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Abstract

We give a shorter proof of the fact that the Jiang-Su algebra is strongly self-absorbing. This is achieved by introducing and studying so-called unitarily suspended endomorphisms of generalized dimension drop algebras. Along the way we prove uniqueness and existence results for maps between dimension drop algebras and UHF-algebras, using both elementary methods and more advanced tools like the Cuntz semigroup and uniqueness of the hyperfinite II_1-factor.

Zusammenfassung

In dieser Arbeit präsentieren wir einen kürzeren Beweis dafür, dass die Jiang-Su Algebra stark selbst-absorbierend ist. Als fundamentales Werkzeug definieren wir sogenannte unitär eingehängte Endomorphismen, dessen Existenz aus der Klassifizierung von Abbildungen zwischen Dimension-Drop-Algebren und UHF-Algebren folgt. Diese Klassifikation beweisen wir mit elementaren Mitteln, jedoch werden auch alternative Zugänge mithilfe der Cuntz Halbgruppe oder auch der Eindeutigkeit des hyperendlichen II_1-Faktors dargelegt.

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Introduction

In Elliott's classification program for nuclear C*-algebras, a particularly prominent role is played by strongly self-absorbing C*-algebras (see [Rør02] and [Win14]). A separable and unital C*-algebra $D \not\cong \mathbb{C}$ is called *strongly self-absorbing* if there exists a *-isomorphism $D \to D \otimes D$, which is approximately unitarily equivalent to the first factor embedding $\mathrm{id}_D \otimes 1_D$ ([TW07]). Strongly self-absorbing C*-algebras have approximately inner flip and hence are simple and nuclear ([ER78]). The list of known strongly self-absorbing C*-algebras is quite short and consists of the Cuntz algebras $\mathcal{O}_2, \mathcal{O}_\infty$, UHF-algebras of infinite type, tensor products of UHF-algebras of infinite type with $\mathcal{O}_\infty$ and the Jiang-Su algebra $\mathcal{Z}$.

The Jiang-Su algebra $\mathcal{Z}$ has been introduced by Jiang and Su in their remarkable paper [JS99]. For reasons we explain below, it is today one of the most natural and important objects in the classification theory of C*-algebras. Inspired by the work of Connes on the uniqueness of the hyperfinite II_1-factor $\mathcal{R}$, Jiang and Su already observed that $\mathcal{Z}$ is strongly self-absorbing ([JS99, Proposition 8.3, Corollary 8.8]). Before we discuss their proof of this fact in more detail, let us briefly put $\mathcal{Z}$ into more context.

For the classification of C*-algebras, by what is today known as the Elliott invariant, $\mathcal{Z}$-stability is of fundamental importance. The Elliott invariant of a unital C*-algebra A is given by the six-tupel

$$(\mathrm{K}_0(A), \mathrm{K}_0(A)_+, [1_A]_0, \mathrm{K}_1(A), T(A), r_A).$$

The map r_A is a natural pairing between K-theory and the trace simplex $T(A)$. Since $\mathcal{Z}$ has a unique trace and is KK-equivalent to the complex numbers $\mathbb{C}$, in fact it has the same Elliott invariant as $\mathbb{C}$, one might think of $\mathcal{Z}$ as an infinite dimensional version of the complex numbers. Under a natural restriction on the K-groups of a simple C*-algebra A, the Elliott invariant of A and $A \otimes \mathcal{Z}$ are isomorphic. This suggests that classification is in general only possible up to $\mathcal{Z}$-stability (cf. [Win14, Definition 2.6]).

Intimately related is the Toms-Winter conjecture (cf. [WZ10, Conjecture 9.3]), which predicts that the following regularity conditions for a (unital) separable, simple, nuclear and infinite dimensional C*-algebra A are equivalent:

(i) A has finite nuclear dimension,

(ii) A is $\mathcal{Z}$-stable (i.e. $A \cong A \otimes \mathcal{Z}$),

(iii) A has strict comparison.

The implication (ii) $\Rightarrow$ (iii) is proven in [Rør04, Corollary 4.6], where the strong self-absorption of $\mathcal{Z}$ is used in an essential way. Just recently, in [CET$^+$19], the nuclear dimension of a C*-algebra like A has been computed to be zero or one in the $\mathcal{Z}$-stable case and infinite otherwise. Together with a result by Winter ([Win12, Corollary 6.3]) this proves the equivalence between (ii) and (i). By previous results of many hands and decades of work, one of the most outstanding results in the classification of C*-algebras follows: unital, separable, simple, nuclear and $\mathcal{Z}$-stable C*-algebras satisfying the UCT are classified by the Elliott invariant ([CET$^+$19, Corollary D]). In particular unital, separable, simple and nuclear C*-algebras satisfying the UCT are classified up to $\mathcal{Z}$-stability. The UCT is Rosenberg and Schochet's universal coefficient theorem ([RS87]), which roughly says that the KK-theory can be computed in terms of K-theory. A C*-algebra satisfies the UCT precisely if it is KK-equivalent to a commutative C*-algebra.

Getting back to the strong self-absorption of $\mathcal{Z}$, let us discuss the proof of Jiang and Su. In [JS99, Section 2] the algebra $\mathcal{Z}$ is constructed as an inductive limit of dimension drop algebras Z_{p_n,q_n} with p_n, q_n coprime and tending to infinity, where $Z_{p,q}$ is the C*-algebra of continuous functions $f \colon [0,1] \to M_p \otimes M_q$ such that $f(0) \in M_p \otimes 1_q$ and $f(1) \in 1_p \otimes M_q$. Fundamental to their proof that $\mathcal{Z}$ is strongly self-absorbing is a classification machinery for *-homomorphisms between dimension drop algebras ([JS99, Corollary 5.6, Theorem 6.2]). These results rely on a careful analysis of maps between K-homology induced by morphisms between dimension drop algebras.

Although the notion of a strongly self-absorbing C*-algebra was first introduced in [TW07], Jiang and Su already proved two abstract properties of $\mathcal{Z}$, showing that $\mathcal{Z}$ is strongly self-absorbing:

1. $\mathcal{Z}$ has approximately inner half flip, i.e. the first and the second factor embedding of $\mathcal{Z}$ into $\mathcal{Z} \otimes \mathcal{Z}$ are approximately unitarily equivalent,

2. there exists a *-homomorphism $\varphi \colon \mathcal{Z} \otimes \mathcal{Z} \to \mathcal{Z}$ such that $\varphi \circ (\mathrm{id}_{\mathcal{Z}} \otimes 1_{\mathcal{Z}})$ is approximately inner.

The first property is proven in [JS99, Proposition 8.3], only using basic properties of the construction of $\mathcal{Z}$. The second property is proven in [JS99, Proposition 8.5] and relies heavily on the classification machinery just mentioned. Jiang and Su first show that there exists a *-homomorphism ψ from $\mathcal{Z} \otimes \mathcal{Z}$ to B, where B is a simple and unital inductive limit of dimension drop algebras having the same Elliott invariant as $\mathcal{Z}$. Now their existence result ([JS99, Theorem 6.2]) provides a unital *-homomorphism $B \to \mathcal{Z}$. By composing with ψ one gets the desired map φ. Using that any unital *-endomorphism of $\mathcal{Z}$ is approximately inner ([JS99, Theorem 7.6]), it follows that $\varphi \circ (\mathrm{id}_{\mathcal{Z}} \otimes 1_{\mathcal{Z}})$ is approximately inner. Here the uniqueness result for maps between dimension drop algebras is used once more.

The goal of this thesis is to prove that the Jiang-Su algebra is strongly self-absorbing in an as self-contained and elementary as possible way. To do so, we use a different picture of $\mathcal{Z}$. Today, there are many descriptions and characterizations of $\mathcal{Z}$, for example as universal C*-algebra ([JW14]) or as the initial object in the category of strongly self-absorbing C*-algebras ([Win11]). For us however, a construction of $\mathcal{Z}$ as an inductive limit of generalized dimension drop algebras will be most suitable. A generalized dimension drop algebra $Z_{\mathfrak{p},\mathfrak{q}}$ is defined just as before, but now with M_p and M_q replaced by UHF-algebras $M_{\mathfrak{p}}$ respectively $M_{\mathfrak{q}}$, associated to supernatural numbers $\mathfrak{p}$ and $\mathfrak{q}$. More precisely, Rørdam and Winter show in [RW10, Theorem 3.4] that for all coprime supernatural numbers of infinite type $\mathfrak{p}$ and $\mathfrak{q}$, there exists a trace collapsing[1] *-homomorphism $\mu \colon Z_{\mathfrak{p},\mathfrak{q}} \to Z_{\mathfrak{p},\mathfrak{q}}$ and that the stationary inductive limit of $Z_{\mathfrak{p},\mathfrak{q}}$ along μ is isomorphic to $\mathcal{Z}$, i.e.

$$ Z_{\mathfrak{p},\mathfrak{q}} \xrightarrow{\ \mu\ } Z_{\mathfrak{p},\mathfrak{q}} \xrightarrow{\ \mu\ } Z_{\mathfrak{p},\mathfrak{q}} \xrightarrow{\ \mu\ } \cdots \longrightarrow \mathcal{Z} \ . $$

For classification, this picture of $\mathcal{Z}$ has already been proven to be very useful, most notably in [Win14]. Also the main theorem of [GLN14, Theorem 29.5] relies on this picture of $\mathcal{Z}$. However, showing that the limit of $Z_{\mathfrak{p},\mathfrak{q}}$ along μ is strongly self-absorbing still requires comparing it to the original construction of Jiang and Su.

Using the picture of Rørdam and Winter and ideas of Winter (cf. [Win14, Section 4, Definition 4.2]), we introduce so-called *unitarily suspended* *-endomorphisms of generalized dimension drop algebras (Definition IV.2.5). Built into the definition is a unitary path, which in some sense untwists

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[1]Trace collapsing means that $\tau \circ \mu = \tau' \circ \mu$, for all tracial states τ and τ' on $Z_{\mathfrak{p},\mathfrak{q}}$.

the trace collapsing *-endomorphism. From the definition of a unitarily suspended *-endomorphism, it is then possible to prove that the stationary inductive limit along such a map is strongly self-absorbing. We thus arrive at the following theorem (Theorem IV.2.6, Theorem VI.2.2):

Main Theorem: Let $\mathfrak{p}$ and $\mathfrak{q}$ be coprime supernatural numbers of infinite type. Then,

(i) there exists a unitarily suspended *-endomorphism $\rho\colon Z_{\mathfrak{pq}} \to Z_{\mathfrak{p,q}}$,

(ii) for any unitarily suspended *-endomorphism σ of $Z_{\mathfrak{p,q}}$, the stationary inductive limit

$$\overrightarrow{Z_{\mathfrak{p,q}}^{\sigma}} := \varinjlim(Z_{\mathfrak{p,q}} \xrightarrow{\ \sigma\ } Z_{\mathfrak{p,q}} \xrightarrow{\ \sigma\ } Z_{\mathfrak{p,q}} \xrightarrow{\ \sigma\ } \cdots)$$

is strongly self-absorbing.

We will also prove that $\overrightarrow{Z_{\mathfrak{p,q}}^{\rho}}$ enjoys all classification relevant properties which $\mathcal{Z}$ has (Theorem VI.2.9) without comparing it to $\mathcal{Z}$. For example, $\overrightarrow{Z_{\mathfrak{p,q}}^{\rho}}$ is the initial object in the category of strongly self-absorbing C*-algebras (hence the limit is independent of ρ and the pair $\mathfrak{p}, \mathfrak{q}$).

This also confirms that the difficulty of showing that $\mathcal{Z}$ is strongly self-absorbing lies between that for UHF-algebras of infinite type and $\mathcal{O}_2$ or $\mathcal{O}_\infty$ (cf. [JW14, 5.2]). Let us also mention that very recently yet another proof of the strong self-absorption of $\mathcal{Z}$ has been found in [Gha19], using the theory of Fraïssé limits. This supports the previous statement.

Finally, I want to highlight that in Theorem VI.2.2 no classification theory beyond the classification of UHF-algebras is used. In particular, we do not rely on the classification of morphisms between dimension drop algebras or K-homological results. This is possible by proving existence and uniqueness results for the very specific class of maps between generalized prime dimension drop algebras and certain UHF-algebras.

Chapter I

Preliminaries

I.1 Notation

We denote by $\mathbb{N}$ the set of natural numbers, by $\mathbb{Z}$ the set of integers and by $\mathbb{C}$ the set of complex numbers. A C*-algebra A is a complex Banach *-algebra together with an antilinear involution such that $\|a^*a\| = \|a\|^2$, i.e. the C*-identity holds. Usually, C*-algebras will be denoted by the letters $A, B, C, \cdots$. If A is a C*-algebra, we will denote by A_+ the set of positive elements in A, by $A_{\leq 1}$ the closed unit ball of A, by A^+ the forced unitization of A and by $M(A)$ the multiplier algebra of A. A tracial state τ on A is a positive linear functional of norm one such that $\tau(ab) = \tau(ba)$, for all $a, b \in A$. The set of tracial states on A is denoted by $T(A)$. If $\phi\colon A \to B$ is a *-homo-morphism, we denote by $\phi^*\colon T(B) \to T(A)$ the induced map on trace spaces. By M_n we denote the C*-algebra of $n \times n$-matrices. For $\varepsilon > 0$ and $a, b \in A$ we denote $a \approx_\varepsilon b$ if and only if $\|a - b\| < \varepsilon$. If $\varphi, \psi\colon A \to B$ are maps and $\mathcal{F} \subseteq A$ is a finite subset, we write $\varphi \approx_{(\mathcal{F},\varepsilon)} \psi$ if and only if $\|\varphi(a) - \psi(a)\| < \varepsilon$, for all $a \in \mathcal{F}$. If $S, T \subseteq A$, we denote $[S, T] = \{st - ts : s \in S,\ t \in T\}$. If not specified otherwise we will denote by ι the canonical generator of $C_0(0, 1]$, i.e. $\iota(t) = t$, for $t \in (0, 1]$.

I.2 Order zero maps

In this section we recall some basic facts about order zero maps, which have been defined and studied in [WZ09].

I.2.1 Definition. Let $\phi\colon A \to B$ be a linear map. We say that ϕ is *completely*

positive (c.p.) if

$$\phi \otimes \mathrm{id}_{M_n} \colon A \otimes M_n \to B \otimes M_n$$

is positive, for all $n \in \mathbb{N}$. After identifying $A \otimes M_n$ with $M_n(A)$, we see that ϕ is c.p. if and only if for each $n \in \mathbb{N}$ the following holds:

$$M_n(A) \ni (a_{ij})_{i,j=1}^n \geq 0 \quad \Rightarrow \quad M_n(B) \ni (\phi(a_{ij}))_{i,j=1}^n \geq 0.$$

If in addition $\|\phi\| \leq 1$, we say that ϕ is *completely positive contractive* (c.p.c.).

I.2.2 Definition. Let $\phi \colon A \to B$ be a c.p. map. We say that ϕ has *order zero* if ϕ preserves orthogonality, i.e. $\phi(xy) = 0$, for all $x, y \in A$ such that $xy = 0$.

The following is a version of Stinespring's dilation theorem for order zero maps.

I.2.3 Theorem. *Let $\phi \colon A \to B$ be a c.p. order zero map and define $C :=$ $\mathrm{C}^*(\phi(A)) \subseteq B$. Then, there exists a positive element $h_\phi \in C^{**}$, with $\|h_\phi\| = \|\phi\|$, and a *-homomorphism*

$$\pi_\phi \colon A \to C^{**} \cap \{h_\phi\}',$$

such that

$$\phi(x) = \pi_\phi(x)h_\phi = h_\phi \pi_\phi(x) \qquad (x \in A).$$

If A is unital, then $h_\phi = \phi(1_A)$.

I.2.4 Remark. Let $\phi \colon A \to B$ be a c.p. order zero map. Using the structure theorem one may define an order zero functional calculus via

$$f(\phi) := f(h_\phi)\pi_\phi \qquad (f \in C_0(0, \|\phi\|]_+).$$

If ϕ is c.p.c. and $\|f\| \leq 1$, then $f(\phi)$ is again a c.p.c. order zero map.

I.2.5 Lemma. *There exists a bijection between the set of c.p.c. order zero maps from A to B and the set of *-homomorphisms from $C_0(0, 1] \otimes A$ to B.*

More precisely, let $\phi \colon A \to B$ be a c.p.c. order zero map. Then, the associated *-homomorphism $\overline{\phi} \colon C_0(0, 1] \otimes A \to B$ is given by

$$\overline{\phi}(f \otimes a) = f(h_\phi)\pi_\phi(a) \qquad (f \in C_0(0, 1], \ a \in A).$$

where h_ϕ and π_ϕ are as in Theorem I.2.3. Conversely, if $\varphi \colon C_0(0, 1] \otimes A \to B$ is a *-homomorphism, then the associated c.p.c. order zero map $\overline{\varphi}$ is given by

$$\overline{\varphi}(a) = \varphi(\iota \otimes a) \qquad (a \in A).$$

I.2.6 Remark. Let $\phi\colon A \to B$ be a c.p.c. order zero map. It follows immediately from Lemma I.2.5, that the order zero functional calculus for ϕ is given by

$$f(\phi)(a) = \overline{\phi}(f \otimes a) \qquad (a \in A,\ f \in C_0(0,1]_+).$$

I.2.7 Lemma. *Let $\phi\colon A \to B$ be a c.p.c. order zero map, $\tau \in T(B)$ and $f \in C_0(0,1]_+$. Then $\tau \circ f(\phi)$ is a bounded trace on A, i.e. a positive linear functional satisfying the trace property.*

The following definitions will be at the heart of our theory.

I.2.8 Definition. Let $h \in A$ be a positive contraction and assume $T(A) \neq \emptyset$. Then h is called a *Lebesgue contraction* if

$$\tau(f(h)) = \tau_{\mathrm{Leb}}(f) \overset{\mathrm{def}}{=} \int_0^1 f(t)\ dt \qquad (f \in C_0(0,1],\ \tau \in T(A)).$$

We also say that h has *Lebesgue spectral measure*, which refers to the fact that the measure corresponding to the functional $f \mapsto \tau(f)$ is the Lebesgue measure.

I.2.9 Definition. Let A be a C*-algebra such that $T(A) \neq \emptyset$. A *-homomorphism $\varphi\colon C_0(0,1] \to A$ is called *standard* if

$$\tau \circ \varphi = \tau_{\mathrm{Leb}} \qquad (\tau \in T(A)).$$

Equivalently, $\mathrm{Im}\big(\varphi^*\colon T(A) \to T(C_0(0,1])\big) = \{\tau_{\mathrm{Leb}}\}.$

I.2.10 Lemma. *Let A be a unital C*-algebra and let $\phi\colon A \to B$ be a c.p.c. order zero map. Then $h_\phi = \phi(1_A)$ is a Lebesgue contraction if and only if $\overline{\phi}$ is standard when restricted[1] to $C_0(0,1]$.*

I.3 Generalized dimension drop algebras

A *supernatural number* $\mathfrak{p}$ is a formal product

$$\mathfrak{p} = \prod_{p\ \mathrm{prime}} p^{\nu_p(\mathfrak{p})}, \tag{I.3.1}$$

where $\nu_p(\mathfrak{p}) \in \mathbb{N} \cup \{\infty\}$. Note that every natural number is also a supernatural number. There is an obvious way to define the product of two

---

[1]The restriction map is given by $\mathrm{id}_{C_0(0,1]} \otimes 1_A\colon C_0(0,1] \to C_0(0,1] \otimes A$.

supernatural numbers and we say that $\mathfrak{p}$ is of *infinite type* if $\mathfrak{p}^2 = \mathfrak{p}$ and $\mathfrak{p} \neq 1$. This happens precisely when $\mathfrak{p} \neq 1$ and $\nu_p(\mathfrak{p}) \in \{0, \infty\}$, for all primes p. If $(P_n)_{n=1}^{\infty}$ is a sequence of natural numbers such that P_n divides P_{n+1}, for every $n \in \mathbb{N}$, we say that the sequence $(P_n)_{n=1}^{\infty}$ converges to $\mathfrak{p}$ if for every prime number p the following holds:

$$\sup\{\nu \in \mathbb{N} : p^\nu \text{ divides } P_n, \text{ for some } n \in \mathbb{N}\} = \nu_p(\mathfrak{p}).$$

We then define the UHF-algebra $M_{\mathfrak{p}}$ by

$$M_{\mathfrak{p}} := \varinjlim(M_{P_n}, i_n),$$

where $i_n(x)$ is the block diagonal matrix in $M_{P_{n+1}}$ with $\frac{P_{n+1}}{P_n}$ copies of x on the diagonal. It is a basic result in the classification of UHF-algebras that the limit does not depend on the sequence $(P_n)_{n=1}^{\infty}$ and the unital maps i_n, see for example [RLL00, Section 7.4] and Example II.2.1. Furthermore, we say that $\mathfrak{p}$ and $\mathfrak{q}$ are *coprime* if

$$\min(\nu_p(\mathfrak{p}), \nu_p(\mathfrak{q})) = 0,$$

for all primes p.

I.3.1 Definition. ([Win14, Definition 3.1]) Let $\mathfrak{p}$ and $\mathfrak{q}$ be supernatural numbers. We then define

$$Z_{\mathfrak{p},\mathfrak{q}} := \left\{ f \in C([0,1], M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}) : \begin{array}{l} f(0) \in M_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}, \\ f(1) \in 1_{\mathfrak{p}} \otimes M_{\mathfrak{q}} \end{array} \right\}.$$

We call $Z_{\mathfrak{p},\mathfrak{q}}$ a *generalized dimension drop* algebra. If $\mathfrak{p}$ and $\mathfrak{q}$ are coprime, we say that $Z_{\mathfrak{p},\mathfrak{q}}$ is a *prime* dimension drop algebra.

I.3.2 Remark. The center of $Z_{\mathfrak{p},\mathfrak{q}}$ is given by $C([0,1])$, where we identify

$$i\colon C([0,1]) \hookrightarrow Z_{\mathfrak{p},\mathfrak{q}} : f \mapsto \big(t \mapsto f(t) \cdot 1_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}\big).$$

Furthermore, this inclusion induces an isomorphism of tracial state spaces (cf. [JS99, Lemma 2.4]):

$$i^*\colon T(Z_{\mathfrak{p},\mathfrak{q}}) \stackrel{\cong}{\to} T(C([0,1])).$$

For convenience of the reader, let us also compute the K-theory of prime dimension drop algebras. By $\mathrm{K}_*(A)$ we denote the group $\mathrm{K}_0(A) \oplus \mathrm{K}_1(A)$. Furthermore, we denote $V(A) \to \mathrm{K}_0(A) : [p] \mapsto [p]_0$, where $V(A)$ is the Murray-von Neumann semigroup.

I.3.3 Lemma. *Let $p, q \in \mathbb{N}$ be coprime. Then $\mathrm{K}_*(Z_{p,q}) \cong \mathrm{K}_*(\mathbb{C}) \cong (\mathbb{Z}, 0)$. Furthermore, this isomorphism is unital, i.e. $[1_{Z_{p,q}}]_0$ corresponds to $1 \in \mathbb{Z}$.*

Proof. Let us write $Z_{p,q}$ as a pullback:

$$
\begin{array}{ccc}
Z_{p,q} & \xrightarrow{\;\;\pi_1\;\;} & M_p \oplus M_q \\
{\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \phi_0 \oplus \phi_1} \\
C([0,1], M_{pq}) & \xrightarrow[\partial_0 \oplus \partial_1]{} & M_{pq} \oplus M_{pq}
\end{array}
$$

The maps ∂_i denote the evaluations at the endpoints. The maps ϕ_i are given by $\phi_0(x) = x \otimes 1_q$ and $\phi_1(y) = 1_p \otimes y$. Now, we get a six-term exact Mayer-Vietoris sequence ([Bla98, Theorem 21.2.2]):

$$
\begin{array}{ccc}
\begin{array}{c} \mathrm{K}_0(M_p \oplus M_q) \\ \oplus \mathrm{K}_0(C([0,1], M_{pq})) \end{array} & \xrightarrow{\begin{pmatrix} \mathrm{K}_0(\phi_0 \oplus \phi_1) \\ -\mathrm{K}_0(\partial_0 \oplus \partial_1) \end{pmatrix}} & \mathrm{K}_0(M_{pq} \oplus M_{pq}) \\[2ex]
{\scriptstyle \mathrm{K}_0(\pi_1) \oplus \mathrm{K}_0(\pi_2)}\uparrow & & \downarrow{\scriptstyle \gamma} \\[1ex]
\mathrm{K}_0(Z_{p,q}) & & \mathrm{K}_1(Z_{p,q}) \\[2ex]
{\scriptstyle \gamma}\uparrow & & \downarrow \\[1ex]
\mathrm{K}_1(M_{pq} \oplus M_{pq}) & \longleftarrow & \begin{array}{c} \mathrm{K}_1(M_p \oplus M_q) \\ \oplus \mathrm{K}_1(C([0,1], M_{pq})) \end{array}
\end{array}
$$

After identifications, this reduces to

$$
0 \longrightarrow \mathrm{K}_0(Z_{p,q}) \longrightarrow \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} q & 0 & -1 \\ 0 & p & -1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\;\gamma\;} \mathrm{K}_1(Z_{p,q}) \longrightarrow 0 \;.
$$

Let us denote the map from $\mathbb{Z}^3$ to $\mathbb{Z}^2$ by A. Then A is surjective, because p and q are coprime. Indeed, we may write $1 = \alpha p + \beta q$ for some $\alpha, \beta \in \mathbb{Z}$. Then $A(\beta, -\alpha, -\alpha p) = (1, 0)$ and $A(0, 0, -1) = (1, 1)$. By exactness, it follows immediately that $\gamma = 0$ and hence $\mathrm{K}_1(Z_{p,q}) = 0$. Next, $\mathrm{K}_0(Z_{p,q}) \cong \mathrm{Ker}(A) \cong \mathbb{Z}$, with generator (p, q, pq). Finally, $[1_{Z_{p,q}}]_0 \in \mathrm{K}_0(Z_{p,q})$ corresponds to the generator (p, q, pq) of $\mathrm{Ker}(A)$, showing that our isomorphism is unital. $\qquad\square$

I.3.4 Lemma. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be coprime supernatural numbers. Then, there is a unital isomorphism*

$$
\mathrm{K}_*(Z_{\mathfrak{p},\mathfrak{q}}) \cong \mathrm{K}_*(\mathbb{C})
$$

and $Z_{\mathfrak{p},\mathfrak{q}}$ is KK-equivalent to $\mathbb{C}$.

Proof. As in Convention IV.2.3, write $Z_{\mathfrak{p},\mathfrak{q}}$ as an inductive limit of prime dimension drop algebras. Then, continuity of K_* and Lemma I.3.3 show that there is a unital isomorphism $\mathrm{K}_*(Z_{\mathfrak{p},\mathfrak{q}}) \cong \mathrm{K}_*(\mathbb{C})$. Furthermore, note that $Z_{\mathfrak{p},\mathfrak{q}}$ satisfies the UCT. Indeed, there is an obvious short exact sequence

$$0 \to C_0(0,1) \to Z_{\mathfrak{p},\mathfrak{q}} \to M_{\mathfrak{p}} \oplus M_{\mathfrak{q}} \to 0.$$

Since $C_0(0,1)$ and $M_{\mathfrak{p}} \oplus M_{\mathfrak{q}}$ satisfy the UCT, a two-out-of-three type of argument ([Rør02, Theorem 2.4.7 (i)]) shows that $Z_{\mathfrak{p},\mathfrak{q}}$ satisfies the UCT, too. The statement about KK-equivalence then follows from the fact that the inclusion $\mathbb{C} \hookrightarrow Z_{\mathfrak{p},\mathfrak{q}}$ defines a KK-equivalence in $\mathrm{KK}(\mathbb{C}, Z_{\mathfrak{p},\mathfrak{q}})$, cf. [Rør02, Theorem 2.4.6 (ii)]. $\qquad\square$

The following definition is a generalization of the one given in [Rør04, Section 2].

I.3.5 Definition. Let $\mathfrak{p}$ and $\mathfrak{q}$ be supernatural numbers and let $\varphi \colon Z_{\mathfrak{p},\mathfrak{q}} \to A$ be a unital $*$-homomorphism, where A is a unital C*-algebra with $T(A) \neq \emptyset$. We say that φ is *standard* if

$$(\tau \circ \varphi)(f) = \int_0^1 \tau_{M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}}(f(t))\ dt \qquad (\tau \in T(A),\ f \in Z_{\mathfrak{p},\mathfrak{q}}).$$

In other words φ is standard if

$$\tau \circ \varphi = i^*(\tau_{\mathrm{Leb}}) \qquad (\tau \in T(A)),$$

where we recall that $i \colon C([0,1]) \to Z_{\mathfrak{p},\mathfrak{q}}$ is the canonical inclusion.

I.3.6 Definition. Let $Z_{\mathfrak{p},\mathfrak{q}}$ be a generalized dimension drop algebra. We then identify $C_0[0,1) \otimes M_{\mathfrak{p}}$ and $C_0(0,1] \otimes M_{\mathfrak{q}}$ with subalgebras of $Z_{\mathfrak{p},\mathfrak{q}}$ via the inclusions

$$\grave{\imath} \colon C_0[0,1) \otimes M_{\mathfrak{p}} \hookrightarrow Z_{\mathfrak{p},\mathfrak{q}}; \quad \grave{\imath}(f \otimes x)(t) = f(t)(x \otimes 1_{\mathfrak{q}}),$$
$$\acute{\imath} \colon C_0(0,1] \otimes M_{\mathfrak{q}} \hookrightarrow Z_{\mathfrak{p},\mathfrak{q}}; \quad \acute{\imath}(g \otimes y)(t) = g(t)(1_{\mathfrak{p}} \otimes y).$$

Whenever $\varphi \colon Z_{\mathfrak{p},\mathfrak{q}} \to A$ is a $*$-homomorphism, we denote

$$\grave{\varphi} = \varphi \circ \grave{\imath} \colon C_0[0,1) \otimes M_{\mathfrak{p}} \to A,$$
$$\acute{\varphi} = \varphi \circ \acute{\imath} \colon C_0(0,1] \otimes M_{\mathfrak{q}} \to A.$$

Remember, that these maps correspond uniquely to c.p.c. order zero maps

$$\overline{(\check{\varphi})'} \colon M_{\mathfrak{p}} \to A,$$
$$\overline{\check{\varphi}} \colon M_{\mathfrak{q}} \to A,$$

where $(\cdot)'$ means that we have reversed the orientation of the interval. More precisely,

$$(\check{\varphi})' \colon C_0(0,1] \otimes M_{\mathfrak{p}} \to A : f \otimes x \mapsto \check{\varphi}(\check{f} \otimes x),$$

where $\check{f}(x) := f(1-x)$.

I.3.7 Remark. Note that a *-homomorphism $\varphi \colon Z_{\mathfrak{p},\mathfrak{q}} \to A$ is standard if and only if $\varphi \circ i \colon C([0,1]) \to A$ is standard. This follows immediately from the fact that i^* is an isomorphism (see Remark I.3.2).

I.3.8 Lemma. *Let $\acute{\imath}$ and $\grave{\imath}$ be as in Definition I.3.6. Then the images of $\grave{\imath}$ and $\acute{\imath}$ commute and*

$$\mathrm{C}^*\big(\mathrm{Im}(\grave{\imath}), \mathrm{Im}(\acute{\imath})\big) = Z_{\mathfrak{p},\mathfrak{q}},$$

i.e. $Z_{\mathfrak{p},\mathfrak{q}}$ is canonically generated by two commuting cones over $M_{\mathfrak{p}}$ resp. $M_{\mathfrak{q}}$.

Proof. This follows easily by an approximation argument. □

The following theorem by Rørdam and Winter ([RW10, Theorem 2.5]) shows that any prime dimension drop algebra $Z_{p,q}$ can be seen as the universal C*-algebra generated by two c.p.c. order zero maps on M_p resp. M_q, which have commuting ranges and are "large".

