Manuel Amann

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Manuel Amann
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Dekan:	Prof. Dr. Dr. h.c. Joachim Cuntz
Erster Gutachter:	Prof. Dr. Burkhard Wilking
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Für meine Eltern

Rita und Paul

und meinen Bruder

Benedikt

When the hurlyburly's done,
When the battle's lost and won.

William Shakespeare, "Macbeth"

Preface

Positive Quaternion Kähler Geometry lies in the intersection of very classical yet rather different fields in mathematics. Despite its geometrical setting which involves fundamental definitions from Riemannian geometry, it was soon discovered to be accessible by methods from (differential) topology, symplectic geometry and complex algebraic geometry even. Indeed, it is an astounding fact that the whole theory can entirely be encoded in terms of Fano contact geometry. This approach led to some highly prominent and outstanding results.

Aside from that, recent results have revealed in-depth connections to the theory of positively curved Riemannian manifolds. Furthermore, the conjectural existence of symmetry groups, which is confirmed in low dimensions, sets the stage for equivariant methods in cohomology, homotopy and Index Theory.

The interplay of these highly dissimilar theories contributes to the appeal and the beauty of Positive Quaternion Kähler Geometry. Indeed, once in a while one may gain the impression of having a short glimpse at the dull flame of real mathematical insight.

Quaternion Kähler Manifolds settle in the highly remarkable class of special geometries. Hereby one refers to Riemannian manifolds with special holonomy among which Kähler manifolds, Calabi–Yau manifolds or Joyce manifolds are to be mentioned as the most prominent examples. Whilst the latter—i.e. manifolds with $\mathbf{G}_2$-holonomy or $\mathbf{Spin}(7)$-holonomy—seem to be extremely far from being symmetric in general, it is probably the central question in Positive Quaternion Kähler Geometry whether every such manifold is a symmetric space. This question was formulated in the affirmative in a fundamental conjecture by LeBrun and Salamon. It forms the basic motivation for the thesis.

The conjectural rigidity of the objects of research seems to be reflected by the variety of methods that can be applied—especially if these methods are rather "far away from the original definition". For example, the existence of symmetries on the one hand contributes to the structural regularity of the manifolds; on the other hand does it permit a less analytical and more topological approach. The existence of various rigidity theorems and topological recognition theorems backs the idea that the structure imposed by special holonomy and positive scalar curvature is restrictive enough to permit a classification of Positive Quaternion Kähler Manifolds. We shall contribute some more results of this kind.

Conceptually, the thesis splits into two parts: On the one hand we are interested in classification results, (mainly) in low dimensions, which has led to chapters 2 and 4. On the other hand we extend the spectrum of methods used to study Positive Quaternion Kähler Manifolds by an approach via Rational Homotopy Theory. The outcome of this enterprise is depicted in chapter 3.

To be more precise, results obtained feature

- the formality of Positive Quaternion Kähler Manifolds—which is obtained by an in-depth analysis of spherical fibrations in general,
- the discovery and documentation (with counter-examples) of a crucial mistake in the existing "classification" in dimension 12,
- new techniques of how to detect plenty of new examples of non-formal homogeneous spaces,
- recognition theorems for $\mathbb{H}\mathbf{P}^{20}$ and $\mathbb{H}\mathbf{P}^{24}$,
- a partial classification result in dimension 20,
- a recognition theorem for the real Grassmannian $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$, which proves that the main conjecture, which suggests the symmetry of Positive Quaternion Kähler Manifolds, can (almost always) be decided from the dimension of the isometry group
- and results on rationally elliptic Positive Quaternion Kähler Manifolds and rationally elliptic Joyce manifolds.

Moreover, we depict a method of how to find an upper bound for the Euler characteristic of a Positive Quaternion Kähler Manifold under the isometric action of a sufficiently large torus. This makes use of a classification of $\mathbb{S}^1$-fixed-point components and $\mathbb{Z}_2$-fixed-point components of Wolf spaces which we provide. Furthermore, we reprove the vanishing of the elliptic genus of the real Grassmannian $\mathbf{Gr}_4(\mathbb{R}^{n+4})$ for n odd.

The methods applied comprise Rational Homotopy Theory, Index Theory, elements from the theory of transformation groups and equivariant cohomology as well as Lie theory.

In the first chapter we give an elementary introduction to the subject, we recall basic notions and we recapitulate known facts of Positive Quaternion Kähler Geometry, Index Theory and Rational Homotopy Theory. As we want to keep the thesis accessible to a larger audience ranging from Riemannian geometers to algebraic topologists, we shall be concerned to give an easily comprehensible and detailed outline of the concepts. In order to keep the proofs of the main theorems of subsequent chapters compact and comparatively short, we establish and elaborate some crucial but not standard theory in the introductory chapter already. We shall eagerly provide detailed proofs whenever different concepts are brought together or when ambiguities arise (in the literature).

Chapter 2 is devoted to a depiction of an error that was committed by Herrera and Herrera within the process of classifying 12-dimensional Positive Quaternion Kähler Manifolds. As this classification was highly accredited, and since not only the result but also the erroneous method of proof were used several times since then, it seems to be of importance to document the mistake in all adequate clarity. This will be done by giving various counter-examples for the different stages in the proof—varying from purely algebraic examples to geometric ones—as well as for the result, i.e. for the main tool used by Herrera and Herrera, itself. So we shall derive

Theorem. *There is an isometric involution on* $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)$ *having* $\widetilde{\mathbf{Gr}}_2(\mathbb{R}^5)$ *and* $\mathbb{H}\mathbf{P}^1$ *as fixed-point components. The difference in dimension of the fixed-point components is not divisible by* 4 *which implies that the* $\mathbb{Z}_2$*-action is* neither even nor odd.

We shall see that this will contradict what was asserted by Herrera and Herrera. It is the central counter-example to an argument which resulted in the assertion that the $\hat{A}$-genus of a π_2-finite oriented compact connected smooth manifold M^{2n} with an effective smooth $\mathbb{S}^1$-action vanishes. We even obtain

Theorem. *For any* $k > 1$ *there exists a smooth simply-connected* $4k$*-dimensional* π_2*-finite manifold* M^{4k} *with smooth effective* $\mathbb{S}^1$*-action and* $\hat{A}(M^{4k})[M^{4k}] \neq 0$.

This chapter is closely connected to the chapters B and D of the appendix, where not only a more global view on the main counter-example is provided (cf. chapter B), but where we also discuss what still might be true within the setting of Positive Quaternion Kähler Manifolds (cf. chapter D)—judging from the symmetric examples. (The main tool of Herrera and Herrera was formulated in a more general context.) This will also serve as a justification for some assumptions we shall make in chapter 4.

In the third chapter we establish the formality of Positive Quaternion Kähler Manifolds.

Theorem. *A Positive Quaternion Kähler Manifold is a formal space. The twistor fibration is a formal map.*

For this we shall investigate under which circumstances the formality of the total space of a spherical fibration suffices to prove formality for the base space. An application to the twistor fibration proves the main geometric result. The formality of Positive Quaternion Kähler Manifolds can be seen as another indication for the main conjecture, as symmetric spaces are formal. Besides, the question of formality seems to be a recurring topic of interest within the field of special holonomy.

This discussion will be presented in a far more general context than actually necessary for the main result: We shall investigate how formality properties of base space and total space are related for both even-dimensional and odd-dimensional fibre spheres. Moreover, we shall discuss the formality of the fibration itself. It will become clear that the case of even-dimensional fibres is completely distinct from the one with odd-dimensional fibres, where hardly no relations appear at all.

This topic will naturally lead us to the problem of finding construction principles for non-formal homogeneous spaces. Although a lot of homogeneous spaces were identified as being formal, there seems to be a real lack of non-formal examples. As we do not want to get too technical at this point of the thesis let us just mention a few of the most prominent examples we discovered:

Theorem. *The spaces*

$$\frac{\mathbf{Sp}(n)}{\mathbf{SU}(n)} \qquad \frac{\mathbf{SO}(2n)}{\mathbf{SU}(n)} \qquad \frac{\mathbf{SU}(p+q)}{\mathbf{SU}(p)\times\mathbf{SU}(q)}$$

are non-formal for $n \geq 7$*,* $n \geq 8$ *and* $p+q \geq 4$ *respectively.*

The techniques elaborated and the many examples will then also permit to find an (elliptic) non-formal homogeneous spaces in each dimension greater than or equal to 72.

Apart from simple reproofs of the formality of Positive Quaternion Kähler Manifolds in low dimensions, the following sections are mainly dedicated to elliptic manifolds. We shall prove

Theorem. *An elliptic* 16*-dimensional Positive Quaternion Kähler Manifold is rationally a homology Wolf space.*

Theorem. *There are no compact elliptic* $\mathbf{Spin}(7)$*-manifolds. An elliptic simply-connected compact irreducible* $\mathbf{G}_2$*-manifold is a formal space.*

In chapter 4 we provide several classification theorems: First of all, we prove

Theorem. *A* 20*-dimensional Positive Quaternion Kähler Manifold* M^{20} *is homothetic to* $\mathbb{H}\mathbf{P}^5$ *provided* $b_4(M) = 1$*. A* 24*-dimensional Positive Quaternion Kähler Manifold* M^{24} *is homothetic to* $\mathbb{H}\mathbf{P}^6$ *provided* $b_4(M) = 1$*.*

Next we establish a partial classification result for 20-dimensional Positive Quaternion Kähler Manifolds.

Theorem. *A* 20*-dimensional Positive Quaternion Kähler Manifold* M *with* $\hat{A}(M)[M] = 0$ *satisfying* $\dim \mathrm{Isom}(M) \notin \{15, 22, 29\}$ *is a Wolf space.*

This will be done in two major steps. First we combine several known relations from Index Theory and twistor theory to get a grip on possible isometry groups. As a second step we shall use Lie theory to determine the isometry type of the Positive Quaternion Kähler Manifold from there. The techniques in the second step permit a generalisation to nearly arbitrary dimensions. So we shall prove a recognition theorem for the real Grassmannian—a first one of its kind. This will make clear that in order to see whether a Positive Quaternion Kähler Manifold is symmetric or not it suffices to compute the dimension of its isometry group—which itself permits an interpretation as an index of a twisted Dirac operator.

Theorem. *For almost every n it holds: A Positive Quaternion Kähler Manifold M^{4n} is symmetric if and only if* $\dim \mathrm{Isom}(M) > \frac{n^2+5n+12}{2}$.

We remark that the mistakes in the literature which we spotted—cf. chapter 2 and page 21—turned out to be extremely detrimental to the classification result in dimension 20—not exclusively but mainly. As we did not only use the main and most prominent results that might still be true in the special case of Positive Quaternion Kähler Manifolds, but also intermediate conclusions which are definitely wrong, it was no longer possible to sustain a more elaborate form of the classification theorem without unnatural assumptions. So what was true before, now remains a conjecture: If the Euler characteristic of M^{20} is restricted from above (by a suitable bound which probably needs not be very small), then M^{20} is symmetric.

In the appendix we compute the rational cohomology structure of Wolf spaces. Moreover, we provide a classification of $\mathbb{S}^1$-fixed point components and $\mathbb{Z}_2$-fixed point components of Wolf spaces, which illustrates and generalises the examples in chapter 2. It seems that such a classification can not be found in the literature. This may pave the way for equivariant methods.

As a first application of these data we give a technique of how to restrict the Euler characteristic of a Positive Quaternion Kähler Manifold from above under the assumption of a sufficiently large torus rank of the isometry group. This leads to optimal bounds in low dimensions and we do a showcase computation for dimension 16 with an isometric four-torus action. Due to the fact that the classification in dimension 12 does no longer exist, we need to suppose that $\hat{A}(M)[M] = 0$ for all π_2-finite 12-dimensional Positive Quaternion Kähler Manifolds. Then we obtain

Theorem. *A* 16*-dimensional Positive Quaternion Kähler Manifold admitting an isometric action of a* 4*-torus satisfies* $\chi(M) \in \{12, 15\}$ *and* $(b_4, b_6, b_8) \in \{(3, 0, 4), (3, 2, 3)\}$ *unless* $M \in \{\mathbb{H}\mathbf{P}^4, \mathbf{Gr}_2(\mathbb{C}^6)\}$.

This kind of result can be obtained under a smaller torus rank and we shall also indicate how to generalise it to arbitrary dimensions.

We then compute the elliptic genus of the real Grassmannian, which will serve as a justification for the assumptions we make in chapter 4. In particular we see that the elliptic genus vanishes on all 20-dimensional Wolf spaces and so does the $\hat{A}$-genus in particular. This also clarifies the situation on the existing examples of Positive Quaternion Kähler Manifolds after the general results have proved to be wrong (cf. chapter 2). This result, however, is interesting for its own sake, as the method of proof seems to be more direct than a recently published one (cf. [36]).

Finally, we provide preparatory computations in dimension 16—which served to prove recognition theorems for $\mathbb{H}\mathbf{P}^4$—and we comment on further known relations that can easily be reproved.

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Contents

1

Introduction

This chapter is devoted to a brief depiction of Positive Quaternion Kähler Geometry and of Rational Homotopy Theory. We shall give basic definitions and we shall sketch concepts and techniques that arise. We shall outline properties of the objects in question that turn out to be of importance in the thesis. Definitions that appear in different forms in the literature will be discussed and several proofs that clarify the situation are provided. Furthermore, we shall comment on ambiguities we found during the study of the literature. Moreover, we shall elaborate several arguments that will simplify the proofs of the main theorems in the subsequent chapters.

For the general facts stated in the next section we recommend the textbooks [18], [45], [46], [5] and the survey articles [66], [67].

1.1. All the way to Positive Quaternion Kähler Geometry

Let (M, g) be a Riemannian manifold, i.e. a smooth manifold M with Riemannian metric g. One may generalise the intuitive notion of "parallel transport" known from Euclidean space to *parallel transport* on M: Given a (piecewise) smooth curve $\gamma : [0, 1] \to M$ with starting point $\gamma(0) = x$, end point $\gamma(1) = y$ and starting vector $v \in T_xM$ in the tangent space at x one obtains a unique *parallel vector field* s along γ. This yields a vector $P_\gamma(v) = s(1)$ in T_yM, the translation of v.

Denote by $\mathcal{X}(M)$ the smooth vector fields on M, i.e. the smooth sections of the tangent bundle TM. The notion of parallel transport is made precise by means of the concept of a *connection*, an $\mathbb{R}$-bilinear map

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

tensorial in the first component

$$\nabla_{fX} Y = f \nabla_X Y$$

for $f \in C^\infty(M)$ and derivative in the second one

$$\nabla_X (fY) = X(f) \cdot Y + f \nabla_X Y$$

One may now show that there is exactly one connection that additionally is metric

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

and torsion-free

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

This connection is called the *Levi–Civita* connection ∇^g. Since ∇^g is tensorial in the first component, its value depends on the vector in the point only. Thus we may call a vector field $s : [0,1] \to TM \quad , \quad t \mapsto s(t) \in T_{c(t)}M$ along a curve $\gamma : [0,1] \to M$ *parallel* if

$$\nabla^g_{\dot\gamma} s = 0$$

for the velocity field $\dot\gamma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \in T_{\gamma(t)}M$. A starting vector $v \in T_x(M)$ may always be extended along the curve γ to a unique parallel vector field s, since the condition for this may be formulated as a first-order ordinary differential equation.

Let us now focus on closed loops based at some $x \in M$—cf. figure 1.1. Then parallel transport is a linear transformation of T_xM. We may concatenate two loops γ_1, γ_2 to $\gamma_1 * \gamma_2$ as usual. We obtain:

$$P_{\gamma_1 * \gamma_2} = P_{\gamma_2} \circ P_{\gamma_1}$$

Parallel transport along the "inverse loop" $\gamma^{-1}(t) := \gamma(1-t)$ leads to the inverse transformation:

$$P_{\gamma^{-1}} = P_\gamma^{-1}$$

Parallel transport along the constant curve is the identity transformation.

This leads to the following crucial definition:

Definition 1.1. The group

$$\mathrm{Hol}_x(M, g) = \{P_\gamma \mid \gamma \text{ is a closed loop based at } x\} \subseteq \mathbf{GL}(T_xM)$$

is called the *holonomy group* of M (at x).

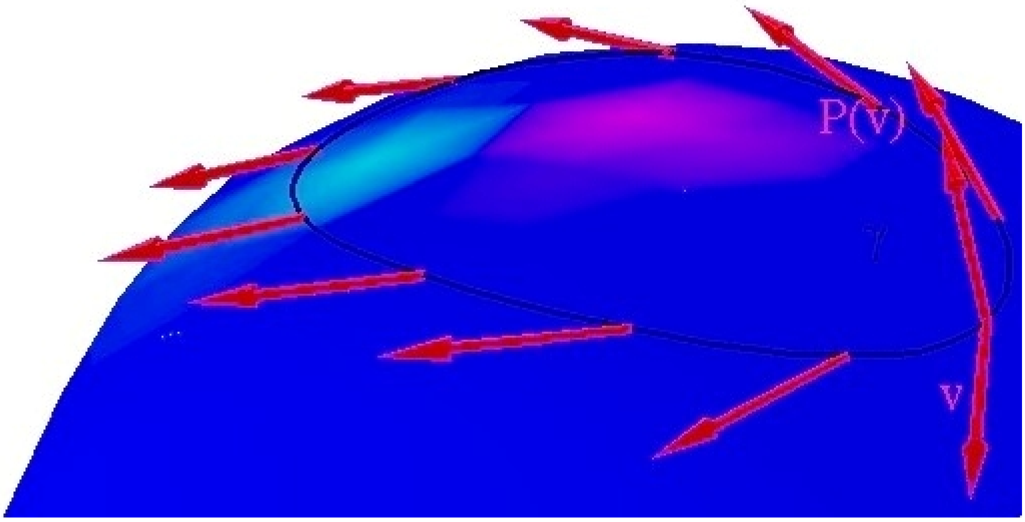


Figure 1.1.: Parallel transport along a loop

Since a change of base-point within the same path-component produces holonomy groups that are isomorphic by an inner automorphism, we may suppress the point.

Clearly, the holonomy group is trivial on the flat $\mathbb{R}^n$, but it already is not on the round sphere $\mathbb{S}^2$:

Example 1.2. We have $\mathrm{Hol}(\mathbb{S}^2) = \mathbf{SO}(2)$. This can be seen as follows: Parallel transport is a linear transformation that preserves angles and lengths, i.e. it is a transformation in the orthogonal group. Moreover, it preserves orientation. Thus we have that $\mathrm{Hol}(\mathbb{S}^2) \subseteq \mathbf{SO}(2)$. For each rotation in $\mathbf{SO}(2)$ we shall now construct a piecewise smooth closed loop with the property that parallel transport along this loop is the given rotation. For this we embed the sphere as well as its tangent spaces at north and south pole into $\mathbb{R}^3$. Thus, without restriction, $\mathbb{S}^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$. We consider loops based at the south pole $x = (0, 0, -1)$. We focus on loops that are concatenations of two segments of two different great circles. Indeed, we consider the geodesics, i.e. the great circles, generated via the exponential map by the tangent vectors $(1, 0, 0) \in T_{(0,0,-1)}\mathbb{S}^2$ and $v \in \mathbb{S}^1 \subseteq \mathbb{R}^2 \cong T_{(0,0,-1)}\mathbb{S}^2$ in the south pole. These great circles intersect in the north pole $y = (0, 0, 1)$. So following the first geodesic until we reach the north pole and then following the second one in the opposite direction until we are back in the south pole yields a closed piecewise smooth curve.

Since the velocity field along a geodesic is parallel by definition, we deduce that parallel transport of the tangent vector $(1, 0, 0) \in T_x(\mathbb{S}^2)$ yields the tangent vector

$(-1,0,0)$ in y. (Parallel transport of v along the second geodesic yields $-v$.) As parallel transport P preserves angles we see that parallel transport of $(-1,0,0) \in T_y(\mathbb{S}^2)$ along the inverse second great circle yields the vector $P(1,0,0) = v^2 \in \mathbb{S}^1 \subseteq \mathbb{C} \cong T_x(\mathbb{S}^2)$, i.e. the vector in $\mathbb{S}^1$ determined by the fact that $\sphericalangle(P(1,0,0),(1,0,0)) = 2 \cdot \sphericalangle(v,(1,0,0))$. Thus by a suitable choice of v we may realise any given rotation in $\mathbf{SO}(2)$. ⊡

Recall that a simply-connected Riemannian manifold is *irreducible* if it does not decompose as a Cartesian product. It is *non-symmetric* if it does not admit an involutive isometry s_p, a *geodesic symmetry*, for every point $p \in M$ having this point as an isolated fixed-point.

We cite the following celebrated theorem due to Berger, 1955—cf. corollary [5].10.92, p. 300:

Theorem 1.3 (Berger)**.** *The holonomy group of a simply-connected, irreducible and non-symmetric Riemannian manifold is one of those listed in table 1.1.*

Table 1.1.: Holonomy types

$\mathrm{Hol}(M,g)$	$\dim M$	Type of manifold	Properties
$\mathbf{SO}(n)$	n	generic	
$\mathbf{U}(n)$	$2n$	Kähler manifold	Kähler
$\mathbf{SU}(n)$	$2n$	Calabi–Yau manifold	Ricci-flat, Kähler
$\mathbf{Sp}(n)\mathbf{Sp}(1)$	$4n$	Quaternion Kähler Manifold	Einstein
$\mathbf{Sp}(n)$	$4n$	hyperKähler manifold	Ricci-flat, Kähler
$\mathbf{G}_2$	7	$\mathbf{G}_2$ manifold (imaginary Cayley)	Ricci-flat
$\mathbf{Spin}(7)$	8	$\mathbf{Spin}(7)$ manifold (octonionic)	Ricci-flat

□

Since there is a classification of symmetric spaces due to Cartan—which yields direct insight into the holonomy groups—and since the holonomy of a product is the product holonomy, we hence do have a rather complete picture.

Originally, there was another case in Berger's list, namely $\mathrm{Hol}(M,g) = \mathbf{Spin}(9)$ in dimension 16. However, this case was shown to imply the symmetry of (M,g) by Alekseevskii. Each of the remaining cases really is established by certain manifolds. (Observe that due to this theorem any reduced holonomy group $\mathrm{Hol}^0(M,g)$ of a Riemannian manifold (M,g) may be written as a direct product of groups from the list and holonomy groups of Riemannian symmetric spaces.)

As the structure of $\mathbf{Sp}(n)\mathbf{Sp}(1)$ is of essential importance, we remark: The symplectic group

$$\mathbf{Sp}(n) = \{A \subseteq \mathbf{GL}_n(\mathbb{H}) \mid A\bar{A}^t = I\}$$

(with quaternionic conjugation defined by $(i,j,k) \mapsto (-i,-j,-k)$) is the quaternionic analogue of the real respectively complex groups $\mathbf{O}(n)$ and $\mathbf{U}(n)$. By $\mathbf{Sp}(n)\mathbf{Sp}(1)$ we denote the group $\phi(\mathbf{Sp}(n) \times \mathbf{Sp}(1))$, where $\phi : \mathbf{Sp}(n) \times \mathbf{Sp}(1) \to \mathbf{GL}_{2n}(\mathbb{C})$ is the representation given by $\phi((A,h))(x) = Axh^{-1}$ with kernel $\ker \phi = \langle -\mathrm{id}, -1 \rangle$. Thus we have:

$$\mathbf{Sp}(n)\mathbf{Sp}(1) := \phi(\mathbf{Sp}(n) \times \mathbf{Sp}(1)) = \mathbf{Sp}(n) \times_{\mathbb{Z}_2} \mathbf{Sp}(1) = (\mathbf{Sp}(n) \times \mathbf{Sp}(1))/_{\langle(-\mathrm{id},-1)\rangle}$$

Definition 1.4. A *Quaternion Kähler Manifold* is a connected oriented Riemannian manifold (M^{4n}, g) with holonomy group contained in $\mathbf{Sp}(n)\mathbf{Sp}(1)$. If $n = 1$ we additionally require M to be Einstein and self-dual.

One may use the Levi–Civita connection to define different notions of curvature. Classically, there are three different concepts: *sectional curvature*, *Ricci curvature* and *scalar curvature*, which arise as contractions of the curvature tensor. Indeed, one may define the Ricci tensor as the first contraction

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{4n} R(E_i, X, Y, E_i)$$

of the Riemannian curvature tensor

$$R(X,Y,Z,W) = g(\nabla^g_X \nabla^g_Y Z - \nabla^g_Y \nabla^g_X Z - \nabla^g_{[X,Y]} Z, W)$$

(The vector fields E_i form a local orthonormal basis.) Scalar curvature is the second contraction, i.e. the trace

$$\mathrm{scal}(p) = \sum_{i=1}^{4n} \mathrm{Ric}_p(E_i, E_i)$$

of the Ricci tensor, where the E_i are orthonormal coordinate vector fields. (By the argument and index p we stress the fact that scalar curvature depends on the point $p \in M$ only.) The sectional curvature along a plane generated by two orthonormal vectors v and w is given by

$$K(v,w) = R(v,w,w,v)$$

(This is well-defined, as $R(X,Y)Z$ in a point depends on the values of X, Y, Z in the point only.) Ricci-curvature is defined as

$$\mathrm{ric}(v) = \mathrm{Ric}(v,v)$$

Scalar curvature is the weakest notion, sectional curvature the strongest one. A nice description of scalar curvature $\mathrm{scal}(p)$ is the following one:

$$\frac{\mathrm{Vol}(B_\varepsilon(p))}{\mathrm{Vol}(B_\varepsilon^{\mathrm{Eucl}}(0))} = 1 - \frac{\mathrm{scal}(p)}{6(n+2)}\varepsilon^2 + O(\varepsilon^4)$$

That is, scalar curvature measures volume distortion to a certain degree.

Quaternion Kähler Manifolds are *Einstein* (cf. [5].14.39, p. 403), i.e. the Ricci tensor is a multiple of the metric tensor. Thus we compute

$$\mathrm{scal}(p) = \sum_{i=1}^{4n} \mathrm{Ric}(E_i, E_i) = k \cdot \sum_{i=1}^{4n} g_p(E_i, E_i) = 4k \cdot n$$

and the scalar curvature is constant. This leads to the following crucial definition.

Definition 1.5. A *Positive Quaternion Kähler Manifold* is a Quaternion Kähler Manifold with complete metric and positive scalar curvature.

We present some clarifying facts:

- Just to avoid confusion: A Quaternion Kähler Manifold needs not be Kählerian in general, since $\mathbf{Sp}(n)\mathbf{Sp}(1) \nsubseteq \mathbf{U}(2n)$. No Positive Quaternion Kähler Manifold admits a compatible complex structure (cf. [67].1.3, p. 88).
- The structure group reduces to $\mathbf{Sp}(n)$, i.e. M is (locally) *hyperKähler* if and only if the scalar curvature vanishes (cf. [5].14.45.a, p. 406 or [67].1.2, p. 87). Little is known in the case of negative scalar curvature.
- A Positive Quaternion Kähler Manifold is necessarily compact and simply-connected (cf. [66], p. 158 and [66].6.6, p. 163).
- A compact orientable (locally) Quaternion Kähler Manifold with positive sectional curvature is isometric to the canonical quaternionic projective space (cf. theorem [5].14.43, p. 406).

The only known examples up to now are given by the so-called *Wolf spaces*, which are all symmetric and which are the only homogeneous examples as a result of Alekseevskii shows (cf. [5].14.56, p. 409). There is an astonishing relation between Wolf spaces and complex simple Lie algebras. Indeed, there is a well-understood construction principle which involves Lie groups. So there are three infinite series of Wolf spaces and some further spaces corresponding to the exceptional Lie algebras (cf. table 1.2). We now state the crucial conjecture that motivates the biggest part of our work in this field.

Conjecture 1.6 (LeBrun, Salamon)**.** *Every Positive Quaternion Kähler Manifold is a Wolf space.*

This conjecture is backed by the following remarkable theorem by Salamon and LeBrun.

Table 1.2.: Wolf spaces

Wolf space M		$\dim M$
$\mathbb{H}\mathbf{P}^n = \mathbf{Sp}(n+1)/\mathbf{Sp}(n) \times \mathbf{Sp}(1)$		$4n$
$\mathbf{Gr}_2(\mathbb{C}^{n+2}) = \mathbf{U}(n+2)/\mathbf{U}(n) \times \mathbf{U}(2)$	ordinary type	$4n$
$\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4}) = \mathbf{SO}(n+4)/\mathbf{SO}(n) \times \mathbf{SO}(4)$		$4n$
$\mathbf{G}_2/\mathbf{SO}(4)$		8
$\mathbf{F}_4/\mathbf{Sp}(3)\mathbf{Sp}(1)$		28
$\mathbf{E}_6/\mathbf{SU}(6)\mathbf{Sp}(1)$	exceptional type	40
$\mathbf{E}_7/\mathbf{Spin}(12)\mathbf{Sp}(1)$		64
$\mathbf{E}_8/\mathbf{E}_7\mathbf{Sp}(1)$		112

Theorem 1.7 (Finiteness)**.** *There are only finitely many Positive Quaternion Kähler Manifolds in each dimension.*

Proof. See theorem [54].0.1, p. 110, which itself makes use of a classification result by Wisniewski—see below. □

A confirmation of the conjecture (cf. 1.6) has been achieved in dimensions four (Hitchin) and eight (Poon–Salamon, LeBrun–Salamon); if the fourth Betti number equals one it was shown that in dimensions 12 and 16 the manifold is a quaternionic projective space—cf. theorem [67].2.1, p. 89.

As we already remarked, it is an astounding fact that the theory of Positive Quaternion Kähler Manifolds may completely be transcribed to an equivalent theory in complex geometry. This is done via the *twistor space* Z of the Positive Quaternion Kähler Manifold M. As this connection will be of importance for us, we shall depict some possible constructions of Z.

Locally the structure bundle with fibre $\mathbf{Sp}(n)\mathbf{Sp}(1)$ may be lifted to its double covering with fibre $\mathbf{Sp}(n) \times \mathbf{Sp}(1)$. The only space that permits such a global lift is the quaternionic projective space; and, in general, the obstruction to the lifting is a class in second integral homology generating a $\mathbb{Z}_2$-subgroup.

So locally one may use the standard representation of $\mathbf{Sp}(1)$ on $\mathbb{C}^2$ to associate a vector bundle H. In general H does not exist globally but its complex projectivisation $Z = \mathbf{P}_\mathbb{C}(H)$ does. In particular, we obtain the *twistor fibration*

$$\mathbb{C}\mathbf{P}^1 \hookrightarrow \mathbf{P}_\mathbb{C}(H) \to M$$

An alternative construction of the same manifold Z, which we may also use to shed some more light on the structure of M, is the following: A hyperKähler manifold may be defined by three complex structures I, J, K behaving like the unit quaternions i, j,

k, i.e. $IJ = -JI = K$ and $I^2 = J^2 = -\mathrm{id}$ and such that the manifold is Kählerian with respect to each of them. On a Quaternion Kähler Manifold M we do not have these structures globally but only locally. That is, we have a subbundle of the bundle $\mathrm{End}(TM)$ locally generated by the almost complex structures I, J and K. This can be seen as follows:

Again locally we lift the $\mathbf{Sp}(n)\mathbf{Sp}(1)$-principal bundle to an $\mathbf{Sp}(n) \times \mathbf{Sp}(1)$ bundle. Now we may use the *adjoint representation* to associate a vector bundle to the principal bundle: Indeed, we have the map $\phi : \mathbf{Sp}(1) \to \mathrm{Aut}(\mathbf{Sp}(1))$ given by $g \mapsto \phi_g$ with $\phi_g(h) = ghg^{-1}$. Since each ϕ_g is an automorphism of $\mathbf{Sp}(1)$, it induces a map Ad_g on the tangent space of the identity element preserving the Lie bracket, i.e. an automorphism of the Lie algebra $\mathfrak{sp}(1)$. The map $\mathrm{Ad} : \mathbf{Sp}(1) \to \mathrm{Aut}(\mathfrak{sp}(1))$ given by $g \mapsto \mathrm{Ad}_g$ is the requested representation.

Note that the center of $\mathbf{Sp}(1)$ is ± 1. Hence we have $\phi_{+1} = \phi_{-1}$ and $\ker \mathrm{Ad} = \{\pm 1\}$. This permits to associate a bundle with respect to the hence well-defined action

$$(g_1, g_2) \cdot (p, s) = ((g_1, g_2) \cdot p, \mathrm{Ad}_{g_2} s)$$

with $p \in P$, the $\mathbf{Sp}(n)\mathbf{Sp}(1)$-principal bundle, $s \in \mathfrak{sp}(1)$ and $(g_1, g_2) \in \mathbf{Sp}(n)\mathbf{Sp}(1)$. So this globally associates the (three-dimensional) real vector bundle E', the quotient of $P \times \mathfrak{sp}(1)$ by this action, to the adjoint representation of $\mathbf{Sp}(1)$.

Moreover, the Lie algebra of $\mathbf{Sp}(1)$ is just

$$\mathfrak{sp}(1) = \{h \in \mathbb{H} \mid h + \bar{h} = 0\}.$$

So $\mathfrak{sp}(1)$ has the structure of the imaginary quaternions. This means that locally E' has three sections I, J and K as requested, i.e. three endomorphisms of $\mathbb{H}$ with the respective commutating properties. (One may view $E' \subseteq \mathrm{End}(TM)$ as the subbundle generated by the three locally defined almost complex structures I, J, K—cf. [5], p. 412.) The twistor space Z of M now is just the unit sphere bundle $\mathbb{S}(E')$ associated to E'. The twistor fibration is just

$$\mathbb{S}^2 \hookrightarrow \mathbb{S}(E') \to M$$

(Comparing this bundle to its version above we need to remark that clearly $\mathbb{C}\mathbf{P}^1 \cong \mathbb{S}^2$.)

A similar construction leads to a 3-Sasakian manifold $\mathbb{S}^1$-fibring over Z (cf. [28]).

As an example one may observe that on $\mathbb{H}\mathbf{P}^n$ we have a global lift of $\mathbf{Sp}(n)\mathbf{Sp}(1)$ and that the vector bundle associated to $\mathbf{Sp}(1)$ is just the tautological bundle. Now complex projectivisation of this bundle yields the complex projective space $\mathbb{C}\mathbf{P}^{2n+2}$ and the twistor fibration is just the canonical projection.

More generally, on Wolf spaces one obtains the following: The Wolf space may be written as $G/K\mathbf{Sp}(1)$ (cf. the table on [5], p. 409) and its corresponding twistor space is given as $G/K\mathbf{U}(1)$ with the twistor fibration being the canonical projection.

For the construction of the twistor space we used local lifts of $\mathbf{Sp}(n)\mathbf{Sp}(1)$ to $\mathbf{Sp}(n) \times \mathbf{Sp}(1)$. Then there are the standard complex representations of $\mathbf{Sp}(n)$ on $\mathbb{C}^{2n}$ and of $\mathbf{Sp}(1)$ on $\mathbb{C}^2$. The bundles associated to these actions will be called E respectively H. That is, define a right action of $\mathbf{Sp}(n)$ on the direct product $P_{\mathbf{Sp}(n)} \times \mathbb{C}^{2n}$ of the (local) principal $\mathbf{Sp}(n)$-bundle $P_{\mathbf{Sp}(n)}$ with $\mathbb{C}^{2n}$ by the pointwise construction

$$(p, v) \cdot g := (p \cdot g, \varrho(g^{-1})v)$$

where ϱ is the standard representation of $\mathbf{Sp}(n)$. Then form the quotient

$$E = (P_{\mathbf{Sp}(n)} \times \mathbb{C}^{2n})/\mathbf{Sp}(n)$$

Do the same for

$$H = (P_{\mathbf{Sp}(1)} \times \mathbb{C}^{2})/\mathbf{Sp}(1).$$

Moreover, we obtain the following formula for the complexified tangent bundle $T_{\mathbb{C}}M$ of the Positive Quaternion Kähler Manifold M (cf. [67], p. 93):

$$T_{\mathbb{C}}M = E \otimes H$$

The bundles E and H arise from self-dual representations and so their odd-degree Chern classes vanish. The Chern classes of E will be denoted by

$$c_{2i} := c_{2i}(E) \in H^{2i}(M)$$

and

$$u := -c_2(H) \in H^4(M)$$

We explicitly draw the reader's attention to the definition of u as the *negative* second Chern class of H. We fix this notation once and for all. (By abuse of notation we shall sometimes equally denote a representative of the cohomology class u by u.)

Cohomology will always be taken with rational coefficients unless specified differently.

The *quaternionic volume*

$$v = (4u)^n \in H^{4n}(M^{4n})$$

is integral and satisfies $1 \leq v \leq 4^n$—cf. [67], p. 114 and corollary [68].3.5, p. 7.

Let us now state some important facts about twistor spaces.

Theorem 1.8. *The twistor space Z of a Positive Quaternion Kähler Manifold is a* simply-connected compact Kähler Einstein Fano contact manifold*.*

Proof. See theorem [54].1.2, p. 113. □

The classification of twistor spaces is equivalent to the classification of the manifolds themselves. For this recall that two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are called *homothetic* if there exists a diffeomorphism $\phi : M_1 \to M_2$ such that the induced metric satisfies $\phi^* g_2 = c g_1$ for some constant $c > 0$. The map ϕ then is a *homothety.*

Theorem 1.9. *Two Positive Quaternion Kähler Manifolds are homothetic if and only if their twistor spaces are biholomorphic.*

Proof. See theorems [54].3.2, p. 119 and [66].4.3, p. 155. □

Furthermore, note the following theorem:

Theorem 1.10. *If Z is a compact Kähler Einstein manifold with a holomorphic contact structure, then Z is the twistor space of some Quaternion Kähler Manifold.*

Proof. See theorem [67].5.3, p. 102. □

Twistor theory has proved to be very fruitful. Apart from the finiteness result (cf. 1.7) above, many properties of Positive Quaternion Kähler Manifolds were gained via Fano contact geometry. In the same vein the fact that M is simply-connected follows from the result that $\pi_1(Z) = 0$ via the long exact homotopy sequence: Indeed, the manifold Z has a finite fundamental group—cf. the theorem of Myers, corollary [18].3.2, p. 202—and, as it is Fano, it does not possess finite coverings by the Kodaira vanishing theorem—cf. [32], p. 154 and [54], p. 114. Thus we have that $\pi_1(Z) = 0$. Let us now illustrate the strength of this theory with yet another example.

For Fano manifolds Z there is a contraction theorem which guarantees that there is always a map of varieties $Z \to X$ decreasing the second Betti number by one whilst the kernel of $H_2(Z, R) \to H_2(X, R)$ is generated by the class of a rational holomorphic curve $\mathbb{C}\mathbf{P}^1 \subseteq Z$. If the second Betti number is one, i.e. $b_2(Z) = 1$, this theorem virtually does not provide any information at all, since X may be taken to be a point. A famous theorem by Wisniewski now classifies the cases in which $b_2(Z) > 1$ and, surprisingly, it only yields three possibilities for Z. In the case of contact varieties Z we even obtain

Theorem 1.11. *Let Z be a Fano contact manifold satisfying $b_2(Z) > 1$, then $Z = \mathbf{P}(T^* \mathbb{C}\mathbf{P}^{n+1})$.*

Proof. See corollary [54].4.2, p. 122. □

This is a major tool for the following corollary which, in particular, can be considered a recognition theorem for the complex Grassmannian.

Corollary 1.12 (Strong rigidity)**.** *Let (M, g) be a Positive Quaternion Kähler Manifold. Then either*

$$\pi_2(M) = \begin{cases} 0 & \text{iff } M \cong \mathbb{H}\mathbf{P}^n \\ \mathbb{Z} & \text{iff } M \cong \mathbf{Gr}_2(\mathbb{C}^{n+2}) \\ \text{finite with } \langle \varepsilon \rangle \cong \mathbb{Z}_2\text{-torsion contained in } \pi_2(M) & \text{otherwise} \end{cases}$$

PROOF. See theorems [54].0.2, p. 110, and [67].5.5, p. 103. □

Via the Hurewicz theorem we may identify $\pi_2(M)$ with $H_2(M, \mathbb{Z})$. Using universal coefficients we obtain that $H^2(M, \mathbb{Z}_2)$ is the two-torsion in $H_2(M, \mathbb{Z})$, as M is simply-connected. The element ε now corresponds to the obstruction to a global lifting of $\mathbf{Sp}(n)\mathbf{Sp}(1)$ to $\mathbf{Sp}(n) \times \mathbf{Sp}(1)$ as defined on [66], p. 149. If such a lifting exists, it is shown in theorem [66].6.3, p. 160–162, that the isometry group of the manifold has to be very large; large enough to identify the manifold as the quaternionic projective space.

Let us now collect cohomological properties of Positive Quaternion Kähler Manifolds M^{4n}. By $b_i = \dim H^i(M)$ we shall denote the Betti numbers of M.

Recall the class $u = -c_2(H)$. There is a cohomology class $l \in H^2(Z)$—represented by a Kähler form—which generates $H^*(Z)$ as an $H^*(M)$-module under the restriction that $l^2 = u$ (cf. [66], p. 148 (and (2.6) on that page), [35], p. 356). This class is given by $l = \frac{1}{2}c_1(L^2)$ for a certain bundle L^2 as on [66], p. 148.

We shall mainly use the terminology $z = l$. With respect to this form, the manifold Z—as it is a Kähler manifold—satisfies the Hard-Lefschetz property: The morphism

$$L^k : H^{n-k}(Z, \mathbb{R}) \to H^{n+k}(\mathbb{Z}, \mathbb{R}) \qquad L^k([\alpha]) = [z^k \wedge \alpha]$$

is an isomorphism.

Theorem 1.13 (Cohomological properties)**.** *A Positive Quaternion Kähler Manifold M satisfies:*

- *Odd-degree Betti numbers vanish, i.e. $b_{2i+1} = 0$ for $i \geq 0$.*
- *The identity*

$$\sum_{p=0}^{n-1} \big(6p(n-1-p) - (n-1)(n-3)\big) b_{2p} = \frac{1}{2} n(n-1) b_{2n}$$

holds and specialises to

$$2b_2 = b_6 \tag{1.1}$$

$$-1 + 3b_2 + 3b_4 - b_6 = 2b_8 \tag{1.2}$$

$$-4 + 5b_2 + 8b_4 + 5b_6 - 4b_8 = 5b_{10} \tag{1.3}$$

in dimension 12, 16 *and* 20 *respectively.*

- *The twistor fibration yields an isomorphism of $H^*(M)$-modules*

$$H^*(Z) \cong H^*(M) \otimes H^*(\mathbb{S}^2) \cong H^*(M) \oplus H^*(M)z$$

- *A Positive Quaternion Kähler Manifold $M^{4n} \not\cong \mathbf{Gr}_2(\mathbb{C}^{n+2})$ is rationally* 3*-connected.*

- *The rational/real cohomology algebra possesses an analogue of the* Hard-Lefschetz property*, i.e. with the four-form $u \in H^4(M)$ from above the morphism*

$$L^k : H^{n-k}(M, \mathbb{R}) \to H^{n+k}(M, \mathbb{R}) \qquad L^k(\alpha) = u^k \wedge \alpha$$

  is an isomorphism. In particular, we obtain

$$b_{i-4} \leq b_i$$

  for (even) $i \leq 2n$. A generator in top cohomology $H^{4n}(M)$ is given by u^n. This defines a canonical orientation.

Proof. The first point is proven in theorem [66].6.6, p. 163, where it is shown that the Hodge decomposition of the twistor space is concentrated in terms $H^{p,p}(Z)$. The second item is due to [65].5.4, p. 403. The third assertion follows from our remark before stating the theorem. It is actually a consequence of the Hirsch lemma, since the spectral sequence of the fibration obviously degenerates at the E_2-term (due to vanishing of odd-degree cohomology groups of M).

The manifold M is simply-connected. Due to corollary 1.12 we have that $b_2 = 0$ for $M^n \not\cong \mathbf{Gr}_2(\mathbb{C}^{n+2})$. By the first point of this theorem we obtain that $b_3 = 0$. So $M^n \not\cong \mathbf{Gr}_2(\mathbb{C}^{n+2})$ is rationally 3-connected by the Whitehead theorem.

The Hard-Lefschetz property of M follows from the Hard-Lefschetz property of Z: Indeed, the class $z \in H^2(Z)$ is a Kähler class and thus M has the Hard-Lefschetz property with respect to the class $0 \neq u = z^2 \in H^4(M)$.

□

So for a Positive Quaternion Kähler Manifold M it is equivalent to demand that M be rationally 3-connected—i.e. to have that $\pi_1(M)\otimes\mathbb{Q} = \pi_2(M)\otimes\mathbb{Q} = \pi_3(M)\otimes\mathbb{Q} = 0$— and to require that M be π_2-finite—i.e. to suppose that $\pi_2(M) < \infty$.

The following consequence is as simple as it is astonishing.

Lemma 1.14. *Let M be rationally* 3*-connected of dimension* 20 *with $b_4 \leq 5$. Then it holds:*

$$b_6 = b_{10} \quad \vee \quad (b_4, b_8) \in \{(1,1), (2,3), (3,5), (4,7), (5,9)\} \tag{1.4}$$

Proof. By assumption $b_2 = 0$. Equation (1.3) becomes

$$4(2b_4 - b_8 - 1) = 5(b_{10} - b_6)$$

where the right hand side is non-negative due to Hard-Lefschetz (cf. 1.13). Hence the term $2b_4 - b_8 - 1$ must either be a positive multiple of 5 or zero. Since also $b_8 \geq b_4$, the first case may not occur for $b_4 \leq 5$. Thus it holds that $b_6 = b_{10}$. The existence of the form $0 \neq [u] \in H^4(M)$ show that $b_4 \geq 1$. □

A recognition theorem which relates cohomology to isometry type is the following:

Theorem 1.15. *Let M be a Positive Quaternion Kähler Manifold. If* $\dim M \leq 16$ *and* $b_4(M) = 1$*, then M is homothetic to the quaternionic projective space.*

Proof. See theorem [67].2.1.ii, p. 89. □

We remark that after corollary 1.12—the "strong rigidity" theorem—this is the second type of theorem that relates information from algebraic topology to the isometry type of M. If we wanted to be laconic, we could say that this gives us a feeling of even "stronger rigidity". Section 4.2 of chapter 4 is devoted to this kind of recognition theorems. In particular, we shall generalise the theorem in corollary 4.4 respectively theorem 4.5 to dimensions 20 and 24 respectively.

Another property of Positive Quaternion Kähler Manifolds related to their cohomology is the definiteness of the generalised intersection form, which is a symmetric bilinear form. This result was obtained by Fujiki (cf. [27]). He asserted *negative or positive* definiteness depending on degrees. Nagano and Takeuchi claim the *positive* definiteness of the intersection form in [61]. So we see that the sign depends on choices.

We shall reprove a precise formulation with our conventions. The theorem will be a consequence of the Hodge–Riemann bilinear relations on the twistor space Z. Again we shall use (a representative of) the form $u = -c_2(H)$ and the Kähler form $l = c_1(L)$. The orientations are naturally given by u^n on M respectively by l^{2n+1} on Z.

Theorem 1.16. *The generalised intersection form*

$$Q(x, y) = (-1)^{r/2} \int_M x \wedge y \wedge u^{n-r/2}$$

for $[x], [y] \in H^r(M^{4n}, \mathbb{R})$ *with even* $r \geq 0$ *is positive definite. In particular, the signature of the manifold satisfies*

$$\operatorname{sign}(M) = (-1)^n b_{2n}(M)$$

Proof. Recall that the Hodge decomposition of the twistor space Z of real dimension $4n + 2$ is given by

$$H^r(Z, \mathbb{C}) = H^{r/2, r/2}(Z),$$

for all $r \geq 0$; i.e. all the groups $H^{p,q}(Z)$ with $p \neq q$ vanish—cf. the proof of theorem [66].6.6, p. 163, respectively formula (6.4) on that page. We consider the Lefschetz decomposition (cf. [32], p. 122) on the complex cohomology of Z:

$$H^r(Z,\mathbb{C}) = H^{r/2,r/2}(Z) = \sum_{s \geq \max\{r-(2n+1),0\}} L^s H_0^{r/2-s,r/2-s}(Z)$$

where $H_0^*(\cdot)$ denotes primitive cohomology (cf. [32], p. 122) and $L^s(x) = l^s \wedge x$.

Recall the twistor fibration $\mathbb{C}\mathbf{P}^1 \hookrightarrow Z \xrightarrow{\pi} M$ with the Leray-Serre spectral sequence degenerating at the E_2-term. Thus for $r \in \mathbb{Z}$ we obtain that

$$H^r(Z,\mathbb{R}) = \pi^* H^r(M,\mathbb{R}) \oplus \pi^* H^{r-2}(M,\mathbb{R}) \cdot l$$

This formula holds with real coefficients. Tensoring with $\mathbb{C}$ makes it valid with complex coefficients. More precisely, we obtain the subalgebra $(\pi^* H^r(M,\mathbb{R})) \otimes \mathbb{C}$ of (complexified) *real* forms. That is, an element in $(\pi^* H^r(M,\mathbb{R})) \otimes \mathbb{C}$ is of the form $\eta + \bar{\eta}$ for $\eta \in H^r(Z,\mathbb{R})$.

We combine this with the Lefschetz decomposition of Z and compute

$$\begin{aligned} H^r(Z,\mathbb{C}) = &\left(\sum_{\substack{s \geq \max\{r-(2n+1),0\} \\ s \text{ even}}} L^s H_0^{r/2-s,r/2-s}(Z) \right) \\ &+ \left(\sum_{\substack{s \geq \max\{r-(2n+1),0\} \\ s \text{ even}}} L^s H_0^{r/2-s,r/2-s}(Z) \right) \cdot l \end{aligned}$$

Since $l^2 = u$ this grading actually enforces

$$\pi^* H^r(M,\mathbb{C}) = \sum_{\substack{s \geq \max\{r-(2n+1),0\} \\ s \text{ even}}} L^s H_0^{r/2-s,r/2-s}(Z) \tag{1.5}$$

From theorem [75].V.6.1, p. 203, we cite that—on a compact Kähler manifold X (of real dimension $2k$)—the form

$$\tilde{Q}(\eta,\mu) = \sum_{\max\{s \geq (r-k),0\}} (-1)^{[r(r+1)/2]+s} \int_X L^{k-r+2s}(\eta_s \wedge \mu_s)$$

with Lefschetz decompositions $\eta = \sum L^s \eta_s \in H^r(X,\mathbb{C})$ and $\mu = \sum L^s \mu_s \in H^r(X,\mathbb{C})$, i.e. with primitive η_s, μ_s, satisfies

$$\tilde{Q}(\eta, J\bar{\eta}) > 0$$

for $\eta \neq 0$ and $J = \sum_{p,q} i^{p-q}\Pi_{p,q}$ with canonical projections $\Pi_{p,q} : H^{p+q}(X,\mathbb{C}) \to H^{p,q}(X)$.

So in the case of the twistor space $X = Z$ we have $J = \mathrm{id}$, since $H^{p,p}(Z) = H^{2p}(Z,\mathbb{C})$. Let $0 \neq \eta = L^s\eta'$ be a real form in $H^{2p}(Z,\mathbb{C})$, i.e. $\eta = \eta' + \bar{\eta}'$ as above and $\bar{\eta} = \eta$, with primitive η'. Then we have that $\eta \in H^{p,p}(Z)$, which is equivalent to $\eta' \in H_0^{p-s,p-s}(Z) = H_0^{2(p-s)}(Z,\mathbb{C})$. Hence it holds that

$$\tilde{Q}(\eta,\eta) = \tilde{Q}(\eta, J\bar{\eta}) > 0$$

So we obtain (with $r = 2p$):

$$\begin{aligned} 0 &< \tilde{Q}(\eta,\eta) \\ &= (-1)^{[2p(2p+1)/2]+s} \int_Z L^{2n+1-2p+2s}(\eta' \wedge \eta') \\ &= (-1)^{s+p} \int_Z l^{2(n-p+s)+1} \wedge \eta' \wedge \eta' \\ &= (-1)^{s+p} \int_Z l^{2(n-p)+1} \wedge \eta \wedge \eta \end{aligned} \tag{1.6}$$

Now let η be an arbitrary element in $\pi^*H^r(M,\mathbb{R}) \otimes \mathbb{C}$. Then η is real and according to (1.5) we have

$$\eta = \sum_{s \text{ even}} l^s \wedge \eta_s$$

with primitive η_s.

Hence by (1.6) we derive that

$$\begin{aligned} 0 &< \sum_{s \text{ even}} (-1)^{s+p} \int_Z l^{2(n-p+s)+1} \wedge \eta_s \wedge \eta_s \\ &= \sum_{s \text{ even}} (-1)^{p} \int_Z l^{2(n-p+s)+1} \wedge \eta_s \wedge \eta_s \\ &= (-1)^{r/2} \int_Z l^{2n-r+1} \wedge \eta \wedge \eta \end{aligned}$$

for $0 \neq \eta \in \pi^*H^r(M,\mathbb{C})$. The twistor transform finally yields the assertion as we have:

$$(-1)^{r/2} \int_M u^{n-r/2} \wedge \eta \wedge \eta = (-1)^{r/2} \int_Z l^{2n-r+1} \wedge \eta \wedge \eta > 0$$

□

We remark that in low dimensions ($\dim M \leq 20$) this result fits to the computation of the signature of the Positive Quaternion Kähler Manifold M via the L-genus (combined with further equations obtained by Index Theory)—cf. chapter 4.

An oriented compact manifold M^{4n} is called *spin* if its $\mathbf{SO}(4n)$-structure bundle lifts to a $\mathbf{Spin}(4n)$-bundle, or equivalently, its second Stiefel-Whitney class $w_2(M) = 0$ vanishes. A vector bundle $E \to M$ is called *spin* if $w_1(E) = w_2(E) = 0$. (Thus a manifold is spin iff its tangent bundle is spin.)

Positive Quaternion Kähler Manifolds $M^{4n} \neq \mathbb{H}\mathbf{P}^n$ are spin if and only if n is even. Indeed, the second Stiefel-Whitney class $w_2(M)$ satisfies

$$w_2(M) = \begin{cases} \varepsilon & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

where ε is the obstruction class to a global lifting of $\mathbf{Sp}(n)\mathbf{Sp}(1)$—cf. proposition [66].2.3, p. 148.

Now Index Theory enters the stage. This approach led to some striking results and is likely to be fruitful in the future, too: Via Index Theory an alternative proof of the classification in dimension 8 was obtained (cf. theorem [54].5.4, p. 129), theorem 1.15 was established and the relations on Betti numbers in theorem 1.13 were found. The results on the isometry groups in dimensions 12 and 16 in theorem 1.18 below are also a consequence of Index Theory.

Let us briefly mention some basic notions from Index Theory. For general reference we recommend the textbooks [40] and [52].

A *genus* ϕ is a ring homomorphism $\phi : \Omega \otimes \mathbb{Q} \to R$ from the rationalised oriented cobordism ring $\Omega \otimes \mathbb{Q}$ into an integral domain R over $\mathbb{Q}$.

Let $Q(x) = 1 + a_2x^2 + a_4x^4 + \ldots$ be an even power series with coefficients in R. Assume the x_i to have degree 2. The product $Q(x_1) \cdot \cdots \cdot Q(x_n)$ is symmetric in the x_i^2. Therefore it may be expressed as a series in the elementary symmetric polynomials p_i of the x_i^2. The term of degree $4r$ is given by $K_r(p_1, \ldots, p_r)$, i.e.

$$\begin{aligned} Q(x_1) \cdot \cdots \cdot Q(x_n) =& 1 + K_1(p_1) + K_2(p_1, p_2) + \ldots \\ & + K_n(p_1, \ldots, p_n) + K_{n+1}(p_1, \ldots, p_n, 0) + \ldots \end{aligned}$$

On a compact oriented differentiable manifold M^{4n} the genus $\phi_Q : \Omega \otimes \mathbb{Q} \to R$ asociated to the power series Q is defined by

$$\phi_Q(M) = K_n(p_1, \ldots, p_n)[M]$$

where $p_i = p_i(M) \in H^{4i}(M, \mathbb{Z})$ are the Pontryagin classes of M. (On such manifolds with dimension not divisible by four the genus is set to zero.)

Prescribing the values of a genus on the complex projective spaces leads to a power series Q as above. As a consequence we see that there is a one-to-one correspondence between such power series and genera.

The genus ϕ_Q corresponding to the power series $Q(x) = \frac{x/2}{\sinh(x/2)}$ is called the *$\hat{A}$-genus*; the genus ϕ_Q corresponding to $Q(x) = \frac{x}{\tanh x}$ is the *L-genus*.

On a $4n$-dimensional spin manifold M one may define an elliptic differential operator

$$\not{D} \colon \Gamma(S^+) \to \Gamma(S^-)$$

called the *Dirac operator*. (The *spinor bundle* S of TM splits into the eigenbundles S^+ and S^- of a certain involution in the Clifford bundle $\mathrm{Cl}(M)$ of the tangent bundle TM.) The index of $\not{D}$ is given by

$$\mathrm{ind}(\not{D}) = \ker \not{D} - \mathrm{coker} \not{D}$$

In the same way one may construct *twisted Dirac operators*: In order to obtain a twisted Dirac operator locally defined by

$$\not{D}(E) : \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$$

we may use a similar definition for certain vector bundles E.

As a consequence of the famous *Atiyah–Singer Index Theorem* we obtain

$$\mathrm{ind}(\not{D}(E)) = \langle \hat{A}(M) \cdot \mathrm{ch}(E), [M] \rangle$$

Thus we see that the Index Theorem interlinks the concept of indices of differential operators with the notion of genera.

On a Positive Quaternion Kähler Manifold M^{4n} we have the locally associated bundles E and H from above. Now form the (virtual) bundles

$$\textstyle\bigwedge_0^k E := \bigwedge^k E - \bigwedge^{k-2} E$$

of exterior powers and the bundles

$$S^l H := \mathrm{Sym}^l H$$

of symmetric powers. Set

$$i^{k,l} := \mathrm{ind} \not{D}\big(\textstyle\bigwedge_0^k E \otimes S^l H \big)$$

The bundles $S^\pm \otimes \bigwedge_0^k E \otimes S^l H$ exist globally if and only if $n + k + l$ is even. In this case, using the index theorem one obtains the following relations (cf. [67], p. 117):

Theorem 1.17. *It holds:*

$$i^{k,l} = \begin{cases} 0 & \text{if } k+l<n \\ (-1)^k(b_{2p}(M)+b_{2p-2}(M)) & \text{if } k+l=n \\ d & \text{if } k=0,\ l=n+2 \end{cases}$$

where $d = \dim \mathrm{Isom}(M)$ *is the dimension of the isometry group of* M *and the* $b_i(M)$ *are the Betti numbers of* M *as usual.*

There is the following information on isometry groups:

Theorem 1.18. *Let* M^{4n} *be a Positive Quaternion Kähler Manifold with isometry group* $\mathrm{Isom}(M)$*. We obtain:*

- *The rank* $\mathrm{rk}\,\mathrm{Isom}(M)$ *may not exceed* $n+1$*. If* $\mathrm{rk}\,\mathrm{Isom}(M) = n+1$*, then* $M \in \{\mathbb{H}\mathbf{P}^n, \mathbf{Gr}_2(\mathbb{C}^{n+2})\}$*.*
- *If* $\mathrm{rk}\,\mathrm{Isom}(M) \geq \frac{n}{2}+3$*, then* M *is isometric to* $\mathbb{H}\mathbf{P}^n$ *or to* $\mathbf{Gr}_2(\mathbb{C}^{n+2})$*.*
- *It holds that* $\dim \mathrm{Isom}(M^{4n}) \leq \dim \mathbf{Sp}(n+1) = (n+1)(2n+3)$*. Equality holds if and only if* $M \cong \mathbb{H}\mathbf{P}^n$*.*
- *If* $n=3$*, then* $\dim \mathrm{Isom}(M) \geq 5$*; if* $n=4$*, then* $\dim \mathrm{Isom}(M) \geq 8$*.*

Proof. The first assertion is due to theorem [67].2.1, p. 89. The second item is theorem [20].1.1, p. 642. The inequality in the third assertion follows from corollary [68].3.3, p. 6. In case $\dim \mathrm{Isom}(M) = (n+1)(2n+3)$ it was already observed on [66], p. 161 that M is homothetic to the quaternionic projective space. The fourth point is due to theorem [66].7.5, p. 169. □

The isometry group $\mathrm{Isom}(M)$ of M is a compact Lie group. The subsequent lemma permits a better understanding of $\mathrm{Isom}(M)$.

Lemma 1.19. *Let* G *be a compact connected Lie group. Then we obtain:*

- *The group* G *possesses a finite covering which is isomorphic to the direct product of a simply-connected Lie group* $\tilde{G}$ *and a torus* T*. In particular,* $\tilde{G}$ *is also compact.*
- *The group* G *is semi-simple if and only if its fundamental group* $\pi_1(G)$ *is finite.*

Proof. See theorems [11].V.8.1, p. 233, and [11].V.7.13, p. 229. □

Table 1.3.: Classification of simple Lie groups up to coverings

type	corresponding Lie group (not necessarily simply-connected)	dimension
$\mathbf{A}_n$, $n = 1, 2, \ldots$	$\mathbf{SU}(n+1)$	$n(n+2)$
$\mathbf{B}_n$, $n = 1, 2, \ldots$	$\mathbf{SO}(2n+1)$	$n(2n+1)$
$\mathbf{C}_n$, $n = 1, 2, \ldots$	$\mathbf{Sp}(n)$	$n(2n+1)$
$\mathbf{D}_n$, $n = 3, 4, \ldots$	$\mathbf{SO}(2n)$	$n(2n-1)$
$\mathbf{G}_2$	$\mathrm{Aut}(\mathbb{O})$	14
$\mathbf{F}_4$	$\mathrm{Isom}(\mathbb{O}\mathbf{P}^2)$	52
$\mathbf{E}_6$	$\mathrm{Isom}((\mathbb{C} \otimes \mathbb{O})\mathbf{P}^2)$	78
$\mathbf{E}_7$	$\mathrm{Isom}((\mathbb{H} \otimes \mathbb{O})\mathbf{P}^2)$	133
$\mathbf{E}_8$	$\mathrm{Isom}((\mathbb{O} \otimes \mathbb{O})\mathbf{P}^2)$	248
the index n denotes the rank		

Consequently, up to finite coverings we may assume that $\mathrm{Isom}_0(M)$ is the product of a simply-connected semi-simple Lie group—i.e. the product of simply-connected simple Lie groups—and a torus.

For the convenience of the reader we give a table of simple Lie groups by 1.3, which will support our future arguments involving dimensions and ranks of Lie groups. From theorem [48].2.2, p. 13, we cite tables 1.4, 1.5, 1.6 of maximal connected subgroups (up to conjugation) of the classical Lie groups. (By $\mathrm{Irr}_\mathbb{R}$, $\mathrm{Irr}_\mathbb{C}$, $\mathrm{Irr}_\mathbb{H}$ real, complex and quaternionic irreducible representations are denoted. The tensor product "$\otimes$"y of matrix Lie groups is induced by the Kronecker product of matrices.)

Table 1.4.: Maximal connected subgroups of $\mathbf{SO}(n)$

subgroup	for
$\mathbf{SO}(k) \times \mathbf{SO}(n-k)$	$1 \leq k \leq n-1$
$\mathbf{SO}(p) \otimes \mathbf{SO}(q)$	$pq = n$, $3 \leq p \leq q$
$\mathbf{U}(k)$	$2k = n$
$\mathbf{Sp}(p) \otimes \mathbf{Sp}(q)$	$4pq = n$
$\varrho(H)$	H simple, $\varrho \in \mathrm{Irr}_\mathbb{R}(H)$, $\deg \varrho = n$

From [8], p. 219, we cite table 1.7 of maximal rank maximal connected subgroups. From table [48].2.1 we cite subgroups of maximal dimension in table 1.8.

Moreover, we briefly state

Lemma 1.20 (Connectivity lemma)**.** *Let $i : N^{4n} \to M^{4m}$ be an embedding of compact Quaternion Kähler Manifolds. Then the map i is $(2n-m+1)$-connected.*

Table 1.5.: Maximal connected subgroups of $\mathbf{SU}(n)$

subgroup	for
$\mathbf{SO}(n)$	
$\mathbf{Sp}(m)$	$2m = n$
$\mathbf{S}(\mathbf{U}(k) \times \mathbf{U}(n-k))$	$1 \leq k \leq n-1$
$\mathbf{SU}(p) \otimes \mathbf{SU}(q)$	$pq = n$, $p \geq 3$, $q \geq 2$
$\varrho(H)$	H simple, $\varrho \in \mathrm{Irr}_{\mathbb{C}}(H)$, $\deg \varrho = n$

Table 1.6.: Maximal connected subgroups of $\mathbf{Sp}(n)$

subgroup	for
$\mathbf{Sp}(k) \times \mathbf{Sp}(n-k)$	$1 \leq k \leq n-1$
$\mathbf{SO}(p) \otimes \mathbf{Sp}(q)$	$pq = n$, $p \geq 3$, $q \geq 1$
$\mathbf{U}(n)$	
$\varrho(H)$	H simple, $\varrho \in \mathrm{Irr}_{\mathbb{H}}(H)$, $\deg \varrho = 2n$

Proof. This follows directly from theorem [19].A3, p. 150, applied to the case when $N_1 = N_2 = N$. □

This lemma is a vital tool in the proof of the second point of theorem 1.18. We shall apply it mainly to fixed-point components. So let us comment briefly on the structure of $\mathbb{S}^1$-fixed-point components and of $\mathbb{Z}_2$-fixed-point components of a Positive Quaternion Kähler Manifold M^{4n}.

Let $H \subseteq \mathrm{Isom}(M)$ be either $\mathbb{S}^1$ or $\mathbb{Z}_2$. Consider the isotropy representation at an H-fixed-point $x \in M$ composed with the canonical projection $\mathbf{Sp}(1) \to \mathbf{SO}(3)$:

$$\varphi : H \hookrightarrow \mathbf{Sp}(n)\mathbf{Sp}(1) \to \mathbf{SO}(3)$$

Theorems [15].4.4, p. 602, and [15].5.1, p. 606, (together with [15], p. 600) show that the type of the fixed-point component F of H around x depends on the image of φ: If $\varphi(H) = 1$, the component F is quaternionic for $H \in \{\mathbb{S}^1, \mathbb{Z}_2\}$. If $\varphi(H) \neq 1$, the component F is locally Kählerian for $H = \mathbb{Z}_2$ and Kählerian for $H = \mathbb{S}^1$. A result by Gray (cf. [30]) shows that a quaternionic submanifold is totally geodesic. Thus the quaternionic components are again Positive Quaternion Kähler Manifolds.

It is easy to see that the dimension of F is exactly $2n$ if $H = \mathbb{Z}_2$ and given that F is locally Kählerian. The dimension of F is smaller than or equal to $2n$ if $H = \mathbb{S}^1$ and provided that F is Kählerian.

Table 1.7.: Maximal rank maximal connected subgroups

ambient group	subgroup
$\mathbf{SU}(n)$	$\mathbf{S}(\mathbf{U}(i)\times\mathbf{U}(n-i-1))$ for $i\geq 1$
$\mathbf{SO}(2n+1)$	$\mathbf{SO}(2n)$, $\mathbf{SO}(2i+1)\times\mathbf{SO}(2(n-i))$ for $1\leq i\leq n-1$
$\mathbf{Sp}(n)$	$\mathbf{Sp}(i)\times\mathbf{Sp}(n-i)$ for $i\geq 1$, $\mathbf{U}(n)$
$\mathbf{SO}(2n)$	$\mathbf{SO}(2i)\times\mathbf{SO}(2(n-i))$ for $i\geq 1$, $\mathbf{U}(n)$
$\mathbf{G}_2$	$\mathbf{SO}(4)$, $\mathbf{SU}(3)$

Table 1.8.: Subgroups of maximal dimension

ambient group	subgroup
$\mathbf{SU}(n)$, $n\neq 4$	$\mathbf{S}(\mathbf{U}(1)\times\mathbf{U}(n+1))$
$\mathbf{SU}(4)$	$\mathbf{Sp}(2)$
$\mathbf{SO}(n)$	$\mathbf{SO}(n-1)$
$\mathbf{Sp}(n)$, $n\geq 2$	$\mathbf{Sp}(n-1)\times\mathbf{Sp}(1)$
$\mathbf{G}_2$	$\mathbf{SU}(3)$

What's wrong with Positive Quaternion Kähler Geometry?

Let us finally mention some irritations we encountered when studying the established theory. This section is intended to give the interested reader the possibility to successfully cope with certain ambiguities that she or he might face.

Foremost we shall not miss to draw the reader's attention to a classification paper by Kobayashi (and originally also by Onda) in which the author(s) claim(s) to have proved the main conjecture (cf. 1.6) in all dimensions (≥ 8) via Ricci flow. Until today this article has appeared in several versions with minor changes on `arxiv.org`. However, experts on Ricci flow seem to be rather sceptical as far as the correctness of the proof is concerned.

In theorem [43].2.6 the author asserts to have found a generalisation of the connectivity lemma (cf. lemma 1.20) for fixed-point components of isometric Lie group actions which involves the dimension of a principal orbit of the group. The result is supposed to follow in the same fashion as the analogous result in positive sectional curvature (cf. theorem [76].2.1, p. 263). There is at least one more article by the same author building upon this result (cf. [44]). However, personal communication with Fuquan Fang lets us doubt the existence of well-elaborated proofs. Moreover, Burkhard Wilking convinced us that the lack of written proof certainly is not for reasons of triviality of the result—if it is correct at all.

In the survey article [67] on page 117 it is remarked that the index $i^{1,n+1}$ vanishes on a Positive Quaternion Kähler Manifold other than the quaternionic projective space.

The given reference (cf. [53]), however, only seems to prove $i^{1,n+1} \leq 0$. Unfortunately, also the powerful methods developed by Uwe Semmelmann and Gregort Weingart did not result in a confirmation of the assertion. As Uwe Semmelmann found out, originally also Simon Salamon considered the statement to be rather of conjectural nature. We want to encourage the reader to prove the result himself, as it has immense consequences at least in low dimensions—see for example theorem 4.11.

Note that theorem [35].1.1, p. 2, asserts that a 16-dimensional Positive Quaternion Kähler Manifold has fourth Betti number $b_4 \neq 2$. In the proof of the theorem $b_4 = 2$ is led to a contradiction. In claim 1 on page 4 it is asserted that c_2 is not proportional to u as this would lead to $M \cong \mathbb{H}\mathbf{P}^4$. This is supposed to follow directly from a result in [28]. Indeed, theorem [28].5.1, p. 62, states that if $b_4 = 1$, which implies that c_2 necessarily is a multiple of u in particular, then $M \cong \mathbb{H}\mathbf{P}^4$. However, the computations in this proof make use of $b_4 = 1$ and one has to do separate computations for the case $b_4 = 2$ to confirm the result.

Finally, we point to chapter 2, where we explain why the classification of 12-dimensional Positive Quaternion Kähler Manifolds established in [35] is erroneous and does no longer persist.

1.2. A brief history of Rational Homotopy Theory

Homotopy groups are an intriguing field of study: They may be defined in a very basic way. Yet, their computation turns out to be extremely difficult. So till today there is not even a complete picture of the homotopy groups of spheres. It was a striking observation by Quillen and Sullivan that "rationalising the groups" made the situation much simpler. This can be observed on spheres already as for example

$$\pi_{2n}(\mathbb{S}^{2n}) \otimes \mathbb{Q} = \pi_{4n-1}(\mathbb{S}^{2n}) \otimes \mathbb{Q} = \mathbb{Q}$$

and all the other groups vanish rationally. Moreover, Quillen and Sullivan managed a translation into algebra. This gave birth to *Rational Homotopy Theory*. Over the years this theory has been well-elaborated and it has become a powerful and highly elegant tool.

As a main reference we want to recommend the book [22]. Throughout the dissertation we shall follow its conceptual approach and its notation unless indicated differently. Let us now review some basic notions we shall need in the main part of the thesis. We do not even attempt to provide a "complete"—of any kind—introduction to the topic. Likewise, we shall not cite lengthy but standard definitions, which all can be found in the literature, for the general theory. Our goal is to motivate and to guide through the well-established notions and to be more precise when concepts are of greater importance for our present work or if they are more particular in nature. We shall give references whenever parts of the theory are not covered by [22].

Although the theory may be presented in larger generality (e.g. for nilpotent spaces), we make the following general assumption:

Throughout this chapter the spaces in consideration are assumed to be simply-connected (which comprises being path-connected). So are the commutative differential graded algebras A, i.e. A^0 is one-dimensional and $A^1 = 0$. Cohomology will always be taken with rational coefficients whenever coefficients are suppressed.

Part 1. . . . wherein we shall describe some basic concepts of Rational Homotopy Theory.

The *rational homotopy type* of a simply-connected topological space X is the homotopy type of a *rationalisation* $X_\mathbb{Q}$ of X. By the definition of rationalisation there is a *rational homotopy equivalence* $\phi : X \xrightarrow{\simeq} X_\mathbb{Q}$, i.e. a morphism inducing isomorphisms

$$\pi_*(X) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_*(X_\mathbb{Q})$$

on homotopy groups. The rationalisation $X_\mathbb{Q}$ can always be constructed, it can be given the structure of a CW-complex and it is unique up to homotopy (cf. [22].9.7). By the Whitehead-Theorem $\pi_*(\phi)$ being an isomorphism is equivalent to $H^*(\phi)$ being an isomorphism. Spaces X and Y have the same rational homotopy type, i.e. $X \simeq_\mathbb{Q} Y$, iff there is a finite chain of rational homotopy equivalences (cf. [22].9.8)

$$X \xrightarrow{\simeq} \ldots \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} \ldots \xleftarrow{\simeq} Y$$

The transition into algebra is mainly given by the functor $\mathrm{A_{PL}}$ which replaces the space X by the *commutative differential graded algebra* $\mathrm{A_{PL}}(X)$ (cf. [22].3, [22].10). This functor is a *local system* $\mathrm{A_{PL}}(\cdot)$ on the simplicial set of singular simplices of X. On a smooth manifold M it is sufficient to consider the commuative differential graded algebra—over the reals(!)—$\mathrm{A_{DR}}(M)$ of ordinary differential forms instead of the polynomial differential forms $\mathrm{A_{PL}}(M)$. Yet observe that the ordinary cochain algebra of singular simplices is not commutative so that this new concept really is necessary. A very brief description of $\mathrm{A_{PL}}$ could be that to first construct the algebra of polynomial differential forms $\mathrm{A_{PL}}_n$ on an ordinary affine n-simplex by simply restricting ordinary differential forms (with $\mathbb{Q}$-coefficients). Then the polynomial p-forms of X are the simplicial set morphisms assigning to a singular n-simplex in X a polynomial p-form in $\mathrm{A_{PL}}_n$.

A morphism ϕ of commutative differential graded algebras (A, d) and (B, d) is called a *quasi-isomorphism* if $H^*(\phi) : H(A) \to H(B)$ is an isomorphism. A *weak equivalence* between A and B, i.e. $A \simeq B$, is a finite chain of quasi-isomorphisms:

$$A \xrightarrow{\simeq} \ldots \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} \ldots \xleftarrow{\simeq} B$$

This may be regarded as an algebraic analogue of a rational homotopy equivalence.

Indeed, the functor $\mathrm{A_{PL}}$ now connects the topological world to the algebraic one in a very beautiful way: Focussing on simply-connected spaces and algebras both having additionally rational cohomology of finite type, there is a 1-1-correspondence (cf. [22].10):

$$\left\{ \begin{array}{c} \text{rational homotopy} \\ \text{types} \end{array} \right\} \xrightarrow{\mathrm{A_{PL}}} \left\{ \begin{array}{c} \text{weak equivalence classes of} \\ \text{commutative cochain algebras} \end{array} \right\}$$

Recall that a graded vector space $V = \bigoplus_i V^i$ is called of *finite type* if each V^i is finite-dimensional. A *cochain algebra* is a differential graded algebra concentrated in non-negative degrees. There is a procedure of *spatial realisation* (cf. [22].17) which reverses $\mathrm{A_{PL}}$ to a certain degree by constructing a CW-complex from a commutative cochain algebra.

By this bijection we need no longer differentiate between (weak equivalence classes of) commutative cochain algebras and (homotopy types of) topological spaces.

As a preliminary conclusion we observe that the rational homotopy type of a space X is encoded in $\mathrm{A_{PL}}(X)$ up to weak equivalence and so is the real homotopy type of a smooth manifold M with respect to $\mathrm{A_{DR}}(M)$.

In order to be able to do effective computations one may find some sort of "standard form" within the weak equivalence class, a so-called *minimal Sullivan algebra*—cf. [22].12. This commutative differential graded algebra is built upon the tensor product $\bigwedge V = \mathrm{Sym}(V^{\mathrm{even}}) \otimes \bigwedge(V^{\mathrm{odd}})$ of the polynomial algebra in even degrees with the exterior algebra in odd degrees of the graded vector space V. Minimality is established by requiring that the differential may not hit an element that is not *decomposable*. An element in $\bigwedge V$ is called *decomposable* if it is a linear combination of non-trivial products of elements of lower degree. More precisely, minimality is given by

$$\operatorname{im} \mathrm{d} \subseteq \bigwedge^{>0} V \cdot \bigwedge^{>0} V$$

A *(minimal) Sullivan model* for a commutative cochain algebra (A, d) respectively a topological space X is a quasi-isomorphism

$$m : \left(\bigwedge V, \mathrm{d}\right) \xrightarrow{\simeq} (A, \mathrm{d})$$

respectively

$$m\colon \big(\bigwedge V, \mathrm{d}\big) \xrightarrow{\simeq} (\mathrm{A}_{\mathrm{PL}}(X), \mathrm{d})$$

with a (minimal) Sullivan algebra $\big(\bigwedge V, \mathrm{d}\big)$.

Before we shall continue to describe the theory we shall comment briefly on notation. Otherwise, there might be room for equivocation. Although the following does not differ from the standard notation in [22], it might be adequate to state it at this initial point of the introduction in a condensed form.

Let $V = \bigoplus_i V^i$ be a graded vector space. By V^k we denote its k-th degree subspace and we shall use the terminology $V^{\leq k} = \bigoplus_{i \leq k} V^i$, $V^{>0} = \bigoplus_{i>0} V^i$, $\ldots$. By $\bigwedge V$ we denote, as described, the free commutative graded algebra (cf. example [22].3.6, p. 45) over V. We shall use the term $\bigwedge^k V$ to denote all the elements of $\bigwedge V$ that have word-length k in V. So, for example, $\bigwedge^{>0} V$ refers to all the elements that can be written as sums of real products of elements in V. Finally, it is not surprising that the grading $\big(\bigwedge V\big)^k = \{x \in \bigwedge V \mid \deg x = k\}$ again refers to degree.

Let us give two examples of constructions of Sullivan models in special situations. We stress that the models will not be minimal in general.

First, we shall deal with fibrations. Here a Sullivan model for the total space is constructed out of the Sullivan models of fibre and base space. Let

$$F \hookrightarrow E \to B$$

be a fibration of path-connected spaces. Suppose that B is simply-connected and that one of $H_*(F)$, $H_*(B)$ has finite type. Suppose that $\big(\bigwedge V_B, \mathrm{d}\big) \to (\mathrm{A}_{\mathrm{PL}}(B), \mathrm{d})$ is a Sullivan model and that $\big(\bigwedge V_F, \bar{\mathrm{d}}\big) \to (\mathrm{A}_{\mathrm{PL}}(F), \mathrm{d})$ is a minimal Sullivan model. We then have

Theorem 1.21. *There is a Sullivan model* $\big(\bigwedge V_B \otimes \bigwedge V_F, \mathrm{d}\big) \to (\mathrm{A}_{\mathrm{PL}}(E), \mathrm{d})$ *such that* $\big(\bigwedge V_B, \mathrm{d}\big)$ *is a subcochain algebra of* $\big(\bigwedge V_B \otimes \bigwedge V_F, \mathrm{d}\big)$*. Moreover, it holds for* $v \in V_B$ *that* $\mathrm{d}v - \bar{\mathrm{d}}v \in \bigwedge^{>0} V_B \otimes \bigwedge V_F$*.*

PROOF. See the corollary on [22], p. 199. □

Even if we choose minimal Sullivan models for base and fibre the model of the total space needs not be minimal. We shall refer to this model as the *model of the fibration*.

An application of this model to homogeneous spaces provides a model for G/H based upon models for the groups G and H—the second example we shall display.

Let H be a closed connected subgroup of a connected Lie group G. Denote by $j\colon H \hookrightarrow G$ the inclusion and by $\mathbf{B}H$ and $\mathbf{B}G$ the corresponding classifying spaces of

H and G. It is known (cf. [22], example 12.3, p. 143) that H-spaces—and Lie groups in particular—admit minimal models of the form

$$\big(\bigwedge V_G, 0\big) \to (\mathrm{A_{PL}}(G), \mathrm{d}) \qquad \text{and} \qquad \big(\bigwedge V_H, 0\big) \to (\mathrm{A_{PL}}(H), \mathrm{d})$$

A direct consequence are the minimal models

$$\big(\bigwedge V_G^{+1}, 0\big) \to (\mathrm{A_{PL}}(\mathbf{B}G), \mathrm{d}) \qquad \text{and} \qquad \big(\bigwedge V_H^{+1}, 0\big) \to (\mathrm{A_{PL}}(\mathbf{B}H), \mathrm{d})$$

which arise from a degree-shift of $+1$ applied to the graded vector spaces V_G and V_H lying at the basis of the minimal models of G and H (cf. [22].15.15, p. 217). That is, $V_{\mathbf{B}G} := V_G^{+1}$ and $V_{\mathbf{B}H} := V_H^{+1}$. Let $x_1, \ldots, x_r$ be a homogeneous basis of V_G. Thus we have a corresponding basis $\bar{x}_1, \ldots, \bar{x}_r$ of $V_{\mathbf{B}G}$ by degree-shift.

Theorem 1.22. *The Sullivan algebra* $\big(\bigwedge V_{\mathbf{B}G} \otimes \bigwedge V_G, \mathrm{d}\big)$ *defined by* $\mathrm{d}|_{V_{\mathbf{B}H}} = 0$ *and by* $\mathrm{d}x_i = H^*(\mathbf{B}(j))\bar{x}_i$ *for* $1 \le i \le r$ *is a Sullivan model for* G/H*.*

Proof. See proposition [22].15.16, p. 219. □

Again, this model by no reasons has to be minimal. This approach can be generalised to the case of *biquotients*:

Let G be a compact connected Lie group and let $H \subseteq G \times G$ be a closed Lie subgroup. Then H acts on G on the left by $(h_1, h_2) \cdot g = h_1 g h_2^{-1}$. The orbit space of this action is called the *biquotient* $G /\!\!/ H$ of G by H. If H is of the form $H = H_1 \times H_2$ with inclusions given by $H_1 \subseteq G \times \{1\} \subseteq G \times G$ and $H_2 \subseteq \{1\} \times G \subseteq G \times G$, we shall also use the notation $H_1 \backslash G / H_2$ instead of $G /\!\!/ H$. If the action of H on G is free, then $G /\!\!/ H$ possesses a manifold structure. This is the only case we shall consider. Clearly, the category of biquotients contains the one of homogeneous spaces.

In the notation from above we see that a minimal model for $G \times G$ is given by $\big(\bigwedge V_{G\times G}, 0\big)$ defined by

$$V_{G\times G} = \langle x_1, \ldots, x_r, y_1, \ldots y_r\rangle$$

where the y_i satisfy $\deg x_i = \deg y_i$. Thus we obtain $V_{\mathbf{B}G\times\mathbf{B}G} = \langle \bar{x}_1, \ldots, \bar{x}_r, \bar{y}_1, \ldots \bar{y}_r\rangle$. Denote by j the inclusion $H \hookrightarrow G \times G$. Then we obtain

Theorem 1.23. *The Sullivan algebra*

$$\big(\bigwedge V_{\mathbf{B}H} \otimes \bigwedge \langle q_1, \ldots, q_r\rangle, \mathrm{d}\big)$$

with $\mathrm{d}|_{\bigwedge V_{\mathbf{B}H}} = 0$ *and defined by* $\mathrm{d}(q_i) = H^*(\mathbf{B}j)(\bar{x}_i - \bar{y}_i)$ *for* $1 \le i \le r$ *is a Sullivan model for* $G /\!\!/ H$*.*

Proof. See proposition [42].1, p. 2. □

Now we present an algorithm that iteratively constructs a minimal model in the case of a trivial differential. This algorithm is adapted from [22], p. 144–145 from the general case to the case $\mathrm{d} = 0$ that will be of interest for us.

Algorithm 1.24. Let $(A, 0)$ be a commutative differential graded algebra with $A^0 = \mathbb{Q}$ and $A^1 = 0$. Then the following procedure produces a minimal Sullivan model for $(A, 0)$.

Choose $m_2 : (\bigwedge V^2, 0) \to (A, 0)$ such that $H^2(m_2) : V^2 \xrightarrow{\cong} A^2$ is an isomorphism.
Suppose now $m_k : (\bigwedge V^{\leq k}, d) \to (A, 0)$ has been constructed for some $k \geq 2$. Then we may extend this morphism to $m_{k+1} : (\bigwedge V^{\leq k+1}, d) \to (A, 0)$ by the following procedure:
Choose elements $a_i \in A^{k+1}$ such that

$$A^{k+1} = \operatorname{im} H^{k+1}(m_k) \oplus \bigoplus_i \mathbb{Q} \cdot [a_i]$$

and choose cocyles $z_j \in (\bigwedge V^{\leq k})^{k+2}$ such that

$$\ker H^{k+2}(m_k) = \bigoplus_j \mathbb{Q} \cdot [z_j]$$

Let V^{k+1} be a vector space (in degree $k+1$) with basis $\{v_i, \dots, w_j, \dots\}$ in one-to-one correspondence with the elements $\{a_i\}$ respectively $\{z_j\}$. Write $\bigwedge V^{\leq k+1} = \bigwedge V^{\leq k} \otimes \bigwedge V^{k+1}$. Extend d to a derivation in $\bigwedge V^{\leq k+1}$ and m_k to a morphism $m_{k+1} : \bigwedge V^{\leq k+1} \to A$ of graded algebras by setting

$$\mathrm{d}v_i = 0, \qquad \mathrm{d}w_j = z_j \qquad \text{and} \qquad m_{k+1}v_i = a_i, \qquad m_{k+1}w_j = 0.$$

Observe that the a_i and z_j can be interpreted in the following way: One may regard $\operatorname{im} H^{k+1}(m_k)$ as the "surviving products" of lower degrees and thus the a_i represent "new" generators, i.e. generators of A^{k+1} that are not a linear combination of products of lower degree. Conversely, the z_j are placed—so to say—one for each vanishing linear combination; i.e. if a linear combination of products in degree $k+2$ vanishes (in A), one sets a generator in degree $k+1$ (in V), which corrects this deviation from freeness by means of d. ⊡

Finally note the following simple property.

Proposition 1.25. *Let a cochain algebra* $\left(\bigwedge V, \mathrm{d}\right)$ *satisfy the properties*

$$V = V^{\geq 2} \qquad \text{and} \qquad \operatorname{im} \mathrm{d} \subseteq \bigwedge^{>0} V \cdot \bigwedge^{>0} V$$

Then this algebra is necessarily a minimal Sullivan algebra.

