X)$ with $f(x) = 0$ and let $a \in A$. It suffices to show that $\overline{\mu}(f)a \in J_x$. Let $\epsilon > 0$. There is a compact subset K of $\mathrm{Prim}(A)$ such that $\|(a+P)\| < \epsilon/(1+\|f\|)$ for all $P \in \mathrm{Prim}(A) \setminus K$. There exists $g \in C_0(X)$ with $0 \le g \le 1$ such that the restriction of g to the compact set $\phi(K)$ is 1. Then $h := (f|_X)g \in C_0(X)$ and $h(x) = 0$.

For $P \in \mathrm{Prim}(A)$,

$$\overline{\mu}(f)a + P = \theta_A(f \circ \phi)a + P = f(\phi(P))(a + P)$$

and $\mu(h)a + P = f(\phi(P))g(\phi(P))(a + P)$. So for $P \in K$, $\overline{\mu}(f)a - \mu(h)a \in P$. For $P \in \mathrm{Prim}(A) \setminus K$,

$$\|(\overline{\mu}(f)a - \mu(h)a) + P\| = |f(\phi(P))|(1 - g(\phi(P)))\|a + P\| < \epsilon.$$

So $\|\overline{\mu}(f)a - \mu(h)a\| < \epsilon$. Hence $\overline{\mu}(f)a \in J_x$.

(ii) It follows from (i) and the definition of H_x that $J_x \subseteq H_x$. Let $b \in H_x$ and $a \in A$. Then $ab, ba \in J_x$ by (i). Hence $b \in \tilde{J}_x$. It now follows from Proposition 1.2 (ii) that $H_x \cap A = J_x$.

(iii) Suppose that $c \in \tilde{J}_x$ for all $x \in X$ and that $a \in A$. Then $ac, ca \in J_x$ for all $x \in X$ and so $ac = ca = 0$. Hence $c = 0$. Thus the canonical *-homomorphism from $M(A)$ into $\Pi_{x \in X} M(A)/\tilde{J}_x$ is injective and hence isometric, establishing the first equality. The second follows from the fact that $\|b + \tilde{J}_x\| \le \|b + H_x\| \le \|b\|$ $(x \in X)$. □

The next lemma establishes a crucial link between the ideal H_x and the ideals $\tilde{J}_y$ for y close to x.

Lemma 1.5. *Let A be a $C_0(X)$-algebra with structure map μ. Let $x \in \beta X$ and $b \in M(A)$.*

(i) Let W be a neighborhood of x in βX. Then

$$\|b + H_x\| \le \sup\{\|b + \tilde{J}_y\| \mid y \in W \cap X\}.$$

(ii) Taking the infimum over neighborhoods W of x in βX, we have

$$\|b + H_x\| = \inf_W \sup\{\|b + \tilde{J}_y\| \mid y \in W \cap X\}.$$

Proof. (i) Choose $f \in C(\beta X)$ with $0 \le f \le 1$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in \beta X \setminus W$. Then $\overline{\mu}(1 - f)b \in H_x$ and so $b - \overline{\mu}(f)b \in H_x$. So

$$\|b + H_x\| = \|\overline{\mu}(f)b + H_x\| \le \|\overline{\mu}(f)b\| \le \sup\{\|\overline{\mu}(f)b + \tilde{J}_y\| \mid y \in X\}$$

by Proposition 1.4 (iii).

Suppose that $y \in X \setminus W$ and $a \in A$. Then

$$(\overline{\mu}(f)ba)_y = f(y)(ba)_y = 0$$

by Proposition 1.2. So $\overline{\mu}(f)ba \in J_y$ and similarly $a\overline{\mu}(f)b = \overline{\mu}(f)ab \in J_y$. Hence $\overline{\mu}(f)b \in \tilde{J}_y$. It follows that

$$\begin{aligned}\|b + H_x\| &\leq \sup\{\|\overline{\mu}(f)b + \tilde{J}_y\| \mid y \in W \cap X\}\\ &\leq \sup\{\|b + \tilde{J}_y\| \mid y \in W \cap X\}.\end{aligned}$$

(ii) By (i), the norm of $b + H_x$ is majorized by the infimum. So let $\epsilon > 0$. By upper semi-continuity, there is a neighborhood W of x in βX such that, for $y \in W \cap X$,

$$\|b + \tilde{J}_y\| \leq \|b + H_y\| \leq \|b + H_x\| + \epsilon.$$

Hence

$$\inf_W \sup\{\|b + \tilde{J}_y\| \mid y \in W \cap X\} \leq \|b + H_x\| + \epsilon.$$

Since ϵ was arbitrary, the result follows. □

2. Sufficient conditions for continuity

In this section we establish conditions which are sufficient for the continuity of norm functions of elements of $M(A)$ on βX (Theorem 2.9). In the next section we shall show that these conditions are also necessary for continuity.

Let A be a $C_0(X)$-algebra with structure map μ and define

$$U = \{x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x\}.$$

Note that if $x \in U$ then $J_x \neq A$ and so $x \in \mathrm{Im}(\phi)$. It will follow from Lemma 2.1 (ii) that $\mathrm{Im}(\phi) \setminus U$ is closed in $\mathrm{Im}(\phi)$ and hence U is an open subset of $\mathrm{Im}(\phi)$. If A is a continuous $C_0(X)$-algebra then $\mathrm{Im}(\phi)$ is open, as we noted in Section 1, and so in this case U is open in X.

There are three subsets of X which require separate consideration. The first is the set U itself, which consists of the easiest points to deal with (Proposition 2.3). The next is the set $X \setminus \mathrm{cl}(U)$, where $\mathrm{cl}(U)$ is the closure of U in X. These points, too, are fairly tractable (Proposition 2.4). The third, and the most difficult to deal with, consists of those points which lie in the boundary of U (Proposition 2.7).

To illustrate two elementary examples, first let A be the C^*-algebra of all sequences $x = (x_n)_{n\geq 1}$ of 2×2 complex matrices such that $x_n \to \mathrm{diag}(\lambda(x), 0)$ as $n \to \infty$. Set $P_n = \{x \in A \mid x_n = 0\}$ $(n \geq 1)$ and $P_\infty = \ker \lambda$. Then $\mathrm{Prim}(A) = \{P_n \mid n \geq 1\} \cup \{P_\infty\}$ with the topology induced from the space $X = \mathbb{N} \cup \{\infty\}$ (the 1-point compactification of $\mathbb{N}$) by the map $\phi : \mathrm{Prim}(A) \to X$ for which $\phi(P_n) = n$ and $\phi(P_\infty) = \infty$. Then $U = \mathbb{N}$ and the point ∞ lies in the boundary of U. Note that U is not C^*-embedded in X (see Lemma 2.6).

Next, let $A = C[0, 1] \otimes K(H)$, where $K(H)$ is the algebra of compact linear operators on an infinite-dimensional Hilbert space H. For $x \in X = [0, 1]$, set $P_x = \{f \in A \mid f(x) = 0\}$. Then $\mathrm{Prim}(A) = \{P_x \mid x \in X\}$ with the topology induced from X by the map $\phi : \mathrm{Prim}(A) \to X$ for which $\phi(P_x) = x$. In this case the set U is empty because $ZM(A) \cap A$ (the center of A) is $\{0\}$.

We begin with a simple lemma.

Lemma 2.1. *Let A be a $C_0(X)$-algebra with structure map μ and base map ϕ and let $x \in \mathrm{Im}(\phi)$. The following are equivalent.*

(i) $\mu(C_0(X)) \cap A \subseteq J_x$;

(ii) $\{f \in C_0(X) \mid \mu(f) \in A\} \subseteq \{f \in C_0(X) \mid f(x) = 0\}$;

(iii) there exists $R \in \mathrm{Prim}(M(A))$ such that $R \supseteq A$ and $\overline{\phi}(R) = x$.

Proof. (i) $\Longrightarrow$ (ii). Assume (i) and let $P \in \mathrm{Prim}(A/J_x)$. Suppose that $f \in C_0(X)$ and that $\mu(f) \in A$. Choose $a \in A \setminus P$. Then

$$0 = \mu(f)a + P = \theta_A(f \circ \phi)a + P = f(\phi(P))a + P = f(x)a + P.$$

Hence $f(x) = 0$ as required.

(ii) $\Longrightarrow$ (i). Assuming (ii), we have

$$\begin{aligned}\mu(C_0(X)) \cap A &= \mu(\{f \in C_0(X) \mid \mu(f) \in A\}) \\ &\subseteq \mathrm{cl}(\mu(\{f \in C_0(X) \mid \mu(f) \in A\})A) \subseteq J_x.\end{aligned}$$

(iii) $\Longrightarrow$ (ii). Assume (iii) and let $f \in C_0(X)$ with $\mu(f) \in A$. Then

$$0 = \mu(f) + R = \overline{\mu}(\overline{f}) + R = \theta_{M(A)}(\overline{f} \circ \overline{\phi}) + R = \overline{f}(\overline{\phi}(R)1 + R = f(x)1 + R$$

and so $f(x) = 0$ as required.

(ii) $\Longrightarrow$ (iii). Suppose that (iii) fails, so that x is not contained in the compact subset $\overline{\phi}(\{R \in \mathrm{Prim}(M(A)) \mid R \supseteq A\})$ of βX. Then there exists $g \in C(\beta X)$ such that $g(x) = 1$ and $g(\overline{\phi}(R)) = 0$ for all $R \in \mathrm{Prim}(M(A))$ such that $R \supseteq A$. Then $\overline{\mu}(g) + R = g(\overline{\phi}(R))1 + R = 0$ for all such R and so $\overline{\mu}(g) \in A$.

Choose $f \in C_0(X)$ such that $f(x) = 1$. Then $f(g|_X) \in C_0(X)$ and takes the value 1 at x. On the other hand, $\mu(f(g|_X)) = \overline{\mu}(\overline{f}g) = \mu(f)\overline{\mu}(g) \in A$. Thus (ii) fails to hold. □

Proposition 2.2. *Let A be a $C_0(X)$-algebra with structure map μ and let*

$$U = \{x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x\}.$$

Let $x \in U$. Then

(i) $H_x = \tilde{J}_x$ and H_x is strictly closed in $M(A)$;

(ii) A/J_x is unital and $\mu(f) + J_x = f(x)1_{A/J_x}$ for all $f \in C_0(X)$ such that $\mu(f) \in A$;

(iii) A/J_x is canonically isomorphic to $M(A)/H_x$ via the map $a + J_x \to a + H_x$ ($a \in A$).

