Reine Mathematik

Twisted Spin cobordism

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Contents

Overview

The present thesis is concerned with a particular area of interplay between differential geometry, geometric topology, and algebraic topology, namely that of positive scalar curvature. What that is and how it connects the various areas is explained in the first half of the first chapter. It turns out that a twisted, generalised homology theory plays a crucial role: Twisted *Spin*-cobordism. We construct such a theory in the second half of the first chapter and then start to analyse its structure, in particular we compare it to twisted, real K-theory, to which it is connected by index theory.
In the second chapter we then make a first step, following a program set up by S. Stolz, towards the following conjecture:

Conjecture (Stolz 1995). If for a connected, closed, smooth n-manifold M, whose universal cover is spin, we have

$$0 = \hat{\alpha}(M) \in ko_n(B\pi_1(M), w^\nu(M))$$

then M carries a metric of positive scalar curvature.

This conjecture is approached using a certain transfer map in twisted *Spin*-cobordism. The homological analysis of this transfer makes up the second chapter, with the last section providing some concluding remarks and future directions. For more detailed information about the goings-on of both chapters we refer the reader to their respective introductions.

CHAPTER 1

Twisted Spin cobordism

1. Introduction

1.1. History and Motivation. The problem of classifying manifolds admitting Riemannian metrics with special features, e.g. certain kinds of symmetry or curvature, is one of the core interests in differential geometry. The present work is concerned with this classification for the case of metrics with positive scalar curvature. Of the three classical types of curvature (sectional, Ricci-, and scalar) the scalar curvature is the weakest, given by averaging processes from the other two, and thus the most robust against manipulation of the metric and even the underlying manifold. The following theorem is arguably the most prominent example of this and forms the cornerstone of current work on the existence of positive scalar curvature metrics.

THEOREM (Gromov-Lawson 1980). *Let (M, g) be a smooth n-dimensional Riemannian manifold of positive scalar curvature and $\varphi : S^k \times D^{n-k} \hookrightarrow M$ an embedding of a k-sphere with trivialised normal bundle. Then the manifold arising from M by surgery along φ again carries a positive scalar curvature metric as long as $n - k \geq 3$. Indeed, the metric g can be extended to a metric with positive scalar curvature on the trace of the surgery.*

Combined with the standard techniques of surgery and handlebody theory, this result shows that in order to prove the existence of a metric of positive scalar curvature on a given closed, smooth manifold M one need only exhibit such a manifold cobordant to M, provided one can bound the dimensions of the surgeries occuring as one moves through the cobordism. Since manifolds up to certain types of cobordism often are explicitly classified, this sometimes allows immediate conclusions about the desired geometric classification.

1.2. Organisation of the chapter and statement of results. The dimensions of the surgeries occuring in a cobordism can be controlled by requiring reference maps to some background space, usually one classifying additional structure (e.g. orientations or *Spin*-structures) on all occuring manifolds. Precisely how this works is explained in section 2, which recalls and extends results from the literature. Thus, no claim of originality is made for this section, and the results presented have been used in various special cases and are easily deduced by any surgeon. Continuing the line of investigation we consider several bordism groups that occur when following the program of Gromov and Lawson, in particular, we explain that for every closed, smooth, connected manifold M of dimension greater than four its 1-type can be used as the background space. We go on to identify the arising cobordism groups as either twisted oriented or twisted *Spin*-cobordism, corresponding as to whether the universal cover of M is spinnable or not. Similar results first appeared in Stolz' preprint [**St ??**] using a slightly different language. We conclude the second section with a short discussion of various known results and conjectures.

The second part of the chapter specifically studies twisted *Spin*-cobordism groups and the underlying representing parametrised spectrum, which we name M_2O. To this end, section 3 reviews the category of parametrised spectra due to May and Sigurdsson, in the process giving a few easy results that seem to have not been written down before. In this section we also construct the parametrised spectrum M_2O and another one, K_2O, representing twisted, real K-theory. Section 4 constructs a twisted version $\hat{\alpha}$ of the Atiyah-Bott-Shapiro orientation. Such an orientation first appeared in [**Jo 97**] using the model for K-theory, which later appeared in [**Jo 01**].
In section 5 we finally begin the analysis of twisted *Spin*-cobordism: A slight variation of the KO-valued Pontryagin classes of [**AnBrPe 66**] produces (homotopy classes of) maps of parametrised spectra $\overline{\theta}_J : M_2O \to K_2O$, that admit lifts to certain connective covers of K_2O. Using these we obtain the following generalisation of the Anderson-Brown-Peterson splitting:

Theorem (5.2.5). *For any choice of lifts $p_j : M_2O \to k_2o\langle n_J\rangle$ there exist maps $x_i : M_2O \longrightarrow K \times sh^i H\mathbb{Z}/2$, such that the combined map*

$$M_2O \longrightarrow \left[\prod_J k_2o\langle n_J\rangle\right] \times \left[\prod_i K \times sh^i H\mathbb{Z}/2\right]$$

induces an isomorphism of parametrised homology theories after localisation at 2.

Section 6 uses this splitting to describe the mod 2 cohomology of M_2O and k_2o (the connective version of K_2O) as modules over the twisted Steenrod algebra $\underline{\mathcal{A}}$. The main structural result here is that there is an embedding $\varphi : \mathcal{A}(1) \hookrightarrow \underline{\mathcal{A}}$, different from the obvious one, that all modules in question are induced along.

Theorem (6.2.4). *Evaluation at the unique non-trivial class in lowest degree gives isomorphisms*

$$\underline{\mathcal{A}}_\varphi \otimes_{\mathcal{A}(1)} \mathbb{Z}/2 \longrightarrow H^*(k_2o, \mathbb{Z}/2)$$
$$sh^{-2}\underline{\mathcal{A}}_\varphi \otimes_{\mathcal{A}(1)} \mathcal{A}(1)/Sq^3 \longrightarrow H^*(k_2o\langle 2\rangle, \mathbb{Z}/2)$$

where $sh^?$ just denotes a shift and $\underline{\mathcal{A}}_\varphi$ denotes the twisted Steenrod algebra viewed as a right module over $\mathcal{A}(1)$ via φ.

Corollary (6.3.1). *$H^*(M_2O, \mathbb{Z}/2)$ is an extended $\mathcal{A}(1)$-module.*

As a final observation we show that the map in homology induced by the twisted orientation $\hat{\alpha}$ does not split as a map of rings, contrary to the untwisted case. This implies that a splitting of the (co)homology of M_2O along the lines of [**St 92**, Corollary 5.5] is not possible; a fact that plays a significant role in the second chapter.

2. The relation of positive scalar curvature to cobordism

2.1. From surgery to bordism. Given the surgery theorem of Gromov and Lawson we want to investigate the class of manifolds from which a given manifold

M arises by surgeries in codimension ≥ 3. The basis for this is the following fundamental result of handlebody theory:

THEOREM (Smale 1962). *Let W be a cobordism from a smooth, closed manifold M to another smooth, closed manifold N, both of dimension n, and let the inclusion of N into W be a k-equivalence. Then as long as $n \geq 5$ and $k < \frac{n}{2}$, W arises from $N \times I$ by attaching handles of dimensions greater than k and thus M from N by surgeries of dimensions greater than k.*

To exploit this we introduce a background space B that is equipped with a stable vector bundle ξ. A ξ-structure on a manifold M is by definition a choice of (stable) normal bundle ν of M, together with a homotopy class of maps of stable bundles $\nu \to \xi$.

2.1.1. PROPOSITION. *Let $n \geq 5$ and M be a smooth, closed n-manifold with a ξ-structure such that the underlying map $M \to B$ is a k-equivalence where $k < \frac{n}{2}$.Let furthermore N be another n-dimensional, closed ξ-manifold that is ξ-bordant to M. Then M arises from N by a sequence of surgeries of codimension $> k$.*

2.1.2. COROLLARY. *Let M be a smooth, closed ξ-manifold of dimension ≥ 5, whose underlying structure map $M \to B$ is a 2-equivalence and where B admits a 2-equivalence from a finite cell complex. If the ξ-cobordism class of M contains a manifold admitting a metric of positive scalar curvature, then also M admits such a metric.*

This corollary is explicitely stated in [**RoSt 94**, Bordism Theorem 3.3], yet the proof is deferred to [**Ro 86**, Theorem 2.2], where only a special case is proved. For a specific choice of B the corollary is also stated in [**Kr 99**, Theorem 1], however, the proof seems to be invalid due to a number shift occuring in a crucial step, as we explain in the proof below.

PROOF OF PROPOSITION 2.1.1. Let W be a ξ-bordism from M to N. One can now do sugery on the interior of W to produce an new ξ-bordism W' such that the ξ-structure map $W' \to B$ is a k-equivalence:
In classical surgery theory this is usually done only for B a Poincaré complex and ξ a reduction of the Spivak normal fibration, but this restriction is only needed to formulate the notion of *degree one* normal maps. Achieving connectivity below the middle dimension only requires a normal map and a finiteness assumption on the target, which is precisely what we have with a ξ-bordism. In principle this is covered in [**Kr 99**, Proposition 4], however, the proof given in [**Kr 99**] is (also) incorrect. It claims (and uses) that the lowest higher homotopy group of a finite cell complex is finitely generated over the group ring of the fundamental group, but this need not be so. A counterexample is given in [**Ha**, Section 4A].
Given the assumptions on B, however, one can perform finitely many $0, 1$ and 2 surgeries in the interior of W by hand to make the map $W \to B$ a 2-equivalence and then the inductive scheme from e.g. [**CrLüMa**, Theorem 3.61] makes this into a k-equivalence $W' \to B$. It is not true, however, that this map can be made into a $k+1$-equivalence without a 'one-higher' finiteness assumption on B, i.e. that there exists a finite cell complex together with a $k+1$-equivalence to B. That this can nevertheless be achieved is claimed in the proof of [**Kr 99**, Theorem 1] for $k = 2$ and a choice of B which only has finite second, but not necessarily third skeleton.