Proof. We elaborate the arguments in [22], example 12.5, p. 144: As a filtration on V we use the filtration on degrees: $V(k) := \bigoplus_{i=1}^{k} V^k$. Then trivially $V = \bigcup_{k=0}^{\infty} V(k)$, $V(0) \subseteq V(1) \subseteq V(2) \subseteq \ldots$ and $\mathrm{d} = 0$ in $V(0) = V^0 = \mathbb{Q}$. We have $\mathrm{d} : V^k \to \bigwedge^{k+1} V$ by definition and thus $\mathrm{d} : V^k \to \big(\bigwedge^{>0} V \cdot \bigwedge^{>0} V\big)^{k+1}$ by assumption. Moreover,

$$\big(\bigwedge^{>0} V \cdot \bigwedge^{>0} V\big)^{k+1} \subseteq \bigwedge V^{\leq k-1}$$

since $V^1 = 0$. Hence

$$\mathrm{d} : V(k) = \bigoplus_{i=0}^{k} V^i \to \bigoplus_{i=0}^{k+1} \big(\bigwedge^{>0} V \cdot \bigwedge^{>0} V\big)^i \subseteq \bigwedge \Big(\bigoplus_{i=0}^{k-1} V^i\Big) = \bigwedge V(k-1)$$

for $k \geq 1$. This reveals $\big(\bigwedge V, \mathrm{d}\big)$ as a Sullivan algebra. Minimality is clear by assumption. □

The most prominent theorem of Rational Homotopy Theory now connects the graded vector space V upon which the minimal model $\big(\bigwedge V, \mathrm{d}\big)$ of a topological space X is based with the rational homotopy groups of X. Let $\mathbb{K}$ be a field of characteristic zero.

Theorem 1.26. *Suppose that X is simply-connected and $H_*(X, \mathbb{K})$ has finite type. Then there is an isomorphism of graded vector spaces*

$$\nu_X : V \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(\pi_*(X), \mathbb{K})$$

Proof. See theorem [22].15.11, p. 208. □

In particular, up to duality, the rational vector space $\pi_i(X) \otimes \mathbb{Q}$ in degree $i \geq 2$ corresponds exactly to V^i. This provides a tremendously effective way of computing rational homotopy!

A topological space X is called n-connected if for $0 \leq i \leq n$ it holds $\pi_i(X) = 0$. In this vein we shall call X *rationally n-connected* if for $0 \leq i \leq n$ it holds $\pi_i(X) \otimes \mathbb{Q} = 0$. Thus for the minimal model of a simply-connected space X with homology of finite type this is equivalent to $V^i = 0$ for $0 \leq i \leq n$.

Remark 1.27. Recall the Hurewicz homomorphism $\mathrm{hur} : \pi_*(X) \to H_*(X, \mathbb{Z})$ (cf. [22], p. 58). We have $H^{>0}\big(\bigwedge V, \mathrm{d}\big) = \ker \mathrm{d} / \mathrm{im}\, \mathrm{d}$, where $\mathrm{im}\, \mathrm{d} \subseteq \bigwedge^{\geq 2} V$ by minimality. Let $\zeta : H^{>0}\big(\bigwedge V, \mathrm{d}\big) \to V$ be the linear map defined by forming the quotient with $\bigwedge^{\geq 2} V$, i.e. the natural projection

$$\zeta : \ker \mathrm{d} / \mathrm{im}\, \mathrm{d} \to \ker \mathrm{d} \Big/ \Big(\Big(\bigwedge^{\geq 2} V\Big) \cap \ker \mathrm{d}\Big) \subseteq V$$

There is a commutative square (cf. [22], p. 173 and the corollary on [22], p. 210)

$$\begin{array}{ccc} H^{>0}\big(\bigwedge V, \mathrm{d}\big) & \xrightarrow{\cong} & H^{>0}(X) \\ \zeta \big\downarrow & & \big\downarrow \mathrm{hur}^* \\ V & \xrightarrow[\nu_X]{\cong} & \mathrm{Hom}(\pi_{>0}(X), \mathbb{Q}) \end{array}$$

where hur^* is the dual of the rationalised Hurewicz homomorphism. Thus by this square ζ and hur^* are identified.

In particular, we see that if $\mathrm{hur} \otimes \mathbb{Q}$ is injective, then hur^* is surjective and so is ζ. In such a case every element $x \in V$ is closed and defines a homology class. ⊡

Part 2. . . . wherein we shall outline the notion of formality.

As we have seen, the rational (respectively real) homotopy type of a space X (respectively a smooth manifold M) is given by $\mathrm{A_{PL}}(X)$ (respectively $\mathrm{A_{DR}}(M)$). A prominent question in Rational Homotopy Theory is the following: When can we replace $\mathrm{A_{PL}}(X)$ by the cohomology algebra $H^*(X)$ without losing information? When is the rational homotopy type of X encoded in its rational cohomology algebra?

Let $\mathbb{K}$ be a field of characteristic zero and (A, d) a connected commutative cochain algebra over $\mathbb{K}$.

Definition 1.28. The commutative differential graded algebra (A, d) is called *formal* if it is weakly equivalent to the cohomology algebra $(H(A, \mathbb{K}), 0)$ (with trivial differential).

We call a path-connected topological space *formal* if its rational homotopy type is a formal consequence of its rational cohomology algebra, i.e. if $(\mathrm{A_{PL}}(X), \mathrm{d})$ is formal. In detail, the space X is formal if there is a weak equivalence $(\mathrm{A_{PL}}(X), \mathrm{d}) \simeq (H^*(X), 0)$, i.e. a chain of quasi-isomorphisms

$$(\mathrm{A_{PL}}(X), \mathrm{d}) \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} \ldots \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} (H^*(X), 0)$$

Theorem 1.29. *Let X have rational homology of finite type. The algebra $(\mathrm{A_{PL}}(X, \mathbb{K}), \mathrm{d})$ is formal for any field extension $\mathbb{K} \supseteq \mathbb{Q}$ if and only if X is a formal space.*

Proof. See [22], p. 156 and theorem 12.1, p. 316. □

Corollary 1.30. *A compact connected smooth manifold is formal iff there is a weak equivalence $(\mathrm{A_{DR}}(M), \mathrm{d}) \simeq (H^*(M, \mathbb{R}), 0)$.*

Example 1.31. Suppose that $n \geq 0$. A minimal model for the odd-dimensional sphere is given by

$$\bigwedge(e, \mathrm{d} = 0) \simeq \mathrm{A_{PL}}(\mathbb{S}^{2n+1}) \qquad \deg e = 2n + 1$$

A minimal model for the even-dimensional sphere is given by

$$\bigwedge(e, e', \mathrm{d}) \simeq \mathrm{A_{PL}}(\mathbb{S}^{2n}) \qquad \deg e = 2n,\ \deg e' = 4n-1,\ \mathrm{d}e = 0,\ \mathrm{d}e' = e^2$$

Since the maps

$$\bigwedge(\langle e\rangle, 0) \xrightarrow{\simeq} H^*(\mathbb{S}^{2n+1})$$
$$\bigwedge(\langle e, e'\rangle, \mathrm{d}) \xrightarrow{\simeq} H^*(\mathbb{S}^{2n})$$

induced by $e \mapsto [\mathbb{S}^{2n+1}]$, respectively by $e \mapsto [\mathbb{S}^{2n}], e' \mapsto 0$ are quasi-isomorphisms, the spheres are formal spaces. Since the k-th-degree subspace V^k of the graded vector space underlying the minimal model is dual to $\pi_k(X) \otimes \mathbb{Q}$ of the respective topological space (cf. [22].15.11, p. 208), we may read off the rational homotopy groups of spheres which we presented initially in this chapter. ⊡

Let us give another example quite similar in nature.

Example 1.32. Suppose the finite-dimensional homology algebra $H(A, \mathrm{d})$ of a commutative differential graded algebra (A, d) has exactly one generator $0 \neq e$ with $\deg e = n$ being even. Then this algebra is formal.

This may be seen as follows: The homology algebra of A necessarily is a truncated polynomial algebra of the form $H(A, \mathrm{d}) = \mathbb{Q}[e]/e^d$ for some $d > 1$. The minimal model $(\bigwedge V, \mathrm{d})$ of (A, d) can be constructed by means of the algorithm on [22] pages 144 and 145—the general version of algorithm 1.24. So we obtain

$$V^n = H^n(A, \mathrm{d}) = \langle e\rangle$$
$$V^{n\cdot d-1} = \langle e'\rangle$$
$$V^i = 0 \qquad \text{for } i \notin \{n, n\cdot d - 1\}$$

The differential of the minimal model then is defined by $\mathrm{d}e = 0$ and $\mathrm{d}e' = e^d$. The morphism $\mu : \left(\bigwedge V, \mathrm{d}\right) \to (H(A, \mathrm{d}), 0)$ of commutative differential graded algebras defined by $\mu(e) = e$ and $\mu(e') = 0$ is a quasi-isomorphism and the algebra (A, d) is formal. ⊡

Using these minimal models of spheres we shall state the following remark illustrating example [22].15.4, p. 202. For this we make

Definition 1.33. A fibration

$$F \hookrightarrow E \xrightarrow{p} B$$

is called a *spherical fibration* if the fibre F is path-connected and has the rational homotopy type of a connected sphere, i.e. $F \simeq_{\mathbb{Q}} \mathbb{S}^n$ for $n \geq 1$.

Remark 1.34. We shall see that a spherical fibration admits a pretty simple model:

Suppose first that n is odd. By the model of the fibration and example 1.31 we form the quasi-isomorphism

$$\big(\bigwedge V \otimes \bigwedge\langle e\rangle, \mathrm{d}\big) \xrightarrow{\simeq} (\mathrm{A_{PL}}(E), \mathrm{d})$$

with $\mathrm{d}e =: u \in \bigwedge V$.

Let us now suppose n to be even and let $\big(\bigwedge V, \mathrm{d}\big) \xrightarrow{\simeq} (\mathrm{A_{PL}}(B), \mathrm{d})$ be a minimal Sullivan model. Using the model of the fibration and example 1.31 we can form a quasi-isomorphism

$$\big(\bigwedge V \otimes \bigwedge\langle e, e'\rangle, \mathrm{d}\big) \xrightarrow{\simeq} (\mathrm{A_{PL}}(E), \mathrm{d})$$

with $\mathrm{d}e \in \bigwedge V$ and $\mathrm{d}e' = ke^2 + a \otimes e + b$ for $k \in \mathbb{Q}$, $a, b \in \bigwedge V$. Without restriction we may assume $k = 1$. Since $\mathrm{d}^2 e' = 0$, we compute:

$$0 = \mathrm{d}(\mathrm{d}e') = \mathrm{d}(e^2 + ae + b) = 2(\mathrm{d}e)e + (\mathrm{d}a)e + a(\mathrm{d}e) + \mathrm{d}b$$

As $\big(\bigwedge V, \mathrm{d}\big)$ forms a differential subalgebra in the model of the fibration, we obtain $\mathrm{d}a = -2\mathrm{d}e$. Set $\tilde{e} := e + \frac{a}{2} \in \bigwedge(V \oplus \langle e\rangle)$ and obtain $\mathrm{d}\tilde{e} = 0$ and $\mathrm{d}e' = \tilde{e}^2 + (b - \frac{a^2}{4})$ with $\mathrm{d}u = 0$ for $u := (\frac{a^2}{4} - b) \in \bigwedge V$. So form the (abstract) commutative differential graded algebra

$$\big(\bigwedge V \otimes \bigwedge\langle \tilde{e}, e'\rangle, \tilde{\mathrm{d}}\big)$$

with the inherited grading and where the differential satisfies that $\tilde{\mathrm{d}}|_{\bigwedge V} = \mathrm{d}|_{\bigwedge V}$, $\tilde{\mathrm{d}}\tilde{e} = 0$ and that $\tilde{\mathrm{d}}e' = \tilde{e}^2 + (b - \frac{a^2}{4})$. (For this notation we identify the $\bigwedge V$-factors. In this newly constructed algebra the element $\tilde{e}$ denotes an abstract element of the graded vector space by which the algebra is generated whereas before—by abuse of notation—we used to consider $\tilde{e}$ as an element of $V \oplus \langle e\rangle$.)

Thus the "identity", i.e. the assignment $e \mapsto \tilde{e} - \frac{a}{2}$, $e' \mapsto e'$ together with the identity on V, induces an isomorphism σ of graded algebras

$$\big(\bigwedge V \otimes \bigwedge\langle e, e'\rangle, \mathrm{d}\big) \xrightarrow{\simeq} \big(\bigwedge V \otimes \bigwedge\langle \tilde{e}, e'\rangle, \tilde{\mathrm{d}}\big)$$

which is compatible with the differential:

$$\begin{aligned}
\mathrm{d}(\sigma(e)) &= \tilde{\mathrm{d}}\left(\tilde{e} - \frac{a}{2}\right) = -\frac{\mathrm{d}a}{2} = \mathrm{d}e = \sigma(\mathrm{d}(e)) \\
\mathrm{d}(\sigma(e')) &= \tilde{\mathrm{d}}(e') \\
&= \tilde{e}^2 + \left(b - \frac{a^2}{4}\right) \\
&= \sigma\left(\left(e + \frac{a}{2}\right)^2 + \left(b - \frac{a^2}{4}\right)\right) \\
&= \sigma(e^2 + ae + b) \\
&= \sigma(\mathrm{d}(e'))
\end{aligned}$$

(Indeed, we know that $\mathrm{d}e \in \bigwedge V$ which implies that $\sigma(\mathrm{d}e') = \mathrm{d}e'$.)

Summarising these considerations we have proved that there is a quasi-isomorphism of cochain algebras

$$\left(\bigwedge V \otimes \bigwedge\langle \tilde{e}, e'\rangle, \tilde{\mathrm{d}}\right) \xrightarrow{\simeq} (\mathrm{A_{PL}}(E), \mathrm{d})$$

with $\mathrm{d}\tilde{e} = 0$, $\mathrm{d}e' = \tilde{e}^2 - u$, $u \in \bigwedge V$, $\mathrm{d}u = 0$.

Note further that the cohomology class $[u]$ is determined by the fibration. It vanishes iff $E \simeq_{\mathbb{Q}} B \times F$. Nonetheless, on the level of $H^*(B)$-modules we always have

$$H^*(E) \cong H^*(B) \otimes H^*(F)$$

This is due to the following arguments: We have seen that the restriction of the total space to the fibre is surjective in cohomology, since we have $[\tilde{e}] \mapsto [e]$. This is what is meant when calling the fibration *totally non-cohomologous to zero*. So by the Hirsch lemma the cohomology of the total space is a free module over the cohomology of the base space. ⊡

With the notation of remark 1.34 we state

Definition 1.35. A fibration p is called *primitive*, if u is not decomposable, i.e. $u \notin \bigwedge^{>0} V \cdot \bigwedge^{>0} V$. Otherwise, the fibration p will be called *non-primitive*.

Let us illustrate primitivity of a spherical fibration with odd-dimensional fibre—the even-dimensional case is not admissible to this description as we changed the algebra—by means of a better known description: The linear part of the differential in the model of the fibration—we use minimal models for base and fibre—can be identified up to duality with the transgression of the long exact homotopy sequence of the fibration. Minimality of this model corresponds to a trivial linear part of the differential. So the fibration is non-primitive iff the transgression is trivial (cf. [22].15 (e), p. 214).

Let us now return to the context of formal spaces by citing some important examples.

Example 1.36.
- H-spaces are formal (cf. [22], example 12.3, p. 143).
- Symmetric spaces of compact type are formal (cf. [22].12.3, p. 162).
- N-symmetric spaces are formal (cf. [71], Main Theorem, p. 40, for the precise statement, [50]).
- Compact Kähler manifolds are formal (cf. [17], Main Theorem, p. 270).

⊡

In the quoted example it is shown that H-spaces admit minimal Sullivan models of the form $\left(\bigwedge V, 0\right)$. So we obtain $H\left(\bigwedge V, 0\right) = \bigwedge V$ and in particular we have $\left(\bigwedge V, 0\right) \xrightarrow{\simeq} \left(H\left(\bigwedge V, 0\right), 0\right)$ which yields formality.

The formality of compact Kähler manifolds is a celebrated theorem by Deligne, Griffiths, Morgan and Sullivan. We want to sketch the idea of the proof very briefly: Let J be the complex structure of the compact Kähler manifold Z. Set $\mathrm{d}_c = J^{-1}\mathrm{d}J : \mathrm{A}_{\mathrm{DR}}(Z) \to \mathrm{A}_{\mathrm{DR}}(Z)$ which again is a differential commuting with d. It is then proved that compact Kähler manifolds satisfy

Lemma 1.37 (dd_c-lemma)**.** *If α is a differential form with $\mathrm{d}\alpha = 0 = \mathrm{d}_c\alpha$ and such that $\alpha = \mathrm{d}\gamma$, then $\alpha = \mathrm{dd}_c\beta$ for some differential form β.*

It follows as an easy conclusion that d restricts to a map $\mathrm{d} : \ker \mathrm{d}_c \to \mathrm{im}\, \mathrm{d}_c$ and that the obvious inclusion and projection

$$\mathrm{A}_{\mathrm{DR}}(Z) \hookleftarrow (\ker \mathrm{d}_c, \mathrm{d}) \to (\ker \mathrm{d}_c/\mathrm{im}\, \mathrm{d}_c, 0)$$

are quasi-isomorphisms. Since the differential on the right hand side of the chain is trivial, this yields formality.

Formality is a highly important concept in Rational Homotopy Theory and serves as a vital tool for several geometric applications. Since symplectic manifolds need not be formal (cf. [22], example 12.5, p. 163) formality for example may be used to find symplectic manifolds that are not Kählerian—see for example the introduction of [23] for more references and illustrations. A possible way of showing non-formality hereby is the existence of a non-vanishing Massey product. As remarked on [17], p. 262, formality is equivalent to "uniform choices so that the forms representing all Massey products and higher order Massey products are exact".

Let us now mention some more properties and characterisations of formality.

Lemma 1.38. *Let*

$$\left(\bigwedge V, \mathrm{d}\right) \xleftarrow{\simeq} (A_1, \mathrm{d}) \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} \ldots \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} (A_n, \mathrm{d})$$

be a chain of quasi-isomorphism of commutative differential graded algebras (A_i, d). Suppose further that $\left(\bigwedge V, \mathrm{d}\right)$ is a minimal Sullivan algebra. Then we obtain a quasi-isomorphism

$$\left(\bigwedge V, \mathrm{d}\right) \xrightarrow{\simeq} (A_n, \mathrm{d})$$

of commutative differential graded algebras, i.e. a minimal model.

Proof. Whenever we have a diagram of quasi-isomorphisms

$$\begin{array}{ccc} \left(\bigwedge V, \mathrm{d}\right) & \xrightarrow{\simeq} & (A, \mathrm{d}) \\ & & \downarrow \simeq \\ & & (B, \mathrm{d}) \end{array}$$

with commutative differential graded algebras (A, d), (B, d) and $\left(\bigwedge V, \mathrm{d}\right)$, composition yields a quasi-isomorphism $\left(\bigwedge V, \mathrm{d}\right) \xrightarrow{\simeq} (B, \mathrm{d})$. In case we have a diagram

$$\begin{array}{ccc} & & (A,\mathrm{d}) \\ & & \downarrow{\scriptstyle \simeq} \\ \left(\bigwedge V, \mathrm{d}\right) & \xrightarrow{\simeq} & (B,\mathrm{d}) \end{array}$$

with a Sullivan algebra $\left(\bigwedge V, \mathrm{d}\right)$, proposition [22].12.9, p. 153, yields the existence of a lifting $\left(\bigwedge V, \mathrm{d}\right) \to (A, \mathrm{d})$ which is unique up to homotopy (cf. [22], p. 149) and which makes the diagram commute up to homotopy. Moreover, this morphism is then necessarily a quasi-isomorphism.

If the situation is given by a diagram

$$\begin{array}{ccc} & & (A,\mathrm{d}) \\ & & \downarrow{\scriptstyle \simeq} \\ \left(\bigwedge V, \mathrm{d}\right) & \xleftarrow[\simeq]{} & (B,\mathrm{d}) \end{array}$$

we extend the diagram by a minimal model $\psi : \left(\bigwedge W, \mathrm{d}\right) \xrightarrow{\simeq} (A, 0)$. Hence we obtain

$$\begin{array}{ccc} \left(\bigwedge W, \mathrm{d}\right) & \xrightarrow[\psi]{\simeq} & (A,\mathrm{d}) \\ & & \downarrow{\scriptstyle \simeq} \\ \left(\bigwedge V, \mathrm{d}\right) & \xleftarrow[\simeq]{} & (B,\mathrm{d}) \end{array}$$

Composition yields a quasi-isomorphism $\phi : \left(\bigwedge W, \mathrm{d}\right) \xrightarrow{\simeq} \left(\bigwedge V, \mathrm{d}\right)$ of minimal Sullivan algebras. By proposition [22].14.13, p. 191, a quasi-isomorphism of minimal Sullivan algebras is an isomorphism. So $\psi \circ \phi^{-1} : \left(\bigwedge V, \mathrm{d}\right) \xrightarrow{\simeq} (A, \mathrm{d})$ is the quasi-isomorphism we have been looking for.

Finally, we treat the situation given by the diagram

$$\begin{array}{ccc} & & (A,\mathrm{d}) \\ & & \uparrow{\scriptstyle \simeq} \\ \left(\bigwedge V, \mathrm{d}\right) & \xleftarrow[\simeq]{} & (B,\mathrm{d}) \end{array}$$

in a similar fashion: Use the minimal model $\psi : \left(\bigwedge W, \mathrm{d}\right) \xrightarrow{\simeq} (A, \mathrm{d})$ and lift this morphism to a morphism $\tilde{\psi} : \left(\bigwedge W, \mathrm{d}\right) \xrightarrow{\simeq} (B, \mathrm{d})$. By composition this morphism again yields a quasi-isomorphism $\phi : \left(\bigwedge W, \mathrm{d}\right) \xrightarrow{\simeq} \left(\bigwedge V, \mathrm{d}\right)$ of minimal Sullivan algebras. Hence we proceed as in the last case.

Iteratively stepping through the chain (1.7) now yields the result. □

This will be a vital tool in the proof of

Theorem 1.39. • *A minimal model* $\big(\bigwedge V, \mathrm{d}\big)$ *of a commutative differential graded algebra* (A, d) *is formal iff* (A, d) *is formal.*

- *A minimal Sullivan algebra* $\big(\bigwedge V, \mathrm{d}\big)$ *is formal if and only if there is a quasi-isomorphism*

$$\mu : \big(\bigwedge V, \mathrm{d}\big) \xrightarrow{\simeq} \big(H\big(\bigwedge V, \mathrm{d}\big), 0\big)$$

- *The quasi-isomorphism* μ *from the last point may be taken to be the identity in cohomology.*
- *This quasi-isomorphism then has the property that*

$$\mu : x \mapsto [x]$$

for d*-closed elements* x.

Proof. The first part is apparent from the definition of formality as a chain of quasi-isomorphisms between algebra and homology and from the definition of minimal model given by a quasi-isomorphism $\big(\bigwedge V, \mathrm{d}\big) \xrightarrow{\simeq} (A, \mathrm{d})$.

Let us now prove the second part. One implication is trivial. Suppose now that $\big(\bigwedge V, \mathrm{d}\big)$ is formal. We then have a weak equivalence

$$\big(\bigwedge V, \mathrm{d}\big) \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} \ldots \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} \big(H\big(\bigwedge V, \mathrm{d}\big), 0\big) \tag{1.7}$$

Applying 1.38 to this chain yields the quasi-isomorphism between the Sullivan algebra and its cohomology algebra.

This quasi-isomorphism μ without restriction may be assumed to induce the identity in cohomology. We give two different lines of argument for this:

As for the first one we observe that the quasi-isomorphism μ induces

$$\mu_* : H\big(\bigwedge V, \mathrm{d}\big) \xrightarrow{\cong} H\big(H\big(\bigwedge V, \mathrm{d}\big), 0\big)$$

which canonically may be considered as an automorphism

$$\mu_* : \big(H\big(\bigwedge V, \mathrm{d}\big), 0\big) \xrightarrow{\cong} \big(H\big(\bigwedge V, \mathrm{d}\big), 0\big)$$

of commutative differential graded algebras. With this canonical identification it is obvious that the morphism induced by μ_* in cohomology is just μ_* itself, i.e. $(\mu_*)_* = \mu_*$. This amounts to the fact that the morphism induced in homology by the chain of morphisms

$$\big(\bigwedge V, \mathrm{d}\big) \xrightarrow{\mu} \big(H\big(\bigwedge V, \mathrm{d}\big), 0\big) \xrightarrow{(\mu_*)^{-1}} \big(H\big(\bigwedge V, \mathrm{d}\big), 0\big)$$

is just the identity.

Alternatively, following algorithm 1.24 we can construct a minimal model

$$\big(\bigwedge V', \mathrm{d}\big) \xrightarrow{\simeq} \big(H\big(\bigwedge V, \mathrm{d}\big), 0\big) = \big(H\big(\bigwedge V', \mathrm{d}\big), 0\big)$$

where the morphism induces the identity in cohomology. Since we may lift morphisms between Sullivan algebras we have a diagram

$$\begin{array}{ccc} \big(\bigwedge V, \mathrm{d}\big) & \xrightarrow{\simeq} & \big(H\big(\bigwedge V, \mathrm{d}\big), 0\big) = \big(H\big(\bigwedge V', \mathrm{d}\big), 0\big) \\ & \nwarrow^{\cong} & \uparrow \simeq \\ & & \big(\bigwedge V', \mathrm{d}\big) \end{array}$$

which is commutative up to homotopy (cf. [22], p. 149). The diagonal morphism is then a quasi-isomorphism by commutativity. Note that a quasi-isomorphism of simply-connected minimal Sullivan algebras is an isomorphism (cf. [22].12.10.i, p. 154).

Thus up to isomorphism of minimal Sullivan algebras we may assume the quasi-isomorphism between minimal model and its cohomology algebra to be the identity.

Let us finally prove the last assertion: A d-closed element $x \in \big(\bigwedge V, \mathrm{d}\big)$ defines a homology class which is mapped to itself under μ_* by the last paragraph. That is, $[x] = \mu_*([x]) = [\mu(x)]$. We now identify $\big(H\big(\bigwedge V, \mathrm{d}\big), 0\big)$ with its own homology algebra in the canonical way. This yields $\mu(x) = [x]$. □

Remark 1.40. In [17] and in [23] the existence of a morphism between the minimal Sullivan algebra $\big(\bigwedge V, \mathrm{d}\big)$ and its cohomology algebra (with zero differential) inducing the identity in cohomology is taken as a definition of the formality of $\big(\bigwedge V, \mathrm{d}\big)$. Furthermore, a commutative differential graded algebra is said to be formal in these articles if so is its minimal model.

The third point in theorem 1.39 tells us that up to isomorphism of minimal models, we may assume the minimal model of a formal—in the sense of definition 1.28—algebra to map into its cohomology algebra whilst inducing the *identity* in cohomology. Since minimal Sullivan models are unique only up to isomorphism, the points two and three from theorem 1.39 thus reconcile the definitions in [17] and in [23] with ours. ⊡

Let us mention the following essential characterisation of formality. This theorem was also established in the famous article [17] by Deligne, Griffiths, Morgan and Sullivan.

Theorem 1.41. *A minimal model* $(\bigwedge V, \mathrm{d})$ *is formal if and only if there is in each* V^i *a complement* N^i *to the subspace of* d*-closed elements* C^i *with* $V^i = C^i \oplus N^i$ *and such that any closed form in the ideal* $I(\bigoplus N^i)$ *generated by* $\bigoplus N^i$ *in* $\bigwedge V$ *is exact.*

Proof. See [17], theorem 4.1, p. 261. □

Set $C := \bigoplus C^i$ and $N := \bigoplus N^i$. This theorem has undergone a certain refinement. Now there is a degreewise formulation of formality:

Definition 1.42. A topological space with minimal model $(\bigwedge V, \mathrm{d})$ is called s-formal ($s \geq 0$) if the space V^i of generators in degree i for each $i \leq s$ decomposes as a direct sum $V^i = C^i \oplus N^i$ with the C^i and N^i satisfying the following conditions:

- $\mathrm{d}(C^i) = 0$,
- $\mathrm{d} : N^i \to \bigwedge V$ is injective.
- Any closed element in the ideal $I_s = N^{\leq s} \cdot (\bigwedge V^{\leq s})$ generated by $N^{\leq s} = \bigoplus_{i \leq s} N^i$ in the free algebra $\bigwedge V^{\leq s} = \bigwedge(\bigoplus_{i \leq s} V^i)$ is exact in $\bigwedge V$.

The full power of this definition becomes visible by

Theorem 1.43. *A connected compact orientable differentiable manifold of dimension $2n$ or $2n - 1$ is formal if and only if it is $(n - 1)$-formal.*

Proof. See [23].3.1, p. 8. □

The next lemma is similar to theorem [17].4.1, p. 261. It can be considered as a refinement of theorem 1.39.

Lemma 1.44. *Suppose $\big(\bigwedge V, \mathrm{d}\big) \xrightarrow{\simeq} (A, \mathrm{d})$ is a minimal model. Let (A, d) be a formal algebra. Let $V = C \oplus N$ be the decomposition from theorem 1.41 with $C = \ker \mathrm{d}|_V$. Then there is a quasi-isomorphism*

$$\mu_A : \big(\bigwedge V, \mathrm{d}\big) \xrightarrow{\simeq} \big(H\big(\bigwedge V, \mathrm{d}\big), 0\big)$$

with the property that $\mu_A(x) = [x]$ if $[x]$ is closed and with $\mu(x) = 0$ for $x \in N$.

Proof. Choose a basis $\{b_i\}|_{i \in J}$ of V that is subordinate to the decomposition $C \oplus N$; i.e. there is a subset $J' \subseteq J$ such that $\{b_i\}|_{i \in J'}$ is a basis of C and such that $\{b_i\}|_{i \in J \setminus J'}$ is a basis of N. Then define

$$\mu_A(b_i) := \begin{cases} [b_i] & \text{for } i \in J' \\ 0 & \text{for } i \in J \setminus J' \end{cases}$$

and extend this linearly and multiplicatively to a morphism of graded algebras. This morphism obviously vanishes on N.

It also maps a closed element to its homology class: The decomposition of V induces the decomposition

$$\bigwedge V = \bigwedge C \oplus I(N)$$

Hence an element from $\bigwedge C$ is mapped to its homology class by the definition of μ. The map μ even vanishes on $I(N)$. Moreover, by formality and theorem 1.41 every closed element in $I(N)$ is exact. So μ maps every closed element to its homology class.

In particular, μ is compatible with differentials. This is due to $\mu(\mathrm{d}x) = [0]$. Indeed, the element $\mathrm{d}x$ is exact and therefore defines the zero homology class.

As μ maps a closed element to its homology class, it induces the identity in homology. In particular, it is a quasi-isomorphism. □

On homogeneous spaces there is the following remarkable theorem:

Theorem 1.45. *Let H be a connected Lie subgroup of a connected compact Lie group G. Let T be a maximal torus of H. Then G/H is formal if and only if G/T is formal.*

PROOF. See the remark on [64], p. 212. □

Let us eventually define the notion of a *formal map*. We shall present the version of formality of a map which was proposed and used in [1]—cf. definition [1].2.7.20, p. 123—as it turns out to be most suitable for our purposes. (Contrast this version with the definition in [17].4, p. 260.)

Let (A, d), (B, d) be commutative differential algebras with minimal models

$$m_A : \big(\bigwedge V, \mathrm{d}\big) \xrightarrow{\simeq} (A, \mathrm{d})$$
$$m_B : \big(\bigwedge W, \mathrm{d}\big) \xrightarrow{\simeq} (B, \mathrm{d})$$

A morphism $f : (A, \mathrm{d}) \to (B, \mathrm{d})$ induces the map $f_* : H(A) \to H(B)$ and the *Sullivan representative* $\hat{f} : \big(\bigwedge V, \mathrm{d}\big) \to \big(\bigwedge W, \mathrm{d}\big)$ uniquely defined up to homotopy (cf. [22].12, p. 154). Indeed, the morphism $\hat{f}$ is defined by lifting the diagram

$$\begin{array}{ccc} (\bigwedge V, \mathrm{d}) & \overset{\hat{f}}{\dashrightarrow} & (\bigwedge W, \mathrm{d}) \\ \simeq\downarrow & & \simeq\downarrow \\ (A, \mathrm{d}) & \xrightarrow{f} & (B, \mathrm{d}) \end{array}$$

Completing the square by $\hat{f}$ makes it a diagram which is commutative up to homotopy (cf. [22], p. 149).

Suppose now that A and B are formal. By theorem 1.39 there are quasi-isomorphisms

$$\phi_A : \big(\bigwedge V, \mathrm{d}\big) \xrightarrow{\simeq} (H^*(A), 0)$$
$$\phi_B : \big(\bigwedge W, \mathrm{d}\big) \xrightarrow{\simeq} (H^*(B), 0)$$

By composing such a model with a suitable automorphism of the cohomology algebra (analogous to what we did in the proof of theorem 1.39) we may suppose that $(m_A)_* = (\mu_A)_*$ and $(m_B)_* = (\mu_B)_*$.

Then the morphism f is said to be *formal* if there is a choice of such quasi-isomorphisms μ_A and μ_B (with $(m_A)_* = (\mu_A)_*$ and that $(m_B)_* = (\mu_B)_*$) which makes the diagram

$$\begin{array}{ccc} \left(\bigwedge V, \mathrm{d}\right) & \xrightarrow{\hat{f}} & \left(\bigwedge W, \mathrm{d}\right) \\ \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle \mu_B} \\ (H(A), 0) & \xrightarrow{f_*} & (H(B), 0) \end{array}$$

commute up to homotopy.

Call a continuous map $f : M \to N$ between two formal topological spaces M, N *formal* if the map $\mathrm{A_{PL}}(f)$ induced on polynomial differential forms is formal.

Let us illustrate this concept in an elementary case first.

Proposition 1.46. *A continuous map of H-spaces is formal.*

Proof. Suppose X and Y are H-spaces with minimal models $\left(\bigwedge V, 0\right)$ and $\left(\bigwedge W, 0\right)$ respectively (cf. example 1.36). So we have the identifications $\left(\bigwedge V, 0\right) = \left(H\left(\bigwedge V, 0\right), 0\right)$ and $\left(\bigwedge W, 0\right) = \left(H\left(\bigwedge W, 0\right), 0\right)$. A continuous map $f : X \to Y$ induces $\hat{f}$ and f_* as above. Since $(\hat{f})_* = f_*$, we necessarily see that—under the identification of the minimal models with their respective cohomology algebras—we obtain the equality $f_* = \hat{f}$. Thus f is formal. □

We shall prove another property of formal spaces that can be found in the literature with varying formulation and more or less detailed proofs (cf. proposition [6].5, p. 335 or lemma [23].2.11, p. 7 for example).

Theorem 1.47. *Let (A, d) and (B, d) be differential graded algebras. The product algebra $(A \otimes B, \mathrm{d})$ is formal if and only if so are both A and B. In this case the canonical inclusion $(A, \mathrm{d}) \hookrightarrow (A \otimes B, \mathrm{d})$ given by $a \mapsto (a, 1)$ is a formal map.*

Proof. First assume A and B to be formal with minimal models $\left(\bigwedge V, \mathrm{d}\right)$ and $\left(\bigwedge W, \mathrm{d}\right)$. Then

$$\left(\bigwedge V, \mathrm{d}\right) \otimes \left(\bigwedge W, \mathrm{d}\right) \cong \left(\bigwedge (V \oplus W), \mathrm{d}\right)$$

is again a minimal Sullivan algebra; it is a model for the product algebra $(A \otimes B, \mathrm{d})$, since there is an isomorphism $H(A) \otimes H(B) \cong H(A \otimes B)$ (cf. [22], example 12.2, p. 142–143).

By theorem 1.39 there are quasi-isomorphisms

$$\big(\bigwedge V, \mathrm{d}\big) \xrightarrow{\phi_1} (H(A, \mathrm{d}), 0) \qquad \text{and} \qquad \big(\bigwedge W, \mathrm{d}\big) \xrightarrow{\phi_2} (H(B, \mathrm{d}), 0)$$

Hence the product morphism

$$\phi_1 \otimes \phi_2 : \big(\bigwedge (V \oplus W), \mathrm{d}\big) \to (H(A \otimes B, \mathrm{d}), 0)$$

is a quasi-isomorphism. By theorem 1.39 the algebra $(A \otimes B, \mathrm{d})$ is formal.

Suppose now that $A \otimes B$ is formal. With the minimal models $\big(\bigwedge V, \mathrm{d}\big)$ and $\big(\bigwedge W, \mathrm{d}\big)$ for (A, d) respectively (B, d) the minimal Sullivan algebra $\big(\bigwedge (V \oplus W), \mathrm{d}\big)$ is a model for $(A \otimes B, \mathrm{d})$. Consider the composition

$$\psi : \big(\bigwedge V, \mathrm{d}\big) \hookrightarrow \big(\bigwedge (V \oplus W), \mathrm{d}\big) \xrightarrow{\simeq} (H(A \otimes B, \mathrm{d}), 0) \to (H(A, \mathrm{d}), 0)$$

(The first map is the inclusion of minimal models induced by the inclusion $V \hookrightarrow V \oplus W$. The second map comes from theorem 1.39; we may suppose it to induce the identity in cohomology. The third map is the canonical projection.) Since the second map induces the identity in cohomology, we see that it factorises as the product map

$$\phi_1 \otimes \phi_2 : \big(\bigwedge V, \mathrm{d}\big) \otimes \big(\bigwedge W, \mathrm{d}\big) \xrightarrow{\simeq} (H(A, \mathrm{d}), 0) \otimes (H(B, \mathrm{d}), 0)$$

with two morphisms ϕ_1 and ϕ_2 that induce the identity in cohomology. In particular, the morphism ψ factors and we obtain $\psi = \phi_1$, which is a quasi-isomorphism.

Let us now see that the inclusion $(A, \mathrm{d}) \overset{i}{\hookrightarrow} (A, \mathrm{d}) \otimes (B, \mathrm{d}) = (A \otimes B, \mathrm{d})$ is formal if so are (A, d) and (B, d). The diagram

$$\begin{array}{ccc} \big(\bigwedge V, \mathrm{d}\big) & \xrightarrow{\hat{i}} & \big(\bigwedge V, \mathrm{d}\big) \otimes \big(\bigwedge W, \mathrm{d}\big) \\ \downarrow \simeq & & \downarrow \simeq \\ (A, \mathrm{d}) & \underset{i}{\xhookrightarrow{a \mapsto (a,1)}} & (A, \mathrm{d}) \otimes (B, \mathrm{d}) \end{array}$$

commutes if $\hat{i}$ is the canonical inclusion $x \mapsto (x, 1)$. By definition this inclusion serves as a Sullivan representative for i.

The map $i_* : H(A, \mathrm{d}) \to H(A \otimes B, \mathrm{d}) = H(A, \mathrm{d}) \otimes H(B, \mathrm{d})$ clearly is also just the canonical inclusion. Thus, in total, these observations combine to yield the commutativity of the diagram

$$\begin{array}{ccc} \big(\bigwedge V, \mathrm{d}\big) & \underset{\hat{i}}{\xhookrightarrow{x \mapsto (x,1)}} & \big(\bigwedge V, \mathrm{d}\big) \otimes \big(\bigwedge W, \mathrm{d}\big) \\ \phi_1 \downarrow \simeq & & \simeq \downarrow \phi_1 \otimes \phi_2 \\ (H(A, \mathrm{d}), 0) & \underset{i_*}{\xhookrightarrow{a \mapsto (a,1)}} & (H(A, \mathrm{d}), 0) \otimes (H(B, \mathrm{d}), 0) \end{array}$$

Hence the map i is formal. $\square$

As a direct consequence one sees that a space that is (rationally) a product $X \times Y$ of topological spaces X and Y is formal if and only if so are both factors X and Y.

Part 3. . . . wherein rational ellipticity is dealt with.

In the following we shall briefly introduce the notion of *rational ellipticity*. See [22], p. 370 for the definition of the *rational Lusternik–Schnirelmann category* cat_0. Suppose now that X is a simply connected topological space with rational homology of finite type and with finite rational Lusternik–Schnirelmann category $\mathrm{cat}_0 X < \infty$. Then we have the so-called *rational dichotomy*, i.e. exactly one of the following two cases applies (cf. [22], p. 452):

- The space X is *(rationally) elliptic.*
- The space X is *(rationally) hyperbolic.*

Definition 1.48. A simply-connected topological space X is called *(rationally) elliptic* if it satisfies both

$$\dim H_*(X, \mathbb{Q}) < \infty \qquad \text{and} \qquad \dim \pi_*(X) \otimes \mathbb{Q} < \infty$$

Note that finite rational Lusternik–Schnirelmann category is equivalent to finite-dimensional rational homology (cf. [22].32.4, p. 438).

According to the dichotomy a simply-connected space X with rational homology of finite type is called *(rationally) hyperbolic* if $\mathrm{cat}_0(X) < \infty$ and $\dim \pi_*(X) \otimes \mathbb{Q} = \infty$. More can be said in this case: If X is hyperbolic then $\sum_{i=2}^{k} \dim \pi_i(X) \otimes \mathbb{Q}$ grows exponentially in k. If X additionally has finite-dimensional rational homology and *formal dimension* d, then for each $k \geq 1$ there is an $i \in (k, k+d)$ with the property that $\pi_i(X) \otimes \mathbb{Q} \neq 0$—cf. [22], p. 452. (By *formal dimension* we denote rational homological dimension.)

Let us now focus on properties of elliptic spaces X. From cf. [22].32, p. 434, we cite several relations on rational homotopy groups satisfied by elliptic spaces. By d we shall denote the formal dimension of X. (Clearly, in case X is a manifold both notions of dimension coincide.)

Let the x_i form a basis of $\pi_{\mathrm{odd}}(X) \otimes \mathbb{Q}$ and let the y_i form a basis of $\pi_{\mathrm{even}}(X) \otimes \mathbb{Q}$. Moreover set the *homotopy Euler characteristic*

$$\chi_\pi(X) := \dim \pi_{\mathrm{odd}}(X) \otimes \mathbb{Q} - \dim \pi_{\mathrm{even}}(X) \otimes \mathbb{Q}$$

on the analogy of the ordinary Euler characteristic

$$\chi(X) := \sum_i (-1)^i \dim H^i(X)$$

We obtain (cf. [22], p. 434):

(1.8) $$\sum_i \deg x_i \leq 2d - 1$$

(1.9) $$\sum_i \deg y_i \leq d$$

(1.10) $$\dim \pi_i(X) \otimes \mathbb{Q} = 0 \qquad \text{for } i \geq 2d$$

(1.11) $$\dim \pi_*(M) \otimes \mathbb{Q} \leq d$$

(1.12) $$\sum_i \deg x_i - \sum_j (\deg y_j - 1) = d$$

(1.13) $$\chi_\pi(X) \geq 0$$

(1.14) $$\chi(X) \geq 0$$

(1.15) $$\chi_\pi(X) > 0 \iff \chi(X) = 0 \qquad \chi_\pi(X) = 0 \iff \chi(X) > 0$$

(1.16) $$\dim \pi_{\mathrm{odd}}(X) \otimes \mathbb{Q} \leq \mathrm{cat}_0 X$$

Important examples of elliptic spaces are closed simply-connected Dupin hypersurfaces in $\mathbb{S}^{n+1}$, closed simply-connected manifolds admitting strict cohomogeneity one actions—both being asserted in [33], theorem B—and biquotients:

Proposition 1.49. *A biquotient is an elliptic space.*

PROOF. As biquotients are compact manifolds, their homology is of finite type. Lie groups are elliptic. This is due to the fact that path-connected H-spaces X with rational homology of finite type satisfy $\bigwedge V \cong H^*(X)$ for some graded vector space V (cf. [22], example 12.3, p. 143). That is, cohomology is a free commutative graded algebra. Since cohomology of a manifold vanishes above its dimension, we see $V = V^{\mathrm{odd}}$ and $\dim V < \infty$.

Now use the long exact sequence of homotopy groups associated to the fibration

$$H \hookrightarrow G \times G \to G /\!\!/ H$$

to see that rational homotopy groups of $G/\!\!/H$ vanish from a certain degree on. □

Note that biquotients constitute a natural class of manifolds with non-negative curvature which attributes to their importance in Riemannian geometry. (In fact, Lie groups admit metrics of non-negative curvature. By the submersion theorem so does a biquotient). The fact that ellipticity and non-negative curvature appear together on a space was speculated to be not accidental:

Conjecture 1.50 (Bott)**.** *Simply-connected compact Riemannian manifolds with non-negative sectional curvature are rationally elliptic.*

Recall that we constructed the model $\big(\bigwedge V_{\mathbf{B}H} \otimes \bigwedge\langle q_1, \ldots, q_r\rangle, \mathrm{d}\big)$ for biquotients (cf. 1.23). Since H-spaces rationally are a direct product of odd-dimensional spheres, the algebra $\bigwedge V_{\mathbf{B}H}$ is a polynomial algebra generated in even degrees. By construction the q_i form relations and have odd degree. So biquotients are examples of elliptic spaces that admit a very special (minimal) Sullivan model, a so-called *pure model*.

Definition 1.51. A Sullivan algebra $\big(\bigwedge V, \mathrm{d}\big)$ is called *pure* if V is finite dimensional, if $\mathrm{d}|_{V^{\mathrm{even}}} = 0$ and if $\mathrm{d}(V^{\mathrm{odd}}) \subseteq \bigwedge V^{\mathrm{even}}$.

In a pure Sullivan algebra the differential d is homogeneous of degree -1 with respect to word-length in V^{odd}:

$$0 \leftarrow \bigwedge V^{\mathrm{even}} \leftarrow \bigwedge V^{\mathrm{even}} \otimes V^{\mathrm{odd}} \leftarrow \dots \leftarrow \bigwedge V^{\mathrm{even}} \otimes \bigwedge^k V^{\mathrm{odd}} \leftarrow \dots$$

Denote by $H_k\big(\bigwedge V, \mathrm{d}\big)$ the subspace representable by cocycles in $\bigwedge V^{\mathrm{even}} \otimes \bigwedge^k V^{\mathrm{odd}}$. Thus we have that $H\big(\bigwedge V, \mathrm{d}\big) = \bigoplus_k H_k\big(\bigwedge V, \mathrm{d}\big)$; this is called the *lower grading* of $H\big(\bigwedge V, \mathrm{d}\big)$.

A simply-connected elliptic space X of positive Euler characteristic is called an F_0-space. These spaces have remarkable properties (cf. [22].32.10, p. 444):

- They admit pure models $\big(\bigwedge V, \mathrm{d}\big)$ (cf. [22], p. 437).
- Their cohomology is concentrated in even degrees only.
- Their cohomology is the quotient of a polynomial algebra in even degrees and a regular sequence (cf. [22], p. 437).

So let $\big(\bigwedge V, \mathrm{d}\big) \xrightarrow{\simeq} \mathrm{A_{PL}}(X)$ be a pure Sullivan model of the F_0-space X.

- It holds: $\dim V^{\mathrm{odd}} = \dim V^{\mathrm{even}}$.
- It holds: $H\big(\bigwedge V, \mathrm{d}\big) = H_0\big(\bigwedge V, \mathrm{d}\big)$ (cf. [22].32.3, p. 437).

Another very interesting property due to Halperin—originally proved in [34]—follows:

Theorem 1.52. *F_0-spaces are formal.*

Proof. We prove this using that an F_0-space X has the properties mentioned above. Our main tool will be theorem 1.41:

The space X admits a pure Sullivan model and hence also a pure minimal Sullivan model $\big(\bigwedge V, \mathrm{d}\big)$. By definition of a pure Sullivan algebra we have $\mathrm{d}(V^{\mathrm{even}}) = 0$. The cohomology $H^*(X)$ is concentrated in even degrees only. So by the minimality of the model there cannot be an element $0 \neq a \in V^{\mathrm{odd}}$ with $\mathrm{d}a = 0$. Thus we have a uniquely determined homogeneous decomposition $C = V^{\mathrm{even}}$ and $N = V^{\mathrm{odd}}$ with $V = C \oplus N$ and $C = \ker \mathrm{d}$. We have to show that every closed element $x \in I(N)$ is exact. However, every such element $x \in I(N)$ determines a cohomology class in $H_{>0}\big(\bigwedge V, \mathrm{d}\big)$. Yet, as we have seen, it holds that $H_{>0}\big(\bigwedge V, \mathrm{d}\big) = 0$, which implies that x is exact. $\square$

We shall apply this to biquotients generalising a result by Halperin for homogeneous spaces:

Proposition 1.53. • *A biquotient $G/\!/H$ admits a pure model.*

- *A biquotient $G/\!/H$ with $\operatorname{rk} G = \operatorname{rk} H$ is an F_0-space. (In particular, it has positive Euler characteristic and it is formal.)*

PROOF. The first assertion was proven in proposition [4].7.1, p. 269 and proposition [42].1, p. 2.

In the latter paper on the same page the formula

$$\chi(G/\!/H) = \frac{|W(G)|}{|W(H)|}$$

for the Euler characteristic of $G/\!/H$ in terms of the orders of the Weyl groups of G and H under the assumption that $\operatorname{rk} G = \operatorname{rk} H$ is given. See theorem [70].5.1 for the first result of this kind. As G is a compact Lie group, we know that up to finite coverings it is a product of simply-connected simple Lie groups and a torus (cf. theorem [11].8.1, p. 233). In particular, its Weyl group is trivial if and only if it is a torus up to finite coverings.

If the Weyl group $\mathrm{W}(G)$ is not trivial, the formula yields positive Euler characteristic and thus $G/\!/H$ is an F_0-space. If the Weyl group is trivial, up to finite coverings G is a torus. Thus by $\operatorname{rk} G = \operatorname{rk} H$ we see that $G/\!/H$ is a point. Formality is due to theorem 1.52. This proves the second assertion. □

2

Dimension Twelve—a “Disclassification”

As we pointed out in the introduction, Positive Quaternion Kähler Manifolds are classified in dimensions four (Hitchin) and eight (Poon–Salamon, LeBrun–Salamon). In 2002 Haydeé and Rafael Herrera claimed to have proved the main conjecture (cf. 1.6) for 12-dimensional Positive Quaternion Kähler Manifolds in [35]. According to this article in dimension 12 every Positive Quaternion Kähler Manifold was one of the Wolf spaces $\mathbb{H}\mathbf{P}^3$, $\mathbf{Gr}_2(\mathbb{C}^5)$ and $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)$. The result was commonly approved and held in high esteem. Recently, however, we found the proof to be erroneous and intermediate results to be wrong. Due to this unfortunate development, the classification in dimension 12 can no longer be maintained.

This chapter is intended to provide a brief outline of the problems that appear in the original reasoning. Haydeé and Rafael Herrera showed that any 12-dimensional Positive Quaternion Kähler Manifold M is symmetric if the $\hat{A}$-genus of M vanishes. If M is a spin manifold this condition is always fulfilled by a classical result of Lichnerowicz, since a Positive Quaternion Kähler Manifold has positive scalar curvature. As we have seen on page 16, the manifold M^{12} is spin if and only if $M \cong \mathbb{H}\mathbf{P}^3$.

One also knows that $\hat{A}(M)$ vanishes on the symmetric examples with finite second homotopy group (cf. theorem [7].23.3.iii, p. 332). Atiyah and Hirzebruch showed that the $\hat{A}$-genus vanishes on spin manifolds with smooth effective S^1-action (cf. [3]).

In [35] Haydeé and Rafael Herrera offered a proof for the vanishing of the $\hat{A}$-genus on any π_2-finite manifold with smooth effective S^1-action (cf. theorem [35].1, p. 341). Since one knows from the work of Salamon that the isometry group of a 12-dimensional Positive Quaternion Kähler Manifold is of dimension 5 at least (cf. 1.18), this would have been sufficient to prove the LeBrun–Salamon conjecture (cf. 1.6) in this dimension.

The argument in [35] essentially consists of three parts:

- In the first part Haydeé and Rafael Herrera argue that any smooth S^1-action on a π_2-finite manifold is of even or of odd type. (This condition means that the sum of rotation numbers at the S^1-fixed-points is always even or always odd.)
- In the second step they argue that the proof of Bott–Taubes [9] for the rigidity of the elliptic genus may be adapted to non-spin manifolds if the S^1-action is of even or of odd type.
- Finally, they use an argument by Hirzebruch–Slodowy [39] to derive the vanishing of the $\hat{A}$-genus from the rigidity of the elliptic genus.

Unfortunately, the first part of their argument cannot be correct. The real Grassmannian $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$ is a simply-connected non-spin Positive Quaternion Kähler Manifold with finite second homotopy group $\pi_2(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})) = \mathbb{Z}_2$—cf. p. 16 and corollary 1.12. However, for any odd $n \geq 3$ the real Grassmannian $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$ admits a smooth effective $\mathbb{S}^1$-action that is *neither odd nor even*:

We shall describe such an action on $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)$ in detail. We refer the reader to chapter B of the appendix for a far more general treatise on fixed-point components of isometric actions of either $\mathbb{S}^1$ or $\mathbb{Z}_2$ on Wolf spaces. The following computations can be regarded as a showcase computation for the classification given in the appendix.

The symmetric space $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)$ is the oriented real Grassmannian of 4-planes in $\mathbb{R}^7$ counted with orientations.

On $\mathbb{C}$ we denote the standard action of the unit circle $\mathbb{S}^1 \subseteq \mathbb{C}$ by

$$t : \mathbb{S}^1 \to \mathrm{Aut}(\mathbb{C}) = \mathbb{C}^* \qquad t(s) \cdot z = s \cdot z$$

By abuse of notation we shall also denote the standard action on $\mathbb{R}^2 \cong \mathbb{C}$ by t.

We consider the $\mathbb{S}^1$-action $r = (t, 1, 1)$ on $\mathbb{C}^3 \cong \mathbb{R}^6$; i.e. the standard action on the first factor of $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$ and the trivial action on the other two ones. We regard r as an action on $\mathbb{R}^7$ by the canonical inclusion $\mathbb{R}^6 \subseteq \mathbb{R}^7$. Thus r induces an action on $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)$. For this we fix an orthonormal basis v, w, x, y of a plane $\langle v, w, x, y\rangle$ and an orientation $\det(v, w, x, y) := v \wedge w \wedge x \wedge y \in \bigwedge^4 \langle v, w, x, y\rangle$. The image of this pair under a certain orthogonal transformation of $\mathbb{R}^7$ is the plane generated by the images of v, w, x, y with the induced orientation.

In our situation the image of the rotation $r(s)$ with $s \in \mathbb{S}^1$ is the plane

$$\langle (r(s))(v), (r(s))(w), (r(s))(x), (r(s))(y)\rangle$$

with the orientation

$$\begin{aligned} & r(s))(v) \wedge (r(s))(w) \wedge (r(s))(x) \wedge (r(s))(y) \\ \in & \textstyle\bigwedge^4 \langle (r(s))(v), (r(s))(w), (r(s))(x), (r(s))(y)\rangle \end{aligned}$$

Hence the subgroup $\langle -1\rangle = \mathbb{Z}_2 \subseteq \mathbb{S}^1$ acts by $(-1,1,1)$ on $\mathbb{C}^3 \cong \mathbb{R}^6$. Let us write this action as a real action on $\mathbb{R}^7$. Then it is given by

$$r(-1) = (-1,-1,1,1,1,1,1)$$

The fixed-point components of this involution are the spaces $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^5)$ and $\widetilde{\mathbf{Gr}}_2(\mathbb{R}^5)$, the oriented Grassmannian of oriented two-planes in $\mathbb{R}^5$. This can be seen as follows: Let V_1 be the five-dimensional vector space on which the action $r(-1)$ is given by $+1$ and let V_2 be the two-dimensional space with (-1)-action. So suppose $v = (v_1, v_2)$, $w = (w_1, w_2)$, $x = (x_1, x_2)$, $y = (y_1, y_2)$ are vectors in $V_1 \oplus V_2$ which generate a 4-plane $P = \langle v, w, x, y\rangle$ that is fixed by the involution. Thus

$$(v_1, -v_2) = (r(-1))(v_1, v_2) \in P$$

lies in P and so do $(v_1, 0), (0, v_2) \in P$. The same works for the other vectors and we see that P is already generated by vectors from V_1 and from V_2 only, i.e.

$$P = \langle (v_1,0),(0,v_2),(y_1,0),(0,y_2),(y_1,0),(0,y_2),(y_1,0),(0,y_2)\rangle$$

As P is four-dimensional, there is a choice

$$\tilde{v},\tilde{w},\tilde{x},\tilde{y} \in \{(v_1,0),(0,v_2),(y_1,0),(0,y_2),(y_1,0),(0,y_2),(y_1,0),(0,y_2)\}$$

with the property that

$$P = \langle \tilde{v},\tilde{w},\tilde{x},\tilde{y}\rangle$$

(Moreover, by orthonormalisation, we then may assume the $\tilde{v},\tilde{w},\tilde{x},\tilde{y}$ to form an orthonormal basis.) We may now step through the different cases according to how many of generating vectors $\tilde{v},\tilde{w},\tilde{x},\tilde{y}$ are in V_1 (and how many of them lie in V_2). As $\dim V_2 = 2$, there can at most two of them be in V_2.

Assume first that $\tilde{v},\tilde{w},\tilde{x},\tilde{y} \in V_1$. Then the vectors generate a plane that may be regarded as an element in

$$\widetilde{\mathbf{Gr}}_4(V_1) \cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^5)$$

when we count it with orientation. Conversely, every element in this subspace is fixed by $r(-1)$. (Here orientations are not altered by $r(-1)$ as every generating vector $\tilde{v},\tilde{w},\tilde{x},\tilde{y}$ of P is fixed by $r(-1)$.)

As a second case suppose $\tilde{v},\tilde{w},\tilde{x} \in V_1$ and $\tilde{y} \in V_2$, i.e. three generators are from V_1 and the fourth generator is from V_2. This case cannot occur, as P will not be fixed under $r(-1)$. Indeed, the orientation of P is reversed in that case, since an odd numbers of directions is multiplied by (-1), i.e. we apply the matrix

$$r(-1)|_P = \mathrm{diag}(1,1,1,-1) \notin \mathbf{SO}(P) \cong \mathbf{SO}(4)$$

with determinant -1.

Finally, only the case with two generators in both V_1 and V_2 remains, i.e. $\tilde{v}, \tilde{w} \in V_1$ and $\tilde{x}, \tilde{y} \in V_2$. Thus V_2 is entirely contained in P. Hence P (together with an orientation upon it) is a point in

$$\overline{\mathbf{Gr}_2(\mathbb{R}^2) \times \mathbf{Gr}_2(\mathbb{R}^5)}$$

—the space of all oriented four-planes which can be generated by two basis vectors in both V_1 and V_2. This is the same space as

$$\widetilde{\mathbf{Gr}}_2(\mathbb{R}^5)$$

Conversely, every element in $\overline{\mathbf{Gr}_2(\mathbb{R}^2) \times \mathbf{Gr}_2(\mathbb{R}^5)}$ stays fixed under the involution. This is due to the fact that with respect to the basis $\tilde{v}, \tilde{w}, \tilde{x}, \tilde{y}$ we now obtain

$$r(-1)|_P = \mathrm{diag}(-1,-1,1,1) \in \mathbf{SO}(P) \cong \mathbf{SO}(4)$$

Thus the orientation of P is preserved.

Both fixed-point components have positive Euler characteristics

$$\begin{aligned}\chi(\widetilde{\mathbf{Gr}}_2(\mathbb{R}^5)) &= \chi\left(\frac{\mathbf{SO}(5)}{\mathbf{SO}(3) \times \mathbf{SO}(2)}\right) = \frac{|W(\mathbf{SO}(5))|}{|W(\mathbf{SO}(3))|} = \frac{2^2 \cdot 2!}{2^1 \cdot 1!} = 4 \\ \chi(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^5)) &= \chi(\mathbb{H}\mathbf{P}^1) = \chi(\mathbb{S}^4) = 2\end{aligned}$$

(This fits to the fact that $\chi(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)) = 6 = 4+2$.) The $\mathbb{S}^1$-action restricts to an action on each of the fixed-point components. Thus by the Lefschetz fixed-point theorem there are $\mathbb{S}^1$-fixed-points in each respective component.

We can easily compute the dimensions of the fixed-point components:

$$\begin{aligned}\dim \widetilde{\mathbf{Gr}}_2(\mathbb{R}^5) &= \dim \frac{\mathbf{SO}(5)}{\mathbf{SO}(3) \times \mathbf{SO}(2)} = 6 \\ \dim \widetilde{\mathbf{Gr}}_4(\mathbb{R}^5) &= \dim \mathbb{S}^4 = 4\end{aligned}$$

Thus the difference in dimension

$$\dim \widetilde{\mathbf{Gr}}_2(\mathbb{R}^5) - \dim \widetilde{\mathbf{Gr}}_4(\mathbb{R}^5) = 6 - 4 = 2 \not\equiv 0 \mod 4$$

of the fixed-point components is not divisible by 4. Consequently, the $\mathbb{S}^1$-action r can be neither even nor odd.

Clearly, this example may be generalised to arbitrary dimensions: For any odd $n \geq 3$ we obtain an $\mathbb{S}^1$-action on $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$ that is neither even nor odd in the same way as above, i.e. by setting $r := (t, 1, \ldots, 1)$. The fixed-point components are then given by

$$\widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n+2}) = \frac{\mathbf{SO}(n+2)}{\mathbf{SO}(n) \times \mathbf{SO}(2)}$$
$$\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+2}) = \frac{\mathbf{SO}(n+2)}{\mathbf{SO}(n-2) \times \mathbf{SO}(4)}$$

Thus their dimensions can be computed as

$$\dim \widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n+2}) = \frac{n+1}{2}(n+2) - \frac{n-1}{2}n - 1 = 2n$$
$$\dim \widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+2}) = \frac{n+1}{2}(n+2) - \frac{n-3}{2}(n-2) - 6 = 4(n-2)$$

Again the difference in dimension satisfies $4(n-2) - 2n = 2n - 8 \not\equiv 0 \mod 4$, since n is odd.

Several further examples may be given on $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$ (for odd $n \geq 3$) by varying the $\mathbb{S}^1$-action r. So consider $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^9)$ equipped with the action $(t, t, 1, 1, 1, 1, 1)$. Here, for example, the spaces $\overline{\mathbf{Gr}_2(\mathbb{R}^4) \times \mathbf{Gr}_2(\mathbb{R}^5)}$ of dimension 10 and $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^5)$ of dimension 4 occur as $\mathbb{Z}_2$-fixed-point components.

Let us also mention a slightly different type of space on which we may easily produce analogous examples. The real Grassmannian $\widetilde{\mathbf{Gr}}_3(\mathbb{R}^9)$ of real oriented three-planes in $\mathbb{R}^9$ is orientable and has finite second homotopy. Let it be equipped with an $\mathbb{S}^1$-action of the form $(t, t, t, 1, 1, 1)$ yielding an embedded involutive action by $(-1, -1, -1, -1, -1, -1, 1, 1, 1)$. The same arguments as above show that we obtain three fixed-point components of this action, namely a double point, i.e. twice the space $\widetilde{\mathbf{Gr}}_3(\mathbb{R}^3)$, and the space $\overline{\mathbf{Gr}_1(\mathbb{R}^3) \times \mathbf{Gr}_2(\mathbb{R}^6)}$. (Further components do not arise due to orientation issues.) The latter component has dimension 10, whence the difference in dimension is congruent to two modulo four. (Clearly, all components involved have $\mathbb{S}^1$-fixed-points.) One may extend this example to all higher even dimensions by considering $\widetilde{\mathbf{Gr}}_3(\mathbb{R}^{9+2k})$ for $k \in \mathbb{N}$. More examples can easily be derived from the classification given in chapter B of the appendix.

The error in [35] can be traced back to an application of a result of Bredon on the representations at different fixed-points which requires that $\pi_2(M)$ *as well as* $\pi_4(M)$ be finite—cf. theorem [35].4, p. 351 and the paragraph below it. We suppose that this error may have occurred for confusion of real and complex representation rings. Indeed, the computations below [35].4 are correct applied to the *complexified* representations. The computations in [35] rely heavily on the property that a representation is divisible by $(1-t)^2$. Yet, in contrast to the real case, this property is nothing special for complexified representations:

Let $\phi_1(\mathbb{S}^1)$ and $\phi_2(\mathbb{S}^1)$ be real representations of the circle group $\mathbb{S}^1$ on a $2n$-dimensional real vector space. Again we make no difference between the standard complex representation t on $\mathbb{C}$ and the standard real representation of $\mathbb{S}^1$. The complex representation ring of $\mathbb{S}^1$ is just the ring of integer-valued Laurent polynomials $\mathbb{Z}[t,t^{-1}]$. The complexification of $\phi_1(\mathbb{S}^1)$ respectively of $\phi_2(\mathbb{S}^1)$ is denoted by $\phi_1(\mathbb{S}^1)\otimes\mathbb{C} \in \mathbb{Z}[t,t^{-1}]$ respectively by $\phi_2(\mathbb{S}^1)\otimes\mathbb{C} \in \mathbb{Z}[t,t^{-1}]$.

Proposition 2.1. *In* $\mathbb{Z}[t,t^{-1}]$ *we have the divisibility relation:*

$$(1-t)^2 \mid (\phi_1(\mathbb{S}^1)\otimes\mathbb{C} - \phi_2(\mathbb{S}^1)\otimes\mathbb{C})$$

PROOF. Real representations of $\mathbb{S}^1$ split as sums of the t^k and the real-one-dimensional trivial representation. Complexification replaces the real-one-dimensional representation 1 by the complex-one-dimensional representation 1. The difference of two representations on a real-even-dimensional vector space splits into a sum of representations of the form t^k (with possibly $k=0$). Representations on a real-odd-dimensional vector space necessarily have a summand 1 in the real representation ring. Hence their complexifications have a summand 1 in the complex representation ring. These two summands cancel out when taking the difference of two representations on the same odd-dimensional vector space. Thus we conclude that it is sufficient to prove the divisibility result for representations that split as sums of the t^k.

Complexification replaces a representation of the form t^k with $k\in\mathbb{Z}$ by t^k+t^{-k}. The difference $\phi_1(\mathbb{S}^1)\otimes\mathbb{C} - \phi_2(\mathbb{S}^1)\otimes\mathbb{C}$ therefore splits as a finite sum with summands of the form

$$\pm(t^k+t^{-k}-(t^l+t^{-l}))$$

Without restriction we assume $k>l\geq 0$. We show that every such summand is divisible by $(1-t)^2$ in $\mathbb{Z}[t,t^{-1}]$, whence the result follows. Indeed, we have

$$\begin{aligned} t^k+t^{-k}-(t^l+t^{-l}) &= (t^{k-l}-1)(t^l-t^{-k}) \\ &= t^{-k}(t^{k-l}-1)(t^{k+l}-1) \\ &= t^{-k}(1-t)^2(1+t+t^2+\ldots+t^{k-l-1})(1+t+t^2+\ldots+t^{k+l-1}) \end{aligned}$$

and $(1-t)^2 \mid (t^k+t^{-k}-(t^l+t^{-l}))$. □

For the sake of completeness we compute the rotation numbers in our main geometric example: Given the $\mathbb{S}^1$-action $r=(t,1,1,1,1,1)$ on $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)$ we choose $\mathbb{S}^1$-fixed-points x respectively y in the $\mathbb{Z}_2$-fixed-point components $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^5)$ respectively $\widetilde{\mathbf{Gr}}_2(\mathbb{R}^5)$. The situation is simplified by the observation that obviously in our case the $\mathbb{Z}_2$-fixed-point components are exactly the $\mathbb{S}^1$-components. Now we use charts on the real Grassmannian which are given by matrices of the form

$$\begin{pmatrix} * & \ldots & * & 1 & 0 & 0 & 0 \\ * & \ldots & * & 0 & 1 & 0 & 0 \\ * & \ldots & * & 0 & 0 & 1 & 0 \\ * & \ldots & * & 0 & 0 & 0 & 1 \end{pmatrix}^T$$

(together with an orientation) up to reordering. Let us compute the representation in x first. Since the plane corresponding to x lies entirely in V_1, we may choose a chart around x with the property that its identity-block also lies entirely in $(V_1)^4$. For this we clearly use that V_1 is of dimension $\dim V_1 = 5 > 4$. The non-identity part of the chart is a real matrix $T \in \mathbb{R}^{3\times 4}$ and corresponds to the 12-dimensional tangent space of $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)$ at x. Its (1×4)-submatrix contained in $(V_1)^4$ corresponds to the tangent space of $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^5)$.

The matrix T is acted upon by left matrix-multiplication of the (3×3)-real matrix $\operatorname{diag}(t,1)$. So on the tangent space this produces the real-12-dimensional $\mathbb{S}^1$-representation $(t,t,t,t,1,1,1,1)$ —clearly, the action is trivial on the 4-dimensional tangent space of $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^5)$.

In order to compute the representation in y we need to use a chart that is the identity on V_2. Thus the left-action of $\mathbb{S}^1$ is given by left-matrix-multiplication with $\operatorname{diag}(t,1,1,1,1,1)$. Since it changes the chart, we have to correct this by right-multiplication with its inverse (which just replaces generating vectors of the plane by certain linear combinations of them). So we obtain the action on the non-identity part corresponding to the tangent space by

$$\operatorname{diag}((t,1,1),1,1,1)\cdot \begin{pmatrix} 1 & 0 & 0 & 0 & * & \dots & * \\ 0 & 1 & 0 & 0 & * & \dots & * \\ 0 & 0 & 1 & 0 & * & \dots & * \\ 0 & 0 & 0 & 1 & * & \dots & * \end{pmatrix}^T \cdot \operatorname{diag}(t^{-1},1,1)$$

This directly yields the $\mathbb{S}^1$-representation on y which is given by $(t,t,t,1,1,1,1,1,1)$. (This again corresponds to the fact that the real dimension of $\widetilde{\mathbf{Gr}}_2(\mathbb{R}^5)$ is six.) The difference of the complexified representations in x and y now is given by $(4(t+t^{-1})+4)-(3(t^{-1}+t)+6) = t+t^{-1}-2 = t^{-1}(1-t)^2$. The difference of real representations is $4t-3t-2 = t-2$, which is not divisible by $(1-t)^2$ in the real representation ring $\mathbb{Z}[t]$.

All this prompts the question whether one can prove the vanishing of the $\hat{A}$-genus on π_2-finite manifolds with smooth effective $\mathbb{S}^1$-action by other means. This question must be answered in the negative. More precisely, we shall construct counter-examples in each dimension $4k \geq 8$. (In dimension 4 the $\hat{A}$-genus does vanish on a simply connected π_2-finite manifold, since it is a multiple of the signature.)

Together with Anand Dessai we provide a construction for every dimension $4k \geq 8$ of a simply-connected π_2-finite manifold with non-vanishing $\hat{A}$-genus in [2]. The construction is a straight-forward adaption of the classical elementary surgery theory (cf. [12], chapter IV) to the equivariant setting and relies heavily on

Lemma 2.2 (Surgery Lemma)**.** *Let G be a compact Lie group and let M be a smooth simply-connected G-manifold. Suppose the fixed-point manifold M^G contains* *a submanifold N of dimension $\dim N \geq 5$ such that the inclusion map $N \hookrightarrow M$ is 2-connected. Then M is G-equivariantly bordant to a simply connected G-manifold M' with $\pi_2(M') \subseteq \mathbb{Z}_2$.*

PROOF. Let $f : M \to \mathbf{BSO}$ be a classifying map for the stable normal bundle of M. We fix a finite set of generators for the kernel of $\pi_2(f) : \pi_2(M) \to \pi_2(\mathbf{BSO}) \cong \mathbb{Z}_2$. Since the inclusion map $N \hookrightarrow M$ is 2-connected and since $\dim N \geq 5$, we may represent these generators by disjointly embedded 2-spheres in N. By construction the normal bundle in M of each such 2-sphere is trivial as a non-equivariant bundle and equivariantly diffeomorphic to a G-equivariant vector bundle over the trivial G-space $\mathbb{S}^2$. For each embedded 2-sphere we identify the normal bundle G-equivariantly with a tubular neighbourhood of the sphere and perform G-equivariant surgery for all of these 2-spheres. The result of the surgery is a simply connected G-manifold M' with $\pi_2(M') \subseteq \mathbb{Z}_2$. (If M is a spin manifold, then M' is actually 2-connected.) □

Theorem 2.3. *For any $k > 1$ there exists a smooth simply-connected $4k$-dimensional π_2-finite manifold M^{4k} with smooth effective $\mathbb{S}^1$-action and $\hat{A}(M^{4k})[M^{4k}] \neq 0$.*

PROOF. We begin with some effective linear $\mathbb{S}^1$-action on the complex projective space $M = \mathbb{C}\mathbf{P}^{2k}$ such that the fixed-point manifold $M^{\mathbb{S}^1}$ contains a component N which is diffeomorphic to $\mathbb{C}\mathbf{P}^l$ for some $l \geq 3$. Since $N \hookrightarrow \mathbb{C}\mathbf{P}^{2k}$ is 2-connected, the manifold $\mathbb{C}\mathbf{P}^{2k}$ is $\mathbb{S}^1$-equivariantly bordant to a simply-connected $\mathbb{S}^1$-manifold M' with finite $\pi_2(M')$ by the surgery lemma 2.2. (In fact, $\pi_2(M') \cong \mathbb{Z}_2$, since $\mathbb{C}\mathbf{P}^{2k}$ is not a spin manifold.) It is well-known that the $\hat{A}$-genus does not vanish on $\mathbb{C}\mathbf{P}^{2k}$. Since M' is bordant to $\mathbb{C}\mathbf{P}^{2k}$, we obtain $\hat{A}(M') = \hat{A}(\mathbb{C}\mathbf{P}^{2k}) \neq 0$. □

It is straight-forward to produce examples with much larger symmetry using the construction above. Higher torus symmetries follow right in the same way: Let $\mathbb{S}^1 \subseteq T$ be a subgroup of a torus $T \subseteq \mathbf{U}(2k+1)$, where $\mathbb{C}\mathbf{P}^{2k} = \frac{\mathbf{U}(2k+1)}{\mathbf{U}(2k)\times\mathbf{U}(1)}$. Let some $\mathbb{C}\mathbf{P}^l$ be fixed by $\mathbb{S}^1$. The torus action restricts to an action on $\mathbb{C}\mathbf{P}^l$ and we may proceed iteratively. If k is large enough and the torus is chosen in an appropriate manner, there will be a suitable $\mathbb{C}\mathbf{P}^{l'}$ in M^T.

Let $\mathbf{U}(n) \subseteq \mathbf{U}(2k+1)$ be a blockwise included canonical subgroup which acts on $\frac{\mathbf{U}(2k+1)}{\mathbf{U}(2k)\times\mathbf{U}(1)}$ by left multiplication. Thus this subgroup fixes the component

$$\frac{\mathbf{U}(2k-n+1)}{\mathbf{U}(2k-n)\times\mathbf{U}(1)} \cong \mathbb{C}\mathbf{P}^{2k-n}$$

This enables us to find examples with $\mathbf{U}(n)$-symmetry in dimension $4k$ provided $2k - n \geq 3$.

Clearly, in theorem 2.3 one may replace the complex projective space by other suitable spaces with non-vanishing $\hat{A}$-genus. One possible choice is $\mathbf{Gr}_2(\mathbb{C}^{n+2})$.

Moreover, the proof of the surgery lemma 2.2 leads to the following observation:

Remark 2.4. Let M be a manifold of dimension at least 10 with vanishing first Pontrjagin class $p_1(M)$. Let M admit an action by a compact Lie group with a fixed-point component of dimension greater than or equal to 9 the inclusion of which is 4-connected. The proof of the lemma shows that M may be altered to a bordant M' with $\pi_2(M')$ and $\pi_4(M')$ finite.

Indeed, as the first Pontrjagin class $p_1(M)$ of M vanishes, we see that the morphism

$$\pi_4(f)\otimes\mathbb{Q} : \pi_4(M)\otimes\mathbb{Q}\to\pi_4(\mathbf{BSO})\otimes\mathbb{Q}$$

induced by the classifying map $f : M\to\mathbf{BSO}$ is the zero-map. Thus “up to a finite group” the group $\pi_4(M)$ lies in the kernel of $\pi_4(f)$ which can be “removed” by surgery. Due to the dimension of M surgery does not produce new homotopy generators in degrees smaller than or equal to 4.

Thus it follows that M' is π_2-finite and π_4-finite. Hence Bredon’s divisibility result applies and the proof in [35] yields that $\hat{A}(M)[M]=\hat{A}(M')[M']=0$. ⊡

It remains a challenging task to determine whether the $\hat{A}$-genus vanishes on π_2-finite Positive Quaternion Kähler Manifolds as predicted by the LeBrun–Salamon conjecture (cf. 1.6).

Moreover, this project naturally extends to a description of the elliptic genus on such manifolds, which would be very welcome. Indeed, on spin manifolds there is the following theorem:

Theorem 2.5. *Let g be an involution contained in the compact connected Lie group G acting differentiably on the $4n$-dimensional spin manifold X. If the action is odd, then $\Phi(X)=0$. Suppose the action is even and* $\operatorname{codim} X^g\geq 4r$. *If $r>0$, then $\hat{A}(X)=0$. If $r>1$, then $\hat{A}(X,T_{\mathbb{C}}X)=0$. If $r>n/2$, then $\Phi(X)=0$.*

Proof. See the corollary on [39], p. 317. □

This theorem is a refinement of the vanishing theorem of the $\hat{A}$-genus. It relies on the rigidity of the elliptic genus, i.e. $\Phi(X,g)=\Phi(X)$ and the formula

$$\Phi(X,g)=\Phi(X^g\pitchfork X^g)$$

which relates the equivariant elliptic genus to the transversal self-intersection of the fixed-point set X^g—independent of whether the manifold is spin or not. So having proved the rigidity of the elliptic genus, one may not only derive the vanishing of the $\hat{A}$-genus but a theorem analogous to 2.5—cf. theorem [37].2.1, p. 254. This would have enabled us to prove the vanishing of the whole elliptic genus—and the signature $\operatorname{sign}(M)$, in particular—on certain π_2-finite Positive Quaternion Kähler Manifolds. For a treatise on the elliptic genus of Wolf spaces we refer to chapter D of the appendix.

Unfortunately, the erroneous reasoning in [35] has further negative impact: Not only was it used for the classification of 12-dimensional Positive Quaternion Kähler Manifolds, it also appears as a key argument in subsequent work by Haydeé and Rafael Herrera.

Furthermore, for example the classification in dimension 12 was used in lemma [19].2.5, which served to prove theorem B in the article.

3

Rational Homotopy Theory

This chapter provides some results that arise from an application of Rational Homotopy Theory to the field of Positive Quaternion Kähler Manifolds. To the author's knowledge this is the first attempt in this direction. Despite the geometric intentions we have managed to keep this chapter highly algebraical. Thus our results can be stated in a rather general context.

Readers without experience in the field of Rational Homotopy Theory are recommended to initially have a short glimpse at the introductory section 1.2 of chapter 1, where all the necessary concepts, tools, supporting reasonings and notations are provided.

As a main result we shall prove the formality of Positive Quaternion Kähler Manifolds. Although this theorem motivates our efforts, this chapter is devoted to an in-depth analysis of spherical fibrations—far beyond of what turns out to be necessary for the geometric result.

Moreover, we shall find new techniques of how to produce non-formal homogeneous spaces. Furthermore, in low-dimensions we shall provide very simple direct proofs for the formality of Positive Quaternion Kähler Manifolds. This chapter is completed by some results under the assumption that the spaces in consideration are rationally elliptic. In this vein we shall give a discussion for

- low-dimensional Positive Quaternion Kähler Manifolds and
- manifolds the cohomology of which admits a Lefschetz-like property. (Applications can be found in the field of biquotients, symplectic manifolds and Joyce manifolds.)

3.1. Formality and spherical fibrations

Before we shall begin to establish the main result let us just mention the following proposition which may serve as a motivation. Indeed, despite its complete simplicity, its proof illustrates beautifully the interplay between base space and total space of a fibration, which we shall benefit from.

Proposition 3.1. *Let M be a Positive Quaternion Kähler Manifold with $b_4(M) = 1$. Then its rational homotopy groups are a formal consequence of its rational cohomology algebra.*

Proof. By 1.12 and 1.13 we may assume M to be rationally 3-connected as the symmetric space $\mathbf{Gr}_2(\mathbb{C}^{n+2})$ is formal (cf. 1.36). Since $b_4 = 1$ there is a unique class $u \in H^4(M)$ (up to multiples) with respect to which $H^*(M)$ has the Hard-Lefschetz property (cf. 1.13). This implies that the cohomology ring of the twistor space Z is uniquely determined by $H^*(M)$, as we necessarily have $z^2 = u$ (up to multiples). Due to the formality of the compact Kähler manifold Z (cf. 1.36) we obtain that the rational homotopy type of Z is a formal consequence of $H^*(Z)$. In particular, so are the rational homotopy groups $\pi_*(Z) \otimes \mathbb{Q}$. The long exact homotopy sequence applied to the twistor fibration

$$\mathbb{S}^2 \hookrightarrow Z \to M$$

splits and yields the following results: We have

$$\begin{aligned} &\pi_2(Z) \otimes \mathbb{Q} \cong \mathbb{Q} \\ &\pi_i(M) \otimes \mathbb{Q} \cong \pi_i(Z) \otimes \mathbb{Q} \qquad \text{for } i \geq 5 \end{aligned}$$

and

$$0 \to \pi_4(Z) \otimes \mathbb{Q} \to \pi_4(M) \otimes \mathbb{Q} \xrightarrow{\partial} \mathbb{Q} \to \pi_3(Z) \otimes \mathbb{Q} \to 0$$

is exact. Up to duality the transgression ∂ can be identified with the linear part of the differential in the model of the fibration (cf. [22].15 (e), p. 214). In the notation of remark 1.34 we obtain $\mathrm{d}e' = z^2 - u$ with linear part $\mathrm{d}_0 e' = -u \neq 0$. The element e' generates $\pi_3(\mathbb{S}^2) \otimes \mathbb{Q}$. Thus the morphism d_0 is injective and its dual morphism ∂ is surjective. We conclude that

$$\begin{aligned} &\pi_3(Z) \otimes \mathbb{Q} = 0 \\ &\pi_4(M) \otimes \mathbb{Q} \cong (\pi_4(Z) \otimes \mathbb{Q}) \oplus \mathbb{Q} \end{aligned}$$

Hence the rational homotopy groups of Z completely determine the groups $\pi_*(M) \otimes \mathbb{Q}$.

In summary, we have proved that the rational cohomology algebra of M determines the one of Z, which itself determines the rational homotopy groups of Z and consequently also $\pi_*(M) \otimes \mathbb{Q}$. □

This result was intended to serve as a showcase computation. Clearly, once having established the formality of Positive Quaternion Kähler Manifolds, this proposition will be obsolete. Let us now start to work out the main topological results.

According to remark 1.33 we shall call a fibration

$$F \hookrightarrow E \to B$$

of topological spaces a *spherical fibration* provided the fibre F is path-connected and $F \simeq_{\mathbb{Q}} \mathbb{S}^n$ for $n \geq 1$.

The discussion of spherical fibrations mainly splits into two parts: Either the rational homological dimension of the fibre sphere is even or it is odd.

3.1.1. Even-dimensional fibres

In this section we shall see that under slight technical prerequisites it is possible to relate the formality of the base space to the formality of the total space when the (rational) fibre sphere is (homologically) even-dimensional. In particular, we shall prove our main result on Positive Quaternion Kähler Manifolds that will arise basically as a consequence of the far more general

Theorem 3.2. *Let*

$$F \hookrightarrow E \xrightarrow{p} B$$

be a spherical fibration of simply-connected (and path-connected) topological spaces with rational homology of finite type. Let the fibre satisfy $F \simeq_{\mathbb{Q}} \mathbb{S}^n$ for an even $n \geq 2$. Suppose further

- *that the rationalised Hurewicz homomorphism $\pi_*(B) \otimes \mathbb{Q} \to H_*(B)$ is injective in degree n,*
- *that the rationalised Hurewicz homomorphism $\pi_*(B) \otimes \mathbb{Q} \to H_*(B)$ is injective in degree $2n$.*

Then we obtain: If the space E is formal, so is B.

Proof. Choose a minimal Sullivan model $\big(\bigwedge V_B, \mathrm{d}_B\big) \xrightarrow{\simeq} \mathrm{A}_{\mathrm{PL}}(B)$. Recall (cf. 1.34) the following quasi-isomorphism of cochain algebras:

$$\Big(\bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, \mathrm{d}\Big) \xrightarrow{\simeq} (\mathrm{A}_{\mathrm{PL}}(E), \mathrm{d}) \tag{3.1}$$

with $\mathrm{d}z = 0$, $\mathrm{d}z' = z^2 - u$, $u \in \bigwedge V_B$, $\mathrm{d}u = 0$ and $\mathrm{d}|_{\bigwedge V_B} = \mathrm{d}_B$. We call p primitive if u is not decomposable.

We shall prove the theorem in the following manner:

Case 1. The fibration is primitive.

Step 1. We construct a minimal Sullivan algebra from (3.1).

Step 2. We show that it extends to a minimal Sullivan model of E by proving

Surjectivity and

Injectivity of the morphism induced in cohomology.

Step 3. We show the formality of the model for B by means of the one for E.

Case 2. The fibration is non-primitive.

Step 1. We prove that (3.1) is a minimal Sullivan model of E.

Step 2. We show the formality of the model.

Let us commence the proof:

Case 1. We suppose the fibration to be primitive.

Step 1. If the fibration satisfies $\mathrm{d}z' = z^2$, i.e. $u = 0$, then remark 1.34 yields that the fibration is rationally trivial. This just means that $E \simeq_{\mathbb{Q}} B \times F$ or, equivalently, that there is a weak equivalence $\mathrm{A_{PL}}(E) \simeq \mathrm{A_{PL}}(B) \otimes \mathrm{A_{PL}}(F)$. Theorem 1.47 then applies and yields that the formality of E is equivalent to the formality of B, since spheres are formal due to example 1.31.

Therefore it suffices to assume $\mathrm{d}z' \neq z^2$, i.e. $u \neq 0$. Since the fibration is primitive, we may write

$$u = u' + v$$

with $0 \neq u' \in V_B$, $v \in \bigwedge^{>0} V_B \cdot \bigwedge^{>0} V_B$ both homogeneous with respect to degree.

The condition of primitivity contradicts the property of minimality for Sullivan algebras. Thus we shall construct a minimal Sullivan algebra $\big(\bigwedge V_E, \mathrm{d}\big)$ out of $\big(\bigwedge V_B \otimes \bigwedge \langle z, z'\rangle, \mathrm{d}\big)$.

This will be done as follows: Let $u', \{b_i\}_{i\in J}$ denote a (possibly infinite yet countable) basis of V_B homogeneous with respect to degree. Set $V_E := \langle z, \{b_i\}_{i\in J}\rangle$ with the previous grading. Let

$$\phi : \bigwedge V_B \otimes \bigwedge \langle z, z'\rangle \to \bigwedge V_E \tag{3.2}$$

be the (surjective) morphism of commutative graded algebras defined by $\phi(b_i) := b_i$ for $i \in J$, $\phi(z) := z$, $\phi(z') := 0$, $\phi(u') := z^2 - \phi(v)$ and extended linearly and multiplicatively. (This is well-defined, since there is a subset $J' \subseteq J$ such that $v \in \bigwedge \langle \{b_i\}_{i\in J'}\rangle$ by degree. Thus when we define $\phi(u')$, the element $\phi(v)$ is already defined.)

