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Free products of sofic groups with amalgamation over monotileably amenable groups

Benoît Collins and Kenneth J. Dykema

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Abstract. We show that free products of sofic groups with amalgamation over monotileably amenable subgroups are sofic. Consequently, so are HNN extensions of sofic groups relative to homomorphisms of monotileably amenable subgroups. We also show that families of independent uniformly distributed permutation matrices and certain families of nonrandom permutation matrices (essentially, those coming from quasi-actions of a sofic group) are asymptotically $*$-free as the matrix size grows without bound.

1. Introduction

Sofic groups were introduced by M. Gromov [9] and named by B. Weiss [20]. In short, a group is sofic if it can be approximated (in a certain weak sense) by permutations. All amenable and residually amenable groups are sofic. Due in large part to work of Elek and Szabó [6], the class of sofic groups is known to be closed under taking direct products, subgroups, inverse limits, direct limits, free products, and extensions by amenable groups. See also [18] and [3] for recent interesting examples. It is unknown whether all groups are sofic, though Gromov's famous paradoxical dictum ("any statement about all countable groups is either trivial or false") would argue against it.

Several results illustrate the utility of knowing that a given group is sofic. Gromov [9] proved that Gottschalk's Surjunctivity Conjecture holds for the groups now called sofic. Elek and Szabó [4] proved that Kaplansky's Direct Finiteness Conjecture holds for sofic groups. In [5] they gave a description of sofic groups in terms of ultrapowers and proved that sofic groups are hyperlinear, which entails that their group von Neumann algebras embed in R^ω; thus, the topic of sofic groups makes contact with Connes' Embedding Problem,

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which is a fundamental open problem in the theory of von Neumann algebras. See the survey articles [15] and [16] for more on hyperlinear and sofic groups. A. Thom [17] proved some interesting results about the group rings of sofic groups. L. Bowen [1] classified the Bernoulli shifts of a sofic group, provided that the group is also Ornstein (e.g., if it contains an infinite amenable group as a subgroup).

Now we recall a few basic notions and give a definition of sofic groups. (See [4] for a proof that the definition in [20], which was for finitely generated groups, agrees with the one found below if the group is finitely generated.) The *normalized Hamming distance* $\mathrm{dist}(\sigma,\tau)$ between two permutations σ and τ, both elements of the symmetric group S_n, is defined to be the number of points not fixed by $\sigma^{-1}\tau$, divided by n. Note that if we consider S_n as acting on an n-dimensional complex vector space as permutation matrices then this normalized Hamming distance is equal to $1-\mathrm{tr}_n(\sigma^{-1}\tau)$, where tr_n is the trace on $M_n(\mathbb{C})$ normalized so that the identity has trace 1.

A group Γ is *sofic* if for every finite subset F of Γ and every $\epsilon>0$, there exist an integer $n\geq 1$ and a map $\phi:\Gamma\to S_n$ such that

(i) for every $g\in F\backslash\{e\}$, $\mathrm{dist}(\phi(g),\mathrm{id})>1-\epsilon$, where e is the identity element of Γ,

(ii) for all $g_1,g_2\in F$, $\mathrm{dist}(\phi(g_1^{-1}g_2),\phi(g_1)^{-1}\phi(g_2))<\epsilon$.

We will call a map ϕ satisfying these properties an (F,ϵ)-*quasi-action* of Γ.

Since a group is sofic if and only if all of its finitely generated subgroups are sofic, it will suffice to consider countable groups, and it will be convenient to have the elementary reformulation of soficity contained in the following proposition, whose proof is an easy exercise. Given positive integers $n(k)$, we let $\bigoplus_{k=1}^{\infty}(S_{n(k)},\mathrm{dist})$ denote the normal subgroup of $\prod_{k=1}^{\infty}S_{n(k)}$ consisting of all sequences $(\sigma_k)_{k=1}^{\infty}$ such that $\lim_{k\to\infty}\mathrm{dist}(\sigma_k,\mathrm{id}_{n(k)})=0$, where $\mathrm{id}_{n(k)}$ is the identity element of the permutation group $S_{n(k)}$.

Proposition 1.1. *Let Γ be a countable group. Then Γ is sofic if and only if for some sequence of positive integers $n(k)$, there is a group homomorphism*

$$\psi:\Gamma\to\left(\prod_{k=1}^{\infty}S_{n(k)}\right)\bigg/\left(\bigoplus_{k=1}^{\infty}(S_{n(k)},\mathrm{dist})\right),$$

given by $\psi(g)=[(\psi_k(g))_{k=1}^{\infty}]$ for some maps $\psi_k:\Gamma\to S_{n(k)}$ so that

$$\lim_{k\to\infty}\mathrm{dist}(\psi_k(g),\mathrm{id}_{n(k)})=1$$

for all nontrivial elements g of Γ.

In this paper, we prove that the class of sofic groups is closed under taking free products with amalgamation over monotileably amenable subgroups. Recall that a group G is amenable if and only if for every finite set K and every $\epsilon>0$, there is a (K,ϵ)-invariant set, namely, a finite set $F\subseteq G$ such that $|KF\setminus F|<\epsilon|F|$. A *tile* (or monotile) for a group G is a finite set $T\subseteq G$ such that G is a disjoint union of right translates of T. We may choose a set $C\subseteq G$

of *centers*, so that the map $T \times C \to G$ given by multiplication $(t, c) \mapsto tc$ is a bijection. Clearly, a translate of a tile is a tile, so we may assume $e \in T$. We will say a group G is *monotileably amenable* if for every finite set $K \subseteq G$ and every $\epsilon > 0$, there is a tile T for G that is (K, ϵ)-invariant. This notion was introduced (though not named with quite the same words we use here) by B. Weiss in his paper [21], where he proved that every residually finite amenable group and every solvable group is monotileably amenable. This class of groups includes, in addition to the solvable groups, all linear amenable groups and Grigorchuk's groups [8] of intermediate growth. It is an open problem whether all amenable groups are monotileably amenable, and this is not even known for the elementary amenable groups. However, as shown by Ornstein and Weiss [13], all amenable groups do admit quasitilings, involving finite sets of quasitiles and approximations, and this circle of ideas, as further developed by Kerr and Li [10], plays an important role in our proof.

All sofic groups are hyperlinear. An application of results of [2] is that the class of hyperlinear groups is closed under taking free products with amalgamation over amenable subgroups, and this result inspired our effort in this paper. The techniques of [2] do not appear adapted to prove that a group is sofic. The proof in [2] relied on approximation of group von Neumann algebras of amenable groups by finite dimensional algebras, which is not helpful in the context of this paper. However, one aspect of the proof found here is reminiscent of the proof in [2]: the use of independent random unitaries to model freeness with amalgamation. In [2], the random unitaries were distributed according to Haar measure in the group of unitary matrices that commute with a certain finite dimensional subalgebra, whereas here we use uniformly distributed random permutation matrices. See Remark 3.5 for more about this.

To be more precise, our construction of quasi-actions of amalgamated free product groups $\Gamma_1 *_H \Gamma_2$ where H is monotileably amenable goes by proving asymptotic vanishing of certain moments involving random permutation matrices.

