Mathematik

A Description of the Jacobson Topology on the Spectrum of Transformation Group C^*-algebras by Proper Actions

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Abstract

We analyze the Jacobson topology on the spectrum of transformation group C^*-algebras for proper G-spaces X in terms of the space $\mathrm{Stab}(X)\widehat{\ } = \{(x, G_x, \sigma) \mid x \in X,\ \sigma \in \widehat{G_x}\}$. We show that $\mathrm{Stab}(X)\widehat{\ }$ can be topologized and equipped with a G-action in such a way that the quotient space $G\backslash\mathrm{Stab}(X)\widehat{\ }$ is homeomorphic to $(C_0(X) \rtimes G)\widehat{\ }$ via the well-known bijection between these spaces from the Mackey-Rieffel-Green theorem.

We discuss several approaches to define such a topology on $\mathrm{Stab}(X)\widehat{\ }$ and show that the resulting topologies coincide.

Zusammenfassung

Diese Arbeit liefert eine Beschreibung der Jacobson-Topologie auf dem Spektrum von Transformationsgruppen-C^*-algebren für eigentliche G-Räume X vermittels des Raums $\mathrm{Stab}(X)\widehat{\ } = \{(x, G_x, \sigma) \mid x \in X,\ \sigma \in \widehat{G_x}\}$. Es wird gezeigt, daß auf $\mathrm{Stab}(X)\widehat{\ }$ eine Topologie und eine G-Wirkung derart definiert werden können, daß die aus dem Satz von Mackey-Rieffel-Green bekannte Bijektion zwischen dem Bahnenraum $G\backslash\mathrm{Stab}(X)\widehat{\ }$ und $(C_0(X) \rtimes G)\widehat{\ }$ ein Homöomorphismus ist.

Wir diskutieren verschiedene Ansätze zur Topologisierung des Raums $\mathrm{Stab}(X)\widehat{\ }$ und zeigen, daß sie dieselbe Topologie liefern.

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Contents

Introduction

The purpose of this thesis is to describe the spectrum of transformation group C^*-algebras $C_0(X) \rtimes G$ for proper actions. In particular, we want to describe the Jacobson topology on these spaces in terms of the action of G on X.

Crossed products of the form $C_0(X) \rtimes G$ for proper G-spaces X are very important in the theory of operator algebras because their K-theory appears in certain special cases of the Baum-Connes conjecture with coefficients. The topological structure of $(C_0(X) \rtimes G)\widehat{\ }$ is of interest because it corresponds to the ideal structure of $C_0(X) \rtimes G$, the knowledge of which can be helpful for the computation of the K-theory of this C^*-algebra.

This work is based on a well-known result, going back to work of Mackey, Rieffel, and Green, which states that, for a proper G-space X, there exists a bijection

$$\operatorname{ind}^G \colon G\backslash \operatorname{Stab}(X)\widehat{\ } \to (C_0(X) \rtimes G)\widehat{\ },$$

where the space $\operatorname{Stab}(X)\widehat{\ } = \{(x, G_x, \sigma) \mid x \in X,\ \sigma \in \widehat{G_x}\}$ is equipped with a suitable action of G. This shows that it is possible to describe the spectrum of $C_0(X) \rtimes G$ in terms of the spectra of the stabilizer subgroups G_x.

We present several ways to equip $\operatorname{Stab}(X)\widehat{\ }$ with a topology such that the quotient topology on $G\backslash \operatorname{Stab}(X)\widehat{\ }$ coincides with the topology induced by the bijection ind^G. The nicest one, based on an idea by Siegfried Echterhoff, is to define a C^*-algebra A such that the spectrum $\widehat{A}$ is in a natural set bijection with $\operatorname{Stab}(X)\widehat{\ }$. Then $\operatorname{Stab}(X)\widehat{\ }$ can simply be topologized by transferring the Jacobson topology from $\widehat{A}$.

To actually prove that $\operatorname{Stab}(X)\widehat{\ }$ carries a topology with the required properties we need, however, a more tangible definition. This is obtained by viewing $\operatorname{Stab}(X)\widehat{\ }$ as a subset of $X \times S(G)\widehat{\ }$, where $S(G)\widehat{\ }$ is the space of irreducible subgroup representations of G equipped with the Fell topology, which can be described very concretely in terms of functions of positive type. We use this topology to define a closure operation on $\operatorname{Stab}(X)\widehat{\ }$, but the corresponding topology does not in general coincide with the topology induced from $X \times S(G)\widehat{\ }$. In fact, it requires a good deal of work to show that our closure operation satisfies the Kuratowski axioms and therefore induces a topology on $\operatorname{Stab}(X)\widehat{\ }$ at all. But this can be done, and we also show that this topology coincides with the topology induced from the spectrum of A as above.

The definition of this closure operation is inspired by Baggett's work in [Bag68], where he analyzes the topological structure of the spectrum of semidirect product groups $N \rtimes K$ with N abelian and K compact. Since

$$(N \rtimes K)\widehat{\ } \cong C^*(N \rtimes K)\widehat{\ } \cong (C^*(N) \rtimes K)\widehat{\ } \cong (C_0(\widehat{N}) \rtimes K)\widehat{\ },$$

and since compact groups automatically act properly, this is a special case of our situation. In his description of $(N \rtimes K)\widehat{\ }$, Baggett works with cataloguing triples which correspond to the elements of $\mathrm{Stab}(X)^\frown$. He also uses a convergence-like notion based on $S(G)^\frown$, but he does not supply a proof that this notion does actually yield a topology for the set of cataloguing triples. Nonetheless, our proof that the map ind^K is open with respect to our topology in case of a compact group K is based on his ideas.

To prove that openness of ind^K can be used to prove openness of ind^G in the general situation of a properly acting group G we use a theorem of Abels', which states that every proper G-space is locally induced from compact subgroups. This approach has also been used by Echterhoff and Emerson in [EE], where they define a topology on $\mathrm{Stab}(X)^\frown$ for proper G-spaces X that satisfy Palais' slice property, denoted by (SP), which means that X is even locally induced from stabilizer subgroups. The topology defined by Echterhoff and Emerson under the assumption of (SP) is very convenient, because it is completely determined by the G-action and does not require the space $S(G)^\frown$.

We prove that, provided that (SP) is satisfied, their and our topology coincide, and we also discuss why it is not possible to transfer their definition, nor their proof that ind^G is a homeomorphism with respect to their topology, to the general case.

This thesis is organized as follows: In Chapter 1 we present some basic material which will be needed all the time. Chapter 2 begins with a short presentation of the different approaches to defining a topology on $\mathrm{Stab}(X)^\frown$, continues with background material for and the definition of our topology on $\mathrm{Stab}(X)^\frown$, and closes with the proof that this topology makes the map ind^G continuous, and a description of $\mathrm{Stab}(X)^\frown$ in the special case that the map $x \mapsto G_x$ is continuous.

In Chapter 3 we use induction via bimodules to show how the fact that a proper G-space is locally induced from compact subgroups can be used to reduce the analysis of $(C_0(X) \rtimes G)^\frown$ and $\mathrm{Stab}(X)^\frown$ to the situation that G is compact. Chapter 4 contains some results on representations of compact groups K and the structure of $L^2(K)$, which will be used in Chapter 5 to prove that ind^K is open with respect to our topology. That chapter also contains the proof of openness in the general case, which completes the proof that ind^G is a homeomorphism.

We then go back to the definition of our topology on $\mathrm{Stab}(X)^\frown$ and show in Chapter 6 that it coincides with the topology induced from $\widehat{A}$. In Chapter 7 we then turn to proper G-spaces with Palais' slice property, briefly present Echterhoff and Emerson's methods, and compare them with our approach. The Appendix A contains some additional definitions and proofs.

Chapter 1

Definitions and tools

In this chapter we collect most of the basic material which is needed for this thesis. Everything is kept quite short and restricts to the results which we actually need, but references for background reading are given. The topics we discuss here are proper G-spaces, locally compact groups and C^*-algebras and their representations, crossed products and generalized fixed point algebras, and induced representations of covariant systems.

All through this thesis we work with locally compact spaces and locally compact groups. Recall that a topological space is said to be locally compact if every point has a neighborhood basis consisting of compact sets. If the space is Hausdorff, this is equivalent to the condition that every point has a compact neighborhood. As we deal with Hausdorff spaces most of the time, we use the term "locally compact" when we mean "locally compact and Hausdorff". If a space is locally compact, but not necessarily Hausdorff, we will mention this explicitly.

If X is a locally compact space, then we denote by $C_c(X)$ the compactly supported continuous functions on X, and by $C_0(X)$ the continuous functions vanishing at infinity.

1.1 Proper G-spaces

In this section we introduce the notion of (proper) G-spaces, list some important properties, and explain what it means that a proper G-space is locally induced from compact subgroups.

Locally compact G-spaces

Most of the following definitions are also valid for general topological groups and spaces, but we don't need that generality. See [Wil07] for a more extended treatment of this material.

Let G be a locally compact group, and let X be a locally compact Hausdorff space. Then X is a locally compact G-space if there exists a continuous map

$$G \times X \to X,\ (g, x) \mapsto g \cdot x$$

such that $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all $x \in X$ and all $g, h \in G$. We will usually write gx instead of $g \cdot x$ for all $g \in G$ and all $x \in X$. For every $x \in X$, the orbit through x and the stabilizer subgroup at x are defined by

$$Gx := \{gx \in X \mid g \in G\},$$

and

$$G_x := \{g \in G \mid gx = x\},$$

respectively. The set of orbits is called the orbit space and is denoted by $G\backslash X$. It is equipped with the usual quotient topology, which is defined to be the final topology induced by the orbit map $X \to G\backslash X,\ x \mapsto Gx$. The orbit map is continuous (by definition of the topology) and open (see Lemma 3.25 in [Wil07]). The openness of the orbit map yields the following useful description of convergence in $G\backslash X$:

Remark 1.1.1 (Lemma 3.38 in [Wil07])**.** Suppose that X is a locally compact G-space, that $x \in X$, and that $(x_\nu)_{\nu\in N}$ is a net in X such that $([x_\nu])_{\nu\in N}$ converges to $[x]$ in $G\backslash X$. Then there are a subnet $(x_j)_{j\in J}$ and elements $s_j \in G$ for all $j \in J$ such that $s_j x_j \to x$ in X.

If G acts on X and $(x_\nu)_{\nu\in N}$ is a net in X, then we denote the corresponding net of stabilizer subgroups by $(G_\nu)_{\nu\in N}$ instead of $(G_{x_\nu})_{\nu\in N}$. We modify the notation analogously for several other objects associated to a net $(x_\nu)_{\nu\in N}$ in X, and it will hopefully always be clear what we mean.

Proper G-spaces

The following lemma lists several properties that might be used to define proper G-spaces. It is a slight extension of Lemma 3.42 in [Wil07].

Lemma 1.1.2. Let X be a locally compact G-space. Then the following conditions are equivalent:

(i) The map $G \times X \to X \times X,\ (g, x) \mapsto (gx, x)$ is proper in the sense that preimages of compact subsets of $X \times X$ are compact in $G \times X$;

(ii) If $(x_\nu)_{\nu\in N}$ and $(g_\nu)_{\nu\in N}$ are nets in X and G, respectively, and if $x, y \in X$ are such that $x_\nu \to x$ and $g_\nu x_\nu \to y$, then the net $(g_\nu)_{\nu\in N}$ has a convergent subnet;

(iii) For every compact subset $K \subseteq X$ the set $\{g \in G \mid g^{-1}K \cap K \neq \emptyset\}$ is compact in G.

Definition 1.1.3. A locally compact G-space X is said to be proper if it satisfies any, and hence every condition of the previous lemma.

It is not hard to deduce the following important properties of proper G-spaces from Lemma 1.1.2. The proofs of (i)-(iii) can also be found in [Wil07]; statement (iv) follows directly from Lemma 1.1.2 (ii) or (iii).

Proposition 1.1.4. Let X be a proper G-space. Then the following statements are true:

(i) The stabilizer group G_x is compact for every $x \in X$.

(ii) The orbit space $G\backslash X$ is a locally compact Hausdorff space.

(iii) For every $x \in X$, the map $G/G_x \to Gx,\ gG_x \mapsto gx$ is a homeomorphism.

(iv) Let Y be locally compact (not necessarily proper) G-space. Then the action of G on $X \times Y$ given by $g(x, y) = (gx, gy)$ for all $g \in G$, $(x, y) \in X \times Y$ is proper.

It follows immediately from the definition that actions of compact groups are always proper. To get a slightly more general example, let H be a closed subgroup of a locally compact group G. Then the action $H \times G \to G,\ (h, g) \mapsto (gh^{-1})$ is proper, because the map

$$G \times G \to G \times G,\ (h, g) \mapsto (gh^{-1}, g)$$

is a homeomorphism, which implies that its restriction to $H \times G$ is proper.

Induced G-spaces

Definition 1.1.5. Let H be a closed subgroup of a locally compact group G, and let Y be a locally compact H-space. Define an action of H on $G \times Y$ by $h(g, y) := (gh^{-1}, hy)$ for all $h \in H$, $g \in G$, and $y \in Y$. The quotient $G \times_H Y := H \backslash (G \times Y)$ is then a G-space with respect to the action $g'[g, y] := [g'g, y]$ for all $g' \in G$, $[g, y] \in G \times_H Y$.

As shown in the end of the previous paragraph, the action of H on G is proper. Hence it follows that H acts properly on $G \times Y$, and thus the induced space $G \times_H Y$ is a locally compact Hausdorff space, see Proposition 1.1.4 (ii),(iv).

The following proposition gives a criterion for when a given G-space is induced from a closed subgroup H. It is a corollary to a similar statement on general covariant systems, which is proven in [Ech90a]. Recall that, if X and Y are locally compact G-spaces, then a continuous map $\varphi\colon X \to Y$ is called a G-map if $\varphi(gx) = g\varphi(x)$ for all $g \in G$, $x \in X$.

Proposition 1.1.6. Let X be a locally compact G-space and let H be a closed subgroup of G. The following statements are equivalent:

(i) There exists a locally compact H-space Y such that X is G-homeomorphic to $G \times_H Y$.

(ii) There exists a continuous G-map $\varphi\colon X \to G/H$.

If (i) holds, then the corresponding G-map $\varphi\colon G \times_H Y \to G/H$ is given by $\varphi([g, y]) = gH$ for all $g \in G$, $y \in Y$. Given a map φ as in (ii), the corresponding H-space Y is the closed subset $Y := \varphi^{-1}(\{eH\})$ of X, and the G-homeomorphism $\Phi\colon G \times_H Y \to X$ is given by $\Phi([g, y]) = gy$ for all $g \in G$, $y \in Y$.

It is shown in Lemma 1.2 in [EE] that, if H is a closed subgroup of a locally compact group G, and Y is a locally compact H-space, then G acts properly on $G \times_H Y$ if and only if H acts properly on Y. It follows in particular that every G-space which is induced from a compact subgroup must be proper. Locally, the converse of this is also true, which is shown in the following theorem of Abels:

Theorem 1.1.7 (Theorem 3.3 in [Abe78])**.** Let X be a proper G-space. Then for every $x \in X$ there exist a G-invariant open neighborhood U_x of x and a compact subgroup L_x of G such that there is a continuous G-map $\varphi_x\colon U_x \to G/L_x$ with $\varphi_x(x) = eL_x$.

Remarks 1.1.8. (i) By the preceding theorem and by Proposition 1.1.6 there is, for every $x \in X$, a G-homeomorphism $\Phi_x\colon G \times_{L_x} Y_x \to U_x$, where $Y_x := \varphi_x^{-1}(\{eL_x\})$. Notice that $x \in Y_x$.

(ii) If X is a proper G-space and L and Y are such that the statements of Proposition 1.1.6 hold, i.e., $Y \subseteq X$ is an L-invariant subset and $X \cong G \times_L Y$, then Y is called a global L-slice of X. If $U \subseteq X$ is a G-invariant open subset, then $Y \subseteq U$ is a local L-slice for X if it is a global L-slice for the G-space U.

(iii) Let $x \in X$, and let U_x, L_x, and φ_x be as in Theorem 1.1.7. Let $y \in Y_x$. Then $G_y \leq L_x$, because for every $g \in G_y$ we have that

$$eL_x = \varphi_x(y) = \varphi_x(gy) = g\varphi_x(y) = gL_x,$$

and hence $g \in L_x$.

(iv) Palais proved in [Pal61] that if a Lie group G acts properly on a locally compact space X, then the group L_x from Theorem 1.1.7 can be chosen to be the stabilizer subgroup G_x for every $x \in X$. Then every $x \in X$ has a G-invariant open neighborhood U_x which is homeomorphic to $G \times_{G_x} Y_x$ for a suitable G_x-slice Y_x. In [EE], Echterhoff and Emerson say that a proper G-space X satisfies Palais' slice property (SP) if it has just this property. If now Y_x is a G_x-slice through some $x \in X$, then (iii) implies that $G_y \leq G_x$ for all $y \in Y_x$. Under this assumption, Echterhoff and Emerson can formulate a topology on the space $\mathrm{Stab}(X)\widehat{\ }$ such that the map $\mathrm{ind}^G \colon G\backslash\mathrm{Stab}(X)\widehat{\ } \to (C_0(X) \rtimes G)\widehat{\ }$ becomes a homeomorphism, see Chapter 7.

(v) Not all proper G-spaces satisfy (SP). Suppose, for instance, that a compact group K acts on a compact space X. Then the componentwise action of the product $\prod_{n\in\mathbb{N}} K$ on $\prod_{n\in\mathbb{N}} X$ is automatically proper, but does in general not satisfy (SP).

In our applications of Abels' Theorem we will often need the following statement which, loosely speaking, says that it does not matter if we move a net in U_x which converges to a point in Y_x into Y_x.

Lemma 1.1.9. Let X be a proper G-space, let $x \in X$ and let U_x, φ_x, L_x, and Y_x be as in Abels' Theorem. Suppose that $(y_\nu)_{\nu\in N}$ in U_x converges to $y \in Y_x$. Then there exists a subnet $(y_j)_{j\in J}$ such that there is a net $(g_j)_{j\in J}$ in G which satisfies $g_j \to e$ and $g_j y_j \in Y_x$ for all $j \in J$.

Proof. Let $\mathcal{U}(e)$ denote the collection of symmetric open neighborhoods of e in G. Then WY_x is open in X for every $W \in \mathcal{U}(e)$ because the composition

$$G \times Y_x \xrightarrow{q} G \times_{L_x} Y_x \xrightarrow{\Phi_x} U_x$$

of the quotient map q and the map Φ_x from Remark 1.1.8 (i) is open, and WY_x is the image of the relatively open subset $W \times Y_x$ of $G \times Y_x$. So WY_x is relatively open in U_x and, as U_x is open, WY_x is open in X.

It follows that for every $W \in \mathcal{U}(e)$ there exists $\nu \in N$ such that $y_\mu \in WY_x$ for all $\mu \in N_{\geq\nu}$. Hence we can define

$$J := \{(W, \nu) \mid \forall \mu \geq \nu : \ y_\mu \in WY_x\},$$

and J becomes an upwards directed ordered set with $(W_1, \nu_1) \leq (W_2, \nu_2)$ defined by $W_2 \subseteq W_1$ and $\nu_2 \geq \nu_1$. The map $j = (W, \nu) \mapsto \nu$ defines a subnet $(y_j)_{j\in J}$ of $(y_\nu)_{\nu\in N}$ which satisfies $y_j \in WY_x$ for every $j = (W, \nu) \in J$. For every $j = (W, \nu) \in J$ we can thus choose an element $g_j \in W$ such that $g_j y_j \in Y_x$. To see that $g_j \to e$ let $W_0 \in \mathcal{U}(e)$ and choose $\nu_0 \in N$ such that $j_0 := (W_0, \nu_0) \in J$. Then, for all $j = (W, \nu) \in J_{\geq j_0}$ we have that $g_j \in W \subseteq W_0$, and we are done. □

1.2 Locally compact groups and their representations

In this section we want to set up the notation and definitions on locally compact groups and their representations which we will need later on. For details, proofs and background to this material the reader can, for instance, consult [DE09] or [Fol95]; some of this is also covered in [Wil07].

Haar measure and the modular function

Let G be a locally compact group. Then G has a (left) Haar measure, that is, a left-invariant Radon measure, which is unique up to a strictly positive scalar. Equivalently, there exists a non-zero positive integral on $C_c(G)$ which satisfies

$$\int_G f(gx)dx = \int_G f(x)dx \qquad \text{for all } f \in C_c(G),\ g \in G,$$

which is called the (left) Haar integral, and is also unique up to multiplication with a strictly positive scalar.

Since, for every $g \in G$, the map

$$C_c(G) \to \mathbb{C},\ f \mapsto \int_G f(xg)dx$$

also defines a Haar integral, uniqueness implies that there exists a number $\Delta(g) \in \mathbb{R}_{>0}$ such that

$$\int_G f(x)dx = \Delta(g) \int_G f(xg)dx \qquad \text{for all } f \in C_c(G).$$

The map $\Delta\colon G \to \mathbb{R}_{>0}$ is called the modular function on G. It is independent of the choice of Haar measure and is a continuous homomorphism. A group for which $\Delta(g) = 1$ for all $g \in G$ is called unimodular; this is the case for all abelian, all discrete, and all compact groups.

Whenever we consider a compact group K, we suppose that it comes equipped with the normalized Haar measure such that $\int_K 1dk = 1$. In case that we have several groups and Haar measures around we sometimes use the notation $\int_G f(g)d\mu_G(g)$ instead of $\int_G f(g)dg$.

Unitary representations

We now come to the definition of unitary representations of locally compact groups. The nice representation theory of compact groups will play such an important role in this work that it deserves some extra treatment — see Chapter 4 for this.

Again, let G be a locally compact group. A unitary representation of G is a homomorphism σ from G into the group $U(V_\sigma)$ of unitary operators on a Hilbert space V_σ that is continuous with respect to the strong operator topology, which means that $G \to V_\sigma,\ x \mapsto \sigma(x)\xi$ is continuous for every $\xi \in V_\sigma$. The dimension of the space V_σ is denoted by $\dim_\sigma$ and is called the dimension of σ. We often omit the term "unitary".

The most important examples are the left and right regular representation, denoted by λ^G and ϱ^G, respectively, which represent G on the unitary operators on $L^2(G)$ and are defined by

$$(\lambda^G_x f)(g) = f(x^{-1}g) \qquad \text{for all } f \in L^2(G),\ x, g \in G,$$

and

$$(\varrho_x^G f)(g) = \Delta(x)^{\frac{1}{2}} f(gx) \qquad \text{for all } f \in L^2(G),\ x, g \in G.$$

Two representations σ, π of G are said to be unitarily equivalent (notation: $\sigma \approx \pi$) if there exists a unitary operator $U\colon V_\sigma \to V_\pi$ such that

$$U\sigma(x)U^{-1} = \pi(x) \qquad \text{for all } x \in G.$$

The collection of equivalence classes of unitary representations of G (with limited cardinality of the involved Hilbert spaces) is denoted by $\mathrm{Rep}(G)$. For the sake of readability we will not make a notational difference between a representation σ of G and its class in $\mathrm{Rep}(G)$.

Subrepresentations and irreducible representations

When working with group representations it is often desirable to split a given representation up into "smaller" ones, and of course it is also interesting to know which ones are "smallest". This is implemented with the following notion of subrepresentations and irreducible representations. If σ is a unitary representation of G, then a closed subspace M of V_σ is called invariant for σ if $\sigma(x)M \subseteq M$ for all $x \in G$. If M is a nonzero invariant subspace for σ, then there is a representation $\sigma|_M$ of G on M given by $\sigma|_M(x) = \sigma(x)|_M$ for all $x \in G$.

A unitary representation π of G is said to be a subrepresentation of σ, or to be contained in σ, notated by $\pi \leq \sigma$, if there exists an invariant subspace $M \leq V_\sigma$ such that $\pi = \sigma|_M$. We are often a bit sloppy and also use this notation if π is just equivalent to $\sigma|_M$. A unitary representation σ of a locally compact group G is said to be irreducible if it does not admit any invariant subspaces other than V_σ and $\{0\}$. Otherwise, σ is called reducible. If G is compact, then every irreducible representation of G is finite-dimensional, and every representation of G is a direct sum of irreducibles.

The collection of unitary equivalence classes of irreducible representations of a locally compact group G is called the spectrum of G and is denoted by $\widehat{G}$.

Schur's Lemma

The following criterion for irreducibility is fundamental, and many of the results which we will present in Chapter 4 are based on it. Whenever π and σ are unitary representations of a locally compact group G, then $\mathcal{C}(\pi, \sigma)$ denotes the set of all $T \in B(V_\pi . V_\sigma)$ that satisfy $T\pi(g) = \sigma(g)T$ for all $g \in G$. If $\sigma = \pi$ we just write $\mathcal{C}(\pi)$ instead of $\mathcal{C}(\pi, \sigma)$.

Lemma 1.2.1 (Schur's lemma)**.** Let π and σ be irreducible representations of a locally compact group G.

(i) π is irreducible if and only if $\mathcal{C}(\pi)$ contains only scalar multiples of the identity.

(ii) If π and σ are equivalent, then $\mathcal{C}(\pi, \sigma)$ is one-dimensional; otherwise, $\mathcal{C}(\pi, \sigma) = \{0\}$.

The dual of a representation

We will later, when we want to decompose the space $L^2(K)$ for a compact group K, need the following notion of the dual of a given representation: Let σ be a unitary representation of G. Recall that the Hilbert space dual of the space V_σ of σ is given by

$$V_\sigma^* = \{\xi^* \mid \xi \in V_\sigma\}$$

with the linear operations and the inner product given by

$$\lambda\xi^* + \mu\eta^* = (\bar{\lambda}\xi + \bar{\mu}\eta)^*, \qquad \langle \xi^* | \eta^* \rangle_{V_\sigma^*} = \langle \eta | \xi \rangle_{V_\sigma}$$

for all $\lambda, \mu \in \mathbb{C}$ and $\xi, \eta \in V_\sigma$. The dual representation $\sigma^* \colon G \to U(V_\sigma^*)$ of σ is now given by $\sigma^*(g)\xi^* = (\sigma(g)\xi)^*$ for all $g \in G$ and $\xi \in V_\sigma$.

1.3 C^*-algebras and their representations

We now introduce some material on representations of C^*-algebras and the topologies of the spaces $\mathrm{Rep}(A)$ and $\widehat{A}$. For precise definitions and proofs the reader may consult, for instance, [Dix77] or [Mur90]. Some of the terms and notations of Section 1.2 on page 7, like equivalence of representations or subrepresentations, can be transferred directly to C^*-algebras, and we do so without further notice.

The spaces $\mathrm{Rep}(A)$ and $\widehat{A}$

In the following let A be a C^*-algebra. A representation of A is a *-homomorphism $A \to B(V)$ for some Hilbert space V. We denote by $\mathrm{Rep}(A)$ the set of all unitary equivalence classes of nondegenerate *-representations, and by $\widehat{A}$ the spectrum of A, i.e., the subset of all unitary equivalence classes of irreducible *-representations of A.

To define topologies on these sets, we first introduce topologies on the sets $\mathcal{I}(A)$ of closed two-sided ideals in A, and on $\mathrm{Prim}(A) := \{\ker \pi \mid \pi \in \widehat{A}\}$, the primitive ideal space of A.

For every $I \in \mathcal{I}(A)$ let $U(I) := \{J \in \mathcal{I}(A) \mid J \setminus I \neq \emptyset\}$, then $\{U(I) \mid I \in \mathcal{I}(A)\}$ forms a sub-basis for the Fell topology on $\mathcal{I}(A)$. The Fell topology on $\mathrm{Rep}(A)$ is defined as the initial topology via the map $\mathrm{Rep}(A) \to \mathcal{I}(A)$, $\pi \mapsto \ker \pi$. Restriction to $\mathrm{Prim}(A)$ respectively $\widehat{A}$ yields the usual Jacobson topology. Recall that the Jacobson topology can also be defined directly via the closure operation given by

$$\overline{R} := \left\{ J \in \mathrm{Prim}(A) \mid \bigcap_{I \in R} I \subseteq J \right\}$$

for every $R \subseteq \mathrm{Prim}(A)$. This leads us to the concept of weak containment, which will give a convenient description of convergence with respect to the Fell topology:

Definition 1.3.1. An element $\pi \in \mathrm{Rep}(A)$ is said to be weakly contained in a subset R of $\mathrm{Rep}(A)$ if

$$\bigcap_{\varrho \in R} \ker \varrho \subseteq \ker \pi.$$

This is denoted by $\pi \prec R$. If $R = \{\varrho\}$, we simply write $\pi \prec \varrho$. Two subsets R, S of $\mathrm{Rep}(A)$ are said to be weakly equivalent if $\varrho \prec S$ for all $\varrho \in R$ and $\sigma \prec R$ for all $\sigma \in S$.

The following two results are Propositions 1.2 and 1.3 in [Fel64], and they characterize convergence in $\mathrm{Rep}(A)$ in terms of weak containment:

Proposition 1.3.2. Suppose that A is a C^*-algebra, that $(\pi_\nu)_{\nu \in N}$ is a net in $\mathrm{Rep}(A)$ and that $\pi \in \mathrm{Rep}(A)$.

(i) Then $\pi_\nu \to \pi$ in the Fell topology if and only if π is weakly contained in every subnet of $(\pi_\nu)_{\nu \in N}$.

(ii) If $\pi_\nu \to \pi$, then $\pi_\nu \to \varrho$ for every $\varrho \in \mathrm{Rep}(A)$ which is weakly contained in π.

It is useful to keep the following fact in mind, a proof of which can, for instance, be found in Paragraph 3.3.8 of [Dix77].

Remark 1.3.3. For every C^*-algebra A the spectrum $\widehat{A}$ is a locally compact (not necessarily Hausdorff) space.

The spaces $\mathrm{Rep}(G)$ and $\widehat{G}$

If G is a locally compact group, then the unitary representations of G correspond to the non-degenerate representations of the group C^*-algebra $C^*(G)$ (see page 12), and so $\mathrm{Rep}(G)$ and $\widehat{G}$ can be identified with $\mathrm{Rep}(C^*(G))$ and $C^*(G)\widehat{\ }$, respectively. Hence, the preceding concepts of the Fell topology, the Jacobson topology, and weak containment all carry over to the group situation.

In case of a compact group there is a nice criterion for convergence of a net of representations to an irreducible representation. It is taken from [EE], but since it is very important to some of our work, we will also present the proof in the end of Chapter 4.

Lemma 1.3.4 (Lemma 4.8 in [EE])**.** Let K be a compact group, let $(\pi_\nu)_{\nu \in N}$ be a net in $\mathrm{Rep}(K)$, and let $\sigma \in \widehat{K}$. The following statements are equivalent:

(i) $\pi_\nu \to \sigma$;

(ii) There exists an index $\nu_0 \in N$ such that $\sigma \leq \pi_\nu$ for all $\nu \in N_{\geq \nu_0}$.

Weak containment in terms of positive functionals

A very useful characterization of weak containment for representations of C^*-algebras can be obtained using positive functionals which are associated with *-representations. Analogous results can be obtained for group representations (without passing to the group C^*-algebra) using functions of positive type, but we will only need and discuss these in the context of subgroup representations, see Section 2.2. The following material is based on [Fel60].

Definition 1.3.5. Let A be a C^*-algebra. Then a positive functional φ is said to be associated with an element $\pi \in \mathrm{Rep}(A)$ if there exists an element $\xi \in V_\pi$ such that

$$\varphi(a) = \langle \pi(a)\xi, \xi \rangle$$

for all $a \in A$. If $R \subseteq \mathrm{Rep}(A)$, then φ is associated with R if there exists $\varrho \in R$ such that φ is associated with ϱ.

Remark 1.3.6. If φ, π, and ξ are as in the previous definition, then $\|\varphi\| = \|\xi\|^2$, because if $(u_\nu)_{\nu \in N}$ is an approximate unit for A, then

$$\|\varphi\| = \lim_\nu \varphi(u_\nu) = \lim_\nu \langle \pi(u_\nu)\xi, \xi \rangle = \langle \xi, \xi \rangle = \|\xi\|^2.$$

The following result is a special case of Fell's Equivalence Theorem for convergence of representations, which is Theorem 1.2 in [Fel60].

Proposition 1.3.7 (Lemma 2.2 in [Fel62b])**.** Let A be a C^*-algebra, let $B \subseteq \mathrm{Rep}(A)$ and let $\pi \in \widehat{A}$. Then the following conditions are equivalent:

(i) π is weakly contained in B;

(ii) There exists a nonzero positive functional associated with π which is a weak* limit of finite linear combinations of positive functionals associated with B;

(iii) Every nonzero positive functional φ associated with π is a weak* limit of a net of positive functionals $(\varphi_\nu)_{\nu \in N}$ associated with B such that $\|\varphi_\nu\| \leq \|\varphi\|$ for all $\nu \in N$.

In case that π is a general element of $\mathrm{Rep}(A)$, one has to work with "a net of finite sums of positive functionals" in (iii).

1.4 Crossed products and generalized fixed point algebras

We give a short account of some aspects of covariant systems, in particular of transformation group C^*-algebras, which we need in this work. Background material can be found in [Ech] or [Wil07]. We also discuss generalized fixed point algebras and some of their properties; this treatment is based on [EE].

Covariant systems, covariant representations, and crossed products

A covariant system (A, G, α) consists of a C^*-algebra A, a locally compact group G, and a strongly continuous homomorphism $\alpha\colon G \to \mathrm{Aut}(A)$, i.e., the map $g \mapsto \alpha_g(a)$ is continuous for every $a \in A$. A covariant representation of (A, G, α) is a pair (ϱ, U), where $\varrho\colon A \to B(V)$ is a *-representation and U is a unitary representation of G on the same Hilbert space, such that

$$\varrho(\alpha_s(a)) = U_s \pi(a) U_s^{-1}$$

for all $s \in G$ and $a \in A$. We say that (ϱ, U) is nondegenerate if ϱ is nondegenerate, and denote the set of all nondegenerate covariant representations of (A, G, α) by $\mathrm{Rep}(A, G)$.

For the construction of the crossed product $A \rtimes_\alpha G$ we start with $C_c(G, A)$ and turn this space into a *-algebra by defining convolution and involution by

$$f * g(s) := \int_G f(t)\alpha_t(g(t^{-1}s))d\mu(t), \qquad f^*(s) := \Delta(s^{-1})\alpha_s(f(s^{-1}))^*$$

for all $f, g \in C_c(G, A)$ and all $s \in G$. If now (ϱ, U) is a covariant representation of (A, G, α) on some Hilbert space V, then we can construct a *-representation of $C_c(G, A)$ on the same Hilbert space by defining the integrated form

$$\varrho \rtimes U(f) := \int_G \varrho(f(s)) U_s d\mu(s)$$

for every $f \in C_c(G, A)$.

Since every *-representation $\varrho\colon A \to B(V_\varrho)$ induces a covariant representation of (A, G, α) on the Hilbert space $L^2(G, V_\varrho)$, we are not talking about the empty set here, and can define

$$\|f\| := \sup\{\|\varrho \rtimes U(f)\| \mid (\varrho, U) \text{ is a covariant representation of } (A, G, \alpha)\} \qquad (1.4.1)$$

for every $f \in C_c(G, A)$. This defines a norm on $C_c(G, A)$, which is called the universal norm. The completion of $C_c(G, A)$ with respect to this norm is a C^*-algebra, which is called the crossed product of A by G and is denoted by $A \rtimes_\alpha G$. It is a useful fact that the crossed product $A \rtimes_\alpha G$ contains $C_c(G, A)$ as a dense subset.

Another possibility to obtain the crossed product is to define

$$\|f\|_1 := \int_G \|f(s)\| d\mu(s)$$

for every $f \in C_c(G, A)$. This also is a norm on $C_c(G, A)$, and the crossed product is the enveloping C^*-algebra of the completion $L_1(G, A)$ of $C_c(G, A)$ with respect to $\|\cdot\|_1$.

Using the first definition, it follows directly from (1.4.1) that every integrated form $\varrho \rtimes U$ of a covariant representation (ϱ, U) can be extended to a *-representation of $A \rtimes_\alpha G$.

For nondegenerate *-representations, there is also a converse to this statement which uses the canonical inclusions ι_A and ι_G of A and G into the multiplier algebra $M(A \rtimes_\alpha G)$ given by

$$\begin{aligned} (\iota_A(a)f)(s) &= af(s), & (f\iota_A(a))(s) &= f(s)\alpha_s(a) \\ (\iota_G(t)f)(s) &= \alpha_t(f(t^{-1}s)), & (f\iota_G(t))(s) &= \Delta(t^{-1})f(st^{-1}) \end{aligned} \tag{1.4.2}$$

for all $f \in C_c(G, A)$, $a \in A$, and $s, t \in G$. These formulas extend to $A \rtimes_\alpha G$.