Proof. (i) Since $J_x \neq A$, it follows that $\tilde{J}_x$ is a proper ideal of $M(A)$ and hence so is H_x. Let $R \in \mathrm{Prim}(M(A))$ and suppose that $R \supseteq H_x$ (equivalently, $\overline{\phi}(R) = x$). By Lemma 2.1, R does not contain A and so $R = \tilde{P}$ for some $P \in \mathrm{Prim}(A)$. By Proposition 1.2, $\phi(P) = \overline{\phi}(\tilde{P}) = x$ and so $P \supseteq J_x$. Hence

$R = \tilde{P} \supseteq \tilde{J}_x$. It follows that $H_x \supseteq \tilde{J}_x$ and the reverse inclusion always holds (Proposition 1.4 (ii)). So H_x is strictly closed in $M(A)$ by Proposition 1.1 (i).

(ii) Suppose that $f \in C_0(X)$ satisfies $\mu(f) \in A$. Note that, by hypothesis, $J_x \neq A$. Let $P \in \mathrm{Prim}(A)$ with $P \supseteq J_x$. Then, for all $a \in A$,

$$\mu(f)a + P = a\mu(f) + P = f(x)a + P.$$

Hence $\mu(f)a - f(x)a, a\mu(f) - f(x)a \in J_x$. All that remains is to show that A/J_x is unital. By Lemma 2.1, we may choose f such that $f(x) = 1$ and then $\mu(f) + J_x$ is an identity element for A/J_x.

(iii) Since $\tilde{J}_x \cap A = J_x$, the map $a + J_x \to a + \tilde{J}_x$ $(a \in A)$ gives a *-isomorphism of A/J_x onto $(A + \tilde{J}_x)/\tilde{J}_x$. By Proposition 1.1 (v), $(A + \tilde{J}_x)/\tilde{J}_x$ is a unital, essential ideal of $M(A)/\tilde{J}_x$ and hence must equal $M(A)/\tilde{J}_x$. Since $\tilde{J}_x = H_x$, the result follows. □

Proposition 2.3. *Let A be a continuous $C_0(X)$-algebra with structure map μ and let $U = \{x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x\}$. Let $x \in U$. Then for all $b \in M(A)$, the norm function $y \to \|b + H_y\|$ $(y \in X)$ is continuous at x.*

Proof. Since $J_x \neq A$, $x \in \mathrm{Im}(\phi)$ and it follows from Lemma 2.1 that there exists $f \in C_0(X)$ such that $\mu(f) \in A$ and $f(x) = 1$. Replacing f by $|f|^2$, we may assume that $f \geq 0$. There is an open neighborhood V of x in X, contained in the open subset $\mathrm{Im}(\phi)$ of X, such that $f(y) \geq \frac{1}{2}$ for all $y \in V$. Let $g : [0, \infty) \to [0, 1]$ be the continuous function defined by $g(t) = 2t$ $(0 \leq t \leq \frac{1}{2})$ and $g(t) = 1$ $(t > \frac{1}{2})$. Applying functional calculus, we may form $h := g(f) = g \circ f \in C_0(X)$. Then $\mu(h) = g(\mu(f)) \in A$ and $h(y) = 1$ for all $y \in V \subseteq \mathrm{Im}(\phi)$. For all $y \in V$, $\mu(C_0(X)) \cap A \not\subseteq J_y$ by Lemma 2.1 and so $\mu(h) + H_y$ is the identity of $M(A)/H_y$ by Proposition 2.2.

Now let $b \in M(A)$ and set $a = \mu(h)b \in A$. By hypothesis, the function $y \to \|a + J_y\|$ is continuous on X. For $y \in V$, we have

$$\|a + J_y\| = \|\mu(h)b + H_y\| = \|b + H_y\|.$$

So the function $y \to \|b + H_y\|$ $(y \in X)$ is continuous on V and in particular at x. □

For the next class of points we need some definitions. Recall that a subset U of a topological space X is a *cozero set* if there is a continuous real-valued function f on X which vanishes precisely on the complement of U in X. Now let X be a completely regular topological space. A point $x \in X$ is a *BD-point* (standing for basically disconnected) if whenever U is a cozero set in X and V an open set in X such that $x \in \mathrm{cl}(U) \cap \mathrm{cl}(V)$ then $x \in \mathrm{cl}(U \cap V)$. If each point in X is a BD-point then X is *basically disconnected*.

Before establishing a connection between BD-points and continuity of norm functions, we make an observation on open sets and cozero sets. Let A be a continuous $C_0(X)$-algebra and let $b \in M(A)$. Then the set $Y = \{x \in X \mid \|b + \tilde{J}_x\| > 0\}$ is open in X, being the image under the open mapping ϕ of the open set $\{P \in \mathrm{Prim}(A) \mid \|b + \tilde{P}\| > 0\}$ by Proposition 1.1 (iv). Now suppose furthermore that A is σ-unital with a strictly positive element u. Then $bu \in A$ and, for $P \in \mathrm{Prim}(A)$, $b \in \tilde{P}$ if and only if $bu \in \tilde{P}$ (to see this, use the notation of the proof of Proposition 1.1 (iii) and note that if $0 = \tilde{\pi}(bu) = \tilde{\pi}(b)\pi(u)$ then $b \in \ker \tilde{\pi} = \tilde{P}$ because the operator $\pi(u)$ has dense range). Hence for $x \in X$, $b \in \tilde{J}_x$ if and only if $bu \in \tilde{J}_x$. Thus, in this case, the set Y is the cozero set of a continuous function on X, namely the function $x \to \|bu + \tilde{J}_x\|$ $(x \in X)$.

Up till now we have worked with general C*-algebras A but for many of the subsequent results we have to assume that A is σ-unital. When A is a σ-unital C*-algebra which is also a $C_0(X)$-algebra we shall say that A is a *σ-unital $C_0(X)$-algebra*.

In the next proposition, we have chiefly in mind points $x \in X \setminus \mathrm{cl}(U)$ but we do not require this restriction.

Proposition 2.4. *Let A be a σ-unital continuous $C_0(X)$-algebra and let x be a BD-point in X. Then for all $b \in M(A)$, the norm function $y \to \|b + H_y\|$ $(y \in X)$ is continuous at x.*

Proof. By the C^*-condition, it suffices to consider $b \in M(A)^+$. Suppose that there exists $b \in M(A)^+$ such that the norm function of b is discontinuous at x. Since the function $y \to \|b + H_y\|$ $(y \in \beta X)$ is upper semi-continuous on βX, its restriction to X must fail to be lower semi-continuous at x. Hence, by scaling b, we may suppose that $\|b + H_x\| = 1$ and that there exists $\delta \in (0, 1)$ such that x lies in the closure of the set $V = \{y \in X \mid \|b + H_y\| < \delta\}$, a set which is open in X by upper semi-continuity.

On the other hand, by Lemma 1.5 (i), x lies in the closure of the set $W = \{y \in X \mid \|b + \tilde{J}_y\| > \frac{1+\delta}{2}\}$. Let $g : [0, \infty) \to [0, \infty)$ be the continuous function defined by $g(t) = 0$ for $0 \le t \le \frac{1+\delta}{2}$ and $g(t) = t - \frac{1+\delta}{2}$ for $t > \frac{1+\delta}{2}$. Then $W = \{y \in X \mid \|g(b) + \tilde{J}_y\| > 0\}$ and so W is a cozero set in X by the discussion above. Evidently V and W are disjoint and x is in the closure of each of them, contradicting the fact that x is a BD-point. □

The final set of points consists of those in the boundary of U. For these, we need the following two lemmas.

Lemma 2.5. *Let A be a σ-unital continuous $C_0(X)$-algebra with structure map μ and let*

$$U = \{x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x\}.$$

Suppose that $\mathrm{cl}(U)$, the closure of U in X, is clopen in X and that $x \in \mathrm{cl}(U) \setminus U$. Let $b \in M(A)$ and let V be any neighborhood of x in X. Then

$$\|b + H_x\| \le \sup\{\|b + \tilde{J}_y\| \mid y \in U \cap V\}.$$

Proof. Replacing V by its interior, we may assume that V is open. Since $\mathrm{cl}(U) \cap V$ is a neighborhood of x in X, there exists $f \in C_0(X)$ such that $0 \le f \le 1$, $f(x) = 1$ and $f(y) = 0$ for all $y \in X \setminus (\mathrm{cl}(U) \cap V)$. Then

$$\mu(f) + H_x = \overline{\mu}(\overline{f}) + H_x = \overline{f}(x)1 + H_x = 1 + H_x.$$

On the other hand, if $y \in X\backslash(\mathrm{cl}(U)\cap V)$ and $a \in A$ then $(\mu(f)a)_y = f(y)a_y = 0$ so that $\mu(f)a \in J_y$ and hence $\mu(f) \in \tilde{J}_y$. Since V is open, $U \cap V$ is dense in $\mathrm{cl}(U)\cap V$, and since A is continuous it follows that $\bigcap_{y\in U\cap V} J_y = \bigcap_{y\in \mathrm{cl}(U)\cap V} J_y$ and hence $\bigcap_{y\in U\cap V} \tilde{J}_y = \bigcap_{y\in \mathrm{cl}(U)\cap V} \tilde{J}_y$. Using Proposition 1.4 (iii), we now have

$$\begin{aligned}\|b+H_x\| = \|\mu(f)b+H_x\| \leq \|\mu(f)b\| &= \sup\{\|\mu(f)b+\tilde{J}_y\| \mid y \in \mathrm{cl}(U)\cap V\}\\ &= \sup\{\|\mu(f)b+\tilde{J}_y\| \mid y \in U\cap V\}\\ &\leq \sup\{\|b+\tilde{J}_y\| \mid y \in U\cap V\}.\end{aligned}$$

□

Lemma 2.6. *Let V be a completely regular space and let W be a dense subset of V. Then the following are equivalent:*

(i) disjoint zero sets in W have disjoint closures in V;
(ii) W is C^*-embedded in V;
(iii) V is canonically homeomorphic (i.e. homeomorphic under a map which extends the identity map on W) to a subset of βW.