Asymptotic freeness of independent matrices (of various sorts) as the matrix size grows without bound is one of the mainstays of free probability theory, going back to seminal work [19] of Voiculescu, and has been a key element in applications of free probability theory to operator algebras and elsewhere. Asymptotic freeness of independent random permutation matrices was proved by A. Nica [12]. By combining Nica's result with our vanishing of moments result, we are able to extend Nica's asymptotic freeness result to the case of independent random permutation matrices *and* certain sequences of nonrandom permutation matrices; these are essentially sequences that arise from quasi-actions of sofic groups.

The organization of the rest of this paper is as follows: in Section 2, we prove our main technical result on asymptotic vanishing of certain moments in random permutation matrices and certain nonrandom matrices; in Section 3, we apply this asymptotic vanishing theorem to prove our main result, that the class of sofic groups is closed under taking free products with amalgamation over monotileably amenable subgroups; in Section 4, we combine the result of Section 2 with Nica's asymptotic freeness result and extend Nica's result to handle certain nonrandom permutation matrices too.

2. Asymptotic vanishing of certain moments

The main result of this section (Theorem 2.1) is an asymptotic vanishing of moments result involving uniformly distributed random permutation matrices and (sequences of) nonrandom permutation matrices whose traces approach zero as matrix size increases. Actually, a broader class than permutation matrices is considered here, which is needed for applications. The theorem is used in the next section to prove the main result of the paper.

We begin by fixing some notation and definitions. If Z is a finite set then a *partition* of Z is a set $p = \{X_1, \ldots, X_n\}$ of pairwise disjoint, nonempty subsets X_j of Z whose union is all of Z. These sets X_j are called the *blocks* of the partition, and the number of blocks of p is denoted simply $|p|$. We then have the equivalence relation $\overset{p}{\sim}$ on Z defined by $z_1 \overset{p}{\sim} z_2$ if and only if z_1 and z_2 belong to the same block of p.

If $Y \subset Z$ is a nonempty subset then we let $p\restriction_Y$ denote the restriction of p to Y, namely

$$p\restriction_Y = \{X \cap Y \mid X \in p,\, X \cap Y \neq \varnothing\}.$$

We let $\mathcal{P}(n)$ denote the set of all partitions of $\{1, \ldots, n\}$ and let $\leq$ be the usual ordering of $\mathcal{P}(n)$ given by $r \leq s$ if and only if every block of r is contained in some block of s. This makes $\mathcal{P}(n)$ into a lattice, and we use $\vee$ and $\wedge$ for the join and meet operations in this lattice.

If $i = (i_1, \ldots, i_n)$ be a multi index with values in $\{1, \ldots, d\}$ and $p \in \mathcal{P}(n)$ then we define

$$\delta_{i,p} = \begin{cases} 1, & \text{if } k \overset{p}{\sim} \ell \text{ implies } i_k = i_\ell, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Let U be a random $d \times d$ permutation matrix that is uniformly distributed and let us write $U = (u_{i_1,i_2})_{1 \leq i_1,i_2 \leq d}$, keeping in mind the dependence of everything on d. We let Tr denote the usual trace on complex matrix algebras (normalized so that projections of rank 1 have trace 1) and $\mathrm{tr}_d = \frac{1}{d}\mathrm{Tr} : M_d(\mathbb{C}) \to \mathbb{C}$.

Suppose for every $j, d \in \mathbb{N}$, $B_j^{(d)}$ is a $d \times d$ matrix, all of whose entries are 0 and 1, with each row and each column having at most one nonzero entry. For example, $B_j^{(d)}$ could be permutation matrices. We will write $B_j^{(d)} = (b_{i_1,i_2}^{(j,d)})_{1 \leq i_1,i_2 \leq d}$ and often simply $B_j^{(d)} = B_j = (b_{i_1,i_2}^{(j)})_{1 \leq i_1,i_2 \leq d}$, keeping in mind the dependence on d.

Theorem 2.1. *With $B_1, \ldots, B_{2n}$ and U as above, there are constants C_n and D_n depending only on n such that, letting*

$$f(d) = \max_{1 \leq j \leq 2n} \mathrm{tr}_d(B_j), \tag{2}$$

we have

$$\int \Big(\mathrm{tr}_d\big(B_1(UB_2U^*)B_3(UB_4U^*)\cdots B_{2n-1}(UB_{2n}U^*)\big)\Big)\,dU \tag{3}$$
$$\leq C_n f(d) + D_n d^{-1}.$$

Thus, if $\lim_{d\to\infty} \mathrm{tr}_d(B_j^{(d)}) = 0$ *for all* j *then we have*

$$\lim_{d\to\infty}\int \Big(\mathrm{tr}_d\big(B_1(UB_2U^*)B_3(UB_4U^*)\cdots B_{2n-1}(UB_{2n}U^*)\big)\Big)\,dU = 0.$$

Proof. We have

$$\int \Big(\mathrm{tr}_d\big(B_1(UB_2U^*)B_3(UB_4U^*)\cdots B_{2n-1}(UB_{2n}U^*)\big)\Big)\,dU$$
$$= \frac{1}{d}\sum_{1\leq i_1,\ldots,i_{4n}\leq d} b^{(1)}_{i_1,i_2} b^{(2)}_{i_3,i_4}\cdots b^{(2n)}_{i_{4n-1},i_{4n}} \cdot \int u_{i_2,i_3}u_{i_5,i_4}u_{i_6,i_7}u_{i_9,i_8}\cdots u_{i_{4n-2},i_{4n-1}}u_{i_1,i_{4n}}\,dU.$$

Moreover, as is easily verified, for $k_1,\ldots,k_m,\ell_1,\ldots,\ell_m \in \{1,\ldots,d\}$, we have

$$\int u_{k_1,\ell_1}u_{k_2,\ell_2}\cdots u_{k_m,\ell_m}\,dU = \begin{cases} \frac{(d-|r|)!}{d!}, & r = s,\\ 0, & r\neq s,\end{cases}$$

where r and s are the partitions of $\{1,\ldots,m\}$ defined by $i \overset{r}{\sim} j$ if and only if $k_i = k_j$ and $i \overset{s}{\sim} j$ if and only if $\ell_i = \ell_j$. Therefore, we have

$$\left|\int \Big(\mathrm{tr}_d\big(B_1(UB_2U^*)B_3(UB_4U^*)\cdots B_{2n-1}(UB_{2n}U^*)\big)\Big)\,dU\right| \tag{4}$$
$$\leq \frac{1}{d}\sum_{r\in\mathcal{P}(2n)} \frac{(d-|r|)!}{d!} \sum_{i\in I(r)} b^{(1)}_{i_1,i_2} b^{(2)}_{i_3,i_4}\cdots b^{(2n)}_{i_{4n-1},i_{4n}},$$

where $I(r) = I(r,d)$ is the set of all $i = (i_1,\ldots,i_{4n}) \in \{1,\ldots,d\}^{4n}$ such that $i_a = i_b$ whenever $a \overset{p}{\sim} b$, where $p = p(r) \in \mathcal{P}(4n)$ is the partition that is the union of f applied to r and g applied to r, where $f, g : \{1,\ldots,2n\} \to \{1,\ldots,4n\}$ are given by

$$f(j) = \begin{cases} 2j, & j \text{ odd},\\ 2j+1, & j \text{ even and } j < 2n,\\ 1, & j = 2n,\end{cases}$$

$$g(j) = \begin{cases} 2j+1, & j \text{ odd},\\ 2j, & j \text{ even}.\end{cases}$$

These functions are presented in Table 1.