If now Φ is a nondegenerate *-representation of $A \rtimes_\alpha G$, then ι_A and ι_G can be used to induce the representations

$$\varrho := \Phi \circ \iota_A, \qquad U := \Phi \circ \iota_G$$

of A and G, respectively. A calculation shows that (ϱ, U) is then a nondegenerate covariant representation of (A, G, α) and that $\Phi = \varrho \rtimes U$.

Remark 1.4.1. Since this is a special case of induction via bimodules as described in Section 3.3 below, the maps

$$\operatorname{Rep}(A \rtimes_\alpha G) \to \operatorname{Rep}(A),\ \varrho \rtimes U \mapsto \varrho$$

and

$$\operatorname{Rep}(A \rtimes_\alpha G) \to \operatorname{Rep}(G),\ \varrho \rtimes U \mapsto U$$

are continuous.

The group C^*-algebra $C^*(G)$ of a locally compact group G

To every locally compact group G we can associate the covariant system $(\mathbb{C}, G, \mathrm{id})$, and the resulting crossed product $\mathbb{C} \rtimes_{\mathrm{id}} G$ is nothing but the group C^*-algebra $C^*(G)$. The covariant representations of $(\mathbb{C}, G, \mathrm{id})$ correspond to the unitary representations of G, and the integrated form of a unitary representation U of G is just given by

$$\mathrm{id} \rtimes U(f) = \int_G f(s) U_s d\mu(s)$$

for every $f \in C_c(G)$.

Transformation group C^*-algebras

Transformation group C^*-algebras constitute an important class of examples for crossed products, and in the special case that they arise from a proper G-space X, they are our main object of study in this work.

Suppose now that X is a G-space. Then the action of G on X gives rise to an action of G on $C_0(X)$ via left translation defined by $s \cdot f(t) := f(s^{-1}t)$, and we can form the crossed product $C_0(X) \rtimes_{\mathrm{lt}} G$. Since we won't consider any other action on $C_0(X)$ we omit the lt and simply write $C_0(X) \rtimes G$.

It is shown in Section 2.3 in [Wil07] that $C_c(G \times X)$ is a dense *-subalgebra of $C_c(G, C_0(X))$. The formulas for convolution and involution then read

$$\begin{aligned} f * g(s,x) &= \int_G f(t,x) g(t^{-1}s, t^{-1}x) d\mu(t), \\ f^*(s,x) &= \Delta(s^{-1}) \overline{f(s^{-1}, s^{-1}x)} \end{aligned}$$

for all $f \in C_c(G \times X)$, $s \in G$, and $x \in X$. In case of a proper G-space X it is well known, and we mention it in Example A.1.1, that $C_0(X) \rtimes G$ is a $C_0(G \backslash X)$-algebra with fibre over each Gx given by $C_0(Gx) \rtimes G$. In the following we show how this fact can be obtained from a very general construction of $C_0(G \backslash X)$-algebras.

Generalized fixed point algebras

The material of this and the following section is taken from [EE], where one can also find references to older and partly more general treatments of generalized fixed point algebras. These algebras are the basis of our description of $\mathrm{Stab}(X)^\frown$ as the spectrum of a C^*-algebra, and they are also essential for Echterhoff and Emerson's work on proper G-spaces with Palais' slice property.

Assume that X is a proper G-space and that G acts on a C^*-algebra B with an action $\beta \colon G \to \mathrm{Aut}(B)$. If the set

$$\{F \in C_b(X, B) \mid \forall g \in G,\, x \in X : \, F(gx) = \beta_g(F(x)) \text{ and } Gx \mapsto \|F(x)\| \in C_0(G \backslash X)\}$$

is equipped with pointwise addition, multiplication, involution, and the supremum norm, it becomes a C^*-algebra which is denoted by $C_0(X \times_G B)$.

Lemma 1.4.2 (Lemma 2.2 in [EE])**.** The C^*-algebra $C_0(X \times_G B)$ is a $C_0(G \backslash X)$-algebra, where the fibre over each orbit Gx is given by the fixed point algebra B^{G_x}.

In the proof it is shown that $C_0(X \times_G B)$ is a $C_0(G \backslash X)$-algebra with fibre over Gx given by $C_0(Gx \times_G B)$, and that such a fibre is isomorphic to B^{G_x} via the evaluation map $F \mapsto F(x)$.

To use this for a description of $C_0(X) \rtimes G$ we consider the following situation: Let a locally compact group G act on itself by left translation and equip $\mathcal{K} := \mathcal{K}(L^2(G))$ with the G-action given by $\mathrm{Ad}(\varrho) \colon G \to \mathrm{Aut}(\mathcal{K})$, where ϱ is the right regular representation from Example 1.2. Then the crossed product $C_0(G) \rtimes_{\mathrm{lt}} G$ can be described as follows:

Remark 1.4.3. The extended version of the Stone-von Neumann Theorem (see Theorem C.34 in [RW98]) yields an isomorphism

$$M \rtimes \lambda^G \colon C_0(G) \rtimes_{\mathrm{lt}} G \to \mathcal{K},$$

where M is the representation by multiplication operators. Since the right translation action rt of G on $C_0(G)$ commutes with lt, it induces an action $\widetilde{\mathrm{rt}}$ of G on $C_0(G) \rtimes_{\mathrm{lt}} G$ which is given by $\widetilde{\mathrm{rt}}_g(\varphi)(h) = \mathrm{rt}_g(\varphi(h))$ for all $\varphi \in C_c(G, C_0(G))$, $g, h \in G$. A calculation shows that

$$M \rtimes \lambda^G(\widetilde{\mathrm{rt}}_g(\varphi)) = \varrho^G(g) \left(M \rtimes \lambda^G(\varphi)\right) \varrho^G(g^{-1})$$

for all $g \in G$ and $\varphi \in C_0(G) \rtimes_{\mathrm{lt}} G$. If K is a compact subgroup of G, then it follows that an element $\varphi \in C_0(G) \rtimes_{\mathrm{lt}} G \cong \mathcal{K}$ is contained in the fixed point algebra $\mathcal{K}^K$ if and only if $\varphi \in C_0(G/K) \rtimes G$, so we obtain an isomorphism

$$C_0(G/K) \rtimes G \cong \mathcal{K}^K, \tag{1.4.3}$$

which will be used in the explicit description of $\mathcal{K}^K$ in Lemma 7.3.1. See [EE] for details.

If now X is a proper G-space and the C^*-algebra B from Lemma 1.4.2 and the text preceding it is replaced by $\mathcal{K}$ with the G-action given by $\mathrm{Ad}(\varrho)$, then Echterhoff and Emerson prove that

$$C_0(X) \rtimes G \cong C_0(X \times_{G,\mathrm{Ad}\varrho} \mathcal{K}). \tag{1.4.4}$$

Combined with Lemma 1.4.2 this yields that $C_0(X) \rtimes G$ is a $C_0(G\backslash X)$-algebra with fibre at an orbit Gx given by $C_0(Gx) \rtimes G \cong \mathcal{K}^{G_x}$. Echterhoff and Emerson also give a concrete description of these fibres, see Lemma 7.3.1.

We will consider a special case where there is an easier description of the generalized fixed point algebras $C_0(X \times_G B)$:

Topological fundamental domains

For reasons of compatibility with later usage of this material, it is for the moment inconvenient to call our G-space X. So we name it Y for now.

Definition 1.4.4. Let Y be a (not necessarily locally compact) G-space. A closed subspace $Z \subseteq Y$ is said to be a topological fundamental domain for Y if the map $Z \to G\backslash Y,\ z \mapsto Gz$ is a homeomorphism.

Echterhoff and Emerson prove the following nice consequence of Lemma 1.4.2:

Proposition 1.4.5 (Proposition 2.7 in [EE])**.** If $Z \subseteq Y$ is a topological fundamental domain for a proper G-space Y, and if B is any G-algebra, then there is an isomorphism

$$C_0(Y \times_G B) \cong \{f \in C_0(Z, B) \mid \forall z \in Z : f(z) \in B^{G_z}\}$$

given by $F \mapsto F|_Z$.

We are of course interested in the following special case, where G acts on $\mathcal{K} = \mathcal{K}(L^2(G))$ via $\mathrm{Ad}\varrho$ as above.

Corollary 1.4.6. Suppose again that Z is a topological fundamental domain for a proper G-space Y. Then

$$C_0(Y) \rtimes G \cong C_0(Y \times_{G,\mathrm{Ad}\varrho} \mathcal{K}) \cong \{f \in C_0(Z, \mathcal{K}) \mid \forall z \in Z : f(z) \in \mathcal{K}^{G_z}\}. \tag{1.4.5}$$

Proof. The first isomorphism is as in (1.4.4), the second one comes from Proposition 1.4.5 above. □

1.5 Induced representations of covariant systems

The concept of induced representations of covariant systems is of central importance in this work because it gives rise to the bijection ind^G between our cataloguing space $G\backslash\operatorname{Stab}(X)\widehat{\ }$ and the space we want to analyze, namely $(C_0(X) \rtimes G)\widehat{\ }$.

Suppose that (A, G, α) is a covariant system and that H is a closed subgroup of G. There exists a right-Hilbert $A \rtimes_\alpha G$–$A \rtimes_{\alpha|_H} H$ bimodule (see, for instance, Appendix B.1 of [EKQR06]). Using the methods of Section 3.3 on induction via bimodules, this can be used to induce representations of $A \rtimes_{\alpha|_H} H$ up to representations of $A \rtimes_\alpha G$. In our situation it will be more useful to have a concrete realization of induced representations of crossed products, as given by Blattner in [Bla61]. To avoid technicalities, we restrict our treatment to the special cases we actually need, although most of the statements below are also true in more general situations.

Nice treatments of this material which deal both with the bimodule approach and with Blattner's realization, are given in [Ech] and [Wil07]. The case of group representations is also dealt with in [Fol95].

Proposition/Definition 1.5.1. Let (A, G, α) be a covariant system, let H be a compact subgroup of G, and let (ϱ, U) be a covariant representation of $(A, H, \alpha|_H)$ on a Hilbert space V_U. Then

$$H_{\operatorname{ind} U} := \left\{\xi \in L^2(G, V_U) \mid \forall g \in G, h \in H :\ \xi(gh) = U_{h^{-1}}\xi(g)\right\}$$

defines a Hilbert space with inner product given by

$$\langle \xi_1, \xi_2\rangle := \int_G \langle \xi_1(s), \xi_2(s)\rangle_{V_U} ds \qquad \text{for all } \xi_1, \xi_2 \in H_{\operatorname{ind} U}. \tag{1.5.1}$$

On this space we can define a covariant representation $(\pi, \operatorname{ind}_H^G U)$ of (A, G, α) by

$$\begin{aligned}(\pi(a)\xi)(g) &:= \varrho(\alpha_{g^{-1}}(a))\xi(g) \qquad \text{for all } a \in A,\ \xi \in H_{\operatorname{ind} U},\ g \in G,\\ ((\operatorname{ind}_H^G U)_g)\xi(s) &:= \xi(g^{-1}s) \qquad \text{for all } g, s \in G,\ \xi \in H_{\operatorname{ind} U}.\end{aligned}$$

We often denote the representation $\pi \rtimes \operatorname{ind}_H^G U$ by $\operatorname{ind}_H^G(\varrho \rtimes U)$.

This construction is a special case of induction of covariant representations from a closed (not necessarily compact) subgroup H of G. In that more general situation one has to work in some modular functions and has to make some precautions to ensure that the integral in (1.5.1) exists. For us, the compact case will be sufficient.

We continue with the presentation of several important and useful results on induced representations.

Proposition 1.5.2. Induction of representations preserves unitary equivalence, direct sums, and containment of representations.

The following result on induction and containment of irreducible representations of compact groups will be used many times in this work and plays a crucial role both in the proof that our definition of convergence in $\operatorname{Stab}(X)\widehat{\ }$ is valid, and also in the proof of openness of ind^G.

Theorem 1.5.3 (Frobenius Reciprocity Theorem)**.** Let K be a compact group, let H be a closed subgroup of K, let σ be an irreducible representation of K, and let τ be an irreducible representation of H. Then $\tau \leq \sigma|_H$ if and only if $\sigma \leq \operatorname{ind}_H^K \tau$. More precisely, the multiplicity of τ in $\sigma|_H$ equals the multiplicity of σ in $\operatorname{ind}_H^K \tau$.

The following theorem says that induction can be performed in "steps" or "stages".

Theorem 1.5.4. Let (A, G, α) be a covariant system, let $L \leq H \leq G$ be compact subgroups, and let $\varrho \rtimes U$ be a representation of $A \rtimes L$. Then

$$\operatorname{ind}_L^G(\varrho \rtimes U) \approx \operatorname{ind}_H^G(\operatorname{ind}_L^H(\varrho \rtimes U)).$$

The following statement is an immediate consequence of the fact that induction via bimodules is continuous (see [Ech]).

Theorem 1.5.5. Let (A, G, α) be a covariant system and let $H \leq G$ be a compact subgroup. Then the map

$$\operatorname{Rep}(A \rtimes H) \to \operatorname{Rep}(A \rtimes G),\ \varrho \rtimes U \mapsto \operatorname{ind}_H^G(\varrho \rtimes U)$$

is continuous with respect to the Fell topologies.

Since we will have to deal with varying subgroups, we need a refined version of this, see Proposition 2.2.5 in the section on subgroup representations. We close this section with some examples which we will need later on. For more general examples the reader may consult the literature.

Examples 1.5.6.

(i) In case of a trivial covariant system $(\mathbb{C}, G, \mathrm{id})$, the construction from Proposition/Definition 1.5.1 reduces to the case of induced group representations.

(ii) If G is a locally compact group acting properly on a locally compact space X, then, for every $x \in X$, every compact subgroup H of G_x, and every unitary representation σ_H of H, induction of the representation $\mathrm{ev}_x \rtimes \sigma_H$ of $C_0(X) \rtimes H$ yields the representation

$$\pi^{x,\sigma_H} := \operatorname{ind}_H^G(\mathrm{ev}_x \rtimes \sigma_H) = P^x \rtimes \operatorname{ind}_H^G \sigma_H$$

of $C_0(X) \rtimes G$, where P^x is given by

$$P^x(\varphi)\xi(g) = \mathrm{ev}_x(g^{-1} \cdot \varphi)\xi(g) = \varphi(gx)\xi(g)$$

for all $\varphi \in C_0(X)$, $\xi \in H_{\operatorname{ind}\sigma_H}$, and $g \in G$. If we only induce up to G_x, then it follows from the definition of P^x that

$$P^x \rtimes \operatorname{ind}_H^{G_x} \sigma_H = \mathrm{ev}_x \rtimes \operatorname{ind}_H^{G_x} \sigma_H.$$

Notice that we write ev_x not only for the point evaluation representation of $C_0(X)$ on $\mathbb{C}$, but also if ev_x acts as multiples of the identity on any Hilbert space.

(iii) Suppose again that X is a proper G-space, that $x \in X$ and that H is a closed subgroup of G_x. Let $\sigma_H \in \widehat{H}$ and $\sigma \in \widehat{G_x}$ be such that $\sigma_H \leq \sigma|_H$. By Frobenius reciprocity theorem, this implies that $\sigma \leq \operatorname{ind}_H^{G_x} \sigma_H$. If we only induce up to G_x, we have

$$P^x(\varphi)\xi = \mathrm{ev}_x(\varphi)\xi = \varphi(x)\xi$$

for all $\varphi \in C_0(X)$ and all $\xi \in H_{\operatorname{ind}\sigma_H}$. This shows that P^x leaves subspaces of $H_{\operatorname{ind}\sigma_H}$ invariant, and thus $\sigma \leq \operatorname{ind}_H^{G_x} \sigma_H$ implies that

$$\mathrm{ev}_x \rtimes \sigma \leq \mathrm{ev}_x \rtimes \operatorname{ind}_H^{G_x} \sigma_H.$$

We will use this observation in the proof of Proposition 2.4.2.

Chapter 2

The space $\operatorname{Stab}(X)\widehat{\ }$ and continuity of ind^G

This chapter begins with a short motivation why we consider the space $\operatorname{Stab}(X)\widehat{\ }$ at all and a short overview of several ways to obtain a topology on it. In Section 2.2 we introduce the concepts of subgroup C^*-algebras and their representation spaces, which in Section 2.3 will be used to define a topology and a G-action on $\operatorname{Stab}(X)\widehat{\ }$. That section closes with a short discussion of the Mackey-Rieffel-Green theorem which states that induction of representations yields a bijective map ind^G between the orbit space $G\backslash\operatorname{Stab}(X)\widehat{\ }$ and $(C_0(X)\rtimes G)\widehat{\ }$. To show that there is hope that the map ind^G becomes a homeomorphism with respect to the quotient topology on $G\backslash\operatorname{Stab}(X)\widehat{\ }$ and the Jacobson topology on $(C_0(X)\rtimes G)\widehat{\ }$, we give in Section 2.4 the proof of continuity of ind^G. To see that ind^G is also open we will have to work much more, which we will do in the following chapters.

Section 2.5 contains some statements on $\operatorname{Stab}(X)\widehat{\ }$ in case that the stabilizer map $x \mapsto G_x$ is continuous.

2.1 The space $\operatorname{Stab}(X)\widehat{\ }$

Why $\operatorname{Stab}(X)\widehat{\ }$?

Suppose that X is a proper G-space. It is our aim to find a description for the spectrum of $C_0(X)\rtimes G$ and its Jacobson topology. A classical theorem based on results by Mackey, Rieffel, and Green states that, if $\mathcal{S}$ is a section for the orbit space $G\backslash X$, then there exists a bijection

$$\bigcup_{x\in\mathcal{S}} \widehat{G_x} \to (C_0(X)\rtimes G)\widehat{\ },\ \sigma \mapsto \operatorname{ind}_{G_x}^G(\operatorname{ev}_x \rtimes \sigma) = \pi^{x,\sigma}, \tag{2.1.1}$$

see Example 1.5.6 (ii) for the notation and, for instance, [Ech] for a proof. This motivates us to make the following definition:

Definition 2.1.1. Let X be a proper G-space. Define

$$\operatorname{Stab}(X)\widehat{\ } := \{(x, G_x, \sigma) \mid x \in X,\ \sigma \in \widehat{G_x}\}.$$

There is some redundancy in writing down the stabilizer subgroup G_x here, but this notation will be useful when we view $\operatorname{Stab}(X)\widehat{\ }$ as a subset of $X \times S(G)\widehat{\ }$, where $S(G)\widehat{\ }$ is the space of irreducible subgroup representations from Definition 2.2.4 below.

Notice that the definition of $\mathrm{Stab}(X)^\frown$ could be made for general locally compact G-spaces X, but since all our applications need the assumption of properness, we restrict ourselves to the treatment of proper G-actions.

To get closer to the domain of the map in (2.1.1), we define an action of G on $\mathrm{Stab}(X)^\frown$ by $g(x, G_x, \sigma) = (gx, G_{gx}, g\sigma)$ for all $g \in G$ and $(x, G_x, \sigma) \in \mathrm{Stab}(X)^\frown$, where $g\sigma$ is given by $g\sigma(h) := \sigma(g^{-1}hg)$ for all $h \in G_{gx} = gG_xg^{-1}$. At this point, this is not really a G-action in our sense because we lack a topology on $\mathrm{Stab}(X)^\frown$, but this will be mended in Section 2.3, where this material will be discussed in detail. We denote the class of an element $(x, G_x, \sigma) \in \mathrm{Stab}(X)^\frown$ in the orbit space $G\backslash\mathrm{Stab}(X)^\frown$ by $[x, G_x, \sigma]$.

Combining this with the map in (2.1.1), we obtain a bijection

$$G\backslash\mathrm{Stab}(X)^\frown \to (C_0(X) \rtimes G)^\frown,\ [x, G_x, \sigma] \to \pi^{x,\sigma},$$

which we denote by ind^G.

Some ideas for defining a topology on $\mathrm{Stab}(X)^\frown$

Of course, the knowledge of the bijection ind^G motivates us to define a topology on $\mathrm{Stab}(X)^\frown$ in such a way that the quotient topology on $G\backslash\mathrm{Stab}(X)^\frown$ coincides with the topology induced by ind^G from the Jacobson topology on $(C_0(X) \rtimes G)^\frown$. As mentioned before there are several ways to do this. The most elegant is certainly the following approach, the idea of which was communicated to me by Siegfried Echterhoff.

Let X be a proper G-space and suppose that G acts on $\mathcal{K} := \mathcal{K}(L^2(G))$ via $\mathrm{Ad}(\varrho)$ as in the text preceding Remark 1.4.3. Then define

$$A := \{f \in C_0(X, \mathcal{K}) \mid \forall x \in X : f(x) \in \mathcal{K}^{G_x}\}. \tag{2.1.2}$$

We will show in Chapter 6 that A is a $C_0(X)$-algebra with fibre $\mathcal{K}^{G_x}$ at every $x \in X$. Using Proposition A.1.3 on the structure of the spectrum of a $C_0(X)$-algebra and Echterhoff and Emerson's decomposition of the fibres $\mathcal{K}^{G_x}$, which we present in Lemma 7.3.1, we obtain

$$\widehat{A} = \coprod_{x\in X} (\mathcal{K}^{G_x})^\frown = \coprod_{x\in X} \bigoplus_{\sigma\in\widehat{G_x}} \mathcal{K}(H_{\mathrm{ind}\,\sigma})^\frown = \coprod_{\substack{x\in X \\ \sigma\in\widehat{G_x}}} \mathcal{K}(H_{\mathrm{ind}\,\sigma})^\frown = \coprod_{\substack{x\in X \\ \sigma\in\widehat{G_x}}} \{i_\sigma\}, \tag{2.1.3}$$

where i_σ is the unique irreducible representation of $\mathcal{K}(H_{\mathrm{ind}\,\sigma})$, namely the inclusion into $B(H_{\mathrm{ind}\,\sigma})$. It is clear that the right hand side equals $\mathrm{Stab}(X)^\frown$ as a set, so we can equip $\mathrm{Stab}(X)^\frown$ with a topology by transferring the Jacobson topology from $\widehat{A}$. This procedure shows that $\mathrm{Stab}(X)^\frown$ can be topologized in a very natural way.

However, to prove that $\mathrm{Stab}(X)^\frown$ carries a topology as desired, we will use the space $S(G)^\frown$ of irreducible subgroup representations of G equipped with the Fell topology, as indicated in the introduction. The preliminaries for the construction of this topology and its definition are given in the following sections. In Chapter 6 we will prove that this topology coincides with the topology obtained via $\widehat{A}$ as above.

Echterhoff and Emerson show in [EE] that, under the assumption of Palais' slice property, one can define a topology on $\mathrm{Stab}(X)^\frown$ in terms of open neighborhood bases. We present their approach in Chapter 7 and also show that their and our topology are the same if Palais' slice property holds.

2.2 Subgroup C^*-algebras and subgroup representations

In this section we give a short account of the concept of the subgroup C^*-algebra $C^*(S(G), A)$ associated to a covariant system (A, G, α), its representation space $\operatorname{Rep}(S(G), A)$, and of a useful description of the topology on $\operatorname{Rep}(S(G), A)$. This is the stuff that our topology on $\operatorname{Stab}(X)^\frown$ will be made of. First, we need a topology on the set of closed subgroups of a given locally compact group.

The space $\mathscr{K}(G)$

The space $\mathscr{K}(G)$ of closed subgroups of a given locally compact group G can be equipped with the Fell topology, which can be more generally defined on the set of closed subsets of a topological space X. Fell introduced this topology in [Fel62a].

Let X be any topological space and let $\mathscr{C}(X)$ denote the set of all closed subsets of X. For every finite collection $\mathcal{F}$ of open subsets of X and for every compact $C \subseteq X$ define

$$U(\mathcal{F}, C) := \{K \in \mathscr{C}(X) \mid \forall F \in \mathcal{F} : F \cap K \neq \emptyset \text{ and } C \cap K = \emptyset\}.$$

These sets form a basis for the so-called compact-open topology or Fell topology on $\mathscr{C}(X)$, which turns $\mathscr{C}(X)$ into a compact (not necessarily Hausdorff) space.

If X is locally compact there is the following nice description of convergence in $\mathscr{C}(X)$:

Lemma 2.2.1 (Lemma H.2 in [Wil07])**.** Let X be a locally compact (not necessarily Hausdorff) space. A net $(K_\nu)_{\nu \in N}$ in $\mathscr{C}(X)$ converges to $K \in \mathscr{C}(X)$ if and only if the following conditions hold:

(i) if $k_\nu \in K_\nu$ and $k \in X$ satisfy $k_\nu \to k$, then $k \in K$;

(ii) for every $k \in K$ there are a subnet $(K_j)_{j \in J}$ and elements $k_j \in K_j$ such that $k_j \to k$ in X.

Combined with the above, it now follows that $\mathscr{C}(X)$ is a compact Hausdorff space whenever X is a locally compact (not necessarily Hausdorff) space. If we consider the situation of closed subgroups or closed subspaces of locally compact groups or spaces, respectively, we get the following easy, but important result:

Corollary 2.2.2. Let G be a locally compact group and let V be a finite-dimensional, hence also locally compact $\mathbb{K}$-vector space. Then the spaces

$$\mathscr{K}(G) := \{K \in \mathscr{C}(G) \mid K \text{ is a closed subgroup of } G\}$$

and

$$\mathscr{K}(V) := \{W \in \mathscr{C}(V) \mid W \text{ is a subspace of } V\}$$

are closed in $\mathscr{C}(G)$ and $\mathscr{C}(V)$, respectively. In particular, $\mathscr{K}(G)$ and $\mathscr{K}(V)$ are compact Hausdorff spaces.

Proof. The proof is a direct application of Lemma 2.2.1. In the group case the details are given in Corollary H.4 in [Wil07]. We illustrate the idea in the vector space situation. Suppose that $(W_\nu)_{\nu \in N}$ is a net in $\mathscr{K}(V)$ which converges to an element $W \in \mathscr{C}(V)$. Then W is a closed subset of V, and it is left to show that W is a vector subspace of V. It already follows from the group case that W is a subgroup of V. Let now $w \in W$ and $\lambda \in \mathbb{K}$. By Lemma 2.2.1(ii), there exists a subnet $(W_j)_{j\in J}$ of $(W_\nu)_{\nu\in N}$ such that there are elements $w_j \in W_j$ for every $j \in J$ such that $w_j \to w$ in V. Since $\lambda w_j \in W_j$ for every $j \in J$ and $\lambda w_j \to \lambda w$ in V, it follows from Lemma 2.2.1(i) that $\lambda w \in W$. Thus, $W \in \mathscr{K}(V)$, and we have proven that $\mathscr{K}(V)$ is a closed subset of the compact Hausdorff space $\mathscr{C}(V)$, as required. □

The following application of the previous results to nets of stabilizer subgroups will be important when we define convergence in the space $\mathrm{Stab}(X)^\frown$ later on.

Lemma 2.2.3. Let G be a locally compact group that acts on a locally compact Hausdorff space X. Let $(x_\nu)_{\nu\in N}$ be a net in X which converges to an element $x \in X$. Then the net $(G_\nu)_{\nu\in N}$ of stabilizer subgroups has a subnet which converges to a subgroup of G_x.

Proof. Since the space $\mathscr{K}(G)$ is compact, we can, after passing to a subnet and relabeling, assume that $(G_\nu)_{\nu\in N}$ converges to an element H in $\mathscr{K}(G)$. We show that $H \le G_x$. Let $g \in H$. By Lemma 2.2.1(ii) we can, after passing to a subnet and relabeling again, assume that there are elements $g_\nu \in G_\nu$ for every $\nu \in N$ such that $g_\nu \to g$. Now, as $g_\nu x_\nu = x_\nu$ for all $\nu \in N$, the net $(g_\nu x_\nu)_{\nu\in N}$ converges both to gx and to x. Since X is Hausdorff, this implies that $gx = x$ and thus $g \in G_x$, as required. □

Subgroup C^*-algebras of covariant systems and subgroup representations

The following description of subgroup C^*-algebras and subgroup representations is based on Section 2 of [Ech92], but to keep this exposition short, we reduce the general treatment which is given there to the setting we need. The case $A = \mathbb{C}$ is also treated in [Fel64].

In the following let (A, G, α) be a covariant system. We start by defining the set

$$S(G) := \{(H, h) \in \mathscr{K}(G) \times G \mid h \in H\}.$$

It follows from Lemma 2.2.1 on convergence in $\mathscr{K}(G)$ that $S(G)$ is a closed subset of $\mathscr{K}(G)\times G$ and thus is a locally compact Hausdorff space. It is shown in [Ech92] that $C_c(S(G), A)$ becomes a normed *-algebra if convolution, involution, and the norm are pointwise given by the operations on $C_c(H, A)$ (for every $H \in \mathscr{K}(G)$) as in the definition of crossed products on page 11. Explicitly, this gives

$$\begin{aligned}
(f * g)(H, h) &= (f(H, \cdot) * g(H, \cdot\cdot))(h) = \int_H f(H, t)\alpha_t(g(H, t^{-1}h))dt, \\
f^*(H, h) &= f(H, \cdot)^*(h) = \Delta_H(h^{-1})\alpha_h(f(h^{-1})^*), \\
\|f\|_1 &= \sup_{L\in\mathscr{K}(G)} \|f(L, \cdot)\|_1 = \sup_{L\in\mathscr{K}(G)} \int_L \|f(L, l)\|_A dl
\end{aligned}$$

for all $f, g \in C_c(S(G), A)$ and for all $(H, h) \in S(G)$. Note that in the third line the symbol $\|\cdot\|_1$ stands both for the norm on $C_c(S(G), A)$ and the one on $C_c(L, A)$ for every $L \in \mathscr{K}(G)$. Let $L_1(S(G), A)$ denote the completion of $C_c(S(G), A)$ with respect to $\|\cdot\|_1$.

We now show that every pair (H, ϱ), with $H \in \mathscr{K}(G)$ and $\varrho \in \mathrm{Rep}(A, H)$, can be interpreted as a *-representation of $L_1(S(G), A)$. Let (H, ϱ) be such a pair. For every function $F \in C_c(S(G), A)$ define $F_H \in C_c(H, A)$ by $F_H(h) := F(H, h)$ for all $h \in H$. Then the map $F \mapsto F_H$ extends to a norm decreasing *-homomorphism of $L_1(S(G), A)$ onto a dense subalgebra of $L_1(H, A)$, and

$$\widetilde{\varrho}(F) := \varrho(F_H) \qquad \text{for all } F \in L_1(S(G), A)$$

defines the desired *-representation of $L_1(S(G), A)$.

Definition 2.2.4. The enveloping C^*-algebra of $L_1(S(G), A)$ is called the subgroup C^*-algebra of (A, G, α) and is denoted by $C^*(S(G), A)$.

The pairs (H, ϱ) with $H \in \mathscr{K}(G)$ and $\varrho \in \mathrm{Rep}(A, H)$ are called subgroup representations of $C^*(S(G), A)$. We denote the space of all subgroup representations of $C^*(S(G), A)$ by $\mathrm{Rep}(S(G), A)$ and give it the topology induced from $\mathrm{Rep}(C^*(S(G), A))$.

The subspace of all subgroup representations (H, ϱ) with $\varrho \in \widehat{H}$ will be denoted by $(S(G), A)\widehat{\ }$.

It is shown in Proposition 2 in [Ech92] that the irreducible representations of $C^*(S(G), A)$ all come from elements of $(S(G), A)\widehat{\ }$.

The topology on $\mathrm{Rep}(S(G), A)$ makes it possible to investigate the continuity of induction processes where all involved subgroups can vary. The following Proposition, which is part of Corollary 2 in [Ech92], will be essential in our proof that the map ind^G is continuous.

Proposition 2.2.5. Let (A, G, α) be a dynamical system. The map

$$\begin{aligned} \mathrm{ind}\colon \{(K, (H, \varrho)) \in \mathscr{K}(G) \times \mathrm{Rep}(S(G), A) \mid H \le K\} &\to \mathrm{Rep}(S(G), A), \\ (K, (H, \varrho)) &\mapsto (K, \mathrm{ind}_H^K \varrho) \end{aligned}$$

is continuous.

A description of the topology on $\mathrm{Rep}(S(G))$

In case of a trivial covariant system (i.e., if $A = \mathbb{C}$) the spaces $\mathrm{Rep}(S(G))$ and $S(G)\widehat{\ }$ consist of pairs (K, σ), where $K \in \mathcal{K}(G)$ and $\sigma \in \mathrm{Rep}(K)$ or $\sigma \in \widehat{K}$, respectively. In [Fel64], Fell characterizes the topology on $\mathrm{Rep}(S(G))$ in terms of a topology on the function space $\mathscr{F}(G) := \bigcup_{H \in \mathscr{K}(G)} C(H)$. We skip the formal definition of the Fell topology on $\mathscr{F}(G)$ and restrict ourselves to the following criterion for convergence:

Lemma 2.2.6 (Lemma 3.2 in [Fel64])**.** Let G be a locally compact group. Let $(f_\nu)_{\nu \in N}$ be a net in $\mathscr{F}(G)$ and let $f \in \mathscr{F}(G)$. Let H and H_ν be the domains of f and f_ν for every $\nu \in N$, respectively. Then $f_\nu \to f$ if and only if the following conditions hold:

(i) $H_\nu \to H$ in $\mathscr{K}(G)$;

(ii) for each subnet $(f_j)_{j \in J}$ of $(f_\nu)_{\nu \in N}$ and each choice of $x_j \in H_j$ and $x \in H$ with $x_j \to x$ it follows that $f_j(x_j) \to f(x)$.

A function $f \in \mathscr{F}(G)$ is said to be of positive type and associated with an element $(H, \sigma) \in \mathrm{Rep}(S(G))$ if the domain of f is H and if there exists a vector $\xi \in V_\sigma$ such that $f(h) = \langle \sigma(h)\xi, \xi \rangle$ for all $h \in H$.

Theorem 2.2.7 (Theorem 3.1.′ in [Fel64])**.** Let $(H_\nu, \sigma_\nu)_{\nu \in N}$ be a net in $\mathrm{Rep}(S(G))$ and let $(H, \sigma) \in \mathrm{Rep}(S(G))$. Then $(H_\nu, \sigma_\nu) \to (H, \sigma)$ if and only if the following holds: For every function φ of positive type associated with (H, σ) and each subnet of $(H_\nu, \sigma_\nu)_{\nu \in N}$ there exist a subnet $(H_j, \sigma_j)_{j \in J}$ of that subnet and for each $j \in J$ a finite sum φ_j of functions of positive type associated with (H_j, σ_j), such that $\varphi_j \to \varphi$ in $\mathscr{F}(G)$.

Remarks 2.2.8. (i) If (H,σ) is irreducible, then each φ_j can be chosen to be a function of positive type associated with (H_j,σ_j) instead of a finite sum of such functions.

(ii) If $H \leq G$ and $(\sigma_\nu)_{\nu\in N}$ and σ are representations of H, then $(H,\sigma_\nu) \to (H,\sigma)$ in $\mathrm{Rep}(S(G))$ if and only if $\sigma_\nu \to \sigma$ in $\mathrm{Rep}(H)$. This follows directly from Theorem 1.5 in [Fel60], which is basically a "constant group version" of Theorem 2.2.7 above.

(iii) Suppose that $(H_\nu,\sigma_\nu) \to (H,\sigma)$ in $\mathrm{Rep}(S(G))$. Then the above theorem combined with Proposition 1.3.2 (ii) implies that $(H_\nu,\sigma_\nu) \to (H,\varrho)$ for every $\varrho \in \mathrm{Rep}(H)$ which is weakly contained in σ.

Compact subspaces of $\mathrm{Rep}(S(G))$

When we define our topology on the space $\mathrm{Stab}(X)\widehat{\ }$ it will be very important to control subgroup representations which are in some sense dominated by a fixed subgroup representation. For this we introduce the following notation, which will be used all over this thesis: If $(L,\varrho),(H,\sigma) \in \mathrm{Rep}(S(G))$, then

$$(L,\varrho) \leq (H,\sigma) \;:\Longleftrightarrow\; L \leq H \text{ and } \varrho \leq \sigma|_L.$$

Proposition 2.2.9. Let G be a locally compact group. Let $(H,\sigma) \in \mathrm{Rep}(S(G))$ be such that σ is finite-dimensional. Then the space

$$\mathrm{Rep}(S(G))_{\leq(H,\sigma)} := \{(L,\varrho) \in \mathrm{Rep}(S(G)) \mid (L,\varrho) \leq (H,\sigma)\}$$

is compact (not necessarily Hausdorff) in $\mathrm{Rep}(S(G))$.