We now define a differential d_E on $\bigwedge V_E$ by $\mathrm{d}_E z := 0$, by $\mathrm{d}_E b_i := \phi(\mathrm{d}(b_i))$ for $i \in J$ and by making it a derivation. We shall see that this really defines a differential, i.e. that $\mathrm{d}_E^2 = 0$. This will be a direct consequence of the property

$$\mathrm{d}_E \circ \phi = \phi \circ \mathrm{d} \tag{3.3}$$

which we shall prove first: It suffices to do so on the b_i, on u', z and z'. We compute

$$\begin{aligned}
\mathrm{d}_E(\phi(b_i)) &= \mathrm{d}_E(b_i) = \phi(\mathrm{d}(b_i)) \\
\mathrm{d}_E(\phi(u')) &= \mathrm{d}_E(z^2 - \phi(v)) = -\mathrm{d}_E(\phi(v)) = -\phi(\mathrm{d}(v)) = \phi(\mathrm{d}(u')) \\
\mathrm{d}_E(\phi(z)) &= \mathrm{d}_E(z) = 0 = \phi(\mathrm{d}(z)) \\
\mathrm{d}_E(\phi(z')) &= \mathrm{d}_E(0) = 0 = \phi(z^2 - u' - v) = \phi(\mathrm{d}(z'))
\end{aligned}$$

As for the second line we remark that the next to last equality is a direct consequence of the first line, since $v \in \bigwedge\langle\{b_i\}_{i\in J'}\rangle$. The last equality follows from $u = u' + v$ and $\mathrm{d}(u) = 0$.

For $x \in \bigwedge V_E$ there is $y \in V_B \otimes \bigwedge\langle z, z'\rangle$ with $\phi(y) = x$ by the surjectivity of ϕ. Applying (3.3) twice yields

$$\mathrm{d}_E^2(x) = \mathrm{d}_E(\phi(\mathrm{d}(y))) = \phi(\mathrm{d}^2(y)) = 0$$

which proves that $\big(\bigwedge V_E, \mathrm{d}_E\big)$ is a commutative differential graded algebra, even a commutative cochain algebra.

We shall now prove that it is even a minimal Sullivan algebra. For this we show that $\mathrm{d}_E z, \mathrm{d}_E b_i \in \bigwedge^{>0} V_E \cdot \bigwedge^{>0} V_E$ for $i \in J$, i.e. that these are decomposable. This will establish minimality.

The minimality of $\big(\bigwedge V_B, \mathrm{d}_B\big)$, which is a differential subalgebra of $\big(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle, \mathrm{d}\big)$, together with property (3.3) yields

$$\mathrm{d}_E(b_i) = \mathrm{d}_E(\phi(b_i)) = \phi(\mathrm{d}(b_i)) = \phi(\mathrm{d}_B(b_i))$$

with $\mathrm{d}_B(b_i)$ being decomposable. Thus so is $\phi(\mathrm{d}_B(b_i))$, as ϕ is multiplicative.

Additionally, we have $\mathrm{d}_E(z) = 0$. Thus d_E satisfies the minimality condition on V_E, whence so it does on $\bigwedge V_E$. As B is simply-connected, it holds that $V_B^1 = 0$. Thus proposition 1.25 implies that $\big(\bigwedge V_E, \mathrm{d}_E\big)$ is a minimal Sullivan algebra.

Step 2. We shall now exhibit ϕ as a quasi-isomorphism. This will make $\big(\bigwedge V_E, \mathrm{d}_E\big)$ a minimal Sullivan model of $\mathrm{A_{PL}}(E)$ due to step 1, the quasi-isomorphism (3.1) and the weak equivalence

$$\Big(\bigwedge V_E, \mathrm{d}_E\Big) \xleftarrow{\phi} \Big(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle, \mathrm{d}\Big) \xrightarrow{\simeq} (\mathrm{A_{PL}}(E), \mathrm{d})$$

In fact, by lemma 1.38 we then obtain a quasi-isomorphism

$$\big(\bigwedge V_E, \mathrm{d}_E\big) \xrightarrow{\simeq} (\mathrm{A_{PL}}(E), \mathrm{d})$$

We shall prove that ϕ is a quasi-isomorphism by separately proving the surjectivity and the injectivity of

$$\phi_* : H\big(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle, \mathrm{d}\big) \to H\big(\bigwedge V_E, \mathrm{d}_E\big)$$

Surjectivity. Let $x \in \big(\bigwedge V_E, \mathrm{d}_E\big)$ be an arbitrary closed element. Thus it defines a homology class. We shall construct a d-closed preimage of x; this will prove surjectivity.

Define a map of commutative graded algebras

$$\psi : \bigwedge V_E \to \bigwedge V_B \otimes \bigwedge\langle z, z'\rangle$$

by $b_i \mapsto b_i$ for $i \in J$, by $z \mapsto z$ and by extending it linearly and multiplicatively. Then ψ is injective, since

$$\phi \circ \psi = \mathrm{id} \tag{3.4}$$

The main idea to prove surjectivity will be the following: We have $\phi(\psi(x)) = x$. Now vary $\psi(x)$ additively by a form $\tilde{x} \in \ker\phi$ such that $\mathrm{d}(\psi(x) + \tilde{x}) = 0$.

Let $I \in \big(\mathbb{Q} \times \mathbb{N}^{\mathbb{N}}\big)^{\mathbb{N}}$ be a finite tuple $I = (a_1, \ldots, a_s)$ of finite tuples $a_i = (k_i, a_{(i,1)}, \ldots, a_{(i,t(i))})$. We shall use the multi-multiindex notation

$$b_I := \sum_{i=1}^{s} k_i \prod_{j=1}^{t(i)} b_{a_{(i,j)}}$$

in order to denote arbitrary linear combinations of products in the b_i. We construct a set $\tilde{I}$ from I in such a way that summands of b_I change sign according to their degree: Let $\delta(i) := \deg \prod_{j=1}^{t(i)} b_{a_{(i,j)}}$. With the coefficients of b_I we obtain

$$b_{\tilde{I}} = \sum_{i=1}^{s} (-1)^{\delta(i)} k_i \prod_{j=1}^{t(i)} b_{a_{(i,j)}}$$

With this terminology we come back to our problem. There is a finite family $(I_{(i,j)})_{i,j}$ of such tuples with the property that

$$\begin{aligned}\mathrm{d}(\psi(x)) =& (b_{I_{(0,0)}} + b_{I_{(1,0)}} u + b_{I_{(2,0)}} u^2 + \ldots) \\ &+ (b_{I_{(0,1)}} + b_{I_{(1,1)}} u + b_{I_{(2,1)}} u^2 + \ldots) z \\ &+ (b_{I_{(0,2)}} + b_{I_{(1,2)}} u + b_{I_{(2,2)}} u^2 + \ldots) z^2 \\ &+ \ldots\end{aligned}$$

Note that every coefficient $b_{I_{(i,j)}}$ is uniquely determined. Set

$$\tilde{x}_{i,j} := b_{\tilde{I}_{(i,j)}} \cdot z' \cdot z^j \cdot \sum_{l+2m=2i-2} z^l u^m$$

and

$$\tilde{x} := \sum_{i>0,j} \tilde{x}_{i,j} \quad \in \big(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle, \mathrm{d}\big) \tag{3.5}$$

We shall prove in the following that $\tilde{x}$ is indeed the element we are looking for, i.e. $\psi(x) + \tilde{x}$ is d-closed and maps to x under ϕ. This last property actually is self-evident, since

$$\tilde{x} \in z' \bigwedge\langle \{b_i\}_{i\in J}, z, u'\rangle$$

and since $\phi(z') = 0$ by definition. Thus $\phi(\psi(x) + \tilde{x}) = \phi(\psi(x)) = x$.

Let us now see that $\psi(x) + \tilde{x}$ is d-closed: For this we shall prove first that the term $\mathrm{d}(\psi(x) + \tilde{x})$ does not possess any summand that has $u^i z^j$ as a factor for any $i, j \in \mathbb{N} \times \mathbb{N}_0$. (Actually, such a summand is replaced by one in z^{2i+j} and by one more in z'.) This is due to the computation

$$\begin{aligned} &u^i z^j + (z^2 - u) \cdot z^j \cdot \sum_{l+2m=2i-2} z^l u^m \\ =&u^i z^j + z^j \cdot (z^2 - u) \cdot (u^{i-1} + z^2 u^{i-2} + z^4 u^{i-3} + \cdots + z^{2(i-1)}) \\ =&z^{2i+j} \end{aligned} \tag{3.6}$$

for $i > 0$ and finally—the sums run over $(i,j) \in \mathbb{N}_0 \times \mathbb{N}_0$—by

$$\begin{aligned}
\mathrm{d}(\psi(x)+\tilde{x}) &= \sum_{i,j} b_{I_{(i,j)}} u^i z^j + \sum_{i>0,j} \mathrm{d}\Big(b_{\tilde{I}_{(i,j)}} \cdot z' \cdot z^j \cdot \sum_{l+2m=2i-2} z^l u^m\Big) \\
&= \sum_{i,j} b_{I_{(i,j)}} u^i z^j + \sum_{i>0,j} \Big(b_{I_{(i,j)}} \cdot (z^2-u) \cdot z^j \cdot \sum_{l+2m=2i-2} z^l u^m \\
&\quad + \mathrm{d}\big(b_{\tilde{I}_{(i,j)}}\big) \cdot z' \cdot z^j \cdot \sum_{l+2m=2i-2} z^l u^m\Big) \\
&= \sum_{j} b_{I_{(0,j)}} z^j + \sum_{i>0,j} b_{I_{(i,j)}} \Big(u^i z^j + (z^2-u) \cdot z^j \cdot \sum_{l+2m=2i-2} z^l u^m\Big) \\
&\quad + \sum_{i>0,j} \mathrm{d}\big(b_{\tilde{I}_{(i,j)}}\big) \cdot z' \cdot z^j \cdot \sum_{l+2m=2i-2} z^l u^m \\
&\overset{(3.6)}{=} \sum_{j} b_{I_{(0,j)}} z^j + \sum_{i>0,j} \Big(b_{I_{(i,j)}} z^{2i+j} + \mathrm{d}\big(b_{\tilde{I}_{(i,j)}}\big) \cdot z' \cdot z^j \cdot \sum_{l+2m=2i-2} z^l u^m\Big) \\
&= \sum_{i,j} b_{I_{(i,j)}} z^{2i+j} + \sum_{i>0,j} \Big(\mathrm{d}\big(b_{\tilde{I}_{(i,j)}}\big) \cdot z' \cdot z^j \cdot \sum_{l+2m=2i-2} z^l u^m\Big)
\end{aligned} \tag{3.7}$$

Consequently, the term $\mathrm{d}(\psi(x)+\tilde{x})$ is in the "affine subalgebra"

$$\bigwedge \langle \{b_i\}_{i\in J}, z\rangle \oplus z' \bigwedge \langle \{b_i\}_{i\in J}, z, u'\rangle$$

More precisely, we see

$$\begin{aligned}
\big(\mathrm{d}(\psi(x)+\tilde{x})\big)|_{\bigwedge\langle\{b_i\}_{i\in J},z\rangle} &\overset{(3.7)}{=} \sum_{i,j} b_{I_{(i,j)}} z^{2i+j} \\
&= \phi(\mathrm{d}(\psi(x))) \\
&\overset{(3.3)}{=} \mathrm{d}_E(\phi(\psi(x))) \\
&= \mathrm{d}_E(x) \\
&= 0
\end{aligned} \tag{3.8}$$

The last equation holds by assumption, the second one by the definition of $\mathrm{d}(\psi(x))$. Thus it suffices to prove that also the second direct summand

$$\big(\mathrm{d}(\psi(x)+\tilde{x})\big)|_{z' \bigwedge\langle\{b_i\}_{i\in J},z,u'\rangle}$$

equals zero. Using the vanishing of the first one we compute

$$\begin{aligned}
0 = \mathrm{d}^2(\psi(x)+\tilde{x}) &= \mathrm{d}\big(\mathrm{d}(\psi(x)+\tilde{x})|_{z' \bigwedge\langle\{b_i\}_{i\in J},z,u'\rangle}\big) \\
&\in (z^2-u) \bigwedge \langle \{b_i\}_{i\in J}, z, u'\rangle \oplus z' \bigwedge \langle \{b_i\}_{i\in J}, z, u'\rangle
\end{aligned}$$

So let $z'y := \big(\mathrm{d}(\psi(x)+\tilde{x})\big)|_{z' \bigwedge\langle \{b_i\}_{i\in J}, z, u'\rangle})$. We have

$$0 = \mathrm{d}(z'y) = (z^2-u)y - z'\mathrm{d}y$$

Since $y, \mathrm{d}y \in \bigwedge\langle \{b_i\}_{i\in J}, z, u'\rangle$, we have $(z^2-u)y \neq z'\mathrm{d}y$ unless both sides vanish. The term

$$z^2 - u = z^2 - u' - v \in \bigwedge V_B \otimes \bigwedge\langle z, z'\rangle$$

is not a zero-divisor, since $u' \in V_B^{\mathrm{even}}$ is not. This necessarily implies $y = 0$. So we obtain

$$\mathrm{d}(\psi(x)+\tilde{x})|_{z' \bigwedge\langle \{b_i\}_{i\in J}, z, u'\rangle} = 0$$

Combined with (3.8) this yields

$$\mathrm{d}(\psi(x)+\tilde{x}) = 0$$

This finishes the proof of the surjectivity of ϕ_*.

Injectivity. We shall now prove that ϕ_* is also injective, i.e. an isomorphism in total. For this we prove that its kernel is trivial. So let $y \in \big(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle, \mathrm{d}\big)$ be a closed element with $\phi_*([y]) = 0$. Thus there is an element $x \in (\bigwedge V_E, \mathrm{d}_E)$ with $\mathrm{d}_E(x) = \phi(y)$. We shall construct an element $\tilde{x} \in \big(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle, \mathrm{d}\big)$ with the property that $\mathrm{d}(\psi(x)+\tilde{x}) = \bar{y}$. Hereby $\bar{y}$ will be a newly constructed closed element with the property that $[y] = [\bar{y}]$. This will show that $[\bar{y}] = 0$ and will prove injectivity.

There are families $(I_{(i,j)})_{i,j}$ and $(I'_{(i,j)})_{i,j}$ as above—cf. the surjectivity part—such that the element y may be written in the form

$$y = \sum_{i,j} b_{I_{(i,j)}} u^i z^j + z' \cdot \sum_{i,j} b_{I'_{(i,j)}} u^i z^j$$

Again coefficients are uniquely determined. Once more we form elements

$$\tilde{y}_{i,j} := b_{\tilde{I}_{(i,j)}} \cdot z' \cdot z^j \cdot \sum_{l+2m=2i-2} z^l u^m$$

(for $i, j \geq 0$) and

$$\tilde{y} := \sum_{i>0,j} \mathrm{d}\tilde{y}_{i,j}$$

On the analogy of (3.7) this leads to

$$\begin{aligned} y+\tilde{y} &= \sum_{i,j} b_{I_{(i,j)}} u^i z^j + z' \cdot \sum_{i,j} b_{I'_{(i,j)}} u^i z^j \\ &\quad + \sum_{i>0,j} \Big(b_{I_{(i,j)}} (z^2-u) z^j \cdot \sum_{l+2m=2i-2} z^l u^m + \mathrm{d}\big(b_{\tilde{I}_{(i,j)}}\big) \cdot z' z^j \cdot \sum_{l+2m=2i-2} z^l u^m \Big) \\ &= \sum_j b_{I_{(0,j)}} z^j + \sum_{i>0,j} b_{I_{(i,j)}} \cdot \Big(u^i z^j + (z^2-u) z^j \cdot \sum_{l+2m=2i-2} z^l u^m \Big) \\ &\quad + z' \cdot \Big(\sum_{i,j} b_{I'_{(i,j)}} u^i z^j - \mathrm{d}\big(b_{I_{(i,j)}}\big) \cdot z^j \cdot \sum_{l+2m=2i-2} z^l u^m \Big) \\ &\overset{(3.6)}{=} \sum_{i,j} b_{I_{(i,j)}} z^{2i+j} + z' \cdot \Big(\sum_{i,j} b_{I'_{(i,j)}} u^i z^j - \mathrm{d}\big(b_{I_{(i,j)}}\big) \cdot z^j \cdot \sum_{l+2m=2i-2} z^l u^m \Big) \end{aligned}$$

Since $\tilde{y}$ is an exact form by construction, we see that this additive variation has no effect on the homology class of y, i.e. $y+\tilde{y}$ is d-closed with $[y]=[y+\tilde{y}]$.

Consequently, by this reasoning we have found a representative $y+\tilde{y}$ of the homology class of y with

$$\bar{y} := y+\tilde{y} \in \bigwedge\langle \{b_i\}_{i\in J}, z\rangle \oplus z' \bigwedge\langle \{b_i\}_{i\in J}, z, u'\rangle$$

Since ϕ commutes with differentials (cf. (3.3)), we have

$$\mathrm{d}_E\Big(x + \phi\Big(\sum_{i>0,j} \tilde{y}_{i,j} \Big)\Big) = \mathrm{d}_E(x) + \phi(\tilde{y}) = \phi(\bar{y})$$

Moreover, since for every $i,j \in \mathbb{N}_0$ the term $\tilde{y}_{i,j}$ has z' as a factor, we see that $\phi\big(\sum_{i>0,j} \tilde{y}_{i,j}\big) = 0$. This is due to $\phi(z')=0$ and ϕ being multiplicative by definition. So we obtain

$$\phi(y) = \mathrm{d}_E(x) = \phi(\bar{y})$$

We shall now prove that a d-closed element

$$a \in \bigwedge\langle \{b_i\}_{i\in J}, z\rangle \oplus z' \bigwedge\langle \{b_i\}_{i\in J}, z, u'\rangle$$

is completely determined by its first direct summand. Equivalently, we prove that there is no non-trivial closed element of this kind with vanishing first summand. So we suppose additionally that $a|_{\bigwedge\langle\{b_i\}_{i\in J},z\rangle} = 0$ and that $a =: z'\bar{a}$ for some $\bar{a} \in \bigwedge\langle \{b_i\}_{i\in J}, z, u'\rangle$. Thus we compute

$$\begin{aligned} 0 = \mathrm{d}(a) = \mathrm{d}(z'\bar{a}) &= (z^2-u)\cdot \bar{a} - z' \cdot \mathrm{d}\bar{a} \\ &\in (z^2-u) \cdot \bigwedge\langle \{b_i\}_{i\in J}, z, u'\rangle \oplus z' \cdot \bigwedge\langle \{b_i\}_{i\in J}, z, u'\rangle \end{aligned}$$

Thus again we see that $\mathrm{d}(a)$ equals zero if and only if both (direct) summands vanish. As $(z^2 - u)$ is not a zero divisor in $\bigwedge\langle\{b_i\}_{i\in J}, z, u'\rangle$, we derive that $\bar{a} = 0$. Consequently, we obtain $a = 0$.

So let us eventually prove that the cohomology class $[y] = [y + \tilde{y}]$ equals zero by constructing an element that maps to $y + \tilde{y}$ under d. As in the surjectivity part we may deform $\psi(x)$ additively by the special element

$$\tilde{x} \in z' \bigwedge\langle\{b_i\}_{i\in J}, z, u'\rangle \subseteq \ker\phi$$

from (3.5). Since $\phi \circ \psi = \mathrm{id}$ by (3.4), once more we obtain

$$\phi(\psi(x) + \tilde{x}) = x$$

and computation (3.7) shows that

$$\mathrm{d}(\psi(x) + \tilde{x}) \in \bigwedge\langle\{b_i\}_{i\in J}, z\rangle \oplus z' \bigwedge\langle\{b_i\}_{i\in J}, z, u'\rangle$$

So we obtain a commutative diagram

$$\begin{array}{ccccc}
\big(\bigwedge\langle\{b_i\}_{i\in J}, z, z', u'\rangle\big)|_{\psi(x)+\tilde{x}} & & \xrightarrow{\phi} & & \bigwedge\langle\{b_i\}_{i\in J}, z\rangle \\
 & \psi(x) + \tilde{x} & \longmapsto & x & \\
\Big\downarrow \mathrm{d} & \Big\downarrow ??? & & \Big\downarrow & \Big\downarrow \mathrm{d}_E \\
 & \bar{y} & \longmapsto & \phi(\bar{y}) & \\
\bigwedge\langle\{b_i\}_{i\in J}, z\rangle \oplus z' \cdot \bigwedge\langle\{b_i\}_{i\in J}, \ldots, z, u'\rangle & & \xrightarrow[\phi]{(\mathrm{id},0)} & & \bigwedge\langle\{b_i\}_{i\in J}, z\rangle
\end{array}$$

where the commutativity in the upper left corner of the outer square has to be interpreted as

$$\phi(\psi(x) + \tilde{x}) \in \bigwedge\langle\{b_i\}_{i\in J}, z\rangle$$
$$\mathrm{d}(\psi(x) + \tilde{x}) \in \bigwedge\langle\{b_i\}_{i\in J}, z\rangle \oplus z' \cdot \bigwedge\langle\{b_i\}_{i\in J}, z, u'\rangle$$

It remains to prove the correctness of the arrow tagged by question marks: We know that $\bar{y}|_{\bigwedge\langle\{b_i\}_{i\in J},z\rangle} = \psi(\phi(\bar{y}))$, since $\phi|_{\bigwedge\langle\{b_i\}_{i\in J},z\rangle}$ is the "identity" by definition. This leads to

$$\begin{aligned}
\mathrm{d}(\psi(x) + \tilde{x})|_{\bigwedge\langle\{b_i\}_{i\in J},z\rangle} &= \psi(\phi(\mathrm{d}(\psi(x) + \tilde{x}))) \\
&= \psi(\mathrm{d}_E(\phi(\psi(x) + \tilde{x}))) \\
&= \psi(\mathrm{d}_E(x)) \\
&= \psi(\phi(\bar{y})) \\
&= \bar{y}|_{\bigwedge\langle\{b_i\}_{i\in J},z\rangle}
\end{aligned}$$

by the same argument when additionally using the commutativity of the square.

The class $\bar{y}$ is closed by construction, the class $\mathrm{d}(\psi(x)+\tilde{x})$ is closed as it is exact. Since both classes coincide on the first direct summand, the uniqueness property we proved previously yields

$$\bar{y} = \mathrm{d}(\psi(x)+\tilde{x})$$

So $\bar{y}$ is exact. Hence we have that $[y]=[\bar{y}]=0$ and that ϕ_* is injective.

Step 3. Let us now work with the minimal model $\left(\bigwedge V_E, \mathrm{d}\right)$ in order to prove formality for B. We shall make essential use of the characterisation given in theorem 1.41. So we shall decompose V_E as in the theorem, which will yield a similar decomposition for V_B. This will enable us to derive the formality of the minimal model of B. By theorem 1.39 this is equivalent to the formality of B itself.

Since E is formal, we may decompose $V_E = C_E \oplus N_E$ with $C_E = \ker \mathrm{d}_E|_{V_E}$ and N_E^i being a complement of C_E^i in V_E^i with the property that every closed element in the ideal $I(N_E)$ generated by N_E in $\bigwedge V_E$ is exact (cf. 1.41).

Without restriction we may assume that there is a subset $J' \subseteq J$ such that

$$C_E = \langle z, \{b_i\}_{i\in J'}\rangle \qquad (3.9)$$

where $\{b_i\}_{i\in J}, u'$ is the homogeneous basis we chose for V_B.

We now prove that the morphism

$$\phi : \left(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle, \mathrm{d}\right) \to \left(\bigwedge V_E, \mathrm{d}_E\right)$$

restricts to an isomorphism of commutative differential graded algebras

$$\phi|_{\bigwedge V_B} : \left(\bigwedge\langle\{b_i\}_{i\in J}, u'\rangle, \mathrm{d}_B\right) \to \left(\bigwedge\langle z^2, \{b_i\}_{i\in J}\rangle, \mathrm{d}_E\right)$$

Note that clearly $\mathrm{d}|_{\bigwedge V_B} = \mathrm{d}_B$ which makes the left hand side a well-defined differential subalgebra of the domain of ϕ. By abuse of notation—the element z^2 has word-length 2—we write $\bigwedge\langle z^2, \{b_i\}_{i\in J}\rangle$ in order to denote the subalgebra generated by $z^2, \{b_i\}_{i\in J}$ in $\bigwedge\langle z, \{b_i\}_{i\in J}\rangle$.

Foremost we shall see that the morphism is a well-defined and bijective morphism of commutative graded algebras: The right hand side obviously possesses an algebra structure and a grading as a subalgebra of $\bigwedge V_E$. Moreover, it holds that $\phi|_{\bigwedge V_B}(b_i) = b_i$ for $i \in J$ and $\phi|_{\bigwedge V_B}(u') = z^2 - \phi(v) \in \bigwedge\langle z^2, \{b_i\}_{i\in J}\rangle$, as $v \in \bigwedge\langle\{b_i\}_{i\in J}\rangle$. In particular, we derive that $\phi|_{\bigwedge V_B}(u) = z^2$, which shows that $\phi|_{\bigwedge V_B}$ is surjective.

It is also injective: Again we may write an arbitrary element $a \in \bigwedge\langle \{b_i\}_{i\in J}, u'\rangle$ as a uniquely determined polynomial in u, i.e. $a = \sum_i b_{I_i} u^i$ (with the notation introduced previously). Suppose $\phi|_{\bigwedge V_B}(a) = 0$ and compute

$$0 = \phi|_{\bigwedge V_B}(a) = \phi|_{\bigwedge V_B}\Big(\sum_i b_{I_i} u^i\Big) = \sum_i b_{I_i}(z^2)^i$$

However, the element $\sum_i b_{I_i} z^{2i} \in \bigwedge V_E$ vanishes if and only if all the b_{I_i} equal zero. Thus we obtain that $a = 0$.

The morphism ϕ is compatible with differentials (cf. (3.3)). So as soon as we have identified $\big(\bigwedge\langle z^2, \{b_i\}_{i\in J}\rangle, \mathrm{d}_E\big)$ as a *differential* subalgebra, we have also proved that $\phi|_{\bigwedge V_B}$ is a morphism of differential algebras. So it remains to see that $\bigwedge\langle z^2, \{b_i\}_{i\in J}\rangle$ is respected by d_E. This, however, is due to the fact that $\phi|_{\bigwedge V_B}$ is surjective, commutes with differentials (cf. (3.3)) and that $\big(\bigwedge V_B, \mathrm{d}_B\big)$ is a differential algebra. That is, for an element $y \in \bigwedge\langle z^2, \{b_i\}_{i\in J}\rangle$ there is an element $x \in \big(\bigwedge V_B, \mathrm{d}_B\big)$ with $\phi|_{\bigwedge V_B}(x) = y$ and

$$\mathrm{d}_E(y) = \phi|_{\bigwedge V_B}(\mathrm{d}_B(x)) \in \phi|_{\bigwedge V_B}\big(\bigwedge V_B\big) = \bigwedge\langle z^2, \{b_i\}_{i\in J}\rangle$$

This proves that $\phi|_{\bigwedge V_B}$ is an isomorphism of commutative differential graded algebras.

We shall make use of this by constructing a decomposition for V_B. We start by setting

$$C_B := \ker \mathrm{d}_B|_{V_B}$$

In the assertion we assume that the rationalised Hurewicz homomorphism $\pi_*(B) \otimes \mathbb{Q} \to H_*(B)$ is injective in degree n. This guarantees (cf. remark 1.27) that there are no relations in this degree, i.e. $V_B^n = C_B^n$, which itself implies that $N_E^n = 0$ by construction of $\big(\bigwedge V_E, \mathrm{d}_E\big)$. So every complement of C_E in V_E necessarily lies in $\langle \{b_i\}_{i\in J}\rangle$. Hence we may assume without restriction that

$$N_E = \langle \{b_i\}_{i\in J\setminus J'}\rangle$$

(For this it might be necessary to change the basis $\langle \{b_i\}_{i\in J}\rangle$.) Set

$$N_B := \psi|_{\bigwedge\langle \{b_i\}_{i\in J}\rangle} N_E = \langle \{b_i\}_{i\in J\setminus J'}\rangle \subseteq V_B$$

Let us now prove that $V_B = C_B \oplus N_B$ and that every closed element in $I(N_B) \subseteq \bigwedge V_B$ is exact: Due to (3.9), since $\phi|_{\bigwedge V_B}$ commutes with differentials (cf. (3.3)) and by our results on $\phi|_{\bigwedge V_B}$, we know that $\langle \{b_i\}_{i\in J'}\rangle \subseteq \ker \mathrm{d}_B$. Indeed,

$$\mathrm{d}_B a = \big(\phi|_{\bigwedge V_B}\big)^{-1}(\mathrm{d}_E(\phi|_{\bigwedge V_B}(a))) = 0$$

for $a \in \bigwedge\langle \{b_i\}_{i\in J'}\rangle$.

Moreover, the second (non-technical) assumption asserts that the rationalised Hurewicz homomorphism $\pi_*(B)\otimes\mathbb{Q}\to H_*(B)$ is injective in degree $2n$. This again guarantees—cf. 1.27—that $V_B^{2n}=C_B^{2n}$, which itself implies that $u'\in C_B^{2n}$, as $u'\in V_B^{2n}$.

Thus we obtain that $\langle\{b_i\}_{i\in J'},u'\rangle\subseteq C_B$. If there was an element $0\neq a\in\langle\{b_i\}_{i\in J\setminus J'}\rangle$ with $\mathrm{d}_B a=0$, then $\mathrm{d}_E(\phi(a))=0$, as ϕ commutes with differentials (cf. (3.3)). However, $\phi(a)\in N_E$ by construction. This yields a contradiction. Hence we really do obtain a direct sum decomposition

$$V_B = C_B\oplus N_B$$

with

$$C_B=\langle\{b_i\}_{i\in J'},u'\rangle \qquad\text{and}\qquad N_B=\langle\{b_i\}_{i\in J\setminus J'}\rangle$$

Now assume there is a closed element $y\in I(N_B)$. We shall construct an element $\bar{x}\in\big(\bigwedge V_B,\mathrm{d}_B\big)$ with $\mathrm{d}_B\bar{x}=y$. This will finally prove formality in case 1.

The element $\phi|_{\bigwedge V_B}(y)\in\big(\bigwedge\langle z^2,\{b_i\}_{i\in J}\rangle,\mathrm{d}_E\big)$ equally satisfies $\mathrm{d}_E\big(\phi|_{\bigwedge V_B}(y)\big)=0$. Moreover, it lies in $I(N_E)$: This follows from the fact that $\phi|_{\bigwedge V_B}(b_i)=b_i$ for $i\in J$ and that ϕ is multiplicative. So every monomial with factor b_i will be mapped to a product with b_i as a factor. Every monomial of y contains a factor $b_I\in N_B$. Hence every factor of $\phi|_{\bigwedge V_B}(y)$ contains the factor $b_I\in N_E$ and $\phi|_{\bigwedge V_B}(y)\in I(N_E)$.

Thus by formality of E there is an element $x\in\big(\bigwedge V_E,\mathrm{d}_E\big)$ with $\mathrm{d}_E x=\phi|_{\bigwedge V_B}(y)$. As $x\in\big(\bigwedge V_E,\mathrm{d}_E\big)$, we may write

$$x=x_0+x_1z+x_2z^2+x_3z^3+\ldots$$

as a finite sum with $x_i\in\bigwedge\langle\{b_i\}_{i\in J}\rangle$. Since z is d_E-closed, we compute

$$\langle z^2,\{b_i\}_{i\in J}\rangle\supseteq\phi|_{\bigwedge V_B}(y)=\mathrm{d}_E(x)=\mathrm{d}_E(x_0)+\mathrm{d}_E(x_1)z+\mathrm{d}_E(x_2)z^2+\ldots$$

As $\big(\bigwedge\langle z^2,\{b_i\}_{i\in J}\rangle,\mathrm{d}_E\big)$ is a differential subalgebra of $\big(\bigwedge V_E,\mathrm{d}_E\big)$, we see that a term $\mathrm{d}_E(x_i)z^i$ lies in $z^i\cdot\bigwedge\langle z^2,\{b_i\}_{i\in J}\rangle$ again. In particular, the parity of the power of z is preserved by differentiation. Thus—by comparison of coefficients in z—we obtain

$$\sum_{i\geq 0}\mathrm{d}_E(x_{2i+1})=0$$

In particular, we see that already

$$\mathrm{d}_E(x_0+x_2z^2+x_4z^4+\ldots)=\phi|_{\bigwedge V_B}(y)$$

Hence, without restriction, we may assume x to consist of monomials with even powers of z only, i.e. $x_{2i+1}=0$ for $i\geq 0$. Thus it holds that $x\in\langle z^2,\{b_i\}_{i\in J}\rangle$. As $\phi|_{\bigwedge V_B}$ is bijective, there is the well-defined element $\bar{x} = \big(\phi|_{\bigwedge V_B}\big)^{-1}(x) \in \bigwedge V_B$. As ϕ commutes with differentials (cf. (3.3)), we eventually obtain

$$\mathrm{d}_B(\bar{x}) = \big(\phi|_{\bigwedge V_B}\big)^{-1}(\mathrm{d}_E(x)) = \big(\phi|_{\bigwedge V_B}\big)^{-1}\big(\phi|_{\bigwedge V_B}(y)\big) = y$$

Case 2. We shall now prove the assertion under the assumption that the fibration is non-primitive. Again we shall rely on the quasi-isomorphism

$$\big(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle, \mathrm{d}\big) \xrightarrow{\simeq} (\mathrm{A}_{\mathrm{PL}}(E), \mathrm{d})$$

where the left hand side contains the minimal Sullivan algebra $\big(\bigwedge V_B, \mathrm{d}_B\big)$ as a differential subalgebra. Again this last algebra is supposed to be a minimal model for the space B.

Step 1. We show that the algebra generated by $V_E := V_B \oplus \langle z, z'\rangle$ is a minimal Sullivan algebra: We have

$$\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle = \bigwedge V_E$$

Set $\mathrm{d}_E := \mathrm{d}$. So $\big(\bigwedge V_E, \mathrm{d}_E\big)$ is a commutative differential graded algebra. Moreover, it holds that $V_E^1 = 0$, since B is simply-connected by assumption.

By the definition of (non-)primitivity $u \in \bigwedge V_B$ is decomposable, i.e. $u \in \bigwedge^{>0} V_B \cdot \bigwedge^{>0} V_B$. Thus we have that $\mathrm{d}_E z' = z^2 - u \in \bigwedge^{>0} V_E \cdot \bigwedge^{>0} V_E$. Clearly, $\mathrm{d}_E z = \mathrm{d}_E u = 0$ and $\mathrm{d}_E b_i \in \bigwedge^{>0} V_B \cdot \bigwedge^{>0} V_B$ for $i \in J$, as $\big(\bigwedge V_B, \mathrm{d}_B\big)$ is a minimal differential subalgebra of $\big(\bigwedge V_E, \mathrm{d}_E\big)$. As d_E is a derivation, we obtain $\operatorname{im} \mathrm{d}_E \in \bigwedge^{>0} V_E \cdot \bigwedge^{>0} V_E$.

These properties taken together with proposition 1.25 yield that $\big(\bigwedge V_E, \mathrm{d}_E\big)$ is a minimal Sullivan model of E.

Step 2. By assumption E is formal. Theorem 1.41 then yields the existence of a decomposition $V_E = C_E \oplus N_E$ with $C_E = \ker \mathrm{d}_E|_{V_E}$. Clearly, we have $z \in C_E$ and without restriction we may assume that

$$C_E = \langle z, \{b_i\}_{i\in J'}\rangle$$

for a subset $J' \subseteq J$, where $\{b_i\}_{i\in J}$ now is a homogeneous basis of V_B. (Recall that non-primitivity implies that the element u lies in $\bigwedge^{\geq 2}\langle\{b_i\}_{i\in J}\rangle$.)

Since $\mathrm{d}_E z' \neq 0$ and since $z' \in V_E$, there is an element $c \in C$ with the property that $z' + c \in N_E$. Set $\tilde{z}' := z' + c$. Thus we have that $\tilde{z}' \in N_E$. The change of basis caused by $z' \mapsto \tilde{z}'$ induces a linear automorphism of the vector space V_E which is the identity on $V_B \oplus \langle z \rangle$. This automorphism itself induces an automorphism σ of the commutative differential graded algebra $\bigwedge V_E$. By construction this automorphism has the property that $\sigma|_{\bigwedge(V_B \oplus \langle z \rangle)} = \mathrm{id}$. Moreover, we know that $\mathrm{d}_E(\bigwedge(V_B \oplus \langle z \rangle)) \subseteq \bigwedge(V_B \oplus \langle z \rangle)$. Therefore σ is also compatible with the differential, since

$$\begin{aligned}
&\mathrm{d}_E(\sigma(x)) = \sigma(\mathrm{d}_E(x)) \qquad \text{for } x \in \bigwedge(V_B \oplus \langle z \rangle) \\
&\mathrm{d}_E(\sigma(z')) = \mathrm{d}_E(\tilde{z}') = \mathrm{d}_E(z' + c) = \mathrm{d}_E(z') = z^2 - u = \sigma(z^2 - u) = \sigma(\mathrm{d}_E(\tilde{z}'))
\end{aligned}$$

Thus up to this isomorphism of differential graded algebras we may assume that there is a decomposition of $V_E = C_E \oplus N_E$ with $C_E = \ker \mathrm{d}_E|_{V_E}$ and with $z' \in N_E$. Moreover, up to a change of basis we hence may assume that

$$N_E = \langle \{b_i\}_{i \in J \setminus J'}, z' \rangle$$

We use this to split V_B in the following way: Set $C_B := \ker \mathrm{d}_B|_{V_B}$. We recall that $\big(\bigwedge V_B, \mathrm{d}_B\big)$ is a differential subalgebra of $\big(\bigwedge V_E, \mathrm{d}_E\big)$. Thus we realise that

$$C_B = \langle \{b_i\}_{i \in J'} \rangle$$

and we set

$$N_B := \langle \{b_i\}_{i \in J \setminus J'} \rangle$$

We directly obtain $V_B = C_B \oplus N_B$.

As for formality of B, by theorem 1.41 it remains to prove that every closed element in the ideal $I(N_B)$ generated by N_B in $\big(\bigwedge V_B, \mathrm{d}_B\big)$ is exact in $\big(\bigwedge V_B, \mathrm{d}_B\big)$. So let $y \in I(N_B)$ with $\mathrm{d}_B y = 0$. Since $N_B \subseteq N_E$ we obtain that $I(N_B) \subseteq I(N_E)$. Thus by the formality of E we find an element $x \in \bigwedge V_E$ with $\mathrm{d}_E x = y$. Recall the notation we introduced in the surjectivity part of case 1, step 2 and write x as a finite sum

$$x = b_{I_0} + b_{I_1} z + b_{I_2} z^2 + \cdots + b_{I'_0} z' + b_{I'_1} z z' + b_{I'_2} z^2 z' + \ldots$$

with $b_{I_i}, b_{I'_i} \in \bigwedge V_B$. We compute

$$\begin{aligned}
y = \mathrm{d}_E x = &\mathrm{d}_E b_{I_0} + (\mathrm{d}_E b_{I_1}) z + (\mathrm{d}_E b_{I_2}) z^2 + \ldots \\
&+ \big((\mathrm{d}_E b_{I'_0}) z' + b_{\tilde{I}'_0}(z^2 - u)\big) + \big((\mathrm{d}_E b_{I'_1}) z z' + b_{\tilde{I}'_1} z (z^2 - u)\big) \\
&+ \big((\mathrm{d}_E b_{I'_2}) z^2 z' + b_{\tilde{I}'_2} z^2 (z^2 - u)\big) + \ldots \\
= &\sum_{i \geq 0} \big((\mathrm{d}_E b_{I_i}) - b_{\tilde{I}'_i} u + b_{\tilde{I}'_{i-2}}\big) z^i + \sum_{i \geq 0} (\mathrm{d}_E b_{\tilde{I}'_i}) z' z^i
\end{aligned} \tag{3.10}$$

Since $\mathrm{d}_E(b_i) \in \bigwedge V_B$ for $1 \leq i \leq m$ and since $u \in \bigwedge V_B$, we shall proceed with our reasoning by comparing coefficients in z^i respectively in $z' z^i$. That is, the element y

lies in $\bigwedge V_B$. Thus it cannot contain a summand with z^i (for $i > 0$) or with $z'z^i$ (for $i \geq 0$) as a factor. So we see that

$$y = \mathrm{d}_E x = \mathrm{d}_E b_{I_0} - b_{\tilde{I}'_0} u$$

In particular, $(\mathrm{d}_E b_{I_2}) - b_{\tilde{I}'_2} u + b_{\tilde{I}'_0} = 0$. Thus the element $b_{I_0} + b_{I_2} u \in \bigwedge V_B$ satisfies

$$\mathrm{d}_E(b_{I_0} + b_{I_2} u) = \mathrm{d}_E b_{I_0} + \big(\mathrm{d}_E b_{I_2}\big) u = \mathrm{d}_E b_{I_0} + b_{\tilde{I}'_2} u^2 - b_{\tilde{I}'_0} u = y + b_{\tilde{I}'_2} u^2$$

Below we prove that $b_{\tilde{I}'_2}$ is exact, i.e. there is some $a \in \bigwedge V_B$ with $\mathrm{d}_E a = b_{\tilde{I}'_2}$. This yields the exactness of $b_{\tilde{I}'_2} u^2$ by $\mathrm{d}_E(au^2) = b_{\tilde{I}'_2} u^2$. So we finally shall obtain

$$\mathrm{d}_E(b_{I_0} + b_{I_2} u - au^2) = \big(y + b_{\tilde{I}'_2} u^2\big) - b_{\tilde{I}'_2} u^2 = y$$

Since this is a computation in $\bigwedge V_B$ and since d_E restricts to d_B on $\bigwedge V_B$ we obtain that y is exact in $\bigwedge V_B$. This will provide formality.

So let us finally prove that $b_{\tilde{I}'_2}$ is exact. Since we are dealing with finite sums only, there is an even $i_0 \geq 2$ with $b_{\tilde{I}'_{i_0}} = 0$. From (3.10) we again see that

$$0 = (\mathrm{d}_E b_{I_{i_0}}) - b_{\tilde{I}'_{i_0}} u + b_{\tilde{I}'_{i_0-2}} = (\mathrm{d}_E b_{I_{i_0}}) + b_{\tilde{I}'_{i_0-2}}$$

This implies $\mathrm{d}_B(-b_{I_{i_0}}) = b_{\tilde{I}'_{i_0-2}}$ and $b_{\tilde{I}'_{i_0-2}}$ is exact in $\big(\bigwedge V_B, \mathrm{d}_B\big)$. We shall continue this iteratively. That is, as a next step we have

$$0 = (\mathrm{d}_E b_{I_{i_0-2}}) - b_{\tilde{I}'_{i_0-2}} u + b_{\tilde{I}'_{i_0-4}} = (\mathrm{d}_E b_{I_{i_0-2}}) - \mathrm{d}_B\big(-b_{I_{i_0}}\big) + b_{\tilde{I}'_{i_0-4}}$$

and $b_{\tilde{I}'_{i_0-4}}$ is d_B-exact. As i_0 is even, this iterative procedure will finally end up with proving that $b_{\tilde{I}'_2}$ is exact.

This finishes the proof of the formality of B.

□

We already did so in the proof and we shall continue to do so: We refer to the two conditions in the assertion of the theorem that involve the Hurewicz homomorphism as “non-technical” conditions. (The other prerequisites are mainly due to the general setting of the theory.)

It is of importance to remark that $\mathbb{S}^{2n}$ ($n \geq 1$) is an F_0-space. Thus the question whether formality of the base space implies formality for the total space is related to the Halperin conjecture—cf. [56] for an introduction to this topic and theorem [56].3.4 in particular. The Halperin conjecture has been established on these spheres.

Reformulations (cf. [58]) of the conjecture permit extremely short proofs for this. For example it is trivial to see that the cohomology algebra $H^*(\mathbb{S}^{2n})$ does not admit non-trivial derivations of negative degree. A direct proof needs to show that every spherical fibration with fibre of even rational homological dimension is totally non-cohomologous to zero, i.e. that the Serre spectral sequence degenerates at the E_2-term; respectively that the inclusion of the fibre into the total space induces a surjective map in rational cohomology. For this see remark 1.34.

A consequence of theorem [56].3.4 now is that if the fibre of a fibration is an F_0-space, then the formality of the base space implies the formality of the total space. In particular, this proves

Theorem 3.3. *Let*

$$F \hookrightarrow E \xrightarrow{p} B$$

be a spherical fibration of simply-connected (and path-connected) topological spaces. Let the fibre satisfy $F \simeq_{\mathbb{Q}} \mathbb{S}^n$ for an even $n \geq 2$.

Then it holds: If B is a formal space, then so is E.

□

Nonetheless, we shall give a direct proof of this theorem, as this proof will provide some observations and tools which we shall need in the following.

Proof of Theorem 3.3. Choose a minimal model

$$m_B : \left(\bigwedge V_B, \mathrm{d}_B\right) \xrightarrow{\simeq} (\mathrm{A_{PL}}(B), \mathrm{d})$$

As B is formal we have the quasi-isomorphism

$$\mu_B : \left(\bigwedge V_B, \mathrm{d}_B\right) \xrightarrow{\simeq} (H^*(B), 0)$$

from lemma 1.44. For this we fix a complement N of $\ker \mathrm{d}|_{V_B}$ in V_B. We obtain that $\mu_B(N)=0$. Form the model of the fibration p

$$\tilde{m}_E : \left(\bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, \mathrm{d}\right) \xrightarrow{\simeq} (\mathrm{A_{PL}}(E), \mathrm{d})$$

as in remark 1.34. We use the terminology of theorem 3.2 for the model of the fibration, i.e. $\mathrm{d}(z') = u \in \bigwedge V_B$ and $\{b_i\}_{i\in J}$ denotes a homogeneous basis of V_B. (Here we do not make use of a distinguished basis element u'.) We now define a morphism of commutative graded algebras

$$\tilde{\mu}_E : \left(\bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, \mathrm{d}\right) \to (H^*(E), 0) \cong (H^*(B) \oplus H^*(B) \cdot [z], 0)$$

$$\begin{aligned} b_i &\mapsto \mu_B(b_i) \\ z &\mapsto [z] \\ z' &\mapsto 0 \end{aligned} \tag{3.11}$$

by extending this assignement linearly and multiplicatively. We shall prove that this morphism of commutative graded algebras is actually a quasi-isomorphism. The weak equivalence

$$(\mathrm{A_{PL}}(E), \mathrm{d}) \xleftarrow{\tilde m_E} \big(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle, \mathrm{d}\big) \xrightarrow{\tilde\mu_E} (H^*(E), 0)$$

will then yield the formality of E.

Our main tool for proving this will be that every closed element of $\big(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle, \mathrm{d}\big)$ maps to its homology class under $\tilde\mu_E$. Let us establish this result first. So suppose

$$y \in \big(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle, \mathrm{d}\big)$$

is such a d-closed element. It then may be written as

$$y = y_0 + z'y_1$$

with $y_0, y_1 \in \bigwedge\big(V_B \oplus \langle z\rangle\big)$.

First we shall prove that y_1 is exact. We have

$$0 = \mathrm{d}y = \mathrm{d}y_0 + (z^2 - u)y_1 - z'\mathrm{d}y_1$$

By construction of $\big(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle, \mathrm{d}\big)$ we know that

$$\mathrm{d}\big(\bigwedge V_B \oplus \langle z\rangle\big) \subseteq \big(\bigwedge V_B \oplus \langle z\rangle\big)$$

This implies that $\mathrm{d}y_1 = 0$. Thus we have

$$\mathrm{d}y_0 = -(z^2 - u)y_1 \tag{3.12}$$

Hence we write

$$y_0 = y_{0,0} + y_{0,1}z + y_{0,2}z^2 + y_{0,3}z^3 + \cdots + y_{0,k}z^k$$

and

$$y_1 = y_{1,0} + y_{1,1}z + y_{1,2}z^2 + y_{1,3}z^3 + \cdots + y_{1,k-2}z^{k-2}$$

as z-graded elements with $y_{0,i}, y_{1,i} \in \bigwedge V_B$. As z is d-closed, the grading is preserved under differentiation. Equation (3.12) implies in this graded setting that

$$\mathrm{d}_B y_{0,i} = uy_{1,i} - y_{1,i-2}$$

Starting in top degree k we see that $\mathrm{d}_B y_{0,k} = -y_{1,k-2}$. Set $x_{k-2} := -y_{0,k}$. This directly leads to

$$\mathrm{d}_B y_{0,k-2} = u y_{1,k-2} - y_{1,k-4}$$

where $u y_{1,k-2}$ is d_B-exact by $\mathrm{d}_B(-u y_{0,k}) = u y_{1,k-2}$. Thus $y_{1,k-4}$ is exact. Set

$$x_{k-4} := -y_{0,k-2} - u y_{0,k}$$

Hence, inductively continuing in this fashion, we obtain that each $y_{1,i}$ with $i \equiv k \mod 2$ is exact. The same line of argument applied from degree $k-1$ downwards yields finally that also all the $y_{1,i}$ with $i \equiv k-1 \mod 2$ are d_B-exact. In total, for each $0 \le i \le k-2$ there exists an $x_i \in \bigwedge V_B$ with $\mathrm{d}x_i = y_{1,i}$. So the element y_1 itself is exact: Set

$$x := x_0 + x_1 z + x_2 z^2 + \cdots + x_{k-2} z^{k-2}$$

and obtain

$$\begin{aligned} \mathrm{d}(x) &= \mathrm{d}\big(x_0 + x_1 z + x_2 z^2 + \cdots + x_{k-2} z^{k-2}\big) \\ &= \mathrm{d}_B x_0 + \mathrm{d}_B(x_1) z + \mathrm{d}_B(x_2) z^2 + \cdots + \mathrm{d}_B(x_{k-2}) z^{k-2} \\ &= y_{1,0} + y_{1,1} z + y_{1,2} z^2 + y_{1,3} z^3 + \cdots + y_{1,k-2} z^{k-2} \\ &= y_1 \end{aligned} \tag{3.13}$$

Moreover, the x_i are constructed iteratively within the algebra $\bigwedge V_B$. So we obtain $x_i \in \bigwedge V_B$ for $0 \le i \le k-2$ and

$$x \in \bigwedge (V_B \oplus \langle z \rangle) \tag{3.14}$$

This will enable us to see that y is mapped to its homology class. Due to the formality of B we may decompose $V_B = C \oplus N$ homogeneously with the property that $C = \ker \mathrm{d}_B|_V$ and such that every closed element in $I(N)$, the ideal generated by N in $\bigwedge V_B$, is exact—cf. theorem 1.41. This complement N is supposed to be identical to the one we chose for defining μ_B at the beginning of the proof. This decomposition induces a decomposition

$$\bigwedge V_B = \bigwedge C \oplus I(N)$$

For each $1 \le i \le k$ we split $y_{0,i} =: y'_{0,i} + y''_{0,i}$ with $y'_{0,i} \in \bigwedge C$ (being d-closed) and $y''_{0,i} \in I(N)$. The morphism μ_B was chosen to map a closed element to its homology class and to vanish on $I(N)$—cf. lemma 1.44. Thus we see that $\tilde{\mu}_E(y'_{0,i}) = [y'_{0,i}]$ and that $\tilde{\mu}_E(y''_{0,i}) = 0$. We then need to show that

$$\tilde{\mu}_E(y) = \sum_{i\geq 0}[y'_{0,i}][z]^i \overset{!}{=} [y]$$

This will be a consequence of the following reasoning: We suppose $y_{0,i} = y''_{0,i}$ for all $0 \leq i \leq k$ and show that

$$\tilde{\mu}_E(y) = [0] \overset{!}{=} [y]$$

By construction we have

$$y_0 \in N \cdot \bigwedge(V \oplus \langle z \rangle) \tag{3.15}$$

We shall now show that the element

$$y + \mathrm{d}(z'x) = y_0 + z'y_1 + (z^2 - u)x - z'y_1 = y_0 + (z^2 - u)x$$

is exact, which will imply the exactness of y itself.

The whole commutative algebra splits as a direct sum

$$\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle = \Big(\bigwedge(C \oplus \langle z \rangle)\Big) \oplus \Big((N \oplus \langle z' \rangle) \cdot \big(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle\big)\Big)$$

Thus the element x may be written as $x = c + n$ with $c \in \bigwedge(C \oplus \langle z \rangle)$ and $n \in (N \oplus \langle z' \rangle) \cdot \big(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle\big)$. We conclude that $\mathrm{d}(c) = 0$ and that $\mathrm{d}(x - c) = \mathrm{d}n = \mathrm{d}x$. Hence, without restriction, we may assume $x = n$. Combining this with (3.14) thus yields

$$x \in N \cdot \bigwedge \big(V_B \oplus \langle z \rangle\big)$$

Thus, writing x in its z-graded form, we see that each coefficient x_i (for $0 \leq i \leq i - 2$) lies in $I(N) = N \cdot \bigwedge V_B$. By (3.15) every coefficient $y_{0,i}$ (for $0 \leq i \leq k$) in the z-grading of y_0 equally lies in $I(N)$. Hence so does every z-coefficient

$$y_{0,i} - ux_i + x_{i-2} \in I(N)$$

of

$$y_0 + (z^2 - u)x = \sum_{i\geq 0}(y_{0,i} - ux_i + x_{i-2})z^i$$

Since y was assumed to be d-closed and since $y_0 + (z^2 - u)x = y + \mathrm{d}(z'x)$, the element $y_0 + (z^2 - i)x$ also is d-closed. In particular, every z-coefficient is a d_B-closed element in $I(B)$. By the formality of B and by theorem 1.41 every such coefficient is d_B-exact then. An analogous argument to (3.13) then shows that y is d-exact. In total, this proves that μ maps a closed element to its cohomology class.

As for the properties of $\tilde{\mu}_E$, which now follow easily from this feature, we observe: The morphism is compatible with differentials as $\mu(\mathrm{d}(x)) = 0$. (The closed and exact class $\mathrm{d}(x)$ maps to its cohomology class.) Eventually, it is a quasi-isomorphism as the morphism induced in homology is just the identity. This proves formality for E. $\square$

We remark that making extensive use of theorem 1.41 permits a shorter but similar proof. Yet, for us it was important to establish the quasi-isomorphism (3.11). Indeed, it will be a vital tool in the next theorem.

As we have seen, it becomes pretty clear that the formality of the base space and the formality of the total space are highly interconnected. This is illustrated once more by the next theorem.

Theorem 3.4. *Let*

$$F \hookrightarrow E \xrightarrow{p} B$$

be a spherical fibration of simply-connected (and path-connected) topological spaces. Let the fibre satisfy $F \simeq_{\mathbb{Q}} \mathbb{S}^n$ for an even $n \geq 2$.

If B and (consequently) E are formal spaces, then the fibration p is a formal map.

Proof. We shall use the terminology and results from theorem 3.2. By

$$m_B \colon \left(\bigwedge V_B, \mathrm{d}_B\right) \xrightarrow{\simeq} \mathrm{A_{PL}}(B, \mathrm{d})$$

we denote the minimal model. We let $\{b_i\}_{i \in J}$ denote a basis of V_B (and do not make use of a distinguished element u').

Case 1. Suppose the fibration p to be primitive. In the notation of theorem 3.2 we then have $\mathrm{d}z' = z^2 - u$ with primitive u in the model of the fibration. In the case $u = 0$ we obtained the weak equivalence

$$E \simeq_{\mathbb{Q}} B \times F$$

and we have seen the model of the fibration to be just the product model of fibre and base. The diagram

$$\begin{array}{ccc} \mathrm{A_{PL}}(B) & \xrightarrow{\mathrm{A_{PL}}(p)} & \mathrm{A_{PL}}(E) \cong (\mathrm{A_{PL}}(B), \mathrm{d}) \otimes (\mathrm{A_{PL}}(F), \mathrm{d}) \\ \simeq \uparrow & & \uparrow \simeq \\ \left(\bigwedge V_B, \mathrm{d}_B\right) & \xhookrightarrow{x \mapsto (x,1)} & \left(\bigwedge V_B, \mathrm{d}_B\right) \otimes \left(\bigwedge V_F, \mathrm{d}_F\right) \end{array}$$

commutes (cf. [22], p. 198), where $\left(\bigwedge V_F, \mathrm{d}_F\right)$ is the minimal model of the fibre F. So also the induced map $\mathrm{A}_{\mathrm{PL}}(p)$ is just the canonical inclusion into the first factor. Thus theorem 1.47 applies and yields the formality of p in this case.

So we may assume $u \neq 0$. We then have constructed the minimal model

$$m_E : \left(\bigwedge V_E, \mathrm{d}_E\right) \xrightarrow{\simeq} \mathrm{A}_{\mathrm{PL}}(E, \mathrm{d})$$

in case 1, step 2 of theorem 3.2. By lemma 1.38 we have a quasi-isomorphism

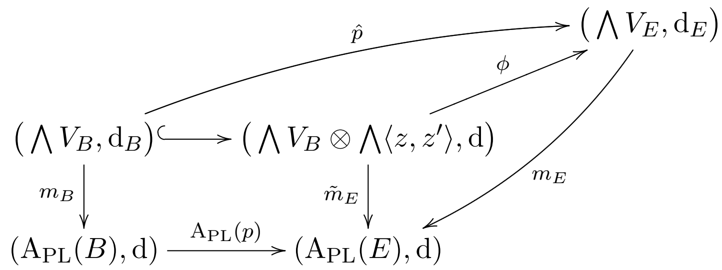
$$\mu_B : \left(\bigwedge V_B, \mathrm{d}_B\right) \xrightarrow{\simeq} (H^*(B), 0)$$

which we may assume to satisfy $(m_B)_* = (\mu_B)_*$. Once more, for a spherical fibration we form the model of the fibration

$$\tilde{m}_E : \left(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle, \mathrm{d}\right) \xrightarrow{\simeq} (\mathrm{A}_{\mathrm{PL}}(E), \mathrm{d})$$

in the altered version of remark 1.34. Consider the following diagram

(3.16)



where ϕ is the quasi-isomorphism from (3.2), which we construct in the proof of theorem 3.2, case 1, step 2. (Note that until case 1, step 3 of the theorem we did not need any of the “non-technical” assumptions which we did not require for this theorem.)

We shall now prove that diagram (3.16) commutes up to homotopy. The minimal model m_E was constructed by means of a diagram of liftings and isomorphisms extending

$$(\mathrm{A}_{\mathrm{PL}}(E), \mathrm{d}) \xleftarrow{\tilde{m}_E} \left(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle, \mathrm{d}\right) \xrightarrow{\phi} \left(\bigwedge V_E, \mathrm{d}_E\right)$$

(cf. the proof of lemma 1.38). Thus this diagram is commutative up to homotopy (cf. [22].12.9, p. 153) and so is the right hand triangle of (3.16).

Since the model of the fibration comes out of a relative Sullivan algebra (cf. [22].14 and [22].15, p. 196) for $\mathrm{A_{PL}}(p)$—composed with the minimal model $(\bigwedge V_B, \mathrm{d}_B) \xrightarrow{m_B} \mathrm{A_{PL}}(B, \mathrm{d})$—the algebra $(\bigwedge V_B, \mathrm{d}_B)$ includes into $(\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle, \mathrm{d})$ by $x \mapsto (x, 1)$ as a differential subalgebra (cf. definition [22], p. 181) and the square in the diagram commutes. (This holds although we altered the model of the fibration as in remark 1.34.)

As a consequence, both the following diagrams commute up to homotopy—the second one as the Sullivan representative $\hat{p}$ is constructed as a lifting (cf. section 1.2).

$$\begin{array}{ccc} (\bigwedge V_B, \mathrm{d}_B) & \xrightarrow{\phi|_{\bigwedge V_B}} & (\bigwedge V_E, \mathrm{d}_E) \\ {\scriptstyle m_B}\downarrow & & \downarrow{\scriptstyle m_E} \\ (\mathrm{A_{PL}}(B), \mathrm{d}) & \xrightarrow{\mathrm{A_{PL}}(p)} & (\mathrm{A_{PL}}(E), \mathrm{d}) \end{array} \qquad \begin{array}{ccc} (\bigwedge V_B, \mathrm{d}_B) & \xrightarrow{\hat{p}} & (\bigwedge V_E, \mathrm{d}_E) \\ {\scriptstyle m_B}\downarrow & & \downarrow{\scriptstyle m_E} \\ (\mathrm{A_{PL}}(B), \mathrm{d}) & \xrightarrow{\mathrm{A_{PL}}(p)} & (\mathrm{A_{PL}}(E), \mathrm{d}) \end{array}$$

Since the Sullivan representative $\hat{p}$ is uniquely defined up to homotopy, we know that $\hat{p}$ and $\phi|_{\bigwedge V_B}$ are homotopic morphisms (cf. [22], p. 149). Thus—in order to prove the assertion in this case—it is sufficient to find a minimal model μ_E with $(\mu_E)_* = (m_E)_*$ for which the diagram

$$\begin{array}{ccc} (\bigwedge V_B, \mathrm{d}_B) & \xrightarrow{\phi|_{\bigwedge V_B}} & (\bigwedge V_E, \mathrm{d}_E) \\ {\scriptstyle \mu_B}\downarrow & & \downarrow{\scriptstyle \mu_E} \\ (H^*(B), 0) & \xrightarrow{p^*} & (H^*(E), 0) \end{array}$$

commutes up to homotopy.

We shall start to do so by foremost recalling the quasi-isomorphism

$$\begin{aligned} \tilde{\mu}_E : (\bigwedge V_B \otimes \bigwedge\langle z, z'\rangle, \mathrm{d}) &\xrightarrow{\simeq} (H^*(E), 0) \\ b_i &\mapsto \mu_B(b_i) \\ z &\mapsto [z] \\ z' &\mapsto 0 \end{aligned}$$

from (3.11). (See the rest of the proof of theorem 3.3 for the fact that this really defines a quasi-isomorphism.) For the definition of the morphism and in the following we shall identify $H^*(E)$ with $p^*H^*(B) \oplus p^*H^*(B)[z]$. Note that this morphism has the property that $(\tilde{\mu}_E)_* = (\tilde{m}_E)_*$. This can be derived as follows: We assumed the analogous property to hold true for μ_B. Moreover, we have

$$(m_E)_* = ((m_B)_*, (m_B)_*) : H^*(B) \oplus H^*(B)[z] \xrightarrow{\cong} H^*(E)$$

by the construction of the model of the fibration as a relative Sullivan model over the basis $(\bigwedge V_B, \mathrm{d}_B)$. This, however, coincides with the definition of $(\tilde{\mu}_E)_*$.

Now form the diagram

$$\begin{array}{ccccc} (\bigwedge V_B, \mathrm{d}_B) & \overset{\hat{p}}{\dashrightarrow} & \cdots & \overset{\phi}{\nearrow} & (\bigwedge V_E, \mathrm{d}_E) \\ (\bigwedge V_B, \mathrm{d}_B) & \hookrightarrow & (\bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, \mathrm{d}) & \overset{\mu_E}{\swarrow} & \\ \mu_B \downarrow & & \downarrow \tilde{\mu}_E & & \\ (H^*(B), 0) & \xrightarrow{p^*} & (H^*(E), 0) & & \end{array} \tag{3.17}$$

where the map μ_E is defined by lemma 1.38, i.e. by means of liftings and isomorphism that make the right hand triangle in (3.17) homotopy commute. As $\tilde{\mu}_E$ and ϕ are quasi-isomorphisms, so becomes μ_E. Again the square in the diagram commutes. So the whole diagram commutes up to homotopy. Since $(\tilde{\mu}_E)_* = (\tilde{m}_E)_*$, we derive that $(\mu_E)_* = (m_E)_*$ by commutativity. This combines with the homotopy commutativity of diagram (3.17) and yields that p is a formal map.

Case 2. We treat the case when p is not primitive. In case 2, step 1 of the proof of theorem 3.2 we have seen that the model of the fibration

$$m_E : \left(\bigwedge V_E, \mathrm{d}_E\right) \xrightarrow{\simeq} (\mathrm{A_{PL}}(E), \mathrm{d})$$

where $V_E = V_B \oplus \langle z, z' \rangle$ and $\mathrm{d}_E|_{V_B} = \mathrm{d}_B$, $\mathrm{d}_E z = 0$, $\mathrm{d}_E z' = z^2 - u$ is minimal already. As this model comes out of a relative Sullivan algebra over $\left(\bigwedge V_B, \mathrm{d}_B\right)$, the induced map $\hat{p}$ is just the inclusion

$$\begin{aligned} \hat{p} : \left(\bigwedge V_B, \mathrm{d}_B\right) &\hookrightarrow \left(\bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, \mathrm{d}\right) = \left(\bigwedge V_E, \mathrm{d}_E\right) \\ x &\mapsto (x, 1) \end{aligned}$$

as a differential graded subalgebra.

As in case 1 we use the quasi-isomorphism

$$\begin{aligned} \tilde{\mu}_E : \left(\bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, \mathrm{d}\right) &\xrightarrow{\simeq} (H^*(E), 0) \\ b_i &\mapsto \mu_B(b_i) \\ z &\mapsto [z] \\ z' &\mapsto 0 \end{aligned}$$

from (3.11) with the property that $(\tilde{\mu}_E)_* = (\tilde{m}_E)_*$—as seen in case 1.

Thus it is obvious that the diagram

$$\begin{array}{ccc} (\bigwedge V_B, \mathrm{d}_B) & \overset{\hat{p}}{\hookrightarrow} & (\bigwedge V_E, \mathrm{d}_E) \\ \mu_B \downarrow & & \downarrow \tilde{\mu}_E \\ (H^*(B), 0) & \xrightarrow{p^*} & (H^*(E), 0) \end{array}$$

commutes. So p is formal in this case, too.

□

We shall now state some results related to the previous theorems. We begin with the following apparent consequence:

Corollary 3.5. *Let*

$$F \hookrightarrow E \xrightarrow{p} B$$

be a spherical fibration of simply-connected (and path-connected) topological spaces with rational homology of finite type. Let the fibre satisfy $F \simeq_{\mathbb{Q}} \mathbb{S}^n$ for an even $n \geq 2$. Suppose further that the space B is rationally $(2n-1)$-connected.

Then we derive:

- *The space E is formal if and only if so is B.*
- *If this is the case, the map p is formal, too.*

Proof. Since B is rationally $(2n-1)$-connected, we have

$$\pi_i(B) \otimes \mathbb{Q} = 0 = H^i(B) \qquad \text{for } i \leq 2n-1$$

So we obtain in particular that the rationalised Hurewicz homomorphism $\pi_*(B) \otimes \mathbb{Q} \to H_*(B)$ is injective in degree n. By the Hurewicz theorem we also obtain that the rationalised Hurewicz homomorphism $\pi_*(B) \otimes \mathbb{Q} \to H_*(B)$ is an isomorphism in degree $2n$—and injective in this degree, in particular. (We may realise this directly by building the minimal model $\left(\bigwedge V_B, \mathrm{d}_B\right)$ of B, which has the property that $V^{\leq 2n-1} = 0$. As the model is minimal, every element in V^{2n} is closed and $H_{2n}(B) \cong H^{2n}(B) = V^{2n} \cong \pi_{2n}(B) \otimes \mathbb{Q}$ by the Hurewicz homomorphism—cf. remark 1.27.)

Thus all the prerequisites of theorem 3.2 are satisfied and the formality of E implies the one of B. By theorem 3.3 we have the reverse implication and theorem 3.4 finishes the proof. □

We remark that this corollary only needs case 1 of theorem 3.2, since by the connectivity-assumption the fibration necessarily is primitve.

Let us formulate an observation from the proofs of the different theorems in this chapter.

Theorem 3.6. *Let*

$$F \hookrightarrow E \xrightarrow{p} B$$

be a fibration of simply-connected (and path-connected) topological spaces with rational homology of finite type. Suppose the rational cohomology algebra $H^*(F)$ is generated (as an algebra) by exactly one (non-trivial) element of even degree n. Suppose further that the space B is rationally n'-connected, where

$$n' := n \cdot (\dim_{\mathbb{Q}} H^*(F)) - 1$$

Then we obtain: The space E is formal if and only if so is B. In this case the map p is formal, too.

Proof. Set $d := \dim_{\mathbb{Q}} H^*(F)$. The cohomology algebra of the fibre F is given by the polynomial algebra $H^*(F) = \mathbb{Q}[z]/z^d$. The minimal model $\left(\bigwedge V_F, \mathrm{d}_F\right)$ of F is given by $V_F = V^n \oplus V^{n \cdot d-1}$ with $V^n = \langle z \rangle$ and $V^{n \cdot d - 1} = \langle z' \rangle$ and by the differentials $\mathrm{d}_F z = 0$ and $\mathrm{d}_F z' = z^d$. See example 1.32 for details.

Let $\left(\bigwedge V_B, \mathrm{d}_B\right)$ be a minimal model of the base space B. We form the model of the fibration:

$$\left(\bigwedge V_B \otimes \bigwedge V_F, \mathrm{d}\right) \xrightarrow{\simeq} (\mathrm{A_{PL}}(E), \mathrm{d})$$

The differential d of the model equals d_B on the subalgebra $\bigwedge V_B$. It remains to describe the differentials $\mathrm{d}(z)$ and $\mathrm{d}(z')$. As B is n'-connected, we obtain $V^{\leq n'} = 0$. Thus, due to theorem 1.21, we derive that $\mathrm{d}z = \mathrm{d}_F z = 0$ and $\mathrm{d}z' = \mathrm{d}_F z' = z^d - u$ for some $u \in V_B^{d \cdot n}$. As $\left(\bigwedge V_B, \mathrm{d}_B\right)$ is minimal and since $V^{<d \cdot n} = 0$, we necessarily obtain $\mathrm{d}u = \mathrm{d}_B u = 0$.

The result now follows by imitating the proofs of theorems 3.2, 3.3 and 3.4 in a completely analogous fashion. Just replace z^2 by z^d and adapt the calculations. Some iterative constructions will need to be done not only on even and odd indices separately, but for each class modulo d. In theorem 3.2 clearly only case 1 has to be considered. Neither are new ideas required nor has there any essential change to be made to the proofs. As we have already seen in the proof of corollary 3.5 the connectedness assumption is sufficient to derive all the "non-technical" conditions in theorem 3.2. This combines to yield the result. □

A crucial example enters the stage: This theorem directly applies to fibrations with fibre F having the rational homotopy type of either $\mathbb{C}\mathbf{P}^n$ or $\mathbb{H}\mathbf{P}^n$—always supposed that all the other assumptions (including the connectivity-assumption) are fulfilled. In such a case the cohomology ring is given by $H^*(F) = \mathbb{Q}[z]/z^{n+1}$ with $\deg z = 2$ or $\deg z = 4$ respectively. This leads us to

Remark 3.7. Let $E \to M$ be a complex vector bundle of rank n over a simply-connected smooth (closed) manifold M. Let

$$\mathbb{C}\mathbf{P}^{n-1} \hookrightarrow \mathbf{P}_{\mathbb{C}}(E) \xrightarrow{p} M$$

be the associated fibre-wise complex projectivisation. By the Hirsch lemma we obtain the isomorphism of $H^*(M)$-modules $H^*(\mathbf{P}_\mathbb{C}(E)) \cong H^*(M) \otimes H^*(\mathbb{C}\mathbf{P}^{n-1})$ which forces the cohomology algebra to be of the form

$$H^*(\mathbf{P}_\mathbb{C}(E)) \cong H^*(M)[z]/(z^n + c_1(E)z^{n-1} + \cdots + c_n(E))$$

for certain coefficients, the Chern classes, $c_i(E)$ (for $1 \leq i \leq n$). The Chern classes identify with the cohomology classes of the obstructions u_i which we obtain in the model of the fibration:

$$\mathrm{d}z' = z^n + u_1 z^{n-1} + \cdots + u_n$$

where $\big(\bigwedge\langle z, z'\rangle, \mathrm{d}\big)$ denotes a minimal model for $\mathbb{C}\mathbf{P}^{n-1}$. So for example a vanishing of the Chern classes $c_i(E)$ for $1 \leq i \leq n-1$ yields a situation similar in nature to what we dealt with.

Note that unit sphere bundles associated to real vector bundles over a manifold provide another class of examples to which theorems 3.2, 3.3 and 3.4 may be applicable. The cohomology class $[u]$ of the obstruction u in the differential $\mathrm{d}z' = z^2 - u$ we used throughout the proofs of the theorems again may be linked to the characteristic classes of the real vector bundle itself (cf. example [22].15.4, p. 202). ⊡

The following proposition is a specialised version of theorem 3.2.

Proposition 3.8. *Let*

$$F \hookrightarrow E \xrightarrow{p} B$$

be a spherical fibration of simply-connected (and path-connected) topological spaces with rational homology of finite type. Let the fibre satisfy $F \simeq_\mathbb{Q} \mathbb{S}^n$ for an even $n \geq 2$. Suppose further that the rationalised Hurewicz homomorphism $\pi_*(B) \otimes \mathbb{Q} \to H_*(B)$ is injective in degree n. Additionally assume the space B to be k-connected with $3k + 2 \geq 2n$.

Then we obtain: If the space E is formal, so is B.

Proof. In order to be able to apply theorem 3.2 we need to slightly modify its proof: In case 1, step 3 we used that the rationalised Hurewicz homomorphism was supposed to be injective in degree $2n$. The purpose of this assumption was to show that $V_B^{2n} = C_B^{2n}$ or actually merely that $u' \in C_B^{2n}$.

We know that $\mathrm{d}z' = z^2 - u$ with $u - u' \in \bigwedge^{\geq 2}\langle\{b_i\}_{i\in J}\rangle$ in the model of the fibration (cf. 1.34) and with u being d_B-closed. We have to show that our new condition is apt to replace the former one, i.e. to equally yield $u' \in C_B^{2n}$.

Since B is k-connected, it holds that $V_B^{\leq k} = 0$. Moreover, this also leads to $\mathrm{d}_B V^{\leq 2k} = 0$ by the minimality of $\big(\bigwedge V_B, \mathrm{d}_B\big)$. As $(2k+1) + (k+1) = 3k+2$, this itself implies that

$$\mathrm{d}_B\Big(\Big(\bigwedge^{\geq 2} V_B\Big)^{\leq 3k+2}\Big) = 0$$

Thus by the assumption $3k + 2 \geq 2n$ we derive that the element $u - u' \in \big(\bigwedge V^{\geq 2}\big)^{2n}$ is closed, i.e. $\mathrm{d}_B(u - u') = 0$. However, we know that $\mathrm{d}_B u = 0$, whence $\mathrm{d}_B u' = 0$ and $u' \in C_B^{2n}$. From here the proof of the theorem can be continued without any change. □

The connectedness-assumption of the proposition is automatically satisfied if $F \simeq_{\mathbb{Q}} \mathbb{S}^2$. So we may formulate very catchy special versions of the proposition:

Corollary 3.9. *Let*

$$\mathbb{S}^2 \hookrightarrow Z \xrightarrow{p} M$$

be a fibre bundle of simply-connected smooth closed manifolds.
Then we obtain:

- *If Z is formal, so is M.*
- *If Z is a complex manifold satisfying the* dd_c*-lemma, then M is formal.*

Proof. Smooth manifolds have (rational) homology of finite type. In the terminology of proposition 3.8 we have $n = 2$ and $k = 1$, since M is simply-connected. So the condition $5 = 3k + 2 \geq 2n = 4$ is satisfied. A model $\big(\bigwedge V, \mathrm{d}\big)$ of a simply-connected space X satisfies $\mathrm{d}V^2 = 0$ and $H^2(X) \cong V^2$ if it is minimal. We identify $V_M^2 \cong \mathrm{Hom}(\pi_2(M), \mathbb{Q})$ and $H_2(M) \cong \mathrm{Hom}(H^2(M), \mathbb{Q})$ by universal coefficients. By remark 1.27 this amounts to the fact that the rationalised Hurewicz homomorphism for M is given by an isomorphism

$$V_M^2 \cong \pi_n(M) \otimes \mathbb{Q} \xrightarrow{\cong} H_*(M) \cong V_M^2$$

(Clearly, we just could have cited the Hurewicz theorem once more.) In particular, it is injective. Now proposition 3.8 applies. If Z is complex satisfying the dd_c-lemma, then it is formal—cf. lemma 1.37 and the reasoning below it. □

This leads us to the main geometric result:

Theorem 3.10. *A Positive Quaternion Kähler Manifold is a formal space. The twistor fibration is a formal map.*

We shall give two slightly different proofs of the theorem. The first one basically relies on the recognition theorem for the complex Grassmannian (cf. 1.12); the second one will need clearly less structure theory of Positive Quaternion Kähler Geometry. It is not surprising that both proofs make essential use of the twistor fibration.

First proof of theorem 3.10. For every Positive Quaternion Kähler Manifold M there is the twistor fibration

$$\mathbb{S}^2 \hookrightarrow Z \to M \tag{3.18}$$

with Z a compact Kähler manifold. Positive Quaternion Kähler Manifolds are simply-connected. Thus so is Z. (As compact manifolds both M and Z have finite-dimensional (rational) homology.) The twistor space Z is a compact Kähler manifold (cf. theorem 1.8) so that it satisfies the dd_c-lemma. Thus Z is a formal space.

Positive Quaternion Kähler Manifolds have vanishing odd-degree Betti numbers, i.e. $b_{2i+1}(M) = 0$ (for $i \geq 0$)—cf. theorem 1.13. In particular, $b_3(M) = 0$.

A Positive Quaternion Kähler Manifold M^n satisfying $b_2(M) > 0$ is necessarily isometric to the complex Grassmannian $\mathbf{Gr}_2(\mathbb{C}^{n+2})$—cf. corollary 1.12. The latter is a symmetric space. Consequently, it is formal (cf. example 1.36). Thus, without restriction, we may focus on the case when $b_2(M) = 0$.

In total, we may assume M to be rationally 3-connected. Now corollary 3.5 applies and yields the formality of M. So both M and Z are formal and theorem 3.4 tells us that the twistor fibration is a formal map. □

Second proof of theorem 3.10. As in the first proof we use the twistor fibration (3.18) and apply corollary 3.9 to it. Since Z is a formal space, the corollary directly yields the result. □

Note that we do not use $b_2(M) = b_3(M) = 0$ in the second proof!

Needless to mention that formality may be interpreted as some sort of rigidity result: Whenever M has the rational cohomology type of a Wolf space, it also has the rational homotopy type of the latter.

For the definition of the following numerical invariants see the definition on [22].28, p. 370, which itself relies on various definitions on the pages 351, 360 and 366 in [22].27.

Corollary 3.11. *A rationally 3-connected Positive Quaternion Kähler Manifold M satisfies*

$$\frac{\dim M}{4} = \mathrm{c}_0(M) = \mathrm{e}_0(M) = \mathrm{cat}_0(M) = \mathrm{cl}_0(M)$$

Proof. On a Positive Quaternion Kähler Manifold M^n there is a class $u \in H^4(M)$ with respect to which M satisfies the analogue of the Hard-Lefschetz theorem (cf. theorem 1.13). In particular, there is the volume form $u^n \neq 0$. Thus the rational cup-length of a rationally 3-connected Positive Quaternion Kähler Manifold M is $\mathrm{c}_0(M) = \frac{\dim M}{4}$. The rest of the equalities is due to formality (cf. [22], example 29.4, p. 388). □

Clearly, Positive Quaternion Kähler Manifolds are necessarily simply-connected and compact. This leads to

Corollary 3.12. *A compact simply-connected Non-Negative Quaternion Kähler Manifold is a formal space. The twistor fibration is formal.*

Proof. Quaternion Kähler Manifolds are Einstein and therefore can be distinguished by their scalar curvature being positive, zero or negative. The case of positive scalar curvature now follows from theorem 3.10. A simply-connected Quaternion Kähler Manifold with vanishing scalar curvature is hyperKählerian and Kählerian in particular. So it is a formal space. The twistor fibration in this case is the canonical projection $M \times \mathbb{S}^2 \to M$. □

Let us end this section by presenting some vague ideas and speculations that are rather thought of as a motivation for the reader to get involved in the subject than to be a sound outline of what is possible.

Before we do so we shall hint to the fact that the question of formality is a prominent and recurring topic in the field of special geometries (like $\mathbf{G}_2$-holonomy).

Let us venture to give a daring interpretation of the formality of Positive Quaternion Kähler Manifolds: The Bott conjecture speculates that simply-connected compact Riemannian manifolds with non-negative sectional curvature are rationally elliptic. In the quaternionic setting theorems as the connectivity lemma (cf. theorem A on [19], p. 150) might suggest that scalar curvature could be seen as an adequate conceptual substitute for sectional curvature. (The only Positive Quaternion Kähler Manifold with positive sectional curvature is $\mathbb{H}\mathbf{P}^n$.) Furthermore, we know that the sectional curvature "along a quaternionic plane" is positive (cf. formula [5].14.42b, p. 406), which again relates quaternionic geometry to positive sectional curvature.

In the case of Positive Quaternion Kähler Manifolds—mainly because rational cohomology is concentrated in even degrees only—we suggest to see formality as a very weak substitute for ellipticity: In the rational elliptic case, we are dealing with F_0-spaces, which admit pure models. Thus the images of the relations—the odd degree generators of the vector space underlying the minimal model— under the differential in the minimal model are very well-behaved as they correspond to a regular sequence. In particular, there are no algebraic combinations of these images that correspond to non-vanishing cohomology classes.

In the case of formality we do not necessarily have this regularity of these images. However, the very effect in cohomology is enforced by an additional generator of the minimal model—corresponding to a homotopy group—which is supposed to map to such a non-trivial algebraic combination under the differential. The effect in cohomology then clearly is the same: The cohomology class of the algebraic combination is zero.

If one is willing to engage with this point of view, the formality of Positive Quaternion Kähler Manifolds might be seen as heading towards a quaternionic Bott conjecture.

There is a notion of *geometric formality* introduced by Kotschick (cf. [49]) consists in the property that the product of harmonic forms is harmonic again. Geometric formality enforces strong restrictions on the topological structure of the underlying manifold. For example, the Betti numbers of the manifold M^n are restricted from above by the Betti numbers of the n-dimensional torus—cf. [49], theorem 6. This result was even improved on Kähler manifolds by Nagy—cf. [62], corollary 4.1.

Symmetric spaces are geometrically formal (cf. [50], [71]). Thus it is tempting to conjecture the same for Positive Quaternion Kähler Manifolds. This would lead to previously unknown bounds for Betti numbers and is of high interest not only therefore. Unfortunately, there are spaces—N-symmetric spaces already (cf. [50])—which are formal but not geometrically formal. Nonetheless, we would like to see our result on formality as a first step in this direction.

Moreover, one might speculate that the twistor space Z is geometrically formal if (and only if) so is the Positive Quaternion Kähler Manifold—which would give direct access to Nagy's improved estimates. This might be motivated by the splitting of modules $H^*(Z) = H^*(M) \oplus H^*(M)[z]$, where z corresponds to the Kähler form of Z. The Kähler form, however, is harmonic. So up to pullback—which constitutes the main problem—$H^*(Z)$ as an algebra would be described by harmonic forms only.

A mere application to Wolf spaces would result in a class of geometrically formal homogeneous non-symmetric (twistor) spaces. According to [50] such spaces are really sought after.

3.1.2. Odd-dimensional fibres

In the following we shall deal with the case of a spherical fibration

$$F \hookrightarrow E \to B \tag{3.19}$$

where the fibre has odd (homological) dimension. We shall see that a priori no general statement can be made on how the formality of the base space and the total space are related. That is, we shall see that the formality of E does not necessarily imply the one of B, neither does the formality of B necessarily imply formality for E. The examples arise as geometric realisations of algebraic constructions.

A disadvantage of these conclusions is that our approach will probably not permit us to prove formality of compact regular 3-Sasakian manifolds (cf. [28] for the definition of the latter). Remark 3.15 will be the crucial one as far as this topic is concerned.

Before we shall start to construct the appropriate negative examples, we shall call the reader's attention to the following direct observation. For this, recall that a fibration is totally non-cohomologous to zero if the induced map in cohomology of the fibre-inclusion is surjective.

Proposition 3.13. *Let*

$$F \overset{i}{\hookrightarrow} E \overset{p}{\to} B$$

be a spherical fibration of simply-connected (and path-connected) topological spaces with rational homology of finite type. Let the fibre satisfy $F \simeq_{\mathbb{Q}} \mathbb{S}^n$ for an odd $n \geq 3$. Suppose further that the fibration is totally non-cohomologous to zero.

Then we have that $E \simeq_{\mathbb{Q}} B \times F$. In particular, the space E is formal if and only if so is B. If this is the case the map p is formal.

Proof. Let $\big(\bigwedge\langle z\rangle, \mathrm{d}\big)$ be the minimal model of $\mathbb{S}^n \simeq_{\mathbb{Q}} F$ defined by $\deg z = n$ and $\mathrm{d}z = 0$. Let $\big(\bigwedge V_B, \mathrm{d}_B\big)$ be the minimal model of the base space. Form the model of the fibration

$$\big(\bigwedge V_B \otimes \bigwedge\langle z\rangle, \mathrm{d}\big) \xrightarrow{\simeq} (\mathrm{A}_{\mathrm{PL}}(E), \mathrm{d})$$

where $\big(\bigwedge V_B, \mathrm{d}_B\big)$ includes as a differential subalgebra and where $\mathrm{d}z = y \in \bigwedge V_B$. The last property is due to theorem 1.21.

Since the fibration is totally non-cohomologous to zero, there is a cohomology class $[\bar{z}] \in H^n(E)$ with the property that $i^*([\bar{z}]) = [z]$. This implies that there is a closed form $\bar{z} \in \bigwedge V_B \otimes \bigwedge\langle z\rangle$ which maps to z under the canonical projection onto $\bigwedge\langle z\rangle$ in the model of the fibration. Hence we have that $\bar{z} = -x + z$ with $x \in \big(\bigwedge V_B\big)^n$. Since $\bar{z}$ is closed, we obtain that $\mathrm{d}x = \mathrm{d}z = y$.

This reveals an isomorphism

$$\phi : \big(\bigwedge V_B \otimes \bigwedge\langle z\rangle, \mathrm{d}\big) \xrightarrow{\cong} \big(\bigwedge V_B \otimes \bigwedge\langle \bar{z}\rangle, \mathrm{d}\big)$$

defined by $\phi|_{\bigwedge V_B} = \mathrm{id}$, by $\phi(z) = \bar{z} + x$ and by linear and multiplicative extension. (We regard the element $\bar{z}$ in the algebra on the right hand side as an abstract element in the vector space upon which the algebra is built.) Indeed, this morphism of commutative graded algebras evidently is bijective. It is also compatible with differentials: This is trivial on $\bigwedge V_B$ and thus follows from

$$\phi(\mathrm{d}(z)) = \phi(y) = y = \mathrm{d}\bar{z} + \mathrm{d}x = \mathrm{d}(\bar{z} + x) = \mathrm{d}(\phi(z))$$

This enables us to use the adapted algebra $\big(\bigwedge V_B \otimes \bigwedge\langle \bar{z}\rangle, \mathrm{d}\big)$ in order to encode the rational homotopy type of E. However, since $\mathrm{d}\bar{z} = 0$, we identify

$$\big(\bigwedge V_B \otimes \bigwedge\langle \bar{z}\rangle, \mathrm{d}\big) \xrightarrow{\cong} \big(\bigwedge V_B, \mathrm{d}_B\big) \otimes \big(\bigwedge\langle \bar{z}\rangle, 0\big)$$

Thus $E \simeq_{\mathbb{Q}} B \times F$. The map $\mathrm{A_{PL}}(p)$ is just the inclusion of $\mathrm{A_{PL}}(B, \mathrm{d})$ into $\mathrm{A_{PL}}(B) \otimes \mathrm{A_{PL}}(F)$ as a differential subalgebra—see the initial arguments in the proof of case 1 of theorem 3.4. Theorem 1.47 then yields the result. $\square$

We remark that in the proof we established the Hirsch lemma in our case before we worked with the adapted algebra. Thus it is not surprising that the proposition can easily be deduced from the (refined version of the) Hirsch lemma given in lemma [17].3.1, p. 257.