Table 1. The functions f and g, used to form the partition p from r.

f maps	$2n$	1			2	3			4	5			$\cdots$
g maps			1	2			3	4			5	6	$\cdots$
to	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$

f maps	$\cdots$	$2n-2$	$2n-1$		
g maps	$\cdots$			$2n-1$	$2n$
to	$\cdots$	$4n-3$	$4n-2$	$4n-1$	$4n$

An upper bound for the right-hand side of (4) when $d \geq 4n$ is

$$2^{2n} \sum_{r\in\mathcal{P}(2n)} d^{-|r|-1} \sum_{1\leq i_1,\ldots,i_{4n}\leq d} \delta_{i,p(r)} b^{(1)}_{i_1,i_2} b^{(2)}_{i_3,i_4} \cdots b^{(2n)}_{i_{4n-1},i_{4n}}, \tag{5}$$

where $\delta_{i,p(r)}$ is as defined in (1).

We will need the following result, which is purely about partitions:

Lemma 2.2. *Let $n \geq 1$ and suppose $r \in \mathcal{P}(2n)$ satisfies $2j-1 \overset{r}{\not\sim} 2j$ for all $j \in \{1,\ldots,n\}$. Let $\eta = \{\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}\} \in \mathcal{P}(2n)$. Then $|r \vee \eta| \leq |r|/2$.*

Proof. Each block X of $r \vee \eta$ contains at least two blocks of r, because if X were equal to a block of r then it would also be a union of blocks of η, which is impossible by the hypothesis on r. This finishes the proof of Lemma 2.2. □

The following lemma will be used to handle the right-most sum in (5).

Lemma 2.3. *Let $n \in \mathbb{N}$ and let p be a partition of $\{1,2,\ldots,2n\}$. Consider $(0,1)$-matrices B_j having at most one nonzero entry per row and column, (as in Theorem 2.1). Let*

$$S(p,d) = S(B_1,\ldots,B_n;p,d) := \sum_{1\leq i_1,\ldots,i_{2n}\leq d} \delta_{i,p}\, b^{(1)}_{i_1,i_2} b^{(2)}_{i_3,i_4} \cdots b^{(n)}_{i_{2n-1},i_{2n}}. \tag{6}$$

Consider $p \vee \eta$ where $\eta = \{\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}\} \in \mathcal{P}(2n)$. Then $S(p,d) \leq d^{|p\vee\eta|}$. Moreover, if

$$2j-1 \overset{p}{\sim} 2j \text{ for some } j \in \{1,\ldots,n\}, \tag{7}$$

then letting $f(d) = \max_{1\leq j\leq n} \mathrm{tr}_d(B_j)$, we have $S(p,d) \leq f(d) d^{|p\vee\eta|}$.

Proof. Writing $p \vee \eta = \{X_1,\ldots,X_m\}$, we have that p is the disjoint union $p_1 \cup \cdots \cup p_m$, where p_k is a partition of X_k. Then

$$S(B_1,\ldots,B_n;p,d) = \prod_{k=1}^{m} S(B_{i(k,1)}, B_{i(k,2)}, \ldots, B_{i(k,\ell_k)}; \tilde{p}_k, d),$$

where $X_k = \{i(k,1),\ldots,i(k,\ell_k)\}$ for $i(k,1) < i(k,2) < \cdots < i(k,\ell_k)$ and where $\tilde{p}_k$ is the appropriate renumbering of p_k. Since the condition (7) holds for p if and only if it holds for some p_k, and since $f(d) \leq 1$ for all d, it will suffice to prove the lemma in the case that $p \vee \eta$ has only one block.

Suppose $p \vee \eta$ has only one block. Fix $i_1, \ldots, i_{2n} \in \{1, \ldots, d\}$ and suppose we have

$$\delta_{i,p}\, b^{(1)}_{i_1,i_2} b^{(2)}_{i_3,i_4} \cdots b^{(n)}_{i_{2n-1},i_{2n}} \neq 0. \tag{8}$$

Since each row and column of each B_j has at most one nonzero entry, for any given $k \in \{1, \ldots, d\}$ there is at most one value of $k' \in \{1, \ldots, d\}$ such that $b^{(j)}_{k,k'} \neq 0$. Since $p \vee \eta$ has only one block, for any given $i_1 \in \{1, \ldots, d\}$ there is at most one choice of $i_2, \ldots, i_{2n}$ such that (8) holds. This implies $S(p, d) \leq d$, as required.

Now suppose $p \vee \eta$ (still) has only one block and $2j - 1 \overset{p}{\sim} 2j$ for some $j \in \{1, \ldots, n\}$. For any choice of $i_{2j} \in \{1, \ldots, d\}$, there is at most one choice of $i_1, \ldots, i_{2j-1}, i_{2j+1}, \ldots, i_{2n}$ such that (8) holds; in this choice, we must have $i_{2j-1} = i_{2j}$, because $\delta_{i,p} \neq 0$. Therefore,

$$0 \leq S(p, d) \leq \sum_{i_{2j}=1}^{d} b^{(j)}_{i_{2j},i_{2j}} = \mathrm{Tr}(B_j) = \mathrm{tr}_d(B_j)d \leq f(d)d$$

This finishes the proof of Lemma 2.3. □

Completion of the proof of Theorem 2.1. Consider $p \vee \gamma$, where

$$\gamma = \{\{1, 2\}, \{3, 4\}, \ldots, \{4n - 1, 4n\}\} \in \mathcal{P}(4n).$$

Then $|p \vee \gamma| = |r \vee \eta| + |r \vee \eta'|$, where $\eta = \{\{1, 2\}, \{3, 4\}, \ldots, \{2n - 1, 2n\}\}$ and $\eta' = \{\{2n, 1\}, \{2, 3\}, \{4, 5\}, \ldots, \{2n - 2, 2n - 1\}\}$ are from $\mathcal{P}(2n)$. By Lemma 2.3, we have

$$S(B_1, \ldots, B_{2n}; p, d) \leq d^{|r \vee \eta| + |r \vee \eta'|}.$$

Furthermore, if $2j - 1 \overset{p}{\sim} 2j$ for some $j \in \{1, \ldots, 2n\}$ then we have

$$S(B_1, \ldots, B_{2n}; p, d) \leq f(d)\, d^{|r \vee \eta| + |r \vee \eta'|},$$

where $f(d)$ as in (2).