Proof. The equivalence of (i) and (ii) is established in [20, Thm. 6.4, (2)⇔(3)].

(ii)⇒ (iii) By [20, Thm. 6.4 (1)] the identity map on W extends to a continuous map Θ from V into βW. Suppose that g is a continuous bounded function on V. Then, by continuity and by agreement on W, we have

$$g = \overline{(g|_W)} \circ \Theta. \qquad (*)$$

Since V is completely regular, any two points of V can be separated by a continuous bounded function g, so Θ is injective. Now let (v_i) be a net in V and suppose that $\Theta(v_i) \to \Theta(v)$ for some $v \in V$. Then $(*)$ gives $g(v_i) \to g(v)$ for all continuous bounded functions g on V, and hence $v_i \to v$ since V is completely regular. Thus Θ is a homeomorphism.

(iii)⇒(i) Disjoint zero sets in W have disjoint closures in βW [20, Thm. 6.5], and hence have disjoint closures in $\Theta(V)$. □

Proposition 2.7. *Let A be a σ-unital continuous $C_0(X)$-algebra and let*

$$U = \{x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x\}.$$

Suppose that $\mathrm{cl}(U)$, the closure of U in X, is clopen in X and that $\mathrm{cl}(U)$ is canonically homeomorphic to a subset of βU. Then for each $b \in M(A)$, the norm function $x \to \|b+H_x\|$ $(x \in X)$ is continuous at each point of $\mathrm{cl}(U)$.

Proof. Let $b \in M(A)$ and suppose that there exists $y \in \mathrm{cl}(U)$ such that the function $x \to \|b+H_x\|$ $(x \in X)$ is not continuous at y. Since the function is continuous at all points of U (Proposition 2.3), it follows that $y \in \mathrm{cl}(U) \setminus U$. Furthermore, since the function is upper semi-continuous on X and $\mathrm{cl}(U)$ is open in X, we may suppose by scaling b that $\|b+H_y\| = 1$ and that y lies in the closure of the open set $V = \{x \in \mathrm{cl}(U) \mid \|b+H_x\| < \delta\}$ for some $\delta \in (0,1)$. Since V is open in X, $V \cap U$ is dense in V and so y lies in the closure in X of the set $Y = \{x \in U \mid \|b+H_x\| \le \delta\}$. Since the norm function of b is continuous on U, Y is a zero set of U (for the function $x \to \max\{\|b+H_x\|, \delta\} - \delta$).

On the other hand, it follows from Lemma 2.6 and Lemma 2.2 (i) that y lies in the closure in X of the set $Z = \{x \in U \mid \|b+H_x\| \ge \frac{1+\delta}{2}\}$, which is also a zero set of U. This contradicts the fact that the disjoint zero sets Y and Z of U have disjoint closures in $\mathrm{cl}(U)$ (Lemma 2.6). □

Next we need to know that continuous norm-functions on X extend continuously to βX.

Proposition 2.8. *Let A be a $C_0(X)$-algebra with structure map μ such that, for each $b \in M(A)$, the norm function $x \to \|b + H_x\|$ $(x \in X)$ is continuous. Then $M(A)$ is a continuous $C(\beta X)$-algebra with structure map $\overline{\mu}$.*

Proof. For each $b \in M(A)$, let $f_b : X \to [0,\infty)$ be the bounded function defined by $f_b(x) = \|b + H_x\|$ $(x \in X)$. By hypothesis, f_b is continuous and so it suffices to show that $\overline{f_b}(y) = \|b + H_y\|$ for all $y \in \beta X \setminus X$ and all $b \in M(A)$.

Let $y \in \beta X \setminus X$ and let $\mathcal{F}$ be a z-ultrafilter on X with limit y. If $b \in H_y$ then $\overline{f_b}(y) = 0$ by the upper semi-continuity of the norm function $x \to \|b + H_x\|$ $(x \in \beta X)$. So we may now restrict to the case where $H_y \ne M(A)$. Since $b \to \overline{f_b}(y) = \lim_{\mathcal{F}} \|b + H_x\|$ defines a C^*-seminorm on $M(A)$, it follows from the uniqueness of the C^*-norm on $M(A)/H_y$ that it suffices to show that if $\overline{f_b}(y) = 0$ then $b \in H_y$.

Let $b \in M(A)$ and suppose that $\overline{f_b}(y) = 0$. Let $\epsilon > 0$. Then there exists $Z \in \mathcal{F}$ such that $\|b + H_x\| < \epsilon/2$ for all $x \in Z$. So Z is disjoint from the set $W = \{x \in X \mid \|b + H_x\| \ge \epsilon\}$. Since W is the zero set for the continuous function $\min\{f_b, \epsilon\} - \epsilon$, it follows from [20, 1.15] that there exists $f \in C^b(X)$ such that $0 \le f \le 1$, $f(Z) = \{0\}$ and $f(W) = \{1\}$. Since $Z \in \mathcal{F}$ and $f(Z) = \{0\}$, $\overline{f}(y) = 0$. Thus $\overline{\mu}(\overline{f})b \in H_y$ by definition of H_y. On the other hand, it follows from Proposition 1.4 (iii) that

$$\|b - \overline{\mu}(\overline{f})b\| = \sup_{x\in X} \|(b - \overline{\mu}(\overline{f})b) + H_x\| = \sup_{x\in X} (1 - f(x))\|b + H_x\| \le \epsilon.$$

Since ϵ was arbitrary, $b \in H_y$. □

Finally, we summarize the work of this section in the following theorem.

Theorem 2.9. *Let A be a σ-unital continuous $C_0(X)$-algebra with structure map μ and base map ϕ, let*

$$U = \{x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x\}$$

and let $\mathrm{cl}(U)$ and $\mathrm{cl}(\mathrm{Im}(\phi))$ be the closures of U and $\mathrm{Im}(\phi)$ in X. Then $M(A)$ is a continuous $C(\beta X)$-algebra with structure map $\overline{\mu}$ if

(i) $\mathrm{cl}(U)$ is clopen in X;
(ii) $\mathrm{cl}(U)$ is canonically homeomorphic to a subset of βU;
(iii) every point of $\mathrm{cl}(\mathrm{Im}(\phi)) \setminus \mathrm{cl}(U)$ is a BD-point of X.

Proof. By Proposition 2.8, it suffices to show that, for each $b \in M(A)$, the norm function $x \to \|b + H_x\|$ $(x \in X)$ is continuous at each point $x \in X$. This continuity was established in Proposition 2.7 for x in $\mathrm{cl}(U)$ and in Proposition 2.4 for x in $\mathrm{cl}(\mathrm{Im}(\phi)) \setminus \mathrm{cl}(U)$.

Finally, since $\{\tilde{P} \mid P \in \mathrm{Prim}(A)\}$ is dense in the compact space $\mathrm{Prim}(M(A))$, $\mathrm{Im}(\overline{\phi})$ is the closure of $\mathrm{Im}(\phi)$ in βX. Hence $\mathrm{Im}(\overline{\phi}) \cap X = \mathrm{cl}(\mathrm{Im}(\phi))$. If x belongs to the open set $X \setminus \mathrm{cl}(\mathrm{Im}(\phi))$ then $x \notin \mathrm{Im}(\overline{\phi})$ and so $\|b + H_x\| = 0$. Thus x is a point of continuity for the norm function. □

3. Necessary conditions of continuity and the main theorem

In this section we prove the converse of Theorem 2.9, thus establishing our main result, Theorem 3.8, which characterizes, for A a σ-unital continuous $C_0(X)$-algebra, when $M(A)$ is a continuous $C(\beta X)$-algebra. The main technical result along the way is Theorem 3.2 which constructs a useful multiplier in $M(A)$.

Proposition 3.1. *Let A be a $C_0(X)$-algebra and set $B = \Pi_{x \in X} M(A)/\tilde{J}_x$. Define $\Phi : M(A) \to B$ by $\Phi(b) = (b + \tilde{J}_x)_x$ and set $\iota = \Phi|_A : A \to B$. Then Φ is a *-isomorphism from $M(A)$ onto B_{id}, the idealizer of $\iota(A)$ in B.*

Proof. It is evident that $\Phi(M(A)) \subseteq B_{\mathrm{id}}$. Moreover, Φ is injective by Proposition 1.4 (iii). It follows from Proposition 1.1 (v) that $\iota(A)$ is an essential ideal of B_{id} and so there exists an injective *-homomorphism $\theta : B_{\mathrm{id}} \to M(A)$ such that $\theta(\iota(a)) = a$ for all $a \in A$ [28, 3.12.8].

Let $b = (b_x)_x \in B_{\mathrm{id}}$. We claim that $\Phi(\theta(b)) = b$. To see this, first note that for each $a \in A$, $b\iota(a) = \iota(c)$ for some $c \in A$. Hence for each $x \in X$,

$$\begin{aligned} \theta(b)a + \tilde{J}_x &= \theta(b)\theta(\iota(a)) + \tilde{J}_x \\ &= \theta(b\iota(a)) + \tilde{J}_x = \theta(\iota(c)) + \tilde{J}_x \\ &= c + \tilde{J}_x = b_x(a + \tilde{J}_x), \end{aligned}$$

the final equality holding because $b_x(a + \tilde{J}_x)$ is the x-component of $\iota(c)$. Similarly, $a\theta(b) + \tilde{J}_x = (a + \tilde{J}_x)b_x$. Since a was arbitrary and $(A + \tilde{J}_x)/\tilde{J}_x$ is essential in $M(A)/\tilde{J}_x$ (Proposition 1.1 (v)), it follows that $\theta(b) + \tilde{J}_x = b_x$. Hence $\Phi(\theta(b)) = b$, as required. □

We now define a function g from the unit interval $[0,1]$ to the space $C[0,1]$ as follows (where for $r \in [0,1]$, g_r is the continuous function on $[0,1]$ corresponding to r):

$$g_0(x) = 1 \text{ for all } x \in [0,1];$$

$$\text{for } 0 < r \leq 1/2, \quad g_r(x) = \begin{cases} 0 & (0 \leq x \leq r/2) \\ (2x/r) - 1 & (r/2 \leq x \leq r) \\ 1 & (r \leq x \leq 1); \end{cases}$$

$$g_r = g_{1/2} \text{ for } r \geq 1/2.$$

For an element a in a C*-algebra A, let $\mathrm{sp}(a)$ denote the spectrum of a. For $a \geq 0$ let $\min \mathrm{sp}(a)$ be the smallest number in $\mathrm{sp}(a)$. Note that the arbitrary cozero set U in the following theorem is not to be confused with the set U defined at the start of Section 2. We will see in Lemma 3.3 (i) and (ii), however, that the set U defined at the start of Section 2 is indeed a cozero set in certain circumstances.

