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Projective C^*-algebras and boundary maps

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Abstract. Both boundary maps in K-theory are expressed in terms of surjections from projective C^*-algebras to semiprojective C^*-algebras.

1. Noncommutative Cells and Boundaries

Cells are absolute retracts that tie together spheres of different dimensions. The analog of an absolute retract for a C^*-algebra is being projective. For better or worse, in the category of all C^*-algebras, we lose the projectivity of $C_0\left(\mathbb{D} \setminus \{-1\}\right)$, so we cannot generally use the exactness of

$$0 \to C_0\left(\mathbb{R}^2\right) \to C_0\left(\mathbb{D} \setminus \{-1\}\right) \to C_0\left(\mathbb{R}\right) \to 0$$

to explain the index map in K-theory. Another difficulty is that we need asymptotic morphisms to obtain the natural isomorphism

$$\left[\left[C_0\left(\mathbb{R}^2\right), D \otimes \mathbb{K}\right]\right] \cong K_0(D).$$

The name "index map" is related to the Toeplitz algebra $\mathcal{T}$ and the exact sequence

$$0 \to \mathbb{K} \to \mathcal{T} \to C(S^1) \to 0.$$

We might prefer to use $\mathcal{T}_0$, generated by the shift minus one, and

$$0 \to \mathbb{K} \to \mathcal{T}_0 \to C_0\left(\mathbb{R}\right) \to 0,$$

but still we may have trouble since $\mathcal{T}_0$ is not projective and $\mathbb{K}$ is not semiprojective.

The "second standard picture of the index map" in [8, Proposition 9.2.2] and the picture of the exponential map presented in [6] both use what might be called noncommutative cells. In both cases, there is a diagram

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "\iota"] & P \arrow[r, "\eta"] & R \arrow[r] & 0 \\ & Q \arrow[u, "\psi_0"] & & & \end{tikzcd} \tag{1}$$

with an exact row and where P is projective. Moreover, R, Q and ψ_0 have enough nice properties to ensure that

$$[R, D \otimes \mathbb{K}] \cong K_i(D), \tag{2}$$

$$[Q, D \otimes \mathbb{K}] \cong K_{i+1}(D), \tag{3}$$

and $K_{i+1}(\psi_0)$ is an isomorphism. The projectivity of P then leads to an implementation of the boundary map as a sequence of maps

$$\partial^{(n)} : [R, \mathbf{M}_n(A/I)] \to [Q, \mathbf{M}_n(I)].$$

There are other examples where we don't have the isomorphisms (2) and (3). What we minimally require is the following.

Definition 1.1. If the row in the diagram (1) is exact, P is projective,

$$K_i(R) = K_{i+1}(Q) = \mathbb{Z},$$

$$K_{i+1}(R) = K_i(Q) = 0,$$

and $K_{i+1}(\psi_0)$ is an isomorphism, then we will call (1) a *cell diagram.*

2. THE INDEX MAP

The noncommutative Grassmannians are unital C^*-algebras with universal properties. One gets easier statements of results if one works with a nonunital variation. For now, we stick with the two-by-two version, as this is the one most closely related to $q\mathbb{C}$.

We use $\tilde{A}$ to denote the unitization of A, where a unit $\mathbb{1}$ is always added.

For a set of relations $\mathcal{R}$ on a set $\mathcal{G}$ we use the notation

$$\iota : \mathcal{G} \to C^* \langle \mathcal{G} \,|\mathcal{R} \rangle$$

to denote the function into a C^*-algebra that is the universal representation of $\mathcal{R}$. See [5] for information on what relations are allowed. The definition of universal representation requires that $\varphi \mapsto \varphi \circ \iota$ determines a natural bijection between

$$\hom \left(C^* \langle \mathcal{G} \,|\mathcal{R} \rangle , B \right)$$

and the set

$$\{ f : \mathcal{G} \to B \,| f \text{ is a representation of } \mathcal{R} \} .$$

We can similarly work with relations in unital C^*-algebras, and denote the universal representation in a unital C^*-algebra by

$$\iota : \mathcal{G} \to C^*_1 \langle \mathcal{G} \,|\mathcal{R} \rangle .$$

In all the examples considered, ι will be an inclusion and so we will identify $\mathcal{G}$ with $\iota(\mathcal{G})$.

Define G_2^{nc}, c.f. [2], as

$$G_2^{\mathrm{nc}} = C^*_1 \left\langle a, b, c \,\middle|\, P^2 = P^* = P \text{ for } P = \begin{bmatrix} a & c^* \\ c & b \end{bmatrix} \right\rangle$$

and define

$$G_2^{\mathrm{st}} = C^* \left\langle h, k, x \,\middle|\, P^2 = P^* = P \text{ for } P = \begin{bmatrix} \mathbb{1} - h & x^* \\ x & k \end{bmatrix} \right\rangle .$$

The "st" is to stand for "standard," as in the standard picture of K_0. The fact that a projection has norm at most one means these relations are bounded, so this universal C^*-algebra does exist.