By the commutativity of

$$\begin{array}{ccc} \pi_*(M) & \longrightarrow & \pi_*(W') \\ & \searrow \quad \swarrow & \\ & \pi_*(B) & \end{array}$$

we find that the inclusion $M \to W'$ is an isomorphism on the first $k-1$ homotopy groups. In order to continue we will, however, need to have the map from M into the cobordism a k-equivalence. One therefore has to argue as Rosenberg does in [**Ro 86**], namely directly perform surgeries in the interior of W'. To this end consider the following part of the long exact sequence of homotopy groups

$$\pi_k(M) \to \pi_k(W') \to \pi_k(W', M) \stackrel{0}{\to} \pi_{k-1}(M) \stackrel{\cong}{\to} \pi_{k-1}(W')$$

Because of the surjectivity of the map $\pi_k(M) \to \pi_k(B)$ we can represent any element in $\pi_k(W', M)$ by an element of $\pi_k(W')$ that lies in the kernel of $\pi_k(W') \to \pi_k(B)$. Any such sphere $S^k \to W'$ has trivial stable normal bundle by construction and can be represented by a framed embedding since we are working below the middle dimension. Picking a finite generating system of $\pi_k(W', M)$ over $\mathbb{Z}[\pi_1(M)]$ (which is possible by [**CrLüMa**, Lemma 3.55] and has also been used several times above) we obtain finitely many disjoint embeddings of $S^k \times D^{n+1-k} \to W'$ on which we can perform surgery to obtain a new bordism W'', which now has $\pi_k(M) \to \pi_k(W)$ surjective. As we are killing spheres that go to zero in B the new cobordism indeed has a B-structure once more, so we could also have proceeded inductively, but the B-structure will be irrelevant from this point on.
By Smale's theorem W'' arises from $M \times I$ by attaching handles of dimensions $\geq k+1$ and thus N from M by surgeries of dimensions $\geq k$. Turning this around we find that M arises from N by surgeries of dimension $< n-k$ (note that the surgery dual to an l-surgery is an $n-l-1$-surgery) and hence of codimensions $> k$. □

2.2. A choice of bordism groups. For a given connected manifold M and a number k we always have Moore-Postnikov decompositions

$$M \to B_k M \to BO$$

of the map classifying the stable normal bundle of M. Here the map $M \to B_k M$ is a $k+1$-equivalence and $B_k M \to BO$ is injective on π_{k+1} and an isomorphism on higher homotopy groups. The spaces $B_k M$ are often called normal k-types of M (e.g. [**Kr 99**]). Pulling back the universal stable bundle over BO along the map $B_k M \to BO$, we arrive at a stable bundle ξ, such that M admits a ξ-structure, whose underlying map we can choose as connected as we need it to be.
For our purposes we of course want to choose $k = 1$ and thus need to study the 1-type of a given manifold. There are essentially two separate cases, one for manifolds, whose universal covers admit *Spin*-structures, and one for the others. Manifolds, whose universal covers are non-spin, are sometimes referred to as totally non-spinnable and we will call the others almost spinnable, or hardly spinnable, if they do not admit a *Spin*-structure themselves. Let us consider the totally non-spinnable manifolds first. Given connected M with fundamental group π there is a unique class in $H^1(B\pi, \mathbb{Z}/2)$ that pulls back to $w_1(M) \in H^1(M, \mathbb{Z}/2)$ under the

map classifying of the universal cover of M (because this map induces an isomorphism on H^1). Since $B\pi$ has the homotopy type of a cell complex, this class can be represented by a map $B\pi \to K(\mathbb{Z}/2, 1)$ which we name $w^\nu(M)$ (ν denoting the fact, that we really care about the Stiefel-Whitney classes of the normal bundle of M). We can then form the following diagram

$$\begin{CD} B_1M @>>> BO \\ @VVV @VV{w_1}V \\ B\pi @>>{w^\nu(M)}> K(\mathbb{Z}/2,1) \end{CD}$$

where the left vertical map is a first Postnikov section of B_1M inducing the isomorphism on π_1 that makes the composition $M \to B_1M \to B\pi$ the canonical one. For each choice of spaces and maps involved the diagram is homotopy commutative, since both possible compositions represent $w_1(M) \in H^1(M, \mathbb{Z}/2)$ when pulled back along the canonical map $M \to B_1M$.

2.2.1. Lemma. *In case M is totally non-spinnable, this diagram expresses B_1M as a homotopy pullback.*

We will use this information after 2.3.2 to express the cobordism groups corresponding to B_1M using the language of parametrised spectra. Before doing that we shall however construct the analogous diagram for hardly spinnable manifolds. Note that the determination of all possible normal 1-types is carried out in [**Kr 99**, Proposition 2]; however, the description given there differs from ours in the case of hardly spinnable manifolds. It is trivial to verify though that the results coincide. For the reader's convenience we include a proof of our description.

Proof. Denote by P a homotopy pullback of the diagram above (with B_1M deleted). By definition of the homotopy pullback we obtain a map $B_1M \to P$ for any choice of homotopy in the above diagram. Any of these is a homotopy equivalence: Since the homotopy fibre of $BO \xrightarrow{w_1} K(\mathbb{Z}/2, 1)$ is a model for BSO, we have a fibre sequence $BSO \to P \to B\pi$. From this we see that P is path-connected, the map $P \to B\pi$ is an isomorphism on π_1 and $BSO \to P$ is an isomorphism on higher homotopy groups. By definition of B_1M the map $B_1M \to B\pi$ induces an isomorphism on π_1 and the map $B_1M \to BO$ is injective on π_2 and an isomorphism on higher groups. Since the appropriate diagrams commute we conclude that our maps $B_1M \to P$ induce isomorphism on homotopy groups except maybe for the second. Here the assumption on M enters: Choosing an orientation on the normal bundle of $\widetilde{M}$, we have a diagram

$$\begin{CD} \widetilde{M} @>>> BSO \\ @VVV @VVV \\ M @>>> BO \end{CD}$$

where $\widetilde{M}$ denotes a universal cover of M (note that the composition $\widetilde{M} \to M \to BO$ classifies the stable normal bundle of $\widetilde{M}$). By assumption $\widetilde{M}$ does not admit a *Spin*-structure and thus neither does its stable normal bundle. This in turn implies that the map $\widetilde{M} \to BSO$ induces an injective map on second cohomology with $\mathbb{Z}/2$ coefficients. Using the Hurewicz theorem we conclude that it induces a surjection on π_2. Since both vertical maps induce isomorphisms on π_2, we find the composition $M \to B_1M \to BO$ to also induce a surjection on π_2. This is only possible if indeed $B_1M \to BO$ induces an isomorphism on π_2. We thus obtain a weak equivalence $B_1M \to P$ and since both spaces have the homotopy type of cell complexes we are done. □

To obtain a similar description for hardly spin manifolds we need the following:

2.2.2. Lemma. *Given a hardly spin manifold M there is a unique class in $H^2(B\pi, \mathbb{Z}/2)$ pulling back to $w_2(M) \in H^2(M, \mathbb{Z}/2)$ under the canonical map $M \to B\pi$.*

Proof. Consider the Serre spectral sequence of the map $M \to B\pi$. Using the fact that $H^0(B\pi, \mathcal{H}^t(\widetilde{M})) \cong H^t(\widetilde{M})^\pi$ (here the upper index denotes fixed points, compare e.g. [**Wh**, Theorem VI.3.2]), we find a short exact sequence

$$0 \longrightarrow H^2(B\pi) \to H^2(M) \longrightarrow H^2(\widetilde{M})^\pi \longrightarrow 0$$

whose maps are the canonical ones. This immediately yields the claim since the assumption precisely says that $w_2(M)$ maps to 0 on the right. □

Let us denote by $w^\nu(M)$ a pair of representing maps for the lift to $B\pi$ of the class $(w_1(M), w_2(M) + w_1(M)^2)$. We can then produce a similar diagram:

$$\begin{array}{ccc} B_1M & \longrightarrow & BO \\ \downarrow & & \downarrow{\scriptstyle (w_1,w_2)} \\ B\pi & \xrightarrow[w^\nu(M)]{} & K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2) \end{array}$$

2.2.3. Lemma. *In case M is hardly spin, this diagram expresses B_1M as a homotopy pull-back.*

Proof. The proof is essentially the same as that of 2.2.1, or we can refer to [**Kr 99**, Proposition 2]. □

We will later choose specific models for BO and $K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2)$ together with a representing map for the first Stiefel-Whitney classes, that is a fibre bundle. For some choice of $B\pi$ and $w^\nu(M)$ we can then take B_1M to be the honest pull-backs of the diagrams above.

2.3. Interpretation via twists. We now set out to compute the Thom spectrum corresponding to these choices for B_1M. The following is an easily verified generalisation of the property $M(\xi \times \xi') \cong M(\xi) \wedge M(\xi')$ for vector bundles. We need, however, a bit of notation. Given a vector bundle ξ over a space X that comes equipped with a map $q : X \to K$ we denote by $M_q(\xi)$ the *fibrewise Thom space*, that is given by first one-point compactifying each fibre of ξ as usual but then only identifying points in the section at infinity if they lie in the same fibre of q. Instead of a basepoint $M_q(\xi)$ comes equipped with an obvious map to K that admits a section by sending $k \in K$ to the points at infinity of the fibre over k. We shall sometimes identify K with this subspace.

2.3.1. Observation. Given stable vector bundles ξ and ξ' over spaces X and X' and a pull-back diagram

$$\begin{array}{ccc} B & \xrightarrow{p} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle q} \\ X' & \xrightarrow[q']{} & K \end{array}$$

there is a canonical isomorphism

$$M(p^*\xi \oplus {p'}^*\xi') = \big[M_q(\xi) \wedge_K M_{q'}(\xi')\big]/K$$

given by the identity before passing to quotient spaces.

Another bit of notation: We shall refer to any space coming from an honest pull-back as described above as BM and the pullback bundle as ξ_M. Similarly K shall from now on refer to either $K(\mathbb{Z}/2,1)$ or $K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2)$ depending on M, and the same goes for $w^\nu(M) : B\pi \to K$, by which we denote the first or first and second normal Stiefel-Whitney class(es) of M, respectively, lifted to $B\pi$. Furthermore we shall denote the fibrewise Thom space of the universal bundle over $BO(n)$ along the map $BO(n) \to K$ given by the first or first and second Stiefel-Whitney classes by $M_K O(n)$.

2.3.2. Corollary. *We have*

$$M(\xi_M, n) \cong \big[(B\pi + K) \wedge_K M_K O(n)\big]/K$$

where $M(\xi_M, n)$ denotes the Thom space of the n-stage of ξ_M and we have suppressed the map $w^\nu(M) : B\pi \to K$ from notation.

For good choices of universal vector bundles the spaces $M_K O(n)$ fit together into a parametrised orthogonal spectrum over K, as shall be explained in the next section. Any parametrised spectrum gives rise to a twisted homology theory (which in our case we denote by $\Omega_n^K(-,-)$) in such a way that (almost by definition, see 3.3.6) we find the (stable) homotopy groups of the right hand side to be $\Omega_*^K(B\pi, w^\nu(M))$. Since by the Pontryagin-Thom theorem we have $\Omega_*^{\xi_M} \cong \pi_*(M(\xi_M))$, we conclude:

2.3.3. Theorem. *A smooth, closed, connected manifold M of dimension ≥ 5 admits a metric of positive scalar curvature if and only if for some (and then every) choice of ξ_M-structure on M, we can represent the bordism class*

$$[M] \in \Omega_n^K(B\pi, w^\nu(M))$$

by some manifold admitting a positive scalar curvature metric.

By various properties of twisted homology theories described in the next section, this result specialises to the well-known cases where either M is oriented and totally non-spinnable (here we find $\Omega_n^K(B\pi, w^\nu(M)) \cong \Omega_n(B\pi)$) or where it is spin (in which case $\Omega_n^K(B\pi, w^\nu(M)) \cong \Omega_n^{Spin}(B\pi)$). Indeed all other cases we know of in the literature are special cases of this general result.

2.3.4. Remark. In [**St ??**] Stolz constructed Lie groups $G(n, \gamma)$ associated to a natural number n and a 'supergroup' γ. A supergroup he describes as a triple (P, w, c), where P is a group, $w : P \to \mathbb{Z}/2$ is a group homomorphism and $c \in P$

lies in the kernel of w. Any manifold, and indeed every bundle E, determines a supergroup $\underline{\pi_1(E)}$ with P a certain extension of the fundamental group π of the base space, and w the orientation character composed with the projection of P onto π. Both c and the extension are related to the spinnability of the bundle and its universal cover. Following Stolz' construction it is not hard to see that the 'stage' of our space BM that pulls back from $BO(n)$ is indeed a classifying space of $G(n, \underline{\pi_1(M)})$. This gives a geometric interpretation of the bordism groups we have just described as $G(-, \underline{\pi_1(M)})$-manifolds modulo $G(-, \underline{\pi_1(M)})$-bordisms. Except in the following comments, which do not connect directly to anything we prove here, we make no use of this identification and thus leave its proof to the reader.

