Münster J. of Math. **9** (2016), 93–118
DOI 10.17879/45209421960
urn:nbn:de:hbz:6-45209431256

Münster Journal of Mathematics

Conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ are not Borel

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(Communicated by Katrin Tent)

Abstract. The group of automorphisms of the Cuntz algebra $\mathcal{O}_2$ is a Polish group with respect to the topology of pointwise convergence in norm. Our main result is that the relations of conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ are not Borel. Moreover, we show that from the point of view of Borel complexity theory, classifying automorphisms of $\mathcal{O}_2$ up to conjugacy or cocycle conjugacy is strictly more difficult than classifying any class of countable structures with Borel isomorphism relation. In fact, the same conclusions hold even if one only considers automorphisms of $\mathcal{O}_2$ of finite order. We moreover show that for any prime number p, the relation of isomorphism of simple purely infinite crossed products $\mathcal{O}_2 \rtimes \mathbb{Z}_p$ (with trivial K_1-group and satisfying the Universal Coefficient Theorem) is not Borel. Moreover, it is strictly more difficult to classify such crossed products than classifying any class of countable structures with Borel isomorphism relation.

1. Introduction

The *Cuntz algebra* $\mathcal{O}_2$ is the universal C*-algebra generated by two isometries with complementary ranges [7]. The key role of $\mathcal{O}_2$ in the classification program of C*-algebras—see [40, Chap. 2]—has served as motivation for an intensive study of the structural properties of $\mathcal{O}_2$ and its automorphism group, as in [30, 44, 5, 6]. In particular, considerable effort has been put into trying to classify several important classes of automorphisms; see for example [23].

In this work, we study the group of automorphisms of $\mathcal{O}_2$ from the perspective of Borel complexity theory. Our main result is the following.

Eusebio Gardella was supported by the US National Science Foundation through his thesis advisor's grant DMS-1101742. Martino Lupini was supported by the York University Elia Scholars Program. This work was completed when the authors were attending the Thematic Program on Abstract Harmonic Analysis, Banach and Operator Algebras at the Fields Institute, in January–July of 2014. The hospitality of the Fields Institute is gratefully acknowledged.

Theorem 1.1. *The relations of conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ are complete analytic sets when regarded as subsets of* $\mathrm{Aut}(\mathcal{O}_2) \times \mathrm{Aut}(\mathcal{O}_2)$*, and in particular, are not Borel. Furthermore if $\mathcal{C}$ is any class of countable structures such that the corresponding isomorphism relation $\cong_{\mathcal{C}}$ is Borel, then $\cong_{\mathcal{C}}$ is Borel reducible to both conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$.*

See [24, Def. 26.7] for the notion of complete analytic set, and [17, Chap. 5] for an introduction to Borel complexity theory and the definition of Borel reducibility. In particular, Theorem 1.1 rules out any classification that uses as invariant Borel measures on a Polish space (up to measure equivalence) or unitary operators on the Hilbert space (up to conjugacy).

The fact that conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ are not Borel should be compared with the fact that for any separable C*-algebra A, the relation of unitary equivalence of automorphisms of A is Borel. This is because the relation of coset equivalence modulo the Borel subgroup $\mathrm{Inn}(A)$ of $\mathrm{Aut}(A)$ is Borel. Similarly, the spectral theorem for unitary operators on the Hilbert space shows that the relation of conjugacy of unitary operators is Borel.

We will show that Theorem 1.1 holds even if one only considers automorphisms of finite order whose induced finite group action has Rokhlin dimension at most one in the sense of [20]. Moreover, it will follow from our proof that the same assertions hold for the relation of isomorphism of simple purely infinite crossed products $\mathcal{O}_2 \rtimes \mathbb{Z}_p$ (with trivial K_0-group and satisfying the Universal Coefficient Theorem), where p is any prime number.

It should be mentioned that, by the main result of [26], the automorphisms of $\mathcal{O}_2$ are not classifiable up to conjugacy by countable structures. This means that there is no explicit way to assign a countable structure to every automorphism of $\mathcal{O}_2$, in such a way that two automorphisms are conjugate if and only if the corresponding structures are isomorphic. Moreover, the same conclusions hold for any set of automorphisms of $\mathcal{O}_2$ which is not meager in the topology of pointwise convergence.

The strategy of the proof of the main theorem is as follows. Using techniques from [9, 22], we show that the relation of isomorphism of countable 2-divisible torsion-free abelian groups is a complete analytic set, and it is strictly more complicated than the relation of isomorphism of any class of countable structures with Borel isomorphism relation. We then show that the relation of isomorphism of 2-divisible abelian groups is Borel reducible to the relations of conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ of order 2. This is achieved by showing that there is a Borel way to assign to a countable abelian group G a Kirchberg algebra A_G with trivial K_1-group, K_0-group isomorphic to G, and with the class of the unit in K_0 being the zero element. We then use a result of Izumi from [23] asserting that there is an automorphism ν of $\mathcal{O}_2$ of order 2 with the following property: Tensoring the identity automorphism of A_G by ν, and identifying $A_G \otimes \mathcal{O}_2$ with $\mathcal{O}_2$ by Kirchberg's absorption theorem, gives a reduction of isomorphism of Kirchberg algebras with 2-divisible K_0-group and with the class of the unit being the zero element in K_0, to conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ of order 2. The proof is concluded by showing—using results from [14]—that such reduction is implemented by a Borel map. A suitable modification of this argument yields the same result when 2 is replaced by an arbitrary prime number.

The present paper is organized as follows. Section 2 exhibits a functorial version of the notion of standard Borel parametrization of a category as defined in [14]. Several functorial parametrizations for the category of C*-algebras are then presented and shown to be equivalent. Section 3 provides a Borel version of the correspondence between unital AF-algebras and dimension groups established in [10, 11]. Finally, Section 4 contains the proofs of the main results.

In the following, all C*-algebras and Hilbert spaces are assumed to be *separable*, and all discrete groups are assumed to be *countable*. We denote by ω the set of natural numbers *including* 0. An element $n \in \omega$ will be identified with the set $\{0, 1, \ldots, n-1\}$ of its predecessors. We will therefore write $i \in n$ to mean that i is a natural number and $i < n$. For $n \geq 1$, we write $\mathbb{Z}_n$ for the cyclic group $\mathbb{Z}/n\mathbb{Z}$. If X is a Polish space and D is a countable set, we endow the set X^D of D-indexed sequences of elements of X with the product topology. Likewise, if X is a standard Borel space, then we give X^D the product Borel structure. In the particular case where $X = 2 = \{0, 1\}$, we identify 2^D with the set of subsets of D with its Cantor set topology, and the corresponding standard Borel structure. We will use throughout the paper the fact that a G_δ subspace of a Polish space is Polish in the subspace topology [24, Thm. 3.11], and that a Borel subspace of a standard Borel space is standard with the inherited Borel structure [24, Prop. 12.1].

2. Parametrizing the category of C*-algebras

2.1. **Functorial parametrization.** Recall that a *(small) semigroupoid* is a quintuple $(X, \mathcal{C}_X, s, r, \cdot)$, where X and $\mathcal{C}_X$ are sets, s, r are functions from $\mathcal{C}_X$ to X, and $\cdot$ is an associative partially defined binary operation on $\mathcal{C}_X$ with domain

$$\{(x, y) \in \mathcal{C}_X \times \mathcal{C}_X \mid s(x) = r(y)\}$$

such that $r(x \cdot y) = r(x)$ and $s(x \cdot y) = s(y)$ for all x and y in X. The elements of X are called objects, the elements of $\mathcal{C}_X$ morphisms, the map $\cdot$ composition, and the maps s and r source and range map. In the following, a semigroupoid $(X, \mathcal{C}_X, s, r, \cdot)$ will be denoted simply by $\mathcal{C}_X$. Note that a (small) category is precisely a (small) semigroupoid, where moreover the identity arrow $\mathrm{id}_x \in \mathcal{C}_X$ is associated with the element x of X. A morphism between semigroupoids $\mathcal{C}_X$ and $\mathcal{C}_{X'}$ is a pair (f, F) of functions $f\colon X \to X'$ and $F\colon \mathcal{C}_X \to \mathcal{C}_{X'}$ such that

- $s \circ F = f \circ s$,
- $r \circ F = f \circ r$, and
- $F(a \cdot b) = F(a) \cdot F(b)$ for every a and $b \in \mathcal{C}_X$.

In the case of categories, a morphism of semigroupoids is just a functor.

A *standard Borel semigroupoid* is a semigroupoid $\mathcal{C}_X$ such that X and $\mathcal{C}_X$ are endowed with standard Borel structures making the composition function $\cdot$ and the source and range functions s and r Borel.

Definition 2.2. Let $\mathcal{D}$ be a category, let $\mathcal{C}_X$ be a standard Borel semigroupoid, and let (f, F) be a morphism from $\mathcal{C}_X$ to $\mathcal{D}$. We say that $(\mathcal{C}_X, f, F)$ is a *good parametrization* of $\mathcal{D}$ if

- (f, F) is *essentially surjective*, that is, if every object of $\mathcal{D}$ is isomorphic to an object in the range of f,
- (f, F) is *full*, that is, if for every $x, y \in X$ the set $\mathrm{Hom}(f(x), f(y))$ is contained in the range of F, and
- the set Iso_X of elements of $\mathcal{C}_X$ that are mapped by F to isomorphisms of $\mathcal{D}$ is Borel.

Observe that if $(\mathcal{C}_X, f, F)$ is a *good parametrization* of $\mathcal{D}$, then (X, f) is a good parametrization of $\mathcal{C}$ in the sense of [14, Def. 2.1].

Definition 2.3. Let $\mathcal{D}$ be a category and let $(\mathcal{C}_X, f, F)$ and $(\mathcal{C}_{X'}, f', F')$ be good parametrizations of $\mathcal{D}$. A *morphism* from $(\mathcal{C}_X, f, F)$ to $(\mathcal{C}_{X'}, f', F')$ is a triple (g, G, η) of maps $g\colon X \to X'$, $G\colon \mathcal{C}_X \to \mathcal{C}_{X'}$, and $\eta\colon X \to \mathcal{D}$, satisfying the following conditions:

(1) The functions g and G are Borel.
(2) $\eta(x)$ is an isomorphism from $f(x)$ to $(f' \circ g)(x)$ for every $x \in X$.
(3) The pair $(f' \circ g, F' \circ G)$ is a semigroupoid morphism $\mathcal{C}_X \to \mathcal{D}$.
(4) We have $s_{X'} \circ G = g \circ s_X$ and $r_{X'} \circ G = g \circ r_X$.
(5) For every $a \in \mathcal{C}_X$,

$$F'(G(a)) \circ \eta(s(a)) = \eta(r(a)) \circ F(a).$$

Two good parametrizations $(\mathcal{C}_X, f, F)$ and $(\mathcal{C}_{X'}, f', F')$ of $\mathcal{C}$ are said to be *equivalent* if there are isomorphisms from $(\mathcal{C}_X, f, F)$ to $(\mathcal{C}_{X'}, f', F')$ and vice-versa. It is not difficult to verify that if $(\mathcal{C}_X, f, F)$ and $(\mathcal{C}_{X'}, f', F')$ are equivalent parametrizations of $\mathcal{D}$, then (X, f) and (X', f') are weakly equivalent parametrizations of $\mathcal{D}$ in the sense of [14, Def. 2.1].

