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A GOING-DOWN PRINCIPLE FOR AMPLE GROUPOIDS AND APPLICATIONS

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A GOING-DOWN PRINCIPLE FOR AMPLE GROUPOIDS AND APPLICATIONS

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Abstract. We study a Going-Down (or restriction) principle for ample groupoids and its applications. The Going-Down principle for locally compact groups was developed by Chabert, Echterhoff and Oyono-Oyono and allows to study certain functors, that arise in the context of the topological K-theory of a locally compact group, in terms of their restrictions to compact subgroups. We extend this principle to the class of ample Hausdorff groupoids using Le Gall's groupoid equivariant version of Kasparov's bivariant KK-theory. Moreover, we provide a number of applications in connection with the Baum-Connes conjecture for ample groupoids.

Zusammenfassung. Diese Arbeit beschäftigt sich mit einem sogenannten Going-Down Prinzip für total unzusammenhängende étale Gruppoide und seinen Anwendungen. Das Going-Down Prinzip für lokalkompakte Gruppen wurde von Chabert, Echterhoff und Oyono-Oyono entwickelt und erlaubt es gewisse Funktoren, die im Zusammenhang mit der topologischen K-theorie einer lokalkompakten Gruppe stehen, mithilfe ihrer Einschränkung auf kompakte Untergruppen zu studieren. In dieser Arbeit wird Le Galls äquivariante Version von Kasparovs bivarianter KK-Theorie verwendet, um dieses Prinzip auf die Klasse der total unzusammenhängenden étalen Gruppoide auszudehnen. Darüberhinaus werden eine Reihe von Anwendungen dieses Prinzips präsentiert, die größtenteils im Zusammenhang zur Baum-Connes Vermutung für Gruppoide stehen.

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Contents

Introduction

From its very beginning the theory of operator algebras has benefited from influences from the theory of dynamical systems. The foundations have been laid already in the work of Murray and von Neumann, through the classical *group measure space construction*, which associates a von Neumann algebra to every measure preserving action of a countable group on a probability space. Similarly, to a topological dynamical system, i.e. a continuous action of a locally compact group on a locally compact Hausdorff space, one can associate C^*-algebras in a canonical way using the crossed product construction. The study of crossed products yields a vast pool of examples of C^*-algebras exhibiting different behaviours, which are closely linked to the properties of the underlying dynamical system. But the study of these C^*-algebras is also interesting in its own right, as they fit into the framework of Connes' noncommutative geometry (confer [**Con94**]). The idea behind this program is to extend the classical link between geometric spaces and commutative algebras to cases, where the classical theory is no longer fruitful. Connes suggested that badly behaved spaces (often arising as some kind of quotient space) should be studied by replacing the space by naturally associated noncommutative operator algebras. One example comes from abstract harmonic analysis, where one replaces the space of irreducible unitary representations by the (full) group C^*-algebra. More generally, following Connes' philosophy, the crossed product associated to a topological dynamical system should replace the orbit space of the system. By now there is a whole zoo of constructions associating C^*-algebras to different kinds of mathematical objects, including groups, semigroups, dynamical systems, (higher rank) graphs, coarse spaces, and quasicrystals. For the most part the C^*-algebras associated to these various objects have been defined and studied separately. Only after Renault's work [**Ren80**] they were slowly realized to fit into a more general framework: Groupoids and their C^*-algebras.

Groupoids simultaneously generalize groups and dynamical systems. In fact, they are much more powerful than that: While groups can be thought of as *global* symmetries of a set, space or geometric object, groupoids can describe its *local* symmetries. Consequently, it is not surprising that one can naturally associate groupoids to partial actions and inverse semigroups. As mentioned before, groupoids have also been used to study aperiodic tilings, (higher rank) graphs (see [**KP00, KPRR97**]) and large scale geometry (see [**STY02**]). The fact that there are so many different classes of examples has led to a fruitful back and forth dynamic between the general theory and the different applications over the last years.

One important step in the study of C*-algebras is the computation of its K-theory. This is a notoriously difficult problem, especially for group C*-algebras and crossed products. Baum, Connes, and Higson present in [**BCH94**] a general method to attack this problem:

If G is a locally compact, second countable group and A is a C*-algebra equipped with a strongly continuous action of G by $*$-automorphisms, the topological K-theory of G with coefficients in A is defined as

$$\mathrm{K}_*^{\mathrm{top}}(G; A) := \lim_{X \subseteq \mathcal{E}(G)} \mathrm{KK}_*^G(C_0(X), A),$$

where X runs through the G-compact (i.e. the quotient space X/G is compact) subspaces of a universal proper G-space $\mathcal{E}(G)$ ordered by inclusion, and KK_*^G denotes Kasparov's equivariant KK-theory. The authors in [**BCH94**] then proceed to construct a group homomorphism

$$\mu_A : \mathrm{K}_*^{\mathrm{top}}(G; A) \to \mathrm{K}_*(A \rtimes_r G).$$

This map is usually called the *assembly map* and the Baum-Connes conjecture asserts, that μ_A is an isomorphism. By work of Higson, Lafforgue and Skandalis (see [**HLS02**]) it is now known, that the conjecture is false in this generality. It has however been proven to be true for large classes of groups including the class of amenable groups and the conjecture with trivial coefficients (i.e. $A = \mathbb{C}$) is still open.

In [**LG94**], Le Gall introduced a groupoid equivariant version of Kasparov's KK-theory, which was subsequently used to define a version of the Baum-Connes assembly map for groupoids. The question, when this map is an isomorphism has been investigated by Tu in [**Tu99a,**

Tu99b]. He proves that the Baum-Connes conjecture is true for every locally compact, σ-compact Hausdorff groupoid acting continuously and isometrically on a continuous field of affine Euclidean spaces. The latter condition is fulfilled in particular by all amenable groupoids. On the other hand, the groupoid version of the Baum-Connes conjecture is known to be false even in the case of trivial coefficients (again by results in [**HLS02**]).

In the case of locally compact groups, Chabert started in [**Cha00**] to study permanence properties of the Baum-Connes conjecture for the case of semi-direct products. In subsequent work of Chabert and Echterhoff (see [**CE01**]) these methods were refined and it was proved that the class of groups satisfying the conjecture is stable under taking subgroups, Cartesian products, and certain group extensions. A similar approach was used in [**CEOO03**] to prove that the topological K-theory of a transformation groupoid $G \ltimes X$ does not depend on X, i.e. that the canonical forgetful map $\mathrm{K}_*^{\mathrm{top}}(G \ltimes X; A) \to \mathrm{K}_*^{\mathrm{top}}(G; A)$ is an isomorphism. Finally, in [**CEOO04**], the authors formalize the methods used to prove the main results in all of the above mentioned work and abstractly develop the so called Going-Down principle, which allows to analyse certain functors connected to the topological K-theory of a locally compact group in terms of their restrictions to compact subgroups. The Going-Down principle has turned out to be very useful in the computation of the K-theory of certain C*-algebras, for example crossed products of the irrational rotation algebras by finite subgroups of $SL_2(\mathbb{Z})$ (see [**ELPW10**]) or the C*-algebras associated to a large class of semigroups (see [**CEL13, CEL15**]).

The starting point of this thesis is the work of Tu, who proves in [**Tu12**] an analogue of the main result of [**CEOO03**] for second countable, locally compact, étale groupoids and uses it to show that satisfying the Baum-Connes conjecture passes to subgroupoids (within this class). Inspired by the ideas in this work we set out to develop a general Going-Down principle in the spirit of [**CEOO04**] for the class of ample groupoids. Although it seems plausible, that similar results can be obtained for all étale groupoids, there are a lot of topological difficulties yet to overcome. In the case of ample groupoids however these difficulties disappear and the theory can be developed beautifully.

Many interesting examples studied in the literature fall naturally into the class of ample groupoids.

At this point, let me summarize the main results obtained in this thesis and simultaneously give an overview of how this dissertation is organized.

The first chapter is dedicated to remind the reader of the basic notions in the theory of groupoids and their actions. We give all the definitions and results that are needed to understand the rest of this work.

Besides reviewing some basics about groupoid dynamical systems and crossed products, the second chapter mainly focuses on a detailed study of induced algebras. One way to look at the induced algebras we are interested in is to use the picture of pullbacks along generalized morphisms of groupoids as developed by Le Gall in [**LG94**]. We however chose to develop the theory in analogy to the classical approach in the group case, which seems to be more useful for our purposes. To the best of our knowledge this approach has not been carried out before in the literature.

Chapter three is dedicated to the study of Le Gall's groupoid equivariant version of Kasparov's KK-theory. We prove a generalization of a result of Meyer (see [**Mey00**]) on when the operator in an equivariant Kasparov tripel can be chosen to be invariant for the action of the groupoid. We then proceed to prove one of the main technical ingredients in the proof of the Going-Down principle. It says that a canonically defined compression homomorphism comp_H^G is an isomorphism:

Theorem A. *(see Theorem 3.6.2) Let G be an étale, locally compact Hausdorff groupoid with a clopen, proper subgroupoid $H \subseteq G$. Let $X := G_{H^{(0)}}$. If A is an H-algebra and B is a G-algebra, then*

$$\mathsf{comp}_H^G : \mathrm{KK}^G(Ind_H^X A, B) \to \mathrm{KK}^H(A, B_{|H^{(0)}})$$

is an isomorphism.

The fourth chapter focuses solely on the proof of the Going-Down principle for ample groupoids. For convenience we first prove the following special case to illustrate the necessary steps in the proof, before we proceed to the abstract (and most general) categorical picture of Going-Down functors:

THEOREM B. *(see Theorem 4.3.7) Suppose G is an ample, locally compact Hausdorff groupoid and A and B are G-algebras. Suppose there is an element $x \in \mathrm{KK}^G(A,B)$ such that*

$$\mathrm{KK}^H(C(H^{(0)}), A_{|H}) \overset{\cdot\otimes res_H^G(x)}{\rightarrow} \mathrm{KK}^H(C(H^{(0)}), B_{|H})$$

is an isomorphism for all compact open subgroupoids $H \subseteq G$. Then the Kasparov-product with x induces an isomorphism

$$\cdot \otimes x : \mathrm{K}_*^{\mathrm{top}}(G;A) \to \mathrm{K}_*^{\mathrm{top}}(G;B).$$

The proof proceeds by reducing the statement in three steps:

(1) In the first step we use a specific approximation of the universal proper G-space by finite dimensional G-simplicial complexes.
(2) Using the existence of long exact sequences in KK^G-theory and Bott periodicity we reduce our problem to the case of zero-dimensional G-simplicial complexes.
(3) In the zero-dimensional case one can use a Mayer-Vietoris type argument to restrict attention to the even more special case that the space in question is the G-saturation of a compact open set. To this compact open set we can canonically associate a compact open subgroupoid H of G such that X is the induced space of an H-space. Finally, we apply the compression isomorphism to obtain the result.

The final chapter of the thesis is dedicated to several applications of the Going-Down principle: The first application concerns the continuity of topological K-theory with respect to the coefficient algebra and is inspired by [**CE01**, §7]:

THEOREM C. *(see Theorem 5.1.2) Let G be an ample groupoid and (A_n, φ_n) an inductive sequence of G-algebras. If we let $A = \lim A_n$, then the maps $\psi_{n,*} : \mathrm{K}_*^{\mathrm{top}}(G;A_n) \to \mathrm{K}_*^{\mathrm{top}}(G;A)$ induced by the canonical maps $\psi_n : A_n \to A$, give rise to an isomorphism*

$$\lim_{n\to\infty} \mathrm{K}_*^{\mathrm{top}}(G;A_n) \cong \mathrm{K}_*^{\mathrm{top}}(G;A).$$

As an immediate consequence we obtain the following permanence property for the Baum-Connes conjecture:

Corollary D. *Let G be an ample groupoid and (A_n, φ_n) an inductive sequence of G-algebras with $A = \lim_{n\to\infty} A_n$. Suppose G satisfies the Baum-Connes conjecture with coefficients in A_n for all $n \in \mathbb{N}$. Assume further, that G is exact, or that all the connecting homomorphisms φ_n are injective. Then G satisfies the Baum-Connes conjecture with coefficients in A.*

The second application revolves around the recent notion of (strong) amenability at infinity for étale groupoids as introduced by Lassagne in [**Las14**] (see also [**AD16**]). Based on ideas of Higson (see [**Hig00**]) we prove the following result:

Theorem E. *(see Theorem 5.2.3) Let G be a second countable ample groupoid, which is strongly amenable at infinity and let A be a G-algebra. Then the Baum-Connes assembly map*

$$\mu_A : \mathrm{K}_*^{\mathrm{top}}(G; A) \to \mathrm{K}_*(A \rtimes_r G)$$

is split injective.

As an application of this result we study the Baum-Connes conjecture for ample group bundles and relate the Baum-Connes conjecture for each of the fibre groups to the Baum-Connes conjecture for the whole group bundle. More precisely, we prove the following:

Theorem F. *(see Theorem 5.2.11) Let G be a second countable ample group bundle, which is strongly amenable at infinity. Suppose A is a G-algebra such that the associated C^*-bundle is continuous, and G_u^u satisfies the Baum-Connes conjecture with coefficients in A_u for all $u \in G^{(0)}$. Then G satisfies the Baum-Connes conjecture with coefficients in A.*

In a third application we study the effect of a continuous deformation of a 2-cocycle on an ample groupoid G on the K-theory of its associated twisted groupoid C*-algebra. This question has been addressed in the case of groups in [**ELPW10**] and for different classes of groupoids in [**Gil15a, Gil15b, Gil16**]. Using the machinery developed in this thesis we can prove:

Theorem G. *(see Theorem 5.3.8) Let G be a second countable ample groupoid, which satisfies the Baum-Connes conjecture with coefficients and let Σ be a continuous homotopy of twists for G. Then the* *canonical map* $q_t : C_r^*(G \times [0,1]; \Sigma) \to C_r^*(G, \Sigma_t)$ *given by evaluation induces an isomorphism*

$$(q_t)_* : K_*(C_r^*(G \times [0,1]; \Sigma)) \to K_*(C_r^*(G, \Sigma_t)).$$

Finally, for the last application of our results we study the relation between the Baum-Connes conjecture for an ample groupoid G with coefficients in A and the Künneth formula for the K-theory of tensor products by $A \rtimes_r G$. Following the strategy of [**CEOO04**], we define a mixed Künneth formula

$$0 \to \mathrm{K}_*^{\mathrm{top}}(G; A) \otimes \mathrm{K}_*(B) \overset{\alpha_G}{\to} \mathrm{K}_*^{\mathrm{top}}(G; A \otimes B) \overset{\beta_G}{\to} \mathrm{Tor}(\mathrm{K}_*^{\mathrm{top}}(G; A), \mathrm{K}_*(B)) \to 0$$

and introduce the class $\mathcal{N}_G$ of all separable exact C*-algebras, for which the above sequence is exact. Using the Baum-Connes assembly map, one can relate this sequence to the ordinary Künneth formula for $A \rtimes_r G$ and B. We show the following:

Theorem H. *(see Theorem 5.4.11 and Corollary 5.4.12) Let* G *be an ample groupoid and* A *a separable and exact* G*-algebra. Suppose that* $A_{|K} \rtimes K$ *satisfies the (ordinary) Künneth formula for all compact open subgroupoids* $K \subseteq G$*. Then* $A \in \mathcal{N}_G$*. In particular, if the fibre* A_x *is type I for all* $x \in G^{(0)}$*, then* $A \in \mathcal{N}_G$*.*

Apart from the results mentioned above, I have also engaged in collaboration with others resulting in the preprints [**BL17, BCHL17**]. Although the results obtained in [**BL17**] also deal with groupoids and their associated C*-algebras, they are not directly related to the contents of this dissertation, as they deal with the structural properties of these algebras rather than K-theory. To keep this dissertation self-contained we will not discuss these results here and refer the interested reader to the corresponding preprints.

CHAPTER 1

Groupoids

This chapter is dedicated to the basic notions of groupoid theory. In section 1.1 we will review the definition of a groupoid, recall some basic, yet important, facts and examples. We will also discuss some properties a groupoid can enjoy. Note, that none of the results in this section is new. We mostly follow the books [**Ren80**] and [**Pat99**] for our exposition and have included proofs of several basic facts to keep this thesis self-contained. The second section reviews actions of groupoids on spaces.

1.1. Basics

There are at least two ways to define the concept of a groupoid and view it as a generalization of a group: The first option, is to consider a group as a category with only one object, the singleton set containing the unit, where the morphisms are the elements of the group. By the very definition of a group every morphism in this category is invertible. The definition of a groupoid just replaces the single object by a set of objects. Thus, one definition of groupoids reads: A *groupoid* is a small category in which every morphism is invertible.

We however, will use another definition, which we believe is more suitable for the 'working mathematician', although it might seem more complicated at first sight. This definition takes the point of view, that groupoids are like groups, where the multiplication is only partially defined. Our definition is taken from [**Pat99**, Page 7], but it first appeared in a paper of Hahn, who in turn attributes it to a conversation with Mackey.

Definition 1.1.1. A *groupoid* is a set G together with a subset $G^{(2)} \subseteq G\times G$, called the set of *composable pairs,* a *product* map $(g,h) \mapsto gh$ from $G^{(2)}$ to G and an *inverse* map $g \mapsto g^{-1}$ from G onto G, such that the following hold:

(1) The product is associative: If $(g_1, g_2), (g_2, g_3) \in G^{(2)}$ for some $g_1, g_2, g_3 \in G$, then we also have $(g_1g_2, g_3), (g_1, g_2g_3) \in G^{(2)}$ and
$$(g_1g_2)g_3 = g_1(g_2g_3)$$
(2) The inverse map is involutive, i.e. $(g^{-1})^{-1} = g$ for all $g \in G$.
(3) $(g, g^{-1}) \in G^{(2)}$ for all $g \in G$ and if $(g, h) \in G^{(2)}$, then
$$g^{-1}(gh) = h \quad \text{and} \quad (gh)h^{-1} = g.$$

The fact that multiplication is partially defined implies that multiple elements may act as (partial) units:

DEFINITION 1.1.2. The set $G^{(0)} := \{g \in G \mid g = g^{-1} = g^2\}$ is called the *set of units* in G. There are canonical maps $d : G \to G^{(0)}$ given by $d(g) = g^{-1}g$ and $r : G \to G^{(0)}$ given by $r(g) = gg^{-1}$, called the *domain* and *range map* respectively.

It is straightforward to show the following basic properties concerning the interplay of domain and range maps with the multiplication (see for example [**Sim17**, Lemmata 2.1.2, 2.1.3, 2.1.4] for details):

LEMMA 1.1.3. *Let G be a groupoid. Then the following hold:*
(1) $(g, h) \in G^{(2)}$ if and only if $d(g) = r(h)$.
(2) For $(g, h) \in G^{(2)}$ we have $d(gh) = d(h)$ and $r(gh) = r(g)$.
(3) We have $d(g^{-1}) = r(g)$ and $r(g^{-1}) = d(g)$ for all $g \in G$.
(4) If $(g, h) \in G^{(2)}$, then $(h^{-1}, g^{-1}) \in G^{(2)}$ and $(gh)^{-1} = h^{-1}g^{-1}$.
(5) $d(G), r(G) \subseteq G^{(0)}$ and $d(u) = u = r(u)$ for all $u \in G^{(0)}$.
(6) We have $r(g)g = g$ and $gd(g) = g$ for all $g \in G$.

For subsets $A, B \in G^{(0)}$ we will write $G_A := d^{-1}(A)$, $G^B := r^{-1}(B)$ and $G_A^B := G_A \cap G^B$. If A (and/or B) consists just of a single unit $u \in G^{(0)}$ we will omit the braces (e.g.: we will write $G^u := r^{-1}(\{u\})$).

In this thesis we will be concerned with topological groupoids: We say that G is a locally compact Hausdorff groupoid, if G is a groupoid, which is equipped with a locally compact Hausdorff topology, such that the multiplication and inversion map are continuous.

LEMMA 1.1.4. *Let G be a locally compact Hausdorff groupoid. Then the following hold:*
(1) The maps $d, r : G \to G^{(0)}$ are continuous.

(2) $G^{(0)}$ *is closed in* G.

(3) $G^{(2)}$ *is closed in* $G \times G$.

PROOF. Recall that $d(g) = g^{-1}g$ and $r(g) = gg^{-1}$. Hence part (1) follows directly from the fact that multiplication and inversion are continuous.

For the second assertion consider the map $r \times id_G : G \to G \times G$. Then $G^{(0)} = (r \times id_G)^{-1}(\Delta_G)$, where Δ_G denotes the diagonal in $G \times G$. Since we assumed G to be Hausdorff, Δ_G is closed in $G \times G$. The claim follows from the continuity of $r \times id$.

The last part is an easy consequence of the continuity of the range and domain maps and Lemma 1.1.3(1). □

We will be mostly concerned with a certain subclass of locally compact Hausdorff groupoids:

DEFINITION 1.1.5. A locally compact groupoid is called *étale*, if $d : G \to G$ is a local homeomorphism, i.e. every point $g \in G$ has an open neighbourhood $U \subseteq G$, such that $d(U)$ is open in G and $d_{|U} : U \to d(U)$ is a homeomorphism.

Note, that a local homeomorphism is automatically an open map. The facts contained in the following lemma are well-known and follow easily from the definition.

LEMMA 1.1.6. *Let G be a locally compact Hausdorff groupoid. Then G is étale if and only if the range map $r : G \to G$ is a local homeomorphism. In that case the following are true:*

(1) $G^{(0)}$ *is open in* G.

(2) *For each* $u \in G^{(0)}$ *the sets* G^u *and* G_u *are discrete (in the subspace topology).*

(3) *For open subsets* $U, V \subseteq G$ *their product* $UV := \{uv \in G \mid (u,v) \in G^{(2)} \cap (U \times V)\}$ *is open.*

PROOF. The first part is easy: The inversion map $i : G \to G$ is a homeomorphism, since it is continuous and involutive. By Lemma 1.1.3 we have $r = d \circ i$ and $d = r \circ i$. As compositions of local homeomorphisms are again local homeomorphisms, the claim follows.

To see part *(1)* just note, that d is an open map and $G^{(0)} = \bigcup_{U \subseteq G \text{ open}} d(U)$.

To prove *(2)* let $g \in G_u$ be given. Since d is a local homeomorphism, there exists an open subset $U \subseteq G$ containing g, such that d is injective on U. Thus $G_u \cap U = \{g\}$ is open in G_u. Applying the first part of the Lemma, the same argument works for G^u using the range map instead of the domain map.

Finally, for *(3)* let U, V be open subsets of G and $(u, v) \in G^{(2)} \cap (U \times V)$ be given. Then there exists an open subset $W \subseteq G$ such that $uv \in W$ and r is a homeomorphism onto its image, when restricted to W. By continuity of the product there exist $U', V' \subseteq G$ open with $u \in U'$ and $v \in V'$ such that $U'V' \subseteq W$. By intersecting with U or V respectively we can assume, that $U' \subseteq U$ and $V' \subseteq V$. We can also suppose that $U' \subseteq d^{-1}(r(V))$. Thus $r(U'V') = r(U')$ is open. But then $U'V' = r^{-1}(r(U'V')) \cap W$ (where the inclusion $\supseteq$ follows from r being injective on W) is open as well and we have $uv \in U'V' \subseteq UV$. $\square$

REMARK 1.1.7. We want to remark here, that neither *(1)* nor *(2)* in the lemma above are equivalent to G being étale. A counterexample to both is the following groupoid: Let $G = \mathbb{R} \coprod \{\infty\}$ as topological space. Define $G^{(2)} = \{(g, g) \in G \times G \mid g \in G\} \cup \{(0, \infty), (\infty, 0)\}$ and the multiplication by $x \cdot x := x$ for $x \in \mathbb{R}$, $\infty \cdot \infty := 0$ and $0 \cdot \infty := \infty =: \infty \cdot 0$. The inverse map is given by the identity map on G. Then G is a locally compact Hausdorff groupoid with $G^{(0)} = \mathbb{R} \subseteq G$ open and each fibre is discrete, since $G_u = G^u = \{u\}$ for $u \in \mathbb{R} \setminus \{0\}$ and $G_0 = G^0 = \{0, \infty\}$. However, the domain map is not open, since $d(\{\infty\}) = \{0\}$ is not open in $\mathbb{R}$.

In the literature étale groupoids are often called *r-discrete*, refering to the discreteness of the range fibres G^u. The above remark explains, why we prefer the term étale.

The following proposition will give other useful characterizations for a groupoid to be étale. We will need the following notion:

DEFINITION 1.1.8. Let G be a locally compact Hausdorff groupoid. An *open bisection* is an open subset $U \subseteq G$ such that the domain map d and the range map r are homeomorphisms onto open subsets of $G^{(0)}$ respectively. The set of all open bisections will be denoted by G^{op}.

The following lemma says, that a groupoid G is étale if and only if the collection of all open bisection is large.

LEMMA 1.1.9. *Let G be a locally compact Hausdorff groupoid. The following are equivalent:*

(1) G is étale.
(2) The product map $m : G^{(2)} \to G$ is a local homeomorphism.
(3) G^{op} contains a basis for the topology of G.

PROOF. For (1) $\Rightarrow$ (2) let $(g,h) \in G^{(2)}$ and let U, V be open bisections with $g \in U$ and $h \in V$ respectively. Then $W := G^{(2)} \cap (U \times V)$ is open in $G^{(2)}$ and $(g,h) \in W$. By Lemma 1.1.6 we know that m is open. Thus, it remains to check that m is injective on W: If $gh = g'h'$, then $r(g) = r(gh) = r(g'h') = r(g')$. Since $g, g' \in U$ and U is a bisection we obtain $g = g'$. The same argument using the domain map yields $h = h'$.

Let us show (2) $\Rightarrow$ (1): Let $g \in G$ be given. Then there exists $W \subseteq G^{(2)}$ open such that $(g, g^{-1}) \in W$ and m restricted to W is a homeomorphism onto its image. Then we can find $U \subseteq G$ open with $g \in U$ and $G^{(2)} \cap (U \times U^{-1}) \subseteq W$. Then r is injective on U, since if $r(g_1) = r(g_2)$ for some $g_1, g_2 \in U$ then $g_1 g_1^{-1} = r(g_1) = r(g_2) = g_2 g_2^{-1}$. But since multiplication is injective on W we have $g_1 = g_2$. Similarly we can find an open neighbourhood of g such that the domain map is injective on it. By intersecting, we can assume that U has this property too. But injectivity of d on U implies $r(U) = UU^{-1}$, so $r(U)$ is open, since the multiplication map is open. Thus the result follows. The implication (3) $\Rightarrow$ (1) is obvious and (1) $\Rightarrow$ (3) follows easily once we observe, that the intersection of two open bisections is an open bisection. $\square$

Let us look at some examples.

EXAMPLES 1.1.10. On one end of the range of examples, every locally compact group G is an locally compact groupoid with $G^{(0)} = \{1_\Gamma\}$. It is étale if and only if G is discrete.

On the other hand, every locally compact Hausdorff space X is an étale groupoid. The set of composable pairs is the diagonal $\Delta_X = \{(x,x) \in X \times X \mid x \in X\}$ and the product and inversion maps are trivial, in the sense that $x \cdot x = x$ and $x^{-1} = x$. Thus, the groupoid X just consists of its unit space.

The two examples above form the extreme cases of the following, more general construction:

EXAMPLE 1.1.11. Let Γ be a discrete group, which acts on a locally compact Hausdorff space X (by homeomorphisms). We construct a groupoid $\Gamma \ltimes X$ out of this data as follows: As a topological space $\Gamma \ltimes X$ is just the product space $\Gamma \times X$. We define two elements $(\gamma, x), (\eta, y) \in \Gamma \ltimes X$ to be composable, if $\gamma^{-1}x = y$, and then define the product and inverse as

$$(\gamma, x)(\eta, \gamma^{-1}x) := (\gamma\eta, x) \text{ and } (\gamma, x)^{-1} := (\gamma^{-1}, \gamma^{-1}x).$$

It is not hard to check, that $\Gamma \ltimes X$ satisfies the groupoid axioms. We have $(\Gamma \ltimes X)^{(0)} = \{(1_\Gamma, x) \in \Gamma \ltimes X \mid x \in X\}$ and will thus identify it with X. Under this identification, the domain and range map are given by $d(\gamma, x) = \gamma^{-1}x$ and $r(\gamma, x) = x$.

Moreover, $\Gamma \ltimes X$ is an étale groupoid: For every open set $U \subseteq X$ and every $\gamma \in \Gamma$ the set $\{\gamma\} \times U$ is clearly an open bisection and since Γ is discrete, the collection of these sets form a basis for the topology of $\Gamma \ltimes X$.

One of the most powerful tools in the study of locally compact groups is the existence of the Haar measure. The following is a groupoid analogue of this concept:

DEFINITION 1.1.12. Let G be a locally compact Hausdorff groupoid. A (*left*) *Haar system* for G is a collection $(\lambda^u)_{u \in G^{(0)}}$ of positive regular Borel measures on G such that the following hold:

(1) The support of each λ^u is G^u.
(2) For any $f \in C_c(G)$ the function $\lambda(f) : G^{(0)} \to \mathbb{C}$, given by
$$\lambda(f)(u) := \int_{G^u} f \, d\lambda^u$$
is continuous (and hence belongs to $C_c(G^{(0)})$).
(3) For any $g \in G$ and $f \in C_c(G)$,
$$\int_{G^{d(g)}} f(gh) d\lambda^{d(g)}(h) = \int_{G^{r(g)}} f(h) d\lambda^{r(g)}(h)$$

Note that the integral in (2) makes sense, since $f_{|G^u} \in C_c(G^u)$. Also the formula in (3) is well-defined since $d(g) = r(h)$ on the left hand

side and thus the product gh exists. Item (2) expresses that $(\lambda^u)_u$ is a continuous family of measures over $G^{(0)}$ and item (3) expresses the invarinace of $(\lambda^u)_u$ under multiplication.
In the case of a locally compact group the above definition reduces to the definition of (the) Haar measure. One should note that in contrast to the group case, locally compact groupoids neither necessarily admit a Haar measure (see **[Sed86]** for a counterexample), nor is it unique.

As we have $(G^u)^{-1} = G_u$ and the inversion map is a homeomorphism from G onto itself, we associate with λ^u the measure $\lambda_u := (\lambda^u)^{-1}$ on G_u, given by $\lambda_u(A) = \lambda^u(A^{-1})$ for a Borel subset $A \subseteq G_u$. Consequently, we get the formula

$$\int\limits_{G_u} f(g)d\lambda_u(g) = \int\limits_{G^u} f(g^{-1})d\lambda^u(g).$$

EXAMPLE 1.1.13. Let us revisit the transformation groups presented in Example 1.1.11. For each $x \in X = (\Gamma \ltimes X)^{(0)}$ the r-fibre over x is just given by a copy of Γ: $(\Gamma \ltimes X)^x = \Gamma \times \{x\}$. Let μ be the counting measure on Γ and δ_x the Dirac measure on X concentrated in $\{x\}$. Then the family $(\mu \times \delta_x)_{x\in X}$ defines a Haar system for $\Gamma \ltimes X$, where $\mu \times \delta_x$ denotes the product measure on $\Gamma \ltimes X$. For every $x \in X$ and $f \in C_c(\Gamma \ltimes X)$ one has the formula

$$\int\limits_{(\Gamma\ltimes X)^x} f(\gamma, y)d(\mu \times \delta_x)(\gamma, y) = \int\limits_{\Gamma} f(\gamma, x)d\mu(\gamma) = \sum_{\gamma\in\Gamma} f(\gamma, x).$$

Since f is compactly supported, the sum on the right is finite and hence it is obvious, that $x \mapsto \sum_{\gamma\in\Gamma} f(\gamma, x)$ is continuous, which verifies the continuity of $(\mu \times \delta_x)_{x\in X}$. To see the invariance we compute

$$\begin{aligned}
&\int\limits_{(\Gamma\ltimes X)^{d(\gamma,x)}} f((\gamma, x)(\eta, y))d(\mu \times \delta_{d(\gamma,x)})(\eta, y)\\
&= \sum_{\eta\in\Gamma} f(\gamma\eta, x)\\
&= \sum_{\eta\in\Gamma} f(\eta, x)\\
&= \int\limits_{(\Gamma\ltimes X)^{r(\gamma,x)}} f(\eta, y)d(\mu \times \delta_x)(\eta, y)
\end{aligned}$$

The existence of a Haar system on a locally compact groupoid has strong topological consequences:

Proposition 1.1.14. [**Ren80**, Proposition 2.4] *If G is a locally compact groupoid and admits a Haar system, then both the range and the domain map are open maps from G onto $G^{(0)}$.*

The domain and range maps being open is reminiscent of étale groupoids, which always have this property. Indeed, every étale groupoid admits a particularly nice canonical Haar system:

Proposition 1.1.15. [**Pat99**, Proposition 2.2.5] *Let G be an étale groupoid. For each $u \in G^{(0)}$ let λ^u be the counting measure G^u. Then $(\lambda^u)_{u\in G^{(0)}}$ is a Haar system for G such that for any $f \in C_c(G)$ we have*

$$\int_{G^u} f d\lambda^u = \sum_{g\in G^u} f(g).$$

Proof. By definition we have $supp(\lambda^u) = G^u$, so it remains to check continuity and invariance: For continuity consider first a function $f \in C_c(U)$, where $U \subseteq G$ is an open bisection. Now if $u \in G^{(0)}$ then $G^u \cap U$ consists of a single element, namely $r_{|U}^{-1}(u)$, where $r_{|U}^{-1}$ is the continuous inverse of $r_{|U} : U \to r(U)$. Then we have

$$\lambda(f)(u) = \sum_{g\in G^u} f(g) = f(r_{|U}^{-1}(u)).$$

Since f and $r_{|U}^{-1}$ are continuous, so is $\lambda(f)$. Now suppose $f \in C_c(G)$ is arbitrary. Since G is étale, we can find a finite open covering $(U_i)_{i=1}^n$ of $supp(f)$ by open bisections. Let $(\varphi_i)_{i=1}^n$ be a partition of unity subordinate to this covering. Then $\varphi_i f \in C_c(U_i)$ for each $1 \leq i \leq n$, and hence $\lambda(\varphi_i f) \in C_c(G^{(0)})$ by what we have shown above. But since $f = \sum_{i=1}^n \varphi_i f$ we have that $\lambda(f) = \sum_{i=1}^n \lambda(\varphi_i f) \in C_c(G^{(0)})$.

For the invariance we note that $h \mapsto gh$ is a bijection $G^{d(g)} \to G^{r(g)}$ and thus we obtain

$$\sum_{h\in G^{d(g)}} f(gh) = \sum_{h\in G^{r(g)}} f(h).$$

□

Convention: From now on, when talking about étale groupoids, we will always take this family of counting measures as the canonical Haar system.

The following well-known basic result will be needed later:

LEMMA 1.1.16. *Let G be a locally compact Hausdorff groupoid with a Haar system $\{\lambda^u\}_{u \in G^{(0)}}$. If $K \subseteq G$ is compact, the set $\{\lambda^u(K) \mid u \in G^{(0)}\}$ is bounded.*

PROOF. Let $f \in C_c(G)$ with $0 \le f \le 1$ and $f = 1$ on K. Then

$$\lambda^u(K) \le \int_{G^u} f(x) d\lambda^u(x)$$

for all $u \in G^{(0)}$. The result follows from axiom (2) of the definition of a Haar system. □

For later purposes it will also be important to note, that the set of functions f for which $\lambda(f)$ as in the definition of the Haar system is continuous, is not limited to functions with compact support.

DEFINITION 1.1.17. A function $\varphi \in C(G)$ is said to have *proper support*, if for every compact subset $K \subseteq G^{(0)}$ the intersection $supp(\varphi) \cap r^{-1}(K)$ is compact.

LEMMA 1.1.18. *If $\varphi \in C(G)$ has proper support, then $\lambda(\varphi) : G^{(0)} \to \mathbb{C}$ given by*

$$\lambda(\varphi)(u) = \int_{G^u} \varphi(x) d\lambda^u(x)$$

is continuous and bounded.

PROOF. We will show that $\lambda(\varphi)$ looks like a continuous function locally. More precisely given any $u \in G^{(0)}$ we can pick a relatively compact neighbourhood V of u. Then choose a function $\psi \in C_c(G^{(0)})$ such that $\psi = 1$ on $\overline{V}$. Then $f(x) := \varphi(x)\psi(r(x))$ is a continuous function with compact support since $supp(f) \subseteq supp(\varphi) \cap r^{-1}(supp(\psi))$ and φ has proper support. Thus $\lambda(f)$ is continuous. But for all $v \in V$ we clearly have

$$\lambda(f)(v) = \int_{G^v} \varphi(x)\psi(v) d\lambda^v(x) = \lambda(\varphi)(v).$$

Thus $\lambda(\varphi)_{|V}$ is continuous. Since u was chosen arbitrary $\lambda(\varphi)$ must be continuous. □

There is an important subclass of the class of étale groupoids, which is of particular interest to us:

Definition 1.1.19. A locally compact Hausdorff groupoid G is called *ample*, if the set $G^a := \{A \in G^{op} \mid A \text{ is compact}\}$ forms a basis for the topology of G.

It follows directly from the definition, that every ample groupoid is étale. Recall, that a topological space X is called *totally disconnected*, if the connected components in X are the one-point sets. The following proposition, a proof of which can be found in [**AT08**, Proposition 3.1.7], gives an alternative description in the locally compact Hausdorff case.

Proposition 1.1.20. *Let X be a locally compact Hausdorff space. If X is totally disconnected, then X has a basis consisting of sets, which are both closed and open in X.*

Using this result, one can easily characterize the ample groupoids among the étale groupoids, as was first noted by Exel:

Proposition 1.1.21. [**Exe10**] *Let G be an étale groupoid. Then G is ample if and only if $G^{(0)}$ is totally disconnected.*

Proof. If G is ample, then G has a basis of compact open subsets. So does $G^{(0)}$, since it is open and closed in G. Now let $X \subseteq G^{(0)}$ be a connected component and assume that there are $x, y \in X$ with $x \neq y$. Then there is a compact open subset $A \subseteq G^{(0)}$ such that $x \in A$ and $y \notin A$. Thus X is the disjoint union of the closed and open sets A and $X \setminus A$. Since X is connected this is a contradiction.

Since G^a is clearly closed under (finite) intersections, it is enough to show that for each $g \in G$ there exists a compact open bisection A such that $g \in A$. Since G is étale we can first choose an open bisection $U \in G^{op}$ such that $g \in U$. Using that G is locally compact, we can find an open subset V of G with $\overline{V}$ compact and $g \in V \subseteq \overline{V} \subseteq U$. Then $r(g)$ is contained in the open set $r(V)$, which is contained in the compact set $r(\overline{V})$. By Proposition 1.1.20 there exists a compact and open subset $A \subseteq r(V)$, such that $r(g) \in A$. Since $r_{|U}$ is a homeomorphism onto its image, $r^{-1}(A) \subseteq U$ is a compact open bisection. □

The remainder of this subsection is dedicated to give more examples of étale, and more specifically ample groupoids.

The non-commutative geometry of tilings. An interesting application of groupoid theory comes from the physics of quasicrystals. In classical crystallography periodic tilings or lattices and their symmetry groups are used to describe the structure of solid materials. In the early 1980s Shechtman et al discovered a real, physical material defying the laws of the classical theory. The materials they produced exhibited rotational symmetry that is forbidden in periodic solids (this is called crystallographic restriction). Soon after this discovery the mathematical theory of quasicrystals and aperiodic tilings developed rapidly. The modern approach uses groupoids and their C*-algebras. A direct link back to physics is given by Bellissard's gap labeling conjecture about the image of the K-theory of the C*-algebras in question under the canonical trace. Here, we just want to describe the basic construction of the groupoid associated to an aperiodic tiling, following the exposition in [**BJS10**] (see also the references therein): A *tile* is a compact subset of $\mathbb{R}^d$ which is the closure of its interior and a *tiling* is a countable collection of tiles covering all of $\mathbb{R}^d$ such that the interiors of the tiles are pairwise disjoint. A *punctured tile* is a pair (T, x), where T is a tile and $x \in T$ and a tiling is called *punctured* if each of its tiles is punctured. Given a tiling $\mathcal{T}$ of $\mathbb{R}^d$ one considers the tiling space Ω of $\mathcal{T}$, which is the closure of $\{\mathcal{T} + a \mid a \in \mathbb{R}^d\}$ in a certain topology. The *canonical transversal*, denoted Ξ, is the subset of Ω consisting of tilings having the origin $0 \in \mathbb{R}^d$ as the puncture of one of its tiles. Under suitable conditions (aperiodicity, repetitivity and finite local complexity) on the tiling the canonical transversal Ξ is a Cantor set. The groupoid of the tiling space is the groupoid associated to the equivalence relation

$$R_\Xi = \{(\mathcal{T}_1, \mathcal{T}_2) \in \Xi \times \Xi \mid \exists a \in \mathbb{R}^d : \mathcal{T}_2 = \mathcal{T}_1 + a\}.$$

The correct topology on R_Ξ is however not the relative topology of $R_\Xi \subseteq \Xi \times \Xi$. Instead, one uses the topology, where $(\mathcal{T}_n, \mathcal{T}_n + a_n)$ converges to $(\mathcal{T}, \mathcal{T} + a)$ if and only if $\mathcal{T}_n \to \mathcal{T}$ in Ξ and $a_n \to a$ in $\mathbb{R}^d$. With this topology R_Ξ is an étale groupoid and since Ξ is totally disconnected it is actually an ample groupoid.

Directed graphs. Following the construction in [**KPRR97**], let $E = (E^0, E^1, r, s)$ be a directed graph consisting of a countable set of vertices E^0, a set of edges E^1 and maps $r, s : E^1 \to E^0$ describing the range and source of a given edge. Assume that E has *no sinks*, meaning that $s : E^1 \to E^0$ is surjective and that E is *row finite*, meaning that

$s^{-1}(v)$ is finite for every $v \in E^0$. Let $E^\infty := \{(x_n)_n \in \prod_{n\in\mathbb{N}} E^1 \mid s(x_{n+1}) = r(x_n)\}$ denote the set of all infinite paths in E equipped with the topology induced from the product topology on $\prod_{n\in\mathbb{N}} E^1$. A basis for this topology is given by the cylinder sets

$$Z(\alpha) = \{\alpha x \mid x \in E^\infty\},$$

where α is a finite path in E. In this topology E^∞ is a locally compact, totally disconnected Hausdorff space and the sets $Z(\alpha)$ are compact open. There is a canonical map $\sigma : E^\infty \to E^\infty$, called the *shift map* given by the formula $\sigma(x)_n := x_{n+1}$. We define the groupoid G_E associated to E to be the set

$$\{(x, k, y) \in E^\infty \times \mathbb{Z} \times E^\infty \mid \exists n, m \in \mathbb{N}_0 : k = n - m \textit{ and } \sigma^n(x) = \sigma^m(y)\}.$$

Then G_E can be equipped with a groupoid structure with product and inverse given by the formulas

$$(x, k, y)(y, l, z) := (x, k + l, z) \text{ and } (x, k, y)^{-1} := (y, -k, x).$$

One easily checks that $G_E^{(0)} = \{(x, 0, x) \in G_E \mid x \in E^\infty\}$ and hence we will identify it with E^∞. Then range and domain maps are then given by $r(x, k, y) = x$ and $d(x, k, y) = y$. One can show, that the sets

$$Z(\alpha, \beta) := \{(x, k, y) \in G_E \mid x \in Z(\alpha), y \in Z(\beta), k = |\beta| - |\alpha|\},$$

where α andβ are finite paths in E with $r(\alpha) = r(\beta)$, form a basis for a locally compact Hausdorff topology on G_E. With this topology G_E is a second countable, ample groupoid in which each $Z(\alpha, \beta)$ is a compact open bisection. The topology that E^∞ inherits by viewing it as a subset of G_E coincides with the topology coming from the product topology as described above.

Ample groupoids associated to inverse semigroups. An inverse semigroup is a semigroup S, such that for each element $s \in S$ there exists a unique element $t \in S$ such that $sts = s$ and $tst = t$. One usually writes $s^* := t$. One can show that $s \mapsto s^*$ is an involution on S. If $e = e^2 \in S$ then by uniqueness we immediately see $e^* = e$. So we can think of the idempotents as projections. Let $E(S)$ be the set of idempotents. This set plays an important role in the theory of inverse semigroups. It is not hard to see that ss^* and s^*s are idempotents for every $s \in S$.

Moreover $E(S)$ is commutative and has a canonical order: $e \leq f$ if and only if $ef = e$.

The following exposition is based on [**Pat99**]: To construct a groupoid out of a given inverse semigroup S, consider the set of characters $\widehat{E(S)}$ on S, i.e. the set of non-zero multiplicative maps $\chi : E(S) \to \{0,1\}$. We equip it with the topology of pointwise convergence. Then we can embed $\widehat{E(S)}$ into the product $\{0,1\}^{E(S)}$. It follows immediately, that $\widehat{E(S)}$ is a totally disconnected locally compact Hausdorff space. For $e \in E(S)$ consider the set $D_e = \{\chi \in \widehat{E(S)} \mid \chi(e) = 1\}$. To construct a groupoid out of S consider the set $\Sigma = \{(s,\chi) \in S \times \widehat{E(S)} \mid \chi \in D_{s^*s}\}$ and define an equivalence relation on Σ by requiring $(s,\chi) \sim (t,\mu)$ if and only if $\chi = \mu$ and there exists an $e \in E(S)$ such that $\chi \in D_e$ and $se = te$. Then, as a topological space the universal groupoid $G(S)$ associated to S is just the quotient $G(S) = \Sigma/\sim$. To define the product we need the following notation: For $s \in S$ and $\chi \in \widehat{E(S)}$ we let $s.\chi$ be the character given by $s.\chi(e) = \chi(s^*es)$. We define a pair of elements $[s,\chi],[t,\mu] \in G(S)$ to be composable if and only if $\mu = s^*.\chi$ and then define their product to be

$$[s,\chi][t,s^*.\chi] = [st,\chi]$$

and the inverse by

$$[s,\chi]^{-1} = [s^*, s^*.\chi]$$

Then $G(S)$ is a groupoid. It is easy to see that $G(S)^{(0)} = \{[e,\chi] \in G(S) \mid e \in E(S)\}$ and hence we can identify the unit space with $\widehat{E(S)}$. Under this identification the domain and range maps $r,d : G(S) \to \widehat{E(S)}$ are given by $d([s,\chi]) = s^*.\chi$ and $r([s,\chi]) = \chi$. We want to see that $G(S)$ is étale. Then it is automatically ample by Proposition 1.1.21, since $G(S)^{(0)} \cong \widehat{E(S)}$ is totally disconnected. For $s \in S$ and an open subset $U \subseteq D_{s^*s}$ let $\Theta(s,U) = \{[s,\chi] \mid \chi \in U\}$. It is shown for example in [**Exe08**, Section 4] (see also [**Pat99**]), that the collection of all $\Theta(s,U)$ form the basis of a topology on $G(S)$. With respect to this topology $G(S)$ is an étale groupoid such that the identification of the unit space $G(S)^{(0)}$ with $\widehat{E(S)}$ described above is a homeomorphism. One should note however, that $G(S)$ need not be Hausdorff in general. Indeed, Steinberg showed in [**Ste10**, Theorem 5.17], that $G(S)$ is Hausdorff if and only if S is a weak semilattice (confer [**Ste10**] for the definition and an easy example of an inverse semigroup that is not a weak semilattice).