Theorem 3.2. *Let A be a σ-unital $C_0(X)$-algebra with base map ϕ and let u be a strictly positive element in A with $\|u\| = 1$. Let $f \in C^b(X)$ with $0 \leq f \leq 1$, let U be the cozero set of f and let $V = \{x \in U \cap \mathrm{Im}(\phi) \mid 2 \min \mathrm{sp}(u_x) \leq f(x)\}$. Let $\overline{U}$ and $\overline{V}$ be the closures of U and V in βX, respectively. Then there exists $b \in M(A)$ with $0 \leq b \leq 1$ such that*

(i) $b + \tilde{J}_x = g_{f(x)}(u + \tilde{J}_x) \quad (x \in X)$*;*

(ii) $b \in A + H_x \subseteq A + \tilde{J}_x$ *for all* $x \in U$*;*

(iii) $1 - b \in \tilde{J}_x$ *for all* $x \in X \setminus U$ *and* $1 - b \in H_x$ *for all* $x \in \beta X \setminus \overline{U}$*;*

(iv) $\|(1 - b) + \tilde{J}_x\| = 1$ *for all* $x \in V$ *and* $\|(1 - b) + H_x\| = 1$ *for all* $x \in \overline{V}$*.*

Furthermore,

(v) H_x *is not strictly closed in* $M(A)$ *for all* $x \in (\overline{V} \cap X) \setminus U$*.*

Proof. (i) Let $B = \Pi_{x \in X} M(A)/\tilde{J}_x$ and define $d \in B$ by

$$d_x = g_{f(x)}(u + \tilde{J}_x) \qquad (x \in X).$$

We wish to show that $d \in B_{\mathrm{id}}$. Let $a \in A$ with $\|a\| = 1$, and let $\epsilon > 0$. We first seek $c \in A$ such that $\|d\iota(a) - \iota(c)\| \leq \epsilon$.

Let $Y = \{P \in \mathrm{Prim}(A) \mid \|a + P\| \geq \epsilon\}$, a compact subset of $\mathrm{Prim}(A)$. Then $Z := \phi(Y)$ is a compact subset of X and, for $x \in X \setminus Z$,

$$\|d_x(a + \tilde{J}_x)\| \leq \|a + \tilde{J}_x\| = \|a_x\| < \epsilon$$

(for, if $P \in \mathrm{Prim}(A)$ and $P \supseteq J_x$ then $\phi(P) = x$ and so $P \notin Y$). For $x \in X$, set $c^x = a$ if $f(x) = 0$ and set $c^x = g_{f(x)}(u)a \in A$ otherwise.

Case 1: $x \in X$ with $f(x) = 0$. We claim that there exists $\delta > 0$ such that $\|a - g_r(u)a\| < \epsilon$ for all $0 < r < \delta$. For, if not, there exists a sequence (r_k) tending to zero such that $\|a - g_{r_k}(u)a\| \geq \epsilon$ for all k, contradicting the fact that $g_{r_k}(u)$ is an approximate identity for A (see the proof of [28, 3.10.5]). Hence the claim holds.

Set $N_x = f^{-1}([0, \delta))$, an open neighborhood of x in X. Then for all $y \in N_x$,

$$\begin{aligned}\|d_y(a + \tilde{J}_y) - (c^x + \tilde{J}_y)\| &= \|g_{f(y)}(u + \tilde{J}_y)(a + \tilde{J}_y) - (a + \tilde{J}_y)\| \\ &\leq \|g_{f(y)}(u)a - a\| < \epsilon\end{aligned}$$

(note that if $f(y) = 0$ then $g_{f(y)}(u) = 1$).

Case 2: $x \in X$ with $f(x) \neq 0$. Set $r = f(x)$ and let

$$N_x = \left\{ y \in X \;\middle|\; r/2 < f(y) < 2r \text{ and } \frac{2|f(y) - r|}{r} < \epsilon \right\},$$

an open neighborhood of x in X. Then for all $y \in N_x$

$$\begin{aligned}\|d_y(a+\tilde{J}_y)-(c^x+\tilde{J}_y)\| &= \|g_{f(y)}(u+\tilde{J}_y)(a+\tilde{J}_y)-g_{f(x)}(u+\tilde{J}_y)(a+\tilde{J}_y)\| \\ &\leq \|g_{f(y)}-g_r\|_\infty \leq \frac{2|f(y)-r|}{r} < \epsilon.\end{aligned}$$

Since Z is compact, there exist $x_1,\dots,x_n \in Z$ such that the open sets N_{x_i} $(1 \leq i \leq n)$ cover Z. Since X is a locally compact Hausdorff space, there exist $h_i \in C_0(X)^+$ $(1 \leq i \leq n)$, with each h_i vanishing off N_{x_i}, such that $\sum_i h_i(x) = 1$ for all $x \in Z$ and $\sum_i h_i(x) \leq 1$ for all $x \in X \setminus Z$. Let $c = \sum_{i=1}^n \mu(h_i)c^{x_i} \in A$.

For all $x \in X$, $c + J_x = \sum_i h_i(x)(c^{x_i} + J_x)$ and so, since $(A+\tilde{J}_x)/\tilde{J}_x$ is canonically isomorphic to A/J_x, $c + \tilde{J}_x = \sum_i h_i(x)(c^{x_i} + \tilde{J}_x)$. For $x \in Z$,

$$\begin{aligned}\|d_x(a+\tilde{J}_x)-(c+\tilde{J}_x)\| &= \|\sum_{i=1}^n h_i(x)(d_x(a+\tilde{J}_x)-(c^{x_i}+\tilde{J}_x))\| \\ &\leq \sum_{i=1}^n h_i(x)\|(d_x(a+\tilde{J}_x)-(c^{x_i}+\tilde{J}_x))\| \leq \epsilon,\end{aligned}$$

and for $x \in X \setminus Z$,

$$\begin{aligned}\|d_x(a+\tilde{J}_x)-(c+\tilde{J}_x)\| &= \|d_x(a+\tilde{J}_x)-\sum_{i=1}^n h_i(x)(c^{x_i}+\tilde{J}_x)\| \\ \leq (1-\sum_{i=1}^n h_i(x))\|d_x(a+\tilde{J}_x)\| &+ \sum_{i=1}^n h_i(x)\|d_x(a+\tilde{J}_x)-(c^{x_i}+\tilde{J}_x)\| \\ \leq (1-\sum_{i=1}^n h_i(x))\epsilon &+ \sum_{i=1}^n h_i(x)\epsilon = \epsilon.\end{aligned}$$

Hence

$$\|d\iota(a)-\iota(c)\| = \sup_{x\in X}\|d_x(a+\tilde{J}_x)-(c+\tilde{J}_x)\| \leq \epsilon.$$

Since ϵ was arbitrary and $\iota(A)$ is norm-closed in B, it follows that $d\iota(a) \in \iota(A)$. Similarly $\iota(a)d \in \iota(A)$ and so $d \in B_{\mathrm{id}}$. Let $b = \Phi^{-1}(d) \in M(A)$ (where $\Phi : M(A) \to B_{\mathrm{id}}$ is the *-isomorphism of Proposition 3.1). Then, for all $x \in X$,

$$b + \tilde{J}_x = d_x = g_{f(x)}(u+\tilde{J}_x).$$

(ii) Let $x \in U$ and set $r = f(x) > 0$ and $a = g_r(u) \in A$. Let $\epsilon > 0$. As in *Case 2* above, let

$$N_x = \left\{ y \in X \;\middle|\; r/2 < f(y) < 2r \text{ and } \frac{2|f(y)-r|}{r} < \epsilon \right\},$$

an open neighborhood of x in X. Then for all $y \in N_x$

$$\begin{aligned}\|(b-a)+\tilde{J}_y\| &= \|(g_{f(y)}(u)-g_{f(x)}(u))+\tilde{J}_y\| \\ &\leq \|g_{f(y)}-g_r\|_\infty \leq \frac{2|f(y)-r|}{r} < \epsilon.\end{aligned}$$

Hence $\|(b-a)+H_x\| \leq \epsilon$ by Lemma 1.5 (i). Since ϵ was arbitrary, $b-a \in H_x \subseteq \tilde{J}_x$.

(iii) Let $x \in X \setminus U$. Then $f(x) = 0$ and so $b + \tilde{J}_x = 1 + \tilde{J}_x$.

Let $x \in W := \beta X \setminus \overline{U}$. By Lemma 1.5 (i),

$$\|(1-b)+H_x\| \leq \sup\{\|(1-b)+\tilde{J}_y\| \mid y \in W \cap X\}.$$

Since $W \cap X \subseteq X \setminus U$, it follows that $\|(1-b)+H_x\| = 0$.

(iv) Let $x \in V$. Then $\min \operatorname{sp}(u+\tilde{J}_x) = \min \operatorname{sp}(u+J_x) \leq f(x)/2$ and so

$$0 = g_{f(x)}(\min \operatorname{sp}(u+\tilde{J}_x) \in \operatorname{sp}(b+\tilde{J}_x)$$

by the spectral mapping theorem. Hence

$$1 = \|(1-b)+\tilde{J}_x\| \leq \|(1-b)+H_x\| \leq 1.$$

By upper semi-continuity, $\|(1-b)+H_x\| = 1$ for all $x \in \overline{V}$.

(v) Let $x \in (\overline{V} \cap X) \setminus U$. Then $1-b \in \tilde{J}_x \setminus H_x$ by (iii) and (iv). Since $J_x \subseteq H_x \subseteq \tilde{J}_x$ and $\tilde{J}_x$ is the strict closure of J_x in $M(A)$, H_x cannot be strictly closed in $M(A)$. □

The next three results go towards establishing conditions (i) and (ii) of Theorem 3.8 when $M(A)$ is a continuous $C(\beta X)$-algebra.