In the following, a good parametrization $(\mathcal{C}_X, f, F)$ of $\mathcal{D}$ will be denoted by $\mathcal{C}_X$ for short. The arguments in the proof of [14, Prop. 2.7] can be easily adapted to show that the good parametrizations $\mathcal{C}_{\widehat{\Xi}}$, $\mathcal{C}_{\Xi}$, and $\mathcal{C}_{\Gamma}$ of the category of C*-algebras with *-homomorphisms defined in Sections 2.6, 2.4, and 2.8 are equivalent in the sense of Definition 2.3.

2.4. **The space $\mathcal{C}_{\widehat{\Xi}}$.** We follow the notation in [14, §2.2], and denote by $\mathbb{Q}(i)$ the field of complex rationals. A $\mathbb{Q}(i)$-$*$-algebra is an algebra over the field $\mathbb{Q}(i)$ endowed with an involution $x \mapsto x^*$. We define $\mathcal{U}$ to be the $\mathbb{Q}(i)$-$*$-algebra of noncommutative $*$-polynomials with coefficients in $\mathbb{Q}(i)$ and without constant term in the formal variables X_k for $k \in \omega$. If A is a C*-algebra, $\gamma = (\gamma_n)_{n\in\omega}$ is a sequence of elements of A, and $p \in \mathcal{U}$, we define $p(\gamma)$ to be the element of A obtained by evaluating p in A, where for every $k \in \omega$, the formal variables X_k and X_k^* are replaced by the elements γ_k and γ_k^* of A.

We denote by $\widehat{\Xi}$ the set of elements

$$A = (f, g, h, k, r) \in \omega^{\omega\times\omega} \times \omega^{\mathbb{Q}(i)\times\omega} \times \omega^{\omega\times\omega} \times \omega^{\omega} \times \mathbb{R}^{\omega}$$

that code on ω a structure of $\mathbb{Q}(i)$-$*$-algebra A endowed with a norm satisfying the C*-identity. The completion $\widehat{A}$ of ω with respect to such norm is a C*-algebra (denoted by $B(A)$ in [14, §2.4]). It is not hard to check that $\widehat{\Xi}$ is a Borel subset of $\omega^{\omega\times\omega} \times \omega^{\mathbb{Q}(i)\times\omega} \times \omega^{\omega\times\omega} \times \omega^{\omega} \times \mathbb{R}^{\omega}$. As observed in [14, §2.4], $\widehat{\Xi}$ can be thought of as a natural parametrization for *abstract C*-algebras*. We use the notation of [14, §2.4] to denote the operations on ω coded by an element $A = (f, g, h, k, r)$ of $\widehat{\Xi}$. We denote by d_A the metric on ω coded by A, which is given by

$$d_A(n, m) = \|n +_f (-1) \cdot_g m\|_r$$

for $n, m \in \omega$. We will also write $n +_A m$ for $n +_f m$, and similarly for g, h, k, r.

Definition 2.5. Suppose that $A = (f, g, h, k, r)$ and $A' = (f', g', h', k', r')$ are elements of $\widehat{\Xi}$, and that $\Phi = (\Phi_n)_{n\in\omega} \in (\omega^{\omega})^{\omega}$ is a sequence of functions from ω to ω. We say that Φ is a *code for a *-homomorphism* from $\widehat{A}$ to $\widehat{A}'$ if the following conditions hold:

(1) The sequence $(\Phi_n(k))_{n\in\omega}$ is Cauchy uniformly in $k \in \omega$ with respect to the metric d_A, and in particular converges to an element $\widehat{\Phi}(k)$ of $\widehat{A}$.
(2) The map $k \mapsto \widehat{\Phi}(k)$ is a contractive *-homomorphism of $\mathbb{Q}(i)$-$*$-algebras, and hence it induces a *-homomorphism $\widehat{\Phi}$ from $\widehat{A}$ to $\widehat{A}'$.

We say that Φ is a *code for an isomorphism* from $\widehat{A}$ to $\widehat{A}'$ if Φ is a code for a *-homomorphism from $\widehat{A}$ to $\widehat{A}'$, and $\widehat{\Phi}$ is an isomorphism. If Φ and Φ' are codes for *-homomorphisms from $\widehat{A}$ to $\widehat{A}'$ and from $\widehat{A}'$ to $\widehat{A}''$, respectively, we define their composition $\Phi' \circ \Phi$ by $(\Phi' \circ \Phi)_n = \Phi'_n \circ \Phi_n$ for $n \in \omega$.

It is easily checked that $\Phi' \circ \Phi \in (\omega^{\omega})^{\omega}$ is a code for the *-homomorphism $\widehat{\Phi}' \circ \widehat{\Phi}$ from $\widehat{A}$ to $\widehat{A}''$. One can verify that the set $\mathcal{C}_{\widehat{\Xi}}$ of triples $(A, A', \Phi) \in \widehat{\Xi} \times \widehat{\Xi} \times (\omega^{\omega})^{\omega}$, such that Φ is a code for a *-homomorphism from $\widehat{A}$ to $\widehat{A}'$, is Borel. We can regard $\mathcal{C}_{\widehat{\Xi}}$ as a standard semigroupoid having $\widehat{\Xi}$ as set of objects, where the composition of (A, A', Φ) and (A', A'', Φ') is $(A', A'', \Phi' \circ \Phi)$, and the source and range of (A, A', Φ) are A and A', respectively. The semigroupoid morphism $(A, A', \Phi) \mapsto (\widehat{A}, \widehat{A}', \widehat{\Phi})$ defines a parametrization of the category of C*-algebras with *-homomorphisms. It is easy to see that this is a good parametrization in the sense of Definition 2.3. In particular, the set $\mathrm{Iso}_{\widehat{\Xi}}$ of elements (A, A', Φ) of $\widehat{\Xi} \times \widehat{\Xi} \times (\omega^{\omega})^{\omega}$, such that Φ is a code for an isomorphism from $\widehat{A}$ to $\widehat{A}'$, is Borel.

2.6. **The space $\mathcal{C}_{\Xi}$.** We denote by Ξ the G_δ subset of $\mathbb{R}^{\mathcal{U}}$ consisting of the nonzero functions $\delta\colon \mathcal{U} \to \mathbb{R}$ such that there exists a C*-algebra A and a dense subset $\gamma = (\gamma_n)_{n\in\omega}$ of A, such that

$$\delta(p) = \|p(\gamma)\|.$$

It could be observed that, differently from [14, §2.3], we are not considering the function constantly equal to zero as an element of Ξ; this choice is just for convenience and will play no role in the rest of the discussion. Observe that any element δ of Ξ determines a semi-norm on the $\mathbb{Q}(i)$-$*$-algebra $\mathcal{U}$; therefore one can consider the corresponding Hausdorff completion of $\mathcal{U}$. Denote by I_δ the ideal of $\mathcal{U}$ given by $I_\delta = \{p \in \mathcal{U} \mid \delta(p) = 0\}$. Then $\mathcal{U}/I_\delta$ is a normed $\mathbb{Q}(i)$-$*$-algebra. Its completion is a C*-algebra, which we shall denote by $\widehat{\delta}$. (What we denote by $\widehat{\delta}$ is denoted by $B(\delta)$ in [14, §2.3].)

Definition 2.7. Let δ and δ' be elements in Ξ, and let $\Phi = (\Phi_n)_{n\in\omega} \in (\mathcal{U}^{\mathcal{U}})^\omega$ be a sequence of functions from $\mathcal{U}$ to $\mathcal{U}$. We say that Φ is a *code for a *-homomorphism* from $\widehat{\delta}$ to $\widehat{\delta}'$, if

(1) for every $p \in \mathcal{U}$, the sequence $(\Phi_n(p))_{n\in\omega}$ is Cauchy uniformly in $p \in \mathcal{U}$, with respect to the pseudometric $(q, q') \mapsto \delta(q - q')$ on $\mathcal{U}$, and in particular converges in $\widehat{\delta}$ to an element $\widehat{\Phi}(p)$, and
(2) $p \mapsto \widehat{\Phi}(p)$ is a morphism of $\mathbb{Q}(i)$-$*$-algebras such that $\|\widehat{\Phi}(p)\| \leq \delta(p)$, and hence induces a *-homomorphism from $\widehat{\delta}$ to $\widehat{\delta}'$.

Writing down explicit formulas defining a code for a *-homomorphism makes it clear that the set $\mathcal{C}_\Xi$ of triples $(\delta, \delta', \Phi) \in \Xi \times \Xi \times (\mathcal{U}^{\mathcal{U}})^\omega$, such that Φ is a code for a *-homomorphism from $\widehat{\delta}$ to $\widehat{\delta}'$, is Borel. Suppose that Φ, Φ' are codes for *-homomorphisms from δ to δ' and from δ' to δ''. Similarly as in Section 2.4, it is easy to check that defining $(\Phi' \circ \Phi)_n = \Phi'_n \circ \Phi_n$, for $n \in \omega$, gives a code for a *-homomorphism from δ to δ''. This defines a standard Borel semigroupoid structure on $\mathcal{C}_\Xi$, such that the map $(\delta, \delta', \Phi) \mapsto (\widehat{\delta}, \widehat{\delta}', \widehat{\Phi})$ is a good standard Borel parametrization of the category of C*-algebras.

2.8. **The space $\mathcal{C}_{\Gamma(H)}$.** Denote by $B_1(H)$ the unit ball of $B(H)$ with respect to the operator norm. Recall that $B_1(H)$ is a compact Hausdorff space when endowed with the weak operator topology. The standard Borel structure generated by the weak operator topology on $B_1(H)$ coincides with the Borel structure generated by several other operator topologies on $B_1(H)$, such as the σ-weak, strong, σ-strong, strong-*, and σ-strong-* operator topology; see [2, Def. I.3.1.1 and §I.3.1.4]. Denote by $B_1(H)^\omega$ the product of countable many copies of $B_1(H)$, endowed with the product topology, and define $\Gamma(H)$ to be the Polish space obtained by removing from $B_1(H)^\omega$ the sequence constantly equal to 0. (The space $\Gamma(H)$ is defined similarly in [14, §2.1]; the only difference is that here the sequence constantly equal to 0 is excluded for convenience.) Given an element γ in $\Gamma(H)$, denote by $C^*(\gamma)$ the C*-subalgebra of $B(H)$ generated by $\{\gamma_n \mid n \in \omega\}$. As explained in [14, §2.1 and Rem. 2.3], the space $\Gamma(H)$ can be thought of as a natural parametrization of *concrete C*-algebras.*

Definition 2.9. Let γ and γ' be elements in $\Gamma(H)$, and let $\Phi = (\Phi_n)_{n\in\omega} \in (\mathcal{U}^{\mathcal{U}})^\omega$ be a sequence of functions from $\mathcal{U}$ to $\mathcal{U}$. We say that Φ is a *code for a *-homomorphism* from $C^*(\gamma)$ to $C^*(\gamma')$, if

(1) the sequence $(\Phi_n(p)(\gamma'))_{n\in\omega}$ of elements of $C^*(\gamma')$ is Cauchy uniformly in p, and hence converges to an element $\widehat{\Phi}(p(\gamma))$ of $C^*(\gamma')$,
(2) the function $p(\gamma) \mapsto \widehat{\Phi}(p(\gamma))$ extends to a *-homomorphism from $C^*(\gamma)$ to $C^*(\gamma')$.