2.4. Obstructions, conjectures and known results. While 2.3.3 gives a method for proving existence of metrics with positive scalar curvature, there are also well-known obstructions coming from indices of Dirac-type operators. Given a spin manifold M of dimension n with fundamental group π these were coalesced into a single invariant $A(M) \in KO_n(C^*\pi)$ by Rosenberg in [**Ro 83**]. This invariant was further generalised to an invariant $A(M) \in KO_n(C^*\underline{\pi})$ for almost spin manifolds by Stolz in [**St ??**]; here $C^*\pi$ denotes the reduced group C^*-algebra of the fundamental group (or even fundamental supergroup in the latter case, see [**St ??**]) of M and KO_n the topological KO-homology. Improving upon a conjecture of Gromov and Lawson, Rosenberg stated the untwisted version of the following:

2.4.1. Conjecture (Gromov-Lawson-Rosenberg 1983). An almost spin manifold M supports a metric of positive scalar curvature if and only if

$$0 = A(M) \in KO_n(C^*\underline{\pi_1(M)})$$

While false in general (see [**Sc 98**]), it is open for manifolds with finite fundamental groups and has been verified for many cases of spin manifolds. Rosenberg furthermore showed that his invariant depends only on the *Spin*-cobordism class of M in $\Omega_n^{Spin}(B\pi)$ and factors as

$$\Omega_n^{Spin}(B\pi) \xrightarrow{\alpha} ko_n(B\pi) \xrightarrow{per} KO_n(B\pi) \xrightarrow{ass} KO_n(C^*\pi)$$

where α denotes the Atiyah-Bott-Shapiro orientation, *per* the canonical periodisation map and *ass* a certain assembly map. A similar decomposition using twisted *Spin*-cobordism and KO-theory works for the case of hardly spin manifolds (but we will not pursue this here). This decomposition connects Rosenberg's conjecture to the bordism-invariance theorem 2.3.3 on the one hand and on the other to the conjecture from the introduction (compare 4.3.2) by the following observation:

2.4.2. Observation. Given a stable vector bundle $\xi : E \to B$ and a group homomorphism $a : \Omega_n^\xi \to A$, $n \geq 5$, such that every n-dimensional ξ-manifold in the kernel of a ($n \geq 5$), whose structure map to B is a 2-equivalence, admits a metric of positive scalar curvature, one can conclude that an n-dimensional ξ-manifold, whose structure map is a 2-equivalence, admits a positive scalar curvature metric if and only if $a([M]) = a([N])$ for some n-dimensional ξ-manifold N, that does admit a positive scalar curvature metric:

For the difference $[M] - [N] = [M + \overline{N}]$ goes to 0 under a and by surgery we can produce a ξ-manifold L that is ξ-cobordant to $M + \overline{N}$ and whose structure map is a 2-equivalence (see the proof of 2.1.1). By assumption L then carries a positive scalar curvature metric and since $[M] = [L + N]$ corollary 2.1.2 yields the claim.

The positive results on the Gromov-Lawson-Rosenberg conjecture mentioned above are mostly obtained by explicit comparison of $ko_n(B\pi)$ with $KO_n(C^*\pi)$ using the untwisted version of 4.3.2, which is a theorem due to Führing and Stolz ([**St 94, Fü 13**]). In addition to being useful on its own, one can thus view 4.3.2 as a way of attacking 2.4.1 in the case of hardly spin manifolds.

In comparison, for a totally non-spin manifold M Rosenberg's 'Dirac tells all'-philosophy predicts the existence of a metric with positive scalar curvature in general, since there are no Dirac-type operators associated with M. While this again fails in general, it is known that if the fundamental class of M in $H_n(B\pi)$ (with local coefficients in case M is non-orientable) vanishes, then M admits a metric of positive scalar curvature, compare [**RoSt 01**, Theorem 4.11]. For that reason deciding the existence of a positive scalar curvature metric for a given manifold is in principle amenable to computation for all cases except that of hardly spin manifolds and the second part of this thesis can be regarded as a first step toward completing this picture.

3. Parametrised spectra after May and Sigurdsson

3.1. Parametrised homology theories. Before recalling the basic features of parametrised spectra, we shall discuss the objects they are supposed to produce, namely twisted or parametrised (co)homology theories. To this end let K be a topological space and Top^2_K denote the category of pairs over K, the over-category of (K, K) in Top^2. We will denote an object of Top^2_K by (X, A, ζ), where $A \subseteq X$ and $\zeta : X \to K$.

3.1.1. Definition. A twisted homology theory over K is a functor $h : Top^2_K \to$ gr-R-Mod and a natural transformation δ satisfying the following version of the Eilenberg-Steenrod axioms:

- Normalisation: $h_*(X, X, \zeta) = 0$ for all spaces X and maps $\zeta : X \to K$.
- Homotopy invariance: A homotopy equivalence $f : (X', A') \to (X, A)$ induces an isomorphism
$$h_*(X', A', f^*\zeta) \to h_*(X, A, \zeta)$$
for any $\zeta : X \to K$.
- Mayer-Vietoris sequences: For any decomposition of a space X into subspaces A, B, C, D, such that $C \subseteq A, D \subseteq B$ and $(A, B, A \cup B), (C, D, C \cup D)$ are excisive triads, the sequence
$$\begin{aligned} \ldots \longrightarrow h_k(A \cap B, C \cap D, \zeta) \longrightarrow & h_k(A, C, \zeta) \oplus h_k(B, D, \zeta) \\ \longrightarrow & h_k(A \cup B, C \cup D, \zeta) \xrightarrow{\delta} h_{k-1}(A \cap B, C \cap D, \zeta) \longrightarrow \ldots \end{aligned}$$
is exact.

Furthermore, it may have the following additional properties:

- Additivity: Whenever $\mathcal{U}$ is a partition of a space X into clopen sets, the natural map
$$\bigoplus_{U \in \mathcal{U}} h_*(U, U \cap A, \zeta) \to h_*(X, A, \zeta)$$
is an isomorphism.

- Singularity: A weak homotopy equivalence $f : (X', A') \to (X, A)$ induces an isomorphism
$$h_*(X', A', f^*\zeta) \to h_*(X, A, \zeta)$$
for any $\zeta : X \to K$.
- Compact Supports: Whenever $(U_i)_{i\in\mathbb{N}}$ is an ascending filtration of X such that each compact subset of X is contained in a single U_i and $(V_i)_{i\in\mathbb{N}}$ is a similar filtration for Y, then any filtration preserving map $f : X \to Y$ that induces isomorphisms
$$h_k(U_i, U_0, f^*\zeta) \to h_k(V_i, V_0, \zeta)$$
for all $i \in \mathbb{N}$, also induces an isomorphism $h_k(X, U_0, f^*\zeta) \to h_k(Y, V_0, \zeta)$.

3.1.2. Remark. We have formulated the axioms for a homology theory in a slightly different way from what is usual. The reason is that we need to employ the general Mayer-Vietoris sequence as described above and this is a bit cumbersome to derive from the usual set of axioms, compare [**St 84**]. The proof given there also works in the parametrised case. Note in addition that as usual (countable) additivity and singularity for a theory h together imply that h has compact supports: Firstly, $\operatorname{colim} h_*(U_i, A, \zeta) = h_*(X, A, \zeta)$ for additive h and a filtration by (non fibrewise) cofibrations (the proof via e.g. the infinite cylinder construction still works verbatim ([**tD**, pp. 148 - 151]) and, secondly, the realisation of the singular simplicial complex reduces the general case to this.

The analogous definition for cohomology should be obvious. In the rest of this first section we will establish our notation and state some easy facts about parametrised (co)homology theories, that we shall need, but for which we know no reference.

3.1.3. Proposition. *If $\pi, \rho : X \longrightarrow K$ are two homotopic maps, then any homotopy H between them induces an isomorphism $H_! : h_*(X, A, \pi) \longrightarrow h_*(X, A, \rho)$ constructed in the proof. Two homotopic homotopies give the same isomorphism and the arising assignment $h : \Pi(X; K) \longrightarrow$ gr-R-Mod is functorial.*

Here $\Pi(X; K)$ is the groupoid whose objects are maps $X \longrightarrow K$ and whose morphisms are homotopy classes of homotopies with composition given by concatenation.

Proof. Consider the diagram

$$\begin{array}{ccccc} X & \xhookrightarrow{i_0} & X \times I & \xhookleftarrow{i_1} & X \\ & {\scriptstyle\pi}\searrow & \downarrow{\scriptstyle H} & \swarrow{\scriptstyle\rho} & \\ & & K & & \end{array}$$

Since both horizontal inclusions are (not necessarily fibrewise) homotopy equivalences, we can define the isomorphism $H_!$ by

$$h_*(X, A, \pi) \xrightarrow{(i_0)_*} h_*(X \times I, A \times I, H) \xrightarrow{(i_1)_*^{-1}} h_*(X, A, \rho)$$

The claimed properties are easily verified right from the definition. □

3.1.4. Definition. The system of coefficients of a homology theory h over K is the local system $h_* : \Pi(K) \longrightarrow \text{gr-}R\text{-Mod}$ given by setting $(X, A) = (*, \emptyset)$ in the previous proposition.
A transformation of homology theories h and h' over spaces K and K' is pair consisting of a map $f : K \to K'$ and a collection of linear maps

$$h_*(X, A, \zeta) \to h'_*(X, A, f_*(\zeta))$$

that is natural and commutes with the boundary operator in the obvious way.
In case both are defined over the same space K, we shall implicitly require f to be the identity.

3.1.5. Theorem. *Given a transformation $\theta : h \longrightarrow k$ of twisted homology theories over K, that is an isomorphism $h_q \to k_q$ for all $q < n$ and surjective for $q = n$, then for a pair (X, A, ζ), such that (X, A) is a finite relative cell complex without cells below dimension l, θ gives an isomorphism $h_q(X, A, \zeta) \to k_q(X, A, \zeta)$ for all $q < n + l$ and a surjection for $q = l + n$.*
If both h and k have compact supports, X need not be finite (relative to A) and in case both h and k are singular (X, A) may be an arbitrary l-connected pair.

The statement is true for cohomology upon replacing $<$ by $>$. The reason to include the proof here is that the usual proof in the untwisted case using the cofibration sequence $\sum_i S^n \to X^{(n)} \to X^{(n+1)}$ does not generalise to the twisted case, simply because the mapping cone does not carry any obvious map to K. We will instead have to rely on Mayer-Vietoris-sequences in their most general form as given in our definition of parametrised (co)homology.