Lemma 2.1. *The unitization* $(G_2^{\mathrm{st}})^\sim$ *is isomorphic to* G_2^{nc} *via*

$$\begin{aligned} \mathbb{1} &\mapsto 1 \\ h &\mapsto 1 - a \\ k &\mapsto b \\ x &\mapsto c. \end{aligned}$$

Proof. In terms of $*$-polynomial relations, $(G_2^{\mathrm{st}})^\sim$ has generators $\mathbb{1}$, h, k, x where $\mathbb{1}$ acts as a unit and

$$\begin{aligned} h &= h^* \\ k &= k^* \\ h^2 + x^*x &= h \\ -xh + kx &= 0 \\ k^2 + xx^* &= k. \end{aligned} \tag{4}$$

Clearly

$$\begin{aligned} h = h^* &\iff (\mathbb{1} - h) = (\mathbb{1} - h)^* \\ h^2 + x^*x = h &\iff (\mathbb{1} - h)^2 + x^*x = (\mathbb{1} - h) \\ -xh + kx = 0 &\iff x(\mathbb{1} - h) + kx = x \end{aligned}$$

and the result now follows easily. □

Lemma 2.2. *The* C^**-algebra* G_2^{st} *is semiprojective.*

Proof. This follows from Corollary 2.16 and Proposition 2.17 of [1]. □

Consider the automorphism

$$\eta : G_2^{\mathrm{st}} \to G_2^{\mathrm{st}}$$

defined by $\eta(h) = k$, $\eta(k) = h$ and $\eta(x) = x^*$.

Lemma 2.3. *The* $*$*-homomorphism*

$$id \oplus \eta : G_2^{\mathrm{st}} \to \mathbf{M}_2\left(G_2^{\mathrm{st}}\right)$$

is null-homotopic.

Proof. In terms of the generators, h, k and x are being sent to

$$\begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix}, \quad \begin{bmatrix} k & 0 \\ 0 & h \end{bmatrix}, \quad \begin{bmatrix} x & 0 \\ 0 & x^* \end{bmatrix}.$$

The homotopy is found in two segments.

For $0 \leq \alpha < 1$, let $\beta = \sqrt{1-\alpha^2}$. Let

$$H_\alpha = \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix}, \quad K_\alpha = \begin{bmatrix} k & 0 \\ 0 & h \end{bmatrix},$$

$$X_\alpha = \begin{bmatrix} \alpha x & -\beta\sqrt{xx^*} \\ \beta\sqrt{x^*x} & \alpha x^* \end{bmatrix}.$$

Clearly H_α and K_α are self-adjoint. The commutation relation is easy, since

$$\begin{aligned} \begin{bmatrix} \alpha x & -\beta\sqrt{xx^*} \\ \beta\sqrt{x^*x} & \alpha x^* \end{bmatrix} \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix} &= \begin{bmatrix} \alpha k x & -\beta k\sqrt{k-k^2} \\ \beta h\sqrt{h-h^2} & \alpha h x^* \end{bmatrix} \\ &= \begin{bmatrix} k & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} \alpha x & -\beta\sqrt{xx^*} \\ \beta\sqrt{x^*x} & \alpha x^* \end{bmatrix}. \end{aligned}$$

For the remaining relations, we have

$$\begin{aligned} X_\alpha^* X_\alpha &= \begin{bmatrix} \alpha^2 x^* x + \beta^2 x^* x & \alpha\beta\left(-x^*\sqrt{xx^*} + \sqrt{x^*x}x^*\right) \\ \alpha\beta\left(-\sqrt{xx^*}x + x\sqrt{x^*x}\right) & \alpha^2 xx^* + \beta^2 xx^* \end{bmatrix} \\ &= \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix} - \begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix}^2 \end{aligned}$$

and by symmetry,

$$X_\alpha X_\alpha^* = \begin{bmatrix} k & 0 \\ 0 & h \end{bmatrix} - \begin{bmatrix} k & 0 \\ 0 & h \end{bmatrix}^2.$$

For the second part of the path, for each $0 \leq \gamma \leq 1$, the generators are

$$H_\gamma = \begin{bmatrix} \gamma h & 0 \\ 0 & \gamma k \end{bmatrix}, \quad K_\gamma = \begin{bmatrix} \gamma k & 0 \\ 0 & \gamma h \end{bmatrix},$$

$$X_\gamma = \begin{bmatrix} 0 & -\sqrt{\gamma k - (\gamma k)^2} \\ \sqrt{\gamma h - (\gamma h)^2} & 0 \end{bmatrix}.$$