Again, it is easily checked that the set $\mathcal{C}_{\Gamma(H)}$ of triples (γ, γ', Φ), such that Φ is a code for a *-homomorphism from $C^*(\gamma)$ to $C^*(\gamma')$, is Borel. Moreover, one can define a standard Borel semigroupoid structure on $\mathcal{C}_{\Gamma(H)}$, in such a way that the map $(\gamma, \gamma', \Phi) \mapsto (C^*(\gamma), C^*(\gamma'), \widehat{\Phi})$ is a good parametrization of the category of C*-algebras.

Recall that, consistently with Definition 2.2, $\mathrm{Iso}_{\Gamma(H)}$ denotes the Borel set of elements in $\mathcal{C}_{\Gamma(H)}$ that code an isomorphism. It is not difficult to see that there is a Borel map $\mathrm{Iso}_{\Gamma(H)} \to (\mathcal{U}^{\mathcal{U}})^{\omega}$, assigning to an element (γ, γ', Φ) of $\mathrm{Iso}_{\Gamma(H)}$ a code $\mathrm{Inv}(\gamma, \gamma', \Phi)$ for an isomorphism from $C^*(\gamma')$ to $C^*(\gamma)$ such that

$$\mathrm{Inv}\widehat{(\gamma, \gamma', \Phi)} = \widehat{\Phi}^{-1}.$$

2.10. **Computing reduced crossed products.** We want to remark that the reduced crossed product of a C*-algebra by an action of a discrete group can be computed in a Borel way in any of the parametrizations of C*-algebras introduced so far. For convenience, we will work using the parametrization $\Gamma(H)$ of C*-algebras; similar statements will hold for the parametrizations Ξ and $\widehat{\Xi}$. We refer the reader to [45] for the definitions of full and reduced crossed products for actions of discrete groups.

Definition 2.11. Let γ be an element of $\Gamma(H)$, and G be an element of $\mathcal{G}$. Suppose that $\Phi = (\Phi_{m,n})_{(m,n)\in\omega\times\omega} \in (\mathcal{U}^{\mathcal{U}})^{\omega\times\omega}$ is an $(\omega \times \omega)$-sequence of functions from $\mathcal{U}$ to $\mathcal{U}$. We say that Φ is a *code for an action of G on $C^*(\gamma)$*, if the following conditions hold:

(1) For every $m \in \omega$, the sequence $(\Phi_{m,n})_{n\in\omega} \in (\mathcal{U}^{\mathcal{U}})^{\omega}$ is a code for an automorphism $\widehat{\Phi}_m$ of $C^*(\gamma)$.
(2) $\Phi_{0,n}(m) = m$ for every $n, m \in \omega$.
(3) The function $m \mapsto \widehat{\Phi}_m$ is an action of G on $C^*(\gamma)$, this is, $\widehat{\Phi}_m \circ \widehat{\Phi}_k = \widehat{\Phi}_n$ whenever $(m, k, n) \in G$.

It is easy to verify that any action of G on $C^*(\gamma)$ can be coded in such way. Moreover, the set $\mathrm{Act}_{\Gamma(H)}$ of triples $(G, \gamma, \Phi) \in \mathcal{G} \times \Gamma(H) \times (\mathcal{U}^{\omega})^{\omega\times\omega}$, such that Φ is a code for an action of G on $C^*(\gamma)$, is a Borel subset of $\mathcal{G}\times\Gamma(H)\times(\mathcal{U}^{\omega})^{\omega\times\omega}$. We will regard $\mathrm{Act}_{\Gamma(H)}$ as the standard Borel space of actions of discrete groups on C*-algebras. From the fact that the reduced crossed product can be computed using a single faithful representation of the given algebra [36, Thm. 7.7.5], it is easy to deduce that crossed products are Borel-computable. Precisely, there is a Borel map $(G, \gamma, \Phi) \mapsto \delta_{(G,\gamma,\Phi)}$ from $\mathrm{Act}_{\Gamma(H)}$ to $\Gamma(H)$ such that $C^*(\delta_{(G,\gamma,\Phi)})$ is isomorphic to $C^*(\gamma) \rtimes^{r}_{\widehat{\Phi}} G$. In particular, this shows that crossed products by single automorphisms—which correspond to actions of $\mathbb{Z}$—are Borel-computable.

One can deduce from this that also the computation of crossed products by injective corner endomorphisms [35, 7] is witnessed by a Borel function. This can be easily deduced from Stacey's explicit construction of such a crossed product from [42, Prop. 3.3].

3. Borel selection of AF-algebras

3.1. **Bratteli diagrams, dimension groups and AF-algebras.** We refer the reader to [12] for the standard definition of a Bratteli diagram. We will identify Bratteli diagrams with elements

$$(l,(w_n)_{n\in\omega},(m_n)_{n\in\omega})\in\omega^\omega\times(\omega^\omega)^\omega\times(\omega^{\omega\times\omega})^\omega$$

such that for every $i,j,n,m\in\omega$, the following conditions hold:

(1) $l(0)=1$.
(2) $w_0(0)=1$.
(3) $w_n(i)>0$ if and only if $i\in l(n)$.
(4) $m_n(i,j)=0$ whenever $i\geq l(n)$ or $j\geq l(n+1)$.
(5) Setting $k_n=i$,

$$w_n(i)=\sum_{(k_j)_{j\in n}\in l(j)^n}\prod_{t\in n}m_t(k_t,k_{t+1}).$$

We denote by $\mathcal{BD}$ the Borel set of all elements (l,w,m) in $\omega^\omega\times(\omega^\omega)^\omega\times(\omega^{\omega\times\omega})^\omega$ that satisfy conditions (1)–(5) above. An element (l,w,m) of $\mathcal{BD}$ codes the Bratteli diagram with $l(n)$ vertices at the n-th level of weight

$$w_n(0),\dots,w(l(n)-1)$$

and with $m_n(i,j)$ arrows from the i-th vertex at the n-th level to the j-th vertex and the $(n+1)$-st level for $n\in\omega$, $i\in l(n)$, and $j\in l(n+1)$. We call the elements of $\mathcal{BD}$ simply "Bratteli diagrams".

We refer the reader to [40, §1.4] for a complete exposition on dimension groups. A dimension group can be coded in a natural way as an element of $\omega^{\omega\times\omega}\times 2^\omega\times\omega$. The set $\mathcal{DG}$ of codes for dimension groups is a Borel subset of $\omega^{\omega\times\omega}\times 2^\omega\times\omega$, which can be regarded as the standard Borel space of dimension groups.

One can associate to a Bratteli diagram (l,w,m) the dimension group $G_{(l,w,m)}$ obtained as follows. For n in ω, denote by

$$\varphi_n\colon\mathbb{Z}^{l(n)}\to\mathbb{Z}^{l(n+1)}$$

the homomorphism given on the canonical bases of $\mathbb{Z}^{l(n)}$ by

$$\varphi_n(e_k^{(l(n))})=\sum_{i\in l(n+1)}m_n(i,j)e_j^{(l(n+1))},$$

for all k in $l(n)$. Then $G_{(l,w,m)}$ is defined as the inductive limit of the inductive system

$$\left(\mathbb{Z}^{l(n)},(w_n(0),\dots,w_n(l(n)-1)),\varphi_n\right)_{n\in\omega}.$$

Theorem 2.2 in [10] asserts that any dimension group is in fact isomorphic to one of the form $G_{(l,w,m)}$ for some Bratteli diagram (l,w,m). The key ingredient in the proof of [10, Thm. 2.2] is a lemma due to Shen, see [10, Lem. 2.1] and also [41, Thm. 3.1]. An inspection of its proof (or the application of standard selection theorems from descriptive set theory) shows that Shen's lemma is witnessed by a Borel function. This can be used to conclude that there is a

Borel function that associates to a dimension group $G = (G, G^+, u) \in \mathcal{DG}$ a Bratteli diagram $(l^G, w^G, m^G) \in \mathcal{BD}$ such that the dimension group associated with (l^G, w^G, m^G) is isomorphic to G.

Let (l, w, m) be a Bratteli diagram. We will describe how to canonically associate to it a unital AF-algebra, which we will denote by $A_{(l,w,m)}$. For each n in ω, define a finite-dimensional C*-algebra F_n by $F_n = \bigoplus_{i\in l(n)} \mathbb{M}_{w_n(i)}$. Denote by $\varphi_n \colon F_n \to F_{n+1}$ the unital injective *-homomorphism determined as follows. For every $i \in l(n)$ and $j \in l(n+1)$, the restriction of φ_n to the i-th direct summand of F_n and the j-th direct summand of F_{n+1} is a diagonal embedding of $m_n(i,j)$ copies of $\mathbb{M}_{w_n(i)}$ in $\mathbb{M}_{w_{n+1}(j)}$. Then $A_{(l,w,m)}$ is the inductive limit of the inductive system $(F_n, \varphi_n)_{n\in\omega}$. The K_0-group of $A_{(l,w,m)}$ is isomorphic to the dimension group $G_{l,w,m}$ associated with (l, w, m). The main result of [3] asserts that any unital AF-algebra is isomorphic to the C*-algebra associated with a Bratteli diagram. It is not difficult to see—working for instance in the parametrization $\Gamma(H)$—that the code for such an AF-algebra can be computed in a Borel way. This together with our remarks above about Bratteli diagrams shows that there is a Borel map that assigns to a dimension group D a unital AF-algebra A_D whose ordered K_0-group is isomorphic to D as dimension group. Since by [13, Prop. 3.4] the K_0-group of a C*-algebra can be computed in a Borel way, one can conclude that if $\mathcal{A}$ is any Borel set of dimension groups, then the relation of isomorphisms restricted to $\mathcal{A}$ is Borel bireducible with the relation of isomorphism unital AF-algebras whose K_0-group is isomorphic to an element of $\mathcal{A}$.

Fix $n \in \mathbb{N}$. A dimension group has *rank* n if n is the largest size of a linearly independent subset. Let us denote by $\cong_n^+$ the relation of isomorphism of dimension groups of rank n, and by $\cong_n^{\mathrm{AF}}$ the relation of isomorphism of AF-algebras whose dimension group has rank n. By the previous discussion and the fact that the computation of the K_0-group is given by a Borel function [13, Cor. 3.7], the relations $\cong_n^+$ and $\cong_n^{\mathrm{AF}}$ are Borel bireducible. Moreover, Theorem 1.11 of [12] asserts that $\cong_n^+ <_B \cong_{n+1}^+$ for every $n \in \mathbb{N}$. This means that $\cong_n^+$ is Borel reducible to $\cong_{n+1}^+$, but $\cong_{n+1}^+$ is *not* Borel reducible to $\cong_n^+$. It follows that the same conclusions hold for the relations $\cong_n^{\mathrm{AF}}$: For every $n \in \mathbb{N}$, we have $\cong_n^{\mathrm{AF}} <_B \cong_{n+1}^{\mathrm{AF}}$. This amounts to saying that it is strictly more difficult to classify AF-algebras with K_0-group of rank $n+1$ than classifying AF-algebras with K_0-group of rank n.