I.3.9 Theorem. *Let A be a unital C*-algebra and let $p, q \in \mathbb{N}$ be coprime. If $\alpha \colon M_p \to A$ and $\beta \colon M_q \to A$ are c.p.c. order zero maps such that*

$$[\alpha(M_p), \beta(M_q)] = \{0\}, \quad \alpha(1_p) + \beta(1_q) = 1_A,$$

*then there exists a (unique) unital *-homomorphism*

$$\varphi \colon Z_{p,q} \to A,$$

such that the following diagram commutes:

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The map $(\overline{\alpha})'$ is obtained by reversing the orientation of the unit interval. Furthermore, φ is standard if and only if h_α and[2] h_β are Lebesgue contractions.

Proof. It remains to prove the last statement. Assume h_β is a Lebesgue contraction. To show that φ is standard, it is enough to show that φ is standard on $C([0,1])$, as already observed earlier. Since $C([0,1])$ is the unitization of $C_0(0,1]$, it is enough to check that φ is standard on $C_0(0,1]$. So let $f \in C_0(0,1]$ and $\tau \in T(A)$. Then

$$\tau(\varphi(f)) = \tau(\acute{\phi}(f)) = \tau\big(\overline{\beta}(f)\big) = \tau_{\mathrm{Leb}}(f).$$

Note that we use Lemma I.2.10 to show that $\overline{\beta}$ is standard, when restricted to $C_0(0,1]$.

Conversely, if φ is standard it is easy to check that h_α and h_β are Lebesgue contractions. $\qquad\square$

I.4 Ultrapowers

In this thesis we will stick to ultrapowers instead of sequence algebras. The main reason for this is the existence of induced traces on ultrapowers.

I.4.1 Definition. For a fixed free ultrafilter ω on $\mathbb{N}$, we denote the *uniform ultrapower* of A by

$$A_\omega := \ell^\infty(A)/\left\{(x_n)_{n=1}^\infty : \lim_{n\to\omega} \|x_n\| = 0\right\}.$$

The equivalence class of a sequence $(x_n)_{n=1}^\infty \in \ell^\infty(A)$ will be denoted by $\big[(x_n)_{n=1}^\infty\big] \in A_\omega$. The diagonal inclusion of A into A_ω is denoted by ι_ω. It will be often convenient to identify A in this way with a subalgebra of A_ω.

If $\tau \in T(A)$, we get a trace $\tau_\omega \in T(A_\omega)$ by

$$\tau_\omega([(x_n)_{n=1}^\infty]) := \lim_\omega \tau(x_n).$$

The limit exists because every bounded sequence of real numbers converges along an ultrafilter. The set of induced traces will be denoted by

$$T(A)_\omega := \{\tau_\omega : \tau \in T(A)\} \subseteq T(A_\omega).$$

---

[2]It is enough to require that h_β is a Lebesgue contraction, since $h_\alpha = 1 - h_\beta$ is then a Lebesgue contraction, too.

I.5 Some equivalence relations

I.5.1 Definition. Let A and B be C*-algebras with A separable and B unital. For *-homomorphisms $\varphi, \psi \colon A \to B$ we define the following equivalence relations:

(i) *Unitary equivalence* $\sim_{\mathrm{u}}$: There exists a unitary $u \in B$ with

$$u\varphi(a)u^* = \psi(a) \qquad (a \in A).$$

(ii) *Approximate unitary equivalence* $\approx_{\mathrm{u}}$: There exists a sequence $(u_n)_{n=1}^{\infty}$ of unitaries in B with

$$\lim_{n \to \infty} \|u_n \varphi(a) u_n^* - \psi(a)\| = 0 \qquad (a \in A).$$

(iii) *Strong asymptotic unitary equivalence* $\sim_{\mathrm{asu}}$: There exists a unitary path $(u_t)_{t \in [0,1)}$ in B such that $u_0 = 1_B$ and

$$\lim_{t \to 1} \|u_t \varphi(a) u_t^* - \psi(a)\| = 0 \qquad (a \in A).$$

(iv) *Murray-von Neumann equivalence* $\sim$: There exists a contraction $v \in B$ with

$$v\varphi(a)v^* = \psi(a), \quad v^*\psi(a)v = \varphi(a) \qquad (a \in A).$$

(v) *Approximate Murray-von Neumann equivalence* $\approx$: There exists a sequence $(v_n)_{n=1}^{\infty}$ of contractions in B with

$$\lim_{n \to \infty} \|v_n \varphi(a) v_n^* - \psi(a)\| = 0 \qquad (a \in A),$$
$$\lim_{n \to \infty} \|v_n^* \psi(a) v_n - \varphi(a)\| = 0 \qquad (a \in A).$$

These equivalence relations also make sense for c.p.c. order zero maps, as they are identified with *-homomorphisms on the corresponding cone (see Lemma I.2.5). The same applies to positive contractions, which correspond to c.p.c. order zero maps on $\mathbb{C}$.

Let us also note that our definition of Murray-von Neumann equivalence generalizes the more common Murray-von Neumann equivalence of projections.

I.5.2 Remark. ([Rør02, Lemma 6.2.5]) Let A and B be C*-algebras with A separable and B unital. For *-homomorphisms $\varphi, \psi \colon A \to B$ we have

$$\varphi \approx_{\mathrm{u}} \psi \iff \iota_\omega \circ \varphi \sim_{\mathrm{u}} \iota_\omega \circ \psi \iff \iota_\omega \circ \varphi \approx_{\mathrm{u}} \iota_\omega \circ \psi.$$

Furthermore, if $\rho, \sigma \colon A \to B_\omega$ are *-homomorphisms, a reindexing argument shows that

$$\rho \approx_{\mathrm{u}} \sigma \iff \rho \sim_{\mathrm{u}} \sigma.$$

Chapter II

Strongly self-absorbing C^*-algebras

In this chapter we recall the concept of a strongly self-absorbing C^*-algebra, which was formally introduced and studied by Toms and Winter in [TW07]. Furthermore, we look at the methods used so far to show that $\mathcal{O}_2, \mathcal{O}_\infty$ and $\mathcal{Z}$ are strongly self-absorbing.

II.1 Generalities

II.1.1 Definition. Let D be a unital and separable C^*-algebra such that $D \not\cong \mathbb{C}$. Then D is called *strongly self-absorbing* if there exists an isomorphism $\varphi \colon D \to D \otimes D$ such that $\varphi \approx_{\mathrm{u}} \mathrm{id}_D \otimes 1_D$, i.e. there exists a sequence of unitaries $(u_n)_{n=1}^\infty$ in $D \otimes D$ such that

$$\lim_{n \to \infty} \|u_n \varphi(x) u_n^* - x \otimes 1_D\| = 0 \qquad (x \in D).$$

II.1.2 Definition. Let D be a unital and separable C^*-algebra. We say that D has *approximately inner half flip* if $\mathrm{id}_D \otimes 1_D \approx_{\mathrm{u}} 1_D \otimes \mathrm{id}_D$, i.e. there exists a sequence of unitaries $(u_n)_{n=1}^\infty$ in $D \otimes D$ such that

$$\lim_{n \to \infty} \|u_n (x \otimes 1_D) u_n^* - 1_D \otimes x\| = 0.$$

Similarly, D has approximately *inner flip* if the flip map

$$\sigma_{D,D} \colon D \otimes D \to D \otimes D : x \otimes y \mapsto y \otimes x$$

is approximately inner, i.e. $\sigma_{D,D} \approx_{\mathrm{u}} \mathrm{id}_{D \otimes D}$.

Let us remark on the difference between approximately inner half flip and approximately inner flip.

II.1.3 Remark. It is clear that approximately inner flip implies approximately inner half flip. The converse however is not true. Examples of C*-algebras having approximately inner half flip but not approximately inner flip are $M_{n+1}(\mathcal{O}_n)$ for $2 < n < \infty$, see [FHRT17, Example 3.6].

Another example is the following. Let us look at the unital UCT Kirchberg algebra[1] $\mathcal{O}^\infty$, determined by

$$\mathrm{K}_*(\mathcal{O}^\infty) = (0, \mathbb{Z}).$$

By the Künneth formula, we know that

$$\mathrm{K}_*(\mathcal{O}^\infty \otimes \mathcal{O}^\infty) = (\mathbb{Z}, 0)$$

and the class of the unit is zero, i.e. $\mathcal{O}^\infty \otimes \mathcal{O}^\infty \cong \mathcal{O}_\infty^{\mathrm{st}}$, where st indicates that $\mathcal{O}_\infty$ is in standard form[2]. By the classification of Kirchberg algebras, it follows that the half flip of $\mathcal{O}^\infty$ is approximately inner, see also [FHRT17, Lemma 3.4] . However, the flip of $\mathcal{O}^\infty$ is not approximately inner.

To see this, I refer to an argument[3] due to Aaron Tikuisis, Dominic Enders and myself: The following diagram commutes, where A and B are arbitrary C*-algebras and α is the natural K-theory product as appearing in the Künneth formula ([Bla98, Theorem 23.1.3]).

$$
\begin{array}{ccc}
\mathrm{K}_1(A) \otimes \mathrm{K}_1(B) & \xrightarrow{\alpha_{A,B}} & \mathrm{K}_0(A \otimes B) \\
{\scriptstyle -\sigma_{\mathrm{K}_1(A),\mathrm{K}_1(B)}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{K}_0(\sigma_{A,B})} \\
\mathrm{K}_1(B) \otimes \mathrm{K}_1(A) & \xrightarrow{\alpha_{B,A}} & \mathrm{K}_0(B \otimes A)
\end{array}
$$

By naturality of α and a diagram chasing argument, the general case can be reduced to $A = B = C(\mathbb{T})$. Let $z \in C(\mathbb{T})$ be the canonical generator, so that $[z]_1$ is a generator of $\mathrm{K}_1(C(\mathbb{T})) \cong \mathbb{Z}$. We identify $C(\mathbb{T}) \otimes C(\mathbb{T})$ with $C(\mathbb{T}^2)$, and under this identification, the flip map $\sigma_{C(\mathbb{T}),C(\mathbb{T})}$ corresponds to swapping coordinates:

$$(\sigma_{C(\mathbb{T}),C(\mathbb{T})}(f))(w, z) = f(z, w).$$

---

[1]A *Kirchberg algebra* is a separable, simple, nuclear and purely infinite C*-algebra.

[2]A unital Kirchberg algebra A is said to be in standard form if $[1_A]_0 = 0$. Equivalently, there exists a unital embedding $\mathcal{O}_2 \to A$.

[3]This argument is part of future work and in particular it is a correction of [Tik16, Lemma 4.1].

We have $\mathrm{K}_0(C(\mathbb{T}^2)) \cong \mathbb{Z}^2$, generated by $[1_{C(\mathbb{T}^2)}]_0$ and the Bott element b. Note that $\mathrm{K}_0(\sigma_{C(\mathbb{T}),C(\mathbb{T})})(b) = -b^4$ and $\alpha_{C(\mathbb{T}),C(\mathbb{T})}([z]_1 \otimes [z]_1) = b$. Thus,

$$\alpha_{C(\mathbb{T}),C(\mathbb{T})}(\sigma_{\mathrm{K}_1(C(\mathbb{T})),\mathrm{K}_1(C(\mathbb{T}))}([z]_1 \otimes [z]_1)) = \alpha_{C(\mathbb{T}),C(\mathbb{T})}([z]_1 \otimes [z]_1)$$
$$= b,$$

and

$$\mathrm{K}_0(\sigma_{C(\mathbb{T}),C(\mathbb{T})})(\alpha_{C(\mathbb{T}),C(\mathbb{T})}([z]_1 \otimes [z]_1)) = \mathrm{K}_0(\sigma_{C(\mathbb{T}),C(\mathbb{T})})(b)$$
$$= -b$$

In particular, the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{K}_1(\mathcal{O}^\infty) \otimes \mathrm{K}_1(\mathcal{O}^\infty) & \overset{\cong}{\longrightarrow} & \mathrm{K}_0(\mathcal{O}^\infty \otimes \mathcal{O}^\infty) \\
{\scriptstyle -\sigma_{\mathrm{K}_1(\mathcal{O}^\infty),\mathrm{K}_1(\mathcal{O}^\infty)}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{K}_0(\sigma_{\mathcal{O}^\infty,\mathcal{O}^\infty})} \\
\mathrm{K}_1(\mathcal{O}^\infty) \otimes \mathrm{K}_1(\mathcal{O}^\infty) & \overset{\cong}{\longrightarrow} & \mathrm{K}_0(\mathcal{O}^\infty \otimes \mathcal{O}^\infty)
\end{array}
$$

Note that the horizontal map $\alpha_{\mathcal{O}^\infty,\mathcal{O}^\infty}$ is an isomorphism, by the Künneth formula. After identifying $\mathrm{K}_1(\mathcal{O}^\infty) \otimes \mathrm{K}_1(\mathcal{O}^\infty)$ with $\mathbb{Z}$, we see that

$$\mathrm{K}_0(\sigma_{\mathcal{O}^\infty,\mathcal{O}^\infty}) = -\,\mathrm{id},$$

proving that $\sigma_{\mathcal{O}^\infty,\mathcal{O}^\infty}$ cannot be approximately inner.

Let us recall some standard results about strongly self-absorbing C*-algebras.

II.1.4 Lemma. *Let D be a unital and separable C*-algebra. Then the following holds:*

(i) *if D has approximately inner half flip, then D is simple and nuclear,*

(ii) *if D is strongly self-absorbing, then D has approximately inner flip and D is either purely infinite or stably finite with a unique trace,*

(iii) *if D and E are strongly self-absorbing, then $D \otimes E$ is strongly self-absorbing.*

---

[4]Since the Bott element in $\mathrm{K}_0(C_0((0,1)^2))$ arises from a clutching construction, any reflection on $(0,1)^2$, and in particular the coordinate swap, i.e., the flip map, sends this Bott element to its inverse. Using the canonical embedding $C_0((0,1)^2) \to C(\mathbb{T}^2)$, it follows that the same is true for the flip map and the Bott element for $C(\mathbb{T}^2)$.

Proof. The proofs of (ii) and (iii) are in [TW07, Section 1]. The proof of (i) is a slight modification of [ER78, Proposition 2.7], where the statement is proven for approximately inner flip. Following their argument, let us just prove that approximately inner half flip implies nuclearity. Given $\mathcal{F} \subseteq D$ finite and $\varepsilon > 0$ let $u \in D \otimes D$ be a unitary such that

$$\|u(x \otimes 1_D)u^* - 1_D \otimes x\| < \varepsilon \qquad (x \in \mathcal{F}).$$

Then, find a contraction v in the algebraic tensor product $D \odot D$ such that $\|u - v\| < \frac{\varepsilon}{2}$. Let $\varphi \in S(D)$ be any state on D and consider the slice map[5]

$$\eta_\varphi \colon D \otimes D \to D : d \otimes d' \mapsto \varphi(d)d'.$$

Then η_φ is a c.p.c. map and we get a factorization of id_D on $\mathcal{F}$ up to ε as follows:

$$
\begin{array}{ccc}
D & \xrightarrow{\ \mathrm{id}_D\ } & D \\
{\scriptstyle \mathrm{id}\,\otimes 1}\downarrow & & \uparrow{\scriptstyle \eta_\varphi} \\
D \otimes D & \xrightarrow[v(\cdot)v^*]{} & D \otimes D
\end{array}
$$

Note that the composition is a finite rank operator.[6] It follows that D is nuclear. $\qquad\square$

The following is [Rør02, Proposition 7.2.1].

II.1.5 Proposition. *Let A and B be separable C*-algebras and assume $\varphi \colon A \to B$ is an injective *-homomorphism. Assume that there are unitaries $(u_n)_{n=1}^\infty$ in $M(B)_\omega \cap \varphi(A)'$ such that*

$$\lim_{n\to\infty} \mathrm{dist}(u_n b u_n^*, \varphi(A)_\omega) = 0 \qquad (b \in B).$$

Then, there is an isomorphism $\psi \colon A \to B$, such that $\psi \approx_\mathrm{u} \varphi$.

The next theorem is [Rør02, Theorem 7.2.2] and is sometimes referred to as *ultrapower intertwining*.

II.1.6 Theorem. *Let A and B be separable and unital C*-algebras. Assume B has approximately inner half flip and embeds unitally into $A_\omega \cap A'$. Then, there exists an isomorphism $\varphi \colon A \to A \otimes B$ such that $\varphi \approx_\mathrm{u} \mathrm{id}_A \otimes 1_B$. In particular, if $A \not\cong \mathbb{C}$ and $A = B$, then A is strongly self-absorbing.*

---

[5]The existence of η_φ follows for example from [BO08, Theorem 3.5.3], since $\eta_\varphi = \varphi \otimes \mathrm{id}_D$.

[6]If $v = \sum_{i=1}^n x_i \otimes y_i$, then the image is contained in $\mathrm{span}_{\mathbb{C}}\{x_i y_j^* : i, j = 1, 2, \cdots, n\}$.

Proof. We show that the embedding $\mathrm{id}_A \otimes 1_B \colon A \to A \otimes B$ satisfies the hypothesis of Proposition II.1.5. By assumption there exists a unital *-homomorphism $B \to A_\omega \cap A'$ and hence also

$$\alpha \colon B \to (A \otimes 1_B)_\omega \cap (A \otimes 1_B)' \subseteq (A \otimes B)_\omega \cap (A \otimes 1_B)'.$$

Denote

$$\beta \colon B \to (A \otimes B)_\omega \cap (A \otimes 1_B)' : b \mapsto 1_A \otimes b.$$

Note that α and β are unital and injective *-homomorphisms with commuting ranges, each of which is isomorphic to B. Injectivity of α follows from the fact that B is simple (see Lemma II.1.4 (i)). Hence

$$\mathrm{C}^*(\alpha(B), \beta(B)) \cong B \otimes B.$$

Since B has approximately inner half flip, it follows that there are unitaries $u_n \in (A \otimes B)_\omega \cap (A \otimes 1_B)'$ such that

$$u_n \alpha(b) u_n^* \to \beta(b) \qquad (b \in B).$$

Let $a \in A$ and $b \in B$. Then

$$\begin{aligned}
u_n^*(a \otimes b)u_n &= u_n^*(a \otimes 1_B)\beta(b)u_n \\
&= (a \otimes 1_B)u_n^*\beta(b)u_n \to (a \otimes 1_B)\alpha(b) \in (A \otimes 1_B)_\omega.
\end{aligned}$$

Hence

$$\mathrm{dist}(u_n^* x u_n, (A \otimes 1_B)_\omega) \to 0 \qquad (x \in A \otimes B).$$

By Proposition II.1.5, there exists an isomorphism $\varphi \colon A \to A \otimes B$ such that $\varphi \approx_{\mathrm{u}} \mathrm{id}_A \otimes 1_B$. The last statement follows from the definition of strongly self-absorbing C*-algebras. $\qquad \square$

The next theorem is [TW07, Proposition 1.10] and provides an abstract way to show that a given C*-algebra is strongly self-absorbing.

II.1.7 Theorem. *Let D be a separable and unital* C*-*algebra such that $D \not\cong \mathbb{C}$. Then the following are equivalent:*

(i) *D is strongly self-absorbing,*

(ii) *D has approximately inner half flip and there exists a unital *-homomorphism $\psi \colon D \otimes D \to D$ such that $\psi \circ (\mathrm{id}_D \otimes 1_D) \approx_{\mathrm{u}} \mathrm{id}_D$.*

Proof. (ii) $\Rightarrow$ (i): Let $(u_n)_{n=1}^{\infty}$ be a sequence of unitaries in $D \otimes D$ such that

$$\|u_n\psi(d \otimes 1_D)u_n^* - d\| \to 0 \qquad (d \in D).$$

Define

$$\phi_n \colon D \to D : d \mapsto u_n\psi(1_D \otimes d)u_n^*.$$

Using that $\psi \circ (\mathrm{id}_D \otimes 1_D)$ and $\psi \circ (1_D \otimes \mathrm{id}_D)$ have commuting ranges, it follows that

$$\|\phi_n(d)d' - d'\phi_n(d)\| \to 0 \qquad (d, d' \in D),$$

i.e. D admits an approximately central sequence of unital *-homomorphisms. In particular, there exists a unital *-homomorphism

$$D \to D_\omega \cap D' : d \mapsto [(\phi_n(d))_{n=1}^{\infty}].$$

Since D has approximately inner half flip, Theorem II.1.6 shows that D is strongly self-absorbing.

(i) $\Rightarrow$ (ii): By assumption, there exists an isomorphism $\varphi \colon D \to D \otimes D$ such that $\varphi \approx_{\mathrm{u}} \mathrm{id}_D \otimes 1_D$. So let $(u_n)_{n=1}^{\infty}$ be a sequence of unitaries in $D \otimes D$ such that

$$u_n\varphi(d)u_n^* \to d \otimes 1_D \qquad (d \in D).$$

Define $w_n \coloneqq \varphi^{-1}(u_n)$ and $\psi \coloneqq \varphi^{-1}$. Then

$$w_n^*\psi(d \otimes 1_D)w_n \to d \qquad (d \in D),$$

showing that $\psi \circ (\mathrm{id}_D \otimes 1_D) \approx_{\mathrm{u}} \mathrm{id}_D$. Furthermore, D has approximately inner half flip by Lemma II.1.4 (ii). $\qquad\square$

II.1.8 Remark. Assuming (ii) in Theorem II.1.7, it is also possible to prove that D is strongly self-absorbing without knowing about ultrapowers or central sequence algebras. Indeed, one can perform an ordinary approximate intertwining, by showing that

$$(\mathrm{id}_D \otimes 1_D) \circ \psi \approx_{\mathrm{u}} \mathrm{id}_{D \otimes D}.$$

The approximate intertwining will then look as follows:

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Once this is established, it is entailed in the approximate intertwining that the isomorphism $\varphi\colon D \to D \otimes D$ is approximately unitarily equivalent to $\mathrm{id}_D \otimes 1_D$ (see [Rør02, Corollary 2.3.4]).

Let us denote the factor embeddings by $\iota^{[i]}\colon D \to D \otimes D$, for $i = 1, 2$ (cf. Notation VI.1.1). Using that D has approximately inner half flip, let $(\bar{u}_n)_{n=1}^\infty$ be a sequence of unitaries in $D \otimes D$ such that

$$(\bar{u}_n)^*(1_D \otimes y)\bar{u}_n \to y \otimes 1_D \qquad (y \in D).$$

Define an endomorphism θ of $D \otimes D$ by $\theta = (\psi \circ \iota^{[2]}) \otimes \mathrm{id}_D$ and let $u_n := \theta(\bar{u}_n)$. One then has

$$
\begin{aligned}
u_n^*(1_D \otimes y)u_n &= u_n^*\theta(1_D \otimes y)u_n \\
&\to \theta(y \otimes 1_D) \\
&= \psi(1_D \otimes y) \otimes 1_D,
\end{aligned}
$$

for all $y \in D$. Next, by assumption $\psi \circ \iota^{[1]} \approx_{\mathrm{u}} \mathrm{id}_D$. Let $(v_n)_{n=1}^\infty$ be a sequence of unitaries in D such that

$$v_n^* x v_n \to \psi(x \otimes 1_D) \qquad (x \in D).$$

Define $w_n := (v_n \otimes 1_D)u_n \in D \otimes D$. Let $\varepsilon > 0$ and $x \in D$. Then, for large $n \in \mathbb{N}$ we get:

$$
\begin{aligned}
w_n^*(x \otimes 1_D)w_n &= u_n^*(v_n^* \otimes 1_D)(x \otimes 1_D)(v_n \otimes 1_D)u_n \\
&\approx_\varepsilon u_n^*(\psi(x \otimes 1_D) \otimes 1_D)u_n \\
&= \psi(x \otimes 1_D) \otimes 1_D.
\end{aligned}
$$

The last equality follows because the images of $(\psi \circ \iota^{[1]}) \otimes 1_D$ and $\theta = (\psi \circ \iota^{[2]}) \otimes \mathrm{id}_D$ commute. On the other hand, for every $y \in D$, we have:

$$
\begin{aligned}
w_n^*(1_D \otimes y)w_n &= u_n^*(v_n^* \otimes 1_D)(1_D \otimes y)(v_n \otimes 1_D)u_n \\
&= u_n^*(1_D \otimes y)u_n \\
&\to \psi(1_D \otimes y) \otimes 1_D.
\end{aligned}
$$

It follows that

$$w_n^*(x \otimes y)w_n \to \psi(x \otimes y) \otimes 1_D \qquad (x, y \in D).$$

This completes the proof.

II.2 Examples

In this section we will look at the known examples of strongly self-absorbing C*-algebras and how one proves that these are strongly self-absorbing. In fact, by [TW07, Corollary 5.3], the list of strongly self-absorbing C*-algebras satisfying the UCT consists of the Jiang-Su algebra $\mathcal{Z}$, the Cuntz algebras $\mathcal{O}_2, \mathcal{O}_\infty$, UHF-algebras of infinite type, denoted by UHF^∞, and tensor products of these. Let us denote the *universal* UHF-algebra by $\mathcal{Q}$, i.e. $\mathcal{Q}$ is the unique UHF-algebra such that $(\mathrm{K}_*(\mathcal{Q}), [1_\mathcal{Q}]_0) \cong (\mathbb{Q}, 0, 1)$. These examples follow a certain hierarchy as depicted in the diagram below, where an arrow indicates a unital inclusion:

$$
\begin{array}{ccc}
 & & \mathcal{O}_2 \\
 & \nearrow & \uparrow \\
\mathcal{Q} & \longrightarrow & \mathcal{Q} \otimes \mathcal{O}_\infty \\
\uparrow & & \uparrow \\
\mathrm{UHF}^\infty & \longrightarrow & \mathrm{UHF}^\infty \otimes \mathcal{O}_\infty \\
\uparrow & & \uparrow \\
\mathcal{Z} & \longrightarrow & \mathcal{O}_\infty
\end{array}
$$

As we do not know any nuclear C*-algebra outside the UCT-class, this list contains all known strongly self-absorbing C*-algebras.

Until now it is only the case of UHF-algebras, where an inductive limit structure is used explicitly to prove strong self-absorption. For all other examples one has to establish some kind of elaborate classification to access the abstract characterization of strongly self-absorbing C*-algebras (Theorem II.1.7). At this point I would like to note that the main theorem of this thesis (Theorem VI.2.2) provides a new strategy to show that the Jiang-Su algebra $\mathcal{Z}$ is strongly self-absorbing by actually using an inductive limit structure.

II.2.1 Example. Let U be a UHF-algebra of infinite type. Then U is strongly self-absorbing. Let us do this for the CAR-algebra

$$
M_{2^\infty} = \varinjlim (M_{2^n}, \gamma_n) \quad \text{with} \quad \gamma_n(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.
$$

We can perform an approximate intertwining

$$
\begin{array}{ccccccc}
M_2 \otimes M_2 & \xrightarrow{\gamma_1^{\otimes 2}} & M_4 \otimes M_4 & \xrightarrow{\gamma_2^{\otimes 2}} & M_8 \otimes M_8 & \xrightarrow{\gamma_3^{\otimes 2}} \cdots \longrightarrow & M_{2^\infty} \otimes M_{2^\infty} \\
\uparrow{\scriptstyle \mathrm{id}\,\otimes 1} \quad {\scriptstyle \cong} & & \uparrow{\scriptstyle \mathrm{id}\,\otimes 1} \quad {\scriptstyle \cong} & & \uparrow{\scriptstyle \mathrm{id}\,\otimes 1} & & {\scriptstyle \varphi}\Big\uparrow{\scriptstyle \cong} \\
M_2 & \xrightarrow{\quad\gamma_1\quad} & M_4 & \xrightarrow{\quad\gamma_2\quad} & M_8 & \xrightarrow{\quad\gamma_3\quad}\cdots \longrightarrow & M_{2^\infty}
\end{array}
$$

Indeed, it is easy to see that two unital *-homomorphisms from M_n to M_m must be unitarily equivalent. By [Rør02, Corollary 2.3.3] it then follows that there is an isomorphism $\varphi \colon M_{2^\infty} \to M_{2^\infty} \otimes M_{2^\infty}$ and unitaries $(u_n)_{n=1}^\infty$ in $M_{2^\infty} \otimes M_{2^\infty}$ such that

$$
u_n \varphi(x) u_n^* \to x \otimes 1_{M_{2^\infty}} \qquad (x \in M_{2^\infty}).
$$

Let us check the statement about maps between matrix algebras, since we will come back to it later (in Lemma III.2.1). If $\varphi \colon M_n \to M_m$ is a unital *-homomorphism, then $n \mid m$ and the projections $\varphi(e_{ii})$ all have the same rank $\frac{m}{n}$. In particular, if $\psi \colon M_n \to M_m$ is another unital *-homomorphism, then $\varphi(e_{11})$ and $\psi(e_{11})$ are Murray-von Neumann equivalent, i.e. there exists a partial isometry $v \in M_m$ such that $\varphi(e_{11}) = vv^*$ and $\psi(e_{11}) = v^*v$. Define

$$
u := \sum_{i=1}^n \varphi(e_{i1}) v \psi(e_{1i}).
$$

Then

$$
uu^* = \sum_{i,j=1}^n \varphi(e_{i1}) v \psi(e_{1i}) \psi(e_{j1}) v^* \varphi(e_{1j}) = \sum_{i=1}^n \varphi(e_{i1}) v v^* v v^* \varphi(e_{1i})
$$

$$
= \sum_{i=1}^n \varphi(e_{i1}) v v^* \varphi(e_{1i}) = \sum_{i=1}^n \varphi(e_{ii}) = 1_{M_m}
$$

Similarly, one computes that $uu^* = 1_{M_m}$ and hence u is a unitary. Furthermore, it is easy to check that

$$
u\psi(e_{ij})u^* = \varphi(e_{ij}) \qquad (i,j = 1,2,\cdots,n).
$$

It follows that $\varphi \sim_{\mathrm{u}} \psi$.

II.2.2 Example. For the purely infinite examples, let us start with the Cuntz algebra $\mathcal{O}_2 \cong \mathrm{C}^*(s_1, s_2 : s_i^* s_i = 1,\ s_1 s_1^* + s_2 s_2^* = 1)$. See [Rør02, Chapter 4] for the details. The main tool to study uniqueness is *stability* of unital *-endomorphisms of purely infinite simple C*-algebras, which is the next technical lemma ([Rør02, Lemma 5.1.4]).

II.2.3 Lemma. *Let A be a unital, purely infinite, simple C*-algebra and let t_1, t_2 be isometries in A satisfying $t_1 t_1^* + t_2 t_2^* = 1_A$.[7] Let γ be the unital *-endomorphism of A given by $\gamma(a) = t_1 a t_1^* + t_2 a t_2^*$. Then, for each unitary $u \in A$ and for each $\varepsilon > 0$, there is a unitary $v \in A$ such that*

$$\|v\gamma(v)^* - u\| < \varepsilon.$$

As a consequence we get the following uniqueness result for $\mathcal{O}_2$:

II.2.4 Theorem. *Let A be a unital, purely infinite, simple C*-algebra and let $\varphi, \psi \colon \mathcal{O}_2 \to A$ be unital *-homomorphisms. Then $\varphi \approx_{\mathrm{u}} \psi$.*

Proof. Let $t_i := \varphi(s_i)$, where s_1, s_2 denote the canonical generators of $\mathcal{O}_2$. Define a unitary
$$u := \psi(s_1)\varphi(s_1)^* + \psi(s_2)\varphi(s_2)^*.$$

By Lemma II.2.3, there exists a sequence $(v_n)_{n=1}^\infty$ of unitaries in A such that $v_n \gamma(v_n)^* \to u$, where $\gamma(a) = t_1 a t_1^* + t_2 a t_2^*$. Then, for $i \in \mathbb{Z}/2\mathbb{Z}$, the following holds:

$$\begin{aligned}
v_n \varphi(s_i) v_n^* = v_n t_i v_n^* &= v_n (t_i v_n^* t_i^*) t_i \\
&= v_n (t_i v_n^* t_i^* + t_{i-1} v_n^* t_{i-1}^*) t_i = v_n \gamma(v_n)^* t_i \\
&\to u t_i = \psi(s_i).
\end{aligned}$$

It follows that $v_n \varphi(x) v_n^* \to \psi(x)$, for all $x \in \mathcal{O}_2$. $\qquad\square$

II.2.5 Lemma. *The Cuntz algebra $\mathcal{O}_2$ has approximately inner half flip and admits an asymptotically central sequence of unital *-endomorphisms. In particular, $\mathcal{O}_2$ embeds unitally into its central sequence algebra $(\mathcal{O}_2)_\omega \cap (\mathcal{O}_2)'$.*

Proof. Since $\mathcal{O}_2 \otimes \mathcal{O}_2$ is purely infinite and simple (cf. [Rør02, Theorem 4.1.10]), it follows from Theorem II.2.4 that $\mathcal{O}_2$ has approximately inner half flip. For the second statement, note that a sequence $(x_n)_{n=1}^\infty$ in $\mathcal{O}_2$ is approximately central if and only if $\|\lambda(x_n) - x_n\| \to 0$, where $\lambda(x) = s_1 x s_1^* + s_2 x s_2^*$. Indeed, assume that $\|\lambda(x_n) - x_n\| \to 0$. Let $\varepsilon > 0$. For $i = 0, 1$ and large $n \in \mathbb{N}$ we get:

$$x_n s_i \approx_\varepsilon \lambda(x_n) s_i = (s_1 x_n s_1^* + s_2 x_n s_2^*) s_i = s_i x_n.$$

---

[7]Which implies that $\mathcal{O}_2$ embeds unitally into A.