Therefore, using the upper bound (5) for the integral in (4), in order to finish the proof of (3), it will suffice to prove: for any $r \in \mathcal{P}(2n)$, we have

$$|r \vee \eta'| + |r \vee \eta| \leq |r| + 1 \tag{9}$$

while if, furthermore,

$$j - 1 \overset{r}{\not\sim} j \quad (j \in \{1, \ldots, 2n - 1\}) \quad \text{and } 1 \overset{r}{\not\sim} 2n, \tag{10}$$

then we have

$$|r \vee \eta'| + |r \vee \eta| \leq |r|. \tag{11}$$

Let us first show that (9) holds for all $r \in \mathcal{P}(2n)$. We write $r = \{X_1 \ldots, X_m\}$ for some nonempty sets X_j and $m \geq 1$. If $m = 1$, (9) holds because we have

$|r \vee \eta'| \leq |r|$ and $|s \vee \eta| \leq |s|$. Suppose $m \geq 2$. We claim that there are $a_1, \ldots, a_{m-1}, b_1, \ldots, b_{m-1} \in \{1, \ldots, m\}$ with

$$b_i \notin \{a_1\} \cup \{b_1, \ldots, b_{i-1}\} \quad (1 \leq i \leq m-1)$$
$$a_i \in \{a_1\} \cup \{b_1, \ldots, b_{i-1}\} \quad (2 \leq i \leq m-1)$$

and with some $j_i \in X_{a_i}$, $k_i \in X_{b_i}$ such that either $j_i \overset{\eta}{\sim} k_i$ or $j_i \overset{\eta'}{\sim} k_i$, (i.e., such that j_i and k_i are distance 1 apart, modulo $2n$). Indeed if Y is any proper, nonempty subset of $\{1, \ldots, 2n\}$ then the complement of Y must contain some element that is distance 1 from some element of Y. So to prove the claim about the a_i and b_i, we start with $a_1 = 1$; taking $Y = X_{a_1}$, what we just showed implies that there is $b_1 \neq 1$ and $j_1 \in X_{a_1}$, $k_1 \in X_{b_1}$ so that either $j_1 \overset{\eta}{\sim} k_1$ or $j_1 \overset{\eta'}{\sim} k_1$. If $m = 2$ then we are done. Otherwise, letting $Y = X_{a_1} \cup X_{b_1}$, what we showed implies that there are $a_2 \in \{a_1, b_1\}$ and $b_2 \in \{1, \ldots, m\} \setminus \{a_1, b_1\}$ with $j_2 \in X_{a_2}$ and $k_2 \in X_{b_2}$ such that either $j_1 \overset{\eta}{\sim} k_1$ or $j_1 \overset{\eta'}{\sim} k_1$. If $m = 3$ then we are done. Otherwise, we continue in this manner, letting $Y = X_{a_1} \cup X_{b_1} \cup X_{b_2}$ and finding $a_3 \in \{a_1, b_1, b_2\}$ and $b_3 \notin \{a_1, b_1, b_2\}$ and $j_3 \in X_{a_3}$, $k_3 \in X_{b_3}$ as required. We continue until we have selected $m-1$ such pairs. This proves the claim.

We may form each of $r \vee \eta$ and $r \vee \eta'$ from r by performing identifications one at a time. Thus, we construct successively a sequence $r = r_0 \leq r_1 \leq \cdots \leq r_\ell = r \vee \eta$ such that $|r_i| = |r_{i-1}| - 1$ and r_i is obtained from r_{i-1} by merging two distinct blocks of r_{i-1} that contain, respectively, j and k with $j \overset{\eta}{\sim} k$, and we similarly construct a sequence $r = r'_0 \leq r'_1 \leq \cdots \leq r'_{\ell'} = r \vee \eta'$ from r to $r \vee \eta'$, by performing identifications implied by $\overset{\eta'}{\sim}$. The choice of $j_i \in X_{a_i}$ and $k_i \in X_{b_i}$ with $b_i \notin \{a_1\} \cup \{b_1, \ldots, b_{i-1}\}$ found in the previous claim shows that, by choosing the identifications accordingly, when forming $r \vee \eta$ and $r \vee \eta'$ from r, an aggregate of at least $m-1$ such identifications is made. Therefore, we have $2|r| - |r \vee \eta| - |r \vee \eta'| \geq m - 1$. But we have $|r| = m$, which yields (9).

Now we prove the inequality (11) under the additional hypothesis (10). By applying Lemma 2.2 to r and to a rotation of r, we get $|r \vee \eta| \leq |r|/2$ and $|r \vee \eta'| \leq |r|/2$. This gives immediately (11), and finishes the proof of Theorem 2.1. □

3. Sofic groups

In this section, we apply Theorem 2.1 to prove our main result.

Lemma 3.1. *Let T be a tile for a countable amenable group G, with $e \in T$, and let C be a set of centers. Let $K \subseteq G$ be a finite subset and $\epsilon > 0$. Then there is a (K, ϵ)-invariant set F of the form $F = TD$ for $D \subseteq C$.*

Proof. We may choose a finite subset $E \subseteq G$ that is $(TT^{-1}, \frac{\epsilon}{2|K|})$-invariant and also $(K, \frac{\epsilon}{2})$-invariant. Let $D = (T^{-1}E) \cap C$ and let $F = TD$. Then

$E \subseteq F \subseteq TT^{-1}E$, and

$$KF \setminus F \subseteq (KF \setminus KE) \cup (KE \setminus E) \subseteq K(F \setminus E) \cup (KE \setminus E).$$

Therefore,

$$\frac{|KF \setminus F|}{|F|} \le \frac{|K||F \setminus E| + |KE \setminus E|}{|E|} \le \frac{|K|\,|TT^{-1}E \setminus E| + |KE \setminus E|}{|E|} < \epsilon.$$

□

For a set X, by $\mathrm{Sym}(X)$ we denote the set of all permutations of X. Thus, we have $S_d = \mathrm{Sym}(\{1,\dots,d\})$. For maps $\phi : G \to \mathrm{Sym}(X)$, when G is a group, we will frequently write ϕ_g instead of $\phi(g)$.

Remark 3.2. Let $0 < \delta < 1$ and let X and X' be nonempty finite sets. If $Y \subseteq X$ and $Y' \subseteq X'$ are finite subsets satisfying $|Y| > (1-\delta)|X|$ and $|Y'| > (1-\delta)|X'|$ and if $\alpha : Y \to Y'$ is a bijection, then for G any group and $\phi : G \to \mathrm{Sym}(X)$ any map, we can define $\phi' : G \to \mathrm{Sym}(X')$ by letting $\phi'_g \circ \alpha(x) = \alpha \circ \phi_g(x)$ whenever $x \in Y \cap \phi_g^{-1}(Y)$, which defines ϕ'_g on all but at most $\delta(|X'| + |X|) \le 2\frac{\delta}{1-\delta}|X'|$ of the points of X', and by choosing some values for ϕ'_g on the other points in order to make it a permutation. If F is a finite subset of G and if ϕ is an (F,ϵ)-quasi-action then it follows that ϕ' is an (F,η)-quasi-action of G on X', where $\eta = \epsilon + 6\delta/(1-\delta)$.

Lemma 4.5 of [10] could be described as yielding quasitilings for quasi-actions of amenable groups. The following is an application of it in the case that the group has a tile. In effect, we tile each of the quasitiles with our fixed monotile.