Coarse geometry. In **[STY02]** the authors introduce a groupoid associated to a metric space with bounded geometry. Let us review the construction: Let (X, d) be a metric space. We say that X has *bounded geometry*, if for any $R > 0$ we have $\sup_{x\in X} |B_R(x)| < \infty$, where $B_R(x)$ denotes the ball of radius R around $x \in X$. It follows that the topology on X induced by the metric d is the discrete topology. Let $A, B \subset X$. A map $\varphi : A \to B$ is called a *partial translation* if φ is a bijection with bounded graph in the sense that $\sup_{x\in X} d(x, \varphi(x)) < \infty$. We will write $dom(\varphi) = A$ and $ran(\varphi) = B$. Consider the Stone-Čhech compactification βX of X. Given a partial translation φ, let $\overline{dom}(\varphi)$ and $\overline{ran}(\varphi)$ denote the closures of $dom(\varphi)$ and $ran(\varphi)$ in βX, respectively. Then $\overline{dom}(\varphi)$ and $\overline{ran}(\varphi)$ are compact open subsets of βX and φ extends to a homeomorphism

$$\overline{\varphi} : \overline{dom}(\varphi) \to \overline{ran}(\varphi).$$

Define an equivalence relation on the set of pairs $(\overline{\varphi}, x)$, where φ is a partial translation and $x \in \overline{dom}(\varphi)$ by letting $(\overline{\varphi}, x) \sim (\overline{\psi}, y)$ if and only if $x = y$ and there exists an open neighbourhood U of x in βX such that $\overline{\varphi}_{|U} = \overline{\psi}_{|U}$. We define $G(X)$ to be the set of equivalence classes of all pairs as above. Then $G(X)$ can be equipped with a groupoid structure with product and inverse given by the formulas

$$[\overline{\psi}, \overline{\varphi}(x)][\overline{\varphi}, x] = [\overline{\psi} \circ \overline{\varphi}, \overline{\psi}(\overline{\varphi}(x))] \text{ and } [\overline{\varphi}, x]^{-1} = [\overline{\varphi}^{-1}, \overline{\varphi}(x)].$$

The sets

$$U_\varphi := \{[\overline{\varphi}, x] \mid x \in \overline{dom}(\varphi)\},$$

where φ is a partial translation, form a basis for a locally compact Hausdorff topology on $G(X)$. With this topology $G(X)$ is a σ-compact ample groupoid whose unit space is (homeomorphic to) βX. The sets U_φ are compact open bisections.

1.2. Groupoid Actions

In this section we review the basic notions of groupoid actions and some important properties these can enjoy.

DEFINITION 1.2.1. Let G be a groupoid and X a set. A (left) action of G on X consists of a map $p : X \to G^{(0)}$, called *anchor map* and a map $G * X \to X$, $(g, x) \mapsto gx$, where $G * X = \{(g, x) \mid d(g) = p(x)\}$, such that the following holds:

(1) If $(g, h) \in G^{(2)}$ and $(h, x) \in G * X$, then $(g, hx) \in G * X$ and $(gh)x = g(hx)$.
(2) For all $x \in X$ we have $p(x)x = x$.

If G is a topological groupoid and X a topological space we require the anchor map and the multiplication map to be continuous. In that case we will call X a (left) G-space.

Similarly we can define right actions in the obvious way. From the definition we can directly get the following Lemma:

LEMMA 1.2.2. *Let G be a groupoid acting from the left on X and $(g, x) \in G * X$. Then the following equations hold:*

(1) $p(gx) = r(g)$.
(2) $g^{-1}(gx) = x$.

Let us now look at some examples:

EXAMPLES 1.2.3. Let G be a locally compact Hausdorff groupoid.

(1) G acts on itself, where the anchor map $G \to G^{(0)}$ is the range map and the action is given by the usual multiplication of the groupoid.
(2) If $H \subseteq G$ is a closed subgroupoid, then H acts from the right on $X = d^{-1}(H^{(0)})$, where the anchor map is the restriction of the domain map $d_{|X} : X \to H^{(0)}$ and the action is given by the multiplication in G.
(3) G acts from the left on its unit space, where the anchor map is the identity on $G^{(0)}$ and the action is given by $g \cdot d(g) = r(g)$.
(4) Let $P(G)$ be the space of Borel probability measures on G, such that the support of each probability measure is contained in G^u for some $u \in G^{(0)}$. Then G acts on $P(G)$: The anchor map $P(G) \to G^{(0)}$ sends μ to u, where u is the unit such that $supp(\mu) \subseteq G^u$. If $d(g) = u$ we define the measure $g\mu$ by $(g\mu)(A) = \mu(g^{-1}A)$.

(5) Finally, consider the *isotropy subgroupoid* $Iso(G) = \{s \in G \mid d(s) = r(s)\}$ of G. Then G acts from the left on $Iso(G)$ with anchor map $p(s) = d(s) = r(s)$ and for $g \in G$ and $s \in Iso(G)$ we can define $g \cdot s = gsg^{-1}$.

Similar to the group case, groupoid actions give rise to a transformation groupoid: If G acts on a set X we can form a new groupoid denoted $G \ltimes X$. As a set it is the subspace of $G \times X$ consisting of all pairs such that $r(g) = p(x)$. Two such pairs $(g, x), (h, y)$ are composable if $y = g^{-1}x$ and in that case we define

$$(g, x)(h, y) := (gh, x).$$

Furthermore we define the inverse map by

$$(g, x)^{-1} := (g^{-1}, g^{-1}x).$$

This groupoid is often called the *transformation groupoid* associated to the action of G on X. Its unit space can be identified with X as follows: Given $(g, x) \in G \ltimes X$ we have $r_{G\ltimes X}(g, x) = (g, x)(g, x)^{-1} = (g, x)(g^{-1}, g^{-1}x) = (gg^{-1}, x) = (r_G(g), x) = (p_X(x), x)$. Thus $(G \ltimes X)^{(0)} = \{(p_X(x), x) \mid x \in X\}$ and the identification with X is given by the projection on the second factor. For the domain map we compute analogously $d_{G\ltimes X}(g, x) = (g, x)^{-1}(g, x) = (g^{-1}, g^{-1}x)(g, x) = (g^{-1}g, g^{-1}x) = (d(g), g^{-1}x)$. It follows, that under the above identification of the unit space with X, the range and domain maps are given by the formulas

$$r_{G\ltimes X}(g, x) = x, \qquad d_{G\ltimes X}(g, x) = g^{-1}x.$$

LEMMA 1.2.4. *Let G be a locally compact Hausdorff groupoid and X be a G-space. Then the groupoid $G \ltimes X$ is a locally compact groupoid. The identification of the unit-space with X is a homeomorphism. Furthermore, the range and domain maps of the groupoid $G \ltimes X$ are open if the range and domain maps of G are open.*

PROOF. Since the maps r and p are continuous, $G \ltimes X$ is closed as a subset of $G \times X$. Thus, it follows that $G \ltimes X$ is locally compact if G and X are. That the identification of the unit space is a homeomorphism is obvious.

It remains to show that the range map $r_{G\ltimes X}$ of $G \ltimes X$ is open if $r : G \to G^{(0)}$ is open. For this we apply [**Wil07**, Proposition 1.15] as follows: Let $(g,x) \in G \ltimes X$ and $(x_\lambda)_\lambda$ be a net in X such that $x_\lambda \to r_{G\ltimes X}(g,x) = x$. Since p is continuous we get $p(x_\lambda) \to p(x) = r(g)$. Thus $(p(x_\lambda))_\lambda$ is a net converging to $r(g)$ and if we apply the above mentioned Proposition to r, we can find a subnet $(p(x_{\lambda_\mu}))_\mu$ and a net $(g_\mu)_\mu$ in G such that $g_\mu \to g$ and $r(g_\mu) = p(x_{\lambda_\mu})$. Thus (g_μ, x_{λ_μ}) is a net in $G \ltimes X$ with $(g_\mu, x_{\lambda_\mu}) \to (g,x)$ and $r_{G\ltimes X}(g_\mu, x_{\lambda_\mu}) = x_{\lambda_\mu}$. Using the Proposition again we see that $r_{G\ltimes X}$ is an open map. □

LEMMA 1.2.5. *Let G be a locally compact Hausdorff groupoid and X be a G-space with anchor map $p : X \to G^{(0)}$. If G is étale, then so is $G \ltimes X$.*

PROOF. If U is an open subset of G such that r restricts to a homeomorphism from U onto an open subset of $G^{(0)}$, then $V := (U \times X) \cap G \ltimes X$ has the same property with respect to $r_{G\ltimes X}$. Indeed we have that $r_{G\ltimes X}(V) = p^{-1}(r(U))$ is open in X and the map $x \mapsto (r_{|U}^{-1}(p(x)), x)$ defines a continuous inverse $r_{G\ltimes X}(V) \to V$. The result now follows from this observation. □

If G acts on X, say from the right, we can form the space of orbits X/G. More specifically we can define an equivalence relation $\sim$ on X by declaring $x \sim y$ if and only if there exists a $g \in G$ such that $p(y) = r(g)$ and $x = yg$. We then define $X/G := X/\sim$ to be the quotient of X by the equivalence relation $\sim$. If G was a topological groupoid acting continuously on a space X we equip X/G with the quotient topology. The following result is standard. A proof can be found in [**Tu04**, Lemma 2.30].

PROPOSITION 1.2.6. *Let G be a locally compact Hausdorff groupoid. Then the range and domain maps of G are open if and only if the canonical quotient map $X \to X/G$ is open for every G-space X. In that case X/G is locally compact (not necessarily Hausdorff), if X is locally compact.*

Many properties of dynamical systems can easily be formulated in terms of the corresponding transformation groupoid and thus give a nice way to generalize them to arbitrary groupoids. The following is an example of this: Recall, that a continuous map $f : X \to Y$ between locally compact Hausdorff spaces X and Y is called *proper*, if $f^{-1}(K)$ is compact for all compact subsetes $K \subseteq Y$. If Γ is a discrete group acting on a space X, the action is called proper, if $(g, x) \mapsto (x, g^{-1}x)$ is a proper map $\Gamma \times X \to X \times X$. In terms of the transformation groupoid the latter map is just the map $r \times d : \Gamma \ltimes X \to X \times X$. Thus, for general groupoids, one defines:

Definition 1.2.7. A locally compact Hausdorff groupoid is called *proper*, if $r \times d : G \to G^{(0)} \times G^{(0)}$ is a proper map.

Similarly, we say that X is a *proper (left) G-space*, if the associated transformation groupoid $G \ltimes X$ is proper.

In practice it is useful to have some more equivalent conditions to check properness. These are provided by the following proposition.

Proposition 1.2.8. [**Tu04**, Proposition 2.14] *Let X be a locally compact G-space. Then the following are equivalent:*

(1) X is a proper G-space.
(2) For every compact subset $K \subseteq X$ the set $\mathcal{F}_K = \{g \in G \mid gK \cap K \neq \emptyset\}$ is compact.
(3) There is a family $(A_i)_{i \in I}$ of subspaces of X such that $X = \bigcup_{i \in I} i$ and $\{g \in G \mid gA_i \cap A_j \neq \emptyset\}$ is contained in a compact subset of G for all $i, j \in I$.
(4) If $(x_\lambda)_\lambda$ is a convergent net in X and $(g_\lambda)_\lambda$ is a net in G such that $d(g_\lambda) = p_X(x_\lambda)$ and $(g_\lambda x_\lambda)_\lambda$ is convergent as well, then $(g_\lambda)_\lambda$ has a convergent subnet.

Proof. (1) $\Rightarrow$ (2) follows from the fact that

$$\mathcal{F}_K = pr_1((r, d)^{-1}(K \times K)).$$

For (2) $\Rightarrow$ (3) choose $I = X$ and let A_x be a compact neighbourhood of x for all $x \in X$. Clearly the interiors of these sets cover X and if $x, y \in X$ let K be the compact set $A_x \cup A_y$. Then it is obvious that $\{g \in G \mid gA_x \cap A_y \neq \emptyset\} \subseteq \mathcal{F}_K$ which is compact by (2).

For (3) $\Rightarrow$ (4) let $(x_\lambda)_\lambda$ be a net in X converging to some $x \in X$ and $(g_\lambda)_\lambda$ a net in G such that $d(g_\lambda) = p_X(x_\lambda)$ which converges to some $y \in X$. If $(A_i)_{i \in I}$ is a family of subsets as in (3) we can find $i, j \in I$

such that $x \in A_i$ and $y \in A_j$. Then there exists a λ_0 such that $x_\lambda \in A_i$ and $g_\lambda x_\lambda \in A_j$ for all $\lambda \geq \lambda_0$. Thus $g_\lambda \in \{g \in G \mid gA_i \cap A_j \neq \emptyset\}$ for all $\lambda \geq \lambda_0$. But this set is contained in a compact subset of G by (3) so all g_λ are contained in a compact set for $\lambda \geq \lambda_0$. Thus we can find a convergent subnet of $(g_\lambda)_\lambda$.

Let us now prove (4) $\Rightarrow$ (1). For convenience of notation will write D and R for the domain and range maps of the transformation groupoid $G \ltimes X$. If $K \subseteq X \times X$ is compact we need to see that $(R, D)^{-1}(K)$ is compact. It suffices to show, that every net in $(R, D)^{-1}(K)$ has a convergent subnet. So let $(g_\lambda, x_\lambda)_\lambda$ be such a net. Since $(R, D)(g_\lambda, x_\lambda) = (x_\lambda, g_\lambda^{-1} x_\lambda) \in K \times K$ we may pass to a subnet and relabel in order to assume that $(x_\lambda)_\lambda$ and $(g_\lambda^{-1} x_\lambda)_\lambda$ are convergent. By (4) we can pass to yet another subnet and relabel to assume $(g_\lambda)_\lambda$ is convergent as well. It follows that $(g_\lambda, x_\lambda)_\lambda$ is convergent, as desired. □

Remark 1.2.9. It is useful to note, that the set $\mathcal{F}_K$ defined above for any compact set $K \subseteq X$ is always closed in G. To see this let $(g_\lambda)_\lambda$ be a net in $\mathcal{F}_K$ converging to some $g \in G$. For every λ there exist $k_\lambda, k'_\lambda \in K$ such that $g_\lambda k_\lambda = k'_\lambda$. As K is compact we can pass to a subnet if necessary to assume that $k_\lambda \to k$ and $k'_\lambda \to k'$ for some $k, k' \in K$. By continuity of the action we have $gk = \lim_\lambda g_\lambda k_\lambda = \lim_\lambda k'_\lambda = k'$. Thus, we have $g \in \mathcal{F}_K$, as desired.

Identifying G with the transformation groupoid $G \ltimes G^{(0)}$ in the obvious way we get a similar looking result characterizing properness of the groupoid itself:

Proposition 1.2.10. *Let G be a locally compact Hausdorff groupoid. Then the following are equivalent:*

(1) G is proper.
(2) For every compact subset $K \subseteq G^{(0)}$ the set G_K^K is compact.
(3) There is a family $(A_i)_i$ of subspaces of $G^{(0)}$ such that $G^{(0)} = \bigcup_{i \in I} i$ and $G_{A_i}^{A_j}$ is contained in a compact subset of G for all $i, j \in I$.
(4) If $(g_\lambda)_\lambda$ is a net in G, such that $(d(g_\lambda))_\lambda$ and $(r(g_\lambda))_\lambda$ are convergent, then g_λ has a converging subnet.

One of the features of proper Hausdorff groupoids is the fact, that their orbit space is again Hausdorff:

LEMMA 1.2.11. *Let G be a proper Hausdorff groupoid with open range and domain maps. Then the quotient space $G \setminus G^{(0)}$ for the canonical left action of G on $G^{(0)}$ is Hausdorff.*

PROOF. Suppose $(Gu_\lambda)_\lambda$ is a net in the quotient $G\setminus G^{(0)}$ converging to both Gu and Gv. We claim that $Gu = Gv$. Our assumptions together with Proposition 1.2.6 imply, that the quotient map $G^{(0)} \to G \setminus G^{(0)}$ is open. Thus, we can pass to a subnet, relabel if necessary, and choose new representatives u_λ, to assume that $u_\lambda \to u$. Then we can use openness of the quotient map again to find elements $g_\lambda \in G$, such that $r(g_\lambda) = g_\lambda u_\lambda \to v$. Hence we can use the characterization of properness from the previous proposition to pass to another subnet and relabel, allowing us to assume that $g_\lambda \to g$ for some $g \in G$. But then $v = \lim g_\lambda u_\lambda = gu$, which proves the claim. □

LEMMA 1.2.12. *Let G be a locally compact Hausdorff groupoid. Then G acts properly on itself.*

PROOF. Let $K \subseteq G$ be a compact subset. Then one easily verifies that the (closed) set $\mathcal{F}_K$ as defined in Proposition 1.2.8 is contained in the compact set KK^{-1} and hence compact itself. □

LEMMA 1.2.13. *Let G be a locally compact Hausdorff groupoid and $H \subseteq G$ a subgroupoid with $H^{(0)}$ closed in $G^{(0)}$. If H is proper, then H is closed in G.*

PROOF. Let $(g_\lambda)_\lambda$ be a net in H converging to $g \in G$. Let K be a compact neighbourhood of g. After passing to a subnet if necessary, we can assume $g_\lambda \in K \cap H \subseteq H_{d(K)}^{r(K)}$. Since H is proper, the latter set is compact and hence closed as a subset of G. Thus $g = \lim_\lambda g_\lambda \in H_{d(K)}^{r(K)} \subseteq H$. □

There is a close connection between proper actions and so called induced spaces. Let us review the definition: Let G be a locally compact Hausdorff groupoid and $H \subseteq G$ a closed subgroupoid. Suppose Y is a (left) H-space with anchor map $p : Y \to H^{(0)}$. Consider the set

$$G \times_{G^{(0)}} Y = \{(g, y) \in G \times Y \mid d(g) = p(y)\}$$

There is a canonical action of H on $G \times_{G^{(0)}} Y$: The anchor map $P : G \times_{G^{(0)}} Y \to H^{(0)}$ is given by $P(g, y) = d(g) = p(y)$ and we define $h(g, y) = (gh^{-1}, hy)$.

LEMMA 1.2.14. *The action of H on $G \times_{G^{(0)}} Y$ defined above is proper.*

PROOF. Let $K \subseteq G \times_{G^{(0)}} Y$ be a compact subset. We need to show that $\mathcal{F}_K = \{h \in H \mid hK \cap K \neq \emptyset\}$ is a compact subset of H. If $K_1 = pr_1(K)$ is the image of K under the projection onto G it is not hard to see that $\mathcal{F}_K \subseteq K_1^{-1} K_1 \cap H$. Since the latter set is compact and $\mathcal{F}_K$ is closed in H, the result follows. □

It follows from the above Lemma combined with Lemma 1.2.11 and Proposition 1.2.6 that the quotient space $G \times_H Y := H \setminus (G \times_{H^{(0)}} Y)$ is a locally compact Hausdorff space. This space is called the *induced space*. There is a canonical left action of G on $G \times_H Y$, coming from the action of G on itself. The anchor map $G \times_H Y \to G^{(0)}$ is given by $[g, y] \mapsto r(g)$ and we define $g_1[g_2, y] := [g_1 g_2, y]$. One easily checks, that this gives a well-defined continuous action.

LEMMA 1.2.15. *Let G be a locally compact Hausdorff groupoid with open domain and range maps. If $H \subseteq G$ is a closed subgroupoid and Y is a proper H-space, then $G \times_H Y$ is a proper G-space.*

PROOF. We will check condition (4) in 1.2.8. Let $([g_\lambda, y_\lambda])_\lambda$ be a convergent net in $G \times_H Y$ with limit $[g, y]$ and let $(h_\lambda)_\lambda$ be a net in G with $d(h_\lambda) = r(g_\lambda)$ and such that $(h_\lambda [g_\lambda, y_\lambda])_\lambda$ is convergent as well. We have to check, that $(h_\lambda)_\lambda$ has a convergent subnet. Our assumptions imply, that the quotient map $G \times_{H^{(0)}} Y \to G \times_H Y$ is open. Hence we can pass to a subnet and relabel twice, to assume that $(g_\lambda, y_\lambda) \to (g, y)$ and $(h_\lambda g_\lambda, y_\lambda)$ converges as well. Using the fact, that G acts properly on itself this implies, that $(h_\lambda)_\lambda$ has a convergent subnet, as required. □

CHAPTER 2

Groupoid Dynamical Systems and Crossed Products

The purpose of this chapter is twofold: First, we review the basic theory of groupoid dynamical systems and reduced crossed products. For this, we also have to include a brief overview of the theory of $C_0(X)$-algebras.

In the last section of this chapter we then deal with generalizing the process of induction introduced for spaces in the end of chapter 1 to arbitrary C^*-algebras.

2.1. $C_0(X)$-algebras

Our exposition in this section is based on [**Wil07**, Appendix C] and [**Goe09**, Section 3.1].

DEFINITION 2.1.1. A C^*-algebra A is called a $C_0(X)$-algebra if there exists a $*$-homomorphism $\Phi_A : C_0(X) \to Z(M(A))$ from $C_0(X)$ into the center of the multiplier algebra of A which is non-degenerate in that

$$C_0(X)A := span\{\Phi_A(f)a \mid f \in C_0(X), a \in A\}$$

is dense in A.

From now on, when there is no danger of confusion, we will omit the structure homomorphism from the notation and just write fa for $\Phi_A(f)a$.

REMARK 2.1.2. Note that if A is a $C_0(X)$-algebra and $(\varphi_\lambda)_\lambda$ is a bounded approximate unit for $C_0(X)$, then $\|\varphi_\lambda a - a\| \to 0$. First let $a = \sum_{i=1}^n \varphi_i a_i$. Given $\varepsilon > 0$ we can find λ_0 such that $\|\varphi_\lambda\varphi_i - \varphi_i\| < \frac{\varepsilon}{\|a_i\| n}$ for all $1 \leq i \leq n$ and $\lambda \geq \lambda_0$. Then for $\lambda \geq \lambda_0$ we can compute:

$$\|\varphi_\lambda a - a\| = \|\sum_{i=1}^n \varphi_\lambda\varphi_i a_i - \varphi_i a_i\| \leq \sum_{i=1}^n \|\varphi_\lambda\varphi_i - \varphi_i\|\|a_i\| < \varepsilon$$

The claim for $a \in A$ arbitrary now follows from the fact that $C_0(X)A$ is dense in A, using a straightforward $\frac{\varepsilon}{3}$-argument. It follows, that already $\{fa \mid f \in C_0(X), a \in A\}$ is dense in A.

For each point $x \in X$ the set $\overline{C_0(X \setminus \{x\})A}$ is a closed two-sided ideal in A, which we denote by I_x. The quotient algebra $A_x := A/I_x$ is called the *fibre* of A over $x \in X$ and we will write $a(x)$ for the image of $a \in A$ under the canonical quotient map $A \to A/I_x$. Let us note the following easy facts:

LEMMA 2.1.3. *Let A be a $C_0(X)$-algebra. Then the following hold:*

(1) For every $a \in A$ the map $x \mapsto \|a(x)\|$ is upper-semicontinuous and vanishes at infinity in the sense that $\{x \in X \mid \|a(x)\| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$.

(2) The norm of an element $a \in A$ can be computed as

$$\|a\| = \sup_{x \in X} \|a(x)\|.$$

(3) For $f \in C_0(X)$ and $a \in A$ one has the formula $(fa)(x) = f(x)a(x)$.

PROOF. See [**Wil07**, Proposition C.10]. □

The following density criterion will turn out to be very useful, when working with $C_0(X)$-algebras. The proof can be adapted easily from [**Wil07**, Proposition C.24].

PROPOSITION 2.1.4. *Let A be a $C_0(X)$-algebra and $\Gamma \subseteq A$ be a linear subspace. Assume additionally, that*

(1) Γ is closed under the action of $C_0(X)$, meaning $fa \in \Gamma$ for all $f \in C_0(X)$ and $a \in \Gamma$, and

(2) the image of Γ under the quotient map $A \to A_x$ is dense in A_x for all $x \in X$.

Then Γ is dense in A.

A first easy application of this result is contained in the proof of the next well-known lemma. Before we can state it, we need some more terminology:

DEFINITION 2.1.5. A $*$-homomorphism $\Phi : A \to B$ between two $C_0(X)$-algebras A and B is called *$C_0(X)$-linear* if $\Phi(fa) = f\Phi(a)$ for all $f \in C_0(X)$ and all $a \in A$.

If $\Phi : A \to B$ is a $C_0(X)$-linear homomorphism, it induces $*$-homomorphisms $\Phi_x : A_x \to B_x$ on the level of the fibres given by $\Phi_x(a(x)) = \Phi(a)(x)$. Conveniently, one can check several properties of Φ on the level of the fibres and vice versa:

LEMMA 2.1.6. [**EE11**, Lemma 2.1] *Let $\Phi : A \to B$ be a $C_0(X)$-linear homomorphism. Then Φ is injective (resp. surjective, resp. bijective) if and only if Φ_x is injective (resp. surjective, resp. bijective) for all $x \in X$.*

PROOF. Assume first that Φ is injective and suppose that $a(x) \in ker(\Phi_x)$. Then $\Phi(a)(x) = \Phi_x(a(x)) = 0$ and hence

$$\Phi(a) \in \overline{C_0(X \setminus \{x\})B}.$$

To conclude that $a(x) = 0$ we need to show that $a \in \overline{C_0(X \setminus \{x\})A}$. So let $\varepsilon > 0$ be given. Then by Remark 2.1.2 there is a function $f \in C_0(X \setminus \{x\})$ such that $\|f\Phi(a) - \Phi(a)\| < \varepsilon$ and thus using that every injective $*$-homomorphism is isometric, we get

$$\|fa - a\| = \|\Phi(fa - a)\| = \|f\Phi(a) - \Phi(a)\| < \varepsilon.$$

Conversely suppose that Φ_x is injective and hence isometric for all $x \in X$. Then by Lemma 2.1.3 we can compute

$$\|\Phi(a)\| = \sup\|\Phi(a)(x)\| = \sup\|\Phi_x(a(x))\| = \sup\|a(x)\| = \|a\|,$$

and hence Φ is injective.
It is straightforward to see that surjectivity of Φ forces all the Φ_x to be surjective. For the converse note that $\Phi(A)$ is a linear subspace of B satisfying the conditions of Proposition 2.1.4. Consequently, it is dense in - and hence equals B. □

Using the canonical quotient maps $A \to A/I_x$ we can think of an element $a \in A$ as a function from X into the disjoint union $\mathcal{A} := \coprod_{x \in X} A_x$ of the fibres, via $x \mapsto a(x)$. Actually, $\mathcal{A}$ is a "bundle of C^*-algebras over X" in a way that we will now make precise.

DEFINITION 2.1.7. An *upper-semicontinuous C^*-bundle* over a locally compact space X is a topological space $\mathcal{A}$ together with a continuous open surjection $q : \mathcal{A} \to X$ such that each *fibre* $A_x = q^{-1}(x)$ is a C^*-algebra and the following conditions are satisfied:

(1) The map $\mathcal{A} \to \mathbb{R}^+$, $a \mapsto \|a\|$ is upper-semicontinuous.
(2) The map $q^*\mathcal{A} := \{(a,b) \in \mathcal{A} \times \mathcal{A} \mid q(a) = q(b)\} \to \mathcal{A}$ sending (a,b) to $a+b$ is continuous.
(3) For each $\lambda \in \mathbb{C}$ the map $a \mapsto \lambda a$ is continuous from $\mathcal{A}$ to $\mathcal{A}$.
(4) The map $q^*\mathcal{A} \to \mathcal{A}$ sending (a,b) to ab is continuous.
(5) The map $\mathcal{A} \to \mathcal{A}$, $a \mapsto a^*$ is continuous.
(6) If $(a_\lambda)_\lambda$ is a net in $\mathcal{A}$ such that $q(a_\lambda) \to x$ and $\|a_\lambda\| \to 0$, then $a_\lambda \to 0 \in A_x$.

A *continuous* C^*-bundle is an upper-semicontinuous bundle for which the map in (1) is continuous.

The next proposition collects the results from [**Wil07**] C.17 through C.20:

PROPOSITION 2.1.8. *Let $q : \mathcal{A} \to X$ be an upper-semicontinuous C^*-bundle. Then the following hold:*

(1) If $(a_\lambda)_\lambda$ is a net in $\mathcal{A}$ with $a_\lambda \to 0 \in A_x$, then $\|a_\lambda\| \to 0$.
(2) Scalar multiplication is a continuous map $\mathbb{C} \times \mathcal{A} \to \mathcal{A}$.
(3) Let $(a_\lambda)_\lambda$ be a net in $\mathcal{A}$ such that $q(a_\lambda) \to q(a)$ for some $a \in \mathcal{A}$. Suppose that for all $\varepsilon > 0$ there is a net $(u_\lambda)_\lambda$ in $\mathcal{A}$ and $u \in \mathcal{A}$ such that:
   (a) $u_\lambda \to u$ in $\mathcal{A}$,
   (b) $q(u_\lambda) = q(a_\lambda)$,
   (c) $\|a - u\| < \varepsilon$, and
   (d) $\|a_\lambda - u_\lambda\| < \varepsilon$ for large λ.

   Then $a_\lambda \to a$.

DEFINITION 2.1.9. Let $\Gamma(X, \mathcal{A})$ be the set of continuous sections of an upper-semicontinuous C^*-bundle $q : \mathcal{A} \to X$, that is the set of all continuous functions $f : X \to \mathcal{A}$ such that $q \circ f = id_X$. Likewise we write $\Gamma_c(X, \mathcal{A})$ for the continuous sections with compact support and $\Gamma_0(X, \mathcal{A})$ for the continuous sections vanishing at infinity.

It is shown in [**Wil07**, Proposition C.23], that $\Gamma_0(X, \mathcal{A})$ is a C*-algebra with respect to the norm

$$\|f\| := \sup_{x \in X} \|f(x)\|,$$

and moreover a $C_0(X)$-algebra with respect to the canonical action given by pointwise multiplication. For each $x \in X$ the evaluation map $\Gamma_0(X, \mathcal{A}) \to A_x$ induces an isomorphism $\Gamma_0(X, \mathcal{A})_x \cong A_x$.

In the other direction a theorem of Fell shows, that given a $C_0(X)$-algebra A, there is a topology on $\mathcal{A} = \coprod_{x \in X} A_x$ such that the canonical surjection $q : \mathcal{A} \to X$ is an upper-semicontinuous C*-bundle. Furthermore, the map $A \to \Gamma_0(X, \mathcal{A})$ sending $a \in A$ to the function $x \mapsto a(x)$ is a $C_0(X)$-linear $*$-isomorphism (see [**Wil07**, Theorem C.25] for a detailed proof). For further reference let us record, that a basis for the topology of $\mathcal{A}$ is defined by the sets

$$W(a, U, \varepsilon) := \{b \in \mathcal{A} \mid q(b) \in U \ \textit{and} \ \|b - a(q(b))\| < \varepsilon\},$$

where $a \in A$, $U \subseteq X$ is an open subset and $\varepsilon > 0$.

Using these results, we will freely alternate between the $C_0(X)$-algebra picture and the bundle picture, whichever seems more useful in the given situation. Next, we will have a look at various constructions of $C_0(X)$-algebras:

Pullbacks. Let A be a $C_0(X)$-algebra and $f : Y \to X$ be a continuous map. Consider the associated upper-semicontinuous bundle $q : \mathcal{A} \to X$. Then we can form the pullback bundle

$$f^*\mathcal{A} = \{(y, a) \in Y \times \mathcal{A} \mid f(y) = q(a)\}$$

where the bundle map $p : f^*\mathcal{A} \to Y$ is the projection onto Y.

PROPOSITION 2.1.10. *The bundle $p : f^*\mathcal{A} \to Y$ is an upper-semicontinuous C^*-bundle over Y.*

PROOF. It is easy to see that we have an identification $p^{-1}(\{y\}) = A_{f(y)}$ and hence that each fibre is a C^*-algebra. Also p is obviously a continuous surjection. To see that it is open, let $y \in Y$, $a \in A_{f(y)}$ and $y_\lambda \to y = p(a, y)$. Then $f(y_\lambda) \to f(y) = q(a)$. Since q is an open surjection we may pass to a subnet and assume that there exist a net $(a_\lambda)_\lambda$ such that $a_\lambda \to a$ and $q(a_\lambda) = f(y_\lambda)$. Thus $(a_\lambda, y_\lambda) \in f^*\mathcal{A}$ and the claim follows. The rest of the axioms are straightforward to verify. □

DEFINITION 2.1.11. Let A be a $C_0(X)$-algebra and $f : Y \to X$ a continuous map. We define the pull back of A along f to be the $C_0(Y)$-algebra $f^*A := \Gamma_0(Y, f^*\mathcal{A})$.

Note, that we can identify $(f^*A)_y = A_{f(y)}$. The proof of the following lemma is an easy exercise:

LEMMA 2.1.12. *Let A be a $C_0(X)$-algebra and $f : Y \to X$ and $g : Z \to Y$ be two continuous maps. Then the algebras $(f \circ g)^*A$ and $g^*(f^*A)$ are canonically isomorphic as $C_0(Z)$-algebras.*

The following result is an easy consequence of Proposition 2.1.4 and often helpful when working with pullbacks.

PROPOSITION 2.1.13. *Let A be a $C_0(X)$-algebra and $f : Y \to X$ a continuous map. For $\varphi \in C_c(Y)$ and $a \in A$ define a function $\varphi \otimes a \in \Gamma_c(Y, f^*\mathcal{A})$ by*

$$(\varphi \otimes a)(y) := \varphi(y)a(f(y)).$$

Then the linear subspace

$$span\{\varphi \otimes a \mid \varphi \in C_c(Y), a \in A\}$$

is dense in f^*A.

When working with crossed products it is often useful to consider another topology on the algebra of continuous sections $\Gamma(X, \mathcal{A})$ of an upper-semicontinuous C*-bundle. We say that a net $(f_\lambda)_\lambda$ of functions in $\Gamma(X, \mathcal{A})$ converges to $f \in \Gamma(X, \mathcal{A})$ with respect to the *inductive limit topology*, if and only if there exists a compact subset K in X such that f and, eventually, all the f_λ vanish off of K and $\|f_\lambda - f\|_\infty \to 0$.

COROLLARY 2.1.14. [**Goe09**, Corollary 3.45] *Keeping the notation of the previous proposition, the linear subspace*

$$span\{\varphi \otimes a \mid \varphi \in C_c(Y), a \in A\}$$

is dense in $\Gamma_c(Y, f^*\mathcal{A})$ with respect to the inductive limit topology.

The next lemma studies the behaviour of pullbacks with respect to $C_0(X)$-linear $*$-homomorphisms.

LEMMA 2.1.15. *Let A and B be two $C_0(X)$-algebras and $f : Y \to X$ a continuous map. If $\Phi : A \to B$ is a $C_0(X)$-linear homomorphism, then the map*

$$f^*\Phi : f^*A \to f^*B$$

given by $(f^*\Phi)(\psi)(y) = \Phi_{f(y)}(\psi(y))$ is a $C_0(Y)$-linear homomorphism. Moreover, the pullback construction is functorial meaning if $\Psi : B \to C$

is another $C_0(X)$-linear $*$-homomorphism into a $C_0(X)$-algebra C then $f^*\Psi \circ f^*\Phi = f^*(\Psi \circ \Phi)$.

Proof. First of all it is easy to see that given $\psi \in f^*A$ the map $f^*\Phi(\psi)$ is a section vanishing at infinity. The only thing which is not immediately clear is the continuity of $f^*\Phi(\psi)$. To this end first consider elements for the form $\varphi \otimes a \in f^*A$, where $\varphi \in C_c(Y)$ and $a \in A$. Then we compute:

$$\begin{aligned} f^*\Phi(\varphi \otimes a)(y) &= \Phi_{f(y)}(\varphi(y)a(f(y))) \\ &= \varphi(y)\Phi_{f(y)}(a(f(y))) \\ &= \varphi(y)\Phi(a)(f(y)) \\ &= (\varphi \otimes \Phi(a))(y) \end{aligned}$$

Hence we see that $f^*\Phi(\varphi \otimes a) = \varphi \otimes \Phi(a)$ is continuous.

Now let $\psi \in f^*A$ be arbitrary and $(y_\lambda)_\lambda$ a net in Y such that $y_\lambda \to y$ for some $y \in Y$. Let $\varepsilon > 0$ be given. Then there exists $\psi' = \sum_{i=1}^{n} \varphi_i \otimes a_i$ such that $\|\psi - \psi'\| < \varepsilon$. We have

(1) $\Phi_{f(y_\lambda)}(\psi'(y_\lambda)) \to \Phi_{f(y)}(\psi'(y))$ by the first part of this proof,
(2) $\|\Phi_{f(y)}(\psi(y)) - \Phi_{f(y)}(\psi'(y))\| < \varepsilon$, and
(3) $\|\Phi_{f(y_\lambda)}(\psi(y_\lambda)) - \Phi_{f(y_\lambda)}(\psi'(y_\lambda))\| < \varepsilon$

Thus, we can apply Proposition 2.1.8 to conclude that $\Phi_{f(y_\lambda)}(\psi(y_\lambda)) \to \Phi_{f(y)}(\psi(y))$ as desired. Straightforward computations show that $f^*\Phi$ is a $C_0(Y)$-linear $*$-homomorphism and the functoriality of the construction. □

Push forward. Let A be a $C_0(X)$-algebra and $f : X \to Y$ a continuous map. Then we can turn A into a $C_0(Y)$-algebra as follows: Since the action $\Phi : C_0(X) \to Z(M(A))$ is non-degenerate there exists a unique extension

$$\tilde{\Phi} : C_b(X) \cong M(C_0(X)) \to M(A)$$

to the bounded continuous functions on X. We need the following

Lemma 2.1.16. *The image of $\tilde{\Phi}$ is contained in the centre $Z(M(A))$ of $M(A)$.*

Proof. Recall from [**Wil07**, Lemma 8.3], that it suffices to show, that $\tilde{\Phi}(f)ab = a\tilde{\Phi}(f)b$ for all $a, b \in A$ and $f \in C_b(X)$. Furthermore, since Φ is non-degenerate, it suffices to check this for elements of the form $\tilde{a} = \Phi(g)a \in A$ with $g \in C_0(X)$ and $a \in A$. So let $g \in C_0(X)$ and $a, b \in A$ be given. Then we have

$$\begin{aligned}\tilde{\Phi}(f)\tilde{a}b &= \tilde{\Phi}(f)\Phi(g)ab \\ &= \Phi(fg)ab \\ &= a\Phi(fg)b \\ &= a\tilde{\Phi}(f)\Phi(g)b \\ &= a\Phi(g)\tilde{\Phi}(f)b \\ &= \Phi(g)a\tilde{\Phi}(f)b \\ &= \tilde{a}\tilde{\Phi}(f)b,\end{aligned}$$

and the proof is complete. □

If we now consider the induced homomorphism $f^* : C_0(Y) \to C_b(X)$ we can just compose it with $\tilde{\Phi}$ to obtain a homomorphism $C_0(Y) \to Z(M(A))$. In other words: For all $\varphi \in C_0(Y)$ and $a \in A$ we can define $\varphi \cdot a := \tilde{\Phi}(\varphi \circ f)a$. In order to see that this indeed turns A into a $C_0(Y)$-algebra we just need to check the non-degeneracy condition, which is the content of the following Proposition.

PROPOSITION 2.1.17. *Let A be a $C_0(X)$-algebra and $f : X \to Y$ be a continuous map. Then A is a $C_0(Y)$-algebra with respect to the homomorphism $\tilde{\Phi} \circ f^* : C_0(Y) \to Z(M(A))$.*

PROOF. We only need to check, that $\tilde{\Phi} \circ f^*$ is non-degenerate. First observe, that f^* is non-degenerate in the sense that $f^*(C_0(Y))C_0(X)$ is dense in $C_0(X)$. This follows easily from the Stone-Weierstrass theorem since if $x \neq y \in X$ then we can choose a function $\varphi \in C_0(X)$ such that $\varphi(x) = 1$ and $\varphi(y) = 0$. Furthermore let $\psi \in C_0(Y)$ be a function such that $\psi(f(x)) = 1$. Then $(f^*(\psi)\varphi)(x) = \psi(f(x))\varphi(x) = 1 \neq 0 = \psi(f(y))\varphi(y) = (f^*(\psi)\varphi)(y)$.

If $a \in A$ and $\varepsilon > 0$ are given, there exist $\varphi \in C_0(X)$ and $b \in A$ such that $\|\tilde{\Phi}(\varphi)b - a\| < \frac{\varepsilon}{2}$ since Φ is non-degenerate. Since $f^*(C_0(Y))C_0(X)$ is dense in $C_0(X)$ we can find functions $g \in C_0(Y)$ and $h \in C_0(X)$ such that $\|f^*(g)h - \varphi\| < \frac{\varepsilon}{2\|b\|}$. Consequently, we get that

$$\|\tilde{\Phi}(f^*(g))\Phi(h)b - a\| = \|\tilde{\Phi}(f^*(g)h)b - a\|$$

$$\leq \|\tilde{\Phi}(f^*(g)h)b - \tilde{\Phi}(\varphi)b\| + \|\tilde{\Phi}(\varphi)b - a\|$$
$$\leq \|f^*(g)h - \varphi\|\|b\| + \|\tilde{\Phi}(\varphi)b - a\|$$
$$< \varepsilon$$

□

It is important to note, that this construction (in contrast to the pullback) does not change the C*-algebra itself, but just the associated bundle structure as we observe in the following example:

EXAMPLE 2.1.18. Let X, Y be locally compact Hausdorff spaces and $p : X \to Y$ a continuous map. Since $C_0(X)$ is a $C_0(X)$-algebra by the above construction we can turn $C_0(X)$ into a $C_0(Y)$-algebra via p. The action for $\varphi \in C_0(Y)$ and $f \in C_0(X)$ is then given by $(\varphi \cdot f)(x) = \varphi(p(x))f(x)$. The fibre $(C_0(X))_y$ over $y \in Y$ can then be identified with $C_0(X_y)$ where $X_y := p^{-1}(\{y\}) \subseteq X$. The isomorphism is induced by the restriction homomorphism $res : C_0(X) \to C_0(X_y)$. This homomorphism is clearly surjective and it is not hard to see that $ker(res) = I_y$.

We will sometimes write f_*A for the pushforward of A along f. The preceding example also illustrates the following general description of the fibres:

PROPOSITION 2.1.19. *Let A be a $C_0(X)$-algebra and $f : X \to Y$ be a continuous map between locally compact Hausdorff spaces. For $y \in Y$ let $X_y := f^{-1}(\{y\})$. Then, viewing A as a $C_0(Y)$-algebra via pushing forward along f, there is an isomorphism*

$$A_y \to \Gamma_0(X_y, \mathcal{A}_{|X_y}).$$

PROOF. Identify A with the section algebra $\Gamma_0(X, \mathcal{A})$ and consider the restriction homomorphism

$$res : \Gamma_0(X, \mathcal{A}) \to \Gamma_0(X_y, \mathcal{A}_{|X_y}).$$

We will show, that this homomorphism factors through the desired isomorphism. First of all $ker(res)$ can be identified with the ideal I_y: For all $x \in X_y$, $\varphi \in C_0(Y \setminus \{y\})$ and $a \in A$ we clearly have $(\varphi \cdot a)(x) = \varphi(f(x))a(x) = \varphi(y)a(x) = 0$ and thus $I_y \subseteq ker(res)$. If conversely $a \in ker(res)$ and $\varepsilon > 0$ is given then $K := \{x \in X \mid$

$\|a(x)\| \geq \varepsilon\}$ is compact. By continuity $f(K)$ is also compact. Since clearly $y \notin f(K)$ there is a function $\varphi \in C_0(Y)$ with $0 \leq \varphi \leq 1$ such that $\varphi = 1$ on $f(K)$ and $\varphi(y) = 0$. Then $\varphi \cdot a \in I_y$. For $x \in K$ we have $\|a(x) - (\varphi \cdot a)(x)\| = \|a(x) - \varphi(f(x))a(x)\| = 0$ and for $x \notin K$ we have $\|a(x) - \varphi(f(x))a(x)\| = |1 - \varphi(f(x))|\|a(x)\| < \varepsilon$ by construction. Thus, we can conclude $\|a - \varphi \cdot a\| = \sup_{x \in X}\|a(x) - \varphi(f(x))a\| < \varepsilon$ and hence $a \in I_y$. Surjectivity follows from another easy application of Proposition 2.1.4. □

The following describes the interplay of the pushforward and the pullback construction:

PROPOSITION 2.1.20. *Let $f : Y \to X$ and $g : Z \to X$ be continuous maps. Consider also the pullback space $Y \times_X Z = \{(y, z) \in Y \times Z \mid f(y) = g(z)\}$ with the canonical projection maps $\pi_Y : Y \times_X Z \to Y$ and $\pi_Z : Y \times_X Z \to Z$. Suppose A is a $C_0(Z)$-algebra. Then $f^*(g_*A)$ is canonically isomorphic to $(\pi_Y)_*(\pi_Z^*A)$ as $C_0(Y)$-algebras.*

PROOF. We will define a map

$$\Phi : f^*(g_*A) \to (\pi_Y)_*(\pi_Z^*A).$$

Note first, that for $y \in Y$ the fibres of each of these $C_0(Y)$-algebras are given by

$$f^*(g_*A)_y = (g_*A)_{f(y)} = \Gamma_0(Z_{f(y)}, \mathcal{A}_{|Z_{f(y)}}), \text{ and}$$
$$(\pi_Y)_*(\pi_Z^*A)_y = \Gamma_0((Y \times_X Z)_y, \pi_Z^*\mathcal{A}_{|(Y \times_X Z)_y}).$$

For $\varphi \in f^*(g_*A) = \Gamma_0(Y, f^*(g_*\mathcal{A}))$ define $(\Phi(\varphi)(y))\,(y, z) = (\varphi(y))(z)$. It is straightforward to check, that Φ is an isometric, $C_0(Y)$-linear $*$-homomorphism. Surjectivity however is obvious for the homomorphism Φ_y at the level of each fibre, hence an application of Lemma 2.1.6 finishes the proof. □

Tensor Products. Let $\otimes_{max}$ denote the maximal tensor product of C^*-Algebras. If A and B are C^*-algebras then the canonical embeddings $i_A : A \to M(A \otimes_{max} B)$ and $i_B : B \to M(A \otimes_{max} B)$ extend to commuting embeddings $M(A) \to M(A \otimes_{max} B)$ and $M(B) \to M(A \otimes_{max} B)$. One easily checks, that these embeddings take central multipliers to central multipliers. By the universal property of the maximal tensor product, there is a homomorphism

$$ZM(A) \otimes_{max} ZM(B) \to ZM(A \otimes_{max} B),$$

characterized by the formula $(m \otimes n)(a \otimes b) = ma \otimes nb$.

PROPOSITION 2.1.21. [**Bla96**, Corollaire 3.16] *Let A be a $C_0(X)$-algebra and B a $C_0(Y)$-algebra with structure homomorphisms $\Phi : A \to ZM(A)$ and $\Psi : C_0(Y) \to ZM(B)$. Then the composition*

$$C_0(X) \otimes C_0(Y) \stackrel{\Phi\otimes\Psi}{\to} ZM(A) \otimes_{max} ZM(B) \to ZM(A \otimes_{max} B)$$

is a non-degenerate $*$-homomorphism. Hence $A \otimes_{max} B$ is a $C_0(X \times Y)$-algebra. Furthermore, there are canonical isomorphisms

$$(A \otimes_{max} B)_{(x,y)} \cong A_x \otimes_{max} B_y.$$

If A and B are two $C_0(X)$-algebras, we would like to consider a notion of tensor product, which is again a $C_0(X)$-algebra. To this end consider the diagonal map $\Delta : X \to X \times X$ and define the *(maximal) balanced tensor product* of A and B over X to be the pullback $A \otimes_X^{max} B := \Delta^*(A \otimes_{max} B)$. Note that there is a canonical isomorphism $A \otimes_X^{max} B \cong A \otimes_{max} B/I_\Delta$, where $I_\Delta = \overline{C_0((X \times X) \setminus im(\Delta))A \otimes_{max} B}$ is the ideal in $A \otimes_{max} B$ corresponding to the closed subset $im(\Delta) \subseteq X \times X$.