3.2. **Endomorphism of Bratteli diagrams.**

Definition 3.3. Let $T = (l, w, m)$ be a Bratteli diagram. We say that an element $q = (q_n)_{n\in\omega} \in (\omega^{\omega\times\omega})^\omega$ is an *endomorphism* of T, if for every $n \in \omega$, $i \in l(n)$ and $t' \in l(n+1)$, the following identity holds:

$$\sum_{t\in l(n+1)} m_n(i,t) q_{n+1}(t,t') = \sum_{t\in l(n+1)} q_n(i,t) m_{n+1}(t,t').$$

The set $\mathrm{End}_{\mathcal{BD}}$ of pairs $(T, q) \in \mathcal{BD} \times (\omega^{\omega\times\omega})^\omega$, such that T is a Bratteli diagram and q is an endomorphism of T, is Borel.

We proceed to describe how an endomorphism of a Bratteli diagram, in the sense of the definition above, gives rise to an endomorphism of the unital AF-algebra associated with it. Let $(F_n, \varphi_n)_{n\in\omega}$ be the inductive system of finite-dimensional C*-algebras associated with T, and denote by A_T its inductive limit. By repeatedly applying [8, Lem. III.2.1], one can define unital *-homomorphisms $\psi_n\colon F_n \to F_{n+1}$ for n in ω, satisfying the following conditions:

(1) ψ_n is unitarily equivalent to the *-homomorphism from F_n to F_{n+1} such that for every $i \in l(n)$ and $j \in l(n+1)$ the restriction of ψ_n to the i-th direct summand of F_n and the j-th direct summand of F_{n+1} is a diagonal embedding of $q_n(i,j)$ copies of $\mathbb{M}_{w_n(i)}$ in $\mathbb{M}_{w_{n+1}(j)}$.
(2) $\psi_n \circ \varphi_{n-1} = \varphi_n \circ \psi_{n-1}$ whenever $n \geq 1$.

(Notice in particular that ψ_0 is determined solely by condition (1).) One thus obtains a one-sided intertwining $(\psi_n)_{n\in\omega}$ from $(F_n, \varphi_n)_{n\in\omega}$ to itself. We denote by $\psi_{T,q}\colon A_T \to A_T$ the corresponding inductive limit endomorphism. Working in the parametrization $\Gamma(H)$ of C*-algebras and *-homomorphisms and using [14, Prop. 3.5], one can show that, given a Bratteli diagram T and an endomorphism q of T, there is a Borel way to compute a code for the endomorphism $\psi_{T,q}$ of A_T associated with q.

3.4. **Endomorphisms of dimension groups.** Let (G, G^+, u) be a dimension group. Let us denote by $\mathrm{End}_{\mathcal{DG}}$ the set of pairs $(G,\varphi) \in \mathcal{DG} \times \omega^\omega$ such that G is a dimension group and φ is an endomorphism of G.

Let (l, w, m) be a Bratteli diagram, and let

$$\big(\mathbb{Z}^{l(n)}, (w_n(0), \dots, w_n(l(n)-1)), \varphi_n\big)_{n\in\omega}$$

be the inductive system of dimension groups whose inductive limit is the dimension group $G_{l,w,m}$ associated with (l,w,m). Fix an endomorphism q of (l,w,m), and for $n \in \omega$, define a positive homomorphism $\psi_n\colon \mathbb{Z}^{l(n)} \to \mathbb{Z}^{l(n+1)}$ by

$$\psi_n(e_i^{(l(n))}) = \sum_{j\in l(n+1)} q_n(i,j) e_j^{(l(n+1))}.$$

Observe that the sequence $(\psi_n)_{n\in\omega}$ induces an endomorphism $\varphi_{((l,w,m),q)}$ of the inductive limit $G_{(l,w,m)}$. A routine verification shows that there is a Borel map

$$\mathrm{End}_{\mathcal{DG}} \to \mathrm{End}_{\mathcal{BD}}, \quad (G,\varphi) \mapsto (T^G, q^{G,\varphi}),$$

such that the dimension group associated with T^G is isomorphic to G, and the endomorphism of the dimension group associated with T^G corresponding to $q^{G,\varphi}$ is conjugate to φ. Using this and the remarks from Sections 3.1 and 3.4, one can conclude that there is a Borel map that assigns to a dimension group G with a distinguished endomorphism φ, a code for a unital AF-algebra A and a code for an endomorphism ρ of A, such that the K_0-group of A is isomorphic to G as dimension groups with order units, and the endomorphism of the K_0-group of A corresponding to ρ is conjugate to φ.

4. Conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$

4.1. **Background on C*-algebras.** Recall that the set $\mathrm{Aut}(A)$ of all automorphisms of a C*-algebra A is a Polish group under composition when endowed with the topology of pointwise norm convergence.

Definition 4.2. Let G be a discrete group and let A be a C*-algebra. An *action* α of G on A is a group homomorphism $\alpha\colon G \to \mathrm{Aut}(A)$. Two actions α and β of G on A are said to be *conjugate* if there is $\gamma \in \mathrm{Aut}(A)$ such that $\gamma \circ \alpha_g \circ \gamma^{-1} = \beta_g$ for every $g \in G$.

Let A be a unital C*-algebra and let α be an action of G on A. An *α-cocycle* is a function $u\colon G \to U(A)$ satisfying $u_{gh} = u_g\alpha_g(u_h)$ for every $g, h \in G$. If u is an α-cocycle, we define the *u-perturbation* of α, denoted $\alpha^u\colon G \to \mathrm{Aut}(A)$, by $\alpha^u_g = \mathrm{Ad}(u_g) \circ \alpha_g$ for g in G.

Definition 4.3. Two actions α and β of G on A are said to be *cocycle conjugate* if β is conjugate to a perturbation of α by a cocycle.

In the case when G is the group of integers $\mathbb{Z}$, actions of $\mathbb{Z}$ on A naturally correspond to single automorphisms of A. Similarly, if G is the group $\mathbb{Z}_n$, then actions of $\mathbb{Z}_n$ on A correspond to automorphisms of A whose order divides n. We show in Lemma 4.4 below that the notions of conjugacy and cocycle conjugacy for actions and automorphisms are respected by this correspondence when A has trivial center. These observations will be used to infer Corollary 4.26 from Corollary 4.25.

Lemma 4.4. *Let α and β be automorphisms of a unital C*-algebra A.*

(1) *The following statements are equivalent:*
   (a) *The actions $n \mapsto \alpha^n$ and $n \mapsto \beta^n$ of $\mathbb{Z}$ on A are cocycle conjugate.*
   (b) *There are an automorphism γ of A and a unitary u of A such that $\mathrm{Ad}(u) \circ \alpha = \gamma \circ \beta \circ \gamma^{-1}$.*

(2) *Assume moreover that α and β have order $k \geq 2$ and that A has trivial center (for example, if A is simple). Then the following statements are equivalent:*
   (a) *The actions $n \mapsto \alpha^n$ and $n \mapsto \beta^n$ of $\mathbb{Z}_k$ on A are cocycle conjugate.*
   (b) *The actions $n \mapsto \alpha^n$ and $n \mapsto \beta^n$ of $\mathbb{Z}$ on A are cocycle conjugate.*

Proof. (1). To show that (a) implies (b), simply take the unitary $u = u_1$ coming from the α-cocycle $u\colon \mathbb{Z} \to U(A)$. Conversely, if u is a unitary in A as in the statement, we define an α-cocycle as follows. Set $u_0 = 1$ and $u_1 = u$, and for $n \geq 2$ define u_n inductively by $u_n = u_1\alpha(u_{n-1})$. Set $u_{-1} = \alpha^{-1}(u_1^*)$, and for $n \leq -2$, define u_n inductively by $u_n = u_{-1}\alpha^{-1}(u_{n+1})$. It is straightforward to check that $n \mapsto u_n$ is an α-cocycle, and that the automorphism γ in the statement implements the conjugacy between α^u and β.

(2). To show that (a) implies (b), it is enough to note that if $u\colon \mathbb{Z}_k \to U(A)$ is an α-cocycle, when we regard α as a $\mathbb{Z}_k$ action, then the sequence $(v_m)_{m\in\mathbb{N}}$ of unitaries in A given by $v_m = u_n$ if $m = n \bmod k$, is an α-cocycle, when we regard α as a $\mathbb{Z}$ action. Assume that α and β are cocycle conjugate as automorphisms of A. Let $(u_n)_{n\in\mathbb{N}}$ be an α-cocycle and let γ be an automorphism implementing the conjugacy. Fix n in $\mathbb{N}$, and write $n = km + r$ for uniquely determined $k \in \mathbb{Z}$ and $r \in k$. Since α and β have order k, we have

$$\begin{aligned}\mathrm{Ad}(u_{km+r}) \circ \alpha^r &= \mathrm{Ad}(u_{km+r}) \circ \alpha^{km+r}\\ &= \gamma \circ \beta^{km+r} \circ \gamma^{-1}\\ &= \gamma \circ \beta^r \circ \gamma^{-1}\\ &= \mathrm{Ad}(u_r) \circ \alpha^r.\end{aligned}$$

In particular, $\mathrm{Ad}(u_{n+mk}) = \mathrm{Ad}(u_n)$, so u_{n+mk} and u_n differ by a central unitary. Since the center of A is trivial, upon correcting by a scalar, we may assume that $u_{n+mk} = u_n$. Thus, the assignment $v\colon \mathbb{Z}_k \to U(A)$ given by $n \mapsto u_n$ is an α-cocycle, when we regard α as a $\mathbb{Z}_k$ action, and γ implements an conjugacy between the $\mathbb{Z}_k$ actions α^v and β. This finishes the proof. ☐

4.5. **Strongly self-absorbing C*-algebras.** We refer the reader to [43] for the definition and basic results concerning strongly self-absorbing C*-algebras. The particular case of Theorem 4.6 when $\mathcal{D}$ is the Jiang–Su algebra $\mathcal{Z}$ has been proved in [13, Thm. A.1].

Theorem 4.6. *Let $\mathcal{D}$ be a strongly self-absorbing C*-algebra. Then the set of $\gamma \in \Gamma(H)$, such that $C^*(\gamma)$ is unital and $\mathcal{D}$-absorbing, is Borel.*

Proof. By [14, Lem. 3.14], the set $\Gamma_u(H)$ of $\gamma \in \Gamma(H)$, such that $C^*(\gamma)$ is unital, is Borel. Moreover, there is a Borel function $Un\colon \Gamma_u(H) \to B(H)$ such that $Un(\gamma)$ is the unit of $C^*(\gamma)$ for every $\gamma \in \Gamma_u(H)$. Denote as in Section 2.4 by $\mathcal{U}$ the $\mathbb{Q}(i)$-$*$-algebra of polynomials with coefficients in $\mathbb{Q}(i)$ and without constant term in the formal variables X_k for $k \in \omega$. Let $\{d_n \mid n \in \omega\}$ be an enumeration of a dense subset of $\mathcal{D}$ such that $d_0 = 1$, and let $\{p_n \mid n \in \omega\}$ be an enumeration of $\mathcal{U}$. By [43, Thm. 2.2] or [40, Thm. 7.2.2], a unital C*-algebra A is $\mathcal{D}$-absorbing if and only if for every $n, m \in \mathbb{N}$ and every finite subset F of A, there are $a_0, a_1, \ldots, a_n \in A$ such that

- a_0 is the unit of A,
- $\|xa_i - a_ix\| < \frac{1}{m}$ for every $i \in n$ and $x \in F$, and
- $\|p_i(a_0, \ldots, a_n) - p_i(d_0, \ldots, d_n)\| < \frac{1}{m}$ for every $i \in m$.