Lemma 3.3. *Let G be an amenable group and suppose T is a tile of G, with $e \in T$. Then for every $\delta > 0$, there is $\delta' > 0$ and a finite set $F \subseteq G$, with $TT^{-1} \subseteq F$, such that if $\phi : G \to \mathrm{Sym}(X)$ is an (F,δ')-quasi-action of G then there is a set Z and there are subsets $Y \subseteq X$ and $Y' \subseteq T \times Z$ with $|Y| > (1-\delta)|X|$ and $|Y'| > (1-\delta)|T \times Z|$ and there is a bijection $\alpha : Y \to Y'$ such that*

$$\alpha \circ \phi_g \circ \alpha^{-1}(t,z) = (gt,z)$$

whenever $(t,z) \in Y'$, $g \in G$, $gt \in T$ and $\phi_g \circ \alpha^{-1}(t,z) \in Y$.

Proof. Let $C \subseteq G$ be a set of centers for the tile T. Using Lemma 3.1, for any $\ell \in \mathbb{N}$ and $\eta' > 0$ we can find sets $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_\ell$ of the form $F_k = TD_k$ for some nonempty subsets $D_k \subseteq C$, and with $TT^{-1} \subseteq F_1$ and $|F_{k-1}^{-1}F_k \setminus F_k| < \eta'|F_k|$ for $k \in \{2,3,\dots,\ell\}$.

We now apply Lemma 4.5 of [10] with $\tau = 0$ and with some $\eta > 0$ to be specified later. This lemma and its proof imply that there exist $\ell \in \mathbb{N}$ and $\eta', \eta'' > 0$ such that whenever $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_\ell$ are chosen as above then for the finite set $F = F_\ell \cup F_\ell^{-1} \subseteq G$, if X is a finite set and if $\phi : G \to \mathrm{Sym}(X)$ is a map and if $B \subseteq X$ satisfies

(i') $|B| \ge (1-\eta'')|X|$

(ii') $\phi_{st}(a) = \phi_s\phi_t(a)$, $\phi_s(a) \neq \phi_{s'}(a)$ and $\phi_e(a) = a$ for all $a \in B$ and all $s, s', t \in F$ with $s \neq s'$,

then there exist sets $C_1, \dots, C_\ell \subseteq X$ such that

(i) for all $k \in \{1, \dots, \ell\}$ the map $F_k \ni s \mapsto \phi_s(c)$ is injective

(ii) the sets $\phi(F_1)C_1, \dots, \phi(F_\ell)C_\ell$ are pairwise disjoint and the sets

$$(\phi(F_k)c)_{1 \le k \le \ell,\, c \in C_k}$$

are η-disjoint and $(1-\eta)$-cover X.

We will choose δ' so small that $\phi : G \to \mathrm{Sym}(X)$ being an (F, δ')-quasi-action will ensure the existence of B such that the hypotheses (i') and (ii') hold. Then we let X' be the disjoint union $\coprod_{k=1}^{\ell} F_k \times C_k$. We can find subsets $Y' \subseteq X'$ and $Y \subseteq X$ such that $|Y'| \ge (1-\eta)|X'|$ and $|Y| \ge (1-2\eta)|X|$ and a bijection $\alpha : Y \to Y'$ such that whenever $s \in F$, $(t, c) \in Y' \cap (F_k \times C_k)$ and $st \in F_k$, we have $\alpha \circ \phi_s \circ \alpha^{-1}(t, c) = (st, c)$. Since $F_k = TD_k$, we have a natural identification of X' with $T \times Z$, where Z is the disjoint union $\coprod_{k=1}^{\ell} D_k \times C_k$. Since $T \subseteq F_1$ and $TT^{-1} \subseteq F$, by choosing $\eta = \delta/2$, we are done. ☐

Theorem 3.4. *Let $\Gamma = \Gamma_1 *_H \Gamma_2$ be a free product of groups with amalgamation over a subgroup H. Assume that Γ_1 and Γ_2 are sofic and that H is a monotileably amenable group. Then Γ is sofic.*

Proof. We may without loss of generality assume that Γ_1 and Γ_2 are countable. Let $R_{i,1} \subseteq R_{i,2} \subseteq \cdots$ be finite subsets of Γ_i whose union is all of Γ_i. Let $K_p = (R_{1,p} \cup R_{2,p}) \cap H$ and let T_p be a tile for H such that

$$|(K_pT_p) \setminus T_p| < \frac{1}{p}|T_p|. \tag{12}$$

Fix a map $\rho_p : K_p \to \mathrm{Sym}(T_p)$ so that $(\rho_p)_h(t) = ht$ whenever $t \in T_p$, $h \in K_p$ and $ht \in T_p$.

We fix some sequence δ_p tending to 0, to be specified later. Now applying Lemma 3.3 in the case of $T_p \subseteq H$ and δ_p, we find finite sets $F_p \subseteq H$ with $T_pT_p^{-1} \subseteq F_p$ and we find $\delta'_p > 0$ as described there. We assume (without loss of generality) $\delta'_p < \delta_p$. In particular, letting $\phi_{i,p} : \Gamma_i \to \mathrm{Sym}(X_{i,p})$ be an $(F_p \cup R_{i,p}, \delta'_p)$-quasi-action of Γ_i, we find sets $Z_{i,p}$ and subsets $Y_{i,p} \subseteq X_{i,p}$ and $Y'_{i,p} \subseteq T_p \times Z_{i,p}$ with $|Y_{i,p}| > (1-\delta_p)|X_{i,p}|$ and $|Y'_{i,p}| > (1-\delta_p)|T_p \times Z_{i,p}|$ and bijections $\alpha_{i,p} : Y_{i,p} \to Y'_{i,p}$ such that

$$\alpha_{i,p} \circ (\phi_{i,p})_h \circ \alpha_{i,p}^{-1}(t, z) = (ht, z)$$

whenever $(t, z) \in Y'_{i,p}$, $h \in H$, $ht \in T$ and $(\phi_{i,p})_h \circ \alpha_{i,p}^{-1}(t, z) \in Y_{i,p}$. As described in Remark 3.2, we thus obtain $(F_p \cup R_{i,p}, \eta_p)$-quasi-actions $\phi_{i,p} : \Gamma_i \to \mathrm{Sym}(T_p \times Z_{i,p})$ such that

$$(\phi'_{i,p})_h(t, z) = (ht, z) \tag{13}$$

whenever $z \in Z_{i,p}$, $h \in H$ and $ht \in T_p$, where $\eta_p = \delta'_p + 6\delta_p/(1-\delta_p)$. By amplification, if necessary, we may without loss of generality assume $Z_{1,p} =$

$Z_{2,p}$, and we denote this set by Z_p. We may further amplify, if necessary, in order to make the cardinality of Z_p as large as desired.