Proof. Let $(L_\nu,\varrho_\nu)_{\nu\in N}$ be a net in $\mathrm{Rep}(S(G))_{\leq(H,\sigma)}$. We show that this net has a convergent subnet. We know from Corollary 2.2.2 that $\mathscr{K}(G)$ is compact, so we can pass to a subnet and relabel to assume that there is $L \in \mathscr{K}(G)$ such that $L_\nu \to L$. It is clear from the description of convergence in $\mathscr{K}(G)$ from Lemma 2.2.1 that $L_\nu \leq H$ for all $\nu \in N$ implies that $L \leq H$.

For every $\nu \in N$, let V_ν denote the space of ϱ_ν. Since $\varrho_\nu \leq \sigma|_{L_\nu}$ for every $\nu \in N$, we can without loss of generality assume that each space V_ν is a $\sigma|_{L_\nu}$-invariant subspace of V_σ. Since V_σ is finite-dimensional, Corollary 2.2.2 tells us that $\mathscr{K}(V_\sigma)$ is compact. Hence, we can pass to a subnet and relabel to assume that there exists a closed subspace $V \leq V_\sigma$ with $V_\nu \to V$ in $\mathscr{K}(V_\sigma)$. It remains to show that V is $\sigma|_L$-invariant and that $(L_\nu,\varrho_\nu)_{\nu\in N}$ converges to $(L,(\sigma|_L)|_V)$ in $\mathrm{Rep}(S(G))$.

For $\sigma|_L$-invariance of V let $l \in L$ and $\xi \in V$. By the description of convergence in $\mathscr{K}(G)$ and $\mathscr{K}(V)$ in Lemma 2.2.1(ii) we can pass to a subnet and relabel twice to assume that there exist elements $l_\nu \in L_\nu$ and $\xi_\nu \in V_\nu$ for all $\nu \in N$ such that $l_\nu \to l$ in G and $\xi_\nu \to \xi$ in V_σ. Since

$$\begin{aligned}\|\sigma(l_\nu)\xi_\nu - \sigma(l)\xi\| &\leq \|\sigma(l_\nu)\xi_\nu - \sigma(l_\nu)\xi\| + \|\sigma(l_\nu)\xi - \sigma(l)\xi\| \\ &\leq \|\xi_\nu - \xi\| + \|\sigma(l_\nu)\xi - \sigma(l)\xi\| \end{aligned} \tag{2.2.1}$$

in V_σ for every ν it follows from $\xi_\nu \to \xi$ and from strong continuity of σ that $\sigma(l_\nu)\xi_\nu$ converges to $\sigma(l)\xi$ in V_σ. As $\sigma(l_\nu)\xi_\nu$ is contained in V_ν for every $\nu \in N$, it follows from Lemma 2.2.1(i) that $\sigma(l)\xi$ is contained in V, which shows that V is $\sigma|_L$-invariant.

It is clear that $(L,(\sigma|_L)|_V) \leq (H,\sigma)$, and to show that $(L_\nu,\varrho_\nu) \to (L,(\sigma|_L)|_V)$ we apply Fell's characterization of convergence in $\mathrm{Rep}(S(G))$ as given in Theorem 2.2.7. For this suppose that φ is a function of positive type associated with $(L,(\sigma|_L)|_V)$, and choose $\eta \in V$ such that $\varphi(k) = \langle\sigma(k)\eta,\eta\rangle$ for all $k \in L$. Suppose that we have passed to a subnet of $(L_\nu,\varrho_\nu)_{\nu\in N}$, and relabeled. As above, we apply the description of convergence in $\mathscr{K}(V_\sigma)$, so we can pass to a subnet and relabel again to assume that for every $\nu \in N$ there exists an element $\eta_\nu \in V_\nu$ such that $\eta_\nu \to \eta$ in V_σ. For every $\nu \in N$ set

$$\varphi_\nu\colon L_\nu \to \mathbb{C},\ k \mapsto \langle\varrho_\nu(k)\eta_\nu,\eta_\nu\rangle = \langle\sigma(k)\eta_\nu,\eta_\nu\rangle,$$

then every φ_ν is a function of positive type associated with (L_ν,ϱ_ν), and, since $\varrho_\nu \leq \sigma|_{L_\nu}$, it is also associated with $\sigma|_{L_\nu}$. We apply Lemma 2.2.6 to see that $\varphi_\nu \to \varphi$ in $\mathscr{F}(G)$, and since we already know that $L_\nu \to L$, it suffices to show (ii) of Lemma 2.2.6. Suppose that we have passed to a subnet and relabeled. Let $k \in L$ and $k_\nu \in L_\nu$ for all $\nu \in N$ be such that $k_\nu \to k$. We have

$$|\varphi(k) - \varphi_\nu(k_\nu)| = |\langle\sigma(k)\eta,\eta\rangle - \langle\sigma(k_\nu)\eta_\nu,\eta_\nu\rangle|.$$

It follows as in (2.2.1) that $\sigma(k_\nu)\eta_\nu \to \sigma(k)\eta$ in V_σ, so it follows from continuity of the scalar product that $\varphi_\nu(k_\nu) \to \varphi(k)$, as required. □

We will also need the following lemma on convergence in $S(K)^\frown$ for a compact group K:

Lemma 2.2.10. Let K be a compact group. Suppose that $(K_\nu,\sigma_\nu)_{\nu\in N}$, (K',σ'), $(H_\nu,\varrho_\nu)_{\nu\in N}$, and (H',ϱ') in $S(K)^\frown$ satisfy the following conditions:

(i) $(K_\nu,\sigma_\nu) \to (K',\sigma')$,

(ii) $(H_\nu,\varrho_\nu) \to (H',\varrho')$,

(iii) $(H_\nu,\varrho_\nu) \leq (K_\nu,\sigma_\nu)$ for all $\nu \in N$.

Then $(H',\varrho') \leq (K',\sigma')$.

The proof, the idea of which was suggested to me by Siegfried Echterhoff, uses the concept of characters associated to finite-dimensional representations on compact groups, so we include the definition and some properties here. For details, the reader may consult [DE09] or [Fol95]. Let K be a compact group and let σ be a finite-dimensional representation of K. Then the character χ_σ of σ is defined by

$$\chi_\sigma(k) = \operatorname{tr}(\sigma(k))$$

for all $k \in K$, where tr is the usual trace. Notice that χ_σ only depends on the class of σ.

If ϱ is a finite-dimensional unitary representation of K such that $\varrho \leq \sigma$, then $\sigma = \varrho \oplus \sigma|_{V_\varrho^\perp}$, and it follows easily that

$$\chi_\sigma = \chi_\varrho + \chi_{\sigma|_{V_\varrho^\perp}}.$$

Using the fact that every representation of a compact group can be decomposed into irreducibles, this leads to

$$\chi_\sigma = \sum_{\tau\in\widehat{K}} \operatorname{mult}(\tau,\sigma)\chi_\tau, \tag{2.2.2}$$

where $\operatorname{mult}(\tau,\sigma)$ denotes the multiplicity of τ in σ. As $\{\chi_\tau \mid \tau \in \widehat{K}\}$ is an orthonormal system in $L^2(K)$, (2.2.2) implies that

$$\langle\chi_\sigma,\chi_\tau\rangle = \int_K \chi_\sigma(k)\overline{\chi_\tau(k)}dk = \operatorname{mult}(\tau,\sigma) \tag{2.2.3}$$

for every $\tau \in \widehat{K}$.

The following lemma shows that convergence in $S(K)^\frown$ implies convergence of the corresponding characters and is the key to the proof of Lemma 2.2.10.

Lemma 2.2.11 (Lemma 7.1-B in [Bag68])**.** Let K be a compact group, and let $(K_\nu, \sigma_\nu)_{\nu \in N}$ and (K', σ') be in $S(K)^\frown$ be such that $(K_\nu, \sigma_\nu) \to (K', \sigma')$. Then there exists a subnet $(K_j, \sigma_j)_{j \in J}$ of $(K_\nu, \sigma_\nu)_{\nu \in N}$ such that $\chi_{\sigma_j} \to \chi_{\sigma'}$ in $\mathscr{F}(K)$.

The following statement on continuity of integration on $\mathscr{F}(K)$ is a consequence of the more general Proposition 3.1 in [Fel64], but there is a more direct proof in the compact case which we present in Appendix A.2. For every $f \in \mathscr{F}(K)$ let $\mathcal{D}(f)$ denote the domain of f.

Proposition 2.2.12. Let K be a compact group. Then the map

$$\mathscr{F}(K) \to \mathbb{C}, \; f \mapsto \int_{\mathcal{D}(f)} f(s) d\mu_{\mathcal{D}(f)}(s)$$

is continuous.

Now we have collected all ingredients for the proof of Lemma 2.2.10, so let's put them together and see what comes out:

Proof of Lemma 2.2.10. Since $H_\nu \leq K_\nu$ for all $\nu \in N$ and $K_\nu \to K'$ in $\mathscr{K}(K)$, it follows immediately from the characterization of convergence in $\mathscr{K}(K)$ that $H' \leq K'$. It is thus left to show that $\varrho' \leq \sigma'|_{H'}$. By (2.2.3) it suffices to show that $\langle \chi_{\sigma'|_{H'}}, \chi_{\varrho'} \rangle \in \mathbb{N}$. Notice that the character of the restriction $\sigma'|_{H'}$ is just the restriction of the character, i.e., $\chi_{\sigma'|_{H'}} = \chi_{\sigma'}|_{H'}$.

By passing to subnets and relabeling twice, if necessary, we can use Lemma 2.2.11 to assume that $\chi_{\sigma_\nu} \to \chi_{\sigma'}$ and $\chi_{\varrho_\nu} \to \chi_{\varrho'}$ in $\mathscr{F}(K)$. The characterization of convergence in $\mathscr{F}(K)$ from Lemma 2.2.6 implies that $\chi_{\sigma_\nu}|_{H_\nu} \to \chi_{\sigma'}|_{H'}$ and

$$(\chi_{\sigma_\nu}|_{H_\nu})\overline{\chi_{\varrho_\nu}} \to (\chi_{\sigma'}|_{H'})\overline{\chi_{\varrho'}}$$

in $\mathscr{F}(K)$. Now Proposition 2.2.12 yields that

$$\int_{H_\nu} \chi_{\sigma_\nu}(h)\overline{\chi_{\varrho_\nu}(h)} d\mu_{H_\nu}(h) \to \int_{H'} \chi_{\sigma'}(h)\overline{\chi_{\varrho'}(h)} d\mu_{H'}(h),$$

and since $\varrho_\nu \leq \sigma_\nu|_{H_\nu}$ implies that

$$\int_{H_\nu} \chi_{\sigma_\nu}(h)\overline{\chi_{\varrho_\nu}(h)} d\mu_{H_\nu}(h) \in \mathbb{N}$$

for all $\nu \in N$, it follows that

$$\langle \chi_{\sigma'|_{H'}}, \chi_{\varrho'} \rangle = \int_{H'} \chi_{\sigma'}(h)\overline{\chi_{\varrho'}(h)} d\mu_{H'}(h) \in \mathbb{N},$$

and hence $\varrho' \leq \sigma'|_{H'}$. ☐

A G-action on $\mathrm{Rep}(S(G))$

The space $\mathrm{Stab}(X)\widehat{\ }$ for a proper G-space X will be equipped with a G-action. Since $\mathrm{Stab}(X)\widehat{\ }$ will be defined as a subset of $X \times S(G)\widehat{\ }$, it makes sense to start with a G-action on $\mathrm{Rep}(S(G))$. Here, we don't need the assumption of properness of the action yet.

Definition 2.2.13. Let X be a locally compact G-space. For every $g \in G$ and for every $(H, \sigma) \in \mathrm{Rep}(S(G))$ define $g(H, \sigma) := (gH, g\sigma)$, where $gH := gHg^{-1}$ and $g\sigma \in \mathrm{Rep}(gH)$ is given by $g\sigma(h) = \sigma(g^{-1}hg)$ for all $h \in gH$.

Lemma 2.2.14 (Lemma 2.7 in [Fel64])**.** The operation of G on $\mathrm{Rep}(S(G))$ defined in the previous definition is a continuous group action.

Remark 2.2.15. For later use we note that, for all $(H, \varrho), (L, \sigma) \in \mathrm{Rep}(S(G))$ and for all $g \in G$ we have that $(H, \varrho) \leq (L, \sigma)$ implies $g(H, \varrho) \leq g(L, \sigma)$.

2.3 A topology and a G-action for $\mathrm{Stab}(X)\widehat{\ }$

We now come back to the space $\mathrm{Stab}(X)\widehat{\ }$ from Definition 2.1.1. We view $\mathrm{Stab}(X)\widehat{\ }$ as a subset of the space $X \times S(G)\widehat{\ }$ and use this identification to define a closure operation on it, for which we prove that it satisfies the Kuratowski axioms and thus induces a topology. This topology will, however, in general not coincide with the relative topology induced from $X \times S(G)\widehat{\ }$. We will also use the space $X \times S(G)\widehat{\ }$ to define a G-action on $\mathrm{Stab}(X)\widehat{\ }$. Once we have that $\mathrm{Stab}(X)\widehat{\ }$ is a G-space we will discuss the Mackey-Rieffel-Green theorem a bit more thoroughly than we did in the beginning of this chapter.

The topology on $\mathrm{Stab}(X)\widehat{\ }$

We define our topology on $\mathrm{Stab}(X)\widehat{\ }$ in terms of a closure operation. We consider $S(G)\widehat{\ }$ to be equipped with the relative topology induced from $\mathrm{Rep}(S(G))$.

Definition 2.3.1. Let X be a proper G-space and let $A \subseteq \mathrm{Stab}(X)\widehat{\ }$. Define $\overline{A}$ to be the set of all $(x, G_x, \sigma) \in \mathrm{Stab}(X)\widehat{\ }$ for which there exist an element $(H, \sigma_H) \in S(G)\widehat{\ }$ and a net $(x_\nu, G_\nu, \sigma_\nu)_{\nu \in N}$ in A such that $(H, \sigma_H) \leq (G_x, \sigma)$ and $(x_\nu, G_\nu, \sigma_\nu) \to (x, H, \sigma_H)$ in $X \times S(G)\widehat{\ }$, in short,

$$(x_\nu, G_\nu, \sigma_\nu) \to (x, H, \sigma_H) \leq (x, G_x, \sigma).$$

We often write $(x, L, \varrho) \leq (y, H, \sigma)$ for elements (x, L, ϱ) and (y, H, σ) in $X \times \mathrm{Rep}(S(G))$ which satisfy $x = y$ and $(L, \varrho) \leq (H, \sigma)$.

Remark 2.3.2. Retain the notation of the previous definition. Then an element (x, G_x, σ) of $\mathrm{Stab}(X)\widehat{\ }$ is contained in $\overline{A}$ if and only if there exists $(H, \sigma_H) \in S(G)\widehat{\ }$ with $(H, \sigma_H) \leq (G_x, \sigma)$ such that (x, H, σ_H) is contained in the closure of A as a subset of $X \times S(G)\widehat{\ }$, i.e., the closure of A in $\mathrm{Stab}(X)\widehat{\ }$ is

$$\left\{(x, G_x, \sigma) \in \mathrm{Stab}(X)\widehat{\ } \mid \exists (H, \sigma_H) \in S(G)\widehat{\ } :\ (H, \sigma_H) \leq (G_x, \sigma) \ \wedge\ (x, H, \sigma_H) \in \overline{A}^{X \times S(G)\widehat{\ }}\right\}.$$

In this notation it becomes clear immediately that the closure of A in $\mathrm{Stab}(X)\widehat{\ }$ is in general quite different from the closure of A in $X \times S(G)\widehat{\ }$. We will see in Section 2.5 below that things become easier if the stabilizer map $x \mapsto G_x$ is continuous.

Proposition 2.3.3. Let X be a proper G-space. Then the operation $\overline{(\cdot)}$ from the previous definition defines a closure operation on $\mathrm{Stab}(X)\widehat{\ }$, that is, complements of closed sets form a topology on $\mathrm{Stab}(X)\widehat{\ }$.

Proof. We have to show that $\overline{(\cdot)}$ satisfies the Kuratowski closure axioms (see for instance [Kel75], Theorem 8 in Chapter 1), i.e., we have to show that

(i) $\overline{\emptyset} = \emptyset$,

(ii) $A \subseteq \overline{A}$ for all $A \subseteq \mathrm{Stab}(X)\widehat{\ }$,

(iii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$ for all $A, B \subseteq \mathrm{Stab}(X)\widehat{\ }$,

(iv) $\overline{\overline{A}} = \overline{A}$ for all $A \subseteq \mathrm{Stab}(X)\widehat{\ }$.

Condition (i) is obviously satisfied, and Condition (ii) holds because for every element (x, G_x, σ) in $\mathrm{Stab}(X)\widehat{\ }$, the constant net $(x_\nu, G_\nu, \sigma_\nu)_{\nu \in N}$ with $(x_\nu, G_\nu, \sigma_\nu) = (x, G_x, \sigma)$ for every $\nu \in N$ converges to (x, G_x, σ) in $X \times S(G)\widehat{\ }$.

In Condition (iii), the inclusion "$\supseteq$" is clear, and "$\subseteq$" follows because, for all subsets $A, B \subseteq \mathrm{Stab}(X)\widehat{\ }$, every net in $A \cup B$ has a subnet which lies completely in A or completely in B.

In Condition (iv), the inclusion "$\supseteq$" follows from (ii), so it is left to show that $\overline{\overline{A}} \subseteq \overline{A}$ for every $A \subseteq \mathrm{Stab}(X)\widehat{\ }$. Let $A \subseteq \mathrm{Stab}(X)\widehat{\ }$ and let $(x, G_x, \sigma) \in \overline{\overline{A}}$. By definition of $\overline{(\cdot)}$, there exist $(H, \sigma_H) \in S(G)\widehat{\ }$ with $(H, \sigma_H) \leq (G_x, \sigma)$ and a net $(x_\nu, G_\nu, \sigma_\nu)_{\nu \in N}$ in $\overline{A}$ such that

$$(x_\nu, G_\nu, \sigma_\nu) \to (x, H, \sigma_H) \leq (x, G_x, \sigma)$$

in $X \times S(G)\widehat{\ }$. For every $\nu \in N$, there now exist $(H_\nu, \sigma_{H_\nu}) \in S(G)\widehat{\ }$ with $(H_\nu, \sigma_{H_\nu}) \leq (G_\nu, \sigma_\nu)$ and a net $(x_{\nu,\mu}, G_{\nu,\mu}, \sigma_{\nu,\mu})_{\mu \in M_\nu}$ in A such that

$$(x_{\nu,\mu}, G_{\nu,\mu}, \sigma_{\nu,\mu}) \xrightarrow{\mu} (x_\nu, H_\nu, \sigma_{H_\nu}) \leq (x_\nu, G_\nu, \sigma_\nu)$$

in $X \times S(G)\widehat{\ }$. It suffices to prove the following statement:

(🚲) There exists an element $(\widetilde{H}, \sigma_{\widetilde{H}}) \in S(G)\widehat{\ }$ with $(\widetilde{H}, \sigma_{\widetilde{H}}) \leq (G_x, \sigma)$, such that a subnet of $(x_\nu, H_\nu, \sigma_{H_\nu})_{\nu \in N}$ converges to $(x, \widetilde{H}, \sigma_{\widetilde{H}})$ in $X \times S(G)\widehat{\ }$.

Since each $(x_\nu, H_\nu, \sigma_\nu)$ is contained in the closure of A with respect to the topology on $X \times S(G)\widehat{\ }$, (🚲) implies that $(x, \widetilde{H}, \sigma_{\widetilde{H}})$ is contained in the double closure of A in $X \times S(G)\widehat{\ }$. Since the Kuratowski axioms hold for the closure operation of any given topology, it follows that $(x, \widetilde{H}, \sigma_{\widetilde{H}})$ is contained in the closure of A in $X \times S(G)\widehat{\ }$, and by Remark 2.3.2 we then have $(x, G_x, \sigma) \in \overline{A}$ in $\mathrm{Stab}(X)\widehat{\ }$, as required.

We first prove (🚲) in the special case that all elements x_ν, $x_{\nu,\mu}$ lie in a slice at x. To do this, we apply Abels' Theorem (Theorem 1.1.7) and the remarks following it to find an open G-invariant neighborhood U_x of x, a compact subgroup $L_x \leq G$, and a local L_x-slice Y_x for x such that $U_x \cong G \times_{L_x} Y_x$. Suppose now that $x_\nu, x_{\nu,\mu} \in Y_x$ for all $\nu \in N$, $\mu \in M_\nu$. By Remark 1.1.8 (iii) this implies that $G_x \leq L_x$, $G_\nu \leq L_x$, and $G_{\nu,\mu} \leq L_x$ for all $\nu \in N$, $\mu \in M_\nu$.

From $(G_\nu, \sigma_\nu) \to (H, \sigma_H)$ in $S(G)\widehat{\ }$ we have by continuity of induction (Proposition 2.2.5) that

$$\mathrm{ind}_{G_\nu}^{L_x} \sigma_\nu \to \mathrm{ind}_H^{L_x} \sigma_H \tag{2.3.1}$$

in $\operatorname{Rep}(L_x)$. Using Frobenius reciprocity theorem and induction in steps (Theorems 1.5.3 and 1.5.4) we can conclude that $\sigma_H \leq \sigma|_H$ implies $\sigma \leq \operatorname{ind}_H^{G_x} \sigma_H$ and thus $\operatorname{ind}_{G_x}^{L_x} \sigma \leq \operatorname{ind}_H^{L_x} \sigma_H$. Let $\pi \in \widehat{L_x}$ be an irreducible subrepresentation of $\operatorname{ind}_{G_x}^{L_x} \sigma$, then also $\pi \leq \operatorname{ind}_H^{L_x} \sigma_H$, and it follows from (2.3.1) together with Proposition 1.3.2 that

$$\operatorname{ind}_{G_\nu}^{L_x} \sigma_\nu \to \pi$$

in $\operatorname{Rep}(L_x)$. Now Lemma 1.3.4 implies that, without loss of generality, we can assume that $\pi \leq \operatorname{ind}_{G_\nu}^{L_x} \sigma_\nu$ and hence, by the Frobenius reciprocity theorem, that $\sigma_\nu \leq \pi|_{G_\nu}$ for all $\nu \in N$. Since $\sigma_{H_\nu} \leq \sigma_\nu|_{H_\nu}$, we also have $\sigma_{H_\nu} \leq \pi|_{H_\nu}$ for all $\nu \in N$. This shows that $(H_\nu, \sigma_{H_\nu})_{\nu \in N}$ is a net in $\operatorname{Rep}(S(G))_{\leq (L_x, \pi)}$. As π is irreducible and hence finite-dimensional, we know by Proposition 2.2.9 that $\operatorname{Rep}(S(G))_{\leq (L_x, \pi)}$ is compact. By passing to a subnet and relabeling we can assume that there exists $(\widetilde{H}, \varrho)$ in $\operatorname{Rep}(S(G))_{\leq (L_x, \pi)}$ such that $(H_\nu, \sigma_{H_\nu})_{\nu \in N}$ converges to $(\widetilde{H}, \varrho)$ in $\operatorname{Rep}(S(G))$. Let $\sigma_{\widetilde{H}}$ be any irreducible subrepresentation of ϱ, then

$$(H_\nu, \sigma_{H_\nu}) \to (\widetilde{H}, \sigma_{\widetilde{H}})$$

in $S(G)\widehat{}$. Now $(G_\nu, \sigma_\nu)_{\nu \in N}$, (H, σ_H), $(H_\nu, \sigma_{H_\nu})_{\nu \in N}$ and $(\widetilde{H}, \sigma_{\widetilde{H}})$ satisfy the assumptions of Lemma 2.2.10, which then yields that

$$(\widetilde{H}, \sigma_{\widetilde{H}}) \leq (H, \sigma_H) \leq (G_x, \sigma),$$

which proves (🚲) in the special case that all $x_\nu, x_{\nu\mu}$ are contained in a slice Y_x at x.

For the proof of (🚲) in the general case we may without loss of generality assume that $x_\nu, x_{\nu,\mu} \in U_x$ for all $\nu \in N$, $\mu \in M_\nu$. By passing to subnets and relabeling, where necessary, we can by Lemma 1.1.9 assume that there exists a net $(g_\nu)_{\nu \in N}$ in G with $g_\nu \to e$ and $g_\nu x_\nu \in Y_x$ for all $\nu \in N$. Since, for every $\nu \in N$, we have that $g_\nu x_{\nu,\mu} \to g_\nu x_\nu$, we can apply Lemma 1.1.9 again to find nets $(g_{\nu,\mu})_{\mu \in M_\nu}$ in G such that $g_{\nu,\mu} \to e$ and $g_{\nu,\mu} g_\nu x_{\nu,\mu} \in Y_x$ for all $\nu \in N$ and $\mu \in M_\nu$. By continuity of the G-action on $X \times \operatorname{Rep}(S(G))$ we have that

$$g_\nu(x_\nu, G_\nu, \sigma_\nu) \to (x, H, \sigma_H) \leq (x, G_x, \sigma)$$

and

$$g_{\nu,\mu} g_\nu (x_{\nu,\mu}, G_{\nu,\mu}, \sigma_{\nu,\mu}) \xrightarrow{\mu} g_\nu(x_\nu, H_\nu, \sigma_{H_\nu}) \leq g_\nu(x_\nu, G_\nu, \sigma_\nu)$$

for every $\nu \in N$, so the special case of (🚲) implies that, after passing to subnets and relabeling, if necessary, there exists $(\widetilde{H}, \sigma_{\widetilde{H}}) \leq (G_x, \sigma)$ such that

$$g_\nu(x_\nu, H_\nu, \sigma_{H_\nu}) \to (x, \widetilde{H}, \sigma_{\widetilde{H}})$$

in $X \times S(G)\widehat{}$. As $g_\nu \to e$, this implies that

$$(x_\nu, H_\nu, \sigma_{H_\nu}) \to (x, \widetilde{H}, \sigma_{\widetilde{H}}),$$

which completes the proof of (🚲) and of the proposition. □

Convergence of nets in $\operatorname{Stab}(X)\widehat{}$ and $G\backslash\operatorname{Stab}(X)\widehat{}$ can be formulated as follows, where the latter is just an application of the openness of the orbit map $\operatorname{Stab}(X)\widehat{} \to G\backslash\operatorname{Stab}(X)\widehat{}$ as in Remark 1.1.1.

Remark 2.3.4. (i) Let X be a proper G-space. A net $(x_\nu, G_\nu, \sigma_\nu)_{\nu\in N}$ in $\mathrm{Stab}(X)^\frown$ converges to $(x, G_x, \sigma) \in \mathrm{Stab}(X)^\frown$ if and only if every subnet of $(x_\nu, G_\nu, \sigma_\nu)_{\nu\in N}$ has a subnet $(x_j, G_j, \sigma_j)_{j\in J}$ such that there exists an element $(H, \sigma_H) \in S(G)^\frown$ satisfying $(H, \sigma_H) \leq (G_x, \sigma)$ and $(x_j, G_j, \sigma_j) \to (x, H, \sigma_H)$ in $X \times S(G)^\frown$.

Notice that it can in fact happen that different subnets of $(x_\nu, G_\nu, \sigma_\nu)_{\nu\in N}$ converge to different subelements of (x, G_x, σ) in $X \times S(G)^\frown$.

(ii) A net $([x_\nu, G_\nu, \sigma_\nu])_{\nu\in N}$ in $G\backslash\mathrm{Stab}(X)^\frown$ converges to an element $[x, G_x, \sigma]$ with respect to the quotient topology if and only if every subnet of $([x_\nu, G_\nu, \sigma_\nu])_{\nu\in N}$ has a subnet $([x_j, G_j, \sigma_j])_{j\in J}$ such there are representatives (x, G_x, σ) of $[x, G_x, \sigma]$ and (x_j, G_j, σ_j) of $[x_j, G_j, \sigma_j]$ for every $j \in J$ such that $(x_j, G_j, \sigma_j) \to (x, G_x, \sigma)$ in $\mathrm{Stab}(X)^\frown$.

Continuity of the G-action on $\mathrm{Stab}(X)^\frown$

Let X be a proper G-space. Combining this with the G-action on $\mathrm{Rep}(S(G))$ from Definition 2.2.13 we obtain a G-action on $X \times \mathrm{Rep}(S(G))$. As the G-action on $\mathrm{Rep}(S(G))$ respects irreducibility it follows that $S(G)^\frown$ is a G-invariant subset of $\mathrm{Rep}(S(G))$. Since for all $x \in X$ and $g \in G$ we have that $g \cdot G_x = gG_xg^{-1} = G_{gx}$, we get that $g(x, G_x, \sigma) = (gx, G_{gx}, g\sigma)$ is contained in $\mathrm{Stab}(X)^\frown$ for all $g \in G$ and $(x, G_x, \sigma) \in \mathrm{Stab}(X)^\frown$. Hence, in the algebraic sense, $\mathrm{Stab}(X)^\frown$ is a G-invariant subspace of the product G-space $X \times S(G)^\frown$, and the action is the one we already mentioned in Section 2.1. It is left to show that this action is continuous with respect to the topology on $\mathrm{Stab}(X)^\frown$.

Suppose that $(x_\nu, G_\nu, \sigma_\nu)_{\nu\in N}$ and (x, G_x, σ) in $\mathrm{Stab}(X)^\frown$, and $(H, \sigma_H) \in S(G)^\frown$ are such that $(x_\nu, G_\nu, \sigma_\nu) \to (x, H, \sigma_H)$ in $X \times S(G)^\frown$ and $(H, \sigma_H) \leq (G_x, \sigma)$. If now $(g_\nu)_{\nu\in N}$ converges to g in G, then it follows from continuity of the G-action on $X \times \mathrm{Rep}(S(G))$ and from Remark 2.2.15 that

$$g_\nu(x_\nu, G_\nu, \sigma_\nu) \to g(x, H, \sigma_H) \leq g(x, G_x, \sigma),$$

which implies that the G-action respects convergence in $\mathrm{Stab}(X)^\frown$ and is hence continuous.

If now $(x, G_x, \sigma) \in \mathrm{Stab}(X)^\frown$ and $g \in G_x$, then it follows from the definition of the G-action that $g\sigma$ is equivalent to σ via the unitary $\sigma(g^{-1})$. Hence, the stabilizer group $G_{(x,G_x,\sigma)}$ coincides with G_x.

We denote the class of an element $(x, G_x, \sigma) \in \mathrm{Stab}(X)^\frown$ in the orbit space $G\backslash\mathrm{Stab}(X)^\frown$ by $[x, G_x, \sigma]$.

The Mackey-Rieffel-Green theorem

We can now give a more precise formulation of the Mackey-Rieffel-Green theorem than the one we provided in Section 2.1. To every element $(x, G_x, \sigma) \in \mathrm{Stab}(X)^\frown$ we associate the representation $\mathrm{ev}_x \rtimes \sigma$ of $C_0(X) \rtimes G_x$. As shown in Example 1.5.6(ii), the induced representation $\mathrm{ind}_{G_x}^G(\mathrm{ev}_x \rtimes \sigma)$ is given by $\pi^{x,\sigma} = P^x \rtimes \mathrm{ind}_{G_x}^G \sigma$ with

$$P^x(\varphi)\xi(g) = \varphi(gx)\xi(g), \qquad ((\mathrm{ind}_{G_x}^G \sigma)(s))\xi(g) = \xi(s^{-1}g)$$

for all $\varphi \in C_0(X)$, $\xi \in H_{\mathrm{ind}\,\sigma}$, and $g, s \in G$. By Proposition 4.2 in [Wil81], these induced representations are irreducible, so we obtain a map

$$\mathrm{ind}\colon \mathrm{Stab}(X)^\frown \to (C_0(X) \rtimes G)^\frown,\ (x, G_x, \sigma) \mapsto \pi^{x,\sigma}. \tag{2.3.2}$$

It is a consequence of Proposition 2.2.5 in [Ech90b] that induction is compatible with the G-action on $\operatorname{Rep}(S(G))$ from Definition 2.2.13. Here, if (x, G_x, σ) is an element of $\operatorname{Stab}(X)\widehat{}$ and $g \in G$, then a unitary equivalence of $\pi^{x,\sigma}$ and $\pi^{gx,g\sigma}$ is implemented by the operator $W\colon H_{\operatorname{ind}\sigma} \to H_{\operatorname{ind}(g\sigma)}$ with $W\xi(s) = \xi(sg)$ for all $\xi \in H_{\operatorname{ind}\sigma}$, $s \in G$.

It follows that the map ind from (2.3.2) factors through the orbit space $G\backslash\operatorname{Stab}(X)\widehat{}$, and the following theorem states that the resulting map can be used to catalogue the elements of $(C_0(X) \rtimes G)\widehat{}$ using the quotient space $G\backslash\operatorname{Stab}(X)\widehat{}$:

Theorem 2.3.5 (Mackey-Rieffel-Green)**.** Let X be a proper G-space. Then the map in (2.3.2) factors through a bijection

$$G\backslash\operatorname{Stab}(X)\widehat{} \to (C_0(X) \rtimes G)\widehat{},\ [x, G_x, \sigma] \mapsto \pi^{x,\sigma},$$

which we denote by ind^G.

We don't give the classical proof of this result, but indicate how it can be obtained by means of some of the results we encounter in this thesis anyway: Using the fact that $C_0(X) \rtimes G$ is a $C_0(G\backslash X)$-algebra and Echterhoff and Emerson's description of the fibres (see page 14 and Lemma 7.3.1), it follows that

$$(C_0(X) \rtimes G)\widehat{} \cong \coprod_{Gx \in G\backslash X} (\mathcal{K}^{G_x})\widehat{} \cong \coprod_{[x,G_x,\sigma]\in G\backslash\operatorname{Stab}(X)\widehat{}} (\mathcal{K}(H_{\operatorname{ind}\sigma}))\widehat{} \cong \coprod_{[x,G_x,\sigma]\in G\backslash\operatorname{Stab}(X)\widehat{}} \{i_\sigma\},$$

compare (2.1.3) in Section 2.1.

2.4 Continuity of ind^G

The proof of continuity of ind^G in case of a proper G-space X consists basically of an application of Proposition 2.2.5 on continuity of induction and is quite similar to the proof of Theorem 2.5-B in [Bag68]. To be able to apply continuity of induction correctly, we need the following lemma. Its proof is not difficult but slightly technical, and since it is not essential for the general understanding at this point, we transferred it to Appendix A.3.

Lemma 2.4.1. Let X be a proper G-space and suppose that the net $(x_\nu, G_\nu, \sigma_\nu)_{\nu\in N}$ in $\operatorname{Stab}(X)\widehat{}$ converges to an element $(x, H, \sigma_H) \in X \times S(G)\widehat{}$ in $X \times S(G)\widehat{}$. Then $(\operatorname{ev}_\nu \rtimes \sigma_\nu)_{\nu\in N}$ converges to $\operatorname{ev}_x \rtimes \sigma_H$ in $(S(G), C_0(X))\widehat{}$.

Proposition 2.4.2. Let X be a proper G-space. Then the map

$$\operatorname{ind}^G\colon G\backslash\operatorname{Stab}(X)\widehat{} \to (C_0(X) \rtimes G)\widehat{},\ [x, G_x, \sigma] \to \pi^{x,\sigma}$$

is continuous with respect to the quotient topology on $G\backslash\operatorname{Stab}(X)\widehat{}$ and the Jacobson topology on $(C_0(X) \rtimes G)\widehat{}$.

Proof. By the universal property of the quotient topology on $G\backslash\operatorname{Stab}(X)\widehat{}$ it suffices to prove continuity of the map

$$\operatorname{ind}\colon \operatorname{Stab}(X)\widehat{} \to (C_0(X) \rtimes G)\widehat{},\ (x, G_x, \sigma) \mapsto \pi^{x,\sigma}.$$

Let thus $(x_\nu, G_\nu, \sigma_\nu)_{\nu\in N}$ be a net in $\operatorname{Stab}(X)\widehat{}$ which converges to (x, G_x, σ) in $\operatorname{Stab}(X)\widehat{}$. We show that $\pi^{x_\nu,\sigma_\nu} \to \pi^{x,\sigma}$ by proving that every subnet of $(\pi^{x_\nu,\sigma_\nu})_{\nu\in N}$ has a subnet which converges to $\pi^{x,\sigma}$.