It follows that $(x_n)_{n=1}^\infty$ is approximately central. On the other hand, assume $(x_n)_{n=1}^\infty$ is approximately central and let $\varepsilon > 0$. Then, for large $n \in \mathbb{N}$, we get

$$\lambda(x_n) = \sum_{i=1}^2 s_i x_n s_i^* \approx_\varepsilon x_n \sum_{i=1}^2 s_i s_i^* = x_n.$$

Now, let $u \in \mathcal{O}_2$ be a unitary such that $\lambda = \lambda_u$, where λ_u is determined by[8] $\lambda_u(s_i) = u s_i$. Actually,

$$u = \sum_{i=1}^2 \lambda(s_i) s_i^*.$$

By Lemma II.2.3 we again find unitaries $(v_n)_{n=1}^\infty$ in $\mathcal{O}_2$ such that $v_n \lambda(v_n)^* \to u$. Define $\rho_n := \lambda_{v_n}$. We claim that the ρ_n are approximately central. Indeed,

$$\|\lambda(\rho_n(s_i)) - \rho_n(s_i)\| = \|\lambda(v_n s_i) - v_n s_i\| = \|\lambda(s_i) - \lambda(v_n)^* v_n s_i\|$$
$$= \|u s_i - \lambda(v_n)^* v_n s_i\|$$
$$\to \|u s_i - u s_i\| = 0.$$

By our previous observation, it follows that $(\rho_n(x))_{n=1}^\infty$ is approximately central, for every $x \in \mathcal{O}_2$. Furthermore, the ρ_n induce a unital *-homomorphism

$$\mathcal{O}_2 \to (\mathcal{O}_2)_\omega \cap (\mathcal{O}_2)' : x \mapsto [(\rho_n(x))_{n=1}^\infty].$$

$\square$

By Theorem II.1.6 it now follows that $\mathcal{O}_2$ is strongly self-absorbing.

II.2.6 Example. The next purely infinite example is

$$\mathcal{O}_\infty = \mathrm{C}^*\big((s_i)_{i=1}^\infty : s_i^* s_j = \delta_{ij} \cdot 1\big),$$

which behaves quite differently from $\mathcal{O}_2$. Let us briefly recall the two main ingredients for this case. As for $\mathcal{O}_2$, there is a uniqueness result for $\mathcal{O}_\infty$ ([Rør02, Proposition 7.2.5]), showing that $\mathcal{O}_\infty$ has approximately inner half flip. The other main ingredient is that the central sequence algebra of a unital Kirchberg algebra is unital, simple and purely infinite ([Rør02, Proposition 7.1.1]). The proof relies on a detailed study of nuclear maps between purely infinite C*-algebras.

---

[8]The map λ_u exists by the universal property of $\mathcal{O}_2$.

II.2.7 Theorem. *Let A be a unital Kirchberg algebra. Then $A_\omega \cap A'$ is a unital, purely infinite and simple C^*-algebra.*

Applying this theorem to $\mathcal{O}_\infty$, we see that the unit of the central sequence algebra of $\mathcal{O}_\infty$ is properly infinite, showing that there exists a unital embedding $\mathcal{O}_\infty \to (\mathcal{O}_\infty)_\omega \cap (\mathcal{O}_\infty)'$, see [Rør02, Proposition 1.1.2]. Again, by Theorem II.1.6, it follows that $\mathcal{O}_\infty$ is strongly self-absorbing.

II.2.8 Example. Our final example will be the Jiang-Su algebra $\mathcal{Z}$. In [JS99], $\mathcal{Z}$ is constructed as an inductive limit

$$\mathcal{Z} = (Z_{p_n, q_n}, \phi_n),$$

where $p_n, q_n \in \mathbb{N}$ are coprime, finite and $p_n, q_n \to \infty$. The connecting maps $\phi_{m,n}$ have a particularly explicit form and look like

$$\phi_{m,n}(f) = u \operatorname{diag}(f \circ \xi_1, f \circ \xi_2, \cdots, f \circ \xi_k) \, u^* \qquad (m < n),$$

where u is a unitary path in $M_{p_n q_n}$ and each $\xi_i \colon [0,1] \to [0,1]$ is a continuous function looking almost constant for larger n, more precisely:

$$\sup_{x,y \in [0,1]} |\xi_i(x) - \xi_i(y)| \leq \frac{1}{2^{n-m}} \qquad (i = 1, 2, \cdots, k).$$

In this picture, one can prove that $\mathcal{Z}$ is strongly self-absorbing by using Theorem II.1.7. This uses the full power of classification (existence and uniqueness) for unital and simple inductive limits of dimension drop algebras. Let us start with the following existence theorem ([JS99, Theorem 6.2]):

II.2.9 Theorem. *Let A and B be unital, infinite dimensional and simple inductive limits of finite direct sums of dimension drop algebras[9]. If there exists a unital morphism*

$$\alpha \colon \operatorname{Ell}(A) \to \operatorname{Ell}(B),$$

then there exists a unital *-homomorphism $\varphi \colon A \to B$, such that φ induces α.

Recall that $\operatorname{Ell}(A) = (K_0(A), K_0(A)_+, [1_A]_0, K_1(A), T(A))$. A unital morphism $\alpha \colon \operatorname{Ell}(A) \to \operatorname{Ell}(B)$ then consists of a triple $(\alpha_0, \alpha_1, \nu)$, where α_0, α_1

---

[9]Here, a dimension drop algebra is an algebra of the form $Z_{p,q}$ with $p, q \in \mathbb{N}$.

are homomorphisms on the respective K-groups, with α_0 positive and unital, and with $\nu \colon \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ a unital, positive and affine map. Furthermore, there should be a compatibility between K-theory and traces, in the sense that the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{K}_0(A) & \xrightarrow{\ \alpha_0\ } & \mathrm{K}_0(B) \\
{\scriptstyle \rho_A}\downarrow & & \downarrow{\scriptstyle \rho_B} \\
\mathrm{Aff}(T(A)) & \xrightarrow[\ \nu\ ]{} & \mathrm{Aff}(T(B))
\end{array}
$$

The map ρ_A is given by $\rho_A([p]_0)(\tau) = (\tau \otimes \mathrm{Tr})(p)$, where Tr is the standard trace on $\mathbb{K}$, such that $\mathrm{Tr}(e_{11}) = 1$.

The above existence result will be used for the next theorem ([JS99, Proposition 8.5]).

II.2.10 Theorem. *There exists a unital *-homomorphism $\mathcal{Z} \otimes \mathcal{Z} \to \mathcal{Z}$.*

Proof. We only recall the main steps of the construction of Jiang and Su (without proof). Let us denote by A_n the building blocks of $\mathcal{Z}$, i.e. $A_n = Z_{p_n,q_n}$ as described earlier. Let B_n denote the diagonal of $A_n \otimes A_n$, where we identify $A_n \otimes A_n$ with continuous functions on $[0,1]^2$ satisfying certain boundary conditions. Let us denote the restriction to the diagonal by $\rho_n \colon A_n \otimes A_n \to B_n$. The goal is then to construct unital *-homomorphisms $\varphi_n \colon B_n \to B_{n+1}$, such that the following diagram is a (one-sided) approximate intertwining, where $B := \varinjlim(B_n, \varphi_n)$:

$$
\begin{array}{ccccccccc}
A_1 \otimes A_1 & \xrightarrow{\ \phi_1^{\otimes 2}\ } & A_2 \otimes A_2 & \xrightarrow{\ \phi_2^{\otimes 2}\ } & \cdots & \longrightarrow & \mathcal{Z} \otimes \mathcal{Z} \\
{\scriptstyle \rho_1}\downarrow & & {\scriptstyle \rho_2}\downarrow & & & & \downarrow{\scriptstyle \varphi} \\
B_1 & \xrightarrow[\ \varphi_1\ ]{} & B_2 & \xrightarrow[\ \varphi_2\ ]{} & \cdots & \longrightarrow & B
\end{array}
$$

Let us remark that the φ_n are constructed by hand, without using a uniqueness result. Once this is established, we still need a unital *-homomorphism $B \to \mathcal{Z}$. This is done by invoking Theorem II.2.9. Since each B_n is a prime dimension drop algebra, it follows that $(\mathrm{K}_0(B), \mathrm{K}_1(B), [1_B]_0) \cong (\mathbb{Z}, 0, 1)$ and one can show that the limit B is simple and has a unique trace. By writing down a map on the level of the Elliott invariant Ell, we then find a unital *-homomorphism $\sigma \colon B \to \mathcal{Z}$. Now, the composition $\sigma \circ \varphi \colon \mathcal{Z} \otimes \mathcal{Z} \to \mathcal{Z}$ is the required unital *-homomorphism. $\qquad\square$

We can now use Theorem II.1.7 to show that $\mathcal{Z}$ is strongly self-absorbing. In [JS99, Proposition 8.3] it is proven that $\mathcal{Z}$ has approximately inner half flip. Let us mention that the proof of this fact, different from the purely infinite cases, does not rely on any classification machinery. Indeed, the unitaries implementing the approximate inner half flip come from the unitary paths used in the construction of $\mathcal{Z}$. Then, fix a unital *-homomorphism $\psi \colon \mathcal{Z} \otimes \mathcal{Z} \to \mathcal{Z}$. The uniqueness theorem for maps between dimension drop algebras ([JS99, Theorem 7.6, Corollary 5.6]) shows that

$$\psi \circ (\mathrm{id}_{\mathcal{Z}} \otimes 1_{\mathcal{Z}}) \approx_{\mathrm{u}} \mathrm{id}_{\mathcal{Z}} .$$

Therefore, all assumptions in Theorem II.1.7 are satisfied and it follows that $\mathcal{Z}$ is strongly self-absorbing.

We see that Jiang and Su need the full force of classification to show that $\mathcal{Z}$ is strongly self-absorbing. This is because it seems difficult to write down an approximate intertwining between $\mathcal{Z} \otimes \mathcal{Z}$ and $\mathcal{Z}$, only using the construction of $\mathcal{Z}$ by Jiang and Su. In this thesis we will circumvent this problem by using a different picture, due to Rørdam and Winter.

Chapter III

Uniqueness

The aim of this chapter is to prove uniqueness results for standard *-homo-morphisms on $Z_{\mathfrak{p},\mathfrak{q}}$. Using a so-called *Basic Homotopy Lemma* (Lemma III.4.9), we can prove an asymptotic uniqueness result for standard *-homo-morphisms $Z_{\mathfrak{p},\mathfrak{q}} \to M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$ and an approximate uniqueness result for standard *-endomorphisms $Z_{\mathfrak{p},\mathfrak{q}} \to Z_{\mathfrak{p},\mathfrak{q}}$, see Theorem III.5.4 respectively Theorem III.6.1.

III.1 Classification of Lebesgue contractions

This section relies on spectral theory and basic properties of UHF-algebras. Even though the main result of this section (Theorem III.1.5) is well-known, I want to provide an elementary proof, as for example described in [Sko16].

III.1.1 Definition. Let A be a unital C*-algebra and $\tau \in T(A)$. We then define the *dimension function* $d_\tau \colon A_+ \to [0,\infty)$ associated to τ by

$$d_\tau(a) := \lim_{n\to\infty} \tau(a^{\frac{1}{n}}) \qquad (a \in A_+).$$

Note that the limit exists since the sequence $(\tau(a^{\frac{1}{n}}))_{n=1}^{\infty}$ is increasing and bounded by $\|a\|$.

Dimension functions play an important role in the classification of C*-algebras and are closely related to the Cuntz semigroup, as we will see in Chapter V.

III.1.2 Definition. Let A be a unital C*-algebra and $\tau \in T(A)$. Let $a \in A$ be a self-adjoint element. We then define the *eigenvalue function* $\lambda_\tau^a \colon [0,1) \to [0,\infty)$ of a with respect to τ by

$$\lambda_\tau^a(s) = \inf\{t \in \mathbb{R} : d_\tau\big((a-t)_+\big) \le s\}.$$

Here $(a-t)_+$ denotes the positive part of $a - t1_A$.

The next lemma justifies the name *eigenvalue function*.

III.1.3 Lemma. *Let A be a unital C*-algebra and $\tau \in T(A)$. Let $\{p_i\}_{i=1}^n$ be a family of pairwise orthogonal projections in A such that $\sum_{i=1}^n p_i = 1_A$. Let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ be real numbers and define*

$$a := \sum_{i=1}^n \lambda_i p_i.$$

Then, for $k = 1, 2, \cdots, n$,

$$\lambda_\tau^a(s) = \lambda_k \qquad \big(s \in [s_{k-1}, s_k)\big),$$

where $s_k = \sum_{i=1}^k \tau(p_i)$.

Proof. One should think of a as a diagonal operator, with respect to the projections p_i, having the λ_i as diagonal entries. For $t \in \mathbb{R}$, we get

$$(a-t)_+ = \sum_{i=1}^n (\lambda_i - t)_+ p_i.$$

It is clear that $(\lambda_i - t)_+ = 0$ if $t \ge \lambda_i$. It follows that

$$d_\tau\big((a-t)_+\big) = \sum_{\{i : t < \lambda_i\}} \tau(p_i) = s_{k'},$$

where k' is the largest integer such that $t < \lambda_{k'}$. We see that for $s \in [s_{k-1}, s_k)$ we have

$$d_\tau\big((a-t)_+\big) \le s \qquad \Longleftrightarrow \qquad t \ge \lambda_k.$$

Hence $\lambda_\tau^a(s) = \lambda_k$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

III.1.4 Example. Let A be a unital C*-algebra and assume $h \in A_+$ is a Lebesgue contraction (see Definition I.2.8). Then, the eigenvalue function of

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a with respect to any tracial state $\tau \in T(A)$ can be computed very explicitly (where μ is the Lebesgue measure):

$$\begin{aligned}
\lambda_\tau^h(s) &= \inf\{t \in \mathbb{R} : d_\tau\big((h-t)_+\big) \le s\} \\
&= \inf\{t \in \mathbb{R} : \mu(\mathrm{supp}((\iota - t)_+)) \le s\} \\
&= \inf\{t \in \mathbb{R} : 1 - t \le s\} \\
&= 1 - s.
\end{aligned}$$

In particular, any two Lebesgue contractions have the same eigenvalue function.

Using the same idea, the following theorem ([Sko16, Theorem 5.1]) is true in a slightly more general setting. In fact, the proof only requires that the C*-algebra in question has real rank zero and strong comparison of projections w.r.t. a faithful tracial state. Relying on more sophisticated methods, we will also show that U can be replaced by a more complicated C*-algebra, for example the ultrapower U_ω of U, cf. Lemma V.2.7.

III.1.5 Theorem. *Let U be a UHF-algebra and assume that $h, k \in U$ are two Lebesgue contractions[1]. Then $h \approx_{\mathrm{u}} k$.*

Proof. Let $\varepsilon > 0$. Since U has real rank zero, there exist positive contractions $h', k' \in U$ with finite spectrum such that $\|h - h'\| < \frac{\varepsilon}{4}$ and $\|k - k'\| < \frac{\varepsilon}{4}$. Using an iterative argument ([Sko16, Lemma 4.2]) one can show that there exist pairwise orthogonal families of projections $\{p_i\}_{i=1}^n$, $\{q_i\}_{i=1}^n$, each of which adds up to the identity, and decreasing scalars $\{\alpha_i\}_{i=1}^n$, $\{\beta_i\}_{i=1}^n$ such that

$$\tau_U(p_i) = \tau_U(q_i) \qquad (i = 1, 2, \cdots, n) \tag{III.1.1}$$

and

$$h' = \sum_{i=1}^n \alpha_i p_i, \qquad k' = \sum_{i=1}^n \beta_i q_i.$$

Since $\tau_U(p_i) = \tau_U(q_i)$ it follows that $p_i \sim_{\mathrm{u}} q_i$, say $u_i p_i u_i^* = q_i$. Define

$$u := \sum_{i=1}^n q_i u_i p_i.$$

---

[1] The same proof works if we assume that h and k have the same eigenvalue function.

Then u is a unitary in U and $u p_i u^* = q_i$, for all i (see also Lemma III.4.5). From Lemma III.1.3 and (III.1.1), we know that

$$\alpha_i = \lambda_{h'}^{\tau_U}(s_i), \quad \beta_i = \lambda_{k'}^{\tau_U}(s_i), \tag{III.1.2}$$

for some $s_i \in [0,1)$. Furthermore, by [Pet85, Proposition 2 (ii)], it follows that

$$|\lambda_{h'}^{\tau_U}(s) - \lambda_{h}^{\tau_U}(s)| \le \|h' - h\| < \frac{\varepsilon}{4} \qquad (s \in [0,1)).$$

The same holds for k' and k, too. Now, we have:

$$
\begin{aligned}
\|u h u^* - k\| &\le \frac{\varepsilon}{2} + \|u h' u^* - k'\| \\
&= \frac{\varepsilon}{2} + \left\| \sum_{i=1}^{n} (\alpha_i - \beta_i) q_i \right\| \\
&= \frac{\varepsilon}{2} + \max_i |\alpha_i - \beta_i| \\
&= \frac{\varepsilon}{2} + \max_i \{ |\lambda_{h'}^{\tau_U}(s_i) - \lambda_{k'}^{\tau_U}(s_i)| \\
&\le \varepsilon + \max_i \{ |\lambda_{h}^{\tau_U}(s_i) - \lambda_{k}^{\tau_U}(s_i)| \\
&= \varepsilon.
\end{aligned}
$$

In the last line we have an equality, because the eigenvalue function of h and k are equal, as shown in Example III.1.4. It follows that there exists a sequence of unitaries $(u_n)_{n=1}^{\infty}$ in U with $u_n h u_n^* \to k$, as desired. $\qquad \square$

III.2 Uniqueness for order zero maps

The next lemma provides a uniqueness result for c.p.c. order zero maps defined on matrix algebras. Since c.p.c. order zero maps are certain compressions of *-homomorphisms, we will use ideas from the classification of *-homomorphisms, cf. Example II.2.1.

III.2.1 Lemma. *Let A be a unital C^*-algebra, $p \in \mathbb{N}$ and assume that $\varphi, \psi \colon M_p \to A$ are c.p.c. order zero maps, such that $\varphi(e_{11}) \approx \psi(e_{11})$. Then $\varphi \approx \psi$. If in addition A has stable rank one, i.e. the set of invertible elements is dense in A, then $\varphi \approx_{\mathrm{u}} \psi$.*

Proof. Assume $\varphi(e_{11}) \approx \psi(e_{11})$. By a standard argument, there exists a contraction $v \in A_\omega$ with

$$v \varphi(e_{11}) v^* = \psi(e_{11}), \quad v^* \psi(e_{11}) v = \varphi(e_{11}).$$

Consider the elements $e_\varphi := \left[(h_\varphi^{\frac{1}{n}})_{n=1}^\infty\right]$ and $e_\psi := \left[(h_\psi^{\frac{1}{n}})_{n=1}^\infty\right]$ in A_ω, where h_φ and h_ψ are as in Theorem I.2.3. The elements e_φ and e_ψ act as unit on the image of φ respectively ψ. Considering $A_\omega \subseteq (A^{**})_\omega$, we furthermore have that $e_\varphi \pi_\varphi(x)$ and $e_\psi \pi_\psi(x)$ are elements in A_ω, for every $x \in M_p$. One then defines

$$u := \sum_{i=1}^p \pi_\psi(e_{i1}) e_\psi v e_\varphi \pi_\varphi(e_{1i}) \in A_\omega$$

and computes that

$$u\varphi(e_{ij})u^* = \psi(e_{ij}), \quad u^*\psi(e_{ij})u = \varphi(e_{ij}) \qquad (i,j = 1, 2, \cdots, p).$$

This shows that $\varphi \approx \psi$. If A has stable rank one, we have a polar decomposition in A_ω (see the proof of Lemma V.2.5). We thus may write $u = v\,|u|$, for some unitary $v \in A_\omega$. This unitary will then implement the unitary equivalence of φ and ψ in A_ω (cf. [Gab18, Proposition 3.13]). It follows that $\varphi \approx_{\mathrm{u}} \psi$. $\qquad\square$

III.2.2 Remark. Let us recall two standard facts about functional calculus. If x and y are two positive and orthogonal contractions in a C*-algebra, then

$$f(x+y) = f(x) + f(y) \qquad (f \in C_0(0,1]).$$

It is enough to check this on polynomials of the form $z \mapsto z^n$:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} = x^n + y^n.$$

Furthermore, if x and y are equivalent, in the sense that $x = a^*a$ and $y = aa^*$, then

$$\tau(f(x)) = \tau(f(y)) \qquad (f \in C_0(0,1]),$$

where τ is a tracial state. Again, since τ is continuous and linear, it is enough to check this on polynomials of the form $z \mapsto z^n$. Using that τ is tracial, we see that

$$\tau(x^n) = \tau((a^*a)^n) = \tau(a^*a \cdot a^*a \cdots a^*a)$$
$$= \tau(aa^* \cdot aa^* \cdots aa^*) = \tau((aa^*)^n) = \tau(y^n).$$

III.2.3 Lemma. *Let $\mathfrak{p}$ be a supernatural number and let $\varphi, \psi \colon M_\mathfrak{p} \to U$ be two c.p.c. order zero maps, where U is a UHF-algebra. Assume that h_φ and h_ψ are both Lebesgue contractions. Then $\varphi \approx_{\mathrm{u}} \psi$.*

Proof. By an easy approximation argument one may assume that φ and ψ are defined on M_p, where $p \in \mathbb{N}$. Note that the positive contractions $\{\varphi(e_{ii})\}_{i=1}^p$ are mutually orthogonal and equivalent positive elements, in the sense that there exist $x_i \in U$ with $\varphi(e_{11}) = x_i^* x_i$ and $x_i x_i^* = \varphi(e_{ii})$. Using that the $\varphi(e_{ii})$ are mutually equivalent, it follows that $\tau_U(f(\varphi(e_{11}))) = \tau_U(f(\varphi(e_{ii})))$, for $i = 1, 2, \cdots, p$ and $f \in C_0(0,1]$. Using that $h_\varphi = \varphi(1_p)$ is a Lebesgue contraction, it follows that

$$\frac{1}{p} \int_0^1 f(t) \, dt = \frac{1}{p} \, \tau_U(f(\varphi(1_p))) = \frac{1}{p} \sum_{i=1}^p \tau_U(f(\varphi(e_{ii})))$$

$$= \frac{1}{p} \sum_{i=1}^p \tau_U(f(\varphi(e_{11}))) = \tau_U(f(\varphi(e_{11}))),$$

for all $f \in C_0(0,1]$. The same computation works for ψ and we see that both $\varphi(e_{11})$ and $\psi(e_{11})$ have $\frac{1}{p}$ Lebesgue spectral measure. In particular they have the same eigenvalue function, see Definition III.1.2 and [Sko16, Definition 2.6]. It follows by Theorem III.1.5 that $\varphi(e_{11}) \approx_{\mathrm{u}} \psi(e_{11})$ and in particular $\varphi(e_{11}) \approx \psi(e_{11})$. Since U has stable rank one, Lemma III.2.1 shows that $\varphi \approx_{\mathrm{u}} \psi$. $\qquad\square$

III.3 Uniqueness for maps on dimension drop algebras

In this section we prove approximate uniqueness results for standard *-homomorphisms $Z_{\mathfrak{p},\mathfrak{q}} \to M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$. This is done by reducing the problem to the commuting cones generating the dimension drop algebra and using the following rotation trick.

III.3.1 Rotation trick. The following provides a method that allows to reduce many arguments about maps on $Z_{\mathfrak{p},\mathfrak{q}}$ to maps on $C([0,1])$. This ingredient is essential for the proofs of Lemma III.3.2 and Lemma III.4.9.

Let $\varphi \colon Z_{\mathfrak{p},\mathfrak{q}} \to U$ be a unital *-homomorphism, where U is some C*-algebra (in the following a UHF-algebra or an ultrapower of such). Let $\mathfrak{p}$ and $\mathfrak{q}$ be supernatural numbers of infinite type. By [DW09, Theorem 2.2], the flip map on $M_{\mathfrak{p}} \otimes M_{\mathfrak{p}}$ and $M_{\mathfrak{q}} \otimes M_{\mathfrak{q}}$ is strongly asymptotically inner, witnessed by say unitary paths $S_{\mathfrak{p}} \colon [0,1) \to M_{\mathfrak{p}} \otimes M_{\mathfrak{p}}$ and $S_{\mathfrak{q}} \colon [0,1) \to M_{\mathfrak{q}} \otimes M_{\mathfrak{q}}$. For

$t \in [0, 1)$, we define a continuous function

$$\xi_t \colon [0, 1] \to [0, 1) : s \mapsto \begin{cases} t & \text{if } s \in [0, t], \\ \text{linear} & \text{if } s \in [t, 1], \\ 0 & \text{if } s = 1 \end{cases}$$

and unitary paths

$$\acute{W}_t \colon [0, 1] \to M_{\mathfrak{q}} \otimes M_{\mathfrak{q}} : s \mapsto S_{\mathfrak{q}}(\xi_t(1 - s)),$$
$$\grave{W}_t \colon [0, 1] \to M_{\mathfrak{p}} \otimes M_{\mathfrak{p}} : s \mapsto S_{\mathfrak{p}}(\xi_t(s)). \tag{III.3.1}$$
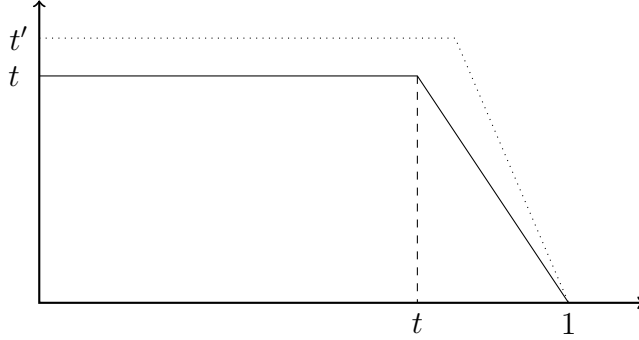

Figure III.1: The function ξ_t and $\xi_{t'}$ for t' slightly larger than t.

The path $\acute{W}_t$, for example, is designed in such a way that it takes most of the time, namely on $[1-t, 1]$, the value $S_{\mathfrak{q}}(t)$, which implements the flip better and better. Furthermore $\acute{W}_t(0) = 1_{\mathfrak{q}} \otimes 1_{\mathfrak{q}}$, showing that $\acute{W}_t$ defines an element of the unitization $(C_0(0, 1] \otimes M_{\mathfrak{q}} \otimes M_{\mathfrak{q}})^+$ and $\acute{W}_t - 1 \in C_0(0, 1] \otimes M_{\mathfrak{q}} \otimes M_{\mathfrak{q}}$, where 1 denotes the adjoined unit. Note that we freely identify $C_0(0, 1] \otimes M_{\mathfrak{q}} \otimes M_{\mathfrak{q}}$ with $\{f \in C([0, 1], M_{\mathfrak{q}} \otimes M_{\mathfrak{q}}) : f(0) = 0\}$ and its unitization is then given by $\{f \in C([0, 1], M_{\mathfrak{q}} \otimes M_{\mathfrak{q}}) : f(0) \in \mathbb{C}\}$. Similarly, $\grave{W}_t \in (C_0[0, 1) \otimes M_{\mathfrak{p}} \otimes M_{\mathfrak{p}})^+$ with scalar part 1. One then easily checks that

$$\lim_{t \to 1} \acute{W}_t(g \otimes r \otimes s)(\acute{W}_t)^* = g \otimes s \otimes r,$$
$$\lim_{t \to 1} \grave{W}_t(f \otimes x \otimes y)(\grave{W}_t)^* = f \otimes y \otimes x, \tag{III.3.2}$$

for $g \in C_0(0, 1]$, $r, s \in M_{\mathfrak{q}}$ and $f \in C_0[0, 1)$, $x, y \in M_{\mathfrak{p}}$. Since the paths $\acute{W}_t$ and $\grave{W}_t$ depend continuously on the parameter t, we get unitary paths

$(\dot{U}_t^\varphi)_{t\in[0,1)}$ and $(\acute{U}_t^\varphi)_{t\in[0,1)}$ defined by

$$\dot{U}_t^\varphi := (\dot{\varphi} \otimes \mathrm{id}_{M_\mathfrak{p}})^+(\dot{W}_t) \in U \otimes M_\mathfrak{p},$$
$$\acute{U}_t^\varphi := (\acute{\varphi} \otimes \mathrm{id}_{M_\mathfrak{q}})^+(\acute{W}_t) \in U \otimes M_\mathfrak{q}. \qquad \text{(III.3.3)}$$

See I.3.6 for the definition of $\acute{\varphi}$ and $\dot{\varphi}$. From (III.3.2), it follows that we may twist the non-central part of each cone separately into an extra tensor factor (as done in Lemma III.3.2):

$$\lim_{t\to 1} \acute{U}_t^\varphi(\acute{\varphi}(g \otimes r) \otimes 1_\mathfrak{q})(\acute{U}_t^\varphi)^* = \acute{\varphi}(g \otimes 1_\mathfrak{q}) \otimes r = \varphi(g) \otimes r,$$
$$\lim_{t\to 1} \dot{U}_t^\varphi(\dot{\varphi}(f \otimes x) \otimes 1_\mathfrak{p})(\dot{U}_t^\varphi)^* = \dot{\varphi}(f \otimes 1_\mathfrak{p}) \otimes x = \varphi(f) \otimes x. \qquad \text{(III.3.4)}$$

If we use $U \otimes M_\mathfrak{p} \otimes M_\mathfrak{q}$ as codomain, as in Lemma III.4.9, we instead define

$$\dot{U}_t^\varphi := (\dot{\varphi} \otimes \imath_{(\mathfrak{p},\mathfrak{q})}^{[1]})^+(\dot{W}_t) \in U \otimes M_\mathfrak{p} \otimes M_\mathfrak{q},$$
$$\acute{U}_t^\varphi := (\acute{\varphi} \otimes \imath_{(\mathfrak{p},\mathfrak{q})}^{[2]})^+(\acute{W}_t) \in U \otimes M_\mathfrak{p} \otimes M_\mathfrak{q}, \qquad \text{(III.3.5)}$$

where $\imath_{(\mathfrak{p},\mathfrak{q})}^{[1]}$ and $\imath_{(\mathfrak{p},\mathfrak{q})}^{[2]}$ denote the obvious embeddings of $M_\mathfrak{p}$ resp. $M_\mathfrak{q}$ into the first resp. second tensor factor of $M_\mathfrak{p} \otimes M_\mathfrak{q}$ (see Notation VI.1.1). In that case

$$[\dot{U}_t^\varphi, \acute{U}_t^\varphi] = 0 \qquad (t \in [0,1)). \qquad \text{(III.3.6)}$$

Then, the obvious adjustment of (III.3.4) holds:

$$\lim_{t\to 1} \quad \acute{U}_t^\varphi(\acute{\varphi}(g \otimes r) \otimes 1_\mathfrak{pq})(\acute{U}_t^\varphi)^* = \acute{\varphi}(g \otimes 1_\mathfrak{q}) \otimes 1_\mathfrak{p} \otimes r = \varphi(g) \otimes 1_\mathfrak{p} \otimes r,$$
$$\lim_{t\to 1} \quad \dot{U}_t^\varphi(\dot{\varphi}(f \otimes x) \otimes 1_\mathfrak{pq})(\dot{U}_t^\varphi)^* = \dot{\varphi}(f \otimes 1_\mathfrak{p}) \otimes x \otimes 1_\mathfrak{q} = \varphi(f) \otimes x \otimes 1_\mathfrak{q},$$

where $1_\mathfrak{pq} := 1_\mathfrak{p} \otimes 1_\mathfrak{q}$. Most importantly, in this situation we may twist the non-central part of both cones simultaneously into commuting subalgebras of $U \otimes M_\mathfrak{p} \otimes M_\mathfrak{q}$:

$$\lim_{t\to 1} \dot{U}_t^\varphi \acute{U}_t^\varphi(\dot{\varphi}(f \otimes x) \otimes 1_\mathfrak{pq})(\acute{U}_t^\varphi)^*(\dot{U}_t^\varphi)^* = \varphi(f) \otimes x \otimes 1_\mathfrak{q},$$
$$\lim_{t\to 1} \dot{U}_t^\varphi \acute{U}_t^\varphi(\acute{\varphi}(g \otimes y) \otimes 1_\mathfrak{pq})(\acute{U}_t^\varphi)^*(\dot{U}_t^\varphi)^* = \varphi(g) \otimes 1_\mathfrak{p} \otimes y, \qquad \text{(III.3.7)}$$

for all $f \in C_0[0,1)$, $x \in M_\mathfrak{p}$ and $g \in C_0(0,1]$, $y \in M_\mathfrak{q}$. Note that (III.3.7) follows directly from (III.3.4) and (III.3.6).