Proof. First we note that if $S^i \xrightarrow{f} K$ is an arbitrary map $h_q(S^i, 1, f) \to k_q(S^i, 1, f)$ is an isomorphism for $q < n + i$ and a surjection for $q = n + i$: The proof is via induction on the subspheres S^j of S^i the case $j = 0$ follows from the automatic finite additivity of both h and k. The induction step is established by the 5-lemma applied to the map of Mayer-Vietoris-sequences as follows:

$$\ldots \longrightarrow h_q(S^{j+1}_+ \cap S^{j+1}_-, 1, f) \longrightarrow h_q(S^{j+1}_+, 1, f) \oplus h_q(S^{j+1}_-, 1, f) \longrightarrow h_q(S^{j+1}, 1, f)$$

$$\xrightarrow{\delta} h_{q-1}(S^{j+1}_+ \cap S^{j+1}_-, 1, f) \longrightarrow h_{q-1}(S^{j+1}_+, 1, f) \oplus h_{q-1}(S^{j+1}_-, 1, f) \longrightarrow \ldots$$

The first and fourth entry receive a fibrewise map and (non-fibrewise) homotopy equivalence from $(S^j, 1) \to K$, hence the induction hypothesis applies, while the second and fifth are simply 0.
Note that while the proof is formally the same as that of the suspension isomorphism in the classical case, a 1-sphere $S^1 \to K$ is not usually a (fibrewise!) suspension at all.
Now for the general case: Again, we carry out an induction on the skeleta of X with induction start the $l-1$-skeleton $X^{(l-1)} = A$, for which there is nothing to do. If now $l = 0$ the induction step from $X^{(-1)}$ to $X^{(0)}$ is again by (finite) additivity.
For the other induction steps we need notation for the standard neighbourhoods for subsets of cell complexes, see [**Ha**, p. 552]: Given a subset $B \subseteq X$ let $N^i_\epsilon(B)$ denote the open neighbourhood of B in $X^{(i)}$ given by successively thickening B by ϵ-tubes cell by cell. Subsequently we can look at the following subsets of $X^{(i+1)}$: M the set of midpoints of $i+1$ cells, $M + \frac{1}{2}$ the set of midpoints shifted by $\frac{1}{2}$ in the first coordinate and consider the Mayer-Vietoris-sequence for $N^{i+1}_{\frac{5}{6}}(X^{(i)}), N^{i+1}_{\frac{4}{6}}(M)$

relative to $N^{i+1}_{\frac{4}{6}}(A), N^{i+1}_{\frac{1}{6}}(M+\frac{1}{2})$. Abbreviating the names of these sets in the obvious way, we first note that $N(A) \subseteq N(X^{(i)}), N(M+) \subseteq N(M)$ as required, and also $N(M+) \subseteq N(A)$ and finally that the inclusion $N(M+) \hookrightarrow N(M)$ is a (non-fibrewise) homotopy equivalence. The sequence then looks as follows:

$$\begin{aligned} \dots \longrightarrow\ & h_q(N(X^{(i)}) \cap N(M), N(M+), \pi) \\ & \longrightarrow h_q(N(X^{(i)}), N(W), \pi) \\ & \quad \longrightarrow h_q(X^{(i+1)}, N(W), \pi) \\ & \quad\quad \xrightarrow{\delta} h_{q-1}(N(X^{(i)}) \cap N(M), N(M+), \pi) \\ & \quad\quad\quad \longrightarrow h_{q-1}(N(X^{(i)}), N(W), \pi) \longrightarrow \dots \end{aligned}$$

On the one hand, $(N(X^{(i)}) \cap N(M), N(M+))$ receives a fibrewise map from a disjoint union of pointed i-spheres that is a homotopy equivalence, and hence θ gives an isomorphism on this term for $q < n+i$ and a surjection for $q = n+i$. On the other hand, $(N(X^{(i)}), N(W))$ deformation retracts onto $(X^{(i)}, A)$ strongly so the inclusion in the other direction certainly is a fibrewise map and a homotopy equivalence, whence the induction hypothesis implies that θ gives an isomorphism here for $q < n+l$ and a surjection for $q = n+l$. An application of the 5-lemma finishes the proof of the first assertion.
The second follows since we have $\operatorname{colim} h_*(X^{(n)}, A, \pi) = h_*(X, A, \pi)$ (for additive h) as stated in 3.1.2. The final assertion is immediate from the usual technique of CW-approximation. $\square$

3.1.6. Definition. An exterior product of (co)homology theories between (co) homology theories h, h' and k over spaces K, K' and L, respectively, refers to a map $f : K \times K' \to L$ and pairings

$$\times : h_*(X, A, \zeta) \times h'_*(Y, B, \eta) \longrightarrow k_*(X \times Y, X \times B \cup A \times Y, f_*(\zeta \times \eta))$$

that are compatible with the boundary map just as in ordinary homology.

3.1.7. Remark. We shall employ this definition only in two trivial, special cases: Firstly, that in which $K = L$, $K' = *$, $h = k$ and $f = id$ giving a notion of a parametrised theory h being a module over an unparametrised one h'. Secondly, that in which $K = K' = L$, f is a monoidal composition law and $h = h' = k$ arises from an unparametrised theory by simply ignoring the twisting data. The reason for this is the following: Asking for a multiplicative, parametrised (co)homology theory only really makes sense over a topological monoid K. The problem is that we know of no example of such a multiplicative theory, aside from the ones that are just given by ignoring the twist. This seems to be an inherent problem of the definition. Naturally given multiplications 'want' to simultaneously produce classes in $h_*(X \times Y, X \times B \cup A \times Y, g_*(\zeta \times \eta))$ for every g homotopic to the multiplication f, satisfying various coherences, instead of e.g. strict associativity. However, only the module structures over unparametrised homology theories will play an essential role for our study and for these the framework is perfectly well suited.

Using multiplicative structures we can now give a generalisation of the Thom-isomorphism:

3.1.8. Definition. A Thom class of a k-dimensional vector bundle $p : X \longrightarrow B$ along a map $\zeta : B \to K$ is a class $t \in h'^k(X, X-0, p^*\eta)$, such that the pairing map

$$h^l(B,\zeta) \to k^{k+l}(X, X-0, p^*(\zeta \oplus \eta)), \quad h \longrightarrow p^*(h) \cdot t$$

restricts to isomorphisms

$$h^l(b,\zeta) \to k^{k+l}(X_b, X_b - 0, p^*(\zeta \oplus \eta))$$

for all $b \in B$. A Thom class is called compatible with a subset $U \subseteq B$, if the map

$$h^*(U,\eta) \to k^{*+k}(X_{|U}, X_{|U} - 0, p^*(\eta \oplus \zeta)), \quad b \longmapsto p^*(b) \cup t_{|U}$$

is an isomorphism. It is called compatible with an atlas of the bundle, if it is compatible with all its covering sets.

3.1.9. Theorem. *Let $p : X \longrightarrow B$ be a k-dimensional vector bundle with a finite atlas $\mathcal{A}$ and $\zeta : B \to K$ a map. If now h, h' and k are as above and $t \in h'^k(X, X-0, p^*\zeta)$ is a Thom class compatible with $\mathcal{A}$, then the map*

$$h^*(B,\eta) \to k^{*+k}(X, X-0, p^*(\zeta \oplus \eta)), \quad b \longmapsto p^*(b) \cup t$$

is an isomorphism.
If h, h' and k admit compact supports, $\mathcal{A}$ need only be countable. If furthermore all are singular and additive, the atlas may be arbitrary and the Thom class is automatically compatible.

Even when restricted to the case of a twisted module theory over an untwisted theory this proposition is more general than that of say [**AnBlGe 10**, Example 5.2], in that it also allows twists for the total space of the bundle instead of the base. It is this version that we shall need.

Proof. The proofs of the unparametrised version here carry over without change: By a Mayer-Vietoris argument the first bit is induction over n for the statement that the Thom class is compatible with $\bigcup_{i=0}^n U_i$ using the usual Mayer-Vietoris-sequence-&-5-lemma-argument, where U is some denumeration of $\mathcal{A}$. The second is immediate from the axiom applied to the filtration

$$\cdots \subseteq \bigcup_{i=0}^{n} U_i \subseteq \bigcup_{i=0}^{n+1} U_i \subseteq \ldots$$

(this filtration is by open subsets). For the last bit one can pull the bundle back along a CW-approximation of B. However, for a cell complex one can argue by induction on the skeleta and then apply the Milnor sequence once more: The 0-skeleton is covered by the additivity axiom and the induction step uses similar sets as in the above proof; compatibility here never becomes an issue because in each step only spheres, disks and lower skeleta appear and the former two have finite atlases of contractible sets. □

3.2. Parametrised spectra. In this section we follow [**MaSi 06**] and give the basics of parametrised spectra. We shall essentially employ orthogonal spectra in their parametrised form as developed in [**MaSi 06**], however, we shall use right instead of left $\mathbb{S}$-modules since this fits better with the usual conventions from bundle theory.

3.2.1. Definition. A parametrised orthogonal sequence over K is a sequence of ex-spaces X over K together with a continuous left $O(n)$-action on X_n by ex-maps for each $n \in \mathbb{N}$. A map between to such objects is a sequence of equivariant ex-maps.
The orthogonal sequence $K \times \mathbb{S}$ is given by $(K \times \mathbb{S})_n = K \times (\mathbb{R}^n)^+$ with the obvious structure maps and $O(n)$-action.
The pre-smash product of two orthogonal sequences X and Y is the sequence given by

$$(X \wedge_K Y)_n = \bigvee_{i=0}^{n} O(n) \underset{O(p)\times O(n-p)}{\wedge} X_p \wedge_K Y_{n-p}$$

with action induced by the left operation on the left factor. There is also an external pre-smash product $\wedge$, which arises simply by substituting the internal smash product of ex-spaces by their external one. Thus it has as input two orthogonal sequences, one over K one over (say) K', and the output is one over $K \times K'$.

The two kinds of pre-smash products are of course related by $X \wedge_K Y = \Delta^*(X \wedge Y)$ where X, Y are sequences over K, Δ is the diagonal map of K and Δ^* means degreewise pullback. As usual the sequence $K \times \mathbb{S}$ comes with a canonical map $(K \times \mathbb{S}) \wedge_K (K \times \mathbb{S}) \to K \times \mathbb{S}$ making it a commutative monoid in the symmetric monoidal structure given by the smash product.

3.2.2. Definition. A parametrised orthogonal spectrum over K is an orthogonal sequence X over K together with a map $X \wedge_K K \times \mathbb{S} \to X$, that makes X into a right $K \times \mathbb{S}$-module. Their category is denoted $\mathcal{S}_K$.

Of course the pre-smash products lift to smash products $\wedge_K : \mathcal{S}_K \times \mathcal{S}_K \to \mathcal{S}_K$ and $\wedge : \mathcal{S}_K \times \mathcal{S}_{K'} \to \mathcal{S}_{K \times K'}$ by the usual coequaliser construction.
May and Sigurdsson then proceed to put two model structures on the arising category of parametrised spectra, namely the *level*- and the *stable* structure ([**MaSi 06**, Theorem 12.1.7 & Theorem 12.3.10]). Except for the proof of proposition 3.3.6 only the weak equivalences of the stable structure will matter. These are the maps $X \to Y$ that induce weak homotopy equivalences of homotopy fibre spectra $X^h_k \to Y^h_k$ for each point $k \in K$; here the homotopy fibre spectrum X^h_k is built from X by using as n-th space the homotopy fibre over k (say the mapping path space for concreteness sake) of the structure map $X_n \to K$. This spaces carry evident structure maps making them orthogonal spectra.

We can now give the usual definitions of the (co)homology theories given by a spectrum, though one comment is in order: Smash products and function spectra are all meant in the derived sense, as is application of any other functor; May and Sigurdsson indeed prove that all functors occuring in the formulas actually are Quillen functors except for F_K, which they construct as a right adjoint to the derived smash product $\wedge_K$. 'Underivedly' it is given by taking fibrewise function spectra. They put

$$E_*(X, A, \zeta) := \pi_*\Theta\big(\operatorname{cofib}_K(X, A) \wedge_K E\big)$$
$$E^*(X, A, \zeta) := \pi_{-*}\Gamma\big(F_K(\operatorname{cofib}_K(X, A), E)\big)$$

where $\Theta : \mathcal{S}_K \to \mathcal{S}$ denotes the degree wise collapse of the base-section, and $\Gamma : \mathcal{S}_K \to \mathcal{S}$ denotes degreewise taking of sections of the structure maps.