Let us now illustrate the appropriate negative examples. First we shall deal with the case of a formal base B and a non-formal total space E. There is an example described on [17], p. 261, which was given there as the simplest non-formal compact manifold. Suppose

$$N^3 := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

and factor out the sublattice

$$\Gamma^3 := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$$

in order to obtain the 3-dimensional compact Heisenberg manifold $M^3 := N^3/\Gamma^3$. We then obtain a fibre bundle

$$\frac{\langle c\rangle}{\Gamma^3} \hookrightarrow \frac{N^3}{\Gamma^3} \to \frac{\langle a, b\rangle}{\Gamma^3}$$

(By abuse of notation we denote by a the set of matrices

$$\left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{R} \right\}$$

The sets b and c are defined in the analogous fashion.) Topologically, this is just

$$\mathbb{S}^1 \hookrightarrow M^3 \to T^2$$

with $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$. However, $E := M$ is proved to be non-formal in [17] whereas $B := T^2$ is formal.

A higher dimensional and simply-connected analogue of this example may be constructed algebraically by the minimal Sullivan algebra $(\bigwedge V, \mathrm{d})$ defined as follows: Let V be generated by elements x, y, z with $\deg x = \deg y = n$, $\deg z = 2n - 1$, n odd, $\mathrm{d}x = \mathrm{d}y = 0$ and $\mathrm{d}z = xy$—cf. [22].12, example 1 on page 157, for a similar example. By [17], lemma 3.2, p. 258, we may give a geometric realisation of this example as a fibration: We may use spatial realisation (cf. [22].17) to construct a CW-complex for the algebra $(\bigwedge\langle x, y\rangle, \mathrm{d})$. The algebra $\bigwedge\langle x, y\rangle \otimes \bigwedge\langle z\rangle$ then forms an elementary extension (cf. [17], p. 249), since $\mathrm{d}z \in \bigwedge\langle x, y\rangle$. So the lemma realises this example with total space having the rational homotopy type of $(\bigwedge V, \mathrm{d})$, base space a realisation of $(\bigwedge\langle x, y\rangle, \mathrm{d})$ and fibre a realisation of $\big(\bigwedge\langle z\rangle, 0\big)$.

In all these examples the fibration is non-primitive as $\mathrm{d}z = xy$ is decomposable. (The algebra $\big(\bigwedge\langle x, y\rangle, \mathrm{d}\big)$ is a minimal Sullivan algebra already.) So we have seen:

Remark 3.14. Assume the fibration (3.19) to be non-primitive: The formality of B does not necessarily imply the formality of E. ⊡

Next we shall give a similar negative example under the assumption that (3.19) is primitive. Consider the commutative differential graded algebra $\big(\bigwedge V, \mathrm{d}\big)$ over the graded vector space V which we define as follows. We shall indicate generators with degree; moreover, we indicate the differential.

$$\begin{aligned} 2: &\quad y \mapsto 0 \\ 3: &\quad b \mapsto 0 \quad c \mapsto 0 \\ 4: &\quad u \mapsto 0 \\ 5: &\quad n \mapsto bc + uy \end{aligned}$$

Extend the differential to $\bigwedge V$ as a derivation. By definition the algebra is minimal and hence a minimal Sullivan algebra, as it is simply-connected (cf. 1.25). Form the algebra $\big(\bigwedge V \otimes \bigwedge\langle z\rangle, \mathrm{d}\big)$ with $\mathrm{d}z = u$, $\deg z = 3$. Since this algebra is an elementary extension of $\big(\bigwedge V, \mathrm{d}\big)$, by lemma [17].3.2 we may realise it as the total space of a fibration with fibre rationally a sphere $\mathbb{S}^3$. Since $\big(\bigwedge V, \mathrm{d}\big)$ is minimal and since z is mapped to u under d, the fibration is primitive.

We shall now show that $\big(\bigwedge V, \mathrm{d}\big)$ is formal whereas $\big(\bigwedge V \otimes \bigwedge\langle z\rangle, \mathrm{d}\big)$ is not. We have a decomposition $C = \langle y, b, c, u\rangle$ and $N = \langle n\rangle$ with $V = N \oplus C$ as in theorem 1.41. The ideal $I(N)$ generated by N in $\bigwedge V$ does not contain any closed form: Every element contained is of the form $n \cdot p(y, b, c, u)$, where p is a rational polynomial. Differentiation yields $\mathrm{d}(n \cdot p(y, b, c, u)) = (bc + uy) \cdot p(y, b, c, u) - n \cdot \mathrm{d}p(y, b, c, u)$. Unless p itself vanishes this term cannot equal zero since uy is not a zero divisor in $\bigwedge V$. In particular, $\big(\bigwedge V, \mathrm{d}\big)$ is formal.

In order to see that $\left(\bigwedge V\otimes\bigwedge\langle z\rangle,\mathrm{d}\right)$ is not formal, we construct a minimal model for it. The latter is given by $\left(\bigwedge\langle y,b,c,n\rangle,\mathrm{d}\right)$, where the differential is uniquely determined by $\mathrm{d}y=\mathrm{d}b=\mathrm{d}c=0$ and $\mathrm{d}(n)=bc$. (Minimality of this algebra is evident. We have a quasi-isomorphism between algebra and minimal model given in the obvious way by $y\mapsto y$, $b\mapsto b$, $c\mapsto c$, $n\mapsto n$.) We have

$$\tilde{C}:=\ker\mathrm{d}|_{\langle y,b,c,n\rangle}=\langle y,b,c\rangle$$

By degree the homogeneous complement $\tilde{N}:=\langle n\rangle$ of $\tilde{C}$ is uniquely determined. Thus C and N satisfy all the prerequisites from theorem 1.41. The element $n\cdot b$ lies in $I(N)$. We have $\mathrm{d}(n\cdot b)=-b^2c=0$, since b has odd degree. Moreover $\deg(n\cdot b)=8$. In degree 7 we compute that $(\bigwedge\langle y,b,c,n\rangle)^7=\langle n\cdot y\rangle$ with $\mathrm{d}(n\cdot y)=(bc)\cdot y$ such that the element $n\cdot b$ may not be exact. Hence by theorem 1.41 the total space of the fibration is not formal and we have proved

Remark 3.15. Assume the fibration (3.19) to be primitive: The formality of B does not necessarily imply the formality of E. ⊡

We shall now present a simple example of a fibration with formal total space and non-formal base space. We construct an algebra the realisation of which will be the base space B. The algebra is $\left(\bigwedge V,\mathrm{d}\right)$, where V will be given by generators with grading; moreover, we indicate the differential which is extended to be a derivation.

$$\begin{aligned} 3:&\quad b\mapsto 0\\ 4:&\quad c\mapsto 0\\ 6:&\quad n\mapsto bc \end{aligned}$$

The algebra is minimal by construction. So it is a minimal Sullivan algebra as it is simply-connected (cf. 1.25). It is rather obvious that the algebra cannot be formal: According to theorem 1.41 there is a unique decomposition $C=\langle b,c\rangle$, $N=\langle n\rangle$ by degree. We compute $\mathrm{d}(nb)=bcb=0$ and $nb\in I(N)$ is a closed but obviously non-exact element.

Form the algebra $\left(\bigwedge V\otimes\bigwedge\langle z\rangle,\mathrm{d}\right)$ with $\mathrm{d}z=c$, $\deg z=3$. Since this algebra is an elementary extension of $\left(\bigwedge V,\mathrm{d}\right)$, by lemma [17].3.2 we may realise it as the total space of a fibration with fibre rationally an $\mathbb{S}^3$. A minimal model for $\left(\bigwedge V\otimes\bigwedge\langle z\rangle,\mathrm{d}\right)$ is given by $\left(\bigwedge\langle b,n\rangle,\mathrm{d}\right)$ with $\deg b=3$, $\deg n=6$, $\mathrm{d}b=\mathrm{d}n=0$ (and where the n here corresponds to $n+zb$ in the model of the fibration). Obviously, this model is formal, as it is free. This proves

Remark 3.16. The formality of E does not necessarily imply the formality of B. ⊡

We observe that this last example leads to a primitive fibration. For the non-primitive case we make the following stunning observation:

Proposition 3.17. *Suppose $n \geq 3$, n odd and let*

$$\mathbb{S}^n \hookrightarrow E \xrightarrow{p} B^m$$

be a non-primitive *spherical fibre bundle of simply-connected (and path-connected) compact manifolds.*

Then it holds: The manifold E is not formal.

PROOF. Clearly, fibre bundles are fibrations. We shall suppose E to be formal. This will lead us to a contradiction.

Choose minimal models

$$\big(\bigwedge V_B, \mathrm{d}_B\big) \xrightarrow{\simeq} (\mathrm{A}_{\mathrm{PL}}(B), \mathrm{d})$$
$$\big(\bigwedge \langle z \rangle, 0\big) \xrightarrow{\simeq} \mathrm{A}_{\mathrm{PL}}(\mathbb{S}^n, \mathrm{d})$$

Form the model of the fibration

$$\big(\bigwedge V_B \otimes \bigwedge \langle z \rangle, \mathrm{d}\big) \xrightarrow{\simeq} (\mathrm{A}_{\mathrm{PL}}(E), \mathrm{d})$$

As the fibration was supposed to be non-primitive, this model is a minimal Sullivan model for E. (Minimality is due to non-primitivity. Then apply proposition 1.25.) Set $V_E := V_B \oplus \langle z \rangle$ with the inherited grading.

As E was supposed to be formal, we may form the decomposition $V_E = C_E \oplus N_E$ with $C_E = \ker \mathrm{d}_E|_{V_E}$ according to theorem 1.41.

For reasons of degree we have $\mathrm{d}z \in \bigwedge V_B$. Due to non-primitivity we see that $\mathrm{d}z \neq 0$. We may assume that $z \in N_E$. This is due to the following arguments: Since $V_E = C_E \oplus N_E$ and since $\mathrm{d}z \neq 0$ by non-primitivity, there is a $c \in C$ such that $\tilde{z} := z + c \in N_E$. The assignment $z \mapsto \tilde{z}$ induces an automorphism of V_E, which itself induces an automorphism σ of the graded algebra $\bigwedge V_E$. This automorphism commutes with differentials, since

$$\mathrm{d}(\sigma(x)) = \mathrm{d}(x) = \sigma(\mathrm{d}(x)) \qquad \text{for } x \in \bigwedge V_B$$
$$\mathrm{d}(\sigma(z)) = \mathrm{d}(\tilde{z}) = \mathrm{d}(z + c) = \mathrm{d}z + \mathrm{d}c = \mathrm{d}z = \sigma(\mathrm{d}(z))$$

This is due to the fact that $\sigma|_{\bigwedge V_B} = \mathrm{id}$.

The manifold E is a compact simply-connected manifold of dimension $n + m$. Thus it satisfies Poincaré Duality. In particular, we have

$$H^{n+m}(E) = \mathbb{Q}$$

Now compute the cohomology of E by means of the minimal model $\big(\bigwedge V_E, \mathrm{d}\big)$. Every closed form in $I(N_E)$ is exact by formality of E and theorem 1.41. Since $z \in N_E$

there is no non-vanishing cohomology class that is represented by an element of the form $z \cdot x + y$ for $0 \neq x \in \bigwedge V_E$ and $y \in I(N_E)$. That is, cohomology is represented by the elements in $\ker \mathrm{d}_B$. Moreover, we have that $\mathrm{d}z \in \bigwedge V_B$ and thus $0 = \mathrm{d}^2 z = \mathrm{d}_B(\mathrm{d}z)$. Hence $\mathrm{d}z \in \ker \mathrm{d}_B$. Thus we compute

$$H^*(E) = \ker \mathrm{d}_B / \operatorname{im} d = (\ker \mathrm{d}_B / \operatorname{im} \mathrm{d}_B)/\mathrm{d}z = H^*(B)/\mathrm{d}z$$

In particular, we obtain that $H^i(E) = 0$ for $i > m$—a contradiction.

□

The fibre bundle $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \to \mathbb{C}\mathbf{P}^n$ (which clearly is non-primitive) shows that the result does not hold in general. Consider the Hopf bundle $\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \to \mathbb{S}^4$ for a simply-connected example.

3.2. Non-formal homogeneous spaces

Homogeneous spaces form an interesting class of manifolds. From the point of view of Rational Homotopy Theory they possess very nice properties: They are elliptic spaces that admit pure models (cf. 1.49 and 1.53). However, they remain complicated enough to constitute a field of study worth-while the attention it is paid. For example, an interesting question comes up: Under which conditions is a homogeneous space formal? An elaborate discussion of this issue was given in the classical book [31]. Amongst others this resulted in long lists of formal homogneous spaces (cf. [31].XI, p. 492-497). Basically, only three examples of non-formal type were given (cf. [31].XI.5, p. 486-491). (One example was generalised to a parametrised family in [74].) So for example the following homogeneous spaces are known to be non-formal for $k \geq 6$ and $n \geq 5$:

$$\frac{\mathbf{SU}(k)}{\mathbf{SU}(3) \times \mathbf{SU}(3)}, \qquad \frac{\mathbf{Sp}(n)}{\mathbf{SU}(n)}$$

A few further examples are cited in [51], example 1.

Homogeneous spaces G/H directly lead to a fibre bundle

$$H \hookrightarrow G \to G/H$$

In the case of a non-formal homogeneous spaces this is a fibration with formal fibre and formal total space but with non-formal base space (cf. example 1.36). Moreover, by theorem 1.45 we know that G/H is formal if and only if G/T is formal, where $T \subseteq H$ is a maximal torus of H. Thus we may successively form the spherical fibre bundles

$$\mathbb{S}^1 \hookrightarrow G/T^k \to G/T^{k+1} \tag{3.20}$$

for $k \geq 0$ and $T^k \subseteq T$ a k-torus. If G/H is non-formal, we end up with a non-formal space G/T as a base space. In particular, we see that there is a choice for k with the property that (3.20) is a fibre bundle with formal total space and non-formal base space. One example of this is

$$\mathbb{S}^1 \hookrightarrow \frac{\mathbf{Sp}(n)}{\mathbf{U}(n)} \to \frac{\mathbf{Sp}(n)}{\mathbf{SU}(n)}$$

for $n \geq 5$. (The total space is formal as $\mathbf{Sp}(n)$and $\mathbf{U}(n)$ form an equal rank pair—cf. 1.53.)

Equally, for example, we may construct a spherical fibre bundle with non-formal total space and formal base space by

$$\mathbb{S}^1 \hookrightarrow \frac{\mathbf{SU}(6)}{\mathbf{SU}(3) \times \mathbf{SU}(3)} \to \frac{\mathbf{SU}(6)}{\mathbf{S}(\mathbf{U}(3) \times \mathbf{U}(3))}$$

The fibration is induced by the canonical inclusion

$$\mathbf{SU}(3) \times \mathbf{SU}(3) \hookrightarrow \mathbf{S}(\mathbf{U}(3) \times \mathbf{U}(3))$$

Note that

$$\frac{\mathbf{SU}(6)}{\mathbf{S}(\mathbf{U}(3) \times \mathbf{U}(3))} \cong \frac{\mathbf{U}(6)}{\mathbf{U}(3) \times \mathbf{U}(3)}$$

which is a complex Grassmannian and a symmetric space. Hence it is formal (cf. 1.36). Alternatively, we directly see that $\operatorname{rk} \mathbf{SU}(6) = \operatorname{rk} \mathbf{S}(\mathbf{U}(3) \times \mathbf{U}(3))$, which also implies formality (cf. 1.53).

These examples form the connection to the topics of the last section. They will be a motivation to find ways of how to construct non-formal homogeneous spaces. Before we do so let us illustrate a showcase computation by which we reprove the non-formality of $\frac{\mathbf{SU}(6)}{\mathbf{S}(\mathbf{U}(3)\times\mathbf{U}(3))}$ in a very direct way. This computation is described explicitly. Thus it will easily be adaptable to several future situations in which we shall merely mention the result of the calculation.

We construct a model for the homogeneous space according to theorem 1.22. A minimal Sullivan model for $\mathbf{SU}(n)$ is given by $\bigwedge(\langle x_2, x_3, \ldots, x_n\rangle, 0)$ with $\deg x_i = 2i-1$ as can be derived directly from formality and its cohomology algebra. Hence a Sullivan model for the classifying space $\mathbf{BSU}(n_k)$ (with $n_k \geq 2$) is given by the polynomial algebra $\big(\bigwedge\langle c_2, \ldots, c_{n_k}\rangle, 0\big)$ with $\deg c_i = 2i$. So we obtain a model

$$\begin{aligned} &\mathrm{A}_{\mathrm{PL}}\left(\frac{\mathbf{SU}(n)}{\mathbf{SU}(n_1) \times \cdots \times \mathbf{SU}(n_k)}, \mathrm{d}\right) \\ (3.21) \qquad &\simeq \bigwedge \big(\langle c_2^1, \ldots, c_{n_1}^1, c_2^2, \ldots, c_{n_2}^2, \ldots, c_2^k, \ldots, c_{n_k}^k\rangle \otimes \langle x_2, \ldots x_n\rangle, \mathrm{d}\big) \end{aligned}$$

with $\deg c_i^j = 2i, \deg x_i = 2i - 1$. The differential vanishes on the c_i^j and $\mathrm{d}x_i = H^*(\mathbf{B}(\phi))y_i$, where y_i is the generator of $\mathbf{BSU}(n)$ corresponding to x_i and ϕ is the blockwise inclusion map $\mathbf{SU}(n_1) \times \cdots \times \mathbf{SU}(n_k) \hookrightarrow \mathbf{SU}(n)$.

Note that we may take the c_i^j for the i-th universal Chern classes of $\mathbf{BSU}(n_j)$. Thus blockwise inclusion of the $\mathbf{SU}(n_j)$ yields the following equations

$$\begin{aligned}
\mathrm{d}(x_2) &= \sum_{i=1}^k c_2^i \\
\mathrm{d}(x_3) &= \sum_{i=1}^k c_3^i \\
\mathrm{d}(x_4) &= \sum_{i=1}^k c_4^i + \sum_{i\neq j} c_2^i c_2^j \\
&\vdots \\
\mathrm{d}(x_i) &= (c^1 \cdot \cdots \cdot c^k)|_{2i} \\
&\vdots \\
\mathrm{d}(x_n) &= c_{n_1}^1 \cdot \cdots \cdot c_{n_k}^k
\end{aligned}$$

which become obvious when writing the Chern classes as elementary symmetric polynomials in elements that generate the cohomology of the classifying spaces of the maximal tori. By c^j we denote the total Chern class $1 + c_2^j + c_3^j + \cdots + c_{n_j}^j$ of $\mathbf{SU}(n_j)$ and by $|_{2i}$ the projection to degree $2i$.

In the case $k = 2$, $n_1 = n_2 = 3$ we obtain a model $\left(\bigwedge V, \mathrm{d}\right)$, where the graded vector space V is generated by

$$\begin{array}{ll}
c, c', d, d' & \deg c = \deg d = 4,\ \deg c' = \deg d' = 6 \\
x_2, x_3, x_4, x_5, x_6 & \deg x_2 = 3,\ \deg x_3 = 5,\ \deg x_4 = 7,\ \deg x_5 = 9,\ \deg x_6 = 11
\end{array}$$

The differential is given by

$$\begin{gathered}
\mathrm{d}c = \mathrm{d}c' = \mathrm{d}d = \mathrm{d}d' = 0 \\
\mathrm{d}x_2 = c + d,\ \mathrm{d}x_3 = c' + d',\ \mathrm{d}x_4 = c \cdot d,\ \mathrm{d}x_5 = c' \cdot d + d' \cdot c,\ \mathrm{d}x_6 = c' \cdot d'
\end{gathered}$$

Thus the cohomology algebra of this algebra is given by generators

$$e, e', y, y' \qquad \deg e = 4,\ \deg e' = 6,\ \deg y = 13,\ \deg y' = 15$$

and by relations

$$e^2 = e \cdot e' = e'^2 = y \cdot e = y^2 = y \cdot y' = y'^2 = y' \cdot e' = 0, \quad y \cdot e' = -y' \cdot e$$

Hence Betti numbers are given by

$$b_0 = b_4 = b_6 = b_{13} = b_{15} = b_{19} = 1$$

with the remaining ones equal to zero.

Using the construction principle from [22].12, p. 144–145, we compute the minimal model for this algebra. On generators it is given by

$$e, e', x, x', x'' \qquad \deg e = 4, \ \deg e' = 6, \ \deg x = 7, \ \deg x' = 9, \ \deg x'' = 11$$

The differential d is given by

$$\mathrm{d}e = \mathrm{d}e' = 0, \ \mathrm{d}x = e^2, \ \mathrm{d}x' = e \cdot e', \ \mathrm{d}x'' = e'^2$$

Finally, cohomology classes are represented by

$$e \leftrightarrow e, \ e' \leftrightarrow e', \ y \leftrightarrow x \cdot e' - x' \cdot e, \ y' \leftrightarrow x' \cdot e' - x'' \cdot e, \ y \cdot e' \leftrightarrow x \cdot e'^2 - x' \cdot e \cdot e'$$

In particular, we see that a splitting of the underlying vector space V of the minimal model into $V = C \oplus N$ as required by theorem 1.41 would necessarily imply $x, x' \in N$ by degree and it would require $x \cdot e' - x' \cdot e$ in the ideal generated by N to be closed but not exact. So $\frac{\mathbf{SU}(6)}{\mathbf{SU}(3)\times\mathbf{SU}(3)}$ is not formal.

We remark that the spaces $\mathbf{SU}(4)/(\mathbf{SU}(2) \times \mathbf{SU}(2))$ and $\mathbf{SU}(5)/(\mathbf{SU}(2) \times \mathbf{SU}(3))$ are formal as a straight-forward computation shows.

In theory, given the groups G, H and the inclusion of H into G it is possible to compute whether the space G/H is formal or not. However, understanding the topological nature of the inclusion may be a non-trivial task.

On the other hand computational complexity theory enters the stage: In [29] the following result was established: Given a rationally elliptic simply-connected space, computing its Betti numbers from its minimal Sullivan model—which serves as an encoding of the space—is an NP-hard problem. So on a general space of large dimension it should be rather challenging to decide whether the space is formal or not.

Moreover, it is desirable (cf. [71], p. 38) to give criteria on invariants of G, H and the embedding $H \hookrightarrow G$ by which a direct identification of the homogeneous space G/H as being formal or not may be achieved. (In the same paper, this was done by identifying N-symmetric spaces as formal. See also [50] for the same result.)

So we may agree that finding a priori arguments which allow to identify infinite non-formal families should be worth-while the effort. Let us present some first observation in this vein.

Proposition 3.18. *Let* $H \subseteq G$ *and* $K \subseteq H \times H$ *be simply-connected compact Lie groups. Suppose that the inclusion of* H *into* G *induces an injective morphism on rational homotopy groups, i.e.* $\pi_*(H) \otimes \mathbb{Q} \hookrightarrow \pi_*(G) \otimes \mathbb{Q}$.

Then $G/\!\!/K$ *is formal if and only if* $H/\!\!/K$ *is formal.*

PROOF. Choose minimal Sullivan models $\big(\bigwedge V_G, 0\big)$, $\big(\bigwedge V_H, 0\big)$ and $\big(\bigwedge V_K, 0\big)$ for G, H and K. Let $x_1, \ldots, x_n$ be a homogeneous basis of V_H. Since the inclusion $H \hookrightarrow G$ induces an injective morphism on rational homotopy groups, we may choose $x'_1, \ldots, x'_k$ such that $x_1, \ldots, x_n, x'_1, \ldots, x'_k$ is a homogeneous basis of V_G. (For this we identify generators of homotopy groups with the x_i, x'_i up to duality.)

So we obtain

$$H^*(G \times G) \cong \bigwedge \langle x_1, \ldots, x_n, x'_1, \ldots, x'_k, y_1, \ldots, y_n, y'_1, \ldots y'_k \rangle$$

(The y_i, y'_i are constructed just like the x_i, x'_i, i.e. as a "formal copy".)

Hence a model for the biquotient $G/\!\!/K$ is given by (cf. 1.21)

$$\big(\bigwedge V_K^{+1} \otimes \bigwedge \langle q_1, \ldots, q_n, q'_1, \ldots, q'_k \rangle, \mathrm{d}\big)$$

(The degrees of V_K are shifted by $+1$.)

Denote the inclusion $K \overset{i_1}{\hookrightarrow} H \times H \overset{i_2}{\hookrightarrow} G \times G$ by $\phi = i_2 \circ i_1$. The differential d on the q_i is given by $\mathrm{d}(q_i) = H^*(\mathbf{B}\phi)(\bar{x}_i - \bar{y}_i)$, where $\bar{x}_i, \bar{y}_i \in H^*(\mathbf{B}G)$ is the class corresponding to $x_i, y_i \in H^*(G)$. (The analogue holds for the q'_i.) By functoriality the morphism $H^*(\mathbf{B}\phi)$ factors over

$$H^*(\mathbf{B}G \times \mathbf{B}G) \xrightarrow{H^*(\mathbf{B}i_2)} H^*(\mathbf{B}H \times \mathbf{B}H) \xrightarrow{H^*(\mathbf{B}i_1)} H^*(\mathbf{B}K)$$

In particular, $H^*(\mathbf{B}\phi) = H^*(\mathbf{B}i_1) \circ H^*(\mathbf{B}i_2)$ and

$$\mathrm{d}(q'_i) = H^*(\mathbf{B}i_1)\big(H^*(\mathbf{B}i_2)\big(\bar{x}'_i - \bar{y}'_i\big)\big)$$

for $1 \le i \le k$. The inclusion i_2 by definition is the product $i_2 = i_2|_H \times i_2|_H$. Thus we obtain a decomposition $H^*\mathbf{B}i_2 = H^*\mathbf{B}(i_2|_H) \otimes H^*\mathbf{B}(i_2|_H)$ and $H^*(\mathbf{B}i_2)\big(\bar{x}'_i\big) \in \bigwedge \langle x_1, \ldots, x_n \rangle$; i.e. it is a linear combination in products of the x_i. By the splitting of $H^*\mathbf{B}i_2$ we obtain that $H^*(\mathbf{B}i_2)\big(\bar{y}'_i\big) \in \bigwedge \langle y_1, \ldots, y_n \rangle$ is the same linear combination with the x_i replaced by the y_i. This implies that

$$\mathrm{d}(q'_i) = H^*(\mathbf{B}i_1)\big(H^*(\mathbf{B}i_2)\big(\bar{x}'_i - \bar{y}'_i\big)\big) \in \operatorname{im} H^*(\mathbf{B}i_1) = \mathrm{d}\big(\bigwedge \langle q_1, \ldots, q_n \rangle\big)$$

(In fact, the very same linear combination as above with the x_i now replaced by the q_i will serve as a preimage of $\mathrm{d}(q'_i)$ under $\mathrm{d}|_{\bigwedge \langle q_1, \ldots, q_n \rangle}$.)

Thus for each $1 \le i \le k$ there is an element $z_i \in \bigwedge \langle q_1, \ldots, q_n \rangle$ with the property that $\mathrm{d}(q'_i - z_i) = 0$. Set $\tilde{q}_i := q'_i - z_i$. We have an isomorphism of commutative differential graded algebras

$$\begin{aligned} \sigma : &\big(\bigwedge V_K^{+1} \otimes \bigwedge \langle q_1, \ldots, q_n, q'_1, \ldots, q'_k \rangle, \mathrm{d}\big) \\ &\xrightarrow{\cong} \big(\bigwedge V_K^{+1} \otimes \bigwedge \langle q_1, \ldots, q_n, \tilde{q}_1, \ldots, \tilde{q}_k \rangle, \mathrm{d}\big) \end{aligned}$$

induced by the "identity" $q_i' \mapsto \tilde{q}_i + z_i$ for all $1 \le i \le k$. (By abuse of notation we now consider the $\tilde{q}_i$ with $\mathrm{d}\tilde{q}_i := \mathrm{d}(q_i' - z_i) = 0$ as abstract elements in the graded vector space upon which the algebra is built.) This morphism is an isomorphism of commutative graded algebras which commutes with differentials:

$$\mathrm{d}(\sigma(q_i')) = \mathrm{d}(\tilde{q}_i + z_i) = \mathrm{d}(q_i') = \sigma(\mathrm{d}(q_i'))$$

as $\sigma|_{\bigwedge\langle q_1,\ldots,q_k\rangle} = \mathrm{id}$.

Thus we obtain a quasi-isomorphism

$$\begin{aligned} \mathrm{A_{PL}}(G/\!\!/K) &\simeq \big(\textstyle\bigwedge V_K^{+1} \otimes \bigwedge\langle q_1,\ldots,q_n,q_1',\ldots,q_k'\rangle, \mathrm{d}\big) \\ &\cong \big(\textstyle\bigwedge V_K^{+1} \otimes \bigwedge\langle q_1,\ldots,q_n,\tilde{q}_1,\ldots,\tilde{q}_k\rangle, \mathrm{d}\big) \\ &= \big(\textstyle\bigwedge V_K^{+1} \otimes \bigwedge\langle q_1,\ldots,q_n\rangle, \mathrm{d}\big) \otimes \big(\textstyle\bigwedge\langle \tilde{q}_1,\ldots,\tilde{q}_k\rangle, 0\big) \end{aligned} \tag{3.22}$$

The algebra $\big(\bigwedge V_K^{+1} \otimes \bigwedge\langle q_1,\ldots,q_n\rangle, \mathrm{d}\big)$ is a model for $H/\!\!/K$, since $x_1,\ldots,x_n,y_1,\ldots,y_n$ is a basis of V_H and since its differential d corresponds to $H^*(\mathbf{B}i_1)$. Hence the last algebra in (3.22) is rationally the product of a model of $H/\!\!/K$ and a formal algebra. Thus it is formal if and only if H/K is formal (cf. theorem 1.47). □

Remark 3.19. We can get rid of the assumption that the Lie groups are simply-connected: This assumption made it possible to identify generators of rational homotopy groups with generators of the vector space generating the minimal model. However, as long as we have the inclusion $V_H \subseteq V_G$ we can proceed as in the proof of proposition 3.18 and the result holds equally. This will be the case in the examples we shall consider later. ⧈

Remark 3.20. For the convenience of the reader we specialise the arguments of the proof to the case when all biquotients are homogeneous spaces, since this will be our main field of application:

Choose minimal Sullivan models $\big(\bigwedge V_G, 0\big)$, $\big(\bigwedge V_H, 0\big)$ and $\big(\bigwedge V_K, 0\big)$ for G, H and K. Let $x_1,\ldots,x_n$ be a homogeneous basis of V_H. Since the inclusion $H \hookrightarrow G$ induces an injective morphism on rational homotopy groups we may choose $x_1',\ldots,x_k'$ such that $x_1,\ldots,x_n,x_1',\ldots,x_k'$ is a homogeneous basis of V_G. (For this we identify generators of homotopy groups with the x_i, x_i' up to duality.)

Hence a model for the homogeneous space G/K is given by (cf. 1.21)

$$\big(\textstyle\bigwedge V_K^{+1} \otimes \bigwedge V_G, \mathrm{d}\big) = \big(\textstyle\bigwedge V_K^{+1} \otimes \bigwedge\langle x_1,\ldots,x_n,x_1',\ldots,x_k'\rangle, \mathrm{d}\big)$$

(The degrees of V_K are shifted by $+1$.)

Denote the inclusion $K \overset{i_1}{\hookrightarrow} H \overset{i_2}{\hookrightarrow} G$ by $\phi = i_2 \circ i_1$. The differential d is given on $\bigwedge V_G$ by $\mathrm{d}(y) = H^*(\mathbf{B}\phi)\bar{y}$, where $\bar{y} \in H^*(\mathbf{B}G)$ is the class corresponding to $y \in H^*(G) = \bigwedge V_G$. By functoriality the morphism $H^*(\mathbf{B}\phi)$ factors over

$$H^*(\mathbf{B}G) \xrightarrow{H^*(\mathbf{B}i_2)} H^*(\mathbf{B}H) \xrightarrow{H^*(\mathbf{B}i_1)} H^*(\mathbf{B}K)$$

In particular, $H^*(\mathbf{B}\phi) = H^*(\mathbf{B}i_1) \circ H^*(\mathbf{B}i_2)$ and

$$\mathrm{d}(x_i') = H^*(\mathbf{B}i_1)\big(H^*(\mathbf{B}i_2)(\bar{x}_i')\big) \in \operatorname{im} H^*(\mathbf{B}i_1) = \mathrm{d}\big(\bigwedge\langle x_1, \ldots, x_n\rangle\big)$$

for $1 \leq i \leq k$. Thus for each $1 \leq i \leq k$ there is an element $y_i \in \bigwedge\langle x_1, \ldots, x_n\rangle$ with the property that $\mathrm{d}(x_i' - y_i) = 0$. Set $\tilde{x}_i := x_i' - y_i$. We have an isomorphism of commutative differential graded algebras

$$\begin{aligned} \sigma :& \big(\bigwedge V_K^{+1} \otimes \bigwedge\langle x_1, \ldots, x_n, x_1', \ldots, x_k'\rangle, \mathrm{d}\big) \\ &\xrightarrow{\cong} \big(\bigwedge V_K^{+1} \otimes \bigwedge\langle x_1, \ldots, x_n, \tilde{x}_1, \ldots, \tilde{x}_k\rangle, \mathrm{d}\big) \end{aligned}$$

induced by the "identity" $x_i' \mapsto \tilde{x}_i + y_i$ for all $1 \leq i \leq k$. (By abuse of notation we now consider the $\tilde{x}_i$ with $\mathrm{d}\tilde{x}_i := \mathrm{d}(x_i' - y_i) = 0$ as abstract elements in the graded vector space upon which the algebra is built.) This morphism is an isomorphism of commutative graded algebras which commutes with differentials:

$$\mathrm{d}(\sigma(x_i')) = \mathrm{d}(\tilde{x}_i + y_i) = \mathrm{d}(x_i') = \sigma(\mathrm{d}(x_i'))$$

as $\sigma|_{\bigwedge\langle x_1,\ldots,x_k\rangle} = \mathrm{id}$.

Thus we obtain a quasi-isomorphism

$$\begin{aligned} \mathrm{A_{PL}}(G/K) &\simeq \big(\bigwedge V_K^{+1} \otimes \bigwedge\langle x_1, \ldots, x_n, x_1', \ldots, x_k'\rangle, \mathrm{d}\big) \\ &\cong \big(\bigwedge V_K^{+1} \otimes \bigwedge\langle x_1, \ldots, x_n, \tilde{x}_1, \ldots, \tilde{x}_k\rangle, \mathrm{d}\big) \\ &= \big(\bigwedge V_K^{+1} \otimes \bigwedge\langle x_1, \ldots, x_n\rangle, \mathrm{d}\big) \otimes \big(\bigwedge\langle \tilde{x}_1, \ldots, \tilde{x}_k\rangle, 0\big) \end{aligned} \tag{3.23}$$

The algebra $\big(\bigwedge V_K^{+1} \otimes \bigwedge\langle x_1, \ldots, x_n\rangle, \mathrm{d}\big)$ is a model for H/K, since $x_1, \ldots, x_n$ is a basis of V_H and since its differential d corresponds to $H^*(\mathbf{B}i_1)$. Hence the last algebra in (3.23) is rationally the product of a model of H/K and a formal algebra. Thus it is formal if and only if H/K is formal (cf. theorem 1.47). ⊡

Corollary 3.21. *Let K be a compact Lie subgroup of the Lie group G, which itself is a Lie subgroup of a Lie group $\tilde{G}$. Table 3.1 gives pairs of groups G and $\tilde{G}$ together with relevant relations and the type of the inclusion such that it holds: The homogenenous space G/K is formal if and only if $\tilde{G}/K$ is formal.*

Proof. For the minimal models of the relevant Lie groups and further reasoning on their inclusions see [22].15, p. 220. Lie groups are formal and so are the depicted inclusions—cf. example 1.36 and proposition 1.46. More precisely, from the proof of proposition 1.46 we see that we may identify minimal models with their cohomology algebra and induced maps on minimal models with induced maps in cohomology. Thus it suffices to see that the given inclusions induce surjective morphisms in cohomology.

The blockwise inclusions are surjective in cohomology due to theorem [57].6.5.(4), p. 148.

The inclusion $\mathbf{SO}(n) \hookrightarrow \mathbf{SU}(n)$ induces a surjective morphism by [57].6.7.(2), p. 149. So does the inclusion $\mathbf{Sp}(n) \hookrightarrow \mathbf{SU}(2n)$ by [57].6.7.(1), p. 149. □

Table 3.1.: Inclusions of Lie groups

G	$\tilde{G}$	with	type of embedding
$\mathbf{SO}(n)$	$\mathbf{SO}(N)$	$n \geq 1$, n odd, $N \geq n$, N odd	blockwise
$\mathbf{SO}(n)$	$\mathbf{SO}(N)$	$n \geq 1$, n odd, $N \geq n+1$, N even	blockwise
$\mathbf{SU}(n)$	$\mathbf{SU}(N)$	$n > 1$, $N \geq n$	blockwise
$\mathbf{Sp}(n)$	$\mathbf{Sp}(N)$	$n \geq 1$, $N \geq n$	blockwise
$\mathbf{SO}(n)$	$\mathbf{SU}(n)$	$n \geq 1$, n odd	induced componentwise by $\mathbb{R} \hookrightarrow \mathbb{C}$
$\mathbf{Sp}(n)$	$\mathbf{SU}(2n)$	$n \geq 1$	induced by the identification $\mathbb{H} \hookrightarrow \mathbb{C}^{2\times 2}$

We remark that the chain of inclusions $\mathbf{SO}(n) \subseteq \mathbf{U}(n) \subseteq \mathbf{Sp}(n)$ does *not* induce a surjective morphism in cohomology (cf. [57].5.8.(1), p. 138 and [57].6.7.(2), p. 149). Neither do the chain $\mathbf{Sp}(n) \subseteq \mathbf{U}(2n) \subseteq \mathbf{SO}(4n)$ nor the chain $\mathbf{Sp}(n) \subseteq \mathbf{U}(2n) \subseteq \mathbf{SO}(4n+1)$ in general by a reasoning taking into account theorem [57].6.11, p. 153, which lets us conclude that the transgression in the rationalised long exact homotopy sequence for $\mathbf{U}(n) \hookrightarrow \mathbf{SO}(2n) \to \mathbf{SO}(2n)/\mathbf{U}(n)$ is surjective.

Example 3.22.

- Using our initial computation and proposition 3.18 we see that the space

$$\frac{\mathbf{SU}(n)}{\mathbf{SU}(3) \times \mathbf{SU}(3)}$$

  is non-formal for $n \geq 6$.

- We also obtain pretty simple proofs for formality: For example, the spaces

$$\frac{\mathbf{U}(n)}{\mathbf{U}(k_1) \times \cdots \times \mathbf{U}(k_l)} \quad \text{and} \quad \frac{\mathbf{SO}(2n+1)}{\mathbf{SO}(2k_1) \times \cdots \times \mathbf{SO}(2k_l)}$$

  with $\sum_{i=1}^{l} k_i \leq n$ are formal: In the equal rank case $\sum_{i=1}^{l} k_i = n'$ formality holds due to proposition 1.53. Then apply proposition 3.18 together with remark 3.19 to the inclusions $\mathbf{U}(n') \hookrightarrow \mathbf{U}(n)$ and $\mathbf{SO}(2n'+1) \hookrightarrow \mathbf{SO}(2n+1)$.

⊡

Remark 3.23. One may present this discussion in a slightly different light: By theorem 1.45 formality of G/H only depends on the way the maximal torus of the denominator is included.

Now fix a maximal torus $T_G := \mathbb{S}^1_1 \times \cdots \times \mathbb{S}^1_l$ of G. Let $T_H \subseteq T_G$ be a maximal torus of H. Set $T_{i_1,i_2,\ldots} := \mathbb{S}^1_{i_1} \times \{1\} \times \cdots \times \{1\} \times \mathbb{S}^1_{i_2} \times \{1\} \times \ldots \subseteq T_G$ (with

$i_1 < i_2 < \cdots \leq l$) to be the subtorus of T_G containing only certain $\mathbb{S}^1$-components. Let further T' with $T_H \subseteq T' \subseteq T_G$ be the smallest subtorus of T_G of this form containing the maximal torus of H, i.e.

$$T' := \bigcap \{T_{i_1,i_2,\ldots} \mid i_1 < i_2 < \cdots \leq l \text{ and } T_H \subseteq T_{i_1,i_2,\ldots}\}$$

Let $\mathrm{W}_{T'}$ be the maximal subgroup of the Weyl group $\mathrm{W}(G)$ leaving T' invariant. The classical Lie groups of types $\mathbf{A}_n$, $\mathbf{B}_n$ and $\mathbf{C}_n$ now have the following property: The morphism

$$H^*(\mathbf{B}G) = H^*(\mathbf{B}T_G)^{\mathrm{W}(G)} \to H^*(\mathbf{B}T')^{\mathrm{W}_{T'}}$$

induced by the inclusion is surjective. The algebra $H^*(\mathbf{B}G)$ is just the minimal model of $\mathbf{B}G$. The algebra $H^*(\mathbf{B}T')^{\mathrm{W}_{T'}}$ is just the cohomology algebra of the subgroup $\tilde{G} \subseteq G$ that corresponds to the block-matrices for which T' is the standard maximal torus. In particular, we have that $\operatorname{rk} \tilde{G} = \operatorname{rk} T'$. Thus the algebra $H^*(\mathbf{B}T')^{\mathrm{W}_{T'}}$ may also be identified with the minimal model of $\mathbf{B}G$. Thus also the map on minimal models induced by the inclusion $\tilde{G} \hookrightarrow G$ is surjective and the model of $\tilde{G}$ lies in the one for G.

So in these cases the question whether G/H is formal or not just depends on $(T', \mathrm{W}_{T'}, T_H)$, the way T_H includes into T' and the way $\mathrm{W}_{T'}$ acts upon T'; or in other words on $(\tilde{G}, T_H, \phi : T_H \hookrightarrow \tilde{G})$ only. ⊡

Let us now present one more result which serves to derive non-formality from some rather easily computable topological structures. This will simplify proofs for some of the known non-formal examples whilst producing a whole variety of further ones. We start with a more general lemma.

Lemma 3.24. *Let E^{2n+1} be a simply-connected compact manifold and let B^{2n} be a simply-connected compact Kähler manifold (for $n \geq 1$). Suppose there is a fibration $\mathbb{S}^1 \hookrightarrow E \xrightarrow{p} B$. Assume further the following to hold true:*

- *The Euler class of p is a non-vanishing multiple of the Kähler class l of B in rational cohomology.*
- *The rational cohomology of B is concentrated in even degrees only.*
- *The rational homotopy groups of B are concentrated in degrees smaller or equal to n.*

Then E is a non-formal space.

Proof. Since the Kähler class and the Euler class rationally are non-trivial multiples, we may formulate the Gysin sequence with rational coefficients for p as follows:

$$\cdots \to H^p(B) \xrightarrow{\cup l} H^{p+2}(B) \to H^{p+2}(E) \to H^{p+1}(B) \xrightarrow{\cup l} \ldots$$

(The fibration is oriented, since B is simply-connected.) The Hard-Lefschetz property of B implies that taking the cup-product with l is injective in degrees $p < n$. So the sequence splits, yielding

$$H^{p+2}(B) = H^p(B) \oplus H^{p+2}(E)$$

for $-2 \leq p \leq n-2$. Since we assumed the odd-dimensional rational cohomology groups of B to vanish we obtain in particular that

$$H^p(E) = 0 \qquad \text{for odd } p \leq n \tag{3.24}$$

Let $\left(\bigwedge V_E, \mathrm{d}_E\right)$ be a minimal model of E. As E was supposed to be simply-connected we clearly have $\pi_p(E) \otimes \mathbb{Q} \cong V_E^p$ up to duality. By the long exact sequence of homotopy groups we obtain

$$\pi_p(E) \cong \pi_p(B) \qquad \text{for } p \geq 3$$

As we assumed

$$\pi_p(B) \otimes \mathbb{Q} = 0 \qquad \text{for } p > n$$

we thus see that $V_E^{>n} = 0$ if $n \geq 2$. If $n = 1$, we clearly still have $V_E^{\geq 3}$=0. So for arbitrary $n \geq 1$ we obtain

$$V_E^p = 0 \qquad \text{for odd } p > n \tag{3.25}$$

Assume now that E is formal. We shall lead this to a contradiction. By theorem 1.41 we may split $V_E = C_E \oplus N_E$ with $C_E = \ker \mathrm{d}_E|_{V_E}$ and with the property that every closed element in $I(N_E)$ is exact.

By observation (3.24) we know that every closed element of odd degree in $\left(\bigwedge V_E\right)^{\leq n}$ is exact. Thus by the minimality of the model we directly derive that

$$(C_E)^{\leq n} = \left(C_E^{\leq n}\right)^{\text{even}} \tag{3.26}$$

Together with (3.25) this implies that

$$C_E = C_E^{\text{even}} \tag{3.27}$$

is concentrated in even degrees only. Hence so is $\bigwedge C_E$. Thus, using the splitting

$$\bigwedge V_E = \bigwedge C_E \oplus I(N_E)$$

we derive from (3.27) and the formality of E that

$$H^{\mathrm{odd}}(E)=\left(\frac{\ker \mathrm{d}_E|_{I(N)}}{\operatorname{im}\mathrm{d}_E}\right)^{\mathrm{odd}}=0 \tag{3.28}$$

However, the manifold E is an odd-dimensional simply-connected compact manifold. Thus it satisfies Poincaré Duality. It is orientable and possesses a volume form which hence generates $H^{2n+1}(E)\cong\mathbb{Q}\neq 0$. This contradicts formula (3.28). Thus E is non-formal. □

Remark 3.25.

- We see that the proof works equally well if the spaces E and B are not simply-connected but (have a finite fundamental group and) are *simple*, i.e. the action of the fundamental group on all homotopy groups (in positive degree) is trivial. Indeed, the spaces then are orientable and the Gysin sequence can be applied. Moreover, Rational Homotopy Theory is applicable to simple spaces without any change.

  From proposition [64].1.17, p. 84, we see that the homogeneous space G/H of a compact connected Lie group G with connected closed Lie subgroup H is simple.

- From the proof of the lemma it is clear that we actually do not need to require B to be Kählerian. It is absolutely sufficient for B to have the Hard-Lefschetz property with respect to the Euler class. For example, this extends the lemma to the case of certain Donaldson submanifolds (cf. [23]) or biquotients (cf. [42]).

  We may relax the prerequisites of the lemma even further: We need not require the rational homotopy groups of B to be concentrated in degrees smaller or equal to n as long as the ones concentrated in odd degrees above n correspond to relations in the minimal model of E via the long exact homotopy sequence; i.e. they are not in the image of the dual of the rationalised Hurewicz homomorphism (cf. 1.27). Then the proof does not undergo any severe adaptation: We just obtain $C_E^p=0$ instead of $V_E^p=0$ for odd $p>n$. This extends the lemma to the case of non-elliptic spaces E and B.

- In theory, this lemma is not restricted to $\mathbb{S}^1$-fibrations. So for example using the Kraines form in degree four instead of the Kähler form in degree 2 one may formulate a version for Positive Quaternion Kähler Manifolds. The formality of the 3-Sasakian manifolds associated to Wolf spaces, however, shows that such an attempt seems to be less fruitful due to the other prerequisites in the lemma. ⊡

See the corollary on [64].13, p. 221 and the examples below for a related result in the category of homogenous spaces. Let us now formulate the theorem that serves as the main motivation for the preceeding lemma.

For this we cite the following simply-connected compact homogeneous Kähler manifolds from [5].8.H, p. 229–234:

$$\frac{\mathbf{Sp}(n)}{\mathbf{U}(p_1)\times\cdots\times\mathbf{U}(p_q)\times\mathbf{Sp}(l)} \quad \text{for } \sum_{i=1}^q p_i + l = n$$

$$\frac{\mathbf{SO}(2n+1)}{\mathbf{U}(p_1)\times\cdots\times\mathbf{U}(p_q)\times\mathbf{SO}(2l+1)} \quad \text{for } \sum_{i=1}^q p_i + l = n$$

$$\frac{\mathbf{SO}(2n)}{\mathbf{U}(p_1)\times\cdots\times\mathbf{U}(p_q)\times\mathbf{SO}(2l)} \quad \text{for } \sum_{i=1}^q p_i + l = n \tag{3.29}$$

$$\frac{\mathbf{SO}(2n)}{\mathbf{U}(p_1)\times\cdots\times\mathbf{U}(p_{q-1})\times\tilde{\mathbf{U}}(p_q)} \quad \text{for } \sum_{i=1}^q p_i = n$$

$$\frac{\mathbf{SU}(n)}{\mathbf{S}(\mathbf{U}(p_1)\times\cdots\times\mathbf{U}(p_q))} \quad \text{for } \sum_{i=1}^q p_i = n$$

where $\tilde{\mathbf{U}}(p_q)\subseteq\mathbf{SO}(2p_q)$ is the unitary group with respect to a slightly altered complex structure of $\mathbb{R}^{2p_q}$ (cf. [5].8.113, p. 231).

In table 3.2 we shall use the convention that $\sum_{i=1}^t k_i = p$, $\sum_i l_i = l$ and $p_1+p_2=p$. Moreover, we also require $p, p_1, p_2 > 0$ in the table.

Theorem 3.26. *The homogeneous spaces in table 3.2 are non-formal.*

Proof. We shall consider a certain subclass of the spaces given in (3.29). Their top rational homotopy group can easily be computed using the long exact homotopy sequence of the fibration formed by denominator, numerator and quotient. For this we shall use that the top rational homotopy of $\mathbf{Sp}(n)$, $\mathbf{SU}(n)$, $\mathbf{SO}(2n)$, $\mathbf{SO}(2n+1)$ lies in degree $4n-1$, $2n-1$, $4n-5$ for $n\geq 2$ and $4n-1$ respectively. So we are able to give the spaces with their dimensions and with the largest degree of a non-vanishing rational homotopy group in table 3.3—as always $p, p_1, p_2 > 0$.

Theses homogeneous spaces have the property that numerator and denominator form an equal rank pair, whence the Euler characteristic is positive and cohomology is concentrated in even degrees only (cf. 1.53).

All these manifolds are compact homogeneous Kähler manifolds by [5].8.H. Since for the following arguments the spaces one to four behave similarly, we shall do showcase computations for the cases one and five.

The fibre bundle

$$\mathbf{SU}(p)\hookrightarrow\mathbf{U}(p)\xrightarrow{\det}\mathbb{S}^1$$

lets us conclude that $\mathbf{U}(p)=\mathbb{S}^1\cdot\mathbf{SU}(p)=\mathbb{S}^1\times_{\mathbb{Z}_p}\mathbf{SU}(p)$, where $\mathbb{Z}_p$ is the (multiplicative) cyclic group of p-th roots of unity acting on each factor by (left) multiplication. So

Table 3.2.: Examples of non-formal homogeneous spaces

homogeneous space	with
$\frac{\mathbf{Sp}(N)}{\mathbf{S}(\mathbf{U}(k_1)\times\cdots\times\mathbf{U}(k_t))\times\mathbf{Sp}(l_1)\times\cdots\times\mathbf{Sp}(l_r)\times\mathbf{U}(l_{r+1})\times\cdots\times\mathbf{U}(l_s)}$	$p+l\leq N,\ r\geq 0,\ s\geq 0$ $\frac{1}{2}p^2+(2l-\frac{7}{2})p-4l+1\geq 0$
$\frac{\mathbf{SU}(N)}{\mathbf{S}(\mathbf{U}(k_1)\times\cdots\times\mathbf{U}(k_t))\times\mathbf{Sp}(l_1)\times\cdots\times\mathbf{Sp}(l_r)\times\mathbf{U}(l_{r+1})\times\cdots\times\mathbf{U}(l_s)}$	$2(p+l)\leq N,\ r\geq 0,\ s\geq 0$ $\frac{1}{2}p^2+(2l-\frac{7}{2})p-4l+1\geq 0$
$\frac{\mathbf{SO}(N)}{\mathbf{S}(\mathbf{U}(k_1)\times\cdots\times\mathbf{U}(k_t))\times\mathbf{SO}(2l_1)\times\cdots\times\mathbf{SO}(2l_r)\times\mathbf{SO}(2l_{r+1}+1)}$	$2(p+l)+1\leq N,\ r+1\geq 0$ $\frac{1}{2}p^2+(2l-\frac{7}{2})p-4l+1\geq 0$
$\frac{\mathbf{SU}(N)}{\mathbf{S}(\mathbf{U}(k_1)\times\cdots\times\mathbf{U}(k_t))\times\mathbf{SO}(2l_1)\times\cdots\times\mathbf{SO}(2l_r)\times\mathbf{SO}(2l_{r+1}+1)}$	$p+l=n,\ N\geq 2n+1,\ r+1\geq 0$ $\frac{1}{2}p^2+(2l-\frac{7}{2})p-4l+1\geq 0$
$\frac{\mathbf{SO}(2n)}{\mathbf{S}(\mathbf{U}(k_1)\times\cdots\times\mathbf{U}(k_t))\times\mathbf{SO}(2l_1)\times\cdots\times\mathbf{SO}(2l_r)}$	$p+l=n,\ r\geq 0,\ n\geq 2,\ l\neq 1$ $\frac{1}{2}p^2+(2l-\frac{9}{2})p-4l+5\geq 0$
$\frac{\mathbf{SO}(2n)}{\mathbf{S}(\tilde{\mathbf{U}}(k_1)\times\cdots\times\tilde{\mathbf{U}}(k_t))}$	$p=n,\ r\geq 0,\ n\geq 2$ $\frac{1}{2}n^2-\frac{9}{2}n+5\geq 0$
$\frac{\mathbf{SU}(N)}{\mathbf{S}(\mathbf{U}(k_1)\times\cdots\times\mathbf{U}(k_s))\times\mathbf{S}(\mathbf{U}(k_{s+1})\times\cdots\times\mathbf{U}(k_t))}$	$p_1+p_2=n\leq N,\ p_1,p_2>0$ $N\geq 2,\ p_1p_2-2(p_1+p_2)+1\geq 0$

the canonical projection yields a fibre bundle

$$\mathbb{S}^1/\mathbb{Z}_p \hookrightarrow \frac{\mathbf{Sp}(n)}{(\mathbb{Z}_p\times_{\mathbb{Z}_p}\mathbf{SU}(p))\times\mathbf{Sp}(l)} \to \frac{\mathbf{Sp}(n)}{(\mathbb{S}_1\times_{\mathbb{Z}_p}\mathbf{SU}(p))\times\mathbf{Sp}(l)}$$

of homogeneous spaces, which clearly is no other than

$$\mathbb{S}^1/\mathbb{Z}_p \hookrightarrow \frac{\mathbf{Sp}(n)}{\mathbf{SU}(p)\times\mathbf{Sp}(l)} \to \frac{\mathbf{Sp}(n)}{\mathbf{U}(p)\times\mathbf{Sp}(l)}$$

since $\mathbb{Z}_p\times_{\mathbb{Z}_p}\mathbf{SU}(p)\xrightarrow{\cong}\mathbf{SU}(p)$—given by left multiplication of $\mathbb{Z}_p$ on $\mathbf{SU}(p)$—is an isomorphism.

In case five we obtain a homomorphism

$$\begin{aligned}\mathbb{S}^1\times\mathbf{SU}(p_1)\times\mathbf{SU}(p_2) &\to \mathbf{S}(\mathbf{U}(p_1)\times\mathbf{U}(p_2))\\ (x,A,B) &\mapsto (x^{p_2}\cdot A, x^{-p_1}\cdot B)\end{aligned}$$

with kernel formed by all the elements of the form $(a,a^{-p_2}I_{p_1},a^{p_1}I_{p_2})$ with $a^{p_1p_2}=1$. Thus the kernel is isomorphic to $\mathbb{Z}_{p_1p_2}$. (It is consequent to propose that

Table 3.3.: Certain homogeneous spaces

homogeneous space	for	dimension	top rat. homotopy
$\frac{\mathbf{Sp}(n)}{\mathbf{U}(p)\times\mathbf{Sp}(l)}$	$p+l=n$	$p^2+4\cdot l\cdot p+p$	$4n-1$
$\frac{\mathbf{SO}(2n+1)}{\mathbf{U}(p)\times\mathbf{SO}(2l+1)}$	$p+l=n$	$p^2+4\cdot l\cdot p+p$	$4n-1$
$\frac{\mathbf{SO}(2n)}{\mathbf{U}(p)\times\mathbf{SO}(2l)}$	$p+l=n$	$p^2+4\cdot l\cdot p-p$	$4n-5$ for $n\geq 2$
$\frac{\mathbf{SO}(2n)}{\tilde{\mathbf{U}}(n)}$		n^2-n	$4n-5$ for $n\geq 2$
$\frac{\mathbf{SU}(n)}{\mathbf{S}(\mathbf{U}(p_1)\times\mathbf{U}(p_2))}$	$p_1+p_2=n$	$2\cdot p_1\cdot p_2$	$2n-1$

$\mathbb{Z}_1 = 1$ be the trivial group.) In particular, this defines an action of $\mathbb{Z}_{p_1p_2} \subseteq \mathbb{S}^1$ on $\mathbf{SU}(p_1)\times\mathbf{SU}(p_2)$ and we obtain

$$\mathbf{S}(\mathbf{U}(p_1)\times\mathbf{U}(p_2)) = \mathbb{S}^1 \times_{\mathbb{Z}_{p_1p_2}} (\mathbf{SU}(p_1)\times\mathbf{SU}(p_2))$$

This leads to the fibre bundle

$$\mathbb{S}^1/\mathbb{Z}_{p_1p_2} \hookrightarrow \frac{\mathbf{SU}(n)}{\mathbb{Z}_{p_1p_2}\times_{\mathbb{Z}_{p_1p_2}}(\mathbf{SU}(p_1)\times\mathbf{SU}(p_2))} \to \frac{\mathbf{SU}(n)}{\mathbb{S}_1\times_{\mathbb{Z}_{p_1p_2}}(\mathbf{SU}(p_1)\times\mathbf{SU}(p_2))}$$

of homogeneous spaces, which clearly is no other than

$$\mathbb{S}^1/\mathbb{Z}_{p_1p_2} \hookrightarrow \frac{\mathbf{SU}(n)}{\mathbf{SU}(p_1)\times\mathbf{SU}(p_2)} \to \frac{\mathbf{SU}(n)}{\mathbf{S}(\mathbf{U}(p)\times\mathbf{SU}(p_2)}$$

Thus, in each case identifying $\mathbb{S}^1/\mathbb{Z}_p$ (for each respective p) with its finite covering $\mathbb{S}^1$, we obtain the following sphere bundles:

$$\mathbb{S}^1 \hookrightarrow \frac{\mathbf{Sp}(n)}{\mathbf{SU}(p)\times\mathbf{Sp}(l)} \to \frac{\mathbf{Sp}(n)}{\mathbf{U}(p)\times\mathbf{Sp}(l)}$$
$$\mathbb{S}^1 \hookrightarrow \frac{\mathbf{SO}(2n+1)}{\mathbf{SU}(p)\times\mathbf{SO}(2l+1)} \to \frac{\mathbf{SO}(2n+1)}{\mathbf{U}(p)\times\mathbf{SO}(2l+1)}$$
$$\mathbb{S}^1 \hookrightarrow \frac{\mathbf{SO}(2n)}{\mathbf{SU}(p)\times\mathbf{SO}(2l)} \to \frac{\mathbf{SO}(2n)}{\mathbf{U}(p)\times\mathbf{SO}(2l)}$$
$$\mathbb{S}^1 \hookrightarrow \frac{\mathbf{SO}(2n)}{\mathbf{S}\tilde{\mathbf{U}}(n)} \to \frac{\mathbf{SO}(2n)}{\tilde{\mathbf{U}}(n)}$$
$$\mathbb{S}^1 \hookrightarrow \frac{\mathbf{SU}(n)}{\mathbf{SU}(p_1)\times\mathbf{SU}(p_2)} \to \frac{\mathbf{SU}(n)}{\mathbf{S}(\mathbf{U}(p_1)\times\mathbf{U}(p_2))}$$

Due to the long exact homotopy sequence we see that the total space E of each bundle has a finite fundamental group, namely 0 or $\mathbb{Z}_2$. Thus the Euler class of each bundle does not vanish, since otherwise the bundle would be rationally trivial. This would imply $E \simeq_{\mathbb{Q}} B \times \mathbb{S}^1$—with the respective base space B—and $\pi_1(E) \otimes \mathbb{Q} \neq 0$.

The long exact homotopy sequence associated to $H \hookrightarrow G \to G/H$—where G is the numerator and H is the denominator group of the base space $B = G/H$ in each respective case—lets us conclude that

$$H^2(B) \cong \pi_2(G/H) \otimes \mathbb{Q} = \mathbb{Q}$$

for $l \neq 1$ and $n \geq 2$ in case three, for $n \geq 2$ in case four and without further restrictions in the other cases. (In each respective case we then have $\pi_2(G) \otimes \mathbb{Q} = 0$, $\pi_1(H) \otimes \mathbb{Q} = \mathbb{Q}$ and $\pi_1(G) \otimes \mathbb{Q} = 0$.)

This implies that the Kähler class and the Euler class—both contained in $H^2(B)$—are non-zero multiples in rational cohomology.

In order to apply lemma 3.24 we need to demand that the top rational homotopy does not lie above half the dimension of the space. This leads to the following restrictions in the respective cases:

$$\begin{aligned}
\frac{1}{2}p^2 + (2l - \frac{7}{2})p - 4l + 1 &\geq 0 \\
\frac{1}{2}p^2 + (2l - \frac{7}{2})p - 4l + 1 &\geq 0 \\
\frac{1}{2}p^2 + (2l - \frac{9}{2})p - 4l + 5 &\geq 0 \\
\frac{1}{2}n^2 - \frac{9}{2}n + 5 &\geq 0 \\
p_1p_2 - 2(p_1 + p_2) + 1 &\geq 0
\end{aligned}$$

By the first point in remark 3.25 we may now apply lemma 3.24 to the depicted fibre bundles. The lemma yields that the total spaces

$$\frac{\mathbf{Sp}(n)}{\mathbf{SU}(p) \times \mathbf{Sp}(l)}, \frac{\mathbf{SO}(2n+1)}{\mathbf{SU}(p) \times \mathbf{SO}(2l+1)}, \frac{\mathbf{SO}(2n)}{\mathbf{SU}(p) \times \mathbf{SO}(2l)},$$
$$\frac{\mathbf{SO}(2n)}{\tilde{\mathbf{SU}}(n)}, \frac{\mathbf{SU}(n)}{\mathbf{SU}(p_1) \times \mathbf{SU}(p_2)}$$

of the respective bundles are non-formal under the given conditions.

Due to theorem 1.45 we may replace the stabilisers of the total spaces of the fibrations by maximal rank subgroups sharing the maximal torus. We apply this first to the $\mathbf{SU}(p)$-factor which arose from the fibre-bundle construction and replace it by a $\mathbf{S}(\mathbf{U}(k_1) \times \cdots \times \mathbf{U}(k_t))$. Then we also substitute the other factors by suitable other ones.

Finally, we are done by an application of proposition 3.18 respectively table 3.1, which allows us to use the described embeddings of the numerator into larger Lie groups to form new homogeneous spaces. The result of this process is depicted in table 3.2:

The first example we considered here leads to the first line in 3.2 by blockwise inclusion $\mathbf{Sp}(n) \hookrightarrow \mathbf{Sp}(N)$ and to the second line by the inclusion

$$\mathbf{Sp}(n) \hookrightarrow \mathbf{SU}(2n) \hookrightarrow \mathbf{SU}(N)$$

(which is identical to $\mathbf{Sp}(n) \hookrightarrow \mathbf{Sp}(N') \hookrightarrow \mathbf{SU}(2N') \hookrightarrow \mathbf{SU}(N)$).

The second example produces lines three and four by the inclusions $\mathbf{SO}(2n+1) \hookrightarrow \mathbf{SO}(N)$ and

$$\mathbf{SO}(2n+1) \hookrightarrow \mathbf{SU}(2n+1) \hookrightarrow \mathbf{SU}(N)$$

(which is the same as $\mathbf{SO}(2n+1) \hookrightarrow \mathbf{SO}(N') \hookrightarrow \mathbf{SU}(2N') \hookrightarrow \mathbf{SU}(N)$).

For examples three and four we do not use any further inclusions. Example five produces line seven by the inclusion $\mathbf{SU}(n) \hookrightarrow \mathbf{SU}(N)$. □

The proof guides the way of how to interpret table 3.2. Nonetheless, let us—exemplarily for the first line of the table—shed some more light upon the used inclusions: The inclusion of the denominator in the first example is given by blockwise inclusions of

$$\begin{aligned}
\mathbf{S}(\mathbf{U}(k_1) \times \cdots \times \mathbf{U}(k_t)) &\hookrightarrow \mathbf{Sp}(p) \hookrightarrow \mathbf{Sp}(n) \hookrightarrow \mathbf{Sp}(N) \\
\mathbf{Sp}(l_1) \times \cdots \times \ldots \mathbf{Sp}(l_r) \hookrightarrow \mathbf{Sp}\Bigg(\sum_{i=1}^{r} l_i\Bigg) &\hookrightarrow \mathbf{Sp}(n) \hookrightarrow \mathbf{Sp}(N) \\
\mathbf{U}(l_{r+1}) \times \cdots \times \ldots \mathbf{U}(l_s) \hookrightarrow \mathbf{Sp}\Bigg(\sum_{i=r+1}^{s} l_i\Bigg) &\hookrightarrow \mathbf{Sp}(n) \hookrightarrow \mathbf{Sp}(N)
\end{aligned}$$

where $n = p + \sum_{i=1}^{s} l_i$.

In examples three and four it does not matter whether or not we have a stabiliser which has—beside the unitary part—special orthogonal groups of the type $\mathbf{SO}(2l_i)$ only, if there is additionally one factor of the form $\mathbf{SO}(2l_{r+1}+1)$ or if there is the factor $\mathbf{SO}(2l_r+1)$ only: All three cases establish maximal rank subgroups of $\mathbf{SO}(2l+1)$, which therefore share the maximal torus with $\mathbf{SO}(2l+1)$.

Moreover, note that the relations in table 3.2 are growing quadratically in p. Thus for a constant l we can always find p and N that will satisfy the restrictions.

Example 3.27. Let us describe some of the simplest and most classical consequences that arise out of our reasoning. We have proved that the spaces

$$\frac{\mathbf{Sp}(n)}{\mathbf{SU}(n)} \qquad \frac{\mathbf{SO}(2n)}{\mathbf{SU}(n)} \qquad \frac{\mathbf{SU}(p+q)}{\mathbf{SU}(p) \times \mathbf{SU}(q)}$$

are non-formal for $n \geq 7$, $n \geq 8$ and $p+q \geq 4$ respectively.

⊡

We observe that the homogeneous 3-Sasakian manifolds—fibring over the twistor space of a Positive Quaternion Kähler Manifold—do not satisfy the assumption on the dimension in lemma 3.24—although they are quite close to it. Their formality shows that the prerequisites in the lemma cannot be relaxed too much. In the same direction goes the following proposition: The prerequisites in the last example in table 3.2 for the space to be non-formal are evidently satisfied if $p_1, p_2 \geq 4$ or if $p \geq 3$ and $p \geq 5$. These are sharp bounds:

Proposition 3.28. *The homogeneous space*

$$\frac{\mathbf{SU}(7)}{\mathbf{SU}(3) \times \mathbf{SU}(4)}$$

is formal.

Proof. Using the model (3.21) we compute a minimal model for the given homogeneous space as generated by

$$c,\ c',\ \tilde{x} \qquad \deg c = 4,\ \ \deg c' = 6,\ \ \deg \tilde{x} = 13$$

and by relations

$$x_1,\ x_2 \qquad \deg x_1 = 9,\ \ \deg x_2 = 11$$

with

$$\mathrm{d}(x_1) = c \cdot c',\ \mathrm{d}(x_2) = c^3 + (c')^2$$

This model is formal. □

As for the interlink between formality and geometry we remark that, for example, from the computations in lemma [4].8.2, p. 272, we directly see that all known odd-dimensional examples—cf. proposition[4].8.1, p. 271—of positively curved Riemannian manifolds are formal spaces.

The importance of formality issues in geometry is also stressed by the following result in [4]. Non-formality of $C := \frac{\mathbf{SU}(6)}{\mathbf{SU}(3)\times\mathbf{SU}(3)}$ plays a crucial role for the construction of a counterexample (cf. [4], theorem 1.4): It gives rise to the existence of a non-negatively curved vector bundle over $C \times T$ (with a torus T of $\dim T \geq 2$) which does not split in a certain sense as a product of bundles over the factors of the base space, but which has the property that its total space admits a complete metric of non-negative sectional curvature with the zero section is a soul.

Moreover, we remark that we have found simply-connected examples of homogeneous spaces M_1 and M_2 with the property that $\chi(M_2) > 0$—i.e. M_2 is formal in particular (cf. 1.52)—and that M_1 is not formal whilst they fibre as

$$\mathbb{S}^1 \hookrightarrow M_1 \to M_2$$

However, it is not possible to find manifolds M_1 and M_2 as described that fibre in the form

$$M_2 \hookrightarrow M_1 \to \mathbb{S}^1$$

This is due to the fact that the Halperin conjecture (cf. p. 71) has been confirmed on homogeneous spaces (cf. [69], theorem A, p. 81) and theorem [56].3.4, p. 12.

It was established that simply-connected compact manifolds of dimension at most six are formal spaces. In [24] Fernández and Muñoz pose the question whether there are non-formal simply-connected compact manifolds in dimensions seven and higher. They answer this in the affirmative by constructing seven-dimensional and eight-dimensional examples. The requested example in a certain dimension above dimension seven is then obviously given by a direct product with the corresponding even-dimensional sphere. This leads to the problem of constructing highly-connected examples (cf. for example [26]). Note that the methods of constructing these manifolds involve surgery theory, in particular.

We shall use a slightly different approach towards this issue by using homogeneous spaces.

Proposition 3.29. *In every dimension $d \geq 72$ there is an irreducible simply connected compact homogeneous space which is not formal. In particular, every such space constitutes a rationally elliptic non-formal example.*

Proof. The example

$$\frac{\mathbf{SU}(N)}{\mathbf{S}(\mathbf{U}(k_1) \times \cdots \times \mathbf{U}(k_s)) \times \mathbf{S}(\mathbf{U}(k_{s+1}) \times \cdots \times \mathbf{U}(k_t))}$$

from table 3.2 will serve us as a main source of further examples. (It is simply-connected as so is the numerator group.)

The restrictions in the table will certainly be satisfied if $p_1, p_2 \geq 4$ or if $p_1 \geq 3$ and $p_2 \geq 5$ (respectively $N \geq p_1 + p_2$). By our initial computations we may also use the case for $p_1 = p_2 = 3$. We use a slightly different terminology: Set $p := p_1$, $k := p_2$ and

$N \geq 0$. Thus whenever $p = k = 3$ or $p, k \geq 4$ or $p \geq 3$ and $k \geq 5$ the spaces

$$\frac{\mathbf{SU}(p+k+N)}{\mathbf{SU}(p) \times \mathbf{SU}(k)} \tag{3.30}$$

$$\frac{\mathbf{SU}(p+k+N)}{\mathbf{SU}(p) \times \mathbf{S}(\mathbf{U}(k-1) \times \mathbf{U}(1))} \tag{3.31}$$

$$\frac{\mathbf{SU}(p+k+N)}{\mathbf{S}(\mathbf{U}(p-1) \times \mathbf{U}(1)) \times \mathbf{SU}(k)} \tag{3.32}$$

$$\frac{\mathbf{SU}(p+k+N)}{\mathbf{S}(\mathbf{U}(p-1) \times \mathbf{U}(1)) \times \mathbf{S}(\mathbf{U}(k-1) \times \mathbf{U}(1))} \tag{3.33}$$

$$\frac{\mathbf{SU}(p+k+N)}{\mathbf{S}(\mathbf{U}(p-2) \times \mathbf{U}(2)) \times \mathbf{SU}(k)} \tag{3.34}$$

$$\frac{\mathbf{SU}(p+k+N)}{\mathbf{S}(\mathbf{U}(p-2) \times \mathbf{U}(2)) \times \mathbf{S}(\mathbf{U}(k-1) \times \mathbf{U}(1))} \tag{3.35}$$

$$\frac{\mathbf{SU}(p+k+N)}{\mathbf{S}(\mathbf{U}(p-2) \times \mathbf{U}(2)) \times \mathbf{S}(\mathbf{U}(k-2) \times \mathbf{U}(1) \times \mathbf{U}(1))} \tag{3.36}$$

$$\frac{\mathbf{SU}(p+k+N)}{\mathbf{S}(\mathbf{U}(p-2) \times \mathbf{U}(1) \times \mathbf{U}(1)) \times \mathbf{SU}(k)} \tag{3.37}$$

$$\frac{\mathbf{SU}(p+k+N)}{\mathbf{S}(\mathbf{U}(p-2) \times \mathbf{U}(1) \times \mathbf{U}(1)) \times \mathbf{S}(\mathbf{U}(k-2) \times \mathbf{U}(2))} \tag{3.38}$$

$$\frac{\mathbf{SU}(p+k+N)}{\mathbf{S}(\mathbf{U}(p-2) \times \mathbf{U}(1) \times \mathbf{U}(1)) \times \mathbf{S}(\mathbf{U}(k-2) \times \mathbf{U}(1) \times \mathbf{U}(1))} \tag{3.39}$$

will be non-formal. In the given order the spaces have the following dimensions:

$$\begin{aligned} 2(k+N)p + (2kN + N^2 + 1) \\ 2(k+N)p + (2kN + N^2 + 2k - 1) \\ 2(k+N+2)p + (2kN + N^2 - 1) \\ 2(k+N+2)p + (2kN + N^2 + 2k - 3) \\ 2(k+N+4)p + (2kN + N^2 - 7) \\ 2(k+N+4p + (2kN + N^2 + 2k - 9) \\ 2(k+N+4)p + (2kN + N^2 + 4k - 13) \\ 2(k+N+4)p + (2kN + N^2 - 5) \\ 2(k+N+4)p + (2kN + N^2 + 4k - 13) \\ 2(k+N+4)p + (2kN + N^2 + 4k - 11) \end{aligned} \tag{3.40}$$

Regard the manifolds as parametrised over $\mathbb{N}$ by the variable p. So we have the infinite sequences $(M_p^{k,N})_{p\in\mathbb{N}}^{(3.30)}, \ldots, (M_p^{k,N})_{p\in\mathbb{N}}^{(3.39)}$. We shall now show that for each congruence class $[m]$ modulo 16 there is a family $(M_p^{k,N})_{p\in\mathbb{N}}^m$ which—for certain numbers p—consists of non-formal manifolds only and which has the property that

$$\dim(M_p^{k,N})^m_{p\in\mathbb{N}} \equiv [m] \mod 16$$

For this we fix the coefficient of p in each dimension in (3.40) to be $2 \cdot 8 = 16$. The following series realise the congruence classes $0, \dots, 15$. In the third column we give the smallest dimension for which there is a $p \in \mathbb{N}$ making the space non-formal. (From this dimension on the series will produce non-formal examples only.) Once we are given p we may compute this dimension. We determine p due to the following rule: If $k = 3$ set $p = 3$, if $k = 4$ set $p = 4$, if $k \geq 5$ set $p = 3$. This guarantees non-formality. The necessary data is given by the subsequent table:

$[m]$	series	starting dimension
0	$(M_p^{3,1})^{(3.34)}_{p\in\mathbb{N}}$	48
1	$(M_p^{8,0})^{(3.30)}_{p\in\mathbb{N}}$	49
2	$(M_p^{5,1})^{(3.37)}_{p\in\mathbb{N}}$	50
3	$(M_p^{4,0})^{(3.38)}_{p\in\mathbb{N}}$	67
4	$(M_p^{3,1})^{(3.35)}_{p\in\mathbb{N}}$	52
5	$(M_p^{4,0})^{(3.39)}_{p\in\mathbb{N}}$	69
6	$(M_p^{3,1})^{(3.36)}_{p\in\mathbb{N}}$	54
7	$(M_p^{6,2})^{(3.31)}_{p\in\mathbb{N}}$	87
8	$(M_p^{3,1})^{(3.39)}_{p\in\mathbb{N}}$	56
9	$(M_p^{4,0})^{(3.34)}_{p\in\mathbb{N}}$	57
10	$(M_p^{5,1})^{(3.32)}_{p\in\mathbb{N}}$	58
11	$(M_p^{4,0})^{(3.37)}_{p\in\mathbb{N}}$	59
12	$(M_p^{7,1})^{(3.31)}_{p\in\mathbb{N}}$	76
13	$(M_p^{6,2})^{(3.30)}_{p\in\mathbb{N}}$	77
14	$(M_p^{6,2})^{(3.33)}_{p\in\mathbb{N}}$	78
15	$(M_p^{6,0})^{(3.32)}_{p\in\mathbb{N}}$	47

From dimension 72 onwards every congruence class modulo 16 can be realised by one of the given spaces. So we are done. $\square$

Clearly, this theorem is far from being optimal and can easily be improved by just stepping through table 3.2 or by doing explicit calculations as we did at the beginning of the section. Indeed, in the proof we have found non-formal homogeneous spaces in dimensions

$$47, 48, 49, 50, 52, 54, 56, 57, 58, 59, 63, 64, 65, 66, 67, 68, 69, 70$$

and from dimension 72 onwards.

We remark that one clearly may use other non-formal series like $\mathbf{Sp}(n)/\mathbf{SU}(n)$ in order to establish similar results. Since the number of partitions of a natural number $n \in \mathbb{N}$ is growing exponentially in n one will find many stabilisers sharing their maximal tori with $\mathbf{SU}(n)$.

As we pointed out, compact homogeneous spaces are rationally elliptic. Thus the question that arises naturally is: In which dimensions are there examples of non-formal elliptic simply-connected irreducible compact manifolds?

We end this section by partly generalising theorem 1.45 which asserts that the formality of a homogeneous space G/H of a connected compact Lie group G and a connected subgroup H is equivalent to the formality of G/T, where $T \subseteq H$ is a maximal torus of H. We shall show that one direction can easily be seen to hold true on a biquotient as well.

For this recall that whenever H acts freely on a topological space X we obtain the fibre bundle:

$$H/T \hookrightarrow X/T \xrightarrow{p} X/H \tag{3.41}$$

where p is the canonical projection $[x] \mapsto [\![x]\!]$. Indeed, the fibre $F_{[\![x]\!]}$ at a point $[\![x]\!]$ is

$$\begin{aligned} F_{[\![x]\!]} &= \{[y] \in X/T \mid p([y]) = [\![x]\!]\} \\ &= \big\{[y] \in X/T \mid \exists h_{[y]} \in H : h_{[y]} \cdot [y] = [x]\big\} \end{aligned}$$

where H acts on X/T by $h \cdot [x] = [h \cdot x]$. The assignment

$$\begin{aligned} F_{[\![x]\!]} &\to H/T \\ [y] &\mapsto h_{[y]}T \end{aligned}$$

is well-defined and a homeomorphism.

Proposition 3.30. *Let G be a connected compact Lie group and let H be a connected closed subgroup of $G \times G$ with maximal torus $T \subseteq H$ of H. Suppose one of $G/\!\!/H$ and $G/\!\!/T$ to be simply-connected.*

Then we obtain: If the biquotient $G/\!\!/H$ is formal, so is $G/\!\!/T$.

PROOF. Due to (3.41) we form the fibre bundle

$$H/T \hookrightarrow G/\!\!/T \to G/\!\!/H$$

of simply-connected spaces. Indeed, by proposition [64].1.19, p. 84, the homogeneous space H/T is simply-connected, as $\operatorname{rk} T = \operatorname{rk} H$. The long exact homotopy sequence then yields $\pi_1(G/\!\!/T) \cong \pi_1(G/\!\!/H)$. One of these two groups vanishes by assumption and so do both of them.

The homogeneous space H/T satisfies the Halperin conjecture (cf. [69], theorem A, p. 81). Thus every fibration of simply-connected spaces with fibre H/T is totally non-cohomologous to zero. Theorem [56].3.4, p. 12, then yields the result. $\square$

3.3. Pure models and Lefschetz-like properties

The last section was dedicated to the interplay of homogeneity and formality. In this section we shall generalise the setting to pure Sullivan models. Recall that a Sullivan algebra $\left(\bigwedge V, \mathrm{d}\right)$ is called *pure* if V is finite-dimensional and if $\bigwedge V = \bigwedge V^{\mathrm{even}} \otimes \bigwedge V^{\mathrm{odd}}$ with $\mathrm{d}|_{V^{\mathrm{even}}} = 0$ and $\mathrm{d}(V^{\mathrm{odd}}) \subseteq \bigwedge V^{\mathrm{even}}$—cf. definition 1.51.

Hence, in geometric terms, we no longer focus on homogeneous spaces, since for example also biquotients enter the stage (cf. 1.53).

A second special feature which we shall take into consideration will be a Lefschetz-like property.

The classical Hard-Lefschetz property for Kähler manifolds M^{2n} states that

$$\cup[\omega]^{n-k} : H^k(M) \xrightarrow{\cong} H^{2n-k}(M) \tag{3.42}$$

is an isomorphism for all $0 \leq k \leq n$. The form ω is the Kähler form of degree two. On Positive Quaternion Kähler Manifolds M^{4n} we have an analogue with the Kraines form ω of degree four (cf. 1.13):

$$\cup[\omega]^{n-k} : H^{2k}(M) \xrightarrow{\cong} H^{4n-2k}(M) \tag{3.43}$$

(which is as powerful as the corresponding property for Kähler manifolds, since odd degree rational cohomology is known to vanish—cf. 1.13.) In each case a direct consequence of the Hard-Lefschetz property is that the volume form v is a multiple of $[\omega]^n$. In terms of a minimal Sullivan model $\left(\bigwedge V, \mathrm{d}\right)$ of M this will imply—at least in the simply-connected case—that v may be regarded as lying in $\bigwedge V^{\mathrm{even}}$, which, in general, is a weaker statement.

Similar but weaker properties than the Hard-Lefschetz properties can be found on Riemannian manifolds with special holonomy—such as Joyce manifolds.

Thus we shall consider special topological properties that have effects in very concrete geometric situations. Consequently, as one possible "dictionary" we suggest

geometry	$\leftrightsquigarrow$	algebra
compact biquotient	$\leftrightsquigarrow$	pure Sullivan model (with non-degenerate cup product pairing)
(non-symmetric) special holonomy	$\leftrightsquigarrow$	(Hard-)Lefschetz-like property

So let us begin with an investigation on formality of spaces with pure models by means of the restrictions imposed by a (Hard-)Lefschetz-like property.

In [42] and [4] the question arises when a biquotient is a Kähler manifold. The answer to that is unknown even for simplest examples as $\mathbb{S}^1\backslash\mathbf{U}(3)/T^2$—which satisfies Hard-Lefschetz. The main motivation for this is that by a result of Meier's (cf. [58], [59]) Hard-Lefschetz spaces—and compact Kähler manifolds, in particular—satisfy the Halperin conjecture (cf. p. 71). Moreover, it is not clear at all whether just simple examples like $\mathbb{S}^1\backslash\mathbf{Sp}(n)/\mathbf{SU}(n)$ have the Hard-Lefschetz property. So the latter deserves some more attention.

We commence our considerations with the following lemma (which is said to hold on homogeneous spaces in [4], p. 280).

Proposition 3.31. *A pure minimal Sullivan algebra* $(\bigwedge V, \mathrm{d})$ *is formal if and only if it is of the form*

$$\big(\bigwedge V, \mathrm{d}\big) \cong \big(\bigwedge V', \mathrm{d}\big) \otimes \big(\bigwedge\langle z_1, \ldots, z_l\rangle, 0\big) \tag{3.44}$$

(for maximal such l) with a pure minimal Sullivan algebra $(\bigwedge V', \mathrm{d})$ *of positive Euler characteristic—which is automatically formal then—and with odd degree generators* z_i.

Proof. Choose a basis $z_1, \ldots, z_m$ of V^{odd} with the property that $\mathrm{d}z_i = 0$ for $1 \leq i \leq l$—and some fixed $1 \leq l \leq m$—and that $\mathrm{d}|_{\langle z_{l+1},\ldots,z_m\rangle}$ is injective. From pureness it follows:

$$\mathrm{d}(\langle z_1, \ldots, z_l\rangle) = 0$$
$$\mathrm{d}(\langle z_{l+1}, \ldots, z_m\rangle) \in \bigwedge V^{\mathrm{even}}$$
$$\mathrm{d}(V^{\mathrm{even}}) = 0$$

Thus we obtain

$$\big(\bigwedge V, \mathrm{d}\big) = \big(\bigwedge(V^{\mathrm{even}} \oplus \langle z_{l+1}, \ldots, z_m\rangle), \mathrm{d}\big) \otimes \big(\bigwedge\langle z_1, \ldots, z_l\rangle, 0\big)$$

Set $V' := V^{\mathrm{even}} \oplus \langle z_{l+1}, \ldots, z_m\rangle$. The minimality of $\big(\bigwedge V, \mathrm{d}\big)$ enforces the minimality of $\big(\bigwedge V', \mathrm{d}\big)$ and $\big(\bigwedge V', \mathrm{d}\big)$ is again a pure minimal Sullivan algebra. (It is a Sullivan algebra as the filtration of V that made $\big(\bigwedge V, \mathrm{d}\big)$ a Sullivan algebra restricts to a filtration on V'—cf. the definition on [22].12, p. 138—since both $\big(\bigwedge V', \mathrm{d}\big)$ and $\big(\bigwedge\langle z_{l+1}, \ldots, z_m\rangle, 0\big)$ are differential subalgebras.)

By theorem 1.47 the formality of $\big(\bigwedge V, \mathrm{d}\big)$ now is equivalent to the formality of $(\bigwedge V', \mathrm{d})$.

If $\big(\bigwedge V', \mathrm{d}\big)$ has positive Euler characteristic, it is formal by theorem 1.52, since it is positively elliptic. Thus $\big(\bigwedge V, \mathrm{d}\big)$ then is formal, too.

For the reverse implication we use contraposition: Suppose the Euler characteristic of $\big(\bigwedge V', \mathrm{d}\big)$ to vanish. We shall show that the algebra is non-formal. The algebra $\big(V', \mathrm{d}\big)$ has the property that $\mathrm{d}((V')^{\mathrm{even}}) = 0$ and that $\mathrm{d}|_{(V')^{\mathrm{odd}}}$ is injective. Thus, setting $C := (V')^{\mathrm{even}}$ and $N := (V')^{\mathrm{odd}}$, yields a decomposition $V' = C \oplus N$ as required in theorem 1.41 and this decomposition is uniquely determined.

Due to $\chi\big(\bigwedge V', \mathrm{d}\big) = 0$ there is a closed and non-exact element x in $\big(\bigwedge V', \mathrm{d}\big)$ that has odd degree. Therefore x necessarily lies in $I(N)$. Thus $\big(\bigwedge V', \mathrm{d}\big)$ is not formal and neither is $\big(\bigwedge V, \mathrm{d}\big)$. $\square$

By abuse of notation we shall speak of volume forms as top cohomology generators of simply-connected elliptic Sullivan algebras. Volume forms then need to be unique up to multiples by [21], theorem A, p. 70. So we shall refer to it as to "the" volume form.

Proposition 3.32. *Let $\big(\bigwedge V, \mathrm{d}\big)$ be a simply-connected pure minimal Sullivan algebra with the property that its volume form $[v] \in H\big(\bigwedge V, \mathrm{d}\big)$ is represented by some closed $v \in \bigwedge V^{even}$.*

Then the following assertions hold: The algebra $\big(\bigwedge V, \mathrm{d}\big)$ is positively elliptic*, i.e. its rational cohomology is concentrated in even degrees only. In particular, it is formal.*

Proof. We split the algebra as in (3.44). Thus a volume form of $\big(\bigwedge V, \mathrm{d}\big)$ necessarily has to be of the form $v = \bar{v} \cdot z_1 \cdot \cdots \cdot z_l \neq 0$ with $\bar{v}$ a volume form of $\big(\bigwedge V', \mathrm{d}\big)$. Recall that the $z_i \in V$ are of odd degree. So $v \in \bigwedge V^{\mathrm{even}}$ requires $l = 0$ and $\big(\bigwedge V, \mathrm{d}\big) = \big(\bigwedge V', \mathrm{d}\big)$.