III.3.2 Lemma. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be supernatural numbers and let U be a UHF-algebra of infinite type, which tensorially absorbs $M_\mathfrak{p}$ or $M_\mathfrak{q}$. Assume that $\beta_0, \beta_1 \colon Z_{\mathfrak{p},\mathfrak{q}} \to U$ are standard *-homomorphisms. Then $\beta_0 \approx_\mathrm{u} \beta_1$.*

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Proof. Let us assume that U tensorially absorbs $M_{\mathfrak{p}}$. The other case is then obtained by reversing the orientation of the unit interval. By assumption,

$$h_{\overline{\beta_0}} = \acute{\beta}_0(\iota \otimes 1_{\mathfrak{q}}) \quad \text{and} \quad h_{\overline{\beta_1}} = \acute{\beta}_1(\iota \otimes 1_{\mathfrak{q}})$$

are both Lebesgue contractions and by Lemma III.2.3 we see that $\acute{\beta}_0 \approx_{\mathrm{u}} \acute{\beta}_1$. By Remark I.5.2, there exists a unitary $u \in U_\omega$, such that $\mathrm{ad}(u) \circ \acute{\beta}_0 = \acute{\beta}_1$, when $\acute{\beta}_0$ and $\acute{\beta}_1$ are considered as maps taking values in U_ω. Let us define $\gamma_i \colon Z_{\mathfrak{p},\mathfrak{q}} \to U_\omega$ by

$$\gamma_0 := \mathrm{ad}(u) \circ \beta_0, \qquad \gamma_1 := \beta_1.$$

Then $\acute{\gamma}_0 = \acute{\gamma}_1$ and one easily checks that

$$\gamma_0(f) = \gamma_1(f) \qquad (f \in C([0,1])). \tag{III.3.8}$$

For $t \in [0,1)$, we define $V_t := (\grave{U}_t^{\gamma_1})^* \grave{U}_t^{\gamma_0} \in U_\omega \otimes M_{\mathfrak{p}} \subseteq (U \otimes M_{\mathfrak{p}})_\omega$, with $\grave{U}_t^{\gamma_i}$ as in (III.3.3). Let $f \in C_0[0,1)$, $x \in M_{\mathfrak{p}}$ and $\varepsilon > 0$. Using (III.3.4), choose $t_0 \in [0,1)$ such that

$$\grave{U}_t^{\gamma_i}(\grave{\gamma}_i(f \otimes x) \otimes 1_{\mathfrak{p}})(\grave{U}_t^{\gamma_i})^* \approx_{\frac{\varepsilon}{2}} \gamma_i(f) \otimes x \qquad (i = 0, 1, \ t \geq t_0). \tag{III.3.9}$$

For $t \geq t_0$, we then get

$$\begin{aligned}
V_t(\grave{\gamma}_0(f \otimes x) \otimes 1_{\mathfrak{p}})V_t^* &= (\grave{U}_t^{\gamma_1})^* \grave{U}_t^{\gamma_0}(\grave{\gamma}_0(f \otimes x) \otimes 1_{\mathfrak{p}})(\grave{U}_t^{\gamma_0})^* \grave{U}_t^{\gamma_1} \\
&\overset{(\text{III.3.9})}{\approx}_{\frac{\varepsilon}{2}} (\grave{U}_t^{\gamma_1})^*(\gamma_0(f) \otimes x)\grave{U}_t^{\gamma_1} \\
&\overset{(\text{III.3.8})}{=} (\grave{U}_t^{\gamma_1})^*(\gamma_1(f) \otimes x)\grave{U}_t^{\gamma_1} \\
&\overset{(\text{III.3.9})}{\approx}_{\frac{\varepsilon}{2}} \grave{\gamma}_1(f \otimes x) \otimes 1_{\mathfrak{p}}.
\end{aligned}$$

Furthermore, V_t commutes with $\acute{\gamma}_0 \otimes 1_{\mathfrak{p}} = \acute{\gamma}_1 \otimes 1_{\mathfrak{p}}$. It follows that

$$\lim_{n \to \infty} \left\| V_{1-\frac{1}{n}}(\gamma_0(g) \otimes 1_{\mathfrak{p}})V_{1-\frac{1}{n}}^* - \gamma_1(g) \otimes 1_{\mathfrak{p}} \right\| = 0 \qquad (g \in Z_{\mathfrak{p},\mathfrak{q}}),$$

and hence $\gamma_0 \otimes 1_{\mathfrak{p}} \approx_{\mathrm{u}} \gamma_1 \otimes 1_{\mathfrak{p}}$. By Remark I.5.2, we have $\gamma_0 \otimes 1_{\mathfrak{p}} \sim_{\mathrm{u}} \gamma_1 \otimes 1_{\mathfrak{p}}$. Let $u' \in (U \otimes M_{\mathfrak{p}})_\omega$, such that

$$\mathrm{ad}(u') \circ (\gamma_0 \otimes 1_{\mathfrak{p}}) = \gamma_1 \otimes 1_{\mathfrak{p}}.$$

Putting $u'' := u'(u \otimes 1_{\mathfrak{p}}) \in (U \otimes M_{\mathfrak{p}})_\omega$, we thus have $\mathrm{ad}(u'') \circ (\beta_0 \otimes 1_{\mathfrak{p}}) = \beta_1 \otimes 1_{\mathfrak{p}}$. By [TW07, Theorem 2.3] and the assumption that $U \cong U \otimes M_{\mathfrak{p}}$, there exists

a *-homomorphism $\varphi \colon U \otimes M_{\mathfrak{p}} \to U_\omega$ with $\varphi(x \otimes 1_{\mathfrak{p}}) = x$, for $x \in U$. This induces a map

$$\varphi_\omega \colon (U \otimes M_{\mathfrak{p}})_\omega \to (U_\omega)_\omega.$$

Let $w \coloneqq \varphi_\omega(u'') \in (U_\omega)_\omega$. Then

$$w\beta_0(x)w^* = \varphi_\omega(u'')\varphi_\omega(\beta_0(x) \otimes 1_{\mathfrak{p}})\varphi_\omega((u'')^*) = \varphi_\omega(\beta_1(x) \otimes 1_{\mathfrak{p}}) = \beta_1(x).$$

It follows that $\beta_0 \sim_{\mathrm{u}} \beta_1$ in $(U_\omega)_\omega$. By Remark I.5.2 again, $\beta_0 \approx_{\mathrm{u}} \beta_1$ as maps into U. $\qquad\square$

III.4   Basic Homotopy Lemmas

In this section we prove a so-called Basic Homotopy Lemma for maps $Z_{\mathfrak{p},\mathfrak{q}} \to M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$. Roughly, it says that a unitary in $M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$, which approximately commutes with the image of such a map, may be connected to the identity by a unitary path, which pointwise approximately commutes with the image very well, too.

Combining this with our approximate uniqueness result (Lemma III.3.2), we arrive at an asymptotic uniqueness result for standard *-homomorphisms $Z_{\mathfrak{p},\mathfrak{q}} \to M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$ and an approximate uniqueness result for standard *-endomorphisms $Z_{\mathfrak{p},\mathfrak{q}} \to Z_{\mathfrak{p},\mathfrak{q}}$ (Theorem III.5.3 respectively Theorem III.6.1).

Let us first collect some basic facts about projections and spectral theory, cf. [BEEK98, Section 2].

III.4.1 Lemma. *Let A be a C^*-algebra and let p, q be projections in A with $\|p - q\| < 1$. Then, there exists a partial isometry $v \in A$ with $vv^* = p$, $v^*v = q$ and $\|v - q\| < 2\|p - q\|$.*

III.4.2 Lemma. *Let $\varepsilon > 0$ and $g \in C([0,1])$. Then, there exists $\delta > 0$ such that the following holds: Whenever h, k are positive contractions in some C^*-algebra A and $\|h - k\| < \delta$, then $\|g(h) - g(k)\| < \varepsilon$.*

III.4.3 Lemma. *Let $\varepsilon > 0$ and assume that $I, J \subseteq [0,1]$ are two intervals such that I is contained in the interior of J. Then, there exists $\delta_0 > 0$ such that the following holds: Whenever h and k are positive contractions with $\|h - k\| < \delta < \delta_0$, then*

$$\|pq - q\| < \varepsilon,$$

where $q = \chi_I(k)$ and $p = \chi_J(h)$ denote the respective spectral projections.

The next lemma is maybe less known and contained in [BEEK98, Lemma 2.4].

III.4.4 Lemma. *Let p, q, e be projections in some unital C^*-algebra A with $e \leq p$ and*

$$\|pq - q\| \leq \varepsilon < \frac{1}{6}.$$

Then, there exists a unitary u in A with

$$e \leq uqu^* \leq p$$

and $\|uqu^* - q\| \leq 10\varepsilon$.

III.4.5 Lemma. *Let $(p_i)_{i=0}^N$ and $(q_i)_{i=0}^N$ be two families of projections in a unital C^*-algebra A, each of which sums up to the identity. If for all $i = 0, 1, 2, \cdots, N$, we have $\|p_i - q_i\| < \varepsilon \leq 1$, then there exists a unitary $u \in A$ with $up_iu^* = q_i$, for $i = 1, 2, \cdots, N$, and $\|u - 1_A\| \leq 2(N + 1)\varepsilon$.*

Proof. Since $\|p_i - q_i\| < 1$, it follows from Lemma III.4.1 that there exist partial isometries $v_i \in A$ such that $v_i v_i^* = p_i$ and $v_i^* v_i = q_i$. Furthermore, $\|v_i - p_i\| \leq 2 \|q_i - p_i\| < 2\varepsilon$. Note that $v_i = p_i v_i = v_i q_i = p_i v_i q_i$. Hence $v_i v_j^* = v_i q_i q_j v_j^* = \delta_{i,j} v_i v_j^* = \delta_{i,j} p_i$. Similarly $v_i^* v_j = \delta_{i,j} q_i$. Define

$$u := \sum_{i=0}^N v_i.$$

Then u is a unitary in A, since

$$uu^* = \sum_{i,j=0}^N v_i v_j^* = \sum_{i,j=0}^N \delta_{i,j} p_i = \sum_{i=0}^N p_i = 1_A.$$

Similarly, we get $u^* u = 1_A$ and it is clear that $up_i u^* = q_i$, for all i. Finally, we have

$$\|u - 1_A\| = \left\| \sum_{i=0}^N v_i - p_i \right\| < 2(N + 1)\varepsilon.$$

$\square$

The following lemma is a reformulation of [BEEK98, Lemma 5.1], which is therein referred to as *Isospectral Homotopy Lemma in Case of a Large Spectral Gap*. In modern work this type of lemma is often called a *Basic Homotopy Lemma* and due to its importance to the classification of C^*-algebras

great effort has been made to generalize the theory of such (cf. [Lin17]). For our purpose however, this elementary result is sufficient.

III.4.6 Lemma. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds: Whenever h is a positive contraction with full spectrum and u is a unitary in some UHF-algebra U such that*

$$\|uhu^* - h\| < \delta,$$

then there exists a positive contraction $h' \in U$ with finite spectrum such that $\|h - h'\| < \varepsilon$ and there exists a unitary path $(w_t)_{t \in [0,1]}$ in U such that $w_0 = 1_U$ and

(i) $\|w_t h' w_t^* - h'\| < \varepsilon$, *for all $t \in [0,1]$,*

(ii) $w_1 h' w_1^* = uh'u^*$.

Proof. **(I)** Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ even with $\frac{1}{N} < \frac{\varepsilon}{10}$ and let $\eta := \frac{\varepsilon}{1000(N+1)}$. Having N, we may define intervals

$$I_k := \left[\frac{N - k + 1}{N}, 1\right] \qquad \left(k = 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots, N + 1\right),$$

so that $\{1\} = I_1 \subseteq I_{\frac{3}{2}} \subseteq I_2 \subseteq \cdots \subseteq I_{N+1} = [0,1]$. By Lemma III.4.3, there exists $0 < \delta < \frac{\varepsilon}{10}$ such that whenever h and k are positive contractions with $\|h - k\| < \delta$, then for $k = 1, \frac{3}{2}, 2, \cdots, N + 1$:

$$\left\|p_k q_{k-\frac{1}{2}} - q_{k-\frac{1}{2}}\right\| < \eta, \quad \left\|q_k p_{k-\frac{1}{2}} - p_{k-\frac{1}{2}}\right\| < \eta, \tag{III.4.1}$$

where $p_k = \chi_{I_k}(h)$ and $q_k = \chi_{I_k}(k)$.

(II) We now show that $\frac{\delta}{3}$ does the job. To this end, let h be a positive contraction with full spectrum in some UHF-algebra U and let $u \in U$ be a unitary with

$$\|uhu^* - h\| < \frac{\delta}{3}.$$

Find a positive contraction $h_0 \in U$ with finite spectrum such that h_0 has no spectral gaps of length greater or equal to $\frac{1}{2N}$ and such that $\|h - h_0\| < \frac{\delta}{3}$. Such h_0 can be found easily since h has full spectrum and U has real rank zero. Then

$$\|uh_0u^* - h_0\| < \delta.$$

Let us look at

$$h' := \frac{1}{N} \sum_{k=1}^{N} p_k \in U,$$

where $p_k := \chi_{I_k}(h_0)$. Then, by functional calculus, $\|h_0 - h'\| \leq \frac{1}{N}$ and hence $\|h - h'\| < \frac{1}{N} + \frac{\delta}{3} < \varepsilon$. Let us define

$$k' := uh'u^* = \frac{1}{N} \sum_{k=1}^{N} up_k u^* = \frac{1}{N} \sum_{k=1}^{N} q_k,$$

where $q_k := up_k u^* = \chi_{I_k}(uh_0 u^*)$. Since $\|uh_0 u^* - h_0\| < \delta$, it follows that (III.4.1) holds for the spectral projections p_k, q_k associated to h_0 resp. $uh_0 u^*$ and we may apply Lemma III.4.4 to get unitaries $(u_k)_{k=1}^N$ and $(v_k)_{k=1}^N$ such that

$$p_{k-1} < p_{k-\frac{1}{2}} \quad \leq u_k q_k u_k^* \quad \leq p_{k+\frac{1}{2}} < p_{k+1}, \qquad \text{(III.4.2)}$$
$$q_{k-1} < q_{k-\frac{1}{2}} \quad \leq v_k p_k v_k^* \quad \leq q_{k+\frac{1}{2}} < q_{k+1} \qquad \text{(III.4.3)}$$

and, with $q_k' := u_k q_k u_k^*$ resp. $p_k' := v_k p_k v_k^*$, we have

$$\|q_k' - q_k\| \leq 10\eta, \qquad \|p_k' - p_k\| \leq 10\eta. \qquad \text{(III.4.4)}$$

After having done this setup, we are now able to find the desired unitary path $(w_t)_{t\in[0,1]}$. We will construct this path in three steps.

(III) Let us first show how to connect

$$h' = \frac{1}{N}(p_1 + p_2 + p_3 + p_4 + \cdots + p_{N-1} + p_N)$$

to

$$h'' := \frac{1}{N}(p_1 + q_2' + p_3 + q_4' + \cdots + p_{N-1} + q_N').$$

Note that (III.4.2) entails that for $k = 1, 2, \cdots, \frac{N}{2}$:

$$p_{2k} - p_{2k-1} \sim_{\mathrm{u}} q_{2k}' - p_{2k-1} \text{ in } (p_{2k+1} - p_{2k-1})U(p_{2k+1} - p_{2k-1}).$$

Since this corner is again a UHF-algebra it follows that there exist unitary paths $(u_{2k,t})_{t\in[0,\frac{1}{3}]}$ inside $(p_{2k+1}-p_{2k-1})U(p_{2k+1}-p_{2k-1})$ connecting $p_{2k}-p_{2k-1}$ and $q_{2k}' - p_{2k-1}$. One now checks that

$$w_t := p_1 + u_{2,t} + u_{4,t} + \cdots + u_{N,t} \qquad (t \in [0, 1/3])$$

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defines a unitary path in U and connects h' to h'', i.e. $w_0 = 1_U$ and $w_{\frac{1}{3}} h' w_{\frac{1}{3}}^* = h''$. Furthermore, it is easy to check that $\|w_t h' w_t^* - h'\| \leq \frac{2}{N} < \frac{\varepsilon}{5} < \varepsilon$, for $t \in [0, 1/3]$.

(IV) In the second step we connect h'' to

$$k'' := \frac{1}{N}(p_1' + q_2 + p_3' + q_4 + \cdots + p_{N-1}' + q_N).$$

Since

$$
\begin{aligned}
1_U &= p_1 + (q_2' - p_1) + (p_3 - q_2') + \cdots + (q_N' - p_{N-1}) + (1_U - q_N') \\
&= p_1' + (q_2 - p_1') + (p_3' - q_2) + \cdots + (q_N - p_{N-1}') + (1_U - q_N)
\end{aligned}
$$

and, by (III.4.4),

$$
\begin{aligned}
\left\| (q_k' - p_{k-1}) - (q_k - p_{k-1}') \right\| &< 2 \cdot 10\eta, \\
\left\| (p_k - q_{k-1}') - (p_k' - q_{k-1}) \right\| &< 2 \cdot 10\eta,
\end{aligned}
$$

it follows together with Lemma III.4.5 that there exists a unitary $z \in U$ which conjugates the $q_{2k}' - p_{2k-1}$ into $q_{2k} - p_{2k-1}'$ and $p_{2k+1} - q_{2k}'$ into $p_{2k+1}' - q_{2k}$, such that

$$\|z - 1_U\| \leq 2(N+1) \cdot 20\eta = 40(N+1)\eta \leq \frac{\varepsilon}{20}.$$

Clearly $zh''z^* = k''$. Since $-1 \notin \sigma(z)$, we may write $z = e^{i\pi a}$ for some self-adjoint element $a \in U$ with $\|a\| < 1$ and one checks that

$$z_t = e^{i3(t-1/3)\pi a} \qquad (t \in [1/3, 2/3])$$

connects 1_U to z such that $\|z_t - 1_U\| \leq \frac{\varepsilon}{20}$. Define

$$w_t := z_t w_{\frac{1}{3}} \qquad (t \in [1/3, 2/3]).$$

For $t \in [1/3, 2/3]$ we then get

$$
\begin{aligned}
\|w_t h' w_t^* - h'\| = \|z_t h'' z_t^* - h'\| &\leq \|z_t h'' z_t^* - h''\| + \|h'' - h'\| \\
&\leq 2 \cdot \frac{\varepsilon}{20} + \left\| w_{\frac{1}{3}} h' w_{\frac{1}{3}}^* - h' \right\| < \frac{\varepsilon}{10} + \frac{\varepsilon}{5} < \varepsilon.
\end{aligned}
$$

(V) In the last step we connect $k'' = w_{\frac{2}{3}} h' w_{\frac{2}{3}}^*$ to $k' = uh'u^*$ similarly as we connected h' to h'', producing a unitary path $(y_t)_{t \in [2/3, 1]}$. We finally define

$$w_t := y_t w_{\frac{2}{3}} \qquad (t \in [2/3, 1]).$$

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Since $\|h' - k'\| \leq \|h' - h_0\| + \|h_0 - uh_0u^*\| + \|uh_0u^* - uh'u^*\| \leq \frac{1}{N} + \delta + \frac{1}{N} = \frac{2}{N} + \delta$, it follows that for $t \in [2/3, 1]$, we also have

$$\|w_t h' w_t^* - h'\| = \|y_t k'' y_t^* - h'\| \leq \|y_t k'' y_t^* - k'\| + \|k' - h'\|$$
$$\leq \frac{2}{N} + \frac{2}{N} + \delta < \frac{2\varepsilon}{5} + \frac{\varepsilon}{3} < \varepsilon.$$

We are now done, since the path $(w_t)_{t \in [0,1]}$ satisfies $w_0 = 1_U$, $w_1 h' w_1^* = k' = uh'u^*$ and $\|w_t h' w_t^* - h'\| < \varepsilon$, for $t \in [0, 1]$. □

III.4.7 Corollary. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds: Whenever h is a positive contraction with full spectrum in a UHF-algebra U and $u \in U$ is a unitary such that*

$$\|uhu^* - h\| < \delta,$$

then there exists a unitary path $(u_t)_{t \in [0,1]}$ in U such that $u_0 = 1_U$, $u_1 = u$ and

$$\|u_t h u_t^* - h\| < \varepsilon \qquad (t \in [0, 1]).$$

Proof. Let $\varepsilon > 0$. Choose $0 < \delta < \varepsilon$ such that the conclusion of Lemma III.4.6 holds for ε replaced by $\frac{\varepsilon}{3}$. We show that $\frac{\delta}{3}$ does the job. To this end let h be a positive contraction in some UHF-algebra U and let u be a unitary in U with

$$\|uhu^* - h\| < \frac{\delta}{3}.$$

By the choice of δ, there exists a positive contraction $h' \in U$ with finite spectrum such that $\|h - h'\| < \frac{\varepsilon}{3}$ and there exists a unitary path $(z_t)_{t \in [0,1]}$ in U such that $z_0 = 1_U$, $z_1 h' z_1^* = uh'u^*$ and

$$\|z_t h' z_t^* - h'\| < \frac{\varepsilon}{3} \qquad (t \in [0, 1]).$$

Let $w := z_1^* u$. Then, the unitary path $(v_t)_{t \in [0,1]}$, given by $v_t := z_t^* u$, connects u with w and one easily checks that

$$\|v_t h v_t^* - h\| < \varepsilon \qquad (t \in [0, 1]).$$

Using that $z_1 h' z_1^* = uh'u^*$, it follows that w actually commutes with h'. Since h' has finite spectrum, we can connect w to 1_U via a unitary path $(w_t)_{t \in [0,1]}$ such that w_t commutes with h', for every $t \in [0, 1]$. Then,

$$\|w_t h w_t^* - h\| < \varepsilon \qquad (t \in [0, 1]).$$

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The desired path $(u_t)_{t \in [0,1]}$ is now given by

$$u_t = \begin{cases} v_{2t} & \text{if } t \in [0, 1/2], \\ w_{2t-1} & \text{if } t \in [1/2, 1]. \end{cases}$$

$\square$

As an easy consequence we get the following Basic Homotopy Lemma for maps on $C([0,1])$:

III.4.8 Lemma. *For every $\varepsilon > 0$ and $\mathcal{F} \subseteq C([0,1])$ finite there exists $\delta > 0$ such that the following holds: Whenever $\varphi \colon C([0,1]) \to U$ is a unital injective[2] *-homomorphism, where U is a UHF-algebra, and $u \in U$ is a unitary such that*

$$\|u\varphi(\iota)u^* - \varphi(\iota)\| < \delta,$$

then there exists a unitary path $(u_t)_{t \in [0,1]}$ in U such that $u_0 = 1_U$, $u_1 = u$ and

$$\|u_t \varphi(f) u_t^* - \varphi(f)\| < \varepsilon \qquad (f \in \mathcal{F},\ t \in [0,1]).$$

Proof. Let $\varepsilon > 0$ and $\mathcal{F} \subseteq C([0,1])$ be finite. By Lemma III.4.2 there exists $\eta > 0$ such that the following holds: Whenever h and k are positive contractions in a C*-algebra with $\|h - k\| < \eta$, then $\|f(h) - f(k)\| < \varepsilon$, for all $f \in \mathcal{F}$. Now Corollary III.4.7 produces some $\delta > 0$ such that the conclusion of the corollary holds for ε replaced by η. We claim that this δ works.

So let $\varphi \colon C([0,1]) \to U$ and $u \in U$ be given, where U is some UHF-algebra, such that

$$\|u\varphi(\iota)u^* - \varphi(\iota)\| < \delta.$$

By the choice of δ we can connect u to the identity by a unitary path $(u_t)_{t \in [0,1]}$ such that for all $t \in [0,1]$:

$$\|u_t \varphi(\iota) u_t^* - \varphi(\iota)\| < \eta. \tag{III.4.5}$$

But then, for $f \in \mathcal{F}$ and $t \in [0,1]$ we get:

$$\begin{aligned} \|u_t \varphi(f) u_t^* - \varphi(f)\| &= \|u_t f(\varphi(\iota)) u_t^* - f(\varphi(\iota))\| \\ &= \|f(u_t \varphi(\iota) u_t^*) - f(\varphi(\iota))\| \\ &< \varepsilon. \end{aligned}$$

---

[2]This assumption ensures that $\varphi(\iota)$ has full spectrum, as needed for Lemma III.4.6. However, one can work around this and remove the assumption of full spectrum so that this lemma also works for φ not necessarily injective. See [BEEK98].

The last inequality follows from the choice of η and (III.4.5). $\qquad\square$

Using our rotation trick and the previous lemma, we are now able to prove a Basic Homotopy Lemma for maps on dimension drop algebras.

III.4.9 Lemma. *For every $\varepsilon > 0$ and $\mathcal{F} \subseteq Z_{\mathfrak{p},\mathfrak{q}}$ finite, where $\mathfrak{p}$ and $\mathfrak{q}$ are supernatural numbers of infinite type, there exists $\delta > 0$ and $\mathcal{G} \subseteq Z_{\mathfrak{p},\mathfrak{q}}$ finite such that the following holds: Whenever $\varphi \colon Z_{\mathfrak{p},\mathfrak{q}} \to U$ is a unital *-homomorphism, where U is a UHF-algebra of infinite type that absorbs $M_{\mathfrak{p}}$ and $M_{\mathfrak{q}}$ tensorially, and $u \in U$ is a unitary such that*

$$\|u\varphi(g)u^* - \varphi(g)\| < \delta \qquad (g \in \mathcal{G}),$$

then, there exists a unitary path $(u_t)_{t \in [0,1]}$ in U with $u_0 = 1_U$, $u_1 = u$ and

$$\|u_t\varphi(f)u_t^* - \varphi(f)\| < \varepsilon \qquad (f \in \mathcal{F},\ t \in [0,1]).$$

Proof. Let $\varepsilon > 0$ and $\mathcal{F} \subseteq Z_{\mathfrak{p},\mathfrak{q}}$ be finite. We may assume that

$$\mathcal{F} = \acute{\mathcal{F}} \cup \grave{\mathcal{F}},$$

where $\acute{\mathcal{F}} = \{g_k \otimes y_k\}_{k=1}^N \subseteq C_0(0,1] \otimes M_{\mathfrak{q}}$ and $\grave{\mathcal{F}} = \{f_k \otimes x_k\}_{k=1}^N \subseteq C_0[0,1) \otimes M_{\mathfrak{p}}$. The two cones are identified with subalgebras of $Z_{\mathfrak{p},\mathfrak{q}}$ as explained in I.3.6. With (III.3.2), we choose $t_0 \in [0,1)$ such that for $k = 1, 2, \cdots, N$:

$$
\begin{aligned}
\acute{W}_{t_0}(g_k \otimes y_k \otimes 1_{\mathfrak{q}})(\acute{W}_{t_0})^* &\approx_{\frac{\varepsilon}{10}} g_k \otimes 1_{\mathfrak{q}} \otimes y_k \\
\grave{W}_{t_0}(f_k \otimes x_k \otimes 1_{\mathfrak{p}})(\grave{W}_{t_0})^* &\approx_{\frac{\varepsilon}{10}} f_k \otimes 1_{\mathfrak{p}} \otimes x_k.
\end{aligned}
\tag{III.4.6}
$$

The unitaries $\acute{W}_{t_0}$ and $\grave{W}_{t_0}$ are defined in (III.3.1). As observed in (III.3.1) the scalar part of $\acute{W}_{t_0}$ and $\grave{W}_{t_0}$ is 1, so that $\acute{W}_{t_0} - 1 \in C_0(0,1] \otimes M_{\mathfrak{q}} \otimes M_{\mathfrak{q}}$ and $\grave{W}_{t_0} - 1 \in C_0[0,1) \otimes M_{\mathfrak{p}} \otimes M_{\mathfrak{p}}$. We thus may approximate

$$
\begin{aligned}
\grave{W}_{t_0} - 1 &\approx_{\frac{\varepsilon}{100}} \sum_{i=1}^K h_i \otimes a_i \otimes b_i, \\
\acute{W}_{t_0} - 1 &\approx_{\frac{\varepsilon}{100}} \sum_{i=1}^K k_i \otimes c_i \otimes d_i,
\end{aligned}
\tag{III.4.7}
$$

where $h_i \in C_0[0,1)$, $a_i, b_i \in M_{\mathfrak{p}}$ and $k_i \in C_0(0,1]$, $c_i, d_i \in M_{\mathfrak{q}}$. Furthermore, we clearly may assume that the elements a_i, b_i, c_i, d_i are normalized. We then define

$$\mathcal{G} := \{h_i \otimes a_i\}_{i=1}^K \cup \{k_i \otimes c_i\}_{i=1}^K \cup \{\iota\} \subseteq Z_{\mathfrak{p},\mathfrak{q}},$$

where ι is the canonical generator of $C_0(0,1]$. Let $0 < \delta < \frac{\varepsilon}{100K}$ be as in Lemma III.4.8, with ε replaced by $\frac{\varepsilon}{100}$ and $\mathcal{F}$ replaced by $\{f_k\}_{k=1}^N \cup \{g_k\}_{k=1}^N$. Let us check that this choice of $\mathcal{G}$ and δ works.