Inductive limits. Let $(A_n, \varphi_n)_{n\in\mathbb{N}}$ be an inductive sequence of C*-algebras, where each A_n is a $C_0(X)$-algebra, such that the connecting homomorphisms φ_n are $C_0(X)$-linear. If $A = \lim_{n\to\infty} A_n$, then A is a $C_0(X)$-algebra in a canonical way. This is surely well-known to the experts, but we could not find a proper reference, so we include the details.

Let us start by recalling the construction of the limit algebra A: Consider the algebra

$$\widetilde{A} = \{(a_n)_n \in \prod_n A_n \mid \exists n_0 : a_{n+1} = \varphi_n(a_n) \forall n \geq n_0\}.$$

Then A is the closure of the image of $\widetilde{A}$ under the quotient map $q : \prod A_n \to \prod A_n / \bigoplus A_n$. Now if $f \in C_0(X)$, then $C_0(X)$-linearity of the φ_n implies, that $\widetilde{A}$ is invariant under component-wise multiplication with f. It also leaves the ideal $\bigoplus A_n$ invariant. Hence we get a well-defined linear map $q(\widetilde{A}) \to q(\widetilde{A})$ by $f \cdot q((a_n)_n) := q((f \cdot a_n)_n)$. Using the equality $\|q((a_n)_n)\| = \lim\|a_n\|$ we get $\|q((f \cdot a_n)_n)\| = \lim\|f \cdot a_n\| \leq$

$\|f\| \lim\|a_n\| = \|f\|\|q((a_n)_n)\|$. Consequently, $f\cdot$ extends to a bounded linear map $A \to A$, actually to an element in $Z(M(A))$, where the adjoint is given by $\overline{f}\cdot$. Thus, we have constructed a $*$-homomorphism $\Phi : C_0(X) \to Z(M(A))$.

LEMMA 2.1.22. *The $*$-homomorphism Φ from above is non-degenerate. Consequently, A is a $C_0(X)$-algebra such that the canonical maps $\psi_n : A_n \to A$ are $C_0(X)$-linear.*

PROOF. Let $a \in A$ and $\varepsilon > 0$ be given. By construction $\bigcup_n \psi(A_n)$ is dense in A, so there exists $b \in A_n$ such that $\|\psi_n(b) - a\| < \frac{\varepsilon}{2}$. Since the structure homomorphism for A_n is non-degenerate we can also find $f \in C_0(X)$ and $c \in A_n$ such that $\|b - fc\| < \frac{\varepsilon}{2\|\psi_n\|}$, and hence $\|\psi_n(b) - f\psi_n(c)\| < \frac{\varepsilon}{2}$. Putting things together we obtain $\|f\psi_n(c) - a\| < \|f\psi_n(c) - \psi_n(b)\| + \|\psi_n(b) - a\| < \varepsilon$. □

We will now identify the fibres of the limit algebra:

LEMMA 2.1.23. *Let (A_n, φ_n) be an inductive sequence of $C_0(X)$-algebras and $A = \lim A_n$. Then, for every $x \in X$, $((A_n)_x, (\varphi_n)_x)$ is an inductive sequence of* C*-*algebras and*

$$\lim_{n\to\infty} (A_n)_x \cong A_x.$$

PROOF. It is immediate, that $((A_n)_x, (\varphi_n)_x)$ is indeed an inductive sequence of C*-algebras. Hence we only need to identify the limit. Let $\pi_{n,x} : A_n \to (A_n)_x$ denote the quotient maps onto the fibres and $\psi_{n,x} : (A_n)_x \to \lim_n (A_n)_x$ the canonical maps. By the universal property of the limit we obtain a surjective $*$-homomorphism

$$\pi : A \to \lim_n (A_n)_x.$$

It remains to show, that the kernel of π coincides with the ideal $I_x = \overline{C_0(X \setminus \{x\})A}$ of A. If $a = \psi_n(b)$ for some $b \in A_n$ and $f \in C_0(X \setminus \{x\})$, then $\pi(fa) = \pi(f\psi_n(b)) = \pi(\psi_n(fb)) = \psi_{n,x}(\pi_{n,x}(fb)) = 0$. By continuity we get $I_x \subseteq ker(\pi)$.

Suppose conversely that $a \in ker(\pi)$ and $\varepsilon > 0$ is given. First we can find $n \in \mathbb{N}$ and $b \in A_n$ such that $\|a - \psi_n(b)\| < \frac{\varepsilon}{3}$. Thus $\|\psi_{n,x}(\pi_{n,x}(b))\| = \|\pi(\psi_n(b))\| = \|\pi(\psi_n(b) - a)\| \leq \|a - \psi_n(b)\| < \frac{\varepsilon}{3}$. Upon replacing b and n by $\varphi_{m,n}(b)$ for m big enough we can actually assume that $\|\pi_{x,n}(b)\| < \frac{\varepsilon}{3}$. Then there exists some $b' \in A_n$ such that

$\|b - b'\| < \frac{\varepsilon}{3}$ and $\pi_{n,x}(b') = 0$. Hence there must be $b'' \in A_n$ and $\varphi \in C_0(X \setminus \{x\})$ such that $\|b' - \varphi b''\| < \frac{\varepsilon}{3}$. Putting things together we obtain

$$\|a-\varphi\psi_n(b'')\| \leq \|a-\psi_n(b)\|+\|\psi_n(b)-\psi_n(b')\|+\|\psi_n(b')-\psi_n(\varphi b'')\| < \varepsilon$$

and hence $ker(\pi) \subseteq I_x$, which completes the proof. □

Next, we want to show that taking the limit of an inductive sequence commutes with pullbacks: Let $(A_n, \varphi_n)_{n\in\mathbb{N}}$ be an inductive sequence of $C_0(X)$-algebras and $f : Y \to X$ a continuous map. Then we get $C_0(Y)$-linear $*$-homomorphisms $f^*\varphi_n : f^*A_n \to f^*A_{n+1}$ by the formula

$$(f^*\varphi_n)(\xi)(y) = (\varphi_n)_{f(y)}(\xi(y)).$$

as in Lemma 2.1.15.

Proposition 2.1.24. *Let $(A_n, \varphi_n)_{n\in\mathbb{N}}$ be an inductive sequence of $C_0(X)$-algebras and $f : Y \to X$ a continuous map. Then $(f^*A_n, f^*\varphi_n)_n$ is an inductive system of $C_0(Y)$-algebras and $f^*(\lim_n A_n)$ is $C_0(Y)$-linearly isomorphic to $\lim_n f^*(A_n)$.*

Proof. Let $A = \lim A_n$ and $\psi_n : A_n \to A$ be the canonical $*$-homomorphisms. Then by Lemma 2.1.15 we obtain $C_0(Y)$-linear $*$-homomorphisms $f^*\psi_n : f^*A_n \to f^*A$ such that $f^*\psi_{n+1} \circ f^*\varphi_n = f^*(\psi_{n+1} \circ \varphi_n) = f^*\psi_n$. Using the universal property of the limit, we obtain a $C_0(Y)$-linear $*$-homomorphism

$$\Psi : \lim_n f^*A_n \to f^*A.$$

To show that it is an isomorphism, it is enough to check that Ψ_y is an isomorphism for all $y \in Y$. But under the identifications

$$(\lim_n f^*A_n)_y \cong \lim_n (A_n)_{f(y)} \text{ and } (f^*A)_y \cong A_{f(y)}$$

the map Ψ_y coincides with the isomorphism

$$\lim_n (A_n)_{f(y)} \to A_{f(y)}$$

from the previous Lemma. □

2.2. Groupoid Dynamical Systems

We are now in a position to define the notion of a groupoid dynamical system. Our exposition follows [**MW08b**]. In order to have any chance of admitting a groupoid action, a C^*-algebra should be fibred over the groupoid's unit space, and in that case an element of the groupoid should give rise to an isomorphism from the fibre over the domain of said element to the fibre over its range. Formally, one makes the following definition:

DEFINITION 2.2.1. A *groupoid dynamical system* (A, G, α) consists of a locally compact Hausdorff groupoid G, a $C_0(G^{(0)})$-algebra A and a family $(\alpha_g)_{g\in G}$ of $*$-isomorphisms $\alpha_g : A_{d(g)} \to A_{r(g)}$ such that $\alpha_{gh} = \alpha_g \circ \alpha_h$ for all $(g, h) \in G^{(2)}$ and such that $g \cdot a := \alpha_g(a)$ defines a continuous action of G on the upper-semicontinuous bundle $\mathcal{A}$ associated to A.

Let us note the following two facts, which follow easily from the definition:

(1) For all $u \in G^{(0)}$ we have $\alpha_u = id_{A_u}$. To see this just compute $\alpha_u = \alpha_{uu} = \alpha_u\alpha_u$. Since α_u is an isomorphism we conclude that $\alpha_u = id_{A_u}$.
(2) For all $g \in G$ we have $\alpha_{g^{-1}} = \alpha_g^{-1}$: Since $gg^{-1} = r(g)$ and using (1) we get $id_{A_{r(g)}} = \alpha_{r(g)} = \alpha_{gg^{-1}} = \alpha_g\alpha_{g^{-1}}$.

We will often omit the action α in our notation and just say that A is a G-algebra.

LEMMA 2.2.2. *Let A be a $C_0(G^{(0)})$-algebra and $\alpha = (\alpha_g)_{g\in G}$ be a family of $*$-isomorphisms $\alpha_g : A_{d(g)} \to A_{r(g)}$, such that $\alpha_{gh} = \alpha_g \circ \alpha_h$ for all $(g, h) \in G^{(2)}$. Then (A, G, α) is a groupoid dynamical system, if and only if for every $a \in A$ the map $g \mapsto \alpha_g(a(d(g)))$ is a continuous section $G \to r^*\mathcal{A}$.*

PROOF. If (A, G, α) is a groupoid dynamcial system, it is clear that the mapping $g \mapsto \alpha_g(a(d(g)))$ is continuous.

For the converse we need to show, that if $(g_\lambda, b_\lambda)_\lambda$ is a net in $G * \mathcal{A}$ converging to some element (g, b), then $\alpha_{g_\lambda}(b_\lambda) \to \alpha_g(b)$ in $r^*\mathcal{A}$. We want to apply Proposition 2.1.8. Choose $a \in A$ with $a(d(g)) = b$. If we put $u_\lambda := \alpha_{g_\lambda}(a(d(g_\lambda)))$ and $u := \alpha_g(a(d(g))) = \alpha_g(b)$, then property (a) holds by our assumption and (b) and (c) are automatically satisfied. It remains to check (d), i.e. that for all $\varepsilon > 0$ we eventually have $\|\alpha_{g_\lambda}(b_\lambda) - u_\lambda\| < \varepsilon$. But $\|\alpha_{g_\lambda}(b_\lambda) - u_\lambda\| = \|b_\lambda - a(d(g_\lambda))\|$ and since

$b_\lambda \to b$ we have that b_λ will eventually be contained in the basic open neighbourhood $W(a, \varepsilon)$ of b, which finishes the proof of (d). □

There is another well-known characterization of groupoid actions on C*-algebras which is often useful (see [**MW08b**, Lemma 4.3] for a proof):

Proposition 2.2.3. *Let (A, G, α) be a groupoid dynamical system. Then the mapping*

$$f \mapsto [g \mapsto \alpha_g(f(g))]$$

defines a $C_0(G)$-linear $*$-isomorphism $d^*A \to r^*A$, also denoted by α.

Conversely, if G is a groupoid, A a $C_0(G^{(0)})$-algebra, and $\alpha : d^*A \to r^*A$ is a $C_0(G)$-linear isomorphism then α induces an isomorphism $\alpha_g : A_{d(g)} \to A_{r(g)}$ for each $g \in G$. If the equation $\alpha_{gh} = \alpha_g \alpha_h$ holds for all $(g, h) \in G^{(2)}$, then (A, G, α) is a groupoid dynamical system.

Example 2.2.4. Let G be a locally compact Hausdorff groupoid, acting on a locally compact Hausdorff space Y with anchor-map $p : Y \to G^{(0)}$. Then $C_0(Y)$ is a $C_0(G^{(0)})$-algebra with respect to the action

$$(\varphi \cdot f)(y) = \varphi(p(y))f(y), \;\; \varphi \in C_0(G^{(0)}), f \in C_0(Y).$$

Note, that this is just the pushforward along p of $C_0(Y)$ viewed as a $C_0(Y)$-algebra. Thus, for $u \in G^{(0)}$ the fibre $(C_0(Y))_u$ is just given by $C_0(Y_u)$. We can now define a G-action on $C_0(Y)$ as follows: For $g \in G$ the isomorphism

$$\alpha_g : (C_0(Y))_{d(g)} = C_0(Y_{d(g)}) \to C_0(Y_{r(g)}) = (C_0(Y))_{r(g)}$$

is just given by

$$\alpha_g(f)(y) = f(g^{-1}y).$$

A proof, that $(C_0(Y), G, \alpha)$ is actually a groupoid dynamical system can be found in [**Goe09**].

We will now study several constructions of groupoid dynamical systems.

Pullbacks. Suppose that $\Phi : H \to G$ is a groupoid homomorphism. Let $\Phi_0 : H^{(0)} \to G^{(0)}$ be the corresponding map between the unit spaces. If (A, G, α) is a groupoid dynamical system, we obtain an isomorphism of $C_0(G)$-algebras:

$$\Phi^*\alpha : \Phi^*(d_G^*A) \to \Phi^*(r_G^*A)$$

by Lemma 2.1.15. Now using the identifications

$$d_H^*(\Phi_0^*A) = (\Phi_0 \circ d_H)^*A = (d_G \circ \Phi)^*A = \Phi^*(d_G^*A)$$

and similarly

$$r_H^*(\Phi_0^*A) = \Phi^*(r_G^*A),$$

we obtain a $C_0(H)$-linear $*$-isomorphism

$$d_H^*(\Phi_0^*A) \to r_H^*(\Phi_0^*A),$$

which defines an action of H on Φ_0^*A by Proposition 2.2.3.

A particular instance of this is given by the inclusion of a closed subgroupoid. Let H be a closed subgroupoid of G and $\iota : H \hookrightarrow G$ the inclusion map. If A is a G-algebra we write $A_{|H} := \iota_0^*A$ and the action of H on $A_{|H}$ is just the restriction of the action of G on A.

Pushforward. Suppose X is a (left) G-space with anchor map $p : X \to G^{(0)}$ and $(A, G \ltimes X, \alpha)$ is a groupoid dynamical system. Then pushing forward along p we can also view A as a $C_0(G^{(0)})$-algebra. Recall that A_u is canonically identified with $\Gamma_0(p^{-1}(u), \mathcal{A})$. We can define a family $(\beta_g)_g$ of $*$-homomorphisms $\beta_g : A_{d(g)} \to A_{r(g)}$ by

$$\beta_g(f)(x) = \alpha_{(g,x)}(f(g^{-1}x)).$$

PROPOSITION 2.2.5. *The tripel (A, G, β) is a groupoid dynamical system.*

PROOF. First of all $\beta_g : A_{d(g)} \to A_{r(g)}$ is an isomorphism, as one easily computes that $\beta_g^{-1} = \beta_{g^{-1}}$ is an inverse. A similar computation yields that $\beta_{gh} = \beta_g \circ \beta_h$ for all $(g, h) \in G^{(2)}$. It remains to check, that the action of G on the bundle $p_*\mathcal{A}$ is continuous. Recall that the action of $G \ltimes X$ is implemented by an isomorphism $\alpha : D^*A \to R^*A$, where $D, R : G \ltimes X \to X$ denote the domain and range maps respectively. Using the pushforward construction along the projection $\pi : G \ltimes X \to G$ onto the first factor, we obtain a $*$-isomorphism

$$\pi_*\alpha : \pi_*(D^*A) \to \pi_*(R^*A).$$

Now an application of Proposition 2.1.20 provides the identifications $\pi_*(D^*A) \cong d^*(p_*A)$ and $\pi_*(R^*A) \cong r^*(p_*A)$. A quick computation reveals that under these identifications we have $(\pi_*\alpha)_g = \beta_g$. □

Tensor products. Given groupoid dynamical systems (A, G, α) and (B, G, β) we want to define the *diagonal action* of G on the balanced tensor product $A \otimes^{max}_{G^{(0)}} B$, following [**LG99**]. Using the canonical identifications of $C_0(G)$-algebras $d^*(A \otimes^{max}_{G^{(0)}} B) = d^*A \otimes^{max}_G d^*B$ and $r^*(A \otimes^{max}_{G^{(0)}} B) = r^*A \otimes^{max}_G r^*B$ the desired action is defined by the isomorphism

$$\alpha \otimes \beta : d^*A \otimes^{max}_G d^*B \to r^*A \otimes^{max}_G r^*B.$$

For $g \in G$ we have $(\alpha \otimes \beta)_g = \alpha_g \otimes \beta_g$.

Inductive limits. Suppose now that $(A_n, \varphi_n)_n$ is an inductive sequence of G-algebras, such that all the connecting homomorphisms are G-equivariant. We have already seen in Lemma 2.1.22, that $A = \lim_n A_n$ is a $C_0(G^{(0)})$-algebra in a canonical way, such that all the homomorphisms $\psi_n : A_n \to A$ are $C_0(G^{(0)})$-linear. The following Proposition shows how we can use the G-actions at each stage of the sequence to obtain a G-action on the limit.

PROPOSITION 2.2.6. *Let $(A_n, \varphi_n)_n$ be an inductive sequence of G-algebras, such that φ_n is G-equivariant for all $n \in \mathbb{N}$. Let $A := \lim_n A_n$ and $\psi_n : A_n \to A$ be the canonical maps. Then there exists a canonical G-action on A, such that ψ_n is G-equivariant for all $n \in \mathbb{N}$.*

PROOF. For each $n \in \mathbb{N}$ let $\alpha_n : d^*A_n \to r^*A_n$ denote the $C_0(G)$-linear isomorphism implementing the action of G on A_n. Since φ_n is G-equivariant for every $n \in \mathbb{N}$ we have commutative diagrams

$$\begin{array}{ccc} d^*A_n & \xrightarrow{\alpha_n} & r^*A_n \\ \downarrow{\scriptstyle d^*\varphi_n} & & \downarrow{\scriptstyle r^*\varphi_n} \\ d^*A_{n+1} & \xrightarrow{\alpha_{n+1}} & r^*A_{n+1} \end{array}$$

By the universal property, we obtain a $C_0(G)$-linear $*$-isomorphism between the respective limits. Combining this with Proposition 2.1.24 we obtain a $C_0(G)$-linear $*$-isomorphism

$$\alpha : d^*A \to r^*A.$$

As each α_n is compatible with the multiplication in G, so is the limit homomorphism α. □

2.3. Crossed Products

In this short section we remind the reader of the definition of reduced crossed products of C*-algebras by étale groupoids roughly following [**KS02**]. Let G be an étale groupoid and (A, G, α) a groupoid dynamical system. Consider the complex vector space $\Gamma_c(G, r^*\mathcal{A})$. It carries a canonical $*$-algebra structure with respect to the following operations:

$$(f_1 * f_2)(g) = \sum_{h \in G^{r(g)}} f_1(h)\alpha_h(f_2(h^{-1}g))$$

and

$$f^*(g) = \alpha_g(f(g^{-1})^*).$$

See for example [**MW08b**, Proposition 4.4] for a proof of this fact. For $u \in G^{(0)}$ consider the Hilbert A_u-module $\ell^2(G^u, A_u)$. It is the completion of the space of finitely supported A_u-valued functions on G^u, with respect to the inner product

$$\langle \xi, \eta \rangle = \sum_{h \in G^u} \xi(h)^* \eta(h).$$

We can then define a $*$-representation $\pi_u : \Gamma_c(G, r^*\mathcal{A}) \to \mathcal{L}(\ell^2(G^u, A_u))$ by

$$\pi_u(f)\xi(g) = \sum_{h \in G^u} \alpha_g(f(g^{-1}h))\xi(h).$$

Using this family of representations, we can define a C*-norm on the convolution algebra $\Gamma_c(G, r^*\mathcal{A})$ by

$$\|f\|_r := \sup_{u \in G^{(0)}} \|\pi_u(f)\|.$$

The reduced crossed product $A \rtimes_r G$ is defined to be the completion of $\Gamma_c(G, r^*\mathcal{A})$ with respect to $\|\cdot\|_r$.

2.4. Induced Algebras

In this section we will define and study a noncommutative analogue of the construction of the induced space, that we studied at the end of chapter 1. The definition is well-known in the group case and has appeared in the literature before also in the groupoid setting (see for example [**Bro12**]), but since we could not find a study of the basic properties, we chose to give a detailed exposition here. Most of our treatment follows ideas quite similar to the group case, which are presented nicely in [**RW98**].

Let (A, G, α) be a groupoid dynamical system and X a right G-space with anchor map $p : X \to G^{(0)}$. Consider the upper-semi-continuous C*-bundle $\mathcal{A}$ over $G^{(0)}$ associated to A. Form the pull-back $p^*(\mathcal{A})$ to obtain an upper-semi-continuous C*-bundle over X. Then define $Ind_G^X(A, \alpha)$ to be the set of all bounded continuous sections $f \in \Gamma_b(X, p^*(\mathcal{A}))$, such that

(1) for all $x \in X$ and $g \in G_{p(x)}$ we have $\alpha_g(f(x)) = f(xg^{-1})$, and
(2) the map $[xG \mapsto \|f(x)\|]$ vanishes at infinity.

As $Ind_G^X(A, \alpha)$ is a closed $*$-subalgebra of $\Gamma_b(X, p^*(\mathcal{A}))$, it is a C*-algebra. If the action of G on X is proper, $Ind_G^X(A, \alpha)$ carries more structure:

PROPOSITION 2.4.1. *Let (A, G, α) be a groupoid dynamical system and X a proper right G-space. Then $Ind_G^X A$ is a $C_0(X/G)$-algebra with respect to the action*

$$(\varphi \cdot f)(x) = \varphi(xG)f(x),$$

for $\varphi \in C_0(X/G)$ and $f \in Ind_G^X A$.

PROOF. First recall that the orbit space for a proper action is a locally compact Hausdorff space, so that our at least claim makes sense. Secondly, using [**Wil07**, Lemma 8.3], we can easily check, that the formula above defines an action of $C_0(X/G)$ as central multipliers: For $f, g \in Ind_G^X(A)$ and $\varphi \in C_0(X/G)$ we have

$$\varphi(ff')(x) = \varphi(xG)f(x)f'(x) = f(x)\varphi(xG)f'(x) = f(\varphi f')(x).$$

It remains to check the non-degeneracy of the action. So let $f \in Ind_X^G A$ and $\varepsilon > 0$ arbitrary. By definition of the induced algebra there exists a compact subset $K \subseteq X/G$ such that $\|f(x)\| < \varepsilon$ for all $xG \notin K$. Choose a function $\varphi \in C_0(X/G)$ with $0 \leq \varphi \leq 1$ such that $\varphi(xG) = 1$ for all $xG \in K$. Then we have $\|\varphi f - f\| < \varepsilon$. □

In what follows we want to identify the fibres of $Ind_G^X A$ with respect to this $C_0(X/G)$-algebra structure. The following result seems to be well-known, but we include a proof anyway, to keep our exposition self-contained.

LEMMA 2.4.2. *Let G be locally compact Hausdorff groupoid with Haar system $(\lambda^u)_{u\in G^{(0)}}$ and let A be a $C_0(G^{(0)})$-algebra. Given an element $f \in \Gamma_c(G, r^*\mathcal{A})$ let*

$$\lambda(f)(u) := \int\limits_{G^u} f(g)d\lambda^u(g). \tag{1}$$

This defines an element $\lambda(f) \in \Gamma_c(G^{(0)}, \mathcal{A})$.

PROOF. Since f is a section of $r^*\mathcal{A}$ the restriction of f to G^u yields an element in $C_c(G^u, A_u)$ the integral above is well-defined and $\lambda(f)$ obviously defines a section with compact support (as $supp(\lambda(f)) \subseteq r(supp(f)))$. It remains to show that $\lambda(f)$ is continuous.

First consider elements of the form $\varphi\otimes a \in \Gamma_c(G, r^*\mathcal{A})$ for $\varphi \in C_c(G)$ and $a \in A$ given by $\varphi \otimes a(g) = \varphi(g)a(r(g))$. Since the restriction of $\varphi \otimes a$ to G^u coincides with the function $\varphi_{|G^u} \otimes a(u) \in C_c(G^u, A_u)$ for all $u \in G^{(0)}$ we deduce that

$$\int\limits_{G^u} \varphi \otimes a(g)\lambda^u(g) = \int\limits_{G^u} (\varphi_{|G^u} \otimes a(u))(g)d\lambda^u(g) = \left(\int\limits_{G^u} \varphi(g)d\lambda^u(g)\right) a(u)$$

From this equation it is now obvious that $\lambda(\varphi \otimes a)$ is continuous.

Now let $f \in \Gamma_c(G, r^*\mathcal{A})$. Let $(u_j)_j$ be a net in $G^{(0)}$ such that $u_j \to u$ for some $u \in G^{(0)}$. We want to show that $\lambda(f)(u_j) \to \lambda(f)(u)$ in $\mathcal{A}$. To this end let $\varepsilon > 0$ be given. Since the span of elements of the form $\varphi \otimes a$ as above forms a dense subset of $\Gamma_c(G, r^*\mathcal{A})$ in the inductive limit topology there exists a net $(f_i)_i$ where each f_i is a finite sum of elementary tensors such that $f_i \to f$. Let $K \subseteq G$ be a compact subset which eventually contains the supports of the f_i and the support of f. Since K is compact there is an $M > 0$ such that $\lambda^v(K) \leq M$ for all $v \in G^{(0)}$. Chose i_0 such that $supp(f_i) \subseteq K$ and $\|f - f_i\| < \frac{\varepsilon}{M}$ for all $i \geq i_0$. Then for all $v \in G^{(0)}$ have

$$\begin{aligned}\|\lambda(f)(v) - \lambda(f_i)(v)\| &= \|\int\limits_{G^v} f(g) - f_i(g)\lambda^v(g)\| \\ &\leq \int\limits_{G^v} \|f(g) - f_i(g)\|\lambda^v(g)\end{aligned}$$

$$\leq \|f - f_i\|\lambda^v(K) < \varepsilon$$

We also know that $\lambda(f_{i_0})(u_j) \to \lambda(f_{i_0})(u)$ from the discussion at the beginning of this proof. Thus, the result follows from Proposition 2.1.8. □

We also need a slight extension of this result:

LEMMA 2.4.3. *For every $f \in \Gamma(G, r^*\mathcal{A})$ such that $supp(f) \cap r^{-1}(K)$ is compact for all compact $K \subseteq G^{(0)}$ the function $\lambda(f)$ is well-defined and continuous.*

PROOF. The proof can be carried out the same way as in the scalar case presented in Lemma 1.1.18. □

The next lemma is a groupoid analogue of [**RW98**, Lemma 6.17], which tells us that there are lots of non-trivial elements in $Ind_G^X(A)$.

LEMMA 2.4.4. *Let G be a locally compact Hausdorff groupoid with Haar system $(\lambda^u)_{u \in G^{(0)}}$. If (A, G, α) is a groupoid dynamical system and X a proper, right G-space with anchor-map $p : X \to G^{(0)}$, then for every $\varphi \in C_c(X)$ and $a \in A$ the formula*

$$\varphi \diamond a(x) := \int_{G^{p(x)}} \varphi(xg)\alpha_g(a(d(g)))d\lambda^{p(x)}(g)$$

gives a well-defined element $\varphi \diamond a \in Ind_G^X(A)$.

PROOF. Since the action of G on X is proper, the set $\{g \in G^{p(x)} \mid x \cdot g \in supp(\varphi)\}$ is compact for each fixed $x \in X$. Thus, the integrand is an element in $C_c(G^{p(x)}, A_{p(x)})$ and we can form the integral. For each $t \in G_{p(x)}$ we have

$$\begin{aligned}
\varphi \diamond a(xt^{-1}) &= \int_{G^{p(xt^{-1})}} \varphi(xt^{-1}g)\alpha_g(a(d(g)))d\lambda^{p(xt^{-1})}(g) \\
&\overset{g \mapsto tg}{=} \int_{G^{p(x)}} \varphi(xg)\alpha_{tg}(a(d(g)))d\lambda^{p(x)}(g) \\
&= \alpha_t\left(\int_{G^{p(x)}} \varphi(xg)\alpha_g(a(d(g)))d\lambda^{p(x)}(g)\right) \\
&= \alpha_t(\varphi \diamond a(x))
\end{aligned}$$

Furthermore $\varphi \diamond a$ is bounded. To see this note that the set $S := \{g \in G \mid supp(\varphi) \cdot g \cap supp(\varphi) \neq \emptyset\}$ is compact. From Lemma 1.1.16 we know that there is a $C > 0$ such that $\lambda^{p(x)}(S) < C$ for all $x \in X$. Then we have $\|\varphi \diamond a(x)\| \leq \int_{G^{p(x)}} |\varphi(xg)| d\lambda^{p(x)}(g) \|a\| \leq \lambda^{p(x)}(S) \|\varphi\| \|a\| \leq C \|\varphi\| \|a\|$. We want to see that $\varphi \diamond a$ is continuous. Note that $(y, g) \mapsto \varphi(y)\alpha_g(a(d(g)))$ is an element in $\Gamma(X \rtimes G, r^*_{X \rtimes G}(p^*\mathcal{A}))$ with proper support and thus by Lemma 2.4.2 the map

$$x \mapsto \int_{(X \rtimes G)^x} \varphi(y)\alpha_g(a(d(g)))d(\delta_x \otimes \lambda^{p(x)})(y, g)$$
$$= \int_{G^{p(x)}} \varphi(xg)\alpha_g(a(d(g)))d\lambda^{p(x)}(g)$$

is continuous. □

We are now ready to identify the fibres. To simplify the notation (and because we are mainly interested in this particular situation) we will now also assume that the action of G on X is *free* in the sense that $xg = x$ implies that g is a unit.

PROPOSITION 2.4.5. *Let G be a locally compact Hausdorff groupoid with Haar system $(\lambda^u)_{u \in G^{(0)}}$. If (A, G, α) is a groupoid dynamical system and X a free and proper, right G-space with anchor map $p : X \to G^{(0)}$, then $Ind_G^X(A, \alpha)$ is a $C_0(X/G)$-algebra, such that the fibre $(Ind_G^X(A, \alpha))_{xG}$ over $xG \in X/G$ is canonically isomorphic to $A_{p(x)}$.*

PROOF. The first part of the assertion has already been dealt with in Proposition 2.4.1. It remains to identify the fibres. For $x \in X$ consider the evaluation map

$$ev_x : Ind_G^X(A, \alpha) \to A_{p(x)}.$$

We will show, that the kernel of ev_x coincides with the ideal

$$I_{xG} = \overline{C_0(X/G \setminus \{xG\}) Ind_G^X(A)}$$

and that ev_x is surjective. Let us start with the kernel. If $\varphi \in C_0(X/G \setminus \{xG\})$ and $f \in Ind_G^X(A)$ we have $ev_x(\varphi \cdot f) = \varphi(xG)f(x) = 0$. Thus $I_{xG} \subseteq ker(ev_x)$. If conversely $f \in ker(ev_x)$ we have $f(xg) = \alpha_{g^{-1}}(f(x)) = 0$ for all $g \in G$. Hence f is zero on the whole orbit of

x. Given $\varepsilon > 0$ the set $K := \{yG \mid \|f(y)\| \geq \varepsilon\}$ is compact by definition of the induced algebra. Since X/G is Hausdorff there exists a $\varphi \in C_c(X/G)$, $0 \leq \varphi \leq 1$ such that $\varphi(xG) = 0$ and $\varphi = 1$ on K. One easily checks that $\varphi \cdot f \in I_{xG}$ and $\|f - \varphi \cdot f\| < \varepsilon$.

To prove surjectivity it suffices to show that ev_x has dense range. So let $a(p(x)) \in A_{p(x)}$ and $\varepsilon > 0$ be given. Choose a neighbourhood U of $p(x)$ in G such that $\|\alpha_g(a(d(g))) - a(p(x))\| < \varepsilon$ for all $g \in G^{p(x)} \cap U$. Choose $V \subseteq X$ open such that $V \cap xG = xU$. If $\phi \in C_c(X)$ is positive and has support contained in V define

$$\varphi(x) := \left(\int_{G^{p(x)}} \phi(xg) d\lambda^{p(x)}(g) \right)^{-1} \phi(x).$$

Then

$$\int_{G^{p(x)}} \varphi(xg) d\lambda^{p(x)}(g) = 1$$

and we have

$$\begin{aligned} \|\varphi \diamond a(x) - a(p(x))\| &= \| \int_{G^{p(x)}} \varphi(xg)\alpha_g(a(d(g)))dg - a(p(x))\| \\ &\leq \int_{G^{p(x)}} \varphi(xg)\|\alpha_g(a(d(g))) - a(p(x))\| d\lambda^{p(x)}(g) \\ &< \varepsilon \end{aligned}$$

□

Remark 2.4.6. Note that it follows from the proof above and Proposition 2.1.4 that

$$\text{span}\{\varphi \diamond a \mid \varphi \in C_c(X), a \in A\}$$

is dense in $Ind_G^X A$.

We will now turn to the situation which is of most interest for our purposes. Let G be a groupoid and $H \subseteq G$ a closed subgroupoid. Set $X := d^{-1}(H^{(0)}) \subseteq G$. Then H acts from the right on X, where the anchor map is the restriction of the domain map to X and the product is just given by multiplication. This action is obviously free and proper since $X \rtimes H$ is a closed subgroupoid of the proper groupoid $G \rtimes G$.

As the restriction of the range map to X is invariant under the H-action, it factors through a continuous map $\tilde{r} : X/H \to G^{(0)}$. This map serves as the anchor map for the canonical action of G on X/H given by multiplication (note that $gx \in X$ for all $g \in G$ and $x \in X$ with $d(g) = r(x)$).

Note that for each $(g, xH) \in G \ltimes X/H$ Proposition 2.4.5 gives us isomorphisms $\widetilde{ev_x} : (Ind_H^X A)_x \to A_{d(x)}$ and $\widetilde{ev_{g^{-1}x}} : (Ind_H^X A)_{g^{-1}x} \to A_{d(x)}$. Hence we get an isomorphism

$$\alpha_{(g,xH)} := \widetilde{ev_x}^{-1} \circ \widetilde{ev_{g^{-1}x}} : (Ind_H^X A)_{g^{-1}xH} \to (Ind_H^X A)_{xH}$$

Let $\alpha = (\alpha_{(g,xH)})_{(g,xH)\in G\ltimes X/H}$ be the family of all these ismorphisms. We want to see that $(Ind_H^X, G \ltimes X/H, \alpha)$ is a groupoid dynamical system. To check continuity of the action we need the following observation:

LEMMA 2.4.7. *Let $q : \mathcal{A} \to X$ be an upper-semicontinuous C^*-bundle. Suppose $(a_\lambda)_\lambda$ and $(b_\lambda)_\lambda$ are nets in $\mathcal{A}$ such that $q(a_\lambda) = q(b_\lambda)$ and $\lim_\lambda a_\lambda = a = \lim_\lambda b_\lambda$. Then*

$$\lim_\lambda \|a_\lambda - b_\lambda\| = 0.$$

PROOF. Let $\varepsilon > 0$ be given. Choose $f \in \Gamma_0(X, \mathcal{A})$ such that $f(q(a)) = a$. Then a is contained in the basic open set

$$W(f, \frac{\varepsilon}{2}) = \{b \in \mathcal{A} \mid \|b - f(q(b))\| < \frac{\varepsilon}{2}\}.$$

By assumption, for large λ we have $a_\lambda, b_\lambda \in W(f, \frac{\varepsilon}{2})$. Consequently, we eventually have

$$\|a_\lambda - b_\lambda\| \leq \|a_\lambda - f(q(a_\lambda))\| + \|f(q(b_\lambda)) - b_\lambda\| < \varepsilon.$$

□

PROPOSITION 2.4.8. *The triple $(Ind_H^X A, G \ltimes X/H, \alpha)$ is a groupoid dynamical system.*

PROOF. Let us first check that α is compatible with the groupoid structure. We compute

$$\begin{aligned}
\alpha_{(g_1,xH)} \circ \alpha_{(g_2,g_1^{-1}xH)} &= \widetilde{ev_x}^{-1} \circ \widetilde{ev_{g_1^{-1}x}} \circ \widetilde{ev_{g_1^{-1}x}}^{-1} \circ \widetilde{ev_{g_2^{-1}g_1^{-1}x}} \\
&= \widetilde{ev_x}^{-1} \circ \widetilde{ev_{(g_1g_2)^{-1}x}} \\
&= \alpha_{(g_1g_2,xH)}
\end{aligned}$$

Next, we have to check continuity. By Lemma 2.2.2, it is enough to check, that for any net $(g_\lambda, x_\lambda H)_\lambda$ in $G \ltimes X/H$ with $(g_\lambda, x_\lambda H) \to (g, xH) \in G \ltimes X/H$ and every $f \in Ind_H^X A$ we have

$$\alpha_{(g_\lambda, x_\lambda H)}(f + I_{g_\lambda^{-1}x_\lambda H}) \to \alpha_{(g,xH)}(f + I_{g^{-1}xH})$$

By definition, we have $\alpha_{(g,xH)}(f + I_{g^{-1}xH}) = \widetilde{ev_x}^{-1}(f(g^{-1}x))$. Suppose that the net $\widetilde{ev_{x_\lambda}}^{-1}(f(g_\lambda^{-1}x_\lambda))$ does not converge to $\widetilde{ev_x}^{-1}(f(g^{-1}x))$. Then, by definition of the topology on the bundle associated to the $C_0(X/H)$-algebra $Ind_H^X A$, there exists $f' \in Ind_H^X A$ such that $f'(x) = f(g^{-1}x)$ and $\varepsilon > 0$, such that after passing to a suitable subnet and relabeling, we can assume for all λ:

$$\|f(g_\lambda^{-1}x_\lambda) - f'(x_\lambda)\| = \|\widetilde{ev_{x_\lambda}}^{-1}(f(g_\lambda^{-1}x_\lambda)) - f' + I_{x_\lambda H}\| \geq \varepsilon$$

After passing to another subnet (and relabeling), we may also assume that $x_\lambda \to x$ by [**Wil07**, Proposition 1.15]. But then, by continuity of f and f' we have $f(g_\lambda^{-1}x_\lambda) \to f(g^{-1}x) = f'(x) \leftarrow f'(x_\lambda)$. Hence Lemma 2.4.7 implies, that

$$\|f(g_\lambda^{-1}x_\lambda) - f'(x_\lambda)\| \to 0,$$

a contradiction. □

REMARK 2.4.9. The dynamical system $(Ind_H^X A, G \ltimes X/H, \alpha)$ can also be obtained using the construction of a pullback along an equivalence of groupoids in the sense of [**LG99**]. Given a closed subgroupoid $H \subseteq G$ the space $X := d^{-1}(H^{(0)}) \subseteq G$ as defined above implements a $G \ltimes X/H - H$-equivalence. One can show that $Ind_H^X A$ and the pullback $X^*(A)$ are isomorphic as $G \ltimes X/H$-algebras.

If A is an H-algebra we can use the pushforward construction along $\tilde{r}$ to turn $Ind_H^X A$ into a $C_0(G^{(0)})$-algebra. Concretely, for $\varphi \in C_0(G^{(0)})$ and $f \in Ind_H^X A$ this action is given by

$$(\varphi \cdot f)(x) = \varphi(r(x))f(x).$$

Let us also identify the fibres of $Ind_H^X A$ with respect to this $C_0(G^{(0)})$-action.

LEMMA 2.4.10. *In the above situation the fibre $(Ind_H^X A)_u$ of $Ind_H^X A$ over $u \in G^{(0)}$ is canonically isomorphic to the algebra $Ind_H^{X^u} A$.*

PROOF. Consider the restriction homomorphism

$$\mathsf{res} : Ind_H^X A \to Ind_H^{X^u} A.$$

The kernel of res can be identified with $I_u = \overline{C_0(G^{(0)} \setminus \{u\}) Ind_H^X A}$ as follows: Let $\varphi \in C_0(G^{(0)} \setminus \{u\})$ and $f \in Ind_H^X A$. Then for all $x \in X^u$ we clearly have $(\varphi \cdot f)(x) = \varphi(r(x))f(x) = \varphi(u)f(x) = 0$. And thus $I_u \subseteq ker(\mathsf{res})$. For the converse inclusion let $f \in Ind_H^X A$ such that $\mathsf{res}(f) = 0$. From the definition of $Ind_H^X A$ we know that for any $\varepsilon > 0$ the set $K = \{xH \in X/H \mid \|f(x)\| \geq \varepsilon\}$ is compact. Since $\tilde{r}$ is continuous $\tilde{r}(K)$ is also compact. Since $u \notin \tilde{r}(K)$ we can find a function $\varphi \in C_c(G^{(0)})$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $\tilde{r}(K)$ and $\varphi(u) = 0$. Then clearly $\varphi \cdot f \in I_u$ and we have $\|f - \varphi \cdot f\| < \varepsilon$ since if $xH \in K$, then $r(x) = \tilde{r}(xH) \in \tilde{r}(K)$ and $\|f(x) - \varphi(r(x))f(x)\| = \|f(x) - f(x)\| = 0$ and if $xH \notin K$ then $\|f(x) - \varphi(r(x))f(x)\| = |1 - \varphi(r(x))| \|f(x)\| < \varepsilon$. Thus, we have $f \in I_u$.

To finish the proof we need to show that res is surjective. To this end it is enough to show that $im(\mathsf{res})$ is dense in $Ind_H^{X^u} A$. It is clear that $im(\mathsf{res})$ is a linear subspace in $Ind_H^{X^u} A$. Moreover, it is closed under the $C_0(X^u/H)$-action since if $\varphi \in C_0(X^u/H)$ and $f \in im(\mathsf{res})$ then we can identify X^u/H with the closed subspace $\tilde{r}^{-1}(\{u\}) \subseteq X/H$ and thus find an element $\tilde{\varphi}$ such that $\tilde{\varphi}_{|X^u/H} = \varphi$. If $\tilde{f}$ with $res(\tilde{f}) = f$ then clearly $\varphi \cdot f = \mathsf{res}(\tilde{\varphi} \cdot \tilde{f}) \in im(\mathsf{res})$. Furthermore, for all $xH \in X^u/H$ we know that $\{\mathsf{res}(f)(x) \mid f \in Ind_H^X A\} = ev_x(Ind_H^X A)$ is dense in $A_{d(x)}$ from the above proposition. Since $A_{d(x)} = (Ind_H^{X^u} A)_{xH}$ we can apply Proposition 2.1.4 to conclude that $im(\mathsf{res})$ is dense in $Ind_H^{X^u} A$ as desired. □

PROPOSITION 2.4.11. *Consider the family of isomorphisms* $(\beta_g)_{g \in G}$*, where*

$$\beta_g : Ind_H^{X^{d(g)}} \to Ind_H^{X^{r(g)}}, \quad \beta_g(f)(x) = f(g^{-1}x).$$

Then $(Ind_H^X A, G, \beta)$ *is a groupoid dynamical system.*

PROOF. Apply Proposition 2.2.5 to $(Ind_H^X A, G \ltimes X/H, \alpha)$. □

For later purposes we want to examine what happens, if we restrict our G-action on $Ind_H^X A$ to the subgroupoid H again. We have the following result:

LEMMA 2.4.12. *The restriction $(Ind_H^X A)_{|H}$ of the G-algebra $Ind_H^X A$ to the subgroupoid H is isomorphic to the induced algebra $Ind_H^{G'} A$, where $G' = G_{H^{(0)}}^{H^{(0)}} \subseteq X$.*

PROOF. Recall that $(Ind_H^X A)_{|H}$ is defined as the algebra of continuous sections of the bundle $\coprod_{u \in H^{(0)}} Ind_H^{X^u} A$ vanishing at infinity. Thus, we can define a map $\Phi : (Ind_H^X A)_{|H} \to Ind_H^{G'} A$ by letting $\Phi(f)(x) = f(r(x))(x)$. One easily checks that this is a $C_0(H^{(0)})$-linear $*$-homomorphism. It is not hard to see that the composition of Φ followed by the restriction map $Ind_H^{G'} A \to Ind_H^{X^u} A$ coincides with the evaluation homomorphism $ev_u : (Ind_H^X A)_{|H} \to Ind_H^{X^u} A$. Hence Φ induces the identity on each fibre, which is an isomorphism. By Lemma 2.1.6 it follows that Φ must be an isomorphism itself. Following the construction of the restricted action it is easy to see that Φ is compatible with the H-actions on both sides. □

Earlier we claimed that the process of induction should generalize the construction of the induced space presented in Chapter 1. The following proposition finally justifies this:

PROPOSITION 2.4.13. *Let G be a locally compact Hausdorff groupoid and $H \subseteq G$ a closed subgroupoid. If Y is a left H-space with anchor map $p : Y \to H^{(0)}$, then $C_0(Y)$ turns into an H-algebra. Consider the right H-space $X := d^{-1}(H^{(0)})$. Then $Ind_H^X(C_0(Y))$ is canonically isomorphic to $C_0(G \times_H Y)$, where $G \times_H Y$ is the classical induced G-space.*

PROOF. We want to define a map from $Ind_H^X(C_0(Y))$ to $C_0(G \times_H Y)$. For this let $\mathcal{B}$ denote the upper-semicontinuous C^*-bundle associated to the $C_0(H^{(0)})$-algebra $C_0(Y)$. Now let $f \in Ind_H^X(C_0(Y))$ be given. Then for each $x \in X$ we have that $f(x) \in (d_{|X}^*(\mathcal{B}))_x = \mathcal{B}_{d(x)} = C_0(Y)_{d(x)} = C_0(Y_{d(x)})$ where $Y_{d(x)} = p^{-1}(\{d(x)\}) \subseteq Y$. Define $\Phi : Ind_H^X(C_0(Y)) \to \ell^\infty(G \times_H Y)$ by

$$\Phi(f)([x, y]) := (f(x))(y).$$

We need to see, that this is well-defined. Recall that the left action of H on $G \times_{G^{(0)}} Y$ is given by $h \cdot (x, y) := (xh^{-1}, hy)$. Then we have $\Phi(f)([xh^{-1}, hy]) = (f(xh^{-1}))(hy) = (lt_h(f(x)))(hy) = (f(x))(y)$. Lets show that Φ has image in $C_0(G \times_H Y)$. First consider functions of the form $\varphi \diamond g$ for $\varphi \in C_c(X)$ and $g \in C_c(Y)$. Let $k : G \times_{G^{(0)}} Y \to \mathbb{C}$ be the function $k(x,y) = \varphi(x)g(y)$. Clearly k has compact support. Combining this with the fact that H acts properly on $G \times_{G^{(0)}} Y$ we obtain that the map $H \ltimes (G \times_{G^{(0)}} Y) \to \mathbb{C}$ given by $(h,x,y) \mapsto k(h^{-1}(x,y))$ is continuous and properly supported. Thus the map

$$(x,y) \mapsto \int_{H \ltimes (G \times_{G^{(0)}} Y)^{(x,y)}} k(h^{-1}(x',y'))d\lambda^{d(x)} \otimes \delta_{(x,y)}(h,x',y')$$

is continuous. But the latter integral equals

$$\int_{H^{d(x)}} \varphi(xh)g(h^{-1}y)d\lambda^{d(x)}(h) = \Phi(\varphi \diamond g)([x,y]).$$

Thus $\Phi(\varphi \diamond g)$ is continuous and compactly supported. Since the linear span of elements of the form $\varphi \diamond g$ is dense in $Ind_H^X C_0(Y)$ and Φ is clearly a $*$-homomorphism and isometric, its image is contained in $C_0(G \times_H Y)$. A quick application of the Stone–Weierstrass theorem gives that $im(\Phi) = C_0(G \times_H Y)$. $\square$

We also have, that the process of induction is compatible with the maximal tensor product in the following sense:

Lemma 2.4.14. *Let G be a locally compact Hausdorff groupoid and $H \subseteq G$ a proper subgroupoid. If A is an H-algebra and B a G-algebra we have a canonical isomorphism of G-algebras*

$$\Phi : (Ind_H^X A) \otimes_{C_0(G^{(0)})} B \to Ind_H^X (A \otimes_{C_0(H^{(0)})} B_{|H})$$

satisfying

$$\Phi(f \otimes b)(g) = f(g) \otimes \beta_{g^{-1}}(b(r(g)))$$

for all $f \in Ind_H^X A$ and $b \in B$.