Again the self-adjoint conditions are clear, and then

$$\begin{aligned} &\begin{bmatrix} 0 & -\sqrt{\gamma k - (\gamma k)^2} \\ \sqrt{\gamma h - (\gamma h)^2} & 0 \end{bmatrix} \begin{bmatrix} \gamma h & 0 \\ 0 & \gamma k \end{bmatrix} \\ &\qquad = \begin{bmatrix} 0 & -\gamma k\sqrt{\gamma k - (\gamma k)^2} \\ \gamma h\sqrt{\gamma h - (\gamma h)^2} & 0 \end{bmatrix} \\ &\qquad = \begin{bmatrix} \gamma k & 0 \\ 0 & \gamma h \end{bmatrix} \begin{bmatrix} 0 & -\sqrt{\gamma k - (\gamma k)^2} \\ \sqrt{\gamma h - (\gamma h)^2} & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} X_\gamma^* X_\gamma &= \begin{bmatrix} \gamma h - (\gamma h)^2 & 0 \\ 0 & \gamma k - (\gamma k)^2 \end{bmatrix} \\ &= \begin{bmatrix} \gamma h & 0 \\ 0 & \gamma k \end{bmatrix} - \begin{bmatrix} \gamma h & 0 \\ 0 & \gamma k \end{bmatrix}^2 \end{aligned}$$

and by symmetry,

$$X_\gamma X_\gamma^* = \begin{bmatrix} \gamma k & 0 \\ 0 & \gamma h \end{bmatrix} - \begin{bmatrix} \gamma k & 0 \\ 0 & \gamma h \end{bmatrix}^2.$$

□

The next result should be compared to the well-known isomorphisms

$$\varinjlim\, [C_0(0,1), \mathbf{M}_n(D)] \cong K_1(D)$$

and

$$\varinjlim\, [q\mathbb{C}, \mathbf{M}_n(D)] \cong K_0(D).$$

Theorem 2.4. *For a C^*-algebra D, there is a natural isomorphism*

$$\varinjlim \left[G_2^{\mathrm{st}}, \mathbf{M}_n(D)\right] \cong K_0(D).$$

Proof. By [4, Theorem 4.3] we know

$$K_0(D) \cong \left[\left[G_2^{\mathrm{st}}, A \otimes \mathbb{K}\right]\right]$$

and by semiprojectivity

$$\left[\left[G_2^{\mathrm{st}}, D \otimes \mathbb{K}\right]\right] \cong \left[G_2^{\mathrm{st}}, D \otimes \mathbb{K}\right] \cong \varinjlim \left[G_2^{\mathrm{st}}, D \otimes \mathbf{M}_n\right].$$

This also follows from standard results in K-theory. □

Recall from [3] that $q\mathbb{C}$ was defined via an exact sequence

$$0 \to q\mathbb{C} \to \mathbb{C} * \mathbb{C} \to \mathbb{C} \to 0.$$

It has the concrete description

$$q\mathbb{C} = \left\{ f \in C_0\left((0,1], \mathbf{M}_2\right) \middle| f(1) \text{ is diagonal} \right\}.$$

as well as being universal on generators h_0, k_0 and x_0 for the relations

$$\mathcal{P} = C^* \left\langle h_0, k_0, x_0 \,\middle|\, h_0 k_0 = 0,\ P_0^2 = P_0^* = P_0 \text{ for } P_0 = \begin{bmatrix} \mathbb{1} - h_0 & x_0^* \\ x_0 & k_0 \end{bmatrix} \right\rangle.$$

Theorem 2.5. *There is a surjection ρ and an inclusion λ so that*

$$\begin{tikzcd} \mathbf{M}_2\left(G_2^{\mathrm{st}}\right) & \\ G_2^{\mathrm{st}} \arrow[u, "\mathrm{id}\otimes e_{11}"] \arrow[r, "\rho"] & q\mathbb{C} \arrow[ul, "\lambda"'] \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathbf{M}_2(q\mathbb{C}) & \\ q\mathbb{C} \arrow[u, "\mathrm{id}\otimes e_{11}"] \arrow[r, "\lambda"] & \mathbf{M}_2\left(G_2^{\mathrm{st}}\right) \arrow[ul, "\rho\otimes\mathrm{id}"'] \end{tikzcd}$$

commute up to homotopy. In terms of generators,

$$\rho(h) = h_0,$$
$$\rho(k) = k_0,$$
$$\rho(x) = x_0$$

and

$$\lambda(h_0) = h \otimes e_{11},$$

$$\lambda(k_0) = k \otimes e_{22},$$
$$\lambda(x_0) = x \otimes e_{21}.$$

Composition with λ leads to a natural isomorphism

$$\lim_{\to} \left[G_2^{st}, D \otimes \mathbf{M}_n \right] \cong \lim_{\to} \left[q\mathbb{C}, D \otimes \mathbf{M}_n \right]$$

(and with $K_0(D)$.)