To this end assume that $\varphi\colon Z_{\mathfrak{p},\mathfrak{q}} \to U$ is a unital $*$-homomorphism, where $U \cong U \otimes M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$, and $u \in U$ is a unitary such that

$$\|u\varphi(g)u^* - \varphi(g)\| < \delta \qquad (g \in \mathcal{G}). \tag{III.4.8}$$

Since $\iota \in \mathcal{G}$, it follows from the choice of δ, that there exists a unitary path $(u_t)_{t \in [0,1]}$ in U, such that $u_0 = 1_U$, $u_1 = u$ and

$$\|u_t\varphi(f)u_t^* - \varphi(f)\| < \frac{\varepsilon}{100} \qquad (f \in \{f_k\}_{k=1}^N \cup \{g_k\}_{k=1}^N). \tag{III.4.9}$$

Let us define a unitary path

$$Z_t := (\acute{U}_{t_0}^\varphi)^*(\grave{U}_{t_0}^\varphi)^*(u_t \otimes 1_{\mathfrak{p}\mathfrak{q}})\acute{U}_{t_0}^\varphi\grave{U}_{t_0}^\varphi, \qquad (t \in [0,1]),$$

with $\grave{U}_{t_0}^\varphi$ and $\acute{U}_{t_0}^\varphi$ as in (III.3.5). We show that Z_t commutes sufficiently well with $\varphi(\mathcal{F})$. Let us check this for $\acute{\mathcal{F}}$. The computation for $\grave{\mathcal{F}}$ is analogous. For $t \in [0,1]$ and $k = 1, 2, \cdots, N$ we get:

$$
\begin{aligned}
&\mathrm{ad}(Z_t) \circ (\acute{\varphi}(g_k \otimes y_k) \otimes 1_{\mathfrak{p}\mathfrak{q}}) \\
&\quad = \quad (\acute{U}_{t_0}^\varphi)^*(\grave{U}_{t_0}^\varphi)^*(u_t \otimes 1_{\mathfrak{p}\mathfrak{q}})\acute{U}_{t_0}^\varphi\grave{U}_{t_0}^\varphi \; (\acute{\varphi}(g_k \otimes y_k) \otimes 1_{\mathfrak{p}\mathfrak{q}}) \cdot \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\grave{U}_{t_0}^\varphi)^*(\acute{U}_{t_0}^\varphi)^*(u_t^* \otimes 1_{\mathfrak{p}\mathfrak{q}})\grave{U}_{t_0}^\varphi\acute{U}_{t_0}^\varphi \\
&\quad \overset{\text{(III.3.7),(III.4.6)}}{\approx_{\frac{\varepsilon}{100}}} (\acute{U}_{t_0}^\varphi)^*(\grave{U}_{t_0}^\varphi)^*(u_t \otimes 1_{\mathfrak{p}\mathfrak{q}})(\varphi(g_k) \otimes 1_{\mathfrak{p}} \otimes y_k)(u_t^* \otimes 1_{\mathfrak{p}\mathfrak{q}})\grave{U}_{t_0}^\varphi\acute{U}_{t_0}^\varphi \\
&\quad \overset{\text{(III.4.9)}}{\approx_{\frac{\varepsilon}{100}}} (\acute{U}_{t_0}^\varphi)^*(\grave{U}_{t_0}^\varphi)^*(\varphi(g_k) \otimes 1_{\mathfrak{p}} \otimes y_k)\grave{U}_{t_0}^\varphi\acute{U}_{t_0}^\varphi \\
&\quad \overset{\text{(III.3.7),(III.4.6)}}{\approx_{\frac{\varepsilon}{100}}} \acute{\varphi}(g_k \otimes y_k) \otimes 1_{\mathfrak{p}\mathfrak{q}}.
\end{aligned}
$$

It follows that

$$\|Z_t\varphi(f)Z_t^* - \varphi(f)\| < \frac{\varepsilon}{30} \qquad (f \in \mathcal{F}, \; t \in [0,1]). \tag{III.4.10}$$

Next, we show that Z_1 is close to $u \otimes 1_{\mathfrak{p}\mathfrak{q}}$. To this end, observe that

$$
\begin{aligned}
\left\|(\grave{U}_{t_0}^\varphi)^*(u \otimes 1_{\mathfrak{p}\mathfrak{q}})\grave{U}_{t_0}^\varphi - u \otimes 1_{\mathfrak{p}\mathfrak{q}}\right\| &< \frac{\varepsilon}{30}, \\
\left\|(\acute{U}_{t_0}^\varphi)^*(u \otimes 1_{\mathfrak{p}\mathfrak{q}})\acute{U}_{t_0}^\varphi - u \otimes 1_{\mathfrak{p}\mathfrak{q}}\right\| &< \frac{\varepsilon}{30}.
\end{aligned}
\tag{III.4.11}
$$

Let us prove that (III.4.11) holds for $\grave{U}_{t_0}^{\varphi}$. The computation for $\acute{U}_{t_0}^{\varphi}$ is similar. Since the image of the scalar part is central it suffices to show that (recall the definition of $\grave{U}_{t_0}^{\varphi}$ from (III.3.5)):

$$(u \otimes 1_{\mathfrak{p}\mathfrak{q}}) \cdot (\grave{\varphi} \otimes \imath_{(\mathfrak{p},\mathfrak{q})}^{[1]})^{+}(\grave{W}_{t_0} - 1) \approx_{\frac{\varepsilon}{30}} (\grave{\varphi} \otimes \imath_{(\mathfrak{p},\mathfrak{q})}^{[1]})^{+}(\grave{W}_{t_0} - 1) \cdot (u \otimes 1_{\mathfrak{p}\mathfrak{q}}).$$

The following calculation shows that this is indeed true:

$$(u \otimes 1_{\mathfrak{p}\mathfrak{q}}) \cdot (\grave{\varphi} \otimes \imath_{(\mathfrak{p},\mathfrak{q})}^{[1]})^{+}(\grave{W}_{t_0} - 1)$$

$$\overset{\text{(III.4.7)}}{\approx_{\frac{\varepsilon}{100}}} (u \otimes 1_{\mathfrak{p}\mathfrak{q}}) \cdot (\grave{\varphi} \otimes \imath_{(\mathfrak{p},\mathfrak{q})}^{[1]})^{+}\left(\sum_{i=1}^{K} h_i \otimes a_i \otimes b_i\right)$$

$$= \sum_{i=1}^{K} u\grave{\varphi}(h_i \otimes a_i) \otimes b_i \otimes 1_{\mathfrak{q}}$$

$$\overset{\text{(III.4.8)}}{\approx_{K\delta}} \sum_{i=1}^{K} \grave{\varphi}(h_i \otimes a_i)u \otimes b_i \otimes 1_{\mathfrak{q}}$$

$$= (\grave{\varphi} \otimes \imath_{(\mathfrak{p},\mathfrak{q})}^{[1]})^{+}\left(\sum_{i=1}^{K} h_i \otimes a_i \otimes b_i\right) \cdot (u \otimes 1_{\mathfrak{p}\mathfrak{q}})$$

$$\overset{\text{(III.4.7)}}{\approx_{\frac{\varepsilon}{100}}} (\grave{\varphi} \otimes \imath_{(\mathfrak{p},\mathfrak{q})}^{[1]})^{+}(\grave{W}_{t_0} - 1) \cdot (u \otimes 1_{\mathfrak{p}\mathfrak{q}}).$$

Since $K\delta < \frac{\varepsilon}{100}$, the claim follows. We are now ready to construct the desired unitary path in U. By [TW07, Remark 2.7], there exists a unital *-homomorphism $\theta \colon U \otimes M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} \to U$ such that

$$\theta(x \otimes 1_{\mathfrak{p}\mathfrak{q}}) \approx_{\frac{\varepsilon}{10}} x \qquad (x \in \varphi(\mathcal{F}) \cup \{u\}).$$

Let $(W_t)_{t\in[0,1]}$ be given by $W_t := \theta(Z_t)$. Then,

$$W_t\varphi(f)W_t^* \approx_{\frac{\varepsilon}{10}} \theta(Z_t(\varphi(f) \otimes 1_{\mathfrak{p}\mathfrak{q}})Z_t^*)$$

$$\approx_{\frac{\varepsilon}{30}} \theta(\varphi(f) \otimes 1_{\mathfrak{p}\mathfrak{q}})$$

$$\approx_{\frac{\varepsilon}{10}} \varphi(f),$$

for all $f \in \mathcal{F}$ and $t \in [0, 1]$. Next, by (III.4.11), it follows that $\|Z_1 - u \otimes 1_{\mathfrak{p}\mathfrak{q}}\| < \frac{\varepsilon}{15}$, so that $W_1 = \theta(Z_1) \approx_{\frac{\varepsilon}{15}} \theta(u \otimes 1_{\mathfrak{p}\mathfrak{q}}) \approx_{\frac{\varepsilon}{10}} u$. By a standard functional calculus argument, we may extend $(W_t)_{t\in[0,1]}$ to a unitary path $(W_t)_{t\in[0,2]}$ such that $W_2 = u$ and $\|W_t - W_1\| < \frac{\varepsilon}{6}$, for $t \in [1, 2]$. Then

$$\|W_t\varphi(f)W_t^* - \varphi(f)\| < \varepsilon \qquad (t \in [0, 2], \ f \in \mathcal{F}).$$

This finishes the proof. $\qquad\square$

III.5   Asymptotic uniqueness

III.5.1 Definition. Assume A is a unital C*-algebra and let $\mathscr{C}$ be a class of unital C*-algebras. We say that A satisfies the *Basic Homotopy Property* for $\mathscr{C}$ if the following holds:

For every $\varepsilon > 0$ and $\mathcal{F} \subseteq A$ finite there exists $\delta > 0$ and $\mathcal{G} \subseteq A$ finite such that: If $\varphi \colon A \to C \in \mathscr{C}$ is a unital *-homomorphism and $u \in C$ is a unitary such that

$$\|u\varphi(g)u^* - \varphi(g)\| < \delta \qquad (g \in \mathcal{G}),$$

then there exists a unitary path $(u_t)_{t\in[0,1]}$ in C such that $u_0 = 1_C$, $u_1 = u$ and

$$\|u_t\varphi(f)u_t^* - \varphi(f)\| < \varepsilon \qquad (f \in \mathcal{F},\ t \in [0,1]).$$

We will abbreviate this and say that A satisfies the (BHP) for $\mathscr{C}$.

III.5.2 Example. It follows from Lemma III.4.8 that $C([0,1])$ satisfies the (BHP) for UHF-algebras. By Lemma III.4.9, any generalized dimension drop algebra $Z_{\mathfrak{p},\mathfrak{q}}$, with $\mathfrak{p}$ and $\mathfrak{q}$ of infinite type, satisfies the (BHP) for the class

$$\mathrm{UHF}^{\infty} \otimes M_{\mathfrak{p}} \otimes M_{\mathfrak{q}},$$

consisting of $M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$-stable UHF-algebras of infinite type. Furthermore, by [DW09, Lemma 2.1], we have that any strongly self-absorbing C*-algebra D satisfies the (BHP) for

$$\{A \otimes D : A \text{ is a unital C*-algebra}\}.$$

The next lemma is a standard tool which combines the Basic Homotopy Property and approximate uniqueness to get an asymptotic uniqueness result (cf. [DW09, Theorem 2.2]).

III.5.3 Lemma. *Let A be a unital and separable* C*-*algebra and assume A satisfies the* (BHP) *for $\mathscr{C}$. Then the following is true: If $\varphi, \psi \colon A \to C \in \mathscr{C}$ are unital *-homomorphisms and $\varphi \approx_{\mathrm{u}} \psi$, then φ and ψ are asymptotically unitarily equivalent. If furthermore the unitary group of C is connected, we get strong asymptotic unitary equivalence, i.e. $\varphi \sim_{\mathrm{asu}} \psi$.*

Proof. Let $\varphi, \psi \colon A \to C \in \mathscr{C}$ be unital *-homomorphisms such that $\varphi \approx_{\mathrm{u}} \psi$. Let $(\mathcal{F}_n)_{n=1}^{\infty}$ be an increasing sequence of finite subsets of A such that their

union is dense. Let us also fix a decreasing sequence $\varepsilon_n > 0$ such that $\varepsilon_n \to 0$. Using the Basic Homotopy Property, let $\mathcal{G}_n \subseteq A$ be finite and $\delta_n > 0$ such that the following holds:

Whenever $u \in C$ is a unitary such that

$$\|u^*\psi(g)u - \psi(g)\| < \delta_n \qquad (g \in \mathcal{G}_n),$$

then u is connected to 1_C via a unitary path that commutes pointwise with $\psi(\mathcal{F}_n)$ up to ε_n.

Since φ and ψ are are approximately unitarily equivalent, we may choose unitaries $(u_n)_{n=1}^\infty$ in C, such that

$$\|u_n^*\varphi(g)u_n - \psi(g)\| < \frac{\delta_n}{2} \qquad (g \in \mathcal{G}_n, \ n \in \mathbb{N}).$$

We may furthermore assume that

$$\mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \mathcal{G}_{n+1} \qquad \text{and} \qquad \delta_{n+1} < \delta_n < \varepsilon_n \qquad (n \in \mathbb{N}).$$

Define

$$w_n := u_{n+1}^* u_n \qquad (n \in \mathbb{N}).$$

One then checks that

$$\|w_n^*\psi(g)w_n - \psi(g)\| < \delta_n \qquad (g \in \mathcal{G}_n).$$

By the choice of $\mathcal{G}_n$ and δ_n, there exists a path $(w_{n,t})_{t\in[0,1]}$ of unitaries in C such that $w_{n,0} = w_n$, $w_{n,1} = 1_C$ and

$$\left\|w_{n,t}^*\psi(f)w_{n,t} - \psi(f)\right\| < \varepsilon_n \qquad (f \in \mathcal{F}_n, \ t \in [0,1]).$$

Let us now define a unitary path $(v_t)_{t\in[0,\infty)}$ in C by

$$v_t = u_{n+1}w_{n,t-n} \qquad (t \in [n, n+1)).$$

One checks that v is continuous and that

$$\|v_t\psi(f)v_t^* - \varphi(f)\| < 2\varepsilon_n \qquad (f \in \mathcal{F}_n, \ t \geq n).$$

By density, it follows that

$$\lim_{t\to\infty} \|v_t\psi(f)v_t^* - \varphi(f)\| = 0 \qquad (f \in A).$$

By defining

$$v'_t = v_{\tan(\pi t/2)} \qquad (t \in [0,1)),$$

we see that φ and ψ are asymptotically unitarily equivalent. Furthermore, if the unitary group of C is connected, we can clearly assume that the unitary path starts at the identity. Hence $\varphi \sim_{\mathrm{asu}} \psi$. $\qquad\square$

III.5.4 Theorem. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be supernatural numbers of infinite type. Assume $\varphi, \psi \colon Z_{\mathfrak{p},\mathfrak{q}} \to U$ are standard *-homomorphisms, where U is a UHF-algebra of infinite type that absorbs $M_{\mathfrak{p}}$ and $M_{\mathfrak{q}}$ tensorially. Then $\varphi \sim_{\mathrm{asu}} \psi$.*

Proof. We know that $\varphi \approx_{\mathrm{u}} \psi$ by Lemma III.3.2 and we saw in Example III.5.2 that $Z_{\mathfrak{p},\mathfrak{q}}$ satisfies the (BHP) for U. By Lemma III.5.3 we get $\varphi \sim_{\mathrm{asu}} \psi$. $\qquad\square$

III.6 Uniqueness for endomorphisms of dimension drop algebras

The next theorem shows how the Basic Homotopy Lemma for maps $Z_{\mathfrak{p},\mathfrak{q}} \to M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$ can be lifted to an approximate uniqueness result for standard *-endomorphisms of $Z_{\mathfrak{p},\mathfrak{q}}$.

III.6.1 Theorem. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be supernatural numbers of infinite type. Assume that $\varphi, \psi \colon Z_{\mathfrak{p},\mathfrak{q}} \to Z_{\mathfrak{p},\mathfrak{q}}$ are standard *-homomorphisms. Then $\varphi \approx_{\mathrm{u}} \psi$.*

Proof. Let $\mathcal{F} \subseteq Z_{\mathfrak{p},\mathfrak{q}}$ be finite and $\varepsilon > 0$. Let $\mathcal{G} \subseteq Z_{\mathfrak{p},\mathfrak{q}}$ be finite and $0 < \delta < \varepsilon$, such that the conclusion of Lemma III.4.9 holds for $\mathcal{F}$ and ε replaced by $\frac{\varepsilon}{10}$. Let $N \in \mathbb{N}$ such that

$$|s - t| \le \frac{1}{N} \quad \Rightarrow \quad \|\gamma_s(f) - \gamma_t(f)\| < \frac{\delta}{10} \qquad (f \in \mathcal{F} \cup \mathcal{G},\ \gamma \in \{\varphi, \psi\}).$$

Since $\varphi_s, \psi_s \colon Z_{\mathfrak{p},\mathfrak{q}} \to M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$ are standard for each $s \in [0,1]$, we know from Lemma III.3.2 that

$$\varphi_{\frac{i}{N}} \approx_{\mathrm{u}} \psi_{\frac{i}{N}} \qquad (i = 0, 1, 2, \cdots, N-1, N).$$

It follows that there exist unitaries $u_0, u_1, \cdots, u_{N-1}, u_N \in M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$, such that

$$\left\| u_i \varphi_{\frac{i}{N}}(g) u_i^* - \psi_{\frac{i}{N}}(g) \right\| < \frac{\delta}{5} \qquad (g \in \mathcal{G}).$$

We may assume that $u_0 \in M_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}$ and $u_N \in 1_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$. Now, observe that for $i = 0, 1, 2, \cdots, N - 1$, we have

$$\left\| u_{i+1}^* u_i \varphi_{\frac{i}{N}}(g) u_i^* u_{i+1} - \varphi_{\frac{i}{N}}(g) \right\| < \delta \qquad (g \in \mathcal{G}).$$

By the choice of $\mathcal{G}$ and δ, it follows that there exist unitary paths $(u_{i,t})_{t \in [0,1]}$ for $i = 0, 1, \cdots, N - 1$, with $u_{i,0} = 1_{\mathfrak{pq}}$, $u_{i,1} = u_{i+1}^* u_i$ and such that:

$$\left\| u_{i,t} \varphi_{\frac{i}{N}}(f) u_{i,t}^* - \varphi_{\frac{i}{N}}(f) \right\| < \frac{\varepsilon}{10} \qquad (t \in [0,1], \ f \in \mathcal{F}).$$

This defines a unitary u in $Z_{\mathfrak{p},\mathfrak{q}}$ by

$$u_t := u_i u_{i,Nt-i}^* \quad \text{for} \quad t \in \left[\frac{i}{N}, \frac{i+1}{N} \right].$$

For $f \in \mathcal{F}$ and $t \in [i/N, (i+1)/N]$, we get

$$u_t \varphi_t(f) u_t^* \approx_{\frac{\delta}{10}} u_t \varphi_{\frac{i}{N}}(f) u_t^* = u_i u_{i,Nt-i}^* \varphi_{\frac{i}{N}}(f) u_{i,Nt-i} u_i^*$$
$$\approx_{\frac{\varepsilon}{10}} u_i \varphi_{\frac{i}{N}}(f) u_i^* \approx_{\frac{\delta}{5}} \psi_{\frac{i}{N}}(f)$$
$$\approx_{\frac{\delta}{10}} \psi_t(f).$$

As this holds on every interval $[i/N, (i+1)/N]$, it follows that

$$\| u \varphi(f) u^* - \psi(f) \| < \varepsilon \qquad (f \in \mathcal{F}).$$

This proves that $\varphi \approx_{\mathrm{u}} \psi$. $\qquad \qquad \square$

Chapter IV

Existence

In this chapter we establish existence results for standard c.p.c. order zero maps and standard *-homomorphisms from generalized prime dimension drop algebras into certain UHF-algebras. At the end of this section we construct so-called *unitarily suspended* *-endomorphisms of generalized prime dimension drop algebras. These are particular instances of trace collapsing *-endomorphisms, which will allow us to show that any stationary inductive limit along a fixed generalized prime dimension drop algebra together with such a unitarily suspended *-endomorphism is strongly self-absorbing.

IV.1   Order zero maps

IV.1.1 Convention. Let $x \in M_n$ and $y \in M_m$. Whenever we identify $M_n \otimes M_m$ with M_{nm}, we do this via

$$x \otimes y = (x \cdot y_{ij})_{i,j=1}^m.$$

For $p \in \mathbb{N}$, define the UHF-algebra M_{p^∞} to be the limit of

$$M_{p^n} \to M_{p^{n+1}} : x \mapsto 1_p \otimes x.$$

IV.1.2 Lemma. *Let A be a unital C^*-algebra. Assume $0 \le \alpha < \beta \le 1$. If $h \in A$ is a Lebesgue contraction, then the element*

$$h(\alpha, \beta) := \alpha 1_A + (\beta - \alpha)h$$

has spectrum $[\alpha, \beta]$ and

$$\tau(f(h(\alpha, \beta))) = \frac{1}{\beta - \alpha} \int_\alpha^\beta f(t) \ dt \qquad \left(f \in C_0(0,1], \ \tau \in T(A) \right).$$

Proof. It is clear that $\sigma(h(\alpha, \beta)) = [\alpha, \beta]$, since $\sigma(h) = [0, 1]$. For the remaining statement consider the function

$$g \colon [0, 1] \to [0, 1] : t \mapsto (\beta - \alpha)t + \alpha.$$

For $f \in C_0(0, 1]$ we then get by a change of variables:

$$\tau(f(h(\alpha, \beta))) = \tau((f \circ g)(h)) = \int_0^1 f(g(t)) \; dt = \frac{1}{\beta - \alpha} \int_\alpha^\beta f(t) \; dt.$$

$\square$

IV.1.3 Remark. ([Win09, 1.2.3]) Let $p \geq 2$ be a natural number. For $i, j = 2, \cdots, p$ we consider the relations

$$\|x_i\| \leq 1, \; x_i x_j = 0 \; (i \neq j), \; x_i^* x_j = \delta_{i,j} \cdot x_2^* x_2. \qquad (\mathcal{R}_p)$$

Then

$$C_0(0, 1] \otimes M_p \cong \mathrm{C}^*(x_2, x_3, \cdots, x_p \mid \mathcal{R}_p).$$

In particular, if $x_2, x_3, \cdots, x_p$ are elements in some C*-algebra A satisfying $(\mathcal{R}_p)$, then the assignment

$$\varphi(e_{ij}) = x_i x_j^* \qquad (i, j = 2, 3, \cdots, p)$$

defines[1] a c.p.c. order zero map $\varphi \colon M_p \to A$. Note that

$$\varphi(e_{11}) = \varphi(e_{i1}^* e_{i1}) = x_i^* x_i = x_2^* x_2 \qquad (i = 2, 3, \cdots, p).$$

IV.1.4 Lemma. *Let $p, q \geq 2$ be natural numbers. Then M_{q^∞} contains elements $x_2, x_3, \cdots, x_p$ satisfying $(\mathcal{R}_p)$ such that $x_2^* x_2$ has $\frac{1}{p}$ Lebesgue spectral measure, i.e. such that*

$$\tau_{M_{q^\infty}}(f(x_2^* x_2)) = \frac{1}{p} \int_0^1 f(t) \; dt \qquad (f \in C_0(0, 1]).$$

Proof. Let us recursively define two sequences of non-negative integers $(\alpha_i)_{i=1}^\infty$ and $(\beta_i)_{i=0}^\infty$. Set $\beta_0 := 1$ and define α_i and β_i for $i \geq 1$ by

$$q\beta_{i-1} = p\alpha_i + \beta_i, \qquad (\mathrm{IV.1.1})$$

---

[1] More precisely: The elements $\{\iota^{\frac{1}{2}} \otimes e_{i1} : i = 2, 3, \cdots, p\}$ satisfy $(\mathcal{R}_p)$ and hence induce a *-homomorphism $C_0(0, 1] \otimes M_p \to A$, which in turn corresponds to a c.p.c. order zero map $\varphi \colon M_p \to A$ satisfying $\varphi(e_{ij}) = x_i x_j^*$.

where $\beta_i \in \{0, 1, 2, \cdots, p-1\}$. A short computation using (IV.1.1) shows that the coefficients $(\alpha_i)_{i=1}^\infty$ are chosen such that

$$\sum_{i=1}^\infty \frac{p\alpha_i}{q^i} = 1. \tag{IV.1.2}$$

For further reference, we denote the partial sums by S_N. We define recursively another sequence $(\delta_i)_{i=0}^\infty$ by $\delta_0 = 0$ and for $i \geq 1$ we let

$$\delta_i := q(\delta_{i-1} + p\alpha_i). \tag{IV.1.3}$$

Using (IV.1.3) one easily checks that

$$q^{i+1} = q\beta_i + \delta_i \qquad (i \geq 0). \tag{IV.1.4}$$

In particular (IV.1.4) implies that

$$\delta_{i-1} + p\alpha_i + \beta_i = q^i \qquad (i \geq 1). \tag{IV.1.5}$$

All this is made in such a way that we can subdivide the matrix algebra M_{q^i} into three orthogonal parts, as indicated in Figure IV.1. Let $h \in M_{q^\infty}$ be a
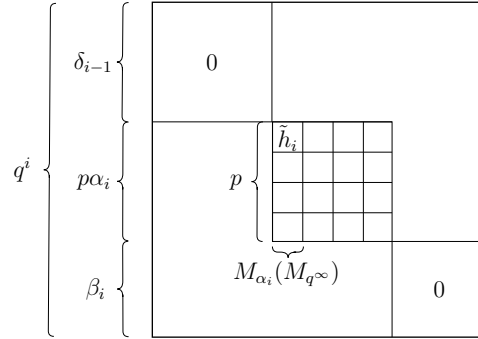
Figure IV.1: Subdivision of M_{q^i} and the element $\bar{h}_i$.

Lebesgue contraction, for example

$$h := \sum_{i=1}^\infty \left(\sum_{k=1}^{q-1} \frac{k}{q^i} 1_q^{\otimes(i-1)} \otimes e_{kk}^{(q)} \otimes 1_q^{\otimes\infty} \right).$$

For $i \in \mathbb{N}$ and $r \in \{1, 2, \cdots, \alpha_i\}$ we set (as in Lemma IV.1.2)

$$\tilde{h}_{i,r} := h\left(1 - S_{i-1} - \frac{rp}{q^i}, 1 - S_{i-1} - \frac{(r-1)p}{q^i} \right) \in M_{q^\infty}$$

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and
$$\tilde{h}_i := \operatorname{diag}(\tilde{h}_{i,1}, \tilde{h}_{i,2}, \cdots, \tilde{h}_{i,\alpha_i}) \in M_{\alpha_i}(M_{q^\infty}).$$

By Lemma IV.1.2, we know

$$\tau_{M_{q^\infty}}(f(\tilde{h}_{i,r})) = \frac{q^i}{p} \int_{1-S_{i-1}-\frac{rp}{q^i}}^{1-S_{i-1}-\frac{(r-1)p}{q^i}} f(t)\, dt \qquad (f \in C_0(0,1],\ 1 \le r \le \alpha_i)$$

and hence

$$(\tau_{M_{q^\infty}} \circ \operatorname{Tr}_{\alpha_i})(f(\tilde{h}_i)) = \frac{q^i}{p} \int_{1-S_i}^{1-S_{i-1}} f(t)\, dt \qquad (f \in C_0(0,1]), \qquad \text{(IV.1.6)}$$

where $\operatorname{Tr}_{\alpha_i}$ denotes the sum over the diagonal in $M_{\alpha_i}(M_{q^\infty})$. Next, put $\tilde{h}_i$ into the upper left corner of $M_p(M_{\alpha_i}(M_{q^\infty}))$, which we denote by

$$e_{11}^{(p)} \otimes \tilde{h}_i,$$

and set

$$\bar{h}_i := 0_{\delta_{i-1}} \oplus (e_{11}^{(p)} \otimes \tilde{h}_i) \oplus 0_{\beta_i} \in M_{q^i}(M_{q^\infty}) \cong M_{q^i} \otimes M_{q^\infty}.$$

The element $\bar{h}_i$ is as depicted in Figure IV.1. Note that by (IV.1.5), the element $\bar{h}_i$ lives in $M_{q^i}(M_{q^\infty})$. Next, we consider the elements $\bar{h}_i$ in

$$M_{q^\infty} \otimes M_{q^\infty} = \varinjlim \left(M_{q^i} \otimes M_{q^\infty}, 1_q \otimes \operatorname{id}_{M_{q^i}} \otimes \operatorname{id}_{M_{q^\infty}} \right).$$

By construction, the $\bar{h}_i$ are mutually orthogonal positive contractions, which are summable in $M_{q^\infty} \otimes M_{q^\infty}$. We therefore may define

$$\bar{h} := \sum_{i=1}^\infty \bar{h}_i \in M_{q^\infty} \otimes M_{q^\infty}.$$

We can now check that $\bar{h}$ has $\frac{1}{p}$ Lebesgue spectral measure. Denote $T_N := \sum_{i=1}^N \bar{h}_i$. For $f \in C_0(0,1]$ and $N \in \mathbb{N}$ we get:

$$\tau_{M_{q^\infty} \otimes M_{q^\infty}}(f(T_N)) = \sum_{i=1}^N (\tau_{M_{q^i}} \otimes \tau_{M_{q^\infty}})(f(\bar{h}_i)) = \sum_{i=1}^N \frac{1}{q^i}(\tau_{M_{q^\infty}} \circ \operatorname{Tr}_{\alpha_i})(f(\tilde{h}_i))$$

$$\overset{\text{(IV.1.6)}}{=} \sum_{i=1}^N \frac{1}{q^i} \frac{q^i}{p} \int_{1-S_i}^{1-S_{i-1}} f(t)\, dt = \frac{1}{p} \int_{1-S_N}^1 f(t)\, dt.$$

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By (IV.1.2) the result follows, if we let N tend to infinity.

In order to finish the proof, instead of $\bar{h}_i$, we look, for $j = 2, 3, \cdots, p$, at the elements

$$\bar{x}_{j,i} := 0_{\delta_{i-1}} \oplus (e_{j1}^{(p)} \otimes (\tilde{h}_i)^{\frac{1}{2}}) \oplus 0_{\beta_i} \in M_{q^i}(M_{q^\infty}) \cong M_{q^i} \otimes M_{q^\infty}$$

and define

$$x_j := \sum_{i=1}^{\infty} \bar{x}_{j,i} \qquad (j = 2, 3, \cdots, p).$$

The elements $\{x_j\}_{j=2}^{p}$ then satisfy $(\mathcal{R}_p)$ and $x_2^* x_2 = \bar{h}$, which has $\frac{1}{p}$ Lebesgue spectral measure. $\qquad\square$

The reader is invited to run this construction for $p = 2$ and $q = 3$, in which case $\alpha_i = \beta_i = 1$, for all $i \in \mathbb{N}$.

IV.1.5 Lemma. *Let $p, q \geq 2$ be natural numbers. Then, there exists a c.p.c. order zero map*

$$\varphi \colon M_p \to M_{q^\infty}$$

such that h_φ is a Lebesgue contraction.

Proof. The elements $x_2, x_3, \cdots, x_p$ produced in Lemma IV.1.4 satisfy $(\mathcal{R}_p)$ and hence define, as described in Remark IV.1.3, a c.p.c. order zero map $\varphi \colon M_p \to M_{q^\infty}$ such that $\varphi(e_{ij}) = x_i x_j^*$, for $i, j = 2, 3, \cdots, p$. Using that $x_2^* x_2$ has $\frac{1}{p}$ Lebesgue spectral measure, it is now easy to see that h_φ is a Lebesgue contraction. $\qquad\square$

IV.2 Unitarily suspended endomorphisms

IV.2.1 Lemma. *Let $p, q \geq 2$ be natural numbers. Then, for $r \in \{p, q\}$, there exist c.p.c. order zero maps $\alpha \colon M_p \to M_{r^\infty}$ and $\beta \colon M_q \to M_{r^\infty}$ such that*

$$\alpha(1_p) + \beta(1_q) = 1, \quad [\alpha(M_p), \beta(M_q)] = \{0\}$$

and such that h_α, h_β are Lebesgue contractions.