Suppose the algebra $\big(\bigwedge V, \mathrm{d}\big)$ is non-formal. Thus there is a closed non-exact element $x \neq 0$ in the ideal $I(N)$ generated by $N = V^{\mathrm{odd}}$—cf. the proof of proposition 3.31—in $\bigwedge V$.

Due to [21], theorem A, p. 70, simply-connected elliptic spaces have the homotopy type of finite Poincaré complexes. Thus by the non-singularity of the cup-product pairing we see that there is an element $x' \in \big(\bigwedge V, \mathrm{d}\big)$ such that $x \cdot x'$ represents a non-vanishing element in (rational) top cohomology. By our assumption and by the uniqueness of the volume form we now derive that

$$[x \cdot x' - kv] = 0 \in H\big(\bigwedge V, \mathrm{d}\big)$$

for some $0 \neq k \in \mathbb{Q}$. Hence there is a form $y \in \big(\bigwedge V, \mathrm{d}\big)$ with $\mathrm{d}y = x \cdot x' - kv$.

Now we use the fact that on a pure algebra we have the lower grading, i.e.:

$$\mathrm{d} : \bigwedge V^{\mathrm{even}} \otimes \bigwedge^l V^{\mathrm{odd}} \to \bigwedge V^{\mathrm{even}} \otimes \bigwedge^{l-1} V^{\mathrm{odd}}$$

Now we shall see that the form $x \cdot x'$ lies in $\bigwedge V^{\text{even}} \otimes \bigwedge^{>0} V^{\text{odd}}$ whereas $v \in \bigwedge V^{\text{even}} \otimes \bigwedge^{0} V^{\text{odd}}$ by assumption. This is due to the fact that the element x is in the ideal $I(N)$ generated by $N = V^{\text{odd}}$ in $\bigwedge V$, i.e. $x \in \bigwedge V^{\text{even}} \otimes \bigwedge^{>0} V^{\text{odd}}$. So we also obtain

$$x \cdot x' \in \bigwedge V^{\text{even}} \otimes \bigwedge^{>0} V^{\text{odd}}$$

By the lower grading we obtain a direct sum decomposition $y = \sum_{i=0}^{l} y_l$ with $y_i \in \bigwedge V^{\text{even}} \otimes \bigwedge^{i} V^{\text{odd}}$. Without restriction we may assume that $y_0 = 0$. Hence we see that $y = y_1 + \cdots + y_l$, i.e. $y = y_1 + \tilde{y}$ with

$$\tilde{y} = y_2 + \cdots + y_l \in \bigwedge V^{\text{even}} \otimes \bigwedge^{>0} V^{\text{odd}}$$

Thus

$$\begin{aligned} \mathrm{d}(y_1) &\in \bigwedge V^{\text{even}} \otimes \bigwedge^{0} V^{\text{odd}} = \bigwedge V^{\text{even}} \ni kv \\ \mathrm{d}(\tilde{y}) &\in \bigwedge V^{\text{even}} \otimes \bigwedge^{>0} V^{\text{odd}} \ni x \cdot x' \end{aligned}$$

Since $\mathrm{d}y_1 + \mathrm{d}\tilde{y} = \mathrm{d}y = -kv + x \cdot x'$, the lower grading directly implies that $\mathrm{d}y_1 = -kv$ and that $\mathrm{d}\tilde{y} = x \cdot x'$. This, however, contradicts the fact that both $[v]$ and $[x \cdot x']$ were volume forms and consequently it contradicts the choice of x. Thus there is no closed non-exact element $x \in I(N)$.

Thus the algebra $\left(\bigwedge V, \mathrm{d}\right) = \left(\bigwedge V', \mathrm{d}\right)$ is formal by 1.41. By proposition 3.31 this is exactly the case if $(\bigwedge V, \mathrm{d})$ has positive Euler characteristic. This, however, is equivalent to the vanishing of odd-degree homology and finishes the proof. □

Recall that *symplectic manifolds* M^{2n}, i.e. smooth manifolds with closed, non-degenerate 2-form ω, have the following property: The *symplectic form* $\omega \in \mathrm{A}^2_{\mathrm{DR}}(M)$ is closed and satisfies $\omega^n(p) \neq 0$ for all $p \in M$. This is equivalent to the non-degeneracy of ω. Compact Kähler manifolds are symplectic in particular.

A weaker version of symplecticity is the following: A manifold M^{2n} is called *cohomologically symplectic* if there is a class $\eta \in H^2(M^{2n})$ with $\eta^n \neq 0$. Clearly, Hard-Lefschetz manifolds—in the version of formula (3.42)—are cohomologically symplectic.

Theorem 3.33. *Let M be a simply-connected manifold which admits a pure Sullivan model. Suppose that* one *of the following statements holds:*

- *The manifold M^{2n} is cohomologically symplectic.*
- *The manifold M^{4n} satisfies Hard-Lefschetz in the version of formula* (3.43)*.*

Then M is formal and has cohomology concentrated in even degrees only.

Proof. We form a pure minimal Sullivan model $\left(\bigwedge V, \mathrm{d}\right)$ for M. We shall show that its non-degeneracy requires the volume form to be represented by an element in the subalgebra generated by even-degree elements:

In the first case of the assertion we shall show that the cohomology-symplectic form $[\omega] \in H^2(M)$ satisfies $\omega \in \bigwedge V^{\mathrm{even}}$. We then derive that ω^n is a volume form in $\bigwedge V^{\mathrm{even}}$.

So we have $\deg \omega = 2$ and we directly see that ω is not decomposable, since the manifolds are simply-connected. This implies that $\omega \in V^2$.

In the second case we consider the form ω in degree four with respect to which we have Hard-Lefschetz. Clearly, ω^n is a volume form. So we shall show that $\omega \in \bigwedge V^{\mathrm{even}}$ which will yield the result.

Every decomposition of ω would have to be done by linear combinations of products of two elements (as there are none in degree one). These two elements may now either be of degree two both or one of degree one and the other of degree three. The latter possibility again is excluded by simply-connectedness. So even if the form ω is decomposable, it still remains in $\bigwedge V^{\mathrm{even}}$.

By proposition 3.32 we may now derive the result on cohomology and formality. □

Remark 3.34.

- In the case of simply-connected (Positive Quaternion) Kähler Manifolds with pure model we might have just proceeded as follows: Use their formality to see that $\left(\bigwedge V', \mathrm{d}\right)$ in decomposition (3.44) is formal, whence it is positively elliptic. Use "cohomologically symplectic" to realise that $V' = V$.

- Theorem 3.33—either applied to the twistor space or to the manifold itself—may serve as a reproof of the vanishing of odd-degree Betti numbers and of formality for Positive Quaternion Kähler Manifolds within the class of simply-connected manifolds admitting pure models.

- Define a *quaternionic cohomologically symplectic manifold* in an analogous fashion. This property instead of the quaternionic Hard-Lefschetz property (3.43) clearly would have been sufficient to conclude the theorem.

- From [60] and [13].5, p. 79–94, we deduce that a generalised dd_c-lemma on a compact symplectic manifold holds if and only if the manifold satisfies Hard-Lefschetz. Nonetheless, the manifold then needs not be formal in general. In the light of theorem 3.33 it is so, however, if the manifold admits a pure model.

- In example [14].4.4, p. 346, an example of a 12-dimensional non-formal simply-connected symplectic Hard-Lefschetz manifold is given.

- We remark that there are several examples of Hard-Lefschetz manifolds that are not Kählerian although they are formal. For example see [23], proposition 6.2, p. 171, where certain four-dimensional Donaldson submanifolds are shown to be symplectic, to satisfy Hard-Lefschetz but to admit no complex structures;

i.e. there is no Kähler structure, in particular. See theorem [25].3.5, p. 3322, for examples of *cohomologically Kähler* manifolds that admit no Kähler metric. (They are formal and satisfy Hard-Lefschetz.)

⧈

We shall now derive topological properties of some particular manifolds with holonomy $\mathbf{G}_2$ or $\mathbf{Spin}(7)$.

In proposition [41].10.2.6, p. 246, it is proved that a compact $\mathbf{G}_2$-manifold M^7, i.e. a manifold M with $\operatorname{Hol}(M) = \mathbf{G}_2$, satisfies

$$\langle a \cup a \cup [\omega], [M]\rangle < 0 \tag{3.45}$$

for every non-zero $a \in H^2(M)$ and with respect to the 3-form ω defining the $\mathbf{G}_2$-structure (cf. [41].10.1.1, p. 242). The analogue holds for compact $\mathbf{Spin}(7)$-manifolds (cf. [41].6.6, p. 261) with the respective 4-form ω (cf. [41].10.5.1, p. 255). So, in particular, combining this with Poincaré Duality one obtains Lefschetz-like properties for manifolds with holonomy $\mathbf{G}_2$ or $\mathbf{Spin}(7)$:

$$\cup\omega : H^2(M) \xrightarrow{\cong} H^{n-2}(M)$$

for the closed 3-form ω defining the $\mathbf{G}_2$-structure and $n = 7$; respectively the 4-form ω defining the $\mathbf{Spin}(7)$-structure and $n = 8$.

If $\operatorname{Hol}(M) = \mathbf{Spin}(7)$, theorem [41].10.6.8, p. 261, gives further strong topological restrictions; there is the following relation on Betti numbers:

$$b_3 + b_4^+ = b_2 + 2b_4^- + 25 \tag{3.46}$$

where $b_4^+ + b_4^- = b_4$ and $b_4^+ \geq 1$. The manifold is simply-connected and spin with $\hat{A}(M)[M] = 1$—cf. [41], equation (10.25) on page 260, together with proposition 10.6.5, p. 260. A direct consequence of the latter result is that there are no smooth effective $\mathbb{S}^1$-actions upon M due to [3]—compare the refined version given by theorem 2.5. In particular, we shall not find any homogeneous space that admits a metric with such holonomy. Using the relation on Betti numbers this can be generalised with ease. For this recall— we adapt the formulas (1.8), (1.9), (1.11) and (1.12) to our case—that for an elliptic orientable compact manifold—these properties imply that the formal/homological dimension equals $\dim M$—the following equations hold:

$$\sum_i \deg x_i \leq 2 \cdot \dim M - 1 \tag{3.47}$$

$$\sum_i \deg y_i \leq \dim M \tag{3.48}$$

$$\dim \pi_*(M) \otimes \mathbb{Q} \leq \dim M \tag{3.49}$$

$$\sum_i \deg x_i - \sum_j (\deg y_j - 1) = \dim M \tag{3.50}$$

Again, the x_i form a homogeneous basis of $\pi_{\mathrm{odd}} \otimes \mathbb{Q}$ and the y_i form a homogeneous basis of $\pi_{\mathrm{even}} \otimes \mathbb{Q}$.

Using these relations we shall prove

Proposition 3.35. *A rationally elliptic compact manifold does not admit a metric g with $\mathrm{Hol}_g(M) = \mathbf{Spin}(7)$.*

Proof. Relation (3.46) yields $b_4 \geq b_4^+ \geq b_2 - b_3 + 25$ and $b_3 + b_4 \geq b_2 + 25$. Let $\left(\bigwedge V, \mathrm{d}\right)$ be the minimal model of M. Since M is simply-connected we obtain that

$$\begin{aligned}
&\dim \pi_2(M) \otimes \mathbb{Q} = \dim V^2 = b_2\\
&\dim \pi_3(M) \otimes \mathbb{Q} = \dim V^3 \geq b_3\\
&\dim \pi_4(M) \otimes \mathbb{Q} = \dim V^4 \geq b_4 - \dim \mathrm{Sym}_2(V^2) = b_4 - \frac{b_2(b_2+1)}{2}
\end{aligned}$$

In particular, we see that

$$\begin{aligned}
\dim \pi_*(M) \otimes \mathbb{Q} &\geq b_2 + b_3 + \left(b_4 - \frac{b_2(b_2+1)}{2}\right)\\
&= b_3 + b_4 + \frac{b_2(1-b_2)}{2}\\
&\geq 25 + b_2 + \frac{b_2(1-b_2)}{2}\\
&= 25 + \frac{b_2(3-b_2)}{2}
\end{aligned}$$

Since we have that $\dim V^2 = b_2$ equation (3.48) yields in particular that $2 \cdot b_2 \leq \dim M = 8$ and that $b_2 \leq 4$. Thus we derive that $\dim \pi_*(M) \otimes \mathbb{Q} \geq 23$, which contradicts (3.49). □

A $\mathbf{G}_2$-manifold is known to have a finite fundamental group (cf. [41].10.2.2, p. 245). We shall now shed a light on the rational homotopy type of an elliptic simply-connected $\mathbf{G}_2$-manifold.

Proposition 3.36. *Let M be a simply-connected compact elliptic manifold that admits a Riemannian metric g with $\mathrm{Hol}_g(M) = \mathbf{G}_2$. Then a minimal model $\left(\bigwedge V, \mathrm{d}\right)$ of M is one of the following models, which we describe by the graded vector space V. We give generators in their respective degree together with their differentials when possible. In the first two cases the model is completely described. In the third case we obtain a family of models depending on the differentials of x and x'.*

$2 : a \mapsto 0$	$3 : \omega \mapsto 0$	$2 : a \mapsto 0, \quad b \mapsto 0$
$3 : \omega \mapsto 0$	$4 : a \mapsto 0$	$3 : \omega \mapsto 0, \quad x \mapsto ?, \quad x' \mapsto ?$
$5 : x \mapsto a^3$	$7 : x \mapsto a^2$	

The first case has the rational homotopy type of $\mathbb{CP}^2 \times \mathbb{S}^3$ *and the second one has the rational homotopy type of* $\mathbb{S}^3 \times \mathbb{S}^4$*. In the third case* $\mathrm{d}|_{\langle x, x' \rangle}$ *is injective.*

Moreover, in any case the minimal model is formal.

Proof. A $\mathbf{G}_2$-manifold is 7-dimensional. In particular, by Poincaré Duality its Euler characteristic vanishes. By equation (1.15) we therefore know that $\dim \pi_{\mathrm{odd}}(M) \otimes \mathbb{Q} > \dim \pi_{\mathrm{even}}(M) \otimes \mathbb{Q}$. Let $\left(\bigwedge V, \mathrm{d}\right)$ be the minimal model of M. Since M is simply-connected, this result on homotopy groups thus translates to $\dim V^{\mathrm{odd}} > \dim V^{\mathrm{even}}$.

We shall now proceed by a case by case check depending on the number and the degrees of even-degree generators. We always have the existence of a closed three-form $\omega \in V^3$—it cannot be decomposable, as M is simply-connected. By Poincaré Duality we therefore obtain a closed four-form. Given a certain configuration of even-degree generators we use equation (3.50) to compute all the possible configurations of odd-degree generators from partitions of $7 + \sum_j (\deg y_j - 1)$ which contain the element 3. Equations (3.47) and (3.48) limit the number of generators of V and their degrees. We shall give the generators and their index will denote their degree. (The class ω is of degree 3 always.)

As a showcase computation we shall do the first case explicitly: Assume there is exactly one even generator in degree 2. Then equation (3.50) yields $\sum_i \deg x_i = 7 + 1$. With the existence of the three-form ω only the partition $(3, 5)$ of 8 is a partition that gives the degrees of odd-degree generators, since M is simply-connected, i.e. $V^1 = 0$. Under all these restrictions these are the remaining possibilities:

$$V = \langle a_2, \omega, x_5 \rangle \tag{3.51}$$

$$V = \langle \omega, a_4, x_7 \rangle \tag{3.52}$$

$$V = \langle a_2, b_2, \omega, x_3, x_3' \rangle \tag{3.53}$$

$$V = \langle a_2, \omega, x_3, b_4, x_5' \rangle \tag{3.54}$$

(There cannot be more than three even-degree generators as this would contradict equation (3.48). If there are three even-degree generators they are all of degree 2. Then there have to be at least four odd-degree generators. Their degree is at least 3 each. However, $4 \cdot 3 = 12 > 10 = 7 + 3 \cdot (2 - 1)$ contradicting equation (3.50).)

Let us step through the different cases. We always have $\mathrm{d}\omega = 0$.

In case (3.51) we obtain $\mathrm{d}a_2 = 0$ by degree. Moreover, $\mathrm{d}x_5 \in (\bigwedge V)^6 = \langle a^3 \rangle$. Thus, up to multiples, i.e. up to isomorphisms of the model, we obtain $\mathrm{d}x_5 = a_2^3$. Indeed, $\mathrm{d}x_5 \neq 0$ as this would admit the closed non-exact class $x_5 \cdot \omega$ in degree 8. This yields the first algebra in the assertion.

In case (3.52) by the analogous argument we have $\mathrm{d}a_4 = 0$ and $\mathrm{d}x_7 = a_4^2$ yielding the second algebra in the assertion.

In case (3.53) we have $\mathrm{d}a_2 = \mathrm{d}b_2 = 0$ by degree. We therefore obtain that $\mathrm{d}x_3, \mathrm{d}x_3' \in \langle a_2^2, b_2^2, a_2b_2\rangle$. None of the differentials may vanish as this would produce—by multiplication with ω—a closed non-exact class in degree 6 which would have to be Poincaré dual to a class in degree one. Since the space is simply-connected, this may not be the case. This argument applied to a linear combination of x_3 and x_3' shows that $\mathrm{d}|_{\langle x_3, x_3'\rangle}$ is injective.

In case (3.54) we have that $\mathrm{d}a_2 = 0$ by degree. Equally, by degree we have $\mathrm{d}x_3 \in \langle a^2, b_4\rangle$; thus by minimality we obtain $\mathrm{d}x_3 \in \langle a^2\rangle$. If $\mathrm{d}x_3 \neq 0$, i.e. $\mathrm{d}x_3 = a^2$ without restriction, we obtain $[a^2] = 0$ and $[a^2 \cdot \omega] = 0$ in cohomology. This contradicts property (3.45). Thus $\mathrm{d}x_3 = 0$ and x_3 defines a non-vanishing class in cohomology. Hence we obtain that $\dim H^3(M) = \dim\langle \omega, x_3\rangle = 2$ and the same holds for $\dim H^4(M) = 2$ by duality. Thus, in particular, we obtain $\mathrm{d}b_4 = 0$ by degree. Hence we compute $\dim H^5(M) = \dim\langle [a_2 \cdot \omega], [a_2 \cdot x_3]\rangle = 2$. Since $\dim H^2(M) = \dim\langle a_2\rangle = 1$, this contradicts Poincaré Duality.

The first two models are easily seen to have the rational homotopy type of $\mathbb{C}\mathbf{P}^2 \times \mathbb{S}^3$ and $\mathbb{S}^3 \times \mathbb{S}^4$ respectively. So they are cartesian products of symmetric spaces. Hence they are formal by 1.36 and theorem 1.47.

Every model in the third family is formal, too: We shall prove this by an application of theorem 1.41. For this split $V = C \oplus N$ with $C := \langle a_2, b_2, \omega\rangle$ and $N := \langle x_3, x_3'\rangle$. Then $\ker \mathrm{d}|_V = C$, since we showed $\mathrm{d}|_{\langle x_3, x_3'\rangle}$ to be injective. Every element in the ideal $I(N)$ lies in degree $5, 6, 7, \ldots$ or higher. In degree six or in degree higher than seven every closed element in the ideal must be exact, since in these degrees rational cohomology vanishes. Assume there is a closed non-exact element $y \in I(N)$ in degree 5 or 7.

Let us first deal with the case when $\deg y = 5$. We have

$$\dim H^5(M) \geq \dim\langle [a_2 \cdot \omega], [b_2 \cdot \omega]\rangle = 2$$

since a_2, b_2, ω are closed and since every element in degree 4 is closed. Thus by duality we see $\dim H^5(M) = \dim H^2(M) = 2$ and $H^5(M) = \langle [a_2 \cdot \omega], [b_2 \cdot \omega]\rangle$. We assumed y to represent a non-vanishing cohomology class $[y] \in H^5(M)$. As there is no relation in degree 4, i.e. no element with non-trivial differential, we obtain that

$$[y] \notin \langle [a_2 \cdot \omega], [b_2 \cdot \omega]\rangle \subseteq \bigwedge C$$

and that $H^5(M) = 3$; a contradiction.

Suppose now that $y \in (I(N))^7$ is closed and non-exact. By reasons of degree we obtain

$$y \in \langle x_3 \cdot a_2^2, x_3 \cdot a_2 \cdot b_2, x_3 \cdot b_2^2, x_3' \cdot a_2^2, x_3' \cdot a_2 \cdot b_2, x_3' \cdot b_2^2\rangle$$

As any cohomology class in degree seven is represented by a multiple of $a^2 \cdot \omega \notin I(N)$ by property (3.45) and by Poincaré Duality, we see that there must be a relation in degree six, i.e. an element $y' \in \left(\bigwedge V\right)^6$ with

$$\mathrm{d}y' = k \cdot y - a_2^2 \cdot \omega \tag{3.55}$$

for some $0 \neq k \in \mathbb{Q}$. By reasons of degree we have

$$y' \in \langle x_3 \cdot x_3', x_3 \cdot \omega, x_3' \cdot \omega \rangle$$

So let $y' = k_1 \cdot x_3 \cdot x_3' + k_2 \cdot x_3 \cdot \omega + k_3 \cdot x_3' \cdot \omega$ for $k_i \in \mathbb{Q}$ and compute

$$\mathrm{d}y' = k_1((\mathrm{d}x_3)x_3' - x_3(\mathrm{d}x_3')) + k_2(\mathrm{d}x_3)\omega + k_3(\mathrm{d}x_3')\omega$$

with $k_1((\mathrm{d}x_3)x_3' - x_3(\mathrm{d}x_3')) \in \bigwedge\langle x_3, x_3', a_2, b_2\rangle$ and $k_2(\mathrm{d}x_3)\omega + k_3(\mathrm{d}x_3')\omega \in \omega \cdot \bigwedge\langle a_2, b_2\rangle$. Combine this with (3.55) to obtain that

$$\mathrm{d}(k_2x_3 + k_3x_3') = k_2(\mathrm{d}x_3) + k_3(\mathrm{d}x_3') = -a_2^2$$

This, however, implies that $\mathrm{d}(-(k_2x_3 + k_3x_3')\omega) = a_2^2\omega$, which contradicts (3.45). Thus we cannot find a suitable element y' and there is no closed non-exact element y in $(I(N))^7$.

Summarising our reasoning, we see that there is no closed non-exact element in $I(N)$ at all. Thus by theorem 1.41 we obtain that M is formal. □

In [13], example 8.5, p. 131, a simply-connected *non-formal* manifold is constructed which has all the known topological properties of a $\mathbf{G}_2$-manifold ([13], p. 128). We now see that such an example is no longer possible in the category of simply-connected elliptic spaces.

A consequence of the preceeding propositions is

Theorem 3.37. *A rationally elliptic compact simply-connected irreducible Riemannian manifold* (M, g) *with* $\mathrm{Hol}_g(M) \subseteq \mathbf{Spin}(7)$ *is formal.*

PROOF. It holds that $\mathbf{G}_2 \subseteq \mathbf{Spin}(7)$. Symmetric spaces are formal (cf. example 1.36). Thus—taking into account all the possible inclusions of Lie groups—Berger's theorem yields that $\mathrm{Hol}(M) \in \{\mathbf{Spin}(7), \mathbf{G}_2\}$ unless $\dim M \leq 6$. If $\dim M \leq 6$, we directly obtain that M is formal—as so is every manifold of that dimension due to proposition [63].4.6, p. 574. Now propositions 3.35 and 3.36 finish the proof. □

3.4. Low dimensions

In low dimensions, i.e. in dimensions 12, 16 and 20, there are comparatively simple and completely different proofs for the formality of Positive Quaternion Kähler Manifolds. Moreover, we shall study these manifolds under the restrictions imposed by the assumption of ellipticity.

3.4.1. Formality

We provide a proof of the formality in low dimensions which relies heavily on the concept of s-formality—cf. 1.42. We then proceed by theorem 1.43.

Again we may focus on rationally 3-connected Positive Quaternion Kähler Manifolds, as the symmetric space $\mathbf{Gr}_2(\mathbb{C}^{n+2})$ is formal (cf. 1.36 and 1.13).

Theorem 3.38. *Let M be a rationally 3-connected Positive Quaternion Kähler Manifold. Suppose $\dim M \leq 20$. Then M is formal.*

First proof of theorem 3.38. Symmetric spaces are formal, hence, in particular, so are Wolf spaces (1.36). The classification of Positive Quaternion Kähler Manifolds of dimensions 4 and 8 yields the result in this case.

Dimension 12. Suppose $\dim M = 12$. By theorem 1.43 we need to show 5-formality: Let $\left(\bigwedge V, \mathrm{d}\right) \xrightarrow{\simeq} (\mathrm{A}_{\mathrm{PL}}(M), \mathrm{d})$ be a minimal model. Since M is 3-connected we obtain that $V^1 = V^2 = V^3 = 0$ and that $V^4 \subseteq \ker \mathrm{d}$ (from the algorithm on [22].12, p. 144–145 for example). Moreover, we obtain $V^5 = 0$, as the cohomology of M is concentrated in even degrees (cf. 1.13) and since $(\bigwedge V)^6 = V^6$, i.e. since we need not place any relations in degree 5. In the terminology of definition 1.42 we choose $C^{\leq 5} := V^4$ and $N^{\leq 5} := 0$. This yields a homogeneous decomposition $V = C \oplus N$ as required in 1.42. In particular, there is no closed non-zero element in I_5 and M is formal.

Dimension 16. In this dimension we need to prove 7-formality. We build the minimal model as above. That is, we obtain $V^1 = V^2 = V^3 = 0$, V^4 is in bijection with $H^4(M)$. For reasons of degree and since $H^5(M) = 0$, we again obtain $V^5 = 0$. As $\left(\bigwedge^{\geq 2} V\right)^{\leq 7} = 0$, we see that V^6 lies in the kernel of d and is in bijection with $H^6(M)$. As $H^7(M) = 0$, we see that $\mathrm{d}|_{V^7}$ is injective and that every element in degree seven represents a relation. We set

$$C^{\leq 7} := V^4 \oplus V^6$$

and

$$N^{\leq 7} := V^7$$

Then clearly $V = C \oplus N$ with $C = \ker \mathrm{d}|_V$ and $\mathrm{d}|_N$ being injective.

We need to see that there is no closed non-exact element in I_7. We obtain that

$$I_7 = N^7 \cdot \bigwedge(C^4 \oplus C^6 \oplus N^7)$$

is concentrated in odd degrees $7, 11, 13, 15$ in the even degree 14 or in degrees strictly larger than 16.

Whenever $H^i(M) = 0$, every closed form in degree i is necessarily exact. As the odd-degree cohomology of M vanishes (cf. 1.13), every closed element in degrees $7, 11, 13, 15$ is exact. As M is rationally 3-connected the same holds in degree 14 due to Poincaré Duality.

Dimension 20. Let M now be of dimension 20. We need to prove 9-formality. In an analogous manner to what we did in the previous cases we see that V^4, V^6 and V^8 are contained in $\ker \mathrm{d}$. Moreover, we see that d is necessarily injective on V^7 and V^9. So set $C^{\leq 9} := V^4 \oplus V^6 \oplus V^8$ and $N^{\leq 9} := V^7 \oplus V^9$. We then obtain

$$V = (V^4 \oplus V^6 \oplus V^8) \oplus (V^7 \oplus V^9) = C^{\leq 9} \oplus N^{\leq 9}$$

with $C = \ker \mathrm{d}|_V$ and d being injective on N.

We need to show that every closed element in $I_{\leq 9}$ is exact. We obtain that

$$I_9 = (N^7 \oplus N^9) \cdot \bigwedge (C^4 \oplus C^6 \oplus N^7 \oplus C^8 \oplus N^9)$$

is concentrated in degrees 7, 9, 11, 13, 14, 15, 16, 17, 18, 19, 20 and degrees strictly larger than 20. For the same reasons as in the case of dimension 16 we may neglect them all except for degrees 14, 16 and 20. We shall show that in these dimensions there is no non-trivial closed form in I_9.

So we compute:

$$\begin{aligned}
(I_9)^{14} &= \textstyle\bigwedge^2 N^7 \\
(I_9)^{16} &= N^7 \cdot N^9 \\
(I_9)^{20} &= N^7 \cdot N^7 \cdot C^6 \oplus N^7 \cdot N^9 \cdot C^4
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathrm{d}(I_9)^{14} &\in \mathrm{d}(N^7 \cdot N^7) \subseteq C^4 \cdot C^4 \cdot N^7 \\
\mathrm{d}(I_9)^{16} &\in \mathrm{d}(N^7 \cdot N^9) \subseteq C^4 \cdot C^4 \cdot N^7 \oplus C^4 \cdot C^6 \cdot N^9 \\
\mathrm{d}(I_9)^{20} &\in \mathrm{d}(N^7 \cdot N^7 \cdot C^6 \oplus N^7 \cdot N^9 \cdot C^4) \subseteq C^4 \cdot C^4 \cdot C^6 \cdot N^7 \oplus C^4 \cdot C^4 \cdot C^4 \cdot N^9
\end{aligned}$$

Let us first deal with the case of degree 14. Every form in $(I_9)^{14}$ is of the form $s = \sum_k a_k n_{i_k} n_{j_k}$ (with $a_k \in \mathbb{Q}$) for $n_{i_k}, n_{j_k} \in N^7$ and $0 \neq n_{i_k} \neq n_{j_k} \neq 0$. Thus

$$\mathrm{d}(s) = \sum_k a_k (\mathrm{d}(n_{i_k}) n_{j_k} - n_{i_k} \mathrm{d}(n_{j_k})) = \sum_\beta \sum_{\substack{\alpha \\ \alpha \mathcal{R} \beta}} b_{\alpha,\beta} \, \mathrm{d}(n_\alpha) n_\beta$$

with $b_{\alpha,\beta} \in \mathbb{Q}$ and $\alpha\mathcal{R}\beta \Leftrightarrow \exists k : (\alpha = i_k \wedge \beta = j_k) \vee (\alpha = j_k \wedge \beta = i_k)$. As $\mathrm{d}|_{N^7}$ is injective with $\mathrm{d}(N^7) \subseteq C^4 \cdot C^4$, we obtain that $\mathrm{d}(s) = 0$ if and only if for each fixed β the sum $\sum_{\substack{\alpha \\ \alpha\mathcal{R}\beta}} b_{\alpha,\beta}\, \mathrm{d}(n_\alpha)$ is zero which itself is equivalent to $b_{\alpha,\beta} = 0$ for all α. This requires $s = 0$.

An analogous reasoning applies to the case of degree 16. The case of degree 20 is only slightly more complicated and follows in a similar fashion: In degree 20 each element in I_9 is a linear combination s of products of the form $n_{i_k} n_{j_k} c_k$ or $\tilde{n}_{i_k}\tilde{n}_{j_k}\tilde{c}_k$, where $\deg n_{i_k} = \deg n_{j_k} = 7$, $\deg c_l = 6$, $\deg \tilde{n}_{i_k} = 7$, $\deg \tilde{n}_{j_k} = 9$ and $\deg \tilde{c}_l = 4$. That is,

$$s = \sum_{k \in K_1} a_k n_{i_k} n_{j_k} c_k + \sum_{k \in K_2} b_k \tilde{n}_{i_k} \tilde{n}_{j_k} \tilde{c}_k$$

with $a_k, b_k \in \mathbb{Q}$. Suppose $\mathrm{d}s = 0$. We compute

$$0 = \mathrm{d}(s) = \sum_{k \in K_1} a_k \big(\mathrm{d}(n_{i_k}) n_{j_k} - n_{i_k} \mathrm{d}(n_{j_k})\big) c_k + \sum_{k \in K_2} b_k \big(\mathrm{d}(\tilde{n}_{i_k}) \tilde{n}_{j_k} - \tilde{n}_{i_k} \mathrm{d}(\tilde{n}_{j_k})\big) \tilde{c}_k$$

We observe that $b_k = 0$ for all $k \in K_2$, since the summands $\mathrm{d}(\tilde{n}_{i_k})\tilde{n}_{j_k}\tilde{c}_k$ contain the factor $\tilde{n}_{j_k} \in (I_9)^9$ and since no other summands contain a factor from $(I_9)^9$. Thus

$$\mathrm{d}(s) = \sum_{k \in K_1} a_k (\mathrm{d}(n_{i_k}) n_{j_k} - n_{i_k} \mathrm{d}(n_{j_k})) c_k,$$

and we may argue in a way completely analogous to the reasoning in degree 14.

□

We can provide yet another proof for the formality of Positive Quaternion Kähler Manifolds in dimensions 12 and 16. This proof uses far more structure theory of Positive Quaternion Kähler Manifolds.

SECOND PROOF OF THEOREM 3.38. By theorem 1.18 Positive Quaternion Kähler Manifolds in dimensions 12 and 16 permit a smooth effective $\mathbb{S}^1$-action. The fixed-point components of such an action are either again Positive Quaternion Kähler Manifolds or Kähler manifolds. Due to theorem [10].VII.2.1, p. 375 and the remark on [10].VII.2, p. 378, we conclude that the rational homology of each fixed-point component is concentrated in even degrees, as so is $H^*(M)$ by 1.13. For the same reason the spectral sequence of the fibration

$$M \hookrightarrow \mathbf{E}\mathbb{S}^1 \times_{\mathbb{S}^1} M \to \mathbf{B}\mathbb{S}^1$$

of the Borel construction degenerates at the E_2-term.

As we may suppose M to be rationally 3-connected, we see that every form of degree smaller than $\dim M$ in V^{even} lies in the kernel of d: In the case of dimension 12 this follows from the fact that there is no non-trivial closed form in V^{11}. (This is due to the fact that—in the notation of the first proof—an element in $\left(\bigwedge V\right)^{11}$ lies in $C^4 \cdot N^7$.) If $\dim M = 16$ we need to observe that there are no non-trivial closed forms in $\left(\bigwedge V\right)^{11}$, $\left(\bigwedge V\right)^{13}$ and $\left(\bigwedge V\right)^{15}$. This follows in the same way.

Using the classification of Positive Quaternion Kähler Manifolds until dimension 8 and the fact that Kähler manifolds and symmetric spaces are formal (cf. 1.36) we are done by an application of corollary [55].5.9, p. 2785.

Thus, given the formality of 12-dimensional Positive Quaternion Kähler Manifolds, we conclude that 16-dimensional Positive Quaternion Kähler Manifolds are formal by means of the corollary. □

3.4.2. Ellipticity

Let us now investigate Positive Quaternion Kähler Manifolds in dimensions 16 and 20 under the hypothesis of rational ellipticity. Since Wolf spaces are elliptic, as they are homogeneous (cf. 1.49), this is a plausible assumption.

We shall first see that Positive Quaternion Kähler Manifolds M of arbitrary dimension satisfy the rational dichotomy, i.e. that they are either elliptic or hyperbolic.

This follows directly: As they are compact manifolds, the (rational) homology of Positive Quaternion Kähler Manifolds is finite-dimensional. Hence a Positive Quaternion Kähler Manifold M admits a model $\left(\bigwedge V, \mathrm{d}\right)$ with a finite-dimensional vector space V—cf. the corollary on page [22], p. 146. As M is simply-connected, thus also its rational Lusternik–Schnirelmann category $\mathrm{cat}_0(M) < \infty$ is finite (cf. [22].32.4, p. 438) and M satisfies the rational dichotomy.

We remark that even more can be said: As M^{4n} additionally is formal (cf. 3.10), its rational cup-length equals its rational Lusternik–Schnirelmann category, i.e. $n = \mathrm{c}_0(M) = \mathrm{cat}_0(M)$—cf. 3.11. Moreover, by the formality of the twistor space together with its Hard-Lefschetz property we obtain

$$2n+1 = \mathrm{c}_0 Z = \mathrm{cat}_0 Z \geq \mathrm{cat}_0 M + \mathrm{c}_0 \mathbb{S}^2 = n+1$$

(cf. example [22].29.4, p. 388). The inequality matches a more general theorem by Jessup (cf. proposition [22].30.8, p. 410).

So from now on we shall assume M to be a rationally elliptic. Since the cohomology of M is concentrated in even degrees (cf. 1.13) we obtain that M then is an F_0-space. In particular, its formality then also follows from theorem 1.52.

We adapt some of the formulas (1.8), ..., (1.16), where the x_i are a homogeneous basis of $\pi_{\mathrm{odd}}(M) \otimes \mathbb{Q}$ and the y_j are a homogeneous basis of $\dim \pi_{\mathrm{even}}(M) \otimes \mathbb{Q}$:

$$\dim \pi_{\mathrm{odd}}(M) \otimes \mathbb{Q} \leq \mathrm{cat}_0 M \tag{3.56}$$

$$\sum \deg x_i \leq 2 \dim M - 1 \qquad\qquad \sum \deg y_j \leq \dim M \tag{3.57}$$

$$\dim M = \sum_i \deg x_i - \sum_j (\deg y_j - 1) \tag{3.58}$$

Since its cohomology is concentrated in even degrees, the manifold M has positive Euler characteristic and we obtain that

$$\dim \pi_{\mathrm{odd}}(M) \otimes \mathbb{Q} = \dim \pi_{\mathrm{even}}(M) \otimes \mathbb{Q} \tag{3.59}$$

We denote by

$$c_i := \dim \pi_i(M) \otimes \mathbb{Q}$$

the "homotopy Betti numbers" of M.

Theorem 3.39. *Suppose M to be a rationally 3-connected 16-dimensional Positive Quaternion Kähler Manifold. Then its non-trivial Betti numbers b_i (up to Poincaré Duality) together with its (non-trivial) homotopy Betti numbers c_i are given by one of the following possibilities:*

$$b_4 = b_8 = 1,\ b_6 = 0 \qquad \textit{and} \qquad c_4 = 1,\ c_{19} = 1$$

or

$$b_4 = 3,\ b_6 = 0,\ b_8 = 4 \qquad \textit{and} \qquad c_4 = 3,\ c_7 = 2,\ c_{11} = 1$$

In particular, we obtain that an elliptic 16-dimensional Positive Quaternion Kähler Manifold is rationally a homology Wolf space.

Proof. Since M is formal, corollary (3.11) yields in particular that $\mathrm{cat}_0 M = \frac{1}{4} \dim M$ and by (3.56) and (3.59) we obtain:

$$\dim \pi_{\mathrm{even}}(M) \otimes \mathbb{Q} = \dim \pi_{\mathrm{odd}}(M) \otimes \mathbb{Q} \leq \tfrac{1}{4} \dim M \tag{3.60}$$

Recall the equation

$$-1 + 3b_4 - b_6 = 2b_8$$

from (1.2). Combining this equation with the Hard-Lefschetz property (cf. 1.13) yields

$$(b_4, b_6, b_8) \in \{(1,0,1), (2,1,2), (3,0,4), (3,2,3), (4,1,5), (4,3,4), (5,0,7), (5,2,6), (5,4,5), \dots\}$$

(The remaining triples satisfy $b_4 \geq 6$.) As in the proof of theorem 3.38 we form the minimal model $\left(\bigwedge V, \mathrm{d}\right)$ of M and we see that $c_1 = c_2 = c_3 = 0$, $c_4 = b_4$, $c_5 = 0$, $c_6 = b_6$, since $c_i = \dim V_i$. Thus by equation (3.60) we have $b_4 + b_6 \leq 4$ in particular, which reduces the list to the triples

$$(b_4, b_6, b_8) \in \{(1,0,1), (2,1,2), (3,0,4)\}$$

If $b_4 = 1$, then M has the rational cohomology algebra of $\mathbb{H}\mathbf{P}_4$. Thus by formality it is weakly equivalent to the quaternionic projective space. In this case we obtain that $c_{19} = 1$ is the only non-trivial homotopy Betti number other than $c_4 = 1$.

For the case $b_4 = 2$ we may cite theorem [38].1.1, p. 2, which excludes exactly a possible occurence of this Betti number.

If $b_4 = 3$, again by (3.57) and (3.59) we have

$$\dim \pi_{\mathrm{odd}}(M) \otimes \mathbb{Q} = \dim \pi_{\mathrm{even}}(M) \otimes \mathbb{Q} = 3$$

We compute

$$\dim \left(\bigwedge V\right)^8 = \dim \left(\bigwedge^2 V^4 \oplus V_8\right) = \dim \mathrm{Sym}_2(V^4) + c_8 = \frac{b_4(b_4+1)}{2} + c_8 = 6 + c_8$$

As $\mathrm{d}|_{V^7}$ is injective, we obtain that

$$4 = \dim H^8(M) = \dim \left(\bigwedge V\right)^8 - \dim V^7 = 6 + c_8 - c_7$$

As $\dim \pi_{\mathrm{even}}(M) \otimes \mathbb{Q} = 3$, we see that $c_8 = 0$ and that $c_7 = 2$. By (3.58) we compute that $c_{11} = 1$ and that there is no further rational homotopy.

Thus there only remain configurations of Betti numbers which are realised by Wolf spaces (cf. corollary A.2), and so M is rationally a homology Wolf space. By corollary 1.12 this clearly extends to the case when $b_2 \neq 0$. □

Similar computations can be done in dimension 20, where still only very few possible configurations for the b_i and c_i exist. We shall not bore the reader with more details and with computations that blow up dramatically.

4

Classification Results

In this chapter we combine several relations from Index Theory with the connectivity lemma, with further properties of Positive Quaternion Kähler Manifolds and with Lie theoretic arguments to obtain new properties of Positive Quaternion Kähler Manifolds and partial classification results in low dimensions. By this we mainly refer to dimension 20 beside dimensions 16 and 24.

Several arguments involve heavy computations. All of these were done with the help of MATHEMATICA 6.01 or MAPLE 9 or later programmme versions respectively.

4.1. Preparations

This section is devoted to a computation of several indices $i^{p,q}$ in terms of the characteristic numbers of the complexified tangent bundle $T_{\mathbb{C}}M$ for a Positive Quaternion Kähler Manifold M of dimension 20. That is, we compute

$$i^{p,q} = \langle \hat{A}(M) \cdot \mathrm{ch}(R^{p,q}), M \rangle$$

with $R^{p,q} = \bigwedge_0^p E \otimes S^q H$ as usual. For the analogous computations in dimension 16 see chapter E of the appendix. The formulas relating these indices to other invariants are given in theorem 1.17. For $5 + p + q$ even they become:

$$i^{p,q} = \begin{cases} 0 & \text{for } p+q < 5 \\ (-1)^p (b_{2p}(M) + b_{2p-2}(M) & \text{for } p+q = 5 \\ d & \text{for } p = 0,\ q = 7 \end{cases}$$

where $d = \dim \mathrm{Isom}(M)$.

Combining these equations with our computations yields the fundamental system of equations we shall mainly be concerned with in the following sections. It is linear in the characteristic numbers of M.

We shall compute these indices in terms of characteristic classes $u = -c_2(H)$ respectively $c_2, c_4, \ldots, c_{10}$ of the well-known bundles H and E. Using the formula $\mathrm{ch}(E) = \sum_{i=1}^{10} e^{x_i}$ (for the formal roots x_i), the analogue for the bundle H and the fact that Chern classes may be described as the elementary polynomials in the formal roots one obtains easily:

$$
\begin{aligned}
\mathrm{ch}(H) =& 2 + u + \frac{u^2}{12} + \frac{u^3}{360} + \frac{u^4}{20160} + \frac{u^5}{1814400} \\
\mathrm{ch}(E) =& 10 - c_2 + \frac{1}{12}\big(c_2^2 - 2c_4\big) + \frac{1}{360}\big(-c_2^3 + 3c_2c_4 - 3c_6\big) \\
&+ \frac{1}{20160}\big(c_2^4 - 4c_2^2c_4 + 2c_4^2 + 4c_2c_6 - 4c_8\big) \\
&+ \frac{1}{1814400}\big(-5c_{10} - c_2^5 + 5c_2^3c_4 - 5c_2c_4^2 - 5c_2^2c_6 + 5c_4c_6 + 5c_2c_8\big)
\end{aligned}
$$

Now use the formula $T_{\mathbb{C}}M = E \otimes H$ to successively compute the Chern classes of the complexified tangent bundle and the Pontryagin classes p_i of M:

$$
\begin{aligned}
p_1 =& -(2c_2 - 10u) \\
p_2 =& c_2^2 + 2c_4 - 14c_2u + 45u^2 \\
p_3 =& -(2c_2c_4 + 2c_6 - 8c_2^2u + 40c_2u^2 - 120u^3) \\
p_4 =& c_4^2 + 2c_2c_6 + 2c_8 - 10c_2c_4u + 22c_6u + 28c_2^2u^2 - 40c_4u^2 - 56c_2u^3 + 210u^4 \\
p_5 =& -(2c_{10} + 2c_4c_6 + 2c_2c_8 - 6c_4^2u + 4c_2c_6u + 52c_8u + 18c_2c_4u^2 - 78c_6u^2 \\
&- 56c_2^2u^3 + 128c_4u^3 + 28c_2u^4 - 252u^5)
\end{aligned}
$$

Filling in these Pontryagin classes into the characteristic series $Q_{\hat{A}}$ of the $\hat{A}$-genus, which is given by

$$
\begin{aligned}
&1 - \frac{p_1}{24} + \frac{1}{5760}\big(7p_1^2 - 4p_2\big) + \frac{1}{967680}\big(-31p_1^3 + 44p_1p_2 - 16p_3\big) \\
&+ \frac{1}{464486400}\big(381p_1^4 - 904p_1^2p_2 + 208p_2^2 + 512p_1p_3 - 192p_4\big) \\
&+ \frac{1}{122624409600}\big(-2555p_1^5 + 8584p_1^2p_2 - 4976p_1p_2^2 - 5856p_1^2p_3 \\
&+ 2688p_2p_3 + 3392p_1p_4 - 1280p_5\big)
\end{aligned}
$$

yields the $\hat{A}$-genus

$$\begin{aligned}\hat{A}(M) =&1 + \frac{1}{12}\big(c_2 - 5u\big) + \frac{1}{720}\big(3c_2^2 - c_4 - 28c_2u + 65u^2\big)\\&+ \frac{1}{60480}\big(10c_2^3 - 9c_2c_4 + 2c_6 - 136c_2^2u + 55c_4u + 570c_2u^2 - 820u^3\big)\\&+ \frac{1}{3628800}\big(21c_2^4 - 34c_2^2c_4 + 5c_4^2 + 13c_2c_6 - 3c_8 - 384c_2^3u + 409c_2c_4u\\&- 113c_6u + 2274c_2^2u^2 - 1060c_4u^2 - 5736c_2u^3 + 5760u^4\big)\\&+ \frac{1}{479001600}\big(90c_2^5 - 219c_2^3c_4 + 87c_2c_4^2 + 109c_2^2c_6 - 32c_4c_6 - 43c_2c_8\\&+ 10c_{10} - 2136c_2^4u + 3990c_2^2c_4u - 675c_4^2u - 1834c_2c_6u + 525c_8u\\&+ 16524c_2^3u^2 - 19740c_2c_4u^2 + 6155c_6u^2 - 57576c_2^2u^3 + 29935c_4u^3 + 98815c_2u^4\\&- 73985u^5\big)\end{aligned}$$

We now compute the Chern characters of the exterior powers of E. For this we use that the roots of $\bigwedge^k E$ are given by $y_{i_1,\ldots,i_k} = x_{i_1} + \cdots + x_{i_k}$ for $1 \leq i_1 < \cdots < i_k \leq 10$.

$$\begin{aligned}\mathrm{ch}(\textstyle\bigwedge^2 E) =&45 - 8c_2 + \frac{1}{3}\big(2c_2^2 - c_4\big) + \frac{1}{180}\big(-4c_2^3 - 3c_2c_4 + 33c_6\big)\\&+ \frac{1}{5040}\big(2c_2^4 + 6c_2^2c_4 + 11c_4^2 - 76c_2c_6 + 118c_8\big)\\&+ \frac{1}{907200}\big(-4c_2^5 - 25c_2^3c_4 - 95c_2c_4^2 + 445c_2^2c_6 + 5c_4c_6 - 1075c_2c_8\\&+ 1255c_{10}\big)\\\mathrm{ch}(\textstyle\bigwedge^3 E) =&120 - 28c_2 + \frac{1}{3}\big(7c_2^2 + 4c_4\big) + \frac{1}{90}\big(-7c_2^3 - 24c_2c_4 + 24c_6\big)\\&+ \frac{1}{720}\big(c_2^4 + 8c_2^2c_4 + 8c_4^2 - 8c_2c_6 - 136c_8\big)\\&+ \frac{1}{453600}\big(-7c_2^5 - 100c_2^3c_4 - 260c_2c_4^2 + 100c_2^2c_6 - 640c_4c_6\\&+ 7460c_2c_8 - 18260c_{10}\big)\\\mathrm{ch}(\textstyle\bigwedge^4 E) =&210 - 56c_2 + \frac{1}{3}\big(14c_2^2 + 17c_4\big) + \frac{1}{180}\big(-28c_2^3 - 141c_2c_4 - 129c_6\big)\\&+ \frac{1}{720}\big(2c_2^4 + 22c_2^2c_4 + 19c_4^2 + 68c_2c_6 + 22c_8\big)\\&+ \frac{1}{907200}\big(-28c_2^5 - 535c_2^3c_4 - 1265c_2c_4^2 - 3245c_2^2c_6 - 5125c_4c_6\\&- 6205c_2c_8 + 220585c_{10}\big)\end{aligned}$$

$$\begin{aligned}\mathrm{ch}(\textstyle\bigwedge^5 E) =& 252 - 70c_2 + \frac{1}{6}\big(35c_2^2 + 50c_4\big) + \frac{1}{36}\big(- 7c_2^3 - 39c_2c_4 - 57c_6\big)\\ &+ \frac{1}{288}\big(c_2^4 + 12c_2^2c_4 + 52c_2c_6 + 10c_4^2 + 140c_8\big)\\ &+ \frac{1}{181440}\big(- 7c_2^5 - 145c_2^3c_4 - 335c_2c_4^2 - 1199c_2^2c_6 - 1537c_4c_6\\ &- 8881c_2c_8 - 78095c_{10}\big)\end{aligned}$$

These exterior powers already determine all the higher ones. For this recall the definition of the Hodge $*$-operator:

$$*: \bigwedge^k E_x \to \bigwedge^{10-k} E_x \qquad e_{i_1} \wedge \cdots \wedge \mathrm{e}_{i_k} \mapsto \pm e_{j_1} \wedge \cdots \wedge e_{j_{10-k}}$$

where the e_i form a basis of E_x and where $\{j_1, \ldots, j_{10-k}\}$ is the complement of $\{i_1, \ldots, i_k\}$ in $\{1, \ldots, 10\}$. The plus sign is chosen if the permutation $\{i_1, \ldots, i_k, j_1, \ldots, j_{10-k}\}$ is even, the minus sign if it is odd. The operator extends to a bundle isomorphism

$$\bigwedge^k E \cong \bigwedge^{10-k} E$$

for $0 \leq k \leq 10$ and

$$\mathrm{ch}\big(\bigwedge^k E\big) = \mathrm{ch}\big(\bigwedge^{10-k} E\big)$$

respectively

$$\begin{aligned}\mathrm{ch}(\textstyle\bigwedge_0^k E) &= \mathrm{ch}(\textstyle\bigwedge^k E) - \mathrm{ch}(\textstyle\bigwedge^{k-2} E)\\ &= \mathrm{ch}(\textstyle\bigwedge^{10-k} E) - \mathrm{ch}(\textstyle\bigwedge^{10-(k-2)} E)\\ &= -\mathrm{ch}(\textstyle\bigwedge_0^{12-k} E)\end{aligned}$$

and

$$i^{p,q} = -i^{12-p,q}$$

In particular, we obtain that $i^{6,q} = 0$ for all $q \in \mathbb{N}_0$, since $\mathrm{ch}(\bigwedge^6 E) = \mathrm{ch}(\bigwedge^4 E)$.

It remains to compute the Chern characters of the symmetric bundles. For this we may use the Clebsch-Gordan formula

$$S^j H \otimes S^k H \cong \sum_{r=0}^{\min(j,k)} S^{j+k-2r} H$$

to obtain

$$
\begin{aligned}
\mathrm{ch}(S^0H) =&1\\
\mathrm{ch}(S^1H) =&\mathrm{ch}(H)\\
\mathrm{ch}(S^2H) =&\mathrm{ch}(H)^2-1\\
\mathrm{ch}(S^3H) =&\mathrm{ch}(H)^3-2\mathrm{ch}(H)\\
\mathrm{ch}(S^4H) =&\mathrm{ch}(H)^4-3\mathrm{ch}(H)^2+1\\
\mathrm{ch}(S^5H) =&\mathrm{ch}(H)^5-4\mathrm{ch}(H)^3+3\mathrm{ch}(H)\\
\mathrm{ch}(S^6H) =&\mathrm{ch}(H)^6-5\mathrm{ch}(H)^4+6\mathrm{ch}(H)^2-1\\
\mathrm{ch}(S^7H) =&\mathrm{ch}(H)^7-6\mathrm{ch}(H)^5+10\mathrm{ch}(H)^3-4\mathrm{ch}(H)
\end{aligned}
$$

since the irreducible representations of $\mathbf{Sp}(1)$ are the symmetric powers of $\mathbb{C}^2$.

Alternatively, we may use a similar approach as for the exterior powers using the roots of the symmetric bundles given by the $y_{i_1,\dots,i_k} = x_{i_1}+\cdots+x_{i_k}$ for $1 \le i_1 \le \cdots \le i_k \le 10$.

This enables us to compute the following indices in terms of characteristic numbers. (The "indices" $i^{p,q}$ with $p+q+5$ odd are to be regarded as formal expressions, as they do not necessarily correspond to twisted Dirac operators.) As we have seen, as for the first parameter p it is sufficient to compute the indices from $p=0$ to $p=5$.

$$
\begin{aligned}
i^{0,0} =&\frac{1}{479001600}\big(10c_{10}+90c_2^5-32c_4c_6-2136c_2^4u-675c_4^2u+525c_8u+6155c_6u^2\\
&+29935c_4u^3-73985u^4-3c_2^3(73c_4-5508u^2)+c_2^2(109c_6+3990c_4u-57576u^3)\\
&+c_2(87c_4^2-43c_8-1834c_6u-19740c_4u^2+98815u^4)\big)\\
i^{0,1} =&\frac{1}{239500800}\big(10c_{10}+90c_2^5-32c_4c_6-750c_2^4u-345c_4^2u+327c_8u-643c_6u^2\\
&-22799c_4u^3+90817u^5-3c_2^3(73c_4+1840u^2)+c_2^2(109c_6+1746c_4u+50400u^3)\\
&+c_2(87c_4^2-43c_8-976c_6u+4284c_4u^2-116543u^4\big)\\
i^{0,3} =&\frac{1}{119750400}\big(10c_{10}+90c_2^5-32c_4c_6+4794c_2^4u+975c_4^2u-465c_8u-4075c_6u^2\\
&+87025c_4u^3+310465u^5-3c_2^3(73c_4-8368u^2)\\
&+c_2^2(109c_6-7230c_4u-135456u^3)\\
&+c_2(87c_4^2-43c_8+2456c_6u-6540c_4u^2-277055u^4)\big)
\end{aligned}
$$

$$
\begin{aligned}
i^{0,5} =& \frac{1}{79833600}\big(10c_{10} + 90c_2^5 - 32c_4c_6 + 14034c_2^4u + 3175c_4^2u - 1785c_8u \\
&+ 74685c_6u^2 - 1546255c_4u^3 + 44944065u^5 + c_2^3(-219c_4 + 498544u^2) \\
&+ c_2^2(109c_6 - 22190c_4u + 6228704u^3) \\
&+ c_2(87c_4^2 - 43c_8 + 8176c_6u - 404740c_4u^2 + 30037185u^4)\big) \\
i^{0,7} =& \frac{1}{59875200}\big(10c_{10} + 90c_2^5 - 32c_4c_6 + 26970c_2^4u + 6255c_4^2u - 3633c_8u \\
&+ 362357c_6u^2 - 18291599c_4u^3 + 3830160577u^5 - 3c_2^3(73c_4 - 682800u^2) \\
&+ c_2^2(109c_6 - 43134c_4u + 61087200u^3) \\
&+ c_2(87c_4^2 - 43c_8 + 16184c_6u - 1760556c_4u^2 + 796656577u^4)\big) \\
i^{1,0} =& \frac{1}{239500800}\big(- 610c_{10} + 450c_2^5 + 1952c_4c_6 - 9294c_2^4u - 49575c_4^2u + 22425c_8u \\
&- 149405c_6u^2 + 690875c_4u^3 - 369925u^5 + c_2^3(-2481c_4 + 60576u^2) \\
&+ c_2^2(-4669c_6 + 43050c_4u - 179904u^3) \\
&+ c_2(4593c_4^2 - 3317c_8 + 56104c_6u - 224760c_4u^2 + 113915u^4)) \\
i^{1,2} =& \frac{1}{79833600}\big(- 610c_{10} + 450c_2^5 + 1952c_4c_6 + 9186c_2^4u + 60425c_4^2u - 43575c_8u \\
&+ 8115c_6u^2 + 750715c_4u^3 - 693765u^5 - c_2^3(2481c_4 + 48544u^2) \\
&- c_2^2(4669c_6 + 39670c_4u + 144704u^3) \\
&+ c_2(4593c_4^2 - 3317c_8 - 101416c_6u + 203800c_4u^2 + 43515u^4)\big) \\
i^{1,4} =& \frac{1}{47900160}\big(- 610c_{10} + 450c_2^5 + 1952c_4c_6 + 46146c_2^4u + 280425c_4^2u - 175575c_8u \\
&- 5810093c_6u^2 - 31189765c_4u^3 - 6917125u^5 - 3c_2^3(827c_4 - 333472u^2) \\
&- c_2(-4593c_4^2 + 3317c_8 + 416456c_6u + 3272904c_4u^2 + 8410117u^4) \\
&- c_2^2(4669c_6 + 4770(43c_4u - 1504u^3))\big) \\
i^{2,1} =& \frac{1}{5443200}\big(15070c_{10} + 90c_2^5 - 2864c_4c_6 - 246c_2^4u + 855c_4^2u + 21567c_8u \\
&- 153103c_6u^2 - 79439c_4u^3 + 90817u^5 - 3c_2^3(241c_4 + 2112u^2) \\
&+ c_2^2(13c_6 + 1146c_4u + 21984u^3) \\
&+ c_2(1599c_4^2 + 8369c_8 - 7840c_6u + 32784c_4u^2 + 56257u^4)\big) \\
i^{2,3} =& \frac{1}{2721600}\big(15070c_{10} + 90c_2^5 - 2864c_4c_6 + 5298c_2^4u + 59775c_4^2u + 530535c_8u \\
&+ 1506665c_6u^2 + 60625c_4u^3 + 310465u^5 + c_2^3(-723c_4 + 53088u^2) \\
&+ c_2^2(13c_6 - 36630c_4u + 109728u^3) \\
&+ c_2(1599c_4^2 + 8369c_8 - 5848c_6u - 307800c_4u^2 + 414145u^4)\big)
\end{aligned}
$$

$$
\begin{aligned}
i^{3,0} =& \frac{1}{21772800}\big(- 876370c_{10} + 450c_2^5 - 38176c_4c_6 - 7278c_2^4u - 73575c_4^2u \\
&+ 1714425c_8u + 571315c_6u^2 - 293125c_4u^3 - 369925u^5 \\
&+ c_2^3(-4497c_4 + 28512u^2) \\
&+ c_2^2(9347c_6 + 55050c_4u - 22848u^3) \\
&+ c_2(10641c_4^2 + 15931c_8 - 121112c_6u - 159720c_4u^2 - 439045u^4)\big) \\
i^{3,2} =& \frac{1}{7257600}\big(- 876370c_{10} + 450c_2^5 - 38176c_4c_6 + 11202c_2^4u + 190025c_4^2u \\
&- 3765975c_8u - 158205c_6u^2 - 636485c_4u^3 - 693765u^5 \\
&- c_2^3(4497c_4 + 3808u^2) \\
&+ c_2^2(9347c_6 - 104470c_4u - 225728u^3) \\
&+ c_2(10641c_4^2 + 15931c_8 + 201368c_6u - 126680c_4u^2 - 1062405u^4)\big) \\
i^{4,1} =& \frac{1}{7257600}\big(3509330c_{10} + 450c_2^5 - 61216c_4c_6 + 282c_2^4u + 46275c_4^2u \\
&+ 10275c_8u + 994705c_6u^2 + 1245365c_4u^3 + 454085u^5 \\
&- 3c_2^3(1709c_4 + 11376u^2) \\
&+ c_2^2(18977c_6 - 15270c_4u + 24672u^3) \\
&+ c_2(12531c_4^2 - 73079c_8 + 101488c_6u + 228300c_4u^2 + 799685u^4)\big) \\
i^{5,0} =& \frac{1}{3628800}\big(- 1415810c_{10} + 90c_2^5 - 15488c_4c_6 - 1254c_2^4u - 13275c_4^2u \\
&- 1020075c_8u - 599905c_6u^2 - 314465c_4u^3 - 73985u^5 \\
&- 3c_2^3(367c_4 - 832u^2) \\
&+ c_2^2(5191c_6 + 10290c_4u + 11136u^3) \\
&- c_2(-2733c_4^2 + 33097c_8 + 25816c_6u + 3360c_4u^2 + 143105u^4)\big)
\end{aligned}
$$

For the sake of completeness we now give a complete description of what results from solving the described linear system of equations:

$$
\begin{aligned}
c_2^5 =& -\frac{1686169}{315} + \frac{14345b_{10}}{12} - \frac{3239b_4}{9} - \frac{8879b_6}{9} + 361b_8 + \frac{19697d}{63} + \frac{19e}{3} \\
&- \frac{50f}{3} + \frac{323450368i^{0,0}}{315} - \frac{8q}{3} - \frac{10724s}{27} - \frac{4894t}{27} - \frac{829913\tilde{v}}{45} \\
c_2^3c_4 =& -\frac{69653}{210} + \frac{1977b_{10}}{4} - \frac{2197b_4}{15} - \frac{950b_6}{3} + \frac{927b_8}{5} + \frac{3161d}{84} + 3e - 6f \\
&+ \frac{21455872i^{0,0}}{105} + 3q - \frac{1544s}{9} - \frac{1249t}{9} - \frac{34967\tilde{v}}{15}
\end{aligned}
$$

$$
\begin{aligned}
c_2^2c_6 =&\frac{9449}{42}+148b_{10}-\frac{3352b_4}{15}-\frac{359b_6}{3}+\frac{912b_8}{5}+\frac{149d}{84}+2f\\
&+\frac{3979264i^{0,0}}{105}-\frac{719s}{9}-\frac{382t}{9}-\frac{3014\tilde{v}}{15}\\
c_2c_8 =&\frac{5129}{70}+\frac{131b_{10}}{2}-\frac{244b_4}{5}-61b_6+\frac{192b_8}{5}-\frac{23d}{28}\\
&+\frac{95744i^{0,0}}{35}-\frac{22s}{3}-\frac{41t}{3}+\frac{165\tilde{v}}{5}\\
c_{10} =&-\frac{181}{70}+\frac{3b_{10}}{4}+\frac{7b_4}{5}+2b_6+\frac{9b_8}{5}+\frac{d}{28}+\frac{512i^{0,0}}{35}-\frac{s}{3}-\frac{2t}{3}-\frac{7\tilde{v}}{5}\\
c_2^4u =&-\frac{37109}{210}-\frac{39b_{10}}{4}+\frac{1079b_4}{15}+\frac{202b_6}{3}-\frac{39b_8}{5}+\frac{929d}{84}\\
&-\frac{14587904i^{0,0}}{105}+3q+\frac{4s}{9}-\frac{394t}{9}-\frac{10631\tilde{v}}{15}\\
c_2^3u^2 =&\frac{33827}{420}-\frac{33b_{10}}{4}+\frac{53b_4}{15}+\frac{35b_6}{6}-\frac{33b_8}{5}-\frac{473d}{168}\\
&+\frac{1443328i^{0,0}}{105}+\frac{40s}{9}+\frac{29t}{9}+\frac{2032\tilde{v}}{15}\\
c_2^2c_4u =&\frac{-2315}{42}-\frac{63b_{10}}{4}+\frac{553b_4}{15}+\frac{128b_6}{3}-\frac{63b_8}{5}+\frac{199d}{84}\\
&-\frac{2768896i^{0,0}}{105}+2q+\frac{74s}{9}-\frac{161t}{9}-\frac{2179\tilde{v}}{15}\\
c_2^2u^3 =&\frac{6347}{420}+\frac{3b_{10}}{4}-\frac{13b_4}{15}-\frac{2b_6}{3}+\frac{3b_8}{5}-\frac{65d}{168}-\frac{19456i^{0,0}}{21}-\frac{2s}{9}-\frac{t}{9}+\frac{50\tilde{v}}{3}\\
c_2c_6u =&\frac{907}{210}-\frac{9b_{10}}{2}-\frac{172b_4}{15}-\frac{23b_6}{3}-\frac{18b_8}{5}+\frac{11d}{84}\\
&-\frac{478208i^{0,0}}{105}+\frac{49s}{9}+\frac{92t}{9}-\frac{212\tilde{v}}{15}\\
c_2c_4u^2 =&\frac{349}{21}-\frac{27b_{10}}{4}+\frac{17b_4}{15}+\frac{13b_6}{3}-\frac{27b_8}{5}-\frac{13d}{21}\\
&+\frac{295936i^{0,0}}{105}+\frac{40s}{9}+\frac{29t}{9}+\frac{439\tilde{v}}{15}\\
c_2u^4 =&-\frac{81}{70}+\frac{3d}{28}+\frac{1536i^{0,0}}{35}-\frac{31\tilde{v}}{5}\\
c_4u^3 =&\frac{257}{35}+\frac{9b_{10}}{4}-\frac{13b_4}{5}-2b_6+\frac{9b_8}{5}-\frac{d}{7}-\frac{2048i^{0,0}}{35}-\frac{2s}{3}-\frac{t}{3}+\frac{33\tilde{v}}{5}\\
b_2 =&\frac{4}{5}-\frac{8b_4}{5}-b_6+\frac{4b_8}{5}+b_{10}
\end{aligned}
$$

The result depends on the variables $e=c_4^2c_2$, $f=c_4c_6$, $q=c_4^2u$, $s=c_6u^2$, $t=c_8u$ and $\tilde{v}=u^5$, on the Betti numbers b_i, on the dimension of the isometry group d and on the $\hat{A}$-genus $i^{0,0}$.

Observe that the last line just reestablishes the well-known relation (1.3) on Betti numbers from theorem 1.13.

Moreover, we may give corresponding expressions for higher indices. So we obtain

$$\begin{aligned} i^{1,6} =& \frac{1}{45}(-2109 - 1395b_{10} + 1752b_4 + 1275b_6 - 1116b_8 - 15d - 98304i^{0,0} + 320s \\ &+ 160t + 5088\tilde{v}) \\ i^{2,5} =& \frac{1}{280}(22661 + 5915b_{10} - 4984b_4 - 5075b_6 + 4732b_8 - 905d - 1015808i^{0,0} \\ &+ 245i^{1,6} - 1120t + 33376\tilde{v}) \end{aligned}$$

The following indices are written in the two variables $i^{1,6}$ and $i^{2,5}$:

$$\begin{aligned} i^{3,4} =& -\frac{1871}{35} - 29b_{10} + \frac{187b_4}{5} + 24b_6 - \frac{116b_8}{5} + \frac{6d}{7} - \frac{131072i^{0,0}}{35} - 2i^{1,6} + i^{2,5} + \frac{v}{10} \\ i^{4,3} =& \frac{1382}{35} + 10b_4 + 2b_8 - \frac{23d}{7} - \frac{262144i^{0,0}}{35} - i^{2,5} + \frac{v}{5} \\ i^{5,2} =& \frac{3001}{35} + 22b_{10} - 25b_4 - 24b_6 + 17b_8 - \frac{22d}{7} - \frac{131072i^{0,0}}{35} + i^{1,6} - i^{2,5} + \frac{v}{10} \end{aligned}$$

Note that their integrality is already covered by known relations. Moreover recall the formula

$$\sum_{p=0} n(-1)^p i^{p,n+2-p} = 2\chi(M) + b_{2n-2} + b_{2n}$$

from proposition [67].84, p. 117, which—in the case $n = 5$—is also a direct consequence of our computations.

Moreover, we know that M has a negative definite intersection form. Hence its signature equals $-b_{10}$ (cf. theorem 1.16). However, we may prove this independently by formulating the L-genus of M as the number

$$\begin{aligned} &-\frac{1}{467775}(2(5110c_{10} - 45c_2^5 + 4438c_4c_6 - 1011c_2^4u - 13275c_4^2u + 142050c_8u \\ &- 89620c_6u^2 + 152260c_4u^3 - 112085u^5 + c_2^3(76c_4 - 1641u^2) - 2c_2^2(613c_6 - 3705c_4u \\ &+ 28473u^3) + c_2(-1083c_4^2 + 3272c_8 - 5584c_6u + 40560c_4u^2 - 64835u^4))) \end{aligned}$$

This expression is a linear combination of certain terms in the system of equations (cf. page 135). The system of equations then directly yields that this term equals $-b_{10}$.

Now compute the Hilbert Polynomial f of M in the parameters d, v and $i^{0,0} \in \mathbb{Q}$. (Assume M to be rationally 3-connected.) The Hilbert Polynomial f on M is given by

$$f(q) = \mathrm{ind}D(S^qH) = \langle \hat{A} \cdot \mathrm{ch}(S^qH), [M] \rangle = i^{0,q}$$

The formula

$$(H-2)^{\otimes m} = \sum_{j=0}^{m} (-1)^j \left(\binom{2m}{j} - \binom{2m}{j-2} \right) S^{m-j} H$$

resulting from the Glebsch-Gordan formula now lets us compute the following list of equations

$$\begin{aligned}
\langle \hat{A}(M) \cdot \mathrm{ch}(H-2)^{\otimes 0}, [M] \rangle =& f(0) \\
\langle \hat{A}(M) \cdot \mathrm{ch}(H-2)^{\otimes 1}, [M] \rangle =& f(1) - 2f(0) \\
\langle \hat{A}(M) \cdot \mathrm{ch}(H-2)^{\otimes 2}, [M] \rangle =& f(2) - 4f(1) + 5f(0) \\
\langle \hat{A}(M) \cdot \mathrm{ch}(H-2)^{\otimes 3}, [M] \rangle =& f(3) - 6f(2) + 14f(1) - 14f(0) \\
\langle \hat{A}(M) \cdot \mathrm{ch}(H-2)^{\otimes 4}, [M] \rangle =& f(4) - 8f(3) + 27f(2) - 48f(1) + 42f(0) \\
\langle \hat{A}(M) \cdot \mathrm{ch}(H-2)^{\otimes 5}, [M] \rangle =& f(5) - 10f(4) + 44f(3) - 110f(2) \\
&+ 165f(1) - 132f(0) \\
\langle \hat{A}(M) \cdot \mathrm{ch}(H-2)^{\otimes 6}, [M] \rangle =& f(6) - 12f(5) + 65f(4) - 208f(3) + 429f(2) \\
&- 572f(1) + 429f(0) \\
\langle \hat{A}(M) \cdot \mathrm{ch}(H-2)^{\otimes 7}, [M] \rangle =& f(7) - 14f(6) + 90f(5) - 350f(4) + 910f(3) \\
&- 1638f(2) + 2002f(1) - 1430f(0) \\
\langle \hat{A}(M) \cdot \mathrm{ch}(H-2)^{\otimes 8}, [M] \rangle =& f(8) - 16f(7) + 119f(6) - 544f(5) + 1700f(4) \\
&- 3808f(3) + 6188f(2) - 7072f(1) + 4862f(0) \\
\langle \hat{A}(M) \cdot \mathrm{ch}(H-2)^{\otimes 9}, [M] \rangle =& f(9) - 18f(8) + 152f(7) - 798f(6) + 2907f(5) \\
&- 7752f(4) + 15504f(3) - 23256f(2) + 25194f(1) \\
&- 16796f(0) \\
\langle \hat{A}(M) \cdot \mathrm{ch}(H-2)^{\otimes 10}, [M] \rangle =& f(10) - 20f(9) + 189f(8) - 1120f(7) + 4655f(6) \\
&- 14364f(5) + 33915f(4) - 62016f(3) + 87210f(2) \\
&- 90440f(1) + 58786f(0) \\
\langle \hat{A}(M) \cdot \mathrm{ch}(H-2)^{\otimes 11}, [M] \rangle =& f(11) - 22f(10) + 230f(9) - 1518f(8) + 7084f(7) \\
&- 24794f(6) + 67298f(5) - 144210f(4) + 245157f(3) \\
&- 326876f(2) + 326876f(1) - 208012f(0)
\end{aligned}$$

The leading term of the power series of the $\hat{A}$-genus is 1 and the first non-zero coefficient of the power series $\mathrm{ch}(H-2)^m$ lies in degree m. Thus we obtain that $\langle \hat{A}(M)\cdot \mathrm{ch}(H-2)^{\otimes 5}, [M]\rangle = u^5$ and all the higher terms vanish.

Moreover, by theorem 1.17 we have $f(0) = i^{0,0}$, $f(1) = f(3) = 0$, $f(5) = 1$ and $f(7) = d$. This yields a system of equations which we may solve by

$$f(0) = 0$$

$$f(1) = i^{0,0}$$

$$f(2) = \frac{-2816 + 128d - 360448 i^{0,0} - 7v}{229376}$$

$$f(3) = 0$$

$$f(4) = \frac{269568 - 7040d + 4685824 i^{0,0} + 273v}{1146880}$$

$$f(5) = 1$$

$$f(6) = \frac{228096 + 18304d - 2342912 i^{0,0} - 273v}{114688}$$

$$f(7) = d$$

$$f(8) = \frac{13(-143616 + 35200d + 3063808 i^{0,0} + 595v)}{114688}$$

$$f(9) = \frac{1}{140}(-10692 + 1760d + 262144 i^{0,0} + 63v)$$

$$f(10) = \frac{13(-4333824 + 598400d + 116424704 i^{0,0} + 33915v)}{229376}$$

$$f(11) = \frac{1}{14}(-9152 + 1144d + 262144 i^{0,0} + 91v)$$

The Hilbert polynomial has degree 11. Thus we have sufficient data to compute it:

$$\begin{aligned}
f(q) =& i^{0,0} + \frac{684288 - 35200d + 50462720i^{0,0} + 2205v}{454164480} q \\
&+ \frac{2909952 - 144000d - 3385851904i^{0,0} + 8575v}{2890137600} q^2 \\
&+ \frac{-41119488 + 2176640d - 3385851904i^{0,0} - 140931v}{26011238400} q^3 \\
&+ \frac{-1059264 + 53280d + 166330368i^{0,0} - 3229v}{928972800} q^4 \\
&+ \frac{28512 - 2640d + 9240576i^{0,0} + 257v}{464486400} q^5 \\
&+ \frac{84096 - 4800d - 4718592i^{0,0} + 329v}{619315200} q^6 \\
&+ \frac{8064 - 320d - 524288i^{0,0} + 7v}{619315200} q^7 \\
&+ \frac{-5184 + 480d + 196608i^{0,0} - 49v}{2167603200} q^8 \\
&+ \frac{-3456 + 320d + 131072i^{0,0} - 21v}{13005619200} q^9 \\
&+ \frac{v}{3715891200} q^{10} \\
&+ \frac{v}{40874803200} q^{11}
\end{aligned}$$

Note that we may substitue $(d, v) = (78, 1024)$ and $(d, v) = (36, 264)$ to obtain the well-known Hilbert Polynomials

$$\begin{aligned}
f_{\mathbb{H}\mathbf{P}^5}(5 + 2q) =& \frac{1}{39916800}(1 + 2q)(2 + 2q)(3 + 2q)(4 + 2q)(5 + 2q)(6 + 2q) \\
&\cdot (7 + 2q)(8 + 2q)(9 + 2q)(10 + 2q)(11 + 2q)
\end{aligned}$$

and

$$f_{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^9)}(5 + 2q) = \frac{(1 + q)(2 + q)^2(3 + q)^2(4 + q)^2(5 + q)(5 + 2q)(6 + 2q)(7 + 2q)}{604800}$$

From theorem [68].1.1, p. 2, we are given the formula

$$0 \leq f_M(5 + 2q) \leq f_{\mathbb{H}\mathbf{P}^5}(5 + 2q) = \binom{11 + 2q}{11}$$

for $q \in \mathbb{N}_0$. So we may compute for each q a lower and an upper bound for $i^{0,0}$—depending on d and v. Unfortunately, with q growing, these bounds seem to become worse so that we use low values of q—i.e. $q = 3$ respectively $q = 2$—to obtain:

$$\frac{1}{14}(-9152 + 262144 i^{0,0} + 1144d + 91v) \leq 12376 \tag{4.1}$$

$$\frac{1}{140}(-10692 + 262144 i^{0,0} + 1760d + 63v) \geq 0 \tag{4.2}$$

Unfortunately, these inequalities are not strong enough to yield the vanishing of the $\hat{A}$-genus (when filing in the known bounds for d and v). Observe that from the integrality of indices used in the Hilbert polynomial or from the integrality of Pontryagin classes one may deduce certain properties on divisibility which, however, do not seem to be strong enough to make essential progress on the topic.

Let us now compute further relations involving the $\hat{A}$-genus of a Positive Quaternion Kähler Manifold M. We adapt lemma [66].7.6, p. 169, to dimension 20 and plug in the expression $-(2c_2 - 10u)$ for the first Pontryagin class p_1:

$$8u^5 - p_1 u^4 \geq 0 \Leftrightarrow 8u^5 + (2c_2 - 10u)u^4 \geq 0 \Leftrightarrow c_2 u^4 - u^5 \geq 0 \tag{4.3}$$

From the solution of the fundamental system of equations (cf. page 135) we cite

$$c_2 u^4 = -\frac{81}{70} + \frac{3d}{28} + \frac{1536\hat{A}(M)[M]}{35} - \frac{31u^5}{5}$$

Combining it with formula (4.3) yields

$$-\frac{81}{70} + \frac{3d}{28} - \frac{36u^5}{5} + \frac{1536\hat{A}(M)[M]}{35} \geq 0 \tag{4.4}$$

Recall that $1 \leq v \leq 1024$. So for $d = 0$ the equation becomes

$$-\frac{81}{70} - \frac{36}{5 \cdot 1024} + \frac{1536\hat{A}(M)[M]}{35} \geq 0 \tag{4.5}$$

4.2. Recognising quaternionic projective spaces

In this section we shall prove recognition theorems of the following kind: Whenever the fourth Betti number of a Positive Quaternion Kähler Manifold M equals one, then M is homothetic to the quaternionic projective space.

This theorem is well-known in dimensions smaller than or equal to 16—cf. 1.15. We shall generalise it to dimensions 20 and 24. We remark that a general theorem of this kind *cannot* hold true, since the exceptional Wolf spaces equally satisfy $b_4 = 1$—cf. the remark on [67], p. 89. Nonetheless, as we were told by Gregor Weingart, in dimension 28 a similar recognition theorem—which also identifies the exceptional Wolf space $\mathbf{F}_4/\mathbf{Sp}(3)\mathbf{Sp}(1)$—seems to be possible.

Let us first give generalisations of the theorem in dimension 16.

4.2.1. Dimension 16

In dimension 16 the relations on Betti numbers given in (1.2) of theorem 1.13 together with the Hard-Lefschetz property have the following consequence: If $b_4 = 1$ and if we assume M to be rationally 3-connected (cf. 1.12), we obtain $b_0 = b_4 = b_8 = b_{12} = b_{16} = 1$ with all the other Betti numbers vanishing. So necessarily every Pontryagin class is a multiple of the corresponding power of the form u. This motivates the following slight improvement.

Proposition 4.1. *If each of the Chern classes c_i of the bundle E over a 16-dimensional Positive Quaternion Kähler Manifold is a (scalar) multiple of the corresponding power of u, then the manifold already is homothetic to $\mathbb{H}\mathbf{P}^4$.*

Proof. We refer the reader to chapter E of the appendix for the computation of all the necessary indices $i^{p,q}$ in dimension 16. We then may form the system of equations as in 1.17. This system is linear in the characteristic numbers.

By assumption we may now replace every Chern class c_i by some $x_i u^{n_i}$ for $x_i \in \mathbb{R}$.

If one focuses on the case $b_2 = 0$ (cf. 1.12), the system of equations can be solved and it yields $d = 55$, $b_4 = b_8 = 1$, $b_6 = 0$, $u^4 = 1$ (with all the factors x_i equal to one). We then observe that in dimension 55 only semi-simple Lie groups of rank at least 5 appear. Theorem 1.18 then yields the assertion; i.e. the isometry group becomes very large and permits to identify M as the quaternionic projective space.

If one does not assume $b_2 = 0$, the list of possible configurations for $(d, b_2, b_4, b_6, b_8, u^4)$ becomes a little larger. However, the configuration from above remains the only one with integral $d \in \mathbb{Z}$. $\square$

Assume $b_2 = 0$. Then the same proof works if one only requires c_2 and c_4 to be scalar multiples of u respectively u^2. In this case a numerical solving procedure leads to six different solutions of which the only one with an integral value for d is the requested one as in proposition 4.1.

Focussing on the case that only c_2 is a multiple $x \in \mathbb{R}$ of u leads to the two equations

$$d = 7 + \frac{v}{6} + \frac{vx}{48} \tag{4.6}$$

$$b_4 = \frac{783}{2} - \frac{7}{8}v - \frac{9}{16}vx - \frac{11}{128}vx^2 - \frac{1}{512}vx^3 \tag{4.7}$$

where $v = (4u)^4$ is the quaternionic volume. The element x is integral by the same reasoning as in the original proof, i.e. the proof of theorem [28].5.1, p. 62. On page 21 we saw that it is plausible to suppose the vanishing of $i^{1,5}$ in the case $M \neq \mathbb{H}\mathbf{P}^4$. This assumption produces

$$d = \frac{7(304 + 56x + 3x^2)}{16 + 20x + 3x^2} \tag{4.8}$$

$$b_4 = -\frac{27(1280 - 304x - 40x^2 + 7x^3)}{8(16 + 20x + 3x^2)} \tag{4.9}$$

$$b_6 = \frac{1}{36}\big(3289 - 294x + 63x^2 - 6(c_4u^2)x^2 - \frac{53200}{16 + 20x + 3x^2} - \frac{93548x}{16 + 20x + 3x^2}\big) \tag{4.10}$$

$$b_8 = \frac{1}{144}\Big(14410 - 1113x - (126x^2 - 12(c_4u^2))x^2 - \frac{1163680}{16 + 20x + 3x^2} + \frac{14728x}{16 + 20x + 3x^2}\Big) \tag{4.11}$$

$$c_4^2 = \frac{1}{16}\Big(- 3546 + 567x - 378x^2 + 44(c_4u^2)x^2 - \frac{821520}{16 + 20x + 3x^2} + \frac{445032x}{16 + 20x + 3x^2}\Big) \tag{4.12}$$

The only integral solution for $8 \leq d < 55 = \dim \mathbf{Sp}(5)$ (cf. theorem 1.18) is given by $x = 4$ and $d = 28$. Then we directly obtain $b_4 = 3$ by (4.9) and also $v = 84$ by (4.6). Indeed, by the relations on Betti numbers in 1.13 only two possibilities for (b_4, b_6, b_8) remain, namely $(3, 0, 4)$ or $(3, 2, 3)$. Equations (4.10) and (4.11) yield $c_4u^2 = \frac{27}{32}$ in the first case. By theorem 1.16 we may use the positive definiteness of the bilinear form Q to see $0 \leq Q(c_4, c_4) = c_4^2$. Yet, in the case $(b_4, b_6, b_8) = (3, 2, 3)$ we obtain the contradiction $c_4^2 = -\frac{75}{16}$ by (4.12). As a consequence, we have the following theorem:

Theorem 4.2. *If M is a rationally 3-connected 16-dimensional Positive Quaternion Kähler Manifold with $i^{1,5} = 0$ and if the class c_2 is a scalar multiple of u, then either $M \cong \mathbb{H}\mathbf{P}^4$ or the datum $(d, v, b_4, b_6, b_8) = (28, 84, 3, 0, 4)$ is exactly the one of $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^8)$.*

□

We remark that the property that c_2 is a multiple of u seems to be a special feature of $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$ for $n = 4$ among the infinite series of Wolf spaces other than the quaternionic projective space.

4.2.2. Dimension 20

Let us now prove the result in dimension 20. Again, without restriction we focus on rationally 3-connected Positive Quaternion Kähler Manifolds M^{20}—cf. 1.12.

Theorem 4.3. *If each of the Chern classes c_{2i} of E over M is a (scalar) multiple of the corresponding power of u, then $M \cong \mathbb{H}\mathbf{P}^5$.*

Proof. We proceed as before in dimension 16; i.e. we solve the system of equations (cf. page 135) setting all characteristic classes to rational multiples of a suitable power of the form u. As a result we obtain $b_4 = b_8 = 1$, $b_6 = b_{10} = 0$, $u^5 = 1$, all the scalars are 1 and $d = 78$. However, such a large isometry group can only occur for the quaternionic projective space—cf. theorem 1.18. □

We remark that for this proof to work we do not need that c_6 is a scalar multiple of u^3. We shall now use the observation we stated in lemma 1.14 to finish the reasoning.

Corollary 4.4. *If $b_4(M^{20}) = 1$, then $M^{20} \cong \mathbb{H}\mathbf{P}^5$.*

Proof. Lemma 1.14 tells us that $b_4 = 1$ implies $b_8 = 1$. By Poincaré Duality we may conclude that all the $c_{2i} \in H^{4i}(M)$ satisfy the condition from theorem 4.3. □

Observe that clearly by this corollary we have ruled out an infinite number of possible configurations of Betti numbers, since it automatically follows that $b_4 = 1$ not only implies $b_8 = 1$ but also $b_6 = b_{10} = 0$.

Observe that one may prove this corollary in a slightly different way: Assume only that c_2 and u are scalar multiples and do the same for monomials containing c_2 and u. Use the equations with the additional information $b_4 = 1$ (and $b_2 = 0$) and the result follows directly.

4.2.3. Dimension 24

We apply similar techniques as before to prove

Theorem 4.5. *A 24-dimensional Positive Quaternion Kähler Manifold M with $b_4(M) = 1$ is homothetic to $\mathbb{H}\mathbf{P}^n$.*

Proof. Again we replace c_2 by a scalar multiple of u and do the same for all the monomials containing c_2 as a factor. This and the additional information $b_4 = 1$ (respectively $b_2 = 0$) simplifies the system of equations (cf. 1.17) that we build up as we did for dimensions 16 and 20. Solving it yields a list of possibilities, which we run through in order to isolate the one leading to $\mathbb{H}\mathbf{P}^6$:

We may rule out the first solution, since it yields $d = \frac{244724}{2891} \notin \mathbb{Z}$ for the dimension of the isometry group. The second solution gives $d = 105 = \dim \mathbf{Sp}(7)$. Thus we directly know that $M \cong \mathbb{H}\mathbf{P}^6$ in this case—cf. theorem 1.18. So what is left is to rule out the following configuration of solutions, which is marked by

$$\begin{aligned} d =& \frac{7937019926774969}{402874803650560} x_1 - \frac{19592196959405797}{2417248821903360} x_2^2 + \frac{457452279096536909}{2417248821903360} \\ &- \frac{263256496233}{805749607301120} x_3^6 + \frac{1176648936457}{402874803650560} x_4^5 + \frac{14142811929437}{161149921460224} x_5^4 \\ &- \frac{282904843313851}{604312205475840} x_6^3 \end{aligned}$$

where each x_i is a root of

$$29223x^7 - 358275x^6 - 6960405x^5 + 67759961x^4 + 579930789x^3 - 4142432537x^2 - 9711667063x + 33284884867$$

Numerically, the roots of this polynomial are given by

$$2.156753156, 7.720829360, 11.12408307, 15.23992325, -10.15093795 + 2.570319306i, -3.679678028, -10.15093795 - 2.570319306i$$

A computer-based check on all the possible combinations now shows that there are no integral solutions for d in all these cases. So we are done. □

4.3. Properties of interest

In this section we shall show that under slight assumptions some surprising results on the degree of symmetry of 20-dimensional Positive Quaternion Kähler Manifolds are obtained.