Let $\gamma \in \Gamma(H)$ be such that $C^*(\gamma)$ is unital. Then $C^*(\gamma)$ is $\mathcal{D}$-absorbing if and only if for every $n, m \in \mathbb{N}$ there are $k_1, \ldots, k_n \in \omega$ such that

- $\|\gamma_i\gamma_{k_j} - \gamma_{k_j}\gamma_i\| < \frac{1}{m}$ for $i \in m$ and $1 \leq j \leq n$,
- $\|p_i(Un(\gamma), \gamma_{k_1}, \ldots, \gamma_{k_n}) - p_i(d_0, \ldots, d_n)\| < \frac{1}{m}$ for every $i \in m$.

This shows that the set of $\gamma \in \Gamma(H)$, such that $C^*(\gamma)$ is unital and $\mathcal{D}$-absorbing, is Borel. ☐

4.7. **Borel spaces of Kirchberg algebras.** Recall that a C*-algebra A is said to be a *Kirchberg algebra* if it is purely infinite, simple, nuclear and separable. We will denote by $\Gamma_{\mathrm{uKir}}(H)$ the set of $\gamma \in \Gamma(H)$ such that $C^*(\gamma)$ is a *unital* Kirchberg algebra.

Proposition 4.8. *The set* $\Gamma_{\mathrm{uKir}}(H)$ *is Borel.*

Proof. Corollary 7.5 of [14] asserts that the set $\Gamma_{\mathrm{uns}}(H)$ of $\gamma \in \Gamma(H)$, such that $C^*(\gamma)$ is unital, nuclear, and simple, is Borel. The result then follows from this fact together with Theorem 4.6. □

Definition 4.9. Fix a projection p in $\mathcal{O}_\infty$ such that $[p] = 0$ in $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$. Define the *standard Cuntz algebra* $\mathcal{O}_\infty^{\mathrm{st}}$ to be the corner $p\mathcal{O}_\infty p$.

The C*-algebra $\mathcal{O}_\infty^{\mathrm{st}}$ is the (unique) unital Kirchberg algebra that satisfies the Universal Coefficient Theorem (UCT), with K-theory given by

$$(K_0(\mathcal{O}_\infty^{\mathrm{st}}), [1_{\mathcal{O}_\infty^{\mathrm{st}}}], K_1(\mathcal{O}_\infty^{\mathrm{st}})) \cong (\mathbb{Z}, 0, 0).$$

In particular, a different choice of the projection p in Definition 4.9 (as long as its class on K-theory is 0) would yield an isomorphic C*-algebra.

We point out that, even though there is an isomorphism $\mathcal{O}_\infty^{\mathrm{st}} \otimes \mathcal{O}_\infty^{\mathrm{st}} \cong \mathcal{O}_\infty^{\mathrm{st}}$ (see comments on [23, p. 262]), the C*-algebra $\mathcal{O}_\infty^{\mathrm{st}}$ is not strongly self-absorbing. Indeed, if $\mathcal{D}$ is a strongly self-absorbing C*-algebra, then the infinite tensor product $\bigotimes_{n=1}^\infty \mathcal{D}$ of $\mathcal{D}$ with itself is isomorphic to $\mathcal{D}$. However, $\bigotimes_{n=1}^\infty \mathcal{O}_\infty^{\mathrm{st}}$ is isomorphic to $\mathcal{O}_2$, and thus $\mathcal{O}_\infty^{\mathrm{st}}$ is not strongly self-absorbing.

We proceed to give a K-theoretic characterization of those unital Kirchberg algebras that absorb $\mathcal{O}_\infty^{\mathrm{st}}$. Our characterization will be used to show that the set of all $\mathcal{O}_\infty^{\mathrm{st}}$-absorbing unital Kirchberg algebras is Borel.

Lemma 4.10. *Let A be a unital Kirchberg algebra. Then the following statements are equivalent:*

(1) *A is $\mathcal{O}_\infty^{\mathrm{st}}$-absorbing.*
(2) *The class $[1_A]$ of the unit of A in $K_0(A)$ is zero.*

Proof. We first show that (1) implies (2). Since $\mathcal{O}_\infty^{\mathrm{st}}$ satisfies the UCT, the Künneth formula applied to $A \otimes \mathcal{O}_\infty^{\mathrm{st}}$ gives

$$K_0(A \otimes \mathcal{O}_\infty^{\mathrm{st}}) \cong K_0(A) \quad \text{and} \quad K_1(A \otimes \mathcal{O}_\infty^{\mathrm{st}}) \cong K_1(A),$$

with $[1_{A\otimes\mathcal{O}_\infty^{\mathrm{st}}}] = 0$ as an element in $K_0(A)$. The claim follows since any isomorphism $A \otimes \mathcal{O}_\infty^{\mathrm{st}} \cong A$ must map the unit of $A \otimes \mathcal{O}_\infty^{\mathrm{st}}$ to the unit of A.

Let us now show that (2) implies (1). Fix a nonzero projection p in $\mathcal{O}_\infty$ such that $[p] = 0$ as an element of $K_0(\mathcal{O}_\infty)$. Then $1_A \otimes 1_{\mathcal{O}_\infty}$ represents the zero element in $K_0(A \otimes \mathcal{O}_\infty)$. Likewise, $1_A \otimes p$ also represents the zero element in $K_0(A \otimes \mathcal{O}_\infty)$. Since any two nonzero projections in a Kirchberg algebra are Murray–von Neumann equivalent if and only if they determine the same class in K-theory, it follows that there is an isometry v in $A \otimes \mathcal{O}_\infty$ such that

$vv^* = 1_A \otimes p$. The universal property of the algebraic tensor product yields a linear map

$$\varphi_0 \colon A \odot \mathcal{O}_\infty \to (1_A \otimes p)(A \otimes \mathcal{O}_\infty)(1_A \otimes p) \cong A \otimes \mathcal{O}_\infty^{\mathrm{st}}$$

with $\varphi_0(a \otimes b) = v(a \otimes b)v^*$ for a in A and b in $\mathcal{O}_\infty$. It is straight-forward to check that φ_0 extends to a *-homomorphism $\varphi \colon A \otimes \mathcal{O}_\infty \to A \otimes \mathcal{O}_\infty^{\mathrm{st}}$. We claim that φ is an isomorphism. For this, it is enough to check that the *-homomorphism

$$\psi \colon (1_A \otimes p)(A \otimes \mathcal{O}_\infty)(1_A \otimes p) \to A \otimes \mathcal{O}_\infty$$

given by $\psi(x) = v^*xv$ for all x in $(1_A \otimes p)(A \otimes \mathcal{O}_\infty)(1_A \otimes p)$ is an inverse for φ. This is immediate since for all x in $(1_A \otimes p)(A \otimes \mathcal{O}_\infty)(1_A \otimes p)$ we have $(1_A \otimes p)x(1_A \otimes p) = x$.

Once we have $A \otimes \mathcal{O}_\infty^{\mathrm{st}} \cong A \otimes \mathcal{O}_\infty$, the result follows from the fact that there is an isomorphism $A \cong A \otimes \mathcal{O}_\infty$ by Kirchberg's $\mathcal{O}_\infty$-isomorphism theorem. This finishes the proof of the lemma. □

Corollary 4.11. *The set of all $\gamma \in \Gamma(H)$, such that $C^*(\gamma)$ is an $\mathcal{O}_\infty^{\mathrm{st}}$-absorbing unital Kirchberg algebra, is Borel.*

Proof. This follows from Lemma 4.10, together with the fact that the K-theory of a C*-algebra and the class of its unit in K_0 can be computed in a Borel fashion; see [13, §3.3]. □

4.12. **Automorphisms of $\mathcal{O}_2$.** Denote by $\mathrm{Aut}(\mathcal{O}_2)$ the Polish group of automorphisms of $\mathcal{O}_2$ with respect to the topology of pointwise convergence. Given a positive integer n, the closed subspace $\mathrm{Aut}_2(\mathcal{O}_2)$ of automorphisms of $\mathcal{O}_2$ of order 2 can be identified with the space of actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$.

Definition 4.13. An action α of $\mathbb{Z}_2$ on $\mathcal{O}_2$ is said to be *approximately representable* if for every $\varepsilon > 0$ and for every finite subset F of $\mathcal{O}_2$, there exists a unitary u of $\mathcal{O}_2$ such that

(1) $\|u^2 - 1\| < \varepsilon$,
(2) $\|\alpha(u) - u\| < \varepsilon$, and
(3) $\|\alpha(a) - uau^*\| < \varepsilon$ for every $a \in F$.

It is clear that the set of approximately representable automorphisms of order 2 of $\mathcal{O}_2$ is a G_δ subset of $\mathrm{Aut}_2(\mathcal{O}_2)$.

We now recall a construction of a *model action* of $\mathbb{Z}_2$ on $\mathcal{O}_2$ from [23, p. 262]. Fix a projection e of $\mathcal{O}_\infty^{\mathrm{st}}$ such that $[e]$ is a generator of $K_0(\mathcal{O}_\infty^{\mathrm{st}})$ and let u be the order 2 unitary $u = 2e - 1$ of $\mathcal{O}_\infty^{\mathrm{st}}$. Identifying $\mathcal{O}_2$ with the infinite tensor product $\bigotimes_{n\in\omega} \mathcal{O}_\infty^{\mathrm{st}}$, one can define the approximately representable action $\nu = \bigotimes_{n\in\omega} \mathrm{Ad}(u)$ of $\mathbb{Z}_2$ on $\mathcal{O}_2$. Lemma 4.7 of [23] asserts that the crossed product $\mathcal{O}_2 \rtimes_\nu \mathbb{Z}_2$ is isomorphic to $\mathcal{O}_\infty^{\mathrm{st}} \otimes \mathbb{M}_{2^\infty}$, which from now on we will denote by D_2.

For a simple nuclear unital C*-algebra A, denote by $\widetilde{\alpha}_A$ the automorphism of $A \otimes \mathcal{O}_2$ defined by $\mathrm{id}_A \otimes \nu$. Using Kirchberg's $\mathcal{O}_2$-isomorphism theorem [28, Thm. 3.8], we fix an isomorphism $\varphi \colon A \otimes \mathcal{O}_2 \to \mathcal{O}_2$. Denote by α_A the automorphism of $\mathcal{O}_2$ given by $\alpha_A = \varphi \circ \widetilde{\alpha}_A \circ \varphi^{-1}$. It is immediate to check that α_A is approximately representable.