Suppose we have passed to a subnet and relabeled. By the definition of convergence in $\mathrm{Stab}(X)^\frown$ we can do this again to assume that there exists an element $(H, \sigma_H) \in S(G)^\frown$ such that $(H, \sigma_H) \le (G_x, \sigma)$ and $(x_\nu, G_\nu, \sigma_\nu) \to (x, H, \sigma_H)$ in $X \times S(G)^\frown$. By Lemma 2.4.1 above this implies that $\mathrm{ev}_\nu \rtimes \sigma_\nu \to \mathrm{ev}_x \rtimes \sigma_H$ in $(S(G), C_0(X))^\frown$, which by continuity of induction implies that

$$\mathrm{ind}_{G_\nu}^G(\mathrm{ev}_\nu \rtimes \sigma_\nu) \to \mathrm{ind}_H^G(\mathrm{ev}_x \rtimes \sigma_H),$$

which, in short notation, is just $\pi^{x_\nu, \sigma_\nu} \to \pi^{x, \sigma_H}$ as representations of $C_0(X) \rtimes G$. By Proposition 1.3.2(ii) it is only left to show that $\pi^{x,\sigma}$ is weakly contained in π^{x,σ_H}. We even get containment: By choice of (H, σ_H) we have that $\sigma_H \le \sigma|_H$, which by the Frobenius reciprocity theorem is equivalent to $\sigma \le \mathrm{ind}_H^{G_x} \sigma_H$. As shown in Example 1.5.6(iii), this implies that

$$\mathrm{ev}_x \rtimes \sigma \le \mathrm{ev}_x \rtimes \mathrm{ind}_H^{G_x} \sigma_H = \mathrm{ind}_H^{G_x}(\mathrm{ev}_x \rtimes \sigma_H),$$

which, after using induction in steps, yields that

$$\pi^{x,\sigma} = \mathrm{ind}_{G_x}^G(\mathrm{ev}_x \rtimes \sigma) \le \mathrm{ind}_H^G(\mathrm{ev}_x \rtimes \sigma_H) = \pi^{x,\sigma_H},$$

which completes the proof. □

2.5 $\mathrm{Stab}(X)^\frown$ for actions with continuous stabilizer map

As mentioned in Remark 2.3.2, our topology on $\mathrm{Stab}(X)^\frown$ is usually quite different from the one induced from $X \times S(G)^\frown$, which makes it difficult to say anything about the topological properties of $\mathrm{Stab}(X)^\frown$. But if the stabilizer map is continuous, the situation becomes much easier. The idea for the proof of "(iii)⇒(i)" has been used several times before, for instance in [Ech94] and [Bag68].

Proposition 2.5.1. Let X be a proper G-space. Then the following statements are equivalent:

(i) The stabilizer map $X \to \mathscr{K}(G),\ x \mapsto G_x$ is continuous;

(ii) Our definition of convergence in $\mathrm{Stab}(X)^\frown$ coincides with the convergence in $\mathrm{Stab}(X)^\frown$ induced from the topology on $X \times S(G)^\frown$;

(iii) $\mathrm{Stab}(X)^\frown$ is Hausdorff;

(iv) $\mathrm{Stab}(X)^\frown$ is a locally compact Hausdorff space and the G-action is proper;

(v) $G\backslash\mathrm{Stab}(X)^\frown$ is a locally compact Hausdorff space.

As indicated in the beginning of this chapter, the space $\mathrm{Stab}(X)^\frown$ is, for any proper G-action on X, homeomorphic to the spectrum of a C^*-algebra A. This implies that $\mathrm{Stab}(X)^\frown$, and hence also $G\backslash\mathrm{Stab}(X)^\frown$, are always locally compact (not necessarily Hausdorff) spaces. Also, once we have proved that $G\backslash\mathrm{Stab}(X)^\frown$ is homeomorphic to $(C_0(X) \rtimes G)^\frown$, we get local compactness (but not the Hausdorff property) of $G\backslash\mathrm{Stab}(X)^\frown$ for free. But in the situation of this proposition these statements can also be obtained independently.

Proof of Proposition 2.5.1. We prove the following implications:

(i)⇒(ii)⇒(iii)⇒(i), (i)+(ii)+(iii)⇒(iv), (iv)⇒(iii), (iv)⇒(v), and (v)⇒(iii).

(i)⇒(ii): Suppose that $x \mapsto G_x$ is continuous. Let $(x_\nu, G_\nu, \sigma_\nu)_{\nu \in N}$ and (x, G_x, σ) be in $\mathrm{Stab}(X)^\frown$. It follows from the definitions that $(x_\nu, G_\nu, \sigma_\nu) \to (x, G_x, \sigma)$ with respect to $X \times S(G)^\frown$ always implies convergence in $\mathrm{Stab}(X)^\frown$.

Suppose now that $(x_\nu, G_\nu, \sigma_\nu) \to (x, G_x, \sigma)$ in $\mathrm{Stab}(X)^\frown$, i.e., for every subnet there exist a subnet $(x_j, G_j, \sigma_j)_{j \in J}$ of that subnet and an element $(H, \sigma_H) \leq (G_x, \sigma)$ such that $(x_j, G_j, \sigma_j) \to (x, H, \sigma_H)$ in $X \times S(G)^\frown$. Since $\mathscr{K}(G)$ is Hausdorff, the continuity assumption implies that $H = G_x$, and by irreducibility we have that $\sigma_H \leq \sigma$ becomes $\sigma_H = \sigma$. This shows that every subnet of $(x_\nu, G_\nu, \sigma_\nu)_{\nu \in N}$ has a subnet which converges to (x, G_x, σ) in $X \times S(G)^\frown$, which means that $(x_\nu, G_\nu, \sigma_\nu) \to (x, G_x, \sigma)$ in $X \times S(G)^\frown$.

(ii)⇒(iii): Suppose that statement (ii) holds. Let $(x_\nu, G_\nu, \sigma_\nu)_{\nu \in N}$ be a net in $\mathrm{Stab}(X)^\frown$ which converges to (x_1, G_1, σ_1) and (x_2, G_2, σ_2) in $\mathrm{Stab}(X)^\frown$ and hence, by (ii), in $X \times S(G)^\frown$. As X is Hausdorff we immediately get that $x_1 = x_2$.

Since the proper G-space X is locally induced from compact subgroups, we can assume that there exist a compact subgroup L_x of G and an L_x-slice Y_x at x, such that $U_x = G \times_{L_x} Y_x$ is an open neighborhood of x. By Lemma 1.1.9 and by continuity of the G-action on $\mathrm{Stab}(X)^\frown$ we can without loss of generality assume that $x_\nu \in Y_x$ and thus $G_\nu \leq L_x$ for all $\nu \in N$. But then we have $(x_\nu, G_\nu, \sigma_\nu) \to (x, G_x, \sigma_i)$ for $i \in \{1, 2\}$ in $X \times S(L_x)^\frown$, which is Hausdorff because L_x is compact, see Theorem 7.2 in [Bag68]. It follows that σ_1 is equivalent to σ_2, so $\mathrm{Stab}(X)^\frown$ is Hausdorff.

(iii)⇒(i): By contraposition. Suppose that $x \mapsto G_x$ is not continuous. By compactness of $\mathscr{K}(G)$ and by Lemma 2.2.3 we can find a net $(x_\nu)_{\nu \in N}$, an element $x \in X$, and a proper subgroup $H \leq G_x$ such that $x_\nu \to x$ in X and $G_\nu \to H$ in $\mathscr{K}(G)$. It follows directly from Fell's characterization of convergence in $\mathrm{Rep}(S(G))$ given in Theorem 2.2.7 that

$$(x_\nu, G_\nu, 1_\nu) \to (x, H, 1_H) \tag{2.5.1}$$

in $X \times S(G)^\frown$, where 1_ν denotes the trivial representation of G_ν for all $\nu \in N$, and 1_H is defined accordingly. By definition of convergence in $\mathrm{Stab}(X)^\frown$ we get that

$$(x_\nu, G_\nu, 1_\nu) \to (x, G_x, \sigma)$$

in $\mathrm{Stab}(X)^\frown$ for every $\sigma \in \widehat{G_x}$ that satisfies $1_H \leq \sigma|_H$. By the Frobenius reciprocity theorem, this is true for all elements of

$$S := \{\sigma \in \widehat{G_x} \mid \sigma \leq \mathrm{ind}_H^{G_x} 1_H\}.$$

We show that S contains at least two distinct elements. It is clear that 1_H is contained in $1_{G_x}|_H$ exactly once, which by Frobenius reciprocity implies that 1_{G_x} is contained in S also exactly once. But since H is a proper subgroup of G_x, the representation $\mathrm{ind}_H^{G_x} 1_H$ is at least two-dimensional, so there has to exist some other element of S.

Thus, we can choose two distinct elements $\sigma, \varrho \in \widehat{G_x}$ such that $(x_\nu, G_\nu, \sigma_\nu)_{\nu \in N}$ converges both to (x, G_x, σ) and to (x, G_x, ϱ) in $\mathrm{Stab}(X)^\frown$, which shows that $\mathrm{Stab}(X)^\frown$ is not Hausdorff.

(i)+(ii)+(iii)⇒(iv): Suppose now that the three equivalent conditions (i), (ii), and (iii) hold. Then $\mathrm{Stab}(X)^\frown$ is Hausdorff. By (ii) we can consider $\mathrm{Stab}(X)^\frown$ to be a topological subspace of $X \times S(G)^\frown$. Recall that, by definition, the space $S(G)^\frown$ is homeomorphic to the spectrum of the subgroup C^*-algebra $C^*(S(G))$ and therefore locally compact (but not necessarily Hausdorff).

We now show that $\mathrm{Stab}(X)\widehat{\ }$ is closed in the locally compact (not necessarily Hausdorff) space $X \times S(G)\widehat{\ }$. Let $(x_\nu, G_\nu, \sigma_\nu)_{\nu \in N}$ be a net in $\mathrm{Stab}(X)\widehat{\ }$ and let $(x, H, \sigma_H) \in X \times S(G)\widehat{\ }$ be such that $(x_\nu, G_\nu, \sigma_\nu) \to (x, H, \sigma_H)$ in $X \times S(G)\widehat{\ }$. By continuity of the stabilizer map and since $\mathscr{K}(G)$ is Hausdorff it follows as above that $H = G_x$ and $\sigma_H \in \widehat{G_x}$. Hence, (x, H, σ_H) is contained in $\mathrm{Stab}(X)\widehat{\ }$, which implies that $\mathrm{Stab}(X)\widehat{\ }$ is a closed subspace of $X \times S(G)\widehat{\ }$ and thus locally compact.

The properness of the G-action on $\mathrm{Stab}(X)\widehat{\ }$ follows from Proposition 1.1.4 (iv) because G acts properly on X and $\mathrm{Stab}(X)\widehat{\ }$ is a closed G-invariant subspace of the product G-space $X \times S(G)\widehat{\ }$.

(iv)⇒(iii): This is trivial.

(iv)⇒(v): Suppose that $\mathrm{Stab}(X)\widehat{\ }$ is a proper G-space. By Proposition 1.1.4 (ii), this implies that the orbit space $G\backslash\mathrm{Stab}(X)\widehat{\ }$ is a locally compact Hausdorff space.

(v)⇒(iii): By contraposition. Suppose that $\mathrm{Stab}(X)\widehat{\ }$ is not Hausdorff. Then there exist a net $(x_\nu, G_\nu, \sigma_\nu)_{\nu \in N}$ in $\mathrm{Stab}(X)\widehat{\ }$ and two distinct elements $\sigma, \varrho \in \widehat{G_x}$ such that $(x_\nu, G_\nu, \sigma_\nu)_{\nu \in N}$ converges both to (x, G_x, σ) and to (x, G_x, ϱ) in $\mathrm{Stab}(X)\widehat{\ }$. Since, for every $g \in G$, the equality $g(x, G_x, \sigma) = (x, G_x, \varrho)$ would imply that $g \in G_x$ and hence equivalence of σ and ϱ, it follows that $[x, G_x, \sigma] \neq [x, G_x, \varrho]$ in $G\backslash\mathrm{Stab}(X)\widehat{\ }$.

This shows that the net $([x_\nu, G_\nu, \sigma_\nu])_{\nu \in N}$ converges to the distinct points $[x, G_x, \sigma]$ and $[x, G_x, \varrho]$ in $G\backslash\mathrm{Stab}(X)\widehat{\ }$, and thus $G\backslash\mathrm{Stab}(X)\widehat{\ }$ is not Hausdorff. □

Chapter 3

Induction via bimodules

In this chapter we give a short introduction to right-Hilbert bimodules and show how they can be used to induce representations between C^*-algebras. Introductions to right-Hilbert bimodules can be found in [Lan95] and [RW98], but our outline is mostly based on [EKQR06], where the categorical approach to the induction procedure is given.

After discussing the basic definitions and the process of inducing representations in Sections 3.1–3.3, we present in Section 3.4 the Morita equivalence of $C_0(G \times_H Y) \rtimes G$ and $C_0(Y) \rtimes H$ for a locally compact group G and a closed subgroup $H \leq G$ which acts on a locally compact space Y. In Section 3.5 we show how this result can be used to pass from $\operatorname{Stab}(X)\widehat{\ }$ for a general proper G-space X to $\operatorname{Stab}(Y)\widehat{\ }$ for a K-space Y, where K is a compact subgroup of G.

3.1 Basic definitions

Hilbert bimodules

Let A and B be C^*-algebras. A (right-) Hilbert B-module is a complex vector space X equipped with a right B-module structure and a B-valued positive definite sesquilinear form $\langle \cdot, \cdot \rangle_B \colon X \times X \to B$ satisfying

$$\langle x, y \cdot b \rangle_B = \langle x, y \rangle_B b, \qquad \langle x, y \rangle_B^* = \langle y, x \rangle_B$$

for all $x, y \in X$, $b \in B$, such that X is complete in the norm defined by $\|x\| = \|\langle x, x \rangle_B\|^{1/2}$ for all $x \in X$. We denote a Hilbert B-module X by X_B and usually write xb instead of $x \cdot b$ for all $x \in X$ and all $b \in B$. Left-Hilbert B-modules are defined accordingly.

A right-Hilbert A–B bimodule is a Hilbert B-module X which also carries a nondegenerate left A-module structure (nondegenerate means that $AX = X$) such that

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b, \qquad \langle a \cdot x, y \rangle_B = \langle x, a^* \cdot y \rangle_B \qquad \text{for all } a \in A,\ b \in B,\ x, y \in X.$$

We denote a right-Hilbert A–B bimodule X by ${}_A X_B$.

Examples 3.1.1. (i) Let A and B be C^*-algebras. Then B is a Hilbert B-module, denoted by B_B, with respect to the operations $b \cdot c = bc$ and $\langle b, c \rangle_B = b^* c$ for all $b, c \in B$. If $\varphi \colon A \to \mathcal{M}(B)$ is a nondegenerate *-homomorphism, then B_B becomes a right-Hilbert A–B bimodule, where the left action is given by

$$a \cdot b = \varphi(a) b \qquad \text{for all } a \in A,\ b \in B.$$

In particular, if φ is the identity map on B, this shows that B can be seen as a right-Hilbert B–B bimodule.

(ii) Assuming the convention that the inner product on a Hilbert space is conjugate linear in the first and linear in the second variable, every Hilbert space H is a Hilbert $\mathbb{C}$-module. If now A is a C^*-algebra and π is a nondegenerate representation of A on a Hilbert space H, then the operation $a \cdot \xi = \pi(a)\xi$ gives H the structure of a right-Hilbert A–$\mathbb{C}$ bimodule.

Definition 3.1.2. Let A and B be C^*-algebras, and let X be a right-Hilbert A–B bimodule which also carries the structure of a left Hilbert A-module, such that

$$_A\langle x, y\rangle \cdot z = x \cdot \langle y, z\rangle_B \qquad \text{for all } x, y, z \in X,$$

and such that both $_A\langle \cdot, \cdot\rangle$ and $\langle \cdot, \cdot\rangle_B$ are full, i.e., $\overline{_A\langle X, X\rangle} = A$ and $\overline{\langle X, X\rangle_B} = B$. Then X is called an A–B imprimitivity bimodule. If such an X exists, then A and B are said to be Morita equivalent.

Isomorphisms of Hilbert bimodules

Suppose that A and B are C^*-algebras, and that X and Y are right-Hilbert A–B bimodules. We say that X and Y are isomorphic if there exists a bijective map $\Phi\colon {_AX_B} \to {_AY_B}$ satisfying

(i) $\Phi(ax) = a\Phi(x)$,

(ii) $\Phi(xb) = \Phi(x)b$,

(iii) $\langle \Phi(x), \Phi(y)\rangle_B = \langle x, y\rangle_B$

for all $a \in A,\ b \in B,\ x, y \in X$. Isomorphism of right-Hilbert A–B bimodules defines an equivalence relation. If $_AX_B$ and $_AY_B$ are imprimitivity bimodules, then an isomorphism Φ between them automatically respects the $_A\langle \cdot, \cdot\rangle$-structure, because for all $x, x', \xi \in X$ we have

$$\begin{aligned} _A\langle x, x'\rangle \cdot \Phi(\xi) &= \Phi(_A\langle x, x'\rangle \cdot \xi) \\ &= \Phi(x \cdot \langle x', \xi\rangle_B) = \Phi(x) \cdot \langle x', \xi\rangle_B \\ &= \Phi(x) \cdot \langle \Phi(x'), \Phi(\xi)\rangle_B =_A \langle \Phi(x), \Phi(x')\rangle \cdot \Phi(\xi) \end{aligned}$$

Example 3.1.3. Let A be a C^*-algebra and let π_1 and π_2 be nondegenerate representations of A on Hilbert spaces H_1 and H_2, respectively. Then it is easily checked that H_1 and H_2 are isomorphic as right-Hilbert A–$\mathbb{C}$ bimodules if and only if π_1 and π_2 are unitarily equivalent.

3.2 The category $\mathcal{C}$

We now show how one can set up a category whose elements are C^*-algebras and whose morphisms are isomorphism classes of right-Hilbert bimodules. The composition of morphisms will be given by balanced tensor products, which we will now define.

Proposition/Definition 3.2.1. Let A, B, and C be C^*-algebras, and let $_AX_B$ and $_BY_C$ be right-Hilbert bimodules. Then the algebraic tensor product $X \odot Y$ is a pre-right-Hilbert A–C bimodule with operations given by

$$a \cdot (x \otimes y) = a \cdot x \otimes y, \qquad (x \otimes y) \cdot b = x \otimes y \cdot b$$

for all $a \in A,\ b \in B,\ x \in X,\ y \in Y$, and

$$\langle x \otimes y, x' \otimes y' \rangle_C = \langle y, \langle x, x' \rangle_B \cdot y' \rangle_C$$

for all $x, x' \in X,\ y, y' \in Y$. The completion is denoted by $X \otimes_B Y$ and is called the balanced tensor product of ${}_AX_B$ and ${}_BY_C$.

Theorem 3.2.2 (Theorem 2.2 and Lemma 2.4 in [EKQR06])**.** There exists a category $\mathcal{C}$ whose objects are C^*-algebras, and in which the morphisms from a C^*-algebra A to a C^*-algebra B are given by the isomorphism classes of right-Hilbert A–B bimodules. The composition of two morphisms $[X]\colon A \to B$ and $[Y]\colon B \to C$ is given by the isomorphism class of the balanced tensor product $X \otimes_B Y$. The identity morphism on a C^*-algebra A is the isomorphism class of the standard right-Hilbert bimodule ${}_AA_A$, and the equivalences in $\mathcal{C}$ are exactly the isomorphism classes of imprimitivity bimodules.

The problem that the morphisms from a C^*-algebra A to a C^*-algebra B do not in general form a set can be handled by limiting the cardinality of the C^*-algebras and modules in question. For example, given two C^*-algebras A and B, one can consider only right-Hilbert A–B bimodules with cardinality smaller than the larger of the cardinalities of A and B.

3.3 Inducing representations via bimodules

As seen in Example 3.1.3, the unitary equivalence classes of nondegenerate representations of a C^*-algebra A coincide with the morphisms from A to $\mathbb{C}$ in the category $\mathcal{C}$. We denote this class by $\mathrm{Rep}(A)$ as on page 9, and, to emphasize that we are dealing with Hilbert spaces in this case, we denote its elements by $[{}_AH_{\mathbb{C}}]$ or just $[H]$. This interpretation of $\mathrm{Rep}(A)$ allows us to use composition of morphisms in $\mathcal{C}$ to induce representations from one C^*-algebra to another.

Proposition/Definition 3.3.1. Let A and B be C^*-algebras and let ${}_AX_B$ be a right-Hilbert A–B bimodule. Then there is a map

$$\mathrm{ind}^X\colon \mathrm{Rep}(B) \to \mathrm{Rep}(A),\ [H] \mapsto [H] \circ [X] = [X \otimes_B H]$$

which is bijective if X is an imprimitivity A–B bimodule.

Proposition 3.3.2 (Proposition 6.26 in [Rie74])**.** If, in the situation of the preceding proposition/definition, $\mathrm{Rep}(A)$ and $\mathrm{Rep}(B)$ are equipped with the Fell topologies, then the map ind^X is continuous. In particular, if X is an imprimitivity A–B bimodule, then ind^X is a homeomorphism.

Remark 3.3.3. By taking direct sums of right-Hilbert bimodules we can define sums of morphisms in $\mathcal{C}$. This operation is commutative and obeys the distributive law with respect to composition, which implies that induction of representations preserves direct sums. Thus, if X is an imprimitivity A–B bimodule, then the map ind^X from Proposition/Definition 3.3.1 restricts to a homeomorphism between $\widehat{B}$ and $\widehat{A}$.

Example 3.3.4. Let A and B be C^*-algebras and let $\varphi\colon A \to \mathcal{M}(B)$ be a nondegenerate *-homomorphism. Then induction via the right-Hilbert A–B bimodule ${}_AB_B$ from Example 3.1.1 (i) coincides with the map

$$\mathrm{Rep}(B) \to \mathrm{Rep}(A),\ \pi \mapsto \overline{\pi} \circ \varphi,$$

where $\overline{\pi}$ is the extension of π to $M(A)$, because for every $[H] \in \operatorname{Rep}(B)$ we have $[B \otimes_B H] = [{}_A H_{\mathbb{C}}]$, where the action of A on H is given by $a\xi = \varphi(a)\xi$ for all $a \in A$, $\xi \in H$.

3.4 The Morita equivalence of $C_0(G \times_H Y) \rtimes G$ and $C_0(Y) \rtimes H$

As seen in Theorem 1.1.7 and Remark 1.1.8 (i), a proper G-space X is locally induced from compact subgroups, that is, for every $x \in X$ there exist a G-invariant open neighborhood U_x, a compact subgroup $L_x \leq G$, and an L_x-invariant subset $Y_x \subseteq U_x$ such that U_x is homeomorphic to $G \times_{L_x} Y_x$. We will show in this section that the C^*-algebras $C_0(Y_x) \rtimes L_x$ and $C_0(U_x) \rtimes G$ are Morita equivalent, which by Remark 3.3.3 implies that they have homeomorphic spectra. This material is based on Section 3.6 in [Wil07].

$C_0(G \times_H Y)$ as an induced algebra

Let H be a closed subgroup of a locally compact group G, and let (A, H, α) be a covariant system. Then the induced algebra of the system (A, H, α) and G is defined as

$$\begin{aligned} \operatorname{ind}_H^G(A) := \operatorname{ind}_H^G(A, \alpha) := \{F \in C_b(G, A) \mid \forall s \in G, h \in H :\ & F(sh) = \alpha_{h^{-1}} F(s) \\ & \text{and } G/H \to \mathbb{R}_{\geq 0},\ sH \mapsto \|F(s)\| \in C_0(G/H)\}, \end{aligned}$$

which coincides with the generalized fixed point algebra $C_0(G \times_H A)$ from page 13 when H acts on G via $hg = gh^{-1}$ for all $h \in H$, $g \in G$ as on page 5. The group G acts on $\operatorname{ind}_H^G(A) = C_0(G \times_H A)$ by left translation.

Example 3.4.1. Let G be a locally compact group and let $H \leq G$ be a closed subgroup which acts on a locally compact Hausdorff space Y. Then the map

$$\delta \colon C_0(G \times_H C_0(Y)) \to C_0(G \times_H Y)$$

given by

$$\delta(F)([s, y]) = F(s)(y)$$

for all $F \in C_0(G \times_H C_0(Y))$, $[s, y] \in G \times_H Y$ is an isomorphism. The calculation

$$\delta(g \cdot F)[s, y] = (g \cdot F)(s)(y) = F(g^{-1}s)(y) = \delta(F)([g^{-1}s, y]) = \delta(F)(g^{-1}[s, y]) = (g \cdot \delta(F))[s, y]$$

for all $F \in C_0(G \times_H C_0(Y))$, $g, s \in G$, and $y \in Y$, shows that δ is covariant with respect to the G-actions on $C_0(G \times_H C_0(Y))$ and $C_0(G \times_H Y)$. It thus follows that

$$\operatorname{ind}_H^G(C_0(Y)) \rtimes G = C_0(G \times_H C_0(Y)) \rtimes G \cong C_0(G \times_H Y) \rtimes G. \tag{3.4.1}$$

Morita equivalence for induced G-spaces

It is proved in Corollary 4.17 in [Wil07] that $\operatorname{ind}_H^G(C_0(Y)) \rtimes G$ and $C_0(Y) \rtimes H$ are Morita equivalent. This is a consequence of Raeburn's symmetric imprimitivity theorem from [Rae88], which was inspired by Green's work in [Gre78], and the proof of which is also given in [Wil07]. We present a modified version of Corollary 4.17 from [Wil07] which is adapted to the identification from (3.4.1) above.

Proposition 3.4.2. Let G be a locally compact group and let H be a closed subgroup of H which acts on a locally compact Hausdorff space Y. View $E_0 := C_c(G \times (G \times_H Y))$ and $B_0 := C_c(H \times Y)$ as dense subalgebras of $C_0(G \times_H Y) \rtimes G$ and $C_0(Y) \rtimes H$, respectively, and let $Z_0 := C_c(G \times Y)$. For all $c \in E_0$, $w, z \in Z_0$, $b \in B_0$, $s, t \in G$, $h \in H$, and $y \in Y$ define

$$\begin{aligned} c \cdot w(s,y) &= \int_G \Delta_G(t)^{\frac{1}{2}} c(t,[s,y]) w(t^{-1}s, y) d\mu_G(t) && (3.4.2) \\ w \cdot b(s,y) &= \int_H \Delta_H(h)^{-\frac{1}{2}} w(sh^{-1}, hy) b(h, hy) d\mu_H(h) \\ {}_{E_0}\langle w, z\rangle(t,[s,y]) &= \Delta_G(t)^{-\frac{1}{2}} \int_H w(sh, h^{-1}y) \overline{z(t^{-1}sh, h^{-1}y)} d\mu_H(h) \\ \langle w, z\rangle_{B_0} &= \Delta_H(h)^{-\frac{1}{2}} \int_G \overline{w(t^{-1}, y)} z(t^{-1}h, h^{-1}y) d\mu_G(t). && (3.4.3) \end{aligned}$$

Then Z_0 is an E_0–B_0 pre-imprimitivity bimodule, and its completion Z is a $C_0(G \times_H Y) \rtimes G$–$C_0(Y) \rtimes H$ imprimitivity bimodule.

As outlined in Section 3.3 on induction of representations via bimodules, the existence of the $C_0(G \times_H Y) \rtimes G$–$C_0(Y) \rtimes H$ imprimitivity bimodule Z implies that there is a homeomorphism

$$\mathrm{ind}^Z \colon (C_0(Y) \rtimes H)\widehat{\ } \to (C_0(G \times_H Y) \rtimes G)\widehat{\ }.$$

For every $\varrho \in (C_0(Y) \rtimes H)\widehat{\ }$, the induced representation is given by

$$\mathrm{ind}^Z \varrho = [V_\varrho] \circ [Z] = [Z \otimes_{C_0(Y) \rtimes H} V_\varrho].$$

By definition of the balanced tensor product, its action on $Z \otimes_{C_0(Y) \rtimes H} V_\varrho$ is given by

$$(\mathrm{ind}^Z \varrho)_c(z \otimes v) = c \cdot z \otimes v \qquad (3.4.4)$$

for all $c \in C_0(G \times_H Y) \rtimes G$, $z \in Z$, and $v \in V_\varrho$, where $c \cdot z$ is the extension of the operation defined in (3.4.2).

3.5 The reduction to the compact case

In this section we present a proposition which allows us to use the fact that proper G-spaces are locally induced from compact subgroups to reduce the proof of openness of ind^G to the situation of compact group actions. The idea for this reduction procedure is taken from [EE], where it is used for proper G-spaces with Palais' slice property. First, we need one more map:

Lemma 3.5.1. Suppose that X is a proper G-space, and that $H \le G$ is a closed subgroup and $Y \subseteq X$ is an H-space such that $G \times_H Y$ is homeomorphic to X via $[g, y] \to gy$ as in Proposition 1.1.6. Then the inclusion $\mathrm{Stab}(Y)\widehat{\ } \hookrightarrow \mathrm{Stab}(X)\widehat{\ }$ induces a continuous bijection

$$\iota \colon H\backslash\mathrm{Stab}(Y)\widehat{\ } \to G\backslash\mathrm{Stab}(X)\widehat{\ },\ [y, H_y, \sigma] \to [y, H_y, \sigma].$$

It can also be shown that the map ι is open, and hence a homeomorphism. But once we have proved that ind^K is a homeomorphism for a compact group K (for which ι is not needed), the openness of ι is a direct consequence of Proposition 3.5.2 below, and so we omit the proof.

Proof of Lemma 3.5.1. Recall from Remark 1.1.8(iii) that $G_y \leq H$, and thus $H_y = G_y$ for all $y \in Y$. By this and by definition of the group actions on the $\mathrm{Stab}(\cdot)\widehat{\ }$-spaces it follows that the canonical map $\mathrm{Stab}(Y)\widehat{\ } \to G\backslash\mathrm{Stab}(X)\widehat{\ }$ factors through the map ι as given above. It is easy to see that ι is injective.

Let now $[x, G_x, \sigma] \in G\backslash\mathrm{Stab}(X)\widehat{\ }$. Find $g \in G$ and $y \in Y$ such that $x = gy$, then

$$[x, G_x, \sigma] = [g(y, G_y, g^{-1}\sigma)] = [y, G_y, g^{-1}\sigma] = \iota([y, H_y, g^{-1}\sigma]),$$

which shows that ι is surjective. Let now $(y_\nu, H_\nu, \sigma_\nu)_{\nu\in N}$ be a net in $\mathrm{Stab}(Y)\widehat{\ }$ which converges to an element (y, H', σ') in $Y \times S(H)\widehat{\ }$. Then, as $Y \subseteq X$, and by the definition of convergence in $S(G)\widehat{\ }$ (see Theorem 2.2.7) it follows that $(y_\nu, H_\nu, \sigma_\nu) \to (y, H', \sigma')$ in $X \times S(G)\widehat{\ }$, too. This implies that the inclusion of $\mathrm{Stab}(Y)\widehat{\ }$ into $\mathrm{Stab}(X)\widehat{\ }$ and hence also the canonical map $\mathrm{Stab}(Y)\widehat{\ } \to G\backslash\mathrm{Stab}(X)\widehat{\ }$ are continuous. It now follows from the universal property of the quotient topology on $H\backslash\mathrm{Stab}(Y)\widehat{\ }$ that ι is continuous, too. □

Proposition 3.5.2 (Proposition 3.13 in [EE])**.** Let X be a proper G-space, let $K \leq G$ be a compact subgroup, and let $Y \subseteq X$ be such that Y is a K-space and $G \times_K Y$ is homeomorphic to X as above. Let Z be the $C_0(X) \rtimes G$–$C_0(Y) \rtimes K$ imprimitivity bimodule from Proposition 3.4.2. Then the diagram of bijective maps

$$\begin{array}{ccc} K\backslash\mathrm{Stab}(Y)\widehat{\ } & \xrightarrow{\ \iota\ } & G\backslash\mathrm{Stab}(X)\widehat{\ } \\ \Big\downarrow{\scriptstyle \mathrm{ind}^K} & & \Big\downarrow{\scriptstyle \mathrm{ind}^G} \\ (C_0(Y) \rtimes K)\widehat{\ } & \xrightarrow[\mathrm{ind}^Z]{} & (C_0(X) \rtimes G)\widehat{\ }, \end{array}$$

commutes.

Proof. Recall that the modular function Δ_K is constantly one because K is compact. We just write Δ instead of Δ_G. Let $(y, K_y, \sigma) \in \mathrm{Stab}(Y)\widehat{\ }$. Let π^K denote the representation $\pi^{y,\sigma} = \mathrm{ind}_{K_y}^K(\mathrm{ev}_y \rtimes \sigma)$ in $(C_0(Y) \rtimes K)\widehat{\ }$, write π^G for $\pi^{y,\sigma} := \mathrm{ind}_{K_y}^G(\mathrm{ev}_y \rtimes \sigma)$ in $(C_0(X) \rtimes G)\widehat{\ }$, and denote their respective Hilbert spaces by H^K and H^G. Recall that

$$H^K = \{\xi \in L^2(K, V_\sigma) \mid \forall l \in K_y, k \in K:\ \xi(kl) = \sigma(l^{-1})\xi(k)\},$$

and that

$$\pi^K(f)\xi(k) = \int_K f(l, ky)\xi(l^{-1}k)dl \tag{3.5.1}$$

for all $f \in C_c(K \times Y)$, $\xi \in H^K$, and all $k \in K$, see Example 1.5.6(ii). The space H^G and the representation π^G are defined accordingly. We have to show that $\mathrm{ind}^Z(\pi^K)$ is unitarily equivalent to π^G, i.e., we need a unitary operator

$$\Phi\colon Z \otimes_{C_0(Y)\rtimes K} H^K \to H^G$$

which intertwines $\mathrm{ind}^Z(\pi^K)$ and π^G. Define

$$\Phi(z \otimes \xi)(g) = \Delta(g)^{-\frac{1}{2}} \int_K z(gk^{-1}, ky)\xi(k)dk \tag{3.5.2}$$

for all $z \in Z_0 = C_c(G \times Y)$, $\xi \in H^K$, and all $g \in G$. We have to show that

(i) Φ is well-defined, i.e., it maps into H^G,

(ii) Φ preserves inner products,

(iii) Φ intertwines $\mathrm{ind}^Z(\pi^K)$ and π^G.

It follows from (i) and (ii) that Φ extends to a norm preserving operator from $Z \otimes_{C_0(Y)\rtimes K} H^K$ to H^G, and since $\mathrm{ind}^Z(\pi^K)$ and π^G are irreducible, (iii) then implies that Φ is a unitary operator, as required (Corollary 6.1.9 in [DE09]).

Proof of (i): Let $z \in C_c(G \times Y)$ and let ξ be a continuous element of H^K. Then $\Phi(z \otimes \xi)$ is a continuous function which has compact support in G, because z has compact support in $G \times Y$, and and for all $g \in G$, $l \in K_y$ we get

$$\begin{aligned}
\Phi(z \otimes \xi)(gl) &= \Delta(gl)^{-\frac{1}{2}} \int_K z(glk^{-1}, ky)\xi(k)dk \\
&= \Delta(g)^{-\frac{1}{2}} \int_K z(gk^{-1}, ky)\xi(kl)dk \\
&= \Delta(g)^{-\frac{1}{2}} \int_K z(gk^{-1}, ky)\sigma(l^{-1})\xi(k)dk \\
&= \sigma(l^{-1})\Delta(g)^{-\frac{1}{2}} \int_K z(gk^{-1}, ky)\xi(k)dk \\
&= \sigma(l^{-1})\Phi(z \otimes \xi)(g),
\end{aligned}$$

where the step from the first to the second line follows by replacing k by kl and using that K is unimodular and that $l \in K_y$.