3.2.3. Theorem (May-Sigurdsson). *Given a parametrised spectrum E over K the above constructions yield an additive and singular homology theory and an additive singular cohomology theory.*

3.2.4. Remark. In [**MaSi 06**] Multiplicative structures are not much investigated. However, just as in the unparametrised case a map of spectra $g : E \wedge F \to f^*G$ (for parametrised spectra E, F and G over spaces K, K' and L and a map $f : K \times K' \to L$) gives rise to an exterior product of (co)homology theories. If then the spectra all agree, f is an associative multiplication on K and g is at least fibre-homotopy associative/commutative, we would obtain multiplications with the same properties. We do not know how to produce such multiplications on parametrised spectra representing twisted KO-theory or spin cobordism.

3.2.5. Remark. Another approach to twisted cohomology based on $(\infty, 1)$-categories is for example given in [**AnBlGeHoRe 09**] and their methods certainly produce functors of the type we have been studying. We have, however, not investigated whether they fit into definition 3.1.1 on the nose, though we believe that they do.

3.3. Our models for twisted *Spin*-bordism. We use a slight variation of the spectra appearing in [**Jo 04**, Chapter 6]. Essentially we translate this model to orthogonal spectra and introduce a second version which trades the strictly associative and commutative multiplication of this spectrum for an action of $PGL^{\pm}(\ell^2)$ by maps of orthogonal spectra. While this to some extend looks unnatural, it illustrates the problems with multiplicative structures mentioned in the last section.

3.3.1. Construction. Let $L^2(\mathbb{R}^n)$ denote the Hilbert space of square integrable, real valued functions on $\mathbb{R}^n$, which we regard as $\mathbb{Z}/2$-graded by even/odd functions, and let Cl_n denote the Clifford algebra of $\mathbb{R}^n$ with $v^2 = \langle v, v\rangle$, where the right hand side denotes the standard scalar product. We further denote $L_n = L^2(\mathbb{R}^n) \otimes Cl_n$ and $L'_n = \ell^2 \otimes L_n$ (ℓ^2 the space of square integrable real sequences, regarded as $\mathbb{Z}/2$-graded by the vanishing of odd/even entries); tensor products will always refer to graded tensor products.
By Kuiper's Theorem both components of $Gl^{\pm}(L_n)$ and $Gl^{\pm}(L'_n)$ (the groups of continuous, linear, even/odd and invertible selfmaps) are contractible, we will abbreviate these groups to $G\ell_n$ and $G\ell'_n$, respectively. The following definitions work equally well for the $'$-case, hence we omit them for legibility. The group $Pin(n) \subseteq Cl(n)$ can be mapped into $G\ell_n$ via

$$j : p \longmapsto (f \otimes c \longmapsto f \circ \rho(p)^{-1} \otimes p \cdot c)$$

where $\rho : Pin(n) \to O(n)$ is the usual surjection.
The element $-1 \in Pin(n)$ thus acts as multiplication by $-1 \in \mathbb{R}$ on L_n; this yields an inclusion $O(n) \hookrightarrow PG\ell_n$. Note firstly that $-id \in O(n)$ does not map to $-id = id \in PG\ell_n$) and secondly that this embedding maps $Spin(n)$ and $SO(n)$ into $G\ell_n^+$ and $PG\ell_n^+$, respectively, where $+$ denotes the subgroup of degree preserving ('even') elements. We thus find

$$ESpin(n) := G\ell_n^+ \longrightarrow G\ell_n^+/Spin(n) = PGl_n/O(n) =: BSpin(n)$$

to be a universal principal $Spin(n)$-bundle.
It is noteworthy that there is a second natural map $i : Pin(n) \to Gl_n$, namely the

one coming from functoriality of the Clifford algebra:

$$p \longmapsto (f \otimes c \longmapsto f \circ \rho(p)^{-1} \otimes Cl_{\rho(p)}(c))$$

This second action obviously factors through ρ, and gives an embedding $O(n) \hookrightarrow G\ell_n$. Coming back to the *Spin*-spectrum: Forming the Thom spaces of the associated vector bundles of the universal *Spin*-bundles above yields two (remember the primed version) nice models of the Thom spectrum $MSpin$ as as orthogonal spectra. The obvious inclusions of $G\ell_n$ into $G\ell'_n$ produce a map of spectra $MSpin \to MSpin'$, that is easily seen to be a weak equivalence.
There is also an obvious concatenation operation $G\ell_n \times G\ell_k \to G\ell_{n+k}$ (factoring through an inclusion $G\ell_n \times_{\mathbb{C}^*} G\ell_k \hookrightarrow G\ell_{n+k}$) that induces a strictly associative and commutative multiplication on $MSpin$. Similarly the analogous map $G\ell'_n \times G\ell_k \to G\ell'_{n+k}$ gives a right $MSpin$-module structure to $MSpin'$. There is however no good choice for a map $G\ell'_n \times G\ell'_k \to G\ell'_{n+k}$ as one would have to merge the two ℓ^2 factors somehow, and there seems to be no associative way to do this. However, any choice of isomorphism $\ell^2 \to \ell^2 \otimes \ell^2$ induces one such concatenation operation. Since by Kuiper's theorem the space of these isomorphisms is contractible the induced mulitplication on $MSpin'$ corresponding to one such choice is homotopy associative and homotopy commutative (and homotopy everything).
The redeeming feature about $MSpin'$ is that it carries an action of $PG\ell$ by maps of orthogonal spectra induced by letting $G\ell$ act by left multiplication on the extra ℓ^2 factor of $G\ell'_n$.

To properly interpret the last few statements there are various structure maps to be defined and even more verifications to be done, but essentially all of them are obvious (and many of the definitions are given in [**Jo 04**]), so we leave them to the reader. We now want to employ the construction of [**MaSi 06**, section 22.1] to produce a parametrised spectrum over $BPG\ell$. Even though we will not endow the arising twisted spectra with multiplicative structures, we need to be picky about which $BPG\ell$ to choose, i.e. we need it to be a topological monoid. To this end, we have:

3.3.2. Lemma. *The space $BPG\ell$ has the homotopy type of a product K of Eilenberg-MacLane spaces one of type $(\mathbb{Z}/2, 1)$, the other of type $(\mathbb{Z}/2, 2)$. Furthermore, there are two (homotopy classes of) choices of such an equivalence and under each the multplication on $BPG\ell$ induced by any choice of isomorphism $i : \ell^2 \to \ell^2 \otimes \ell^2$ translates into the map $K \times K \to K$ represented by*

$$(\iota_1 \otimes 1 + 1 \otimes \iota_1, 1 \times \iota_2 + \iota_2 \otimes 1 + \iota_1 \otimes \iota_1) \in H^1(K \times K, \mathbb{Z}/2) \times H^2(K \times K, \mathbb{Z}/2)$$

or put somewhat loosely

$$(a, b) \cdot (c, d) = (a + c, b + d + ac)$$

By results of Milgram the space K and the multiplication can thus be chosen so as to make it a topological monoid.

Proof. This result is folklore. The complex case is proved for example in [**AtSe 04**, Proposition 2.3] using slightly different language. The non-trivial self-homotopy equivalence on K is given by $(a, b) \longmapsto (a, b + a^2)$ as one can immediately read off from the cohomology of K. □

For our construction of a twisted spin cobordism spectrum, we need to choose the following data: A topological monoid K as above and a universal $PG\ell$-bundle $EPG\ell$ over K. Since there are two isomorphism classes of such bundles, there are two possible conventions. We choose one as follows: Observe that the spaces $EPG\ell_n := EPG\ell \times_{PG\ell} PG\ell_n$ are contractible, so that they can serve as universal $PG\ell_n$- and in particular $O(n)$-spaces (along the homomorphism $j : O(n) \to PG\ell_n$). Altogether find that $BO(n) := EPG\ell_n/j(O(n))$ is a choice of classifying space for $O(n)$ which admits a canonical map to $K = BPG\ell$ by further quotienting. For this map there are two possibilities:

3.3.3. Lemma. *The map $BO(n) \to K$ either represents (w_1, w_2) or $(w_1, w_2 + w_1^2)$ and postcomposing with the non-trivial self-homotopy equivalence of K (which is the same as choosing a universal bundle in the other equivalence class) interchanges these.*

Proof. It follows immediately from the long exact sequence of

$$PG\ell_n/O(n) \to BO(n) \to K$$

(recall that $PG\ell_n/O(n) = BSpin(n)$) that the induced map $\pi_*(BO(n)) \to \pi_*(K)$ is an isomorphism in degrees one and two. Now the Hurewicz theorem applied to this (and the corresponding map of universal covers) gives the first claim. The second follows immediately from the explicit description of the non-trivial self-homotopy equivalence given in the proof of 3.3.2. □

Since choices are unavoidable in the context of parametrised spectra and twisted cohomology, we shall keep the monoid K, and a universal $PG\ell$-bundle, such that the map above represents (w_1, w_2) fixed throughout the remainder of this thesis.

3.3.4. Definition. The parametrised spectrum M_2O given by

$$(M_2O)_n = EPG\ell \times_{PG\ell} MSpin(n)'$$

with structure maps and $O(n)$-actions induced from $MSpin'$ will be called the twisted *Spin*-bordism spectrum.

We define *twisted Spin-cobordism* to be the homology theory represented by this spectrum. Note that the actions of $MSpin$ and $PG\ell$ on $MSpin'$ explained in 3.3.1 commute and thus give a pairing

$$M_2O \wedge MSpin \longrightarrow M_2O$$

making twisted Spin-bordism into a module theory over (non-twisted) Spin-bordism.

3.3.5. Observation. The spaces M_2O_n are on the nose the fibrewise Thom space along the map $BO(n) \to K$ of the universal bundles described above. In particular, we have that $\Theta(M_2O)$ is a model for MO. This justifies the name M_2O.

To fully connect this to the musings from the first part, however, we need the following technical proposition:

3.3.6. Proposition. *Let $\zeta : X \to K$ be a space over K, where X is a cell complex. Then there is a natural isomorphism*

$$\Omega_n^{Spin}(X, \zeta) \cong \pi_n(\Theta(M_2O \wedge_K (X + K)))$$

Indeed such an isomorphism is just a zig-zag arising from fibrant and cofibrant replacements.

The point is that no cofibrant or fibrant resolution has to take place in order to correctly predict the twisted cobordism groups.