Proof. It is easy to check that $\Phi(f \otimes b) \in Ind_H^X (A \otimes_{C_0(H^{(0)})} B_{|H})$. Recall, that we can identify the fibre over $u \in G^{(0)}$ as $((Ind_H^X A) \otimes B)_u \cong Ind_H^{X^u} A \otimes B_u$ and $(Ind_H^X (A \otimes B_{|H}))_u \cong Ind_H^{X^u} (A \otimes B_{|H})$. Using this identification we get that the image of $\Phi(f \otimes b)$ in the fibre $(Ind_H^X (A \otimes B_{|H}))_u$ can be identified with the function $g \mapsto f(g) \otimes \beta_{g^{-1}}(b(u))$. Hence we can compute

$$\|\Phi(f \otimes b)\| = \sup_{u \in G^{(0)}} \|\Phi(f \otimes b)(u)\|$$

$$= \sup_{u \in G^{(0)}} \sup_{g \in X^u} \|f(g) \otimes \beta_{g^{-1}}(b(u))\|$$
$$= \sup_{u \in G^{(0)}} \sup_{g \in X^u} \|f(g)\|\|b(u)\|$$
$$= \sup_{u \in G^{(0)}} \|f_{|X^u}\|\|b(u)\|$$
$$= \sup_{u \in G^{(0)}} \|f_{|X^u} \otimes b(u)\|$$
$$= \|f \otimes b\|$$

Hence Φ extends to an isometric, $C_0(G^{(0)})$-linear $*$-homomorphism. To check it is an isomorphism, it is enough to check, that Φ induces an isomorphism on each fibre. Viewing $Ind_H^{X^u}(A \otimes B_{|H})$ as a $C_0(X^u/H)$-algebra it is also not hard to show that $im(\Phi_u)$ is a $C_0(X^u/H)$-linear subspace such that for each fixed $g \in X^u$ the set

$$\{\Phi_u(\xi)(g) \mid \xi \in Ind_H^{X^u} A \otimes B_u\}$$

is dense in $(Ind_H^{X^u}(A \otimes B_{|H}))_{gH} = A_{d(g)} \otimes B_{d(g)}$. Thus, $im(\Phi_u)$ is dense in $Ind_H^{X^u}(A \otimes B_{|H})$ by Proposition 2.1.4 and hence Φ_u is an isomorphism for all $u \in G^{(0)}$. Consequently, Φ is an isomorphism by Lemma 2.1.6. $\square$

CHAPTER 3

Equivariant KK-Theory

In this chapter we first review the basic constructions of groupoid equivariant KK-Theory and lift some well-known results from the group case to the realm of groupoids. Our exposition is based on the work of Le Gall (cf. [**LG94, LG99**]).

3.1. Preliminaries on Hilbert-modules

Let us start be recalling the basic notions of Hilbert module theory to set up notation. We will not give any proofs here and refer the reader to the detailed exposition in [**RW98**] for details.

Let A be a C*-algebra. Recall, that a right Hilbert A-module is a right A-module E together with an A-valued inner product $\langle\cdot,\cdot\rangle_A : E \times E \to A$ satisfying:

(1) $\langle\cdot,\cdot\rangle_A$ is linear in the second and anti-linear in the first variable.
(2) $\langle x, y\rangle_A^* = \langle y, x\rangle_A$ and $\langle x, y\rangle_A a = \langle x, ya\rangle_A$ for all $x, y \in E$ and $a \in A$.
(3) $\langle x, x\rangle_A \geq 0$ for all $x \in E$ and $\langle x, x\rangle_A = 0$ implies $x = 0$.
(4) E is complete with respect to the norm $\|x\| = \sqrt{\|\langle x, x\rangle_A\|}$.

Recall, that an operator $T : E \to F$ between two Hilbert A-modules is called *adjointable*, if there exists a map $T^* : F \to E$ satisfying $\langle Tx, y\rangle_A = \langle x, T^*y\rangle_A$ for all $x \in E$ and $y \in F$. It is well-known, that every adjointable operator T is automatically A-linear and bounded with respect to the operator norm given by $\|T\| = \sup\{\|Te\| \mid \|e\| \leq 1\}$. We will write $L(E)$ for the C*-algebra of adjointable operators on E. To every pair of elements $x, y \in E$ we can associate an adjointable operator $\Theta_{x,y} \in L(E)$ by setting $\Theta_{x,y}(z) = x\langle y, z\rangle_A$. The operators $\Theta_{x,y}$ are usually called *rank one operators*. The *compact operators* $K(E) \subseteq L(E)$ are then defined to be the closure in $L(E)$ of the linear span of the operators $\Theta_{x,y}$, where x and y range over E. Since $T \circ \Theta_{x,y} = \Theta_{Tx,y}$

and $\Theta_{x,y} \circ T = \Theta_{x,T^*y}$ for all $x, y \in E$ and $T \in L(E)$ the compact operators form a closed two-sided ideal in $L(E)$.

We will frequently use the *internal tensor product* of two Hilbert modules. It is defined as follows: Let A and B be C*-algebras. Given a right Hilbert A-module F, a right Hilbert B-module E and a $*$-homomorphism $\Phi : B \to L(F)$ we can equip the algebraic tensor product $E \odot F$ with an A-valued inner product by

$$\langle x_1 \odot y_1, x_2 \odot y_2 \rangle_A := \langle y_1, \Phi(\langle x_1, x_2 \rangle_B) y_2 \rangle_A.$$

Let $N = \{z \in E \odot F \mid \langle z, z \rangle_A = 0\}$ and let $E \otimes_\Phi F$ denote the completion of $F \odot E/N$ with respect to $\|z\| = \sqrt{\|\langle z, z \rangle_A\|}$. Then $E \otimes_\Phi F$ is a right Hilbert A-module.

3.2. Hilbert Modules over $C_0(X)$-algebras

Let A be a $C_0(X)$-algebra and E a right Hilbert A-module. For $\varphi \in C_0(X)$ we can define an action of $C_0(X)$ on $EA = E$ by adjointable operators by

$$\varphi \cdot (xa) := x(a\varphi)$$

It is straightforward to check, that this action actually takes values in the center $Z(L(E))$ of the adjointable operators on E. Using the canonical isomorphism $M(K(E)) \cong L(E)$ we actually get a $*$-homomorphism $\Phi : C_0(X) \to Z(M(K(E)))$. For rank-one operators this action is given by $\varphi \cdot \Theta_{x,y} = \Theta_{\varphi x,y}$.

Proposition 3.2.1. *Let A be a $C_0(X)$-algebra and E be a right Hilbert A-module. Then $K(E)$ is a $C_0(X)$-algebra with respect to the structure homomorphism Φ described above.*

Proof. We need to show that Φ is non-degenerate. Let $\Theta_{x,y} \in K(E)$ and $\varepsilon > 0$ be given. Then, since $\overline{C_0(X)E} = E$ there exist $\varphi \in C_0(X)$ and $x' \in E$ such that $\|x - \varphi x'\| < \frac{\varepsilon}{\|y\|}$. Thus $\|\Theta_{x,y} - \varphi\Theta_{x',y}\| = \|\Theta_{x,y} - \Theta_{\varphi x',y}\| = \|\Theta_{x-\varphi x',y}\| \leq \|x - \varphi x'\|\|y\| < \varepsilon$. Consequently, all rank-one operators are in $\overline{C_0(X)K(E)}$ and thus $K(E)$ is contained in $\overline{C_0(X)K(E)}$. □

Similar to $C_0(X)$-algebras we can also view E as a fibred object in the following way: For $x \in X$ let E_x be the quotient (as a vector space) of E by the closed subspace $\overline{C_0(X \setminus \{x\})E}$. Denote the image of an element $e \in E$ under the quotient map on E_x by $e(x)$. Then we can define an A_x-valued inner product on E_x by

$$\langle e(x), e'(x)\rangle_{A_x} := \langle e, e'\rangle_A(x).$$

LEMMA 3.2.2. *The map $\langle\cdot,\cdot\rangle_{A_x} : E_x \times E_x \to A_x$ is a well-defined A_x-valued inner product on E_x and E_x is complete with respect to the norm induced by $\langle\cdot,\cdot\rangle_{A_x}$.*

PROOF. The only part of the first assertion which is not completely trivial to check is the positive definiteness. Let $\pi_x : A \to A_x$ denote the quotient map. Then it is clear that $\langle e(x), e(x)\rangle_{A_x} = \langle e, e\rangle_A(x) = \pi_x(\langle e, e\rangle_A) \geq 0$ since $\langle e, e\rangle_A$ is positive and $*$-homomorphisms preserve positivity. Let now $e \in E$ such that $\langle e(x), e(x)\rangle_{A_x} = 0$. We need to show that $e(x) = 0$, or equivalently $e \in \overline{EC_0(X \setminus \{x\})}$. For this let $\varepsilon > 0$ be given. Write $e = e'\langle e', e'\rangle$ for some $e' \in E$. Then we have $0 = \langle e(x), e(x)\rangle_{A_x} = \langle e, e\rangle_A(x) = (\langle e', e'\rangle_A(x))^3$. Since $\langle e', e'\rangle_A(x)$ is positive it follows that $\langle e', e'\rangle_A(x) = 0$ which implies that $\langle e', e'\rangle_A \in \overline{C_0(X \setminus \{x\})A}$. Thus we can find $a \in A$ and $\varphi \in C_0(X \setminus \{x\})$ such that $\|\langle e', e'\rangle_A - a\varphi\| < \frac{\varepsilon}{\|e'\|}$. But then $e'a\varphi \in EC_0(X \setminus \{x\})$ and $\|e - e'a\varphi\| = \|e'\langle e', e'\rangle_A - e'a\varphi\| \leq \|e'\|\|\langle e', e'\rangle_A - a\varphi\| < \varepsilon$.

The second part can be shown by using a 2×2-matrix trick a sketch of which is given below (compare [**RW98**, Proposition 3.25]): Note that $\overline{EC_0(X \setminus \{x\})} = \overline{EI_x}$ where I_x is the ideal $\overline{AC_0(X \setminus \{x\})}$ in A. Then $\overline{EI_x}$ is a Hilbert-I_x-module and the linking algebra L_{I_x} is a closed two-sided ideal in the linking algebra L of E. Thus L/L_J is a C^*-algebra, i.e. complete. But L/L_J can then be written as $L/L_J = \begin{pmatrix} \mathcal{K}(E)/_{\mathcal{K}(E)}\overline{\langle EI_x, EI_x\rangle} & E/\overline{EI_x} \\ E^*/\overline{EI_x}^* & A/J \end{pmatrix}$. Compression to the upper right corner yields the result. □

REMARK 3.2.3. Note that one could also define the fibre E_x as the tensor product $E \otimes_A A_x$. The canonical morphism

$$E \otimes_A A_x \to E_x,$$

sending an elementary tensor $e \otimes a(x)$ to the product $(ea)(x)$, is an isomorphism.

If E, F are two Hilbert A-modules, then every operator $T \in L(E, F)$ is automatically compatible with the $C_0(X)$-structures on E and F.

Hence T factors through a well-defined operator $T_x \in L(E_x, F_x)$ for every $x \in X$. Using [**Wil07**, Lemma C.11] one can show that

$$\|T\| = \sup_{x\in X}\|T_x\|.$$

If $T \in K(E)$ is a compact operator, then so is T_x for every $x \in X$. For a rank one operator $\Theta_{e,f} \in K(E)$ this is obvious since $(\Theta_{e,f})_x = \Theta_{e(x),f(x)}$. The general case follows by approximating $T \in K(E)$ by finite linear combinations of rank one operators. This gives rise to a convenient description of the compact operators on of E_x. Indeed, the canonical map $T \mapsto T_x$ factors through an isomorphism

$$K(E)_x \cong K(E_x),$$

where $K(E)_x$ denotes the fibre of $K(E)$ over x with respect to the $C_0(X)$-structure described in Proposition 3.2.1.

We have the following

LEMMA 3.2.4. *Let A be a $C_0(X)$-algebra and E a right Hilbert A-module.*

(1) For each $e \in E$ the map $x \mapsto \|e(x)\|$ is upper semicontinuous and vanishes at infinity.
(2) For each $e \in E$ we have $\|e\| = \sup_{x\in X}\|e(x)\|$.
(3) For $e \in E$ and $f \in C_0(X)$ we have $(fe)(x) = f(x)e(x)$.

PROOF. Since $\langle e, e\rangle_A \in A$ we have that $x \mapsto \|\langle e, e\rangle_A(x)\|$ is upper semicontinous and vanishes at infinity. Consequently, the mapping $x \mapsto \sqrt{\|\langle e, e\rangle_A(x)\|} = \|e(x)\|$ is upper semicontinuous and vanishes at infinity as well.

Now we can compute

$$\begin{aligned}\|e\|^2 = \|\langle e, e\rangle_A\| &= \sup_{x\in X}\|\langle e, e\rangle_A(x)\| \\ &= \sup_{x\in X}\|\langle e(x), e(x)\rangle_{A_x}\| = \sup_{x\in X}\|e(x)\|^2,\end{aligned}$$

establishing (2).

For the last part note that for $e = e'\psi \in EC_0(X)$ we have $ef(x) - ef = e'\psi f(x) - e'\psi f = e'(\psi f(x) - \psi f)$. But $\psi f(x) - \psi f$ vanishes in x and thus in the quotient $(ef)(x) = e(x)f(x)$. $\square$

Let $\mathcal{E} = \coprod_{x\in X} E_x$ be the disjoint union of the fibres. We want to see, that in analogy to $C_0(X)$-algebras, there is a topology on $\mathcal{E}$ such that E is isomorphic (as a Hilbert-A-module) to $\Gamma_0(X,\mathcal{E})$, where the inner product and A-action on the latter are defined pointwise (using the identification $\Gamma_0(X,\mathcal{A}) \cong A$).

We need some preparations for this: Consider the compact operators $K(E\oplus A)$. Then we have an embedding $i_E : E \to K(E\oplus A)$ given by

$$i_E(e) = \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix}.$$

Analogously, we get embeddings of each fibre $i_{E_x} : E_x \to K(E_x\oplus A_x) \cong K(E\oplus A)_x$. Since $K(E\oplus A)$ is a $C_0(X)$-algebra, there is a topology on $\mathcal{K}(E\oplus A) := \coprod_{x\in X} K(E\oplus A)_x$ such that $K(E\oplus A) \cong \Gamma_0(X,\mathcal{K}(E\oplus A))$. The inclusions i_{E_x} induce an inclusion $i : \mathcal{E} \to \mathcal{K}(E\oplus A)$ and we equip $\mathcal{E}$ with the induced topology. Write $\Gamma_0(X,\mathcal{E})$ for the continuous sections of the bundle $\mathcal{E} \to X$ vanishing at infinity. Then we get a commutative diagram, where the homomorphism at the top is given by $e \mapsto [x \mapsto e(x)]$ and the right vertical map is given by sending $f \in \Gamma_0(X,\mathcal{E})$ to the map $x \mapsto i_{E_x}(f(x))$:

$$\begin{array}{ccc} E & \longrightarrow & \Gamma_0(X,\mathcal{E}) \\ \downarrow{\scriptstyle i_E} & & \downarrow \\ K(E\oplus A) & \xrightarrow{\cong} & \Gamma_0(X,\mathcal{K}(E\oplus A)) \end{array}$$

Thus, the isomorphism in the bottom row restricts to an isomorphism $E \to \Gamma_0(X,\mathcal{E})$ as desired.

In the next step, we want to define pullbacks of Hilbert modules with respect to the $C_0(X)$-action. If $f : Y \to X$ is a continuous map and A is a $C_0(X)$-algebra we can form the pullback f^*A of A under f. We equip it with the canonical right Hilbert f^*A-module structure. Define a left A-action $\Phi : A \to L(f^*A)$ by $(\Phi(a)f)(y) = a(f(y))f(y)$. One easily checks that this is a well-defined $*$-homomorphism.

Definition 3.2.5. (Pullbacks) Suppose A is a $C_0(X)$-algebra and E a right Hilbert A-module. If $f : Y \to X$ is a continuous map we define the pullback f^*E of E as the internal tensor product $f^*E := E \otimes_\Phi f^*A$.

For $y \in Y$ we then have $(f^*E)_y = (E \otimes_\Phi f^*A)_y \cong E \otimes_\Phi f^*A \otimes_{f^*A} (f^*A)_y \cong E \otimes_A A_{f(y)} = E_{f(y)}$. Here we used that for each $C_0(X)$-algebra A there is a canonical isomorphism $A \otimes_A A_x \to A_x$ given by $a \otimes b(x) \mapsto ab(x)$. The following proposition is concerned with the behaviour of the interior tensor product under pullbacks.

PROPOSITION 3.2.6. [**LG94**, Proposition 2.3.3] *Let A, B be two $C_0(X)$-algebras. If E is a Hilbert A-module, F is a Hilbert B-module, and $\Phi : A \to L(F)$ is a $*$-homomorphism, then for every continuous map $f : Y \to X$ there is a canonical isomorphism of Hilbert f^*B-modules*

$$f^*E \otimes_{f^*A} f^*F \to f^*(E \otimes_A F).$$

In particular for each $x \in X$, there is a canonical isomorphism

$$(E \otimes_A F)_x \cong E_x \otimes_{A_x} F_x.$$

3.3. G-Hilbert modules

In this section we want to define what we mean by a groupoid action on a Hilbert-module. For this let (A, G, α) be a groupoid dynamical system and E be a right Hilbert A-module. From the discussion above we know that E is equipped with a $C_0(G^{(0)})$-action arising from the corresponding action on A. Now, if $d, r : G \to G^{(0)}$ denote the domain and range maps respectively, we can form the pullback modules d^*E and r^*E. By construction r^*E is a right Hilbert r^*A-module, but we can also equip it with the structure of a right Hilbert d^*A-module by letting $x \cdot a := x \cdot \alpha(a)$ and $\langle x, y\rangle_{d^*A} := \alpha^{-1}(\langle x, y\rangle_{r^*A})$.

Thus, we can consider elements $T \in L_{d^*A}(d^*E, r^*E)$. For $g \in G$ consider the operator $T_g \in L_{A_{d(g)}}(E_{d(g)}, E_{r(g)})$ induced by T on each fibre. Using Remark 3.2.3 this operator can also be described as

$$T_g = T \otimes \alpha_g : E_{d(g)} = d^*E \otimes_{d^*A} A_{d(g)} \to r^*E \otimes_{d^*A} A_{r(g)} = E_{r(g)}.$$

DEFINITION 3.3.1. Let A be a G-algebra and E a right Hilbert A-module. An action of G on E is a unitary $V \in L_{d^*A}(d^*E, r^*E)$ such that $V_g V_{g'} = V_{gg'}$ for all $(g, g') \in G^{(2)}$.

For every locally compact Hausdorff groupoid G with Haar-system λ there is a canonical G-equivariant Hilbert $C_0(G^{(0)})$-module denoted $L^2(G)$ given as the completion of the complex vector space $C_c(G)$ with respect to the $C_0(G^{(0)})$-valued inner product

$$\langle f_1, f_2\rangle(x) = \int\limits_{G^x} \overline{f_1(g)} f_2(g) d\lambda^x(g),$$

and right $C_0(G^{(0)})$-action

$$(f \cdot \varphi)(g) = f(g)\varphi(r(g)).$$

Now we define a G-action on $L^2(G)$: From [**Goe09**, Lemma 4.37] we know that there are isomorphisms $d^*(C_0(G^{(0)})) \cong C_0(G \times_{d,r} G)$ and $r^*(C_0(G^{(0)})) \cong C_0(G \times_{r,r} G)$. Thus we have

$$d^*(L^2(G)) = L^2(G) \otimes_{C_0(G^{(0)})} d^*(C_0(G^{(0)})) \cong L^2(G) \otimes_{C_0(G^{(0)})} C_0(G \times_{d,r} G)$$

and

$$r^*(L^2(G)) \cong L^2(G) \otimes_{C_0(G^{(0)})} C_0(G \times_{r,r} G)$$

Now we define $V : d^*(L^2(G)) \to r^*(L^2(G))$ as $id_{L^2(G)} \otimes lt$, where $lt : C_0(G \times_{d,r} G) \to C_0(G \times_{r,r} G)$ is given by

$$lt(f)(g, h) = f(g, g^{-1}h).$$

Then V is a unitary with $V_{gg'} = V_g V_{g'}$ for all $(g, g') \in G^{(2)}$.

Remark 3.3.2. Note that $L^2(G)$ is a full Hilbert $C_0(G^{(0)})$-module in the sense that the ideal $\langle L^2(G), L^2(G)\rangle$ is dense in $C_0(G^{(0)})$. To see this we apply the Stone-Weierstraß-Theorem: If $x \in G^{(0)}$ pick a compact neighbourhood $V \subseteq G$ of x. Now let $f \in C_c(G)$ be any function such that $f = 1$ on V. Then we have

$$\langle f, f\rangle(x) = \int\limits_{G^x} |f(g)|^2 d\lambda^x(g) \geq \int\limits_{V} |f(g)|^2 d\lambda^x(g) = \lambda^x(V \cap G^x) > 0$$

since $supp(\lambda^x) = G^x$. If $x, y \in G^{(0)}$ such that $x \neq y$ the set $G \setminus G^y$ is an open neighbourhood of x. Since G is locally compact we can find a compact neighbourhood V of x such that $V \subseteq G \setminus G^y$. Then pick a function $f \in C_c(G)$ such that $f = 1$ on V and $f = 0$ off of $G \setminus G^y$. It follows that $\langle f, f\rangle(x) \neq 0$ by the same computation as above but $\langle f, f\rangle(y) = \int\limits_{G^y} |f(g)|^2 d\lambda^y(g) = 0$.

More generally, if A is any G-algebra we can view it as a $C_0(G^{(0)})-A$ bimodule and form the G-equivariant right Hilbert A-module

$$L^2(G,A) := L^2(G) \otimes_{C_0(G^{(0)})} A.$$

Note that we could also concretely construct $L^2(G,A)$ as the completion of the pre-Hilbert A-module $\Gamma_c(G, d^*\mathcal{A})$ with respect to the inner product

$$\langle f_1, f_2 \rangle_A(x) = \int\limits_{G^x} \alpha_g(f_1(g)^* f_2(g)) d\lambda^x(g)$$

and the right A-action

$$(f \cdot a)(g) = f(g)\alpha_{g^{-1}}(a(r(g))).$$

A canonical isomorphism

$$\Phi : L^2(G) \otimes_{C_0(G^{(0)})} A \to \overline{\Gamma_c(G, d^*\mathcal{A})}$$

is given on elementary tensors by

$$\Phi(f \otimes a)(g) = f(g)\alpha_{g^{-1}}(a(r(g)))$$

for $f \in C_c(G)$ and $a \in A$. One easily checks that Φ is isometric. Thus, it suffices to show that $im(\Phi)$ is dense in $\Gamma_c(G, d^*\mathcal{A})$. But it follows from Corollary 2.1.14 that elements of the form $\Phi(f \otimes a)$ span a dense subset of $\Gamma_c(G, d^*\mathcal{A})$ with respect to the inductive limit topology. So for $\varphi \in \Gamma_c(G, d^*\mathcal{A})$ we can find a net $(\varphi_i)_i$ in $span\{\Phi(f \otimes a) \mid f \in C_c(G), a \in A\}$ and a compact set $K \subseteq G$ such that $supp(\varphi_i) \subseteq K$ eventually and $\|\varphi - \varphi_i\|_\infty \to 0$. But then we eventually have:

$$\begin{aligned}
\|\varphi - \varphi_i\|^2 &= \sup_{x \in G^{(0)}} \|\int\limits_{G^x} \alpha_g((\varphi - \varphi_i)(g)^*(\varphi - \varphi_i)(g)) d\lambda^x(g)\| \\
&\leq \sup_{x \in G^{(0)}} \int\limits_{G^x} \|(\varphi - \varphi_i)(g)\|^2 d\lambda^x(g) \\
&\leq \|\varphi_i - \varphi\|_\infty^2 \lambda^x(supp(\varphi - \varphi_i)) \\
&\leq \|\varphi_i - \varphi\|_\infty^2 \lambda^x(K) \\
&\leq \|\varphi_i - \varphi\|_\infty^2 C \\
&\to 0
\end{aligned}$$

Thus $\varphi_i \to \varphi$ in the norm induced by the inner product. Consequently, we have $\varphi \in \overline{im(\Phi)} = im(\Phi)$. The following result is a special case of [**LG94**, Proposition 2.3.2]:

PROPOSITION 3.3.3. *There is a G-equivariant $*$-isomorphism*

$$\Psi : K(L^2(G)) \otimes_{G^{(0)}}^{max} A \to K(L^2(G, A))$$

given by $\Psi(T \otimes a)(\xi \otimes b) = T\xi \otimes ab$. Consequently, $L^2(G, A)$ implements a G-equivariant Morita-equivalence

$$(K(L^2(G)) \otimes_{G^{(0)}}^{max} A, Ad\ V \otimes \alpha) \sim_M (A, \alpha).$$

Even more generally, let E be a G-equivariant Hilbert A-module. As seen above there is a natural $*$-homomorphism $\Phi : C_0(G^{(0)}) \to L(E)$ induced by the $C_0(G^{(0)})$-structure of A. Thus we can form the tensor product

$$L^2(G, E) := L^2(G) \otimes_\Phi E$$

Again we could also explicitly construct $L^2(G, E)$ as the completion of the pre-Hilbert A-module $\Gamma_c(G, d^*\mathcal{E})$ with respect to the inner product

$$\langle f_1, f_2 \rangle_A(x) = \int_{G^x} \alpha_g(\langle f_1(g), f_2(g) \rangle_{A_{d(g)}}) d\lambda^x(g)$$

equipped with a right A-action given by

$$(f \cdot a)(g) = f(g)\alpha_{g^{-1}}(a(r(g))).$$

Again, an isomorphism

$$\Phi : L^2(G) \otimes_\Phi E \to \overline{\Gamma_c(G, d^*\mathcal{E})}$$

is given on elementary tensors by

$$\Phi(f \otimes e)(g) = f(g)V_{g^{-1}}(e(r(g)))$$

for $f \in C_c(G)$ and $e \in E$.

3.4. KK-theory

We will now recall the definitions of groupoid equivariant KK-theory, as introduced by Le Gall in [**LG94, LG99**]. Throughout we will assume, that G is a locally compact, second countable Hausdorff groupoid.

DEFINITION 3.4.1. Let A and B be two G-algebras. A *G-equivariant Kasparov Triple* for (A, B) is a triple (E, Φ, T), where E is a G-equivariant $\mathbb{Z}/2\mathbb{Z}$-graded right Hilbert B-module, $\Phi : A \to L(E)$ is a graded G-equivariant $*$-homomorphism and $T \in L(E)$ is an adjointable operator of degree 1, such that $\Phi(a)(T - T^*)$, $\Phi(a)(T^2 - 1)$, $[\Phi(a), T] \in K(E)$ for every $a \in A$, and for every element $f \in r^*A \cong \Gamma_0(G, r^*\mathcal{A})$ the mapping

$$g \mapsto \Phi_{r(g)}(f(g))(T_{r(g)} - V_g T_{d(g)} V_g^*)$$

defines and element in $\Gamma_0(G, r^*\mathcal{K}(E)) = r^*(K(E))$.

Two Kasparov triples (E_i, Φ_i, T_i), $i = 1, 2$ for (A, B) are called unitarily equivalent if there exists a G-equivariant unitary $U \in L(E_1, E_2)$ of degree 0, which intertwines the representations Φ_1 and Φ_2 as well as the operators T_1 and T_2. We denote the set of all unitary equivalence classes of such triples by $\mathbb{E}^G(A, B)$. A Kasparov triple (E, Φ, T) is called *essential* if $\overline{\Phi(A)E} = E$.

EXAMPLE 3.4.2. If $\Phi : A \to B$ is a G-equivariant $*$-homomorphism between two G-algebras the triple $(B, \Phi, 0)$ defines an equivariant Kasparov triple for (A, B).

A *homotopy* in $\mathbb{E}^G(A, B)$ is an element in $\mathbb{E}^G(A, C([0, 1], B))$ and the triples in $\mathbb{E}^G(A, B)$ obtained by evaluating at 0 and 1 respectively are called *homotopic*. Homotopy is an equivalence relation on $\mathbb{E}^G(A, B)$.

DEFINITION 3.4.3. The set of homotopy classes of $\mathbb{E}^G(A, B)$ is denoted by $\mathrm{KK}^G(A, B)$.

It is not hard to see, that homotopy respects the operation of taking direct sums of Kasparov triples. Using this one can show that $\mathrm{KK}^G(A, B)$ is an abelian group with respect to taking direct sums of the respresenting Kasparov triples. The same proof as in the non-equivariant setting (see [**Bla98**, Proposition 17.3.3]) works.

REMARK 3.4.4. If G is a locally compact group acting continuously on a locally compact Hausdorff space X, then for all $G \ltimes X$-algebras A and B one has

$$\mathrm{KK}^{G \ltimes X}(A, B) = \mathcal{R}\mathrm{KK}^G(X; A, B).$$

DEFINITION 3.4.5. For $n \in \mathbb{N}$ and two G-algebras A and B, define

$$\mathrm{KK}_n^G(A, B) = \mathrm{KK}^G(A \otimes C_0(\mathbb{R}^n), B)$$

The following proposition says, that KK^G is a functor, which is contravariant in the first, and covariant in the second variable.

PROPOSITION 3.4.6. *Let A_1, A_2 and B be G-algebras, and $\varphi : A_1 \to A_2$ a G-equivariant $*$-homomorphism. If $(E, \Phi, T) \in \mathbb{E}^G(A_2, B)$, then the triple $(E, \Phi \circ \varphi, T) \in \mathbb{E}^G(A_1, B)$ and the mapping $(E, \Phi, T) \mapsto (E, \Phi \circ \varphi, T)$ defines a group homomorphism*

$$\varphi^* : \mathrm{KK}^G(A_2, B) \to \mathrm{KK}^G(A_1, B).$$

If $(E, \Phi, T) \in \mathbb{E}^G(B, A_1)$, then $(E \otimes_\varphi A_2, \Phi \otimes 1, T \otimes 1) \in \mathbb{E}^G(B, A_2)$ and the mapping $(E, \Phi, T) \mapsto (E \otimes_\varphi A_2, \Phi \otimes 1, T \otimes 1)$ defines a group homomorphism

$$\varphi_* : \mathrm{KK}^G(B, A_1) \to \mathrm{KK}^G(B, A_2).$$

The following result of Le Gall extends the Kasparov product to KK^G-theory.

THEOREM 3.4.7. [**LG99**, Theorème 6.3] *Let G be a locally compact σ-compact Hausdorff groupoid and A, B and C be separable G-algebras. Then there exists a bilinear map*

$$\otimes_C : \mathrm{KK}^G(A, C) \times \mathrm{KK}^G(C, B) \to \mathrm{KK}^G(A, B),$$

called the Kasparov product. Moreover, the Kasparov product is associative: If D is another separable G-algebra, and $x \in \mathrm{KK}^G(A, C)$, $y \in \mathrm{KK}^G(C, B)$, and $z \in \mathrm{KK}^G(B, D)$, then

$$(x \otimes_C y) \otimes_B z = x \otimes_C (y \otimes_B z) \in \mathrm{KK}^G(A, D).$$

We shall also use the fact, that the equivariant KK-theory is functorial with respect to groupoid homomorphisms:

PROPOSITION 3.4.8. *Let G and H be two locally compact, σ-compact Hausdorff groupoids and $f : G \to H$ a groupoid homomorphism. Suppose A and B are G-algebras and $(E, \Phi, T) \in \mathbb{E}^H(A, B)$. Then the triple $(f^*E, \Phi \otimes 1, f^*T)$ is an element of $\mathbb{E}^G(f^*A, f^*B)$ and the mapping $(E, \Phi, T) \mapsto (f^*E, \Phi \otimes 1, f^*T)$ defines a group homomorphism*

$$f^* : \mathrm{KK}^H(A, B) \to \mathrm{KK}^G(f^*A, f^*B).$$

Moreover, it is compatible with the Kasparov product in the following sense: If A, B *and* C *are separable* H*-algebras, and* $x \in \mathrm{KK}^H(A, C)$ *and* $y \in \mathrm{KK}^H(C, B)$*, then*

$$f^*(x \otimes_C y) = f^*(x) \otimes_{f^*C} f^*(y) \in \mathrm{KK}^G(f^*A, f^*B).$$

PROOF. See [**LG99**, Propositions 7.1 and 7.2]. □

An important special case of the above construction is given by the inclusion of a subgroupoid $H \hookrightarrow G$. In this case we will also denote the resulting map $\mathrm{KK}^G(A, B) \to \mathrm{KK}^H(A_{|H}, B_{|H})$ by res^G_H and call it the *restriction homomorphism*.

The following proposition extends the pushforward construction for C*-algebras as in Proposition 2.2.5 to Hilbert modules and hence provides a homomorphism on the level of KK^G-theory.

PROPOSITION 3.4.9. *Let* G *be a locally compact Hausdorff groupoid and* X *a* G*-space with anchor map* $p : X \to G^{(0)}$*. For every pair of* $G \ltimes X$*-algebras* A *and* B *the map* p *gives rise to a homomorphism*

$$p_* : KK^{G \ltimes X}(A, B) \to KK^G(A, B),$$

compatible with the Kasparov product in the following sense: If A, B *and* C *are separable* $G \ltimes X$*-algebras and* $x \in KK^{G\ltimes X}(A, C)$ *and* $y \in KK^{G\ltimes X}(C, B)$*, then*

$$p_*(x \otimes_C y) = p_*(x) \otimes_C p_*(y).$$

PROOF. On the level of Kasparov triples $(E, \Phi, T) \in \mathbb{E}^{G\ltimes X}(A, B)$ the desired map is basically given by the identity. Viewing A and B as G-algebras via the pushforward construction (see Proposition 2.2.5) also E inherits a canonical fibration over $G^{(0)}$ and using the same formulas as in the C*-algebraic construction we can push the action of $G \ltimes X$ forward to obtain an action of G on E. Since neither the operator T nor the left action Φ of A on E changed, it follows from the isomorphism $\pi_*(R^*(K(E))) \cong r^*(p_*(K(E)))$, where $R : G \ltimes X \to X$ is the range map and $\pi : G \ltimes X \to G$ is the projection on the first factor (confer Proposition 2.1.20), that (E, Φ, T) equipped with this G-action represents an element in $\mathbb{E}^G(A, B)$. Applying the same arguments to a homotopy gives the desired homomorphism. Using again, that only the action on E changes under p_* it is easy to see, that p_* respects the Kasparov product. $\square$

PROPOSITION 3.4.10. *Let G be a locally compact Hausdorff groupoid admitting a Haar system and $H \subseteq G$ a closed subgroupoid. Suppose, that A and B are separable H-algebras. Then there is an induction homomorphism*

$$\mathsf{Ind}_H^G : \mathrm{KK}^H(A, B) \to \mathrm{KK}^G(Ind_H^X A, Ind_H^X B),$$

where $X := d^{-1}(H^{(0)})$. The homomorphism Ind_H^G is compatible with the Kasparov product in the following sense: If A, B and C are separable H-algebras and $x \in \mathrm{KK}^H(A, C)$ and $y \in \mathrm{KK}^H(C, B)$, then

$$\mathsf{Ind}_H^G(x \otimes_C y) = \mathsf{Ind}_H^G(x) \otimes_{Ind_H^G C} \mathsf{Ind}_H^G(y).$$

PROOF. The space $X = d^{-1}(H^{(0)}) \subseteq G$ with the induced topology implements an equivalence between the groupoids $G \ltimes X/H$ and H. Hence by [**LG99**, Defintion 7.1, Theorem 7.2] there is a canonical homomorphism $X^* : \mathrm{KK}^H(A, B) \to \mathrm{KK}^{G \ltimes X/H}(Ind_H^G A, Ind_H^G B)$ compatible with the Kasparov product (compare Remark 2.4.9). If we now compose this homomorphism with the homomorphism obtained by pushing forward alsong $G \ltimes X/H \to G$ as in Proposition 3.4.9 we obtain the desired map and compatibility with the product follows since both maps in this composition have this property. Alternatively, one could define this map explicitly along the lines of [**Kas95**, §5] as follows: If $x \in \mathrm{KK}^G(A, B)$ is represented by the Kasparov triple (E, Φ, T), then we can form the induced Hilbert $Ind_H^X B$-module $Ind_H^X E$ as the set of all $\xi \in \Gamma_b(X, d^*\mathcal{E})$ such that $V_h(\xi(x)) = \xi(xh^{-1})$ for all $x \in X$ and $h \in H$ and $[xH \mapsto \|\xi(x)\|] \in C_0(X/H)$, equipped with the pointwise actions and inner products. Pointwise action on the left gives a representation $\mathsf{Ind}_H^G \Phi : Ind_H^X A \to L(Ind_H^X E)$. Using a cutoff function $c : X \to \mathbb{R}^+$ for the groupoid $X \rtimes H$ as in Definition 3.5.1 we can define an operator $\widetilde{T} \in L(Ind_H^X E)$ by

$$(\widetilde{T}\xi)(x) = \int_{H^{d(x)}} c(xh)V_h(T(\xi(xh)))d\lambda^{d(x)}(h).$$

Then $(Ind_H^X E, Ind_H^X \Phi, \widetilde{T})$ can be shown to be a Kasparov tripel representing the element $\mathsf{Ind}_H^G(x) \in \mathrm{KK}^G(Ind_H^X A, Ind_H^X B)$.

□

Finally, we would like to link the KK^G-groups of two G-algebras A and B to the (non-equivariant) KK-groups of their corresponding crossed products. Since we only introduced the crossed product construction for étale groupoids, we shall stick to this setting, although the construction works in greater generality (see [**LG94**, Propositions 7.2.1 and 7.2.2]).

PROPOSITION 3.4.11. *Let G be a Hausdroff étale groupoid. Suppose A and B are two G-algebras. Then there exits a canonical homomorphism*

$$j_G : \mathrm{KK}^G(A, B) \to \mathrm{KK}(A \rtimes_r G, B \rtimes_r G).$$

Moreover, j_G is compatible with the Kasparov product in the following sense: If A, B and C are separable G-algebras, and $x \in \mathrm{KK}^G(A, C)$ and $y \in \mathrm{KK}^G(C, B)$, then

$$j_G(x \otimes_C y) = j_G(x) \otimes_{C \rtimes_r G} j_G(y) \in \mathrm{KK}(A \rtimes_r G, B \rtimes_r G).$$

For later reference let us outline the construction of the map j_G: Given a Kasparov triple $(E, \Phi, T) \in \mathbb{E}^G(A, B)$ we can define a right $\Gamma_c(G, r^*\mathcal{B})$-module structure and a $\Gamma_c(G, r^*\mathcal{B})$-valued inner product on $\Gamma_c(G, r^*\mathcal{E})$ by

$$\langle \xi_1, \xi_2 \rangle(g) = \sum_{h \in G_{r(g)}} \beta_{h^{-1}}(\langle \xi_1(h), \xi_2(hg) \rangle)$$

and

$$(\xi f)(g) = \sum_{h \in G^{r(g)}} \xi(h) \beta_h(f(h^{-1}g)).$$

The Hilbert $B \rtimes_r G$-module obtained by completion is denoted by $E \rtimes_r G$. A representation $\widetilde{\Phi} : A \rtimes_r G \to \mathcal{L}(E \rtimes_r G)$ is determined by the formula

$$(\widetilde{\Phi}(f)\xi)(g) = \sum_{h \in G^{r(g)}} \Phi_{r(h)}(f(h)) V_h(\xi(h^{-1}g)),$$

where $f \in \Gamma_c(G, r^*\mathcal{A})$ and $\xi \in \Gamma_c(G, r^*\mathcal{E})$. Finally, one defines an operator $\widetilde{T} \in \mathcal{L}(E \rtimes_r G)$ by

$$(\widetilde{T}\xi)(g) := T_{r(g)}(\xi(g)).$$

Then one can show that $(E \rtimes_r G, \widetilde{\Phi}, \widetilde{T}) \in \mathbb{E}(A \rtimes_r G, B \rtimes_r G)$ and the map j_G is given by $j_G([E, \Phi, T]) = [E \rtimes_r G, \widetilde{\Phi}, \widetilde{T}]$.

REMARK 3.4.12. Equivalently, one can use the canonical representation $B \to M(B \rtimes_r G)$ to define $E \rtimes_r G$ as the tensor product $E \otimes_B (B \rtimes_r G)$.

3.5. Automatic Equivariance

In this section we shall elaborate, when the operator in a Kasparov triple can be chosen in an equivariant way. The main ideas are based on the paper [**Mey00**], which deals with the case of locally compact groups.

Let A and B be (trivially graded) G-algebras and let (E, Φ, T) be an equivariant Kasparov triple for (A, B). We call $T' \in L(E)$ a *compact perturbation* of T if the operators $\Phi(a)(T' - T)$ and $(T' - T)\Phi(a)$ are compact for all $a \in A$. In this case the triples (E, Φ, T) and (E, Φ, T') are *operator homotopic* via the trivial path $T_s := (1 - s)T + sT'$ and hence represent the same element in $\mathrm{KK}^G(A, B)$ (see for example [**Bla98**, Corollary 17.2.6]). To illustrate the usefulness of the above notion, we want to show (the well-known result) that if G is a proper groupoid, then every element in $\mathrm{KK}^G(A, B)$ can be represented by a Kasparov triple with a G-equivariant operator. For the proof we need the following notion:

DEFINITION 3.5.1. [**Tu99b**, Definition 6.7] Let G be a locally compact Hausdorff groupoid equipped with a Haar system $(\lambda^u)_{u \in G^{(0)}}$. A *cutoff* function for G is a continuous map $c : G^{(0)} \to \mathbb{R}^+$ such that

(1) for every $u \in G^{(0)}$ we have $\int_{G^u} c(d(g)) d\lambda^u(g) = 1$, and
(2) the map $r : supp(c \circ d) \to G^{(0)}$ is proper.

The following result is due to Tu (see [**Tu99b**, Propositions 6.10 and 6.11]).

PROPOSITION 3.5.2. *Let G be a locally compact Hausdorff groupoid equipped with a Haar system. Then G admits a cutoff function if and only if G is proper. Moreover, if G is proper and the orbit space $G \backslash G^{(0)}$ is compact, then G admits a cutoff function with compact support.*

We are now ready for the proof of the promised example using compact perturbations.

Proposition 3.5.3. *Let G be a proper groupoid with Haar system $(\lambda^u)_{u\in G^{(0)}}$ and $(E,\Phi,T)\in\mathbb{E}^G(A,B)$ a G-equivariant Kasparov-tripel. Then there is a G-equivariant operator $T^G\in L(E)$ which is a compact pertubation of T.*

Proof. Let $(E,\Phi,T)\in\mathbb{E}^G(A,B)$ be given. Choose a cutoff function c for G. Then for $u\in G^{(0)}$ define

$$(T^G)_u=\int_{G^u} c(d(g))V_gT_{d(g)}V_{g^{-1}}d\lambda^u(g).$$

This clearly defines an operator $T^G\in L(E)$. Let us check that this operator is indeed G-equivariant. For $s\in G$ and $\xi,\eta\in E_{d(s)}$ we compute

$$\begin{aligned}\langle (T^G)_{r(s)}V_s\xi,\eta\rangle_{A_{d(s)}} &= \int_{G^{r(s)}} c(d(g))\langle V_gT_{d(g)}V_{g^{-1}s}\xi,\eta\rangle_{A_{d(s)}}d\lambda^{r(s)}(g)\\ &= \int_{G^{d(s)}} c(d(g))\langle V_{sg}T_{d(g)}V_{g^{-1}}\xi,\eta\rangle_{A_{d(s)}}d\lambda^{d(s)}(g)\\ &= \int_{G^{d(s)}} c(d(g))\langle V_gT_{d(g)}V_{g^{-1}}\xi,V_s^*\eta\rangle_{A_{d(s)}}d\lambda^{d(s)}(g)\\ &= \langle (T^G)_{d(s)}\xi,V_s^*\eta\rangle_{A_{d(s)}}\\ &= \langle V_s(T^G)_{d(s)}\xi,\eta\rangle_{A_{d(s)}}.\end{aligned}$$

It remains to show that T^G is a compact pertubation of T, i.e. we need to see that $\Phi(a)(T^G-T)\in K(E)$ for all $a\in A$. By density we can assume that a viewed as a section $G^{(0)}\to\mathcal{A}$ has compact support. We have

$$\begin{aligned}(\Phi(a)(T^G-T))_u &= \Phi(a)_u\left(\int_{G^u} c(d(g))V_gT_{d(g)}V_{g^{-1}}d\lambda^u(g)-T_u\right)\\ &= \Phi(a)_u\left(\int_{G^u} c(d(g))\left(V_gT_{d(g)}V_{g^{-1}}-T_u\right)d\lambda^u(g)\right)\\ &= \int_{G^u} c(d(g))\Phi(a)_u\left(V_gT_{d(g)}V_{g^{-1}}-T_u\right)d\lambda^u(g)\\ &= \int_{G^u} \Phi_{r(g)}(c(d(g))a(r(g)))\left(V_gT_{d(g)}V_{g^{-1}}-T_{r(g)}\right)dg.\end{aligned}$$

Note that $g \mapsto c(d(g))a(r(g))$ defines an element b in $\Gamma_c(G, r^*\mathcal{A})$ (continuity is obvious and $supp(b) \subseteq supp(c \circ d) \cap r^{-1}(supp(a))$ implies that b has compact support). Since (E, Φ, T) is a G-equivariant Kasparov triple the family

$$(\Phi_{r(g)}(c(d(g))a(r(g))) \left(V_g T_{d(g)} V_{g^{-1}} - T_{r(g)}\right))_{g \in G}$$

defines an element in $r^*K(E)$. Then, by Lemma 2.4.2, integration against the Haar system yields an element in $K(E)$. Consequently, the above computation shows $\Phi(a)(T^G - T) \in K(E)$ as desired. $\square$

Definition 3.5.4. Let E_1 be a graded G-equivariant Hilbert A-module and E_2 be a graded G-equivariant Hilbert $A-B$-bimodule and $E := E_1 \hat{\otimes}_A E_2$. For $x \in E_1$ define an operator $T_x \in L(E_2, E)$ by

$$T_x(y) = x \otimes y.$$

Let $F_2 \in L(E_1)$. An operator $F \in L(E)$ is called an F_2-*connection* if $T_x F_2 - (-1)^{\partial x \partial F_2} F T_x \in K(E_2, E)$ and $F_2 T_x^* - (-1)^{\partial x \partial F_2} T_x^* F \in K(E, E_2)$ for all $x \in E_1$.