Proof of (ii): Let $w, z \in C_c(G \times Y)$ and let ξ and η be continuous elements of H^K. Then

$$\begin{aligned}
&\langle \Phi(z \otimes \xi), \Phi(w \otimes \eta)\rangle_{H^G} \\
&\quad= \int_G \langle \Phi(z \otimes \xi)(g), \Phi(w \otimes \eta)(g)\rangle_{V_\sigma} dg \\
&\quad= \int_G \Delta(g)^{-1} \left\langle \int_K z(gk^{-1}, ky)\xi(k)dk, \int_K w(gl^{-1}, ly)\eta(l)dl \right\rangle_{V_\sigma} dg \\
&\quad= \int_G \Delta(g)^{-1} \int_K \int_K \overline{z(gk^{-1}, ky)} w(gl^{-1}, ly) \langle \xi(k), \eta(l)\rangle_{V_\sigma} dk\, dl\, dg,
\end{aligned} \tag{3.5.3}$$

and, with the action of $C_0(Y) \rtimes K$ on H^K given by π^K,

$$\begin{aligned}
&\langle z \otimes \xi, w \otimes \eta\rangle_{Z \otimes_{C_0(Y)\rtimes K} H^K} \\
&\quad= \left\langle \xi, \langle z, w\rangle_{C_0(Y)\rtimes K} \cdot \eta \right\rangle_{H^K} \\
&\quad= \int_K \left\langle \xi(k), \langle z, w\rangle_{C_0(Y)\rtimes K} \cdot \eta(k) \right\rangle_{V_\sigma} dk \\
&\quad= \int_K \left\langle \xi(k), \pi^K(\langle z, w\rangle_{C_0(Y)\rtimes K})\eta(k) \right\rangle_{V_\sigma} dk \\
&\quad\overset{(3.5.1)}{=} \int_K \left\langle \xi(k), \int_K \langle z, w\rangle_{C_0(Y)\rtimes K} (l, ky)\eta(l^{-1}k)dl \right\rangle_{V_\sigma} dk \\
&\quad\overset{(3.4.3)}{=} \int_K \int_K \left\langle \xi(k), \int_G \overline{z(g^{-1}, ky)} w(g^{-1}l, l^{-1}ky)dg\, \eta(l^{-1}k) \right\rangle_{V_\sigma} dl\, dk
\end{aligned}$$

$$
\begin{aligned}
&= \int_K \int_K \int_G \left\langle \xi(k), \overline{z(g^{-1}, ky)} w(g^{-1}l, l^{-1}ky)\eta(l^{-1}k) \right\rangle_{V_\sigma} dg\, dl\, dk \\
&\overset{l \to kl^{-1}}{=} \int_K \int_K \int_G \overline{z(g^{-1}, ky)} w(g^{-1}kl^{-1}, ly) \left\langle \xi(k), \eta(l) \right\rangle_{V_\sigma} dg\, dl\, dk \\
&\overset{g \to kg^{-1}}{=} \int_K \int_K \int_G \Delta g^{-1} \overline{z(gk^{-1}, ky)} w(gl^{-1}, ly) \left\langle \xi(k), \eta(l) \right\rangle_{V_\sigma} dg\, dl\, dk,
\end{aligned}
$$

which is equal to (3.5.3) by Fubini. This shows (ii), and it follows that Φ extends to the whole space $Z \otimes_{C_0(Y) \rtimes K} H^K$.

Proof of (iii): Let $F \in C_c(G \times (G \times_K Y))$, $z \in Z_0$, $\xi \in H^K$, and $g \in G$. We show that

$$
\left(\Phi \circ \operatorname{ind}^Z(\pi^K)(F)\right)(z \otimes \xi)(g) = \left(\pi^G(F) \circ \Phi\right)(z \otimes \xi)(g).
$$

We have

$$
\begin{aligned}
&\left(\Phi \circ \operatorname{ind}^Z(\pi^K)(F)\right)(z \otimes \xi)(g) \\
&\quad \overset{(3.4.4)}{=} \Phi(F \cdot z \otimes \xi)(g) \\
&\quad \overset{(3.5.2)}{=} \Delta(g)^{-\frac{1}{2}} \int_K (F \cdot z)(gk^{-1}, ky)\xi(k)dk \\
&\quad \overset{(3.4.2)}{=} \Delta(g)^{-\frac{1}{2}} \int_K \int_G F(t, \underbrace{[gk^{-1}, ky]}_{=[g,y]}) z(t^{-1}gk^{-1}, ky)\Delta(t)^{\frac{1}{2}} dt\, \xi(k)dk,
\end{aligned} \tag{3.5.4}
$$

and

$$
\begin{aligned}
&\left(\pi^G(F) \circ \Phi\right)(z \otimes \xi)(g) \\
&\qquad = \pi^G(F)(\Phi(z \otimes \xi))(g) \\
&\qquad \overset{(3.5.1)}{=} \int_G F(t, g[e, y])\Phi(z \otimes \xi)(t^{-1}g)dt \\
&\qquad = \int_G F(t, [g, y])\Delta(t^{-1}g)^{-\frac{1}{2}} \int_K z(t^{-1}gk^{-1}, ky)\xi(k)dk\, dt,
\end{aligned}
$$

which, using multiplicativity of Δ and applying Fubini, equals (3.5.4). This implies (iii), and the proof is complete. □

Chapter 4

Representation theory of compact groups and ideals in $L^2(K)$

In this chapter we will first present the Peter-Weyl Theorem and related results, then we will show how subrepresentations of the left regular representation λ^K of a compact group K correspond to ideals in $L^2(K)$. In Section 4.3 we show that, for a general locally compact group G and a compact subgroup $K \leq G$, the space $L^2(G)$ can be written as a space induced from $L^2(K)$. We apply this result in Section 4.4 to analyze what happens if we restrict continuous elements of $L^2(K)$ to a closed subgroup H of K, which will be very helpful in the proof of openness of ind^K for a compact group K. In Section 4.5 we supply the proof of Lemma 1.3.4 on convergence of representations of a compact group to an irreducible limit.

4.1 The Peter-Weyl Theorem

We start with a short review of some important facts on compact groups, then we present the idea of matrix elements. We close the section with the Peter-Weyl theorem. For a careful treatment of the material of this section the reader may consult [DE09] or [Fol95]; part of the notation is also influenced by [EE].

Basic facts on representations of compact groups

Recall from page 7 that compact groups are always unimodular, that is, the Haar measure is always left and right invariant. We will always work with normalized Haar measure μ, so that $\mu(K) = 1$. Representations of compact groups have several very useful properties:

Theorem 4.1.1. Let K be a compact group. Then every irreducible representation of K is finite-dimensional, and every unitary representation of K is a direct sum of irreducible ones.

Definition 4.1.2. Let K be a compact group, let ϱ and σ be unitary representations of K with σ irreducible. We define the isotype $V_\varrho(\sigma)$ of σ in ϱ to be the closed linear span of all irreducible subspaces of V_ϱ on which ϱ is equivalent to σ.

The following proposition, which is a consequence of Schur's lemma (Lemma 1.2.1), shows that the space of every unitary representation ϱ of K decomposes into the isotypes of the irreducible representations which are contained in ϱ.

Proposition 4.1.3. Let K be a compact group, let ϱ be a unitary representation of K and let σ and σ' be irreducible representations of K. Then $V_\varrho(\sigma) \perp V_\varrho(\sigma')$ whenever σ and σ' are not equivalent. For every irreducible subspace M of $V_\varrho(\sigma)$, the restriction $\varrho|_M$ is equivalent to σ.

Matrix elements

Definition 4.1.4. Let K be a compact group and let σ be a unitary representation of K. For any choice of $u, v \in V_\sigma$ define

$$\varphi^\sigma_{u,v} \colon K \to \mathbb{C},\ x \mapsto \langle \sigma(x^{-1})u, v\rangle.$$

These functions are called matrix elements for σ. If e_i, e_j are members of an orthonormal basis for V_σ, then we write φ^σ_{ij} instead of $\varphi^\sigma_{e_i,e_j}$.

It is easy to check that equivalent representations σ and σ' have the same set of matrix elements. If K is a compact group, σ is a representation of K, and $(e_j)_{j\in J}$ is an orthonormal basis for V_σ, then the entries of the matrices of $\sigma(k)$ and the dual representation $\sigma^*(k)$ with respect to this basis are given by

$$\sigma_{ij}(k) = \langle \sigma(k)e_j, e_i\rangle = \overline{\langle \sigma(k^{-1})e_i, e_j\rangle} = \overline{\varphi^\sigma_{ij}(k)}$$

and

$$\sigma^*_{ij}(k) = \langle \sigma^*(k)e^*_j, e^*_i\rangle_{V^*_\sigma} = \langle (\sigma(k)e_j)^*, e^*_i\rangle_{V^*_\sigma} = \langle e_i, \sigma(k)e_j\rangle_{V_\sigma} = \varphi^\sigma_{ij}(k) \tag{4.1.1}$$

for all $k \in K$ and all $i, j \in J$, see the definition of the dual representation on page 8.

Since matrix elements are continuous functions and thus in particular elements of $L^p(K)$ for every $p \in \mathbb{N}$, we can define a map

$$\phi\colon \bigoplus_{\tau\in\widehat{K}} V_\tau \otimes V^*_\tau \to L^2(K)$$

by

$$\phi(v \otimes w^*) = \sqrt{d_\sigma}\varphi^\sigma_{v,w} \tag{4.1.2}$$

for all elementary tensors $v \otimes w^* \in V_\sigma \otimes V^*_\sigma$ for every $\sigma \in \widehat{K}$.

The matrix elements of irreducible representations satisfy the following relations in $L^2(K)$, which underly most of the following results:

Lemma 4.1.5 (Schur Orthogonality Relations)**.** Let σ and σ' be irreducible representations of a compact group K. Then

(i) If $\sigma \neq \sigma'$ in $\widehat{K}$, then $\phi(V_\sigma \otimes V^*_\sigma)$ and $\phi(V_{\sigma'} \otimes V^*_{\sigma'})$ are orthogonal subspaces of $L^2(K)$;

(ii) If $\{e_j \mid j \in \mathbb{N}_{\leq d_\sigma}\}$ is any orthonormal basis for V_σ, then $\{\sqrt{d_\sigma}\varphi^\sigma_{ij} \mid i, j \in \mathbb{N}_{\leq d_\sigma}\}$ is an orthonormal basis for $\phi(V_\sigma \otimes V^*_\sigma)$.

Corollary 4.1.6. Let σ and σ' be irreducible representations of a compact group K such that $\sigma \neq \sigma'$ in $\widehat{K}$. Let $\{e_j \mid j \in \mathbb{N}_{\leq d_\sigma}\}$ and $\{f_l \mid l \in \mathbb{N}_{\leq d_{\sigma'}}\}$ be orthonormal bases for the spaces V_σ and $V_{\sigma'}$. Then convolution and involution of the corresponding matrix elements as elements of $L^2(K)$ are given by

$$\begin{aligned} \varphi_{ij}^\sigma * \varphi_{kl}^{\sigma'} &= 0 \\ \varphi_{ij}^\sigma * \varphi_{mn}^\sigma &= \frac{1}{d_\sigma}\delta_{jm}\varphi_{in}^\sigma \\ (\varphi_{ij}^\sigma)^* &= \varphi_{ji}^\sigma \end{aligned}$$

for all $i, j, m, n \in \mathbb{N}_{\leq d_\sigma}$ and all $k, l \in \mathbb{N}_{\leq d_{\sigma'}}$.

The proof consists of easy calculations based on the Schur Orthogonality Relations.

The isomorphism $\bigoplus_{\tau\in\widehat{K}} V_\tau \otimes V_\tau^* \cong L^2(K)$

The correspondence between the spaces $\bigoplus_{\tau\in\widehat{K}} V_\tau \otimes V_\tau^*$ and $L^2(K)$ deserves some more attention. A calculation using Schur's Orthogonality Relations shows that the map ϕ from (4.1.2) is norm preserving (this is why we need the factor $\sqrt{d_\sigma}$). It is shown in Chapter 5.2 in [Fol95] that ϕ is also surjective, so altogether it is a unitary isomorphism between $\bigoplus_{\tau\in\widehat{K}} V_\tau \otimes V_\tau^*$ and $L^2(K)$, where $\bigoplus_{\tau\in\widehat{K}} V_\tau \otimes V_\tau^*$ is endowed with the canonical Hilbert space structure.

Combining the formulas from Corollary 4.1.6 with the definition of ϕ we can transfer convolution and involution from $L^2(K)$ to $\bigoplus_{\tau\in\widehat{K}} V_\tau \otimes V_\tau^*$ as follows:

Proposition 4.1.7. Let K be a compact group and suppose that σ and σ' are irreducible representations with $\sigma \neq \sigma'$ in $\widehat{K}$, and that $\{e_j \mid j \in \mathbb{N}_{\leq d_\sigma}\}$ and $\{f_l \mid l \in \mathbb{N}_{\leq d_{\sigma'}}\}$ are orthonormal bases for V_σ and $V_{\sigma'}$. Then the formulas

$$\begin{aligned} (e_i \otimes e_j^*) * (f_k \otimes f_l^*) &= 0, \\ (e_i \otimes e_j^*) * (e_m \otimes e_n^*) &= \frac{1}{\sqrt{d_\sigma}}\delta_{jm} e_i \otimes e_n^*, \\ (e_i \otimes e_j^*)^* &= e_j \otimes e_i^* \end{aligned}$$

for all $i, j, m, n \in \mathbb{N}_{\leq d_\sigma}$ and all $k, l \in \mathbb{N}_{\leq d_{\sigma'}}$ define a convolution and an involution on $\bigoplus_{\tau\in\widehat{K}} V_\tau \otimes V_\tau^*$ such that it is isomorphic to $L^2(K)$ as a *-algebra.

Notice that the * denotes both the elements of the dual space V_σ^* and the involution here, but since we probably won't run into this ambiguity again, we don't worry about dissolving it here.

The formulas in Proposition 4.1.7 imply that, if $\sigma \in \widehat{K}$ and $e \in V_\sigma$ is a unit vector, then $V_\sigma \otimes e^*$ (resp. $e \otimes V_\sigma^*$) are left (resp. right) ideals in $\bigoplus_{\tau\in\widehat{K}} V_\tau \otimes V_\tau^*$, and $V_\sigma \otimes V_\sigma^*$ is a two-sided ideal. We will pursue the issue of ideals in $L^2(K)$ further in Section 4.2 below.

The Peter-Weyl theorem

We now define representations $\lambda^{K,\otimes}$ and $\varrho^{K,\otimes}$ of K on $\bigoplus_{\tau\in\widehat{K}} V_\tau \otimes V_\tau^*$ such that $\lambda^{K,\otimes} \approx \lambda^K$ and $\varrho^{K,\otimes} \approx \varrho^K$ modulo the unitary equivalence ϕ defined above. Let $\mathrm{id}_V : K \to U(V),\ k \mapsto \mathrm{id}_V$ denote the trivial representation on any Hilbert space V.

Proposition 4.1.8. Let K be a compact group and define unitary representations $\lambda^{K,\otimes}$ and $\varrho^{K,\otimes}$ of K on $\bigoplus_{\tau\in\widehat{K}} V_\tau \otimes V_\tau^*$ by

$$\lambda^{K,\otimes} := \bigoplus_{\tau\in\widehat{K}} \tau \otimes \mathrm{id}_{V_{\tau^*}}, \qquad \varrho^{K,\otimes} := \bigoplus_{\tau\in\widehat{K}} \mathrm{id}_{V_\tau} \otimes \tau^*.$$

Then $\lambda^{K,\otimes} \approx \lambda^K$ and $\varrho^{K,\otimes} \approx \varrho^K$, where the unitary equivalence is given by ϕ.

These characterizations of the left and right regular representation are also used in [EE]. We can now state the following version of the Peter-Weyl Theorem:

Theorem 4.1.9 (Peter-Weyl Theorem)**.** Let K be a compact group. Then the following statements are true:

(i) The Hilbert spaces and *-algebras $L^2(K)$ and $\bigoplus_{\tau\in\widehat{K}} V_\tau \otimes V_\tau^*$ are unitarily isomorphic via the map ϕ as defined in (4.1.2). If, for every $\tau \in \widehat{K}$, we have an orthonormal basis $\{e_i^\tau \mid i \in \mathbb{N}_{\leq d_\tau}\}$, then

$$\{\sqrt{d_\tau}\varphi_{ij} \mid \tau \in \widehat{K},\ i,j \in \mathbb{N}_{\leq d_\tau}\}$$

is an orthonormal basis for $L^2(K)$.

(ii) For every $\tau \in \widehat{K}$ and every unit vector $e \in V_\tau$ the following holds: $V_\tau \otimes e^*$ is invariant under $\lambda^{K,\otimes}$, and $\lambda^{K,\otimes}|_{V_\tau\otimes e^*}$ is unitarily equivalent to τ; analogously, $e \otimes V_\tau^*$ is invariant under $\varrho^{K,\otimes}$, and $\varrho^{K,\otimes}|_{e\otimes V_\tau^*}$ is unitarily equivalent to τ^*. The multiplicity of τ (resp. τ^*) in $\lambda^{K,\otimes}$ (resp. $\varrho^{K,\otimes}$) is d_τ.

The proof of (i) follows from the results on $\bigoplus_{\tau\in\widehat{K}} V_\tau \otimes V_\tau^*$ and ϕ above and the fact that

$$\{e_i^\tau \mid \tau \in \widehat{K},\ i,j \in \mathbb{N}_{\leq d_\tau}\}$$

is an orthonormal basis for $\bigoplus_{\tau\in\widehat{K}} V_\tau \otimes V_\tau^*$. The statements of (ii) follow from the definition of $\lambda^{K,\otimes}$ and $\varrho^{K,\otimes}$.

4.2 Minimal ideals in $L^2(K)$

It is now quite easy to analyze the ideal structure of $L^2(K)$ for a compact group K in terms of $\bigoplus_{\tau\in\widehat{K}} V_\tau \otimes V_\tau^*$. As a starting point we take the following proposition:

Proposition 4.2.1. Let K be a compact group and let V be a closed subspace of $L^2(K)$. Then V is a left (right) ideal if and only if it is invariant under λ^K (ϱ^K).

In [Fol95], Theorem 2.43, this proposition is formulated for $L^1(K)$. Since K is compact and we have assumed that $\int_K 1dk = 1$, we have that $L^2(K) \subseteq L^1(K)$, which implies that Folland's proof can be transferred to this situation.

Combining our version of the Peter-Weyl Theorem with Proposition 4.2.1 we see that the minimal left (right) ideals of $L^2(K) \cong \bigoplus_{\tau\in\widehat{K}} V_\tau \otimes V_\tau^*$ are exactly the spaces of the form $V_\sigma \otimes e^*$ (resp. $e \otimes V_\sigma^*$) for every $\sigma \in \widehat{K}$ and every unit vector $e \in V_\sigma$. The minimal two-sided ideals are exactly the spaces $V_\sigma \otimes V_\sigma^*$ for every $\sigma \in \widehat{K}$, and $V_\sigma \otimes V_\sigma^*$ is the isotype $V_{\lambda^K}(\sigma)$ of σ in λ^K.

It also follows that every left (right) ideal $P \in L^2(K)$ is an invariant subspace for λ^K (ϱ^K), and so the Peter-Weyl Theorem implies that P decomposes into the minimal left (right) ideals which correspond to the irreducible subrepresentations of $\lambda^K|_P$ (resp. $\varrho^K|_P$).

The following observation will be important in the proof of openness of ind^K:

Remark 4.2.2. The preceding observations imply in particular that every minimal left, right, or two-sided ideal of $L^2(K)$ is finite-dimensional and contains continuous functions only.

Projections onto minimal ideals

If $\sigma \in \widehat{K}$ and $\{e_i \mid i \in \mathbb{N}_{\leq d_\sigma}\}$ is an orthonormal basis for V_σ, then the projection of the space $\bigoplus_{\tau \in \widehat{K}} V_\tau \otimes V_\tau^*$ onto the minimal two-sided ideal $V_\sigma \otimes V_\sigma^*$ is given by convolution with

$$p_\sigma := \sqrt{d_\sigma} \sum_{i=1}^{d_\sigma} e_i \otimes e_i^*.$$

This is easily verified using the formulas from Proposition 4.1.7. We also give a description of these projections in the $L^2(K)$-notation: Suppose that $\sigma \in \widehat{K}$ and that I is the corresponding minimal ideal of $L^2(K)$, i.e., I is the image of $V_\sigma \otimes V_\sigma^*$ in $L^2(K)$ under the isomorphism ϕ from above. Then the projection onto I is given by convolution with

$$p_I := \phi(p_\sigma) = d_\sigma \sum_{i=1}^{d_\sigma} \varphi_{ii}^\sigma \overset{(4.1.1)}{=} d_\sigma \sum_{i=1}^{d_\sigma} \sigma_{ii}^* = d_\sigma \mathrm{tr}(\sigma^*) = d_\sigma \chi_{\sigma^*}, \tag{4.2.1}$$

where χ_{σ^*} is the character of σ^* as on page 23.

4.3 $L^2(G)$ as an induced space

The following description of $L^2(G)$ will be used in Section 4.4 on restriction of continuous functions, and it also underlies the proof of Lemma 7.3.1 (Lemma 3.1 in [EE]). In their proof of this lemma, Echterhoff and Emerson also give the definition of the isomorphism below, and Baggett uses a similar construction in Paragraph 5.2 in [Bag68].

Let G be a locally compact group and let H be a compact subgroup of G. In this section we want to use the isomorphism $L^2(H) \cong \bigoplus_{\tau \in \widehat{H}} V_\tau \otimes V_\tau^*$ to establish an isomorphism

$$L^2(G) \cong \bigoplus_{\tau \in \widehat{H}} H_{\mathrm{ind}\,\tau} \otimes V_\tau^*. \tag{4.3.1}$$

To this end, we will induce $\lambda^H \approx \lambda^{H,\otimes} = \bigoplus_{\tau \in \widehat{H}} \tau \otimes \mathrm{id}_{V_\tau^*}$ up to G, and then show that $\mathrm{ind}_H^G \lambda^H = \bigoplus_{\tau \in \widehat{H}} \mathrm{ind}_H^G \tau \otimes \mathrm{id}_{V_\tau^*}$ is equivalent to λ^G. To prove the existence of the isomorphism in (4.3.1) we decompose it as follows:

$$L^2(G) \cong H_{\mathrm{ind}\,\lambda^H} \cong H_{\mathrm{ind}\,\lambda^{H,\otimes}} \cong \bigoplus_{\tau \in \widehat{H}} H_{\mathrm{ind}\,\tau} \otimes V_\tau^*. \tag{4.3.2}$$

In fact, all these isomorphisms will be implemented by unitaries, that at the same time implement the corresponding unitary equivalences

$$\lambda^G \approx \mathrm{ind}_H^G \lambda^H \approx \mathrm{ind}_H^G \lambda^{H,\otimes} \approx \bigoplus_{\tau \in \widehat{H}} \mathrm{ind}_H^G \tau \otimes \mathrm{id}_{V_\tau^*}.$$

The isomorphism $H_{\operatorname{ind}\lambda^H} \cong L^2(G)$

According to Proposition/Definition 1.5.1, the induced Hilbert space $H_{\operatorname{ind}\lambda^H}$ is given by

$$H_{\operatorname{ind}\lambda^H} = \{\xi \in L^2(G, L^2(H)) \mid \forall h \in H, g \in G:\ \xi(gh) = \lambda^H_{h^{-1}}\xi(g)\}.$$

We want to define a unitary operator $P\colon H_{\operatorname{ind}\lambda^H} \to L^2(G)$. We do this by giving an explicit formula for $P(\xi)$ for continuous elements ξ of $H_{\operatorname{ind}\lambda^H}$, and by supplying an inverse operator. Set

$$P(\xi)(g) := \xi(g)(e_H) \tag{4.3.3}$$

for all $\xi \in H_{\operatorname{ind}\lambda^H} \cap C(G, C(H))$ and all $g \in G$. To obtain an inverse operator for P, define

$$Q(\eta)(g)(h) := \eta(gh)$$

for all $\eta \in C_c(G)$, and all $g \in G,\ h \in H$. It is easily checked that $Q(\eta) \in H_{\operatorname{ind}\lambda^H}$ for each $\eta \in C_c(G)$. For all $\xi \in H_{\operatorname{ind}\lambda^H} \cap C(G, C(H))$, $g \in G$, and $h \in H$ we compute

$$(Q \circ P)(\xi)(g)(h) = P(\xi)(gh) = \xi(gh)(e_H) = \lambda^H_{h^{-1}}\xi(g)(e_H) = \xi(g)(h),$$

so we have $(Q \circ P)(\xi) = \xi$ for all continuous $\xi \in H_{\operatorname{ind}\lambda^H}$. A similar calculation shows that $(P \circ Q)(\eta) = \eta$ for all $\eta \in C_c(G)$. A computation using the definition of the scalar product on $H_{\operatorname{ind}\lambda^H}$ shows that P and Q are isometric on the dense subsets of continuous elements of $H_{\operatorname{ind}\lambda^H}$ and $L^2(G)$, respectively. It follows that P and Q can be extended to $H_{\operatorname{ind}\lambda^H}$ and $L^2(G)$, respectively, and that $Q \circ P = \mathrm{id}_{H_{\operatorname{ind}\lambda^H}}$ and $P \circ Q = \mathrm{id}_{L^2(G)}$. Being isometric surjections, the operators P and Q are unitary, and since they are inverse to each other it follows that $Q = P^*$.

A short calculation shows that P intertwines λ^G and $\operatorname{ind}_H^G \lambda^H$, as required.

The isomorphism $H_{\operatorname{ind}\lambda^H} \cong H_{\operatorname{ind}\lambda^{H,\otimes}}$

Let $\phi_H\colon \bigoplus_{\tau\in\widehat{H}} V_\tau \otimes V_\tau^* \to L^2(H)$ be the unitary isomorphism defined in (4.1.2) applied to H. This induces a canonical unitary isomorphism $\operatorname{ind}\phi_H$ between

$$H_{\operatorname{ind}\lambda^{H,\otimes}} = \{\xi \in L^2(G, \bigoplus_{\tau\in\widehat{H}} V_\tau \otimes V_\tau^*) \mid \forall h \in H, g \in G:\ \xi(gh) = \lambda^{H,\otimes}_{h^{-1}}\xi(g)\}$$

and the space $H_{\operatorname{ind}\lambda^H}$. For continuous elements of $H_{\operatorname{ind}\lambda^{H,\otimes}}$ this isomorphism is given by

$$((\operatorname{ind}\phi_H)\xi)(g) = \phi_H(\xi(g)) \tag{4.3.4}$$

for all $\xi \in H_{\operatorname{ind}\lambda^{H,\otimes}} \cap C(G, \bigoplus_{\tau\in\widehat{H}} V_\tau \otimes V_\tau^*)$ and all $g \in G$. The unitary $\operatorname{ind}\phi_H$ intertwines λ^H and $\lambda^{H,\otimes}$.

The isomorphism $\bigoplus_{\tau\in\widehat{H}} H_{\operatorname{ind}\tau} \otimes V_\tau^* \cong H_{\operatorname{ind}\lambda^{H,\otimes}}$

By definition, $\lambda^{H,\otimes} = \bigoplus_{\tau\in\widehat{H}} \tau \otimes \mathrm{id}_{V_{\tau^*}}$. Since induction preserves direct sums and commutes with amplifications (Proposition 9.18 in [Ech]) it follows that

$$\operatorname{ind}_H^G \lambda^{H,\otimes} \approx \bigoplus_{\tau\in\widehat{H}} \operatorname{ind}_H^G(\tau \otimes \mathrm{id}_{V_\tau^*}) \approx \bigoplus_{\tau\in\widehat{H}} \operatorname{ind}_H^G \tau \otimes \mathrm{id}_{V_\tau^*}$$

and

$$H_{\operatorname{ind}\lambda^{H,\otimes}} \cong \bigoplus_{\tau\in\widehat{H}} H_{\operatorname{ind}(\tau\otimes\mathrm{id}_{V_\tau^*})} \cong \bigoplus_{\tau\in\widehat{H}} H_{\operatorname{ind}\tau}\otimes V_\tau^*.$$

Explicitly, this isomorphism maps an elementary tensor $\xi\otimes w^*\in H_{\operatorname{ind}\tau}\otimes V_\tau^*$ with continuous ξ to the element

$$G\to V_\tau\otimes V_\tau^*,\ g\mapsto \xi(g)\otimes w^* \tag{4.3.5}$$

of $H_{\operatorname{ind}\lambda^{H,\otimes}}$.

The isomorphism $\bigoplus_{\tau\in\widehat{H}} H_{\operatorname{ind}\tau}\otimes V_\tau^*\cong L^2(G)$

Combining the results of the preceding paragraphs we obtain a unitary isomorphism between $\bigoplus_{\tau\in\widehat{H}} H_{\operatorname{ind}\tau}\otimes V_\tau^*$ and $L^2(G)$. To get an explicit formula, fix $\tau\in\widehat{H}$ and let $\xi\otimes w^*\in H_{\operatorname{ind}\tau}\otimes V_\tau^*$ be such that $\xi\in C(G,V_\tau)$. As seen in (4.3.5), this elementary tensor is mapped to

$$G\to V_\tau\otimes V_\tau^*,\ g\mapsto \xi(g)\otimes w^*$$

in $H_{\operatorname{ind}\lambda^{H,\otimes}}$. To this we can apply the isomorphism $\operatorname{ind}\phi_H$ from (4.3.4) as follows:

$$\begin{aligned}\operatorname{ind}\phi_H(g'\mapsto\xi(g')\otimes w^*)(g) &= \phi_H(\xi(g)\otimes w^*)\\ &= (h\mapsto \sqrt{d_\tau}\langle\tau(h^{-1})\xi(g),w\rangle)\\ &= (h\mapsto \sqrt{d_\tau}\langle\xi(gh),w\rangle)\end{aligned} \tag{4.3.6}$$

for all $g\in G$. Applying the isomorphism $P\colon H_{\operatorname{ind}\lambda^H}\to L^2(G)$ from (4.3.3) to (4.3.6), we obtain the function

$$g\mapsto \sqrt{d_\tau}\langle\xi(g),w\rangle_{V_\tau}$$

in $L^2(G)$. In short, we have a unitary isomorphism

$$\gamma\colon \bigoplus_{\tau\in\widehat{H}} H_{\operatorname{ind}\tau}\otimes V_\tau^*\to L^2(G), \tag{4.3.7}$$

which, for each elementary tensor $\xi\otimes w^*\in H_{\operatorname{ind}\tau}\otimes V_\tau^*$ with continuous ξ, is given by

$$\gamma(\xi\otimes w^*)(g) = \sqrt{d_\tau}\langle\xi(g),w\rangle$$

for all $g\in G$. It also follows that γ intertwines $\bigoplus_{\tau\in\widehat{H}}\operatorname{ind}_H^G\tau\otimes\mathrm{id}_{V_\tau^*}$ and λ^G.

4.4 Restriction of continuous elements of $L^2(K)$

Let now H be a closed subgroup of a compact group K. The results of the previous section, applied to K instead of G, imply in particular that each subspace $H_{\operatorname{ind}\tau}\otimes V_\tau^*$ is invariant for λ^K and hence a left ideal in $L^2(K)$. We now analyze what happens if we restrict continuous elements of $L^2(K)$ to H. The results of this section are based on the work of Baggett in Paragraphs 5.1 and 5.4 in [Bag68].

Restricting ideals in $L^2(K)$ to $L^2(H)$

We start with the following observation:

Lemma 4.4.1. If M is a finite-dimensional left ideal in $L^2(K)$, then all elements of M are continuous, and

$$M|_H := \{f|_H \mid f \in M\}$$

is a closed left ideal in $L^2(H)$.

Proof. Being finite-dimensional, M is a finite sum of minimal left ideals of $L^2(K)$. By Remark 4.2.2, the minimal left ideals contain continuous functions only, so this is also true for M. By Theorem 4.2.1 it now suffices to show that $M|_H$ is a closed invariant subspace for λ^H. Being a finite-dimensional vector space, $M|_H$ is closed, and since M is invariant under λ^K, it follows immediately that

$$\lambda_h^H(f|_H) = (\lambda_h^K(f))|_H \in M|_H$$

for every $h \in H$ and every $f \in M$, which shows that $M|_H$ is invariant under λ^H. □

We can describe the restriction of a continuous function quite explicitly when we use the isomorphisms $\gamma\colon \bigoplus_{\tau\in\widehat{H}} H_{\mathrm{ind}\,\tau} \otimes V_\tau^* \to L^2(K)$ from (4.3.7) and $\phi_H\colon \bigoplus_{\tau\in\widehat{H}} V_\tau \otimes V_\tau^* \to L^2(H)$ from (4.1.2):

Lemma 4.4.2. Let $\tau \in \widehat{H}$, let $\xi \in H_{\mathrm{ind}\,\tau}$ be continuous, and let $w^* \in V_\tau^*$. Then the restriction of $\gamma(\xi \otimes w^*) \in L^2(K)$ to H equals $\phi_H(\xi(e_K) \otimes w^*)$.

Proof. The definitions of the isomorphisms γ and ϕ_H give

$$\gamma(\xi \otimes w^*)|_H(h) = \sqrt{d_\tau}\langle \xi(h), w\rangle = \sqrt{d_\tau}\langle \tau(h^{-1})\xi(e_K), w\rangle = \phi_H(\xi(e_K) \otimes w^*)(h)$$

for all $h \in H$. □

Roughly speaking, the lemma says that restriction of continuous functions from $L^2(K)$ to $L^2(H)$ can be implemented by evaluating elements of $\bigoplus_{\tau\in\widehat{H}} H_{\mathrm{ind}\,\tau} \otimes V_\tau^*$ at the unit element e_K of K, which means that a continuous element $\xi \otimes w^* \in H_{\mathrm{ind}\,\tau} \otimes V_\tau^*$ restricts to the element $\xi(e_K) \otimes w^* \in V_\tau \otimes V_\tau^*$.

The following corollary contains two more precise statements on restriction, which will be used in the proof of openness of ind^K.

Corollary 4.4.3. Let K be a compact group and let H be a closed subgroup.

(i) Let $\sigma \in \widehat{H}$ and suppose that M is a left ideal of $L^2(K)$ which is contained in $H_{\mathrm{ind}\,\sigma} \otimes V_\sigma^*$. Then

$$M|_H := \{f|_H \mid f \in M \text{ is continuous}\}$$

is a closed left ideal in $L^2(H)$ which is contained in $V_\sigma \otimes V_\sigma^*$.

(ii) If M is a minimal left ideal in $L^2(K)$, then $M|_H$ defined as in Lemma 4.4.1 is also minimal in $L^2(H)$.

Proof. (i): It follows directly from Lemma 4.4.2 that $M|_H$ is contained in $V_\sigma \otimes V_\sigma^*$, which implies that $M|_H$ is a finite-dimensional and hence closed subspace of $L^2(H)$. It follows as in the proof of Lemma 4.4.1 that $M|_H$ is invariant for λ^H and thus a left ideal.

(ii): Suppose that M is a minimal left ideal in $L^2(K)$. Then M is finite-dimensional and contains continuous functions only, so $M|_H$ can be defined as in Lemma 4.4.1. It is a consequence of the description of $L^2(K)$ as $\bigoplus_{\tau\in\widehat{K}} V_\tau \otimes V_\tau^*$ that $L^2(K)$ decomposes into left ideals of the form $H_{\operatorname{ind}\tau} \otimes f^*$ with $\tau \in \widehat{H}$ and f^* a unit vector in V_τ^*.

By minimality of M, we can thus choose $\sigma \in \widehat{H}$ and a unit vector $e^* \in V_\sigma^*$ such that $M \subseteq H_{\operatorname{ind}\sigma} \otimes e^*$. It then follows by the above description of restriction that $M|_H \subseteq V_\sigma \otimes e^*$. By Lemma 4.4.1, $M|_H$ is a closed left ideal, so minimality of $V_\sigma \otimes e^*$ implies that $M|_H = V_\sigma \otimes e^*$ is a minimal left ideal in $L^2(H)$. □

Application to representations

Translated into the language of representations, the previous corollary reads as follows:

Remark 4.4.4. If M is as in (i) of Corollary 4.4.3, then there exists $n \in \mathbb{N}$ such that

$$\lambda^H|_{(M|_H)} = n\sigma.$$

If M is minimal, then it follows from (ii) of Corollary 4.4.3 that there exists $\sigma \in \widehat{H}$ such that

$$\lambda^H|_{(M|_H)} = \sigma.$$

The second statement is a slight refinement of Lemma 5.4-B in [Bag68], because we show that $\lambda^H|_{(M|_H)}$ is not just equivalent to a multiple of σ, but to σ itself.

In the proof of openness of ind^K it will be necessary to construct, for given $(K,\sigma) \in S(K)\widehat{\ }$ and $H \leq K$, an element $\sigma_H \in \widehat{H}$ with $\sigma_H \leq \sigma|_H$. For this we need one more lemma, the proof of which is based on the following fact, which is Theorem 1.2 in [Mac55].

Theorem 4.4.5. Let G be a locally compact group and let $\pi, \varrho \in \operatorname{Rep}(G)$. Suppose that $T \in B(V_\pi, V_\varrho)$ intertwines π and ϱ, i.e., $\varrho(g)T = T\pi(g)$ for all $g \in G$. Let $N := (\ker T)^\perp$ and $L := \overline{\operatorname{im} T}$. Then N and L are invariant subspaces for π and ϱ, respectively, and $\pi|_N$ is unitarily equivalent to $\varrho|_L$.