Lemma 3.3. *Let A be a σ-unital $C_0(X)$-algebra with structure map μ and base map ϕ and let u be a strictly positive element of A with $\|u\| = 1$. Suppose that $M(A)$ is a continuous $C(\beta X)$-algebra with base map $\overline{\phi}$. Define $f : X \to [0,1]$ by $f(x) = (1 - \|(1-u)+H_x\|)^{\frac{1}{2}}$ for $x \in \operatorname{Im}(\overline{\phi}) \cap X$ and $f(x) = 0$ otherwise. Then*

(i) $\operatorname{Im}(\overline{\phi}) \cap X$ is clopen in X and f is continuous;
(ii) for $x \in X$, $f(x) > 0$ if and only if $\mu(C_0(X)) \cap A \not\subseteq J_x$;
(iii) if $x \in X$ and $0 < f(x) \leq \frac{1}{2}$ then $2 \min \operatorname{sp}(u_x) \leq f(x)$.

Proof. (i) Since $\operatorname{Prim}(M(A))$ is compact, $\operatorname{Im}(\overline{\phi})$ is compact and hence is the closure of $\operatorname{Im}(\phi)$ in βX. On the other hand,

$$\operatorname{Im}(\overline{\phi}) = \{x \in \beta X \mid H_x \neq M(A)\},$$

which is the union (over $b \in M(A)$) of the cozero sets of the continuous functions $x \to \|b+H_x\|$ $(x \in \beta X)$. Thus $\operatorname{Im}(\overline{\phi})$ is clopen in βX and hence $\operatorname{Im}(\overline{\phi}) \cap X$ (which is the closure of $\operatorname{Im}(\phi)$ in X) is clopen in X. Since $x \to \|(1-u)+H_x\|$ is continuous on βX and hence on $\operatorname{Im}(\overline{\phi}) \cap X$, it follows that f is continuous.

(ii) Let $x \in X$. Suppose that $f(x) > 0$. Then $x \in \operatorname{Im}(\overline{\phi})$ and so $H_x \neq M(A)$ and $u \notin H_x$. Hence $J_x \neq A$ and $x \in \operatorname{Im}(\phi)$. Since $\|(1-u)+H_x\| < 1$, $u + H_x$ is invertible in $M(A)/H_x$. So no primitive ideal of $M(A)$ containing H_x can contain A. It follows from Lemma 2.1 that $\mu(C_0(X)) \cap A \not\subseteq J_x$

Conversely, suppose that $\mu(C_0(X) \cap A \not\subseteq J_x$. Since $u + J_x$ is strictly positive in the unital algebra A/J_x, it is invertible. By Proposition 2.2 (iii), $u + H_x$ is invertible in $M(A)/H_x$ and hence $\|(1-u)+H_x\| < 1$. Thus $f(x) > 0$.

(iii) Let $x \in X$ and suppose that $0 < f(x) \leq \frac{1}{2}$. Then

$$f(x) \geq 2(f(x))^2 = 2 \min \mathrm{sp}(u + H_x).$$

But, since $f(x) > 0$, $\mu(C_0(X)) \cap A \not\subseteq J_x$ and so A/J_x is canonically isomorphic to $M(A)/H_x$ (Proposition 2.2 (iii)). Hence $\mathrm{sp}(u + H_x) = \mathrm{sp}(u_x)$. □

Proposition 3.4. *Let A be a σ-unital $C_0(X)$-algebra with structure map μ and suppose that $M(A)$ is a continuous $C(\beta X)$-algebra with structure map $\overline{\mu}$. Let*

$$U = \{x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x\}$$

and let $\overline{U}$ be the closure of U in βX. Then $\overline{U} \cap X$, the closure $\mathrm{cl}(U)$ *of U in X, is open in X.*

Proof. With u and f as in Lemma 3.3, U is the cozero set of f by Lemma 3.3 (ii). If U is closed in X then there is nothing to prove. So we may assume that $(\overline{U} \cap X) \setminus U$ is nonempty. Let $b \in M(A)$ be constructed as in Theorem 3.2. By Theorem 3.2 (iii), $\|(1 - b) + H_x\| = 0$ for all $x \in \beta X \setminus \overline{U}$ and hence for all $x \in X \setminus (\overline{U} \cap X)$.

Recalling that $U \subseteq \mathrm{Im}(\phi)$), let $V = \{x \in U \mid 2 \min \mathrm{sp}(u_x) \leq f(x)\}$ and let $\overline{V}$ be the closure of V in βX. Then $\{x \in U \mid 0 < f(x) \leq \frac{1}{2}\} \subseteq V$ by Lemma 3.3 (iii). Let $x \in (\overline{U} \cap X) \setminus U$ and let (x_α) be a net in U that is convergent to x. Then $f(x_\alpha) \to f(x) = 0$ and so $x_\alpha \in V$ eventually, from which it follows that $x \in \overline{V}$. Thus $\|(1 - b) + H_x\| = 1$ for all $x \in (\overline{U} \cap X) \setminus U$ by Theorem 3.2 (iv). The function $x \to \|(1 - b) + H_x\|$ is continuous on βX, and hence on X, and takes the value 1 on the nonempty set $(\overline{U} \cap X) \setminus U$ and the value 0 on $X \setminus (\overline{U} \cap X)$. It follows that $X \setminus (\overline{U} \cap X)$ is closed in X and hence $\overline{U} \cap X$ is open in X. □

Proposition 3.5. *Let A be a σ-unital $C_0(X)$-algebra with structure map μ and suppose that $M(A)$ is a continuous $C(\beta X)$-algebra with structure map $\overline{\mu}$. Let*

$$U = \{x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x\}$$

and let $\overline{U}$ be the closure of U in βX. Then $\overline{U} \cap X$, the closure $\mathrm{cl}(U)$ *of U in X, is canonically homeomorphic to a subset of βU.*

Proof. Suppose that $\mathrm{cl}(U)$ is not canonically homeomorphic to a subset of βU. Then by Lemma 2.5 there is a point $y \in \mathrm{cl}(U) \setminus U$ and disjoint zero sets Y and Z of U such that y lies in the closures of both Y and Z. With u and f as in Lemma 3.3, let $b \in M(A)$ be an element with the properties of Theorem 3.2. Recalling that $U \subseteq \mathrm{Im}(\phi)$, set $V = \{x \in U \mid 2 \min \mathrm{sp}(u_x) \leq f(x)\}$. Then $\{x \in U \mid 0 < f(x) \leq \frac{1}{2}\} \subseteq V$ by Lemma 3.3 (iii) and $\|(1 - b) + H_x\| = 1$ for all $x \in V$ by Theorem 3.2 (iv).

By [20, 1.15], there is a continuous function g on U with $0 \leq g \leq 1$ such that $g(Y) = \{0\}$ and $g(Z) = \{1\}$. Let $I_b = \text{norm-cl}(A(1 - b)A)$, a closed two-sided ideal of A. If $P \in \mathrm{Prim}(A)$ and $I_b \not\subseteq P$ then $\phi(P) \in U$ by Theorem 3.2 (iii). It follows that $g \circ \phi$ defines a continuous bounded function on $\mathrm{Prim}(I_b)$

and hence induces a unique central multiplier z_g of I_b via the Dauns-Hofmann isomorphism for $ZM(I_b)$. Extending z_g to be the zero multiplier on $I_b^\perp$, we may regard z_g as a central element of $M(I_b+I_b^\perp)$. Since $I_b+I_b^\perp$ is an essential ideal of A, $M(A)\subseteq M(I_b+I_b^\perp)$. Hence $z_g(1-b)=(1-b)z_g\in M(I_b+I_b^\perp)$. Using an approximate identity for A, we see that $(1-b)a, a(1-b)\in I_b$ for all $a\in A$, and hence that $z_g(1-b)a\in I_b\subseteq A$ and $az_g(1-b)=a(1-b)z_g\in I_b\subseteq A$. So $z_g(1-b)$ is in the idealizer A_{id} of A in $M(I_b+I_b^\perp)$. Since $I_b+I_b^\perp$ is essential in its multiplier algebra, A is essential in A_{id} and so there is a *-isomorphism Φ of A_{id} into $M(A)$ such that $\Phi(a)=a$ for all $a\in A$. It follows that if $a\in I_b+I_b^\perp$ then

$$(z_g(1-b)-\Phi(z_g(1-b)))a=0=a(z_g(1-b)-\Phi(z_g(1-b)))$$

and so $z_g(1-b)=\Phi(z_g(1-b))\in M(A)$.

Let $x\in U$, $a\in A$ and $P\in\mathrm{Prim}(A)$ with $P\supseteq J_x$. If $I_b\not\subseteq P$ then

$$z_g(1-b)a+(P\cap I_b)=g(x)(1-b)a+(P\cap I_b)$$

and so $z_g(1-b)a-g(x)(1-b)a\in P\cap I_b\subseteq P$. On the other hand, if $I_b\subseteq P$ then $z_g(1-b)a-g(x)(1-b)a\in I_b\subseteq P$. Thus in either case $z_g(1-b)a-g(x)(1-b)a\in P$. Since this is true for all such P, $(z_g(1-b)-g(x)(1-b))a\in J_x$ and similarly $a(z_g(1-b)-g(x)(1-b))\in J_x$. Hence, using Proposition 2.2 (i), we obtain that

$$z_g(1-b)-g(x)(1-b)\in\tilde{J}_x=H_x\quad(x\in U).$$

It now follows that $\|z_g(1-b)+H_x\|=\|(1-b)+H_x\|$ for all $x\in Z$ and $\|z_g(1-b)+H_x\|=0$ for all $x\in Y$. Since $M(A)$ is a continuous $C(\beta X)$-algebra and y is in the closure of Y, we obtain that $\|z_g(1-b)+H_y\|=0$. On the other hand, let (x_α) be a net in Z converging to y. Then $f(x_\alpha)\to f(y)=0$ and so we may assume that $f(x_\alpha)\leq\frac{1}{2}$ for all α. Then $x_\alpha\in V$ and

$$\|z_g(1-b)+H_{x_\alpha}\|=\|(1-b)+H_{x_\alpha}\|=1$$

for all α, by the first paragraph of the proof. Since $x_\alpha\to y$, $\|z_g(1-b)+H_y\|=1$ by the (upper semi-)continuity of the norm function. This contradiction establishes the result. □

The next two lemmas are needed in order to establish condition (iii) of Theorem 3.8 when $M(A)$ is a continuous $C(\beta X)$-algebra.