Remark 4.14. If A is a simple nuclear unital C*-algebra, then

$$\mathcal{O}_2 \rtimes_{\alpha_A} \mathbb{Z}_2 \cong A \otimes D_2.$$

Indeed,

$$\mathcal{O}_2 \rtimes_{\alpha_A} \mathbb{Z}_2 \cong (A \otimes \mathcal{O}_2) \rtimes_{\mathrm{id}_A \otimes \nu} \mathbb{Z}_2 \cong A \otimes (\mathcal{O}_2 \rtimes_\nu \mathbb{Z}_2) \cong A \otimes D_2.$$

Proposition 4.15. *Let A and B be simple nuclear unital C*-algebras. The following statements are in decreasing order of strength:*

(1) *A and B are isomorphic.*
(2) *The actions α_A and α_B are conjugate.*
(3) *The actions α_A and α_B are cocycle conjugate.*
(4) *The crossed products $\mathcal{O}_2 \rtimes_{\alpha_A} \mathbb{Z}_2$ and $\mathcal{O}_2 \rtimes_{\alpha_B} \mathbb{Z}_2$ are isomorphic.*
(5) *$A \otimes D_2$ and $B \otimes D_2$ are isomorphic.*

In particular, if A and B are D_2-absorbing unital Kirchberg algebras, then all the statements above are equivalent.

Proof. If $\psi\colon A \to B$ is an isomorphism, then $\psi \otimes \mathrm{id}_{\mathcal{O}_2}\colon A \otimes \mathcal{O}_2 \to B \otimes \mathcal{O}_2$ conjugates $\mathrm{id}_A \otimes \nu_p$ and $\mathrm{id}_B \otimes \nu_p$, and hence $\alpha_{A,p}$ and $\alpha_{B,p}$ are conjugate. This shows that (1) implies (2). It is well known that (2) implies (3) and that (3) implies (4). Remark 4.14 shows that (4) implies (5). ☐

We refer the reader to [20, Def. 1.1] for the definition of the Rokhlin dimension of a finite group action on a unital C*-algebra. We remark that an action has Rokhlin dimension 0 if and only if it has the Rokhlin property in the sense of [23, Def. 3.1].

Proposition 4.16. *Let A be a D_2-absorbing unital Kirchberg algebra not isomorphic to $\mathcal{O}_2$. Then α_A has Rokhlin dimension* 1.

Proof. Since ν is outer, so is α_A. It follows from [18, Thm. 4.19] that α_A has Rokhlin dimension at most 1. Assume now by contradiction that α_A has the Rokhlin property. Corollary 3.4 in [19] asserts that the crossed product $\mathcal{O}_2 \rtimes_{\alpha_A} \mathbb{Z}_2$ is isomorphic to $\mathcal{O}_2$. At the same time $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2$ is isomorphic to A by Remark 4.14. Therefore $A \cong \mathcal{O}_2$, against our assumption. ☐

4.17. **Isomorphism of p-divisible torsion-free abelian groups.**

Definition 4.18. Let G be an abelian group and let n be a positive integer.

(1) We say that G is *n-divisible*, if for every x in G there exists y in G such that $x = ny$.
(2) We say that G is *uniquely n-divisible*, if for every x in G there exists a unique y in G such that $x = ny$.

Given a set S of positive integers, we say that G is (uniquely) S-divisible, if G is (uniquely) n-divisible for every n in S.

It is clear that if n is a positive integer, then any n-divisible torsion-free abelian group is uniquely n-divisible.

It is easily checked that the following classes of abelian groups are Borel subsets of the standard Borel space of countable infinite groups $\mathcal{G}$:

- torsion free groups;
- n-divisible groups, for any positive integer n;
- uniquely n-divisible groups.

The main result of [22] asserts that if $\mathcal{C}$ is any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of isomorphisms of elements of $\mathcal{C}$ is Borel, then $\cong_{\mathcal{C}}$ is Borel reducible to the relation $\cong_{\mathrm{TFA}}$ of isomorphism of torsion-free abelian groups. Moreover, Theorem 1.1 of [9] asserts that $\cong_{\mathrm{TFA}}$ is a complete analytic set and, in particular, not Borel.

Proposition 4.19. *Suppose that $\mathcal{P}$ is a set of prime numbers which is co-infinite in the set of all primes. If $\mathcal{C}$ is any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of isomorphism of elements of $\mathcal{C}$ is Borel, then $\mathcal{C}$ is Borel reducible to the relation of isomorphism of torsion-free $\mathcal{P}$-divisible countable infinite groups. Moreover, the latter equivalence relation is a complete analytic set and, in particular, not Borel.*

Proof. A variant of the argument used in the proof of the main result of [22] can be used to prove the first assertion. Indeed, the only modification needed is in the definition of the *group eplag* associated with an *excellent prime labeled graph* as in [22, §2] (we refer to [22] for the definitions of these notions). Suppose that (V, E, f) is an excellent prime labeled graph such that the range of f is disjoint from $\mathcal{P}$. Denote by $\mathbb{Q}^{(V)}$ the direct sum $\mathbb{Q}^{(V)} = \bigoplus_{v \in V} \mathbb{Q}$ of copies of $\mathbb{Q}$ indexed by V, and identify an element v of V with the corresponding copy of $\mathbb{Q}$ in $\mathbb{Q}^{(V)}$. We define the $\mathcal{P}$-divisible group eplag $\mathcal{G}_{\mathcal{P}}(V, E, f)$ associated with (V, E, f) to be the subgroup of $\mathbb{Q}^{(V)}$ generated by

$$\left\{ \frac{v}{p^n f(v)^m}, \frac{v+w}{p^n f(\{v,w\})} \mid v \in V,\ \{v,w\} \in E,\ n,m \in \omega,\ p \in \mathcal{P} \right\}.$$

It is easy to check that $\mathcal{G}_{\mathcal{P}}(V, E, f)$ is indeed a torsion-free $\mathcal{P}$-divisible abelian group. The group eplag $\mathcal{G}(V, E, f)$ as defined in [22, §2] is the particular case of this definition with $\mathcal{P} = \varnothing$. The same argument as in [22], where

(1) the group eplag $\mathcal{G}(V, E, f)$ is replaced everywhere by $\mathcal{G}_{\mathcal{P}}(V, E, f)$,
(2) all the primes are chosen from the *complement* of $\mathcal{P}$,

gives a proof of the first claim of this proposition.

The second claim follows by modifying the argument in [9] and, in particular, the construction of the torsion-free abelian group associated with a tree on ω as in [9, Thm. 2.1]. Choose injective enumerations $(p_n)_{n \in \omega}$ and $(q_n)_{n \in \omega}$ of disjoint subsets of the complement of $\mathcal{P}$ in the set of all primes, and let T be a tree on ω. Define the excellent prime labeled graph (V_T, E_T, f_T) as follows. The graph (V_T, E_T) is just the tree T, and

$$f \colon V_T \cup E_T \to \{p_n, q_n \mid n \in \omega\}$$

is defined by

$$f(x) = \begin{cases} q_n & \text{if } x \text{ is a vertex in the } n\text{-th level of } T, \\ p_n & \text{if } x \text{ is an edge between the } n\text{-th and the } (n+1)\text{-st level} of \mathrm{T}. \end{cases}$$

Define the $\mathcal{P}$-divisible torsion-free abelian group $G_{\mathcal{P}}(T)$ to be the group eplag $\mathcal{G}_{\mathcal{P}}(V_T, E_T, f_T)$. The same proof as that of [9, Thm. 2.1] shows the following facts: If T and T' are isomorphic trees, then the groups $G_{\mathcal{P}}(T)$ and $G_{\mathcal{P}}(T')$ are isomorphic. On the other hand, if T is well-founded and T' is ill-founded, then $G_{\mathcal{P}}(T)$ and $G_{\mathcal{P}}(T')$ are not isomorphic. The second claim of this proposition can now be proved as [9, Thm. 1.1]. □

4.20. **Constructing Kirchberg algebras with a given K_0-group.** The following is the main result of this subsection.

Theorem 4.21. *There is a Borel map from the Borel space $\mathcal{G}$ of countable infinite groups to the Borel space $\Gamma_{\mathrm{uKir}}(H)$ parametrizing unital Kirchberg algebras, which assigns to every infinite countable abelian group G, a code γ for a unital Kirchberg algebra $C^*(\gamma)$ that satisfies the UCT, and with K-theory given by*

$$\big(K_0(C^*(\gamma)), [1_{C^*(\gamma)}], K_1(C^*(\gamma))\big) \cong (G, 0, \{0\}).$$

Moreover, $C^*(\gamma)$ is D_2-absorbing if and only if G is uniquely 2-divisible.

Proof. An inspection of the proof of [39, Prop. 3.5] shows that one can choose, in a Borel way from G, a torsion-free abelian group H and an automorphism α of H such that

$$H/\mathrm{Im}(\mathrm{id}_H - \alpha) \cong G.$$

Denote by L the dimension group given by $L = \mathbb{Z}[\frac{1}{2}] \oplus H$ with positive cone

$$L^+ = \{(t,h) \in D \mid t > 0\} \cup \{(0,0)\},$$

and order unit $(1,0)$. Consider the endomorphism ρ of L defined by

$$\beta(t,h) = \Big(\frac{t}{2}, \alpha(H)\Big)$$

for (t,h) in L. It is clear that L and β can be computed in a Borel way from H and α. As remarked in Section 3.4, one can obtain in a Borel way from H and β a code for a unital AF-algebra B and a code for an injective corner endomorphism ρ of B such that the K_0-group of B is isomorphic to L, and the endomorphism of the K_0-group of B induced by ρ is conjugate to β. As observed in Section 2.10, one can obtain in a Borel way a code $\gamma_G \in \Gamma(H)$ for the crossed product $B \rtimes_\rho \mathbb{N}$ of B by the endomorphism ρ. It can be shown, as in the proof of [39, Thm. 3.6], that $C^*(\gamma_G)$ is a unital Kirchberg algebra satisfying the UCT, with trivial K_1-group, K_0-group isomorphic to G, and $[1_{C^*(\gamma_G)}] = 0$ in $K_0(C^*(\gamma))$. An easy application of the Pimsner–Voiculescu exact sequence gives the computation of the K-theory. Pure infiniteness of $C^*(\gamma_G)$ is proved in [39, Thm. 3.1]. The map $\mathcal{G} \to \Gamma_{\mathrm{uKir}}(H)$ given by $G \mapsto \gamma_G$ is Borel by construction. □

4.22. **Nonclassification of automorphisms of $\mathcal{O}_2$ of order 2.** Since $\mathbb{M}_{2^\infty}$ is strongly self-absorbing, it follows from Theorem 4.6 and Corollary 4.11 that the set $\mathcal{D}$ of $\gamma \in \Gamma(H)$ such that $C^*(\gamma)$ is a unital D_2-absorbing Kirchberg C*-algebra not isomorphic to $\mathcal{O}_2$ is a Borel subset of $\Gamma(H)$. One can regard $\mathcal{D}$ as the standard Borel space parametrizing D_2-absorbing unital Kirchberg algebras not isomorphic to $\mathcal{O}_2$. Thus, the equivalence relation E on $\mathcal{D}$ defined by $\gamma E \gamma'$ if and only if $C^*(\gamma) \cong C^*(\gamma')$ can be identified with the relation of isomorphism of unital D_2-absorbing Kirchberg algebras not isomorphic to $\mathcal{O}_2$.