Lemma 4.4.6. Let $H \leq K$ be a closed subgroup, let M be a minimal left ideal in $L^2(K)$. Then

$$\lambda^H|_{(M|_H)} \leq (\lambda^K|_M)|_H.$$

Proof. Define $T\colon M \to M|_H,\ f \mapsto f|_H$. Then

$$\left(T \circ \left((\lambda^K|_M)|_H\right)(h)\right)(f)(l) = f(h^{-1}l) = \left((\lambda^H|_{(M|_H)})(h) \circ T\right)(f)(l)$$

for all $f \in M$ and for all $h, l \in H$, which shows that T intertwines $(\lambda^K|_M)|_H$ and $\lambda^H|_{(M|_H)}$. Since T is surjective, the above theorem implies that $\lambda^H|_{(M|_H)}$ is equivalent to the restriction of $(\lambda^K|_M)|_H$ to $(\ker T)^\perp$, which shows that $\lambda^H|_{(M|_H)}$ is a subrepresentation of $(\lambda^K|_M)|_H$, as required. □

4.5 A convergence lemma for nets in $\mathrm{Rep}(K)$ with limit in $\widehat{K}$

As promised, we now include the proof of Lemma 1.3.4, which was used in the proof that our closure operation on $\mathrm{Stab}(X)^\frown$ satisfies the Kuratowski closure axioms, and which will also be used when we compare our topology on $\mathrm{Stab}(X)^\frown$ with the one defined by Echterhoff and Emerson in [EE] under the assumption of Palais' slice property. Notice that its proof is completely independent of the results we obtained for the topology on $\mathrm{Stab}(X)^\frown$. For the convenience of the reader, we restate the lemma here:

Lemma 4.5.1 (Lemma 1.3.4, Lemma 4.8 in [EE])**.** Let K be a compact group, let $(\pi_\nu)_{\nu\in N}$ be a net in $\mathrm{Rep}(K)$, and let $\sigma \in \widehat{K}$. The following statements are equivalent:

(i) $\pi_\nu \to \sigma$;

(ii) There exists an index $\nu_0 \in N$ such that $\sigma \leq \pi_\nu$ for all $\nu \in N_{\geq \nu_0}$.

The proof requires some facts about containment and weak containment, which we put into the following lemma. We will not need all implications stated here, but leaving them out makes things seem incomplete.

Lemma 4.5.2. Let K be a compact group, let $\pi \in \mathrm{Rep}(K)$, and let $\sigma \in \widehat{K}$. The following statements are equivalent:

(i) $\sigma \leq \pi$;

(ii) $V_\sigma \otimes V_\sigma^* \nsubseteq \ker \pi$;

(iii) $\sigma \prec \pi$.

Proof. As seen above, the space $V_\sigma \otimes V_\sigma^*$ can be identified with a subset of $C(K)$ and hence of $C^*(K)$. It is an immediate consequence of the Schur Orthogonality Relations that σ is the only element of $\widehat{K}$ which is nontrivial on $V_\sigma \otimes V_\sigma^*$, i.e., which satisfies $V_\sigma \otimes V_\sigma^* \nsubseteq \ker \sigma$.

(i)⇒(iii): This follows directly from the definitions and requires neither the compactness assumption nor the irreducibility of σ.

(iii)⇒(ii): By contraposition. Suppose that $V_\sigma \otimes V_\sigma^* \subseteq \ker \pi$. It follows immediately that $\ker \pi \nsubseteq \ker \sigma$, because otherwise we had $V_\sigma \otimes V_\sigma^* \subseteq \ker \pi \subseteq \ker \sigma$, which would contradict $V_\sigma \otimes V_\sigma^* \nsubseteq \ker \sigma$.

(ii)⇒(i): Suppose that $V_\sigma \otimes V_\sigma^* \nsubseteq \ker \pi$. Since π is a direct sum of irreducible representations, and since σ is the only element of $\widehat{K}$ which is nonzero on $V_\sigma \otimes V_\sigma^*$, it follows that π contains σ as a direct summand. □

Proof of Lemma 4.5.1. Recall from Proposition 1.3.2 that $\pi_\nu \to \sigma$ if and only if σ is weakly contained in every subnet of $(\pi_\nu)_{\nu\in N}$.

(i)⇒(ii): By contraposition. Suppose that for every $\nu \in N$ there exists $\mu \in N_{\geq \nu}$ such that σ is not contained in π_ν. We construct a subnet $(\pi_j)_{j\in J}$ of $(\pi_\nu)_{\nu\in N}$ which does not weakly contain σ. Set

$$J := \{(\nu, \mu) \in N \times N \mid \mu \geq \nu \text{ and } \sigma \nleq \pi_\mu\},$$

then J is an upwards directed ordered set with respect to componentwise ordering, and

$$J \to N,\ j = (\nu, \mu) \mapsto \mu$$

defines a subnet $(\pi_j)_{j\in J}$ of $(\pi_\nu)_{\nu\in N}$ such that $\sigma \not\leq \pi_j$ for all $j \in J$. By Lemma 4.5.2 this implies that

$$V_\sigma \otimes V_\sigma^* \subseteq \bigcap_{j\in J} \ker \pi_j$$

and hence, as $V_\sigma \otimes V_\sigma^* \not\subseteq \ker \sigma$, it follows that

$$\bigcap_{j\in J} \ker \pi_j \not\subseteq \ker \sigma,$$

i.e., σ is not weakly contained in $(\pi_j)_{j\in J}$.

(ii)⇒(i): Choose $\nu_0 \in N$ such that $\sigma \leq \pi_\nu$ for all $\nu \in N$ with $\nu \geq \nu_0$. By Lemma 4.5.2 and by definition of weak containment it follows that

$$\ker \sigma \supseteq \ker \pi_\nu$$

for every $\nu \in N_{\geq \nu_0}$. Let now $(\pi_{\nu_j})_{j\in J}$ be any subnet of $(\pi_\nu)_{\nu\in N}$. Find $j_0 \in J$ such that $\nu_j \geq \nu_0$ for every $j \geq j_0$. Then

$$\ker \sigma \supseteq \bigcap_{j\geq j_0} \ker \pi_{\nu_j} \supseteq \bigcap_{j\in J} \ker \pi_{\nu_j},$$

i.e., σ is weakly contained in $(\pi_{\nu_j})_{j\in J}$, as required. □

Chapter 5

The map ind^G is a homeomorphism

In this chapter we will prove that the topology on $\mathrm{Stab}(X)\widehat{\ }$ from Definition 2.3.1 and Proposition 2.3.3 makes the map

$$\operatorname{ind}^G \colon G\backslash\mathrm{Stab}(X)\widehat{\ } \to (C_0(X) \rtimes G)\widehat{\ },\ [x, G_x, \sigma] \mapsto \pi^{x,\sigma}$$

a homeomorphism in case of a proper locally compact G-space X. We have already seen in Section 2.4 that ind^G is continuous. It is left to work out the much more complicated proof that ind^G is also open. In Section 5.1 we prove some first results concerning the X- and the $\mathscr{K}(G)$-component of ind^G in case of a proper G-action. In Section 5.2 we consider a compact group K acting on a locally compact space Y and prove that ind^K is open in this situation. Openness of ind^K can then be combined with the results on reduction to the compact case from Section 3.5 to prove openness of ind^G in the general case of a proper group action. This is done in Section 5.3.

We close the chapter with a short discussion of our assumptions and an application of our results on actions with continuous stabilizer map from Section 2.5 to the space $(C_0(X) \rtimes G)\widehat{\ }$.

5.1 First steps towards openness of ind^G

Let G be a locally compact group which acts properly on a locally compact Hausdorff space X. Suppose that $(\pi_\nu)_{\nu\in N}$ converges to π in $(C_0(X) \rtimes G)\widehat{\ }$. By the Mackey-Rieffel-Green Theorem (2.3.5) we can choose $([x_\nu, G_\nu, \sigma_\nu])_{\nu\in N}$ and $[x, G_x, \sigma]$ in $G\backslash\mathrm{Stab}(X)\widehat{\ }$ such that

$$\pi = \pi^{x,\sigma} = P^x \rtimes \operatorname{ind}_{G_x}^G \sigma \qquad \text{and} \qquad \pi_\nu = \pi^{x_\nu,\sigma_\nu} = P^{x_\nu} \rtimes \operatorname{ind}_{G_\nu}^G \sigma_\nu$$

for all $\nu \in N$. We show in this section that there are a subnet $([x_j, G_j, \sigma_j])_{j\in J}$ and representatives $(x_j, G_j, \sigma_j)_{j\in J}$ such that $x_j \to x$ in X and $G_j \to H \leq G_x$ in $\mathscr{K}(G)$. This is a consequence of the following two results:

Lemma 5.1.1. Retain the notation of the preceding text. Then $Gx_\nu \to Gx$ in $G\backslash X$.

Proof. As described in Remark 1.4.1, the canonical inclusion $\iota_{C_0(X)} \colon C_0(X) \to M(C_0(X) \rtimes G)$ induces a continuous map $\mathrm{Rep}(C_0(X) \rtimes G) \to \mathrm{Rep}(C_0(X))$, mapping $\varrho \rtimes U$ to ϱ. Combined with the above assumptions this gives that $P^{x_\nu} \to P^x$ as representations of $C_0(X)$.

Now the canonical inclusion $j \colon C_0(G\backslash X) \to M(C_0(X))$ given by $j(\overline{f})(y) = \overline{f}(Gy)$ for all $\overline{f} \in C_0(G\backslash X)$ and all $y \in X$ induces a continuous map

$$\mathrm{Rep}(C_0(X)) \to \mathrm{Rep}(C_0(G\backslash X)),\ \varrho \to \varrho \circ j.$$

As

$$(P^x \circ j)(\overline{f})\xi(g) = j(\overline{f})(gx)\xi(g) = \overline{f}(Gx)\xi(g) = \mathrm{ev}_{Gx}(\overline{f})\xi(g)$$

for all $\overline{f} \in C_0(G\backslash X)$, $\xi \in H_{\operatorname{ind}\sigma}$ and $g \in G$, and analogously for P^{x_ν} for all $\nu \in N$, it follows that $\mathrm{ev}_{Gx_\nu} \to \mathrm{ev}_{Gx}$ as representations of the commutative C^*-algebra $C_0(G\backslash X)$. Since $G\backslash X$ is a locally compact Hausdorff space, this implies that $Gx_\nu \to Gx$ in $G\backslash X$. □

Combining this result with Lemma 2.2.3 on the convergence of the stabilizer subgroups of a convergent net in X, we obtain the following proposition:

Proposition 5.1.2. Let G be a locally compact group acting properly on a locally compact Hausdorff space X. Let $(\pi_\nu)_{\nu\in N}$ converge to π in $(C_0(X) \rtimes G)\widehat{\ }$. Then there exist a subnet $(\pi_j)_{j\in J}$, elements $(x_j, G_j, \sigma_j)_{j\in J}$ and (x, G_x, σ) in $\mathrm{Stab}(X)\widehat{\ }$, and a subgroup $H \leq G_x$ such that

$$\pi = \pi^{x,\sigma} \qquad \text{and} \qquad \pi_j = \pi^{x_j,\sigma_j}$$

for all $j \in J$, and $x_j \to x$ in X and $G_j \to H$ in $\mathscr{K}(G)$.

Proof. Let (x, G_x, σ) be a representative of $[x, G_x, \sigma]$. By Lemma 5.1.1 above we know that $Gx_\nu \to Gx$, so Remark 1.1.1 yields a subnet $([x_l, G_l, \sigma_l])_{l\in L}$ and representatives $(x_l, G_l, \sigma_l)_{l\in L}$ such that $x_l \to x$. It then follows from Lemma 2.2.3 that there exist a subnet $(x_j, G_j, \sigma_j)_{j\in J}$ and $H \leq G_x$ with $G_j \to H$. □

5.2 Openness of ind^K for a compact group K

This section contains most part of the work for the proof of openness of ind^G. Since we have to use several results from the representation theory of compact groups, we restrict ourselves to actions of compact groups here. The material of this section is based on the work of Baggett in [Bag68], but the reduction to group representations below and our description of restriction of continuous functions in $L^2(K)$ to closed subgroups from Section 4.4 allow some simplifications.

So we suppose now that a compact group K acts on a locally compact Hausdorff space Y. As before, let $(\pi_\nu)_{\nu\in N}$ converge to π in $(C_0(Y) \rtimes K)\widehat{\ }$, and suppose that we have found $(y_\nu, K_\nu, \sigma_\nu)_{\nu\in N}$ and (y, K_y, σ) in $\mathrm{Stab}(Y)\widehat{\ }$ with

$$\pi = \pi^{y,\sigma} = P^y \rtimes \mathrm{ind}_{K_y}^K \sigma \qquad \text{and} \qquad \pi_\nu = \pi^{y_\nu,\sigma_\nu} = P^{y_\nu} \rtimes \mathrm{ind}_{K_\nu}^K \sigma_\nu$$

for all $\nu \in N$. We have already seen that it is not too difficult to obtain some first openness results for the Y- and the $\mathscr{K}(K)$-component of ind^K. We now have to worry about convergence of the subgroup representations $(K_\nu, \sigma_\nu)_{\nu\in N}$.

Reduction to group representations

The integrated form $\iota_{C^*(K)}\colon C^*(K) \to M(C_0(Y) \rtimes K)$ of the canonical inclusion ι_K from Remark 1.4.1 induces a continuous map $\mathrm{Rep}(C_0(Y) \rtimes K) \to \mathrm{Rep}(C^*(K)) = \mathrm{Rep}(K)$, which maps a representation $\tau \rtimes U$ to U. Hence, the above assumption that $\pi^{y_\nu,\sigma_\nu} \to \pi^{y,\sigma}$ implies that $\mathrm{ind}_{K_\nu}^K \sigma_\nu \to \mathrm{ind}_{K_y}^K \sigma$ in $\mathrm{Rep}(K)$.

Let $\varrho \in \widehat{K}$ be such that $\varrho \leq \operatorname{ind}_{K_y}^K \sigma$. It then follows from Proposition 1.3.2 that $\operatorname{ind}_{K_\nu}^K \sigma_\nu \to \varrho$. As seen in section 4.3, the left regular representation λ^K of K is equivalent to $\operatorname{ind}_{K_\nu}^K \lambda^{K_\nu}$ for all $\nu \in N$, and so $\sigma_\nu \leq \lambda^{K_\nu}$ for all $\nu \in N$ implies that $\operatorname{ind}_{K_\nu}^K \sigma_\nu$ is equivalent to a subrepresentation of λ^K for every $\nu \in N$. Hence we can choose closed λ^K-invariant subspaces $(P_\nu)_{\nu \in N}$ of $L^2(K)$ such that $\operatorname{ind}_{K_\nu}^K \sigma_\nu$ is equivalent to $\lambda^K|_{P_\nu}$ for all $\nu \in N$. Let P_ϱ be a minimal left ideal of $L^2(K)$ associated to ϱ, i.e., $P_\varrho = V_\varrho \otimes e^*$ for some unit vector $e \in V_\varrho$. Then ϱ is equivalent to $\lambda^K|_{P_\varrho}$.

We now construct a positive functional associated with ϱ so that we can apply Fell's characterization of convergence in terms of positive functionals from Proposition 1.3.7. Recall that P_ϱ contains continuous elements only, and let h be a nonzero element of P_ϱ. Define

$$\varphi\colon C^*(K) \to \mathbb{C},\ F \mapsto \langle \varrho(F)h, h\rangle = \langle \lambda^K(F)h, h\rangle,$$

then φ is a positive functional associated with ϱ as a representation of $C^*(K)$. By Proposition 1.3.7(iii) we can, after passing to a subnet and relabeling, if necessary, find a net of positive functionals $(\varphi'_\nu)_{\nu \in N}$, each φ'_ν associated to $\operatorname{ind}_{K_\nu}^K \sigma_\nu$ as representations of $C^*(K)$, such that $\|\varphi'_\nu\| \leq \|\varphi\|$ for all $\nu \in N$ and such that $\varphi'_\nu \to \varphi$ in the weak* topology.

Choose, for every $\nu \in N$, an element $h'_\nu \in P_\nu$ such that φ'_ν is given by

$$\varphi'_\nu\colon C^*(K) \to \mathbb{C},\ F \mapsto \langle (\operatorname{ind}_{K_\nu}^K \sigma_\nu)(F)h'_\nu, h'_\nu\rangle = \langle \lambda^K(F)h'_\nu, h'_\nu\rangle.$$

In what follows we work out the following idea: Using the net $(h'_\nu)_{\nu \in N}$ we construct a space $Z \leq L^2(K)$ such that $\lambda^K|_Z$ is equivalent to ϱ and the following holds: For every element $g \in Z$ we find continuous elements $g_\nu \in P_\nu$ which converge uniformly to g. As seen in Section 4.4, the restriction of any continuous element g of Z to K_y lies in the left ideal of $L^2(K_y)$ associated to σ, and in the same way the restrictions $g_\nu|_{K_\nu}$ are related to σ_ν for all $\nu \in N$. We will use this fact combined with the convergence $g_\nu \to g$ to prove that $(K_\nu, \sigma_\nu) \to (H, \sigma_H)$ for a suitable element $(H, \sigma_H) \leq (K_y, \sigma)$. First, we need a modification of the elements h_ν.

Lemma 5.2.1. Retain the above assumptions. Let $I := V_\varrho \otimes V_\varrho^*$ denote the minimal two-sided ideal of $L^2(K)$ which contains h. Let $p_I \in C(K)$ be the projection onto I as defined in (4.2.1). For every $\nu \in N$ define $h_\nu := p_I * h'_\nu$ and

$$\varphi_\nu\colon C^*(K) \to \mathbb{C},\ F \mapsto \langle \lambda^K(F)h_\nu, h_\nu\rangle.$$

Then $h_\nu \in P_\nu \cap I$ and $\|h_\nu\|_2 \leq \|h\|_2$ for every $\nu \in N$, each φ_ν is a positive functional associated with $\operatorname{ind}_{K_\nu}^K \sigma_\nu$, and $\varphi_\nu \to \varphi$ in the weak* topology.

As, on the finite-dimensional ideal I, the L^2-norm is equivalent to the maximum norm, and since all the functions involved here are contained in I and in particular continuous, it follows that there exists a constant $c > 0$ such that $\|h_\nu\|_\infty \leq c\|h\|_\infty$.

Proof of Lemma 5.2.1. Being invariant subspaces for λ^K, all P_ν are closed left ideals in $L^2(K)$ by Theorem 4.2.1. Since I is a two-sided ideal it follows directly that $h_\nu \in P_\nu \cap I$, and hence that φ_ν is associated with $\operatorname{ind}_{K_\nu}^K \sigma_\nu$ for every $\nu \in N$. Moreover we can use that

$$\|h'_\nu\|_2^2 = \|\varphi'_\nu\| \leq \|\varphi\| = \|h\|_2^2$$

to see that

$$\|h_\nu\|_2 = \|p_I * h'_\nu\|_2 \leq \|h\|_2$$

for all $\nu \in N$, because convolution with p_I is a projection. For the proof of convergence of the net $(\varphi_\nu)_{\nu\in N}$ to φ we use the fact that

$$\lambda^K(\eta)\zeta = \eta * \zeta$$

for all $\eta, \zeta \in C(K)$. It follows that, for every $F \in C(K)$:

$$\begin{aligned}
\varphi(F) &= \langle \lambda^K(F)h, h\rangle = \langle F * (p_I * h), p_I * h\rangle \\
&= \langle p_I * (F * (p_I * h)), h\rangle = \langle \lambda^K(p_I * (F * p_I))h, h\rangle \\
&= \varphi(p_I * (F * p_I)) = \lim_\nu \varphi'_\nu(p_I * (F * p_I)) \\
&= \lim_\nu \langle \lambda^K(p_I * (F * p_I))h'_\nu, h'_\nu\rangle = \lim_\nu \langle (p_I * (F * p_I)) * h'_\nu, h'_\nu\rangle \\
&= \lim_\nu \langle F * (p_I * h'_\nu), p * h'_\nu\rangle = \lim_\nu \langle \lambda^K(F)(p * h'_\nu), p_I * h'_\nu\rangle \\
&= \lim_\nu \varphi_\nu(F)
\end{aligned}$$

Since $C(K)$ is dense in $C^*(K)$ and all positive functionals are continuous it follows that $\varphi(F) = \lim_\nu \varphi_\nu(F)$ for all $F \in C^*(K)$, which completes the proof. □

Proposition 5.2.2. Retain the above assumptions. Then there exists a λ^K-invariant subspace Z of $L^2(K)$ such that

(i) $\lambda^K|_Z$ is equivalent to ϱ,

(ii) for every $g \in Z$ there exist continuous elements $g_\nu \in P_\nu$ for all $\nu \in N$ such that $g_\nu \to g$ uniformly.

Proof. As seen above, we can assume that $h_\nu \in P_\nu \cap I$ for all $\nu \in N$, where $I = V_\varrho \otimes V_\varrho^*$, which in particular implies that all h_ν are continuous, and that there is a constant $c > 0$ such that $\|h_\nu\|_\infty \leq c\|h\|_\infty$. Now $\overline{\{h_\nu \mid \nu \in N\}}$ is a closed bounded subset of the finite-dimensional space I, hence it is compact. After passing to a subnet and relabeling we can thus assume that there is $v \in I$ such that $h_\nu \to v$ uniformly. Define

$$Z := \{\lambda^K(F)v \mid F \in C^*(K)\}.$$

Notice that Z is finite-dimensional, because v is contained in the λ^K-invariant finite-dimensional subspace I. For every $F \in C^*(K)$ we have that

$$\begin{aligned}
\langle \lambda^K(F)v, v\rangle &= \lim_\nu \langle \lambda^K(F)h_\nu, h_\nu\rangle = \lim_\nu \varphi_\nu(F) \\
&= \varphi(F) = \langle \lambda^K(F)h, h\rangle.
\end{aligned}$$

Since cyclic representations are equivalent if and only if the positive functionals coming from their cyclic vectors are the same (Theorem 5.1.4 in [Mur90]), it follows that $\lambda^K|_Z$ is equivalent to $\lambda^K|_P$, where

$$P := \{\lambda^K(F)h \mid F \in C^*(K)\}.$$

But as P_ϱ is an invariant subspace for λ^K and $h \in P_\varrho$, we have that $\lambda^K(F)h \in P_\varrho$ for all $F \in C^*(K)$. This shows that P is an invariant subspace of P_ϱ, which implies that $P = P_\varrho$ by minimality of P_ϱ. We now have that $\lambda^K|_Z$ is equivalent to ϱ, which shows (i).

Let now $g \in Z$ and choose $F \in C^*(K)$ with $g = \lambda^K(F)v$. Define the net $(g_\nu)_{\nu\in N}$ by $g_\nu := \lambda^K(F)h_\nu$ for every $\nu \in N$. As $h_\nu \in P_\nu$ for every $\nu \in N$ and as each P_ν is λ^K-invariant, it follows that $g_\nu \in P_\nu$ for all $\nu \in N$. Since $h_\nu \to v$ and everything takes place in I we can conclude that $g_\nu \to g$ uniformly. □

Proof of openness for a compact group K

We now give the proof of openness of the bijection ind^K in case of a compact group K.

Theorem 5.2.3. Suppose that K is a compact group acting on a locally compact Hausdorff space Y. Then the map $\operatorname{ind}^K\colon K\backslash\operatorname{Stab}(Y)\widehat{\ }\to (C_0(Y)\rtimes K)\widehat{\ }$ is open.

Proof. Suppose that $(\pi_\nu)_{\nu\in N}$ converges to π in $(C_0(Y)\rtimes K)\widehat{\ }$, and let $([y_\nu, K_\nu, \sigma_\nu])_{\nu\in N}$ and $[y, K_y, \sigma]$ be in $\operatorname{Stab}(Y)\widehat{\ }$ such that $\pi = \pi^{y,\sigma}$ and $\pi_\nu = \pi^{y_\nu,\sigma_\nu}$ for every $\nu \in N$. To prove that $[y_\nu, K_\nu, \sigma_\nu] \to [y, K_y, \sigma]$ in $K\backslash\operatorname{Stab}(Y)\widehat{\ }$ in the sense of Remark 2.3.4, we first pass to a subnet and relabel.

As seen in Proposition 5.1.2 we can, after passing to a subnet and relabeling again, assume that there exist representatives $(y_\nu, K_\nu, \sigma_\nu)_{\nu\in N}$ and (y, K_y, σ) in $\operatorname{Stab}(Y)\widehat{\ }$, and a subgroup $H \leq K_y$ such that

$$\pi = \pi^{y,\sigma} = P^y \rtimes \operatorname{ind}_{K_y}^K \sigma \qquad \text{and} \qquad \pi_\nu = \pi^{y_\nu,\sigma_\nu} = P^{y_\nu} \rtimes \operatorname{ind}_{K_\nu}^K \sigma_\nu$$

for all $\nu \in N$, and such that $y_\nu \to y$ in Y and $K_\nu \to H$ in $\mathscr{K}(K)$. We have to find an element $\sigma_H \in \widehat{H}$ such that $\sigma_H \leq \sigma|_H$ and $(K_\nu, \sigma_\nu) \to (H, \sigma_H)$ in $S(K)\widehat{\ }$.

As in the beginning of Section 5.2 choose closed λ^K-invariant subspaces $(P_\nu)_{\nu\in N}$ of $L^2(K)$ such that $\operatorname{ind}_{K_\nu}^K \sigma_\nu$ is equivalent to $\lambda^K|_{P_\nu}$ for every $\nu \in N$, and let $\varrho \in \widehat{K}$ be an irreducible subrepresentation of $\operatorname{ind}_{K_y}^K \sigma$.

Let Z be a subspace of $L^2(K)$ as constructed in Proposition 5.2.2. Being equivalent to ϱ, the representation $\lambda^K|_Z$ is an irreducible subrepresentation of $\operatorname{ind}_{K_y}^K \sigma$. Thus we can view Z as a minimal left ideal in $H_{\operatorname{ind}\sigma} \otimes V_\sigma^*$, which by Corollary 4.4.3 and Remark 4.4.4 implies that $Z|_{K_y}$ is a minimal left ideal in $L^2(K_y)$ and

$$\lambda^{K_y}|_{(Z|_{K_y})} = \sigma.$$

Again by Corollary 4.4.3 it follows that $Z|_H$ is a minimal left ideal in $L^2(H)$, so $\sigma_H := \lambda^H|_{(Z|_H)}$ is an element of $\widehat{H}$, and by Lemma 4.4.6 we get that

$$\sigma_H = \lambda^H|_{(Z|_H)} \leq \left(\lambda^{K_y}|_{(Z|_{K_y})}\right)\Big|_H = \sigma|_H.$$

It is now left to show that $(K_\nu, \sigma_\nu) \to (H, \sigma_H)$ in $S(K)\widehat{\ }$. We use Fell's criterion for convergence of subgroup representation pairs given in Theorem 2.2.7. Let $f \in Z|_H$, then

$$\psi\colon H \to \mathbb{C},\ h \mapsto \langle \sigma_H(h) f, f\rangle = \int_H f(h^{-1}x)\overline{f(x)}d\mu_H(x)$$

is a function of positive type associated with (H, σ_H). Suppose that we have passed to a subnet of $(K_\nu, \sigma_\nu)_{\nu\in N}$ and relabeled. To construct functions of positive type associated with the elements (K_ν, σ_ν) we first go back to the level of $L^2(K)$ and apply Proposition 5.2.2. To this end choose $g \in Z$ such that $f = g|_H$. Then there exists a net $(g_\nu)_{\nu\in N}$ of continuous elements $g_\nu \in P_\nu$ such that $g_\nu \to g$ uniformly. For every $\nu \in N$ define

$$M_\nu := \{h|_{K_\nu} \mid h \in P_\nu \text{ is continuous}\},$$

then by Remark 4.4.4 there exists for every $\nu \in N$ a number $n_\nu \in \mathbb{N}$ such that

$$\lambda^{K_\nu}|_{M_\nu} = n_\nu \sigma_\nu.$$

For every $\nu \in N$ set $f_\nu := g_\nu|_{K_\nu}$, then $f_\nu \in M_\nu$ and

$$\psi_\nu \colon K_\nu \to \mathbb{C},\ k \mapsto \langle (n_\nu\sigma_\nu)(k) f_\nu, f_\nu\rangle = \int_{K_\nu} f_\nu(k^{-1}x)\overline{f_\nu(x)}d\mu_{K_\nu}(x)$$

is a function of positive type associated with $n_\nu\sigma_\nu$, and hence a finite sum of functions of positive type associated with σ_ν. To apply Fell's criterion for convergence in $\mathrm{Rep}(S(K))$ from Theorem 2.2.7 we now have to show that $\psi_\nu \to \psi$ in the Fell topology on $\mathscr{F}(K)$ as characterized in Lemma 2.2.6.

We already know that $K_\nu \to H$ in $\mathscr{K}(K)$. Suppose that we have passed to a subnet of $(\psi_\nu)_{\nu\in N}$ and relabeled, and that we have chosen elements $k_\nu \in K_\nu$ for all $\nu \in N$ and $k \in H$ such that $k_\nu \to k$. We have to show that $\psi_\nu(k_\nu) \to \psi(k)$. Define

$$h \colon H \to \mathbb{C},\ l \mapsto f(k^{-1}l)\overline{f(l)}$$

and

$$h_\nu \colon K_\nu \to \mathbb{C},\ l \mapsto f_\nu(k_\nu^{-1}l)\overline{f_\nu(l)}$$

for every $\nu \in N$. We first show that $h_\nu \to h$ in $\mathscr{F}(K)$. Suppose again that we have passed to a subnet and relabeled, and that we have chosen elements $l_\nu \in K_\nu$ for all $\nu \in N$ and $l \in H$ such that $l_\nu \to l$. Using that $f = g|_H$ and $f_\nu = g_\nu|_{K_\nu}$ for all $\nu \in N$ we get that

$$\begin{aligned} |h_\nu(l_\nu) - h(l)| &= |f_\nu(k_\nu^{-1}l_\nu)\overline{f_\nu(l_\nu)} - f(k^{-1}l)\overline{f(l)}| \\ &= |g_\nu(k_\nu^{-1}l_\nu)\overline{g_\nu(l_\nu)} - g(k^{-1}l)\overline{g(l)}|. \end{aligned}$$

This tends to zero because $g_\nu \to g$ uniformly. It now follows from Proposition 2.2.12 on continuity of integration on $\mathscr{F}(K)$ that

$$\begin{aligned} |\psi_\nu(k_\nu) - \psi(k)| &= \left| \int_{K_\nu} f_\nu(k_\nu^{-1}s)\overline{f_\nu(s)}d\mu_{K_\nu}(s) - \int_H f(k^{-1}t)\overline{f(t)}d\mu_H(t) \right| \\ &= \left| \int_{K_\nu} h_\nu(s)d\mu_{K_\nu}(s) - \int_H h(t)d\mu_H(t) \right| \end{aligned}$$

tends to zero. This proves that $\psi_\nu \to \psi$ in $\mathscr{F}(K)$, which implies $(K_\nu, \sigma_\nu) \to (H, \sigma_H)$ in $S(K)^\wedge$ and thus $(y_\nu, K_\nu, \sigma_\nu) \to (y, K_y, \sigma)$ in $\mathrm{Stab}(Y)^\wedge$, which completes the proof. □

5.3 Openness of ind^G

Let G be a locally compact group which acts properly on a locally compact Hausdorff space X. We already saw in Section 2.4 that the bijection

$$\mathrm{ind}^G \colon G\backslash\mathrm{Stab}(X)^\wedge \to (C_0(X) \rtimes G)^\wedge$$

from the Mackey-Rieffel-Green Theorem is continuous with respect to the quotient topology on $G\backslash\mathrm{Stab}(X)^\wedge$ and the Jacobson topology on $(C_0(X) \rtimes G)^\wedge$. We now prove that ind^G is also open, and hence a homeomorphism.

To achieve this, we will use the fact that the proper G-space X is locally induced from compact subgroups, combined with the openness of ind^K in case of a compact group K acting on a locally compact Hausdorff space Y. The following remarks will be useful for the proof of the theorem.

Remarks 5.3.1. (i) If $U \subseteq X$ is a G-invariant subset, then $G\backslash\mathrm{Stab}(U)^\frown \subseteq G\backslash\mathrm{Stab}(X)^\frown$.

(ii) If $U \subseteq X$ is a G-invariant open subset, then $C_0(U)$ is a closed G-invariant ideal in $C_0(X)$. Since taking (full) crossed products gives an exact functor between the category of G-C^*-algebras and the category of C^*-algebras (see Proposition 4.8 in [Ech]), this implies that $C_0(U) \rtimes G$ is a closed ideal in $C_0(X) \rtimes G$. Combined with the fact that closed ideals of a C^*-algebra correspond to open subsets of the spectrum (see Proposition 3.2.2 in [Dix77]), we obtain that $(C_0(U) \rtimes G)^\frown$ is an open subset of $(C_0(X) \rtimes G)^\frown$.

Theorem 5.3.2. Let X be a proper G-space. Then ind^G is open.

Proof. We show that images of open sets are open. Let thus $W \subseteq G\backslash\mathrm{Stab}(X)^\frown$ be an open subset. Let V be the pre-image of W in $\mathrm{Stab}(X)^\frown$. For every $(w, G_w, \sigma) \in V$ suppose that U_w is a G-invariant open neighborhood of w, that $L_w \leq G$ is a compact subgroup, and that $\varphi_w \colon U_w \to G/L_w$ is a continuous G-map as supplied by Abels' Theorem (Theorem 1.1.7). Then, with $Y_w := \varphi_w^{-1}(\{eL_w\})$, we have for every $(w, G_w, \sigma) \in V$ that $w \in Y_w$ and that $G \times_{L_w} Y_w$ is homeomorphic to U_w via $[g, y] \mapsto gy$. We then have that

$$W = \bigcup_{(w, G_w, \sigma) \in V} G\backslash\mathrm{Stab}(U_w)^\frown \cap W$$

and thus

$$\mathrm{ind}^G(W) = \bigcup_{(w, G_w, \sigma) \in V} \mathrm{ind}^G\left(G\backslash\mathrm{Stab}(U_w)^\frown \cap W\right).$$

It thus suffices to show that the sets

$$\mathrm{ind}^G\left(G\backslash\mathrm{Stab}(U_w)^\frown \cap W\right)$$

are open in $(C_0(X) \rtimes G)^\frown$ for every $(w, G_w, \sigma) \in V$. Let thus $(w, G_w, \sigma) \in V$. By Proposition 3.5.2 there is a commutative diagram of bijective maps

$$\begin{array}{ccc} L_w\backslash\mathrm{Stab}(Y_w)^\frown & \xrightarrow{\ \iota\ } & G\backslash\mathrm{Stab}(U_w)^\frown \\ {\scriptstyle \mathrm{ind}^{L_w}}\downarrow & & \downarrow{\scriptstyle \mathrm{ind}^G} \\ (C_0(Y_w) \rtimes L_w)^\frown & \xrightarrow[\mathrm{ind}^Z]{} & (C_0(U_w) \rtimes G)^\frown, \end{array}$$

where ind^{L_w} and ind^G are the usual Mackey-Rieffel-Green maps, ind^Z is the homeomorphism coming from the Morita equivalence of $C_0(Y_w) \rtimes L_w$ and $C_0(U_w) \rtimes G$ from Section 3.4, and ι is the bijection from Lemma 3.5.1. Since ι is continuous, ind^Z is a homeomorphism and ind^{L_w} is open by compactness of L_w, it follows immediately that the restriction of ind^G to $G\backslash\mathrm{Stab}(U_w)^\frown$ is open.

Since W is open in $G\backslash\mathrm{Stab}(X)^\frown$, the intersection $G\backslash\mathrm{Stab}(U_w)^\frown \cap W$ is relatively open in $G\backslash\mathrm{Stab}(U_w)^\frown$, hence $\mathrm{ind}^G\left(G\backslash\mathrm{Stab}(U_w)^\frown \cap W\right)$ is relatively open in $(C_0(U_w) \rtimes G)^\frown$. But as $(C_0(U_w) \rtimes G)^\frown$ is open in $(C_0(X) \rtimes G)^\frown$ by Remark 5.3.1(ii), it already follows that $\mathrm{ind}^G\left(G\backslash\mathrm{Stab}(U_w)^\frown \cap W\right)$ is open in $(C_0(X) \rtimes G)^\frown$, which completes the proof. □

Altogether we have now obtained our main result:

Theorem 5.3.3. If X is a proper G-space, then the bijection

$$\mathrm{ind}^G \colon G\backslash\mathrm{Stab}(X)^\frown \to (C_0(X) \rtimes G)^\frown$$

from the Mackey-Rieffel-Green Theorem is a homeomorphism with respect to the quotient topology induced from $\mathrm{Stab}(X)^\frown$ on $G\backslash\mathrm{Stab}(X)^\frown$ and the Jacobson topology on $(C_0(X) \rtimes G)^\frown$.