Proof. Let us do the case $r = q$. By Lemma IV.1.5, there exists a c.p.c. order zero map $\phi \colon M_p \to M_{q^\infty}$ such that h_ϕ is a Lebesgue contraction. We then define c.p.c. order zero maps

$$\alpha \colon M_p \to M_{q^\infty} \otimes M_q : x \mapsto \phi(x) \otimes 1_q,$$
$$\beta \colon M_q \to M_{q^\infty} \otimes M_q : y \mapsto (1 - h_\phi) \otimes y.$$

Since h_ϕ commutes with the image of ϕ, we see that the images of α and β commute. Furthermore, by definition $\alpha(1_p) + \beta(1_q) = 1$ and it is clear that h_α and h_β are Lebesgue contractions. $\qquad\square$

IV.2.2 Corollary. *Let $p, q \geq 2$ be coprime natural numbers. Then, for $r \in \{p, q\}$, there exists a standard *-homomorphism*

$$\varphi\colon Z_{p,q} \to M_{r^\infty}.$$

Proof. Let $r \in \{p, q\}$ and let $\alpha\colon M_p \to M_{r^\infty}$ and $\beta\colon M_q \to M_{r^\infty}$ be as in Lemma IV.2.1. By Theorem I.3.9 these maps induce a standard *-homomorphism $\varphi\colon Z_{p,q} \to M_{r^\infty}$ such that $\dot\varphi = (\overline{\alpha})'$ and $\acute\varphi = \overline{\beta}$, where $(\cdot)'$ means that we have reversed the orientation of the unit interval. $\qquad\square$

IV.2.3 Convention. Let $\mathfrak{p}$ and $\mathfrak{q}$ be supernatural numbers. We then write $Z_{\mathfrak{p},\mathfrak{q}}$ as an inductive limit as follows: Fix sequences $(P_n)_{n=1}^\infty$, $(Q_n)_{n=1}^\infty$ converging (see Section I.3) to $\mathfrak{p}$ respectively $\mathfrak{q}$ such that P_n divides P_{n+1} and Q_n divides Q_{n+1}. This defines unital inclusions $M_{P_n} \subseteq M_{P_{n+1}}$, $M_{Q_n} \subseteq M_{Q_{n+1}}$ and $\gamma_{n,n+1}\colon Z_{P_n,Q_n} \to Z_{P_{n+1},Q_{n+1}}$ such that

$$Z_{\mathfrak{p},\mathfrak{q}} = \varinjlim(Z_{P_n,Q_n}, \gamma_{n,n+1}).$$

Let us denote the limit maps by $\gamma_n\colon Z_{P_n,Q_n} \to Z_{\mathfrak{p},\mathfrak{q}}$. Furthermore, we denote by

$$\kappa_n\colon Z_{\mathfrak{p},\mathfrak{q}} \to Z_{P_n,Q_n}$$

the induced conditional expectations.

IV.2.4 Lemma. *Le $\mathfrak{p}$ and $\mathfrak{q}$ be coprime supernatural numbers. Then, for $\mathfrak{r} \in \{\mathfrak{p}, \mathfrak{q}\}$, there exists a standard *-homomorphism*

$$\varphi\colon Z_{\mathfrak{p},\mathfrak{q}} \to M_{\mathfrak{r}}.$$

Proof. Let $(\mathcal{F}_n)_{n=1}^\infty$ be an increasing sequence of finite subsets of $Z_{\mathfrak{p},\mathfrak{q}}$ with dense union, such that $\mathcal{F}_n \subseteq Z_{P_n,Q_n}$ and $\gamma_{n,n+1}(\mathcal{F}_n) \subseteq \mathcal{F}_{n+1}$. Furthermore, let $\varepsilon_n > 0$ with $\sum \varepsilon_n < \infty$. By Corollary IV.2.2, for each $n \in \mathbb{N}$, there exists a standard *-homomorphism $\varphi_n\colon Z_{P_n,Q_n} \to M_{\mathfrak{r}}$. Note that $\varphi_{n+1} \circ \gamma_{n,n+1}$ is also standard. Hence, by Lemma III.3.2, there exists a unitary $u_{n+1} \in M_{\mathfrak{r}}$ such that

$$\big\|\varphi_n(x) - u_{n+1}\varphi_{n+1}(\gamma_{n,n+1}(x))u_{n+1}^*\big\| < \varepsilon_n \qquad (x \in \mathcal{F}_n).$$

Define $\tilde{\varphi}_1 \coloneqq \varphi_1$ and for $n \geq 2$ we let

$$\tilde{\varphi}_n \colon Z_{P_n, Q_n} \to M_{\mathfrak{r}} : x \mapsto u_2 u_3 \cdots u_n \varphi_n(x) u_n^* \cdots u_3^* u_2^*.$$

One then checks that for each $x \in Z_{P_n, Q_n}$ the sequence $(\tilde{\varphi}_k(\gamma_{n,k}(x)))_{k=n}^\infty$ is Cauchy and we may define

$$\varphi(x) \coloneqq \lim_{k \to \infty} \tilde{\varphi}_k(\gamma_{n,k}(x)) \qquad (x \in Z_{P_n, Q_n}).$$

This extends to a *-homomorphism $\varphi \colon Z_{\mathfrak{p},\mathfrak{q}} \to M_{\mathfrak{r}}$, which is easily seen to be standard. $\qquad\square$

IV.2.5 Definition. Let $\mathfrak{p}$ and $\mathfrak{q}$ be supernatural numbers of infinite type. We say that a *-homomorphism $\mu \colon Z_{\mathfrak{p},\mathfrak{q}} \to Z_{\mathfrak{p},\mathfrak{q}}$ is *unitarily suspended* if

$$\mu_t = \begin{cases} \mathrm{ad}(u_t) \circ (\alpha_0 \otimes 1_{\mathfrak{q}}) & \text{if } t \in [0,1), \\ 1_{\mathfrak{p}} \otimes \alpha_1 & \text{if } t = 1, \end{cases}$$

where $\alpha_0 \colon Z_{\mathfrak{p},\mathfrak{q}} \to M_{\mathfrak{p}}$ and $\alpha_1 \colon Z_{\mathfrak{p},\mathfrak{q}} \to M_{\mathfrak{q}}$ are unital *-homomorphisms and $(u_t)_{t \in [0,1)}$ is a unitary path in $M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$ starting at the identity.

IV.2.6 Theorem. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be coprime supernatural numbers of infinite type. Then, there exists a unitarily suspended standard *-homomorphism $\mu \colon Z_{\mathfrak{p},\mathfrak{q}} \to Z_{\mathfrak{p},\mathfrak{q}}$.*

Proof. By Lemma IV.2.4, there exist standard *-homomorphisms $\alpha_0 \colon Z_{\mathfrak{p},\mathfrak{q}} \to M_{\mathfrak{p}}$ and $\alpha_1 \colon Z_{\mathfrak{p},\mathfrak{q}} \to M_{\mathfrak{q}}$. By Theorem III.5.4, we have $\alpha_0 \otimes 1_{\mathfrak{q}} \sim_{\mathrm{asu}} 1_{\mathfrak{p}} \otimes \alpha_1$, witnessed by a unitary path $(u_t)_{t \in [0,1)}$ starting at the identity. Define a *-homomorphism $\mu \colon Z_{\mathfrak{p},\mathfrak{q}} \to Z_{\mathfrak{p},\mathfrak{q}}$ by $\mu_t \coloneqq \mathrm{ad}(u_t) \circ (\alpha_0 \otimes 1_{\mathfrak{p}})$ for $t \in [0,1)$ and $\mu_1 \coloneqq 1_{\mathfrak{p}} \otimes \alpha_1$. Then μ is standard and unitarily suspended in the sense of Definition IV.2.5. $\qquad\square$

Chapter V

Alternative approaches to classification

This chapter explains alternative routes leading to classification, as established in Chapter III and Chapter IV. More precisely, we will reprove Lemma IV.1.5 and Lemma III.2.3, i.e.

(i) there exists a c.p.c. order zero map $\varphi\colon M_p \to U$ such that h_φ is a Lebesgue contraction,

(ii) any two c.p.c. order zero maps as in (i) are approximately unitarily equivalent,

where U is a UHF-algebra of infinite type.[1] Once these results are established, one proceeds as before to get uniqueness and existence results for maps on generalized dimension drop algebras.

To get (i) and (ii) in a more general and quick fashion, we rely on classification of *-homomorphisms from $C_0(0,1]$ into stable rank one C*-algebras by means of the Cuntz semigroup ([CE08]). Furthermore, we provide a proof for (ii) using (i) and the uniqueness of the hyperfinite II_1-factor. These von Neumann algebraic methods are inspired by recent methods in classifcation, e.g. [Sch18].

In the last section we show how classification by the Cuntz semigroup, as in [Rob12], can be used to circumvent all of these methods and how to recover Lemmas IV.2.4 and III.3.2:

---

[1]We also prove (i) and (ii) for more complicated codomains, e.g. ultrapowers of UHF-algebras.

(iii) there exists a standard *-homomorphism $Z_{\mathfrak{p},\mathfrak{q}} \to U$,

(iv) any two standard *-homomorphisms $Z_{\mathfrak{p},\mathfrak{q}} \to U$ are approximately unitarily equivalent.

However, note that the results of Section III.4 and in particular the asymptotic uniqueness result (Theorem III.5.4) cannot be proved by the methods used in this chapter.

V.1 The Cuntz semigroup

Let us briefly recall the Cuntz semigroup of a C*-algebra and the category **Cu**, which has been introduced in [CEI08]. See also [APT11] for an excellent survey and connections to classification.

V.1.1 Definition. Let A be a C*-algebra and $a, b \in A_+$. We say that a is *Cuntz below* b if and only if there exists a sequence $(x_n)_{n=1}^\infty$ in A such that

$$\lim_{n \to \infty} \|x_n^* b x_n - a\| = 0.$$

We denote this relation by $a \preccurlyeq b$. We say that a is *Cuntz equivalent* to b if and only if $a \preccurlyeq b$ and $b \preccurlyeq a$. This is denoted by $a \sim_{\mathrm{Cu}} b$ and we note that $\sim_{\mathrm{Cu}}$ is an equivalence relation.

V.1.2 The Cuntz semigroup of a C*-algebra. Let A be a C*-algebra. We define
$$\mathrm{Cu}(A) := (A \otimes \mathbb{K})_+ / \sim_{\mathrm{Cu}},$$

where $\mathbb{K}$ denotes the compact operators on $\ell^2(\mathbb{N})$. The class of an element $a \in (A \otimes \mathbb{K})_+$ is denoted by $[a]$. Then $\mathrm{Cu}(A)$ becomes a positively ordered abelian monoid, with $[0]$ as neutral element and where

$$[a] \leq [b] \qquad \overset{\mathrm{def}}{\Longleftrightarrow} \qquad a \preccurlyeq b \qquad (a, b \in (A \otimes \mathbb{K})_+).$$

Addition is defined by
$$[a] + [b] := [\varphi(a \oplus b)],$$

where $a \oplus b \in M_2(A \otimes \mathbb{K})$ denotes the diagonal sum and $\varphi \colon M_2(A \otimes \mathbb{K}) \to A \otimes \mathbb{K}$ is any *-isomorphism induced by a *-isomorphism $\mathbb{K} \otimes M_2 \to \mathbb{K}$. This addition does not depend on the choice of isomorphism, since any two such

isomorphisms are unitarily equivalent.[2] Then, $\mathrm{Cu}(A)$ is a positively ordered abelian monoid, called the *Cuntz semigroup* of A.[3] For a $*$-homomorphism $\phi\colon A \to B$ we define

$$\mathrm{Cu}(\phi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B) : [a] \mapsto [\phi(a)],$$

where ϕ is understood to be $\phi \otimes \mathrm{id}_{\mathbb{K}}$. This makes Cu into a functor from the category of C*-algebras into a certain category **Cu**, which is explained below and was identified in [CEI08].

V.1.3 The category Cu. Let S be a positively ordered abelian monoid. For $x, y \in S$ we say that x is *way below* y if and only if the following holds: Whenever $(y_n)_{n=1}^{\infty}$ is an increasing sequence in S such that $\sup_n y_n$ exists and $y \leq \sup_n y_n$, then there exists some $m \in \mathbb{N}$ such that $x \leq y_m$. This will be denoted by $x \ll y$. We say that a sequence $(x_n)_{n=1}^{\infty}$ in S is *rapidly increasing* if $x_n \ll x_{n+1}$, for all $n \in \mathbb{N}$.

The objects of **Cu** are positively ordered abelian monoids S such that the following four axioms are satisfied:

(O1) Any increasing sequence in S has a supremum in S,

(O2) every element in S is the supremum of a rapidly increasing sequence in S,

(O3) if $x_1, x_2, y_1, y_2 \in S$ and $x_i \ll y_i$, for $i = 1, 2$, then $x_1 + x_2 \ll y_1 + y_2$,

(O4) if $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are increasing sequences in S, then

$$\sup_n (x_n + y_n) = \sup_n x_n + \sup_n y_n.$$

A morphism in **Cu** is a map $\alpha\colon S \to T$, such that

(M1) α is additive and preserves 0,

(M2) α is order preserving,

---

[2] By a unitary in $M(A \otimes \mathbb{K})$.

[3] There is also a more classical version called the Cuntz semigroup, usually denoted by $W(A)$, which is given by those equivalence classes in $\mathrm{Cu}(A)$ that are represented by an element in $M_n(A)_+$, for some $n \in \mathbb{N}$. It is true that $W(A)$ is dense in $\mathrm{Cu}(A)$ and $W(A \otimes \mathbb{K}) \cong \mathrm{Cu}(A)$. See [APT11, Theorem 4.33]

(M3) α preserves suprema of increasing sequences,

(M4) α preserves the way below relation.

If α only satisfies (M1),(M2) and (M3), we say that α is a *generalized* **Cu**-*morphism.*

V.1.4 Example. The Cuntz semigroup of $\mathbb{K} = \mathbb{K}(\ell^2(\mathbb{N}))$ is isomorphic to $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. The semigroup structure on $\overline{\mathbb{N}}$ is the obvious one. Furthermore, the isomorphism is given by

$$\mathrm{Cu}(\mathbb{K}) \to \overline{\mathbb{N}} : [a] \mapsto \mathrm{rank}(a),$$

where $\mathrm{rank}(a) := \dim(\mathrm{Im}(a))$.

V.1.5 Definition. Let S be a **Cu**-semigroup and note that $[0, \infty]$ is in a natural way a **Cu**-semigroup. Then, a *functional* on S is a generalized **Cu**-morphism $\lambda \colon S \to [0, \infty]$, i.e. λ is additive and preserves the zero element, order, addition and suprema of increasing sequences. The set of functionals on S will be denoted by $\mathrm{F}(S)$.

The next theorem shows that functionals on $\mathrm{Cu}(A)$ can be described very nicely.

V.1.6 Theorem. *There is a bijection*

$$\mathrm{QT}_2(A) \to \mathrm{F}(\mathrm{Cu}(A)) : \tau \mapsto d_\tau$$

between the set of lower semicontinuous 2-quasitraces on A and functionals on $\mathrm{Cu}(A)$. *The functional[4] d_τ is given by*

$$d_\tau([a]) = \lim_{n \to \infty} (\tau \otimes \mathrm{Tr}_{\mathbb{K}})(a^{\frac{1}{n}}) \qquad (a \in (A \otimes \mathbb{K})_+).$$

If A is exact, then

$$\mathrm{T}(A) \cong \mathrm{F}(\mathrm{Cu}(A)),$$

where $\mathrm{T}(A)$[5] denotes the non-cancellative cone of lower semicontinuous traces[6] on A. If in addition A is simple, then

$$\mathrm{T}_d(A) \cup \{\tau_\infty\} = \mathrm{T}(A) \cong \mathrm{F}(\mathrm{Cu}(A)),$$

---

[4]Note that one of the main parts of the proof consists of showing that d_τ is indeed a functional.

[5]Note the subtle difference between the set of tracial states $T(A)$ and the set of lower semicontinuous traces $\mathrm{T}(A)$.

[6]Trace here means a homogeneous map $\tau \colon A_+ \to [0, \infty]$ such that $\tau(x^*x) = \tau(xx^*)$, for all $x \in A$.

where $\mathrm{T}_d(A) \subseteq \mathrm{T}(A)$ *denotes the set of densely defined lower semicontinuous traces. Moreover, if A is unital we may identify $\mathrm{T}_d(A) \cong T(A)$.*

Proof. The bijective correspondence is proven in [ERS11, Proposition 4.2]. The statement about exactness follows from [ERS11, Remark 4.5] and essentially relies on a result by Haagerup that 2-quasitraces on exact C*-algebras are traces. For the remaining statement, let us show that if A is simple and $\tau(a) < \infty$, for some $a \in A_+$, then τ is densely defined. We still assume that τ is a lower semicontinuous trace in the sense of [ERS11, Definition 3.1]. Let $x \in A_+$ and $\varepsilon > 0$. Since A is assumed to be simple, a is full and we know that there exist $x_1, \cdots, x_n \in A$ with

$$x \approx_\varepsilon \sum_{i=1}^{n} x_i^* a x_i.$$

Using [APT11, Theorem 2.13], we may assume that

$$(x - \varepsilon)_+ = \sum_{i=1}^{n} x_i^* a x_i.$$

Now

$$\tau\big((x - \varepsilon)_+\big) = \sum_{i=1}^{n} \tau\big(a^{\frac{1}{2}}(x_i^* x_i) a^{\frac{1}{2}}\big) \leq \sum_{i=1}^{n} \|x_i\|^2 \, \tau(a) < \infty.$$

This proves the claim that τ is densely defined and it immediately follows that

$$\mathrm{T}(A) = \mathrm{T}_d(A) \cup \{\tau_\infty\},$$

where

$$\tau_\infty(x) = \begin{cases} 0 & \text{if } x = 0, \\ \infty & \text{otherwise.} \end{cases}$$

If A is unital then $\mathrm{T}_d(A) \cong T(A)$, because densely defined traces are finite on 1_A and hence must be bounded traces. The isomorphism is then given by $\tau \mapsto \frac{\tau}{\|\tau\|}$. $\qquad\square$

V.1.7 Definition. Let A be a C*-algebra. We then denote

$$\mathrm{Cu}(A)_+ := \{[a] \in \mathrm{Cu}(A) : [a] \text{ is not represented by a projection in } A \otimes \mathbb{K}\}.$$

V.1.8 Remark. Let A be a C*-algebra of stable rank one. Then $V(A)$, the Murray-von Neumann semigroup of A, sits canonically inside $\mathrm{Cu}(A)$ (see [APT11, Lemma 2.20]). Furthermore, by [APT11, Corollary 2.24], $\mathrm{Cu}(A)_+$ is an absorbing subsemigroup of $\mathrm{Cu}(A)$ and it follows that

$$\mathrm{Cu}(A) = V(A) \sqcup \mathrm{Cu}(A)_+,$$

where $\sqcup$ means disjoint union. Absorbing means that $V(A) + \mathrm{Cu}(A)_+ \subseteq \mathrm{Cu}(A)_+$.

The next lemma will be used to show that on $\mathrm{Cu}(A)_+$ certain functionals behave like a **Cu**-morphism.

V.1.9 Lemma. *Let A be a C*-algebra and let $x \in \mathrm{Cu}(A)$ and $y \in \mathrm{Cu}(A)_+$ such that $x \ll y$. Then, there exists $z \in \mathrm{Cu}(A)$ such that $z \neq 0$ and*

$$x + z \leq y.$$

Proof. Write $x = [a]$ and $y = [b]$, where $a, b \in (A \otimes \mathbb{K})_+$. We may assume that $\|a\| = \|b\| = 1$. Since $x \ll y$, there exists $\varepsilon > 0$ such that $[a] \leq [(b - \varepsilon)_+]$.[7] Let $c_\varepsilon \in C_0(0, 1]$ be such that $\|c_\varepsilon\| = 1$ and $c_\varepsilon(t) = 0$ for $t \in [\varepsilon, 1]$. Since 0 is not an isolated point in the spectrum of b, it follows that $c_\varepsilon(b) \neq 0$.[8] Furthermore $c_\varepsilon(b)$ is orthogonal to $(b - \varepsilon)_+$. We see that

$$[a] + [c_\varepsilon(b)] \leq [(b - \varepsilon)_+] + [c_\varepsilon(b)] = [(b - \varepsilon)_+ + c_\varepsilon(b)] \leq [b].$$

Let $z := [c_\varepsilon(b)]$. Then $z \neq 0$ and by the above calculation $x + z \leq y$. $\qquad\square$

V.1.10 Definition. Let A be a unital C*-algebra. By $\mathrm{LAff}_{++}(T(A))$ we denote the semigroup of lower semicontinuous affine real-valued (not necessarily finite) strictly positive functions on $T(A)$. Addition and order is pointwise. Then, there is a map of semigroups

$$\mathrm{Cu}(A) \to \mathrm{LAff}_{++}(T(A)) : [a] \mapsto \widehat{[a]},$$

where

$$\widehat{[a]}(\tau) := d_\tau([a]).$$

---

[7] This is because $[b] = \sup_{n \in \mathbb{N}}[(b - 1/n)_+]$.

[8] If 0 is isolated in $\sigma(b)$, we easily see that b is represented by a projection, which contradicts the assumption that $y = [b] \in \mathrm{Cu}(A)_+$.

Let us define another semigroup (cf. [APT11, Definition 5.4]). The underlying set is

$$\widetilde{W}(A) := V(A) \sqcup \mathrm{LAff}_{++}(T(A)).$$

We extend the addition of $V(A)$ and $\mathrm{LAff}_{++}(T(A))$ by setting

$$[p] + f := \widehat{[p]} + f \in \mathrm{LAff}_{++}(T(A)),$$

where $[p] \in V(A)$ and $f \in \mathrm{LAff}_{++}(T(A))$. The order is extended by

- $f \leq [p]$ if and only if $f(\tau) \leq \tau(p)$, for all $\tau \in T(A)$ and

- $[p] \leq f$ if and only if $\tau(p) < f(\tau)$, for all $\tau \in T(A)$.

V.1.11 Example. Let $M_{\mathfrak{p}}$ be an infinite dimensional UHF-algebra and denote $\mathrm{K}_0(M_{\mathfrak{p}}) \cong \mathbb{Q}(\mathfrak{p}) \subseteq \mathbb{Q}$.[9] Then

$$\widetilde{W}(M_{\mathfrak{p}}) \cong \mathbb{Q}(\mathfrak{p})_+ \sqcup (0, \infty],$$

by identifying $f \in \mathrm{LAff}_{++}(T(M_{\mathfrak{p}}))$ with $f(\tau_{M_{\mathfrak{p}}}) \in (0, \infty]$ and $[p] \in V(M_{\mathfrak{p}})$ with $(\tau_{M_{\mathfrak{p}}} \otimes \mathrm{Tr}_{\mathbb{K}})(p)$, where $\tau_{M_{\mathfrak{p}}}$ is the unique trace on $M_{\mathfrak{p}}$. In order to distinguish elements in $\mathbb{Q}(\mathfrak{p})_+$ and $(0, \infty]$, it is convenient to denote elements in $(0, \infty) \subseteq \widetilde{W}(M_{\mathfrak{p}})$ by α', where $\alpha \in (0, \infty)$. Then the order is given by the usual order on the two distinguished subsets and for $\alpha \in (0, \infty)$ and $\beta \in \mathbb{Q}(\mathfrak{p})_+$, we have

- $\alpha' \leq \beta$ if and only if $\alpha \leq_{\mathbb{R}} \beta$ and

- $\beta \leq \alpha'$ if and only if $\beta <_{\mathbb{R}} \alpha$.

The next lemma gives a more explicit description of the Cuntz semigroup of certain C*-algebras, cf. [APT11, Proposition 5.26].

V.1.12 Lemma. *Let A be a simple, unital, exact C*-algebra with stable rank one and strict comparison (see Definition V.2.1 for strict comparison). Then, the map*[10]

$$\Gamma \colon \mathrm{Cu}(A) \to V(A) \sqcup \mathrm{LAff}_{++}(T(A)) \ (= \widetilde{W}(A))$$

---

[9] Recall that $\mathbb{Q}(\mathfrak{p}) = \left\{ \frac{k}{\prod_{p \in \mathcal{F}} p^{\kappa_p}} : k \in \mathbb{Z}, \ \mathcal{F} \subseteq \mathbb{P} \text{ finite}, \ \kappa_p < \infty, \ \kappa_p \leq \nu_p(\mathfrak{p}) \right\}$, where $\mathbb{P}$ denotes the set of prime numbers.

[10] To define Γ we use that $\mathrm{Cu}(A) = V(A) \sqcup \mathrm{Cu}(A)_+$.

given by

$$\Gamma([a]) = \begin{cases} [a] & \text{if } [a] \in V(A), \\ \widehat{[a]} & \text{if } [a] \in \mathrm{Cu}(A)_+ \end{cases}$$

is an order embedding.

V.1.13 Corollary. *Let U be a* UHF-*algebra of infinite type. Then*

$$\mathrm{Cu}(U) \cong V(U) \sqcup (0, \infty] = \widetilde{W}(U).$$

The isomorphism is the map Γ from Lemma V.1.12.

Proof. It is a standard result that UHF-algebras have strict comparison, cf. [Rør92]. By Lemma V.1.12, we know that Γ is an order embedding, so it suffices to show that Γ is surjective. As before, we identify $\mathrm{LAff}_{++}(T(U))$ with $(0, \infty]$, by identifying f with $f(\tau_U)$, where τ_U is the unique trace on U. To show that Γ is surjective, it remains to show that the map

$$\kappa : \mathrm{Cu}(U)_+ \to (0, \infty] : [x] \mapsto d_{\tau_U}([x])$$

is surjective. To see this let $h \in U$ be a Lebesgue contraction[11] and $\alpha \in (0, 1]$. Let $f \in C_0(0, 1]_+$ be such that $\|f\| = 1$ and $\mu(\mathrm{supp}(f)) = \alpha$, where μ is the Lebesgue measure on $(0, 1]$. Then

$$d_{\tau_U}(f(h)) = \lim_{n \to \infty} \tau_U(f(h)^{\frac{1}{n}}) = \lim_{n \to \infty} \int_0^1 f^{\frac{1}{n}}(t) \, dt = \mu(\mathrm{supp}(f)) = \alpha.$$

Note that 0 is not an isolated point in $\sigma(f(h))$, because $\sigma(f(h)) = f(\sigma(h)) = f([0, 1]) = [0, 1]$. By [APT11, Proposition 2.23] we have $[f(h)] \in \mathrm{Cu}(U)_+$ and it follows that $\alpha \in \mathrm{Im}(\kappa)$. If now $\beta \in (0, \infty)$, let $n \in \mathbb{N}$ with $\frac{\beta}{n} \in (0, 1]$. Let $x \in \mathrm{Cu}(U)_+$ with $\kappa(x) = \frac{\beta}{n}$. Then $nx \in \mathrm{Cu}(U)_+$ and $\kappa(nx) = n\kappa(x) = \beta$. The element $\infty \in \widetilde{W}(U)$ is given by $\kappa([1_U \otimes k])$, where k is the compact diagonal operator with diagonal $(\frac{1}{n})_{n=1}^{\infty}$. This finishes the proof. $\qquad\square$

The next theorem is due to Ciuperca and Elliott ([CE08]) and is proven to hold for a more general class of target algebras in [RS10, Theorem 2].

V.1.14 Theorem. *Assume A is a* C^*-*algebra that has stable rank one. Then, for any* **Cu**-*morphism*

$$\alpha \colon \mathrm{Cu}(C_0(0, 1]) \to \mathrm{Cu}(A),$$

---

[11]See the proof of Lemma IV.1.4 for an explicit formula for such a Lebesgue contraction.

such that $\alpha([\iota]) \leq [h_A]$, *where $h_A \in A$ is strictly positive, there exists a* *-*homomorphism $\varphi\colon C_0(0,1] \to A$ such that* $\mathrm{Cu}(\varphi) = \alpha$. *Furthermore, α is unique up to approximate unitary equivalence by unitaries in* $(A \otimes \mathbb{K})^+$.[12]

Building on this Robert proved the following[13] in [Rob12]:

V.1.15 Theorem. *Let A be a unital one dimensional non-commutative* CW *complex with trivial* K_1*-group or a unital sequential inductive limit of such. Let B be a unital stable rank one* C*-algebra. Then, for every* **Cu***-morphism*

$$\alpha\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$$

such that $\alpha([1_A]) = [1_B]$, there exists a unital *-homomorphism $\varphi\colon A \to B$ such that $\mathrm{Cu}(\varphi) = \alpha$. Furthermore, φ is unique up to approximate unitary equivalence.*

V.1.16 Remark. By definition a one dimensional non-commutative CW complex A is given by a pullback

$$
\begin{array}{ccc}
A & \xrightarrow{\ \pi_1\ } & E \\
{\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \phi} \\
C([0,1], F) & \xrightarrow[\partial_0 \oplus \partial_1]{} & F \oplus F
\end{array}
$$

where E and F are finite dimensional C*-algebras and $\phi\colon E \to F \oplus F$ is a *-homomorphism. The maps ∂_i are the evaluations at the endpoints. We already saw in Lemma I.3.3 that $Z_{p,q}$ is such a pullback, for $p, q \in \mathbb{N}$. Hence Theorem V.1.15 also applies to generalized dimension drop algebras.

V.2 Uniqueness for order zero maps

V.2.1 Definition. Let A be a unital C*-algebra. We say that A has *strict comparison of positive elements by bounded traces* if for all $a, b \in M_n(A)_+$, we have

$$d_\tau(a) < d_\tau(b) \qquad (\tau \in T(A)),$$

---

[12]If A is unital, we may assume that the unitaries implementing the approximate unitary equivalence live in A.

[13]He actually proved a more general result for possibly non-unital C*-algebras. However, we will not need this version and hence omit it.

implies that $a \precsim b$ in $M_n(A)$. Recall that the functional d_τ is given by

$$d_\tau(x) = \lim_{n \to \infty} (\tau \otimes \mathrm{Tr}_n)(x^{\frac{1}{n}}) \qquad (x \in M_n(A)_+),$$

where Tr_n is the sum over the diagonal on M_n.

The following is [TWW17, Lemma 2.1 (ii)].

V.2.2 Lemma. *Let A be a unital C^*-algebra that has strict comparison of positive elements by bounded traces. If $h, k \in A$ are two Lebesgue contractions, then*

$$\mathrm{Cu}(\phi_h) = \mathrm{Cu}(\phi_k),$$

where

$$\phi_h, \phi_k \colon C_0(0, 1] \to A$$

are the *-homomorphisms induced by the given contractions.

Proof. It suffices to show that $f(h) \sim_{\mathrm{Cu}} f(k)$, for every non-zero $f \in C_0(0, 1]_+$. Let $\varepsilon > 0$. Then, there exists a small open interval, say $I = (a, b) \cap (0, 1]$ such that $f(t) \in (0, \varepsilon)$, for $t \in I$. In particular $(f - \varepsilon)_+$ vanishes on I. Let $c_I \in C_0(0, 1]_+$ be of norm one such that $c_I(t) = 0$ for $t \notin I$, see Figure V.1. Then

$$d_\tau(c_I(h)) > 0 \qquad (\tau \in T(A)) \tag{V.2.1}$$

Furthermore, by the choice of I we have that

$$(f - \varepsilon)_+ + c_I \precsim f. \tag{V.2.2}$$

This is because in commutative C^*-algebras, Cuntz subequivalence is determined by the support of functions (cf. [APT11, Proposition 2.5]). Using that
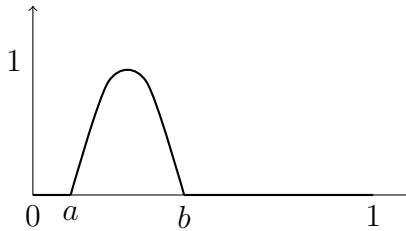

Figure V.1: The function c_I.

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$c_I(h)$ and $(f(h) - \varepsilon)_+$ are orthogonal, we get for every $\tau \in T(A)$:

$$d_\tau\big((f(h) - \varepsilon)_+\big) \overset{\text{(V.2.1)}}{<} d_\tau\big((f(h) - \varepsilon)_+\big) + d_\tau\big(c_I(h)\big)$$
$$= d_\tau\big((f(h) - \varepsilon)_+ + c_I(h)\big)$$
$$\overset{\text{(V.2.2)}}{\leq} d_\tau(f(h))$$
$$= d_\tau(f(k)).$$

The last equality follows, because h and k are both Lebesgue contractions. Now, by strict comparison, it follows that

$$(f(h) - \varepsilon)_+ \precsim f(k)$$

and hence

$$[f(h)] = \sup_n \left[\left(f(h) - \frac{1}{n}\right)_+\right] \leq [f(k)].$$

Reversing the roles of h and k, we see that $[f(k)] \leq [f(h)]$ and thus $f(h) \sim_{\text{Cu}} f(k)$. $\qquad\square$

V.2.3 Definition. Let us denote by $\mathscr{A}$ the class of unital C*-algebras that have stable rank one and strict comparison of positive elements by bounded traces.