Theorem 4.23. *There are Borel reductions:*

(1) *From the relation of isomorphism of D_2-absorbing unital Kirchberg algebras not isomorphic to $\mathcal{O}_2$, to the relation of cocycle conjugacy of approximately representable actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$ that have Rokhlin dimension* 1.
(2) *From the relation of isomorphism of D_2-absorbing unital Kirchberg algebras, to the relation of conjugacy of approximately representable actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$ that have Rokhlin dimension* 1.

Proof. In view of Propositions 4.15, 4.16 and Elliott's theorem $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, it is enough to find a Borel function $\Gamma_{\mathrm{uKir}}(H) \to \mathrm{Aut}_2(\mathcal{O}_2 \otimes \mathcal{O}_2)$ that assigns to every $\gamma \in \Gamma_{\mathrm{uKir}}(H)$ an automorphism α_γ of $\mathcal{O}_2 \otimes \mathcal{O}_2$ which is conjugate to $\mathrm{id}_{C^*(\gamma)} \otimes \nu$.

We follow the notation of [14, §6.1], and denote by $SA(\mathcal{O}_2)$ the space of C*-subalgebras of $\mathcal{O}_2$. Then $SA(\mathcal{O}_2)$ is a Borel subset of the Effros Borel space of closed subsets of $\mathcal{O}_2$, as defined in [24, §12.C]. It follows from [14, Thm. 6.5] that the set $SA_{\mathrm{uKir}}(\mathcal{O}_2)$ of C*-subalgebras of $\mathcal{O}_2$ isomorphic to a unital Kirchberg algebra is Borel. Moreover, again by [14, Thm. 6.5], there is a Borel function $\Gamma_{\mathrm{uKir}}(\mathcal{O}_2) \to SA_{\mathrm{uKir}}(\mathcal{O}_2)$ that assigns to an element γ of $\Gamma_{\mathrm{uKir}}(\mathcal{O}_2)$ a subalgebra of $\mathcal{O}_2$ isomorphic to $C^*(\gamma)$. It is therefore enough to show that there is a Borel function $SA_{\mathrm{uKir}}(\mathcal{O}_2) \to \mathrm{Aut}_2(\mathcal{O}_2 \otimes \mathcal{O}_2)$ that assigns to $A \in SA_{\mathrm{uKir}}(\mathcal{O}_2)$ an automorphism α_A of $\mathcal{O}_2 \otimes \mathcal{O}_2$ conjugate to $\mathrm{id}_A \otimes \nu$.

Denote by $\mathrm{End}(\mathcal{O}_2 \otimes \mathcal{O}_2)$ the space of endomorphism of $\mathcal{O}_2 \otimes \mathcal{O}_2$. By [14, Thm. 7.6], there is a Borel map $SA_{\mathrm{uKir}}(\mathcal{O}_2) \to \mathrm{End}(\mathcal{O}_2 \otimes \mathcal{O}_2)$ that assigns to an element A in $SA_{\mathrm{uKir}}(\mathcal{O}_2)$ a unital injective endomorphism η_A of $\mathcal{O}_2 \otimes \mathcal{O}_2$ with range $A \otimes \mathcal{O}_2$. In particular, η_A is an isomorphism between $\mathcal{O}_2 \otimes \mathcal{O}_2$ and $A \otimes \mathcal{O}_2$. For A in $SA_{\mathrm{uKir}}(\mathcal{O}_2)$, define

$$\alpha_A = \eta_A^{-1} \circ (\mathrm{id}_A \otimes \nu) \circ \eta_A,$$

and note that the map $A \mapsto \alpha_A$ is Borel.

It is enough to show that for every $x, y \in \mathcal{O}_2$ and every $\varepsilon > 0$, the set of C*-algebras A in $SA_{\mathrm{uKir}}(\mathcal{O}_2)$, such that $\|\alpha_A(x) - y\| < \varepsilon$, is Borel. Fix x and y in $\mathcal{O}_2$. By [24, Thm. 12.13], there is a sequence $(a_n)_{n\in\omega}$ of Borel functions from $SA_{\mathrm{uKir}}(\mathcal{O}_2)$ to $\mathcal{O}_2$ such that for A in $SA_{\mathrm{uKir}}(\mathcal{O}_2)$, the set $\{a_n^A \mid n \in \omega\}$ is an enumeration of a dense subset of A.

Fix a countable dense subset $\{b_n \mid n \in \omega\}$ of $\mathcal{O}_2$. Then

$$\|\alpha_A(x) - y\| = \|(\mathrm{id}_A \otimes \nu)(\eta_A(x)) - \eta_A(y)\|,$$

and thus $\|\alpha_A(x) - y\| < \varepsilon$ if and only if there are positive integers $k \in \omega$ and $n_0, \ldots, n_{k-1}, m_0, \ldots, m_{k-1} \in \omega$, and scalars $\lambda_0, \ldots, \lambda_{k-1} \in \mathbb{Q}(i)$, such that

$$\Big\|\eta_A(x) - \sum_{i\in k} \lambda_i a^G_{n_i} \otimes b_{m_i}\Big\| < \frac{\varepsilon}{2} \quad \text{and} \quad \Big\|\sum_{i\in k} \lambda_i a^G_{n_i} \otimes \nu(b_{m_i}) - \eta_A(y)\Big\| < \frac{\varepsilon}{2}.$$

Since the map $A \mapsto \eta_A$ is Borel, it follows that the set of all C*-algebras A in $SA_{\mathrm{uKir}}(\mathcal{O}_2)$, such that $\|\alpha_A(x) - y\| < \varepsilon$, is Borel. The result follows. ☐

Theorem 4.24. *There are Borel reductions:*

(1) *From the relation of isomorphism of infinite countable abelian groups, to the relation of isomorphism of $\mathcal{O}_\infty^{\mathrm{st}}$-absorbing unital Kirchberg algebras satisfying the UCT with infinite K_0-group and with trivial K_1-group.*
(2) *From the relation of isomorphism of uniquely 2-divisible infinite countable abelian groups, to the relation of isomorphism of D_2-absorbing unital Kirchberg algebras satisfying the UCT with infinite K_0-group and with trivial K_1-group that are a crossed product of $\mathcal{O}_2$ by an action of $\mathbb{Z}_2$ of Rokhlin dimension 1.*

Proof. Both results follow from Remark 4.14, Proposition 4.16, Theorem 4.23, and Theorem 4.21, together with the Kirchberg–Phillips classification theorem (see [27] and [37]). ☐

Corollary 4.25. *Let $\mathcal{C}$ be any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of isomorphism of elements of $\mathcal{C}$ is Borel. Assume that F is any of the following equivalence relations:*

- *isomorphism of simple $\mathcal{O}_2 \rtimes \mathbb{Z}_2$ (with infinite K_0-group and trivial K_1-group);*
- *conjugacy of approximately representable actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$ that have Rokhlin dimension 1,*
- *cocycle conjugacy of approximately representable actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$ that have Rokhlin dimension 1.*

Then $\cong_{\mathcal{C}}$ is Borel reducible to F, and moreover F is complete analytic.

Corollary 4.26. *The relations of isomorphism of simple crossed products $\mathcal{O}_2 \rtimes \mathbb{Z}_2$ satisfying the UCT, conjugacy of automorphisms of $\mathcal{O}_2$, and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ are complete analytic sets and, in particular, not Borel.*

Recall that by [23, Thm. 3.5], if G is a finite group, then any two actions of G on $\mathcal{O}_2$ with the Rokhlin property are conjugate. On the other hand there are at the moment no classification results for actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$, even in the case of Rokhlin dimension 1. Corollary 4.25 shows that such classification problem is rather complicated.

4.27. **Nonclassification of automorphisms of $\mathcal{O}_2$ of order p.** Corollary 4.25 can be in fact generalized to automorphisms of order p for any prime number p. This is the content of the following theorem.

Theorem 4.28. *Fix a prime number p. Let $\mathcal{C}$ be any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of isomorphism of elements of $\mathcal{C}$ is Borel. Let F be any of the following equivalence relations:*

- *isomorphism of simple purely infinite crossed products $\mathcal{O}_2 \rtimes \mathbb{Z}_p$ (with infinite K_0-group and trivial K_1-group);*
- *conjugacy of approximately representable actions of $\mathbb{Z}_p$ on $\mathcal{O}_2$ that have Rokhlin dimension 1,*
- *cocycle conjugacy of approximately representable actions of $\mathbb{Z}_p$ on $\mathcal{O}_2$ that have Rokhlin dimension 1.*

Then $\cong_{\mathcal{C}}$ is Borel reducible to F. Moreover, F is a complete analytic set.

We explain here how to adapt the arguments in Sections 4.17, 4.20, and 4.22 to obtain Theorem 4.28. In the rest of this subsection we suppose that p is a fixed prime number.

An analog of the model action $\nu\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{O}_2)$, for actions of $\mathbb{Z}_p$, has been obtained in [1, Prop. 4.15], and we thank Selçuk Barlak and Gabor Szabo for calling our attention to it. We recall its construction in a form which is suitable for our purposes. Denote by D_p the (unique) unital Kirchberg algebra satisfying the UCT whose K-theory is given by

$$(K_0(D_p), [1_{D_p}], K_1(D_p)) \cong \Big(\mathbb{Z}\Big[\frac{1}{p}\Big] \oplus \cdots \oplus \mathbb{Z}\Big[\frac{1}{p}\Big], 0, \{0\}\Big),$$

where the nontrivial group on the right-hand side has $p-1$ direct summands. Note that when $p=2$, the algebra D_p is isomorphic to $\mathbb{M}_{2^\infty} \otimes \mathcal{O}_\infty^{\mathrm{st}}$.

Proposition 4.29. *Let p be a prime number. Then there exists an approximately representable action $\nu_p\colon \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{O}_2)$ such that $\mathcal{O}_2 \rtimes_{\nu_p} \mathbb{Z}_2 \cong D_p$.*

Proof. The proof is essentially the same as that of [1, Prop. 4.15]. Set ζ_p to be a primitive p-th root of unity. One may identify the group $K_0(D_p)$ with the additive group of the ring $\mathbb{Z}[\frac{1}{p}, \zeta_p]$. Under this identification, the automorphism of $K_0(D_p)$ determined by multiplication by ζ_p can be lifted to an action $\mu_p\colon \mathbb{Z}_p \to \mathrm{Aut}(D_p)$ with the Rokhlin property, by [1, Thm. 2.10]. The crossed product of such action is easily seen to be a Kirchberg algebra with trivial K-theory. Since it satisfies the UCT by [34, Prop. 3.7], it follows from the Kirchberg–Phillips classification theorem that

$$D_p \rtimes_{\mu_p} \mathbb{Z}_p \cong \mathcal{O}_2.$$

The desired action is then the dual action $\nu_p = \widehat{\mu}_p\colon \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{O}_2)$. □

If A is a simple nuclear unital C*-algebra, then one can define the action $\alpha_{A,p}$ of $\mathbb{Z}_p$ on A analogously as in Section 4.12, after replacing ν with ν_p. The proof of Proposition 4.16 goes through without changes when α_A is replaced with $\alpha_{A,p}$. Theorem 4.23 is then generalized after replacing D_2 with D_p and $\mathbb{Z}_2$ with $\mathbb{Z}_p$.