Lemma 3.6. *Let A be a σ-unital continuous $C_0(X)$-algebra with structure map μ and base map ϕ and let V be a nonempty open subset of X such that A/J_x is unital for each $x\in V$. Then there exists $x\in V$ such that $\mu(C_0(X)))\cap A\not\subseteq J_x$.*

Proof. For all $x\in V$, $(A+\tilde{J}_x)/\tilde{J}_x$ is canonically isomorphic to A/J_x and so, being a unital essential ideal, must equal $M(A)/\tilde{J}_x$. Let u be a strictly positive element of A with $\|u\|=1$. Then, for all $x\in V$, u_x is invertible and so $\|(1-u)+\tilde{J}_x\|<1$. For every $\epsilon\geq 0$, the set $\{x\in X\mid \|(1-u)+\tilde{J}_x\|>\epsilon\}$ is open, being the image under the open map ϕ of the open set $\{P\in\mathrm{Prim}(A)\mid \|(1-u)+\tilde{P}\|>\epsilon\}$ by Proposition 1.1 (iv) (note that if $\|(1-u)+\tilde{J}_x\|>0$ then $J_x\neq A$). Hence the function $x\to\|(1-u)+\tilde{J}_x\|$ $(x\in X)$ is lower semi-continuous on X. Since V is open in X, V is a locally compact Hausdorf space, hence a Baire space, and so any lower semi-continuous function on V has a point of continuity [13, B18]. Thus there exists $x \in V$ with an open neighborhood $W \subseteq V$ and $\epsilon > 0$ such that $\|(1-u)+\tilde{J}_y\| < 1-\epsilon$ for all $y \in W$. Hence $\min \mathrm{sp}(u+\tilde{J}_y) > \epsilon$ for all $y \in W$.

Let $g : [0,\infty) \to [0,1]$ be a continuous function such that $g(0) = 0$ and $g(t) = 1$ for all $t \geq \epsilon$. Let $w = g(u) \in A$ and observe that $w + J_y$ is the identity of A/J_y for all $y \in W$. Choose $f \in C_0(X)$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in X \setminus W$. For all $a \in A$ and all $y \in X$,

$$(\mu(f)a - \mu(f)wa)_y = f(y)(a_y - w_y a_y) = 0.$$

Thus $\mu(f)a = \mu(f)wa$ and similarly $a\mu(f) = a\mu(f)w$. Hence $\mu(f) = \mu(f)w \in A$. Since A/J_x is unital, $J_x \neq A$ and so Lemma 2.1 ((i) implies (ii)) yields the result. □

In the next lemma we do not require the $C_0(X)$-algebra A to be continuous.

Lemma 3.7. *Let A be a σ-unital $C_0(X)$-algebra and let*

$$Y = \{x \in \mathrm{Im}(\phi) \mid A/J_x \text{ is nonunital}\}.$$

Suppose that z is a non-BD-point in X and that z has a neighborhood N in X such that $Y \cap N$ is dense in N. Then there exists $c \in M(A)$ such that the norm function $x \to \|c + H_x\|$ $(x \in X)$ is discontinuous at z.

Proof. Since z is not a BD-point in X, there exists a cozero set S and an open set T such that $z \in \mathrm{cl}(S) \cap \mathrm{cl}(T)$ but $z \notin \mathrm{cl}(S \cap T)$. Replacing T by $T \setminus \mathrm{cl}(S)$, we may assume that $S \cap T = \varnothing$. Let u be a strictly positive element of A with $\|u\| = 1$. Then for all $x \in Y$, $\min \mathrm{sp}(u_x) = 0$ since A/J_x is nonunital. Applying Theorem 3.2 to the cozero set S, we obtain $b \in M(A)$ with $0 \leq b \leq 1$ such that $1 - b \in \tilde{J}_x$ for all $x \in X \setminus S$ and $\|(1-b)+H_x\| = 1$ for all $x \in S \cap Y$.

Let V be an arbitrary open neighborhood of z contained in N. Since $z \in \mathrm{cl}(S)$, $V \cap S$ is a nonempty open set contained in N and hence contains some point $y \in Y$. Then $y \in S \cap Y$ and so $\|(1-b)+H_y\| = 1$. Since $y \in V$ and V was arbitrary, it follows from the upper semi-continuity of the norm function that $\|(1-b)+H_z\| = 1$. On the other hand, applying Lemma 1.5 (i) to any open subset of βX whose intersection with X is T, we obtain that $\|(1-b)+H_x\| = 0$ for all $x \in T$ since $T \subseteq X \setminus S$. Since $z \in \mathrm{cl}(T)$, it follows that the norm function of $1 - b$ is discontinuous at z. □

We are now in a position to obtain the main result of the paper which is the converse of Theorem 2.9.

Theorem 3.8. *Let A be a σ-unital continuous $C_0(X)$-algebra with structure map μ and base map ϕ, let*

$$U = \{x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x\},$$

and let $\mathrm{cl}(U)$ be the closure of U in X. Then $M(A)$ is a continuous $C(\beta X)$-algebra with structure map $\overline{\mu}$ if and only if

(i) $\mathrm{cl}(U)$ *is clopen in* X*;*
(ii) $\mathrm{cl}(U)$ *is canonically homeomorphic to a subset of* βU*;*
(iii) every point of $\mathrm{cl}(\mathrm{Im}(\phi)) \setminus \mathrm{cl}(U)$ *is a BD-point of* X*.*

Moreover, when these conditions hold, $\mathrm{cl}(\mathrm{Im}(\phi))$ *is clopen in* X *and* $\mathrm{cl}(\mathrm{Im}(\phi)) \setminus \mathrm{cl}(U)$ *is basically disconnected.*

Proof. The "if" part of the result is Theorem 2.9. Conversely, suppose that $M(A)$ is a continuous $C(\beta X)$-algebra with structure map $\overline{\mu}$. Then $\mathrm{cl}(\mathrm{Im}(\phi))$ is clopen in X (see the proof of Lemma 3.3 (i)). Also, $\mathrm{Im}(\phi)$ is open in X since A is a continuous $C_0(X)$-algebra.

Conditions (i) and (ii) follow from Propositions 3.4 and 3.5 respectively. For condition (iii), we may suppose that $\mathrm{cl}(\mathrm{Im}(\phi)) \setminus \mathrm{cl}(U)$ is nonempty (for otherwise there is nothing to prove). Let $Y = \{x \in \mathrm{Im}(\phi) \mid A/J_x \text{ is nonunital}\}$. If V is any nonempty open subset of the clopen set $\mathrm{cl}(\mathrm{Im}(\phi)) \setminus \mathrm{cl}(U)$ then $V \cap \mathrm{Im}(\phi)$ is also a nonempty open subset. Since it is disjoint from U, it must contain an element of Y by Lemma 3.6. Thus $Y \cap (\mathrm{cl}(\mathrm{Im}(\phi)) \setminus \mathrm{cl}(U))$ is dense in $\mathrm{cl}(\mathrm{Im}(\phi)) \setminus \mathrm{cl}(U)$. Since $M(A)$ is a continuous $C(\beta X)$-algebra, it follows from Lemma 3.7 that every $x \in \mathrm{cl}(\mathrm{Im}(\phi)) \setminus \mathrm{cl}(U)$ is a BD-point in X and hence is a BD-point of the clopen set $\mathrm{cl}(\mathrm{Im}(\phi)) \setminus \mathrm{cl}(U)$. □

If A is separable then we can extract some further information from Theorem 3.8.

Corollary 3.9. *Let* A *be a separable continuous* $C_0(X)$*-algebra with structure map* μ *and base map* ϕ *and let*

$$U = \{x \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_x\}.$$

If $M(A)$ *is a continuous* $C(\beta X)$*-algebra with structure map* $\overline{\mu}$ *then*
(i) U *is clopen in* $\mathrm{Im}(\phi)$*;*
(ii) every point of $\mathrm{Im}(\phi) \setminus U$ *is an isolated point of* X*.*

Conversely, if (i) and (ii) hold and $X = \mathrm{Im}(\phi)$ *then* $M(A)$ *is a continuous* $C(\beta X)$*-algebra with structure map* $\overline{\mu}$*.*

Proof. Suppose first that $M(A)$ is a continuous $C(\beta X)$-algebra with structure map $\overline{\mu}$. Then A satisfies conditions (i), (ii), and (iii) of Theorem 3.8. Furthermore, $\mathrm{Im}(\phi)$ is second countable, being the image of the second countable space $\mathrm{Prim}(A)$ [13, 3.3.4] under the continuous open map ϕ.

In particular, each point $x \in \mathrm{cl}(U) \cap \mathrm{Im}(\phi)$ has a countable neighborhood base in $\mathrm{Im}(\phi)$, and hence in X since $\mathrm{Im}(\phi)$ is open in X. Thus each $x \in \mathrm{cl}(U) \cap \mathrm{Im}(\phi)$ has a countable neighborhood base in the subspace $\mathrm{cl}(U)$ of X. But by condition (ii) of Theorem 3.8 we have the inclusions $U \subseteq \mathrm{cl}(U) \cap \mathrm{Im}(\phi) \subseteq \mathrm{cl}(U) \subseteq \beta U$. It follows, since $\mathrm{cl}(U)$ is dense in the compact space βU, that each $x \in \mathrm{cl}(U) \cap \mathrm{Im}(\phi)$ has a countable neighborhood base in βU, cp. [20, 9.7]. But no point of $\beta U \setminus U$ has a countable neighborhood base in βU [20, Cor. 9.6]. Hence $\mathrm{cl}(U) \cap \mathrm{Im}(\phi) = U$, so U is clopen in $\mathrm{Im}(\phi)$, establishing (i).

Similarly, each point $x \in \mathrm{Im}(\phi) \setminus \mathrm{cl}(U) = \mathrm{Im}(\phi) \setminus U$ has a countable neighborhood base in X, and hence in the basically disconnected space $\mathrm{cl}(\mathrm{Im}(\phi)) \setminus \mathrm{cl}(U)$.

But in a basically disconnected space, a point with a countable neighborhood base is isolated [20, 14N]. Thus each $x \in \mathrm{Im}(\phi) \setminus U$ is isolated in $\mathrm{cl}(\mathrm{Im}(\phi)) \setminus \mathrm{cl}(U)$, which is a clopen subset of X, and is therefore isolated in X itself. This establishes (ii).