Remark 3.5.5. Suppose $(E, \Phi, F) \in \mathbb{E}^G(A, B)$ is an essential triple. Then we have a canonical identification $E \cong A \otimes_\Phi E$.

(1) Under the above identification the operator T_a is just given by $\Phi(a)$ and since $[\Phi(a), F] \in K(E)$ for all $a \in A$ we have that F is an F-connection.
(2) The operator $F \in L(E)$ is a 0-connection if and only if both $F\Phi(a)$ and $\Phi(a)F$ are in $K(E)$. Consequently, an operator $F' \in L(E)$ is a compact perturbation of F if and only if $F - F'$ is a 0-connection.

The following Lemma is a groupoid equivariant version of [**Bla98**, Proposition 18.3.4] and proved in the same way.

Lemma 3.5.6. *Let E_1 be a G-equivariant Hilbert-A-module, E_2 be a G-equivariant Hilbert $A-B$-bimodule and E_3 a G-equivariant Hilbert $B-C$ bimodule.*

(1) If F is an F_2-connection and F' is an F_2'-connection, then $F + F'$ is an $(F_2 + F_2')$-connection.
(2) Let $F_3 \in L(E_3)$ such that $[F_3, B] \in K(E_3)$. Now if F_{23} is an F_3-connection on $E_2 \otimes_B E_3$, and F is an F_{23}-connection

on $E_1 \otimes_A (E_2 \otimes_B E_3)$*, then* F *is an* F_3*-connection on* $(E_1 \otimes_A E_2) \otimes_B E_3$*.*

Now we prove a generalization of [**Mey00**, Lemma 3.1].

LEMMA 3.5.7. *Let* G *be a* σ*-compact locally compact groupoid with Haar system, and let* A *and* B *be* σ*-unital* G*-algebras and* $(E, \Phi, T) \in \mathbb{E}^G(A, B)$ *an essential Kasparov tripel. Then there is a* G*-equivariant* T*-connection* T' *on* $L^2(G, E) \cong L^2(G, A) \otimes_\Phi E$*. If* T *is a self-adjoint contraction, then so is* T'*.*

PROOF. Consider the space $\Gamma_c(G, d^*\mathcal{E})$ of continuous sections of $d^*\mathcal{E}$ with compact support. The inner product

$$\langle f_1, f_2 \rangle_B(u) = \int_{G^u} \beta_g(\langle f_1(g), f_2(g) \rangle_{B_{d(g)}}) d\lambda^u(g)$$

together with the right B-action

$$(f \cdot b)(g) = f(g)\beta_{g^{-1}}(b(r(g)))$$

turns $\Gamma_c(G, d^*\mathcal{E})$ into a pre-Hilbert B-module. Its completion is canonically identified with $L^2(G, E)$ via the isomorphism which sends an elementary tensor $f \otimes e \in L^2(G) \otimes_{C_0(G^{(0)})} E$ to the function $g \mapsto f(g)V_{g^{-1}}e(r(g))$. Since Φ is essential, we have

$$L^2(G, E) \cong L^2(G, A) \otimes_\Phi E.$$

Now define $T' : \Gamma_c(G, d^*\mathcal{E}) \to \Gamma_c(G, d^*\mathcal{E})$ by

$$(T'f)(g) = T_{d(g)}(f(g)).$$

We have

$$\begin{aligned}
\|T'f\|^2 &= \|\langle T'f, T'f \rangle_B\| \\
&= \sup_{u \in G^{(0)}} \|\langle T'f, T'f \rangle_B(u)\| \\
&= \sup_{u \in G^{(0)}} \|\int_{G^u} \beta_g(\langle (T'f)(g), (T'f)(g) \rangle_{B_{d(g)}}) d\lambda^u(g)\| \\
&= \sup_{u \in G^{(0)}} \|\int_{G^u} \beta_g(\underbrace{\langle T_{d(g)}(f(g)), T_{d(g)}(f(g)) \rangle_{B_{d(g)}}}_{\leq \|T\|^2 \langle f(g), f(g) \rangle}) d\lambda^u(g)\| \\
&\leq \|T\|^2 \|f\|^2
\end{aligned}$$

Thus, T' is bounded with $\|T'\| \leq \|T\|$. Let us check that T' is indeed G-equivariant. If V' denotes the unitary implementing the G-action on $L^2(G, E)$, then we have

$$\begin{aligned}(T'_{r(g)}V'_g f)(s) &= T_{d(s)}(V'_g f(s)) \\ &= T_{d(s)}(f(g^{-1}s)) \\ &= (T'_{d(g)}f)(g^{-1}s) \\ &= (V'_g T'_{d(g)} f)(s).\end{aligned}$$

An easy computation reveals that self-adjointness of T implies self-adjointness of T'.

We claim that T' is a T-connection. To show this we have to check that $K := T_\xi T - T' T_\xi \in K(E, L^2(G, E))$ for all $\xi \in L^2(G, A)$. Let us first take a closer look at the rank one operators in $K(E, L^2(G, E))$. For $x, y \in E$ and $\xi \in L^2(G, E)$ of the form $\xi(g) = f \otimes e(g) = f(g)V_{g^{-1}}e(r(g))$ for $f \in C_c(G)$ and $e \in E$ we have

$$\begin{aligned}\theta_{\xi,x}(y)(g) &= (\xi \cdot \langle x, y\rangle_A)(g) \\ &= \xi(g)\alpha_{g^{-1}}(\langle x, y\rangle_A(r(g))) \\ &= f(g)V_{g^{-1}}(e \cdot \langle x, y\rangle_A(r(g))) \\ &= (f \otimes \theta_{e,x}(y))(g).\end{aligned}$$

Back to the operator K: Since elements of the form $f \otimes a$, where

$$(f \otimes a)(g) = f(g)\alpha_{g^{-1}}(a(r(g))),$$

form a dense subset of $L^2(G, A)$ we can restrict to ξ of this form. Recall that the canonical isomorphism $L^2(G, A) \otimes_\Phi E \cong L^2(G, E)$ sends $\xi \otimes e$ to the function $g \mapsto \Phi_{d(g)}(\xi(g))V_{g^{-1}}(e(r(g)))$. Thus, for all $e \in E$ and $g \in G$ we can compute

$$\begin{aligned}(Ke)(g) &= (T_\xi Te)(g) - (T' T_\xi e)(g) \\ &= (\xi \otimes Te)(g) - T_{d(g)}(T_\xi e(g)) \\ &= \Phi_{d(g)}(\xi(g))V_{g^{-1}}T_{r(g)}(e(r(g))) - T_{d(g)}\Phi_{d(g)}(\xi(g))V_{g^{-1}}e(r(g)) \\ &= f(g)\Phi_{d(g)}(\alpha_{g^{-1}}(a(r(g))))V_{g^{-1}}T_{r(g)}(e(r(g))) \\ &\quad - f(g)T_{d(g)}\Phi_{d(g)}(\alpha_{g^{-1}}(a(r(g))))V_{g^{-1}}e(r(g)) \\ &= f(g)V_{g^{-1}}\Phi_{r(g)}(a(r(g)))T_{r(g)}(e(r(g)))\end{aligned}$$

$$- f(g)T_{d(g)}V_{g^{-1}}\Phi_{r(g)}(a(r(g)))e(r(g))$$
$$= (*)$$

By adding and substracting the term $f(g)V_{g^{-1}}T_{r(g)}\Phi_{r(g)}(a(r(g)))e(r(g))$ in the last line we get

$$(*) = (f \otimes [\Phi(a), T]e)(g) + f(g)(V_{g^{-1}}T_{r(g)} - T_{d(g)}V_{g^{-1}})\Phi(a(r(g)))e(r(g)).$$

Now approximating $[\Phi(a), T]$ by sums of rank one operators and using our description of these it is not hard to see that $e \mapsto f \otimes [\Phi(a), T]e \in K(E, L^2(G, E))$. The second summand in $(*)$ can be rewritten as

$$V_{g^{-1}}(T_{r(g)} - V_g T_{d(g)} V_{g^{-1}})\Phi(f(g)a(r(g))) \cdot e(r(g)).$$

Since (E, Φ, T) is a G-equivariant Kasparov triple, the family

$$(T_{r(g)} - V_g T_{d(g)} V_{g^{-1}})\Phi(f(g)a(r(g))))_{g \in G}$$

defines an element in $r^*(K(E))$ and since f has compact support it can be approximated by finite sums of elements of the form $\psi \otimes F$ for $\psi \in C_c(G)$ and $F \in K(E)$ where $(\psi \otimes F)(g) = \psi(g)F_{r(g)}$. Passing to such elements we are left with the term

$$\psi(g)V_{g^{-1}}F_{r(g)}e(r(g)) = \psi(g)V_{g^{-1}}(Fe(r(g))) = (\psi \otimes Fe)(g)$$

But $e \mapsto \psi \otimes Fe$ can be approximated by rank-one operators as above and thus we have shown that $K \in K(E, L^2(G, E))$. □

Now we can use the exact same arguments as in [**Mey00**, Proposition 3.2] to show:

PROPOSITION 3.5.8. *Suppose A and B are σ-unital G-algebras and (E, Φ, T) is an essential Kasparov triple in $\mathbb{E}^G(K(L^2(G)) \otimes^{max}_{G^{(0)}} A, B)$. Then there exists a G-equivariant compact perturbation of T.*

3.6. The Compression Isomorphism

Before we can construct the compression isomorphism we need the following preliminary observation:

LEMMA 3.6.1. *Let G be an étale, locally compact Hausdorff groupoid and $H \subseteq G$ a clopen subgroupoid, such that $H^{(0)} = G^{(0)}$. If A is an H-algebra, then there is an H-equivariant embedding*

$$i_A : A \to Ind_H^G A$$

given by the formula

$$i_A(a)(g) = \left\{ \begin{array}{ll} \alpha_{g^{-1}}(a(r(g))) & , g \in H \\ 0_{d(g)} & , else \end{array} \right\}$$

Proof. First, we check that $i_A(a)$ is indeed an element in $Ind_H^G A$. The continuity of $i_A(a)$ is clear, as H is clopen in G. Now let $h \in H$ and $g \in G$ such that $d(g) = d(h)$. Then we clearly have $g \in H \Leftrightarrow gh^{-1} \in H$ and thus in this case we can compute

$$i_A(a)(gh^{-1}) = \alpha_{hg^{-1}}(a(r(gh^{-1}))) = \alpha_h(\alpha_{g^{-1}}(a(r(g))) = \alpha_h(i_A(a)(g)).$$

If $g \notin H$ we have $i_A(a)(gh^{-1}) = 0_{A_{r(h)}} = \alpha_h(i_A(a)(g))$. It remains to verify that $gH \mapsto \|i_A(a)(g)\|$ vanishes at infinity. Given $\varepsilon > 0$ there exists a compact subset $K \subseteq H^{(0)}$ such that $\|a(u)\| < \varepsilon$ for all $u \notin K$. Let C be the image of K in the quotient space G/H. Now if $gH \notin C$, then either $g \in G \setminus H$, in which case $\|i_A(a)(g)\| = 0$, or $g \in H$, in which case $r(g)H = gH \notin C$. But then $r(g) \notin K$, which implies $\|i_A(a)(g)\| = \|a(r(g))\| < \varepsilon$.

It is straightforward to see that i_A is an isometric $*$-homomorphism. For the H-equivariance we compute for all $h \in H$ and $g \in H^{r(g)}$: $\beta_h((i_A)_{d(h)}(a(d(h))))(g) = (i_A)_{d(h)}(a(d(h)))(h^{-1}g) = \alpha_{g^{-1}h}(a(d(h))) = (i_A)_{r(h)}(\alpha_h(a(d(h))))(g)$. □

Let us proceed with the construction of the compression homomorphism: Consider an étale, locally compact Hausdorff groupoid G with an étale subgroupoid $H \subseteq G$. Let $X := G_{H^{(0)}}$ and $G' := G_{H^{(0)}}^{H^{(0)}}$. Suppose, that H is clopen in G'. Now if A is an H-algebra and B is a G-algebra we define the compression homomorphism

$$\mathsf{comp}_H^G : \mathrm{KK}^G(Ind_H^X A, B) \to \mathrm{KK}^H(A, B_{|H})$$

as the composition

$$\begin{array}{ccc} \mathrm{KK}^G(Ind_H^X A, B) & \xrightarrow{\mathsf{res}_H^G} & \mathrm{KK}^H(Ind_H^{G'} A, B_{|H}) \\ & \overset{\mathsf{comp}_H^G}{\searrow} & \downarrow i_A^* \\ & & \mathrm{KK}^H(A, B_{|H}) \end{array}$$

Here res_H^G is the homomorphism induced by the inclusion map $H \hookrightarrow G$ (cf. [**LG99**, Proposition 7.1]), and i_A is the inclusion map from Lemma 3.6.1. We are now proceeding to prove the main theorem of this section:

THEOREM 3.6.2. *Let G be an étale locally compact Hausdorff groupoid with a clopen, proper subgroupoid $H \subseteq G$. Let $X := G_{H^{(0)}}$. If A is an H-algebra and B is a G-algebra, then*

$$\mathsf{comp}_H^G : \mathrm{KK}^G(Ind_H^X A, B) \to \mathrm{KK}^H(A, B_{|H^{(0)}})$$

is an isomorphism.

In order to prove the above theorem we will construct an inverse. Let (E, Φ, T) be a Kasparov triple representing an element in the group $\mathrm{KK}^H(A, B_{|H^{(0)}})$ and let V denote the unitary operator implementing the action of H on E. Since H is proper, we can assume that T is H-equivariant by Proposition 3.5.3. Consider the complex vector space $\widetilde{E}_c$ consisting of bounded continuous sections $\xi : X \to d^*_{|X}(\mathcal{E})$ such that

- $\xi(gh^{-1}) = V_h(\xi(g))$ for all $g \in X$ and $h \in H$ with $d(g) = d(h)$, and
- the map $gH \mapsto \|\xi(g)\|$ has compact support in X/H.

Then $\widetilde{E}_c$ becomes a G-equivariant pre-Hilbert B-module as follows. Using the identification $B \cong \Gamma_0(G^{(0)}, \mathcal{B})$ we define a B-valued inner product by letting

$$\langle \xi, \eta \rangle_B(u) := \sum_{gH \in X^u/H} \beta_g(\langle \xi(g), \eta(g) \rangle_{B_{d(g)}}).$$

The second condition on the elements of $\widetilde{E}_c$ guarantees that the sum in the formula above is finite (since X^u/H is discrete). Let us check that $\langle \xi, \eta \rangle_B$ defines an element in $\Gamma_c(G^{(0)}, \mathcal{B})$: Consider the map

$$gH \mapsto \beta_g(\langle \xi(g), \eta(g) \rangle_{B_{d(g)}}.$$

This map is clearly continuous and hence an element in $\Gamma(X/H, \tilde{r}^*(\mathcal{B}))$, where $\tilde{r} : X/H \to G^{(0)}$ is the map induced by the restriction of the range map of G to X. Moreover, its support is easily checked to be contained in the intersection of the compact supports of the maps $gH \mapsto \|\xi(g)\|$ and $gH \mapsto \|\eta(g)\|$, and hence compact as well. Thus, our claim follows from the following Lemma:

LEMMA 3.6.3. *Let G, H, X be as above and $f \in C_c(X/H)$. Then the map*

$$u \mapsto \sum_{gH \in X^u/H} f(gH)$$

is continuous.

PROOF. For this we only need to note, that the map $\tilde{r} : X/H \to G^{(0)}$ is a local homeomorphism. Then the same proof, that shows continuity for the system of counting measures on an étale groupoid (see Proposition 1.1.15), gives the desired result. But if U is an open r-section of G, then $\tilde{r}$ will be a homeomorphism onto an open set, when restricted to the image of $U \cap X$ in X/H. □

The right B-action on $\widetilde{E}_c$ is defined by the formula

$$(\xi \cdot b)(g) := \xi(g)\beta_{g^{-1}}(b(r(g))).$$

The following computation shows, that $\xi \cdot b$ is indeed an element of $\widetilde{E}_c$ again:

$$\begin{aligned}(\xi \cdot b)(gh^{-1}) &= \xi(gh^{-1})\beta_{hg^{-1}}(b(r(gh^{-1}))) \\ &= V_h(\xi(g)) \cdot \beta_h(\beta_{g^{-1}}(b(r(g)))) \\ &= V_h(\xi(g) \cdot \beta_{g^{-1}}(b(r(g)))) \\ &= V_h((\xi \cdot b)(g)).\end{aligned}$$

The support of the map $gH \mapsto \|(b \cdot \xi)(g)\|$ is clearly compact since the support of ξ is. Let us check that with the above defined inner product and B-action $\widetilde{E}_c$ is indeed a pre-Hilbert B-module: It is straightforward to check that the inner product is linear in the second and conjugate linear in the first variable. Also, we clearly have $\langle \xi, \xi \rangle_B \geq 0$ for all $\xi \in \widetilde{E}_c$. Now if $\langle \xi, \xi \rangle_B(u) = 0$ for all $u \in G^{(0)}$ then $\langle \xi(g), \xi(g) \rangle_{B_{d(g)}} = 0$ for all $g \in X$ and thus $\xi = 0$. It remains to verify that the B-action is compatible with the inner product:

$$\begin{aligned}\langle \xi, \eta \cdot b \rangle_B(u) &= \sum_{gH} \beta_g(\langle \xi(g), (\eta \cdot b)(g) \rangle_{B_{d(g)}}) \\ &= \sum_{gH} \beta_g(\langle \xi(g), \eta(g)\beta_{g^{-1}}(b(r(x))) \rangle_{B_{d(g)}}) \\ &= \sum_{g\mathcal{F}_U} \beta_g(\langle \xi(g), \eta(g) \rangle_{B_{d(g)}} \beta_{g^{-1}}(b(r(g))))\end{aligned}$$

$$= \sum_{gH} \beta_g(\langle \xi(g), \eta(g)\rangle_{B_{d(g)}}) b(r(g))$$

$$= \left(\sum_{gH} \beta_g(\langle \xi(g), \eta(g)\rangle_{B_{d(g)}}) \right) b(u)$$

$$= (\langle \xi, \eta\rangle_B b)(u)$$

Let $\widetilde{E}$ be the completion of $\widetilde{E}_c$ with respect to the norm induced by the inner product.

To define the G-action on $\widetilde{E}$, let us identify the fibres. For $u \in G^{(0)}$ consider the complex vector space of bounded continuous sections $\xi : X^u \to d^*\mathcal{E}$ such that

- $\xi(gh^{-1}) = V_h(\xi(g))$ for all $g \in X^u$ and $h \in H$ such that $d(g) = d(h)$, and
- the map $gH \mapsto \|\xi(g)\|$ has compact support in X^u/H.

We can turn this into a pre-Hilbert B_u-module by defining

$$\langle \xi, \eta\rangle_{B_u} := \sum_{gH \in X^u/H} \beta_g(\langle \xi(g), \eta(g)\rangle_{B_{d(g)}})$$

and

$$(\xi \cdot b(u))(g) := \xi(g) \cdot \beta_{g^{-1}}(b(u)).$$

Let F_u denote the completion with respect to this inner product.

LEMMA 3.6.4. *For $u \in G^{(0)}$ the restriction map $\mathsf{res} : \widetilde{E}_c \to F_u$, $\xi \mapsto \xi_{|X^u}$ factors through an isomorphism between the Hilbert B_u-modules $\widetilde{E}_u$ and F_u.*

PROOF. It is clear that $C_0(G^{(0)} \setminus \{u\})\widetilde{E}_c \subseteq ker(res)$ and by continuity we get

$$\overline{C_0(G^{(0)} \setminus \{u\})\widetilde{E}} \subseteq ker(res).$$

For the converse inclusion let $\xi \in ker(res)$ and $\varepsilon > 0$ be given. First, find $\eta \in \widetilde{E}_c$ such that $\|\eta - \xi\| < \frac{\varepsilon}{2}$. Then, for all $x \in X^u$ we have $\|\eta(x)\| \leq \|res(\eta)\| = \|res(\eta) - res(\xi)\| \leq \|\eta - \xi\| < \frac{\varepsilon}{2}$. Now let $K = \{xH \in X/H \mid \|\eta(x)\| \geq \frac{\varepsilon}{2}\}$. Then $u \notin \tilde{r}(K)$. Thus, we can find $\varphi \in C_c(G^{(0)})$ such that $\varphi \equiv 1$ on $\tilde{r}(K)$ and $\varphi(u) = 0$. Then, by construction we have $\|\eta - \varphi \cdot \eta\| < \frac{\varepsilon}{2}$ and thus $\|\xi - \varphi \cdot \eta\| \leq \|\xi - \eta\| + \|\eta - \varphi \cdot \eta\| < \varepsilon$. Consequently, ξ is contained in the closure of $C_0(G^{(0)} \setminus \{u\})\widetilde{E}$. Hence res does indeed factor through an injective map $\widetilde{E}_u \to F_u$ and it follows directly from the definition of the respective inner products, that this map is isometric. Consequently, to complete the proof it is enough to show that the image is dense. This however can be done in analogy to the case of induced C*-algebras. □

Let us now define the G-action on $\widetilde{E}$: For $g \in G$ define an operator $\widetilde{V}_g \in L(\widetilde{E}_{d(g)}, \widetilde{E}_{r(g)})$ by

$$(\widetilde{V}_g \xi)(s) := \xi(g^{-1}s) \ \forall s \in X^{r(g)}.$$

With this action $\widetilde{E}$ is a G-equivariant Hilbert B-module. Define a $*$-homomorphism $\widetilde{\Phi} : Ind_H^X A \to L(\widetilde{E})$ by the formula

$$(\widetilde{\Phi}(f) \cdot \xi)(g) := \Phi_{d(g)}(f(g)) \cdot \xi(g).$$

Last but not least define an operator $\widetilde{T} \in L(\widetilde{E})$ by the formula

$$(\widetilde{T}\xi)(g) = T_{d(g)}(\xi(g)).$$

We need to check that $\widetilde{T}\xi \in \widetilde{E}_c$: For $g \in X$ and $h \in H$ such that $d(g) = d(h)$ we have

$$\begin{aligned}
(\widetilde{T}\xi)(gh^{-1}) &= T_{d(gh^{-1})}(\xi(gh^{-1})) \\
&= T_{r(h)}(\xi(gh^{-1})) \\
&= T_{r(h)}(V_h(\xi(g))) \\
&= V_h(T_{d(h)}(\xi(g))) \\
&= V_h(T_{d(g)}(\xi(g))) \\
&= V_h((\widetilde{T}\xi)(g)).
\end{aligned}$$

To see that $\widetilde{T}$ is bounded and hence extends to an operator on $\widetilde{E}$ note the following two general facts:

(1) If $a, b \in A$ are positive elements with $a \leq b$, then $\|a\| \leq \|b\|$.
(2) If E is a right Hilbert A-module, then

$$\langle Tx, Tx\rangle_A \leq \|T\|^2 \langle x, x\rangle_A$$

for all $x \in E$ and $T \in L(E)$ (see [**RW98**, Corollary 2.22]).

Because of the above facts and using that the positive elements form a cone we have that

$$\|\sum_{gH} \beta_g(\langle T_{d(g)}(\xi(g)), T_{d(g)}(\xi(g))\rangle_{B_{d(g)}})\| \leq \|\sum_{gH} \beta_g(\|T_{d(g)}\|^2 \langle \xi(g), \xi(g)\rangle)\|,$$

where the sum is over X^u/H. Thus, we can compute:

$$\begin{aligned}
\|\widetilde{T}\xi\|^2 &= \|\langle\widetilde{T}\xi,\widetilde{T}\xi\rangle_B\| \\
&= \sup_{u\in G^{(0)}} \|\langle\widetilde{T}\xi,\widetilde{T}\xi\rangle_B(u)\| \\
&= \sup_{u\in G^{(0)}} \|\sum_{gH\in X^u/H} \beta_g(\langle T_{d(g)}(\xi(g)), T_{d(g)}(\xi(g))\rangle_{B_{d(g)}})\| \\
&\leq \sup_{u\in G^{(0)}} \|\sum_{gH\in X^u/H} \beta_g(\|T_{d(g)}\|^2\langle\xi(g),\xi(g)\rangle)\| \\
&\leq \|T\|^2 \sup_{u\in G^{(0)}} \|\sum_{gH\in X^u/H} \beta_g(\langle\xi(g),\xi(g)\rangle)\| \\
&= \|T\|^2 \sup_{u\in G^{(0)}} \|\langle\xi,\xi\rangle_B(u)\| \\
&= \|T\|^2\|\xi\|^2
\end{aligned}$$

Hence $\widetilde{T}$ extends to a bounded operator on $\widetilde{E}$. It is clearly adjointable with $(\widetilde{T})^* = \widetilde{T^*}$. We want to show that $(\widetilde{E},\widetilde{\Phi},\widetilde{T})$ is a G-equivariant Kasparov-tripel for $Ind_H^X A$ and B. To this end we will need some helpful Lemmas. Note that for every $u \in G^{(0)}$ we also have a homomorphism

$$i_A^u : A \to Ind_H^{X^u} A$$

from A into each fibre of $Ind_H^X A$, given by the same formulas as i_A. Here, continuity of $i_A^u(a)$ is not a problem as X^u carries the discrete topology.

LEMMA 3.6.5. *Let $u \in G^{(0)}$. Consider the set*

$$A_0 = \{\sum_{i=1}^{n} \tilde{\alpha}_{g_i}(i_A^{d(g_i)}(a_i)) \mid n \in \mathbb{N}, g_i \in X^u, a_i \in A\},$$

where $\tilde{\alpha}$ is the action of G on $Ind_H^X A$ defined in section 2.4. Then A_0 is dense in $Ind_H^{X^u} A$.

PROOF. We want to apply Proposition 2.1.4 to A_0. To this end let us first note that A_0 is a linear subspace of $Ind_H^{X^u} A$ and moreover it is $C_0(X^u/H)$-invariant. To see this let $a \in A$, $g \in X^u$ and $\varphi \in C_0(X^u/H)$. Then for every $s \in X^u$ such that $g^{-1}s \in H$ we have $gH = sH$ and can compute:

$$(\varphi \cdot (\tilde{\alpha}_g(i_A^{d(g)}(a))))(s) = \varphi(sH)\tilde{\alpha}_g(i_A^{d(g)}(a))(s)$$

$$= \varphi(gH) i_A^{d(g)}(a)(g^{-1}s)$$
$$= i_A^{d(g)}(\varphi(gH)a)(g^{-1}s)$$
$$= \tilde{\alpha}_g(i_A^{d(g)}(\varphi(gH)a))(s)$$

Since $i_A(a)$ vanishes if $g^{-1}s$ is not in H, we can conclude:

$$\varphi \cdot (\tilde{\alpha}_g(i_A^{d(g)}(a))) = \tilde{\alpha}_g(i_A^{d(g)}(\varphi(gH)a)) \in A_0.$$

So to see that A_0 is dense we just need to show that for any fixed $g \in X^u$ we have $\{f(g) \mid f \in A_0\} = A_{d(g)} (\cong (Ind_H^{X^u} A)_{gH})$. But since for any $a \in A$ we have $\tilde{\alpha}_g(i_A^{d(g)}(a))(g) = i_A(a)(g^{-1}g) = i_A(a)(d(g)) = \alpha_{d(g)}(a(d(g))) = a(d(g))$ this is obvious. $\square$

Next, we use this result to identify a nice dense subset of the whole algebra $Ind_H^X A$. For this write $Ind_H^X \mathcal{A}$ for the upper semi-continuous C^*-bundle associated to the $C_0(G^{(0)})$-algebra $Ind_H^X A$. Let us recall some notation: For $\varphi \in C_c(G)$ and $a \in A$ we can define $\varphi \otimes i_A(a) \in \Gamma_c(G, d^*(Ind_H^X \mathcal{A}))$ by

$$(\varphi \otimes i_A(a))(g) = \varphi(g) i_A(a)_{|X^{d(g)}} = \varphi(g) i_A^{d(g)}(a).$$

Furthermore, let

$$\lambda : \Gamma_c(G, r^*(Ind_H^X \mathcal{A})) \to Ind_H^X A$$

be the continuous map from Lemma 2.4.2 given by the formula

$$\lambda(f)(u) = \sum_{g \in G^u} f(g), \ \forall u \in G^{(0)}.$$

Lemma 3.6.6. *The set*

$$\Gamma = \{\lambda(\tilde{\alpha}(\varphi \otimes i_A(a))) \mid a \in A, \varphi \in C_c(G)\}$$

is dense in $Ind_H^X A$.

Proof. First we note that Γ is a $C_0(G^{(0)})$-invariant linear subspace of $Ind_H^X A$, since for $\psi \in C_0(G^{(0)})$ we have

$$\begin{aligned}
\psi \cdot \lambda(\tilde{\alpha}(\varphi \otimes i_A(a)))(u) &= \sum_{g \in G^u} \psi(u)\varphi(g) \tilde{\alpha}_g(i_A^{d(g)}(a)) \\
&= \sum_{g \in G^u} (\psi \otimes \varphi)(g) \tilde{\alpha}_g(i_A^{d(g)}(a)) \\
&= \lambda(\tilde{\alpha}(\psi \otimes \varphi) \otimes i_A(a))(u),
\end{aligned}$$

where $\psi \otimes \varphi \in C_c(G)$ is given by $(\psi \otimes \varphi)(g) = \psi(r(g))\varphi(g)$. Then note that for fixed $u \in G^{(0)}$ we have $A_0 \subseteq \{\lambda(\tilde{\alpha}(\varphi \otimes i_A(a)))(u) \mid \varphi \in C_c(G), a \in A\} \subseteq Ind_H^{X^u} A$. By the previous lemma A_0 is dense in $Ind_H^{X^u} A$ and thus, so is the middle set. Consequently, Γ is dense in $Ind_H^X A$ by yet another application of Proposition 2.1.4. $\square$

We are now prepared for:

LEMMA 3.6.7. $(\widetilde{E}, \widetilde{\Phi}, \widetilde{T}) \in \mathbb{E}^G(Ind_H^X A, B)$.

PROOF. As a first step we check that $\widetilde{T}$ is G-equivariant. For this note that for $u \in G^{(0)}$ the operator $\widetilde{T}_u : \widetilde{E}_u \to \widetilde{E}_u$ is given by the same formula as $\widetilde{T}$ itself. Thus for all $g \in G$, $\xi \in \widetilde{E}_{d(g)}$ and $s \in X^{r(g)}$ we can compute:

$$\begin{aligned}(\widetilde{T}_{r(g)}\widetilde{V}_g\xi)(s) &= T_{d(s)}((\widetilde{V}_g \cdot \xi)(s)) \\ &= T_{d(s)}(\xi(g^{-1}s)) \\ &= (\widetilde{T}\xi)(g^{-1}s) \\ &= (\widetilde{V}_g\widetilde{T}_{d(g)}\xi)(s)\end{aligned}$$

Consequently, it is enough to check that $[\widetilde{T}, \widetilde{\Phi}(f)]$, $(\widetilde{T}^2 - 1)\widetilde{\Phi}(f)$ and $(\widetilde{T}^* - \widetilde{T})\widetilde{\Phi}(f)$ are compact operators on $\widetilde{E}$ for all $f \in Ind_H^X A$. We will do this in two steps:

Step 1: $f = i_A(a)$:

For this we note that there is an embedding $i_E : E \hookrightarrow \widetilde{E}$ of E as a direct summand of $\widetilde{E}$ given by the formula

$$i_E(e)(g) = \left\{ \begin{array}{ll} V_{g^{-1}}(e(r(g))) & , g \in H \\ 0_{d(g)} & , else \end{array} \right\}.$$

This embedding induces a corresponding embedding $i_{K(E)} : K(E) \to K(\widetilde{E})$. By checking on rank-one operators and going through the formulas we can see that for $F \in K(E)$ we have the following equation:

$$(i_{K(E)}(F)\xi)(g) = \left\{ \begin{array}{ll} (V_{g^{-1}}F_{r(g)}V_g) \cdot \xi(g) & , g \in H \\ 0_{d(g)} & , else \end{array} \right\}$$

Note also that for $a \in A$ we have $(\widetilde{\Phi}(i_A(a))\xi)(g) = 0$ if $g \notin H$. For $g \in H$ we can use the H-equivariance of T to compute:

$$(i_{K(E)}([T, \Phi(a)])\xi)(g) = (V_{g^{-1}}[T_{r(g)}, \Phi_{r(g)}(a(r(g)))]V_g)(\xi(g))$$

$$\begin{aligned}&= [T_{d(g)}, \Phi_{d(g)}(\alpha_{g^{-1}}(a(r(g))))]\xi(g)\\ &= [T_{d(g)}, \Phi_{d(g)}(i_A(a)(g))]\xi(g)\\ &= ([\widetilde{T}, \widetilde{\Phi}(i_A(a))]\xi)(g).\end{aligned}$$

Consequently, we have $i_{K(E)}([T, \Phi(a)]) = [\widetilde{T}, \widetilde{\Phi}(i_A(a))]$ for all $a \in A$. Similar computations show that $i_{K(E)}((T^2-1)\Phi(a)) = (\widetilde{T}^2-1)\widetilde{\Phi}(i_A(a))$ and $i_{K(E)}((T - T^*)\Phi(a)) = (\widetilde{T} - \widetilde{T}^*)\widetilde{\Phi}(i_A(a))$ for all $a \in A$.

Step 2: $f = \lambda(\tilde{\alpha}(\varphi \otimes i_A(a)))$

Since $(\widetilde{T} - \widetilde{T}^*)\widetilde{\Phi}(i_A(a)) \in K(\widetilde{E})$ by the first step, we have

$$\widetilde{V}(\varphi \otimes (\widetilde{T} - \widetilde{T}^*)\widetilde{\Phi}(i_A(a)))\widetilde{V}^* \in \Gamma_c(G, r^*\mathcal{K}(\widetilde{E}))$$

for all $\varphi \in C_c(G)$ and hence

$$\lambda(\widetilde{V}(\varphi \otimes (\widetilde{T} - \widetilde{T}^*)\widetilde{\Phi}(i_A(a)))\widetilde{V}^*) \in K(\widetilde{E})$$

by Lemma 2.4.2. Let us show that

$$(\widetilde{T} - \widetilde{T}^*)\widetilde{\Phi}(\lambda(\tilde{\alpha}(\varphi \otimes i_A(a)))) = \lambda(\widetilde{V}(\varphi \otimes (\widetilde{T} - \widetilde{T}^*)\widetilde{\Phi}(i_A(a)))\widetilde{V}^*))$$

For $f = \lambda(\tilde{\alpha}(\varphi \otimes i_A(a)))$ we compute:

$$\begin{aligned}&((\widetilde{T} - \widetilde{T}^*)\widetilde{\Phi}(f) \cdot \xi)(s)\\ &\quad= (T_{d(s)} - T^*_{d(s)})\Phi_{d(s)}(\lambda(\tilde{\alpha}(\varphi \otimes i_A(a)))(s)) \cdot \xi(s)\\ &\quad= \sum_{g \in G^{r(s)}} \varphi(g)(T_{d(s)} - T^*_{d(s)})\Phi_{d(s)}(\tilde{\alpha}_g(i_A^{d(g)}(a)(s)))\xi(s)\\ &\quad= \sum_{g \in G^{r(s)}} \varphi(g)((\widetilde{T}_{r(g)} - \widetilde{T}^*_{r(g)})\widetilde{\Phi}_{r(g)}(\tilde{\alpha}_g(i_A^{d(g)}(a))\xi))(s)\\ &\quad= \sum_{g \in G^{r(s)}} \widetilde{V}_g\varphi(g)(\widetilde{T}_{d(g)} - \widetilde{T}^*_{d(g)})\widetilde{\Phi}_{d(g)}(i_A^{d(g)}(a))\widetilde{V}_g^*\xi)(s)\\ &\quad= \sum_{g \in G^{r(s)}} (((\widetilde{V}(\varphi \otimes (\widetilde{T} - \widetilde{T}^*)\widetilde{\Phi}(i_A(a)))\widetilde{V}^*))(g)\xi)(s)\\ &\quad= \left(\lambda(\widetilde{V}(\varphi \otimes (\widetilde{T} - \widetilde{T}^*)\widetilde{\Phi}(i_A(a)))\widetilde{V}^*)(r(s)) \cdot \xi_{|r(s)}\right)(s)\\ &\quad= (\lambda(\widetilde{V}(\varphi \otimes (\widetilde{T} - \widetilde{T}^*)\widetilde{\Phi}(i_A(a)))\widetilde{V}^*) \cdot \xi)(s)\end{aligned}$$

Similarly, we compute

$$[\widetilde{T}, \widetilde{\Phi}(\lambda(\tilde{\alpha}(\varphi \otimes i_A(a))))] = \lambda(\widetilde{V}(\varphi \otimes [\widetilde{T}, \widetilde{\Phi}(i_A(a))])\widetilde{V}^*)$$

and

$$(\widetilde{T}^2 - 1)\widetilde{\Phi}(\lambda(\tilde{\alpha}(\varphi \otimes i_A(a)))) = \lambda(\widetilde{V}(\varphi \otimes (\widetilde{T}^2 - 1)\widetilde{\Phi}(i_A(a)))\widetilde{V}^*)).$$

From the previous lemma we know that elements of the form $\lambda(\tilde{\alpha}(\varphi \otimes i_A(a)))$ form a dense subset of $Ind_H^X A$ and thus the result follows by continuity. □

Applying the same constructions to a homotopy we conclude that the mapping $(E, \Phi, T) \mapsto (\widetilde{E}, \widetilde{\Phi}, \widetilde{T})$ induces a map in equivariant KK-theory, which we call the inflation map:

$$\mathsf{inf}_H^G : \mathrm{KK}^H(A, B_{|H}) \to \mathrm{KK}^G(Ind_H^X A, B)$$

Proof of Theorem 3.6.2. As a first step we claim that the result is invariant under passing to a Morita-equivalent algebra in the first variable. Indeed if A' is Morita-equivalent to A and if we let $x \in \mathrm{KK}^H(A', A)$ be the corresponding invertible KK^H-element, the claim will follow from the commutativity of the following diagram:

$$\begin{array}{ccc}
\mathrm{KK}_*^G(Ind_H^X\ A, B) & \xrightarrow{\mathsf{comp}_H^G} & \mathrm{KK}_*^H(A, B_{|H^{(0)}}) \\
{\scriptstyle \mathsf{Ind}_H^G(x) \otimes \cdot}\downarrow & & \downarrow{\scriptstyle x \otimes \cdot} \\
\mathrm{KK}_*^G(Ind_H^X\ A', B) & \xrightarrow{\mathsf{comp}_H^G} & \mathrm{KK}_*^H(A', B)
\end{array}$$

Here $\mathsf{Ind}_H^G(x)$ denotes the image of x under the induction homomorphism

$$\mathsf{Ind}_H^G : \mathrm{KK}^H(A', A) \to \mathrm{KK}^G(Ind_H^X A', Ind_H^X A),$$

from Proposition 3.4.10. Commutativity of the above diagram follows from the equation

$$[i_{A'}] \otimes \mathsf{res}_H^G(\mathsf{Ind}_H^G(x)) = x \otimes [i_A],$$

since then for any $y \in \mathrm{KK}^G(Ind_H^X\ A, B)$ we have

$$\begin{aligned}
x \otimes \mathsf{comp}_H^G(y) &= x \otimes i_A^*(\mathsf{res}_H^G(y)) \\
&= x \otimes [i_A] \otimes \mathsf{res}_H^G(y) \\
&= [i_{A'}] \otimes \mathsf{res}_H^G(\mathsf{Ind}_H^G(x)) \otimes \mathsf{res}_H^G(y) \\
&= [i_{A'}] \otimes \mathsf{res}_H^G(\mathsf{Ind}_H^G(x) \otimes y)
\end{aligned}$$

$$= \mathsf{comp}_H^G(\mathsf{Ind}_H^G(x) \otimes y).$$

We will now show that the inflation map constructed above is inverse to the compression homomorphism. We will begin with the easier direction: Let (E, Φ, T) represent an element in $\mathrm{KK}^H(A, B_{|H})$. We need to see that $\mathsf{comp}_H^G([\widetilde{E}, \widetilde{\Phi}, \widetilde{T}]) = [E, \Phi, T]$. By definition the element $\mathsf{comp}_H^G([\widetilde{E}, \widetilde{\Phi}, \widetilde{T}])$ can be represented by the triple $(\widetilde{E}_{|H}, \widetilde{\Phi}_{|A_{|H}} \circ i_A, \widetilde{T}_{|\widetilde{E}_{|H}})$. It is not too hard to see that $\widetilde{E}_{|H}$ can be obtained by the same definitions as $\widetilde{E}$ if we just consider bounded continuous functions $\xi : G' \to d^*_{|G'}\mathcal{E}$, where $G' = G_{H^{(0)}}^{H^{(0)}}$. Consider the split-exact sequence coming from the restriction map $\mathsf{res} : \widetilde{E} \to \Gamma_0(H^{(0)}, \mathcal{E}) \cong E; \xi \mapsto \xi_{|H^{(0)}}$. The split is then given by the map i_E and thus $\widetilde{E} = i_E(E) \oplus ker(\mathsf{res})$. Now for $a \in A$ and $\xi \in ker(\mathsf{res}) \subseteq \widetilde{E}$ we have

$$\widetilde{\Phi}(i_A(a))(\xi)(g) = \Phi_{d(g)}(i_A(a)(g))(\xi(g)) = 0,$$

since for $g \in H$ we have $\xi(g) = V_{g^{-1}}(\xi(r(g))) = 0$ and for $g \in G \setminus H$ we have that $i_A(a)(g) = 0$.

On the other hand given $e \in E$, $a \in A$ and $g \in H$ we compute

$$\begin{aligned}(\widetilde{\Phi}(i_A(a))i_E(e))(g) &= \Phi_{d(g)}(i_A(a)(g))(i_E(e)(g)) \\ &= \Phi_{d(g)}(\alpha_{g^{-1}}(a(r(g))))V_{g^{-1}}e(r(g)) \\ &= V_{g^{-1}}\Phi_{r(g)}(a(r(g)))e(r(g)) \\ &= V_{g^{-1}}(\Phi(a)e)(r(g)) \\ &= i_E(\Phi(a)e)(g).\end{aligned}$$

Since both sides are clearly zero for $g \notin H$, we have

$$\widetilde{\Phi}(i_A(a))i_E(e) = i_E(\Phi(a)e).$$

Combining these results we get that under the identification $E \cong i_E(E)$ and for all $a \in A$ we have

$$\widetilde{\Phi}(i_A(a))(e + \xi) = \Phi(a)(e),$$

and thus $\widetilde{\Phi} \circ i_A$ decomposes as $\Phi \oplus 0$ under the decomposition $\widetilde{E} = i_E(E) \oplus ker(\mathsf{res})$. Similar (but even easier) computations yield that $\widetilde{T} = T \oplus \widetilde{T}_{|ker(\mathsf{res})}$. We conclude that

$$\mathsf{comp}_H^G([\widetilde{E}, \widetilde{\Phi}, \widetilde{T}]) = [(\widetilde{E}_{|H}, \widetilde{\Phi}_{|A_{|H}} \circ i_A, \widetilde{T}_{|\widetilde{E}_{|H}})]$$

$$= [(E, \Phi, T)] + \underbrace{[(ker(\mathsf{res}), 0, \widetilde{T}_{|ker(\mathsf{res})})]}_{=0}.$$

This completes the proof of

$$\mathsf{comp}_H^G \circ \mathsf{inf}_H^G = id_{\mathrm{KK}^H(A, B_{|H})}.$$

For the converse we make use of the first paragraph of this proof and pass to the stabilization $A \otimes_{H^{(0)}} K(L^2(G^{H^{(0)}}))$ of A (if necessary) which is Morita-equivalent to A via the imprimitivity bimodule $L^2(G^{H^{(0)}}, A) = L^2(G^{H^{(0)}}) \otimes_{C_0(H^{(0)})} A$. Using the identification $K(L^2(G))_{|H} \cong K(L^2(G^{H^{(0)}}))$, we have a canonical isomorphism

$$Ind_H^X(A \otimes_{H^{(0)}} K(L^2(G^{H^{(0)}}))) \cong (Ind_H^X\, A) \otimes_{G^{(0)}} K(L^2(G))$$

by Lemma 2.4.14. Thus, given a representative (F, Ψ, S) of an element in the group $\mathrm{KK}^G(Ind_H^X\, A, B)$, we may assume that Ψ is essential and S is G-equivariant by Proposition 3.5.8.

Since X^u/H is discrete for every $u \in G^{(0)}$ the characteristic function χ_{gH} is an element in $C_0(X^{r(g)}/H)$. Using these functions we can define a family of pairwise orthogonal projections $\{p_{gH} \mid gH \in X^u/H\}$ on the Hilbert $Ind_H^{X^u} A$-B_u-module F_u by letting

$$p_{gH}(\Psi_u(f)e(u)) = \Psi_u(\chi_{gH} f)e(u).$$

Let us check that this definition is actually continuous in gH or in other words, that $gH \mapsto p_{gH}$ defines an element in $L(\tilde{r}^*(F))$:

For this it is enough to show that for each $\varphi \in C_c(X/H)$, $f \in Ind_H^X A$ and $e \in F$ we have that

$$gH \mapsto P(\varphi \otimes \Psi(f)e)(gH) := \varphi(gH) p_{gH}(\Psi(f)e)$$

is continuous, since elements of the form $\varphi \otimes \Psi(f)e$ are dense in $\tilde{r}^*(F)$.

By density, it is enough to consider $f \in Ind_H^X A$ such that $gH \mapsto \|f(g)\|$ has compact support and using a partition of unity argument, we can assume that this support is actually contained in an open set $U \subseteq X/H$ on which $\tilde{r}$ is injective. But then for any $gH \in U$ we have

$$\chi_{gH} f_{|X^{r(g)}} = f_{|X^{r(g)}}$$

since $f_{|X^{r(g)}}(x) \neq 0$ implies $xH \in X^{r(g)} \cap U$. But of course we have $gH \in X^{r(g)} \cap U$ as well and since $\tilde{r}(xH) = \tilde{r}(gH)$ we must have $gH =$

xH by injectivity of $\tilde{r}_{|U}$. Thus, we have

$$f_{|X^{r(g)}}(x) = \left\{ \begin{array}{ll} f(x) & gH = xH \\ 0 & , else \end{array} \right\} = \chi_{gH} f_{|X^{r(g)}}(x).$$

It follows that $gH \mapsto \chi_{gH} f_{|X^{r(g)}}$ is a compactly supported continuous section of the bundle over X/H associated to $Ind_H^X A$. Consequently, for each $\varphi \in C_c(X/H)$ and $e \in F$ we have that

$$gH \mapsto \varphi(gH) p_{gH}(\Psi_{r(g)}(f)e(r(g))) = \varphi(gH)\Psi_{r(g)}(\chi_{gH} f_{|X^{r(g)}})e(r(g))$$

is a compactly supported continuous section of $\tilde{r}^*(\mathcal{F})$, as desired.

It is not hard to check that the following equality holds

$$V_g p_{sH} = p_{gsH} V_g \ \forall (g,s) \in G^{(2)}. \tag{2}$$

Define an operator S' on F by

$$S'_u := \sum_{gH \in X^u/H} p_{gH} S_u p_{gH}$$

Since for all $e \in F$ and $f \in (Ind_H^X A)_c$ the map

$$gH \mapsto p_{gH} S_{r(g)} p_{gH} \Psi_{r(g)}(f_{|X^{r(g)}})e(r(g)$$

is continuous and compactly supported, integrating against the counting measures on the fibres of X/H yields a well-defined operator $S' \in L(F)$. Using equation (2) from above one easily verifies that S' is still G-equivariant but additionally satisfies the relation $S'_{r(g)} p_{gH} = p_{gH} S'_{r(g)}$ for all $g \in X$. We will show that S' is a compact perturbation of S which allows us two assume that any element in $\mathrm{KK}^G(Ind_H^X A, B)$ can be represented by an essential Kasparov triple with an equivariant operator, which commutes with the families of projections defined above.