Let us say that a countable infinite group G is self-absorbing if $G \oplus G \cong G$. It follows from the Kirchberg–Phillips classification theorem that, if A is a unital

Kirchberg algebra with K-theory $(G, 0, \{0\})$ and G is a *self-absorbing* torsion-free p-divisible abelian group, then A is D_p-absorbing. Thus, Theorem 4.24 still holds after replacing D_2 with D_p, and uniquely 2-divisible infinite countable abelian groups with *self-absorbing* uniquely p-divisible infinite countable abelian groups.

Finally, one needs to modify the construction in Proposition 4.19 to obtain *self-absorbing* p-divisible torsion-free abelian groups. This is accomplished by considering the following modification in the definition of the group eplag associated with an excellent prime labeled graph. Suppose that (V, E, f) is an excellent prime labeled graph such that the range of f is disjoint from $\mathcal{P}$. Denote by $\mathbb{Q}^{(V\times\omega)}$ the direct sum

$$\mathbb{Q}^{(V\times\omega)} = \bigoplus_{(v,n)\in V\times\omega} \mathbb{Q}$$

of copies of $\mathbb{Q}$ indexed by $V \times \omega$, and identify an element (v, n) of $V \times \omega$ with the corresponding copy of $\mathbb{Q}$ in $\mathbb{Q}^{(V)}$. We define the p-divisible *self-absorbing* group eplag $\mathcal{G}_p^{\mathrm{sa}}(V, E, f)$ associated with (V, E, f) to be the subgroup of $\mathbb{Q}^{(V\times\omega)}$ generated by

$$\left\{ \frac{(v,k)}{p^n f(v)^m}, \frac{(v,k)+(w,k)}{p^n f(\{v,w\})} \mid v, w \in V,\, \{v,w\} \in E,\, n, m, k \in \omega,\, p \in \mathcal{P} \right\}.$$

It is easy to check that $\mathcal{G}_p^{\mathrm{sa}}(V, E, f)$ is indeed a self-absorbing p-divisible torsion-free abelian group. The proof of Theorem 4.28 is thus complete.

4.30. **Actions of discrete groups on $\mathcal{O}_2$.** Let G be a countable (discrete) group. Denote by $\mathrm{Act}(G, A)$ the space of actions of G on A endowed with the topology of pointwise convergence in norm. It is clear that $\mathrm{Act}(G, A)$ is homeomorphic to a G_δ subspace of the product of countably many copies of A and, in particular, is a Polish space.

Let G and H be discrete groups, and let $\pi\colon G \to H$ be a surjective homomorphism from G to H. Define the Borel map $\pi^*\colon \mathrm{Act}(H, A) \to \mathrm{Act}(G, A)$ by $\pi^*(\alpha) = \alpha \circ \pi$ for α in $\mathrm{Act}(H, A)$. It is easy to check that π^* is a Borel reduction from the relation of conjugacy of actions of H to the relation of conjugacy of actions of G. The following proposition is then an immediate consequence of this observation together with Theorem 4.28.

Proposition 4.31. *Let G be a discrete group with a nontrivial cyclic quotient. If $\mathcal{C}$ is any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of isomorphism of elements of $\mathcal{C}$ is Borel, then $\cong_{\mathcal{C}}$ is Borel reducible to the relation of conjugacy of actions of G on $\mathcal{O}_2$.*

Moreover, the latter equivalence relation is a complete analytic set as a subset of $\mathrm{Act}(G, A) \times \mathrm{Act}(G, A)$ *and, in particular, is not Borel.*

The situation for cocycle conjugacy is not as clear. It is not hard to verify that if $G = H \times N$ and $\pi\colon G \to H$ is the canonical projection, then π^*, as defined before, is a Borel reduction from the relation of cocycle conjugacy in $\mathrm{Act}(H, A)$ to the relation of cocycle conjugacy in $\mathrm{Act}(G, A)$. Using this observation and the structure theorem for finitely generated abelian groups, one obtains the following fact as a consequence of Theorem 4.28.

Proposition 4.32. *Let G be any finitely generated abelian group. If $\mathcal{C}$ is any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of isomorphism of elements of $\mathcal{C}$ is Borel, then $\cong_{\mathcal{C}}$ is Borel reducible to the relation of conjugacy of actions of G on $\mathcal{O}_2$.*

Moreover, the latter equivalence relation is a complete analytic set as a subset of $\mathrm{Act}(G, A) \times \mathrm{Act}(G, A)$ *and, in particular, not Borel.*

5. Final comments and remarks

Recall that an automorphism of a C*-algebra A is said to be *pointwise outer* (or *aperiodic*) if none of its nonzero powers is inner. By [33, Thm. 1], an automorphism of a Kirchberg algebra is pointwise outer if and only if it has the Rokhlin property. Moreover, it follows from this fact, together with [38, Cor. 5.14], that the set $\mathrm{Rok}(A)$ of pointwise outer automorphisms of a Kirchberg algebra A is a dense G_δ subset of $\mathrm{Aut}(A)$, which is moreover easily seen to be invariant by cocycle conjugacy.

It is an immediate consequence of [31, Thm. 5.2] that aperiodic automorphisms of $\mathcal{O}_2$ form a single cocycle conjugacy class. In particular, and despite the fact that the relation of cocycle conjugacy of automorphisms of $\mathcal{O}_2$ is not Borel, its restriction to the comeager subset $\mathrm{Rok}(\mathcal{O}_2)$ of $\mathrm{Aut}(\mathcal{O}_2)$ has only one class and, in particular, is Borel. This can be compared with the analogous situation for the group of ergodic measure preserving transformations of the Lebesgue space: The main result of [15] asserts that the relation of conjugacy of ergodic measure preserving transformations of the Lebesgue space is a complete analytic set. On the other hand, the restriction of such relation to the comeager set of ergodic *rank one* measure preserving transformations is Borel.

It is conceivable that similar conclusions might hold for the relation of conjugacy of automorphisms of $\mathcal{O}_2$. We therefore suggest the following.

Question 5.1. Consider the relation of conjugacy of automorphisms of $\mathcal{O}_2$, and restrict it to the invariant dense G_δ set of aperiodic automorphisms. Is this equivalence relation Borel?

By [26, Thm. 4.5], the automorphisms of $\mathcal{O}_2$ are not classifiable up to conjugacy by countable structures. It would be interesting to know if one can obtain a similar result for the relation of cocycle conjugacy.

Question 5.2. Is the relation of cocycle conjugacy of automorphisms of $\mathcal{O}_2$ classifiable by countable structures?

Theorem 4.5 of [26] shows that the relation of conjugacy of automorphisms is not classifiable for a large class of C*-algebras, including all C*-algebras that are classifiable according to the Elliott classification program. It would be interesting to know if the same holds for the relation of cocycle conjugacy. More generally, it would be interesting to draw similar conclusions about the complexity of the relation of cocycle conjugacy for automorphisms of other simple C*-algebras. This problem seems to be wide open.

Problem 5.3. Find an example of a simple unital nuclear separable C*-algebra for which the relation of cocycle conjugacy of automorphisms is not classifiable by countable structures.

Recall that an equivalence relation on a standard Borel space is said to be *smooth*, or *concretely classifiable*, if it is Borel reducible to the relation of equality in some standard Borel space. A smooth equivalence relation is in particular Borel and classifiable by countable structures. Thus, it is a consequence of Corollary 4.26 that the relation of cocycle conjugacy of automorphisms of $\mathcal{O}_2$ is not smooth.

If X is a compact Hausdorff space, we denote by $C(X)$ the unital commutative C*-algebra of complex-valued continuous functions on X. It is a classical result of Gelfand and Naimark that any unital commutative C*-algebra is of this form; see [2, Thm. II.2.2.4]. Moreover, by [2, §II.2.2.5], the group $\mathrm{Aut}(C(X))$ of automorphisms of $C(X)$ is canonically isomorphic to the group $\mathrm{Homeo}(X)$ of homeomorphisms of X. It is clear that in this case the relations of conjugacy and cocycle conjugacy of automorphisms coincide. By [4, Thm. 5], if X is the Cantor set, then the relation of (cocycle) conjugacy of automorphisms of $C(X)$ is not smooth (but classifiable by countable structures). On the other hand, when X is the unit square $[0,1]^2$, then the relation of cocycle conjugacy of automorphisms of $C(X)$ is not classifiable by countable structures in view of [21, Thm. 4.17]. This addresses Problem 5.3 in the case of abelian unital C*-algebras. No similar examples are currently known for simple unital C*-algebras.

It is worth mentioning here that if one considers instead the relation of *unitary conjugacy* of automorphisms, then there is a *strong dichotomy* in the complexity. Recall that two automorphisms α, β of a unital C*-algebra are unitarily conjugate if $\alpha \circ \beta^{-1}$ is an inner automorphism, that is, implemented by a unitary element of A. Theorem 1.2 in [29] shows that whenever this relation is not smooth, then it is even not classifiable by countable structures. The same phenomenon is shown to hold for unitary conjugacy of *irreducible representations* in [25, Thm. 2.8.]; see also [36, §6.8]. It is possible that similar conclusions might hold for the relation of conjugacy or cocycle conjugacy of automorphisms of simple C*-algebras.

Question 5.4. Is it true that, whenever the relation of (cocycle) conjugacy of automorphisms of a simple unital C*-algebra A is not smooth, then it is not even classifiable by countable structures?

The Kirchberg–Phillips classification theorem asserts that Kirchberg algebras satisfying the UCT are classified up to isomorphism by their K-theory. By [13, §3.3], the K-theory of a C*-algebra can be computed in a Borel way. It follows that Kirchberg algebras satisfying the UCT are classifiable up to isomorphism by countable structures. Conversely, by Corollary 4.25, if $\mathcal{C}$ is any class of countable structure *with Borel isomorphism relation*, then the relation of isomorphism of elements of $\mathcal{C}$ is Borel reducible to the relation of isomorphism of Kirchberg algebras satisfying the UCT. It is natural to ask whether the same conclusion holds for *any* class of countable structures $\mathcal{C}$.

Question 5.5. Suppose that $\mathcal{C}$ is a class of countable structures. Is the relation of isomorphism of elements of $\mathcal{C}$ Borel reducible to the relation of isomorphism of Kirchberg algebras with the UCT?

A class $\mathcal{D}$ of countable structures is *Borel complete* if the following holds: For any class of countable structures $\mathcal{C}$ the relation of isomorphism of elements of $\mathcal{C}$ is Borel reducible to the relation of isomorphism of elements of $\mathcal{D}$. Theorems 1, 3, and 10 of [16] assert that the classes of countable trees, countable linear orders, and countable fields of any fixed characteristic are Borel complete. Theorem 7 of [16] shows, using results of Mekler from [32], that the relation of isomorphism of countable groups is Borel complete. A long standing open problem—first suggested in [16]—asks whether the class of (torsion-free) abelian groups is Borel complete. In view of Theorem 4.24, a positive answer to such problem would settle Question 5.5 affirmatively.

Acknowledgments. The authors would like to thank Samuel Coskey and Ilijas Farah for several helpful conversations. We also thank Selçuk Barlak and Gabor Szabo for pointing out a mistake in our original construction of the action ν_p in Section 4 and for referring us to their work [1]. Finally, we thank the referee for suggestions and comments that led to several improvements.

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Received November 4, 2014; accepted April 6, 2016

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