Proof. We can define the homotopy φ_t from $\lambda \circ \rho$ to $\mathrm{id} \otimes e_{11}$ on generators by

$$\varphi_t(h) = h \otimes |w_t|,$$
$$\varphi_t(k) = k \otimes |w_t^*|,$$
$$\varphi_t(x) = x \otimes w_t$$

for some homotopy of partial isometries w_t from e_{21} to e_{11}. The homotopy from $(\rho \otimes \mathrm{id}) \circ \lambda$ to $\mathrm{id} \otimes e_{11}$ is found in a similar manner. □

Now we look at an extension that is somehow universal for the index map. See [8] and "the second standard picture of the index map," (proposition 9.2.2). Recall

$$C_0(0,1) = C^* \left\langle x \,\middle|\, (\mathbb{1} + x)^* = (\mathbb{1} + x)^{-1} \right\rangle$$

and define

$$\mathcal{D} = C^* \left\langle y \,\middle|\, \|\mathbb{1} + y\| \le 1 \right\rangle .$$

Sending y to x gives a surjection. Let $\mathcal{V}$ be the kernel, so that we have the exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{D} \longrightarrow C_0(0,1) \longrightarrow 0 .$$

Lemma 2.6. *The C^*-algebra $\mathcal{D}$ is projective and $C_0(0,1)$ is semiprojective.*

Proof. These have unitization the universal unital C^*-algebra generated by a contraction and $C(S^1)$, respectively. The usual facts about unitaries and contractions tell us these are semiprojective, or in the first case projective, in the unital category. We are done, by Theorem 10.1.9 and Lemma 14.1.6 of [5]. □

Consider $a = \mathbb{1} + x$ and

$$h_1 = \mathbb{1} - a^*a$$
$$k_1 = \mathbb{1} - aa^*$$
$$x_1 = a\sqrt{\mathbb{1} - a^*a}.$$

These are all elements of $\mathcal{V}$ and it is easy to see that

$$P_1 = \begin{bmatrix} \mathbb{1} - h_1 & x_1^* \\ x_1 & k_1 \end{bmatrix}$$

is a projection. This determines a $*$-homomorphism ψ_0 from G_2^{st} to $\mathcal{V}$.

Lemma 2.7. *The diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{V} \arrow[r] & \mathcal{D} \arrow[r] & C_0(0,1) \arrow[r] & 0 \\ & G_2^{\mathrm{st}} \arrow[u, "\psi_0"] & & & \end{tikzcd}$$

is a cell diagram.

Proof. Since $K_*(\mathcal{D}) = 0$ we know

$$\partial : K_1(C_0(0,1)) \to K_0(\mathcal{V})$$

is an isomorphism. The fact that $K_1(\psi_0)$ is an isomorphism follows from the definition of the boundary map ([8]). □

3. The exponential Map

Consider a short exact sequence

$$0 \to I \to A \to A/I \to 0$$

and the associated boundary map

$$\partial : K_0(A/I) \to K_1(I).$$

In [6] we showed that if x in $K_0(A/I)$ is realized by φ in $\hom(q\mathbb{C}, A/I)$ then $\partial(x)$ is realized by some ψ in $\hom(C_0(0,1), I)$. Equivalently, $\partial(x)$ is realized as a unitary in $\tilde{I}$.

We want to verify that the construction of ψ is well defined up to homotopy and that

$$[q\mathbb{C}, A/I] \to [C_0(0,1), I]$$

is natural. This can be done by an examination of the proof of [6, Theorem 6], but is more clearly taken care of by a cell diagram and Theorem 4.2.

In this approach to the exponential map, the key point is the projectivity of

$$\mathcal{P} = C^* \left\langle h, k, x \,\middle|\, hk = 0,\ 0 \leq P \leq 1 \text{ for } P = \begin{bmatrix} \mathbb{1} - h & x^* \\ x & k \end{bmatrix} \right\rangle.$$

We will not reprove that, but refer the reader to [6, Theorem 9]. What we will do is give a second approach to the K-theory calculations related to $\mathcal{P}$ based on finding an embedding

$$\mathcal{P} \hookrightarrow \mathbb{C} * C_0(0,1].$$

Let η be surjection $\eta : \mathcal{P} \to q\mathbb{C}$ onto

$$q\mathbb{C} = C^* \left\langle h_0, k_0, x_0 \,\middle|\, P_0^* = P_0^2 = P_0 \text{ for } P_0 = \begin{bmatrix} \mathbb{1} - h_0 & x_0^* \\ x_0 & k_0 \end{bmatrix} \right\rangle$$

defined by $\eta(h) = h_0$, etc. Let $\mathcal{U}$ denote the kernel of η and ι denote the inclusion. Recall that $K_0(q\mathbb{C})$ is a copy of $\mathbb{Z}$ generated by the class of the projection P_0 in $\mathbf{M}_2\left((q\mathbb{C})^\sim\right)$.