Proof. We will freely use the language of [**MaSi 06**] in this proof. We have to construct a zig-zag of weak equivalences between

$$\Theta(M_2O \wedge_K (X+K))$$

and

$$\Theta c\Delta^* f(cM_2O \wedge c(X+K))$$

where c and f denote cofibrant and fibrant resolutions in the appropriate model structures. Since the external smash product preserves homotopy equivalences between well-sectioned spaces (by the same proof as [**MaSi 06**, Proposition 8.2.6]), we have a homotopy equivalence

$$cM_2O \wedge c(X+K) \to M_2O \wedge X + K$$

since all spaces in sight have the homotopy type of cell complexes (see the exposition in [**MaSi 06**, Section 9.1] for the resolved objects). We therefore obtain a diagram

$$f(cM_2O \wedge c(X+K)) \to f(M_2O \wedge X + K) \to M_2O \wedge LX + K$$

where L is the Moore-mapping-path-space functor. The rightmost map exists since $LX + K$ and M_2O_n are ex-fibrations, so is their smash product by [**MaSi 06**, Proposition 8.2.3], and spectra consisting of ex-fibrations are level-fibrant. As a right Quillen functor Δ^* preserves weak equivalences between fibrant objects and therefore

$$\Delta^* f(cM_2O \wedge c(X+K)) \to M_2O \wedge_K LX + K$$

is a weak equivalence. Now both sides are well-sectioned: For the right side this follows by inspection and for the left it follows since cofibrant spaces are well-sectioned by [**MaSi 06**, Theorem 6.2.6], and Δ^* preserves 'well-sectionedness' by [**MaSi 06**, Proposition 8.2.2]. Since Θ preserves weak equivalences between well-sectioned ex-spaces we obtain a weak equivalence

$$\Theta c\Delta^* f(cM_2O \wedge c(X+K) \to \Theta(M_2O \wedge_K (LX+K))$$

The right hand side here is the Thom-spectrum associated to the pullback of

$$\begin{array}{ccc} & & BO(i) \\ & & \downarrow \\ LX & \longrightarrow & K \end{array}$$

Since the canonical map $X \to LX$ is a homotopy equivalence and the map $BO(i) \to K$ a fibration, this pullback is homotopy equivalent to that of

$$\begin{array}{ccc} & & BO(i) \\ & & \downarrow \\ X & \longrightarrow & K \end{array}$$

which concludes the proof. □

3.3.7. REMARK. Choosing an isomorphism $i : \ell^2 \to \ell^2 \otimes \ell^2$ one obtains a map $PG\ell \times PG\ell \to PG\ell$ and we can ask for a compatible bundle map

$$\begin{array}{ccc} EPG\ell \times EPG\ell & \longrightarrow & EPG\ell \\ \downarrow & & \downarrow \\ K \times K & \longrightarrow & K \end{array}$$

covering the multiplication of K. Since $EPG\ell \to K$ is universal, the space of these bundle maps were contractible if we left the multiplication map variable (within its homotopy class) as well. However, fixing it there is more than a single fibre homotopy class of such maps. While every one of them induces a multiplication $M_2O \wedge M_2O \to M_2O$, none of them will be (homotopy-)associative. They do however produce a single homotopy-class of maps $\Theta(M_2O) \wedge \Theta(M_2O) \to \Theta(M_2O)$ making $\Theta(M_2O)$ an orthogonal spectrum with a homotopy ring structure, which is somewhat disappointing given that MO can easily be made into an honest orthogonal ring spectrum.

3.4. Our model for twisted K-theory. We again use a slight variation of the spectra appearing in [**Jo 04**, Chapter 6], introducing a version which trades the nice multiplication for an action by $PG\ell$.

3.4.1. CONSTRUCTION. Consider the spaces $KO_n = Hom(C_0(\mathbb{R}), K(L_n))$ and $KO'_n = Hom(C_0(\mathbb{R}), K(L'_n))$, where C_0 denotes the algebra of real valued functions vanishing at infinity ('suspension algebra'), K denotes the space of right-Clifford-linear compact operators and Hom denotes the space of degree preserving C^*-homomorphisms, pointed by the null map. It is shown in [**Jo 04**], that these spaces represent KO-Theory. Both sequences form orthogonal spectra as follows:
The orthogonal group acts on KO_n by conjugation using the map $i : O(n) \to G\ell_n$ given by $p \longmapsto (f \otimes c \mapsto f \circ p^{-1} \otimes Cl(p)c)$ (using functoriality of the Clifford algebras). Note that also under this map $-id \in O(n)$ does not map to $-id \in G\ell_n$. Again a similar action works in the primed case.
The structure maps will arise via the unit of a commutative multiplication on KO and a KO-module structure on KO'. These multiplications come from the coproduct Δ on $C_0(\mathbb{R})$ and the isomorphism $K(L_n) \otimes K(L_m) \to K(L_{n+m})$ and $K(L_n) \otimes K(L'_m) \to K(L'_{n+m})$ induced by the isomorphisms $L_n \otimes L^{(\prime)}_m \to L^{(\prime)}_{n+m}$. Unwinding these definitions immediately gives that the product maps are $O(n) \times O(m)$-equivariant. The unit for the product on KO is given by extending

$$\mathbb{R}^n \to KO(n), \quad w \mapsto \Big(f \mapsto p_n \otimes f(w) \cdot -\Big)$$

by zero, where p_n denotes the projection operator of $L^2(\mathbb{R}^n)$ onto the subspace generated by the function $v \mapsto e^{-|v|^2}$ and $f(w)$ may be interpreted as functional calculus. This map is equivariant because $Cl(p)(f(w)) = f(p(w))$ for all $f \in C_0(\mathbb{R}), w \in R^n, p \in O(n)$ and the subspace generated by e_w is $O(n)$-invariant.
As before we also have a weak equivalence $KO \to KO'$, however, this time it depends upon a choice of rank 1 projection operator p on ℓ^2 and is then given by sending

$$\varphi : C_0(\mathbb{R}) \to KO'(n) \text{ to } f \longmapsto p \otimes \varphi(f)$$

Note that the space of rank 1 operators is path-connected so the homotopy class of this equivalence is uniquely determined; it is a weak equivalence, since the map on compact operators $K(H) \to K(H \otimes H')$ given by tensoring an operator with a rank 1 projection on H' is a homotopy equivalence of C^*-algebras, being homotopic to the isomorphism arising from any identification $H \cong H \otimes H'$, see e.g. [**Me 00**]. The spectrum KO' can furthermore be given an action by $PG\ell$ via conjugating a homomorphism $\varphi : C_0(\mathbb{R}) \to KO'(n)$ with $p \in PG\ell$ to

$$f \longmapsto (-1)^{|p|}(p \otimes id_{L_n}) \circ \varphi(f) \circ (p^{-1} \otimes id_{L_n})$$

The twist is necessary to make the next paragraph work.

Again, the checks that need to be performed are partially given in [**Jo 04**] and the rest is left to the reader. From this data we produce a twisted spectrum just as above.

3.4.2. Definition. The spectrum K_2O given by

$$K_2O = EPG\ell \times_{PG\ell} KO'(n)$$

with structure maps and $O(n)$-actions induced form KO' we call the twisted, real K-theory spectrum.

As in the case of the *Spin*-spectrum, this spectrum K_2O is a module spectrum over KO and the spectrum $\Theta(K_2O)$ inherits at least a homotopy multiplication.

4. Fundamental classes and orientations

4.1. The twisted Atiyah-Bott-Shapiro orientation. Recall that *Spin*-bundles have Thom classes in real K-theory by the work of Atiyah, Bott and Shapiro. This gives rise to a (multiplicative) transformation of homology theories $\Omega_*^{Spin} \to KO_*$, represented by some map of spectra $MSpin \to KO$. In [**Jo 04**] M. Joachim has constructed such a (ring-)map explicitly using the spectra described above (implying for one that this orientation is an E_∞-map). We will modify his construction slightly to obtain a map $M_2O \to K_2O$, which will provide us with Thom classes in twisted K-theory.

4.1.1. Construction. First note that the action of $O(n)$ on $KO(n)$ extends to an action of all $G\ell_n$ by conjugation twisted by degree and this factors through $PG\ell_n$; as the $O(n)$ action on L_n is by even transformations only, it also extends to the $G\ell_n$-action by untwisted conjugation, but then the map below would not be $O(n)$ equivariant.
We now have the following composite α induced by the unit of the KO-spectrum

$$\begin{aligned} MSpin(n) = PG\ell_{n+} \wedge_{O(n)} S^n & \\ \longrightarrow PG\ell_{n+} \wedge_{O(n)} KO(n) & \\ \longrightarrow PG\ell_{n+} \wedge_{PG\ell_n} KO(n) & \\ \xrightarrow{\cong} KO(n) & \end{aligned}$$

Checking that this map is indeed equivariant is a bit more involved than before since the action of $O(n)$ on $MSpin(n)$ was given through multiplying from the left on $PG\ell_n$ using the map $j : O(n) \to PG\ell_n$ whereas the action of $O(n)$ on $KO(n)$ was given by conjugation using the map $i : O(n) \to G\ell_n$. Tracing through the definitions

shows that the equivariance of α is equivalent to both these maps inducing the same conjugation on $K(L_n)$, which they do since $Cl_{\rho(g)}(c) = (-1)^{|g|}gcg^{-1}$ for $g \in Pin(n)$ and $c \in Cl_n$.
We similarly have a map $\alpha' : MSpin' \to KO'$ (depending on our fixed rank 1 operator p on ℓ^2). These maps make the following diagram commute:

$$\begin{array}{ccc} MSpin & \xrightarrow{\alpha} & KO \\ \downarrow & & \downarrow \\ MSpin' & \xrightarrow{\alpha'} & KO' \end{array}$$

where both maps to KO' come from the same choice of p. The crucial observation is that α' is also equivariant under the action of $PG\ell$, which was the reason for introducing the sign in the definition of the $PG\ell$-action.

Again the remaining checks we leave to the reader.

4.1.2. Theorem (Theorem 6.9 [**Jo 04**]). *The map $\alpha : MSpin \to KO$ induces the KO-theory Thom classes for Spin-bundles of Atiyah, Bott and Shapiro. The same is therefore true of the primed version.*

4.1.3. Definition. We call the map $\hat{\alpha} : M_2O \to K_2O$ induced by α' the *twisted Atiyah-Bott-Shapiro-orientation.*

4.1.4. Remark. We remark that the existence of such a transformation on the level of homotopy categories can be deduced along the lines of [**AnBlGe 10**], by considering the fibre sequence $\mathbb{Z}/2 \times K(\mathbb{Z}/2,1) \to BSpin \to BO$ and studying the induced map of Thom spectra. One obtains a map $\Sigma^\infty(K + *) \to MSpin$ that is adjoint to a map $K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2) \to Bgl_1(MSpin)$. The E_∞ ring map $MSpin \to KO$ then transports these twists to KO-theory and the result is a map of twisted spectra. However, from this description it is not clear how to get at the geometric content of the induced transformation; something that (although not explicitly considered in this thesis) is very relevant to our work.

Let us now explain how this map gives rise to Thom classes. Recall that by proposition 3.3.3 we have $M_2O(n)$ identified with the fibrewise Thom space of $EO(n) \to BO(n)$ along a fibration $BO(n) \to K$ representing the first and second Stiefel-Whitney classes. Given now an n-dimensional bundle $V \to B$ together with a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & EO(n) \\ {\scriptstyle p}\downarrow & & \downarrow \\ B & \longrightarrow & BO(n) \\ & {\scriptstyle w}\searrow \quad \swarrow & \\ & K & \end{array}$$

we obtain a canonical fibre homotopy class $(DV, SV) \to (M_2O(n), K)$ over K representing a class

$$t \in \Omega^n_{Spin}(V, V-0, p^*w)$$

and thus a class $\hat{\alpha}(t) \in KO^n(V, V-0, p^*w)$.