V.2.4 Example. The easiest and for us most interesting class of C*-algebras that belong to $\mathscr{A}$ is the class of UHF-algebras. The stable rank one property is easy to check. Strict comparison is proven in [Rør92] and relies on the fact that matrix algebras enjoy this property.

V.2.5 Lemma. *Let $A \in \mathscr{A}$ and assume $T(A_\omega) = T(A)_\omega$, where we recall that $T(A)_\omega$ denotes the set of limit traces, i.e. those of the form*

$$\tau_\omega([(x_n)_{n=1}^\infty]) = \lim_\omega \tau(x_n) \qquad (\tau \in T(A)).$$

Then $A_\omega \in \mathscr{A}$.

Proof. Using that $T(A_\omega) = T(A)_\omega$ and [BBS+19, Lemma 1.23], it follows that A_ω has strict comparison. To see that A_ω has stable rank one, let $x \in A_\omega$, say $x = [(x_n)_{n=1}^\infty]$. Since A has stable rank one, we find $g_n \in A$ such that $\|x_n - g_n\| < \frac{1}{n}$ and g_n is invertible. Look at the polar decomposition $g_n = u_n |g_n|$. Let $u = [(u_n)_{n=1}^\infty]$ and $g = [(g_n)_{n=1}^\infty]$. Then

$$x = [(x_n)_{n=1}^\infty] = [(g_n)_{n=1}^\infty] = u \, |g| \, ,$$

where $u \in A_\omega$ is a unitary and $|g| \in A_\omega$ is positive. Then

$$x \approx_\varepsilon u(|g| + \varepsilon)$$

and $u(|g| + \varepsilon)$ is invertible. $\qquad\square$

V.2.6 Remark. To get examples of C*-algebras to which Lemma V.2.5 applies, let us briefly recall a result on limit traces established by Ozawa. In [Oza13, Theorem 8] he shows (in a more general fashion) that

$$\overline{T(A)_\omega}^{\text{weak}^*} = T(A_\omega)$$

if A has *commutator bounds* (m, C) for some $m \in \mathbb{N}$ and $C > 0$ in the sense of [NR16, Remark 2.2]:

For every $h \in A_+$ and $\varepsilon > 0$, there exist $x_1, y_1, \cdots, x_m, y_m \in A$ such that

$$\left\| h - \sum_{i=1}^m [x_i, y_i] \right\| \leq \sup_{\tau \in T(A)} \tau(h) + \varepsilon$$

and

$$\sum_{i=1}^m \|x_i\| \, \|y_i\| \leq C \, \|h\| \, .$$

Using [NR16, Proposition 2.4] we know that commutator bounds pass to inductive limits. Since matrix algebras have commutator bounds $(1, 2)$ ([Fac82, Lemma 3.5]) it follows that any UHF-algebra has commutator bounds $(1, 2)$ and we see that

$$T(U)_\omega = T(U_\omega),$$

where U is a UHF-algebra. In particular, it follows that $U_\omega \in \mathscr{A}$.

V.2.7 Lemma. *(cf. Theorem III.1.5) Let $A \in \mathscr{A}$. If h and k are Lebesgue contractions in A, then $h \approx_{\mathrm{u}} k$.*

Proof. By Lemma V.2.2 we see that $\mathrm{Cu}(\phi_h) = \mathrm{Cu}(\phi_k)$, where $\phi_h, \phi_k \colon C_0(0, 1] \to A$ denote the *-homomorphisms corresponding to h resp. k. Furthermore, $\mathrm{Cu}(\phi_h)([\iota]) = [h] \leq [1_A]$ and similarly $\mathrm{Cu}(\phi_k)([\iota]) \leq [1_A]$. By Theorem V.1.14 it follows that $\phi_h \approx_{\mathrm{u}} \phi_k$ and hence $h \approx_{\mathrm{u}} k$. $\qquad\square$

V.2.8 Lemma. *(cf. Lemma III.2.3) Let $\mathfrak{p}$ be a supernatural number and let $\varphi, \psi \colon M_{\mathfrak{p}} \to A$ be two c.p.c. order zero maps such that h_φ and h_ψ are Lebesgue contractions. Assume also that $A \in \mathscr{A}$. Then $\varphi \approx_{\mathrm{u}} \psi$.*

Proof. We may assume that $\mathfrak{p}$ is finite, say equal to $p \in \mathbb{N}$. We saw in the proof of Lemma III.2.3 that $\varphi(e_{11})$ and $\psi(e_{11})$ both have $\frac{1}{p}$ Lebesgue spectral measure. By Lemma V.2.7 it follows that $\varphi(e_{11}) \approx \psi(e_{11})$. By Lemma III.2.1 and the fact that A has stable rank one it now follows that $\varphi \approx_{\mathrm{u}} \psi$. $\qquad \square$

V.2.9 Definition. Let $p \in \mathbb{N}$. Fix $\varepsilon > 0$ and $\mathcal{F} \subseteq C_0(0,1] \otimes M_p$ finite. A c.p.c. map $\varphi \colon C_0(0,1] \otimes M_p \to A$ is called

- $(\mathcal{F}, \varepsilon)$-*multiplicative* if

$$\|\varphi(xy) - \varphi(x)\varphi(y)\| < \varepsilon \qquad (x, y \in \mathcal{F}),$$

- $(\mathcal{F}, \varepsilon)$-*standard* if

$$|(\tau \circ \varphi)(x) - \tau_{\mathrm{Leb}}(x)| < \varepsilon \qquad (x \in \mathcal{F},\ \tau \in T(A)).$$

The following lemma is inspired by [Sch18, Lemma 5.2].

V.2.10 Lemma. *Let $p \in \mathbb{N}$ and assume U is a UHF-algebra. Then, for every finite set $\mathcal{F} \subseteq C_0(0,1] \otimes M_p$ and $\varepsilon > 0$, there exists a finite set $\mathcal{G} \subseteq C_0(0,1] \otimes M_p$ and $\delta > 0$ such that the following holds: Whenever $\varphi, \psi \colon C_0(0,1] \otimes M_p \to U$ are $(\mathcal{G}, \delta)$-multiplicative and $(\mathcal{G}, \delta)$-standard c.p.c. maps, there exists a unitary $u \in U$ such that*

$$\|u\varphi(f)u^* - \psi(f)\| < \varepsilon \qquad (f \in \mathcal{F}).$$

Proof. Assume the result is false. Then, there exist $\mathcal{F} \subseteq C_0(0,1] \otimes M_p$ finite and $\varepsilon > 0$ and c.p.c. maps $\varphi_n, \psi_n \colon C_0(0,1] \otimes M_p \to U$ which are $(\mathcal{G}_n, \delta_n)$-multiplicative and $(\mathcal{G}_n, \delta_n)$-standard, where $\mathcal{G}_n$ is an increasing sequence of finite subsets of $C_0(0,1] \otimes M_p$ with dense union and $\delta_n \downarrow 0$, but such that for every unitary $u \in U$ we have

$$\|u\varphi_n(f)u^* - \psi_n(f)\| \geq \varepsilon,$$

for some $f \in \mathcal{F}$. By assumption, the φ_n and ψ_n induce standard *-homomorphisms $\varphi, \psi \colon C_0(0,1] \otimes M_p \to U_\omega$. By Lemma V.2.8 and the fact that $U_\omega \in \mathscr{A}$, there exists a unitary $u \in U_\omega$ with $\mathrm{ad}(u) \circ \varphi = \psi$[14]. We can write

---

[14]As U_ω is an ultrapower, a reindexing argument can be used to turn $\approx_{\mathrm{u}}$ into $\sim_{\mathrm{u}}$, see Remark I.5.2.

$u = [(u_n)_{n=1}^\infty]$ for a sequence of unitaries $u_n \in U$.[15] As

$$\lim_{n \to \omega} \|u_n \varphi_n(f) u_n^* - \psi_n(f)\| = 0 \qquad (f \in \mathcal{F}),$$

there exists for every $f \in \mathcal{F}$ a set $X_f \in \omega$ with $\|u_n \varphi_n(f) u_n^* - \psi_n(f)\| < \varepsilon$, for $n \in X_f$. Now, any $n \in \bigcap_{f \in \mathcal{F}} X_f \in \omega$ satisfies

$$\|u_n \varphi_n(f) u_n^* - \psi_n(f)\| < \varepsilon \qquad (f \in \mathcal{F}),$$

contradicting our assumption. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

V.3   Existence of order zero maps

V.3.1 Lemma. *Let $p \in \mathbb{N}$ and assume U is a UHF-algebra of infinite type. Then, for every finite set $\mathcal{F} \subseteq C_0(0,1] \otimes M_p$ and $\varepsilon > 0$, there exists a $(\mathcal{F}, \varepsilon)$-multiplicative and $(\mathcal{F}, \varepsilon)$-standard c.p.c. map $\phi \colon C_0(0,1] \otimes M_p \to U$.*

Proof. Let $h \in Q$ be a Lebesgue contraction, where Q is the universal UHF-algebra. See the proof of Lemma IV.1.4 for the construction of such a Lebesgue contraction. Since Q is strongly self-absorbing, we may fix an isomorphism $\varphi \colon Q \otimes Q \to Q$. After identifying M_p with a unital subalgebra of Q, we consider $\psi \colon M_p \to Q : x \mapsto \varphi(h \otimes x)$, which is a c.p.c. order zero map such that $h_\psi = \varphi(h \otimes 1_p)$ is a Lebesgue contraction. Next, we look at the following diagram:

$$
\begin{array}{ccc}
 & & U_\omega \\
 & \overset{\phi}{\nearrow} & \downarrow \\
C_0(0,1] \otimes M_p & \underset{\overline{\psi}}{\longrightarrow} & \mathcal{R}^\omega
\end{array}
$$

By viewing U as a weak*-dense subalgebra of $\mathcal{R}$, the unique separably acting hyperfinite II$_1$-factor, the inclusion $U \hookrightarrow \mathcal{R}$ induces a trace preserving surjection $U_\omega \twoheadrightarrow \mathcal{R}^\omega$, by [KR14, Theorem 3.3]. To get a *-homomorphic lift ϕ of $\overline{\psi}$,[16] we use that the cone over M_p is projective. Next, by the Choi-Effros

---

[15] Look at any lift $(x_n)_{n=1}^\infty$ of u in $\ell^\infty(U)$. Then the x_n are approximately unitary and in particular eventually invertible. By replacing x_n with $u_n := x_n |x_n|^{-1}$ we get a unitary lift.

[16] Since Q has weak closure $\mathcal{R}$ (in its GNS-representation), we regard $\overline{\psi}$ as a map taking values in $\mathcal{R}$, which then sits canonically inside $\mathcal{R}^\omega$. Recall also that $\mathcal{R}^\omega$ is the tracial ultrapower of $\mathcal{R}$, given by $\ell^\infty(\mathcal{R})$ modulo sequences $(x_n)_{n=1}^\infty$ such that $\lim_\omega \tau_{\mathcal{R}}(x_n^* x_n) = 0$.

lifting theorem, there are c.p.c. maps $\phi_n\colon C_0(0,1]\otimes M_p\to U$ such that

$$\phi(x)=[(\phi_n(x))_{n=1}^\infty]\qquad(x\in C_0(0,1]\otimes M_p).$$

Then, the ϕ_n are approximately multiplicative and approximately standard, since the lift ϕ is a standard *-homomorphism. The result follows. $\qquad\square$

We can now present two alternative proofs of Lemma IV.1.5.

V.3.2 Lemma. *(cf. Lemma IV.1.5) Let $p\in\mathbb{N}$ and let U be a UHF-algebra of infinite type. Then, there exists a c.p.c. order zero map $\psi\colon M_p\to U$ such that h_ψ is a Lebesgue contraction.*

Proof using Lemma V.3.1. Let $(\mathcal{F}_n)_{n=1}^\infty$ be an increasing sequence of finite subsets of $C_0(0,1]\otimes M_p$ with dense union and let $\varepsilon_n>0$ such that $\sum\varepsilon_n<\infty$. Find finite subsets $\mathcal{G}_n\subseteq C_0(0,1]\otimes M_p$ and $\delta_n>0$ accordingly to Lemma V.2.10. We may assume that $\delta_n\downarrow 0$ and $(\mathcal{G}_n)_{n=1}^\infty$ is increasing with dense union. By Lemma V.3.1, for each $n\in\mathbb{N}$, there exists a $(\mathcal{G}_n,\delta_n)$-multiplicative and $(\mathcal{G}_n,\delta_n)$-standard c.p.c. map $\phi_n\colon C_0(0,1]\otimes M_p\to U$. By Lemma V.2.10, for each $n\geq 1$, there exists a unitary $u_{n+1}\in U$ such that

$$\left\|\phi_n(x)-u_{n+1}\phi_{n+1}(x)u_{n+1}^*\right\|<\varepsilon_n\qquad(x\in\mathcal{F}_n).$$

Define $\psi_1:=\phi_1$ and

$$\psi_n:=u_2 u_3\cdots u_n\phi_n(\cdot)u_n^*\cdots u_3^* u_2^*\qquad(n\geq 2).$$

Then it is easy to see that $(\psi_n(x))_{n=1}^\infty$ is Cauchy for all $x\in\bigcup_{n=1}^\infty\mathcal{F}_n$ and hence for all $x\in C_0(0,1]\otimes M_p$. Finally, define

$$\psi\colon C_0(0,1]\otimes M_p\to U:x\mapsto\lim_{n\to\infty}\psi_n(x).$$

Then ψ is a standard *-homomorphism, which is the same as a c.p.c. order zero map $\overline{\psi}\colon M_p\to U$ such that $h_{\overline{\psi}}$ is a Lebesgue contraction. $\qquad\square$

Proof using Theorem V.1.14. Let $\Gamma\colon\mathrm{Cu}(U)\to V(U)\sqcup(0,\infty]$ be the isomorphism from Corollary V.1.13. We claim that the functional

$$d_{\tau_{\mathrm{Leb}}}\colon\mathrm{Cu}(C_0(0,1]\otimes M_p)\to[0,\infty]$$

is actually a **Cu**-morphism. To this end it suffices to show that it preserves the way below relation. Since $C_0(0,1]$ is stably projectionless, we see that

$$\mathrm{Cu}(C_0(0,1]\otimes M_p)=\{0\}\sqcup\mathrm{Cu}(C_0(0,1]\otimes M_p)_+.\qquad(\text{V.3.1})$$

Let $x, y \in \mathrm{Cu}(C_0(0,1] \otimes M_p)$ such that $x \ll y$. We may assume that $x, y \neq 0$ (because $0 \ll 0$ in $[0, \infty]$). Furthermore, we note that $d_{\tau_{\mathrm{Leb}}}(x) < \infty$.[17] By (V.3.1) and Lemma V.1.9 it follows that there exists $0 \neq z \in \mathrm{Cu}(C_0(0,1] \otimes M_p)$ such that $x + z \leq y$. Since τ_{Leb} is faithful we see that $d_{\tau_{\mathrm{Leb}}}(z) > 0$ and hence

$$d_{\tau_{\mathrm{Leb}}}(x) < d_{\tau_{\mathrm{Leb}}}(x) + d_{\tau_{\mathrm{Leb}}}(z) = d_{\tau_{\mathrm{Leb}}}(x + z) \leq d_{\tau_{\mathrm{Leb}}}(y).$$

The calculation makes sense, because $d_{\tau_{\mathrm{Leb}}}(x) < \infty$.

Let now $\alpha \colon \mathrm{Cu}(C_0(0,1] \otimes M_p) \to \mathrm{Cu}(U)$ be such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Cu}(C_0(0,1] \otimes M_p) & \xrightarrow{\ \ \alpha\ \ } & \mathrm{Cu}(U) \\
{\scriptstyle d_{\tau_{\mathrm{Leb}}}}\big\downarrow & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \Gamma} \\
[0, \infty] & \xhookrightarrow{\ \ j\ \ } & V(U) \sqcup (0, \infty]
\end{array}
$$

Note that $\alpha([\iota \otimes 1_p]) = \Gamma^{-1}(1') \leq \Gamma^{-1}([1_U]) = [1_U]$, where $1'$ denotes the element 1 considered in $(0, \infty]$, cf. Example V.1.11. Now Theorem V.1.14 shows that the **Cu**-morphism α lifts to a *-homomorphism $\varphi \colon C_0(0,1] \otimes M_p \to U$. Let us show that φ is standard. Indeed, denote

$$\tau := \tau_U \circ \varphi.$$

Then τ is a trace on $C_0(0,1] \otimes M_p$ and by Theorem V.1.6 it follows that $\tau = \tau_{\mathrm{Leb}}$ if $d_\tau = d_{\tau_{\mathrm{Leb}}}$. Let us check this:

$$j \circ d_\tau = j \circ d_{\tau_U \circ \varphi} = j \circ d_{\tau_U} \circ \alpha = \Gamma \circ \alpha = j \circ d_{\tau_{\mathrm{Leb}}}.$$

It follows that $d_\tau = d_{\tau_{\mathrm{Leb}}}$, because j is injective. $\qquad\square$

V.4 Existence and uniqueness of maps on dimension drop algebras

Using Robert's classification theorem, we can prove Lemma IV.2.4 immediately, without having to establish existence results for standard c.p.c. order zero maps first.

---

[17]We already noted that $g = \sup_n g_n$, where each g_n lives in some matrix algebra over $C_0(0,1]$, see also [APT11, Lemma 5.23]. But then, since $f \ll g$, we have $f \leq g_n$ for some $n \in \mathbb{N}$ and the claim follows, because $d_{\tau_{\mathrm{Leb}}}(g_n) < \infty$.

V.4.1 Lemma. *(cf. Lemma IV.2.4) Let $\mathfrak{p}$ and $\mathfrak{q}$ be coprime supernatural numbers of infinite type and let U be a UHF-algebra of infinite type. Then, there exists a standard $*$-homomorphism*

$$\varphi \colon Z_{\mathfrak{p},\mathfrak{q}} \to U,$$

which is unique up to approximate unitary equivalence.

Proof. Since $Z_{\mathfrak{p},\mathfrak{q}}$ has stable rank one (see [GLN14, Proposition 3.3]), we have, by Remark V.1.8, that

$$\mathrm{Cu}(Z_{\mathfrak{p},\mathfrak{q}}) = V(Z_{\mathfrak{p},\mathfrak{q}}) \sqcup \mathrm{Cu}(Z_{\mathfrak{p},\mathfrak{q}})_+ \cong \mathbb{N} \sqcup \mathrm{Cu}(Z_{\mathfrak{p},\mathfrak{q}})_+.$$

See Lemma I.3.3 for the computation of $V(Z_{\mathfrak{p},\mathfrak{q}})$. Let us define a map

$$\alpha \colon \mathbb{N} \sqcup \mathrm{Cu}(Z_{\mathfrak{p},\mathfrak{q}})_+ \to \mathbb{N} \sqcup (0, \infty]$$

by

$$\alpha(x) := \begin{cases} x & \text{if } x \in \mathbb{N}, \\ d_{\tau_{\mathrm{Leb}}}(x) & \text{if } x \in \mathrm{Cu}(Z_{\mathfrak{p},\mathfrak{q}})_+. \end{cases}$$

The set $\mathbb{N} \sqcup (0, \infty]$ has the structure of a **Cu**-semigroup as explained in Example V.1.11. Let us check that α is a **Cu**-morphism.

(M1): Clearly α is additive on each component and preserves the zero element. If $n \in \mathbb{N}$ and $x \in \mathrm{Cu}(Z_{\mathfrak{p},\mathfrak{q}})_+$, then $n+x \in \mathrm{Cu}(Z_{\mathfrak{p},\mathfrak{q}})_+$ (since $\mathrm{Cu}(Z_{\mathfrak{p},\mathfrak{q}})_+$ is absorbing) and we get

$$\alpha(n + x) = d_{\tau_{\mathrm{Leb}}}(n + x) = n d_{\tau_{\mathrm{Leb}}}(1) + d_{\tau_{\mathrm{Leb}}}(x) = n + d_{\tau_{\mathrm{Leb}}}(x)$$
$$= \alpha(n) + \alpha(x).$$

Recall that we identify $\mathbb{N} \cong \{n[1_{Z_{\mathfrak{p},\mathfrak{q}}}] : n \in \mathbb{N}\} \subseteq \mathrm{Cu}(Z_{\mathfrak{p},\mathfrak{q}})$.

(M2): Clearly, α is order preserving on each component. Assume $n \in \mathbb{N}$ and $[f] \in \mathrm{Cu}(Z_{\mathfrak{p},\mathfrak{q}})_+$. If $n \leq [f]$, we have to show that $\alpha(n) \leq \alpha([f])$. By definition of α and the order on $\mathbb{N} \sqcup (0, \infty]$, this is true if $n <_{\mathbb{R}} d_{\tau_{\mathrm{Leb}}}([f])$. However, since n is compact[18], we see that $n \ll [f]$. Then Lemma V.1.9 shows that there exists $0 \neq z \in \mathrm{Cu}(Z_{\mathfrak{p},\mathfrak{q}})$ such that $n + z \leq [f]$. It follows that

$$n = d_{\tau_{\mathrm{Leb}}}(n) < d_{\tau_{\mathrm{Leb}}}(n) + d_{\tau_{\mathrm{Leb}}}(z) \leq d_{\tau_{\mathrm{Leb}}}([f]).$$

---

[18]An element $x \in \mathrm{Cu}(A)$ is called compact if $x \ll x$.

If $[f] \leq n$ it is clear that $\alpha([f]) \leq \alpha(n)$.

(M3): This follows from the fact that α preserves suprema of increasing sequences on each component.

(M4) It remains to prove that α preserves the way below relation. It is enough to check this on $\mathrm{Cu}(Z_{\mathfrak{p},\mathfrak{q}})_+$. Again, this follows from Lemma V.1.9.

We see that α is **Cu**-morphism. Using Lemma V.1.13 and composing α with the inclusion

$$\mathbb{N} \sqcup (0, \infty] \to V(U) \sqcup (0, \infty] \cong \mathrm{Cu}(U),$$

it follows by Theorem V.1.15 that α lifts to a unital *-homomorphism $\varphi\colon Z_{\mathfrak{p},\mathfrak{q}} \to U$.[19] Furthermore, it is again easy to check that $d_{\tau_U \circ \varphi} = d_{\tau_{\mathrm{Leb}}}$, showing that $\tau_U \circ \varphi = \tau_{\mathrm{Leb}}$, as desired.

The uniqueness statement is also contained in Theorem V.1.15. $\qquad\square$

---

[19]Note the $\alpha([1_{Z_{\mathfrak{p},\mathfrak{q}}}]) = [1_U]$.

Chapter VI

Strongly self-absorbing stationary inductive limits

In this chapter we show that the stationary inductive limit associated to a unitarily suspended *-homomorphism is strongly self-absorbing (Theorem VI.2.2). We end with some general observations, for example that these limits are all isomorphic and define the initial object in the category of strongly self-absorbing C*-algebras.

VI.1 An approximate intertwining

VI.1.1 Notation. Let $n \in \mathbb{N}$ and $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n$ be supernatural numbers. Let us define

$$M_{(\mathfrak{p}_1,\cdots,\mathfrak{p}_n)} := M_{\mathfrak{p}_1} \otimes M_{\mathfrak{p}_2} \otimes \cdots \otimes M_{\mathfrak{p}_n}.$$

If $i_1, i_2, \cdots, i_k$ are k distinct integers, not necessarily in increasing order, with $i_k \in \{1, 2, \cdots, n\}$, then there is an obvious inclusion

$$\imath^{[i_1,i_2,\cdots,i_k]}_{(\mathfrak{p}_1,\cdots,\mathfrak{p}_n)} \colon M_{(\mathfrak{p}_{i_1},\mathfrak{p}_{i_2},\cdots,\mathfrak{p}_{i_k})} \to M_{(\mathfrak{p}_1,\cdots,\mathfrak{p}_n)}.$$

Let us furthermore denote

$$M^{[i_1,i_2,\cdots,i_k]}_{(\mathfrak{p}_1,\cdots,\mathfrak{p}_n)} := \mathrm{Im}\big(\imath^{[i_1,i_2,\cdots,i_k]}_{(\mathfrak{p}_1,\cdots,\mathfrak{p}_n)}\big).$$

If the n-tupel $(\mathfrak{p}_1, \cdots, \mathfrak{p}_n)$ in question is clear, we will omit the subscripts and simply write $\imath^{[i_1,\cdots,i_k]}$ and $M^{[i_1,\cdots,i_k]}$.

VI.1.2 Example. The map $\iota^{[1]}_{(\mathfrak{p},\mathfrak{q})} \colon M_{\mathfrak{p}} \to M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$ for example is the first factor embedding, whereas $\iota^{[2]}_{(\mathfrak{p},\mathfrak{q})} \colon M_{\mathfrak{q}} \to M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$ denotes the second factor embedding. Another example is the flip map $M_{\mathfrak{p}} \otimes M_{\mathfrak{p}} \to M_{\mathfrak{p}} \otimes M_{\mathfrak{p}}$, which is $\iota^{[2,1]}_{(\mathfrak{p},\mathfrak{p})}$ in our notation.

VI.1.3 Definition. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ and $\mathfrak{q}_1, \cdots, \mathfrak{q}_m$ be supernatural numbers. Let $\mathfrak{r} := (\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n)$ and $\mathfrak{s} := (\mathfrak{q}_1, \mathfrak{q}_2, \cdots, \mathfrak{q}_m)$. We then define

$$
Z_{\mathfrak{r},\mathfrak{s}} = \left\{ f \in C([0,1], M_{\mathfrak{r}} \otimes M_{\mathfrak{s}}) : \begin{array}{l} f(0) \in M_{\mathfrak{r}} \otimes 1_{\mathfrak{s}}, \\ f(1) \in 1_{\mathfrak{r}} \otimes M_{\mathfrak{s}} \end{array} \right\}.
$$

VI.1.4 Convention. Let $\mathfrak{p}$ and $\mathfrak{q}$ be supernatural numbers. To simplify notation a bit, let us define $\mathfrak{p}^2 := (\mathfrak{p},\mathfrak{p})$, $\mathfrak{q}^2 := (\mathfrak{q},\mathfrak{q})$ and $\mathfrak{r} := (\mathfrak{p}^2, \mathfrak{q}^2, \mathfrak{p}^2, \mathfrak{q}^2)$. We then identify $Z_{\mathfrak{p}^2,\mathfrak{q}^2} \otimes Z_{\mathfrak{p}^2,\mathfrak{q}^2}$ with a C*-subalgebra of $C([0,1]^2, M_{\mathfrak{r}})$, where $f \otimes g \in Z_{\mathfrak{p}^2,\mathfrak{q}^2} \otimes Z_{\mathfrak{p}^2,\mathfrak{q}^2}$ gets identified with the function $(f \otimes g)(s,t) := f(s) \otimes g(t)$. Under this identification, one can check that an element $h \in Z_{\mathfrak{p}^2,\mathfrak{q}^2} \otimes Z_{\mathfrak{p}^2,\mathfrak{q}^2}$ must satisfy:

- $h(0,0) \in M_{\mathfrak{r}}^{[1,2,5,6]}$,

- $h(s,0) \in M_{\mathfrak{r}}^{[1,2,3,4,5,6]}$, for $s \in (0,1)$ and

- $h(1,0) \in M_{\mathfrak{r}}^{[3,4,5,6]}$.

Similar boundary conditions must be fulfilled on the remaining sides of the square.

VI.1.5 Notation. Let $\mu \colon Z_{\mathfrak{p},\mathfrak{q}} \to Z_{\mathfrak{p},\mathfrak{q}}$ be a unital *-homomorphism. For $(i,j) \in \{(1,3),(2,4)\}$ we define

$$
\mu^{[i,j]} \colon Z_{\mathfrak{p},\mathfrak{q}} \to Z_{\mathfrak{p}^2,\mathfrak{q}^2}; \quad \mu_t^{[i,j]} := \iota^{[i,j]}_{(\mathfrak{p}^2,\mathfrak{q}^2)} \circ \mu_t.
$$

Furthermore, we define the following *-homomorphism:

$$
\widetilde{\widetilde{\mu}} \colon Z_{\mathfrak{p},\mathfrak{q}} \otimes Z_{\mathfrak{p},\mathfrak{q}} \to Z_{\mathfrak{p}^2,\mathfrak{q}^2}; \quad \widetilde{\widetilde{\mu}}_s(f \otimes g) := \mu_s^{[1,3]}(f)\mu_s^{[2,4]}(g).
$$

Note that $\mu_s^{[1,3]}$ and $\mu_s^{[2,4]}$ are commuting *-homomorphisms.

VI.1.6 Proposition. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be coprime supernatural numbers of infinite type and assume $\mu \colon Z_{\mathfrak{p},\mathfrak{q}} \to Z_{\mathfrak{p},\mathfrak{q}}$ is unitarily suspended. Then*

$$
(\mathrm{id}_{Z_{\mathfrak{p}^2,\mathfrak{q}^2}} \otimes 1_{Z_{\mathfrak{p}^2,\mathfrak{q}^2}}) \circ \widetilde{\widetilde{\mu}} \approx_{\mathrm{u}} \mu^{[1,3]} \otimes \mu^{[1,3]}.
$$

That is, we can make the following diagram commute arbitrary well up to conjugating by a unitary.

$$
\begin{array}{ccc}
Z_{\mathfrak{p},\mathfrak{q}} \otimes Z_{\mathfrak{p},\mathfrak{q}} & \xrightarrow{\;\mu^{[1,3]} \otimes \mu^{[1,3]}\;} & Z_{\mathfrak{p}^2,\mathfrak{q}^2} \otimes Z_{\mathfrak{p}^2,\mathfrak{q}^2} \\
{\scriptstyle \widetilde{\approx}\,\mu} \searrow & & \nearrow {\scriptstyle \mathrm{id}\,\otimes 1} \\
& Z_{\mathfrak{p}^2,\mathfrak{q}^2} &
\end{array}
\tag{VI.1.1}
$$

Proof. **(I)** Let us first do some setup. Recall from Definition IV.2.5 that $\mu\colon Z_{\mathfrak{p},\mathfrak{q}} \to Z_{\mathfrak{p},\mathfrak{q}}$ is given by

$$
\mu_t = \begin{cases} \mathrm{ad}(v_t) \circ (\alpha_0 \otimes 1_{\mathfrak{q}}) & \text{if } t \in [0,1), \\ 1_{\mathfrak{p}} \otimes \alpha_1 & \text{if } t = 1, \end{cases}
\tag{VI.1.2}
$$

where $(v_t)_{t\in[0,1)}$ is a unitary path in $M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$ starting at the identity and where $\alpha_0\colon Z_{\mathfrak{p},\mathfrak{q}} \to M_{\mathfrak{p}}$ and $\alpha_1\colon Z_{\mathfrak{p},\mathfrak{q}} \to M_{\mathfrak{q}}$ are unital *-homomorphisms. During the proof we identify the codomain of $\mu^{[1,3]} \otimes \mu^{[1,3]}$ with functions on the square, accordingly to Convention VI.1.4. Under this identification, the map $\mu^{[1,3]} \otimes \mu^{[1,3]}$ becomes:

$$
(\mu^{[1,3]} \otimes \mu^{[1,3]})(f \otimes g)(s,t) = \imath_{\mathfrak{r}}^{[1,3]}(\mu_s(f)) \cdot \imath_{\mathfrak{r}}^{[5,7]}(\mu_t(g)),
\tag{VI.1.3}
$$

where $\mathfrak{r} = (\mathfrak{p}^2, \mathfrak{q}^2, \mathfrak{p}^2, \mathfrak{q}^2)$ is as in Convention VI.1.4. Finally, we will need to write $Z_{\mathfrak{p},\mathfrak{q}}$ as an inductive limit $\varinjlim(Z_{P_n,Q_n}, \gamma_{n,n+1})$, as explained in Convention IV.2.3.