Lemma 3.1. *There is a $*$-homomorphism ψ_0 so that*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{U} \arrow[r, hook, "\iota"] & \mathcal{P} \arrow[r, "\eta"] & q\mathbb{C} \arrow[r] & 0 \\ & C_0(0,1) \arrow[u, "\psi_0"] & & & \end{tikzcd}$$

is a cell diagram.

Since we know from [6] that $\mathcal{P}$ is projective, we need only find ψ_0 that induces an isomorphism on K-theory. This is easily seen to be equivalent to the following Lemma.

Lemma 3.2. *In $\tilde{\mathcal{P}}$, let*

$$u = -\mathbb{1} + \sum_{i,j} v_{ij}$$

where

$$v = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = e^{2\pi i P}.$$

Then u is a unitary in $\tilde{U}$ that represents $\partial([P_0])$ in $K_1(I)$.

Proof. Theorem 6 in [6], applied to

$$0 \to \mathcal{U} \to \mathcal{P} \to q\mathbb{C} \to 0,$$

tells us that some unitary in $\tilde{U}$ will represent that K_1-class of the boundary $[P_0]$. The proof of that result tells gives us the formula for u, and moreover shows that $e^{2\pi i P_0}$ is homotopic through unitaries to

$$\begin{bmatrix} u & 0 \\ 0 & \mathbb{1} \end{bmatrix}.$$

□

The rest of this section is devoted to an alternative proof of Lemma 3.1.

First we get more specific regarding the exact sequence

$$0 \longrightarrow q\mathbb{C} \xrightarrow{\theta_0} \mathbb{C} * \mathbb{C} \xrightarrow{\rho_0} \mathbb{C} \longrightarrow 0\ .$$

We will use p_0 and q_0 to denote the two generating projections in $\mathbb{C} * \mathbb{C}$. Both of these are sent to 1 by ρ_0. The inclusion θ_0 of $q\mathbb{C}$ in $\mathbb{C} * \mathbb{C}$ is determined on generators by

$$\begin{aligned} \theta(h_0) &= p_0 - p_0 q_0 p_0, \\ \theta(k_0) &= (\mathbb{1} - p_0) q_0 (\mathbb{1} - p_0), \\ \theta(x_0) &= (\mathbb{1} - p_0) q_0 p_0. \end{aligned}$$

There is a similar exact sequence involving $\mathcal{P}$. Let the obvious generators of $\mathbb{C} * C_0(0,1]$ be denoted p and l, so the only relations on them are

$$\begin{aligned} p^2 = p^* = p, \\ 0 \leq l \leq 1. \end{aligned}$$

Theorem 3.3. *There are $*$-homomorphisms θ and ρ defined by*

$$\begin{aligned}\theta(h) &= p - plp\\ \theta(k) &= (\mathbb{1}-p)l(\mathbb{1}-p)\\ \theta(x) &= (\mathbb{1}-p)lp\end{aligned}$$

and

$$\begin{aligned}\rho(p) &= 1,\\ \rho(l) &= 1\end{aligned}$$

so that the sequence

$$0 \longrightarrow \mathcal{P} \xrightarrow{\theta} \mathbb{C} * C_0(0,1] \xrightarrow{\rho} \mathbb{C} \longrightarrow 0$$

is exact.

Proof. Since

$$\begin{aligned}&\begin{bmatrix} \mathbb{1}-(p-plp) & ((\mathbb{1}-p)lp)^* \\ (\mathbb{1}-p)lp & (\mathbb{1}-p)l(\mathbb{1}-p)\end{bmatrix}\\ &\qquad = \begin{bmatrix} p & \mathbb{1}-p \\ \mathbb{1}-p & p\end{bmatrix}\begin{bmatrix} l & 0 \\ 0 & \mathbb{1}-p\end{bmatrix}\begin{bmatrix} p & \mathbb{1}-p \\ \mathbb{1}-p & p\end{bmatrix}\end{aligned}$$

we have

$$0 \leq \begin{bmatrix} \mathbb{1}-(p-plp) & ((\mathbb{1}-p)lp)^* \\ (\mathbb{1}-p)lp & (\mathbb{1}-p)l(\mathbb{1}-p)\end{bmatrix} \leq 1.$$

Since

$$(p-plp)\left((\mathbb{1}-p)l(\mathbb{1}-p)\right) = 0,$$

we see that θ is well-defined.

The unit 1 is both a projection and a positive contraction, so ρ is well-defined.

Exactness at $\mathbb{C}$ is obvious.