4.1.5. PROPOSITION. *The classes t and $\hat{\alpha}(t)$ are Thom-classes for V with respect to the pairings given by the module structures over Spin cobordism and KO-theory, respectively. Therefore, any bundle $V \xrightarrow{p} B$ together with a representing map $B \xrightarrow{w} K$ of its first and second Stiefel-Whitney class admits a Thom class in both twisted Spin-bordism and twisted K-theory.*

PROOF. Given a rank k vector bundle $p : V \to B$ with first and second Stiefel-Whitney classes represented by $w : B \to K$, we find, for every $b \in B$, a commutative diagram

$$\begin{array}{ccc} \Omega^n_{Spin}(b) \times \Omega^k_{Spin}(V_b, V_b - 0, p^*w) & \longrightarrow & \Omega^{n+k}_{Spin}(V_b, V_b - 0, p^*w) \\ \downarrow \cong & & \downarrow \cong \\ \Omega^n_{Spin}(b) \times \Omega^k_{Spin}(V_b, V_b - 0) & \longrightarrow & \Omega^{n+k}_{Spin}(V_b, V_b - 0) \end{array}$$

since the cohomology theories $(X, A) \longmapsto \Omega^*_{Spin}(X, A, c_k)$, where c_k denotes the constant map with value $k \in K$, are all isomorphic to Spin cobordism, albeit non-canonically, since they are represented by the (derived) fibre spectra of M_2O, which are equivalent to $MSpin'$ and thus $MSpin$. Any such equivalence coming from a choice of point in the fibre of $EPG\ell$ over k furthermore preserves the multiplicative structure, so for one such isomorphism the lower hand map is again given by multiplication. In addition such an identification induces a Spin structure on the restriction of V to the fibre over $w(b)$ and our construction then reduces to the standard construction of Thom classes in Spin cobordism. This proves the claim for the case of twisted Spin cobordism.
The KO-theory case follows by the same argument, since the map induced by $\hat{\alpha}$ on the fibres is just α' under an identification as above and theorem 4.1.2 applies. □

4.1.6. REMARK. Note that, even though our cohomology theories are singular, not every bundle admits Thom classes this way, since it does nessecarily admit the twisting maps. Furthermore, the classes constructed in the proof seem to depend on further choices than just the twisting map. We see neither a reason to believe that the classes above are actually independent of the choices made nor even the existence of more canonical Thom classes. Of course we have canonical Thom classes for the universal bundles, because we can then choose the identity as the classifying map.

In particular we find:

4.1.7. PROPOSITION. *There are isomorphisms*

$$KO^{k+n}(M_2O(n), K, w) \cong KO^{k+n}(EO(n), EO(n) - 0, w) \cong KO^k(BO(n))$$

given by excision and the canonical Thom class just mentioned.

This is the twisted analogue of the classical K-theory Thom isomorphism

$$KO^{k+n}(MSpin(n), *) \cong KO^k(BSpin(n))$$

We shall see in the next section that this also works for connective K-theory, but there are a few technical difficulties we have to address first.

4.2. Twisted connective covers. What we need is a construction of connective covers for orthogonal spectra such that all diagrams, which are supposed to commute, do so on the nose and if a G-action on some spectrum E is continuous for a topological group G, then so is the induced action on $E\langle n\rangle$. Covers built using simplicial spaces satisfy this, as we explain below. Assuming all this we can immediately produce twisted connective covers:

$$(K_2O\langle n\rangle)_m := EPG\ell \times_{PG\ell} (KO\langle n\rangle)_m$$
$$(M_2O\langle n\rangle)_m := EPG\ell \times_{PG\ell} (MSpin\langle n\rangle)_m$$

From the functoriality we immediately obtain a lift of $\alpha' : MSpin' \to KO'$ to $MSpin'\langle 0\rangle \to KO'\langle 0\rangle$ and preservation of continuous group action then produces a map $M_2O\langle n\rangle \to K_2O\langle n\rangle$. In accordance with the usual naming scheme we set $K_2O\langle 0\rangle =: k_2o$. However the map $M_2O\langle 0\rangle \to M_2O$ is obviously a weak equivalence and we shall suppress it in much of what follows. All in all we obtain a transformation of twisted homology theories

$$\hat\alpha : \Omega_*^{Spin}(-;-) \longrightarrow ko_*(-;-)$$

Furthermore, we have pairings $ko' \wedge KO\langle n\rangle \to KO'\langle n\rangle$ and consequently

$$k_2o \wedge KO\langle n\rangle \longrightarrow K_2O\langle n\rangle$$

up to homotopy as follows: We have a commutative diagram

$$\begin{array}{ccccc}
(ko' \wedge KO\langle n\rangle)\langle n\rangle & \longrightarrow & (KO' \wedge KO)\langle n\rangle & \longrightarrow & KO'\langle n\rangle \\
\downarrow \simeq & & \downarrow & & \downarrow \\
ko' \wedge KO\langle n\rangle & \longrightarrow & KO' \wedge KO & \longrightarrow & KO'
\end{array}$$

where the first line arises from the second by applying $-\langle n\rangle$. As above we produce a map

$$\begin{aligned}
k_2o \wedge KO\langle n\rangle &\stackrel{\simeq}{\longleftarrow} EPG\ell \times_{PG\ell} (ko' \wedge KO\langle n\rangle)\langle n\rangle \\
&\longrightarrow EPG\ell \times_{PG\ell} (KO' \wedge KO)\langle n\rangle \\
&\longrightarrow K_2O\langle n\rangle
\end{aligned}$$

As in the non-parametrised case, such a 'pairing' is homotopy associative and so on, inducing a pairing of homology and cohomology theories. By inspection we find:

4.2.1. Proposition. *The transformation $\hat\alpha : \Omega_*^{Spin}(-;-) \to ko_*(-;-)$ maps Thom classes to Thom classes for the above pairing.*

Unraveling this we find that for a rank k vector bundle V as above, we have an isomorphism

$$ko\langle l\rangle^{n+l}(V, V-0, p^*w) \cong ko\langle l\rangle^n(B)$$

Finally, we have the construction we used above:

4.2.2. Proposition. *Realising the simplicial* space

$$k \mapsto X\langle n\rangle_k = \{f : \Delta^k \to X \mid f\left((\Delta^k)^n\right) = x\}$$

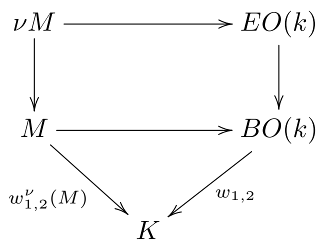
associated to a pointed space (X, x) produces an n-connective cover of X with all the stated properties.

Proof. All the assertions are more or less evident, except for maybe the fact that $|X\langle n\rangle_\bullet|$ is indeed a connective cover. This is wellknown for the untopologised version of the above construction (compare e.g. [**Ma**, Paragraph 8]) and can be reduced to that by the following argument:
The bisimplicial *set*

$$Z_{k,l} = \{f : \Delta^k \times \Delta^l \to X \mid f\left(\Delta^k \times (\Delta^l)^n\right) = x\}$$

in the one direction realises to $|Z_{k,\bullet}| = |Sing_\bullet(C(\Delta^k, X))\langle n\rangle|$, where $C(\Delta^k, X)$ denotes the *space* of continuous maps and $\langle n\rangle$ denotes the simplicial *set* construction of connective covers. The arising simplicial *space* admits the obvious levelwise weak equivalence (by constant maps) from the constant simplicial space given by $|Sing_\bullet(X)\langle n\rangle|$. We conlude that $|Z_{\bullet,\bullet}|$ really is a connective cover of X.
Realising in the other direction first produces $|Z_{\bullet,l}| = |Sing_\bullet(X\langle n\rangle_l)|$, the singularisation of the l-space of the singular space in question. Therefore, we conclude that $|Z_{\bullet,\bullet}|$ and $|X\langle n\rangle_\bullet|$ are weakly equivalent and a quick check of the maps involved yields the proposition. □

4.3. Fundamental classes. Given a closed, connected smooth manifold with a choices of maps



and normal bundle νM of some dimension k, we obtain a class $[M] \in \Omega_n^{Spin}(M, w^\nu_M)$ by the Pontryagin-Thom construction and either 3.3.6 or definition.

4.3.1. Observation. This class $[M] \in \Omega_n^{Spin}(M, w^\nu(M))$ would deserve the name of fundamental class of M in twisted spin bordism except for the fact that it depends on the choice of classifying map for the normal bundle: Its push forward generates $\Omega_*^{Spin}(M, M - x, w^\nu(M))$ freely as a module over Ω_*^{Spin} for every $x \in M$, because firstly the inclusion

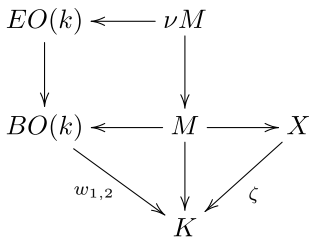
$$\Omega_*^{Spin}(U, U - x, w^\nu(M)) \to \Omega_*^{Spin}(M, M - x, w^\nu(M))$$

induces an isomorphism by excision, secondly for U a ball around x, we find

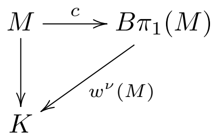
$$\Omega_*^{Spin}(U, U - x, w^\nu(M)) \cong \Omega_*^{Spin}(U, U - x)$$

induced by some nullhomotopy of $w^\nu(M)_{|U}$, thirdly all of this is compatible with the Ω_*^{Spin}-module structure and finally $[M]$ corresponds to the image of a fundamental class in $\Omega_n^{Spin}(U, \partial U)$.

Proposition 3.3.6 even gives the geometric interpretation of twisted cobordism groups: Cycles for $\Omega_n^{Spin}(X, \zeta)$ are given by commutative diagrams:



Now, if the universal cover of M is spin, then we saw in 2.2.1 and 2.2.2, that there is a diagram



where c classifies the universal cover of M. Extending this diagram to a cycle for $(B\pi_1(M), w^\nu(M))$ we obtain a class

$$c_*(M) \in \Omega_n^{Spin}(B\pi, w_M)$$

We are now finally able to formulate the conjecture, that is the long term goal of our project:

4.3.2. Conjecture (Stolz 1995). If for a connected, closed, smooth manifold M, whose universal cover is spin, we have

$$0 = \hat{\alpha}(c_*(M)) \in ko_n(B\pi, w^\nu(M))$$

for some choice of necessary auxiliary data (and then for every), then M carries a metric of positive scalar curvature.