5.4 Discussion of assumptions

We have now proved that $(C_0(X) \rtimes G)^\frown \cong G\backslash\mathrm{Stab}(X)^\frown$ for any proper G-space X, where X is a locally compact Hausdorff space. This might be a good place to contemplate where we actually need these assumptions.

We used properness to make Abels' Theorem available, which tells us that proper G-spaces are locally induced from compact subgroups. The reduction of the proof of openness of ind^G to the case where G is compact relies heavily on this fact.

After reading the proof of openness in the compact case above it is obvious that it only works in case of a compact group K, since we use the representation theory of compact groups like the Peter-Weyl Theorem, the minimal ideal structure, and the Frobenius Reciprocity Theorem all over the place.

The assumption that X is Hausdorff is implicitly used in many places, for instance when we use that the orbit space of a proper G-space X is Hausdorff. It appears explicitly when we prove that whenever $x_\nu \to x$, there is a subnet $(x_j)_{j\in J}$ such that $G_j \to H \le G_x$.

5.5 Proper actions with continuous stabilizer map

A nice application of our result is that it leads to a new proof of the well-known fact that, given a proper G-space X, the spectrum $(C_0(X) \rtimes G)^\frown$ is Hausdorff if and only if the stabilizer map $x \mapsto G_x$ is continuous. Results of this kind have been obtained, for instance, by Echterhoff in [Ech94], by Baggett in [Bag68], and also by Williams in [Wil82].

Theorem 5.5.1. Let X be a proper G-space. Then the stabilizer map $X \to \mathscr{K}(G),\ x \mapsto G_x$ is continuous if and only if $(C_0(X) \rtimes G)^\frown$ is Hausdorff.

Proof. We already proved in Proposition 2.5.1 that continuity of the stabilizer map is equivalent to the fact that $G\backslash\mathrm{Stab}(X)^\frown$ is Hausdorff. As seen above, this is equivalent to $(C_0(X) \rtimes G)^\frown$ being Hausdorff, which completes the proof. □

Chapter 6

A C^*-algebra with spectrum $\mathrm{Stab}(X)\widehat{\ }$

In the beginning of Chapter 2 we mentioned that, for every proper G-space X, there exists a C^*-algebra A, the spectrum of which is in one-to-one correspondence to $\mathrm{Stab}(X)\widehat{\ }$. In Section 6.1 we give the details for the construction of A and show that the spectrum $\widehat{A}$ is then even homeomorphic to $\mathrm{Stab}(X)\widehat{\ }$ with its usual topology as defined in Section 2.3. In Section 6.2 we define a G-action on A such that the induced G-action on $\widehat{A}$ corresponds to the G-action on $\mathrm{Stab}(X)\widehat{\ }$.

The ideas underlying the material of this chapter where communicated to me by Siegfried Echterhoff.

6.1 Construction of A with $\widehat{A} \cong \mathrm{Stab}(X)\widehat{\ }$

Let X be a proper G-space and let G act on $\mathcal{K} := \mathcal{K}(L^2(G))$ via $\mathrm{Ad}(\varrho)\colon G \to \mathrm{Aut}(\mathcal{K})$, where ϱ is the right regular representation of G on $L^2(G)$. As in Section 2.1 we define

$$A := \{f \in C_0(X, \mathcal{K}) \mid \forall x \in X : f(x) \in \mathcal{K}^{G_x}\}. \tag{6.1.1}$$

We mentioned before that A is a $C_0(X)$-algebra with fibre $\mathcal{K}^{G_x}$ over every $x \in X$. This can be shown directly by modifying Echterhoff and Emerson's proof of the bundle structure of generalized fixed point algebras. It is then the main issue to show that the fibres are really all of $\mathcal{K}^{G_x}$ for every $x \in X$.

But, since we have to work with topological fundamental domains anyway, we can use Corollary 1.4.6 from page 14 to obtain the desired statement on A: If

($\star$) we can describe X as a topological fundamental domain for a proper G-space Y in such a way that, for every $x \in X$, the stabilizer subgroup G_x with respect to the G-action on X coincides with G_x with respect to the G-action on Y,

then the just mentioned corollary yields that

$$A \cong C_0(Y) \rtimes G \cong C_0(Y \times_{G,\mathrm{Ad}\varrho} \mathcal{K}),$$

which by Lemma 1.4.2 implies that A is a $C_0(X)$-algebra as described above. We showed in Section 2.1 that then the spectrum $\widehat{A}$ equals $\mathrm{Stab}(X)\widehat{\ }$ as a set.

We will prove in Proposition 6.1.2 that ($\star$) is always satisfied. This is point 1. of our plan below, in which we outline the idea for the proof that $\widehat{A}$ is in fact homeomorphic to $\mathrm{Stab}(X)\widehat{\ }$.

The plan

To avoid too much irritating text we introduce the following notation:

Notation 6.1.1. Suppose that W is a G-space. If there are also other G-spaces around and there is a risk of misunderstanding, then we denote the stabilizer subgroup of an element $w \in W$ with respect to the G-action on W by G_w^W. If V is a G-space such that $G_v^V = G_v^W$ for all $v \in V \cap W$, then we say that V and W have coinciding stabilizers.

To obtain our desired result that, for a proper G-space X, the C^*-algebra A as above is a $C_0(X)$-algebra and that its spectrum is homeomorphic to $\mathrm{Stab}(X)\widehat{\ }$, we will pursue the following strategy:

1. Construct a proper G-space Y which contains X as a topological fundamental domain such that X and Y have coinciding stabilizers, i.e., $G_x^X = G_x^Y$ for every $x \in X$.

2. Prove that this implies that $\mathrm{Stab}(X)\widehat{\ }$ is a topological fundamental domain for the G-space $\mathrm{Stab}(Y)\widehat{\ }$.

3. Use 1., Theorem 5.3.3, and 2., respectively, to obtain the homeomorphisms

$$\widehat{A} \cong (C_0(Y) \rtimes G)\widehat{\ } \cong G\backslash\mathrm{Stab}(Y)\widehat{\ } \cong \mathrm{Stab}(X)\widehat{\ },$$

   and be happy.

Every proper G-space X is a topological fundamental domain for some proper G-space Y

As above, let X be a given proper G-space. As usual we assume that X is a locally compact Hausdorff space. Recall from topology that a topological space S is Hausdorff if and only if the diagonal $\Delta_S := \{(s,s) \mid s \in S\}$ is closed in $S \times S$. Notice that this Δ has nothing to do with the modular function Δ_G of a locally compact group G. Since the diagonal of a topological space and the modular function of a locally compact group will not appear in the same place here, there should be no confusion about the use of Δ.

Proposition 6.1.2. There exists a locally compact Hausdorff space Y equipped with a proper G-action which contains X as a topological fundamental domain such that X and Y have coinciding stabilizers.

Proof. Define

$$Y := \{(x,y) \in X \times X \mid y \in Gx\}$$

and equip Y with the relative topology induced from $X \times X$. Then Y is Hausdorff. To see that Y is locally compact consider the continuous map

$$\varphi\colon X \times X \to G\backslash X \times G\backslash X,\ (x,y) \mapsto Gx \times Gy.$$

Since G acts properly on X, the orbit space $G\backslash X$ is Hausdorff (Proposition 1.1.4 (ii)), which implies that the diagonal $\Delta_{G\backslash X}$ is closed. It is easy to check that $Y = \varphi^{-1}(\Delta_{G\backslash X})$, which implies that Y is a closed, hence locally compact, subspace of the locally compact space $X \times X$.

To define an action of G on Y we first equip the space $X \times X$ with the G-action given by $g(x_1, x_2) = (x_1, gx_2)$ for all $g \in G, (x_1, x_2) \in X \times X$. As X is a proper G-space, it follows from Proposition 1.1.4 (iv) that this action makes $X \times X$ also a proper G-space. Being a closed, G-invariant subspace of $X \times X$, the G-space Y is now also proper.

We have now established that Y is a locally compact Hausdorff space with a proper G-action, and it is left to show that X is a topological fundamental domain for Y. Via the embedding $i\colon X \to Y,\ x \mapsto (x, x)$ we identify X with the diagonal $i(X) \subseteq Y$. Since X is Hausdorff, $i(X) = \Delta_X = \Delta_X \cap Y$ is closed in Y. Moreover, for every $x \in X$, we have that $G^X_x = G^Y_{(x,x)} = G^Y_{i(x)}$, which shows that the stabilizer subgroups of X and Y coincide.

We now show that the map $q\colon X \to G\backslash Y,\ x \mapsto G(x, x)$ is a homeomorphism. To see that q is injective, let $x_1, x_2 \in X$ be such that $G(x_1, x_1) = G(x_2, x_2)$. Then there exists $g \in G$ with $(x_2, x_2) = g(x_1, x_1) = (x_1, gx_1)$, from which it follows that $x_1 = x_2$.

Let now $G(x, y) \in G\backslash Y$. By definition of Y there is $g \in G$ with $y = gx$, which implies that $(x, y) = g(x, x)$ and thus $q(x) = G(x, x) = G(x, y)$. This shows that q is surjective.

Since q is the composition of the continuous inclusion i with the continuous quotient map $p\colon Y \to G\backslash Y$, it is itself continuous.

To see that q is open let $U \subseteq X$ be an open subset. We have to show that

$$q(U) = \{G(x, x) \in G\backslash Y \mid x \in U\}$$

is open in $G\backslash Y$, which is equivalent to the question whether $p^{-1}(q(U))$ is open in Y. We check that

$$\begin{aligned} p^{-1}(q(U)) &= \{(x, y) \in Y \mid \exists x' \in U :\ G(x, y) = G(x', x')\} \\ &= \{(x, gx) \in Y \mid x \in U,\ g \in G\} \\ &= (U \times GU) \cap Y. \end{aligned}$$

Since U is open in X, so is the set $GU = \bigcup_{g \in G} gU$. Thus, $U \times GU$ is open in $X \times X$, which implies that $(U \times GU) \cap Y$ is open in Y, as required. This shows that q is open, which completes the proof. □

Notice that if X is a topological fundamental domain for a G-space Y, if $q\colon X \to G\backslash Y$ denotes the corresponding homeomorphism and $p\colon Y \to G\backslash Y$ is the usual quotient map, then

$$f := q^{-1} \circ p\colon Y \to X,\ y \mapsto q^{-1}(Gy) \tag{6.1.2}$$

is a continuous surjection. This will be useful in the proof of the following lemma, with which we proceed to step 2 of our strategy.

$\mathrm{Stab}(X)^\frown$ as a topological fundamental domain

Lemma 6.1.3. Suppose that a proper G-space X is a topological fundamental domain for a proper G-space Y such that the stabilizer subgroups coincide. Let G act on $\mathrm{Stab}(X)^\frown$ and $\mathrm{Stab}(Y)^\frown$ as usual. Then $\mathrm{Stab}(X)^\frown$ is a topological fundamental domain for the G-space $\mathrm{Stab}(Y)^\frown$, and the stabilizers coincide.

This lemma could also be formulated for a topological fundamental domain X of the G-space Y, where X does not carry a G-action of its own. Then the space $\mathrm{Stab}(X)^\frown$ would be defined with respect to the G-action on Y, and the statement of the lemma would still be true, without major changes in the proof.

But since we consider the situation where we start with a G-space X, the chosen formulation of the lemma seems more convenient to us.

Proof of Lemma 6.1.3. Since $X \subseteq Y$ and since the stabilizers for X and Y coincide, it follows immediately that $\mathrm{Stab}(X)\widehat{\ }$ can be identified with the subset $\{(x, G_x, \sigma) \in \mathrm{Stab}(Y)\widehat{\ } \mid x \in X\}$ of $\mathrm{Stab}(Y)\widehat{\ }$. As X is closed in Y, the definition of convergence in $\mathrm{Stab}(X)\widehat{\ }$ resp. $\mathrm{Stab}(Y)\widehat{\ }$ implies that $\mathrm{Stab}(X)\widehat{\ }$ is closed in $\mathrm{Stab}(Y)\widehat{\ }$. It is left to show that

$$\varphi\colon \mathrm{Stab}(X)\widehat{\ } \to G\backslash\mathrm{Stab}(Y)\widehat{\ },\ (x, G_x, \sigma) \mapsto [x, G_x, \sigma]$$

is a homeomorphism.

If $(x_1, G_{x_1}, \sigma_1), (x_2, G_{x_2}, \sigma_2) \in \mathrm{Stab}(X)\widehat{\ }$ are such that $[x_1, G_{x_1}, \sigma_1] = [x_2, G_{x_2}, \sigma_2]$ in $G\backslash\mathrm{Stab}(Y)\widehat{\ }$, then there exists $g \in G$ with $(gx_1, G_{gx_1}, g\sigma_1) = (x_2, G_{x_2}, \sigma_2)$ with respect to the G-action on $\mathrm{Stab}(Y)\widehat{\ }$. But, if q is the homeomorphism of X and $G\backslash Y$ from above, then $gx_1 = x_2$ in Y implies that $q(x_1) = q(x_2)$ in $G\backslash Y$, and thus $x_1 = x_2$ in X. It follows that $g \in G_{x_1} = G_{(x_1, G_{x_1}, \sigma_1)}$, and hence $(x_2, G_{x_2}, \sigma_2) = g(x_1, G_{x_1}, \sigma_1) = (x_1, G_{x_1}, \sigma_1)$. This proves injectivity of φ.

Let now $[y, G_y, \varrho] \in G\backslash\mathrm{Stab}(Y)\widehat{\ }$. Let $x = f(y)$ with f as in (6.1.2) and find $g \in G$ with $x = gy$ with respect to the G-action on Y. Then $g(y, G_y, \varrho) = (x, G_x, g\varrho) \in \mathrm{Stab}(X)\widehat{\ }$ and $\varphi(x, G_x, g\varrho) = [y, G_y, \varrho]$, which shows that φ is surjective.

Being the composition of the continuous embedding of $\mathrm{Stab}(X)\widehat{\ }$ into $\mathrm{Stab}(Y)\widehat{\ }$ and of the continuous projection of $\mathrm{Stab}(Y)\widehat{\ }$ onto $G\backslash\mathrm{Stab}(Y)\widehat{\ }$, the map φ is continuous.

It is left to show that φ is open. Suppose that $[y_\nu, G_{y_\nu}, \varrho_\nu] \to [y, G_y, \varrho]$ in $G\backslash\mathrm{Stab}(Y)\widehat{\ }$. Let $(x, G_x, \sigma) = \varphi^{-1}([y, G_y, \varrho])$ and $(x_\nu, G_{x_\nu}, \sigma_\nu) = \varphi^{-1}([y_\nu, G_{y_\nu}, \varrho_\nu])$ for all $\nu \in N$. We have to show that $(x_\nu, G_{x_\nu}, \sigma_\nu) \to (x, G_x, \sigma)$ in $\mathrm{Stab}(X)\widehat{\ }$. We pass to a subnet and relabel. After doing this again, if necessary, we can assume that there are representatives (y, G_y, ϱ) of $[y, G_y, \varrho]$ and $(y_\nu, G_{y_\nu}, \varrho_\nu)$ of $[y_\nu, G_\nu, \varrho_\nu]$ for every $\nu \in N$ such that $(y_\nu, G_{y_\nu}, \varrho_\nu) \to (y, G_y, \varrho)$ in $\mathrm{Stab}(Y)\widehat{\ }$. After passing to a subnet and relabeling again, we can assume that there exists $(H, \varrho_H) \le (G_y, \varrho)$ such that $(y_\nu, G_{y_\nu}, \varrho_\nu) \to (y, H, \varrho_H)$ in $Y \times S(G)\widehat{\ }$.

Find $g \in G$ with $(x, G_x, \sigma) = g(y, G_y, \varrho)$ and a net $(g_\nu)_{\nu \in N}$ in G such that $(x_\nu, G_{x_\nu}, \sigma_\nu) = g_\nu(y_\nu, G_{y_\nu}, \varrho_\nu)$ for all $\nu \in N$ with respect to the action on $\mathrm{Stab}(Y)\widehat{\ }$. Since the first component of φ coincides with the homeomorphism $q\colon X \to G\backslash Y$, we get that $x = f(y)$ and $x_\nu = f(y_\nu)$ for all $\nu \in N$. As $y_\nu \to y$ and as f is continuous, we get that $x_\nu \to x$.

To obtain that $(G_{x_\nu}, \sigma_\nu)_{\nu \in N} = (g_\nu(G_{y_\nu}, \varrho_\nu))_{\nu \in N}$ has a convergent subnet, we need a convergent subnet of $(g_\nu)_{\nu \in N}$. By properness of the action of G on Y the map

$$\gamma\colon G \times Y \to Y \times Y,\ (h, z) \mapsto (hz, z)$$

is proper. Let C be a compact neighborhood of (x, y) in $Y \times Y$. Since $x_\nu \to x$ and $y_\nu \to y$ in Y we can without loss of generality assume that $(x_\nu, y_\nu) \in C$ for all $\nu \in N$. Since $g_\nu y_\nu = x_\nu$ for all $\nu \in N$, this implies that (g_ν, y_ν) is contained in the compact subset $\gamma^{-1}(C) \subseteq G \times Y$ for all $\nu \in N$. After passing to a subnet and relabeling we can thus assume that there is $g' \in G$ with $g_\nu \to g'$. Since $(g_\nu y_\nu)_{\nu \in N}$ converges both to x and to $g'y$ it follows that $x = g'y$.

It now follows from $(y_\nu, G_{y_\nu}, \varrho_\nu) \to (y, H, \varrho_H)$ in $Y \times S(G)\widehat{\ }$ that

$$g_\nu(y_\nu, G_{y_\nu}, \varrho_\nu) \to g'(y, H, \varrho_H),$$

i.e., that

$$(x_\nu, G_{x_\nu}, \sigma_\nu) \to (x, g'H, g'\varrho_H) \tag{6.1.3}$$

in $X \times S(G)\widehat{\ }$. It is left to show that $(g'H, g'\varrho_H) \leq (G_x, \sigma)$. We know that $(H, \varrho_H) \leq (G_y, \varrho)$, so it is clear that

$$(g'H, g'\varrho_H) \leq g'(G_y, \varrho) = (G_x, g'\varrho). \tag{6.1.4}$$

From above we already have an element $g \in G$ with $(x, G_x, \sigma) = g(y, G_y, \varrho)$. It follows that $gy = x = g'y$, and hence that $g'g^{-1} \in G_x$. Using this, we get that $g'\varrho = (g'g^{-1})\sigma$ is equivalent to σ. Combined with (6.1.3) and (6.1.4) this shows that $(x_\nu, G_\nu, \sigma_\nu) \to (x, G_x, \sigma)$ in $\mathrm{Stab}(X)\widehat{\ }$, and thus that φ is open. This completes the proof. □

$\mathrm{Stab}(X)\widehat{\ }$ is homeomorphic to $\widehat{A}$

We now have all ingredients to prove the result we wanted:

Theorem 6.1.4. Let X be a proper G-space and let

$$A = \{f \in C_0(X, \mathcal{K}) \mid \forall x \in X : f(x) \in \mathcal{K}^{G_x}\}$$

as in (6.1.1). Then the spectrum $\widehat{A}$ is homeomorphic to $\mathrm{Stab}(X)\widehat{\ }$.

Proof. By Proposition 6.1.2 we can find a proper G-space Y which contains X as a topological fundamental domain such that the stabilizers coincide. By Lemma 6.1.3, this implies that $\mathrm{Stab}(X)\widehat{\ }$ is a topological fundamental domain for the G-space $\mathrm{Stab}(Y)\widehat{\ }$, i.e., $\mathrm{Stab}(X)\widehat{\ }$ is homeomorphic to $G\backslash\mathrm{Stab}(Y)\widehat{\ }$, which by our main result (Theorem 5.3.3) is homeomorphic to $(C_0(Y) \rtimes G)\widehat{\ }$. But, by Corollary 1.4.6, $C_0(Y) \rtimes G$ is isomorphic to A, so that altogether we get

$$\widehat{A} \;\cong\; (C_0(Y) \rtimes G)\widehat{\ } \;\cong\; G\backslash\mathrm{Stab}(Y)\widehat{\ } \;\cong\; \mathrm{Stab}(X)\widehat{\ },$$

which completes the proof. □

6.2 A G-action for A

Let G, X, $\mathcal{K}$, and A be as above. Before we define a G-action on A it will be useful to have a description of the irreducible representations of A in terms of the decompositions of the fibres $\mathcal{K}^{G_x}$ at hand.

Irreducible representations of A

Recall from (2.1.3) that $\widehat{A}$ can be decomposed as

$$\widehat{A} = \coprod_{x \in X} (\mathcal{K}^{G_x})\widehat{\ } = \coprod_{x \in X} \bigoplus_{\sigma \in \widehat{G_x}} \mathcal{K}(H_{\mathrm{ind}\,\sigma})\widehat{\ } = \coprod_{\substack{x \in X \\ \sigma \in \widehat{G_x}}} \mathcal{K}(H_{\mathrm{ind}\,\sigma})\widehat{\ } = \coprod_{\substack{x \in X \\ \sigma \in \widehat{G_x}}} \{i_\sigma\}.$$

It follows that an element $(x, G_x, \sigma) \in \mathrm{Stab}(X)\widehat{\ } \cong \widehat{A}$ can be identified with the irreducible representation $q^{x,\sigma} \circ \mathrm{ev}_x$ of A, where $q^{x,\sigma}$ denotes the projection of $\mathcal{K}^{G_x}$ onto the summand $\mathcal{K}(H_{\mathrm{ind}\,\sigma})$, and ev_x is evaluation at x on A.

A G-action for A and $\widehat{A}$

Recall that the G-action on $\mathcal{K} = \mathcal{K}(L^2(G))$ which appears in the definition of A is given by $\mathrm{Ad}(\varrho)$. The following lemma will be useful:

Lemma 6.2.1. Let $g \in G$ and $x \in X$. Then

$$\mathrm{Ad}(\varrho)(g)(\mathcal{K}^{G_x}) = \mathcal{K}^{G_{gx}}.$$

Proof. "$\subseteq$": Let $T \in \mathcal{K}^{G_x}$, i.e., for every $l \in G_x$ we have that $\mathrm{Ad}(\varrho)(l)T = T$. Let now $k \in G_{gx} = gG_xg^{-1}$ and choose $l \in G_x$ such that $k = glg^{-1}$. Then

$$\mathrm{Ad}(\varrho)(k)\left(\mathrm{Ad}(\varrho)(g)T\right) = \mathrm{Ad}(\varrho)(glg^{-1})\mathrm{Ad}(\varrho)(g)T = \mathrm{Ad}(\varrho)(g)\mathrm{Ad}(\varrho)(l)T = \mathrm{Ad}(\varrho)(g)T,$$

which shows that $\mathrm{Ad}(\varrho)(g)T \in \mathcal{K}^{G_{gx}}$.

"$\supseteq$": This is equivalent to $\mathrm{Ad}(\varrho)(g^{-1})(\mathcal{K}^{G_{gx}}) = \mathcal{K}^{G_x}$, which follows from "$\subseteq$". □

Define now for every $g \in G$ and $f \in A$ a function gf by

$$(gf)(x) = \mathrm{Ad}(\varrho)(g)f(g^{-1}x) \tag{6.2.1}$$

for every $x \in X$. Since, for every $g \in G$ and $x \in X$, we know by definition of A that $f(g^{-1}x) \in \mathcal{K}^{G_{g^{-1}x}}$, the previous lemma implies that $(gf)(x) \in \mathcal{K}^{G_x}$, and hence $gf \in A$. It follows that (6.2.1) defines a G-action on A.

This induces a G-action on $\widehat{A}$ by $(g\pi)(f) = \pi(g^{-1}f)$ for all $g \in G$, $f \in A$, of which we now want to show that it corresponds to the G-action on $\mathrm{Stab}(X)\widehat{\ }$. Let $(x, G_x, \sigma) \in \mathrm{Stab}(X)\widehat{\ }$ and $g \in G$. We saw in the first paragraph of this section that (x, G_x, σ) corresponds to the element $q^{x,\sigma} \circ \mathrm{ev}_x$ of $\widehat{A}$.

A calculation shows that

$$g(q^{x,\sigma} \circ \mathrm{ev}_x)(f) = \left((q^{x,\sigma} \circ \mathrm{Ad}(\varrho)(g^{-1})) \circ \mathrm{ev}_{gx}\right)(f).$$

Since $q^{x,\sigma} \circ \mathrm{Ad}(\varrho)(g^{-1})$ is the projection of $\mathcal{K}^{G_{gx}}$ onto $\mathcal{K}(H_{\mathrm{ind}\,\sigma})$, and $H_{\mathrm{ind}\,\sigma}$ is isomorphic to $H_{\mathrm{ind}(g\sigma)}$ by equivalence of $\mathrm{ind}_{G_x}^G \sigma$ and $\mathrm{ind}_{G_{gx}}^G (g\sigma)$, it follows that $g(q^{x,\sigma} \circ \mathrm{ev}_x)$ is equivalent to $q^{gx,g\sigma} \circ \mathrm{ev}_{gx}$, which corresponds to $g(x, G_x, \sigma)$ in $\mathrm{Stab}(X)\widehat{\ }$.

Combined with the facts that $\widehat{A}$ is homeomorphic to $\mathrm{Stab}(X)\widehat{\ }$ and that the quotient $G\backslash\mathrm{Stab}(X)\widehat{\ }$ is homeomorphic to $(C_0(X) \rtimes G)\widehat{\ }$, we have now obtained that $(C_0(X) \rtimes G)\widehat{\ }$ can be described as the quotient $G\backslash\widehat{A}$ of the spectrum of A.

Chapter 7

Proper G-spaces with Palais' slice property

Recall from Remark 1.1.8 (iv) that a proper G-space X has Palais' slice property (SP) if it is locally induced from the stabilizer subgroups, which means that for every $x \in X$ there exists a G-invariant open neighborhood U_x such that there is a continuous G-map $\varphi_x \colon U_x \to G/G_x$ with $\varphi_x(x) = eG_x$. It then follows from Remark 1.1.8(i) that U_x is G-homeomorphic to $G \times_{G_x} Y_x$ with $Y_x = \varphi_x^{-1}(\{eG_x\})$. Palais proved in [Pal61, Proposition 2.3.1] that (SP) is satisfied whenever the group G is a Lie group.

In this chapter we first show that the assumption of property (SP) leads to a simplification of our result, namely, it allows to compare convergence in $(C_0(X) \rtimes G)^\frown$ directly with convergence in $\mathrm{Stab}(X)^\frown$, and not just in the quotient $G\backslash\mathrm{Stab}(X)^\frown$. We continue in Section 7.2 with a presentation of Echterhoff and Emerson's definition of a topology on $\mathrm{Stab}(X)^\frown$ in the situation that X satisfies (SP). As announced in Section 2.1, we also show that, if X satisfies (SP), their and our definition give the same topology on $\mathrm{Stab}(X)^\frown$.

In Section 7.3 we give a sketch of their proof that ind^G is a homeomorphism, and we also discuss why it is so different from our approach.

7.1 Motivation for the consideration of (SP)

In terms of convergence, the fact that ind^G is a homeomorphism reads as follows:

Corollary 7.1.1. Let $(x_\nu, G_\nu, \sigma_\nu)_{\nu \in N}$ be a net in $\mathrm{Stab}(X)^\frown$ and let $(x, G_x, \sigma) \in \mathrm{Stab}(X)^\frown$. Then the following conditions are equivalent:

(i) $[x_\nu, G_\nu, \sigma_\nu] \to [x, G_x, \sigma]$ in $G\backslash\mathrm{Stab}(X)^\frown$,

(ii) $\pi^{x_\nu,\sigma_\nu} \to \pi^{x,\sigma}$ in $(C_0(X) \rtimes G)^\frown$.

In general we can not conclude that $(x_\nu, G_\nu, \sigma_\nu)_{\nu \in N}$ converges to (x, G_x, σ) in $\mathrm{Stab}(X)^\frown$, because, as seen in Lemma 5.1.1, the convergence $\pi^{x_\nu,\sigma_\nu} \to \pi^{x,\sigma}$ in $(C_0(X) \rtimes G)^\frown$ does not imply $x_\nu \to x$ in X but only $Gx_\nu \to Gx$ in $G\backslash X$.

There is, however, a special case in which we get equivalence of convergence in $\mathrm{Stab}(X)^\frown$ and in $(C_0(X) \rtimes G)^\frown$. The key to this special case is the following lemma, the assumptions of which suggest that it will be useful to consider Palais' slice property:

Lemma 7.1.2. Let K be a compact group, let Y be a K-space and suppose that $y \in Y$ is fixed by K, i.e., that $K_y = K$. Then a net $(y_\nu, K_\nu, \sigma_\nu)_{\nu\in N}$ converges to (y, K, σ) in $\mathrm{Stab}(Y)\widehat{}$ if and only if $([y_\nu, K_\nu, \sigma_\nu])_{\nu\in N}$ converges to $[y, K, \sigma]$ in $K\backslash\mathrm{Stab}(Y)\widehat{}$.

Proof. Let $(y_\nu, K_\nu, \sigma_\nu)_{\nu\in N}$ be a net in $\mathrm{Stab}(Y)\widehat{}$ and let $(y, K, \sigma) \in \mathrm{Stab}(Y)\widehat{}$. It is clear that $(y_\nu, K_\nu, \sigma_\nu) \to (y, K, \sigma)$ in $\mathrm{Stab}(Y)\widehat{}$ implies convergence in the orbit space.

Suppose now that $[y_\nu, K_\nu, \sigma_\nu] \to [y, K, \sigma]$ in $K\backslash\mathrm{Stab}(Y)\widehat{}$, and that we have passed to a subnet of $(y_\nu, K_\nu, \sigma_\nu)_{\nu\in N}$ and relabeled. By Remark 1.1.1 on openness of the orbit map and by definition of convergence in $\mathrm{Stab}(Y)\widehat{}$ we can pass to a subnet $(y_j, K_j, \sigma_j)_{j\in J}$ such that there exist a net $(k_j)_{j\in J}$ in K and an element $(H, \sigma_H) \le (K, \sigma)$ such that

$$k_j(y_j, K_j, \sigma_j) \to (y, H, \sigma_H) \le (y, K, \sigma)$$

in $Y \times S(K)\widehat{}$. By compactness of K we can assume without loss of generality that there is $k \in K$ with $k_j \to k$. By continuity of the K-action on $Y \times \mathrm{Rep}(S(K))$, see Lemma 2.2.14, it follows that

$$(y_j, K_j, \sigma_j) \to k^{-1}(y, H, \sigma_H),$$

so it remains to show that $k^{-1}(y, H, \sigma_H) \le (y, K, \sigma)$. Since $k \in K = K_y$ it follows immediately from $(y, H, \sigma_H) \le (y, K, \sigma)$ that

$$k^{-1}(y, H, \sigma_H) \le k^{-1}(y, K, \sigma) = (y, K, \sigma),$$

as required. □

Corollary 7.1.3. Let K be a compact group, let Y be a K-space and suppose that $y \in Y$ is fixed by K, i.e., that $K_y = K$. Let $(y_\nu, K_\nu, \sigma_\nu)_{\nu\in N}$ be a net in $\mathrm{Stab}(Y)\widehat{}$, and let $(y, K, \sigma) \in \mathrm{Stab}(Y)\widehat{}$. Then the following are equivalent:

(i) $(y_\nu, K_\nu, \sigma_\nu) \to (y, K, \sigma)$ in $\mathrm{Stab}(Y)\widehat{}$,

(ii) $\pi^{y_\nu,\sigma_\nu} \to \pi^{y,\sigma} \approx \sigma$ in $(C_0(Y) \rtimes K)\widehat{}$.

Proof. The equivalence of (i) and (ii) is a direct consequence of Corollary 7.1.1 and Lemma 7.1.2. It is straightforward from the definitions that $\pi^{y,\sigma} = P^y \rtimes \mathrm{ind}_K^K \sigma$ is equivalent to σ. □

The assumptions of the preceding lemma and corollary are locally satisfied if a proper G-space satisfies (SP), and we now continue to have a closer look at such spaces.

7.2 The topology on $\mathrm{Stab}(X)\widehat{}$ if X satisfies (SP)

Definition of neighborhood bases in $\mathrm{Stab}(X)\widehat{}$ with (SP)

In the following we give a short account of Echterhoff and Emerson's approach to describe the topology of $(C_0(X) \rtimes G)\widehat{}$ in case of a proper G-space X with (SP). For proofs and details the reader may consult Section 4 of Echterhoff and Emerson's paper [EE].

Definition 7.2.1. Let X be a proper G-space with (SP). For every $x \in X$ define

$$S_x := \{y \in X \mid G_y \le G_x\}.$$

An almost slice at x is a set of the form $W \cdot V_x$, where W is a symmetric open neighborhood of e in G, and V_x is a relatively open neighborhood of x in S_x. The set of all almost slices at x is denoted by $\mathcal{A}S_x$.

It is only a matter of convenience that we work with symmetric open neighborhoods of e in G, the theory works as well without the assumption of symmetry. Notice that, for every $x \in X$, we have $Y_x \subseteq S_x$ for every G_x-slice Y_x in x, because $G_y \leq G_x$ for every $y \in Y_x$ (see Remark 1.1.8 (iii)).

Echterhoff and Emerson show that, for every $x \in X$, the set $\mathcal{A}S_x$ forms an open neighborhood base at x. As long as we work in their setting for a topology on $\mathrm{Stab}(X)\widehat{\ }$, we will denote the elements of $\mathrm{Stab}(X)\widehat{\ }$ by (x, σ) instead of (x, G_x, σ), because we won't have to work in $S(G)\widehat{\ }$ anyway.

Definition 7.2.2. Suppose that X is a proper G-space which satisfies (SP). For every element $(x, \sigma) \in \mathrm{Stab}(X)\widehat{\ }$ and every almost slice $WV_x \in \mathcal{A}S_x$ define

$$U\left((x, \sigma), WV_x\right) := \left\{(y, \tau) \in \mathrm{Stab}(X)\widehat{\ } \mid \exists g \in W : \; gy \in V_x \;\wedge\; g\tau \leq \sigma|_{G_{gy}}\right\}; \tag{7.2.1}$$

further set

$$\mathcal{U}_{(x,\sigma)} := \left\{U((x, \sigma), WV_x) \mid WV_x \in \mathcal{A}S_x\right\}.$$

Notice that, by symmetry of the neighborhoods W, the existence of an element $g \in W$ with $gy \in V_x$ is equivalent to $y \in WV_x$.

Lemma 7.2.3 (Lemma 4.3 in [EE])**.** Let X be a proper G-space satisfying (SP). Then there is a topology on $\mathrm{Stab}(X)\widehat{\ }$ such that the elements of $\mathcal{U}_{(x,\sigma)}$ form a base of open neighborhoods for every $(x, \sigma) \in \mathrm{Stab}(X)\widehat{\ }$. The canonical action of G on $\mathrm{Stab}(X)\widehat{\ }$ is continuous with respect to this topology.

It should be pointed out that the definition of this topology does not work if we drop the assumption of Palais' slice property. To see this, suppose that $U\left((x, \sigma), WV_x\right)$ is a neighborhood of $(x, \sigma) \in \mathrm{Stab}(X)\widehat{\ }$ as in (7.2.1) and that $(y, \tau) \in U\left((x, \sigma), WV_x\right)$. Without (SP) it would in general not be true that $G_{gy} \leq G_x$, so the statement $g\tau \leq \sigma|_{G_{gy}}$ would not make sense.

In a general proper G-space X, we only know that there exist a compact subgroup L_x of G and a L_x-slice Y_x such that $G_y \leq L_x$ for all $y \in Y_x$ (see Remark 1.1.8 (iii)), but L_x is usually bigger than G_x. So it is essential for Echterhoff and Emerson's topology on $\mathrm{Stab}(X)\widehat{\ }$ that X is locally induced from the stabilizers, not just from arbitrary compact subgroups.

Echterhoff and Emerson's topology on $\mathrm{Stab}(X)\widehat{\ }$ becomes even easier to describe if G is discrete. Since then the set $\{e\}$ is open in G, we can ignore the open neighborhoods W above. It follows that the set S_x is itself open in X for every $x \in X$ and contains an open neighborhood base of x. Hence, an open neighborhood base of an element $(x, \sigma) \in \mathrm{Stab}(X)\widehat{\ }$ is given by

$$\mathcal{U}_{(x,\sigma)} := \left\{U((x, \sigma), V_x) \mid V_x \subseteq S_x \text{ is an open neighborhood of } x\right\},$$

where

$$U((x, \sigma), V_x) = \left\{(y, \tau) \in \mathrm{Stab}(X)\widehat{\ } \mid y \in V_x \;\wedge\; \tau \leq \sigma|_{G_y}\right\}$$

for every open neighborhood V_x of x in S_x.