To prove exactness at $\mathcal{P}$, suppose $\pi : \mathcal{P} \to \mathbb{B}(\mathbb{H})$ is a faithful representation of $\mathcal{P}$, and let $h_1 = \pi(h)$, etc. Let $r = [h_1]$ be the range projection of h_1 and let $q = [k_1]$ be the range projection of k_1. The orthogonality of h and k implies orthogonality for r and q. We established in the proof of Theorem 4.3 of [6] the factorization $x = k^{\frac{1}{8}} y h^{\frac{1}{8}}$ for some y and so

$$rx_1 = x_1 q = 0$$

and

$$qx_1 = x_1 r = x_1,$$

and of course

$$\begin{aligned} rh_1 &= h_1 r = h_1, \\ qh_1 &= h_1 q = rk_1 = k_1 r = 0, \\ qk_1 &= k_1 q = k_1. \end{aligned}$$

We know

$$0 \leq \left[\begin{array}{cc} I - h_1 & x_1^* \\ x_1 & k_1 \end{array} \right] \leq 1$$

and so

$$0 \leq \left[\begin{array}{cc} r & q \end{array} \right] \left[\begin{array}{cc} I - h_1 & x_1^* \\ x_1 & k_1 \end{array} \right] \left[\begin{array}{cc} r & q \end{array} \right]^* \leq 1.$$

This says

$$0 \leq r - h_1 + x_1 + x_1^* + k_1 \leq 1$$

We can define a representation $\overline{\pi}$ of $\mathbb{C} * C_0(0,1]$ on $\mathbb{B}(\mathbb{H})$ by setting $\overline{\pi}(p) = r$ and

$$\overline{\pi}(l) = r - h_1 + x_1 + x_1^* + k_1.$$

This is an extension of π because

$$\overline{\pi} \circ \theta(h) = r - r(r - h_1 + x_1 + x_1^* + k_1)r = h_1$$

and

$$\overline{\pi} \circ \theta(k) = (I - r)(r - h_1 + x_1 + x_1^* + k_1)(I - r) = k_1$$

and

$$\overline{\pi} \circ \theta(x) = (I - r)(r - h_1 + x_1 + x_1^* + k_1)r = x_1.$$

We have shown ι is one-to-one, and so have exactness at $\mathcal{P}$.

Next we show that the image of θ is an ideal. This follows from these equalities:

$$\begin{aligned} (p - plp)\, p &= (p - plp) \\ (p - plp)\, l &= (p - plp)\left((\mathbb{1} - p)lp\right)^* + (p - plp) - (p - plp)^2 \\ ((\mathbb{1} - p)l(\mathbb{1} - p))\, p &= 0 \\ ((\mathbb{1} - p)l(\mathbb{1} - p))\, l &= ((\mathbb{1} - p)l(\mathbb{1} - p))\left((\mathbb{1} - p)lp\right) + \left((\mathbb{1} - p)l(\mathbb{1} - p)\right)^2 \\ ((\mathbb{1} - p)lp)\, p &= ((\mathbb{1} - p)lp) \\ p\left((\mathbb{1} - p)lp\right) &= 0 \\ ((\mathbb{1} - p)lp)\, l &= ((\mathbb{1} - p)lp) - ((\mathbb{1} - p)lp)(p - plp) \\ &\quad + ((\mathbb{1} - p)lp)\left((\mathbb{1} - p)lp\right)^* \\ l\left((\mathbb{1} - p)lp\right) &= ((\mathbb{1} - p)lp)^*\left((\mathbb{1} - p)lp\right) + ((\mathbb{1} - p)l(\mathbb{1} - p))\left((\mathbb{1} - p)lp\right). \end{aligned}$$

As to exactness in the middle, it is clear that $\rho \circ \theta = 0$. We need to show that the induced map

$$\overline{\rho} : (\mathbb{C} * C_0(0,1])/\,\theta(\mathcal{P}) \to \mathbb{C}$$

is an isomorphism. Since

$$\begin{aligned} l - p &= -(p - plp) + ((\mathbb{1} - p)lp) + (pl(\mathbb{1} - p)) + ((\mathbb{1} - p)l(\mathbb{1} - p)) \\ &= -\theta(h) + \theta(x) + \theta(x)^* + \theta(k) \end{aligned}$$

we discover

$$(\mathbb{C} * C_0(0,1])/\,\theta(\mathcal{P})$$

is generated by a single projection. Since $\overline{\rho}$ maps onto $\mathbb{C}$, it must be an isomorphism. ☐

Recall we have the surjection $\eta : \mathcal{P} \to q\mathbb{C}$. Consider also the surjection

$$\eta_1 : \mathbb{C} * C_0(0,1] \to \mathbb{C} * \mathbb{C}$$

defined via

$$\begin{aligned} \eta_1(p) &= p, \\ \eta_1(l) &= q. \end{aligned}$$

Let us use $\mathcal{U}_1$ to denote the kernel of η_1. This gives us the diagram

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & \mathbb{C} \arrow[u] \arrow[r, equal] & \mathbb{C} \arrow[u] & \\
0 \arrow[r] & \mathcal{U}_1 \arrow[r, hook, "\iota_1"] & \mathbb{C} * C_0(0,1] \arrow[u, "\rho"] \arrow[r, "\eta_1"] & \mathbb{C} * \mathbb{C} \arrow[u, "\rho_0"] \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{U} \arrow[u, "\theta_1", "\cong"'] \arrow[r, hook, "\iota"] & \mathcal{P} \arrow[u, hook, "\theta"] \arrow[r, "\eta"] & q\mathbb{C} \arrow[u, hook, "\theta_0"] \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where θ_1 is the restriction of θ. Both rows and both columns are exact, and it follows that θ_1 is an isomorphism.