One easily checks that

$$((S-S')\Psi(f))_u = \sum_{gH \in X^u/H} (S_u - p_{gH} S_u)\Psi_u(\chi_{gH} f_{|X^u}).$$

Using compactness of $[S, \Psi(\chi_{gH} f_{|X^u})]$ we can see that each summand in the above sum is compact. Then we use our standard argument again that the map $gH \mapsto (S_{r(g)} - p_{gH} S_{r(g)})\Psi_{r(g)}(\chi_{gH} f_{|X^{r(g)}})$ defines a compactly supported continuous section $X/H \to \widetilde{r}^*(\mathcal{K}(F))$ and therefore integration with respect to the system of counting measures on X/H yields a continuous section $G^{(0)} \to \mathcal{K}(F)$, i.e. an element in $K(F)$.

Now let χ_H be the characteristic function of the $\pi(H^{(0)}) \subseteq X/H$. The set $\pi(H^{(0)})$ is clopen since the pre-image under the quotient map is just H, which is clopen in X by assumption. Thus $\chi_H \in C_b(X/H)$. Now define a projection $p_H \in L(F)$ on the dense subset $\Psi(Ind_H^X\, A)F \subseteq F$ by

$$p_H(\Psi(f)e) = \Psi(\chi_H \cdot f)e.$$

Then $(E, \Phi, T) := (p_H F, p_H \Psi p_H, p_H S p_H)$ is a representative of the element $\mathsf{comp}_H^G([F, \Psi, S])$.

Now for $\xi \in \widetilde{E}_c$ and $u \in G^{(0)}$ define an element $\Theta(\xi)$ in F by

$$\Theta(\xi)(u) = \sum_{gH \in X^u/H} V_g(\xi(g)).$$

We want to show that this definition extends to a bounded linear map $\Theta : \widetilde{E} \to F$. For this we need the following: Whenever $e \in p_H F$ and $g \in G \setminus H$ we can use equation 2 to see that

$$(p_H)_{r(g)} V_g(e(d(g)) = 0.$$

If $\xi \in \widetilde{E}_c$ and $g, s \in G^x$ for some $x \in G^{(0)}$ such that $gH \neq sH$, i.e. $s^{-1}g \in G \setminus H$ we have by the above result:

$$\langle V_g(\xi(g)), V_s(\xi(s))\rangle = \langle \underbrace{V_{s^{-1}g}(\xi(g))}_{\in (p_H F)^{\perp}_{d(g)}}, \underbrace{\xi(s)}_{\in (p_H F)_{d(g)}} \rangle = 0$$

Now we are ready to prove that Θ extends to an isometry as follows:

$$\begin{aligned}
\|\Theta(\xi)\|^2 &= \sup_{x \in G^{(0)}} \|\langle \Theta(\xi)(x), \Theta(\xi)(x)\rangle\| \\
&= \sup_{x \in G^{(0)}} \|\sum_{gH} \sum_{sH} \langle V_g(\xi(g)), V_s(\xi(s))\rangle\| \\
&= \sup_{x \in G^{(0)}} \|\sum_{gH} \langle V_g(\xi(g)), V_g(\xi(g))\rangle\| \\
&= \sup_{x \in G^{(0)}} \|\sum_{gH} \beta_g(\langle \xi(g), \xi(g)\rangle)\| \\
&= \|\xi\|^2
\end{aligned}$$

Let us also check that Θ is G-equivariant:

$$V_s(\Theta_{d(s)}(\xi)(d(s))) = \sum_{gH \in G^{d(s)}_{H^{(0)}}/H} V_{sg}(\xi(g))$$

$$\begin{aligned}
&= \sum_{gH \in G^{r(s)}_{H^{(0)}}/H} V_g(\xi(s^{-1}g)) \qquad (gH \mapsto s^{-1}gH) \\
&= \sum_{gH \in G^{r(s)}_{H^{(0)}}/H} V_g(\widetilde{V}_s(\xi)(g)) \\
&= (\Theta_{r(s)}(\widetilde{V}_s(\xi)))(r(s))
\end{aligned}$$

Similarly, we can show that Θ intertwines $\widetilde{\Phi}$ with Ψ and $\widetilde{T}$ with S. Now if $e \in F$ is arbitrary we can define $\xi \in \widetilde{E}$ by letting

$$\xi(g) = (p_H)_{d(g)} V_{g^{-1}} \cdot e(r(g)).$$

Then we can compute $\Theta(\xi)(x) = \sum_{gH} V_g((p_H)_{d(g)} V_{g^{-1}} \cdot e(x)) = e(x)$. This completes the proof that

$$\begin{aligned}
\mathsf{inf}_H^G(\mathsf{comp}_H^G([F, \Psi, S])) &= \mathsf{inf}_H^G([E, \Phi, T]) \\
&= [\widetilde{E}, \widetilde{\Phi}, \widetilde{T}] = [F, \Psi, S].
\end{aligned}$$

□

In the next chapter, we shall also need the following compatibility property of the compression homomorphism with respect to taking right Kasparov products:

LEMMA 3.6.8. *Let G be a second countable étale groupoid, $H \subseteq G$ a proper open subgroupoid and let $X := G_{H^{(0)}}$. Let A be an H-algebra and let B and B' be two G algebras. Then, for every $x \in \mathrm{KK}^G(B, B')$ we have a commutative diagram:*

$$\begin{array}{ccc}
\mathrm{KK}^G(Ind_H^X A, B) & \xrightarrow{\cdot \otimes x} & \mathrm{KK}^G(Ind_H^X A, B') \\
\Big\downarrow \mathsf{comp}_H^G & & \Big\downarrow \mathsf{comp}_H^G \\
\mathrm{KK}^H(A, B_{|H}) & \xrightarrow{\cdot \otimes res_H^G(x)} & \mathrm{KK}^H(A, B'_{|H})
\end{array}$$

PROOF. Using the definition of the compression homomorphism, it is enough to prove, that the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{KK}^G(Ind_H^X A, B) & \xrightarrow{res_H^G} & \mathrm{KK}^H(Ind_H^{G'} A, B_{|H}) & \xrightarrow{i_A^*} & \mathrm{KK}^H(A, B_{|H}) \\
\Big\downarrow \cdot \otimes x & & \Big\downarrow \cdot \otimes res_H^G(x) & & \Big\downarrow \cdot \otimes res_H^G(x) \\
\mathrm{KK}^G(Ind_H^X A, B') & \xrightarrow{res_H^G} & \mathrm{KK}^H(Ind_H^{G'} A, B'_{|H}) & \xrightarrow{i_A^*} & \mathrm{KK}^H(A, B'_{|H})
\end{array}$$

Commutativity of the diagram on the right follows from the associativity of the Kasparov product. Using the fact that the map res_H^G is given by pulling back along the inclusion map $\iota : H \hookrightarrow G$, commutativity of the left diagram follows from [**LG94**, Proposition 6.1.3]. □

CHAPTER 4

The Going-Down Principle

In this chapter we state and prove the Going-Down (or restriction) principle for ample groupoids. After reminding the reader about universal spaces for proper actions of groupoids and the formulation of the Baum-Connes conjecture, we first prove a special case of the restriction principle (see Theorem 4.3.7), that can be applied directly in many cases. We then extend the formalism of Going-Down functors as in [**CEOO04**] to our setting and state the main results in full generality.

4.1. Universal Spaces for Proper Actions

Recall the following definition:

Definition 4.1.1. Let G be a locally compact Hausdorff groupoid. A proper G-space Z is called a *universal proper G-space*, if for every proper G-space X there exists a continuous G-equivariant map $\varphi : X \to Z$ which is unique up to G-equivariant homotopy.

Note that a universal proper G-space Z as in the definition above is unique up to G-equivariant homotopy equivalence. A priori it is not clear that a universal proper G-space always exists. Let us elaborate on the existence:

In what follows let G be an étale groupoid and Z be a proper G-space. Let $M(Z)$ denote the space of all finite, positive Radon measures μ on Z with total mass contained in $(\frac{1}{2}, 1]$, such that there exists an element $u \in G^{(0)}$ with $supp(\mu) \subseteq Z_u$. Via the Riezs-Representation Theorem we can identify $M(Z)$ with a subset of the positive linear functionals on $C_c(Z)$, and thus endow it with the weak-$*$-topology. More precisely, a Radon measure μ induces a positive linear map $I_\mu : C_c(Z) \to \mathbb{C}$ given by

$$I_\mu(\varphi) = \int_Z \varphi d\mu.$$

Conversely, every positve linear map I on $C_c(Z)$ induces a unique measure μ such that $I(\varphi) = \int_Z \varphi d\mu$. Furthermore, continuity of I with respect to the supremumnorm on $C_c(Z)$ is equivalent to finiteness of the measure μ that represents I and in that case we have $\mu(Z) = \|I\|$. Thus, the set of finite Radon measures on Z can be identified with $C_c(Z)'$ and we can carry over the weak-$*$-topology of $C_c(Z)'$ to the set of finite Radon measures, which is uniquely determined by the fact that a net $(\mu_i)_i$ of finite Radon measures converges to μ if and only if

$$\int_Z \varphi d\mu_i \to \int_Z \varphi d\mu \quad \forall \varphi \in C_c(Z).$$

The condition, that for every $\mu \in M(Z)$ there exists a $u \in G^{(0)}$ such that $supp(\mu) \subseteq Z_u$ allows us to define a canonical map $M(Z) \to G^{(0)}$, which serves as the anchor map of a canonical action of G on $M(Z)$, given by translation. The following result can be found in [**Tu99b**, Proposition 6.13, Lemma 6.14].

Proposition 4.1.2. *Let Z be a locally compact metrizable space. Then $M(Z)$ is locally compact. Moreover, if G acts properly on Z, then the induced action of G on $M(Z)$ is proper as well.*

We have the following proposition due to Tu:

Proposition 4.1.3. [**Tu99a**, Proposition 11.4] *If Z is a proper G-space, such that the anchor map $p : Z \to G^{(0)}$ is open, then $M(Z)$ is a universal proper G-space.*

Corollary 4.1.4. *Every second countable étale groupoid G admits a locally compact universal proper G-space.*

Proof. Apply Proposition 4.1.3 to the canonical action of G on itself by left multiplication (which is always proper by Lemma 1.2.12), to see that $M(G)$ is a universal proper G-space. □

Example 4.1.5. If G is a proper groupoid, then $G^{(0)}$ is a universal proper G-space. Clearly, it is a proper G-space. If X is any other proper G-space, then the anchor map p of the action is the desired map $X \to G^{(0)}$. Moreover, G-equivariance implies, that p is actually unique: If $p' : X \to G^{(0)}$ is another G-equivariant map, then $p'(x) = p'(p(x)x) = p(x)p'(x) = p(x)$ for all $x \in X$.

4.2. Topological K-theory and the Baum-Connes Assembly Map

In this section we want to recall the definition of the topological K-theory of a groupoid. We also use the opportunity to remind the reader of the definition of the Baum-Connes assembly map. Recall, that a G-space X is called *G-compact (or cocompact)* if there exists a compact subset $K \subseteq X$, such that $X = GK$. We need the following elementary fact:

Lemma 4.2.1. *Let G be a locally compact Hausdorff groupoid. Furthermore, let X be a G-compact G-space and Y be a proper G-space. Then every G-equivariant continuous map $\varphi : X \to Y$ is automatically proper.*

Proof. Let $K \subseteq Y$ be a compact subset. Our goal is to show that $\varphi^{-1}(K)$ is compact. To this end let $(x_\lambda)_\lambda$ be a net in $\varphi^{-1}(K)$. We claim that $(x_\lambda)_\lambda$ has a convergent subnet. Since K is compact, we can pass to a subnet to assume that $\varphi(x_\lambda) \to y$ for some $y \in K \subseteq Y$. Next, we use the G-compactness of X to find a compact subset $C \subseteq X$ such that $X = GC$. Hence, we may write $x_\lambda = g_\lambda c_\lambda$ for some $g_\lambda \in G$ and $c_\lambda \in C$. Passing to a subnet again, we may assume that c_λ converges to some element $c \in C$ (using compactness of C). Using the continuity of φ we have $\varphi(c_\lambda) \to \varphi(c)$. Since φ is G-equivariant we also get $g_\lambda \varphi(c_\lambda) = \varphi(x_\lambda) \to y$. Now we can use properness of Y (see Proposition 1.2.8 (4)) to pass to yet another subnet and relabel, allowing us to assume that $g_\lambda \to g$ for some $g \in G$. But then we have $x_\lambda = g_\lambda c_\lambda \to gc$ proving our claim. □

Let $\mathcal{E}(G)$ denote a universal proper G-space. Then, applying the above lemma, for any two G-compact subsets $X_1 \subseteq X_2 \subseteq \mathcal{E}(G)$ we have a canonical $*$-homomorphism $C_0(X_2) \to C_0(X_1)$ given by restriction. This homomorphism in turn induces a map

$$\mathrm{KK}^G(C_0(X_1), A) \to \mathrm{KK}^G(C_0(X_2), A)$$

for every G-algebra A. Thus, the following definition makes sense:

Definition 4.2.2. Let G be an étale, second countable Hausdorff groupoid and A be a G-algebra. The *topological K-theory of G with* *coefficients in* A is defined as

$$\mathrm{K}_*^{\mathrm{top}}(G;A) := \varinjlim \mathrm{KK}_*^G(C_0(X),A),$$

where the direct limit is taken over all G-comapct, locally compact and second countable subsets $X \subseteq \mathcal{E}(G)$.

Remark 4.2.3. To justify why one considers the group $\mathrm{K}_*^{\mathrm{top}}(G;A)$ instead of simply looking at $\mathrm{KK}^G(C_0(\mathcal{E}(G)),A)$, it makes sense to generalize the above definition to arbitrary proper G-spaces: If Y is any proper G-space, let

$$\mathrm{RK}_*^G(Y;A) = \varinjlim \mathrm{KK}_*^G(C_0(X),A),$$

where again, the limit is taken over all G-comapct, locally compact and second countable subsets $X \subseteq Y$. The main point in taking the limit is, that $\mathrm{KK}^G(C_0(\cdot),A)$ is only functorial for proper G-equivariant maps $X \to X'$, whereas $\mathrm{RK}_*^G(\cdot;A)$ is functorial for arbitrary continuous G-equivariant maps.

Next, we want to define the Baum-Connes assembly map. We shall need the following well-known result. Since we could not find an explicit reference for the general groupoid case, we include a proof for completeness. Recall, that every proper étale groupoid G admits a cutoff function in the sense of Definition 3.5.1 by Proposition 3.5.2.

Lemma 4.2.4. *Let G be a proper étale groupoid with compact orbit space $G \setminus G^{(0)}$ and let $c: G^{(0)} \to \mathbb{R}^+$ be a compactly supported cutoff function for G. Then the function $p_c: G \to \mathbb{C}$, $g \mapsto \sqrt{c(d(g))c(r(g))}$ defines a projection in $C_r^*(G)$. Moreover the class $[p_c] \in \mathrm{K}_0(C_r^*(G)) = \mathrm{KK}(\mathbb{C}, C_r^*(G))$ does not depend on the choice of the cutoff function c.*

Proof. One easily checks that

$$supp(p_c) \subseteq supp(c \circ d) \cap r^{-1}(supp(c)).$$

It follows that p_c is compactly supported by part (2) of the definition of cutoff functions. Consequently, we can view p_c as an element of $C_r^*(G)$. Since p_c only takes real values we have $p_c^*(g) = p_c(g^{-1}) = p_c(g)$ for all $g \in G$. Let us check that p_c is an idempotent: For all $g \in G$ we have

$$p_c * p_c(g) = \sum_{h \in G^{r(g)}} p_c(h)p_c(h^{-1}g)$$

$$= \sum_{h \in G^{r(g)}} \sqrt{c(d(h))c(r(g))}\sqrt{c(d(g))c(d(h))}$$
$$= \sqrt{c(d(g))c(r(g))} \sum_{h \in G^{r(g)}} c(d(h)) = p_c(g).$$

Thus, p_c is a projection in $C_r^*(G)$. It remains to show, that the class of p_c in $\mathrm{K}_0(C_r^*(G))$ is independent of the choice of cutoff function. But if c' is another compactly supported cutoff function for G, then

$$c_t := \sqrt{tc^2 + (1-t)c'^2}$$

defines a continuous path of cutoff functions from c' to c. Thence p_{c_t} defines a continuous path of projections in $C_r^*(G)$ from $p_{c'}$ to p_c. $\square$

We are now in the position to define the Baum-Connes assembly map: Let A be a G-algebra. For every G-compact subspace $X \subseteq \mathcal{E}(G)$ we can consider the composition

$$\mu_X : \mathrm{KK}_*^G(C_0(X), A) \overset{j_G}{\to} \mathrm{KK}_*(C_r^*(G \ltimes X), A \rtimes_r G) \overset{[p_c]\otimes \cdot}{\to} \mathrm{KK}_*(\mathbb{C}, A \rtimes_r G)$$

where j_G is the descent homomorphism defined in Proposition 3.4.11. Note, that we also used the identification $C_0(X) \rtimes_r G \cong C_r^*(G \ltimes X)$. One easily checks, that the maps μ_X give rise to a well-defined homomorphism

$$\mu_A : \mathrm{K}_*^{\mathrm{top}}(G; A) \to \mathrm{KK}_*(\mathbb{C}, A \rtimes_r G) = \mathrm{K}_*(A \rtimes_r G).$$

This is the *Baum-Connes assembly map* for G with coefficients in A.

4.3. The Going-Down Principle

Let $P(G)$ denote the subset of all probability measures in $M(G)$. Recall, that for a measure $\mu \in M(G)$ the support of μ is defined as

$$supp(\mu) = \{g \in G \mid \mu(U) > 0 \text{ for each open neighbourhood } U \text{ of } g\}.$$

Since we are working with the weak-*-topology, a description in terms of continuous functions with compact support would be much more convenient. Such a description is given by the following lemma.

Lemma 4.3.1. *For $\mu \in M(G)$ and $g \in G$ we have that $g \in supp(\mu)$ if and only if $I_\mu(\varphi) > 0$ for each $\varphi \in C_c^+(G)$ such that $\varphi(g) > 0$.*

Proof. Let $g \in supp(\mu)$ and $\varphi \in C_c^+(G)$ such that $\varphi(g) > 0$. Find a $\varphi(g) > \varepsilon > 0$. Since φ is continuous we can find a neighbourhood U

of g such that $\varphi(h) > \varepsilon$ for all $h \in U$. If we define $c := \frac{1}{2}\inf\{\varphi(x) \mid x \in U\} > 0$ then $c\chi_U \leq \varphi$ and thus $0 < c\mu(U) = \int_G c\chi_U d\mu \leq I_\mu(\varphi)$.

For the converse let $U \subseteq G$ be an open neighbourhood of an element $g \in G$. Pick a function $\varphi \in C_c^+(G)$ with $0 \leq \varphi \leq 1$, $\varphi(g) = 1$ and $supp(\varphi) \subseteq U$. Then $\mu(U) = \int_G \chi_U d\mu \geq I_\mu(\varphi) > 0$. □

Let $P(G)$ denote the probability measures on G and for each $K \subseteq G$ compact define

$$P_K(G) = \{\mu \in P(G) \mid \forall g, h \in supp(\mu) : \ r(g) = r(h) \ and \ g^{-1}h \in K\}.$$

Note that there is a canonical left action of G on $P_K(G)$ with respect to the anchor map $P_K(G) \to G^{(0)}$, $\mu \mapsto r(g)$ for any $g \in supp(\mu)$, given by translation. It was shown in [**Tu12**, Proposition 3.1] that $P_K(G)$ is a locally compact, G-compact, proper G-space. Furthermore, if X is any G-compact proper G-space, there exists a compact subset $K \subseteq G$ and a G-equivariant map $X \to P_K(G)$ (see [**Tu12**, Proposition 3.2]). If G is ample we can always choose the set K to be compact and open, since if $K_1 \subseteq K_2$ then obviously $P_{K_1}(G) \subseteq P_{K_2}(G)$ and if K is any compact set it is contained in a compact open set. In the following, we will show that in this case the spaces $P_K(G)$ are geometric realizations of G-simplicial complexes in the following sense (compare [**Tu99b**, Definition 3.1]):

Definition 4.3.2. Let G be an ample groupoid and $n \in \mathbb{N}$. A *G-simplicial complex* of dimension at most n is a pair (X, Δ) consisting of a locally compact G-space X (the set of vertices) and a collection Δ of finite, non-empty subsets of X (called simplices) with at most $n+1$ elements such that:

1. the anchor map $p : X \to G^{(0)}$ has the property, that for every $x \in X$ there exists a compact open neighbourhood $U \subseteq X$ such that $p|_U : U \to p(U)$ is a homeomorphism onto a compact open subset of $G^{(0)}$.
2. for each $\sigma \in \Delta$ we have $\sigma \subseteq p^{-1}(u)$ for some $u \in G^{(0)}$,
3. if $\sigma \in \Delta$, then every non-empty subset of σ is also an element of Δ, and
4. for each $\sigma \in \Delta$, say $\sigma = \{x_1, \ldots, x_n\} \subseteq X_u$, there exists a compact open neighbourhood V of u in $G^{(0)}$ and continuous sections $s_1, \ldots s_n : V \to X$ of p such that $\{s_1(v), \ldots s_n(v)\} \in \Delta$ for all $v \in V$ and $\{s_1(u), \ldots, s_n(u)\} = \sigma$.

The G-simplicial complex is *typed* if there is a discrete set $\mathcal{T}$ and a G-invariant continuous map $X \to \mathcal{T}$ whose restriction to the support of a single simplex in Δ is injective.

The *geometric realization* of a G-simplicial complex (X, Δ) is the set

$$|\Delta| = \{\mu \in P(X) \mid supp(\mu) \in \Delta\}$$

equipped with the weak-$*$-topology. The geometric realization $|\Delta|$ will always be a locally compact space and the action of G on $|\Delta|$, induced by the acion of G on X, is proper if X is a proper G-space.

REMARK 4.3.3. If $\sigma \in \Delta$, say $\sigma = \{x_1, \ldots, x_n\} \subseteq X_u$ as in item (4) above and for each $1 \leq i \leq n$ U_i is a compact open neighourhood of x_i such that the U_i are pairwise disjoint and $p_{|U_i}$ is a homeomorphism onto its image, then we may always assume that the section s_i only takes images in U_i. If not, pass from the domain V of the s_i to

$$\widetilde{V} = V \cap \bigcap_{0 \leq i \leq n} s_i^{-1}(U_i).$$

Note that the realization of a 0-dimensional complex (X, Δ) can be canonically identified with a subset of X. Using the existence of local sections as in item (4) we can show that Δ is actually open in X: Let $x \in \Delta$ be given and U in X be an open neighbourhood of x such that $p_{|U}$ is a homeomorphism onto its image. Furthermore let V be a neighbourhood of $p(x)$ and $s : V \to X$ be a section as in (4). By the above remark we may assume $s(V) \subseteq U$. Then $p^{-1}(V) \cap U$ is an open neighbourhood of x and since $p^{-1}(V) \cap U = s(V \cap p(U))$, it is contained in Δ.

Thus, if we restrict p to the subset Δ, it still has the property, that every point $x \in \Delta$ has a compact open neighbourhood U such that $p_{|U} : U \to p(U)$ is a homeomorphism onto a compact open subset of $G^{(0)}$.

LEMMA 4.3.4. *Let G be an ample groupoid and K be a compact open subset of G. If we define*

$$\Delta_K(G) = \{\sigma \subseteq G \mid \forall g, h \in \sigma : \ r(g) = r(h) \ and \ g^{-1}h \in K\}$$

then $(G, \Delta_K(G))$ is a G-simplicial complex in the sense of Definition 4.3.2 and $P_K(G)$ is its geometric realization. We note that $\Delta_K(G)$ has finite dimension (as a G-simplicial complex).

Proof. We consider the action of G on itself by left multiplication. Hence the anchor map is just the range map of G. Since G is ample, condition (1) of Definition 4.3.2 clearly holds. As axioms (2) and (3) are built into the definition of $\Delta_K(G)$, it remains to prove (4): So let $\sigma = \{g_1, \dots, g_n\} \in \Delta_K(G)$ be given and let $u := r(g_1) = \dots = r(g_n)$. Let $\widetilde{U}_i$ be a compact open neighbourhood of g_i such that $r_{|\widetilde{U}_i} : \widetilde{U}_i \to r(\widetilde{U}_i)$ is a homeomorphism. We would like to take the inverses of these maps on $\bigcap_{i=1}^n r(\widetilde{U}_i)$ as our sections but we need to make sure that images of a point form a simplex again. Thus, we use the continuity of the multiplication and the openness of K to shrink the $\widetilde{U}_i$ appropriately. To be more precise: Consider the continuous map

$$f : G \ltimes G \to G$$

given by $(g, h) \mapsto g^{-1}h$. As K is open and f is continuous, $f^{-1}(K)$ is open. Thus, for all $1 \leq i, j \leq n$ we can find compact open neighbourhoods $U_{i,j}$ of g_i and $V_{j,i}$ of g_j such that $(U_{i,j} \times V_{j,i}) \cap G \ltimes G \subseteq f^{-1}(K)$. Let

$$U_i := \widetilde{U}_i \cap \bigcap_{1 \leq j \leq n} U_{i,j} \cap V_{i,j}.$$

Then each U_i is a compact open neighbourhood of g_i. Let $V := \bigcap r(U_i)$ and define $s_i : V \to U_i \subseteq G$ to be the inverse of the range map restricted to U_i. These are continuous sections by definition and for each $v \in V$ and $1 \leq l, k \leq n$ we have $s_l(v) \in U_{l,k}$ and $s_k(v) \in V_{k,l}$ and thus $s_i(v)^{-1}s_j(v) = f(s_i(v), s_j(v)) \in K$ by construction. Consequently, we get $\{s_1(v), \dots s_n(v)\} \in \Delta_K(G)$ for all $v \in V$.

Let us finally show that $\Delta_K(G)$ has finite dimension. It is not hard to see that $\Delta_K(G) = G \cdot \{\sigma \in \Delta_K(G) \mid \sigma \subseteq K\}$ and since translating a $\sigma \in \Delta_K(G)$ does not increase its cardinality it is enough to show that the cardinalities of elements of $\{\sigma \in \Delta_K(G) \mid \sigma \subseteq K\}$ are bounded. But for such a $\sigma \subseteq G^u$ we have $|\sigma| \leq |K \cap G^u| = \lambda^u(K) \leq \sup\{\lambda^u(K) \mid u \in G^{(0)}\} < \infty$ by Lemma 1.1.16, where λ denotes the Haar system given by the counting measure on each fibre. $\square$

The arguments in [**Tu99b**, Section 3.2] carry over to the G-equivariant setting and show that the barycentric subdivision of a G-simplicial complex (X, Δ) is a typed G-simplicial complex whose geometric realization is G-equivariantly homeomorphic to the original one. However for the sake completeness let us at least recall the construction of the barycentric subdivision and show that it is a G-simplicial complex again.

Definition 4.3.5. Let (X, Δ) be G-simplicial complex. For $\mu \in |\Delta|$ with $supp(\mu) = \{x_1, \dots, x_n\}$ let

$$bc(\mu) = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}$$

denote the *isobarycenter* of the simplex $supp(\mu) \in \Delta$. Let $X' = \{bc(\mu) \mid \mu \in |\Delta|\}$ and define Δ' such that a set $\{\nu_1, \dots, \nu_l\}$ is in Δ' if and only if $\bigcup_{0 \leq j \leq l} supp(\nu_j) \in \Delta$.

Proposition 4.3.6. *The pair (X', Δ') is a G-simplicial complex.*

Proof. We will only show that $p' : X' \to G^{(0)}$ satisfies property (1) from the definition. The other properties follow easily from the construction. Let $\mu \in X'$, say $\mu = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}$ for $x_1, \dots, x_n \in X$ and let U_i be a compact open neighbourhood of x_i such that $p_{|U_i}$ is a homeomorphism onto its image. Since G is Hausdorff we can assume that the U_i are pairwise disjoint. Now from condition (4) of the definition we get continuous sections $s_1, \dots s_n : V \to X$, where V is a compact open neighbourhood of $u := p'(\mu)$. Following Remark 4.3.3 we can assume that $s_i(V) \subseteq U_i$. Consider the sets

$$W_i := \{\nu \in X' \mid supp(\nu) \cap U_i \neq \emptyset\}.$$

Note that the intersection $supp(\nu) \cap U_i$ will contain at most one element, since $supp(\nu)$ is contained in one fibre and U_i is the domain of a local homeomorphism. It follows from Lemma 4.3.1 that W_i is open. Now let

$$W = p'^{-1}(V \cap \bigcap_i p(U_i)) \cap \bigcap_i W_i.$$

It is now easy to see that $p'(W) = V \cap \bigcap_i p(U_i)$ and thus $p'(W)$ is compact and open. Furthermore, the map $p'(W) \to W$ sending an element v to the measure $\frac{1}{n}\sum_{i=1}^{n} \delta_{s_i(v)}$ is a continuous inverse of p'. Hence also W is compact and p' satisfies property (1) from the definition of a G-simplicial complex. □

Let us now proceed to prove one of the main results of this thesis:

THEOREM 4.3.7. *Let G be an ample, second countable, locally compact Hausdorff groupoid and let A and B be separable G-algebras. Suppose there is an element $x \in \mathrm{KK}^G(A, B)$ such that*

$$\mathrm{KK}^H(C(H^{(0)}), A_{|H}) \overset{\cdot\otimes res_H^G(x)}{\to} \mathrm{KK}^H(C(H^{(0)}), B_{|H})$$

is an isomorphism for all compact open subgroupoids $H \subseteq G$. Then the Kasparov-product with x induces an isomorphism

$$\cdot \otimes x : \mathrm{K}_*^{\mathrm{top}}(G; A) \to \mathrm{K}_*^{\mathrm{top}}(G; B).$$

To show the above theorem we will show that for every G-compact subset $X \subseteq \mathcal{E}(G)$ the map

$$\cdot \otimes x : \mathrm{KK}^G(C_0(X), A) \to \mathrm{KK}^G(C_0(X), B)$$

is an isomorphism. Let us first consider the following special case:

PROPOSITION 4.3.8. *Under the assumptions of Theorem 4.3.7 the map*

$$\cdot \otimes x : \mathrm{KK}^G(C_0(X), A) \to \mathrm{KK}^G(C_0(X), B)$$

is an isomorphism for every G-compact proper G-space X whose anchor map $p : X \to G^{(0)}$ has the property, that for every $x \in X$ there exists a compact open neighbourhood U of x in X such that $p_{|U} : U \to p(U)$ is a homeomorphism onto a compact open subset of $G^{(0)}$.

PROOF. Let us first consider the case that X is the orbit of a single compact open subset U such that $p(U)$ is compact and open in $G^{(0)}$ and $p_{|U} : U \to p(U)$ is a homeomorphism, i.e. $X = GU$. Consider the set

$$H = \{g \in G \mid gU \cap U \neq \emptyset\}.$$

Using the fact that $p_{|U}$ is a homeomorphism onto $p(U)$ it is not hard to see, that H is a subgroupoid of G and as such isomorphic to $(G \ltimes X)_U^U$ (the isomorphism $(G \ltimes X)_U^U \to H$ is given by the projection onto the first factor). Since $G \ltimes X$ is proper, the restriction $(G \ltimes X)_U^U$ to U is compact. Clearly, the latter is also open in $G \ltimes X$. Since the anchor map $p : X \to G^{(0)}$ is open, we can deduce that the first projection $pr_1 : G \ltimes X \to G$ is open. Thus, H is a compact open subgroupoid of G. We also have a canonical G-equivariant homeomorphism $G \times_H U \cong GU = X$ and thus an equivariant isomorphism

$$Ind_H^G C(U) \cong C_0(G \times_H U) \cong C_0(X)$$

by Proposition 2.4.13. Using this we can consider the following diagram, which commutes by Lemma 3.6.8.

$$\begin{array}{ccc} \mathrm{KK}^G(C_0(X), A) & \xrightarrow{\cdot \otimes x} & \mathrm{KK}^G(C_0(X), B) \\ \downarrow \mathsf{comp}_H^G & & \downarrow \mathsf{comp}_H^G \\ \mathrm{KK}^H(C(U), A_{|H}) & \xrightarrow{\cdot \otimes res_H^G(x)} & \mathrm{KK}^H(C(U), B_{|H}) \end{array}$$

Since we have an isomorphism $C(U) \cong C(H^{(0)})$, the bottom line in this diagram is an isomorphism. By Theorem 4.3.7 the homomorphism comp_H^G is an isomorphism as well and hence the result follows in this case.

Let us now consider the general case. As X is G-compact it admits a finite cover of the form

$$X = \bigcup_{i=1}^{n} GU_i,$$

where $U_i \subseteq X$ is compact open such that $p_{|U_i}$ is a homeomorphism onto its image. Let us first consider the case $n = 2$. By Mayer-Vietoris we have a commutative diagram with exact columns, where the horizontal maps are all given by taking Kasparov product with x and we write $\mathrm{KK}_*^G(X, A)$ for $\mathrm{KK}_*^G(C_0(X), A)$ for brevity:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathrm{KK}^G_*(X,A) & \longrightarrow & \mathrm{KK}^G_*(X,B) \\
\downarrow & & \downarrow \\
\mathrm{KK}^G_*(GU_1,A)\oplus \mathrm{KK}^G_*(GU_2,A) & \longrightarrow & \mathrm{KK}^G_*(GU_1,B)\oplus \mathrm{KK}^G_*(GU_2,B) \\
\downarrow & & \downarrow \\
\mathrm{KK}^G_*(GU_1\cap GU_2,A) & \longrightarrow & \mathrm{KK}^G_*(GU_1\cap GU_2,B) \\
\downarrow & & \downarrow \\
\mathrm{KK}^G_{*+1}(X,A) & \longrightarrow & \mathrm{KK}^G_{*+1}(X,B) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

Using the first step of this proof we already know, that the second horizontal map is an isomorphism. Consider the set $V = U_1 \cap GU_2$. It is clearly open and using properness of the action one easily verifies that is is also closed (apply Proposition 1.2.8 (4)). Since $V \subseteq U_1$ we have that $p_{|V}$ is also a homeomorphism onto its image. One easily checks that $GV = GU_1 \cap GU_2$. Thus, the third horizontal map is also an isomorphism. Hence the result follows by an application of the Five-Lemma.

If $n > 2$ is arbitrary, use induction and the above Mayer-Vietoris argument on the decomposition $X = GU_1 \cup \bigcup_{i=2}^{n} GU_i$ to complete the proof. □

We are now ready for the proof of Theorem 4.3.7:

Proof of Theorem 4.3.7. As mentioned before, it is enough to show that

$$\cdot \otimes x : \mathrm{KK}^G(C_0(X), A) \to \mathrm{KK}^G(C_0(X), B)$$

is an isomorphism for every G-compact subset $X \subseteq \mathcal{E}(G)$. Our proof consists of a two step reduction, each of which tells us that we can use "more special" spaces X. In the first step we will use the spaces $P_K(G)$ from the beginning of this section: Let X_1 be any G-compact subspace of $\mathcal{E}(G)$. Then X_1 is a proper G-space itself and thus we can find a compact open subset $K_1 \subseteq G$ and a G-equivariant map $\varphi_1 : X_1 \to P_{K_1}(G)$ by the discussion in the beginning of this section. Using the universal property of $\mathcal{E}(G)$ there is also a G-equivariant map $\psi_1 : P_{K_1}(G) \to \mathcal{E}(G)$. Let $X_2 := \psi_1(P_{K_1}(G))$. Then X_2 is a G-compact subspace of $\mathcal{E}(G)$. Now proceed as above to find G-compact subspaces $X_3, X_4, \ldots$.

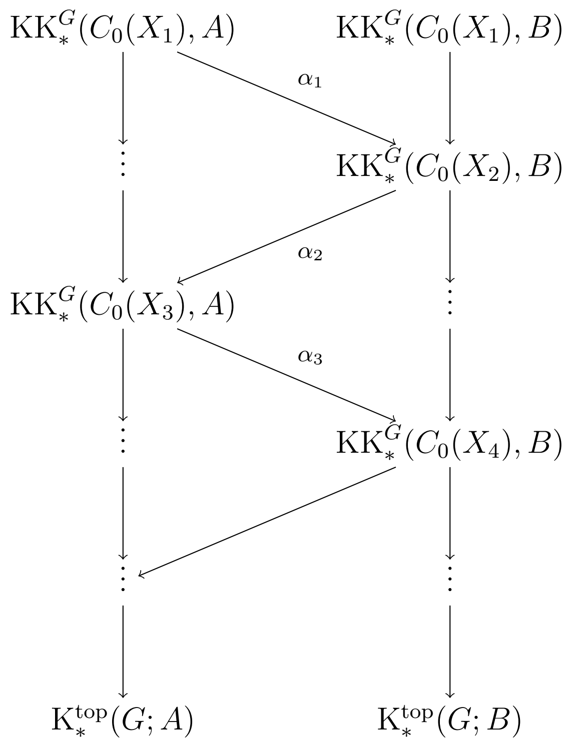
Suppose now that

$$\cdot \otimes x : \mathrm{KK}^G(C_0(P_K(G)), A) \to \mathrm{KK}^G(C_0(P_K(G)), B)$$

is an isomorphism for each compact open set $K \subseteq G$. Since the Kasparov-product is natural, we get a commutative diagram, where all the horizontal arrows are given by taking Kasparov-product with x and the vertical arrows are the maps found by the above arguments.

$$\begin{array}{ccc}
\mathrm{KK}^G_*(C_0(X_1), A) & \longrightarrow & \mathrm{KK}^G_*(C_0(X_1), B) \\
\downarrow & & \downarrow \\
\mathrm{KK}^G_*(C_0(P_{K_1}(G)), A) & \xrightarrow{\cong} & \mathrm{KK}^G_*(C_0(P_{K_1}(G)), B) \\
\downarrow & & \downarrow \\
\mathrm{KK}^G_*(C_0(X_2), A) & \longrightarrow & \mathrm{KK}^G_*(C_0(X_2), B) \\
\downarrow & & \downarrow \\
\mathrm{KK}^G_*(C_0(P_{K_2}(G)), A) & \xrightarrow{\cong} & \mathrm{KK}^G_*(C_0(P_{K_2}(G)), B) \\
\downarrow & & \downarrow \\
\mathrm{KK}^G_*(C_0(X_3), A) & \longrightarrow & \mathrm{KK}^G_*(C_0(X_3), B) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

By going 'zick-zack' in this diagram we get the following diagram:



Whenever we have such a diagram, the inductive limits must be isomorphic, such that the isomorphism commutes with the diagram (i.e. it is exactly the morphism induced by taking Kasparov-product in each step). Consequently, it is enough to show that

$$\cdot \otimes x : \mathrm{KK}^G(C_0(P_K(G)), A) \to \mathrm{KK}^G(C_0(P_K(G)), B)$$

is an isomorphism for each compact open set $K \subseteq G$. Since each $P_K(G)$ is (the geometric realization of) a proper, G-compact finite dimensional G-simplicial complex and its barycentric subdivision is G-equivariantly homeomorphic to it, it is sufficient to show that

$$\cdot \otimes x : \mathrm{KK}_*^G(C_0(X), A) \to \mathrm{KK}_*^G(C_0(X), B)$$

is an isomorphism for every typed, proper, G-compact G-simplcial complex X of finite dimension.

In the second step we will use an induction argument on the dimension n of X to reduce the problem to the zero dimensional case. If X is (the geometric realization) of a 0-dimensional complex it follows from the discussion after Remark 4.3.3, that the anchor map $X \to G^{(0)}$

has the property, that every point in X has a compact open neighbourhood, such that the anchor map restricts to a homeomorphism onto its image on that neighbourhood. Consequently, Proposition 4.3.8 tells us that $\cdot \otimes x : \mathrm{KK}_*^G(C_0(X), A) \to \mathrm{KK}_*^G(C_0(X), B)$ is an isomorphism.

Now let X be a G-simplicial complex of dimension $n > 0$, Y be its $n-1$-skeleton, and $U = X \setminus Y$ the union of all open n-simplices. Then we get a G-equivariant exact sequence

$$0 \longrightarrow C_0(U) \longrightarrow C_0(X) \longrightarrow C_0(Y) \longrightarrow 0.$$

As Y is clearly G-invariant, [**Tu12**, Lemma 3.9] yields the following commutative diagram with exact columns:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathrm{KK}_*^G(C_0(Y), A) & \overset{\cdot \otimes x}{\longrightarrow} & \mathrm{KK}_*^G(C_0(Y), B) \\
\downarrow & & \downarrow \\
\mathrm{KK}_*^G(C_0(X), A) & \overset{\cdot \otimes x}{\longrightarrow} & \mathrm{KK}_*^G(C_0(X), B) \\
\downarrow & & \downarrow \\
\mathrm{KK}_*^G(C_0(U), A) & \overset{\cdot \otimes x}{\longrightarrow} & \mathrm{KK}_*^G(C_0(U), B) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

If we assume inductively, that the upper horizontal map is an isomorphism we only need to show that the lower map is also an isomorphism to invoke the Five-Lemma and conclude the result. But U is equivariantly homeomorphic to $X' \times \mathbb{R}^n$, where X' denotes the barycenters of n-dimensional simplices. Thus, we have $\mathrm{KK}_*^G(C_0(U), A) \cong \mathrm{KK}_{*+n}^G(C_0(X'), A)$. Since taking suspension is compatible with the Kasparov product, it is enough to show that

$$\cdot \otimes x : \mathrm{KK}_*^G(C_0(X'), A) \to \mathrm{KK}_*^G(C_0(X'), B)$$

is an isomorphism. But X' is a G-compact, proper G-space whose anchor map is a local homeomorphism. Hence the result follows from Proposition 4.3.8. □

In the following we briefly discuss the difficulties that arise when one tries to prove an analogue of Theorem 4.3.7 for general étale groupoids. The main difficulties arise from basic point-set topology facts: If G is no longer totally disconnected, then we can still do most of the reduction steps using the simplicial complexes $P_K(G)$. Note however that in this context K cannot be chosen to be open. This leads to the fact, that in the zero-dimensional case (compare Proposition 4.3.8) we may only assume, that the anchor map is locally injective. When defining H as in the proof of Proposition 4.3.8 it is still a subgroupoid, but has relatively bad topological properties as a subset of G. It is neither open nor closed in G, two features we used in the proof Theorem 3.6.2. Even in this general situation one can still show, that H is a proper groupoid, which is open in $G_{H^{(0)}}$ and closed in $G_{H^{(0)}}^{H^{(0)}}$. Hence the compression homomorphism still makes sense in this setting. It is just our method to prove that comp_H^G is an isomorphism, which fails in this generality.

4.4. Going-Down Functors

Theorem 4.3.7 can be applied directly in many situations (see sections 5.2 and 5.3) but oftentimes it is not directly a map on $\mathrm{K}_*^{\mathrm{top}}(G;A)$ one is interested in, but a map on a construction involving this group, which still shares the same basic functorial properties. Moreover, the map in question must not necessarily be given by taking the Kasparov product. A closer inspection of the proof of Theorem 4.3.7 reveals, that we only used the naturality of the Kasparov product. Hence, following **[CEOO04]** we can use the language of category theory to obtain a more general result. To begin with, given a second countable ample groupoid G, we denote by $\mathcal{C}(G)$ the category of separable commutative proper G-algebras, i.e. algebras of the form $C_0(X)$, where X is a second countable proper G-space. Also let $\mathcal{S}(G)$ be the set containing G and all of its compact open subgroupoids.

Definition 4.4.1. Let G be an ample groupoid. A *Going-Down functor* for G is a collection of $\mathbb{Z}$-graded functors $\mathcal{F} = (\mathcal{F}_H^n)_{H\in\mathcal{S}(G)}$, where $\mathcal{F}_H^n$ is a covariant additive functor from the category of second countable, proper, locally compact G-spaces (with morphisms being the proper, continuous G-maps) to the category of abelian groups, such that the following axioms are satisfied:

(1) Cohomology axioms: For every $H \in \mathcal{S}(G)$
   (a) the functor $\mathcal{F}_H^n$ is homotopy invariant;
   (b) the functor $\mathcal{F}_H^n$ is half-exact, i.e. for every short exact sequence
   $$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$
   in $\mathcal{C}(H)$, the sequence
   $$\mathcal{F}_H^n(A/I) \longrightarrow \mathcal{F}_H^n(A) \longrightarrow \mathcal{F}_H^n$$
   is exact in the middle; and
   (c) for each $n \in \mathbb{Z}$ there is a natural equivalence between $\mathcal{F}_H^{n+1}$ and the functor $A \mapsto \mathcal{F}_H^n(A \otimes C_0(\mathbb{R}))$, where H acts trivially on the second tensor factor.
(2) Induction axiom: For every compact open subgroupoid H of G, there are natural equivalences $I_H^G(n)$ between the functors $\mathcal{F}_H^n$ and $\mathcal{F}_G^n \circ Ind_H^G$, compatible with suspension, where $Ind_H^G : \mathcal{C}(H) \to \mathcal{C}(G)$, $A \mapsto Ind_H^{G_{|H^{(0)}}} A$ denotes induction from H-algebras to G-algebras.

If $\mathcal{F}$ is a Going-Down functor for G, we define
$$\mathcal{F}^n(G) := \lim_{X \subseteq \mathcal{E}(G)} \mathcal{F}_G^n(C_0(X)),$$
where X runs through the G-compact subsets of $\mathcal{E}(G)$.

Our main examples of Going-Down functors arise from the topological K-theory of ample groupoids:

Example 4.4.2. Let G be a second countable ample groupoid and A be a fixed G-algebra. Define $\mathcal{F}_H^*(C_0(X)) := \mathrm{KK}_*^H(C_0(X), A_{|H})$ for $H \in \mathcal{S}(G)$ and $C_0(X) \in \mathcal{C}(H)$. Then $\mathcal{F}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Going-Down functor:

(1) Cohomology axioms:
   (a) Homotopy invariance is clear, since groupoid equivariant KK-theory is invariant with respect to equivariant homotopies in the first variable.

(b) Half-exactness follows from [**Tu99b**, Proposition 7.2 and Lemma 7.7].
(c) The suspension axiom is clear from the definition of the higher equivariant KK-groups (see Definition 3.4.5).

(2) The natural equivalence required in the induction axiom is provided by the compression homomorphism defined prior to Theorem 3.6.2 (or rather its inverse, the inflation map). From the definition of the compression homomorphism it is easy to see, that it indeed provides a natural transformation with respect to equivariant $*$-homomorphisms.

The following lemma can be proved using standard homotopy techniques (see for example [**Bla98**, §21.4])

LEMMA 4.4.3. *Let $\mathcal{F}$ be a Going-Down functor. For every short exact sequence*

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

in $\mathcal{C}(H)$ there are natural maps $\partial_n : \mathcal{F}_H^n(I) \to \mathcal{F}_H^{n+1}(A/I)$ providing a long exact sequence

$$\cdots \longrightarrow \mathcal{F}_H^n(A/I) \longrightarrow \mathcal{F}_H^n(A) \longrightarrow \mathcal{F}_H^n(I) \xrightarrow{\partial_n} \mathcal{F}_H^{n+1}(A/I) \longrightarrow \cdots$$

DEFINITION 4.4.4. Let $\mathcal{F}$ and $\mathcal{G}$ be Going-Down functors for the ample groupoid G. A *Going-Down transformation* is a collection $\Lambda = (\Lambda_H^n)_{H\in\mathcal{S}(G)}$ of natural transformations between $\mathcal{F}_H^n$ and $\mathcal{G}_H^n$ compatible with suspension, such that $I_H^G(n) \circ \Lambda_H^n = \Lambda_G^n \circ I_H^G(n)$.