Proposition 4.6. *Unless M^{20} admits an isometric $\mathbb{S}^1$-action, the $\hat{A}$-genus of M^{20} is restricted by*

$$0.0321350097 < \hat{A}(M)[M] < 0.6955146790$$

Proof. The upper bound clearly results from equation (4.1) when substituting in the extremal value $v = 1$. For the lower bound we form the linear combination

$$\frac{1}{448}(-1053 + 136d + 32768i^{0,0}) \geq 0$$

out of equations (4.2) and (4.4). Plug in $d = 0$ and the result follows. □

Let us now use the fact that the terms $f(5 + 2q) = i^{0,5+2q}$ (for $q \geq 0$) are indices of the twisted Dirac operator $D\!\!\!/(S^{5+2q}H)$; i.e. in particular they are integral. This leads to congruence relations for the dimension of the isometry group and the quaternionic volume.

Theorem 4.7. *A 20-dimensional rationally 3-connected Positive Quaternion Kähler Manifold with $\hat{A}(M)[M] = 0$ satisfies*

$$d \equiv 1 \mod 7 \qquad \textit{and} \qquad v \equiv 4 \mod 20$$

PROOF. We use the Hilbert Polynomial f of M. Thus, using the assumption that $\hat{A}(M)[M] = 0$, we obtain

$$\mathbb{Z} \ni i^{0,9} = f(9) = \frac{1}{140}(-10692 + 1760d + 63v).$$

This implies that

$$\begin{aligned} & -10692 + 1760d + 63v \equiv 0 \mod 140 \\ \iff & 88 + 80d + 63v \equiv 0 \mod 140 \\ \iff & (d \equiv 1 \mod 7) \vee (v \equiv 4 \mod 20) \end{aligned}$$

□

Remark 4.8.

- Any computation of further indices seems to result in the fact that only denominators appear that divide $2^2 \cdot 5 \cdot 7$. Since $v_{\mathbb{H}\mathbf{P}^5} = 1024$ and since $v_{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^9)} = 264$, we see that the relations found in the theorem are the only ones that may hold on the quaternionic volume when focussing on congruence modulo m for $m|140$.
- The dimension $d_{\mathbb{H}\mathbf{P}^n}$ of $\mathbb{H}\mathbf{P}^n$ is given by $(n+1)(2n+3)$ with

$$d_{\mathbb{H}\mathbf{P}^n} \equiv (n+1)((n+1)+(n+2)) \equiv (-1)(-1+0) \equiv 1 \mod n+2.$$

The dimension $d_{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})}$ of $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$ is given by

$$d_{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})} = \begin{cases} ((n+3)/2)(2(n+3)/2+1) = \frac{1}{2}(n+3)(n+4) & \text{for } n \text{ odd} \\ ((n+4)/2)(2(n+4)/2-1) = \frac{1}{2}(n+3)(n+4) & \text{for } n \text{ even} \end{cases}$$

So in any case we have

$$d_{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})} \equiv \frac{1}{2} \cdot 1 \cdot 2 \equiv 1 \mod n+2.$$

Thus in dimensions without exceptional Wolf spaces—for these it is not true—by the main conjecture 1.6 it should hold on a rationally 3-connected Positive Quaternion Kähler Manifold that the dimension of the isometry group is congruent 1 modulo $n+2$.

⊡

Corollary 4.9. *A* 20*-dimensional Positive Quaternion Kähler Manifold satisfying* $\hat{A}(M)[M] = 0$ *and possessing an isometry group of dimension greater than* 36 *is isometric to the complex Grassmannian or the quaternionic projective space.*

PROOF. By corollary 1.12 we assume the manifold to be rationally 3-connected. There are no compact Lie groups in dimensions 43, 50, 57, 64 and 71 with rank smaller than or equal to 5. Thus theorems 1.18 and 4.7 yield the result. □

We shall now prove the existence of $\mathbb{S}^1$-actions on 20-dimensional Positive Quaternion Kähler Manifolds which satisfy some slight assumptions. For this we use the definiteness of the intersection form to establish

Lemma 4.10. *We have:*

$$u^5 \geq 0 \qquad c_2^4 u \geq 0 \qquad c_2^2 u^3 \geq 0 \qquad c_4^2 u \geq 0$$

More generally, the same holds for

$$(kc_2^2 + lc_2 u + mu^2 + nc_4)^2 u \geq 0$$

with $k, l, m, n \in \mathbb{R}$.

Proof. Recall the generalised intersection form Q from theorem 1.16. All the classes y from the assertion may be written as $y = Q(x, x)$ for some $x \in H^r(M)$ with $r \in \{4, 8\}$. However, the intersection form Q is positive definite in degrees divisible by 4 and it results that $Q(x, x) \geq 0$. □

Lemma 4.10 yields that $c_2^2 u^3 \geq 0$, which translates to

$$\frac{495392 - 14240d - 35651584 i^{0,0} - 1120 i^{1,6} + 707v}{35840} \geq 0$$

after solving the usual system of equations on indices (cf. page 135) and substituting in the special solution for $i^{1,6}$. So we obtain:

$$i^{1,6} \leq \frac{495392 - 14240d - 35651584 i^{0,0} + 707v}{1120}$$

which already shows that for extremal values of v and d, i.e. $v = 1024$ and $d = 0$, the index $i^{1,6}$ becomes very small.

Now consider the term $(c_2 u + mu^2)^2 u$ together with the solution from the system of equations (cf. page 135) and the solution for $i^{1,6}$. By lemma 4.10 we have $(c_2 u + mu^2)^2 u \geq 0$. Suppose $d = 0$. This has the consequence that

$$i^{1,6} \leq \frac{495392 - 35651584 i^{0,0} - 82944m + 3145728 i^{0,0} m + 707v - 434mv + 35m^2 v}{1120}$$

for all $m \in \mathbb{R}$. The right hand side is a parabola in m. Determine the apex of this parabola as $m_0 = \frac{41472 - 1572864 i^{0,0} + 217v}{35v}$, put it into the inequality and obtain:

$$\begin{aligned} i^{1,6} \leq -\frac{1}{4900v}&(214990848 + 309237645312 (i^{0,0})^2 + 82516v + 2793v^2 \\ &+ 131072 i^{0,0}(-124416 + 539v)) \end{aligned}$$

The right hand side is a function in $i^{0,0}$ and v which has no critical point in the interior of the square $\big[0.0321350097, 0.695514790\big] \times \big[1, 1024\big]$—cf. proposition 4.6. Thus its maximum lies in the boundary of the square. A direct check reveals that on the border of the square the function is decreasing monotonously in $i^{0,0}$ for $v \in \{1, 1024\}$. Analogously, we see that for $i^{0,0} = 0.0321350097$ the function has the only maximum -549.348 for $v = 61$ and for $i^{0,0} = 0.695514790$ it is increasing in $[1, 1024]$. So it takes its maximum -549.348 on the square in $v = 61$ and $i^{0,0} = 0.695514790$. So, in particular, we obtain the following theorem:

Theorem 4.11. *A 20-dimensional Positive Quaternion Kähler Manifold with*

$$i^{1,6} \geq -549$$

admits an effective isometric $\mathbb{S}^1$-action.

□

As we explained on page 21, for $M \not\cong \mathbb{H}\mathbf{P}^n$ the index $i^{1,6}$ is smaller or equal to zero and it is conjectured to equal zero. On $\mathbb{H}\mathbf{P}^n$ it equals $i^{1,n+1} = n(2n + 3)$.

Next we shall link the existence of an isometric $\mathbb{S}^1$-action to the Euler characteristic.

Theorem 4.12. *A 20-dimensional Positive Quaternion Kähler Manifold M with Euler characteristic restricted by*

$$\chi(M) < 16236$$

admits an effective isometric $\mathbb{S}^1$-action. The same holds if the Betti numbers of M satisfy either

$$b_4 - \frac{b_6}{4} < 842.5$$

or

$$\frac{59b_4}{3} - \frac{25b_6}{4} < 3027.93$$

or—as a combination of both inequalities—if

$$b_4 \leq 3381$$

Proof. The theorem is trivial for $\mathbf{Gr}_2(\mathbb{C}^7)$. Thus by 1.12 we assume M to be rationally 3-connected. We give a proof by contradiction and assume $d = \dim \operatorname{Isom}(M) = 0$. We shall choose special values for k, l, m, n from lemma 4.10. These coefficients determine an element y. We shall obtain the contradiction $Q(y, y) < 0$ under the assumptions from the assertion.

Use the linear combination in corollary 4.10 with coefficients $n = -0.168$, $m = 4.99$, $k = -n$, $l = -2\sqrt{-mn - 18n^2}$ under the assumption of $d = 0$, $b_2 = 0$ together with our solution to the system of indices (cf. page 135) and the relation on Betti numbers (1.3). This results in the formula

$$19.9668 + 0.254016b_4 - 0.063504b_6 - 9835.62i^{0,0} + 0.0801763v \geq 0$$

(where coefficients are rounded off.) Substituting in the lower bound $i^{0,0} \geq 0.0321350097$ from proposition 4.6 and the upper bound $v = 1024$ yields $b_4 - \frac{b_6}{4} \geq 842.468$. This contradicts our assumption $b_4 - \frac{b_6}{4} < 842.5$. Thus we obtain $d \neq 0$.

The second formula involving Betti numbers results from similar arguments with coefficients $l = 22k$, $n = -\frac{3l}{22}$, $m = -\frac{239n}{3}$, $n = 1$. This yields

$$-467.202 + \frac{59b_4}{3} - \frac{25b_6}{4} - 104.025i^{0,0} + 4.72222i^{1,6} - 0.439931v \geq 0$$

Once more we assume there is no $\mathbb{S}^1$-action on M. Thus theorem 4.11 gives us $i^{1,6} \leq -551$. So in this case we additionally substitute the other known bounds $i^{0,0} \geq 0.0321350097$ and $v \geq 1$. This eventually yields that $\frac{59b_4}{3} - \frac{25b_6}{4} \geq 3027.93$ contradicting our assumption.

Assume $d = 0$. Thus from the previous two relations on Betti numbers we compute

$$25 \cdot b_4 - \frac{59b_4}{3} \geq 25 \cdot 842.5 - 3027.93 \Leftrightarrow b_4 \geq 3381.48$$

Hence, whenever $b_4 \leq 3381$, we obtain a contradiction and $d \neq 0$.

The result on the Euler characteristic results from the formula $b_4 \leq 3381$ and the Hard-Lefschetz property by a computer-based check on all possible configurations of (b_4, b_6, b_8, b_{10})—in a suitable range—that satisfy relation (1.3). That is, we start with $b_4 = 3382$ and figure out the configuration of (b_4, b_6, b_8, b_{10})—satisfying all the properties from theorem 1.13—with smallest Euler characteristic. This configuration is given by

$$(b_4, b_6, b_8, b_{10}) = (3382, 0, 3383, 2704)$$

and Euler characteristic $\chi(M) = 16236$. So whenever $\chi(M) < 16236$ we necessarily have $b_4 \leq 3381$. The result follows by our previous reasoning. □

Theorem 4.13. *Let $M^{20} \notin \{\mathbb{H}\mathbf{P}^5, \mathbf{Gr}_2(\mathbb{C}^7)\}$ be a (rationally 3-connected) Positive Quaternion Kähler Manifold with $\hat{A}(M)[M] = 0$. Then it holds:*

- *The dimension d of the isometry group of M satisfies*

$$d \in \{15, 22, 29, 36\}$$

- *The pair (d, v) of the dimension of the isometry group and the quaternionic volume is one of*

$$(15,4),(15,24),(15,44),(15,64),(22,24),\dots,(22,164),$$
$$(29,24),\dots,(29,264),(36,24),\dots(36,384)$$

  where v increases by steps of 20.

- *The connected component $\mathrm{Isom}_0(M)$ of the isometry group of M is as given in table 4.1 up to finite coverings.*

Table 4.1.: Possible isometry groups

$\dim \mathrm{Isom}(M)$	type of $\mathrm{Isom}_0(M)$ up to finite coverings
15	$\mathbf{SO}(6)$, $\mathbf{G}_2 \times \mathbb{S}^1$, $\mathbf{SO}(4) \times \mathbf{SO}(4) \times \mathbf{SO}(3)$, $\mathbf{Sp}(2) \times \mathbf{Sp}(1) \times \mathbb{S}^1 \times \mathbb{S}^1$, $\mathbf{SU}(3) \times \mathbf{SO}(4) \times \mathbb{S}^1$
22	$\mathbf{Sp}(3) \times \mathbb{S}^1$, $\mathbf{SO}(7) \times \mathbb{S}^1$, $\mathbf{G}_2 \times \mathbf{SU}(3)$
29	$\mathbf{SO}(8) \times \mathbb{S}^1$, $\mathbf{SO}(6) \times \mathbf{G}_2$, $\mathbf{G}_2 \times \mathbf{G}_2 \times \mathbb{S}^1$, $\mathbf{SO}(7) \times \mathbf{SU}(3)$, $\mathbf{Sp}(3) \times \mathbf{SU}(3)$
36	$\mathbf{SO}(9)$, $\mathbf{Sp}(4)$

Proof. Recall equation (4.4)

$$-\frac{81}{70} + \frac{3d}{28} - \frac{36u^5}{5} + \frac{1536\hat{A}(M)[M]}{35} \geq 0$$

and set the $\hat{A}$-genus to zero. This results in

$$-\frac{162}{35} + \frac{3d}{7} - \frac{144v}{1024 \cdot 5} \geq 0 \tag{4.13}$$

From computations with the Hilbert Polynomial we know that $d \equiv 1 \mod 7$ and $v \equiv 4 \mod 20$ for the dimension of the isometry group and the quaternionic volume—cf. theorem 4.7. Since $\frac{3d}{7} - \frac{162}{35} < 0$ for all values of d with $d \equiv 1 \mod 7$ and $d < 15$, we may thus rule them all out.

Now use the classification of compact Lie groups. The connected component of the identity of Isom(M) permits a finite covering by a product of a semi-simple Lie group and a torus—cf. lemma 1.19. Now we figure out all those products G of simple Lie groups and tori that satisfy

- $\dim G \equiv 1 \mod 7$,
- $15 \leq \dim G \leq 36$ (cf. corollary 4.9),
- $\operatorname{rk} G \leq 5$. (By theorem 1.18 we know that $\operatorname{rk} G \geq 6$ already implies $M \cong \mathbb{H}\mathbf{P}^5$, as M is rationally 3-connected.)

Congruence classes modulo 7 of the dimensions of the relevant different types of Lie groups are as described in table 4.2. The list of Lie groups G then is given as in the assertion.

Table 4.2.: Dimensions modulo 7

type	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$\mathbf{A}_n$	3	1	1	3	0
$\mathbf{B}_n$	3	3	0	1	6
$\mathbf{C}_n$	3	3	0	1	6
$\mathbf{D}_n$	1	6	1	0	3
$\dim \mathbf{G}_2 \equiv 0 \mod 7$					
$\dim \mathbf{F}_4 \equiv 3 \mod 7$					

Let us finally establish the list of pairs (d, v). For this we apply equation (4.13) once more. Additionally, note that the total deficiency

$$\Delta = 2n + 1 + 2v - d \geq 0$$

is non-negative (cf. [38], p. 208), which translates to

$$11 + 2v - d \geq 0$$

for dimension 20. These conditions already reduce the list of pairs (d, v) due to the following restrictions: It holds $4 \leq v \leq 64$, $24 \leq v \leq 164$, $24 \leq v \leq 264$ and $24 \leq v \leq 384$ when $d = 15$, $d = 22$, $d = 29$ and $d = 36$ respectively.

□

We remark that all the Lie groups in the theorem have rank at least 3. Note that if one focusses on simple groups G, then only dimensions 15 and 36 occur. Moreover, we easily derive the following recognition theorem for the quaternionic volume:

Corollary 4.14. *A* 20*-dimensional Positive Quaternion Kähler Manifold with* $\hat{A}(M)[M] = 0$ *and quaternionic volume* $v > 384$ *is symmetric.*

□

In chapter D from the appendix we compute the elliptic genus $\Phi(M)$ of Wolf spaces. In particular, we see that $\Phi(M^{20}) = 0$ for a 20-dimensional Wolf space $M \not\cong \mathbf{Gr}_2(\mathbb{C}^7)$. In particular, this implies that $\hat{A}(M)[M] = \hat{A}(M, T_{\mathbb{C}}M)[M] = 0$, i.e. that $i^{0,0} = i^{1,1} = 0$ in dimension $\dim M = 20$. Equally, this yields $\operatorname{sign}(M) = 0$. This justifies the assumption

$$\hat{A}(M)[M] = \hat{A}(M, T_{\mathbb{C}}M)[M] = \operatorname{sign}(M) = 0$$

We shall now depict very briefly how this assumption—combined with some more—leads to a classification of Positive Quaternion Kähler Manifolds in dimension 20. In this vein we shall see that originally, i.e. mainly before we spotted the mistake in the classification in dimension 12 (cf. chapter 2), it was possible to prove the main conjecture 1.6 in dimension 20 under a slight assumption on the Euler characteristic.

The assumption $\operatorname{sign}(M) = 0$ directly yields $b_2 = b_6 = b_{10} = 0$ by Hard-Lefschetz—cf. 1.13. Thus by equation 1.3 the fourth Betti number b_4 completely determines the Euler characteristic $\chi(M)$. By our main linear system of equations we then see that the index $i^{1,6}$ is divisible by 5 and that it is restricted to a very small set of possible values depending on the dimensions d of the isometry group. Then the computations involving the generalised intersection form (cf. theorem 1.16 and lemma 4.10) yield that $d \in \{15, 22, 29\}$ requires b_4 and thus the Euler characteristic $\chi(M)$ to be relatively large; the smaller is d, the larger is $\chi(M)$. Due to table 4.1 we see that in dimensions 22 and 29 only isometry groups of rank 4 and 5 appear. The methods we present in chapter C of the appendix apply and theorem C.5 and further refinements of it yield an upper bound for the Euler characteristic. (For this bound to hold we need to assume that $\hat{A}(M)[M] = 0$ for all 12-dimensional Positive Quaternion Kähler Manifolds M with $|\pi_2(M)| < \infty$.) This bound contradicts the values obtained from the computations here.

As mentioned above, in the case $d = 15$ we obtain a large Euler characteristic $\chi(M)$ from the computations. So we assume the Euler characteristic to lie below this value; this rules out $d = 15$ and leaves us with $d = 36$. As we shall see in theorem 4.22 this will suffice—cf. table 4.1—to prove the symmetry of M.

Eventually, we remark that before we encountered the error in the classification proof for dimension 12—cf. chapter 2—we were able to derive the assumptions on the $\hat{A}$-genus, the twisted $\hat{A}$-genus and the signature from the theory. The theorem proposed in [43].2.8 directly ruled out an isometry group of dimension 29, as all these groups are of rank 5—cf. table 4.1. However, this theorem is based upon theorem [43].2.6, which lacks a proof—cf. page 21.

So it would have been possible to prove the symmetry of 20-dimensional Positive Quaternion Kähler Manifolds under a slight assumption on the Euler characteristic.

4.4. Classification results in dimension 20

In this section we prove that a Positive Quaternion Kähler Manifold of dimension 20 with isometry group one of $\mathbf{Sp}(4)$ or $\mathbf{SO}(9)$ up to finite coverings is homothetic to the real Grassmannian $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^9)$. We shall combine this result with the computations from section 4.3, especially with theorem 4.13. The arguments given in this section will be Lie theoretic mainly.

In this section we obey the following convention: Every statement on equalities, inclusions or decompositions of groups is a statement up to finite coverings.

First of all note the relation

$$\mathbf{SO}(n) \otimes \mathbf{Sp}(1) \cong (\mathbf{SO}(n) \times \mathbf{Sp}(1))/\langle((-1)^{n+1}, (-\mathrm{id})^{n+1})\rangle$$

This can be seen by computing the Lie algebras (cf. [48], p. 25) and determining the fibre of the covering $\mathbf{SO}(n) \times \mathbf{Sp}(1) \to \mathbf{SO}(n) \otimes \mathbf{Sp}(1)$ over the identity.

From the tables in appendix B in [48], p. 63–68, we cite that all irreducible representations $\varrho(\mathbf{G}_2)$ of $\mathbf{G}_2$ in degrees smaller than or equal to

- $\deg \varrho \leq 30$ if $\varrho \in \mathrm{Irr}_{\mathbb{R}}(\mathbf{G}_2)$
- $\deg \varrho \leq 15$ if $\varrho \in \mathrm{Irr}_{\mathbb{C}}(\mathbf{G}_2)$
- $\deg \varrho \leq 12$ if $\varrho \in \mathrm{Irr}_{\mathbb{H}}(\mathbf{G}_2)$

are real and have degree

$$\deg \varrho \in \{7, 14, 27\} \tag{4.14}$$

This is considered an exemplary citation of the most important case.

We shall use this information in order to shed more light on inclusions of Lie groups. Some of the following lemmas will be generalised in the next section to higher dimensions. Nonetheless, for the convenience of the reader and since some proofs may be given in a shorter form, we shall also prove the special cases relevant for dimension 20.

Lemma 4.15. *Let $G = G_1 \times G_2$ be a decomposition of Lie groups. Let further $H \neq 1$ be a simple Lie subgroup of G. Then (up to finite coverings) H is also a subgroup of one of G_1 and G_2 (by the canonical projection).*

Proof. Compose the inclusion $H \hookrightarrow G_1 \times G_2$ with the canonical projection $G \to G_i$ (with $i \in \{1,2\}$) to obtain a morphism $f_i : H \to G_i$. The kernel of f_i is a normal subgroup of H, i.e. we have $\ker f_i \in \{1, H\}$ (up to finite coverings). If $\ker f_i = 1$, the morphism f_i is an injection and we are done. Otherwise, if $\ker f_i = H$, the morphism f_i is constant. So if both f_1 and f_2 are constant, the original inclusion $H \hookrightarrow G$ is a constant map, too. This contradicts $H \neq 1$. □

Lemma 4.16. *There is no inclusion of Lie groups* $\mathbf{SO}(7) \hookrightarrow \mathbf{Sp}(5)$ *(not even up to finite coverings).*

Proof. By table 1.6 the group $\mathbf{SO}(7)$ has to be contained in one of

$$\mathbf{U}(5),\ \mathbf{Sp}(4) \times \mathbf{Sp}(1),\ \mathbf{Sp}(3) \times \mathbf{Sp}(2),\ \mathbf{SO}(5) \otimes \mathbf{Sp}(1)$$

as there is no quaternionic representation of a simple Lie group H in degree 10 (other than the standard representation of $\mathbf{Sp}(5)$) by the tables in appendix B in [48], p. 63–68. (The tables neglect the cases of Lie groups with dimensions smaller than 11. Clearly, so can we.)

Thus, by lemma 4.15 and for dimension reasons, we see that $\mathbf{SO}(7)$ has to be a subgroup of one of

$$\mathbf{SU}(5),\ \mathbf{Sp}(4)$$

Suppose first that $\mathbf{SO}(7)$ is a subgroup of $\mathbf{SU}(5)$. By table 1.5, lemma 4.15 and for dimension reasons this is not possible. (Again, there are no further irreducible complex representations of degree 5 of interest.)

Assume $\mathbf{SO}(7)$ is a subgroup of $\mathbf{Sp}(4)$. We argue in the analogous way to get a contradiction. Alternatively, one may quote table 1.8 for subgroups of maximal dimension. □

Lemma 4.17. *There is no inclusion of Lie groups* $\mathbf{SU}(5) \hookrightarrow \mathbf{SO}(9)$ *and equally,* $\mathbf{SU}(5)$ *is not a subgroup of* $\mathbf{Sp}(4)$ *either—not even up to finite coverings.*

Proof. **Case 1.** The group $\mathbf{SU}(5)$ cannot be included into $\mathbf{SO}(9)$:

If this was the case, the group $\mathbf{SU}(5)$ would be a maximal rank subgroup of $\mathbf{SO}(9)$. Thus iteratively applying table 1.7 we see that all the subgroups of $\mathbf{SO}(9)$ of maximal rank are again products of special orthogonal groups. Indeed, by dimension, we would obtain that $\mathbf{SU}(5)$ would have to equal $\mathbf{SO}(8)$ up to finite coverings; a contradiction.

Case 2. The group $\mathbf{SU}(5)$ cannot be included into $\mathbf{Sp}(4)$: By table 1.8 a subgroup of maximal dimension of $\mathbf{Sp}(4)$ is given by dimension $21 < 24 = \dim \mathbf{SU}(5)$. □

Lemma 4.18. *There is no inclusion of Lie groups* $\mathbf{Sp}(2) \times \mathbf{SU}(3) \hookrightarrow \mathbf{SO}(9)$ *and equally, the group* $\mathbf{Sp}(2) \times \mathbf{SU}(3)$ *also is not a subgroup of* $\mathbf{Sp}(4)$*—not even up to finite coverings.*

Proof. **Case 1.** The group $\mathbf{Sp}(2) \times \mathbf{SU}(3)$ is not a subgroup of $\mathbf{SO}(9)$:

Again, we may iteratively use table 1.7 in order to see that all the maximal rank subgroups of $\mathbf{SO}(9)$ are products of special orthogonal groups. Since

$$\operatorname{rk} \mathbf{Sp}(2) \times \mathbf{SU}(3) = 4 = \operatorname{rk} \mathbf{SO}(9)$$

this yields the result

Case 2. The group $\mathbf{Sp}(2) \times \mathbf{SU}(3)$ is not a subgroup of $\mathbf{Sp}(4)$: Assume the contrary. Since

$$\operatorname{rk} \mathbf{Sp}(2) \times \mathbf{SU}(3) = \operatorname{rk} \mathbf{Sp}(4) = 4$$

we derive that $\mathbf{Sp}(2) \times \mathbf{SU}(3)$ necessarily includes into one of

$$\mathbf{Sp}(1) \times \mathbf{Sp}(3), \ \mathbf{Sp}(2) \times \mathbf{Sp}(2), \ \mathbf{U}(4)$$

by means of table 1.7. If it included into $\mathbf{Sp}(1) \times \mathbf{Sp}(3)$, then there would even be an inclusion into $\mathbf{Sp}(3)$ (cf. 4.15), which is impossible by rank. If it included into $\mathbf{Sp}(2) \times \mathbf{Sp}(2)$, we would necessarily obtain an inclusion of $\mathbf{SU}(3)$ into $\mathbf{Sp}(2)$ (cf. 4.15), which is impossible by table 1.8, as a subgroup of $\mathbf{Sp}(2)$ of maximal dimension has dimension $6 < 8 = \dim \mathbf{SU}(3)$. An inclusion of $\mathbf{Sp}(2) \times \mathbf{SU}(3)$ into $\mathbf{U}(4)$ equally cannot exist by reasons of dimension. This yields a contradiction.

□

Lemma 4.19. *Suppose that $k \in \{3, 4\}$. The only inclusion of Lie groups $\mathbf{Sp}(k) \hookrightarrow \mathbf{Sp}(5)$ is given by the canonical blockwise inclusion up to conjugation.*

Proof. Unless $\mathbf{Sp}(k)$ is included in the depicted way, it has to be a subgroup of one of

$$\mathbf{SU}(5), \ \mathbf{Sp}(4)$$

by lemma 4.15, by table 1.6 and by dimension. (Again, we use that there are no further quaternionic representations of degree 10 of interest.)

It cannot be a subgroup of $\mathbf{SU}(5)$, as a subgroup of $\mathbf{SU}(5)$ of maximal dimension has dimension $16 < \dim \mathbf{Sp}(3) = 21$ by table 1.8.

If $\mathbf{Sp}(3)$ is a subgroup of $\mathbf{Sp}(4)$, then it has to include blockwise (by our usual arguments). As the inclusion of $\mathbf{Sp}(4)$ into $\mathbf{Sp}(5)$ is also given blockwise, we obtain the blockwise inclusion of $\mathbf{Sp}(3) \hookrightarrow \mathbf{Sp}(5)$. □

Lemma 4.20. *There is no inclusion of Lie groups $\mathbf{Sp}(2) \times \mathbf{Sp}(2) \hookrightarrow \mathbf{Sp}(5)$ unless one of the $\mathbf{Sp}(2)$-factors includes blockwise (up to conjugation).*

Proof. By table 1.6 and by dimension $\mathbf{Sp}(2) \times \mathbf{Sp}(2)$ includes into one of

$$\mathbf{U}(5),\ \mathbf{Sp}(4) \times \mathbf{Sp}(1),\ \mathbf{Sp}(3) \times \mathbf{Sp}(2)$$

Thus by lemma 4.15 we see that $\mathbf{Sp}(2) \times \mathbf{Sp}(2)$ is a subgroup of one of

$$\mathbf{SU}(5),\ \mathbf{Sp}(3) \times \mathbf{Sp}(2)$$

It cannot be a subgroup of $\mathbf{SU}(5)$, as a subgroup of $\mathbf{SU}(5)$ of maximal dimension is of dimension $16 < 20 = \dim \mathbf{Sp}(2) \times \mathbf{Sp}(2)$. Thus $\mathbf{Sp}(2) \times \mathbf{Sp}(2) \subseteq \mathbf{Sp}(3) \times \mathbf{Sp}(2)$ and we need to determine all the possible inclusions i.

The only inclusion of $\mathbf{Sp}(2)$ into $\mathbf{Sp}(3)$ possible is given by the canonical blockwise inclusion (up to conjugation). Suppose now that both $\mathbf{Sp}(2)$-factors do not include blockwise. We shall lead this to a contradiction.

We obtain that the inclusion composed with the canonical projection

$$i_1 : \mathbf{Sp}(2) \times \{1\} \hookrightarrow \mathbf{Sp}(3) \times \mathbf{Sp}(2) \to \mathbf{Sp}(2)$$

again is an inclusion. This is due to the fact that the kernel of this map has to be trivial, as $\mathbf{Sp}(2)$ is simple. Thus by our assumption of non-blockwise inclusion the kernel has to be the trivial group and i_1 is an inclusion. The same holds for

$$i_2 : \{1\} \times \mathbf{Sp}(2) \hookrightarrow \mathbf{Sp}(3) \times \mathbf{Sp}(2) \to \mathbf{Sp}(2)$$

Without restriction, we may suppose that $i_1 = i_2 = \mathrm{id}$. Thus we obtain that i is an inclusion if and only if

$$\mathbf{Sp}(2) \times \mathbf{Sp}(2) \hookrightarrow \mathbf{Sp}(3) \times \mathbf{Sp}(2) \to \mathbf{Sp}(3)$$

is an inclusion. By consideration of rank this is impossible. This yields a contradiction and at least one $\mathbf{Sp}(2)$-factor is canonically included. $\square$

Lemma 4.21. *The only inclusion of Lie groups* $\mathbf{SU}(4) \hookrightarrow \mathbf{Sp}(5)$ *respectively* $\mathbf{SU}(4) \times \mathbf{Sp}(1) \hookrightarrow \mathbf{Sp}(5)$ *is given by the canonical blockwise inclusion up to conjugation.*

Proof. By table 1.6, by the tables in appendix B in [48], p. 63–68, and by dimension we see that $\mathbf{SU}(4)$ respectively $\mathbf{SU}(4) \times \mathbf{Sp}(1)$ lies in one of

$$\mathbf{U}(5),\ \mathbf{Sp}(4) \times \mathbf{Sp}(1),\ \mathbf{Sp}(3) \times \mathbf{Sp}(2)$$

The group $\mathbf{SU}(4)$ is not a subgroup of $\mathbf{Sp}(3)$ by table 1.6 and by dimension. The only inclusion of $\mathbf{SU}(4)$ into $\mathbf{U}(5)$ is given by the canonical blockwise one due to the usual arguments. The group $\mathbf{SU}(4) \times \mathbf{Sp}(1)$ is not a subgroup of $\mathbf{U}(5)$; indeed, this is impossible by table 1.5, the tables in appendix B in [48], p. 63–68, and by dimension.

Case 1. Thus in the case of the inclusion $\mathbf{SU}(4) \hookrightarrow \mathbf{Sp}(5)$ we observe that either the inclusion factors over $\mathbf{SU}(4) \hookrightarrow \mathbf{U}(4) \hookrightarrow \mathbf{Sp}(5)$ and is given blockwise or the inclusion is given via $\mathbf{SU}(4) \hookrightarrow \mathbf{Sp}(4) \hookrightarrow \mathbf{Sp}(5)$. Again, the inclusion necessarily is given blockwise. For this we realise that $\mathbf{SU}(4)$ cannot be included into any maximal subgroup of $\mathbf{Sp}(4)$ other than $\mathbf{U}(4)$.

Case 2. As for the inclusion $\mathbf{SU}(4) \times \mathbf{Sp}(1) \hookrightarrow \mathbf{Sp}(5)$ we note that $\mathbf{SU}(4) \times \mathbf{Sp}(1)$ maps into $\mathbf{Sp}(4) \times \mathbf{Sp}(1)$ by dimension. By lemma 4.15 and by dimension we see that $\mathbf{SU}(4)$ again lies in $\mathbf{Sp}(4)$. As we have seen this inclusion necessarily is blockwise. As $\mathbf{Sp}(4)$ maps into $\mathbf{Sp}(5)$ by blockwise inclusion, the inclusion $\mathbf{SU}(4) \hookrightarrow \mathbf{Sp}(5)$ is the canonical one.

It then remains to see that $\mathbf{Sp}(1)$ includes into the $\mathbf{Sp}(1)$-factor of $\mathbf{Sp}(4) \times \mathbf{Sp}(1)$ (and *not* into the $\mathbf{Sp}(4)$-factor). Assume this is not the case. This means that the group $\mathbf{Sp}(1)$ necessarily does include into the $\mathbf{Sp}(4)$-factor. Thus there is a homomorphism of groups

$$i : \mathbf{SU}(4) \times \mathbf{Sp}(1) \to \mathbf{Sp}(4)$$

with the property that $i|_{\mathbf{SU}(4)}$ as well as $i|_{\mathbf{Sp}(1)}$ are injective. (Clearly, the morphism i itself cannot be injective.) However, as we shall show, this contradicts the fact that i is a homomorphism: For $(x_1, x_2), (y_1, y_2) \in \mathbf{SU}(4) \times \mathbf{Sp}(1)$ we compute

$$\begin{aligned} i((x_1, x_2) \cdot (y_1, y_2)) &= i(x_1 y_1, x_2 y_2) = i(x_1, 1) i(1, y_1) i(1, x_2) i(1, y_2) \\ i((x_1, x_2) \cdot (y_1, y_2)) &= i(x_1, x_2) i(y_1, y_2) = i(x_1, 1) i(1, x_2) i(y_1, 1) i(1, y_2) \end{aligned}$$

Thus we necessarily have that $i(1, y_1) i(1, x_2) = i(1, x_2) i(y_1, 1)$. As $i|_{\mathbf{Sp}(1)}$ is injective, we realise that—whatever the inclusion $i|_{\mathbf{Sp}(1)}$ will be—the group $i(\mathbf{Sp}(1))$ always contains elements that do not commute with every element of $\mathbf{SU}(4) \subseteq \mathbf{Sp}(4)$. (Clearly, $\mathbf{Sp}(1)$ cannot be included into the centre T^4 of $\mathbf{SU}(4)$.)

Consequently, we obtain that the inclusion of $\mathbf{Sp}(1)$ into $\mathbf{Sp}(4) \times \mathbf{Sp}(1)$ maps $\mathbf{Sp}(1)$ to the $\mathbf{Sp}(1)$-factor and the projection $\mathbf{Sp}(1) \to \mathbf{Sp}(4) \times \mathbf{Sp}(1) \to \mathbf{Sp}(4)$ is the trivial map. This proves the assertion. □

Let us now prove the classification result.

Theorem 4.22. *A* 20*-dimensional Positive Quaternion Kähler Manifold M which has an isometry group* $\mathrm{Isom}(M)$ *that satisfies*

$$\mathrm{Isom}_0(M) \in \{\mathbf{SO}(9), \mathbf{Sp}(4)\}$$

up to finite coverings is homothetic to the real Grassmannian

$$M \cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^9)$$

Proof. We proceed in three steps. First we shall establish a list of stabiliser groups in a T^4-fixed-point—where T^4 is the maximal torus of $\mathrm{Isom}_0(M)$—that might occur unless M is a Wolf space. As a second step we reduce the list by inclusions into the isometry group and the holonomy group. Finally, in the third step we show by more distinguished arguments that also the remaining stabilisers from the list cannot occur, whence M has to be symmetric.

Step 1. Both groups $\mathbf{SO}(9)$ as well as $\mathbf{Sp}(4)$ have rank 4, i.e. they contain a 4-torus $T^4 = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. The Positive Quaternion Kähler Manifold M has positive Euler characteristic by theorem 1.13. Thus by the Lefschetz fixed-point theorem we derive that there exists a T^4-fixed-point $x \in M$. Let H_x denote the (identity-component of the) isotropy group of the G-action in x for $G \in \{\mathbf{SO}(9), \mathbf{Sp}(4)\}$. Since G is of dimension 36 and since $\dim M = 20$, we obtain that $\dim H_x \geq 17$ unless the action of G on M is transitive. If this is the case, a result by Alekseevskii (cf. [5].14.56, p. 409) yields the symmetry of M. If M is symmetric, it is a Wolf space and the dimension of the isometry group then yields that $M \cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^9)$. We may even assume $\dim H_x \geq 18$ by the classification of cohomogeneity one Positive Quaternion Kähler Manifolds in theorem [16].7.4, p. 24.

Since $\mathrm{rk}\, G = 4$ and since x is a T^4-fixed-point, we also obtain $\mathrm{rk}\, H_x = 4$. Moreover, H_x is a closed subgroup. Thus, using lemma 1.19, we may give a list of all products of semi-simple Lie groups and tori of dimension $36 \geq \dim H_x \geq 18$ and with $\mathrm{rk}\, H_x = 4$ up to finite coverings:

$$\begin{gathered}\mathbf{SU}(5),\ \mathbf{SO}(9),\ \mathbf{SO}(8),\ \mathbf{Sp}(4),\ \mathbf{Sp}(3)\times\mathbf{Sp}(1),\ \mathbf{Sp}(3)\times\mathbb{S}^1,\ \mathbf{SO}(7)\times\mathbf{Sp}(1),\\ \mathbf{SO}(7)\times\mathbb{S}^1,\ \mathbf{SO}(6)\times\mathbf{Sp}(1),\ \mathbf{Sp}(2)\times\mathbf{Sp}(2),\ \mathbf{Sp}(2)\times\mathbf{SU}(3),\ \mathbf{Sp}(2)\times\mathbf{G}_2,\\ \mathbf{G}_2\times\mathbf{G}_2,\ \mathbf{G}_2\times\mathbf{SU}(3),\ \mathbf{G}_2\times\mathbf{Sp}(1)\times\mathbf{Sp}(1),\ \mathbf{G}_2\times\mathbf{Sp}(1)\times\mathbb{S}^1\end{gathered}$$

Step 2. We now apply two criteria by which we may reduce the list:

- On the one hand we have that H_x is a Lie subgroup of G.
- On the other hand by the isotropy representation H_x is a Lie subgroup of $\mathbf{Sp}(5)\mathbf{Sp}(1)$—cf. theorem [45].VI.4.6, p. 248.

We use lemma 4.15 to see that every group H_x in the list contains a factor that has to include into $\mathbf{Sp}(5)$ up to finite coverings.

An iterative application of table 1.7 yields that every maximal rank subgroup of the classical group $G \in \{\mathbf{SO}(9), \mathbf{Sp}(4)\}$ again is a product of classical groups. Thus H_x may not be one of the groups

$$\mathbf{Sp}(2)\times\mathbf{G}_2,\ \mathbf{G}_2\times\mathbf{G}_2,\ \mathbf{G}_2\times\mathbf{SU}(3),\ \mathbf{G}_2\times\mathbf{Sp}(1)\times\mathbf{Sp}(1),\ \mathbf{G}_2\times\mathbf{Sp}(1)\times\mathbb{S}^1$$

(not even up to finite coverings).

Now apply lemma 4.16 in the respective cases to reduce the list of potential stabilisers to

$$\begin{gathered}\mathbf{SU}(5),\ \mathbf{Sp}(4),\ \mathbf{Sp}(3)\times\mathbf{Sp}(1),\ \mathbf{Sp}(3)\times\mathbb{S}^1,\\ \mathbf{SO}(6)\times\mathbf{Sp}(1),\ \mathbf{Sp}(2)\times\mathbf{Sp}(2),\ \mathbf{Sp}(2)\times\mathbf{SU}(3)\end{gathered}$$

Indeed, this lemma rules out all the groups H_x that contain a factor of the form $\mathbf{SO}(7)$; and as we see that $\mathbf{SO}(7)$ is not a subgroup of $\mathbf{Sp}(5)$, also $\mathbf{SO}(8)$ and $\mathbf{SO}(9)$ cannot be subgroups of $\mathbf{Sp}(5)$.

Now apply lemmas 4.17 and 4.18 by which potential inclusions into the isometry group G are made clearer. That is, they rule out the groups $\mathbf{SU}(5)$ and $\mathbf{Sp}(2)\times\mathbf{SU}(3)$. Thus the list of possible isotropy groups reduces further to

$$\mathbf{Sp}(4),\ \mathbf{Sp}(3)\times\mathbf{Sp}(1),\ \mathbf{Sp}(3)\times\mathbb{S}^1,\ \mathbf{SO}(6)\times\mathbf{Sp}(1),\ \mathbf{Sp}(2)\times\mathbf{Sp}(2)$$

Step 3. Let us consider the inclusion of the groups H_x into the holonomy group $\mathbf{Sp}(5)\mathbf{Sp}(1)$. By 4.15 the largest direct factor of the candidates in our list has to be a subgroup of $\mathbf{Sp}(5)$ (up to finite coverings), as it cannot be included into $\mathbf{Sp}(1)$. By lemma 4.19 we see that the inclusion of $\mathbf{Sp}(4)$ into $\mathbf{Sp}(5)$ and the one of the $\mathbf{Sp}(3)$-factor of $\mathbf{Sp}(3)\times\mathbf{Sp}(1)$ respectively of $\mathbf{Sp}(3)\times\mathbb{S}^1$ has to be blockwise. Due to lemma 4.20 we observe that there is also an $\mathbf{Sp}(2)$-factor of $\mathbf{Sp}(2)\times\mathbf{Sp}(2)$ that includes blockwise into $\mathbf{Sp}(5)$. Thus we obtain that every group from our list which contains a factor of the form $\mathbf{Sp}(k)$ for $k\geq 2$ has a circle subgroup $\mathbb{S}^1\subseteq\mathbf{Sp}(k)$ that includes into $\mathbf{Sp}(5)$ by

$$\operatorname{diag}(\mathbb{S}^1,1,1,1,1)\subseteq\operatorname{diag}(\mathbf{Sp}(k),1,\ldots,1)\subseteq\mathbf{Sp}(5)$$

Thus this circle group fixes a codimension 4 Positive Quaternion Kähler component. Due to theorem [20].1.2, p. 2, we obtain that $M\cong\mathbb{H}\mathbf{P}^5$ or $M\cong\mathbf{Gr}_2(\mathbb{C}^7)$; a contradiction by our assumption on the isometry group.

This leaves us with $H_x=\mathbf{SO}(6)\times\mathbf{SO}(3)$, which is $\mathbf{SU}(4)\times\mathbf{Sp}(1)$ up to $(\mathbb{Z}_2\oplus\mathbb{Z}_2)$-covering. Equivalently, we consider an orbit of the form

$$\begin{aligned}X:&=\frac{\mathbf{SO}(9)}{\mathbf{SO}(6)\times\mathbf{SO}(3)}\\&=\frac{\mathbf{Spin}(9)}{(\mathbf{Spin}(6)\times\mathbf{Spin}(3))/\langle(-\mathrm{id},-\mathrm{id})\rangle}\\&=\frac{\mathbf{Spin}(9)}{(\mathbf{SU}(4)\times\mathbf{Sp}(1))/\langle(-\mathrm{id},-1)\rangle}\end{aligned}$$

Now consider the isotropy representation of the stabiliser group. There are basically two possibilities: Either the whole stabiliser includes into the $\mathbf{Sp}(5)$-factor or the $\mathbf{Sp}(1)$-factor includes into the $\mathbf{Sp}(1)$-factor of $\mathbf{Sp}(5)\mathbf{Sp}(1)$ (and not into the $\mathbf{Sp}(5)$-factor).

In the first case we apply lemma 4.21 to see that the inclusion of $\mathbf{SU}(4) \times \mathbf{Sp}(1)$ into $\mathbf{Sp}(5)$ is blockwise. So is the inclusion of the $\mathbf{Sp}(1)$-factor in particular. Thus again we obtain a sphere which is represented by $\operatorname{diag}(\mathbb{S}^1, 1, 1, 1, 1)$ and which fixes a codimension four quaternionic component. We proceed as above.

Let us now deal with the second case. Again we cite lemma 4.21 to see that the $\mathbf{SU}(4)$-factor includes into $\mathbf{Sp}(5)$ in a blockwise way. Observe now that the tangent bundle TM of M splits as

$$TM = TX \oplus NX$$

over X, where NX denotes the normal bundle. Since

$$\dim X = \dim \mathbf{SO}(9) - \dim \mathbf{SO}(6) \times \mathbf{SO}(3) = 36 - 18 = 18$$

we obtain that the normal bundle is two-dimensional. Thus the slice representation of the isotropy group $(\mathbf{SU}(4) \times \mathbf{Sp}(1))/(-\mathrm{id}, -1)$ at a fixed-point, i.e. the representation on NX, is necessarily trivial. That is, the action of the isotropy group $(\mathbf{SU}(4) \times \mathbf{Sp}(1))/(-\mathrm{id}, -1)$ at a fixed-point has to leave the normal bundle pointwise fixed.

The $\mathbf{Sp}(1)$-factor of $\mathbf{SU}(4) \times \mathbf{Sp}(1)$ maps isomorphically (up to finite coverings) into the $\mathbf{Sp}(1)$-factor of the holonomy group; the $\mathbf{SU}(4)$-factor maps into $\mathbf{Sp}(5)$. Thus the action of this $\mathbf{SU}(4) \times \mathbf{Sp}(1)$ on the tangent space $T_xM \cong \mathbb{H}^5$ at $x \in M$ is given by $(A, h)(v) = Avh^{-1}$. This action, however, has no 18-dimensional (respectively 2-dimensional) invariant subspace as the $\mathbf{Sp}(1)$-factor acts transitively on each $\mathbb{H}$-component. Hence the normal bundle does not remain fixed under the action of the stabiliser. Thus $\mathbf{SU}(4) \times \mathbf{Sp}(1)$ cannot occur as an isotropy group.

Hence we have excluded all the cases that arose from the assumption $\dim H_x \geq 18$. This was equivalent to the action of the isometry group neither being transitive nor of cohomogeneity one. In the latter two cases—as already observed—the manifold M has to be symmetric. More precisely, since there is no 20-dimensional Wolf space with $\mathrm{Isom}_0(M) = \mathbf{Sp}(4)$ (up to finite coverings), we obtain $\mathrm{Isom}_0 M = \mathbf{SO}(9)$ and $M \cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^9)$. □

We combine this result with previous computations.

Theorem 4.23. *A* 20*-dimensional Positive Quaternion Kähler Manifold* M *with* $\hat{A}(M)[M] = 0$ *satisfying*

$$\dim \mathrm{Isom}_0(M) \notin \{15, 22, 29\}$$

(where possible groups $\mathrm{Isom}_0(M)$ *in these dimensions can be read off from table 4.1) is a Wolf space.*

PROOF. This follows by combining theorems 4.13 and 4.22. □

Clearly, as for $\dim \mathrm{Isom}_0(M) \notin \{15, 22, 29\}$ one hopes an approach by similar techniques as in the proof of 4.22 to be likewise successful. Yet, we remark that for example in dimension 29 one will have to cope with five different isometry groups due to table 4.1. All these groups are of rank 5. So one lists all the possible stabilisers at a T^5-fixed-point on M that do not necessarily make the action of $\mathrm{Isom}(M)$ transitive or of cohomogeneity one. That is, one computes all the products H of semi-simple Lie groups and tori that satisfy $\mathrm{rk}\, H = 5$ and $11 \leq \dim H \leq 29$. This results in a list of 45 possible groups H (up to finite coverings). Following our previous line of argument we then try to rule out stabilisers by showing that they either may not include into a respective isometry group or that they may not be a subgroup of $\mathbf{Sp}(5)\mathbf{Sp}(1)$. If both is not the case, as a next step we try to show that the way H includes into the holonomy group already implies the existence of an $\mathbb{S}^1$-fixed-point component of codomension 4. This would imply the symmetry of the ambient manifold M^{20}. We observe that by far the biggest part of this procedure is covered by the arguments we applied before and we encourage the reader to provide the concrete reasoning. Nonetheless, we encounter new difficulties: For example, the group $\mathbf{SU}(4) \times \mathbb{S}^1 \times \mathbb{S}^1$ includes into the isometry group $\mathbf{SU}(4) \times \mathbf{G}_2 \cong \mathbf{SO}(6) \times \mathbf{G}_2$. If its inclusion into $\mathbf{Sp}(5)\mathbf{Sp}(1)$ is induced by the blockwise inclusion of $\mathbf{SU}(4)$ into $\mathbf{Sp}(5)$, the canonical inclusion of $\mathbb{S}^1$ into $\mathbf{Sp}(1)$ and the diagonal inclusion of $\mathbb{S}^1$ into $\mathbf{Sp}(5)$, we realise that there is no codimension four $\mathbb{S}^1$-fixed-point component. Then methods more particular in nature will have to be provided—as we did in step 3 of the proof of theorem 4.22. We leave this to the reader.

4.5. A recognition theorem for the real Grassmannian

We shall prove a theorem for Positive Quaternion Kähler Manifolds M^{4n} that recognises the isometry type of M by means of the dimension of the isometry group. The theorem will determine the isometry type of the manifold when given a large enough isometry group G. Compare the second point in theorem 1.18, which asserts a recognition theorem for the quaternionic projective space and the complex Grassmannian. This result uses the weaker information of the rank of the isometry group. We shall use it as an important tool. Yet, observe that this theorem is *not* capable of detecting the real Grassmannian $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$; indeed, there seems to be no general recognition theorem for Wolf spaces in the literature. (We hint to non-published work by Gregor Weingart for a recognition theorem on all indices $i^{p,q}$—cf. 1.17.) By corollary 1.12 we see that both $\mathbb{H}\mathbf{P}^n$ and $\mathbf{Gr}_2(\mathbb{C}^{n+2})$ are topologically well-recognisable. Thus a main problem with a positive confirmation of conjecture 1.6 lies in identifying the real Grassmannian and the exceptional Wolf spaces. There seem to be no properties known by which these spaces could be identified.

Since one may relate the index $i^{0,n+2}$ (cf. 1.17) directly to the dimension of $\mathrm{Isom}(M^{4n})$, it seems to be pretty natural to try to provide a recognition theorem on this information. The methods applied extend the ones from section 4.4. **Again, we shall neglect finite coverings.**

Lemma 4.24. *For $n \geq 6$ there is no inclusion of Lie groups $\mathbf{SO}(n+1) \hookrightarrow \mathbf{Sp}(n)$, not even up to finite coverings.*

Proof. Due to table 1.6 the group $\mathbf{SO}(n+1)$ either has to be contained in $\mathbf{U}(n)$, $\mathbf{Sp}(k) \times \mathbf{Sp}(n-k)$ with $1 \leq k \leq n-1$, some $\mathbf{SO}(p) \otimes \mathbf{Sp}(q)$ with $pq = n$, $p \geq 3$, $q \geq 1$ or in $\varrho(H)$ for a simple Lie group H and an irreducible quaternionic representation $\varrho \in \mathrm{Irr}_{\mathbb{H}}(H)$ of dimension $\deg \varrho = 2n$. The cases with direct product or tensor product yield an inclusion of $\mathbf{SO}(n+1)$ in either some $\mathbf{SO}(k)$ with $k \geq n$—which is impossible by dimension—or into some smaller symplectic group by lemma 4.15.

Assume there is an inclusion into $\mathbf{U}(n) = (\mathbf{SU}(n) \times \mathbf{U}(1))/\mathbb{Z}_n$. Then again lemma 4.15 yields an inclusion into $\mathbf{SU}(n)$. By table 1.5 the maximal subgroups of $\mathbf{SU}(n)$ are given by $\mathbf{SO}(n)$, $\mathbf{Sp}(m)$ with $2m = n$, $\mathbf{S}(\mathbf{U}(k) \times \mathbf{U}(n-k))$ for $(1 \leq k \leq n-1)$, $\mathbf{SU}(p) \otimes \mathbf{SU}(q)$ with $pq = n$, $p \geq 3$, $q \geq 2$ and by $\varrho(H)$ for a simple Lie group H and an irreducible quaternionic representation $\varrho \in \mathrm{Irr}_{\mathbb{C}}(H)$ of dimension $\deg \varrho = n$. An inclusion in the first case is impossible due to dimension. Cases two to four lead to inclusions into smaller symplectic or special unitary groups by lemma 4.15.

Hence we need to have a closer look at irreducible quaternionic and complex representations of simple Lie groups H. The tables in [48], appendix B, p. 63–68, give all the representations of simple Lie groups satisfying a certain dimension bound, which is given by

$$\begin{aligned} 2\dim H &\geq \deg \varrho - 2 \\ \dim H &\geq \deg \varrho - 1 \\ \dim H &\geq \frac{3}{2} \deg \varrho - 4 \end{aligned}$$

for real, complex and quaternionic representations respectively.

First of all for $n \geq 3$ the tables together with our previous reasoning yield that $k = n$ is the maximal number for which $\mathbf{SU}(k)$ is a maximal subgroup of $\mathbf{Sp}(n)$. Equally, for $n \geq 7$ we obtain that $k = n$ is the maximal number for which $\mathbf{SO}(k)$ is a maximal subgroup of $\mathbf{Sp}(n)$ or of $\mathbf{SU}(n)$. This means in particular that $\mathbf{SO}(n+1)$ cannot be included into $\mathbf{Sp}(n)$ by a chain

$$\mathbf{SO}(n+1) \subseteq G_1 \subseteq \ldots \subseteq G_l \subseteq \mathbf{Sp}(n) \tag{4.15}$$

of (irreducible representations of) classical groups $G_1, \dots, G_l$ for $n \geq 6$.

It remains to prove that there is no such chain involving (representations of) exceptional Lie groups G_i. For this it suffices to realise that there are no exceptional Lie groups H satisfying

$$\dim \mathbf{SO}(n+1) \leq \dim H \leq \dim \mathbf{Sp}(n) \tag{4.16}$$

with H admitting a quaternionic or complex representation of degree smaller than or equal to $2n$ or n respectively. (We clearly may neglect the real representations ϱ of degree k with $k \leq n$, as there evidently cannot be inclusions $\mathbf{SO}(n+1) \subseteq \varrho(H) \subseteq \mathbf{SO}(k)$ by dimension.)

In table 4.3 for each exceptional Lie group H we give the values of n for which the inequalities (4.16) are satisfied. Additionally, we note the corresponding maximal degree $\deg \varrho = 2n$ ($\deg \varrho = n$) of an irreducible quaternionic (complex) representation ϱ by which H might become the subgroup $\varrho(H) \subseteq \mathbf{Sp}(k)$ ($\varrho(H) \subseteq \mathbf{SU}(k)$) with $k \leq n$ for the given values of n. That is, for example in the case of $\mathbf{G}_2$ we see that if there is a quaternionic representation (a complex representation) of degree smaller than or equal to 22 (to 11), then there is an inclusion of $\mathbf{G}_2$ into $\mathbf{Sp}(k)$ (into $\mathbf{SU}(k)$) for $k \leq 11$ and now also conversely: If there is no such representation, then $\mathbf{G}_2$ cannot be a subgroup satisfying $\mathbf{SO}(n+1) \subseteq \mathbf{G}_2 \subseteq \mathbf{Sp}(n)$ for any $n \in \mathbb{N}$.

Now the tables in [48] yield that there are no quaternionic respectively complex representations of H in the degrees depicted in table 4.3. This amounts to the fact that for the relevant values of n from table 4.3 there is no inclusion $\mathbf{SO}(n+1) \subseteq H \subseteq \mathbf{Sp}(n)$. Thus by (4.16) there are no inclusions of exceptional Lie groups H with $\mathbf{SO}(n+1) \subseteq H \subseteq \mathbf{Sp}(n)$ for any $n \in \mathbb{N}$.

Thus we have proved that there cannot be a chain of the form (4.15) with an exceptional Lie group G_i. Combining this with our previous arguments proves the assertion.

Table 4.3.: Degrees of relevant representations

Lie group	$n \in$	$\deg \varrho \leq$
$\mathbf{G}_2$	$\{3, 4\}$	$8, 4$
$\mathbf{F}_4$	$\{5, 6, 7, 8, 9\}$	$18, 9$
$\mathbf{E}_6$	$\{7, 8, 9, 10, 11\}$	$22, 11$
$\mathbf{E}_7$	$\{8, 9, 10, 11, 12, 13, 14, 15\}$	$30, 15$
$\mathbf{E}_8$	$\{11, 12, \dots, 20, 21\}$	$42, 21$

□

Note that the bound $n \geq 7$ in the lemma is necessary since the universal two-sheeted covering of $\mathbf{SO}(6)$ is $\mathbf{SU}(4)$. In higher dimensions no such exceptional identities occur as can be seen from the corresponding Dynkin diagrams.

Lemma 4.25. *For $n \geq 3$ the only inclusion of Lie groups $\mathbf{Sp}(\lfloor \frac{n}{2} \rfloor + 1) \hookrightarrow \mathbf{Sp}(n)$ is given by the canonical blockwise one up to conjugation.*

Proof. We proceed as in lemma 4.24. Indeed, by the same arguments as above we see that every chain of classical groups

$$\mathbf{Sp}\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \subseteq G_1 \subseteq \ldots \subseteq G_l \subseteq \mathbf{Sp}(n) \tag{4.17}$$

involves symplectic or special unitary groups G_i of rank smaller than or equal to n only. For this we use that there is no inclusion $\mathbf{Sp}(\lfloor \frac{n}{2} \rfloor + 1) \subseteq \mathbf{SO}(n)$ by dimension; indeed

$$\dim \mathbf{SO}(n) = \frac{n(n-1)}{2} < \left\{ \begin{array}{ll} \left(\frac{n}{2}+1\right)(n+3) & \text{for } n \text{ even} \\ \frac{(n+1)(n+2)}{2} & \text{for } n \text{ odd} \end{array} \right\} = \dim \mathbf{Sp}\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)$$

More precisely, in such a chain of classical groups the inclusion of $\mathbf{Sp}(\lfloor \frac{n}{2} \rfloor + 1)$ is necessarily blockwise since $n \geq 3$. This is due to the fact that actually only symplectic groups G_i which are included in a blockwise way may appear by table 1.5; i.e. the subgroups of $\mathbf{SU}(n)$ are to small to permit the inclusion of $\mathbf{Sp}(\lfloor \frac{n}{2} \rfloor + 1)$.

We now have to realise that there is no chain as in (4.17) with an exceptional Lie group G_i. As in the proof of lemma 4.24 we depict the values of n for which an inclusion $\mathbf{Sp}(\lfloor \frac{n}{2} \rfloor + 1) \subseteq H \subseteq \mathbf{Sp}(n)$ of an exceptional Lie group H might be possible—when merely considering dimensions—in table 4.4. The table again also yields degree bounds for the degrees of quaternionic and complex representations. Then the tables in

Table 4.4.: Degrees of relevant representations

Lie group	$n \in$	$\deg \varrho \leq$
$\mathbf{G}_2$	$\{3\}$	$6, 3$
$\mathbf{F}_4$	$\{5, 6, 7\}$	$14, 7$
$\mathbf{E}_6$	$\{7, 8, 9\}$	$18, 9$
$\mathbf{E}_7$	$\{8, 9, 10, 11, 12, 13\}$	$26, 13$
$\mathbf{E}_8$	$\{11, 12, 13, 14, 15, 16, 17, 18, 19\}$	$38, 19$

appendix [48].B, p. 63–68, yield that under these respective restrictions no representations of exceptional Lie groups can be found. This implies that each G_i in the chain is classical. Thus the inclusion of $\mathbf{Sp}(\lfloor \frac{n}{2} \rfloor + 1)$ into $\mathbf{Sp}(n)$ is necessarily given blockwise. □

Note that $\mathbf{Sp}(1) \cong \mathbf{SU}(2)$, whence the inclusion $\mathbf{Sp}(1) \hookrightarrow \mathbf{Sp}(2)$ is not necessarily blockwise.

Lemma 4.26. *Let $n \geq 7$ be odd. Every inclusion $\mathbf{Sp}(\frac{n+1}{2} - 1) \times \mathbf{Sp}(2) \hookrightarrow \mathbf{Sp}(n)$ restricts to the canonical blockwise one (up to conjugation and finite coverings) on the first factor.*

PROOF. By table 1.6 and by dimension we see that $\mathbf{Sp}(\frac{n+1}{2}-1)\times\mathbf{Sp}(2)$ lies in one of

$$\mathbf{U}(n),\ \mathbf{Sp}(k)\times\mathbf{Sp}(n-k) \text{ for } 1\le k\le n-1,\ \varrho(H)$$

for a simple Lie group H and an irreducible quaternionic representation ϱ of degree $\deg\varrho = 2n$.

If $\mathbf{Sp}(\frac{n+1}{2}-1)\times\mathbf{Sp}(2)$ should happen to appear as a subgroup of $\mathbf{SU}(n)$, then table 1.5 would show that

$$\mathbf{Sp}\left(\frac{n+1}{2}-1\right)\hookrightarrow\mathbf{SU}(n-1)\hookrightarrow\mathbf{SU}(n)$$

necessarily is included in the standard "diagonal" way induced by the standard inclusion $\mathbb{H}\hookrightarrow\mathbb{C}^{2\times 2}$. For this we observe the following facts that result when additionally taking into account the tables in appendix [48].B, p. 63–68: The largest special orthogonal subgroup (up to finite coverings) of $\mathbf{SU}(n)$ is $\mathbf{SO}(n)$ for $n\ge 6$. The group $\mathbf{SO}(n)$ does not permit $\mathbf{Sp}(\frac{n+1}{2}-1)$ as a subgroup for $n\ge 4$. Moreover, there are no irreducible complex representations by which a simple Lie group H might include into some $\mathbf{SU}(k)$ (for $k\le n$) satisfying $\mathbf{Sp}(\frac{n+1}{2}-1)\subseteq H$.

Now we see that whenever $\mathbf{Sp}(\frac{n+1}{2}-1)$ is included diagonally into $\mathbf{SU}(n)$ as depicted, there is no inclusion of $\mathbf{Sp}(\frac{n+1}{2}-1)\times\mathbf{Sp}(2)$ possible. That is, for the inclusion of this direct product to be a homomorphism we need the group $\mathbf{Sp}(2)$ to map into the centraliser $C_{\mathbf{SU}(n)}(\mathbf{Sp}(\frac{n+1}{2}-1))$ of $\mathbf{Sp}(\frac{n+1}{2}-1)$ in $\mathbf{SU}(n)$. Yet, we obtain

$$C_{\mathbf{SU}(n)}\left(\mathbf{Sp}\left(\frac{n+1}{2}-1\right)\right)\cong\mathbb{S}^1\times\mathbb{S}^1$$

and thus no inclusion of $\mathbf{Sp}(2)$ is possible. Hence $\mathbf{Sp}(\frac{n+1}{2}-1)\times\mathbf{Sp}(2)$ cannot be a subgroup of $\mathbf{U}(n)$.

The tables in [48] again yield that whenever a simple Lie group H is included into $\mathbf{Sp}(k)$ (for $k\le n$) via an irreducible quaternionic representation ϱ, the inclusion is one of

$$\mathbf{SU}(6)\hookrightarrow\mathbf{Sp}(10),\ \mathbf{SO}(11)\hookrightarrow\mathbf{Sp}(16),\ \mathbf{SO}(12)\hookrightarrow\mathbf{Sp}(16),\ \mathbf{E}_7\hookrightarrow\mathbf{Sp}(28)$$

or an inclusion of a symplectic group of rank smaller than k unless the degree of the representation is far too large to be of interest for our purposes. Indeed, already the depicted inclusions are not relevant, since $\mathbf{Sp}(\frac{n+1}{2}-1)$ cannot be included into $\mathbf{SU}(6),\ \mathbf{SO}(11),\ \mathbf{SO}(12),\ \mathbf{E}_7$ respectively when $n\ge 10,16,16,28$.

Thus we see that every inclusion of $\mathbf{Sp}(\frac{n+1}{2}-1)\times\mathbf{Sp}(2)$ has to factor through one of $\mathbf{Sp}(n-k)\times\mathbf{Sp}(k)$ for $1\le k\le n-1$. Hence for $1\le k<\frac{n+1}{2}-1$ we obtain that the only inclusion of $\mathbf{Sp}(\frac{n+1}{2}-1)$ into $\mathbf{Sp}(n)$ factoring through $\mathbf{Sp}(n-k)\times\mathbf{Sp}(k)$ is given by standard blockwise inclusion

$$\mathbf{Sp}\left(\frac{n+1}{2}-1\right)\hookrightarrow\mathbf{Sp}(n-k)\hookrightarrow\mathbf{Sp}(k)$$

Thus we may assume without restriction that $k=\frac{n+1}{2}-1$ (and $n-k=\frac{n+1}{2}$) and that the inclusion of $\mathbf{Sp}(\frac{n+1}{2}-1)\hookrightarrow\mathbf{Sp}(n)$ is not the standard blockwise one. Thus we see that the inclusion necessarily factors over

$$\mathbf{Sp}\left(\frac{n+1}{2}-1\right)\hookrightarrow\mathbf{Sp}\left(\frac{n+1}{2}-1\right)\times\mathbf{Sp}\left(\frac{n+1}{2}\right)\hookrightarrow\mathbf{Sp}(n)$$

where the first inclusion splits as a product of the standard blockwise inclusions

$$\begin{aligned}\mathbf{Sp}\left(\frac{n+1}{2}-1\right)&\overset{\mathrm{id}}{\hookrightarrow}\mathbf{Sp}\left(\frac{n+1}{2}-1\right)\\ \mathbf{Sp}\left(\frac{n+1}{2}-1\right)&\hookrightarrow\mathbf{Sp}\left(\frac{n+1}{2}\right)\end{aligned}$$

So regard $\mathbf{Sp}(\frac{n+1}{2}-1)$ as the subgroup of $\mathbf{Sp}(n)$ given by this inclusion. Again we make use of the fact that the $\mathbf{Sp}(2)$-factor has to include into the centraliser

$$C_{\mathbf{Sp}(n)}\left(\mathbf{Sp}\left(\frac{n+1}{2}-1\right)\right)\cong\mathbb{S}^1\times\mathbb{S}^1\times\mathbf{Sp}(1)$$

Such an inclusion clearly is impossible and we obtain a contradiction. Thus the inclusion of $\mathbf{Sp}(\frac{n+1}{2}-1)\times\mathbf{Sp}(2)$ is the standard blockwise one when restricted to the $\mathbf{Sp}(\frac{n+1}{2}-1)$-factor. □

Lemma 4.27. *In table 4.5 the semi-simple Lie groups of maximal dimension with respect to a fixed rank (from rank* 1 *to rank* 12*) are given up to isomorphisms and finite coverings. From rank* 13 *on the groups that are maximal in this sense are given by the two infinite series* $\mathbf{Sp}(n)$ *and* $\mathbf{SO}(2n+1)$ *only.*

Proof. Table 4.5 results from a case by case check using table 1.3. From dimension 13 on semi-simple Lie groups involving factors that are exceptional Lie groups are smaller than the largest classical groups. Among products of classical groups the ratio between dimension and rank is maximal for the types $\mathbf{B}_n$ and $\mathbf{C}_n$, □

Let us now prove the classification theorem that actually may be regarded as a recognition theorem for the real Grassmannian.

Theorem 4.28. *Let* M^{4n} *be a Positive Quaternion Kähler Manifold. Suppose that the dimension of the isometry group* $\dim\mathrm{Isom}(M^{4n})$ *satisfies the respective condition*

Table 4.5.: Largest Lie groups with respect to fixed rank

$\mathrm{rk}\, G$	extremal Lie groups G	$\dim G$
1	$\mathbf{Sp}(1)$	3
2	$\mathbf{G}_2$	14
3	$\mathbf{Sp}(3), \mathbf{SO}(7)$	21
4	$\mathbf{F}_4$	52
5	$\mathbf{Sp}(5), \mathbf{F}_4 \times \mathbf{Sp}(1), \mathbf{SO}(11)$	55
6	$\mathbf{E}_6, \mathbf{Sp}(6)$	78
7	$\mathbf{E}_7$	133
8	$\mathbf{E}_8$	248
9	$\mathbf{E}_8 \times \mathbf{Sp}(1)$	251
10	$\mathbf{E}_8 \times \mathbf{G}_2$	262
11	$\mathbf{E}_8 \times \mathbf{Sp}(3), \mathbf{E}_8 \times \mathbf{SO}(7)$	269
12	$\mathbf{Sp}(12), \mathbf{SO}(25), \mathbf{E}_8 \times \mathbf{F}_4$	300

depicted in table 4.6. Then M is symmetric and it holds:

$$\begin{array}{lll} M \cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4}) & \Leftrightarrow \dim \mathrm{Isom}(M) = \frac{n^2+7n+12}{2} & \Leftrightarrow \mathrm{rk}\, \mathrm{Isom}(M) = \lfloor \frac{n}{2} \rfloor + 2 \\ M \cong \mathbf{Gr}_2(\mathbb{C}^{n+2}) & \Leftrightarrow \dim \mathrm{Isom}(M) = n^2 + 4n + 3 & \\ M \cong \mathbb{H}\mathbf{P}^n & \Leftrightarrow \dim \mathrm{Isom}(M) = 2n^2 + 5n + 3 & \end{array}$$

In particular, if the isometry group satisfies that

$$\dim \mathrm{Isom}(M) > \frac{n^2 + 5n + 12}{2}$$

for $n \geq 22$ and $n \notin \{27, 28\}$ then we recognise the real Grassmannian by the dimension of its isometry group.

Proof. **Step 1.** By theorem 1.18 we may suppose that $\mathrm{rk}\, \mathrm{Isom}(M^{4n}) \leq \lceil \frac{n}{2} \rceil + 2$, since otherwise $M \in \{\mathbb{H}\mathbf{P}^n, \mathbf{Gr}_2(\mathbb{C}^{n+2})\}$. The dimension bounds in table 4.6 for $4 \leq n \leq 20$ result from table 4.5: That is, for each such n the bound is the dimension of the largest group that satisfies this rank condition. Thus every group with larger dimension has rank large enough to identify M^{4n} as one of $\mathbb{H}\mathbf{P}^n$ and $\mathbf{Gr}_2(\mathbb{C}^{n+2})$.

In degree $n = 3$ we use that there is no semi-simple Lie group of rank smaller than or equal to 4 in dimensions 29 to $36 = \dim \mathbf{Sp}(4)$. By theorem 1.18 we have $\dim \mathrm{Isom}(M^{12}) \leq 36$.

Step 2. Now we determine all (the one-components of) the isometry groups $G = \mathrm{Isom}_0(M^{4n})$ (up to finite coverings) with $\mathrm{rk}\, G \leq \lceil \frac{n}{2} \rceil + 2$ satisfying the dimension bound for $n \geq 22$ and $n \notin \{27, 28\}$ as

$$G \in \left\{ \mathbf{SO}(n+4),\ \mathbf{SO}(n+5),\ \mathbf{Sp}\left(\frac{n}{2} + 2\right) \right\}$$

for n even and as

$$G \in \Big\{ \mathbf{SO}(n+4),\ \mathbf{SO}(n+4)\times\mathbf{SO}(2),\ \mathbf{SO}(n+4)\times\mathbf{SO}(3),\ \mathbf{SO}(n+5),\\ \mathbf{SO}(n+6),\ \mathbf{Sp}\Big(\frac{n+1}{2}+2\Big),\ \mathbf{Sp}\Big(\frac{n+1}{2}+1\Big),\ \mathbf{Sp}\Big(\frac{n+1}{2}+1\Big)\times\mathbf{SO}(2),\\ \mathbf{Sp}\Big(\frac{n+1}{2}+1\Big)\times\mathbf{Sp}(1)\Big\}$$

for n odd. This can be achieved as follows: We see that whenever we have a product of classical groups we may replace it by a simple classical group of the same rank and of larger dimension. The classical groups for which the ratio between dimension and rank is maximal are given by the groups of type $\mathbf{B}$ and $\mathbf{C}$. Moreover, the series $\dim \mathbf{B}_n = \dim \mathbf{C}_n$ is strictly increasing in n. We compute

$$\dim \mathbf{SO}(n+3)\times\mathbf{SO}(3) = \frac{n^2+5n+12}{2}$$

whilst

$$\operatorname{rk} \mathbf{SO}(n+3)\times\mathbf{SO}(3) = \left\lfloor \frac{n+3}{2} \right\rfloor + 1 = \left\lceil \frac{n}{2} \right\rceil + 2$$

Consequently, by our assumption on the dimension of G we need to find all the groups that are larger in dimension but not larger in rank than $\mathbf{SO}(n+3)\times\mathbf{SO}(3)$. This process results in the list we gave.

We still need to see when there are groups G that are larger in dimension than $\mathbf{SO}(n+3)\times\mathbf{SO}(3)$ but not larger in rank and that have exceptional Lie groups as direct factors (up to finite coverings). Clearly, this can only happen in low dimensions. So we use lemma 4.27 and table 4.5 to see that unless $n \in \{27, 28\}$ there do no appear exceptional Lie groups as factors. As for degrees $n \in \{27, 28\}$ the group $\mathbf{E}_8 \times \mathbf{E}_8$ has dimension

$$\dim(\mathbf{E}_8\times\mathbf{E}_8) = 496 > \begin{cases} 438 = \dim \mathbf{SO}(30)\times\mathbf{SO}(3) \\ 468 = \dim \mathbf{SO}(31)\times\mathbf{SO}(3) \end{cases}$$

Therefore in these degrees we want to assume that $\dim \operatorname{Isom}(M^{4n}) > 496$. This will make it impossible to identify the real Grassmannian, since $\dim \mathbf{SO}(31) = 465$ and since $\dim \mathbf{SO}(32) = 496$. Nonetheless the following arguments hold as well.

Step 3. We now prove that whenever G is taken out of the list we gave, then actually $G = \mathbf{SO}(n+4)$ and $M \cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$. In order to establish this we shall have a closer look at orbits around a fixed-point of the maximal torus of G for each respective possibility of G. (Such a point exists due to the Lefschetz fixed-point theorem and the fact that $\chi(M) > 0$—cf. 1.13.) This will lead to the observation that a potential action of G has to be transitive, whence M is homogeneous. Due to Alekseevski homogeneous Positive Quaternion Kähler Manifolds are Wolf spaces—cf. [5].14.56, p. 409.

Since $\dim M = 4n$, we necessarily obtain that the orbit G/H of G has dimension at most $\dim G/H \leq 4n$. Assume first that G is a direct product from the list with $\mathbf{SO}(n+4)$ as a factor (up to finite coverings). Thus all the maximal rank subgroups H of G satisfying $\dim G/H \leq 4n$ for $n \geq 22$, necessarily contain one of the groups

$$\mathbf{SO}(n) \times \mathbf{SO}(4),\ \mathbf{SO}(n+3),\ \mathbf{SO}(n+2) \times \mathbf{SO}(2)$$

as a factor—which includes into $\mathbf{SO}(n+4)$—due to table 1.7. (Note that whether $\mathbf{SO}(n+3)$ is a maximal rank subgroup of $\mathbf{SO}(n+4)$ or not depends on the parity of n being odd or even.) By the same arguments we see that for $G = \mathbf{SO}(n+5)$ only the following subgroups H may appear:

$$\mathbf{SO}(n+4),\ \mathbf{SO}(n+3) \times \mathbf{SO}(2),\ \mathbf{SO}(n+2) \times \mathbf{SO}(2),\ \mathbf{SO}(n+2) \times \mathbf{SO}(3)$$

For $G = \mathbf{SO}(n+6)$ the group H is out of the following list:

$$\mathbf{SO}(n+5),\ \mathbf{SO}(n+4) \times \mathbf{SO}(2),\ \mathbf{SO}(n+3) \times \mathbf{SO}(3),\ \mathbf{SO}(n+3) \times \mathbf{SO}(2)$$

In any of the cases there has to be an inclusion $\mathbf{SO}(n+k)$ for $k \geq 0$ into $\mathbf{Sp}(n)$ by the isotropy representation—cf. [45], theorem VI.4.6.(2). By lemma 4.24, however, this is impossible unless $k = 0$. Thus we see that

$$G/H = \frac{\mathbf{SO}(n+4)}{\mathbf{SO}(n) \times \mathbf{SO}(4)} = M$$

since the action of G thus necessarily is transitive.

Now suppose that $G = \mathbf{Sp}(\frac{n}{2}+2)$ for n even. Again we use table 1.7 to list maximal rank subgroups H with $\dim G/H \leq 4n$:

$$\mathbf{Sp}\left(\frac{n}{2}+1\right) \times \mathbf{Sp}(1),\ \mathbf{Sp}\left(\frac{n}{2}+1\right) \times \mathbf{U}(1),\ \mathbf{Sp}\left(\frac{n}{2}\right) \times \mathbf{Sp}(2)$$

If $H = \mathbf{Sp}(\frac{n}{2}) \times \mathbf{Sp}(2)$, we see that $\dim G/H = 4n$ and that the action of G is transitive. Thus M is homogeneous and symmetric. Yet, by the classification of Wolf spaces we derive that

$$M = \frac{\mathbf{Sp}\left(\frac{n}{2}+2\right)}{\mathbf{Sp}\left(\frac{n}{2}\right) \times \mathbf{Sp}(2)}$$

cannot be the case.

Now apply lemma 4.25 in the other cases and derive that the isotropy representation of H is marked by a blockwise included $\mathbf{Sp}(\frac{n}{2}+1) \hookrightarrow \mathbf{Sp}(n)$ for $k > 0$. This implies that the sphere

$$\mathbb{S}^1 \times \{1\} \times \overset{(n/2)}{\cdots} \times \{1\} \hookrightarrow T^{n/2+1} \hookrightarrow \mathbf{Sp}\left(\frac{n}{2}+1\right)$$

is represented by $\mathbb{S}^1 \times \{1\} \times \overset{(n-1)}{\cdots} \times \{1\}$ in the $\mathbf{Sp}(n)$-factor of the holonomy group $\mathbf{Sp}(n)\mathbf{Sp}(1)$. Thus it fixes a quaternionic fixed-point component of codimension 4. Thus by theorem [20].1.2, p. 2, we obtain that $M^{4n} \in \{\mathbb{H}\mathbf{P}^{4n}, \mathbf{Gr}_2(\mathbb{C}^{n+2})\}$.

If $G = \mathbf{Sp}(\frac{n+1}{2}+2)$ and n is odd, virtually the same arguments apply. That is, the list of isotropy subgroups H is given by

$$\mathbf{Sp}\left(\frac{n+1}{2}+1\right) \times \mathbf{Sp}(1),\ \mathbf{Sp}\left(\frac{n+1}{2}+1\right) \times \mathbf{U}(1)$$

As above this leads to a codimension four quaternionic $\mathbb{S}^1$-fixed-point component.

Finally, suppose n to be odd and the group G to contain a factor of the form $\mathbf{Sp}(\frac{n+1}{2}+1)$. The list of possible stabilisers is given as

$$\begin{aligned}&\mathbf{Sp}\left(\frac{n+1}{2}\right) \times \mathbf{Sp}(1),\ \mathbf{Sp}\left(\frac{n+1}{2}\right) \times \mathbf{U}(1),\ \mathbf{Sp}\left(\frac{n+1}{2}-1\right) \times \mathbf{Sp}(2),\\ &\mathbf{Sp}\left(\frac{n+1}{2}-1\right) \times \mathbf{Sp}(1) \times \mathbf{Sp}(1)\end{aligned}$$

If $H = \mathbf{Sp}(\frac{n+1}{2}-1) \times \mathbf{Sp}(1) \times \mathbf{Sp}(1)$, the action of G again is transitive which is impossible by the classification of Wolf spaces. If

$$H \in \left\{\mathbf{Sp}\left(\frac{n+1}{2}\right) \times \mathbf{Sp}(1),\ \mathbf{Sp}\left(\frac{n+1}{2}\right) \times \mathbf{U}(1)\right\}$$

we note that its $\mathbf{Sp}(\frac{n+1}{2})$-factor again maps into $\mathbf{Sp}(n)$ in a blockwise way—cf. lemma 4.25—by the isotropy representation. This leads to a codimension four quaternionic $\mathbb{S}^1$-fixed point component once more. Now suppose

$$H = \mathbf{Sp}\left(\frac{n+1}{2}-1\right) \times \mathbf{Sp}(2)$$

Then the holonomy representation necessarily makes H a subgroup of $\mathbf{Sp}(n)$. By lemma 4.26 this can only occur in the standard blockwise way. Again this yields a quaternionic codimension four $\mathbb{S}^1$-fixed-point component which leads to $M \in \{\mathbb{H}\mathbf{P}^n, \mathbf{Gr}_2(\mathbb{C}^{n+2})\}$.

In degree $n = 21$ we see that similar arguments apply as for $n \geq 22$. However, we have that $\dim(\mathbf{SO}(25)) = 300 < 303 = \dim \mathbf{E}_6 \times \mathbf{F}_4 \times \mathbf{Sp}(1)$. Thus due to the assumption that $\dim \mathrm{Isom}(M^{21}) > 303$ we may not identify the real Grassmannian $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{25})$ but only the quaternionic projective space and the complex Grassmannian. □

Table 4.6.: A recognition theorem

$n =$	$\dim \mathrm{Isom}(M^{4n}) >$	recognising
3	28	$\mathbb{H}$
4	52	$\mathbb{H}$
5	55	$\mathbb{H}$
6	55	$\mathbb{H}, \mathbb{C}$
7	78	$\mathbb{H}, \mathbb{C}$
8	78	$\mathbb{H}, \mathbb{C}$
9	133	$\mathbb{H}$
10	133	$\mathbb{H}, \mathbb{C}$
11	248	$\mathbb{H}$
12	248	$\mathbb{H}$
13	251	$\mathbb{H}$
14	251	$\mathbb{H}, \mathbb{C}$
15	262	$\mathbb{H}, \mathbb{C}$
16	262	$\mathbb{H}, \mathbb{C}$
17	269	$\mathbb{H}, \mathbb{C}$
18	269	$\mathbb{H}, \mathbb{C}$
19	300	$\mathbb{H}, \mathbb{C}$
20	300	$\mathbb{H}, \mathbb{C}$
21	303	$\mathbb{H}, \mathbb{C}$
22	303	$\mathbb{H}, \mathbb{C}, \mathbb{R}$
23	328	$\mathbb{H}, \mathbb{C}, \mathbb{R}$
24	354	$\mathbb{H}, \mathbb{C}, \mathbb{R}$
25	381	$\mathbb{H}, \mathbb{C}, \mathbb{R}$
26	409	$\mathbb{H}, \mathbb{C}, \mathbb{R}$
27	496	$\mathbb{H}, \mathbb{C}$
28	496	$\mathbb{H}, \mathbb{C}$
$n \geq 29$	$\frac{n^2+5n+12}{2}$	$\mathbb{H}, \mathbb{C}, \mathbb{R}$

The symbols $\mathbb{H}, \mathbb{C}, \mathbb{R}$ in the column "recognising" in table 4.6 refer to whether we may identify the quaternionic projective space, the complex Grassmannian or the real Grassmannian in this dimension by the theorem.

We remark that in general the dimension of the isometry group $\dim \mathrm{Isom}(M^{4n})$ takes values in

$$[0, \dim \mathbf{Sp}(n+1)] = [0, 2n^2 + 5n + 3]$$

(cf. 1.18). Hence—apart from recognising the real Grassmannian—this theorems rules out approximately three quarters of all possible values. For example when taking into account more structure data for exceptional Lie groups one certainly may generalise the theorem with ease.

This theorem now shows that the whole question whether Positive Quaternion Kähler Manifolds M^{4n} are symmetric or not boils down to a computation of the dimension of the isometry group $\mathrm{Isom}(M^{4n})$.

A Positive Quaternion Kähler Manifold M^{4n} is symmetric (for almost every n) if and only if $\dim \mathrm{Isom}(M) > \frac{n^2+5n+12}{2}$.

Furthermore, by theorem 1.17 one may interpret the dimension of the isometry group as the index of a twisted Dirac operator, i.e. $i^{0,n+2} = \dim \mathrm{Isom}(M)$. As a major consequence of a non-published formula by Gregor Weingart one obtains that a Positive Quaternion Kähler Manifold is a Wolf space if and only if all the indices $i^{p,q}$ and Betti numbers coincide with the ones of the symmetric example. As we have proved by our approach—which is certainly much simpler than the highly elaborated techniques applied by Gregor Weingart—the index $i^{0,n+2}$ already suffices for such a conclusion in almost every dimension. This lets us extend our statement from above:

The question whether a Positive Quaternion Kähler Manifold M^{4n} is symmetric or not can (almost always) be decided from the index

$$\mathrm{ind}(\not{D}(S^{n+2}H)) = \langle \hat{A}(M) \cdot \mathrm{ch}(S^{n+2}H), [M] \rangle$$

(For the bundle H and the indices $i^{p,q}$ we refer to the notation from the introduction, section 1.1.)

A

Cohomology of Wolf Spaces

Although the results are well-known, for the convenience of the reader we shall compute the rational cohomology of the three infinite series of Wolf spaces up to degree 8. Recall that the series are given by the quaternionic projective space, the complex Grassmannian and the oriented real Grassmannian:

$$\mathbb{H}\mathbf{P}^n = \frac{\mathbf{Sp}(n+1)}{\mathbf{Sp}(n) \times \mathbf{Sp}(1)} \tag{A.1}$$

$$\mathbf{Gr}_2(\mathbb{C}^{n+2}) = \frac{\mathbf{U}(n+2)}{\mathbf{U}(n) \times \mathbf{U}(2)} \tag{A.2}$$

$$\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4}) = \frac{\mathbf{SO}(n+4)}{\mathbf{SO}(n) \times \mathbf{SO}(4)} \tag{A.3}$$

The computation of the cohomology of $\mathbb{H}\mathbf{P}^n$ will illustrate the method.

We refer to the "stable case" as to the cohomology of $\mathbf{Gr}_2(\mathbb{C}^\infty)$ and of $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^\infty)$ respectively.