The K-theory of the middle row is easy to work out, and so we see that $K_1(\mathcal{U}_1) \cong \mathbb{Z}$ and has generator represented by the unitary $e^{2\pi i l}$ in $\mathcal{U}_1^{\sim}$. This completes the second proof of Lemma 3.1.

4. Projectives Determine Boundary Maps

Lemma 4.1. *Suppose P is projective, $\rho : A \to B$ is a surjective $*$-homomorphism and $\varphi_t : P \to B$ is a homotopy of $*$-homomorphisms. Given $*$-homomorphisms ψ_0 and ψ_1 from P to A that are lifts of φ_0 and φ_1, there exists a homotopy of $*$-homomorphisms $\bar{\psi}$ so that $\bar{\psi}_t$ is a lift of φ_t and $\bar{\psi}_0 = \psi_0$ and $\bar{\psi}_1 = \psi_1$.*

Proof. This proof is a standard argument using a mapping cylinder. ☐

Theorem 4.2. *Suppose*

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "\iota"] & P \arrow[r, "\eta"] & R \arrow[r] & 0 \\ & Q \arrow[u, "\psi_0"] & & & \end{tikzcd}$$

is a cell diagram. Suppose that α is a fixed generator of $K_i(R)$ and that β in $K_{i+1}(Q)$ is defined so that $\partial(\alpha) = (\psi_0)_*(\beta)$. Given an ideal I in A, there are natural maps $\partial^{(n)}$ so that

$$\begin{tikzcd} {[R, \mathbf{M}_n(A/I)]} \arrow[r, "\partial^{(n)}"] \arrow[d] \arrow[dd, bend right, "\Xi^{(n)}"'] & {[Q, \mathbf{M}_n(I)]} \arrow[d] \arrow[dd, bend right, "\Lambda^{(n)}"'] \\ {[R, \mathbf{M}_{n+1}(A/I)]} \arrow[r, "\partial^{(n+1)}"] \arrow[d, "\Xi^{(n+1)}"] & {[Q, \mathbf{M}_{n+1}(I)]} \arrow[d, "\Lambda^{(n+1)}"] \\ K_i(A/I) \arrow[r, "\partial"] & K_{i+1}(I) \end{tikzcd}$$

commutes, where ∂ is the boundary map in K-theory. Here $\Xi^{(n)}(\psi) = \psi_*(\alpha)$ and $\Lambda^{(n)}(\varphi) = \varphi_*(\beta)$.

Proof. The naturality of $\partial^{(1)}$ allows us construct $\partial^{(n)}$ out of $\partial^{(1)}$ so we only concern ourselves with finding $\partial^{(1)}$ with $\delta \circ \Xi^{(1)} = \Lambda^{(1)} \circ \delta^{(1)}$.

Suppose

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\kappa"] & A \arrow[r, "\pi"] & B \arrow[r] & 0 \end{tikzcd}$$

is exact and we are given $\varphi : R \to B$. Projectivity tells us there exists $\bar{\varphi}$ with $\pi \circ \bar{\varphi} = \varphi \circ \theta$. This restricts to a map $\hat{\varphi}$ between ideals. In terms of a commuting diagram with exact rows, we are here:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\kappa"] & A \arrow[r, "\pi"] & B \arrow[r] & 0 \\ 0 \arrow[r] & U \arrow[r, "\iota"] \arrow[u, "\hat{\varphi}"] & P \arrow[r, "\theta"] \arrow[u, "\bar{\varphi}"] & R \arrow[r] \arrow[u, "\varphi"'] & 0 \\ & Q \arrow[u, "\psi_0"'] & & & \end{tikzcd}$$

We wish to define

$$\partial^{(1)}([\varphi]) = [\hat{\varphi} \circ \psi_0]\,.$$

This composition depends on our choice of $\bar{\varphi}$ as well as the choice of representative of the homotopy class $[\varphi]$. By the Lemma 4.1, we get a well defined map from $[R, B]$ to $[Q, I]$. The naturality of the boundary map in K-theory shows $\partial^{(1)}$ implements the boundary map. The flexibility in choosing the lift $\bar{\varphi}$ makes it easy to show that $\partial^{(1)}$ is natural. □