Convergence in $\mathrm{Stab}(X)^\frown$ with (SP)

We proceed with a formulation of convergence in this topology, and then show that in proper G-spaces with Palais' slice property, this convergence coincides with our convergence in $\mathrm{Stab}(X)^\frown$ from Remark 2.3.4.

Remark 7.2.4. Let X be a proper G-space with (SP). Then a net $(x_\nu, \sigma_\nu)_{\nu\in N}$ converges to (x, σ) in $\mathrm{Stab}(X)^\frown$ in the sense of Lemma 7.2.3 if and only if $x_\nu \to x$ and every subnet of $(x_\nu, \sigma_\nu)_{\nu\in N}$ has a subnet $(x_j, \sigma_j)_{j\in J}$ such that there exists a net $(g_j)_{j\in J}$ with $g_j \to e$ in G, and such that $g_j x_j \in S_x$ and $g_j\sigma_j \leq \sigma|_{G_{g_j x_j}}$ for all $j \in J$.

Proposition 7.2.5. Let X be a proper G space which satisfies Palais' slice property. Then the notions of convergence given in Remark 2.3.4 and in Remark 7.2.4 coincide.

The idea of the proof is based on the same tools as the proof that our closure operation on $\mathrm{Stab}(X)^\frown$ for a general proper G-space X satisfies the Kuratowski axioms (Proposition 2.3.3):

Proof of Proposition 7.2.5. Let $(x_\nu, G_\nu, \sigma_\nu)_{\nu\in N}$ be a net in $\mathrm{Stab}(X)^\frown$ and let (x, G_x, σ) be in $\mathrm{Stab}(X)^\frown$. We use Palais' slice property to choose an open neighborhood U_x of x and a G_x-slice Y_x at x such that $U_x \cong G \times_{G_x} Y_x$.

Suppose first that $(x_\nu, G_\nu, \sigma_\nu)_{\nu\in N}$ converges to (x, G_x, σ) in the sense of Remark 2.3.4, i.e., for every subnet there exist a subnet $(x_j, G_j, \sigma_j)_{j\in J}$ and an element $(H, \sigma_H) \in S(G)^\frown$ such that $(H, \sigma_H) \leq (G_x, \sigma)$ and $(x_j, G_j, \sigma_j) \to (x, H, \sigma_H)$ in $X \times S(G)^\frown$. This implies in particular that $x_\nu \to x$.

To prove that the other conditions from Remark 7.2.4 hold, suppose that we have passed to a subnet of $(x_\nu, G_\nu, \sigma_\nu)_{\nu\in N}$ and choose a subnet $(x_j, G_j, \sigma_j)_{j\in J}$ of that subnet satisfying $(x_j, G_j, \sigma_j) \to (x, H, \sigma_H)$ in $X \times S(G)^\frown$ for some $(H, \sigma_H) \leq (G_x, \sigma)$. By Lemma 1.1.9 we can without loss of generality assume that there is a net $(g_j)_{j\in J}$ in G such that $g_j \to e$ and $g_j x_j \in Y_x \subseteq S_x$, and hence $g_j G_j = G_{g_j x_j} \leq G_x$ for all $j \in J$.

By continuity of the G-action on $X \times S(G)^\frown$ we then also have $g_j(x_j, G_j, \sigma_j) \to (x, H, \sigma_H)$, which by continuity of induction implies that

$$\mathrm{ind}_{G_{g_j x_j}}^{G_x} (g_j \sigma_j) \to \mathrm{ind}_H^{G_x} \sigma_H.$$

By choice of σ_H we know that $\sigma_H \leq \sigma|_H$, which by Frobenius reciprocity is equivalent to $\sigma \leq \mathrm{ind}_H^{G_x} \sigma_H$, so we also get

$$\mathrm{ind}_{G_{g_j x_j}}^{G_x} (g_j \sigma_j) \to \sigma.$$

Now Lemma 1.3.4 yields that, by passing to a subnet and relabeling if necessary, we can assume that

$$\sigma \leq \mathrm{ind}_{G_{g_j x_j}}^{G_x} (g_j \sigma_j)$$

and thus, by Frobenius reciprocity,

$$g_j \sigma_j \leq \sigma|_{G_{g_j x_j}}$$

for all $j \in J$, as required.

Conversely, suppose now that $(x_\nu, \sigma_\nu)_{\nu\in N}$ converges to (x, σ) in the sense of Remark 7.2.4. Suppose that we have passed to a subnet of $(x_\nu, \sigma_\nu)_{\nu\in N}$, and choose a subnet $(x_j, \sigma_j)_{j\in J}$ of that subnet and a net $(g_j)_{j\in J}$ in G such that $g_j \to e$, $g_j x_j \in S_x$, and $g_j\sigma_j \leq \sigma|_{G_{g_j x_j}}$ for all $j \in J$.

Now, the net $(G_{g_j x_j}, g_j\sigma_j)_{j\in J}$ lies in the space $\mathrm{Rep}(S(G))_{\leq(G_x,\sigma)}$, which by Lemma 2.2.9 is compact. Hence we can, after passing to a subnet and relabeling, assume that there exist a subgroup $H \leq G_x$ and a subrepresentation $\varrho \leq \sigma|_H$ such that

$$g_j(x_j, G_j, \sigma_j) \to (x, H, \varrho)$$

in $X \times \mathrm{Rep}(S(G))$. Let now σ_H be any irreducible subrepresentation of ϱ, then we have $(H, \sigma_H) \leq (G_x, \sigma)$ and $g_j(G_j, \sigma_j) \to (H, \sigma_H)$ in $S(G)\widehat{\ }$. Since $g_j \to e$, continuity of the G-action implies that

$$(x_j, G_j, \sigma_j) \to (x, H, \sigma_H) \leq (x, G_x, \sigma),$$

which completes the proof. □

This shows that, if X is a proper G-space with Palais' slice property, our topology and the one defined by Echterhoff and Emerson are the same. Hence, the fact that the map

$$\mathrm{ind}^G\colon G\backslash\mathrm{Stab}(X)\widehat{\ } \to (C_0(X) \rtimes G)\widehat{\ }$$

is a homeomorphism with respect to Echterhoff and Emerson's topology, as proved in [EE], can be seen as a special case of our result from Theorem 5.3.3. The methods employed by Echterhoff and Emerson are, however, quite different from ours, and have several advantages.

7.3 Different settings — different proofs

To be able to compare Echterhoff and Emerson's proof that ind^G is a homeomorphism if the proper G-space X satisfies (SP) with our proof in the general case, we present a very short version of some of their notation and results. The reader may refer to [EE] for details.

As mentioned before, Echterhoff and Emerson show that $C_0(X) \rtimes G$ is a $C_0(G\backslash X)$-algebra and give quite a concrete description of the fibres. This description is essential for their proof of continuity of ind^G in the (SP)-situation.

The fibres of $C_0(X) \rtimes G$ as a bundle of fixed point algebras

Recall from page 14 that if X is a proper G-space and $\mathcal{K} := \mathcal{K}(L^2(G))$ is equipped with the G-action given by $\mathrm{Ad}(\varrho)$, then

$$C_0(X) \rtimes G \cong C_0(X \times_{G,\mathrm{Ad}\varrho} \mathcal{K}),$$

which implies that $C_0(X) \rtimes G$ is a $C_0(G\backslash X)$-algebra with fibre at an orbit Gx given by $\mathcal{K}^{G_x}$. The structure of these fibres is described explicitly in the following lemma.

Recall from (4.2.1) that, if K is a compact group and $\sigma \in \widehat{K}$, then $p_\sigma := d_\sigma \chi_{\sigma^*}$ is the projection onto the minimal two-sided ideal in $L^2(K)$ corresponding to σ.

Lemma 7.3.1 (Lemma 3.1 in [EE]). Let G be a locally compact group and let K be a compact subgroup of G which acts on $\mathcal{K} := \mathcal{K}(L^2(G))$ via $k \mapsto \mathrm{Ad}\varrho(k)$. For every $\sigma \in \widehat{K}$ define $L^2(G)_\sigma := \varrho(p_\sigma)L^2(G)$. Then the following statements hold:

(i) $L^2(G) = \bigoplus_{\sigma\in\widehat{K}} L^2(G)_\sigma$,

(ii) For every $\sigma \in \widehat{K}$, the space $L^2(G)_\sigma$ is $\varrho(K)$-invariant and is isomorphic to $H_{\operatorname{ind}\sigma} \otimes V_\sigma^*$ such that $\varrho(k)|_{L^2(G)_\sigma} = 1_{H_{\operatorname{ind}\sigma}} \otimes \sigma^*(k)$ for all $k \in K$,

(iii) $\mathcal{K}^K \cong \bigoplus_{\sigma\in\widehat{K}} \mathcal{K}(H_{\operatorname{ind}\sigma})$, where the isomorphism is given by mapping $T \in \mathcal{K}(H_{\operatorname{ind}\sigma})$ to the operator $T \otimes 1_{V_\sigma^*} \in \mathcal{K}(L^2(G)_\sigma)$ under the decomposition of (ii),

(iv) The projection of $C_0(G/K) \rtimes G \cong \mathcal{K}^K$ (see (1.4.3)) onto the factor $\mathcal{K}(H_{\operatorname{ind}\sigma})$ in the decomposition in (iii) is equal to the representation $P^\sigma \rtimes \operatorname{ind}_K^G \sigma$ of $C_0(G/K) \rtimes G$ on $H_{\operatorname{ind}\sigma}$, where P^σ is given by

$$P^\sigma(\varphi)\xi(g) = \varphi(gK)\xi(g)$$

for all $\varphi \in C_0(G/K)$, $\xi \in H_{\operatorname{ind}\sigma}$, and $g \in G$.

We give a very short indication of the ideas of the proof, the details of which can be found in [EE]. Statement (i) of the lemma holds because the projections p_σ add up to the unit in $M(C^*(K))$ with respect to the strict topology. Statement (ii) is a consequence of the isomorphism $L^2(G) \cong \bigoplus_{\tau\in K} H_{\operatorname{ind}\tau} \otimes V_\tau^*$ from (4.3.7), and part (iii) follows from a calculation using the definition of $\mathcal{K}^K$, the decomposition from (ii), and Schur's lemma. Statement (iv) also follows from the definitions.

Proof of "ind^G is a homeomorphism" in the (SP)-case

Suppose now that X is a proper G-space with (SP). We will now outline how Echterhoff and Emerson's analysis of the structure of $C_0(X) \rtimes G$ can be used to prove that ind^G is a homeomorphism. Proposition 3.5.2 implies that it is possible to reduce the proof to the situation where a compact group — here even a stabilizer subgroup — acts on a corresponding slice. The transition to the general case works exactly as in our situation (see the proof of Theorem 5.3.2), so we don't discuss it any further here. Restricted to a stabilizer subgroup acting on a slice the main result reads as follows:

Proposition 7.3.2 (Proposition 4.9 in [EE]). Let K be a compact group which acts on a locally compact Hausdorff space Y, and suppose that $y \in Y$ is such that $K_y = K$, i.e., y is fixed by K. Let $\sigma \in \widehat{K}$ and let $(y_\nu, \sigma_\nu)_{\nu\in N}$ be a net in $\operatorname{Stab}(Y)\widehat{\ }$. We identify $\pi^{y,\sigma} = P^y \rtimes \operatorname{ind}_K^K \sigma$ with σ. The following statements are equivalent:

(i) The net $(y_\nu, \sigma_\nu)_{\nu\in N}$ converges to (y, σ) in $\operatorname{Stab}(Y)\widehat{\ }$,

(ii) The net $(\pi^{y_\nu,\sigma_\nu})_{\nu\in N}$ converges to $\pi^{y,\sigma}$ in $(C_0(Y) \rtimes K)\widehat{\ }$.

Sketch of proof. (i)⇒(ii) (continuity): By Proposition 1.3.2 it suffices to show that $\sigma = \pi^{y,\sigma}$ is weakly contained in every subnet of $(\pi^{y_\nu,\sigma_\nu})_{\nu\in N}$. Suppose that we have passed to a subnet and relabeled. We have to show that every $a \in C_0(Y) \rtimes K$ with $\pi^{y_\nu,\sigma_\nu}(a) = 0$ for all $\nu \in N$ satisfies $\sigma(a) = 0$.

Using the structure of $C_0(Y) \rtimes K$ as a $C_0(K\backslash Y)$-algebra and the decomposition of the fibres from Lemma 7.3.1 we get that every $a \in C_0(Y) \rtimes K$ is represented on the fibres by a net $(a_{y_\nu})_{\nu\in N}$ and an element a_y such that $a_{y_\nu} \to a_y$ in norm and

$$a_{y_\nu} \in \mathcal{K}(L^2(K))^{K_\nu} \cong \bigoplus_{\tau\in\widehat{K_\nu}} \mathcal{K}(H_{\operatorname{ind}\tau}) \otimes 1_{V_\tau^*}$$

for every $\nu \in N$ and

$$a_y \in \mathcal{K}(L^2(K))^K \cong \bigoplus_{\tau \in \widehat{K}} \mathcal{K}(H_{\mathrm{ind}\,\tau}) \otimes 1_{V_\tau^*} \overset{\tau \approx \mathrm{ind}_K^K \tau}{\cong} \bigoplus_{\tau \in \widehat{K}} \mathcal{K}(V_\tau) \otimes 1_{V_\tau^*}.$$

Moreover, by a similar argument as in Remark 1.4.3, we get that

$$\mathcal{K}(L^2(K))^K \cong C^*(K).$$

In what follows we write 1_ν instead of $1_{V_{\sigma_\nu}^*}$ for every $\nu \in N$. The projections of the fibres $C_0(Ky_\nu) \rtimes K \cong \mathcal{K}(L^2(K))^{K_\nu}$ onto the factors $\mathcal{K}(H_{\mathrm{ind}\,\sigma_\nu}) \otimes 1_\nu$ are just the representations $\pi^{y_\nu,\sigma_\nu} \otimes 1_\nu$ for every $\nu \in N$. Since $\sigma(p_\sigma) = 1_{V_\sigma}$ we can replace any $a \in C_0(Y) \rtimes K$ by $p_\sigma a p_\sigma$. Using the embedding $C^*(K) \to M(C_0(Y) \rtimes K)$ we can calculate for every $a \in C_0(Y) \rtimes K$ and every $\nu \in N$:

$$\pi^{y_\nu,\sigma_\nu}(p_\sigma a p_\sigma) = (\mathrm{ind}_{K_\nu}^K \sigma_\nu(p_\sigma) \otimes 1_\nu)(\pi^{y_\nu,\sigma_\nu}(a) \otimes 1_\nu)(\mathrm{ind}_{K_\nu}^K \sigma_\nu(p_\sigma) \otimes 1_\nu),$$

where $(\mathrm{ind}_{K_\nu}^K \sigma_\nu(p_\sigma) \otimes 1_\nu)$ is the projection of $H_{\mathrm{ind}\,\sigma_\nu} \otimes V_{\sigma_\nu}^*$ onto the isotype

$$W_\nu := (H_{\mathrm{ind}\,\sigma_\nu} \otimes V_{\sigma_\nu}^*) \cap (V_\sigma \otimes V_\sigma^*)$$

of σ in $\mathrm{ind}_{K_\nu}^K \sigma_\nu$. By assumption and by the definition of convergence in $\mathrm{Stab}(Y)\widehat{\ }$ as in Remark 7.2.4 we know that $\sigma_\nu \leq \sigma|_{K_\nu}$ and thus by Frobenius reciprocity $\sigma \leq \mathrm{ind}_{K_\nu}^K \sigma_\nu$, so the spaces W_ν are nonzero for all $\nu \in N$. As $V_\sigma \otimes V_\sigma^*$ is finite-dimensional, we can without loss of generality assume that the projections $q_\nu \colon V_\sigma \otimes V_\sigma^* \to W_\nu$ converge to a projection q of $V_\sigma \otimes V_\sigma^*$ onto a nonzero subspace W.

If now $a \in C_0(Y) \rtimes K$ is such that

$$\pi^{y_\nu,\sigma_\nu}(a) \otimes 1_\nu = q_\nu a_{y_\nu} q_\nu = 0$$

for all $\nu \in N$, then it follows from the norm convergence $a_{y_\nu} \to a_y$ and $q_\nu \to q$ that we get $q_\nu a_{y_\nu} q_\nu \to q a_y q$, and thus that $q a_y q = 0$. Since W is a K-invariant subspace of $V_\sigma \otimes V_\sigma^*$, it follows that $\sigma(a_y) = 0$, as required.

(ii)⇒(i) (openness): As shown in Lemma 5.1.1, it follows from $\pi^{y_\nu,\sigma_\nu} \to \pi^{y,\sigma}$ that $Ky_\nu \to Ky$. But as $Ky = \{y\}$ and K is compact, it follows easily that every subnet of $(y_\nu)_{\nu \in N}$ has a subnet which converges to y, and thus that $y_\nu \to y$. Additionally, $\pi^{y_\nu,\sigma_\nu} \to \pi^{y,\sigma}$ implies that $\mathrm{ind}_{K_\nu}^K \sigma_\nu \to \mathrm{ind}_K^K \sigma = \sigma$ in $\mathrm{Rep}(K)$, as shown in the beginning of Section 5.2. As shown in Lemma 1.3.4, this implies that there exists an index $\nu_0 \in N$ such that $\sigma \leq \mathrm{ind}_{K_\nu}^K \sigma_\nu$, and thus by Frobenius reciprocity, that $\sigma_\nu \leq \sigma|_{K_\nu}$ for all $\nu \geq \nu_0$. This shows that $(y_\nu, \sigma_\nu) \to (y, \sigma)$ in $\mathrm{Stab}(Y)\widehat{\ }$, and the proof is complete. □

Discussion

The most apparent difference between Echterhoff and Emerson's proof and ours is the following: In their situation, it is easy to show openness of ind^G, but difficult to prove continuity. With our topology on $\mathrm{Stab}(X)\widehat{\ }$, continuity of ind^G is an easy consequence of the continuity of induction, but the proof of openness requires a lot of work.

We saw in Proposition 7.2.5 that, in the (SP)-situation, Echterhoff and Emerson's and our topologies coincide. Hence, one could also prove continuity of ind^G in the (SP)-case by passing from Echterhoff and Emerson's notion of convergence to ours, and then use continuity of induction. But this procedure would spoil a significant advantage of Echterhoff and Emerson's topology, namely, that it does not require the space $\operatorname{Rep}(S(G))$ with its somewhat difficult to grasp topology.

It is, however, not possible to transfer Echterhoff and Emerson's proof of continuity of ind^G to the situation of a general proper G-space X. First of all, the definition of the topology would not make sense, because we can not assume that the space X is locally induced from the stabilizers. Moreover, it is very important that $K_y = K$ in the notation of Proposition 7.3.2, because otherwise we don't get the identification of $\operatorname{ind}_{K_y}^K \sigma$ and σ. But $K_y = K$ can only be assumed if the space X satisfies (SP).

The two features mentioned in the previous sentence are also crucial for the nice and easy proof of openness provided by Echterhoff and Emerson. In our case, we also obtain convergence of the form $\operatorname{ind}_{K_\nu}^K \sigma_\nu \to \operatorname{ind}_{K_y}^K \sigma$, but in general we have that K_y is a proper subgroup of K, so it is not possible to use the argument from the proof of Proposition 7.3.2 above.

Appendix A

Some useful results and complementary proofs

In this appendix we give some supplements to the contents of this thesis. We start with a brief introduction to $C_0(Z)$-algebras and present an important result on the structure of their spectra.

In Section A.2, we show that, if K is a compact group, then integration on $\mathscr{F}(K)$ is continuous. The proof is based on Fell's proof of Proposition 3.1 in [Fel64], but it becomes much more readable in the compact case.

As a technical detail we deliver the proof of Lemma 2.4.1, which was used in the proof of continuity of ind^G in Section A.3.

A.1 $C_0(Z)$-algebras

Most of the following material on $C_0(Z)$-algebras is taken from [Wil07]. Let A be a C^*-algebra and let Z be a locally compact Hausdorff space. Then A is said to be a $C_0(Z)$-algebra if there exists a *-homomorphism Φ from $C_0(Z)$ into the center $ZM(A)$ of the multiplier algebra of A, which is nondegenerate, which means that the ideal

$$\Phi(C_0(Z))A = \operatorname{span}\{\Phi(f)a \mid f \in C_0(Z),\ a \in A\}$$

is dense in A. We often write $f \cdot a$ instead of $\Phi(f)a$. The C^*-algebra A is then fibred over the space Z in the following way: Whenever J is an ideal in $C_0(Z)$, then the closure of $\Phi(J)A$ is an ideal in A. For every $z \in Z$ let J_z be the ideal of functions vanishing at z, and let $I_z := \overline{\Phi(J_z)A}$ denote the corresponding ideal in A. Then the quotient $A(z) = A/I_z$ is called the fibre of A over z. For every $a \in A$ and $z \in Z$ we define $a(z) := a + I_z \in A(z)$. This allows us to identify every $a \in A$ with the function $z \mapsto a(z)$ from Z into the disjoint union $\coprod_{z\in Z} A(z)$; which can be seen as a section of the bundle of C^*-algebras $\{A(z) \mid z \in Z\}$.

Example A.1.1. Let X be a proper G-space. We already know from the identification of $C_0(X) \rtimes G$ with $C_0(X \times_G \mathcal{K})$ that the former is a $C_0(G\backslash X)$-algebra with fibres $C_0(Gx) \rtimes G$, see page 14. This can also be obtained directly by considering the map

$$\Phi\colon C_0(G\backslash X) \to ZM(C_0(X) \rtimes G)$$

given by

$$(\Phi(f)F)(g)(x) = f(Gx)F(g)(x)$$

for all $f \in C_0(G\backslash X)$, $F \in C_c(G, C_0(X))$, $g \in G$, and $x \in X$.

We now give a result which allows a very useful description of the algebra $ZM(A)$. A proof of this is given in [RW98], Appendix A.3.

Theorem A.1.2 (Dauns-Hofmann-Theorem)**.** Let A be a C^*-algebra. For every $P \in \text{Prim}(A)$ let $\pi_P : A \to A/P$ denote the quotient map. Then there is an isomorphism Ψ from $C_b(\text{Prim}(A))$ to $ZM(A)$ such that

$$\pi_P(\Psi(f)a) = f(P)\pi_P(a) \qquad \text{for all } P \in \text{Prim}(A),\ f \in C_b(\text{Prim}(A)),\ a \in A. \tag{A.1.1}$$

We often write $f \cdot a$ instead of $\Psi(f)a$.

In terms of irreducible representations, (A.1.1) gives

$$\pi(\Psi(f)a) = f(\ker\pi)\pi(a) \qquad \text{for all } \pi \in \widehat{A},\ f \in C_b(\text{Prim}(A)),\ a \in A.$$

The following result,which is Proposition C.5 in [Wil07], shows how the existence of a continuous map $\sigma_A : \text{Prim}(A) \to Z$ makes A a $C_0(Z)$-algebra and vice versa.

Proposition A.1.3. Let A be a C^*-algebra and let Z be a locally compact Hausdorff space. If there exists a continuous map $\gamma_A : \text{Prim}(A) \to Z$, then A is a $C_0(Z)$-algebra, where the map $\Phi : C_0(Z) \to C_b(\text{Prim}(A)) \cong ZM(A)$ is given by

$$\Phi(f) = f \circ \gamma_A \qquad \text{for all } f \in C_0(Z). \tag{A.1.2}$$

Conversely, if $\Phi : C_0(Z) \to C_b(\text{Prim}(A)) \cong ZM(A)$ turns A into a $C_0(Z)$-algebra, then there is a continuous map $\gamma_A : \text{Prim}(A) \to Z$ such that (A.1.2) holds.

In particular, every $\pi \in \widehat{A}$ is lifted from a fibre $A(z)$ for some $z \in Z$ as follows: If $\pi \in \widehat{A}$, then the ideal $I_{\gamma_A(\ker\pi)}$ is contained in $\ker\pi$, and π is lifted from an irreducible representation of the fibre $A(\gamma_A(\ker\pi))$. Thus, $\widehat{A}$ can be identified with the disjoint union $\coprod_{z \in Z} A(z)\widehat{\ }$.

A.2 Continuity of integration on $\mathscr{F}(G)$

The proof of continuity of integration on $\mathscr{F}(G)$ is based on the following fact, which is proved in the appendix of [Gli62].

Proposition/Definition A.2.1. Let G be a locally compact group. It is possible to assign to every $K \in \mathscr{K}(G)$ a left Haar measure μ_K such that the map

$$\mathscr{K}(G) \to \mathbb{C},\ K \mapsto \int_K f(k)d\mu_K(k)$$

is continuous for every $f \in C_c(G)$. Such an assignment is called a continuous choice of Haar measures.

If G is compact, then the assignment of normalized Haar measure to every $K \in \mathscr{K}(G)$ is a continuous choice. To see this, note that the function given by $f(g) = 1$ for all $g \in G$ is in $C_c(G) = C(G)$. Then take any continuous choice of Haar measures $(\mu_K)_{K \in \mathscr{K}(G)}$ and set $c_K := \int_K f(k) d\mu_K(k)$ for every $K \in \mathcal{K}(G)$. Then $K \mapsto c_K$ is continuous and $(c_K^{-1}\mu_K)_{K \in \mathscr{K}(G)}$ is a continuous choice of normalized Haar measures.

We continue with a lemma:

Lemma A.2.2. Let K be a compact group and equip $\mathscr{K}(K)$ with a (general) continuous choice of Haar measures. Then the map

$$\mathscr{K}(K) \to \mathbb{C},\ H \mapsto \int_H F(H, x) d\mu_H(x)$$

is continuous for every $F \in C(\mathscr{K}(K) \times K)$.

Proof. Let $F \in C(\mathscr{K}(K) \times K)$, and let $(H_\nu)_{\nu \in N}$ and H be in $\mathscr{K}(K)$ such that $H_\nu \to H$. Then

$$\begin{aligned}&\left| \int_{H_\nu} F(H_\nu, x) d\mu_{H_\nu}(x) - \int_H F(H, x) d\mu_H(x) \right| \\ &\leq \left| \int_{H_\nu} \left(F(H_\nu, x) - F(H, x)\right) d\mu_{H_\nu}(x) \right| + \left| \int_{H_\nu} F(H, x) d\mu_{H_\nu}(x) - \int_H F(H, x) d\mu_H(x) \right|\end{aligned} \tag{A.2.1}$$

for every $\nu \in N$. By continuity of F it is clear that $F(H_\nu, \cdot)$ converges to $F(H, \cdot)$ pointwise and hence uniformly in $C(K)$. It follows from the observations preceding this lemma that the map

$$\varphi\colon \mathscr{K}(K) \to \mathbb{C},\ L \mapsto c_L$$

is continuous on the compact space $\mathscr{K}(K)$, and hence bounded. So we have

$$\left| \int_{H_\nu} \left(F(H_\nu, x) - F(H, x)\right) d\mu_{H_\nu}(x) \right| \leq \|\varphi\|_\infty \|F(H_\nu, \cdot) - F(H, \cdot)\|_\infty$$

for all $\nu \in N$, which implies that the first summand in (A.2.1) tends to zero. By applying the fact that we have a smooth choice of Haar measures to the function

$$K \to \mathbb{C},\ k \mapsto F(H, k)$$

it follows that the second summand tends to zero, too. Thus,

$$\int_{H_\nu} F(H_\nu, x) d\mu_{H_\nu}(x) \to \int_H F(H, x) d\mu_H(x),$$

as required. □

Proposition A.2.3 (Proposition 2.2.12, Proposition 3.1 in [Fel64])**.** Let K be a compact group. Then the map

$$\mathscr{F}(K) \to \mathbb{C},\ f \mapsto \int_{\mathcal{D}(f)} f(s) d\mu_{\mathcal{D}(f)}(s)$$

is continuous.

Proof. Fix a smooth choice of Haar measures on $\mathscr{K}(K)$. Let $(f_\nu)_{\nu\in N}$ and f be in $\mathscr{F}(K)$ such that $f_\nu \to f$. Let $H_\nu := \mathcal{D}(f_\nu)$ and $H := \mathcal{D}(f)$ denote the domains of the f_ν and of f, respectively. By the characterization of convergence in $\mathscr{F}(K)$ we know that $H_\nu \to H$ in $\mathscr{K}(K)$. To prove convergence of the integrals we first notice that

$$\tilde{f}\colon \{H\} \times H \to \mathbb{C},\ (H, h) \mapsto f(h)$$

is a continuous function on a closed and therefore compact subset of $\mathscr{K}(K)\times K$, which implies that there exists a continuous extension $F \in C(\mathscr{K}(K) \times K)$ of $\tilde{f}$.

We now prove the following claim: For every $\varepsilon > 0$ there exists $\nu_\varepsilon \in N$ such that for all $\nu \in N$ with $\nu \geq \nu_\varepsilon$ and for all $x \in H_\nu$ we have

$$|f_\nu(x) - F(H_\nu, x)| < \varepsilon. \tag{A.2.2}$$

Assume that this is false. Then we can choose $\varepsilon > 0$, a subnet $(f_j)_{j\in J}$ of $(f_\nu)_{\nu\in N}$ and for every $j \in J$ an element $x_j \in H_j$ such that

$$|f_j(x_j) - F(H_j, x_j)| \geq \varepsilon.$$

By compactness of K and since $H_j \to H$, we can, after passing to a subnet and relabeling, assume that there is $x \in H$ with $x_j \to x$. But now we have

$$|f_j(x_j) - F(H_j, x_j)| \leq |f_j(x_j) - f(x)| + |f(x) - F(H_j, x_j)|$$

for each $j \in J$, which is a contradiction because the left hand side is bounded below by ε, and the right hand side tends to zero because $f_j(x_j) \to f(x)$ and $F(H_j, x_j) \to F(H, x) = f(x)$. This proves the claim.

Consider now

$$\begin{aligned}
&\left|\int_{H_\nu} f_\nu(s)d\mu_{H_\nu}(s) - \int_H f(s)d\mu_H(s)\right| \\
&\leq \int_{H_\nu} |f_\nu(s) - F(H_\nu, s)|\, d\mu_{H_\nu}(s) + \left|\int_{H_\nu} F(H_\nu, s)d\mu_{H_\nu}(s) - \int_H F(H, s)d\mu_H(s)\right|,
\end{aligned}$$

then the first summand tends to zero by the claim above and the second one tends to zero by Lemma A.2.2. This completes the proof. □

A.3 Proof of Lemma 2.4.1

Lemma A.3.1 (Lemma 2.4.1)**.** Let X be a proper G-space. Let $(x_\nu, G_\nu, \sigma_\nu)_{\nu\in N}$ in $\mathrm{Stab}(X)^\frown$ be a net which converges to an element $(x, H, \sigma_H) \in X \times S(G)^\frown$ in $X \times S(G)^\frown$. Then $\mathrm{ev}_\nu \rtimes \sigma_\nu \to \mathrm{ev}_x \rtimes \sigma_H$ in $(S(G), C_0(X))^\frown$.

Proof. As outlined in the text before Definition 2.2.4, convergence of $(\mathrm{ev}_\nu \rtimes \sigma_\nu)_{\nu\in N}$ to $\mathrm{ev}_x \rtimes \sigma_H$ in $(S(G), C_0(X))^\frown$ means that $(\mathrm{ev}_\nu \rtimes \sigma_\nu)^\sim \to (\mathrm{ev}_x \rtimes \sigma_H)^\sim$ as representations of the subgroup C^*-algebra $C^*(S(G), C_0(X))$. Recall that, for every pair (L, ϱ) with $L \in \mathscr{K}(G)$ and ϱ in $\mathrm{Rep}(C_0(X), L)$, the representation $\widetilde{\varrho}$ is defined as follows: For every $F \in C_c(S(G), C_0(X))$ define $F_L \in C_c(L, C_0(X))$ by $F_L(l) = F(L, l)$ for all $l \in L$, and let $\widetilde{\varrho}(F) = \varrho(F_H)$. Then $\widetilde{\varrho}$ extends to a representation of $C^*(S(G), C_0(X))$.

Since σ_H is irreducible, so is $(\mathrm{ev}_x \rtimes \sigma_H)^{\sim}$, and so it suffices by Proposition 1.3.7 to show that there exists a positive functional associated with $(\mathrm{ev}_x \rtimes \sigma_H)^{\sim}$ which is a weak* limit of positive functionals associated with the representations $(\mathrm{ev}_\nu \rtimes \sigma_\nu)^{\sim}$. We use this criterion to prove that every subnet of $((\mathrm{ev}_\nu \rtimes \sigma_\nu)^{\sim})_{\nu \in N}$ has a subnet which converges to $(\mathrm{ev}_x \rtimes \sigma_H)^{\sim}$.

Since $(G_\nu, \sigma_\nu) \to (H, \sigma_H)$ in $S(G)^{\wedge}$, we have that $\widetilde{\sigma_\nu} \to \widetilde{\sigma_H}$ as representations of $C^*(S(G))$. Let ψ be a positive functional associated with $\widetilde{\sigma_H}$ and, after passing to a subnet and relabeling, assume that there is a net $(\psi_\nu)_{\nu \in N}$ of positive functionals, each ψ_ν associated with $\widetilde{\sigma_\nu}$, such that $\psi_\nu \to \psi$ in the weak* topology. Choose vectors $\xi \in V_{\sigma_H}$ and $\xi_\nu \in V_\nu$ for every $\nu \in N$ such that

$$\psi(F) = \langle \widetilde{\sigma_H}(F)\xi, \xi \rangle$$

and

$$\psi_\nu(F) = \langle \widetilde{\sigma_\nu}(F)\xi_\nu, \xi_\nu \rangle$$

for all $F \in C^*(S(G))$. We use these vectors to define positive functionals associated with $(\mathrm{ev}_x \rtimes \sigma_H)^{\sim}$ and $(\mathrm{ev}_\nu \rtimes \sigma_\nu)^{\sim}$ for all $\nu \in N$, respectively, as follows:

$$\varphi\colon C^*(S(G), C_0(X)) \to \mathbb{C},\ F \mapsto \langle (\mathrm{ev}_x \rtimes \sigma_H)^{\sim}(F)\xi, \xi \rangle$$

and

$$\varphi_\nu\colon C^*(S(G), C_0(X)) \to \mathbb{C},\ F \mapsto \langle (\mathrm{ev}_\nu \rtimes \sigma_\nu)^{\sim}(F)\xi_\nu, \xi_\nu \rangle$$

for all $\nu \in N$. Notice that $C_c(S(G)) \odot C_0(X)$ can be considered as a dense subset (with respect to the inductive limit topology) of $C_c(S(G), C_0(X))$, when we identify an elementary tensor $F \otimes f$ with the function $(L, l) \mapsto F(L, l)f = F_L(l)f$. We prove below that

$$\varphi(F \otimes f) = \mathrm{ev}_x(f)\psi(F) \quad \text{and} \quad \varphi_\nu(F \otimes f) = \mathrm{ev}_\nu(f)\psi_\nu(F) \tag{A.3.1}$$

for all $F \otimes f \in C_c(S(G)) \odot C_0(X)$ and for all $\nu \in N$. It follows that

$$\varphi_\nu(F \otimes f) \to \varphi(F \otimes f)$$

for all $F \otimes f \in C_c(S(G)) \odot C_0(X)$, and by linearity and continuity it follows that $\varphi_\nu \to \varphi$ in the weak* topology. This proves that the net $((\mathrm{ev}_\nu \rtimes \sigma_\nu)^{\sim})_{\nu \in N}$ converges to $(\mathrm{ev}_x \rtimes \sigma_H)^{\sim}$, as required.

It is left to check (A.3.1). Let $F \otimes f \in C_c(S(G)) \odot C_0(X)$, then

$$\begin{aligned}
(\mathrm{ev}_x \rtimes \sigma_H)^{\sim}(F \otimes f) &= (\mathrm{ev}_x \rtimes \sigma_H)(F_H \otimes f) \\
&= \int_H \mathrm{ev}_x((F_H \otimes f)(h))\sigma_H(h)dh \\
&= \int_H F(H, h)f(x)\sigma_H(h)dh \\
&= f(x) \int_H F(H, h)\sigma_H(h)dh = \mathrm{ev}_x(f)\sigma_H(F_H),
\end{aligned}$$

which implies that

$$\begin{aligned}
\varphi(F \otimes f) &= \langle (\mathrm{ev}_x \rtimes \sigma_H)^{\sim}(F \otimes f)\xi, \xi \rangle \\
&= \langle \mathrm{ev}_x(f)\sigma_H(F_H)\xi, \xi \rangle \\
&= \mathrm{ev}_x(f)\langle \widetilde{\sigma_H}(F)\xi, \xi \rangle = \mathrm{ev}_x(f)\psi(F).
\end{aligned}$$

The equation for the φ_νs and ψ_νs follows in the same way. □

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