EXAMPLE 4.4.5. Let G be a second countable ample groupoid and A and B be separable G-algebras. Let $\mathcal{F}$ be the Going-Down functor defined by $\mathcal{F}_H^*(C_0(X)) = \mathrm{KK}_*^H(C_0(X), A_{|H})$ and let $\mathcal{G}$ be the Going-Down functor defined by $\mathcal{G}_H^*(C_0(X)) = \mathrm{KK}_*^H(C_0(X), B_{|H})$ as in Example 4.4.2. Suppose that $x \in \mathrm{KK}^G(A, B)$. Then we can define a Going-Down transformation Λ from $\mathcal{F}$ to $\mathcal{G}$ by letting $\Lambda_H^*(C_0(X))$ be the map

$$\mathcal{F}_H^*(C_0(X)) = \mathrm{KK}_*^H(C_0(X), A_{|H}) \stackrel{\cdot\otimes x}{\to} \mathrm{KK}_*^H(C_0(X), B_{|H}) = \mathcal{G}_H^*(C_0(X)).$$

By associativity of the Kasparov product, Λ_H^* is a natural transformation, which is clearly compatible with suspension. Compatibility with I_H^G follows from Lemma 3.6.8.

Using the naturality, a Going-Down transformation Λ between two Going-Down functors $\mathcal{F}$ and $\mathcal{G}$ induces morphisms $\Lambda^n(G) : \mathcal{F}^n(G) \to \mathcal{G}^n(G)$ in the limit.

THEOREM 4.4.6. *Let $\mathcal{F}$ and $\mathcal{G}$ be two Going-Down functors for an ample groupoid G and let Λ be a Going-Down transformation between $\mathcal{F}$ and $\mathcal{G}$. Suppose that $\Lambda^n_H(C(H^{(0)})) : \mathcal{F}^n_H(C(H^{(0)})) \to \mathcal{G}^n_H(C(H^{(0)}))$ is an isomorphism for all compact open subgroupoids H of G. Then $\Lambda^n(G) : \mathcal{F}^n(G) \to \mathcal{G}^n(G)$ is an isomorphism.*

PROOF. The proof is essentially the same as that of Theorem 4.3.7, replacing $\mathrm{KK}^H_*(C_0(X), A_{|H})$ by $\mathcal{F}^*_H(C_0(X))$ and $\mathrm{KK}^H_*(C_0(X), B_{|H})$ by $\mathcal{G}^*_H(C_0(X))$, and the map $\cdot \otimes res^G_H(x)$ by Λ^*_H, once we note, that all we used in that proof are precisely the properties we ask for in the definition of Going-Down functors and transformations. $\square$

CHAPTER 5

Applications

In this final chapter we provide applications of the restriction principle in several different directions.

5.1. Continuity of Topological K-theory

In this section we will show, that the topological K-theory of an ample groupoid is continuous with respect to the coefficient algebra. The following is an analogue of [**CEN03**, Lemma 2.5] for étale groupoids:

LEMMA 5.1.1. *Let G be an étale groupoid and (A_n, φ_n) an inductive sequence of G-algebras with limit $A = \lim_n A_n$. Then $(A_n \rtimes_r G, \varphi_n \rtimes G)$ is an inductive sequence of* C*-*algebras. Suppose additionally, that either one of the following conditions hold:*

1. *All the connecting maps φ_n are injective.*
2. *The groupoid G is exact.*

Then $A \rtimes_r G = \lim_n A_n \rtimes_r G$ with respect to the connecting homomorphisms $\varphi_n \rtimes G$.

PROOF. It is clear that $(A_n \rtimes_r G, \varphi_n \rtimes G)$ is an inductive sequence of C*-algebras. For the second statement we follow the argument in [**CEN03**, Lemma 2.5]: In the case of (1) we may regard each $A_n \rtimes_r G$ as a subalgebra of $A \rtimes_r G$ and hence also the inductive limit $\overline{\bigcup_{n\in\mathbb{N}} A_n \rtimes_r G}$ is contained in $A \rtimes_r G$. Let us check that $\bigcup_{n\in\mathbb{N}} \Gamma_c(G, r^*\mathcal{A}_n) \subseteq \overline{\bigcup_{n\in\mathbb{N}} A_n \rtimes_r G}$ is dense in $A \rtimes_r G$. First, consider elements of the form $f \otimes a \in \Gamma_c(G, r^*\mathcal{A})$ for $f \in C_c(G)$ and $a \in A$. Let $\varepsilon > 0$ be given. Then, by (1) we can find $n \in \mathbb{N}$ and $b \in A_n$ such that $\|a - b\| < \frac{\varepsilon}{\|f\|}$. It follows, that $f \otimes b \in \Gamma_c(G, r^*\mathcal{A}_n)$ with $\|f \otimes a - f \otimes b\| \leq \|f\|\|a - b\| < \varepsilon$. Since finite sums of elements of the form $f \otimes a$ are dense in $\Gamma_c(G, r^*\mathcal{A})$ in the inductive limit topology, it follows that $\bigcup_{n\in\mathbb{N}} \Gamma_c(G, r^*\mathcal{A}_n)$ is dense in $\Gamma_c(G, r^*\mathcal{A})$ with respect to the inductive limit topology and hence also with respect to the reduced norm topology. For the proof of (2) we make use of the following general fact: If (B_n, ψ_n) is an inductive sequence of C*-algebras, then so is $(B_n/ker(\psi_n), \widetilde{\psi_n})$, where $\widetilde{\psi}_n$ are the maps induced by ψ_n on the quotients. Then it is easy to check, that all the maps $\widetilde{\psi}_n$ are injective and $B = \lim_n B_n/ker(\psi_n)$. Returning to the proof of (2), let $I_n = ker(\varphi_n)$. Using the exactness of G now, we see that $I_n \rtimes G$ is precisely the kernel of the map $\varphi_n \rtimes G : A_n \rtimes_r G \to A \rtimes_r G$. By the above remark we have $A = \lim_n A_n/I_n$ and since the connecting maps are all injective we get $A \rtimes_r G = \lim_n (A_n/I_n) \rtimes_r G$ by (1). Using the exactness of G now, we see that $I_n \rtimes G$ is precisely the kernel of the map $\varphi_n \rtimes G : A_n \rtimes_r G \to A \rtimes_r G$, hence $(A_n/I_n) \rtimes_r G = A_n \rtimes_r G / I_n \rtimes_r G$ and another application of the above mentioned fact together with the identity $I_n \rtimes_r G = ker(\varphi_n \rtimes_r G)$ yields

$$A \rtimes_r G = \lim_n A_n/I_n \rtimes_r G = \lim_n A_n \rtimes_r G/I_n \rtimes_r G = \lim_n A_n \rtimes_r G.$$

□

THEOREM 5.1.2. *Let G be an ample groupoid and (A_n, φ_n) an inductive sequence of G-algebras. If we let $A = \lim A_n$, then the maps $\psi_{n,*} : \mathrm{K}_*^{\mathrm{top}}(G; A_n) \to \mathrm{K}_*^{\mathrm{top}}(G; A)$ induced by the canonical maps $\psi_n : A_n \to A$, give rise to an isomorphism*

$$\lim_{n\to\infty} \mathrm{K}_*^{\mathrm{top}}(G; A_n) \cong \mathrm{K}_*^{\mathrm{top}}(G; A).$$

PROOF. Let $\psi^* : \lim_{n\to\infty} \mathrm{K}_*^{\mathrm{top}}(G; A_n) \to \mathrm{K}_*^{\mathrm{top}}(G; A)$ be the homomorphism induced by the morphisms $\psi_n : A_n \to A$. Our aim is to show that ψ^* is an isomorphism. For every proper G-space X let

$$\psi_X^* : \lim_{n\to\infty} \mathrm{KK}_*^G(C_0(X), A_n) \to \mathrm{KK}_*^G(C_0(X), A)$$

be the morphism induced by ψ_n at the level of X. Now the structure maps for taking the limit over X are given by left Kasparov products, whereas the structure maps for taking the limit over the A_n is given by right Kasparov products. Since the Kasparov product is associative, the limits can be permuted and we get

$$\lim_{n\to\infty} \mathrm{K}_*^{\mathrm{top}}(G; A_n) \cong \lim_X \left(\lim_n \mathrm{KK}_*^G(C_0(X), A_n) \right).$$

The map ψ^* can then be computed via the maps ψ_X^* by

$$\lim_X \left(\lim_n \mathrm{KK}_*^G(C_0(X), A_n) \right) \to \lim_X \mathrm{KK}_*^G(C_0(X), A).$$

We define a contravariant functor

$$\mathcal{F}_H^*(C_0(X)) := \lim_n \mathrm{KK}_*^H(C_0(X), A_{n|H}).$$

Then $\mathcal{F}$ is a Going-Down functor. Let $\mathcal{G}$ denote the Going-Down functor $C_0(X) \mapsto \mathrm{KK}_*^H(C_0(X), A_{|H})$ from Example 4.4.2. Then the maps ψ_X define a Going-Down transformation $\Psi : \mathcal{F} \to \mathcal{G}$, such that $\Psi^*(G) = \psi^*$. By Theorem 4.4.6 it is hence enough to prove, that

$$\lim_n \mathrm{KK}_*^H(C(H^{(0)}), A_{n|H}) \to \mathrm{KK}_*^H(C(H^{(0)}), A_{|H})$$

is an isomorphism for all compact open subgroupoids H in G. For every $n \in \mathbb{N}$ we have a commutative diagram

$$\begin{array}{ccc} \mathrm{KK}_*^H(C(H^{(0)}), A_{n|H}) & \xrightarrow{(\psi_n)_*} & \mathrm{KK}_*^H(C(H^{(0)}), A_{|H}) \\ \Big\downarrow{\scriptstyle \mu_n} & & \Big\downarrow{\scriptstyle \mu} \\ \mathrm{K}_*(A_{n|H} \rtimes H) & \xrightarrow{(\psi_n \rtimes H)_*} & \mathrm{K}_*(A_{|H} \rtimes H) \end{array},$$

where μ and μ_n are the isomorphisms coming from the groupoid version of the Green-Julg theorem (see [**Tu99b**, Proposition 6.25]). By commutativity of the above diagrams it is hence enough to prove, that the maps $(\psi_n \rtimes H)_*$ induce an isomorphism

$$\lim_n \mathrm{K}_*(A_{n|H} \rtimes H) \to \mathrm{K}_*(A_{|H} \rtimes H).$$

Using the continuity of K-theory, the result follows from Lemma 5.1.1. □

COROLLARY 5.1.3. *Let G be an ample groupoid and (A_n, φ_n) an inductive sequence of G-algebras with $A = \lim_{n\to\infty} A_n$. Suppose G satisfies the Baum-Connes conjecture with coefficients in A_n for all $n \in \mathbb{N}$. Assume further, that G is exact, or that all the connecting homomorphisms φ_n are injective. Then G satisfies the Baum-Connes conjecture with coefficients in A.*

5.2. Amenability at Infinity and the Baum-Connes Conjecture

As another application of Theorem 4.3.7 we will show that for ample groupoids, which are strongly amenable at infinity, the Baum-Connes assembly map is split-injective. Let us first recall the definitions:

DEFINITION 5.2.1 ([**Las14**],[**AD16**]). A locally compact Hausdorff groupoid G is called *amenable at infinity*, if there exists a G-space Y such that the anchor map $p : Y \to G^{(0)}$ is proper and $G \ltimes Y$ is amenable (i.e. G acts amenably on Y).

We call G *strongly amenable at infinity*, if in addition the anchor map $p : Y \to G^{(0)}$ admits a continuous (not necessarily equivariant) section.

Note, that every amenable groupoid is strongly amenable at infinity by taking $Y = G^{(0)}$ with the canonical G-action. Furthermore, by results of [**Las14**], if Y is a G-space witnessing amenability at infinity of G, such that the anchor map p is also open, then G is strongly amenable at infinity.

Now if G is (strongly) amenable at infinity and Y is a G-space witnessing this, the properness of $p : Y \to G^{(0)}$ implies that we get an induced map

$$p^* : C_0(G^{(0)}) \to C_0(Y)$$

and consequently, for every G-algebra A, we get a G-equivariant $*$-homomorphism

$$id_A \otimes p^* : A \cong A \otimes_{G^{(0)}} C_0(G^{(0)}) \to A \otimes_{G^{(0)}} C_0(Y).$$

This homomorphism in turn induces a map on the level of topological K-theory, which we - by slight abuse of notation - also denote by

$$p_* : \mathrm{K}_*^{\mathrm{top}}(G; A) \to \mathrm{K}_*^{\mathrm{top}}(G; A \otimes_{G^{(0)}} C_0(Y))$$

By results in [**AD16**] and [**Las14**] we can always find Y with the following additional properties:

- Y is second countable.
- Each Y_u is a convex space and G acts by affine transformations on Y.

If we fix Y with these properties we can show:

PROPOSITION 5.2.2. *Let Y be a G-space with the properties listed above. If $K \subseteq G$ is a proper, open subgroupoid, then $Y_K = p^{-1}(K) \subseteq Y$ is K-equivariantly homotopy-equivalent to $K^{(0)}$.*

PROOF. We will construct a K-equivariant continuous section $\tilde{s}: K^{(0)} \to Y_K$ as follows: Let $c : K^{(0)} \to [0,1]$ be a cut-off function for K, i.e.

(1) $\sum_{k \in K^u} c(d(k)) = 1$ for all $u \in K^{(0)}$, and
(2) $r : supp(c \circ d) \to K^{(0)}$ is proper.

We define

$$\tilde{s}(u) := \sum_{k \in K^u} c(d(k))k \cdot s(d(k)),$$

where $s : G^{(0)} \to Y$ is the continuous section from above. Note that by (2) the sum in the definition is finite for each fixed $u \in K^{(0)}$, and hence (1) and the convexity of Y_u imply that $\tilde{s}(u) \in Y_u$. Thus $\tilde{s}$ is a well-defined section.

The following calculation shows that $\tilde{s}$ is K-equivariant:

$$\begin{aligned}
\tilde{s}(k' \cdot u) &= \tilde{s}(r(k')) \\
&= \sum_{k \in K^{r(k')}} c(d(k))k \cdot s(d(k)) \\
&= \sum_{k \in K^u} c(d(k'k))k'k \cdot s(d(k'k)) \\
&= k' \cdot \left(\sum_{k \in K^u} c(d(k))k \cdot s(d(k)) \right) \\
&= k' \tilde{s}(u)
\end{aligned}$$

It remains to show that $\tilde{s}$ is continuous. We prove this along the lines of Lemma 1.1.18: Fix a $u \in K^{(0)}$ and let V be an open neighbourhood of u such that $\overline{V}$ is compact. Let $\psi \in C_c(K^{(0)})$ be a positive function with $\psi \equiv 1$ on $\overline{V}$. Then $f(k) := c(d(k))\psi(r(k))$ has compact support and for all $v \in V$ we still have $\sum_{k \in K^v} f(k) = 1$ and hence $\tilde{s}(v) = \sum_{k \in K^v} f(k)k \cdot s(d(k)) \in Y_v$. Now we use compactness of $supp(f)$ to cover it with a finite number of open bisections $(U_i)_i$ and use a partition of unity subordinate to this covering to write f as a finite sum $f = \sum f_i$. Then we get

$$\tilde{s}(v) = \sum_i \sum_{k \in K^v} f_i(k)k \cdot s(d(k)) = \sum_i f_i(r_{|U_i}^{-1}(v)) r_{|U_i}^{-1}(v) \cdot s(d(r_{|U_i}^{-1}(v))).$$

The latter expression in this equation is obviously continuous in v since all the functions and operations used are continuous. Hence $\tilde{s}$ must be continuous.

Now by construction we have $p \circ \tilde{s} = id_{K^{(0)}}$ and by convexity the linear homotopy gives $\tilde{s} \circ p \simeq id_{Y_K}$. This homotopy is equivariant since the action of K on Y_K is affine. $\square$

We can now prove the following extention of results from [**Hig00**] and [**CEOO04**] to ample groupoids:

THEOREM 5.2.3. *Let G be a second countable ample groupoid which is strongly amenable at infinity. Then, for any separable G-algebra A the Baum-Connes assembly map*

$$\mu_A : \mathrm{K}_*^{\mathrm{top}}(G; A) \to \mathrm{K}_*(A \rtimes_r G)$$

is split injective.

PROOF. Consider the homomorphism

$$p_* : \mathrm{K}_*^{\mathrm{top}}(G; A) \to \mathrm{K}_*^{\mathrm{top}}(G; A \otimes_{G^{(0)}} C_0(Y))$$

induced by the anchor map $p : Y \to G^{(0)}$ as explained prior to Proposition 5.2.2. As explained there, we can also assume that Y is second countable, each fibre Y_u is convex and G acts by affine transformations. Furthermore we may assume that p admits a continuous section. Thus for every proper, open subgroupoid $K \subseteq G$ we can apply Proposition 5.2.2 to see that the restriction of $p_K : Y_K \to K^{(0)}$ of p induces an isomorphism

$$\mathrm{KK}^K(C_0(K^{(0)}), A_K) \to \mathrm{KK}^K(C_0(K^{(0)}), A_K \otimes_{K^{(0)}} C_0(Y_K)).$$

Thus we have checked the conditions of Theorem 4.3.7 and can deduce that p_* is an isomorphism. By naturality of the assembly map, p_* fits into the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{K}_*^{\mathrm{top}}(G; A) & \xrightarrow{\mu_A} & \mathrm{K}_*(A \rtimes_r G) \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle (p \rtimes G)_*} \\
\mathrm{K}_*^{\mathrm{top}}(G; A \otimes_{G^{(0)}} C_0(Y)) & \xrightarrow{\mu_{A \otimes C_0(Y)}} & \mathrm{K}_*((A \otimes_{G^{(0)}} C_0(Y)) \rtimes_r G)
\end{array}$$

By [**STY02**, Lemma 4.1] the Baum-Connes assembly map for G with coefficients in $A \otimes_{G^{(0)}} C_0(Y)$ is an isomorphism if and only if the assembly map for $G \ltimes Y$ with coefficients in $A \otimes_{G^{(0)}} C_0(Y)$ is. Since $G \ltimes Y$ is amenable by assumption, we can apply the results in [**Tu99a**] to conclude, that the lower horizontal map in the above diagram is an isomorphism. Thus, μ_A is injective with splitting homomorphism $\sigma_A := p_*^{-1} \circ \mu_{A\otimes C_0(Y)} \circ (p \rtimes_r G)_*$. □

We will now apply Theorem 5.2.3 to relate the Baum-Connes conjecture for an ample, strongly amenable at infinity groupoid group bundle to the Baum-Connes conjecture for each of its isotropy groups. This generalizes part (b) of [**CEN03**, Proposition 3.1], which treats the case of a trivial group bundle (i.e. $G = \Gamma \times X$ for some discrete group Γ and a totally disconnected space X). We also make use of ideas from the recent paper [**ELN18**] to avoid γ-elements.

We shall need the notion of an exact groupoid:

Definition 5.2.4. A locally compact groupoid G with Haar system is called *exact* (in the sense of Kirchberg and Wassermann), if for every G-equivariant exact sequence

$$0 \to I \to A \to B \to 0$$

of G-algebras, the corresponding sequence

$$0 \to I \rtimes_r G \to A \rtimes_r G \to B \rtimes_r G \to 0$$

of reduced crossed products is exact.

The following result is a part of [**AD16**, Proposition 6.7]:

Proposition 5.2.5. *Let G be an étale groupoid. If G is amenable at infinity, then G is exact.*

Let us now focus on group bundles: For a start let us observe, that if G is an étale groupoid group bundle, and (A, G, α) is a groupoid dynamical system, then (A_u, G_u^u, α_u) is a (group) dynamical system for every $u \in G^{(0)}$. The following proposition describes the relation of the crossed product $A \rtimes_r G$ with the crossed products corresponding to the fibres:

Proposition 5.2.6. *Let G be an étale groupoid group bundle and A be a G-algebra. Then the following hold:*

(1) The reduced crossed product $A \rtimes_r G$ is a $C_0(G^{(0)})$-algebra.
(2) If G is exact, then the fibres are given by $(A \rtimes_r G)_u = A_u \rtimes_r G_u^u$.
(3) If in addition the C^*-bundle $\mathcal{A}$ associated to A is continuous, then so is the C^*-bundle associated to $A \rtimes_r G$.

Proof. For $\varphi \in C_0(G^{(0)})$ and $f \in \Gamma_c(G, r^*\mathcal{A})$ define a linear map $\Phi(\varphi) : \Gamma_c(G, r^*\mathcal{A}) \to \Gamma_c(G, r^*\mathcal{A})$ by

$$(\Phi(\varphi)f)(g) := \varphi(r(g))f(g)$$

We want to show, that $\Phi(\varphi)$ extends to an element of the multiplier algebra of $A \rtimes_r G$. To this end let $u \in G^{(0)}$. Then, for $\varphi \in C_0(G^{(0)})$, $f \in \Gamma_c(G, r^*\mathcal{A})$ and $\xi \in C_c(G_u^u, A_u)$, we compute

$$\begin{aligned}(\pi_u(\Phi(\varphi)f)\xi)(g) &= \sum_{h \in G_u^u} \alpha_g^{-1}((\Phi(\varphi)f)(g^{-1}h))\xi(h) \\ &= \sum_{h \in G_u^u} \varphi(u)\alpha_g(f(g^{-1}h))\xi(h) \\ &= (\varphi(u)\pi_u(f)\xi)(g)\end{aligned}$$

Hence we have $\pi_u(\Phi(\varphi(f))) = \varphi(u)\pi_u(f)$ and applying this equality we obtain

$$\|\Phi(\varphi)f\|_r = \sup_{u \in G^{(0)}} \|\pi_u(\Phi(\varphi)f)\| = \sup_{u \in G^{(0)}} |\varphi(u)| \|\pi_u(f)\| \leq \|\varphi\|_\infty \|f\|_r$$

Thus, $\Phi(\varphi)$ extends to a bounded linear map $\Phi(\varphi) : A \rtimes_r G \to A \rtimes_r G$. One easily computes on the dense subalgebra $\Gamma_c(G, r^*\mathcal{A})$, that $\Phi(\varphi)$ is adjointable with $\Phi(\varphi)^* = \Phi(\overline{\varphi})$. We have thus defined a $*$-homomorphism $\Phi : C_0(G^{(0)}) \to M(A \rtimes_r G)$. Next, we would like to show that Φ takes its image in the center of the multiplier algebra. By [**Wil07**, Lemma 8.3] it is enough to show, that $\Phi(\varphi)(f_1 * f_2) = f_1 * \Phi(\varphi)f_2$ for all $f_1, f_2 \in \Gamma_c(G, r^*\mathcal{A})$ and $\varphi \in C_0(G^{(0)})$. For $g \in G$ and $u := r(g) = d(g)$ we compute

$$\begin{aligned}(\Phi(\varphi)(f_1 * f_2)(g) &= \varphi(u)(f_1 * f_2)(g) \\ &= \sum_{h \in G_u^u} \varphi(u)f_1(h)\alpha_h(f_2(h^{-1}g)) \\ &= \sum_{h \in G_u^u} f_1(h)\alpha_h(\varphi(u)f_2(h^{-1}g))\end{aligned}$$

$$= \sum_{h\in G_u^u} f_1(h)\alpha_h((\Phi(\varphi)f_2)(h^{-1}g))$$
$$= (f_1 * \Phi(\varphi)f_2)(g)$$

It remains to show that Φ is non-degenerate. Given $x \in A \rtimes_r G$ and $\varepsilon > 0$, find $f \in \Gamma_c(G, r^*\mathcal{A})$ such that $\|x - f\|_r < \varepsilon$. Choose a function $\varphi \in C_c(G^{(0)})$, $0 \leq \varphi \leq 1$ with $\varphi = 1$ on $r(supp(f))$. Then $\Phi(\varphi)f = f$ and hence $x \in \overline{C_0(G^{(0)})A \rtimes_r G}$. We have thus established the first part of the proposition, namely that $A \rtimes_r G$ is a $C_0(G^{(0)})$-algebra.

For the second part, we want to analyze the fibres: We always have a canonical family of surjective $*$-homomorphisms defined as follows: For each $u \in G^{(0)}$, there is a canonical map $q_u : \Gamma_c(G, r^*\mathcal{A}) \to C_c(G_u^u, A_u)$ given by restriction. This map extends to a surjective $*$-homomorphism $A \rtimes_r G \to A_u \rtimes_r G_u^u$, still denoted by q_u. Let J_u denote the ideal $\overline{C_0(G^{(0)} \setminus \{u\})A \rtimes_r G}$ of $A \rtimes_r G$. We clearly have $J_u = A_{|G^{(0)}\setminus\{u\}} \rtimes_r G_{|G^{(0)}\setminus\{u\}}$. Now if G is exact, the sequence

$$0 \to A_{|G^{(0)}\setminus\{u\}} \rtimes_r G_{|G^{(0)}\setminus\{u\}} \to A \rtimes_r G \xrightarrow{q_u} A_u \rtimes_r G_u^u \to 0$$

is exact for every $u \in G^{(0)}$. Hence $ker(q_u) = J_u$. It follows that $(A \rtimes_r G)_u = A_u \rtimes_r G_u^u$.

Finally, for part (3), we have to show continuity of the C^*-bundle associated to the $C_0(G^{(0)})$-algebra $A \rtimes_r G$, provided the continuity of $\mathcal{A}$. For this we have to prove, that $u \mapsto \|q_u(x)\|$ is lower semicontinuous for every $x \in A \rtimes_r G$. Recall that we have a representation $\pi : \Gamma_c(G, r^*\mathcal{A}) \to \mathcal{L}_A(L^2(G, A))$. We can compute

$$\begin{aligned}\|q_u(f)\|_r &= \|\pi_u(f)\| \\ &= \sup\{\|\langle \pi(f)\xi, \eta\rangle_A(u)\| \mid \xi, \eta \in \Gamma_c(G, r^*\mathcal{A}), \|\xi\|, \|\eta\| \leq 1\}.\end{aligned}$$

The latter expression however is lower semicontinuous as a function in u, since it is the supremum of the continuous functions

$$u \mapsto \|\langle \pi(f)\xi, \eta\rangle_A(u)\|.$$

□

LEMMA 5.2.7. *Let G be an étale groupoid group bundle. If G is amenable at infinity, then so is G_u^u for each $u \in G^{(0)}$.*

PROOF. By assumption there exists a locally compact space X and an action of G on X with proper anchor map $p : X \to G^{(0)}$, such that $G \ltimes X$ is amenable. Then $X_u := p^{-1}(\{u\})$ is a compact subspace of X and the action of G on X restricts to an action of G_u^u on X_u. In particular $G_u^u \ltimes X_u$ is a closed subgroupoid of $G \ltimes X$. Hence it is amenable by [**ADR00**, Proposition 5.1.1]. □

Next, we would like to turn to KK-theory. We will start with the following observation:

LEMMA 5.2.8. *If G is a second countable étale groupoid group bundle and (A, G, α) and (B, G, β) are separable groupoid dynamical systems, then the descent map $j_{G,r}$ actually takes values in the group $\mathcal{R}\mathrm{KK}(G^{(0)}; A \rtimes_r G, B \rtimes_r G)$.*

PROOF. Let $(E, \Phi, T) \in \mathbb{E}^G(A, B)$. It is enough to show, that for all $\varphi \in C_0(G^{(0)})$, $f \in \Gamma_c(G, r^*\mathcal{A})$, $f' \in \Gamma_c(G, r^*\mathcal{B})$ and $\xi \in \Gamma_c(G, r^*\mathcal{E})$ we have

$$(\varphi f)\xi f' = f\xi(\varphi f').$$

Hence we compute for all $g \in G$:

$$\begin{aligned}
((\varphi f)\xi f')(g) &= \sum_{h \in G^{r(g)}} ((\varphi f)\xi)(h)\beta_h(f'(h^{-1}g)) \\
&= \sum_{h \in G^{r(g)}} \sum_{s \in G^{r(h)}} \varphi(r(s))f(s)V_s(\xi(s^{-1}h))\beta_h(f'(h^{-1}g)) \\
&= \sum_{h \in G^{r(g)}} \sum_{s \in G^{r(h)}} f(s)V_s(\xi(s^{-1}h))\beta_h((\varphi f')(h^{-1}g)) \\
&= (f\xi(\varphi f'))(g).
\end{aligned}$$

□

LEMMA 5.2.9. *Let G be a second countable exact étale groupoid group bundle and A be a separable G-algebra. For each $u \in G^{(0)}$ the inclusion map $i_u : G_u^u \to G$ induces a group homomorphism $i_u^* : \mathrm{K}_*^{\mathrm{top}}(G; A) \to \mathrm{K}_*^{\mathrm{top}}(G_u^u; A_u)$, such that the following diagram commutes:*

$$\begin{array}{ccc}
\mathrm{K}_*^{\mathrm{top}}(G; A) & \xrightarrow{\mu_A} & \mathrm{K}_*(A \rtimes_r G) \\
\Big\downarrow{\scriptstyle i_u^*} & & \Big\downarrow{\scriptstyle q_{u,*}} \\
\mathrm{K}_*^{\mathrm{top}}(G_u^u; A_u) & \xrightarrow{\mu_{A_u}} & \mathrm{K}_*(A_u \rtimes_r G_u^u)
\end{array}$$

Proof. It follows from Proposition 3.4.8, that the inclusion map i_u induces group homomorphisms

$$i^*_{X,u} : \mathrm{KK}^G(C_0(X), A) \to \mathrm{KK}^{G^u_u}(C_0(X_u), A_u)$$

for every locally compact G-space X. If X is proper and cocompact, then X_u is a proper and cocompact G^u_u-space. Hence we obtain maps $\mathrm{KK}^G(C_0(X), A) \to \mathrm{K}^{\mathrm{top}}_*(G^u_u; A_u)$. One easily checks, that these commute with the connecting maps coming from continuous G-maps $X \to Y$ for two proper G-compact G-spaces X and Y. Consequently, taking the limit over all proper and G-compact subspaces $X \subseteq \mathcal{E}(G)$, we obtain the desired homomorphism $i^*_u : \mathrm{K}^{\mathrm{top}}_*(G; A) \to \mathrm{K}^{\mathrm{top}}_*(G^u_u; A_u)$. In order to obtain commutativity of the diagram in the proposition, it is enough to observe that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{KK}^G(C_0(X), A) & \xrightarrow{\quad i^*_{X,u} \quad} & \mathrm{KK}^{G^u_u}(C_0(X_u), A_u) \\
\big\downarrow {\scriptstyle j_G} & & \big\downarrow {\scriptstyle j_{G^u_u}} \\
\mathcal{R}\mathrm{KK}(G^{(0)}; C_0(X) \rtimes_r G, A \rtimes_r G) & \xrightarrow{(i^{(0)}_u)^*} & \mathrm{KK}(C_0(X_u) \rtimes_r G^u_u, A_u \rtimes_r G^u_u) \\
\big\downarrow {\scriptstyle p_{G \ltimes X} \otimes \cdot} & & \big\downarrow {\scriptstyle p_{G^u_u \ltimes X_u} \otimes \cdot} \\
\mathrm{K}_0(A \rtimes_r G) & \xrightarrow{\quad q_{u,*} \quad} & \mathrm{K}_0(A_u \rtimes_r G^u_u)
\end{array}$$

The middle vertical map is induced by the inclusion map $i^{(0)}_u : \{u\} \hookrightarrow G^{(0)}$. Let us deal with the left square first: Let (E, Φ, T) be a Kasparov triple in $\mathbb{E}^G(C_0(X), A)$. Recall, that $j_{G,r}$ sends the class of (E, Φ, T) to the class represented by $(E \rtimes_r G, \widetilde{\Phi}, \widetilde{T})$. Applying Proposition 5.2.6 and Proposition 3.2.6 we obtain a canonical isomorphism

$$\begin{aligned}
(E \rtimes_r G)_u &= (E \otimes_A (A \rtimes_r G))_u \\
&\cong E_u \otimes_{A_u} (A \rtimes_r G)_u \\
&\cong E_u \otimes_{A_u} (A_u \rtimes G^u_u) \\
&= E_u \rtimes_r G^u_u,
\end{aligned}$$

which intertwines the representations $(\widetilde{\Phi})_u$ and $\widetilde{\Phi_u}$ and the operators $(\widetilde{T})_u$ and $\widetilde{T_u}$. In order to prove commutativity of the second square we first fix a cut-off function c for $G \ltimes X$. Then its restriction to the subspace X_u is easily checked to be a cut-off function for $G_u^u \ltimes X_u$. It follows, that if $p := p_{G \ltimes X}$ is the canonical projection associated to c, then $p(u) \in C_0(X_u) \rtimes_r G_u^u$ is the projection associated to the restriction of c to X_u. Now let (E, Φ, T) be the representative of an element $x \in \mathcal{R}\mathrm{KK}(G^{(0)}, C_0(X) \rtimes_r G, A \rtimes_r G)$. Recall, that under the identification $\mathrm{K}_0(C_0(X) \rtimes_r G) \cong \mathrm{KK}(\mathbb{C}, C_0(X) \rtimes_r G)$ the class of p is represented by the Kasparov tripel $(C_0(X) \rtimes_r G, \Phi_p, 0)$, where $\Phi_p : \mathbb{C} \to C_0(X) \rtimes_r G$ is given by $\Phi_p(1) = p$. Then the Kasparov product $p \otimes x \in \mathrm{KK}(\mathbb{C}, A \rtimes_r G)$ can be represented by the tripel $(E \otimes_{q_u} (A_u \rtimes_r G_u^u), (\Phi \circ \Phi_p) \otimes 1, T \otimes 1)$. On the other hand $(i_u^{(0)})^*(x)$ is represented by the tripel (E_u, Φ_u, T_u) and hence the product $p(u) \otimes (i_u^{(0)})^*(x)$ is represented by the tripel $(E_u, \Phi_u \circ \Phi_{p(u)}, T_u)$, where $\Phi_{p(u)} : \mathbb{C} \to C_0(X_u) \rtimes_r G_u^u$ is again given by $1 \mapsto p(u)$. But by Remark 3.2.3 there is a canonical isomorphism $E \otimes_{q_u} (A_u \rtimes_r G_u^u) \to E_u$ and one easily checks on elementary tensors, that this isomorphism intertwines $(\Phi \circ \Phi_p) \otimes 1$ with $\Phi_u \circ \Phi_{p(u)}$ and $T \otimes 1$ with T_u. $\square$

Let G be an ample groupoid group bundle, which is strongly amenable at infinity and let A be a G-algebra. Let $\sigma_A : \mathrm{K}_*(A \rtimes_r G) \to \mathrm{K}_*^{\mathrm{top}}(G; A)$ be the splitting homomorphism provided by Theorem 5.2.3. Then $\gamma_A := \mu_A \circ \sigma_A$ is an idempotent endomorphism of $\mathrm{K}_*(A \rtimes_r G)$ such that $im(\gamma_A) = im(\mu_A)$. In particular, it follows that G satisfies the Baum-Connes conjecture for A if and only if $(1 - \gamma_A)\mathrm{K}_*(A \rtimes_r G) = \{0\}$.

Since G is strongly amenable at infinity, it is exact. Hence the reduced crossed product $A \rtimes_r G$ is the algebra of C_0-sections of a continuous bundle of C^*-algebras over $G^{(0)}$ with fibres $(A \rtimes_r G)_u = A_u \rtimes_r G_u^u$. Let $q_u : A \rtimes_r G \to A_u \rtimes_r G_u^u$ be the corresponding quotient map. Likewise, every group G_u^u of the bundle G is amenable at infinity. Hence by the same reasoning, we obtain idempotents $\gamma_{A_u} \in End(\mathrm{K}_*(A_u \rtimes_r G_u^u))$. We shall need the observation, that the elements γ_A and γ_{A_u} are compatible:

Lemma 5.2.10. *Let G be a second countable ample groupoid group bundle, which is strongly amenable at infinity. If A is a separable G-algebra and $q_u : A \rtimes_r G \to A_u \rtimes_r G_u^u$ denotes the canonical quotient map, then $q_{u,*} \circ \gamma_A = \gamma_{A_u} \circ q_{u,*}$.*

PROOF. Let $\pi_u : (A \otimes_{G^{(0)}} C_0(Y)) \rtimes_r G \to (A_u \otimes C(Y_u)) \rtimes_r G_u^u$ be the canonical quotient map. Then we have a commutative diagram:

$$\begin{array}{ccc}
\mathrm{K}_*(A \rtimes_r G) & \xrightarrow{q_{u,*}} & \mathrm{K}_*(A_u \rtimes_r G_u^u) \\
\downarrow (p_A \rtimes_r G)_* & & \downarrow ((id_{A_u} \otimes 1) \rtimes_r G)_* \\
\mathrm{K}_*((A \otimes_{G^{(0)}} C_0(Y)) \rtimes_r G) & \xrightarrow{\pi_{u,*}} & \mathrm{K}_*((A_u \otimes C(Y_u)) \rtimes_r G_u^u) \\
\downarrow (\mu_{A \otimes C_0(Y)})^{-1} & & \downarrow (\mu_{A_u \otimes C(Y_u)})^{-1} \\
\mathrm{K}_*^{\mathrm{top}}(G; A \otimes_{G^{(0)}} C_0(Y)) & \xrightarrow{i_u^*} & \mathrm{K}_*^{\mathrm{top}}(G_u^u; A_u \otimes C(Y_u))
\end{array}$$

Here, the first square commutes already at the level of the $*$-homomorphisms, since $p_A \rtimes_r G$ is a $C_0(G^{(0)})$-linear map with $(p_A \rtimes_r G)_u = (id_{A_u} \otimes 1_{C(Y_u)}) \rtimes_r G_u^u$. The second square commutes by Lemma 5.2.9 applied to the G-algebra $A \otimes_{C_0(G^{(0)})} C_0(Y)$. For similar reasons, each square in the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{K}_*^{\mathrm{top}}(G; A \otimes_{G^{(0)}} C_0(Y)) & \xrightarrow{(p_A)_*^{-1}} & \mathrm{K}_*^{\mathrm{top}}(G; A) & \xrightarrow{\mu_A} & \mathrm{K}_*(A \rtimes_r G) \\
\downarrow i_u^* & & \downarrow i_u^* & & \downarrow q_{u,*} \\
\mathrm{K}_*^{\mathrm{top}}(G_u^u; A_u \otimes C(Y_u)) & \xrightarrow{(p_{A_u})_*^{-1}} & \mathrm{K}_*^{\mathrm{top}}(G_u^u; A_u) & \xrightarrow{\mu_{A_u}} & \mathrm{K}_*(A_u \rtimes_r G_u^u)
\end{array}$$

Since the composition of the upper (respective lower) rows of these diagrams is by definition γ_A (respective γ_{A_u}), the result follows. □

THEOREM 5.2.11. *Let G be a second countable ample group bundle, which is strongly amenable at infinity. Suppose A is a separable G-algebra such that the associated C^*-bundle is continuous, and G_u^u satisfies the Baum-Connes conjecture with coefficients in A_u for all $u \in G^{(0)}$. Then G satisfies the Baum-Connes conjecture with coefficients in A.*

PROOF. By the above considerations, it is enough to show, that $(1-\gamma_A)\mathrm{K}_*(A \rtimes_r G) = \{0\}$. To this end, let $x \in (1-\gamma_A)\mathrm{K}_*(A \rtimes_r G)$. By Lemma 5.2.10 we have $q_{u,*}(x) = q_{u,*}(1-\gamma_A)(x) = (1-\gamma_{A_u})(q_{u,*}(x)) \in (1-\gamma_{A_u})\mathrm{K}_*(A_u \rtimes_r G_u^u)$. But the latter group is zero by our assumption, hence $q_{u,*}(x) = 0$ for all $u \in G^{(0)}$. By [**CEN03**, Lemma 3.4] every $u \in G^{(0)}$ admits a compact neighbourhood C of u, such that $q_{C,*}(x) = 0$, where $q_C : A \rtimes_r G \to A_{|C} \rtimes_r G_{|C}$ denotes the map induced by restriction. Since $G^{(0)}$ is assumed to be totally disconnected, we can find a partition $G^{(0)} = \coprod_{i\in I} C_i$ into compact open sets C_i such that $q_{C_i,*}(x) = 0$ for all $i \in I$. As the cover is disjoint, we obtain a decompositon $A \rtimes_r G = \bigoplus_{i\in I} A_{|C_i} \rtimes_r G_{|C_i}$. Using the additivity of K-theory, we see that the maps q_{C_i} induce an isomorphism $\mathrm{K}_*(A \rtimes_r G) \cong \bigoplus_{i\in I} \mathrm{K}_*(A_{|C_i} \rtimes_r G_{|C_i})$. Since $q_{C_i,*}(x) = 0$ for all $i \in I$, we conclude $x = 0$ as desired. □

5.3. K-theory of Twisted Groupoid C*-algebras

In this section we will study the K-theory of twisted groupoid C*-algebras. We are particularly interested in the effect of a continuous deformation of the twist on the K-theory groups. This topic has been studied previously by Gillaspy in a series of papers (see [**Gil15a, Gil15b, Gil16**]). To start of let us recall the notion of twisted groupoids.

Definition 5.3.1. Let G be a topological groupoid. A *twist* Σ over G is a central groupoid extension

$$G^{(0)} \times \mathbb{T} \xrightarrow{i} \Sigma \xrightarrow{j} G,$$

by which we mean:

(1) The map i is a homeomorphism onto $j^{-1}(G^{(0)}) \subseteq \Sigma$,
(2) the map j is a continuous and open surjection, and
(3) the extension is central meaning that $i(r(\sigma), z)\sigma = \sigma i(d(\sigma), z)$ for all $\sigma \in \Sigma$ and $z \in \mathbb{T}$.

We say that Σ is a *continuous twist* over G, if j admits a continuous cross section.

Note, that we can canonically identify $\Sigma^{(0)}$ with $G^{(0)}$ and for all $u \in G^{(0)}$ we have $j(i(u, z)) = u$. Moreover, Σ admits a canonical left action of $\mathbb{T}$ given by $z \cdot \sigma := i(r(\sigma), z)\sigma$. If $s : G \to \Sigma$ is a continuous cross section for j, then s is automatically compatible with the range and domain maps in the sense that $s(d(g)) = d(s(g))$ and $s(r(g)) = r(s(g))$.

Following [**MW92**], we associate a C*-algebra to a twist Σ over an étale groupoid G as follows: Consider

$$C_c(G; \Sigma) := \{f \in C_c(\Sigma) \mid f(z\sigma) = zf(\sigma)\}.$$

Then $C_c(G;\Sigma)$ becomes a $*$-algebra with repect to the operations

$$f_1 * f_2(\sigma) = \sum_{j(\tau)\in G^{r(\sigma)}} f_1(\tau)f_2(\tau^{-1}\sigma) \text{ and } f^*(\sigma) = \overline{f(\sigma^{-1})}.$$

Observe that the sum makes sense, since the expression $f_1(\tau)f_2(\tau^{-1}\sigma)$ only depends on $j(\tau) \in G$. For each $u \in G^{(0)}$ let E_u be the Hilbert space consisting of functions $\xi : \Sigma_u \to \mathbb{C}$ such that $\xi(z\sigma) = z\xi(\sigma)$ and $\sum_{j(\sigma)\in G_u} |\xi(\sigma)|^2 < \infty$, with the inner product $\langle \xi, \eta\rangle = \sum_{j(\sigma)\in G_u} \overline{\xi(\sigma)}\eta(\sigma)$. Then, for $f \in C_c(G;\Sigma)$ we can define an operator $\pi_u(f)$ on E_u by $\pi_u(f)\xi) = f*\xi$. The operator $\pi_u(f)$ is bounded and we define $C_r^*(G;\Sigma)$ to be the completion of $C_c(G;\Sigma)$ with respect to the norm

$$\|f\|_r := \sup_{u\in G^{(0)}} \|\pi_u(f)\|.$$

Recall, that a 2-cocycle for G is a map $\omega : G^{(2)} \to \mathbb{T}$, such that

$$\omega(g_1, g_2)\omega(g_1g_2, g_3) = \omega(g_1, g_2g_3)\omega(g_2, g_3)$$

for all $g_1, g_2, g_3 \in G$ with $(g_1, g_2), (g_2, g_3) \in G^{(2)}$, and

$$\omega(g, d(g)) = 1 = \omega(r(g), g)$$

for all $g \in G$.

Given a 2-cocycle ω on G we can define a groupoid structure on $\Sigma_\omega := G \times \mathbb{T}$ as follows: Two pairs $(g_1, s_1), (g_2, s_2)$ are composable if $(g_1, g_2) \in G^{(2)}$ and their product is defined as

$$(g_1, s_1)(g_2, s_2) := (g_1g_2, s_1s_2\omega(g_1, g_2))$$

The inverse of $(g, s) \in \Sigma_\omega$ is given by

$$(g, s)^{-1} := (g^{-1}, \overline{s\omega(g^{-1}, g)})$$

If ω is continuous, it is not hard to check that Σ_ω is a locally compact Hausdorff groupoid in the product topology. Thus we obtain a central extension of groupoids

$$G^{(0)} \times \mathbb{T} \xrightarrow{i} \Sigma_\omega \xrightarrow{j} G,$$

where the first map is the canonical inclusion and the second map is the projection onto the first factor. Note, that j has a canonical continuous cross section s given by $s(g) = (g, 1)$. Conversely, starting with a twist Σ over G with continuous section $s : G \to \Sigma$ we note that $j(s(gh)^{-1}s(g)s(h)) \in G^{(0)}$. Hence by exactness $s(gh)^{-1}s(g)s(h) \in i(G^{(0)} \times \mathbb{T})$. Since i is a homeomorphism onto its image we obtain a continuous map $\omega : G^{(2)} \to \mathbb{T}$ by letting $\omega(g,h) = i^{-1}(s(gh)^{-1}s(g)s(h))$. It is then routine to check, that ω satisfies the cocycle identity (it is not normalized however).

REMARK 5.3.2. In the literature one often finds a direct construction of the twisted groupoid C*-algebra associated to a continuous 2-cocycle, that does not pass through the canonical extension Σ_ω explained above. It is defined as a completion of the convolution algebra $C_c(G)$ with product and involution given by

$$f_1 *_\omega f_2(g) = \sum_{h \in G^{r(g)}} f_1(h) f_2(h^{-1}g)\omega(h, h^{-1}g) \text{ and}$$

$$f^*(g) = \overline{f(g^{-1})\omega(g, g^{-1})},$$

and we will denote it by $C_r^*(G,\omega)$. Note, that both constructions yield the same C*-algebras, since there is a canonical isomorphism $\Phi : C_r^*(G,\Sigma_\omega) \to C_r^*(G,\omega)$, given by $\Phi(f)(g) = f(g,1)$. One can easily define an inverse map Ψ by $\Psi(f)(g,z) = zf(g)$.

Let E be a Hilbert $C_0(X)$-module. In **[vEW14]** van Erp and Williams introduced the groupoid

$$Iso(E) := \{(x,V,y) \mid V : E_y \to E_x \text{ is a unitary}\},$$

with the obvious operations $(x,V,y)(y,W,z) = (x,VW,z)$.