Applying this to the examples in Sections 2 and 3 we get the somewhat unified picture of the boundary maps summarized in three diagrams:

$$\begin{tikzcd}
{[C_0(0,1), \mathbf{M}_n(A/I)]} \arrow[r, "\partial^{(n)}"] \arrow[d] & {[G_2^{\mathrm{st}}, \mathbf{M}_n(I)]} \arrow[d] \\
\varinjlim\, [C_0(0,1), \mathbf{M}_k(A/I)] \arrow[r] \arrow[d, "\cong"] & \varinjlim \left[G_2^{\mathrm{st}}, \mathbf{M}_k(I)\right] \arrow[d, "\cong"] \\
K_1(A/I) \arrow[r, "\partial"] & K_0(I)
\end{tikzcd}$$

$$\begin{tikzcd}
{[G_2^{\mathrm{st}}, \mathbf{M}_n(D)]} \arrow[r] \arrow[d] & {[q\mathbb{C}, \mathbf{M}_{2n}(D)]} \arrow[d] \\
\varinjlim \left[G_2^{\mathrm{st}}, \mathbf{M}_k(D)\right] \arrow[r, "\cong"] \arrow[d, "\cong"] & \varinjlim\, [q\mathbb{C}, \mathbf{M}_k(D)] \arrow[dl, "\cong"] \\
K_0(D) &
\end{tikzcd}$$

$$\begin{tikzcd}
{[q\mathbb{C}, \mathbf{M}_n(A/I)]} \arrow[r, "\partial^{(n)}"] \arrow[d] & {[C_0(0,1), \mathbf{M}_n(I)]} \arrow[d, "\Lambda^{(n)}"] \\
\varinjlim\, [q\mathbb{C}, \mathbf{M}_k(A/I)] \arrow[r] \arrow[d, "\cong"] & \varinjlim\, [C_0(0,1), \mathbf{M}_k(I)] \arrow[d, "\cong"] \\
K_0(A/I) \arrow[r, "\partial"] & K_1(I)
\end{tikzcd}$$

The jump up in the size of matrices in the middle diagram is the one we saw in Theorem 2.5.

5. Further Examples

The list of projective C^*-algebras is still growing. For example, Shulman ([9]) has recently shown that a nilpotent contraction lifts to a nilpotent contraction of the same order. It should be fruitful to search for semiprojective quotients of projective C^*-algebras that can also serve as “boundaries of nonstandard cells.” To illustrate, we now look at two more applications of Theorem 4.2.

Recall that the cone

$$\mathbf{CM}_n = C_0\left((0,1], \mathbf{M}_n\right)$$

is projective and fits in a nice exact sequence

$$0 \to \mathbf{SM}_n \to \mathbf{CM}_n \to \mathbf{M}_n \to 0$$

with the suspension $\mathbf{SM}_n$ of $\mathbf{M}_n$. The larger cone

$$\mathbf{C}\mathbb{K} = C_0\left((0,1],\mathbb{K}\right)$$

is not residually finite dimensional, so it is not projective. We can instead consider the mapping telescope

$$\mathbf{T}(\mathbb{K}) = \{f \in C_0\left((0,\infty],\mathbb{K}\right) \mid t \le n \implies f(t) \in \mathbf{M}_n\}$$

which is projective ([7]) and maps onto $\mathbb{K}$ by evaluation at ∞. The kernel of this map is a bit awkward, being

$$\mathbf{I}(\mathbb{K}) = \{f \in C_0\left((0,\infty),\mathbb{K}\right) \mid t \le n \implies f(t) \in \mathbf{M}_n\}.$$

Theorem 5.1. *Let x be the standard generator of $K_0\left(\mathbf{M}_n\right)$ and y be the standard generator of $K_1\left(\mathbf{SM}_n\right)$. Given an ideal I in a C^*-algebra A, for any $*$-homomorphism*

$$\varphi : \mathbf{M}_n \to A/I$$

there is a $*$-homomorphism

$$\psi : \mathbf{SM}_n \to I$$

so that

$$\psi_*(y) = \partial(\varphi_*(x)).$$

The mapping $\varphi \mapsto \psi$ can be chosen to be natural and well-defined up to homotopy.

By the "standard generator" of $K_1\left(\mathbf{I}(\mathbb{K})\right)$ we mean the push-forward of the standard generator of $K_1\left(C_0(0,1)\right)$ by the obvious inclusion

$$C_0(0,1) = C_0\left((0,1),\mathbf{M}_1\right) \hookrightarrow \mathbf{I}(\mathbb{K}).$$

Theorem 5.2. *Let x be the standard generator of $K_0\left(\mathbf{M}_n\right)$ and y be the standard generator of $K_1\left(\mathbf{I}(\mathbb{K})\right)$. Given an ideal I in a C^*-algebra A, for any $*$-homomorphism*

$$\varphi : \mathbb{K} \to A/I$$

there is a $*$-homomorphism

$$\psi : \mathbf{I}(\mathbb{K}) \to I$$

so that

$$\psi_*(y) = \partial(\varphi_*(x)).$$

Notice that from ψ we get realizations of the boundary of $\varphi_*(x)$ via a map

$$\psi_n : \mathbf{SM}_n \to I$$

for any n.

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