Thinking of elements of $\mathrm{Sym}(T_p \times Z_p)$ as permutation matrices and elements of $M_{|T_p|\,|Z_p|}(\mathbb{C})$, making the obvious identification of this matrix algebra with $M_{|T_p|}(\mathbb{C}) \otimes M_{|Z_p|}(\mathbb{C})$ and letting $(e_{t,t'})_{t,t' \in T_p}$ be the usual system of matrix units for $M_{|T_p|}(\mathbb{C})$, we have for each $g \in \Gamma_i$

$$(\phi'_{i,p})_g = \sum_{t,t' \in T_p} e_{t,t'} \otimes B^{(i)}_{g,t,t'} \in M_{|T_p|}(\mathbb{C}) \otimes M_{|Z_p|}(\mathbb{C}), \tag{14}$$

where each $B^{(i)}_{g,t,t'}$ is a $(0,1)$-matrix having at most one 1 in each row and column. Fixing any $t, t' \in T_p$ and letting $h = t(t')^{-1}$, from (13) we see that $B^{(i)}_{h,t,t'}$ is the identity matrix. Using that $\phi'_{i,p}$ is an $(F_p \cup R_{i,p}, \eta_p)$-quasi-action and that $T_p T_p^{-1} \subseteq F_p$, we see that for every $g \in R_{i,p} \setminus H$, the permutation $(\phi'_{i,p})_g (\phi'_{i,p})^{-1}_{t(t')^{-1}}$ has at most $2\eta_p |T_p \times Z_p|$ fixed points; this implies that $B^{(i)}_{g,t,t'}$ has at most $2\eta_p |T_p \times Z_p|$ diagonal entries that are equal to 1. In other words, for $g \in R_{i,p} \setminus H$ and all $t, t' \in T_p$, we have

$$\mathrm{tr}_{|Z_p|}(B^{(i)}_{g,t,t'}) \leq 2\eta_p |T_p|. \tag{15}$$

Let U_p be a uniformly distributed random $|Z_p| \times |Z_p|$ permutation matrix, and let $V_p = 1 \otimes U_p$, taking values in $M_{|T_p|}(\mathbb{C}) \otimes M_{|Z_p|}(\mathbb{C})$. Take $n \in \mathbb{N}$ and take

$$g_j \in \begin{cases} R_{1,p} \setminus H, & j \text{ odd}, \\ R_{2,p} \setminus H, & j \text{ even}, \end{cases}$$

and consider the moment

$$\mathrm{tr}_{|T_p \times Z_p|}\big(\phi'_{1,p}(g_1)\big(V_p \phi'_{2,p}(g_2) V_p^*\big) \cdots \phi'_{1,p}(g_{2n-1})\big(V_p \phi'_{2,p}(g_{2n}) V_p^*\big)\big), \tag{16}$$

thought of as a random variable. Writing out $V_p = 1 \otimes U_p$ and using (14), we find that the moment (16) equals the sum

$$\frac{1}{|T_p|} \sum_{t_1, t_2, \ldots, t_{2n} \in T_p} \mathrm{tr}_{|Z_p|}\Bigg(B^{(1)}_{g_1,t_1,t_2}\big(U_p B^{(2)}_{g_2,t_2,t_3} U_p^*\big) \cdots B^{(1)}_{g_{2n-1},t_{2n-1},t_{2n}}\big(U_p B^{(2)}_{g_{2n},t_{2n},t_1} U_p^*\big)\Bigg).$$

Using Theorem 2.1 and (15), we find an upper bound for the expectation of the above sum to be

$$|T_p|^{2n-1}\bigg(C_n(2\eta_p|T_p|) + \frac{D_n}{|Z_p|}\bigg), \tag{17}$$

where C_n and D_n are the constants from Theorem 2.1. Since $\eta_p \leq \delta_p + 6\delta_p/(1-\delta_p)$ can be made arbitrarily small by choosing δ_p small enough, and since $|Z_p|$ can be made as large as needed, we choose δ_p and $|Z_p|$ so that for every n, the upper bound (17) tends to zero as $p \to \infty$.

Now we modify $\phi'_{i,p}$ on K_p so that they agree for $i=1,2$. By the estimate (12) and the formula (13), letting

$$(\phi''_{i,p})_g = \begin{cases} (\rho_p)_g \times \mathrm{id}_{Z_p}, & g \in K_p, \\ (\phi'_{i,p}(g))_g, & \text{otherwise,} \end{cases}$$

we see that $\phi''_{i,p}$ is an $(R_{i,p} \cup F_p, \eta_p + \frac{6}{p})$-quasi-action of Γ_i. Moreover, since $(\phi''_{i,p})_g$ agrees with $(\phi'_{i,p})_g$ if $g \notin H$, the moment (16) still tends to zero as $p \to \infty$ when $\phi''_{i,p}$ replaces $\phi'_{i,p}$. Note that V_p commutes with $\phi''_{i,p}(h)$ for all $h \in K_p$.

Now we will change our random permutation matrix V_p to a nonrandom permutation matrix, at the cost of increasing the matrix size. Indeed, V_p takes on $|Z_p|!$ different values in $M_{|T_p|\,|Z_p|}(\mathbb{C})$, each with equal probability. So define $\tilde{\phi}_{i,p} : \Gamma_i \to M_{|T_p|\,|Z_p|(|Z_p|!)}(\mathbb{C})$ by letting $\tilde{\phi}_{i,p}(g)$ be the block diagonal permutation matrix consisting of $|Z_p|!$ copies of $\phi''_{i,p}(g)$ down the diagonal, and let $\widetilde{V}_p$ be the block diagonal permutation matrix consisting of the $|Z_p|!$ different values taken by V_p repeated one after the other down the diagonal. Now it is clear that the expectation of the trace $\mathrm{tr}_{|T_p|\,|Z_p|}$ applied to a word with letters taken from $\phi''_{1,p}(\Gamma_1)$, $\phi''_{2,p}(\Gamma_2)$ and $\{V_p, V_p^*\}$ equals the trace $\mathrm{tr}_{|T_p|\,|Z_p|(|Z_p|!)}$ applied to the corresponding word of letters taken from $\tilde{\phi}_{1,p}(\Gamma_1)$, $\tilde{\phi}_{2,p}(\Gamma_2)$ and $\{\widetilde{V}_p, \widetilde{V}_p^*\}$. Upon identifying permutation matrices with permutations, we have that $\tilde{\phi}_{i,p}$ is an $(R_{i,p} \cup K_p, \eta_p + \frac{6}{p})$-quasi-action of Γ_i on the set $T_p \times Z_p \times \mathrm{Sym}(Z_p)$ and that $(\tilde{\phi}_{i,p})_h = (\rho_p)_h \times \mathrm{id}_{Z_p} \times \mathrm{id}_{\mathrm{Sym}(Z_p)}$ is independent of $i \in \{1,2\}$ and commutes with $\widetilde{V}_p$ for every $h \in K_p$.