For the converse, suppose that (i) and (ii) hold and that $X = \mathrm{Im}(\phi)$. Then it is trivial that conditions (i), (ii), and (iii) of Theorem 3.8 hold. Hence $M(A)$ is a continuous $C(\beta X)$-algebra with structure map $\overline{\mu}$. □

4. Applications

In this section we give some applications of Theorem 3.8. Our first application is to C*-algebras A for which the locally compact space $\mathrm{Prim}(A)$ is Hausdorff. It is well-known that A is then a continuous $C_0(\mathrm{Prim}(A))$-algebra [13, 3.9.11]. We may take $X = \mathrm{Prim}(A)$ and $\phi = \mathrm{id}$, so that μ is the restriction to $C_0(\mathrm{Prim}(A))$ of the Dauns-Hofmann isomorphism θ_A. In this case, $J_P = P$ for all $P \in \mathrm{Prim}(A)$ and the mapping $\overline{\phi} : \mathrm{Prim}(M(A)) \to \beta X$ satisfies $\overline{\phi}(\tilde{P}) = \phi(P) = P$ $(P \in \mathrm{Prim}(A))$. Since $\theta_A^{-1}(Z(A)) \subseteq C_0(\mathrm{Prim}(A))$, the set

$$U := \{P \in X \mid \mu(C_0(X)) \cap A \not\subseteq J_P\}$$

takes the form $U = \{P \in \mathrm{Prim}(A) \mid Z(A) \not\subseteq P\}$, where $Z(A)$ is the center of A. With this notation, we immediately obtain the following corollary from Theorem 3.8.

Theorem 4.1. *Let A be a σ-unital C^*-algebra with Hausdorff primitive ideal space $X = \mathrm{Prim}(A)$ and let $U = \{P \in \mathrm{Prim}(A) \mid Z(A) \not\subseteq P\}$. Then $M(A)$ is a continuous $C(\beta X)$-algebra with base map $\overline{\mathrm{id}} : \mathrm{Prim}(M(A)) \to \beta X$ if and only if*

(i) $\mathrm{cl}(U)$ *is clopen in* $\mathrm{Prim}(A)$*;*
(ii) $\mathrm{cl}(U)$ *is canonically homeomorphic to a subset of* βU*;*
(iii) every point of $\mathrm{Prim}(A) \setminus \mathrm{cl}(U)$ *is a BD-point of* $\mathrm{Prim}(A)$*.*

The conditions in Theorem 4.1 simplify substantially in the case $U = \varnothing$, which holds if and only if $Z(A) = \{0\}$. In particular this applies when A is a stable C*-algebra.

Corollary 4.2. *Let A be a σ-unital C^*-algebra with Hausdorff primitive ideal space $X = \mathrm{Prim}(A)$ and suppose that $Z(A) = \{0\}$ (e.g. if A is stable). Then $M(A)$ is a continuous $C(\beta X)$-algebra with base map $\overline{\mathrm{id}} : \mathrm{Prim}(M(A)) \to \beta X$ if and only if $\mathrm{Prim}(A)$ is basically disconnected.*

A second countable, basically disconnected space is discrete [20, 14.N], so Corollary 4.2 implies the following.

Corollary 4.3. *Let A be a stable separable C^*-algebra with Hausdorff primitive ideal space $X = \mathrm{Prim}(A)$. Then $M(A)$ is a continuous $C(\beta X)$-algebra with base map $\overline{\mathrm{id}} : \mathrm{Prim}(M(A)) \to \beta X$ if and only if $\mathrm{Prim}(A)$ is discrete.*

Theorem 4.1 raises the question of characterizing the space βX and investigating the nature of the ideals H_x of $M(A)$. In fact, it is a consequence of the Dauns-Hofmann theorem that βX in Theorem 4.1 is homeomorphic to the maximal ideal space Δ of $ZM(A)$ which is in turn homeomorphic to the complete regularization Glimm$(M(A))$ of Prim$(M(A))$. The ideals H_x of $M(A)$ ($x \in \beta X$) are the Glimm ideals of M(A) (generated by the ideals in Δ [21]).

It is easy to see that the ideals H_x need not be maximal ideals in $M(A)$ even when Prim(A) is Hausdorff. For example, if $A = LC(H)$, the algebra of compact linear operators on an infinite-dimensional Hilbert space, then X (= βX) is a singleton set containing the zero ideal $\{0\}$, and $H_{\{0\}} = \{0\}$ which is a primitive but nonmaximal ideal of $M(A) = B(H)$. This phenomenon occurs whenever A has a nonunital primitive quotient.

Even when all the primitive quotients of A are unital and Prim(A) is Hausdorff, it is still possible for Prim$(M(A))$ to be non-Hausdorff.

Example 4.4. Let $B = C_r^*(F_2)$, where F_2 is the free group on two generators, and let $A = c_0 \otimes B$. Then Prim(A) is homeomorphic to $\mathbb{N}$ and hence is Hausdorff, and the set U of Theorem 4.1 is equal to Prim(A) so $M(A)$ is a continuous $\beta\mathbb{N}$-algebra. For $x \in \beta\mathbb{N} \setminus \mathbb{N}$, the quotient $M(A)/H_x$ is an ultrapower of B, and hence is a primitive but nonsimple C*-algebra [19, Thm. 5.4, Cor. 5.5]. Thus Prim$(M(A))$ is non-Hausdorff.

Our next application is to quasi-standard C*-algebras. These can be defined in various equivalent ways but perhaps the easiest one for our present purposes is that A is *quasi-standard* if A is a continuous $C_0(X)$-algebra with $X = \mathrm{Im}(\phi)$ such that J_x is a primal ideal of A for each $x \in \mathrm{Im}(\phi)$ [6, Thm. 3.4]. Recall that a closed two-sided ideal J of a C^*-algebra A is *primal* if, whenever $n \geq 2$ and $I_1, \dots, I_n$ are closed two-sided ideals of A with product zero, then there exists $j \in \{1, \dots, n\}$ such that $I_j \subseteq J$ [5]. Every primitive ideal is prime and hence primal, so the algebra $M(A)$ in Example 4.4 is quasi-standard. Similarly, if Prim(A) is Hausdorff then A is quasi-standard. Von Neumann algebras are quasi-standard and so too are many group C*-algebras, for example those of the discrete and continuous Heisenberg groups [21], [26], [1], [23].

The main reason for considering primal ideals in this context is that they are the limits of nets of primitive ideals in an appropriate topology [4]. Thus, if A is a continuous $C_0(X)$-algebra and J_x is primitive for x in a dense subset of X then J_x will be primal for all $x \in X$, and a converse statement holds if A is separable [6, 3.4, 3.5]. Primal ideals have found a number of other applications in the theory of C^*-algebras. It was shown in [5] that a state of a C^*-algebra is a weak*-limit of factorial states if and only if the kernel of the GNS representation is a primal ideal. Primal ideals play a crucial role in the solution of the isometry problem for the central Haagerup tensor product [9] and in the study of norms of inner derivations [29], [7], [8].

Lemma 4.5. *Let J be a proper, closed, two-sided ideal of a C^*-algebra A and suppose that J is a primal ideal of A. Then $\tilde{J}$ is a primal ideal of $M(A)$.*

Proof. Suppose that $n \geq 2$ and $I_1, \dots, I_n$ are closed two-sided ideals of $M(A)$ such that $I_1 I_2 \dots I_n = \{0\}$. Then

$$(I_1 \cap A)(I_2 \cap A) \dots (I_n \cap A) = \{0\}$$

and so there exists j such that $I_j \cap A \subseteq J$. For $b \in I_j$ and $a \in A$, we have $ab, ba \in I_j \cap A \subseteq J$ and so $b \in \tilde{J}$. Thus $I_j \subseteq \tilde{J}$. □

Proposition 4.6. *Let A be a continuous $C_0(X)$-algebra with base map ϕ such that J_x is a primal ideal of A for all $x \in X$. Let $y \in \beta X$ and suppose that for all $b \in M(A)$ the function $x \to \|b + H_x\|$ $(x \in \beta X)$ is continuous at y. Then H_y is a primal ideal of $M(A)$.*

Proof. Suppose that $n \geq 2$ and $b_1, \dots, b_n \in M(A) \setminus H_y$. There exists an open neighborhood V of y in βX such that $\|b_j + H_x\| > 0$ for $1 \leq j \leq n$ and all $x \in V$. For $1 \leq j \leq n$, let

$$U_j := \{x \in X \mid \|b_j + \tilde{J}_x\| > 0\} = \phi(\{P \in \mathrm{Prim}(A) \mid \|b_j + \tilde{P}\| > 0\}).$$

Since A is a continuous $C_0(X)$-algebra, ϕ is open and so U_j is an open subset of X. But since X is locally compact, it is open in βX [20, 3.15(d)] and so U_j is open in βX.

Let W be a nonempty open subset of V and let $x \in W$. By Lemma 1.5 (i),

$$0 < \|b_j + H_x\| \leq \sup\{\|b_j + \tilde{J}_t\| \mid t \in W \cap X\}.$$

So $U_j \cap W$ is nonempty and hence $U_j \cap V$ is a dense open subset of V. It follows that $\bigcap_{j=1}^n U_j$ is a nonempty subset of X. So there exists $x \in X$ such that, for $1 \leq j \leq n$, $b_j \notin \tilde{J}_x$.

By Lemma 4.5, $\tilde{J}_x$ is a primal ideal of $M(A)$ and so

$$b_1 M(A) b_2 M(A) \dots b_{n-1} M(A) b_n \neq \{0\}.$$

It follows that H_y is a primal ideal of $M(A)$. □

One important fact about a quasi-standard C*-algebra A is that the space X such that A is a continuous $C_0(X)$-algebra with $X = \mathrm{Im}(\phi)$ and J_x primal for all $x \in X$ is unique. Indeed X is the complete regularization of $\mathrm{Prim}(A)$ [20, 3.9], [6, 3.3 and 3.4]. For a general C*-algebra A, let $\phi_A : \mathrm{Prim}(A) \to X$ denote the complete regularization map. If A is not quasi-standard then X need not be locally compact. However, it is always possible to form $J_x := \bigcap\{P \in \mathrm{Prim}(A) \mid \phi_A(P) = x\}$ for each $x \in X$. The ideals J_x $(x \in X)$ are called the *Glimm ideals* of A and we set $\mathrm{Glimm}(A) = \{J_x \mid x \in X\}$, with the complete regularization topology.