(II) Let us fix a normalized and finite set $\mathcal{F} \subseteq Z_{\mathfrak{p},\mathfrak{q}}$ and let $\varepsilon > 0$. The goal is to find a unitary $W \in Z_{\mathfrak{p}^2,\mathfrak{q}^2} \otimes Z_{\mathfrak{p}^2,\mathfrak{q}^2}$ making the diagram (VI.1.1) commute up to ε on $\mathcal{F} \otimes \mathcal{F} := \{f \otimes g : f, g \in \mathcal{F}\}$. For convenience we may assume that the unitary path $(v_t)_{t\in[0,1)}$ is constant in a small open neighborhood of zero, i.e. we may assume there exists $0 < \eta < 1$ such that $v_t = 1_{\mathfrak{p}\mathfrak{q}}$, for $t \in [0,\eta]$. Let us fix some $0 < \delta < \eta$, such that

$$
|s - s'| \leq \delta \quad \Rightarrow \quad \|\mu_s(f) - \mu_{s'}(f)\| \leq \frac{\varepsilon}{10} \qquad (f \in \mathcal{F})
\tag{VI.1.4}
$$

and let $n \in \mathbb{N}$ such that

$$
\|\kappa_n(\mu(f)) - \mu(f)\| \leq \frac{\varepsilon}{10} \qquad (f \in \mathcal{F}),
\tag{VI.1.5}
$$

where κ_n is the conditional expectation as defined in Convention IV.2.3. By possibly enlarging n we can ensure the existence[1] of a unitary path $(\bar{v}_t)_{t\in[0,1]}$

---

[1]This is done by approximating the unitary $(v_t)_{t\in[0,1-\delta]} \in C([0,1-\delta], M_{\mathfrak{p}} \otimes M_{\mathfrak{q}})$ by a unitary in $C([0,1-\delta], M_{P_n} \otimes M_{Q_n})$.

such that

$$
\left[
\begin{array}{ll}
\bar{v}_t = 1_{P_n} \otimes 1_{Q_n} & \text{if } t \in [0, \delta], \\
\|\bar{v}_t - v_t\| \le \frac{\varepsilon}{10} & \text{if } t \in [\delta, 1 - \delta], \\
\bar{v}_t = \bar{v}_{1-\delta} & \text{if } t \in [1 - \delta, 1].
\end{array}
\right. \qquad \text{(VI.1.6)}
$$

Let us denote $p := P_n$, $q := Q_n$ and $r := (p, p, q, q, p, p, q, q)$. We then identify $M_r \subseteq M_{\mathfrak{r}}$ as unital subalgebra, via our fixed inclusions[2]. Next, let

$$
S_p \in M_p \otimes M_p \quad \text{and} \quad S_q \in M_q \otimes M_q
$$

denote the self-adjoint flip unitaries. This means $S_p(x \otimes y)S_p^* = y \otimes x$, for all $x, y \in M_p$ and similarly for M_q. Then define

$$
\left[
\begin{array}{l}
\widetilde{S}_p := \imath_r^{[2,5]}(S_p), \\
\widetilde{S}_q := \imath_r^{[4,7]}(S_q), \\
\widetilde{S} := \widetilde{S}_p \widetilde{S}_q = \widetilde{S}_q \widetilde{S}_p \in M_r^{[2,4,5,7]}
\end{array}
\right. \qquad \text{(VI.1.7)}
$$

and for $t \in [0, 1]$ we let

$$
\begin{aligned}
u_t &:= \imath_{\mathfrak{r}}^{[2,4]}\big(v_{\min(t, 1-\delta)}\big), \\
\bar{u}_t &:= \imath_r^{[2,4]}(\bar{v}_t).
\end{aligned} \qquad \text{(VI.1.8)}
$$

By (VI.1.6) we see that $\|u_t - \bar{u}_t\| \le \frac{\varepsilon}{10}$, for all $t \in [0, 1]$. We are now ready to define a first approximation to the desired unitary W:

$$
U \colon [0, 1]^2 \to M_r : (s, t) \mapsto \widetilde{S}\bar{u}_t \bar{u}_s^*. \qquad \text{(VI.1.9)}
$$

Note that U does not yet define an element of $Z_{\mathfrak{p}^2, \mathfrak{q}^2} \otimes Z_{\mathfrak{p}^2, \mathfrak{q}^2}$. However, let us show that the unitary U makes diagram (VI.1.1) commute approximately, i.e.

$$
U(\widetilde{\bar{\mu}}(f \otimes g) \otimes 1_{Z_{\mathfrak{p}^2, \mathfrak{q}^2}})U^* \approx_{\frac{\varepsilon}{2}} \mu^{[1,3]}(f) \otimes \mu^{[1,3]}(g) \qquad (f, g \in \mathcal{F}). \qquad \text{(VI.1.10)}
$$

First, observe the following identities:

(a) $\operatorname{ad}(\widetilde{S}_p) \circ \imath_r^{[2,4]} = \imath_r^{[5,4]}$,

(b) $\operatorname{ad}(\widetilde{S}_q) \circ \imath_r^{[5,4]} = \imath_r^{[5,7]}$,

(c) $\operatorname{ad}(\widetilde{S}_q) \circ \imath_r^{[2,4]} = \imath_r^{[2,7]}$,

---

[2]Remember that M_r and $M_{\mathfrak{r}}$ are a tensor product of eight matrix respectively UHF-algebras, see Notation VI.1.1.

(d) $\mathrm{ad}(\widetilde{S}) \circ \imath_r^{[2,4]} = \imath_r^{[5,7]}$,

(e) $u_t u_s^* \mu_s^{[2,4]}(g) u_s u_t^* = \mu_t^{[2,4]}(g)$, for any $g \in Z_{\mathfrak{p},\mathfrak{q}}$ and $s,t \in [0, 1-\delta]$.

The identities (a)-(d) are easily proven by looking at elementary tensors, whereas (e) follows from the definition of μ. We are now able to prove (VI.1.10). Let $f,g \in \mathcal{F}$ and $(s,t) \in [0,1]^2$. Filling in the definitions gives

$$
\begin{aligned}
&U_{s,t}(\widetilde{\mu}(f \otimes g) \otimes 1_{Z_{\mathfrak{p}^2,\mathfrak{q}^2}})(s,t)U_{s,t}^* \\
&= \widetilde{S}\bar{u}_t \bar{u}_s^* \imath_{\mathfrak{r}}^{[1,3]}(\mu_s(f))\imath_{\mathfrak{r}}^{[2,4]}(\mu_s(g))\bar{u}_s \bar{u}_t^* \widetilde{S}^*.
\end{aligned}
\tag{VI.1.11}
$$

Note that $\bar{u}_t$ and $\widetilde{S}$ commute with $M_{\mathfrak{r}}^{[1,3]}$. For $(s,t) \in [0, 1-\delta]^2$, for example, one checks that (VI.1.11) becomes

$$
\imath_{\mathfrak{r}}^{[1,3]}(\mu_s(f))\widetilde{S}\bar{u}_t \bar{u}_s^* \imath_{\mathfrak{r}}^{[2,4]}(\mu_s(g))\bar{u}_s \bar{u}_t^* \widetilde{S}^*
$$

$$
\overset{\text{(VI.1.6)}}{\underset{\frac{2\varepsilon}{10}}{\approx}} \quad \imath_{\mathfrak{r}}^{[1,3]}(\mu_s(f))\widetilde{S}u_t u_s^* \imath_{\mathfrak{r}}^{[2,4]}(\mu_s(g))u_s u_t^* \widetilde{S}^*
$$

$$
\overset{\text{(e)}}{=} \quad \imath_{\mathfrak{r}}^{[1,3]}(\mu_s(f))\widetilde{S}\imath_{\mathfrak{r}}^{[2,4]}(\mu_t(g))\widetilde{S}^*
$$

$$
\overset{\text{(d),(VI.1.5)}}{\underset{\frac{2\varepsilon}{10}}{\approx}} \quad \imath_{\mathfrak{r}}^{[1,3]}(\mu_s(f))\imath_{\mathfrak{r}}^{[5,7]}(\mu_t(g))
$$

$$
\overset{\text{(VI.1.3)}}{=} \quad (\mu^{[1,3]}(f) \otimes \mu^{[1,3]}(g))(s,t).
$$

For $(s,t) \in ([1-\delta,1] \times [0,1-\delta]) \cup ([0,1-\delta] \times [1-\delta,1]) \cup [1-\delta,1]^2$ one proceeds similarly and uses in particular (VI.1.4). This proves the claim (VI.1.10).

(III) The next step is to perturb U on the boundary. Since S_p and S_q are connected to the identity in $M_p \otimes M_p$ respectively $M_q \otimes M_q$ we can find a unitary path $(\nu_t)_{t \in [0,1]}$ inside M_r such that the following holds:

$$
\begin{bmatrix}
\nu_0 = \widetilde{S}_q, \\
\nu_t \in \mathrm{C}^*(\widetilde{S}_q) \subseteq M_r^{[4,7]} & \text{if } t \in [0,\delta], \\
\nu_t = 1_r & \text{if } t \in [\delta, 1-\delta], \\
\nu_t \in \mathrm{C}^*(\widetilde{S}_p) \subseteq M_r^{[2,5]} & \text{if } t \in [1-\delta, 1], \\
\nu_1 = \widetilde{S}_p.
\end{bmatrix}
\tag{VI.1.12}
$$

Observe that

(f) $[\widetilde{S}, \nu_t] = 0$, for every $t \in [0,1]$,

(g) $[\nu_s, \imath_{\mathfrak{r}}^{[i,j]}(\mu_s(f))] \approx_{\frac{\varepsilon}{10}} 0$, for all $s \in [0,1]$, $(i,j) \in \{(2,4),(5,7)\}$ and $f \in \mathcal{F}$.

For (f), let $t \in [0, \delta]$. Then, $\nu_t \in M_r^{[4,7]}$. Since $\widetilde{S}_p \in M_r^{[2,5]}$ we see that $\widetilde{S}\nu_t\widetilde{S}^* = \widetilde{S}_q\nu_t\widetilde{S}_q^*$. Note that $\widetilde{S}_q$ commutes with ν_t for $t \in [0, \delta]$, since the C*-algebra generated by $\widetilde{S}_q$ is commutative. For $t \in [\delta, 1 - \delta]$ nothing is to check and for $t \in [1 - \delta, 1]$ we argue similarly as before. Let us check (g) for $(i, j) = (2, 4)$ and $s \in [0, \delta]$:

$$\nu_s \imath_{\mathfrak{r}}^{[2,4]}(\mu_s(f))\nu_s^* \overset{(\text{VI.1.4})}{\underset{\frac{\varepsilon}{10}}{\approx}} \nu_s \imath_{\mathfrak{r}}^{[2,4]}(\mu_0(f))\nu_s^* = \imath_{\mathfrak{r}}^{[2,4]}(\mu_0(f)).$$

The last equality follows from the fact that $\mu_0(f) = \alpha_0(f) \otimes 1_{\mathfrak{q}} \in M_{\mathfrak{p}} \otimes 1_{\mathfrak{q}}$ so that $\imath_{\mathfrak{r}}^{[2,4]}(\mu_0(f)) \in M_{\mathfrak{r}}^{[2]}$, whereas $\nu_s \in M_{\mathfrak{r}}^{[4,7]}$ for $s \in [0, \delta]$. A similar observation applies to the case $s \in [1 - \delta, 1]$. We can now define the desired unitary W by

$$W_{s,t} := \begin{cases} \nu_{\min(s,t)}U_{s,t} & \text{if } (s,t) \in [0, \delta]^2, \\ \nu_{\max(s,t)}U_{s,t} & \text{if } (s,t) \in [1 - \delta, 1]^2, \\ \nu_t U_{s,t}\nu_s & \text{else.} \end{cases}$$

This should be compared to the construction of Jiang and Su in [JS99, Proposition 8.3]. Now, we have to check the following:

(i) W is continuous,

(ii) $W \in Z_{\mathfrak{p}^2, \mathfrak{q}^2} \otimes Z_{\mathfrak{p}^2, \mathfrak{q}^2}$,

(iii) $\mathrm{ad}(W) \circ (\widetilde{\widetilde{\mu}} \otimes 1_{Z_{\mathfrak{p}^2,\mathfrak{q}^2}}) \approx_{(\mathcal{F} \otimes \mathcal{F}, \varepsilon)} \mu^{[1,3]} \otimes \mu^{[1,3]}$.

Without reference, we will use that

$$U_{s,t} = \widetilde{S} \qquad ((s,t) \in [0, \delta]^2 \cup [1 - \delta, 1]^2).$$

We note that (i) follows easily from (f). In order to show (ii), we have to verify that W satisfies all boundary conditions, as explained in Convention VI.1.4. The computation is not difficult but tedious, so as an example we show how to handle the case $t = 0$, which splits up into the following three cases:

- $s \in [0, \delta] : W_{s,t} = \nu_{\min(s,t)}U_{s,t} = \nu_0\widetilde{S} = \widetilde{S}_q\widetilde{S} = \widetilde{S}_p \in M_r^{[2,5]}$,

- $s \in [\delta, 1] : W_{s,t} = \nu_t U_{s,t}\nu_s = \nu_0 U_{s,0}\nu_s = \widetilde{S}_q(\widetilde{S}\bar{u}_s^*)\nu_s = \widetilde{S}_p\bar{u}_s^*\nu_s$, which lives in $M_r^{[2,5]}M_r^{[2,4]}M_r^{[2,5]} \subseteq M_r^{[2,4,5]}$,

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- $s = 1 : W_{s,t} = \widetilde{S}_p \bar{u}_s^* \nu_s = \widetilde{S}_p \bar{u}_{1-\delta}^* \widetilde{S}_p \in \widetilde{S}_p M_r^{[2,4]} \widetilde{S}_p \underset{\text{(a)}}{\subseteq} M_r^{[4,5]}.$

It follows that W is a unitary in $Z_{\mathfrak{p}^2,\mathfrak{q}^2} \otimes Z_{\mathfrak{p}^2,\mathfrak{q}^2}$ and it remains to prove (iii).
For $(s,t) \in [0,\delta]^2$ and $f, g \in \mathcal{F}$ we compute

$$W_{s,t}(\widetilde{\bar{\mu}}(f \otimes g) \otimes 1_{Z_{\mathfrak{p}^2,\mathfrak{q}^2}})(s,t)W_{s,t}^*$$

$$= \nu_{\min(s,t)} \widetilde{S} \imath_{\mathfrak{r}}^{[1,3]}(\mu_s(f)) \imath_{\mathfrak{r}}^{[2,4]}(\mu_s(g)) \nu_{\min(s,t)}^* \widetilde{S}^*$$

$$\overset{\text{(f),(VI.1.12)}}{=} \widetilde{S} \imath_{\mathfrak{r}}^{[1,3]}(\mu_s(f)) \nu_{\min(s,t)} \imath_{\mathfrak{r}}^{[2,4]}(\mu_s(g)) \nu_{\min(s,t)}^* \widetilde{S}^*$$

$$\overset{\text{(g)}}{\underset{\frac{\varepsilon}{10}}{\approx}} \widetilde{S} \imath_{\mathfrak{r}}^{[1,3]}(\mu_s(f)) \imath_{\mathfrak{r}}^{[2,4]}(\mu_s(g)) \widetilde{S}^*$$

$$= U_{s,t}(\widetilde{\bar{\mu}}(f \otimes g) \otimes 1_{Z_{\mathfrak{p}^2,\mathfrak{q}^2}})(s,t)U_{s,t}^*$$

$$\overset{\text{(VI.1.10)}}{\underset{\frac{\varepsilon}{2}}{\approx}} (\mu^{[1,3]}(f) \otimes \mu^{[1,3]}(g))(s,t).$$

The computation for $(s,t) \in [1-\delta,1]^2$ is similar. For the remaining part, we get

$$W_{s,t}(\widetilde{\bar{\mu}}(f \otimes g) \otimes 1_{Z_{\mathfrak{p}^2,\mathfrak{q}^2}})(s,t)W_{s,t}^*$$

$$= \nu_t U_{s,t} \nu_s \imath_{\mathfrak{r}}^{[1,3]}(\mu_s(f)) \imath_{\mathfrak{r}}^{[2,4]}(\mu_s(g)) \nu_s^* U_{s,t}^* \nu_t^*$$

$$\overset{\text{(g)}}{\underset{\frac{2\varepsilon}{10}}{\approx}} \nu_t U_{s,t} \imath_{\mathfrak{r}}^{[1,3]}(\mu_s(f)) \imath_{\mathfrak{r}}^{[2,4]}(\mu_s(g)) U_{s,t}^* \nu_t^*$$

$$\overset{\text{(VI.1.10)}}{\underset{\frac{\varepsilon}{2}}{\approx}} \nu_t \imath_{\mathfrak{r}}^{[1,3]}(\mu_s(f)) \imath_{\mathfrak{r}}^{[5,7]}(\mu_t(g)) \nu_t^*$$

$$\overset{\text{(g)}}{\underset{\frac{2\varepsilon}{10}}{\approx}} \imath_{\mathfrak{r}}^{[1,3]}(\mu_s(f)) \imath_{\mathfrak{r}}^{[5,7]}(\mu_t(g))$$

$$\overset{\text{(VI.1.3)}}{=} (\mu^{[1,3]}(f) \otimes \mu^{[1,3]}(g))(s,t).$$

This finishes the proof. $\qquad\square$

VI.2   Proof of the main theorem

VI.2.1 Definition. Let $\mathfrak{p}$ and $\mathfrak{q}$ be coprime supernatural numbers of infinite type. By Theorem IV.2.6, there exists a unitarily suspended standard *-homomorphism $\mu \colon Z_{\mathfrak{p},\mathfrak{q}} \to Z_{\mathfrak{p},\mathfrak{q}}$, which we fix for the rest of this section. Define

$$\bar{\mu} \colon Z_{\mathfrak{p},\mathfrak{q}} \to Z_{\mathfrak{p},\mathfrak{q}}; \quad \bar{\mu} := \Phi \circ \mu^{[1,3]},$$

where $\Phi \colon Z_{\mathfrak{p}^2,\mathfrak{q}^2} \to Z_{\mathfrak{p},\mathfrak{q}}$ is the canonical isomorphism, induced fiberwise by isomorphisms $M_{\mathfrak{p}^2} \to M_{\mathfrak{p}}$ and $M_{\mathfrak{q}^2} \to M_{\mathfrak{q}}$. We then define

$$\overrightarrow{Z_{\mathfrak{p},\mathfrak{q}}^{\mu}} := \varinjlim(Z_{\mathfrak{p},\mathfrak{q}}, \bar{\mu}).$$

That is, $\overrightarrow{Z_{\mathfrak{p},\mathfrak{q}}^{\mu}}$ is the stationary inductive limit of $Z_{\mathfrak{p},\mathfrak{q}}$ along $\bar{\mu}$.

VI.2.2 Theorem. *The C*-algebra $\overrightarrow{Z_{\mathfrak{p},\mathfrak{q}}^{\mu}}$ is strongly self-absorbing. Furthermore, for any sequence $(\varphi_n)_{n=1}^{\infty}$ of standard *-endomorphisms of $Z_{\mathfrak{p},\mathfrak{q}}$ we have*

$$\overrightarrow{Z_{\mathfrak{p},\mathfrak{q}}^{\mu}} \cong \varinjlim(Z_{\mathfrak{p},\mathfrak{q}}, \varphi_n).$$

Proof. Let us consider the following diagram, where $A := \varinjlim(Z_{\mathfrak{p},\mathfrak{q}}, \varphi_n)$:

Let us first have a look at row (I). By Proposition VI.1.6, we have $(\mathrm{id}\otimes 1) \circ \tilde{\tilde{\mu}} \approx_{\mathrm{u}} \mu^{[1,3]} \otimes \mu^{[1,3]}$. By composing with $\Phi \otimes \Phi$ it follows that

$$(\mathrm{id}\otimes 1) \circ (\Phi \circ \tilde{\tilde{\mu}}) \approx_{\mathrm{u}} \bar{\mu} \otimes \bar{\mu}.$$

Furthermore, one easily computes that

$$(\Phi \circ \tilde{\tilde{\mu}}) \circ (\mathrm{id}\otimes 1) = \Phi \circ \mu^{[1,3]} = \bar{\mu}.$$

This shows that each triangle in (I) commutes approximately up to conjugating by a unitary. By [Rør02, Corollary 2.3.3] it follows that there exist unitaries $(u_n)_{n=1}^{\infty}$ in $Z_{\mathfrak{p},\mathfrak{q}} \otimes Z_{\mathfrak{p},\mathfrak{q}}$ such that if $\mathrm{id}\otimes 1$ is replaced by $\mathrm{ad}(u_n) \circ (\mathrm{id}\otimes 1)$, then the above diagram is an approximate intertwining (in the sense of [Rør02, Definition 2.3.1]). By the same corollary the induced isomorphism $\varphi \colon \overrightarrow{Z_{\mathfrak{p},\mathfrak{q}}^{\mu}} \to \overrightarrow{Z_{\mathfrak{p},\mathfrak{q}}^{\mu}} \otimes \overrightarrow{Z_{\mathfrak{p},\mathfrak{q}}^{\mu}}$ satisfies

$$\varphi(x) = \lim_{n\to\infty} u_n(x \otimes 1)u_n^* \qquad (x \in \overrightarrow{Z_{\mathfrak{p},\mathfrak{q}}^{\mu}}),$$

where now each u_n is considered as an element of $\overrightarrow{Z_{\mathfrak{p},\mathfrak{q}}^{\mu}} \otimes \overrightarrow{Z_{\mathfrak{p},\mathfrak{q}}^{\mu}}$. By definition, it follows that $\overrightarrow{Z_{\mathfrak{p},\mathfrak{q}}^{\mu}}$ is strongly self-absorbing.

Next, we look at row (II). Clearly, all lower triangles commute and by Theorem III.6.1 we have that $\varphi_n \approx_{\mathrm{u}} \bar{\mu}$, for each $n \in \mathbb{N}^3$. Again, by an approximate intertwining argument, it follows that $A \cong \overrightarrow{Z_{\mathfrak{p},\mathfrak{q}}^{\mu}}$. $\qquad\square$

---

[3]Note that $\bar{\mu}$ is still standard.

VI.2.3 Remark. The previous theorem picks up the spirit of [RW10, Theorem 3.4] and shows that any stationary inductive limit with a standard (hence trace collapsing) *-endomorphism is strongly self-absorbing. In our approach however, we do not have to compare the limit to the Jiang-Su algebra $\mathcal{Z}$.

VI.2.4 Lemma. *Let $n \in \mathbb{N}$. Then, there exists $k \in \mathbb{N}$ and a unital *-homomorphism $Z_{n,n+1} \to Z_{2^k,2^k+1}$.*

Proof. Choose $k \in \mathbb{N}$ such that $2^k \geq n(n+1)$ and write $2^k = nd + r$, where $r \in \{0, 1, \cdots, n-1\}$. Then clearly $d \geq n+1 > r+1$. By [Sat10, Proposition 2.1], there exist elements $s, c_1, c_2, \cdots, c_{2^k}$ in $Z_{2^k,2^k+1}$ such that

$$c_1 \geq 0, \quad c_i c_j^* = \delta_{i,j} c_1^2, \quad s^* s + \sum_{j=1}^{2^k} c_j^* c_j = 1 \quad \text{and} \quad c_1 s = s.$$

For $i = 1, 2, \cdots, n$ define

$$x_i := \sum_{j=(i-1)\cdot d+1}^{i\cdot d} c_j \quad \text{and} \quad b_i := x_i^* x_i = \sum_{j=(i-1)\cdot d+1}^{i\cdot d} c_j^* c_j.$$

Note that the b_i are mutually equivalent orthogonal positive elements. By the relations on the c_j and s it is easy to see that

(a) $1 = b_1 + b_2 + \cdots + b_n + (c_{nd+1}^* c_{nd+1} + \cdots + c_{nd+r}^* c_{nd+r}) + s^* s$,

(b) $[s^* s] \leq [c_1]$ and $[c_1] = [c_1^2] = [c_1^* c_1]$,

(c) $d[c_1] = [b_1]$.

Here $[a]$ denotes the class of an element a in $W(Z_{2^k,2^k+1})$, the Cuntz semi-group of $Z_{2^k,2^k+1}$. Now we have

$$
\begin{aligned}
[1 - (b_1 + \cdots + b_n)] &\overset{(a)}{=} [(c_{nd+1}^* c_{nd+1} + \cdots + c_{nd+r}^* c_{nd+r}) + s^* s] \\
&\leq [c_{nd+1}^* c_{nd+1}] + \cdots + [c_{nd+r}^* c_{nd+r}] + [s^* s] \\
&= r[c_1^* c_1] + [s^* s] \overset{(b)}{\leq} r[c_1^* c_1] + [c_1] = (r+1)[c_1] \\
&\ll d[c_1] \overset{(c)}{=} [b_1].
\end{aligned}
$$

The last line follows since $r + 1 < d$. Hence, there exists $\varepsilon > 0$ such that

$$1 - (b_1 + \cdots + b_n) \precsim (b_1 - \varepsilon)_+.$$

By [RW10, Proposition 5.1 (ii)], it follows that $Z_{n,n+1}$ embeds unitally into $Z_{2^k,2^k+1}$. $\qquad \square$

VI.2.5 Proposition. *Let $p, q \in \mathbb{N}$ be coprime. Then $Z_{p,q}$ maps unitally into any strongly self-absorbing C*-algebra.*

Proof. Let $p, q \in \mathbb{N}$ be coprime. We can find $k, l \in \mathbb{N}$ such that $lq - kp = 1$ and $2l > p$, $2k > q$. With $n := kp$, there exists by [JS99, Proposition 2.5] a unital *-homomorphism $Z_{p,q} \to Z_{kp,lq} = Z_{n,n+1}$. By Lemma VI.2.4, $Z_{n,n+1}$ embeds unitally into $Z_{2^k,2^k+1}$, for some large enough k. Finally, by [Win11, Theorem 3.1], $Z_{2^k,2^k+1}$ maps unitally into any strongly self-absorbing C*-algebra. $\qquad\square$

VI.2.6 Lemma. *Let A be a strongly self-absorbing C*-algebra that is locally approximated[4] by prime dimension drop algebras. Then A embeds unitally into any strongly self-absorbing C*-algebra.*

Proof. Let D be a strongly self-absorbing C*-algebra. Since A is locally approximated by prime dimension drop algebras $Z_{p,q}$ with $p, q \in \mathbb{N}$, it follows by [Lor93, Theorem 3.8] that A is an inductive limit of such dimension drop algebras. By Proposition VI.2.5, each of these embed unitally into D and hence into the central sequence algebra $D_\omega \cap D'$, see [TW07, Theorem 2.2]. By [TW08, Proposition 2.2][5] we see that $D \cong A \otimes D$ and hence A embeds unitally into D. $\qquad\square$

For the next theorem we first recall Kirchberg's definition of a central sequence algebra (cf. [Kir06, Definition 1.1]).

VI.2.7 Definition. Let A be a C*-algebra. We then define

$$F(A) := \frac{A_\omega \cap A'}{\mathrm{Ann}(A, A_\omega)},$$

where $\mathrm{Ann}(A, A_\omega) := \{x \in A_\omega : xA = Ax = \{0\}\}$.

VI.2.8 Remark. If A is σ-unital and $(h_n)_{n=1}^\infty$ is an approximate unit for A, then $F(A)$ is unital with unit $[(h_n)_{n=1}^\infty] + \mathrm{Ann}(A, A_\omega)$.

VI.2.9 Theorem. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be coprime supernatural numbers of infinite type and let $(\varphi_n)_{n=1}^\infty$ be a sequence of standard *-endomorphisms of $Z_{\mathfrak{p},\mathfrak{q}}$. Let us denote*

$$A := \varinjlim(Z_{\mathfrak{p},\mathfrak{q}}, \varphi_n).$$

---

[4]By definition, this means that there exists a sequence $A_n \subseteq A$ of prime dimension drop algebras, such that for any finite set $\mathcal{F} \subseteq A$ and $\varepsilon > 0$, there exists some $n \in \mathbb{N}$ such that $\mathcal{F} \subseteq_\varepsilon A_n$, i.e. for every $x \in \mathcal{F}$ there exists $y \in A_n$ such that $\|x - y\| < \varepsilon$.

[5]In this proposition, the assumption of K_1-injectivity is superfluous.

Then the following holds:

(i) *A is the initial object in the category[6] of strongly self-absorbing* C*-*algebras,*

(ii) *If B is any separable* C*-*algebra such that there exists a unital* *-*homomorphism* $Z_{2,3} \to F(B)$*, then* $B \cong B \otimes A$*,*

(iii) *If B is any separable* C*-*algebra and* $Z_{2,3}$ *maps unitally into* $M(B)_\omega \cap B'$*, then* $B \cong B \otimes A$*.*

Proof.   (i) By Theorem VI.2.2 it follows that A is strongly self-absorbing. Then Lemma VI.2.6 implies that A is the initial object in the category of strongly self-absorbing C*-algebras.

(ii) By [Kir06, Corollary 1.13], there is a unital *-homomorphism $Z_{2,3}^{\otimes \infty} \to F(B)$ and hence any dimension drop algebra $Z_{2^n,3^n} \subseteq Z_{2,3}^{\otimes n}$ maps unitally into $F(B)$. Let $(\mathfrak{r}, \mathfrak{s}) := (2^\infty, 3^\infty)$. By a similar argument as in [TW08, Proposition 2.2], it follows that $\overrightarrow{Z_{\mathfrak{r},\mathfrak{s}}^{\mu'}}$ (see Definition VI.2.1) maps unitally into $F(B)$. By [Jia97], the algebra $\overrightarrow{Z_{\mathfrak{r},\mathfrak{s}}^{\mu'}}$ is K_1-injective[7]. Then the same proof as in [Naw13, Proposition 5.1] applies to show that $B \cong B \otimes \overrightarrow{Z_{\mathfrak{r},\mathfrak{s}}^{\mu'}}$. See also [Kir06, Proposition 4.11 (1)]. By (i) we know that $A \cong \overrightarrow{Z_{\mathfrak{r},\mathfrak{s}}^{\mu'}}$ and hence $B \cong B \otimes A$.

(iii) Note that there is a unital *-homomorphism

$$M(B)_\omega \cap B' \to F(B) : [(x_n)_{n=1}^\infty] \mapsto [(x_n h_n)_{n=1}^\infty] + \mathrm{Ann}(B, B_\omega),$$

where $(h_n)_{n=1}^\infty$ is an approximate unit for B. It follows that $Z_{2,3}$ maps unitally into $F(B)$ and (ii) applies. $\square$

VI.2.10 Remark. It is now particularly easy to see that any prime dimension drop algebra $Z_{\mathfrak{p},\mathfrak{q}}$ embeds unitally into any A, where A is as in Theorem VI.2.9. Furthermore, it is encoded in the construction, that these embeddings are standard.

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[6]The morphisms in this category are approximate unitary equivalence classes of *-homomorphisms. This ensures that the initial object is unique, cf. [TW07, Proposition 5.12].

[7]The result there is stated for the Jiang-Su algebra $\mathcal{Z}$, however it only uses that $\mathcal{Z}$ is a simple inductive limit of prime dimension drop algebras.

List of Symbols

$A, B, C, \cdots$	C*-algebras
M_n	complex $n \times n$ matrices
1_n	unit of M_n
Tr_n	the unnormalized trace on M_n
$M_{\mathfrak{p}}$	UHF-algebra associated to $\mathfrak{p}$
$1_{\mathfrak{p}}$	unit of $M_{\mathfrak{p}}$
$\tau_{M_{\mathfrak{p}}}$	the unique tracial state on $M_{\mathfrak{p}}$
$C_0(X)$	cts. functions on X that vanish at infinity
A_+	positive elements
A^+	forced unitization
$T(A)$	tracial state space
φ^*	induced map on the tracial state spaces
$\sim_{\mathrm{u}}$	unitary equivalence
$\approx_{\mathrm{u}}$	approximate unitary equivalence
$\sim$	Murray-von Neumann equivalence
$\approx$	Approximate Murray-von Neumann equivalence
$\sim_{\mathrm{asu}}$	strong asymptotic unitary equivalence
$\precsim$	Cuntz below
$\sim_{\mathrm{Cu}}$	Cuntz equivalent
$x \approx_{\varepsilon} y$	$\|x - y\| < \varepsilon$
$[x, y]$	the commutator $xy - yx$
ad	$\mathrm{ad}(u)(x) = uxu^*$
$\grave{\imath}, \acute{\imath}, \grave{\varphi}, \varphi$	Definition I.3.6
$\sigma(x)$	spectrum of an element x

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