Theorem A.1. *For the quaternionic projective space it holds:*

$$H^*(\mathbb{H}\mathbf{P}^n, \mathbb{Q}) = \mathbb{Q}[u]/u^{n+1}$$

Iff $n \geq 4$ we are in the stable case and obtain

$$H^i(\mathbf{Gr}_2(\mathbb{C}^{n+2})) = \begin{cases} 0 & \textit{for } i \textit{ odd} \\ \mathbb{Q} & \textit{for } i = 0 \\ \langle l \rangle_{\mathbb{Q}} & \textit{for } i = 2 \\ \langle l^2, x \rangle_{\mathbb{Q}} & \textit{for } i = 4 \\ \langle l^3, lx \rangle_{\mathbb{Q}} & \textit{for } i = 6 \\ \langle l^4, l^2x, x^2 \rangle_{\mathbb{Q}} & \textit{for } i = 8 \end{cases}$$

For the complex Grassmannian we have: If n is odd and iff $n \geq 5$ we are in the stable case and obtain

$$H^i(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})) = \begin{cases} 0 & \text{for } i \in \{2,6\} \text{ or } i \text{ odd} \\ \mathbb{Q} & \text{for } i = 0 \\ \langle u, x\rangle_{\mathbb{Q}} & \text{for } i = 4 \\ \langle u^2, ux, x^2\rangle_{\mathbb{Q}} & \text{for } i = 8 \end{cases}$$

For the real Grassmannian we obtain: If n is even and iff $n \geq 10$ we are in the stable case and obtain

$$H^i(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})) = \begin{cases} 0 & \text{for } i \in \{2,6\} \text{ or } i \text{ odd} \\ \mathbb{Q} & \text{for } i = 0 \\ \langle l^2, x\rangle_{\mathbb{Q}} & \text{for } i = 4 \\ \langle l^4, l^2x, x^2\rangle_{\mathbb{Q}} & \text{for } i = 8 \end{cases}$$

In lower dimensions we have the following cohomology algebras:

$$\begin{aligned}
H^*(\mathbf{Gr}_2(\mathbb{C}^3)) &= \mathbb{Q}[l]/l^3 \quad \deg l = 2 \\
H^*(\mathbf{Gr}_2(\mathbb{C}^4)) &= \mathbb{Q}[l,x]/\{lx = 0, x^2 = l^4\} \quad \deg l = 2, \deg x = 4 \\
H^*(\mathbf{Gr}_2(\mathbb{C}^5)) &= \mathbb{Q}[l,x]/\{\mathcal{R}_3\} \quad \deg l = 2, \deg x = 4 \\
H^*(\mathbf{Gr}_2(\mathbb{C}^6)) &= \mathbb{Q}[l,x]/\{\mathcal{R}_4\} \quad \deg l = 2, \deg x = 4 \\
H^*(\mathbf{Gr}_2(\mathbb{C}^7)) &= \mathbb{Q}[l,x]/\{\mathcal{R}_5\} \quad \deg l = 2, \deg x = 4
\end{aligned}$$

$$\begin{aligned}
H^*(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^5)) &= \mathbb{Q}[u]/u^2 \quad \deg u = 4 \\
H^*(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6)) &= \mathbb{Q}[l,x]/\{l^3 = 2lx, x^2 = l^2x\} \quad \deg l = 2, \deg x = 4 \\
H^*(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)) &= \mathbb{Q}[l,x]/\{\mathcal{R}_1\} \quad \deg l = \deg x = 4 \\
H^*(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^8)) &= \mathbb{Q}[l,x,y]/\{\mathcal{R}_2, \mathcal{R}_2'\} \quad \deg l = \deg x = \deg y = 4 \\
H^*(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^9)) &= \mathbb{Q}[l,x]/\{\mathcal{R}_2\} \quad \deg l = \deg x = 4
\end{aligned}$$

In order to complete the list of low-dimensional Wolf spaces we compute

$$H^*(\mathbf{G}_2/\mathbf{SO}(4)) = \mathbb{Q}[u]/u^3 \quad \deg u = 4$$

The relations $\mathcal{R}_n$ consist of setting $\sigma_i = 0$ for $i > n$ in the following list and the *relations* $\mathcal{R}'_i$ *require* $\sigma_i = y^2$.

$$\begin{aligned}
\sigma_1 &= -l \\
\sigma_2 &= l^2 - x \\
\sigma_3 &= 2lx - l^3 \\
\sigma_4 &= l^4 - 3l^2x + x^2 \\
\sigma_5 &= -l^5 + 4l^3x - 3lx^2 \\
\sigma_6 &= l^6 - 5l^4x + 6l^2x^2 - x^3 \\
\sigma_7 &= -l^7 + 6l^5x - 10l^3x^2 + 4lx^3 \\
\sigma_8 &= l^8 - 7l^6x + 15l^4x^2 - 10l^2x^3 + x^4 \\
\sigma_9 &= l^9 + 8l^7x - 21l^5x^2 + 20l^3x^3 - 5lx^4 \\
\sigma_{10} &= l^{10} - 9l^8x + 28l^6x^2 - 35l^4x^3 + 15l^2x^4 - x^5
\end{aligned}$$

This yields the following cohomology groups with relations (without linear dependencies between generators).

$$H^i(\mathbf{Gr}_2(\mathbb{C}^4)) = \begin{cases} \langle\tilde\sigma_1\rangle_{\mathbb{Q}} & \text{for } i=2 \\ \langle\tilde\sigma_1^2, \tilde\sigma_2\rangle_{\mathbb{Q}} & \text{for } i=4 \\ \langle\tilde\sigma_1^3\rangle_{\mathbb{Q}} & \text{for } i=6 \\ \langle\tilde\sigma_1^4\rangle_{\mathbb{Q}} & \text{for } i=8 \end{cases} \qquad \tilde\sigma_1\tilde\sigma_2 = \tilde\sigma_1^3/2,\ \tilde\sigma_2^2 = \tilde\sigma_1^4/2$$

$$H^i(\mathbf{Gr}_2(\mathbb{C}^5)) = \begin{cases} \langle\tilde\sigma_1\rangle_{\mathbb{Q}} & \text{for } i=2 \\ \langle\tilde\sigma_1^2, \tilde\sigma_2\rangle_{\mathbb{Q}} & \text{for } i=4 \\ \langle\tilde\sigma_1^3, \tilde\sigma_1\tilde\sigma_2\rangle_{\mathbb{Q}} & \text{for } i=6 \\ \langle\tilde\sigma_1^4, \tilde\sigma_1^2\tilde\sigma_2\rangle_{\mathbb{Q}} & \text{for } i=8 \end{cases} \qquad \tilde\sigma_2^2 = -\tilde\sigma_1^4 + 3\tilde\sigma_1^2\tilde\sigma_2$$

$$H^i(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6)) = \begin{cases} \langle\tilde\sigma_1'\rangle_{\mathbb{Q}} & \text{for } i=2 \\ \langle\tilde\sigma_1'^2, \tilde\sigma_2'\rangle_{\mathbb{Q}} & \text{for } i=4 \\ \langle\tilde\sigma_1'^3\rangle_{\mathbb{Q}} & \text{for } i=6 \\ \langle\tilde\sigma_1'^4\rangle_{\mathbb{Q}} & \text{for } i=8 \end{cases} \qquad \tilde\sigma_1'\tilde\sigma_2' = 0,\ (\tilde\sigma_2')^2 = (\tilde\sigma_1')^4$$

$$H^i(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)) = \begin{cases} \langle\tilde\sigma_1, \tilde\sigma_2'\rangle_{\mathbb{Q}} & \text{for } i=4 \\ \langle(\tilde\sigma_1)^2, \tilde\sigma_1\tilde\sigma_2'\rangle_{\mathbb{Q}} & \text{for } i=8 \end{cases} \qquad (\tilde\sigma_2')^2 = (\tilde\sigma_1)^2$$

$$H^i(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^8)) = \begin{cases} \langle\sigma_2', \tilde\sigma_1, \tilde\sigma_2'\rangle_{\mathbb{Q}} & \text{for } i=4 \\ \langle(\tilde\sigma_1)^2, \tilde\sigma_1\sigma_2', \tilde\sigma_1\tilde\sigma_2', \tilde\sigma_2^2\rangle_{\mathbb{Q}} & \text{for } i=8 \end{cases} \quad (\sigma_2')^2 = \tilde\sigma_1^2 - (\tilde\sigma_2')^2,\ \sigma_2'\tilde\sigma_2' = 0$$

Observe that these results reflect the ring structure in low degrees and the well-known isometries $\mathbf{Gr}_2(\mathbb{C}^3) \cong \mathbb{C}\mathbf{P}^2$, $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^5) \cong \mathbb{H}\mathbf{P}^1 \cong \mathbb{S}^4$, $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6) \cong \mathbf{Gr}_2(\mathbb{C}^4)$.

Proof. It holds

$$
\begin{aligned}
H^*(\mathbb{H}\mathbf{P}^n) &= H^*(\mathbf{B}(\mathbf{Sp}(n)\times\mathbf{Sp}(1))/H^{>0}(\mathbf{BSp}(n+1)) \\
&= H^*(\mathbf{B}T)^{\mathrm{W}(\mathbf{Sp}(n)\times\mathbf{Sp}(1))}/H^{>0}(\mathbf{B}T)^{\mathrm{W}(\mathbf{Sp}(n+1))}
\end{aligned} \tag{A.4}
$$

where T denotes the maximal torus of $\mathbf{Sp}(n+1)$ and W is the Weyl group. Now recall that the Weyl group of $\mathbf{Sp}(n)$ is given by $G(n)$, the group of permutations ϕ of the set $\{-n,\dots,-1,1,\dots,n\}$ satisfying $\phi(-v)=-\phi(v)$ (cf. [11], p. 171). The group acts on the maximal torus by $\phi^{-1}(\theta_1,\dots,\theta_n)=(\theta_{\phi(1)},\dots,\theta_{\phi(n)})$, where the θ_i denote the angles of the corresponding rotations. By convention we write $\theta_{-v}=-\theta_v$. Recall that the cohomology of the classifying space $\mathbf{B}T$ is the polynomial algebra on $n+1$ generators x_i in degree 2 corresponding to the θ_i. Due to these remarks it becomes obvious that the cohomology of $\mathbb{H}\mathbf{P}^n$ is generated by those polynomials symmetric in $x_1,\dots,x_n$ which are invariant under $x_i\mapsto -x_i$ for all $1\le i\le n$ respectively by the polynomials in x_{n+1} which are invariant under $x_{n+1}\mapsto -x_{n+1}$. These are exactly the elementary symmetric polynomials $\sigma_1,\dots,\sigma_n$ in the x_i^2 for $1\le i\le n$ and $\tilde\sigma_1=x_{n+1}^2$. So (A.4) equals

$$
\begin{aligned}
&\mathbb{Q}[\sigma_1,\dots\sigma_n,\tilde\sigma_1]/\{\sigma\tilde\sigma=1\} \\
=&\mathbb{Q}[\sigma_1,\dots\sigma_n,\tilde\sigma_1])/\{\sigma_1=-\tilde\sigma_1,\sigma_2=\tilde\sigma_1^2,\dots,\sigma_n=(-1)^n\tilde\sigma_1^n,\tilde\sigma_1^{n+1}=0\} \\
=&\mathbb{Q}[\tilde\sigma_1]/\tilde\sigma_1^{n+1}
\end{aligned}
$$

where $\sigma=1+\sigma_1+\dots\sigma_n$ respectively $\tilde\sigma=1+\tilde\sigma_1$. (Recall that the elementary symmetric polynomials in $x_1,\dots,x_{n+1}$ can be described as $\sigma\tilde\sigma$.)

Now let us compute the cohomology of the complex Grassmannian. As above we obtain

$$
\begin{aligned}
H^*(\mathbf{Gr}_2(\mathbb{C}^{n+2})) &= H^*(\mathbf{B}T)^{\mathrm{W}(\mathbf{U}(n)\times\mathbf{U}(2))}/H^{>0}(\mathbf{B}T)^{\mathrm{W}(\mathbf{U}(n+2))} \\
&= \mathbb{Q}[\sigma_1,\dots,\sigma_n,\tilde\sigma_1,\tilde\sigma_2]/\{\sigma\tilde\sigma=1\}
\end{aligned}
$$

where the σ_i are the elementary symmetric polynomials in the x_i for $1\le i\le n$ and the $\tilde\sigma_i$ are the elementary symmetric polynomials in x_{n+1},x_{n+2}. (Recall that the Weyl group of $\mathbf{U}(n)$ is just the symmetric group S_n.) From the relation $\sigma\tilde\sigma=1$ we compute in each degree:

$$
\text{(A.5)} \qquad
\begin{aligned}
\sigma_1 &= -\tilde\sigma_1\\
\sigma_2 &= \tilde\sigma_1^2 - \tilde\sigma_2\\
\sigma_3 &= 2\tilde\sigma_1\tilde\sigma_2 - \tilde\sigma_1^3\\
\sigma_4 &= \tilde\sigma_1^4 - 3\tilde\sigma_1^2\tilde\sigma_2 + \tilde\sigma_2^2\\
\sigma_5 &= -\tilde\sigma_1^5 + 4\tilde\sigma_1^3\tilde\sigma_2 - 3\tilde\sigma_1\tilde\sigma_2^2\\
\sigma_6 &= \tilde\sigma_1^6 - 5\tilde\sigma_1^4\tilde\sigma_2 + 6\tilde\sigma_1^2\tilde\sigma_2^2 - \tilde\sigma_2^3\\
\sigma_7 &= -\tilde\sigma_1^7 + 6\tilde\sigma_1^5\tilde\sigma_2 - 10\tilde\sigma_1^3\tilde\sigma_2^2 + 4\tilde\sigma_1\tilde\sigma_2^3\\
\sigma_8 &= \tilde\sigma_1^8 - 7\tilde\sigma_1^6\tilde\sigma_2 + 15\tilde\sigma_1^4\tilde\sigma_2^2 - 10\tilde\sigma_1^2\tilde\sigma_2^3 + \tilde\sigma_2^4\\
\sigma_9 &= \tilde\sigma_1^9 + 8\tilde\sigma_1^7\tilde\sigma_2 - 21\tilde\sigma_1^5\tilde\sigma_2^2 + 20\tilde\sigma_1^3\tilde\sigma_2^3 - 5\tilde\sigma_1\tilde\sigma_2^4\\
\sigma_{10} &= \tilde\sigma_1^{10} - 9\tilde\sigma_1^8\tilde\sigma_2 + 28\tilde\sigma_1^6\tilde\sigma_2^2 - 35\tilde\sigma_1^4\tilde\sigma_2^3 + 15\tilde\sigma_1^2\tilde\sigma_2^4 - \tilde\sigma_2^5
\end{aligned}
$$

This yields the first cohomology groups

$$
\begin{aligned}
H^2(\mathbf{Gr}_2(\mathbb{C}^{n+2})) &= \langle\tilde\sigma_1\rangle_{\mathbb{Q}}\\
H^4(\mathbf{Gr}_2(\mathbb{C}^{n+2})) &= \langle\tilde\sigma_1^2, \tilde\sigma_2\rangle_{\mathbb{Q}}\\
H^6(\mathbf{Gr}_2(\mathbb{C}^{n+2})) &= \langle\tilde\sigma_1^3, \tilde\sigma_1\tilde\sigma_2\rangle_{\mathbb{Q}}\\
H^8(\mathbf{Gr}_2(\mathbb{C}^{n+2})) &= \langle\tilde\sigma_1^4, \tilde\sigma_1^2\tilde\sigma_2, \tilde\sigma_2^2\rangle_{\mathbb{Q}}
\end{aligned}
$$

for $n \geq 4$ (without any linear relations between the generators). In low dimensions we obtain additional restrictions by setting the corresponding equations in (A.5) to zero, i.e. $\sigma_i = 0$ for $i > n$.

The same procedure for the real Grassmannian yields the following: The Weyl group of $\mathbf{SO}(n)$ for odd n is $G(n)$. For even n it is $\mathbf{S}G(n)$ consisting of even permutations of $G(n)$. So in the odd case invariant polynomials again are generated by elementary symmetric polynomials in the x_i. In the even case the n-th elementary symmetric polynomial in the x_i^2 can be replaced by the n-th elementary symmetric polynomial $\sigma_n' = x_1 \cdot x_2 \cdots x_n$. (A permuation is in $SG(n)$ iff it causes a change of sign on an even number of x_i. Thus it has no effect on σ_n'.) In the case that n is odd we obtain

$$
\begin{aligned}
H^*(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})) &= H^*(\mathbf{B}T)^{\mathrm{W}(\mathbf{SO}(n)\times\mathbf{SO}(4))}/H^{>0}(\mathbf{B}T)^{\mathrm{W}(\mathbf{SO}(n+4))}\\
&= \mathbb{Q}[\sigma_1, \ldots, \sigma_{(n-1)/2}, \tilde\sigma_1, \tilde\sigma_2']/\{\sigma\tilde\sigma = 1\}
\end{aligned}
$$

where the σ_i are the elementary symmetric polynomials in the x_i^2 for $1 \leq i \leq (n-1)/2$, the $\tilde\sigma_i$ are the elementary symmetric polynomials in $x_{(n+1)/2}^2, x_{(n+3)/2}^2$ and $\tilde\sigma_2' = x_{(n+1)/2} \cdot x_{(n+3)/2}$ with $(\tilde\sigma_2')^2 = \tilde\sigma_2$. So we obtain the relations (A.5) in our terminology. For $n \geq 5$ this yields the following cohomology groups (without any linear relations between generators).

$$H^4(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})) = \langle \tilde{\sigma}_1, \tilde{\sigma}_2' \rangle_{\mathbb{Q}}$$
$$H^8(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})) = \langle \tilde{\sigma}_1^2, \tilde{\sigma}_1\tilde{\sigma}_2', \tilde{\sigma}_2 \rangle_{\mathbb{Q}}$$

In low dimensions we obtain cohomology algebras as in the assertion by setting the corresponding σ_i to zero.

If n is even we have

$$\begin{aligned} H^*(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})) &= H^*(\mathbf{B}T)^{\mathrm{W}(\mathbf{SO}(n)\times\mathbf{SO}(4))}/H^{>0}(\mathbf{B}T)^{\mathrm{W}(\mathbf{SO}(n+4))} \\ &= \mathbb{Q}[\sigma_1, \ldots, \sigma_{(n-2)/2}, \sigma'_{n/2}, \tilde{\sigma}_1, \tilde{\sigma}_2']/\{\sigma\tilde{\sigma} = 1, \sigma'_{n/2}\tilde{\sigma}_2' = 0\} \end{aligned}$$

where $\sigma'_{n/2} = x_1 \cdots x_{n/2}$. Using the relations (A.5) we obtain for $n \geq 10$ that

$$H^4(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})) = \langle \tilde{\sigma}_1, \tilde{\sigma}_2' \rangle_{\mathbb{Q}}$$
$$H^8(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})) = \langle \tilde{\sigma}_1^2, \tilde{\sigma}_1\tilde{\sigma}_2', \tilde{\sigma}_2 \rangle_{\mathbb{Q}}$$

and, as usual, the cohomology groups in degrees 1 to 3 and 5 to 7 vanish. A computation of the relations in the particular cases yields the corresponding results.

The ring structure of $\mathbf{G}_2/\mathbf{SO}(4)$ can be derived in the same way: The Weyl group of $\mathbf{G}_2$ is the symmetry group of the regular hexagon, i.e. the dihedral group D_6. The maximal torus of $\mathbf{G}_2$ can be regarded as the quotient of the full hexagon by identifying opposite sides and collapsing one such pair. Thus the Weyl group acts on T as a symmetry group, i.e. given generating loops x, y and

$$D_6 = \mathbb{Z}_6 \rtimes \mathbb{Z}_2 = \langle d, s \mid d^6 = 1, s^2 = 1, dsd = d^{-1} \rangle$$

we obtain the relations:

$$dx = y, d^2x = y - x, d^3x = -x, d^4x = -y, d^5x = x - y, d^6x = x, sx = y, s^2x = x$$

A polynomial in x and y invariant under D_6 must be symmetric due to the action of s, so again it must be generated by the elementary symmetric polynomials σ_1' and σ_2' in x and y. Now a simple calculation using the invariance under the action of d shows that the polynomial $(\sigma_1')^2 - 3\sigma_2'$ is (up to multiples) the only polynomial in degree 4 invariant under the action of D_6. In degree 8 no such polynomial exists. Set $\sigma_1 = x^2 + y^2$ (the first elementary symmetric polynomial in x^2, y^2) and obtain $(\sigma_1')^2 - 3\sigma_2' = \sigma_1 - \sigma_2'$. So the following computations yield the desired result:

$$H^*(\mathbf{G}_2/\mathbf{SO}(4)) = H^*(\mathbf{B}T)^{\mathrm{W}(\mathbf{SO}(4))}/H^{>0}(\mathbf{B}T)^{\mathrm{W}(\mathbf{G}_2)}$$

Set $u = \sigma_1$. The relation $\sigma_1 = \sigma_2'$ yields $H^4(\mathbf{G}_2/\mathbf{SO}(4)) = \langle u \rangle_{\mathbb{Q}}$. The non-existence of relations in degree eight yields $H^8(\mathbf{G}_2/\mathbf{SO}(4)) = \langle u^2 \rangle_{\mathbb{Q}}$. □

Observe that the isomorphism $H^*(\mathbf{Gr}_2(\mathbb{C}^4)) \cong H^*(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6))$ is given by $l \mapsto l$ and $x \mapsto l^2 + 2x$.

As a corollary we obtain the Betti numbers in low dimensions.

Corollary A.2. *Betti numbers are concentrated in degrees divisible by two for the complex Grassmannian, in degrees divisible by four for the other Wolf spaces—except for* $\mathbf{Gr}_2(\mathbb{C}^4) \cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^6)$*. Non-trivial Betti numbers* b_i *are given (mostly) up to duality. The Euler characteristic* χ *is computed.*

$$\mathbb{H}\mathbf{P}^n: \quad b_0 = b_4 = \cdots = b_n = 1, \qquad \chi = n+1$$

$$\begin{aligned}
\mathbf{Gr}_2(\mathbb{C}^3): &\quad b_0 = 1, b_2 = 1 & \chi = 3\\
\mathbf{Gr}_2(\mathbb{C}^4): &\quad b_0 = 1, b_2 = 1, b_4 = 2 & \chi = 6\\
\mathbf{Gr}_2(\mathbb{C}^5): &\quad b_0 = 1, b_2 = 1, b_4 = 2, b_6 = 2 & \chi = 10\\
\mathbf{Gr}_2(\mathbb{C}^6): &\quad b_0 = 1, b_2 = 1, b_4 = 2, b_6 = 2, b_8 = 3 & \chi = 15\\
\mathbf{Gr}_2(\mathbb{C}^7): &\quad b_0 = 1, b_2 = 1, b_4 = 2, b_6 = 2, b_8 = 3, b_{10} = 3 & \chi = 21
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathbf{Gr}}_4(\mathbb{R}^5): &\quad b_0 = 1, b_4 = 1 & \chi = 2\\
\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6): &\quad b_0 = 1, b_2 = 1, b_4 = 2 & \chi = 6\\
\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7): &\quad b_0 = 1, b_4 = 2 & \chi = 6\\
\widetilde{\mathbf{Gr}}_4(\mathbb{R}^8): &\quad b_0 = 1, b_4 = 3, b_8 = 4 & \chi = 12\\
\widetilde{\mathbf{Gr}}_4(\mathbb{R}^9): &\quad b_0 = 1, b_4 = 2, b_8 = 3 & \chi = 12
\end{aligned}$$

$$\mathbf{G}_2/\mathbf{SO}(4): \quad b_0 = b_4 = b_8 = 1 \qquad \chi = 3$$

□

B

Isometric Group Actions on Wolf Spaces

In this chapter we shall describe the fixed-point components of isometric circle-actions and certain isometric involutions on the infinite series of Wolf-spaces.

B.1. Isometric circle actions

Let us classify isometric circle actions together with their fixed-point components on the three infinite series. Observe that each element of the isometry group may be taken to lie in a maximal torus and any two maximal tori are conjugated. Since a one-sphere admits a topological generator, we may thus consider the sphere to be part of the maximal torus. Equally, this does not impose any restriction on the structure of possible fixed-point components, since conjugation satisfies $tst^{-1}(t(x)) = t(x) \Leftrightarrow s(x) = x$ (for any two isometries s and t acting on $M \ni x$). We have

$$\begin{aligned} \operatorname{rk}\operatorname{Isom}(\mathbb{H}\mathbf{P}^n) &= n+1 \\ \operatorname{rk}\operatorname{Isom}(\mathbf{Gr}_2(\mathbb{C}^{n+2})) &= n+1 \\ \operatorname{rk}\operatorname{Isom}(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})) &= \left\lfloor \frac{n+4}{2} \right\rfloor \end{aligned}$$

(Note that we have to write $\mathbf{Gr}_2(\mathbb{C}^{n+2}) = \mathbf{SU}(n+2)/(\mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(2)))$ in order to reveal a group acting almost effectively.)

We shall bear in mind the following line of argument: The description of a Grassmannian as a homogeneous space may be interpreted as follows. The isometry group acts transitively on the manifold. So every plane can be obtained by multiplying a basis of a standard coordinate plane in the ambient vector space with a suitable element of the isometry group and eventually by forming the quotient with all those actions that leave invariant the coordinate plane and its complement; i.e. those actions that transform a basis of the plane to just another basis of the very same plane, a basis of the complement to another basis of the complement. Thus, in particular, it is obvious that the maximal torus of the isometry group just acts by ordinary rotations in the corresponding coordinate planes, i.e. its action on planes is the one induced by the standard action on Euclidean space.

Theorem B.1. *Every isometric circle action on* $\mathbb{H}\mathbf{P}^n$ *is given in homogeneous coordinates and up to reordering by*

$$(t^{k_1}z_1 : t^{k_2}z_2 : \ldots : t^{k_j}z_j : z_{j+1} : \ldots : z_{n+1})$$

with $k \in \mathbb{Z}\backslash\{0\}$. *So every isometric circle action has at most one fixed-point component equal to* $\mathbb{H}\mathbf{P}^{n'}$. *All the other fixed-point components are of the form* $\mathbb{C}\mathbf{P}^{m_i}$ *and it holds*

$$n' + \sum_i (m_i + 1) = n$$

(Take $n' = -1$ *in the formula if there is no quaternionic component.)*

Proof. As remarked above, the i-th factor $\mathbb{S}^1$ of the standard maximal torus T^{n+1} of $\mathrm{Isom}(\mathbb{H}\mathbf{P}^n) = \mathbf{Sp}(n+1)/\mathbb{Z}_2$ acts on $\mathbb{H}\mathbf{P}^n$ by $(z_1 : \ldots : z_{i-1} : tz_i : z_{i+1} : \ldots : z_{n+1})$. Up to conjugation every circle acting isometrically on M must factor over its image in this maximal torus. Since all the group endomorphisms of $\mathbb{S}^1$ are given by $s \mapsto s^k$ for $k \in \mathbb{Z}$, its action is exactly as described in the assertion.

As for fixed-point components observe that the elements $(z_{j+1} : \ldots : z_{n+1})$ again form an $\mathbb{H}\mathbf{P}^{n-j}$. A maximal sequence of the form $t^k z_{i_1} : \ldots : t^k z_{i_m}$ fixes exactly a $\mathbb{C}\mathbf{P}^{m-1}$, namely the set of all elements with strictly complex coefficients z_i. These are mapped to a common complex multiple t^k. Rotations with different exponents t^{k_1} and t^{k_2} clearly cannot fix common points. So the result follows. □

Clearly, every configuration of fixed-point manifolds permitted by the theorem can be realised.

Lemma B.2. *The isometric* $\mathbb{S}^1$*-fixed-point components of* $\mathbb{C}\mathbf{P}^n$ *are again* $\mathbb{C}\mathbf{P}^i$. *In particular, on* $\mathbb{C}\mathbf{P}^3$ *any such action either fixes the whole space, a* $\mathbb{C}\mathbf{P}^2$ *and a fixed-point or three fixed-points.*

Proof. We shall only prove the assertion for $\mathbb{C}\mathbf{P}^3$, since that is all we shall make use of. The general case follows by similar arguments.

For $\mathbb{C}\mathbf{P}^3$ we observe that in homogeneous coordinates the action of the standard torus is given by $(t^{k_1}z_1 : t^{k_2}z_2 : t^{k_3}z_3)$. There are three cases corresponding to $k_1 = k_2 = k_3$, $k_1 = k_2 \neq k_3$ (respectively $k_1 = k_3 \neq k_2$ or $k_2 = k_3 \neq k_1$) and to k_1, k_2, k_3 being pairwise distinct. □

Given an $\mathbb{S}^1$-action on a (complex) vector space of the form $(t^{k_1}, \ldots, t^{k_{n+1}}, t^{-\sum_{1\leq j\leq n+1} k_j})$ we set V_i to equal the (maximal) space on which the action is given by t^{k_i}. Let d_i denote its (complex) dimension. Let I be the set of all the V_i.

Theorem B.3. *Every isometric circle action on $\mathbf{Gr}_2(\mathbb{C}^{n+2})$ is induced up to conjugation by an action on the surrounding vector space given by some $(t^{k_1}, \ldots, t^{k_{n+1}}, t^{-\sum_{1\leq j\leq n+1} k_j})$ with integer k_i. The fixed-point components of this action are given by the set*

$$\{\mathbf{Gr}_2(\mathbb{C}^{d_i})\}_{i\in I} \cup \{\mathbb{C}\mathbf{P}^{d_i-1} \times \mathbb{C}\mathbf{P}^{d_j-1}\}_{i\neq j\in I}$$

Proof. Again the standard maximal torus $T^{n+1} \subseteq \mathbf{SU}(n+2)$ given by all matrices of the form $\operatorname{diag}(e^{l_1 i}, \ldots, e^{l_{n+1} i}, e^{-\sum_{1\leq j\leq n+1} l_j i})$ acts on $\mathbb{C}^{n+2}$ in the standard way, i.e. by rotations in the coordinate planes. Thus a circle subgroup of this torus acts by a rotation of the form $(t^{k_1}, \ldots, t^{k_{n+1}}, t^{-\sum_{1\leq j\leq n+1} k_j})$ inducing its action on contained 2-planes.

Without restriction, the action has the form $(t^{k_1}, \ldots, t^{k_1}, t^{k_2}, \ldots, t^{k_2}, \ldots, t^{k_j})$ and pairwise it holds $0 \neq k_{i_1} \neq k_{i_2}$. Assume now a certain plane is fixed. Since projections commute with rotations, the image of each projection to some $\langle z_1, \ldots, \hat{z}_i, \ldots, z_n\rangle$ remains fixed under the action. So we may consider all the projections into complex 3-spaces $\langle z_{i_1}, z_{i_2}, z_{i_3}\rangle$ with corresponding actions given by rotation numbers k_1, k_2 and k_3. However, the Grassmannian $\mathbf{Gr}_2(\mathbb{C}^3)$ is canonically isometric to $\mathbb{C}\mathbf{P}^2$—by taking orthogonal complements. Thus we may reduce the situation to an action on the latter space. Due to lemma B.2 we may now discern the following cases:

First assume $k_1 = k_2 = k_3$. Then the image of the projection is fixed, no matter what it looks like.

In the case $k_1 = k_2 \neq k_3$ we obtain: If the plane is mapped to a plane, in $\langle z_{i_1}, z_{i_2}, z_{i_3}\rangle$ it must be of the form $\langle (x_1, x_2, 0), (0, 0, x_3)\rangle$. If it is mapped to a line, the image looks like $\langle (x_1, x_2, 0), (kx_1, kx_2, 0)\rangle$ (with $k \in \mathbb{C}$) or $\langle (0, 0, x_3), (0, 0, kx_3)\rangle$. (The case of the image being a point can be neglected.)

In the case that all three rotation numbers are pairwise distinct the image cannot be a plane. So—unless it is a point—we obtain that it has the shape $\langle (x_1, 0, 0), (kx_1, 0, 0)\rangle$, $\langle (0, x_2, 0), (0, kx_2, 0)\rangle$ or $\langle (0, 0, x_3), (0, 0, kx_3)\rangle$.

Thus we make the following observation: Whenever we project to a 3-space the image is spanned by eigenvectors of the rotation. Successively applying this result to all coordinate 3-spaces lets us conclude that a 2-plane in $\mathbb{C}^{n+2}$ which remains fixed is spanned by eigenvectors of the action. So the fixed-point components are now either made up of those planes generated by two eigenvectors corresponding to the the same eigenvalue or by two ones corresponding to two different eigenvalues. In the first case we clearly obtain the complex Grassmannian, in the other cases we have the products of complex projective spaces as asserted. □

Observe that all possibilities can be realised indeed.

In the following, we use the notation introduced above and let V_i denote the maximal complex vector subspace on which the rotation number of the circle action on $\mathbb{R}^{n+4}$ is given by $k_i > 0$. That is, we define the complex structure of V_i such that the rotation number is positive. Again we use the set I as an index set for the V_i. Moreover, we denote the complex dimension of V_i by d_i. Let $\tilde{V}$ be the real vector space on which the rotation number vanishes with real dimension $\tilde{d} = \dim \tilde{V}$. By the "dual space", i.e. by the notation $(X)^*$ we denote a copy of the subspace $X \subseteq \widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$ with all planes $x \in X$ in the opposite orientation.

Theorem B.4. *Every isometric circle action on $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$ is induced by an action on the surrounding vector space given by some $(t^{k_1}, \ldots, t^{k_{\lfloor n/2 \rfloor+2}})$ with $k_i > 0$. The fixed-point components of this action are given by the set*

$$\begin{aligned}&\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{\tilde{d}})\} \cup \{\mathbb{C}\mathbf{P}^{d_i-1} \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^{\tilde{d}}) \mid i \in I\} \cup \{\mathbb{C}\mathbf{P}^{d_i-1} \times \mathbb{C}\mathbf{P}^{d_j-1} \mid i \neq j \in I\}\\ &\cup \{(\mathbb{C}\mathbf{P}^{d_i-1} \times \mathbb{C}\mathbf{P}^{d_j-1})^* \mid i \neq j \in I\} \cup \{\mathbf{Gr}_2(\mathbb{C}^{d_i}) \mid i \in I\} \cup \{(\mathbf{Gr}_2(\mathbb{C}^{d_i}))^* \mid i \in I\}\end{aligned}$$

if $\tilde{d} > 4$. Otherwise, if $\tilde{d} = 2$, there is additionally the component

$$\{(\mathbb{C}\mathbf{P}^{d_i-1} \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^{\tilde{d}}))^* \mid i \in I\}$$

If $\tilde{d} = 4$ there is additionally the component

$$\{(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^4))^*\}$$

Proof. We start by considering planes H that lie in at most two of the spaces V_i, i.e. $H \in V_1 \oplus V_2$. First we consider planes $H \subseteq V_1 \oplus \tilde{V}$, i.e. $V_2 = \tilde{V}$.

Let $v = v_1 + v_2$ be a vector in $\mathbb{R}^{n+4}$ with components $v_i \in V_i$ contained in just two of the spaces V_i. Assume—without restriction—that k_2 is equal to zero, whence $V_2 = \tilde{V}$. Then $k_1 \neq 0$ by construction. So all the rotations of v generate the affine plane

$$\{c \cdot v_1 + v_2 \mid c \in \mathbb{C}\} = v_2 + \mathbb{C}v_1$$

Thus any oriented real 4-plane $H \subseteq V_1 \oplus \tilde{V}$ invariant under the circle action—i.e. any fixed-point of the induced action on the Grassmannian—which contains v also contains this affine 2-plane. In particular, it contains the vector v_2 and the plane $\mathbb{C}v_1$. This implies that $\langle \mathbb{C}v_1, v_2 \rangle \subseteq H$ is a subspace.

By dimension there is a $v' = v_1' + v_2' \in H$ with $v_1' \in V_1$, $v_2' \in V_2$ and $v' \notin \langle \mathbb{C}v_1, v_2 \rangle$. If $v_1' \in \mathbb{C}v_1$, we thus derive that $v_2' \notin \langle v_2 \rangle$. Otherwise, if $v_1' \notin \mathbb{C}v_1$, then $\dim \langle \mathbb{C}v_1, \mathbb{C}v_1' \rangle = 4$ (unless $v_1 = 0$ which may be neglected) and we obtain $v_2 = v_2' = 0$. Summing this up, in any case H (without restriction) can be generated by either two complex planes in V_1, a complex plane in V_1 and a real 2-plane in $\tilde{V}$ or it equals a real 4-plane in $\tilde{V}$.

As a second case we assume now that neither k_1 nor k_2 equals zero and that $H \subseteq V_1 \oplus V_2$. Yet k_1 and k_2 may be identical. Then rotate $v = v_1 + v_2$ by $e^{2\pi i/k_1} \in \mathbb{S}^1$ and obtain $v_1 + e^{2\pi i \cdot k_2/k_1}$. As above we note that H contains the affine plane $v_1 + \mathbb{C}v_2$.

A rotation with $e^{2\pi i/k_2} \in \mathbb{S}^1$ then reveals that actually $H = \mathbb{C}v_1 + \mathbb{C}v_2$, if v_1 and v_2 are non-zero.

For a general plane $H \subseteq \mathbb{R}^{n+4}$ consider its projections onto "pairs" of subspaces $V_i \oplus V_j$ as described and use the reasoning applied above. This finally yields that H corresponds to either a complex plane in one V_i and a real 2-plane in $\tilde{V}$, two complex planes each contained in some V_i, a real 4-plane in $\tilde{V}$ or a complex 2-plane in some V_i. (The complex structure is induced by the rotation number.) Since we are dealing with oriented planes, we get two copies of every such plane with fixed opposite orientations. This yields the assertion. □

Observe that—due to the second factor—the dual of $\{\mathbb{C}\mathbf{P}^{d_i-1} \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^{\tilde{d}}) \mid i \in I\}$ already is identical to the space itself if $\tilde{d} > 2$.

Corollary B.5. *On low-dimensional spaces M we obtain the possibilities for the fixed-point set $F_{\mathbb{S}^1}(M)$ of an isometric $\mathbb{S}^1$-action as depicted in table B.1. (There is redundancy contained in the list due to exceptional isomorphisms of Wolf spaces.)*

□

The next corollary can be derived by means of Weyl groups. Nonetheless, we can give a very elementary proof now.

Corollary B.6. *The Euler characteristics are given by*

$$\begin{aligned}\chi(\mathbb{H}\mathbf{P}^n) &= n+1\\ \chi(\mathbf{Gr}_2(\mathbb{C}^{n+2})) &= \binom{n+2}{2}\\ \chi(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})) &= (\lfloor n/2\rfloor+2)(\lfloor n/2\rfloor+1)\end{aligned}$$

Proof. The quaternionic projective space admits an isometric circle action with $n+1$ fixed-points only. So does the complex Grassmannian with exactly $\binom{n+2}{2}$ fixed-points and the real Grassmannian with exactly $(\lfloor n/2\rfloor+2)(\lfloor n/2\rfloor+1)$ ones. (The actions are just those with different rotation numbers everywhere. In the case of the complex Grassmannian this actions fixes $\binom{n+2}{2}$ fixed-points of the form $\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^0$. In the case of the real Grassmannian the fixed-point set consists of $\binom{\lfloor n/2\rfloor+2}{2}$ points of the form $\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^0$ and the same amount of points identical to $(\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^0)^*$. So there are exactly $2 \cdot \binom{\lfloor n/2\rfloor+2}{2} = (\lfloor n/2\rfloor+2)(\lfloor n/2\rfloor+1)$ fixed-points.) Since the Euler characteristic of the fixed-point set equals the one of the manifold, we are done. □

In order to give a complete description of fixed-point components in terms of cohomology, we still have to take a look at

$$\widetilde{\mathbf{Gr}}_2(\mathbb{R}^n) = \frac{\mathbf{SO}(n+2)}{\mathbf{SO}(n) \times \mathbf{SO}(2)}$$

Considering this space as a complex quadric gives insight into its structure. For the convenience of the reader we reprove

Theorem B.7. *The space* $\widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n+2})$ *is a simply-connected Kähler manifold of dimension* $2n$ *and its rational cohomology algebra is given by*

$$H^*(\widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n+2})) = \mathbb{Q}[x]/x^{n+1} \qquad \deg x = 2$$

if n *is odd. In the case that* n *is even we obtain*

$$H^*(\widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n+2})) = \mathbb{Q}[x,y]/\{x^{n+1} = 0, xy = 0, y^2 = (-1)^{n/2}x^n\}$$

with

$$\deg x = 2, \ \deg y = n$$

Moreover, we know $\widetilde{\mathbf{Gr}}_2(\mathbb{R}^3) \cong \mathbb{C}\mathbf{P}^1$ *and* $\widetilde{\mathbf{Gr}}_2(\mathbb{R}^4) \cong \mathbb{C}\mathbf{P}^1 \times \mathbb{C}\mathbf{P}^1$.

Proof. From the general theory (cf. page 20) we know that $\widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n+2})$ is locally Kählerian. We shall now compute the fundamental group, the vanishing of which will prove that it is actually Kählerian. First of all, the space $\widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n+2})$ is connected, since so is $\mathbf{SO}(n+2)$.

The space is even simply-connected, as the inclusion of $\mathbf{SO}(n) \times \mathbf{SO}(2)$ induces a surjective morphism

$$\pi_1(\mathbf{SO}(n) \times \mathbf{SO}(2)) \to \pi_1(\mathbf{SO}(n+2))$$

(This morphism is already surjective when restricted to the $\mathbf{SO}(2)$-factor). The long exact homotopy sequence then yields the result.

We compute

$$\dim \widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n+2}) = \dim \mathbf{SO}(n+2) - \dim \mathbf{SO}(n) \times \mathbf{SO}(2) = 2n$$

Furthermore, we have

$$\widetilde{\mathbf{Gr}}_2(\mathbb{R}^3) = \frac{\mathbf{SO}(3)}{\mathbf{SO}(2)} = \mathbb{S}^2 \cong \mathbb{C}\mathbf{P}^1$$

and

$$\begin{aligned}
\mathbb{C}\mathbf{P}^1 \times \mathbb{C}\mathbf{P}^1 &= \frac{\mathbf{U}(2)}{\mathbf{U}(1) \times \mathbf{U}(1)} \times \frac{\mathbf{U}(2)}{\mathbf{U}(1) \times \mathbf{U}(1)} \\
&= \frac{\mathbf{SU}(2)}{\mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(1))} \times \frac{\mathbf{SU}(2)}{\mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(1))} \\
&= \frac{\mathbf{SU}(2)}{\mathbf{U}(1)} \times \frac{\mathbf{SU}(2)}{\mathbf{U}(1)} \\
&= \frac{\mathbf{Sp}(1)}{\mathbf{SO}(2)} \times \frac{\mathbf{Sp}(1)}{\mathbf{SO}(2)} \\
&= \frac{\mathbf{SO}(4)}{\mathbf{SO}(2) \times \mathbf{SO}(2)} \\
&= \widetilde{\mathbf{Gr}}_2(\mathbb{R}^4)
\end{aligned}$$

For this we note that the inclusion of the denominator group is the inclusion of the standard maximal torus in each respective case. The used identifications preserve standard tori.

As we did with the Wolf spaces we may now compute the rational cohomology of $\widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n+2})$ in the terminology introduced before. So for n odd we derive:

$$\begin{aligned} H^*(\widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n+2})) &= H^*(\mathbf{B}T)^{\mathrm{W}(\mathbf{SO}(n)\times\mathbf{SO}(2))}/H^{>0}(\mathbf{B}T)^{\mathrm{W}(\mathbf{SO}(n+4))} \\ &= \mathbb{Q}[\sigma_1,\ldots,\sigma_{(n-1)/2},\tilde{\sigma}_1']/\{\sigma\tilde{\sigma}=1\} \end{aligned}$$

where $\tilde{\sigma}_1'$ denotes the first elementary symmetric polynomial in the formal roots of $\mathbf{SO}(2)$, i.e. just in x_1. The relations $\sigma\tilde{\sigma}=1$ are just $\sigma_i = -\tilde{\sigma}_1\sigma_{i-1}$ and can successively be reduced from $\sigma_1+\tilde{\sigma}_1 = 0 \Leftrightarrow \sigma_1 = -(\tilde{\sigma}_1')^2$ to finally $\sigma_i = (-1)^i(\tilde{\sigma}_1')^{2i}$. So actually the cohomology is generated by $x=\tilde{\sigma}_1'$ and the only relation that remains is

$$\sigma_{(n+1)/2} = 0 \Leftrightarrow (-1)^{(n+1)/2}(\tilde{\sigma}_1')^{n+1} = 0 \Leftrightarrow x^{n+1} = 0$$

If n is even we obtain

$$\begin{aligned} H^*(\widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n+2})) &= H^*(\mathbf{B}T)^{\mathrm{W}(\mathbf{SO}(n)\times\mathbf{SO}(2))}/H^{>0}(\mathbf{B}T)^{\mathrm{W}(\mathbf{SO}(n+4))} \\ &= \mathbb{Q}[\sigma_1,\ldots,\sigma_{(n-2)/2},\sigma_{n/2}',\tilde{\sigma}_1']/\{\sigma\tilde{\sigma}=1, \sigma_{n/2}'\tilde{\sigma}_1' = 0\} \end{aligned}$$

As in the odd case we derive $\sigma_i = (-1)^i(\tilde{\sigma}_1')^{2i}$. Set $x := \tilde{\sigma}_1'$ and obtain as before that $x^{n+1}=0$. This is the only relation on x. Set $y=\sigma_{n/2}'$ and derive $xy=0$. Finally, recall

$$(\sigma_{n/2}')^2 = \sigma_{n/2} = (-1)^{n/2}(\tilde{\sigma}_1')^n$$

which implies $y^2 = (-1)^{n/2}x^n$. □

Observe that the fact that $\widetilde{\mathbf{Gr}}_2(\mathbb{R}^4) \cong \mathbb{C}\mathbf{P}^1 \times \mathbb{C}\mathbf{P}^1$ reflects the isometry $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6) \cong \mathbf{Gr}_2(\mathbb{C}^4)$ on the level of $\mathbb{S}^1$-fixed-point components.

Corollary B.8. *The Euler characteristic is given by*

$$\chi(\widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n+2})) = \begin{cases} n+2 & \text{for } n \text{ even} \\ n+1 & \text{for } n \text{ odd} \end{cases}$$

□

B.2. Isometric involutions

Let us now classify the fixed-point components of $\mathbb{Z}_2$-actions on the infinite series of Wolf spaces under the condition that $\mathbb{Z}_2 \subseteq \mathbb{S}^1 \subseteq T^{\max}$ is a subgroup of the standard maximal torus of the isometry group. For this, we use our knowledge of how the maximal torus acts. However, observe that using the description of the spaces as homogeneous spaces as given in (A.1) we may have to pass over to certain finite quotients of the numerator in order to obtain the effectively acting isometry group.

Note further that our assumptions on the $\mathbb{Z}_2$-actions are not really restrictive: Every $\mathbb{Z}_2$-subgroup in the connected component $\mathrm{Isom}_0(M)$ of the isometry group of M is contained in some maximal torus and every two maximal tori are conjugated. Conjugation, however, has no effect on the structure of the fixed-point sets; i.e. conjugation with $g \in \mathrm{Isom}(M)$ of different $\mathbb{Z}_2$-subgroups yields that g is an isometry between the different fixed-point sets. However, for example on the real Grassmannian we have the isometric involution which reverses all the orientations of planes. This involution acts without fixed-points.

Theorem B.9. *The fixed-point set of such an (effective) involutive action on $\mathbb{H}\mathbf{P}^n$ can be described as follows: Either are there $n_1 > 0$, $n_2 > 0$ with $n_1 + n_2 + 1 = n$ and the involution fixes*

$$\mathbb{H}\mathbf{P}^{n_1},\ \mathbb{H}\mathbf{P}^{n_2}$$

or the action fixes a

$$\mathbb{C}\mathbf{P}^n$$

Proof. Let $\mathbb{Z}_2 = \langle g \rangle$. Since $g^2 = \mathrm{id}$, in homogeneous coordinates of $\mathbb{H}\mathbf{P}^n$ the element g may act by $\pm 1, \pm i$ in each coordinate only. (This is due to the fact that the center of $\mathbb{H}$ is given by $C(\mathbf{Sp}(1)) = \pm 1$.) That is, the action of $\mathbb{Z}_2$ in homogeneous coordinates is given by

$$(i^{k_1} z_1 : i^{k_1} z_2 : \ldots : i^{k_1} z_l : i^{k_2} z_{l+1} : \ldots : i^{k_2} z_n : i^{k_2} z_{n+1})$$

where $k_1, k_2 \in \{0,1,2,3\}$ and $k_1 \in \{1,3\} \Leftrightarrow k_2 \in \{1,3\}$, since the action of the square of the generator must equal multiplication with a real scalar. So there are the following cases: If $k_1 = k_2 \in \{0,2\}$, then the action is the identity—not effective—and the whole $\mathbb{H}\mathbf{P}^n$ is fixed. If $k_1 \neq k_2$ and $k_1, k_2 \in \{0,2\}$, an $\mathbb{H}\mathbf{P}^{l-1}$ and an $\mathbb{H}\mathbf{P}^{n-l}$ are fixed. If $k_1 = k_2 \in \{1,3\}$, then the standard complex projective subspace $\mathbb{C}\mathbf{P}^n$ is fixed. If $k_1 \neq k_2 \in \{1,3\}$ (without restriction $i^{k_1} = i$ and $i^{k_2} = -i$), then the space of all points of the form $(z_1 : \ldots z_l : z_l j : \ldots : z_{n+1} j)$ with $z_i \in \mathbb{C}$ are fixed. These points also form a $\mathbb{C}\mathbf{P}^n$. □

Assume a $\mathbb{Z}_2$-action on $\mathbf{Gr}_2(\mathbb{C}^{n+2})$ as depicted at the beginning of the section.

Theorem B.10. *There are $k_1 \geq 0$ and $k_2 \geq 0$ with $k_1 + k_2 = n + 2$ with the property that the fixed-point set of the involution on $\mathbf{Gr}_2(\mathbb{C}^{n+2})$ is given by*

$$\{\mathbf{Gr}_2(\mathbb{C}^{k_1}),\ \mathbf{Gr}_2(\mathbb{C}^{k_2}),\ \mathbb{C}\mathbf{P}^{k_1-1} \times \mathbb{C}\mathbf{P}^{k_2-1}\}$$

Proof. Recall that the action of a circle of the maximal torus is induced by the action

$$(t^{k_1}, \ldots, t^{k_{n+1}}, t^{-\sum_{1\leq j\leq n+1} k_j})$$

on $\mathbb{C}^{2n}$. Now assume an involutive action on $\mathbb{C}^{n+2}$ contained in this torus to be given by multiplication of the form

$$(s_1 z_1, s_2 z_2, \ldots, s_{n+2} z_{n+2})$$

where $s_{n+2} = (\prod_{j=1}^{n+1} s_j)^{-1}$. As a first step we shall prove that this action induces the identity on the Grassmannian, i.e. on 2-planes, if and only if $s_1 = \cdots = s_{n+2}$ (for $n \geq 1$). Suppose that s_1, s_2, s_3 are not identical, without restriction. We consider the action $(s_1 z_1, s_2 z_2, s_3 z_3)$ induced on $\mathbb{C}^3$. This action induces an action on 2-planes in $\mathbb{C}^3$. Division by $s_1 \neq 0$ shows that, without restriction, we may assume the action to be induced by $(1 \cdot z_1, s_2 z_2, s_3 z_3)$. Consider the 2-plane P generated by $(1, 1, 0)$ and $(1, 0, 1)$. Suppose this plane is fixed under the involution. Thus we obtain that the image of the generating vectors lies in P again, i.e. $(1, s_2, 0) \in P$ and $(1, 0, s_3) \in P$. By forming suitable linear combinations we see that $(0, s_1 - 1, 0) \in P$ and $(0, 0, s_2 - 1) \in P$. As we assumed $s_1 = 1$, at least one of s_2, s_3 cannot equal one by assumption. Thus, without restriction, we also obtain $(0, 1, 0) \in P$. In total, we see that P is generated by the vectors $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 0)$. As these vectors are linearly independent, this contradicts $\dim P = 2$. Thus we obtain $s_1 = \cdots = s_{n+2}$.

Hence the identity action on 2-planes is given by multiplication with a certain $(n+2)$-th root of unity, since $s_1 = s_2 = \cdots = s_{n+2} = (\prod_{j=1}^{n+1} s_j)^{-1} = s_1^{-n-1}$ implies $(s_1)^{-n-2} = 1$. (This corresponds to the fact that the isometry group is a finite quotient of $\mathbf{SU}(n+2)$.) Without restriction, i.e. after division with s_1 we may assume that $s_1 = 1$.

Hence any $\mathbb{Z}_2$ contained in the torus acts by $(s, \ldots, s, -s, \ldots, -s)$ for a certain $s \in \mathbb{S}^1$ and again, without restriction, we suppose it to act by

$$(1, \ldots, 1, -1, \ldots, -1)$$

As a next step we shall identify the fixed-point components of such an action. Let us denote the subspace of $\mathbb{C}^{2n}$ with a $(+1)$-action by V_1, the one with a (-1)-action by V_2. So let

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle$$

be a complex 2-plane with $x_i, y_i \in V_i$ which is fixed by the action. So, clearly, the plane also contains the vectors $(x_1, -x_2)$ and $(y_1, -y_2)$. Thus it is generated by $(x_1, 0)$, $(0, x_2)$, $(y_1, 0)$ and $(0, y_2)$. This implies that any complex 2-plane fixed under the action is of the form $\langle x, y \rangle$ with either $x, y \in V_1$, $x, y \in V_2$ or (without restriction) $x \in V_1$, $y \in V_2$ and the result on fixed-point components follows. $\square$

Observe that the Kähler component $\mathbb{C}\mathbf{P}^{k_1-1} \times \mathbb{C}\mathbf{P}^{k_2-1}$ indeed has real dimension $2n$ as required by the general theory (cf. page 20).

Again we denote by $(X)^*$ the "dual space" of X with reversed orientations. To finish off our discussion of this sort of $\mathbb{Z}_2$-actions we suppose

$$\overline{\mathbf{Gr}_2(\mathbb{R}^{d_1}) \times \mathbf{Gr}_2(\mathbb{R}^{d_2})} = \frac{\mathbf{S}(\mathbf{O}(d_1) \times \mathbf{O}(d_2))}{\mathbf{S}(\mathbf{O}(d_1-2) \times \mathbf{O}(d_2-2)) \times \mathbf{S}(\mathbf{O}(2) \times \mathbf{O}(2))}$$

and prove

Theorem B.11. *A $\mathbb{Z}_2$-action (as specified) on $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$ has one of the following fixed-point sets: One possibility for the set is given in table B.2. (These components depend on the dimensions d_1 and d_2 of the subspaces on which the action on $\mathbb{R}^{n+4}$ inducing the action on the Grassmannian is given by multiplication with $+1$ respectively with -1. For low dimensional versions of this type see also table B.3.)*

The second type of fixed-point set can only appear if n is even. It is of the form

$$\{\mathbf{Gr}_2(\mathbb{C}^{n/2+2}), (\mathbf{Gr}_2(\mathbb{C}^{n/2+2}))^*\}$$

PROOF. We shall now apply a reasoning similar to the one used in the case of the complex Grassmannian to reveal the structure of possible $\mathbb{Z}_2$-actions. As in the complex case we see that the only actions on $\mathbb{R}^{n+4}$ inducing the identity on real 4-planes are given by $\pm\mathrm{id}$.

This will be done as follows: The torus $T = T^{\lfloor n/2 \rfloor+2}$ acts on $\mathbb{R}^{n+4}$ by left-multiplication

$$(s_1(x_1,x_2), s_2(x_3,x_4), \ldots, s_{\lfloor n/2\rfloor+2}(x_{2\lfloor n/2\rfloor+3}, x_{2\lfloor n/2\rfloor+4}), x_{n+4})$$

for $(s_1,\ldots,s_{\lfloor n/2\rfloor+2}) \in T$. (Whether the last coordinate x_{n+4} is acted on trivially or not depends on the parity of the dimension. We described the case for n odd.) Suppose this action induces the identity on 4-planes.

First we observe that consequently all the s_i have to be real. Assume the contrary. Without restriction $s_1 \notin \mathbb{R}$. Consider the 4-plane

$$P = \left\langle \begin{pmatrix}1\\0\\0\\0\\0\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\\0\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\\0\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\1\\0\\\vdots\\0\end{pmatrix} \right\rangle$$

As it is left invariant by $(s_1, \ldots, s_{\lfloor n/2 \rfloor + 2})$, we see that $(s_1(1,0), 0, \ldots, 0) \in P$. Thus $(\mathbb{R}^2, 0, \ldots, 0) \in P$, which contradicts $\dim P = 4$.

As a next step we obtain that all the s_i have to be identical. For this we proceed as in the case of the complex Grassmannian—cf. the proof of theorem B.10.

Thus we see that the identity on 4-planes is induced by $\pm\mathrm{id}$ on $\mathbb{R}^{n+4}$ only. Hence we see that the $\mathbb{Z}_2$-action is given by either

$$(1, \ldots, 1, -1, \ldots, -1) \tag{B.1}$$

or

$$(i, \ldots, i, -i, \ldots, -i) \tag{B.2}$$

Let V_1 be the vector space on which the action is given by $+1$ in the first case respectively by $+i$ in the second one. Let V_2 be the space with (-1)- respectively $(-i)$-action. Let $d_i := \dim V_i$. Observe that d_2 is even in any case, since the (-1)-action respectively the $(-i)$-action is induced by an $\mathbb{S}^1$-action on $\mathbb{R}^2$.

The fixed-point components in the case (B.1) can be detected as follows: Let (v_1, v_2), (w_1, w_2), (x_1, x_2), (y_1, y_2) be vectors in $V_1 \oplus V_2$ generating a 4-plane $P = \langle v, w, x, y \rangle$ that is invariant under the involution. Thus $(v_1, -v_2)$ lies in P and so do $(v_1, 0)$, $(0, v_2)$. The same works for the other vectors and we see that P is already generated by vectors from V_1 and from V_2 only, i.e.

$$P = \langle (v_1, 0), (0, v_2), (y_1, 0), (0, y_2), (y_1, 0), (0, y_2), (y_1, 0), (0, y_2) \rangle$$

As P is four-dimensional, there is a choice

$$\tilde{v}, \tilde{w}, \tilde{x}, \tilde{y} \in \{(v_1, 0), (0, v_2), (y_1, 0), (0, y_2), (y_1, 0), (0, y_2), (y_1, 0), (0, y_2)\}$$

with the property that

$$P = \langle \tilde{v}, \tilde{w}, \tilde{x}, \tilde{y} \rangle$$

We may now step through the different cases according to how many of the generating vectors $\tilde{v}, \tilde{w}, \tilde{x}, \tilde{y}$ are in V_1 (and how many of them lie in V_2).

Assume first that $\tilde{v}, \tilde{w}, \tilde{x}, \tilde{y} \in V_1$. Then the vectors generate a plane that may be regarded as an element in $\widetilde{\mathbf{Gr}}_4(V_1)$ when we count it with orientation. Conversely, every element in this subspace is fixed by the involution. (Here orientations are not altered as every generating vector $\tilde{v}, \tilde{w}, \tilde{x}, \tilde{y}$ of P is fixed by the involution.)

As a second case suppose $\tilde{v}, \tilde{w}, \tilde{x} \in V_1$ and $\tilde{y} \in V_2$, i.e. three generators are from V_1 and the fourth generator is from V_2. This case cannot occur, as P will not be fixed under the involution. Indeed, the orientation of P is reversed in that case, since an odd numbers of directions is multiplied by (-1).

Finally only the case with two generators in both V_1 and V_2 remains, i.e. $\tilde{v}, \tilde{w} \in V_1$ and $\tilde{x}, \tilde{y} \in V_2$. Thus V_2 is entirely contained in P. Hence P (endowed with an orientation) is a point in

$$\overline{\mathbf{Gr}_2(V_1) \times \mathbf{Gr}_2(V_2)}$$

—the space of all oriented four-planes which can be generated by two basis vectors in both V_1 and V_2. Conversely, every element in $\overline{\mathbf{Gr}_2(V_1) \times \mathbf{Gr}_2(V_2)}$ stays fixed under the involution. This is due to the fact that the orientation of P is preserved, as we multiply twice by (-1).

Let us now deal with the case when the involution is given by (B.2). This case may only occur when n is even, since otherwise there is a coordinate not affected by multiplication with an element from the maximal torus, i.e. the action is the identity on there. The action defines a complex structure I on $\mathbb{R}^{2(n/2+2)} \cong \mathbb{C}^{n/2+1}$ given by $I = (+i, \ldots, +i, -i, \ldots, -i)$, i.e. by multiplication with $(+i)$ on V_1 and by multiplication with $(-i)$ on V_2. Due to that definition the image of the real vector $(v_1, v_2) \in V_1 \oplus V_2$ lies in P again—i.e. P is invariant—if and only if the complex plane $\langle (v_1, v_2) \rangle_{\mathbb{C}}$ (with complex structure I) is invariant under the involution. So, clearly, the planes P invariant under the action are just the complex 2-planes (with respect to I) that are generated by one vector in V_1 and by one more in V_2. For an *oriented* plane P to stay invariant we need the action restricted to P to be orientation-preserving. This, however, is always the case since $\det_{\mathbb{R}}(\pm i) = 1$. So again we obtain two dual components with opposite orientations. □

In order to shed more light on the structure of the fixed-point set we prove

Theorem B.12. *We have*

$$\overline{\mathbf{Gr}_2(\mathbb{R}^{n_1+2}) \times \mathbf{Gr}_2(\mathbb{R}^{n_2+2})} = (\widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n_1+2}) \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n_2+2}))/\mathbb{Z}_2$$

(for $n_1, n_2 > 0$) and

$$\overline{\mathbf{Gr}_2(\mathbb{R}^{n_1+2}) \times \mathbf{Gr}_2(\mathbb{R}^{n_2+2})} = \widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n_1+2})$$

for $n_2 = 0$. Further

$$\overline{\mathbf{Gr}_2(\mathbb{R}^3) \times \mathbf{Gr}_2(\mathbb{R}^3)} = (\mathbb{C}\mathbf{P}^1 \times \mathbb{C}\mathbf{P}^1)/\mathbb{Z}_2$$

For $n_1, n_2 > 0$ the space $\overline{\mathbf{Gr}_2(\mathbb{R}^{n_1+2}) \times \mathbf{Gr}_2(\mathbb{R}^{n_2+2})}$ is locally Kähler, has dimension $2(n_1+n_2)$, fundamental group $\mathbb{Z}_2$ and Euler characteristic

$$\chi((\widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n_1+2}) \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n_2+2}))/\mathbb{Z}_2) = \begin{cases} (n_1+2)(n_2+2)/2 & \textit{for } n_1 \textit{ and } n_2 \textit{ even} \\ (n_1+2)(n_2+1)/2 & \textit{for } n_1 \textit{ even and } n_2 \textit{ odd} \\ (n_1+1)(n_2+2)/2 & \textit{for } n_1 \textit{ odd and } n_2 \textit{ even} \\ (n_1+1)(n_2+1)/2 & \textit{for } n_1 \textit{ and } n_2 \textit{ odd} \end{cases}$$

Proof. The first equalities follow directly from the description as homogenous spaces of the Grassmannians involved. Alternatively, they are directly due to a geometric reasoning concerning subplanes of Euclidean space. Exemplarily, we compute the least obvious case:

$$\begin{aligned} \overline{\mathbf{Gr}_2(\mathbb{R}^3) \times \mathbf{Gr}_2(\mathbb{R}^3)} &= \frac{\mathbf{S}(\mathbf{O}(3) \times \mathbf{O}(3))}{\mathbf{S}(\mathbf{O}(1) \times \mathbf{O}(1)) \times \mathbf{S}(\mathbf{O}(2) \times \mathbf{O}(2))} \\ &= \frac{\mathbf{S}(\mathbf{O}(3) \times \mathbf{O}(3))}{\mathbb{Z}_2 \times \mathbf{S}(\mathbf{O}(2) \times \mathbf{O}(2))} \\ &= \left(\frac{\mathbf{SO}(3)}{\mathbf{SO}(2)} \times \frac{\mathbf{SO}(3)}{\mathbf{SO}(2)} \right) \Big/ \mathbb{Z}_2 \\ &= (\mathbb{C}\mathbf{P}^1 \times \mathbb{C}\mathbf{P}^1)/\mathbb{Z}_2 \end{aligned}$$

In general, the space is a $\mathbb{Z}_2$-quotient of a simply connected space. So its fundamental group equals $\mathbb{Z}_2$. Its dimension is

$$\dim \overline{\mathbf{Gr}_2(\mathbb{R}^{n_1+2}) \times \mathbf{Gr}_2(\mathbb{R}^{n_2+2})} = \dim \widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n_1+2}) + \dim \widetilde{\mathbf{Gr}}_2(\mathbb{R}^{n_2+2}) = 2(n_1+n_2)$$

The space is locally Kähler by the general theory (cf. page 20).

The Euler characteristic is multiplicative and satisfies $\chi(X) = k \cdot \chi(Y)$ for a k-fold covering $X \to Y$. Corollary B.8 yields the result. □

Corollary B.13. *In low dimensions, possible configurations of fixed-point sets $F_{\mathbb{Z}_2}$ of the considered $\mathbb{Z}_2$-actions are given as depicted in table B.3.*

□

B.3. Clearing ambiguities

Observe that there is a certain ambiguity in the terminology: Recall from page 20 that a fixed-point component of an element $g \in \mathrm{Isom}(M)$—be it an involution or a topological generator of an $\mathbb{S}^1$—is a quaternionic submanifold if the isotropy representation of g composed with the canonical projection

$$\mathbf{Sp}(n)\mathbf{Sp}(1) \to \mathbf{SO}(3)$$

(on the second factor) is trivial. If it is not, the component is a locally Kähler submanifold. Conversely, every quaternionic respectively locally Kähler submanifold is a Positive Quaternion Kähler Manifold respectively a locally Kähler manifold.

In general, we directly see whether the components are quaternionic or (locally) Kählerian as submanifolds. Indeed, ambiguity can appear only if the component is simply-connected, Kählerian and Positive Quaternion Kählerian as an abstract manifold. The only Positive Quaternion Kähler Manifold with this property is the complex Grassmannian $\mathbf{Gr}_2(\mathbb{C}^{n+2})$. Moreover, in small dimensions we have $\mathbb{C}\mathbf{P}^2 \cong \mathbf{Gr}_2(\mathbb{C}^3)$ for example. It remains to see when the complex Grassmannian respectively certain complex projective spaces appear as quaternionic submanifolds and when they appear as Kähler submanifolds.

(Moreover, it is easy to see that (locally) Kähler components of an involution are of at most half the dimension of the ambient manifold. In the case of the involution they are of exactly half the dimension; this provides another criterion.)

Theorem B.14. • *Given an $\mathbb{S}^1$-action—without restriction assume it to be of the form $(t^{k_1}, \ldots, t^{k_{n+1}}, t^{-\sum_{1\leq j\leq n+1} k_j})$—on $\mathbf{Gr}_2(\mathbb{C}^{n+2})$, for each rotation number k_i there is exactly one quaternionic component, namely a complex Grassmannian as depicted in theorem B.3—provided the complex dimension of the subspace on which the action is given by $(t^{k_i}, \ldots, t^{k_i})$ is greater than or equal to 2.*

The products of complex projective spaces that arise as components in the theorem are Kähler submanifolds.

- *Given a $\mathbb{Z}_2$-action—without restriction by $(1, \ldots, 1, -1, \ldots, -1)$—on $\mathbf{Gr}_2(\mathbb{C}^{n+2})$, there are*
  - *exactly two quaternionic components, if the complex dimension of the space on which the action is given by 1 is at least 2 and if the same holds for the dimension of the space on which the action is given by -1.*
  - *exactly one quaternionic component, if exactly one of the spaces with ± 1-action is at least 2-dimensional.*

  The components are given by the complex Grassmannians corresponding to the subspaces on which the involution is given by $+1$ respectively by -1 only (cf. theorem B.10).

  Thus the products of complex projective spaces are Kähler components. (They are never quaternionic submanifolds, not even for low dimensions.)

PROOF. Elements of $\mathbf{Gr}_2(\mathbb{C}^{n+2})$ are given as complex 2-planes of $\mathbb{C}^{n+2}$, i.e. they are represented by $((n+2) \times 2)$-matrices. Recall that charts are hence given by matrices of the form

$$\begin{pmatrix} * & * & * & \dots & * & 1 & 0 \\ * & * & * & \dots & * & 0 & 1 \end{pmatrix}^T \tag{B.3}$$

when having applied a certain permutation to line vectors (cf. [32], p. 193). Recall that the description of $\mathbf{Gr}_2(\mathbb{C}^{n+2})$ as the homogeneous space $\mathbf{Gr}_2(\mathbb{C}^{n+2}) = \frac{\mathbf{SU}(n+2)}{\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(2))}$ corresponds to the one presented by identifying a generating special unitary 2-frame of a plane with the special unitary matrix that maps the last two vectors e_{n+1}, e_{n+2} of the standard basis of $\mathbb{C}^{n+2}$ to it. Indeed, the identification of special unitary matrices with complex 2-planes is given by $U \mapsto \langle Ue_{n+1}, Ue_{n+2}\rangle$.

The action of the isometry group is induced by left multiplication with $\mathbf{SU}(n+2)$. On a fixed-point $x \in \mathbf{Gr}_2(\mathbb{C}^{n+2})$ of a certain transformation $A \in T^{n+1} \subseteq \mathbf{SU}(n+2)$ we shall compute the isotropy representation of A. For this we shall consider the action of A on a chart like (B.3). For this let x be represented by the matrix $X \in \mathbf{Gr}_2(\mathbb{C}^{n+2})$. Since A is in the standard maximal torus, it is of the form $A = \operatorname{diag}(\prod_{2\le i\le n+2}(a_i)^{-1}, a_2, \dots, a_{n+2})$ for certain complex units $a_i \in \mathbb{S}^1$. Now multiplication by $\operatorname{diag}(a_{n+1}^{-1}, a_{n+2}^{-1}) \in \mathbf{U}(2)$ from the right does not alter the coset of x and maps AX back to the standard form (B.3), i.e.

$$AX\operatorname{diag}(a_{n+1}^{-1}, a_{n+2}^{-1})$$

is the identity on $\langle e_{n+1}, e_{n+2}\rangle$. Thus the action of A on its tangent space $T_x\mathbf{Gr}_2(\mathbb{C}^{n+2}) \cong \mathbb{C}^{n\times 2} \cong \mathbb{C}^{2n} \cong \mathbb{H}^n$ at x is given by $\mathrm{d}A : T_x\mathbf{Gr}_2(\mathbb{C}^{n+2}) \to T_x\mathbf{Gr}_2(\mathbb{C}^{n+2})$ with

$$\mathrm{d}A(y) = \operatorname{diag}\left(\prod_{2\le i\le n+2}(a_i)^{-1}, a_2, \dots, a_n\right) \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^T \cdot \begin{pmatrix} (a_{n+1})^{-1} & 0 \\ 0 & (a_{n+2})^{-1} \end{pmatrix}$$

where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^T \in \mathbb{C}^{n\times 2}$.

Now recall that

$$\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(2)) = \{a(U_1,0) + b(0,U_2) \mid U_1 \in \mathbf{SU}(n); U_2 \in \mathbf{SU}(2); a,b \in \mathbb{S}^1; ab = 1\}$$

So the action on $T_x\mathbf{Gr}_2(\mathbb{C}^{n+2})$ in these terms is given by

$$\mathrm{d}A(y) = (aA_1) \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^T \cdot (bA_2)^{-1} = ((ab^{-1})A_1) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^T A_2^{-1}$$

where $A = (aA_1, bA_2) \in \mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(2))$ with $(A_1, A_2) \in \mathbf{SU}(n)\times\mathbf{SU}(2)$ and $((ab^{-1})A_1) \in \mathbf{U}(n)$. By [5], p. 409, the inclusion of the $\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(2))$-structure bundle into the standard $\mathbf{Sp}(n)\mathbf{Sp}(1)$-bundle now is induced by the canonical inclusion $\mathbf{U}(n) \hookrightarrow \mathbf{Sp}(n)$ and the canonical identification of $\mathbf{SU}(2)$ with $\mathbf{Sp}(1)$.

So the type of a fixed-point, i.e. whether it is quaternionic or (locally) Kählerian, is determined by the rotation numbers of A on the $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$-part of the chart it belongs to, i.e. by the matrix $A_2 = b^{-1} \begin{pmatrix} a_{n+1}^{-1} & 0 \\ 0 & a_{n+2}^{-1} \end{pmatrix}$.

Let us step through the different cases: Assume first that the rotation numbers are constant on there. So the matrix $\begin{pmatrix} (a_{n+1})^{-1} & 0 \\ 0 & (a_{n+2})^{-1} \end{pmatrix}$ can be written as $b \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in the terminology from above—this b indeed is a central factor. Thus the representation in the $\mathbf{Sp}(1)$-factor is the identity and so is the projection onto $\mathbf{SO}(3)$. Hence the point x—and the fixed-point component it is contained in—is quaternionic.

Suppose now that the rotation numbers are different. Hence the $\mathbf{Sp}(1)$-representation has the form $\left(\begin{smallmatrix} b' & 0 \\ 0 & \bar{b}' \end{smallmatrix}\right)$ with $b' \in \mathbb{S}^1 \setminus \{\pm 1\}$. So the projection of this matrix to $\mathbf{SO}(3)$ is not trivial. Hence the fixed-point is (locally) Kählerian.

Thus we see that whenever a plane X that is fixed by the action does not lie in a subspace of $\mathbb{C}^{n+2}$ on which the action of A is given by one rotation number only, then the plane belongs to a (locally) Kähler component. Conversely, if there is only one rotation number upon it, it is quaternionic. Thus—irrespective of whether we consider $\mathbb{S}^1$-actions or involutions—the components we denoted by $\mathbf{Gr}_2(\mathbb{C}^k)$ are quaternionic, the ones involving $\mathbb{C}\mathbf{P}^k$-factors are (locally) Kählerian.

$\square$

Note that alternatively we might have used that the isotropy representation is given by the adjoint map Ad which would permit a more general but less illustrative approach.

Table B.1.: $\mathbb{S}^1$-fixed-point components of low-dimensional Wolf spaces

Wolf space	possible fixed-point sets
$F_{\mathbb{S}^1}(\mathbb{H}\mathbf{P}^1)$	$\{\mathbb{H}\mathbf{P}^1\}, \{\mathbb{H}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0\}, \{\mathbb{C}\mathbf{P}^1\}$
$F_{\mathbb{S}^1}(\mathbb{H}\mathbf{P}^2)$	$\{\mathbb{H}\mathbf{P}^2\}$, $\{\mathbb{H}\mathbf{P}^1, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{H}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{C}\mathbf{P}^2\}$, $\{\mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0\}$
$F_{\mathbb{S}^1}(\mathbb{H}\mathbf{P}^3)$	$\{\mathbb{H}\mathbf{P}^3\}$, $\{\mathbb{H}\mathbf{P}^2, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{H}\mathbf{P}^1, \mathbb{C}\mathbf{P}^1\}$, $\{\mathbb{H}\mathbf{P}^1, \mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{H}\mathbf{P}^0, \mathbb{C}\mathbf{P}^2\}$, $\{\mathbb{H}\mathbf{P}^0, \mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{H}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{C}\mathbf{P}^3\}$, $\{\mathbb{C}\mathbf{P}^2, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^1\}$, $\{\mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0\}$
$F_{\mathbb{S}^1}(\mathbb{H}\mathbf{P}^4)$	$\{\mathbb{H}\mathbf{P}^4\}$, $\{\mathbb{H}\mathbf{P}^3, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{H}\mathbf{P}^2, \mathbb{C}\mathbf{P}^1\}$, $\{\mathbb{H}\mathbf{P}^2, \mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{H}\mathbf{P}^1, \mathbb{C}\mathbf{P}^2\}$, $\{\mathbb{H}\mathbf{P}^1, \mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{H}\mathbf{P}^1, (\mathbb{C}\mathbf{P}^0)^{\sqcup 3}\}$, $\{\mathbb{H}\mathbf{P}^0, \mathbb{C}\mathbf{P}^3\}$, $\{\mathbb{H}\mathbf{P}^0, \mathbb{C}\mathbf{P}^2, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{H}\mathbf{P}^0, \mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^1\}$, $\{\mathbb{H}\mathbf{P}^0, \mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{H}\mathbf{P}^0, (\mathbb{C}\mathbf{P}^0)^{\sqcup 3}\}$, $\{\mathbb{C}\mathbf{P}^4\}$, $\{\mathbb{C}\mathbf{P}^3, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{C}\mathbf{P}^2, \mathbb{C}\mathbf{P}^1\}$, $\{\mathbb{C}\mathbf{P}^2, \mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbb{C}\mathbf{P}^1, (\mathbb{C}\mathbf{P}^0)^{\sqcup 3}\}$, $\{(\mathbb{C}\mathbf{P}^0)^{\sqcup 5}\}$
$F_{\mathbb{S}^1}(\mathbf{Gr}_2(\mathbb{C}^3))$	$\{\mathbf{Gr}_2(\mathbb{C}^3)\}$, $\{\mathbf{Gr}_2(\mathbb{C}^2), \mathbb{C}\mathbf{P}^1\}$, $\{\mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0\}$
$F_{\mathbb{S}^1}(\mathbf{Gr}_2(\mathbb{C}^4))$	$\{\mathbf{Gr}_2(\mathbb{C}^4)\}$, $\{\mathbf{Gr}_2(\mathbb{C}^3), \mathbb{C}\mathbf{P}^2\}$, $\{\mathbf{Gr}_2(\mathbb{C}^2), \mathbf{Gr}_2(\mathbb{C}^2), \mathbb{C}\mathbf{P}^1 \times \mathbb{C}\mathbf{P}^1\}$, $\{\mathbf{Gr}_2(\mathbb{C}^2), \mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^0\}$, $\{(\mathbb{C}\mathbf{P}^0)^{\sqcup 6}\}$
$F_{\mathbb{S}^1}(\mathbf{Gr}_2(\mathbb{C}^5))$	$\{\mathbf{Gr}_2(\mathbb{C}^5)\}$, $\{\mathbf{Gr}_2(\mathbb{C}^4), \mathbb{C}\mathbf{P}^3\}$, $\{\mathbf{Gr}_2(\mathbb{C}^3), \mathbf{Gr}_2(\mathbb{C}^2), \mathbb{C}\mathbf{P}^2 \times \mathbb{C}\mathbf{P}^1\}$, $\{\mathbf{Gr}_2(\mathbb{C}^3), \mathbb{C}\mathbf{P}^2, \mathbb{C}\mathbf{P}^2, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbf{Gr}_2(\mathbb{C}^2), \mathbf{Gr}_2(\mathbb{C}^2), \mathbb{C}\mathbf{P}^1 \times \mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^1\}$, $\{\mathbf{Gr}_2(\mathbb{C}^2), \mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0, \mathbb{C}\mathbf{P}^0\}$, $\{(\mathbb{C}\mathbf{P}^0)^{\sqcup 10}\}$
$F_{\mathbb{S}^1}(\mathbf{Gr}_2(\mathbb{C}^6))$	$\{\mathbf{Gr}_2(\mathbb{C}^6)\}$, $\{\mathbf{Gr}_2(\mathbb{C}^5), \mathbb{C}\mathbf{P}^4\}$, $\{\mathbf{Gr}_2(\mathbb{C}^4), \mathbf{Gr}_2(\mathbb{C}^2), \mathbb{C}\mathbf{P}^3 \times \mathbb{C}\mathbf{P}^1\}$, $\{\mathbf{Gr}_2(\mathbb{C}^4), \mathbb{C}\mathbf{P}^3, \mathbb{C}\mathbf{P}^3, \mathbb{C}\mathbf{P}^0\}$, $\{\mathbf{Gr}_2(\mathbb{C}^3), \mathbf{Gr}_2(\mathbb{C}^3), \mathbb{C}\mathbf{P}^2 \times \mathbb{C}\mathbf{P}^2\}$, $\{\mathbf{Gr}_2(\mathbb{C}^3), \mathbf{Gr}_2(\mathbb{C}^2), \mathbb{C}\mathbf{P}^2 \times \mathbb{C}\mathbf{P}^1, \mathbb{C}\mathbf{P}^2, \mathbb{C}\mathbf{P}^1\}$, $\{\mathbf{Gr}_2(\mathbb{C}^3), (\mathbb{C}\mathbf{P}^2)^{\sqcup 3}, (\mathbb{C}\mathbf{P}^0)^{\sqcup 3}\}$, $\{(\mathbf{Gr}_2(\mathbb{C}^2))^{\sqcup 3}, (\mathbb{C}\mathbf{P}^1 \times \mathbb{C}\mathbf{P}^1)^{\sqcup 3}\}$ $\{(\mathbf{Gr}_2(\mathbb{C}^2))^{\sqcup 2}, \mathbb{C}\mathbf{P}^1 \times \mathbb{C}\mathbf{P}^1, (\mathbb{C}\mathbf{P}^1)^{\sqcup 4}, \mathbb{C}\mathbf{P}^0\}$ $\{\mathbf{Gr}_2(\mathbb{C}^2), (\mathbb{C}\mathbf{P}^1)^{\sqcup 4}, (\mathbb{C}\mathbf{P}^0)^{\sqcup 6}\}$, $(\mathbb{C}\mathbf{P}^0)^{\sqcup 15}$
$F_{\mathbb{S}^1}(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^5))$	$\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^5)\}$, $\{\mathbb{C}\mathbf{P}^0 \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^3)\}$, $\{2(\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^0)\}$
$F_{\mathbb{S}^1}(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6))$	$\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6)\}$, $\{2\widetilde{\mathbf{Gr}}_4(\mathbb{R}^4), \mathbb{C}\mathbf{P}^0 \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^4)\}$, $\{2\mathbf{Gr}_2(\mathbb{C}^2), 2(\widetilde{\mathbf{Gr}}_2(\mathbb{R}^2) \times \mathbb{C}\mathbf{P}^1)\}$, $\{2\mathbf{Gr}_2(\mathbb{C}^3)\}$, $\{2(\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^0), 2(\mathbb{C}\mathbf{P}^0 \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^2)), 2(\mathbb{C}\mathbf{P}^0 \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^2))\}$, $\{2\mathbf{Gr}_2(\mathbb{C}^2), 2(\mathbb{C}\mathbf{P}^1 \times \mathbb{C}\mathbf{P}^0)\}$, $\{2(\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^0), 2(\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^0), 2(\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^0)\}$
$F_{\mathbb{S}^1}(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7))$	$\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)\}$, $\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^5), \mathbb{C}\mathbf{P}^0 \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^5)\}$, $\{2(\mathbf{Gr}_2(\mathbb{C}^2)), \widetilde{\mathbf{Gr}}_2(\mathbb{R}^3) \times \mathbb{C}\mathbf{P}^1\}$, $\{2(\mathbf{Gr}_2(\mathbb{C}^3))\}$, $\{2(\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^0), (\mathbb{C}\mathbf{P}^0 \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^3))^{\sqcup 2}\}$, $\{2(\mathbf{Gr}_2(\mathbb{C}^2)), 2(\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^1)\}$, $\{(2(\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^0))^{\sqcup 3}\}$
$F_{\mathbb{S}^1}(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^8))$	$\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^8)\}$, $\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6), \mathbb{C}\mathbf{P}^0 \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^6)\}$, $\{2(\mathbf{Gr}_2(\mathbb{C}^2)), \mathbb{C}\mathbf{P}^1 \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^4), 2(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^4))\}$ $\{2(\mathbf{Gr}_2(\mathbb{C}^3)), 2(\mathbb{C}\mathbf{P}^2 \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^2))\}$, $\{2(\mathbf{Gr}_2(\mathbb{C}^4))\}$, $\{2(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^4), (\mathbb{C}\mathbf{P}^0 \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^4))^{\sqcup 2}, 2(\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^0)\}$, $\{2(\mathbf{Gr}_2(\mathbb{C}^2)), 2(\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^1), 2(\mathbb{C}\mathbf{P}^0 \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^2)), 2(\mathbb{C}\mathbf{P}^1 \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^2))\}$, $\{2(\mathbf{Gr}_2(\mathbb{C}^3)), 2(\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^2)$, $\{(2(\mathbf{Gr}_2(\mathbb{C}^2)))^{\sqcup 2}, 2(\mathbb{C}\mathbf{P}^1 \times \mathbb{C}\mathbf{P}^1)\}$, $\{(2(\mathbb{C}\mathbf{P}^0 \times \widetilde{\mathbf{Gr}}_2(\mathbb{R}^2)))^{\sqcup 3}, (2(\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^0))^{\sqcup 3}\}$, $\{2(\mathbf{Gr}_2(\mathbb{C}^2)), (2(\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^1))^{\sqcup 2}, 2(\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^0)\}$, $\{(2(\mathbb{C}\mathbf{P}^0 \times \mathbb{C}\mathbf{P}^0))^{\sqcup 6}\}$

In the case of the real Grassmannian as ambient space multiplication of a set by 2 denotes that this set appears once as noted and once in its dual form as appearing in theorem B.4.

Table B.2.: Real-Grassmannian-$\mathbb{Z}_2$-fixed-point components of $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$

	possible fixed-point set
$d_1, d_2 > 4,$	$\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{d_1}), \widetilde{\mathbf{Gr}}_4(\mathbb{R}^{d_2}), \overline{\mathbf{Gr}_2(\mathbb{R}^{d_1}) \times \mathbf{Gr}_2(\mathbb{R}^{d_2})}\},$
$4 = d_1 < d_2$	$\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^4),\ (\widetilde{\mathbf{Gr}}_4(\mathbb{R}^4))^*,\ \widetilde{\mathbf{Gr}}_4(\mathbb{R}^{d_2}),\ \overline{\mathbf{Gr}_2(\mathbb{R}^4) \times \mathbf{Gr}_2(\mathbb{R}^{d_2})}\}$
$d_1 = 3,\ d_2 > 4,$	$\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{d_2}),\ \overline{\mathbf{Gr}_2(\mathbb{R}^3) \times \mathbf{Gr}_2(\mathbb{R}^{d_2})}\}$
$d_1 = 2,\ d_2 > 4,$	$\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{d_2}),\ \widetilde{\mathbf{Gr}}_2(\mathbb{R}^{d_2})\}$
$d_1 < 2,\ d_2 > 4,$	$\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{d_2})$
This is symmetric in d_1 and d_2. We have $d_1 + d_2 = n + 4$, and d_2 is even. The results for small d_i are special cases of the general one just interpreted in the right way.	