Let $n(p) = |T_p|\,|Z_p|(|Z_p|!)$. For $i \in \{1,2\}$ we define the maps

$$\psi_i : \Gamma_i \to \left(\prod_{p=1}^{\infty} S_{n(p)}\right) \Big/ \left(\bigoplus_{p=1}^{\infty} (S_{n(p)}, \mathrm{dist})\right)$$

by

$$\psi_1(g) = \left[(\tilde{\phi}_{1,p}(g))_{p=1}^{\infty}\right]$$
$$\psi_2(g) = \left[(\widetilde{V}_p \tilde{\phi}_{2,p}(g) \widetilde{V}_p^*)_{p=1}^{\infty}\right].$$

Since the $R_{i,p}$ are increasing in p and exhaust Γ_i, and since the K_p are increasing in p and exhaust H, it follows that ψ_1 and ψ_2 are group homomorphisms that agree on H. The universal property for amalgamated free products yields a group homomorphism

$$\psi : \Gamma \to \left(\prod_{p=1}^{\infty} S_{n(p)}\right) \Big/ \left(\bigoplus_{p=1}^{\infty} (S_{n(p)}, \mathrm{dist})\right)$$

that extends ψ_1 and ψ_2. To be able to apply Proposition 1.1 to conclude that Γ is sofic, it remains to see that for every $g \in \Gamma \setminus \{e\}$, there are $\psi_p(g) \in S_{n(p)}$

such that $\psi(g) = [(\psi_p(g))_{p=1}^{\infty}]$ and

$$\lim_{p\to\infty} \mathrm{dist}(\psi_p(g), \mathrm{id}_{n(p)}) = 1. \tag{18}$$

For $g \in \Gamma$ a nontrivial group element, either $g \in H$ or we may write g as a reduced word $g = g_1 g_2 \cdots g_n$ with $g_j \in \Gamma_{i_j} \setminus H$ and $i_1 \neq i_2$, $i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$.

(a) If $g \in H$ or if $n = 1$ and $i_1 = 1$ then we may take $\psi_p(g) = \tilde{\phi}_{1,p}(g_1)$ and we get (18) by the corresponding property for the $\tilde{\phi}_{1,p}$.

(b) If $n = 1$ and $i_1 = 2$ then we may take $\psi_p(g) = \widetilde{V}_p \tilde{\phi}_{2,p}(g_1) \widetilde{V}_p^*$ and we get (18) by the corresponding property for the $\tilde{\phi}_{2,p}$, because dist is invariant under left and right multiplication.

(c) If n is even and $i_1 = 1$ then we may take

$$\psi_p(g) = \tilde{\phi}_{1,p}(g_1)\widetilde{V}_p\tilde{\phi}_{2,p}(g_2)\widetilde{V}_p^*\tilde{\phi}_{1,p}(g_3)\widetilde{V}_p\tilde{\phi}_{2,p}(g_4)\widetilde{V}_p^* \cdots \tilde{\phi}_{1,p}(g_{2n-1})\widetilde{V}_p\tilde{\phi}_{2,p}(g_{2n})\widetilde{V}_p^*$$

and the asymptotic vanishing of the moment (16) as $p \to \infty$ implies that (18) holds.

(d) In all other cases, the nontrivial element g is conjugate in Γ to an element g' of the sort considered in parts (a), (b) or (c); say $g = fg'f^{-1}$ for $f \in \Gamma$. Letting $f_p \in S_{n(p)}$ be any elements so that $\psi(f) = [(f_p)_{p=1}^{\infty}]$, we may take $\psi_p(g) = f_p \psi_p(g') f_p^{-1}$. Since dist is invariant under left and right multiplication in symmetric groups, we get (18) from the same property for the lift $(\psi_p(g'))_{p=1}^{\infty}$ of g'.

□

Remark 3.5. Consider the proof of Theorem 3.4 in the case of H a finite group. Here, with a bit of tweaking, we may arrange that $T_p = H$ and that $(\rho_p)_h \in \mathrm{Sym}(H)$ is left multiplication by h, for all p. Now this proof is analogous in spirit to the construction found in [2] of matricial microstates in a tracial free product $A *_D B$ of von Neumann algebras with amalgamation over a finite dimensional subalgebra D: one starts with microstates for generators of A and of B, one arranges that these microstates agree on generators of D, and then one conjugates with a random unitary that is Haar distributed in the group of all unitaries in the commutant of D. Where the analogy breaks down, however is that in the proof of Theorem 3.4, although we do conjugate with a random permutation that commutes with the action of H, we do not require it to take all values in the commutant of H. Thus, we construct the quasi-actions of $\Gamma_1 *_H \Gamma_2$ more cheaply than we would have expected by analogy with the proof found in [2].

From Theorem 3.4, using a well known picture of the HNN extension (which, for convenience, we sketch) and a result of Elek and Szabó about amenable extensions of sofic groups, we obtain the following result for HNN extensions of sofic groups.

Corollary 3.6. *If $\Gamma = G*_\theta$ is an HNN extension of a sofic group G relative to an injective group homomorphism $\theta : H \to G$ where H is a monotileably amenable subgroup of G then Γ is sofic.*

Proof. The group Γ is generated by G and an extra generator t with the added relations $t^{-1}ht = \theta(h)$ for all $h \in H$. As is well known, and as can be proved using Britton's Lemma and the normal form for HNN extensions (see [11]), the group Γ is isomorphic to the crossed product group $K \rtimes_\alpha \mathbb{Z}$, where K is the subgroup of Γ generated by $\bigcup_{k\in\mathbb{Z}} t^{-k}Gt^k$, by the automorphism $\alpha : x \mapsto t^{-1}xt$ of K. Moreover, K is a direct limit of groups that are obtained as free products with amalgamation over H. For integers p and q, let $K_{[p,q]}$ be the subgroup of Γ generated by $\bigcup_{p\le k\le q} t^{-k}Gt^k$. If $p \le k \le q$, let $\lambda_k : G \to K_{[p,q]}$ denote the injective $*$-homomorphism $g \mapsto t^{-k}gt^k$. Then we have

$$K_{[p,q+1]} \cong K_{[p,q]} *_H G,$$

where the amalgamation is with respect to the maps $\lambda_q \circ \theta : H \to K_{[p,q]}$ and the inclusion map $H \to G$, whereas

$$K_{[p-1,q]} \cong G *_H K_{[p,q]},$$

where the amalgamation is with respect to the maps $\lambda_q\restriction_H : H \to K_{[p,q]}$ and $\theta : H \to G$. By repeated application of Theorem 3.4, each $K_{[p,q]}$ is sofic, so their direct limit K is sofic. Since K is a normal subgroup of Γ with infinite cyclic quotient, by Theorem 1 of [6], Γ is sofic. □

4. Asymptotic freeness

In [12], A. Nica proved asymptotic $*$-freeness for independent random permutation matrices. Let I be a set and for each $d \in \mathbb{N}$, let $(U_i)_{i\in I}$ be an independent family of permutation matrix valued random variables, where each $U_i = U_{i,d}$ is a uniformly distributed random $d \times d$ permutation matrix. Let $\mathbf{E}$ denote the expectation of the underlying probability space. Let $F_I = \langle x_i \mid i \in I\rangle$ be the free group with free generators $(x_i)_{i\in I}$ and if $w \in F_I$, let $w(U)$ denote the $d \times d$ permutation matrix obtained by replacing each x_i in w with U_i and each x_i^{-1} with U_i^*. (Of course, if w is the identity element of F_I then $w(U)$ denotes the $d \times d$ identity matrix.) Nica's asymptotic freeness result is that for every nontrivial $w \in F_I$, we have $\lim_{d\to\infty} \mathbf{E}(\mathrm{tr}_d(w(U))) = 0$.