We shall need the following result due to van Erp and Williams:

PROPOSITION 5.3.3. **[vEW14**, Proposition 5.1**]** *Let Σ be a twist over an étale groupoid G. Then there exists a Hilbert $C_0(G^{(0)})$-module E and an action α of G on $K(E)$, such that $K(E) \rtimes_{\alpha,r} G$ is Morita equivalent to $C_r^*(G;\Sigma)$.*

For later reference, let us briefly recall the constructions in the proof: The Hilbert $C_0(G^{(0)})$-module E is obtained as the completion of $C_c(G;\Sigma)$ with respect to the inner product

$$\langle f_1, f_2\rangle_{C_0(G^{(0)})}(u) = \sum_{j(\sigma)\in G_u} \overline{f_1(\sigma)} f_2(\sigma). \tag{3}$$

We want to remark the following:

LEMMA 5.3.4. *If $G^{(0)} \times \mathbb{T} \to \Sigma \xrightarrow{j} G$ is a twist over G with continuous cross section $s : G \to \Sigma$, then the fibre E_u over $u \in G^{(0)}$ can be identified with the Hilbert space obtained by completion of $E_0(u) = \{f \in C_c(\Sigma_u) \mid f(z\sigma) = zf(\sigma)\}$ with respect to the inner product $\langle f_1, f_2\rangle = \sum_{j(\sigma)\in G_u} \overline{f_1(\sigma)} f_2(\sigma)$.*

PROOF. One easily sees that the restriction map $C_c(G;\Sigma) \to E_0(u)$ factors through an isometric linear map $E_u \to \overline{E_0(u)}$. The only issue is the surjectivity of the restriction map. Let $t : \Sigma \to \mathbb{T}$ be the (continuous!) map given by $t(\sigma) = \sigma s(j(\sigma))^{-1}$. Now given $f \in E_0(u)$ choose any extension $f' \in C_c(\Sigma)$. Then define $f''(\sigma) := t(\sigma)f'(s(j(\sigma)))$. We claim that $f'' \in C_c(G;\Sigma)$. Clearly f'' is continuous with $supp(f'') \subseteq s(j(supp(f')))$. Now if $z \in \mathbb{T}$, then we have $t(z\sigma) = (z\sigma)s(j(z\sigma))^{-1} = z\sigma s(j(\sigma))^{-1} = zt(\sigma)$. Consequently, we can compute

$$f''(z\sigma) = t(z\sigma)f'(s(j(z\sigma))) = zt(\sigma)f'(s(j(\sigma))) = zf''(\sigma),$$

and our claim follows. Finally, if $\sigma \in \Sigma_u$, then we have $f''(\sigma) = t(\sigma)f'(s(j(\sigma))) = t(\sigma)f(s(j(\sigma))) = f(t(\sigma)s(j(\sigma))) = f(\sigma)$. $\square$

Let $\rho : \Sigma \to Iso(E)$ be the representation given by $(\rho(\sigma)\xi)(\tau) = \xi(\tau\sigma)$. Then $\rho(z\sigma) = z\rho(\sigma)$ for all $z \in \mathbb{T}$ and $\sigma \in \Sigma$. Consequently, we obtain a well-defined action α of G on $K(E)$ by $\alpha_{j(\sigma)} = \mathrm{Ad}\,\rho(\sigma)$. Let A_0 be the dense subalgebra $\Gamma_c(G, r^*\mathcal{K}) \subseteq K(E) \rtimes_{r,\alpha} G$. Following [**MW08a**, Theorem 6.4] together with the formulas given in the proof of [**vEW14**, Proposition 5.1] one then defines a pre-Hilbert bimodule-structure on $X_0 := \Gamma_c(G, d^*\mathcal{E})$ as follows: For $\xi, \eta \in X_0$ and $f \in A_0$ define

$$(f\xi)(g) = \sum_{h\in G^{r(g)}} \alpha_{g^{-1}h}(f(h^{-1}))\xi(h^{-1}g),$$

$$_{A_0}\langle \xi, \eta\rangle(g) = \sum_{h\in G^{s(g)}} \alpha_{gh}\big(_{K(E_x)}\langle \xi(h), \eta(gh)\rangle\big).$$

Note that in [**vEW14**], the authors construct the crossed product as a completion of $\Gamma_c(G, s^*\mathcal{K})$. Thus, in order to obtain the formulas above we need to pass through the canonical isomorphism, sending $f \in A_0$ to the function $\check{f} \in \Gamma_c(G, s^*\mathcal{K})$, given by $\check{f}(g) := f(g^{-1})$.

For $\xi, \eta \in X_0$ and $f \in C_c(G;\Sigma)$ define

$$(\xi f)(g) = \sum_{j(\sigma)\in G^{d(g)}} f(\sigma^{-1})\rho(\sigma)\xi(gj(\sigma))$$

and a $C_c(G;\Sigma)$-valued inner product by

$$\langle \xi, \eta\rangle(\tau) = \sum_{j(\sigma)\in G^{d(\tau)}} \langle \rho(\sigma^{-1}\tau^{-1})\xi(j(\sigma^{-1}\tau^{-1})), \rho(\sigma^{-1})\eta(j(\sigma^{-1}))\rangle(d(\sigma))$$

The completion X of X_0 then implements a Morita equivalence between $K(E) \rtimes_r G$ and $C_r^*(G;\Sigma)$. With this description at hand we can prove the following technical little lemma, which will turn out useful later:

Lemma 5.3.5. *If $(\xi_i)_i$ is a net in X_0 converging to $\xi \in X_0$ in the inductive limit topology, then $\|\xi_i - \xi\| \to 0$.*

Proof. Let $\eta_i := \xi - \xi_i \in X_0$. We will show, that ${}_{A_0}\langle \eta_i, \eta_i\rangle$ converges to zero in the inductive limit topology. Then it will also converge to zero in the reduced norm and hence $\|\xi - \xi_i\|^2 = \|{}_{A_0}\langle \eta_i, \eta_i\rangle\| \to 0$ as desired. By assumption, there exists a compact subset $K \subseteq G$ such that $supp(\eta_i) \subseteq K$ for all i. Since the action of G on itself by multiplication is always proper, the set $C := \{g \in G \mid g^{-1}K \cap K \neq \emptyset\}$ is also compact. Now if $0 \neq {}_{A_0}\langle \eta_i, \eta_i\rangle(g) = \sum_{h\in G^{d(g)}} \alpha_{gh}(\langle \eta_i(h), \eta_i(gh)\rangle)$, there exists some $h \in G^{d(g)}$ such that $\langle \eta_i(h), \eta_i(gh)\rangle \neq 0$. But then necessarily $h \in g^{-1}K \cap K$, which implies $g \in C$. Thus $supp({}_{A_0}\langle \eta_i, \eta_i\rangle) \subseteq C$ for all i. Now let $\varepsilon > 0$ be given. Choose $M > 0$ such that $\sup_{u\in G^{(0)}}|K^u| \leq M$. Then we have $\sup_{g\in G}\|\langle \eta_i(g), \eta_i(g)\rangle\| < \frac{\sqrt{\varepsilon}}{M}$ for i large enough. For i large enough we can then compute

$$\begin{aligned}\|\langle \eta_i, \eta_i\rangle(g)\| &\leq \sum_{h\in G^{d(g)}} \|\langle \eta_i(h), \eta_i(gh)\|\\ &\leq \sum_{h\in G^{d(g)}} \|\langle \eta_i(h), \eta_i(h)\|\|\langle \eta_i(gh), \eta_i(gh)\| < \varepsilon,\end{aligned}$$

which finishes the proof. □

Remark 5.3.6. Let Σ be a twist over the étale groupoid G and $H \subseteq G$ a compact open subgroupoid. Then $\Sigma' := j^{-1}(H)$ is easily seen to be a twist over H. Let E and α be as above. Then we can restrict the action α to an action of H on $K(E)_{|H}$. We claim that the resulting crossed product $K(E)_{|H} \rtimes_r H$ is then Morita equivalent to $C_r^*(H;\Sigma')$. The proof is basically the same as in [**vEW14**, Proposition 5.1], we just restrict all the appearing bundles to the subgroupoid H and use the fact that H is an (H,H)-equivalence.

Our goal is to prove, that the K-theory of $C_r^*(G;\Sigma)$ only depends on the homotopy class of Σ. We will start by formalizing what we mean by a homotopy: Given a locally compact Hausdorff groupoid G, consider the trivial bundle of groupoids $G \times [0,1]$ with the product topology. This bundle is itself a locally compact groupoid, where (g,s) and (h,t) are composable if g and h are composable in G and $s = t$. In this case we define $(g,s)(h,s) := (gh,s)$ and $(g,s)^{-1} := (g^{-1},s)$. Consequently, the unit space is given by $G^{(0)} \times [0,1]$.

Definition 5.3.7. A (continuous) twist Σ over $G \times [0,1]$ is called a (continuous) *homotopy of twists* for G.

If Σ is a homotopy of twists over G, then Σ is a continuous field of groupoids over $[0,1]$ in the sense of [**AD16**, Definition 8.9], since $pr_{[0,1]}(d(\sigma)) = pr_{[0,1]}(r(\sigma))$. In particular, for each $t \in [0,1]$ we obtain a twist Σ_t over G by letting $\Sigma_t := (pr_{[0,1]} \circ r)^{-1}(\Sigma) = \Sigma_{|G^{(0)}\times\{t\}}$.

For every $t \in [0,1]$ we obtain a canonical $*$-homomorphism

$$q_t : C_r^*(G \times [0,1];\Sigma) \to C_r^*(G;\Sigma_t),$$

which for $f \in C_c(G \times [0,1];\Sigma)$ is given by $q_t(f) = f_{|\Sigma_t}$. An argument very similar to the proof of Lemma 5.3.4 shows, that q_t is surjective, provided that the twist is continuous. The main goal of this section is to prove the following result:

Theorem 5.3.8. *Let G be an ample groupoid, which satisfies the Baum-Connes conjecture with coefficients and let Σ be a continuous homotopy of twists for G. Then*

$$(q_t)_* : K_*(C_r^*(G \times [0,1];\Sigma)) \to K_*(C_r^*(G,\Sigma_t))$$

is an isomorphism.

The following result deals with the case that G is a compact groupoid and is due to Gillaspy:

Proposition 5.3.9. [**Gil15b**, Proposition 3.1] *If Σ is a continuous homotopy of twists on a compact Hausdorff groupoid G, then the canonical $*$-homomorphism $q_t : C_r^*(G \times [0,1];\Sigma) \to C_r^*(G;\Sigma_t)$ is a homotopy equivalence.*

The idea in proving Theorem 5.3.8 is to use Theorem 4.3.7 to reduce to the case of compact groupoids and then apply Proposition 5.3.9 above.

From now on fix a continuous homotopy of twists Σ over an étale Hausdorff groupoid G. Consider the canonical Hilbert-$C_0(G^{(0)} \times [0,1])$-module E, defined as the completion of $C_c(G \times [0,1]; \Sigma)$ with respect to the inner product 3. Now by Proposition 5.3.3 and the discussion thereafter we obtain an action α of $G \times [0,1]$ on $K(E)$. Observe, that there is a canonical action of G on $G^{(0)} \times [0,1]$ given by $g \cdot (d(g), t) = (r(g), t)$, such that $G \times [0,1] \cong G \ltimes (G^{(0)} \times [0,1])$. Taking this point of view we can use the pushforward construction from Proposition 2.2.5 to obtain an action β of G on $K(E)$. One has the following:

Proposition 5.3.10. [**LaL17**, Theorem 3.8] *The canonical map* $\Phi : \Gamma_c(G \times [0,1], r^*\mathcal{K}) \to \Gamma_c(G, r^*\mathcal{K})$ *given by*

$$\Phi(f)(g)(t) = f(g,t)$$

is a $*$-homomorphism, which extends to an isomorphism $K(E) \rtimes_{\alpha,r} (G \times [0,1]) \to K(E) \rtimes_{\beta,r} G$.

On the other hand for each $t \in [0,1]$ we can apply (the proof of) Proposition 5.3.3 to the twist Σ_t over G, in order to obtain a Hilbert $C_0(G^{(0)})$-module E_t and an action α_t of G on $K(E_t)$. Let us make easy observations concerning the relationship between E and E_t:

Lemma 5.3.11. *The restriction map* $C_c(G \times [0,1]; \Sigma) \to C_c(G; \Sigma_t)$, $f \mapsto f_{|\Sigma_t}$ *extends to a surjective bounded linear map* $p_t : E \to E_t$.

Proof. It is routine to check, that the restriction map is bounded and linear. Using an argument similar to the proof of Lemma 5.3.4 one sees that the restriction map $C_c(G \times [0,1]; \Sigma) \to C_c(G; \Sigma_t)$ is surjective. This is not quite enough to conclude that p_t is surjective. However, if $i_t : G^{(0)} \to G^{(0)} \times [0,1]$ denotes the inclusion at $t \in [0,1]$, then p_t factors through an isometric linear map $i_t^* E \to E_t$. Since this map is isometric, it is enough to know that the dense subset $C_c(G; \Sigma_t)$ is contained in the image to conclude surjectivity. Using, that the canonical map $E \cong E \otimes_{C_0(G^{(0)} \times [0,1])} C_0(G^{(0)} \times [0,1]) \to i_t^* E$ is surjective, the result follows. □

In the case that the twist is continuous, we can use an argument similar to the proof of Lemma 5.3.4 again, to show that for $u \in G^{(0)}$ we can canonically identify the Hilbert spaces $E_{(u,t)}$ and $(E_t)_u$ and hence also $K(E)_{(u,t)}$ with $K(E_t)_u$. Let X and X_t be the equivalence bimodules obtained from applying Proposition 5.3.3 to the twists Σ and Σ_t, respectively. We have the following:

PROPOSITION 5.3.12. *The restriction map* $\Gamma_c(G \times [0,1], d^*\mathcal{E}) \to \Gamma_c(G, d^*\mathcal{E}_t)$, $\xi \mapsto \xi_{|G\times\{t\}}$ *extends to a bounded linear map* $\Psi_t : X \to X_t$ *and factors through an isomorphism*

$$\Theta_t : q_t^*(X) \to X_t$$

of Hilbert $C_r^*(G;\Sigma_t)$*-modules.*

PROOF. From the definition of the respective inner products it is quite obvious that $\langle \Psi_t(\xi), \Psi_t(\eta)\rangle = q_t(\langle \xi, \eta\rangle)$ for all $\xi, \eta \in \Gamma_c(G \times [0,1], d^*\mathcal{E})$. It follows that Ψ_t is bounded and hence extends to all of X. Define $\Theta_t : q_t^*X = X \otimes_{q_t} C_r^*(G;\Sigma_t) \to X_t$ on elementary tensors by $\Theta_t(\xi \otimes a) = \Psi_t(\xi)a$. Then Θ_t extends to an isometric map on all of q_t^*X, since for $\xi, \eta \in X$ and $a, b \in C_r^*(G;\Sigma_t)$ we can compute

$$\begin{aligned}\langle \xi \otimes a, \eta \otimes b\rangle &= (q_t(\langle \eta, \xi\rangle)a)^*b \\ &= (\langle \Psi_t(\eta), \Psi_t(\xi)\rangle a)^*b \\ &= \langle \Psi_t(\xi)a, \Psi_t(\eta)b\rangle \\ &= \langle \Theta(\xi \otimes a), \Theta(\eta \otimes b)\rangle.\end{aligned}$$

Finally, to see that Θ_t is surjective it is enough to show, that the image is dense. First, let $\xi \in \Gamma_c(G, d^*\mathcal{E}_t)$ be of the form $\xi = \varphi \otimes e$ with $\varphi \in C_c(G)$ and $e \in E_t$, i.e. $\xi(g) = \varphi(g)e(d(g))$. Since $p_t : E \to E_t$ is surjective, we can find an element $e' \in E$ such that $p_t(e') = e$. Also, pick any map $\varphi' \in C_c(G \times [0,1])$, such that $\varphi'(g,t) = \varphi(g)$. Then $\varphi' \otimes e' \in X$ such that $\Psi_t(\varphi' \otimes e') = \xi$. Now if $\xi \in \Gamma_c(G, d^*\mathcal{E}_t)$ is arbitrary we can approximate it in the inductive limit topology by finite sums of elements of the form $\varphi \otimes e$ as above. An application of 5.3.5 completes the proof. □

Let $x \in KK(K(E) \rtimes_r G, C_r^*(G \times [0,1]; \Sigma))$ and $x_t \in KK(K(E_t) \rtimes_r G, C_r^*(G;\Sigma_t))$ be the canonical KK-equivalences associated to the equivalence bimodules X and X_t respectively.

Lemma 5.3.13. *For each $t \in [0,1]$ restriction of functions induces a G-equivariant $*$-homomorphism $\Phi_t : K(E) \to K(E_t)$, such that the following diagram commutes:*

$$\begin{array}{ccc} K_*(K(E) \rtimes_r G) & \xrightarrow{(\Phi_t \rtimes G)_*} & K_*(K(E_t) \rtimes_{\alpha_t, r} G) \\ \downarrow \cdot \otimes x & & \downarrow \cdot \otimes x_t \\ K_*(C_r^*(G \times [0,1]; \Sigma)) & \xrightarrow{(q_t)_*} & K_*(C_r^*(G; \Sigma_t)) \end{array}$$

Proof. Recall that $K(E)$ is a $C_0(G^{(0)} \times [0,1])$-algebra. Let $\mathcal{K}(E)$ denote the associated bundle. Similarly $K(E_t)$ is a $C_0(G^{(0)})$-algebra with associate bundle $\mathcal{K}(E_t)$. For $f \in \Gamma_0(G^{(0)} \times [0,1], \mathcal{K}(E)) = K(E)$ and $u \in G^{(0)}$ define

$$\Phi_t(f)(u) := f(u,t)$$

Then $\Phi_t(f) \in \Gamma_0(G^{(0)}, \mathcal{K}(E_t)) \cong K(E_t)$ and it is straightforward to verify, that Φ_t is a G-equivariant $*$-homormophism. To see commutativity of the diagram, it is enough to check that $[\Phi_t \rtimes G] \otimes x_t = x \otimes [q_t]$ in $KK(K(E) \rtimes_r G, C_r^*(G; \Sigma_t))$. Since all the elements involved can be represented by Kasparov-triples, where the operator is zero, these products are easy to describe: The element on the left handside can be represented by the tripel $(X_t, \Phi_t \rtimes G, 0)$, while the right handside is given by the class of $(X \otimes_{q_t} C_r^*(G; \Sigma_t), \psi \otimes 1, 0)$, where ψ is the left action of $K(E) \rtimes_r G$ on X. From Proposition 5.3.12 we have an isomorphism of right Hilbert $C_r^*(G; \Sigma_t)$-bimoudles $\Theta : X \otimes_{q_t} C_r^*(G; \Sigma_t) \to X_t$ given on elementary tensors by $\xi \otimes a \mapsto \xi_{|G \times \{t\}} a$. Thus, to complete the proof, we observe that Θ intertwines the left actions of $K(E) \rtimes_r G$. □

Proof of Theorem 5.3.8. Fix $t \in [0,1]$. In light of Lemma 5.3.13 and the fact that G satisfies the Baum-Connes conjecture with coefficients (and the naturality of the Baum-Connes assembly map), it is enough to show that $\Phi_t : K(E) \to K(E_t)$ induces an isomorphism

$$(\Phi_t)_* : K_*^{top}(G; K(E)) \to K_*^{top}(G; K(E_t)).$$

Hence we are in the position to apply Theorem 4.3.7 to deduce, that it is enough to show, that

$$(\Phi_t \rtimes H)_* : K_*(K(E)_{|H} \rtimes H) \to K_*(E_t)_{|H} \rtimes H)$$

is an isomorphism for all compact open subgroupoids $H \subseteq G$. Using Remark 5.3.6 and the same arguments as in Lemma 5.3.13 for H, we conclude that it is enough to prove that

$$(q_t)_* : K_*(C_r^*(H \times [0,1]; \Sigma)) \to K_*(C_r^*(H; \Sigma_t))$$

is an isomorphism. But this is an immediate consequence of Proposition 5.3.9. □

REMARK 5.3.14. Our Theorem 5.3.8 recovers the main result of [**Gil15a**], which states that the K-theory of a twisted higher-rank graph algebra only depends on the homotopy class of the twist. The result is however limited to row-finite higher-rank graphs without sources. In [**RSY04**] the authors describe how one can associate C*-algebras to the much broader class of all finitely aligned higher-rank graphs. Subsequently, these C*-algebras were also found to be realizable as groupoid C*-algebras associated to ample Hausdorff groupoids in [**FMY05**]. Thus, these groupoids naturally fit into the framework of Theorem 5.3.8. The only missing link to generalize [**Gil15a**, Theorem 4.1] as stated, is to make the connection between cocycles on a finitely aligned higher-rank graph and continuous cocycles on the associated groupoid. This analysis has been carried out for the case of row-finite higher-rank graphs without sources in [**KPS15**], and we believe that a similar argument works in the general case.

5.4. A Mixed Künneth Formula

In this final application we study the K-theory of tensor products by crossed products with ample groupoids in analogy with the results from [**CEOO04**]. The main tool is a mixed Künneth formula involving the topological K-theory of the groupoid in question. Under the assumtion that G satisfies the Baum-Connes conjecture with coefficients, one can relate this mixed Künneth formula to the usual Künneth formula for the crossed product. Let us recall the usual Künneth formula: We say that a C*-algebra A satisfies the *Künneth formula* if for all C*-algebras B, there is a canonical short exact sequence

$$0 \longrightarrow \mathrm{K}_*(A) \otimes \mathrm{K}_*(B) \xrightarrow{\alpha} \mathrm{K}_*(A \otimes B) \xrightarrow{\beta} \mathrm{Tor}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \longrightarrow 0.$$

The map $\alpha : \mathrm{K}_*(A) \otimes \mathrm{K}_*(B) \to \mathrm{K}_*(A \otimes B)$ in the above sequence can be obtained using the Kasparov product as the composition

$$\begin{array}{ccc} \mathrm{KK}(\mathbb{C}, A) \otimes \mathrm{KK}_*(\mathbb{C}, B) & \xrightarrow{\mathrm{id} \otimes \sigma_A} & \mathrm{KK}_*(\mathbb{C}, A) \otimes \mathrm{KK}_*(A, A \otimes B) \\ & \searrow^{\alpha} & \downarrow{\scriptstyle \otimes_A} \\ & & \mathrm{KK}_*(\mathbb{C}, A \otimes B) \end{array}$$

where $\sigma_A : \mathrm{KK}_*(\mathbb{C}, B) \to \mathrm{KK}_*(A, A \otimes B)$ is Kasparov's external tensor product in KK-theory. The following result is shown in [**CEOO04**, Proposition 4.2] (extending earlier results by [**Sch82**]):

PROPOSITION 5.4.1. *Let A be a separable* C**-algebra. Then A satisfies the Künneth formula if and only if $\alpha : \mathrm{K}_*(A) \otimes \mathrm{K}_*(B) \to \mathrm{K}_*(A \otimes B)$ is an isomorphism for all separable* C**-algebras B with $\mathrm{K}_*(B)$ free abelian.*

The authors in [**CEOO04**] then define the class $\mathcal{N}$ to be the class of all separable C*-algebras such that $\alpha : \mathrm{K}_*(A) \otimes \mathrm{K}_*(B) \to \mathrm{K}_*(A \otimes B)$ is an isomorphism for all separable C*-algebras B with $\mathrm{K}_*(B)$ free abelian. It turns out that the class $\mathcal{N}$ is quite large and enjoys many nice permanence properties:

(1) The class $\mathcal{N}$ contains the bootstrap class $\mathcal{B}$ (see [**Bla98**, Definition 22.3.4]).
(2) If $A \in \mathcal{N}$ and B is KK-dominated by A (see [**Bla98**, Definition 23.10.6]), then $B \in \mathcal{N}$.
(3) If $0 \to I \to A \to A/I \to 0$ is a semi-split short exact sequence of C*-algebras such that two of them are in $\mathcal{N}$, then so is the third.
(4) If $A, B \in \mathcal{N}$, then $A \otimes B \in \mathcal{N}$.
(5) If $A = \lim_i A_i$ is an inductive limit, such that each $A_i \in \mathcal{N}$ and, such that all the structure maps are injective, then $A \in \mathcal{N}$.

Our first goal is, to replace $\mathrm{K}_*(A)$ by the topological K-theory of an ample groupoid with coefficients in a suitable separable G-algebra A and define an equivariant version of the map α. Before we can get into it, we need some preliminary observations on minimal tensor products of $C_0(X)$-algebras:

Recall, that for arbitrary C*-algebras A and B, their minimal tensor product $A \otimes B$ sits as an essential ideal inside $M(A) \otimes M(B)$, and hence, using the universal property of the multiplier algebra, there exists a unique embedding $\iota : M(A) \otimes M(B) \hookrightarrow M(A \otimes B)$, satisfying $\iota(m \otimes n)(a \otimes b) = ma \otimes nb$ and $(a \otimes b)\iota(m \otimes n) = am \otimes bn$. In particular, we have $\iota(ZM(A) \otimes ZM(B)) \subseteq ZM(A \otimes B)$. In what follows we will suppress ι in our notation and view $ZM(A) \otimes ZM(B)$ as a subalgebra of $ZM(A \otimes B)$:

PROPOSITION 5.4.2. [**McC15**, Proposition 3.4] *Let A be a $C_0(X)$-algebra with structure map Φ_X and B a $C_0(Y)$-algebra with structure map Φ_Y. Then $A \otimes B$ is a $C_0(X \times Y)$-algebra with respect to the map $\Phi_X \otimes \Phi_Y$. Moreover, the fibre over $(x, y) \in X \times Y$ is*

$$(A \otimes B)_{(x,y)} = (A \otimes B)/I_x \otimes B + A \otimes J_y,$$

where I_x and J_y are the ideals corresponding to the fibres A_x and B_y respectively.

In many situations the fibres are much nicer to describe:

PROPOSITION 5.4.3. *Let A be a $C_0(X)$-algebra, and B be a $C_0(Y)$-algebra. If either A or B is separable and exact, then*

$$(A \otimes B)_{(x,y)} = A_x \otimes B_y.$$

PROOF. This is a direct consequence of [**Bla06**, IV.3.4.22, Proposition IV.3.4.23]. □

Now let A and B be $C_0(X)$-algebras over the same space X, and let $\Delta : X \to X \times X$ be the diagonal inclusion. Then we define the minimal balanced tensor product $A \otimes_X B$ of A and B by $\Delta^*(A \otimes B)$. Thus, $A \otimes_X B$ is a $C_0(X)$-algebra by construction. Note, that $A \otimes_X B$ is canonically isomorphic the quotient of $A \otimes B$ by the ideal $\overline{C_0(X \times X \setminus \Delta(X))A \otimes B}$. It follows from Proposition 5.4.3 above, that if either A or B is separable and exact, that for all $x \in X$ we have

$$(A \otimes_X B)_x = A_x \otimes B_x.$$

With this description of the fibres it is not so hard to see the following:

Lemma 5.4.4. *Let A and B be $C_0(X)$-algebras and $f : Y \to X$ a continuous map. If either A or B is separable and exact, we have $f^*(A \otimes_X B) \cong f^*A \otimes_Y f^*B$.*

Proof. Consider the map $f \times f : Y \times Y \to X \times X$. We will first show, that $f^*A \otimes f^*B$ is canonically isomorphic to $(f \times f)^*(A \otimes B)$ as a $C_0(Y \times Y)$-algebra. Consider the map

$$\Phi : f^*A \otimes f^*B \to (f \times f)^*(A \otimes B),$$

which on an elementary tensor $\varphi \otimes \psi \in f^*A \otimes f^*B$ is defined by

$$\Phi(\varphi \otimes \psi)(y, y') = \varphi(y) \otimes \psi(y') \in A_{f(y)} \otimes B_{f(y')} = (f \times f)^*(A \otimes B)_{(y,y')}.$$

Note, that we use the assumption that either A or B is separable and exact here, to identify the fibres in the last equality. Since

$$\begin{aligned}\|\Phi(\varphi \otimes \psi)\| &= \sup_{(y,y')} \|\varphi(y) \otimes \psi(y')\| \\ &= \sup_{(y,y')} \|\varphi(y)\|\|\psi(y')\| \\ &\leq \|\varphi\|\|\psi\| \\ &= \|\varphi \otimes \psi\|,\end{aligned}$$

the map Φ extends to a bounded $C_0(Y \times Y)$-linear $*$-homomorphism, which clearly induces an isomorphism on each fibre. Hence Φ is an isomorphism as desired. Observe, that we have $\Delta_X \circ f = (f \times f) \circ \Delta_Y$, where Δ_X and Δ_Y denote the diagonal inclusions respectively. Hence we have

$$\begin{aligned}f^*(A \otimes_X B) = (\Delta_X \circ f)^*(A \otimes B) &= ((f \times f) \circ \Delta_Y)^*(A \otimes B) \\ &= \Delta_Y^*((f \times f)^*(A \otimes B)) \\ &\cong \Delta_Y^*(f^*A \otimes f^*B) \\ &= f^*A \otimes_Y f^*B.\end{aligned}$$

□

Suppose now, that G is an étale Hausdorff groupoid. Suppose further, that (A, G, α) and (B, G, β) are groupoid dynamical systems. With the above lemma at hand, it is now easy to define a diagonal action. Suppose that either A or B is separable and exact. Then we define the diagonal action of G on $A \otimes_{G^{(0)}} B$ via the composition

$$d^*(A \otimes_{G^{(0)}} B) \cong d^*A \otimes_G d^*B \xrightarrow{\alpha\otimes\beta} r^*A \otimes_G r^*B \cong r^*(A \otimes_{G^{(0)}} B).$$

Note, that if (A, G, α) is a groupoid dynamical system and B is any C*-algebra, such that either A or B is separable and exact, then $(A \otimes B, G, \alpha \otimes id)$ is a groupoid dynamical system. The reduced crossed product is compatible with the minimal balanced tensor product in the following way:

Proposition 5.4.5. [**LaL15**, Theorem 6.1] *There is a natural isomorphism*

$$\Psi : (A \otimes B) \rtimes_{\alpha\otimes id,r} G \to (A \rtimes_{\alpha,r} G) \otimes B.$$

Before we can proceed, we also need the following:

Proposition 5.4.6. *Let A, B and D be separable G-algebras, such that D is exact. Then there is a homomorphism*

$$\sigma_D : \mathrm{KK}^G(A, B) \to \mathrm{KK}^G(A \otimes_{G^{(0)}} D, B \otimes_{G^{(0)}} D),$$

given by associating to an element $(E, \Phi, T) \in \mathbb{E}^G(A, B)$ the triple $(E \otimes_A A \otimes_{G^{(0)}} D, \Phi \otimes id, T \otimes id)$.

Let us now return to the Künneth formula. Fix a second countable ample Hausdorff groupoid G. For ease of notation let us denote its unit space by X. Let A be a separable exact G-algebra and B any C*-algebra. We wish to define a map

$$\alpha_G : \mathrm{K}_*^{\mathrm{top}}(G; A) \otimes \mathrm{K}_*(B) \to \mathrm{K}_*^{\mathrm{top}}(G; A \otimes B).$$

Consider the trivial group denoted by 1. Then the canonical groupoid homomorphism $G \to 1$ induces a homomorphism

$$\mathrm{KK}_*(\mathbb{C}, B) \to \mathrm{KK}_*^G(C_0(X), C_0(X, B)).$$

Let ε denote the composition:

$$\begin{array}{ccccc} \mathrm{K}_*(B) & \xrightarrow{\cong} & \mathrm{KK}_*(\mathbb{C}, B) & \longrightarrow & \mathrm{KK}_*^G(C_0(X), C_0(X, B)) \\ & \searrow^{\varepsilon} & & & \downarrow \sigma_A \\ & & & & \mathrm{KK}_*^G(A \otimes_X C_0(X), A \otimes_X C_0(X, B)) \end{array}$$

Under the canonical identifications of G-algebras $A \otimes_X C_0(X) \cong A$ and $A \otimes_X C_0(X, B) \cong A \otimes B$ we will view ε as a map

$$\varepsilon : \mathrm{K}_*(B) \to \mathrm{KK}^G_*(A, A \otimes B).$$

Now for any proper and G-compact G-space $Y \subseteq \mathcal{E}(G)$ we define a map α_Y as the composition

$$\begin{array}{ccc} \mathrm{KK}^G_*(C_0(Y), A) \otimes \mathrm{K}_*(B) & \xrightarrow{\mathrm{id} \otimes \varepsilon} & \mathrm{KK}^G_*(C_0(Y), A) \otimes \mathrm{KK}^G_*(A, A \otimes B) \\ & \searrow^{\alpha_Y} & \downarrow \otimes_A \\ & & \mathrm{KK}^G_*(C_0(Y), A \otimes B) \end{array}$$

Passing to the limit, the maps α_Y induce the desired map

$$\alpha_G : \mathrm{K}^{\mathrm{top}}_*(G; A) \otimes \mathrm{K}_*(B) \to \mathrm{K}^{\mathrm{top}}_*(G; A \otimes B).$$

Definition 5.4.7. We denote by $\mathcal{N}_G$ the class of all separable exact G-algebras A such that α_G is an isomorphism for all B with $\mathrm{K}_*(B)$ free abelian.

We will now show, that for a G-algebra A to be in $\mathcal{N}_G$ also corresponds to satisfying a G-equivariant version of the Künneth formula:

Proposition 5.4.8. *Let A be a separable and exact G-algebra. Then $A \in \mathcal{N}_G$ if and only if A for every* C*-*algebra B, there exists a canonical homomorphism*

$$\beta_G : \mathrm{K}^{\mathrm{top}}_*(G; A \otimes B) \to \mathrm{Tor}(\mathrm{K}^{\mathrm{top}}_*(G; A), \mathrm{K}_*(B))$$

such that the sequence

$$0 \to \mathrm{K}^{\mathrm{top}}_*(G; A) \otimes \mathrm{K}_*(B) \overset{\alpha_G}{\to} \mathrm{K}^{\mathrm{top}}_*(G; A \otimes B) \overset{\beta_G}{\to} \mathrm{Tor}(\mathrm{K}^{\mathrm{top}}_*(G; A), \mathrm{K}_*(B)) \to 0$$

is exact.

Proof. Let $\mathcal{S}$ denote the category of all separable C*-algebras with $*$-homomorphisms as morphisms, and let **Ab** denote the category of abelian groups. Consider the functor $F_* : \mathcal{S} \to \mathbf{Ab}$, given by $F_*(B) = \mathrm{K}^{\mathrm{top}}_*(G; A \otimes B)$ and $F_*(\Phi) = (\mathrm{id} \otimes \Phi)_*$ for a $*$-homomorphism $\Phi : B_1 \to B_2$. We will show that F_* is a Künneth functor in the sense of [**CEOO04**, Definition 3.1], provided that $A \in \mathcal{N}_G$. It is clear, that F_* is stable and homotopy invariant, since the topological K-theory has these properties. To see (K2), combine [**CEOO04**, Lemma 4.1] with [**Tu99a**, Proposition 5.6]. Item (K3) again follows from the corresponding property of topological K-theory and (K4) is precisely what it means for A to be in the class $\mathcal{N}_G$. Hence an application of [**CEOO04**, Theorem 3.3] completes the proof. □

The class $\mathcal{N}_G$ enjoys many stability properties similar to those of $\mathcal{N}$:

LEMMA 5.4.9. *Let G be a second countable ample groupoid. Then the following hold:*

1. *If $A \in \mathcal{N}_G$ and B is a separable exact C^*-algebra, which is KK^G-dominated by A (i.e. there exist $x \in \mathrm{KK}^G(A,B)$ and $y \in \mathrm{KK}^G(B,A)$ such that $y \otimes x = 1_B \in \mathrm{KK}^G(B,B)$), then $B \in \mathcal{N}_G$.*
2. *If $0 \to I \to A \to A/I \to 0$ is a semi-split short exact sequence of G-algebras such that two of them are in $\mathcal{N}_G$, then so is the third.*
3. *If $A \in \mathcal{N}_G$ and $B \in \mathcal{N}$, then $A \otimes B \in \mathcal{N}_G$, where $A \otimes B$ is equipped with the action $\alpha \otimes id$.*
4. *If $(A_n, \varphi_n)_n$ is an inductive sequence of G-algebras with injective and G-equivariant connecting maps, such that each $A_n \in \mathcal{N}_G$ for all $n \in \mathbb{N}$, then $A \in \mathcal{N}_G$.*

PROOF. For the proof of (1) let D be any C^*-algebra with $\mathrm{K}_*(D)$ free abelian and consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{K}_*^{\mathrm{top}}(G;B) \otimes \mathrm{K}_*(D) & \xrightarrow{(\cdot \otimes y) \otimes \mathrm{id}} & \mathrm{K}_*^{\mathrm{top}}(G;A) \otimes \mathrm{K}_*(D) & \xrightarrow{(\cdot \otimes x) \otimes \mathrm{id}} & \mathrm{K}_*^{\mathrm{top}}(G;B) \otimes \mathrm{K}_*(D) \\
\downarrow{\scriptstyle \alpha_G} & & \downarrow{\scriptstyle \alpha_G} & & \downarrow{\scriptstyle \alpha_G} \\
\mathrm{K}_*^{\mathrm{top}}(G;B \otimes D) & \xrightarrow{\otimes \sigma_D(y)} & \mathrm{K}_*^{\mathrm{top}}(G;A \otimes D) & \xrightarrow{\otimes \sigma_D(x)} & \mathrm{K}_*^{\mathrm{top}}(G;B \otimes D)
\end{array}$$

By assumption, the composition of the horizontal arrows are the identity maps in each row and the middle vertical map is an isomorphism. An easy diagram chase then shows, that the left (and right) vertical arrows must be isomorphisms as well.

For the proof of (2), we first note that exactness passes to ideals (see [**Bla06**, Theorem IV.3.4.3]), quotients by [**Bla06**, Corollary IV.3.4.19] and semi-split extensions (see [**Bla06**, Theorem IV.3.4.20]) by deep results of Kirchberg and Wassermann. By [**CEOO04**, Lemma 4.1] the sequence $0 \to I \otimes B \to A \otimes B \to A/I \otimes B \to 0$ is a semi-split short exact sequence as well, and hence (2) follows from an easy application of the Five Lemma.

For (3) let us first observe, that if A and B are separable and exact C*-algebras, then so is their minimal tensor product $A \otimes B$ by associativity of the minimal tensor product. Now suppose that $A \in \mathcal{N}_G$ and $B \in \mathcal{N}$. Let D be any C*-algebra with $\mathrm{K}_*(B)$ free abelian. As in the proof of [**CEOO04**, Lemma 4.4(iii)] we can use this fact to make the canonical identification

$$\mathrm{Tor}(\mathrm{K}_*^{\mathrm{top}}(G;A), \mathrm{K}_*(B) \otimes \mathrm{K}_*(D)) \cong \mathrm{Tor}(\mathrm{K}_*^{\mathrm{top}}(G;A), \mathrm{K}_*(B)) \otimes \mathrm{K}_*(D).$$

Now consider the following commutative diagram:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{K}_*^{\mathrm{top}}(G;A) \otimes \mathrm{K}_*(B) \otimes \mathrm{K}_*(D) & \xrightarrow{\mathrm{id} \otimes \alpha} & \mathrm{K}_*^{\mathrm{top}}(G;A) \otimes \mathrm{K}_*(B \otimes D) \\
\downarrow & & \downarrow \\
\mathrm{K}_*^{\mathrm{top}}(G;A \otimes B) \otimes \mathrm{K}_*(D) & \xrightarrow{\alpha_G} & \mathrm{K}_*^{\mathrm{top}}(G;A \otimes B \otimes D) \\
\downarrow & & \downarrow \\
\mathrm{Tor}(\mathrm{K}_*^{\mathrm{top}}(G;A), \mathrm{K}_*(B) \otimes \mathrm{K}_*(D)) & \xrightarrow{\mathrm{Tor}(\mathrm{id}, \alpha)} & \mathrm{Tor}(\mathrm{K}_*^{\mathrm{top}}(G;A), \mathrm{K}_*(B \otimes D)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Under the identification of the Tor groups mentioned above, the first column is the equivariant Künneth sequence for (A, B) tensored with $\mathrm{K}_*(D)$. Thus, using our assumption, that $A \in \mathcal{N}_G$, it is exact by Proposition 5.4.8. Similarly, the second column is the equivariant Künneth sequence for $(A, B \otimes D)$, and hence exact, too. Finally, the top and bottom arrows are isomorphisms, since B was assumed to be in $\mathcal{N}$. By the Five Lemma, the middle vertical map α_G must be an isomorphism as well.

Finally, for item (4) note, that separability clearly passes to sequential inductive limits and exactness passes to inductive limits with injective connecting maps (see [**Bla06**, Proposition IV.3.4.4]). Hence the result follows from Theorem 5.1.2. □

Using the Baum-Connes assembly map we can relate the map α_G to the map α for the crossed product as follows:

PROPOSITION 5.4.10. *Let A be a separable exact G-algebra and B be any C^*-algebra. Then the diagram*

$$\begin{array}{ccc} \mathrm{K}_*^{\mathrm{top}}(G;A)\otimes \mathrm{K}_*(B) & \xrightarrow{\mu_A\otimes id} & \mathrm{K}_*(A\rtimes_r G)\otimes \mathrm{K}_*(B) \\ \Big\downarrow \alpha_G & & \Big\downarrow \alpha \\ \mathrm{K}_*^{\mathrm{top}}(G;A\otimes B) & \xrightarrow{\mu_{A\otimes B}} & \mathrm{K}_*((A\otimes B)\rtimes_r G) \end{array}$$

commutes. In particular, if $\mu_{A\otimes B}$ is an isomorphism for all C^*-algebras B, then $A\in\mathcal{N}_G$ if and only if $A\rtimes_r G\in\mathcal{N}$.

PROOF. First, note that for all $x\in \mathrm{K}_*(B)$ we have $j_G(\varepsilon(x)) = \sigma_{A\rtimes_r G}(x)$. Using this, we can easily check commutativity of the above diagram on the level of each G-compact subspace $Y\subseteq\mathcal{E}(G)$ as follows: For $y\in \mathrm{KK}_*^G(C_0(Y),A)$ and $x\in\mathrm{K}_*(B)$ we compute

$$\begin{aligned} \mu_{Y,A\otimes B}(\alpha_Y(y\otimes x)) &= [p_Y]\otimes_{C_0(Y)\rtimes_r G} j_G(\alpha_Y(y\otimes x)) \\ &= [p_Y]\otimes_{C_0(Y)\rtimes_r G} j_G(y\otimes_A \varepsilon(x)) \\ &= [p_Y]\otimes_{C_0(Y)\rtimes_r G} (j_G(y)\otimes_{A\rtimes_r G}\sigma_{A\rtimes_r G}(x)) \\ &= \mu_{Y,A}(y)\otimes\sigma_{A\rtimes_r G}(x) \\ &= \alpha(\mu_{Y,A}(y)\otimes x). \end{aligned}$$

The second statement then follows directly from the commutativity of the diagram. □

We are now ready for the main result of this section:

THEOREM 5.4.11. *Let G be a second countable ample groupoid and A a separable and exact G-algebra. Suppose that $A_{|K}\rtimes K\in\mathcal{N}$ for all compact open subgroupoids $K\subseteq G$. Then $A\in\mathcal{N}_G$.*

PROOF. Let B be a fixed C^*-algebra with $\mathrm{K}_*(B)$ free abelian. For each $H\in\mathcal{S}(G)$ define contravariant functors $\mathcal{F}_H:\mathcal{C}(H)\to\mathbf{Ab}$ and

$\mathcal{G}_H : \mathcal{C}(H) \to \mathbf{Ab}$ by

$$\mathcal{F}_H(C_0(Y)) := \mathrm{KK}_*^H(C_0(Y), A) \otimes \mathrm{K}_*(B),$$

$$\mathcal{G}_H(C_0(Y)) := \mathrm{KK}_*^H(C_0(Y), A \otimes B).$$

Both $(\mathcal{F}_H)_{H\in\mathcal{S}(G)}$ and $(\mathcal{G}_H)_{H\in\mathcal{S}(G)}$ define Going-Down functors in the sense of Definition 4.4.1.

Moreover, for each $H \in \mathcal{S}(G)$ and every proper H-space Y the maps α_Y determine natural transformations $\Lambda_H : \mathcal{F}_H \to \mathcal{G}_H$, which form a Going-Down transformation Λ. Our assumptions then translate to the fact that $\Lambda_K : \mathcal{F}_K(C_0(K^{(0)})) \to \mathcal{G}_K(C_0(K^{(0)}))$ is an isomorphism for every compact open subgroupoid of G. Hence, by Theorem 4.4.6 the result follows. □

The following corollary gives many examples, when the hypothesis of Theorem 5.4.11 are satisfied and thus provides many examples of G-algebras in class $\mathcal{N}_G$.

Corollary 5.4.12. *Let G be a second countable ample groupoid and A be a separable exact G-algebra, such that A_u is type I for all $u \in G^{(0)}$. Then $A \in \mathcal{N}_G$.*

Proof. It follows from [**Tu99a**, Proposition 10.3], that $A_{|K} \rtimes K$ is a type I C*-algebra for all compact subgroupoids $K \subseteq G$, and hence it is contained in the bootstrap class $\mathcal{B} \subseteq \mathcal{N}$. The result then follows from Theorem 5.4.11. □

We conclude this section by pointing out the connections between Theorem 5.4.11 and the Baum-Connes conjecture:

Proposition 5.4.13. *Let G be a second countable ample groupoid and $A \in \mathcal{N}_G$. Consider the following properties:*

(1) G satisfies the Baum-Connes conjecture with coefficients in $A \otimes B$ for all separable C**-algebras B (with respect to the trivial action on the second factor).*
(2) $A \rtimes_r G \in \mathcal{N}$.

Then (1) *implies* (2) *and the converse holds, provided that G satisfies the Baum-Connes conjecture with coefficients in A.*

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{K}_*^{\mathrm{top}}(G;A)\otimes \mathrm{K}_*(B) & \xrightarrow{\mu_A\otimes \mathrm{id}} & \mathrm{K}_*(A\rtimes_r G)\otimes \mathrm{K}_*(B) \\
\downarrow{\scriptstyle \alpha_G} & & \downarrow{\scriptstyle \alpha} \\
\mathrm{K}_*^{\mathrm{top}}(G;A\otimes B) & \xrightarrow{\mu_{A\otimes B}} & \mathrm{K}_*((A\rtimes_r G)\otimes B) \\
\downarrow{\scriptstyle \beta_G} & & \downarrow{\scriptstyle \beta} \\
\mathrm{Tor}(\mathrm{K}_*^{\mathrm{top}}(G;A),\mathrm{K}_*(B)) & \xrightarrow{\mathrm{Tor}(\mu_A,\mathrm{id})} & \mathrm{Tor}(\mathrm{K}_*(A\rtimes_r G),\mathrm{K}_*(B)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

Since $A \in \mathcal{N}_G$ the left column is exact by Proposition 5.4.8. Now in the situation of (1), all the horizontal arrows are isomorphisms. Consequently, the right column is also exact, which establishes (2). If conversely $A \rtimes_r G \in \mathcal{N}$ and moreover G satisfies the Baum-Connes conjecture with coefficients in A, then both columns in the above diagram are exact by Proposition 5.4.8 and [**CEOO04**, Proposition 4.2] respectively. Moreover, the top and bottom horizontal maps are isomorphisms and an application of the Five Lemma completes the proof. □

Combining Theorem 5.4.11 and the preceding proposition we arrive at

COROLLARY 5.4.14. *Let G be a second countable ample groupoid and A a separable exact G-algebra such that $A_{|K} \rtimes K \in \mathcal{N}$ for all compact open subgroupoids $K \subseteq G$. If G satisfies the Baum-Connes conjecture with coefficients in $A \otimes B$ for all separable* C**-algebras B (with respect to the trivial action on the second factor), then $A \rtimes_r G \in \mathcal{N}$. In particular, $C_r^*(G) \in \mathcal{N}$, provided that G satisfies the Baum-Connes conjecture with coefficients in $C_0(G^{(0)}, B)$ for all separable* C**-algebras B (equipped with the trivial action).*

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