Table B.3.: Involutive fixed-point components of low-dimensional Wolf spaces

Wolf space	possible fixed-point sets
$F_{\mathbb{Z}_2}(\mathbb{H}\mathbf{P}^1)$	$\{\mathbb{H}\mathbf{P}^1\}$, $\{\mathbb{H}\mathbf{P}^0, \mathbb{H}\mathbf{P}^0\}$ $\{\mathbb{C}\mathbf{P}^1\}$
$F_{\mathbb{Z}_2}(\mathbb{H}\mathbf{P}^2)$	$\{\mathbb{H}\mathbf{P}^2\}$, $\{\mathbb{H}\mathbf{P}^1, \mathbb{H}\mathbf{P}^0\}$ $\{\mathbb{C}\mathbf{P}^2\}$
$F_{\mathbb{Z}_2}(\mathbb{H}\mathbf{P}^3)$	$\{\mathbb{H}\mathbf{P}^3\}$, $\{\mathbb{H}\mathbf{P}^2, \mathbb{H}\mathbf{P}^0\}$, $\{\mathbb{H}\mathbf{P}^1, \mathbb{H}\mathbf{P}^1\}$ $\{\mathbb{C}\mathbf{P}^3\}$
$F_{\mathbb{Z}_2}(\mathbb{H}\mathbf{P}^4)$	$\{\mathbb{H}\mathbf{P}^4\}$, $\{\mathbb{H}\mathbf{P}^3, \mathbb{H}\mathbf{P}^0\}$, $\{\mathbb{H}\mathbf{P}^2, \mathbb{H}\mathbf{P}^1\}$ $\{\mathbb{C}\mathbf{P}^4\}$
$F_{\mathbb{Z}_2}(\mathbf{Gr}_2(\mathbb{C}^3))$	$\{\mathbf{Gr}_2(\mathbb{C}^3)\}$, $\{\mathbf{Gr}_2(\mathbb{C}^2), \mathbb{C}\mathbf{P}^1 \times \mathbb{C}\mathbf{P}^0\}$
$F_{\mathbb{Z}_2}(\mathbf{Gr}_2(\mathbb{C}^4))$	$\{\mathbf{Gr}_2(\mathbb{C}^4)\}$, $\{\mathbf{Gr}_2(\mathbb{C}^3), \mathbb{C}\mathbf{P}^2 \times \mathbb{C}\mathbf{P}^0\}$, $\{\mathbf{Gr}_2(\mathbb{C}^2), \mathbf{Gr}_2(\mathbb{C}^2), \mathbb{C}\mathbf{P}^1 \times \mathbb{C}\mathbf{P}^1\}$
$F_{\mathbb{Z}_2}(\mathbf{Gr}_2(\mathbb{C}^5))$	$\{\mathbf{Gr}_2(\mathbb{C}^5)\}$, $\{\mathbf{Gr}_2(\mathbb{C}^4), \mathbb{C}\mathbf{P}^3 \times \mathbb{C}\mathbf{P}^0\}$, $\{\mathbf{Gr}_2(\mathbb{C}^3), \mathbf{Gr}_2(\mathbb{C}^2), \mathbb{C}\mathbf{P}^2 \times \mathbb{C}\mathbf{P}^1\}$
$F_{\mathbb{Z}_2}(\mathbf{Gr}_2(\mathbb{C}^6))$	$\{\mathbf{Gr}_2(\mathbb{C}^6)\}$, $\{\mathbf{Gr}_2(\mathbb{C}^5), \mathbb{C}\mathbf{P}^4 \times \mathbb{C}\mathbf{P}^0\}$, $\{\mathbf{Gr}_2(\mathbb{C}^4), \mathbf{Gr}_2(\mathbb{C}^2), \mathbb{C}\mathbf{P}^3 \times \mathbb{C}\mathbf{P}^1\}$, $\{\mathbf{Gr}_2(\mathbb{C}^3), \mathbf{Gr}_2(\mathbb{C}^3), \mathbb{C}\mathbf{P}^2 \times \mathbb{C}\mathbf{P}^2\}$
$F_{\mathbb{Z}_2}(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^5))$	$\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^5)\}$, $\{2(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^4))\}$, $\{\widetilde{\mathbf{Gr}}_2(\mathbb{R}^3)\}$
$F_{\mathbb{Z}_2}(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6))$	$\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6)\}$, $\{2(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^4)), \widetilde{\mathbf{Gr}}_2(\mathbb{R}^4)\}$, $\{2(\mathbf{Gr}_2(\mathbb{C}^3))\}$
$F_{\mathbb{Z}_2}(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7))$	$\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)\}$, $\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6)\}$, $\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^5), \widetilde{\mathbf{Gr}}_2(\mathbb{R}^5)\}$ $\{2(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^4)), \overline{\mathbf{Gr}_2(\mathbb{R}^3) \times \mathbf{Gr}_2(\mathbb{R}^4)}\}$
$F_{\mathbb{Z}_2}(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^8))$	$\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^8)\}$, $\{\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6), \widetilde{\mathbf{Gr}}_2(\mathbb{R}^6)\}$, $\{2(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^4)), 2(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^4)), \overline{\mathbf{Gr}_2(\mathbb{R}^4) \times \mathbf{Gr}_2(\mathbb{R}^4)}\}$, $\{2(\mathbf{Gr}_2(\mathbb{C}^4))\}$
In the case of the real Grassmannian as ambient space multiplication of a set by 2 denotes that this set appears once as noted and once in its dual form as appearing in theorem B.11.	

C

The Euler Characteristic

In this chapter we apply the last two ones, i.e. chapters A and B, in order to find upper bounds for the Euler characteristic of Positive Quaternion Kähler Manifolds under the existence of torus actions. We shall concentrate on low-dimensional cases. Nonetheless the proof for the result in dimension 20 will pave the way towards an inductive approach over all dimensions.

Unfortunately, due to the errors in the classification in dimensions 12—cf. chapter 2—we need to **suppose that every 12-dimensional Positive Quaternion Kähler Manifold is symmetric—or equivalently, that**

$$\hat{A}(M)[M] = 0$$

for every π_2-finite 12-dimensional Positive Quaternion Kähler Manifold M as a general assumption for this chapter.

Thus in order to keep this chapter comparatively short we shall content ourselves with results that are not optimal yet. The proofs given therefore may be considered as showcase computations only preceeding an in-depth investigation. However, although arguments get much more technical and lengthy when optimising the results, the interested reader will certainly be able to adapt the reasoning we give. Indeed, our exemplary line of argument will illustrate the techniques to an extent that makes it possible to continue in this vein. For this we established chapters A and B, which provide all the necessary information—beyond our current use.

Let us begin by a depiction of the main idea we use. We shall do so for the special case of a 16-dimensional Positive Quaternion Kähler Manifold M and a an effective isometric action of a 3-torus $T^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, although the arguments may clearly be adapted to further cases.

Main Idea. The rational cohomology of M is concentrated in the fixed point set M^{T^3}—cf. theorem [10].VII.1.6, p. 374. That is, in particular $\chi(M) = \chi(M^{T^3})$. As $\chi(M) > 0$ (cf. 1.13) there is a T^3-fixed-point $x \in M$. By means of the isotropy representation at x and by the canonical projection $\mathbb{Z}_2 \hookrightarrow \mathbf{Sp}(1) \to \mathbf{SO}(3)$ we obtain the homomorphism

$$\varphi : T^3 \hookrightarrow \mathbf{Sp}(4)\mathbf{Sp}(1) \xrightarrow{p} \mathbf{SO}(3)$$

Thus by consideration of rank we obtain that there is a $T^2 \subseteq \ker \varphi$. Consider the $\mathbb{Z}_2$-subtorus $T^2_{\mathbb{Z}_2} \subseteq T^2$. Hence every involution $\mathbb{Z}_2 \subseteq T^2_{\mathbb{Z}_2}$ fixes a quaternionic component at x. A combinatorial argument yields that there is always an involution in $T^2_{\mathbb{Z}_2}$ that fixes an at least eight-dimensional (quaternionic) component N_x (cf. page 20).

Inductively, we construct a sequence of quaternionic fixed-point components N_i: Choose a T^3-fixed point x_1 and construct the $\mathbb{Z}_2$-fixed point component $N_1 := N_{x_1}$ as above. Now choose a point $x_2 \in M^{T^3} \setminus N_1$ with corresponding component $N_2 := N_{x_2}$. We proceed by iteratively choosing a point $x_i \in M^{T^3} \setminus \{N_1 \cup \cdots \cup N_{i-1}\}$ and by forming the component $N_i := N_{x_i}$.

By a Fraenkel-type argument (cf. theorem [19].0.1, p. 151) we obtain that any two quaternionic submanifolds of M intersect provided that their dimension sum is at least $\dim M$. Thus the involutions that fix N_i and N_j for $i < j$ are distinct elements of T^3: Since $\dim N_i \geq 8$ and $\dim N_j \geq 8$, we clearly obtain that $N_i \cap N_j \neq \emptyset$. So if both components are fixed by the same involution, they are necessarily identical and $x_j \notin M^T \setminus \{N_1 \cup \cdots \cup N_i \cup \cdots \cup N_{j-1}\}$ contradicting the construction.

The number of different involutions in $T^3_{\mathbb{Z}_2}$ is given by $2^3 - 1 = 7$. Hence the sequence of components N_i we constructed above necessarily is finite and consists of the elements

$$N_1,\ N_2,\ N_3,\ N_4,\ N_5,\ N_6,\ N_7$$

at most.

These components are eight-dimensional or twelve-dimensional Positive Quaternion Kähler Manifolds (cf. page 20). By our general assumption for this chapter all these manifolds are symmetric spaces. Consequently, by corollary A.2 we see that their Euler characteristic is restricted by $\chi(N_i) \leq 6$ from above unless $N_i \cong \mathbf{Gr}_2(\mathbb{C}^5)$ for some $1 \leq i \leq 7$. If this is the case, however, the connectivity lemma 1.20 implies that the inclusion $N_i \hookrightarrow M$ is 3-connected, whence $\pi_2(M) = \pi_2(N_i) = \mathbb{Z}$. Thus by corollary 1.12 we deduce that $M \cong \mathbf{Gr}_2(\mathbb{C}^6)$. So, in total, we may suppose that

$$\chi(M) \leq \sum_{i=1}^{7} \chi(N_i) = 7 \cdot 6 = 42$$

One now proceeds by taking into account the various intersections between the N_i-components. This will reduce the Euler characteristic bound even further. ⊡

We remark that a 16-dimensional Positive Quaternion Kähler Manifold admits an effective isometric action of a 2-torus T^2 by theorem 1.18 and by the classification of simple Lie groups (cf. table 1.3).

We now want to illustrate how to reduce the upper bound of the Euler characteristic by taking into account the several intersections of the fixed-point components N_i. We shall do so in the case of an effective isometric 4-torus action in order to keep the reasoning rather short. Similar results may equally be obtained by means of a T^3-action only.

Lemma C.1. *A component of the intersection of two* 12*-dimensional (quaternionic)* $\mathbb{Z}_2$*-fixed-point components* $N_1 \neq N_2$ *(belonging to different involutions contained in* T^3*) of* M^{16} *is a quaternionic isometric* $\mathbb{Z}_2$*-fixed-point component (of* N_1*) of dimension* 8*.*

Proof. Let the components N_1, N_2 belong to the involutions $a, b \in T^3$. As a and b commute, we see that b restricts to an action on the fixed-point set M^a of a. Moreover, the map b preserves path-components: Choose an arbitrary element $g \in T^3$. Then there is a path $s : [0,1] \to T^3$ with $s(0) = \mathrm{id}$, $s(1) = g$. Thus we obtain the path

$$[0,1] \to M^a : \quad t \mapsto (s(t))(x)$$

from x to $g(x)$ for $x \in M^a$. This implies that the action of g—and actually the one of the whole T^3—restricts to an action on N_i. So does the action of b in particular.

Hence a component $\tilde{N}$ of the intersection $N_1 \cap N_2$ is a component of the action of b on N_1. As N_2 is quaternionic, the morphism

$$\mathbb{Z}_2 b \hookrightarrow \mathbf{Sp}(4)\mathbf{Sp}(1) \to \mathbf{SO}(3)$$

(induced by the isotropy representation in a point $x \in \tilde{N} \subseteq M$ and the canonical projection) is trivial. As the inclusion of $\mathbf{Sp}(3)\mathbf{Sp}(1) \hookrightarrow \mathbf{Sp}(4)\mathbf{Sp}(1)$ is the canonical blockwise one, we see that also the map $\mathbb{Z}_2 b \hookrightarrow \mathbf{Sp}(3)\mathbf{Sp}(1) \to \mathbf{SO}(3)$ is trivial in $x \in \tilde{N} \subseteq N_1$. Thus the fixed-point component $\tilde{N}$ of the action of b on N_1 is quaternionic. That is, a component of $N_1 \cap N_2$ is quaternionic.

The isotropy representation (in M) of a respectively b in a point $x \in \tilde{N}$ (without restriction) is given by $(1,1,1,-1)$ and by $(1,1,-1,1)$ respectively; i.e. infinitesimally, the components N_1 and N_2 share two "quaternionic directions". Thus we compute the dimension of the intersection component $\tilde{N}$ as $\dim \tilde{N} = 2 \cdot 4 = 8$. □

Generalisations of this lemma to different components and higher dimensions are obvious. So for example we directly see that the intersection of a 12-dimensional component and an 8-dimensional quaternionic component (in M^{16}) consists of 4-dimensional quaternionic $\mathbb{Z}_2$-fixed-point-components—unless the 8-dimensional component is contained in the 12-dimensional one.

Proposition C.2. *Let* M *be a Positive Quaternion Kähler Manifold with an effective isometric* T^3*-action. If there is a* $\mathbb{Z}_2$*-subgroup of* T^3 *fixing a* 12*-dimensional component, then* $M \in \{\mathbb{H}\mathbf{P}^4, \mathbf{Gr}_2(\mathbb{C}^6)\}$*.*

Proof. In order to prove the result we shall show that if there is a 12-dimensional $\mathbb{Z}_2$-component N^{12}, the Euler characteristic of M is smaller than 12 unless $M \in \{\mathbb{H}\mathbf{P}^4, \mathbf{Gr}_2(\mathbb{C}^6)\}$. This will yield a contradiction.

As we described when depicting the main idea, we cover the T^3-fixed-point set by the components $N_1, \ldots, N_7$. By dimension, the component N^{12} is necessarily quaternionic. So suppose $N_1 = N^{12}$. By our general assumption N_1 is symmetric. We suppose that $N_1 \cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)$ and shall lead this to a contradiction.

Step 1. We shall see that under the assumption $N_1 \cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)$ the Euler characteristic of M is bounded from above by $\chi(M) < 12$.

Case 1. Suppose there is another 12-dimensional component $N_2 \neq N_1$. By lemma C.1 the intersection $N_1 \cap N_2$ is an 8-dimensional $\mathbb{Z}_2$-component of $N_1 \cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)$. We now apply the classification of $\mathbb{Z}_2$-fixed-point components as elaborated in chapter B. By table B.3 we see that an intersection component $\tilde{N} \subseteq N_1 \cap N_2$ is isometric to $\tilde{N} \cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^6)$. Since

$$\chi(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6)) = 6 = \chi(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7))$$

we realise that the cohomology of N_2 is already concentrated in $\tilde{N}$; i.e. that

$$M^{T^3} \cap N_2 \subseteq \tilde{N} \subseteq N_1$$

This implies that the components $N_1, N_3, \ldots, N_7$ already cover M^{T^3}.

Case 2. Thus we may assume that there are at most seven components N_i and at most one of them, namely N_1, is of dimension 12—the other ones satisfying $\dim N_i = 8$ for $i \neq 1$.

Every 8-dimensional component N_i (for $i > 1$) intersects with $N_1 = N^{12}$ in a 4-dimensional $\mathbb{Z}_2$-components unless it is contained in N^{12}. We clearly may neglect the case of the inclusion $N_i \subseteq N_1$.

As such a 4-dimensional component is a $\mathbb{Z}_2$-component of $N_1 \cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)$, table B.3 yields that it is isometric to $\mathbb{H}\mathbf{P}^1$. Consequently, as the 4-dimensional component equally is a $\mathbb{Z}_2$-component of the 8-dimensional component N_i, the table yields that $N_i \cong \mathbb{H}\mathbf{P}^2$. (For this we make use of the additional information—cf. theorem [73].8.2, p. 537—that $\mathbf{Gr}_2(\mathbb{C}^3)$ is the only 4-dimensional quaternionic submanifold that $\mathbf{G}_2/\mathbf{SO}(4)$ admits.)

Thus we compute

$$\chi(M) \leq \sum_{i=1}^{7} \chi(N_i) = \chi(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)) + 6 \cdot (\chi(\mathbb{H}\mathbf{P}^2) - \chi(\mathbb{H}\mathbf{P}^1)) = 12 \tag{C.1}$$

We shall now show that the components $N_i \cong \mathbb{H}\mathbf{P}^2$ (for $i > 1$) do intersect outside of N_1. This will decrease the upper bound of the Euler characteristic below the crucial value of 12. We may assume that there are exactly seven components N_i as otherwise the upper bound for the Euler characteristic equally is smaller than 12. Due to Fraenkel the components N_i (for $1 < i \leq 7$) intersect pairwise.

Suppose all the points of intersection lie in N_1. Hence such a point x is an intersection point of at least two 8-dimensional components and the 12-dimensional component N_1. Thus the isotropy representation of $T^3_{\mathbb{Z}_2}$ in such a point without restriction is given by

$$\text{(C.2)} \qquad \{(1,1,1,-1),(-1,-1,1,1),(-1,1,-1,1),(-1,-1,1,-1),(1,-1,-1,1),\\ (-1,1,-1,-1),(1,-1,-1,-1)\}$$

as a direct check shows. Indeed, we focused on the only case possible under the absence of further 12-dimensional components. Actually, in case 1 we dealt with the situation of more than one 12-dimensional component and we reduced it to the case of one 12-dimensional component and only six 8-dimensional ones; this yields a bound which is smaller than 12. (As for the notation: We consider a vector—i.e. for example $(1,1,1,-1)$—as a diagonal matrix—i.e. in the example as $\operatorname{diag}(1,1,1,-1)$—in $\mathbf{Sp}(4)$. The representation of each involution is trivial on the $\mathbf{Sp}(1)$-factor.)

Hence all these points are isolated fixed-points of $T^3_{\mathbb{Z}_2}$ on M in which exactly three 8-dimensional components intersect. Without restriction, in the point x we assume these three components to be N_2, N_3 and N_4. Let us now consider the intersection of N_5 with N_2. By assumption there is a point $x' \in N_1 \cap N_2 \cap N_5$ with the representation of $T^3_{\mathbb{Z}_2}$ in x' being given by (C.2). Thus in x as well as in x' every involution in $T^3_{\mathbb{Z}_2}$—other than the involution belonging to N_1—restricted to an action on N_1 fixes a four-dimensional component on $N_1 \cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)$. By table B.3 such a component necessarily is isometric to $\mathbb{H}\mathbf{P}^1$. In particular, we obtain the following: The involution belonging to N_5 fixes a four-dimensional component (in M) in x and an eight-dimensional component (in M) in x'. Restricted to an action on N_1 this involution fixes a four-dimensional component in x (in N_1) and another now necessarily distinct four-dimensional component in x' (in N_1). Thus the involution fixes two four-dimensional quaternionic components on N_1. By table B.3 this is not possible.

So there are components $N_i \neq N_j$ (with $i, j \geq 2$) that do intersect outside of N_1 in a component $\tilde{N} \subseteq N_i \cap N_j$. The Euler characteristic of this component was counted twice in C.1. So we obtain a refinement of (C.1):

$$\chi(M) \leq \chi(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)) + 6 \cdot (\chi(\mathbb{H}\mathbf{P}^2) - \chi(\mathbb{H}\mathbf{P}^1)) - \chi(\tilde{N}) \leq 12 - 1 = 11$$

Step 2. So in any case we reduced the Euler characteristic $\chi(M)$ to a value below 12. However, observe that by formula (1.2) combined with Hard-Lefschetz (cf. 1.13) and by excluding the special case $(b_4, b_6, b_8) = (2, 1, 2)$ via theorem [38].1.1, p. 2, the smallest Euler characteristic that may be realised by a rationally 3-connected 16-dimensional Positive Quaternion Kähler Manifold other than $\mathbb{H}\mathbf{P}^4$ comes from the triple $(b_4, b_6, b_8) = (3, 0, 4)$. It is given by $\chi(M) = 12$.

Combining this with the information obtained in step 1 we conclude that $N_1 \not\cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^7)$. By our general assumption on symmetry in dimension 12 this implies that $N_1 \in \{\mathbb{H}\mathbf{P}^3, \mathbf{Gr}_2(\mathbb{C}^5)\}$. By the connectivity lemma 1.20 this implies that $M \in \{\mathbb{H}\mathbf{P}^4, \mathbf{Gr}_2(\mathbb{C}^6)\}$. □

The next proposition is an important step for restricting the Euler characteristic in the case of a T^3-action. If we are even given a T^4-action—we shall content ourselves with this stronger assumption—it will already yield a suitable bound.

Proposition C.3. *Suppose M^{16} admits an effective isometric action by a 3-torus T^3. Assume further that there is a T^3-fixed-point where the map*

$$\varphi : T_3 \hookrightarrow \mathbf{Sp}(4)\mathbf{Sp}(1) \to \mathbf{SO}(3)$$

induced by the isotropy representation is trivial. Then we obtain

$$\chi(M) \leq 15$$

Proof. Following our main idea we consider the isotropy representations of the three-torus T^3 in its fixed-points. In each such point we obtain a T^2-subtorus, the involutions of which fix quaternionic components. We have $|T^3_{\mathbb{Z}_2}| = 8$. So we cover the T^3-fixed-point set by the components $N_1, \ldots, N_7$ (with $\dim N_i \geq 8$) as depicted. By proposition C.2 we may assume that none of the N_i has dimension 12; i.e. we have that $\dim N_i = 8$ for $1 \leq i \leq 7$.

By the assumption in the assertion there is a T^3-fixed-point x in which every involution in $T^3_{\mathbb{Z}_2}$ has a representation in $\mathbf{Sp}(4)$. A combinatorial argument then yields that in x this representation—in the absence of 12-dimensional components—is given by

$$\text{(C.3)} \qquad \{(1,1,-1,-1), (-1,-1,1,1), (-1,1,1,-1), (1,-1,1,-1), (-1,1,-1,1), (1,-1,-1,1), (-1,-1,-1,-1)\}$$

So there are six 8-dimensional components $N_1, \ldots, N_6$ intersecting in x—we let N_1 correspond to $(1,1,-1,-1)$ and so forth.

Let N_1 respectively N_2 correspond to the involution represented by $(1,1,-1,-1)$ respectively by $(-1,-1,1,1)$. The Euler characteristic concentrated in $N_1 \cup N_2$ is restricted by $6+6-1 = 11$, since $\chi(N_i) \leq 6$ by corollary A.2. Every further component

$N_3, \ldots, N_6$ intersects both with N_1 and N_2 in a 4-dimensional component by (C.3). These 4-dimensional pairwise distinct intersection components are represented in x by

$$\{(1,-1,-1,-1),(-1,1,-1,-1),(-1,-1,1,-1),(-1,-1,-1,1)\}$$

and we shall denote them by $\tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \tilde{N}_4$ (in this order).

Case 1. Suppose that there exist i,j for $1 \leq i < j \leq 4$ with $\{x\} \subsetneq \tilde{N}_i \cap \tilde{N}_j$. On the analogy of lemma C.1 we see that the intersection $\tilde{N}_i \cap \tilde{N}_j$ consists of isolated $\mathbb{Z}_2$-fixed-points in $\tilde{N}_i$. That is, the involution corresponding to $\tilde{N}_j$ has at least two isolated fixed-points when restricted to an action on $\tilde{N}_i$. By table B.3 the only 4-dimensional Wolf space that admits an involution (contained in a torus) with two isolated fixed-points is $\mathbb{H}\mathbf{P}^1$. Thus we obtain that $\tilde{N}_i \cong \mathbb{H}\mathbf{P}^1$. By table B.3 we know that the only 8-dimensional Wolf space that admits a $\mathbb{Z}_2$-fixed-point component isometric to $\mathbb{H}\mathbf{P}^1$ is $\mathbb{H}\mathbf{P}^2$. (Again we make use of the fact that $\mathbf{G}_2/\mathbf{SO}(4)$ does not admit $\mathbb{H}\mathbf{P}^1$ as a quaternionic submanifold by theorem [73].8.2, p. 537.)

Without restriction we suppose $i = 1$. Then $\tilde{N}_1 \cong \mathbb{H}\mathbf{P}^1$. Consequently, the component N_1 is isometric to $\mathbb{H}\mathbf{P}^2$. A 4-dimensional quaternionic $\mathbb{Z}_2$-component of $\mathbb{H}\mathbf{P}^2$ is isometric to $\mathbb{H}\mathbf{P}^1$ by table B.3. Thus also $\tilde{N}_2 \cong \mathbb{H}\mathbf{P}^1$. This again implies $N_4 \cong \mathbb{H}\mathbf{P}^2$ and $N_6 \cong \mathbb{H}\mathbf{P}^2$. Continuing in this fashion we obtain that $N_1, \ldots, N_6 \cong \mathbb{H}\mathbf{P}^2$.

Thus we see that the Euler characteristic concentrated in $N_1 \cup N_2$ is restricted from above by $3 + (3-1) = 5$. The components $N_i \cong \mathbb{H}\mathbf{P}^3$ for $3 \leq i \leq 6$ intersect with either N_1 or N_2 in an $\tilde{N}_j \cong \mathbb{H}\mathbf{P}^1$. Thus the Euler characteristic concentrated in $N_1 \cup \cdots \cup N_6$ is restricted from above by $5 + 4 \cdot (3-2) = 9$. Hence the Euler characteristic concentrated in $N_1 \cup \ldots \cup N_7$ is restricted by $9 + 6 = 15$.

Case 2. Suppose that for all i,j with $1 \leq i < j \leq 4$ we have $\{x\} = \tilde{N}_i \cap \tilde{N}_j$. As we have seen the Euler characteristic concentrated in $N_1 \cup N_2$ is restricted by 11. Every further component N_i for $3 \leq i \leq 6$ intersects both with N_1 and N_2 in one of the $\tilde{N}_j$, $\tilde{N}_k$ with $1 \leq j,k \leq 4$ respectively. Since $\tilde{N}_j \cap \tilde{N}_k = \{x\}$, we see that the Euler characteristic concentrated in $N_1 \cup \cdots \cup N_6$ is restricted by

$$\begin{aligned}
&(\chi(N_1) + \chi(N_2) - \chi(\{x\})) + \sum_{i=3}^{6} \chi(N_i) \\
&- (\chi(\tilde{N}_2) + \chi(\tilde{N}_3) - \chi(\{x\})) \\
&- (\chi(\tilde{N}_1) + \chi(\tilde{N}_3) - \chi(\{x\})) \\
&- (\chi(\tilde{N}_2) + \chi(\tilde{N}_4) - \chi(\{x\})) \\
&- (\chi(\tilde{N}_1) + \chi(\tilde{N}_3) - \chi(\{x\})) \\
\leq & 11 + 4 \cdot (6 - (3 + 3 - 1)) \\
= & 15
\end{aligned} \tag{C.4}$$

Indeed, if there is an N_i isometric to $\mathbf{G}_2/\mathbf{SO}(4)$, then it intersects with another eight-dimensional component in an $\tilde{N}_j \cong \mathbf{Gr}_2(\mathbb{C}^3)$. Consequently, its Euler characteristic is already concentrated in the intersection $\tilde{N}_j$. Thus a similar refined inequality holds.

If there is a component $N_i \cong \mathbb{H}\mathbf{P}^2$, then we see that there are two $\tilde{N}_j$ and $\tilde{N}_k$ that are $\mathbb{Z}_2$-fixed point components of N_i (for two different involutions). According to table B.3 they are homothetic to $\tilde{N}_j, \tilde{N}_k \cong \mathbb{H}\mathbf{P}^1$ each. Moreover, table B.3 gives that every 8-dimensional component N_l they are contained in is homothetic to $\mathbb{H}\mathbf{P}^2$. Iteratively continuing in this fashion we see that $N_i \cong \mathbb{H}\mathbf{P}^2$ for all $1 \leq i \leq 6$. Consequently, the Euler characteristic concentrated in $N_1 \cup N_2$ is $3 + (3-1) = 5$. Every further component N_i for $3 \leq i \leq 6$ intersects both with N_1 and with N_2 in one of the $\tilde{N}_j$, $\tilde{N}_k$ with $1 \leq j, k \leq 4$ respectively. Thus the Euler characteristic concentrated in $N_1 \cup \ldots \cup N_6$ is restricted by

$$
\begin{aligned}
&(\chi(N_1) + \chi(N_2) - \chi(\{x\})) + \sum_{i=3}^{6} \chi(N_i) \\
&\quad - (\chi(\tilde{N}_2) + \chi(\tilde{N}_3) - \chi(\{x\})) \\
&\quad - (\chi(\tilde{N}_1) + \chi(\tilde{N}_3) - \chi(\{x\})) \\
&\quad - (\chi(\tilde{N}_2) + \chi(\tilde{N}_4) - \chi(\{x\})) \\
&\quad - (\chi(\tilde{N}_1) + \chi(\tilde{N}_3) - \chi(\{x\})) \\
\leq & 5 + 4 \cdot (3 - (2 + 2 - 1)) \\
= & 5
\end{aligned}
$$

Thus the Euler characteristic concentrated in $N_1 \cup \cdots \cup N_7$, i.e. the number $\chi(M)$ itself, is restricted by $\chi(M) \leq 5 + 6 = 11$ and we are done.

Hence we may assume in the following that $N_i \cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^6)$ for $1 \leq i \leq 6$.

If there is no seventh component N_7, we are done. Otherwise, we now also take intersections with the seventh component N_7 into account.

Clearly, by Fraenkel N_7 intersects with each of $N_1, \ldots, N_6$.

Case 2.1. Suppose there is a point x' with $x' \in N_7 \cap N_i \cap N_j$ with $i < j < 7$. The existence of these three 8-dimensional components requires the isotropy representation in x' to be one out of two possibilities:

Case 2.1.1. Assume the isotropy representation of T^3 to be characterized by $\ker \varphi = T^2$ with the $\mathbb{Z}_2$-subtorus of this T^2 being represented by

$$\{(1, 1, -1, -1), (-1, 1, 1, -1), (-1, 1, -1, 1)\}$$

These representations correspond to the components N_7, N_i and N_j. Thus the common intersection component of N_7, N_i and N_j is 4-dimensional. This implies that the involution corresponding to N_7 fixes an isolated fixed-point on N_i in x

whereas it fixes a four-dimensional quaternionic submanifold of N_i in x'. According to table B.3 there is no classical Wolf space of dimension eight that permits such an involution other than $N_i \cong \mathbb{H}\mathbf{P}^2$—the subcase we ruled out at the beginning of case 2. Equally, the space $\mathbf{G}_2/\mathbf{SO}(4)$ cannot permit such an involution as the 4-dimensional fixed-point component is necessarily $\mathbf{Gr}_2(\mathbb{C}^3)$—cf. [73].8.2, p. 537—whence the Euler characteristic concentrated in the $\mathbb{Z}_2$-fixed-point set would be equal to or bigger than $\chi(\mathbf{Gr}_2(\mathbb{C}^3)) + \chi(\{x\}) = 3 + 1 = 4$.

Thus this case actually cannot occur.

Case 2.1.2. Assume $\ker \varphi = T^3$ and suppose the isotropy representation of $T^3_{\mathbb{Z}_2}$ in x' to be given by

$$\{(1,1,-1,-1),(-1,-1,1,1),(-1,1,1,-1),(1,-1,1,-1),(-1,1,-1,1),\\ (1,-1,-1,1),(-1,-1,-1,-1)\}$$

Let N_k denote the component corresponding to the involution represented by $(-1,1,1,-1)$ in x'. The involution belonging to N_7 is represented by $(-1,-1,-1,-1)$ in x and by $(1,1,-1,-1)$ in x' without restriction. Thus it fixes a 4-dimensional component around x' in N_k and it has x as an isolated fixed-point in N_k. As in case 2.1.1 this leads to $N_k \cong \mathbb{H}\mathbf{P}^2$, which is impossible by assumption. Hence this case cannot occur either.

Case 2.2. So we may assume that there is no triple intersection of the components N_i for $1 \le i \le 7$, i.e. whenever $x' \in N_i \cap N_j$ with $1 \le i < j \le 7$ there is no $k \notin \{i,j\}$ with $1 \le k \le 7$ and $x' \in N_k$.

By Fraenkel the component N_7 intersects with each of the components $N_1, \ldots, N_6$. By assumption these intersections now have to be pairwise disjoint. The Euler characteristic concentrated in each intersection component is at least one. We have at least six different intersection components $N_1 \cap N_7, \ldots, N_6 \cap N_7$. The Euler characteristic of N_7 is at most $\chi(N_7) \le 6$. These facts imply that the Euler characteristic of N_7 already is concentrated in $N_1 \cup \cdots \cup N_6$, i.e. by equation (C.4) we obtain

$$\chi(M) \le 15 + \chi(N_7) - \sum_{i=1}^{6} \chi(N_i \cap N_7) = 15$$

This proves the assertion. □

Together with our general assumption of symmetry in dimension 12 this proposition directly leads to

Theorem C.4. *A* 16*-dimensional Positive Quaternion Kähler Manifold* M *admitting an isometric action of a* 4*-torus satisfies*

$$\chi(M) \in \{12, 15\}$$

and

$$(b_4, b_6, b_8) \in \{(3,0,4), (3,2,3)\}$$

unless it is necessarily symmetric, i.e. $M \in \{\mathbb{H}\mathbf{P}^4, \mathbf{Gr}_2(\mathbb{C}^6)\}$.

Proof. The map

$$\varphi : T^4 \hookrightarrow \mathbf{Sp}(4)\mathbf{Sp}(1) \xrightarrow{p} \mathbf{SO}(3)$$

induced by the isotropy representation at a T^4-fixed-point x in M satisfies $T^3 \subseteq \ker \varphi$ for some subgroup $T^3 \subseteq T^4$. Hence for the action of this T^3 the prerequisites of proposition C.3 are satisfied. Consequently, the Euler characteristic of M is restricted from above by

$$\chi(M) \leq 15$$

By corollary 1.12 we may suppose M to be rationally 3-connected. Hence by theorem 1.13 and in particular by equation (1.2) we obtain the following possibilities for the Betti numbers (b_4, b_6, b_8):

$$(b_4, b_6, b_8) \in \{(1,0,1), (2,1,2), (3,0,4), (3,2,3)\}$$

If $b_4 = 1$ theorem 1.15 yields that $M \cong \mathbb{H}\mathbf{P}^4$. By [38].1.1, p. 2, the configuration $(2,1,2)$ cannot occur. If $(b_4, b_6, b_8) = (3,0,4)$, then $\chi(M) = 12$. In the case $(b_4, b_6, b_8) = (3,2,3)$ the Euler characteristic is given by $\chi(M) = 15$. $\square$

In the case of a T^3-action only a similar theorem can be proved by a deeper analysis of intersections of $\mathbb{Z}_2$-fixed-point components.

We remark that $\chi(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^8)) = 12$ and that $\chi(\mathbf{Gr}_2(\mathbb{C}^6)) = 15$—cf. corollary A.2. This shows that the upper bound we found is very good.

The next theorem paves the way for an inductive generalisation over all dimensions. In order to establish a bound for the Euler characteristic of the Positive Quaternion Kähler Manifold M^{4n} we *shall not* assume all Positive Quaternionin Kähler Manifolds in dimensions smaller than $4n$ to be symmetric. The proof provides an argument that nonetheless makes it possible to find an upper bound.

Theorem C.5. *A Positive Quaternion Kähler Manifold M of dimension* $\dim M = 20$ *that admits an effective isometric action by a 4-torus T^4 satisfies*

$$\chi(M) \leq 225$$

Proof. We adapt the main idea presented above to dimension 20: In a T^4-fixed-point we consider the isotropy representation of $T^3_{\mathbb{Z}_2} \subseteq T^3 \subseteq \ker\varphi$ (with the notation from above). We realise that we can always find an involution that fixes an at least 12-dimensional quaternionic component. Since $|T^4_{\mathbb{Z}_2}| = 2^4$, we thus know that the fixed-point set M^{T^4} can be covered by at most $2^4 - 1 = 15$ quaternionic $\mathbb{Z}_2$-fixed-point components

$$N_1, \ldots, N_{15}$$

satisfying $\dim N_i \geq 12$. Thus the Euler characteristic of M is restricted by

$$\chi(M) \leq \sum_{i=1}^{15} \chi(N_i)$$

If $\dim N_i = 12$, we cite $\chi(N_i) \leq 10$ from corollary A.2 due to our general assumption of symmetry in dimension 12.

Suppose now there is an N_i of dimension $\dim N_i = 16$. Assume that N_i is fixed by the involution $a \in T^3_{\mathbb{Z}_2}$. Since a commutes with T^4, the action of T^4 restricts to an action on the fixed-point set M^a. Moreover, in the proof of lemma C.1 we have seen that the action of T^4 restricts to an action on N_i.

If this action, $T^4 \curvearrowright N_i$, is not almost effective, i.e. if there is an $\mathbb{S}^1$-subgroup of T^4 fixing N_i, then N_i is a codimension four $\mathbb{S}^1$-fixed-point component of M. Thus by theorem [20].1.2, p. 2, we obtain that $M \cong \mathbb{H}\mathbf{P}^5$ or $M \cong \mathbf{Gr}_2(\mathbb{C}^7)$. Otherwise, if T^4 acts almost effectively, we may apply the whole reasoning to dimension 16 with the action of a 4-torus. This was done in theorem C.4, which yields the upper bound $\chi(N_i) \leq 15$. Hence, in total, we obtain:

$$\chi(M) \leq 15 \cdot 15 = 225$$

□

Taking into account the various intersections of the components N_i again permits to reduce the upper bound for $\chi(M)$ tremendously.

For an iterative approach for $\chi(M^{4n})$ we finally notice the following: In each torus-fixed-point we need to find a $\mathbb{Z}_2$-fixed-point component N_i that has dimension at least $2n$. For this it is sufficient for the torus to have a rank lying in $\mathcal{O}(\log_2 n)$—this can easily be made more precise—as a combinatorial argument shows.

D

The Elliptic Genus of Wolf Spaces

As we remarked in chapter 2, it remains a challenging task to prove the vanishing of the $\hat{A}$-genus on Positive Quaternion Kähler Manifolds M that are not necessarily spin but π_2-finite. This question can be seen in the larger context of a description of the elliptic genus $\Phi(M)$. Indeed, $\Phi(M)$ admits an expansion as a power series which contains $\hat{A}(M)[M]$ as a first coefficient (cf. [39]).

We shall provide this description on Wolf spaces M. We ignore the complex Grassmannian, as $b_2(\mathbf{Gr}_2(\mathbb{C}^{n+2})) = 1$. Stepping through the list of Wolf spaces (cf. table 1.2) then basically yields four different cases:

- The Wolf space is $M \cong \mathbb{H}\mathbf{P}^n$.
- The Wolf space is exceptional.
- We have $M \cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$ for even $n \geq 2$.
- We have $M \cong \widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$ for odd $n \geq 1$.

Clearly, the quaternionic projective space is spin. (Moreover, in odd quaternionic dimensions it is the only Positive Quaternion Kähler Manifold that is spin.) The exceptional Wolf spaces—except for $\mathbf{F}_4/\mathbf{Sp}(3)\mathbf{Sp}(1)$—as well as $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$ for even $n \geq 2$ have even quaternionic dimension. Thus they are spin manifolds M.

So a reasoning due to [39] identifies the equivariant elliptic genus $\Phi(M, g)$ of an involution g with the one of the transversal self-intersection, $\Phi(M^g \pitchfork M^g, g)$, of the fixed-point set of g. The rigidity of the elliptic genus on spin manifolds yields

$$\Phi(M) = \Phi(M, g) = \Phi(M^g \pitchfork M^g, g)$$

This permits an approach inductive on dimension. Due to the theorem on [39], p. 321 this leads to

$$\Phi(M) = \operatorname{sign}(M) = b_{2n}(M)$$

(If M is a Positive Quaternion Kähler Manifold of even quaternionic dimension, its intersection form is positive definite–cf. 1.16. Otherwise, if M is spin and of odd quaternionic dimension, we have $M \cong \mathbb{H}\mathbf{P}^n$ and $\operatorname{sign}(M) = -b_{2n} = 0$.) This gives a complete picture in the spin case.

Thus only the cases of the real Grassmannian $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$ with odd n and of the exceptional space $\mathbf{F}_4/\mathbf{Sp}(3)\mathbf{Sp}(1)$ remain. We shall prove the vanishing of $\Phi(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$ in this case. Compare also the result on [36], p. 202–203, which was obtained by computations on the twistor space. However, we shall give a more direct reasoning. For the exceptional Wolf space we quote the computations on [36], p. 204–205, although our method applies to this case as well.

For the convenience of the reader we shall illustrate a method of direct computation—following [39] once more—in the eight-dimensional case. This will serve as a showcase computation for the more general case of arbitrary odd quaternionic dimension, which we shall deal with later. Alongside this model computation we shall establish necessary facts and notation. As a general reference on Index Theory we recommend the textbooks [40] and [52].

We have

$$\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4}) = \frac{\mathbf{SO}(n+4)}{\mathbf{SO}(n) \times \mathbf{SO}(4)}$$

whence $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6) = \frac{\mathbf{SO}(6)}{\mathbf{SO}(2)\times\mathbf{SO}(4)}$. The positive roots of $\mathbf{SO}(6)$ (corresponding to the standard maximal torus) are given by

$$x_1 \mp x_3, x_1 \mp x_2, x_2 \mp x_3$$

where x_i is dual to the i-th factor of the maximal torus. The positive roots of the denominator $\mathbf{SO}(2) \times \mathbf{SO}(4)$ are given by $x_2 \mp x_3$, whence the complementary positive roots are the

$$x_1 \mp x_3, x_1 \mp x_2$$

(Clearly, we have that $2|\{x_1 \mp x_3, x_1 \mp x_2\}| = \dim \widetilde{\mathbf{Gr}}_4(\mathbb{R}^6)$.)

The Weyl group of $\mathbf{SO}(2n+1)$ is given by the group G of permutations of $\{-n, \ldots, -1, 1, \ldots, n\}$ equivariant under multiplication with (-1)—cf. [11], p. 171. The Weyl group of $\mathbf{SO}(2n)$ is the subgroup $\mathbf{S}G$ of even permutations. We obtain the coset

$$\begin{aligned}
\mathrm{W}\left(\frac{\mathbf{SO}(6)}{\mathbf{SO}(2)\times\mathbf{SO}(4)}\right) :=& \frac{\mathrm{W}(\mathbf{SO}(6)}{\mathrm{W}(\mathbf{SO}(2)\times\mathbf{SO}(4))}\\
=& \frac{\mathbf{S}G(3)}{\mathbf{S}G(1)\times\mathbf{S}G(2)}\\
=& \frac{\mathbf{S}G(3)}{\mathbf{S}G(2)}\\
\cong& \left\{ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\right.\\
& \left.\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&-1&0\\0&0&1\\-1&0&0\end{pmatrix} \right\}
\end{aligned}$$

The maximal torus T^3 of $\mathbf{SO}(6)$ acts isometrically on $\frac{\mathbf{SO}(6)}{\mathbf{SO}(2)\times\mathbf{SO}(4)}$ by left-multiplication. We consider the isometric $\mathbb{S}^1$-action induced by the inclusion of $\mathbb{S}^1$ into T^3 by $g \mapsto (g^1, g^2, g^3)$, where g is a topological generator of $\mathbb{S}^1$.

The fixed-points of this action are isolated and in one-to-one correspondence with

$$\mathrm{W}\left(\frac{\mathbf{SO}(6)}{\mathbf{SO}(2)\times\mathbf{SO}(4)}\right)$$

The rotation numbers on the tangent space of such a fixed-point are given by evaluation on roots, i.e. the rotation numbers in a point

$$w \in \mathrm{W}\left(\frac{\mathbf{SO}(6)}{\mathbf{SO}(2)\times\mathbf{SO}(4)}\right)$$

are given by

$$\langle w(\{x_1 \mp x_2, x_1 \mp x_3\}), \phi\rangle = w(\{x_1 \mp x_2, x_1 \mp x_3\})|_{x_1=1,x_2=2,x_3=3}$$

The orientation in a point is compatible with the orientation of the ambient manifold if and only if the determinant of the transformation w corresponding to the fixed-point is positive. (For this consider w as a linear map on the vector space generated by the roots.)

For example, in the identity component we obtain the rotation numbers

$$\{x_1 \mp x_2, x_1 \mp x_3\}|_{x_1=1,x_2=2,x_3=3} = \{-1, 3, -2, 4\}$$

whereas in the fixed-point corresponding to

$$\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$$

they are given by

$$\begin{aligned}&\left(\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}(\{x_1 \mp x_2, x_1 \mp x_3\})\right)\Bigg|_{x_1=1,x_2=2,x_3=3}\\ =&\{-x_1 \pm x_2, -x_1 \mp x_3\}|_{x_1=1,x_2=2,x_3=3}\\ =&\{1,-3,-4,2\}\end{aligned}$$

Both points are oriented in a positive way, i.e. compatible with the orientation of $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6)$. Continuing in this fashion we obtain the complete list depicted in table D.1. A direct consequence of this is the signature of the Grassmannian, which can be

Table D.1.: Rotation numbers on $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6)$

fixed-point	permutation of roots	rotation numbers	orientation
$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$	$x_1 \mp x_2, x_1 \mp x_3$	$-1, 3, -2, 4$	$+$
$\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$	$-x_1 \pm x_2, -x_1 \mp x_3$	$1, -3, -4, 2$	$+$
$\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$	$x_2 \mp x_1, x_2 \mp x_3$	$1, 3, -1, 5$	$-$
$\begin{pmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$	$-x_2 \pm x_1, -x_2 \mp x_3$	$-1, -3, -5, 1$	$-$
$\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}$	$x_3 \mp x_1, x_3 \mp x_2$	$2, 4, 1, 5$	$+$
$\begin{pmatrix}0 & -1 & 0\\ 0 & 0 & 1\\ -1 & 0 & 0\end{pmatrix}$	$-x_3 \pm x_1, -x_3 \mp x_2$	$-2, -4, -5, -1$	$+$

computed as the difference of the number of positively oriented points and the one of negatively oriented points. So $\operatorname{sign}(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6)) = 4 - 2 = 2$.

The signature may be regarded as the constant term of the power series belonging to the elliptic genus of the manifold. So let us compute further coefficients. Denote by

$T_{\mathbb{C}}$ the complexified tangent bundle and set

$$\bigwedge\nolimits_t T_{\mathbb{C}} = \sum_{k=0}^{4n} (\wedge^k T_{\mathbb{C}}) t^k \qquad\qquad S_t T_{\mathbb{C}} = \sum_{k=0}^{\infty} (S^k T_{\mathbb{C}}) t^k$$

Recall that the elliptic genus is given by

$$\Phi(X) = \operatorname{sign}\Big(X, \prod_{m=1}^{\infty} \bigwedge\nolimits_{q^m} T_{\mathbb{C}} \cdot \prod_{m=1}^{\infty} S_{q^m} T_{\mathbb{C}}\Big) \tag{D.1}$$

where X is a compact oriented $4n$-dimensional manifold. In the same way one obtains an equivariant formula when replacing formal roots by equivariant formal roots. One obtains the following form:

$$\Phi(X, g) = \sum_{m=0}^{\infty} \operatorname{sign}(X, R_m, g) q^m$$

where we focus on g being a topological generator of $\mathbb{S}^1$. The bundles R_m are the coefficient bundles arising from (D.1). So we may compute them by means of the subsequent equation.

$$\begin{aligned}
&\prod_{m=1}^{\infty} \bigwedge\nolimits_{q^m} T_{\mathbb{C}} \cdot \prod_{m=1}^{\infty} S_{q^m} T_{\mathbb{C}} \\
=&\prod_{m=1}^{\infty} \sum_{k=0}^{\infty} (\wedge^k T_{\mathbb{C}})(q^m)^k \cdot \prod_{m=1}^{\infty} \sum_{k=0}^{\infty} (S^k T_{\mathbb{C}})(q^m)^k \\
=&(1 + T_{\mathbb{C}} q + T_{\mathbb{C}} \wedge T_{\mathbb{C}} q^2 + \dots) \cdot (1 + T_{\mathbb{C}} q^2 + T_{\mathbb{C}} \wedge T_{\mathbb{C}} q^4 + \dots) \cdot \dots \\
&\cdot (1 + T_{\mathbb{C}} q + S^2 T_{\mathbb{C}} q^2 + \dots) \cdot (1 + T_{\mathbb{C}} q^2 + S^2 T_{\mathbb{C}} q^4 + \dots) \cdot \dots \\
=&1 + 2T_{\mathbb{C}} q + 2(T_{\mathbb{C}} + T_{\mathbb{C}} \otimes T_{\mathbb{C}} + T_{\mathbb{C}} \wedge T_{\mathbb{C}} + S^2 T_{\mathbb{C}}) q^2 \\
&+ (2T_{\mathbb{C}} + 4T_{\mathbb{C}} \otimes T_{\mathbb{C}} + T_{\mathbb{C}} \otimes T_{\mathbb{C}} \otimes T_{\mathbb{C}} + \wedge^3 T_{\mathbb{C}} + S^3 T_{\mathbb{C}}) q^3 + \dots
\end{aligned}$$

This implies in particular that

$$\begin{aligned}
R_0 &= 1 \\
R_1 &= 2T_{\mathbb{C}} \\
R_2 &= 2(T_{\mathbb{C}} + T_{\mathbb{C}} \otimes T_{\mathbb{C}}) \\
R_3 &= 2T_{\mathbb{C}} + 4T_{\mathbb{C}} \otimes T_{\mathbb{C}} + T_{\mathbb{C}} \otimes T_{\mathbb{C}} \otimes T_{\mathbb{C}} + \wedge^3 T_{\mathbb{C}} + S^3 T_{\mathbb{C}}
\end{aligned}$$

(In degree two we have an exceptional identity $T \wedge T + S^2 T = T \otimes T$.)

In general, one may determine $\operatorname{sign}(X, R_m, g)$ via a computation on fixed-point components ν of g:

$$\operatorname{sign}(X, R_m, g) = \sum_{\nu} \operatorname{sign}(X_\nu^g, R_m, g)$$

up to orientation/sign of each summand (cf. [40], 5.8, p. 72). In terms of power series one may compute such an expression by replacing the formal roots of the power series of the L-genus by equivariant formal roots and by twisting the expression with $\mathrm{ch}_g(R_m)$. (The equivariant Chern character $\mathrm{ch}_g(R_m)$ again results from $\mathrm{ch}(R_m)$ by replacing formal roots x_i by the equivariant roots $x_i + t_i z$ of R_m. Here $t_i \in \mathbb{N}_0$ is the rotation number of g on the root x_i of R_m and z is a formal variable.) In our case all the fixed-points of the $\mathbb{S}^1$-action on $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6)$ are isolated, which simplifies the situation, since the topology of the fixed-point components—the part in the formula with rotation number equal to zero—yields a trivial contribution. So we obtain

$$\begin{aligned}\mathrm{sign}(\nu, R_m, g) &= \sum_\nu \left(\left(\prod_{k>0}\prod_{i=1}^{d_k} \frac{1+e^{-(x_i+t_kz)}}{1-e^{-(x_i+t_kz)}}\right)\cdot \mathrm{ch}_g(R_m)\right)[\nu] \\ &= \sum_\nu \pm\left(\prod_{k>0}\prod_{i=1}^{d_k} \frac{1+e^{-t_kz}}{1-e^{-t_kz}}\right)\cdot \mathrm{ch}_g(R_m) \qquad \text{(D.2)}\end{aligned}$$

The last equation holds, since we evaluate on 0-dimensional isolated fixed-points and therefore may set the x_i to zero. (In this formula x_i is always supposed to correspond to a root on which the rotation number is t_k.) The $\pm$-sign is chosen according to whether the orientation of the fixed-point is compatible with the orientation of the ambient manifold or not.

We need to compute the equivariant Chern characters: Complexification of the tangent bundle T induces an action of g on $T_{\mathbb{C}}$ with rotation numbers $(t_i, -t_i)$ when t_i being a rotation number of the action on T at a fixed-point. The Chern character of a complex bundle of rank m is given by

$$\sum_{r=1}^{m} e^{y_r}$$

in terms of its formal roots y_r. Thus the equivariant Chern character becomes

$$\sum_{r=1}^{m} e^{t_r z + y_r} \qquad \text{(D.3)}$$

With this we compute

$$\mathrm{ch}_g(T_{\mathbb{C}}) = \sum_{r=1}^{4n} e^{t_r z + x_r} + e^{-t_r z + x_{r+n}}$$

The additivity and multiplicativity of the Chern character permits us to compute the equivariant Chern characters of the bundles R_m. In order to compute equivariant

Chern characters of symmetric and exterior powers we use formula (D.3) with the formal roots of $S^kT_{\mathbb{C}}$ being given by the

$$y_r = x_{i_1} + \cdots + x_{i_r} \qquad \text{with } 1 \le i_1 \le \cdots \le i_k \le 2n$$

The formal roots of $\wedge^k T_{\mathbb{C}}$ are the

$$y_r = x_{i_1} + \cdots + x_{i_r} \qquad \text{with } 1 \le i_1 < \cdots < i_k \le 2n$$

From now on let us specialise again to the $\mathbb{S}^1$-action ϕ from above which acts with isolated fixed-points only. Once more, in particular, we therefore may also set the roots x_i in the equivariant Chern characters to zero.

We have obtained a formula for the equivariant elliptic genus by means of equivariant twisted signatures. We have further seen how to compute equivariant Chern characters, which enables us to compute the signatures. Table D.1 now tells us whether the orientation of a fixed-point is compatible with the ambient manifold or not and what the rotation numbers on the tangent bundle look like. All this information now is sufficient to compute the equivariant elliptic genus.

So compute the twisted equivariant signatures: By $\nu_1, \ldots, \nu_6$ we denote the different fixed-points in the order according to their order of appearance in table D.1. Since the rotation numbers in ν_1 and ν_2, the ones in ν_3 and ν_4 and the ones in ν_5 and ν_6 coincide up to sign and since these points have the same orientation, we obtain—irrespective of the twisting bundle R_m—that

$$\begin{aligned}
\operatorname{sign}(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6)), R_m, g) &= \operatorname{sign}(\nu_1, R_m, g) + \operatorname{sign}(\nu_2, R_m, g) \\
&\quad - \operatorname{sign}(\nu_3, R_m, g) - \operatorname{sign}(\nu_4, R_m, g) \\
&\quad + \operatorname{sign}(\nu_5, R_m, g) + \operatorname{sign}(\nu_6, R_m, g) \\
&= 2\operatorname{sign}(\nu_1, R_m, g) - 2\operatorname{sign}(\nu_3, R_m, g) + 2\operatorname{sign}(\nu_5, R_m, g)
\end{aligned}$$

By means of (D.2) we compute

$$\begin{aligned}
\operatorname{sign}(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6), g) &= 2 \cdot \left(\frac{1+e^{1\cdot z}}{1-e^{1\cdot z}}\right)\left(\frac{1+e^{2\cdot z}}{1-e^{2\cdot z}}\right)\left(\frac{1+e^{3\cdot z}}{1-e^{3\cdot z}}\right)\left(\frac{1+e^{4\cdot z}}{1-e^{4\cdot z}}\right) \\
&\quad - 2 \cdot \left(\frac{1+e^{1\cdot z}}{1-e^{1\cdot z}}\right)\left(\frac{1+e^{1\cdot z}}{1-e^{1\cdot z}}\right)\left(\frac{1+e^{3\cdot z}}{1-e^{3\cdot z}}\right)\left(\frac{1+e^{5\cdot z}}{1-e^{5\cdot z}}\right) \\
&\quad + 2 \cdot \left(\frac{1+e^{1\cdot z}}{1-e^{1\cdot z}}\right)\left(\frac{1+e^{2\cdot z}}{1-e^{2\cdot z}}\right)\left(\frac{1+e^{4\cdot z}}{1-e^{4\cdot z}}\right)\left(\frac{1+e^{5\cdot z}}{1-e^{5\cdot z}}\right) \\
&= 2
\end{aligned}$$

$$\begin{aligned}
\mathrm{sign}(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6), R_1, g) &= 2\cdot 2\cdot \left[\left(\frac{1+e^{1\cdot z}}{1-e^{1\cdot z}}\right)\left(\frac{1+e^{2\cdot z}}{1-e^{2\cdot z}}\right)\left(\frac{1+e^{3\cdot z}}{1-e^{3\cdot z}}\right)\left(\frac{1+e^{4\cdot z}}{1-e^{4\cdot z}}\right)\right] \\
&\quad \cdot (e^{1\cdot z}+e^{-1\cdot z}+e^{2\cdot z}+e^{-2\cdot z}+e^{3\cdot z}+e^{-3\cdot z}+e^{4\cdot z}+e^{-4\cdot z}) \\
&\quad - 2\cdot 2\cdot \left[\left(\frac{1+e^{1\cdot z}}{1-e^{1\cdot z}}\right)\left(\frac{1+e^{1\cdot z}}{1-e^{1\cdot z}}\right)\left(\frac{1+e^{3\cdot z}}{1-e^{3\cdot z}}\right)\left(\frac{1+e^{5\cdot z}}{1-e^{5\cdot z}}\right)\right] \\
&\quad \cdot (e^{1\cdot z}+e^{-1\cdot z}+e^{1\cdot z}+e^{-1\cdot z}+e^{3\cdot z}+e^{-3\cdot z}+e^{5\cdot z}+e^{-5\cdot z}) \\
&\quad + 2\cdot 2\cdot \left[\left(\frac{1+e^{1\cdot z}}{1-e^{1\cdot z}}\right)\left(\frac{1+e^{2\cdot z}}{1-e^{2\cdot z}}\right)\left(\frac{1+e^{4\cdot z}}{1-e^{4\cdot z}}\right)\left(\frac{1+e^{5\cdot z}}{1-e^{5\cdot z}}\right)\right] \\
&\quad \cdot (e^{1\cdot z}+e^{-1\cdot z}+e^{2\cdot z}+e^{-2\cdot z}+e^{4\cdot z}+e^{-4\cdot z}+e^{5\cdot z}+e^{-5\cdot z}) \\
&= 0 \\
\mathrm{sign}(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6), R_2, g) &= \ \ldots = 0 \\
\mathrm{sign}(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6), R_3, g) &= \ \ldots = 0
\end{aligned}$$

(We suppress the computations in the last two cases, since they are a little lengthy due to the more complicated twisting bundles. Moreover, these computations are straight forward now and a computer-based check using Mathematica 6.01 yields the result.)

In particular, we see that sufficiently many coefficients of the equivariant elliptic genus vanish, whence it is constant. We obtain

$$\Phi(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6), g) = 2$$

In particular, using continuity when $g \to 1$ this yields

$$\Phi(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^6)) = 2$$

Finally, we remark that the exceptional isometry

$$\frac{\mathbf{SO}(6)}{\mathbf{SO}(2)\times\mathbf{SO}(4)} \cong \frac{\mathbf{SU}(4)}{\mathbf{S}(\mathbf{U}(2)\times\mathbf{U}(2))}$$

induced by exceptional isomorphisms of Lie groups makes it possible to do analogous computations on the second form of the space. Again consider the action given by $\phi : g \mapsto (g^1, g^2, g^3, g^{-6})$. We know that $\mathrm{W}(\mathbf{SU}(4)) = S(4)$ and that $\mathrm{W}(\mathbf{S}(\mathbf{U}(2)\times\mathbf{U}(2)) = S(2)\times S(2)$. Thus the fixed-points are in bijection with

$$\left\{ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix}, \right.$$
$$\left. \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix} \right\}$$

The rotation numbers on these respective isolated fixed-points are

$$(-2,7,-1,8),(2,9,1,8),(-9,-7,-1,1),(2,1,-7,-8),(-1,7,1,9),(-2,-1,-9,-8)$$

and we proceed as above.

Let us now compute the equivariant elliptic genus of a regular $\mathbb{S}^1$-action ϕ on $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{2n+5})$ with $n \geq 0$. Let g be a topological generator of $\mathbb{S}^1$. We let $\mathbb{S}^1$ include into the standard maximal torus of $\mathbf{SO}(2n+5)$ by $\phi : g \mapsto (g^{k_1}, g^{k_2}, \dots, g^{k_n})$ with pairwise distinct k_i. This induces an isometric $\mathbb{S}^1$-action on

$$\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{2n+5}) = \frac{\mathbf{SO}(2n+5)}{\mathbf{SO}(2n+1) \times \mathbf{SO}(4)}$$

with isolated fixed-points only.

Theorem D.1. *Every g-equivariant twisted signature on $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{2n+5})$ with $n \geq 0$ vanishes, whence the equivariant elliptic genus*

$$\Phi(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{2n+5}), g) = 0$$

vanishes. In particular, so does the elliptic genus:

$$\Phi(\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{2n+5})) = 0$$

Proof. The Weyl group of $\mathbf{SO}(2n+5)$ is $\mathrm{W}(\mathbf{SO}(2n+5)) = G(n+2)$ in the notation of [11], p. 171. Moreover, we have $\mathrm{W}(\mathbf{SO}(2n+1)) = G(n)$ and $\mathrm{W}(\mathbf{SO}(4)) = \mathbf{S}G(2)$. Let the x_i be the standard dual forms with respect to the standard maximal torus. The positive roots of $\mathbf{SO}(2n+5)$ are the

$$\begin{aligned} & x_1 \pm x_2, \dots, x_1 \pm x_{n+2}, \\ & x_2 \pm x_3, \dots, x_2 \pm x_{n+2}, \\ & \vdots \\ & x_{n+1} \pm x_{n+2}, \\ & x_1, \dots, x_{n+2} \end{aligned}$$

The positive roots of $\mathbf{SO}(2n+1)) \times \mathbf{SO}(4)$ are the

$$\begin{aligned} & x_1 \pm x_2, \dots, x_1 \pm x_n, \\ & x_2 \pm x_3, \dots, x_2 \pm x_n, \\ & \vdots \\ & x_{n-1} \pm x_n, \\ & x_1, \dots, x_n, \\ & x_{n+1} \pm x_{n+2} \end{aligned}$$

Hence the complementary positive roots of the homogeneous space $\mathbf{SO}(2n+5)/(\mathbf{SO}(2n+1)\times\mathbf{SO}(4))$ are the

$$\begin{aligned}
&x_1 \pm x_{n+1}, x_1 \pm x_{n+2},\\
&\vdots\\
&x_n \pm x_{n+1}, x_n \pm x_{n+2},\\
&x_{n+1}, x_{n+2}
\end{aligned}$$

Since all the rotation numbers t_i of ϕ in the maximal torus are pairwise distinct, we see that ϕ acts with isolated fixed-points only. These fixed-points are in one-to-one correspondence with elements of

$$\frac{\mathrm{W}(\mathbf{SO}(2n+5))}{\mathrm{W}(\mathbf{SO}(2n+1))\times\mathrm{W}(\mathbf{SO}(4))}$$

Let

$$w \in \frac{\mathrm{W}(\mathbf{SO}(2n+5))}{\mathrm{W}(\mathbf{SO}(2n+1))\times\mathrm{W}(\mathbf{SO}(4))}$$

be an arbitrary element. The action of the Weyl group $\mathrm{W}(\mathbf{SO}(2n+5))$ on the roots of $\mathbf{SO}(2n+5)$ induces an action of w on the complementary positive roots. Thus the roots "belonging" to w are the $w(y_i)$ with y_i a complementary positive root as above. The reflection $r := \mathrm{diag}(1,\ldots,1,-1)$ is a non-trivial element of

$$\frac{\mathrm{W}(\mathbf{SO}(2n+5))}{\mathrm{W}(\mathbf{SO}(2n+1))\times\mathrm{W}(\mathbf{SO}(4))}$$

Hence we have that $w \neq w\circ r$. The roots corresponding to $w\circ r$ are the roots belonging to w with the exception of $w(x_{n+2})$, which is replaced by $-w(x_{n+2})$. That is, we have a transformation of complementary positive roots induced by the action of $w\circ r$ on roots. This map transforms the complementary roots from above to the roots

$$\begin{aligned}
&w(x_1) \pm w(x_{n+1}), w(x_1) \mp w(x_{n+2}),\\
&\vdots\\
&w(x_n) \pm w(x_{n+1}), w(x_n) \mp w(x_{n+2}),\\
&w(x_{n+1}), -w(x_{n+2})
\end{aligned}$$

Thus we may compare the action on roots induced by w with the one induced by $w\circ r$. Hence on the vector space spanned by the complementary roots the determinants of both transformations satisfy

$$\det(w\circ r) = -\det w \in \{\pm 1\}$$

whence their orientations are once compatible, once not compatible with the orientation of the ambient space. However, as we have seen, the absolute values of their rotation numbers t_i clearly remain the same. Since $r^2 = 1$, the fixed-point set decomposes into pairs $\{\nu, \nu \circ r\}$. It is now easy to see that every equivariant signature twisted with a complex bundle R vanishes. We obtain

$$\operatorname{sign}(X, R, g) = \sum_{\nu} \operatorname{sign}(X^g_\nu, R, g)$$

Since all our fixed-points ν are isolated, this equation becomes

$$\begin{aligned}
\operatorname{sign}(\nu, R, g) =& \sum_{\nu} \left(\left(\prod_{k>0} \prod_{i=1}^{d_k} \frac{1+e^{-t_k z}}{1-e^{-t_k z}} \right) \cdot \operatorname{ch}_g(R) \right) [\nu] \\
=& \sum_{\{\nu,\nu\circ r\}} \left[\left(\prod_{k>0} \prod_{i=1}^{d_k} \frac{1+e^{-t_k z}}{1-e^{-t_k z}} \right) \cdot \operatorname{ch}_g(R) [\nu] \right. \\
&+ \left. \left(\prod_{k>0} \prod_{i=1}^{d_k} \frac{1+e^{-t_k z}}{1-e^{-t_k z}} \right) \cdot \operatorname{ch}_g(R) [r \circ \nu] \right] \\
=& \sum_{\{\nu,\nu\circ r\}} \left[\left(\prod_{k>0} \prod_{i=1}^{d_k} \frac{1+e^{-t_k z}}{1-e^{-t_k z}} \right) \cdot \operatorname{ch}_g(R) - \left(\prod_{k>0} \prod_{i=1}^{d_k} \frac{1+e^{-t_k z}}{1-e^{-t_k z}} \right) \cdot \operatorname{ch}_g(R) \right] \\
=& \, 0
\end{aligned}$$

where d_k is the complex dimension of the space of roots on which g acts with rotation number t_k.

Using the formula

$$\Phi(X, g) = \sum_{m=0}^{\infty} \operatorname{sign}(X, R_m, g) q^m$$

with the special bundles R_m described above, the vanishing of the g-equivariant elliptic genus follows. The limit process $g \to 1$ now yields that the (non-equivariant) elliptic genus vanishes. □

We shall now illustrate this proof by an example. In dimension twelve we consider the $\mathbb{S}^1$-action given by $\phi : g \mapsto (g^1, g^2, g^3)$ in the maximal torus. The quotient of Weyl groups is

$$\begin{aligned}
\frac{G(3)}{G(1) \times \mathbf{S}G(2)} \cong \Bigg\{ & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
& \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \Bigg\}
\end{aligned}$$

The complementary positive roots are given by

$$x_1 \pm x_2, x_1 \pm x_3, x_2, x_3$$

Hence the rotation numbers on the different points turn out to be

$$\begin{array}{ccc} 3,-1,4,-2,2,3 & \leftrightarrow & 3,-1,-2,4,2,-3 \\ 3,1,5,-1,1,3 & \leftrightarrow & 3,1,-1,5,1,-3 \\ 5,1,4,2,1,2 & \leftrightarrow & 1,5,4,2,1,-2 \end{array}$$

The fixed-point set splits into pairs as indicated. Orientations on the respective points in a pair are opposite to one another whereas absolute values of orientation numbers remain identical. Consequently, twisted equivariant signatures vanish and so does the elliptic genus.

E

Indices in Dimension 16

In this chapter we provide index computations in dimension 16. See section 4.1 for the analogous case of dimension 20.

E.1. Preliminaries

On a 16-dimensional Positive Quaternion Kähler Manifold M the locally-defined complex vector bundles E and H are associated to the standard representations of $\mathbf{Sp}(4)$ and $\mathbf{Sp}(1)$ respectively. We twist the Dirac operator with the bundles

$$R^{p,q} = \bigwedge\nolimits_0^p E \otimes S^q H$$

provided $4+p+q$ is even. Again $\bigwedge_0^p E$ is the (virtual) bundle $\bigwedge^p E - \bigwedge^{p-2} E$. Theorem 1.17 yields the relations

$$i^{p,q} = \begin{cases} 0 & \text{for } p+q<4 \\ (-1)^p(b_{2p-2}+b_{2p}) & \text{for } p+q=4 \\ d & \text{for } p=0, q=6 \end{cases}$$

where $d = \dim \operatorname{Isom}(M)$ is the dimension of the isometry group of M. Again we shall compute the indices $i^{p,q}$ for these interesting values of p and q in terms of characteristic classes of the bundles E and H. As in the 20-dimensional case we compute

$$\begin{aligned}\hat{A}(M) =&1+\left(\frac{c_2-4u}{12}\right)+\left(\frac{3c_2^2-c_4-23c_2u+42u^2}{720}\right)+\\&\left(\frac{10c_2^3-9c_2c_4+2c_6-115c_2^2u+48c_4u+388c_2u^2-432u^3}{60480}\right)+\\&\left(\frac{21c_2^4-34c_2^2c_4+5c_4^2+13c_2c_6-3c_8-334c_2^3u+364c_2c_4u}{3628800}+\right.\\&\left.\frac{-103c_6u+1636c_2^2u^2-799c_4u^2-3263c_2u^3+2497u^4}{3628800}\right)\\ \mathrm{ch}(E) =&8-c_2+\frac{c_2^2-2c_4}{12}+\frac{-c_2^3+3c_2c_4-3c_6}{360}+\frac{c_2^4-4c_2^2c_4+2c_4^2+4c_2c_6-4c_8}{20160}\\ \mathrm{ch}(H) =&2+u+\frac{u^2}{12}+\frac{u^3}{360}+\frac{u^4}{20160}\end{aligned}$$

By means of the Clebsch-Gordan formula we compute

$$\begin{aligned}i^{0,0} =&\frac{c_2^4}{172800}-\frac{17c_2^2c_4}{1814400}+\frac{c_4^2}{725760}+\frac{13c_2c_6}{3628800}-\frac{c_8}{1209600}-\frac{167c_2^3u}{1814400}+\frac{13c_2c_4u}{129600}\\&-\frac{103c_6u}{3628800}+\frac{409c_2^2u^2}{907200}-\frac{799c_4u^2}{3628800}-\frac{3263c_2u^3}{3628800}+\frac{2497u^4}{3628800}\\ i^{0,1} =&\frac{c_2^4}{86400}-\frac{17c_2^2c_4}{907200}+\frac{c_4^2}{362880}+\frac{13c_2c_6}{1814400}-\frac{c_8}{604800}-\frac{17c_2^3u}{907200}+\frac{47c_2c_4u}{907200}-\\&\frac{43c_6u}{1814400}-\frac{37c_2^2u^2}{56700}+\frac{431c_4u^2}{1814400}+\frac{3967c_2u^3}{1814400}-\frac{3233u^4}{1814400}\\ i^{0,2} =&\frac{c_2^4}{57600}-\frac{17c_2^2c_4}{604800}+\frac{c_4^2}{241920}+\frac{13c_2c_6}{1209600}-\frac{c_8}{403200}+\frac{233c_2^3u}{604800}-\frac{89c_2c_4u}{302400}+\\&\frac{19c_6u}{403200}-\frac{211c_2^2u^2}{302400}+\frac{89c_4u^2}{134400}-\frac{647c_2u^3}{134400}+\frac{211u^4}{44800}\\ i^{0,3} =&\frac{c_2^4}{43200}-\frac{17c_2^2c_4}{453600}+\frac{c_4^2}{181440}+\frac{13c_2c_6}{907200}-\frac{c_8}{302400}+\frac{583c_2^3u}{453600}-\frac{493c_2c_4u}{453600}+\\&\frac{197c_6u}{907200}+\frac{1277c_2^2u^2}{113400}-\frac{2209c_4u^2}{907200}+\frac{10207c_2u^3}{907200}-\frac{18593u^4}{907200}\\ i^{0,4} =&\frac{c_2^4}{34560}-\frac{17c_2^2c_4}{362880}+\frac{c_4^2}{145152}+\frac{13c_2c_6}{725760}-\frac{c_8}{241920}+\frac{1033c_2^3u}{362880}-\frac{449c_2c_4u}{181440}+\\&\frac{377c_6u}{725760}+\frac{2129c_2^2u^2}{36288}-\frac{12127c_4u^2}{725760}+\frac{263233c_2u^3}{725760}+\frac{86285u^4}{145152}\\ i^{0,6} =&\frac{7c_2^4}{172800}-\frac{17c_2^2c_4}{259200}+\frac{c_4^2}{103680}+\frac{13c_2c_6}{518400}-\frac{c_8}{172800}+\frac{2233c_2^3u}{259200}-\frac{989c_2c_4u}{129600}+\\&\frac{857c_6u}{518400}+\frac{57169c_2^2u^2}{129600}-\frac{71839c_4u^2}{518400}+\frac{4117057c_2u^3}{518400}+\frac{24236737u^4}{518400}\end{aligned}$$

Again the "indices" $i^{0,1}$ and $i^{0,3}$ are to be regarded as formal expressions. Since the sum of their superscripts is not even, they do not necessarily represent indices of twisted Dirac operators. As in the 20-dimensional case we compute

$$\begin{aligned}
\mathrm{ch}(\textstyle\bigwedge^2 E) =& 28 - 6c_2 + \frac{c_2}{2} - \frac{c_2^3}{60} + \frac{c_2^4}{3360} - \frac{c_2c_4}{30} + \frac{c_2^2c_4}{630} + \frac{c_4^2}{504} + \frac{c_6}{5} - \frac{13c_2c_6}{840} + \frac{c_8}{42}\\
\mathrm{ch}(\textstyle\bigwedge^3 E) =& 56 - 15c_2 + \frac{5c_2^2}{4} - \frac{c_2^3}{24} + \frac{c_2^4}{1344} + \frac{3c_4}{2} - \frac{5c_2c_4}{24} + \frac{41c_2^2c_4}{5040} + \frac{71c_4^2}{10080} - \frac{c_6}{8}\\
&+ \frac{11c_2c_6}{560} - \frac{397c_8}{1680}\\
\mathrm{ch}(\textstyle\bigwedge^4 E) =& 70 - 20c_2 + \frac{5c_2^2}{3} - \frac{c_2^3}{18} + \frac{c_2^4}{1008} + \frac{8c_4}{3} - \frac{c_2c_4}{3} + \frac{4c_2^2c_4}{315} + \frac{13c_4^2}{1260} - \frac{2c_6}{3}\\
&+ \frac{89c_2c_6}{1260} + \frac{151c_8}{315}
\end{aligned}$$

Again these exterior powers already determine all the higher ones by Hodge $*$-duality. That is, we obtain

$$\textstyle\bigwedge^k E \cong \bigwedge^{8-k} E$$

for $0 \leq k \leq 8$ and

$$\mathrm{ch}(\textstyle\bigwedge^k E) = \mathrm{ch}(\bigwedge^{8-k} E)$$

respectively

$$\mathrm{ch}(\textstyle\bigwedge_0^k E) = -\mathrm{ch}(\bigwedge_0^{10-k} E)$$

and

$$i^{p,q} = -i^{10-p,q}$$

In particular, $i^{5,q} = 0$ for all $q \in \mathbb{N}_0$, since $\mathrm{ch}(\bigwedge^5 E) = \mathrm{ch}(\bigwedge^3 E)$.

This enables us to compute the following indices in terms of characteristic numbers. (Again the "indices" $i^{1,0}$ and $i^{3,0}$ are to be regarded as formal expressions.) As we have seen, as for the first parameter it suffices to compute from $p = 0$ to $p = 4$.

$$\begin{aligned}
i^{1,0} =& \frac{1}{453600}(21c_2^4 + 155c_4^2 - 93c_8 - 259c_2^3u + 1157c_6u - 5209c_4u^2 + 2497u^4\\
&+ c_2^2(-109c_4 + 931u^2) + c_2(-227c_6 + 1159c_4u - 23u^3))\\
i^{2,0} =& \frac{1}{403200}(63c_2^4 + 815c_4^2 + 9591c_8 - 602c_2^3u - 27189c_6u - 2397c_4u^2 + 7491u^4\\
&+ c_2^2(-502c_4 + 1148u^2) + c_2(439c_6 + 3652c_4u + 7491u^3))\\
i^{3,0} =& \frac{1}{75600}(21c_2^4 + 355c_4^2 - 17853c_8 - 159c_2^3u + 2837c_6u + 6551c_4u^2 + 2497u^4\\
&- c_2^2(209c_4 + 9u^2) + c_2(713c_6 + 959c_4u + 4297u^3))
\end{aligned}$$

$$\begin{aligned}
i^{4,0} =& \frac{1}{86400}(21c_2^4 + 405c_4^2 + 39357c_8 - 134c_2^3u + 24857c_6u + 12641c_4u^2 + 2497u^4 \\
& - 2c_2^2(117c_4 + 122u^2) + c_2(1173c_6 + 684c_4u + 5377u^3)) \\
i^{1,1} =& \frac{1}{226800}(21c_2^4 + 155c_4^2 - 93c_8 + 41c_2^3u - 673c_6u + 5471c_4u^2 - 3233u^4 \\
& - c_2(109c_4 + 944u^2) - c_2(227c_6 + 56c_4u + 353u^3)) \\
i^{1,3} =& \frac{1}{113400}(21c_2^4 + 155c_4^2 - 93c_8 + 1241c_2^3u - 7993c_6u - 72769c_4u^2 - 18593u^4 \\
& + c_2^2(-109c_4 + 14236u^2) - c_2(227c_6 + 4916c_4u + 24353u^3)) \\
i^{2,2} =& \frac{1}{134400}(63c_2^4 + 815c_4^2 + 9591c_8 + 1798c_2^3u + 80811c_6u + 2403c_4u^2 + 17091u^4 \\
& + c_2^2(-502c_4 + 7148u^2) + c_2(439c_6 - 11948c_4u + 26691u^3)) \\
i^{3,1} =& \frac{1}{37800}(21c_2^4 + 355c_4^2 - 17853c_8 + 141c_2^3u - 1513c_6u - 7969c_4u^2 - 3233u^4 \\
& - c_2^2(209c_4 + 624u^2) + c_2(713c_6 - 1516c_4z - 6113u^3))
\end{aligned}$$

These were the better understood ones; let us now compute some more:

$$\begin{aligned}
i^{1,5} =& \frac{1}{75600}(21c_2^4 + 155c_4^2 - 93c_8 + 3241c_2^3u - 20193c_6u - 633249c_4u^2 + 4686687u^4 \\
& + c_2^2(-109c_4 + 120176u^2) - c_2(227c_6 + 13016c_4u + 816993u^3) \\
i^{2,4} =& \frac{1}{80640}(63c_2^4 + 815c_4^2 + 9591c_8 + 6598c_2^3u + 296811c_6u - 36381c_4u^2 + 1294275u^4 \\
& + c_2^2(-502c_4 + 164300u^2) + c_2(439c_6 - 43148c_4u - 1692861u^3)) \\
i^{3,3} =& \frac{1}{18900}(21c_2^4 + 355c_4^2 - 17853c_8 + 1341c_2^3u - 18913c_6u + 115391c_4u^2 - 18593u^4 \\
& + c_2^2(-209c_4 + 19596u^2) + c_2(713c_6 - 11416c_4u - 70433u^3)) \\
i^{4,2} =& \frac{1}{9600}(7c_2^4 + 135c_4^2 + 13119c_8 + 222c_2^3u - 24941c_6u + 4747c_4u^2 + 1899u^4 \\
& + c_2^2(-78c_4 + 1332u^2) + c_2(391c_6 - 2252c_4u + 5419u^3)) \\
i^{2,6} =& \frac{1}{57600}(63c_2^4 + 815c_4^2 + 9591c_8 + 13798c_2^3u + 620811c_6u - 215517c_4u^2 \\
& + 72710211u^4 + c_2^2(-502c_4 + 762908u^2) + c_2(439c_6 - 89948c_4u - 22491069u^3))
\end{aligned}$$

The fundamental system of equations then yields the relation

$$c_2u^3 = \frac{3}{16}(-7 + d) - 8u^4$$

Reducing this system and extending it by one of the equations above lets us conclude:

$$c_4u^2 = -\frac{3}{16}(-33 + 4b_2 + 3d + i^{1,5}) + 34u^4$$

These two equations now lead to

$$\begin{aligned}
i^{2,4} &= 69 + 5b_2 + 5b_4 - 6d + v \\
i^{3,3} &= 62 - 26b_2 - 2b_4 - 2b_6 - 8d - 3i^{1,5} + 2v \\
i^{4,2} &= \frac{1}{2}(-9 - 45b_2 + 3b_4 + 3b_6 - 6d + 2v - 4i^{1,5}) \\
&= -4 - 24b_2 + 2b_6 + b_8 - 3d + v - 2i^{1,5} \\
i^{2,6} &= 798 + 14b_2 + 14b_4 - 87d + 19v
\end{aligned}$$

E.2. Reproofs of known relations

First we remark that the positive definiteness of the intersection form (cf. 1.16) yields the relation on Betti numbers (1.2) and vice versa.

This can be seen via the L-genus, which can be computed as

$$\begin{aligned}
L(M) =& 1 + \frac{1}{3}\big(-2c_2 + 8u\big) + \frac{1}{15}\big(c_2^2 + \frac{14c_4}{3} - \frac{38c_2u}{3} + 44u^2\big) \\
&+ \big(\frac{2c_2^3}{189} - \frac{8c_2c_4}{105} - \frac{124c_6}{945} + \frac{40c_2^2u}{189} - \frac{152c_4u}{315} - \frac{116c_2u^2}{945} + \frac{176u^3}{105}\big) \\
&+ \big(\frac{c_2^4}{675} - \frac{184c_2^2c_4}{14175} + \frac{61c_4^2}{2835} + \frac{478c_2c_6}{14175} + \frac{254c_8}{4725} + \frac{416c_2^3u}{14175} - \frac{338c_2c_4u}{2025} \\
&+ \frac{11042c_6u}{14175} + \frac{3091c_2^2u^2}{14175} - \frac{9994c_4u^2}{14175} + \frac{3502c_2u^3}{14175} + \frac{7102u^4}{14175}\big)
\end{aligned}$$

Since the intersection form is positive definite (cf. 1.16), we obtain

$$\begin{aligned}
&\Big\langle \frac{c_2^4}{675} - \frac{184c_2^2c_4}{14175} + \frac{61c_4^2}{2835} + \frac{478c_2c_6}{14175} + \frac{254c_8}{4725} + \frac{416c_2^3u}{14175} - \frac{338c_2c_4u}{2025} \\
&+ \frac{11042c_6u}{14175} + \frac{3091c_2^2u^2}{14175} - \frac{9994c_4u^2}{14175} + \frac{3502c_2u^3}{14175} + \frac{7102u^4}{14175}, [M]\Big\rangle \\
&= L(M)[M] = \operatorname{sign}(M) = b_8
\end{aligned}$$

Combining this equation with the known ones resulting from the previous index computations and theorem 1.17 we obtain a linear system of equations. We solve this system with formal variables $r = c_4^2$, $s = c_4u^2$, $t = u^4$. This yields

$$
\begin{aligned}
c_2^4 =& 3r + \frac{1}{192}(1586155 - 5964597b_2 - 5923125b_4 + 1982311b_6 + 3945614b_8 \\
&+ 24420d) + \frac{361s}{3} - \frac{15238t}{3} \\
c_2^2c_4 =& 2r + \frac{1}{48}(47249 - 217647b_2 - 213471b_4 + 72149b6 + 141706b_8 + 1356d) \\
&+ \frac{164s}{3} - \frac{3296t}{3} \\
c_4^2 =& r \\
c_2c_6 =& \frac{1}{192}(8197 - 64731b_2 - 67035b_4 + 20809b_6 + 45074b_8 + 732d) + \frac{43s}{3} - \frac{454t}{3} \\
c_8 =& \frac{1}{32}(-335 + 17b_2 + 113b_4 + 37b_6 - 22b_8 + 12d) + s - 13t \\
c_2^3u =& \frac{1}{48}(-9389 + 54099b_2 + 53091b_4 - 17729b_6 - 35458b_8 - 348d) - \frac{44s}{3} + \frac{728t}{3} \\
c_2c_4u =& \frac{1}{192}(15637 + 28341b_2 + 23733b_4 - 8039b_6 - 16078b_8 - 996d) - \frac{47s}{3} + \frac{542t}{3} \\
c_6u =& \frac{1}{64}(917 - 11b_2 - 203b_4 + 89b_6 + 178b_8 - 36d) - 2s + 20t \\
c_2^2u^2 =& \frac{1}{192}(7303 - 8985b_2 - 8985b_4 + 2995b_6 + 5990b_8 - 204d) + \frac{s}{3} + \frac{110t}{3} \\
c_4u^2 =& s \\
c_2u^3 =& \frac{1}{64}(-119 + 105b_2 + 105b_4 - 35b_6 - 70b_8 + 12d) - 8t \\
u^4 =& t
\end{aligned}
$$

More precisely, solving the system consisting of all but the equation for $i^{0,0} = 0$ yields the solutions above, which impose a certain condition in order to hold equally for the first equation: Namely the well-known formula (cf. (1.2))

$$\frac{1}{12}(1 - 3b_2 - 3b_4 + b_6 + 2b_8) = 0 \Leftrightarrow 1 - 3b_2 - 3b_4 + b_6 = -2b_8$$

Conversely, we may reprove the positive definiteness of the intersection form in the following way: From the above equations we see that on the one hand on the level of characteristic numbers we have

$$\sum_{p=0}^{n} i^{p,n-p} = L(M)[M]$$

On the other hand on the level of Betti numbers we obtain:

$$\sum_{p=0}^{n} i^{p,n-p} = 1 - (b_2 + 1) + (b_4 + b_2) - (b_6 + b_4) + (b_8 + b_6) = b_8$$

Thus $L(M)[M] = b_8$ and the intersection form is positive definite.

Due to theorem 1.16 the Euler characteristic of M obviously can be computed by the formula

$$\chi(M) = \sum_{p=0}^{4} (-1)^p i^{p,4-p}$$

We use a different approach and we shall see that both ways of computing $\chi(M)$ are consistent.

Consider the twistor fibration $q : Z = \mathbf{P}_{\mathbb{C}}(H) \to M$ and form the associated, locally-defined tautological line bundle $L^{-1} \to Z|_U$ for some open set $U \subseteq Z$. That is, the fibre at $z = [h] \in U \subseteq \mathbf{P}_{\mathbb{C}}(H)$ is just the subspace $\mathbb{C}h$ of $q^*(H)_z$. Form the anti-canonical bundle L. The bundle L^2 is defined globally on Z—cf. [66], p. 148.

Moreover, observe that

$$q^*H \cong L \oplus L^{-1} \cong L \oplus \bar{L}$$

from which we obtain

$$q^*(T^*_{\mathbb{C}}M) \cong q^*E \otimes q^*H \cong Lq^*E \oplus \overline{Lq^*E}$$

and thus

$$q^*TM \cong Lq^*E$$

(cf. equation (2.8) on [66], p. 148). In particular, this leads to

$$c_8(Lq^*E) = \chi(M)$$

(It is well-known that on a complex bundle the top Chern class and the Euler class coincide.) Now we compute the Chern classes of Lq^*E in terms of the Chern classes of E and H. This results in the expression

$$\chi(M) = c_8 + c_6 u + c_4 u^2 + c_2 u^3 + u^4$$

This, however, turns out to be exactly the formula we obtained by the first approach.

As in section 4.1 we compute the Hilbert Polynomial of M as

$$\begin{aligned} f(q) =& \frac{v}{92897280}q^9 + \frac{v}{10321920}q^8 + \frac{-42+6d-v}{1935360}q^7 + \frac{-28+4d-v}{184320}q^6 \\ &+ \frac{228-12d+v}{276480}q^5 + \frac{520-40d+7v}{92160}q^4 + \frac{882-126d+31v}{725760}q^3 \\ &+ \frac{-812+56d-9v}{40320}q^2 + \frac{-84+6d-v}{5040}q \end{aligned}$$

In order to obtain the form of the Hilbert Polynomial used in [68] we have to index-transform it with $q \to 4+2q$. As an example we obtain

$$f_{\mathbb{H}\mathbf{P}^4}(4+2q) = \binom{9+2q}{9}$$

by substituting in $v = 256$ and $d = 55$. In general, we know that

$$1 \le v \le 256 \qquad\qquad 8 \le d \le 55$$

(cf. chapter 1). The inequality for v shows that the leading coefficient of f is smaller or equal to the one of $f_{\mathbb{H}\mathbf{P}^4}$ (and greater than zero). Thus a check on a finite number of points yields that for all $q \in \mathbb{N}_0$ it holds:

$$0 \le f(4+2q) \le \binom{9+2q}{9}$$

So we see that in dimension 16 the special result ($v \le 256$) from [66] proves the most general result in that article.

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Notation Index

Index

. . .

And on the pedestal these words appear:
"My name is Ozymandias, king of kings:
Look on my works, ye Mighty, and despair!"
Nothing beside remains: Round the decay
Of that colossal wreck, boundless and bare,
The lone and level sands stretch far away.

Percy Bysshe Shelley, "Ozymandias"