The asymptotic vanishing of moments result, Theorem 2.1, is redolent of asymptotic $*$-freeness. We will combine it with Nica's asymptotic freeness result to obtain actual asymptotic $*$-freeness of independent random permutation matrices and certain families of nonrandom permutation matrices. Though, for convenience, our statements are in terms of sequences of $d \times d$ permutation matrices for *all* natural numbers d, of course the analogous statements hold for $d_k \times d_k$ matrices, so long as $d_k \to \infty$ as $k \to \infty$.

We consider certain families of sequences of nonrandom permutation matrices; for example, these can be taken from quasi-actions of a group that are sufficient to demonstrate that the group is sofic. Let J be a set and suppose for each $j \in J$ and $d \in D$, $B_j = B_{j,d}$ is a $d \times d$ (nonrandom) permutation matrix. Suppose

$$\forall j \in J \qquad \lim_{d\to\infty} \mathrm{tr}_d(B_{j,d}) = 0$$

and

$$\forall j_1, j_2 \in J \quad \text{either} \quad \lim_{d\to\infty} \mathrm{dist}(B_{j_1}B_{j_2}, \mathrm{id}_d) = 0 \tag{19}$$

$$\text{or} \quad \exists j_3 \in J \quad \mathrm{dist}(B_{j_1}B_{j_2}, B_{j_3}) = 0, \tag{20}$$

where we are identifying permutation matrices with their corresponding permutations in S_d.

Theorem 4.1. *Let $(U_i)_{i\in I}$ and $(B_j)_{j\in J}$ be as described above. Then the family*

$$\big(\{U_i, U_i^*\}\big)_{i\in I}, \ \{B_j \mid j \in J\}$$

is asymptotically free as $d \to \infty$, meaning, that we have

$$\lim_{d\to\infty} \mathbf{E}\big(\mathrm{tr}_d\big(w_0(U)B_{j_1}w_1(U)B_{j_2}\cdots w_{n-1}(U)B_{j_n}w_n(U)\big)\big) = 0 \tag{21}$$

whenever $n \geq 0$, $j_1, \dots, j_n \in J$, $w_0, w_1, \dots, w_n \in F_I$, $w_1, \dots, w_{n-1}$ are nontrivial words and if $n = 0$ then w_0 is nontrivial.

Proof. Using the properties of the trace and the property (19)–(20) of the family of the B_j, we may cyclically reduce any expression of the form appearing on the left-hand side of (21) and we see that it equals an expression in one of the three forms

$$\lim_{d\to\infty} \mathbf{E}(\mathrm{tr}_d(w_1(U))) \tag{22}$$

$$\lim_{d\to\infty} \mathbf{E}(\mathrm{tr}_d(B_{j_1})) \tag{23}$$

$$\lim_{d\to\infty} \mathbf{E}\big(\mathrm{tr}_d\big(B_{j_1}w_1(U)B_{j_2}\cdots w_{n-1}(U)B_{j_n}w_n(U)\big)\big) \tag{24}$$

where $j_1, \dots, j_n \in J$ and $w_1 \dots, w_n$ are nontrivial elements of F_I. Here we use that if C_d and D_d are permutation matrices and if $\lim_{d\to\infty} \mathrm{dist}(C_d, D_d) = 0$, then for any permutation matrix V, we have $\lim_{d\to\infty} \mathrm{tr}_d(VC_d - VD_d) = 0$.

The limit in (22) vanishes by Nica's asymptotic freeness result. The limit in (23) vanishes by hypothesis. For the limit in (24), we will use Nica's asymptotic freeness result and Theorem 2.1. Let V be a uniformly distributed random permutation matrix that is independent from all the U_i. Since the distribution of the family $(VU_iV^*)_{i\in I}$ is the same as for $(U_i)_{i\in I}$, it will suffice to show

$$\lim_{d\to\infty} \mathbf{E}\big(\mathrm{tr}_d\big(B_{j_1}Vw_1(U)V^*B_{j_2}\cdots Vw_{n-1}(U)V^*B_{j_n}Vw_n(U)V^*\big)\big) = 0. \tag{25}$$

From Nica's asymptotic freeness result, we get for every $\epsilon > 0$

$$\lim_{d\to\infty} \mathbf{P}\Big(\max_{1\leq j\leq n} \mathrm{tr}_d(w_j(U)) \geq \epsilon\Big) = 0,$$

where $\mathbf{P}$ means the probability of the event. Therefore, we can find a sequence $\epsilon_d \searrow 0$ such that $\lim_{d\to\infty} \mathbf{P}(F_d) = 0$, where F_d is the event

$$\max_{1\le j\le n} \mathrm{tr}_d(w_j(U)) \ge \epsilon_d.$$

Since V and $(U_i)_{i\in I}$ are independent, we can evaluate the expectation in (25) by first, for each fixed choice of values for $(U_i)_{i\in I}$, integrating with respect to V, and then integrating with respect to the $(U_i)_{i\in I}$. For any choice of $(U_i)_{i\in I}$, we have by a trivial bound

$$\int \mathrm{tr}_d\big(B_{j_1} V w_1(U) V^* B_{j_2} \cdots V w_{n-1}(U) V^* B_{j_n} V w_n(U) V^*\big)\, dV \le 1.$$

If we choose values of $(U_i)_{i\in I}$ that lie in the complement of the event F_d then by Theorem 2.1, letting $f(d) = \max(\mathrm{tr}_d(B_{j_1}), \mathrm{tr}_d(B_{j_2}), \ldots, \mathrm{tr}_d(B_{j_n}))$, we have

$$\int \mathrm{tr}_d\big(B_{j_1} V w_1(U) V^* B_{j_2} \cdots V w_{n-1}(U) V^* B_{j_n} V w_n(U) V^*\big)\, dV \\ \le C_n \max(f(d), \epsilon_d) + D_n d^{-1},$$

where C_n and D_n are the constants from Theorem 2.1. So we get the upper bound

$$\mathbf{E}\big(\mathrm{tr}_d\big(B_{j_1} V w_1(U) V^* B_{j_2} \cdots V w_{n-1}(U) V^* B_{j_n} V w_n(U) V^*\big)\big) \\ \le C_n \max(f(d), \epsilon_d) + D_n d^{-1} + \mathbf{P}(F_d),$$

which tends to 0 as $d \to \infty$. □

Note added in proof: After this paper was accepted for publication, independent papers by Paunescu [14] and Elek and Szabó [7] appeared, proving that soficity of groups is preserved under taking free products with amalgamation over arbitrary amenable groups. Also (in March, 2011), equation (4) and surrounding description were corrected.

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Benoît Collins
University of Ottawa, Department of Mathematics and Statistics
585 King Edward, Ottawa, ON K1N 6N5 Canada
and
CNRS, Department of Mathematics
Lyon 1, Claude Bernard University, France
E-mail: bcollins@uottawa.ca

Kenneth J. Dykema
Texas A&M University, Department of Mathematics
College Station, TX 77843-3368, USA
E-mail: kdykema@math.tamu.edu