If A is quasi-standard then $\mathrm{Glimm}(A)$ coincides with the space of minimal primal ideals of A. For convenience, we take $X = \mathrm{Glimm}(A)$ in this case. The corresponding structure map $\mu : C_0(X) \to ZM(A)$ is given by $\mu(f) = \theta_A(f \circ \phi_A)$ $(f \in C_0(X))$. For each $G \in X = \mathrm{Glimm}(A)$,

$$J_G = \bigcap\{P \in \mathrm{Prim}(A) \mid \phi_A(P) = G\} = \bigcap\{P \in \mathrm{Prim}(A) \mid P \supseteq G\} = G.$$

Thus the set U of Section 2 is defined by

$$U = \{G \in \mathrm{Glimm}(A) \mid \mu(C_0(\mathrm{Glimm}(A))) \cap A \not\subseteq G\}.$$

Clearly, $\mu(C_0(\mathrm{Glimm}(A))) \cap A \subseteq ZM(A) \cap A = Z(A)$. Conversely, suppose that $z \in Z(A)$ and let $h = \theta_A^{-1}(z) \in C^b(\mathrm{Prim}(A)$. Then $h(P)1 + \tilde{P} = z + \tilde{P}$ for all $P \in \mathrm{Prim}(A)$. The function h induces $f \in C^b(\mathrm{Glimm}(A))$ such that $h = f \circ \phi_A$. Let $\epsilon > 0$. Since $z \in A$, there exists a compact subset K of $\mathrm{Prim}(A)$ such that

$$|h(P)| = \|z + \tilde{P}\| = \|z + P\| < \epsilon \qquad (P \in \mathrm{Prim}(A)).$$

Then $\phi_A(K)$ is a compact subset of $\mathrm{Glimm}(A)$ such that $|f(G)| < \epsilon$ for all $G \in \mathrm{Glimm}(A) \setminus \phi_A(K)$. Thus $f \in C_0(\mathrm{Glimm}(A))$ and

$$\mu(f) = \theta_A(f \circ \phi_A) = \theta(h) = z.$$

It follows that

$$U = \{G \in \mathrm{Glimm}(A) \mid Z(A) \not\subseteq G\}.$$

The existence of a homeomorphism between $\beta\,\mathrm{Glimm}(A)$ and $\mathrm{Glimm}(M(A))$ is well-known (see, for example, [2, p. 88]) but we provide some details in order to establish equation (1) below.

Proposition 4.7. *Let A be a C^*-algebra. Then there is a homeomorphism*

$$\iota : \beta\,\mathrm{Glimm}(A) \to \mathrm{Glimm}(M(A))$$

such that

$$\iota(\phi_A(P)) = \phi_{M(A)}(\tilde{P}) \qquad (P \in \mathrm{Prim}(A)). \tag{1}$$

Proof. Applying the Dauns-Hofmann theorem both to A and to $M(A)$, we obtain *-isomorphisms

$$\Phi : C(\beta\,\mathrm{Glimm}(A)) \to ZM(A) \quad \text{and} \quad \Psi : C(\mathrm{Glimm}(M(A))) \to ZM(A)$$

such that $\Phi(f) = \theta_A(f \circ \phi_A)$ and $\Psi(g) = \theta_{M(A)}(g \circ \phi_{M(A)})$. By the Banach-Stone theorem, there exist homeomorphisms

$$j : \beta\,\mathrm{Glimm}(A) \to \Delta := \mathrm{Max}(ZM(A)) \quad \text{and} \quad k : \Delta \to \mathrm{Glimm}(M(A))$$

such that

$$\Phi(f) + m = f(j^{-1}(m))1 + m \quad \text{and} \quad \Psi(g) + m = g(k(m))1 + m$$

for all $m \in \Delta$, $f \in C(\beta\,\mathrm{Glimm}(A))$ and $g \in C(\mathrm{Glimm}(M(A)))$. We define $\iota = k \circ j$.

Let $P \in \mathrm{Prim}(A)$ and set $m = \tilde{P} \cap ZM(A) \in \Delta$. Let $f \in C(\beta\,\mathrm{Glimm}(A))$ and write $z = \Phi(f)$. Since

$$z - f(\phi_A(P))1 \in \tilde{P} \cap ZM(A) = m$$

and $z - f(j^{-1}(m))1 \in m$, we obtain $f(\phi_A(P)) = f(j^{-1}(m))$. Since f was arbitrary, $\phi_A(P) = j^{-1}(m)$.

Now let $g \in C(\mathrm{Glimm}(M(A)))$ and write $z = \Psi(g)$. Since

$$z - g(\phi_{M(A)}(\tilde{P}))1 \in \tilde{P} \cap ZM(A) = m$$

and $z - g(k(m))1 \in m$, we obtain $g(\phi_{M(A)}(\tilde{P})) = g(k(m))$. Since g was arbitrary, $\phi_{M(A)}(\tilde{P}) = k(m)$. Hence

$$\iota(\phi_A(P)) = (k \circ j)(\phi_A(P)) = k(m) = \phi_{M(A)}(\tilde{P}).$$

□

Theorem 4.8. *Let A be a σ-unital C^*-algebra and let*

$$U = \{G \in \mathrm{Glimm}(A) \mid Z(A) \not\subseteq G\}.$$

Then $M(A)$ is quasi-standard if and only if

(i) A is quasi-standard;
(ii) $\mathrm{cl}(U)$ is clopen in $\mathrm{Glimm}(A)$;
(iii) $\mathrm{cl}(U)$ is canonically homeomorphic to a subset of βU;
(iv) $\mathrm{Glimm}(A) \setminus \mathrm{cl}(U)$ is basically disconnected.

Proof. Suppose that conditions (i)-(iv) hold. By Theorem 3.8, $M(A)$ is a continuous $C(\beta X)$-algebra (with $X = \mathrm{Glimm}(A)$) with respect to the surjective mapping $\overline{\phi_A} : \mathrm{Prim}(M(A)) \to \beta X$. Since A is quasi-standard, every $G \in \mathrm{Glimm}(A)$ is a primal ideal of A. By Proposition 4.6, H_x is a primal ideal of $M(A)$ for all $x \in \beta X$. Since $\overline{\phi_A}$ is surjective, $H_x \neq M(A)$ for all $x \in \beta X$ and so it follows from [6, Thm. 3.4] that $M(A)$ is quasi-standard.

Conversely, suppose that $M(A)$ is quasi-standard, and set $Y = \mathrm{Glimm}(M(A))$. Then $M(A)$ is a continuous $C(Y)$-algebra with respect to the complete regularization map $\phi_{M(A)} : \mathrm{Prim}(M(A)) \to Y$. By Proposition 4.7, there exists a homeomorphism $\iota : \beta\,\mathrm{Glimm}(A) \to \mathrm{Glimm}(M(A))$ such that $\iota(\phi_A(P)) = \phi_{M(A)}(\tilde{P})$ for all $P \in \mathrm{Prim}(A)$. Since $\{\tilde{P} \mid P \in \mathrm{Prim}(A)\}$ is a dense subset of $\mathrm{Prim}(M(A))$, it follows by continuity that $\iota \circ \overline{\phi_A} = \phi_{M(A)}$. Hence $M(A)$ is a continuous $C(\beta\,\mathrm{Glimm}(A))$-algebra with respect to $\overline{\phi_A}$. It follows from Theorem 3.8 that conditions (ii)-(iv) hold. Finally, A is quasi-standard because it is an ideal of $M(A)$ (see [6, p. 356]). □

As with Theorem 4.1, the conditions in Theorem 4.8 simplify substantially in the case $U = \varnothing$, which holds if and only if $Z(A) = \{0\}$.

Corollary 4.9. *Let A be a σ-unital C^*-algebra with $Z(A) = \{0\}$. Then $M(A)$ is quasi-standard if and only if*

(i) A is quasi-standard;
(ii) $\mathrm{Glimm}(A)$ is basically disconnected.

At the other extreme, recall that a C^*-algebra is *quasi-central* if $Z(A) \not\subseteq P$ for all $P \in \mathrm{Prim}(A)$ [12]. It is easily seen that A is quasi-central if and only if $U = \mathrm{Glimm}(A)$. In the following result, we do not need to assume that A is σ-unital.

Corollary 4.10. *Let A be a quasi-central C^*-algebra. Then $M(A)$ is quasi-standard if and only if A is quasi-standard.*

Proof. Suppose that A is quasi-standard. Since $U = \mathrm{Glimm}(A)$, it follows from Propositions 2.3 and 2.8 that $M(A)$ is a continuous $C(\beta X)$-algebra (with $X = \mathrm{Glimm}(A)$) with respect to the surjective mapping $\overline{\phi_A} : \mathrm{Prim}(M(A)) \to \beta X$. Hence, as in the proof of Theorem 4.8, $M(A)$ is quasi-standard. □

In particular, if A is an n-homogeneous C*-algebra then $\mathrm{Prim}(A)$ is Hausdorff and A is quasi-central so it follows from Corollary 4.10 that $M(A)$ is quasi-standard.

Now let A be a σ-unital subhomogeneous C*-algebra. Since every nonzero ideal in A is subhomogeneous, and therefore contains a nonzero homogeneous ideal, it follows that the set $\{P \in \mathrm{Prim}(A) \mid P \not\supseteq Z(A)\}$ is dense in $\mathrm{Prim}(A)$, and hence that the set U of Theorem 4.8 is automatically dense in $\mathrm{Glimm}(A)$. Thus conditions (ii) and (iv) are automatically trivially satisfied and we have the following.

Corollary 4.11. *Let A be a σ-unital subhomogeneous C*-algebra and let $U = \{G \in \mathrm{Glimm}(A) \mid Z(A) \not\subseteq G\}$. Then $M(A)$ is quasi-standard if and only if*

(i) A is quasi-standard;
(ii) $\mathrm{Glimm}(A)$ is canonically homeomorphic to a subset of βU.

If A in Corollary 4.11 is also separable and $M(A)$ is quasi-standard then it follows from condition (i) of Corollary 3.9 that the dense set U equals $\mathrm{Glimm}(A)$, and hence that A is quasi-central. Thus we have that the multiplier algebra of a separable, subhomogeneous, C*-algebra A is quasi-standard if and only if A is quasi-standard and quasi-central.

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