5. The generalised Anderson-Brown-Peterson splitting

5.1. Twisted K-Theory Pontryagin classes. With the Thom classes in place, we can give a generalisation of the Anderson-Brown-Peterson splitting. Note however, that while the focus of the original paper by Anderson, Brown and Peterson [**AnBrPe 67**] lies on the calculation of the spin-bordism ring, their result also facilitated computations of other Spin cobordism groups by reduction to calculations in KO-theory. It is this second part that our result generalises to the case of twisted cobordism groups. We begin by recalling the essentials from [**AnBrPe 67**]. They define characteristic classes π_k for vector bundles of rank n (which is suppressed in the notation because of part (3) of the following proposition) with values in KO^0, such that the following properties hold:

5.1.1. Proposition. *The following hold for vector bundles E, F:*

(1) $\pi_k(E \oplus F) = \sum_{i=0}^k \pi_i(E) \cup \pi_j(F)$
(2) $\pi_k(E \oplus \mathbb{R}^n) = \pi_k(E)$
(3) $\pi_k(E) = 0$*, if* $2k > rank(E)$ *and* E *orientable*

Proof. These properties are all stated right in or around [**AnBrPe 67**, Proposition 5.1]. □

Because of (2) the π_j for varying n (!) induce a compatible system of classes in $KO^0(BO(n))$ when applied to the universal bundle. By the Milnor-sequence one obtains classes $\pi_j \in KO^0(BO)$, since the lim^1-term vanishes by application of the real version of the Atiyah-Segal completion theorem and explicit computation of the representation rings involved. For a finite sequence of natural numbers J denote $\pi_J = \prod_i \pi_{J_i}$ and $n(J) = \sum_i J_i$. Obviously, sequences that arise by rearrangement give the same class. Anderson, Brown and Peterson then prove:

5.1.2. THEOREM. *The class $\pi_J \in KO^0(BSO)$ lies in the image of the map*

$$ko\langle n_J\rangle^0(BSO) \to KO^0(BSO)$$

where

$$n_J = \begin{cases} 4n(J) & n(J) \textit{ even} \\ 4n(J)-2 & n(J) \textit{ odd} \end{cases}$$

The same statement thus also holds for the restrictions to $BSpin$. The indeterminacy of such lifts is also determined in [**AnBrPe 67**], but we shall not make use of that. Using the Thom classes for spin bundles along the pairing $ko^*(-) \times ko\langle n_J\rangle^*(-) \to ko\langle n_J\rangle^*(-)$, that we generalised in the last section, we find

$$\begin{aligned} [MSpin, ko\langle n_J\rangle] &\longrightarrow \lim_n ko\langle n_J\rangle^n(MSpin(n), *) \\ &\cong \lim_n ko\langle n_J\rangle^0(BSpin(n), *) \\ &\longleftarrow ko\langle n_J\rangle^0(BSpin) \end{aligned}$$

where the first map is a surjection by the Milnor-sequence. We thus obtain a transformation $\Omega_*^{Spin}(-) \to ko\langle n_J\rangle_*(-)$ for each J. We will use the obvious analogue of the above calculation

$$\begin{aligned} [M_2O, k_2o\langle n_J\rangle] &\longrightarrow \lim_n ko\langle n_J\rangle^n(M_2O(n), K, w) \\ &\cong \lim_n ko\langle n_J\rangle^0(BO(n)) \\ &\longleftarrow ko\langle n_J\rangle^0(BO) \end{aligned}$$

to analyse the twisted theory $\Omega_*^{Spin}(-;-)$. However, we do not know whether theorem 5.1.2 also holds for BO instead of BSO or $BSpin$. Therefore, we have to modify the π_j's slightly:

5.1.3. DEFINITION. Put $\overline{\pi}_j(E) = \pi_j(E \oplus \Lambda^n E)$ for any vector bundle E of rank n.

By the second stated property of the π_j's we have $\overline{\pi}_j(E) = \pi_j(E)$ for any orientable bundle. With the same discussion as above we thus obtain classes $\overline{\pi}_j \in KO^0(BO)$ and by the comment just made these restrict to our original $\pi_j \in KO^0(BSO)$. A lift of π_J in $ko\langle n_J\rangle^0(BSO)$ thus determines a lift of $\overline{\pi}_j$ in $ko\langle n_J\rangle^0(BO)$ and a transformation $\Omega_*^{Spin} \to ko\langle n_J\rangle_*$.

5.1.4. COROLLARY. *The class $\overline{\pi}_J \in KO^0(BO)$ lies in the image of the map $ko\langle n_J\rangle^0(BO) \to KO^0(BO)$, where n_J is as above.*

Underlying this construction is of course the homotopy equivalence

$$BO \to BSO \times K(\mathbb{Z}/2, 1)$$

on finite steps (i.e. the $BO(n)$) corresponding to adding (in the first component) and remembering (in the second) the top exterior power of a bundle.

5.2. The Anderson-Brown-Peterson splitting and its twisted generalisation. The investigation of the Spin cobordism ring in [**AnBrPe 67**] starts with the following theorem:

5.2.1. Theorem. *If $\theta_J : MSpin \longrightarrow ko\langle n_J\rangle$ corresponds to π_J under the Thom isomorphism, then, as J runs through all non-decreasing sequences with $J_1 > 1$, the induced map*

$$\bigoplus_J H^*(ko\langle n_J\rangle, \mathbb{Z}/2) \longrightarrow H^*(MSpin, \mathbb{Z}/2)$$

is injective with graded-free cokernel over $\mathcal{A}_2$.

Choosing a split over $\mathcal{A}_2$ of the arising short exact sequences and a homogeneous basis of the cokernel we obtain maps $z_i : MSpin \longrightarrow sh^{n_i} H\mathbb{Z}/2$, where we denote the shift of spectra by sh^i (so that $(sh^i E)_k = E_{i+k}$).

5.2.2. Corollary. *Given θ_J and z_i as above the induced map*

$$\bigoplus_J H^*(ko\langle n_J\rangle, \mathbb{Z}/2) \oplus \bigoplus_i H^*(sh^{n_i} H\mathbb{Z}/2, \mathbb{Z}/2) \longrightarrow H^*(MSpin, \mathbb{Z}/2)$$

is an isomorphism.

An easy computation also gives:

5.2.3. Proposition. *Given θ_J and z_i as above the induced map*

$$\bigoplus_J H^*(ko\langle n_J\rangle, \mathbb{Q}) \oplus \bigoplus_i H^*(sh^{n_i} H\mathbb{Z}/2, \mathbb{Q}) \longrightarrow H^*(MSpin, \mathbb{Q})$$

is a bijection.

Since everything in sight is of finite type, one can now apply Serre-class theory to obtain:

5.2.4. Corollary (Anderson-Brown-Peterson-splitting). *The transformation*

$$\Omega_*^{Spin}(-) \longrightarrow \bigoplus_J ko\langle n_J\rangle_*(-) \oplus \bigoplus_i H_{*-n_i}(-, \mathbb{Z}/2)$$

given by a choice of θ_J and z_i as above, is an isomorphism after localisation at 2.

Given such a choice of z_i we obtain (upon further choices) homotopy classes

$$M_2O \longrightarrow K \times sh^{n_i} H\mathbb{Z}/2$$

as follows: Recall the functor $\Theta : \mathcal{S}_K \to \mathcal{S}$, that collapses the base section and that $\Theta(M_2O) = MO$. One then is tempted to proceed as follows: It is well known that the canonical map

$$H^*(MO, \mathbb{Z}/2) \longrightarrow H^*(MSpin, \mathbb{Z}/2)$$

is a surjection, so pick representatives $\hat{z}_i : MO \to sh^{n_i} H\mathbb{Z}/2$ of inverse images of the z_i. Now Θ is left adjoint to $- \times K$ (also on the homotopy category), whence these should correspond to maps $M_2O \to sh^{n_i} H\mathbb{Z}/2 \times K$ as claimed. We do, however, not know whether our spectrum M_2O is cofibrant (meaning that it is not clear that $\Theta(M_2O) = MO$ on homotopy categories). To verify that we still obtain maps as claimed we need to peek into the black box of model structures on parametrised spectra once more. We can cofibrantly resolve M_2O in the level model structure, since it shares cofibrations with the stable structure and has a stronger notion of weak equivalence. However, as we observed in the proof of 3.3.6 the base sections in our spectrum M_2O are cofibrations and so we certainly obtain a levelwise homology equivalence, and hence a weak equivalence.

Note that the parametrised spectrum $K \times H\mathbb{Z}/2$ represents the parametrised homology theory obtained by ignoring the twist and taking singular homology of the total space.

5.2.5. COROLLARY. *If $\overline{\theta}_J : M_2O \longrightarrow k_2o\langle n_J\rangle$ corresponds to a lift of $\overline{\pi}_J \in KO^0(BO)$ in $ko\langle n_j\rangle^0(BO)$ and $\hat{z}_i : M_2O \to K \times sh^{n_i}H\mathbb{Z}/2$ corresponds to a class $\in H^{n_i}(MO, \mathbb{Z}/2)$ that restricts to $z_i \in H^{n_i}(MSpin, \mathbb{Z}/2)$, the induced map*

$$\Omega_*^{Spin}(-,-) \longrightarrow \bigoplus_J ko\langle n_J\rangle_*(-,-) \oplus \bigoplus_i H_{*+n_i}(-,-,\mathbb{Z}/2)$$

is an isomorphism of twisted homology theories after localising at 2.

PROOF. By Theorem 3.1.5 it suffices to verify, is that the statement holds for each point $k \in K$ and as before, for every point the claim reduces to the classical Anderson-Brown-Peterson splitting (albeit in a non-canonical way)! □

6. The cohomology of twisted spin cobordism

6.1. The twisted Steenrod algebra. Using our generalisation of the Anderson-Brown-Peterson splitting we shall now set out to compute the $\mathbb{Z}/2$-cohomology of the twisted spin cobordism spectrum as a module over the twisted Steenrod algebra, which we denote by $\underline{\mathcal{A}}_2$. Before doing so we shall therefore describe $\underline{\mathcal{A}}_2$. We shall from now on suppress $\mathbb{Z}/2$ and related 2's from notation.

6.1.1. DEFINITION. The graded algebra $\underline{\mathcal{A}}$ is defined to be the 'set' of natural transformations $H^*(-,-) \to H^*(-,-)$ commuting with the connecting transformation, where H^* is regarded as a functor $Top_K^2 \to Ab$.

As in the untwisted setting $\underline{\mathcal{A}}$ is an algebra under composition and carries a canonical coproduct coming from the multiplicativity of H^*, making it a cocommutative Hopf algebra. Representability easily yields:

6.1.2. PROPOSITION. *The inclusions $H^*(K) \to \underline{\mathcal{A}}$ (acting via multiplication) and $\mathcal{A} \to \underline{\mathcal{A}}$ induce an isomorphism*

$$H^*(K) \otimes \mathcal{A} \cong \underline{\mathcal{A}}$$

of vector spaces and coalgebras, but not of algebras.

PROOF. What we have to compute is $\lim_n H^{*+n}(K \times K(\mathbb{Z}/2, n), K \times *)$, which by the Künneth-formula is isomorphic to $H^*(K) \otimes \mathcal{A}$, and since the multiplication of $K \times H\mathbb{Z}/2$ is componentwise, this is an isomorphism of coalgebras. The final statement follows from the next proposition. □

The multiplication, however, is also easily described in terms of the isomorphism above.

6.1.3. PROPOSITION. *For $k, l \in H^*(K)$ and $a, b \in \mathcal{A}$, with $\Delta(a) = \sum_i a'_i \otimes a''_i$ we have*

$$(k \otimes a) \cdot (l \otimes b) = \sum_i k \cup a'_i(l) \otimes a''{}_i \circ b$$

PROOF. Given a pair (X, A) with twist ζ, we find for every $x \in H^*(X, A, \zeta)$:

$$\begin{aligned}(k \otimes a)\left((l \otimes b)x\right) &= (k \otimes a)(\zeta^* l \cup b(x)) \\ &= \zeta^* k \cup a(\zeta^* l \cup b(x)) \\ &= \sum_i \zeta^* k \cup a_i'(\zeta^* l) \cup a_i''(b(x)) \\ &= \sum_i \zeta^*(k \cup a_i'(l)) \cup (a_i'' \circ b)(x) \\ &= \sum_i (k \cup a_i'(l) \otimes a_i'' \circ b)\, x\end{aligned}$$

□

6.1.4. REMARK. From these formulas we see that $\underline{A}$ is precisely the semidirect product of A with $H^*(K)$ and the obvious action of A on $H^*(K)$. As usual we conclude that the inclusions $H^*(K) \hookrightarrow \underline{\mathcal{A}}$ and $\mathcal{A} \hookrightarrow \underline{\mathcal{A}}$ are Hopf algebra maps, whereas of the projection maps $\underline{\mathcal{A}} \to \mathcal{A}$ (killing $H^+(K) \otimes \mathcal{A}$) and $\underline{\mathcal{A}} \to H^*(K)$ (killing $H^*(K) \otimes \mathcal{A}_+$) only the first is multiplicative, while both are comultiplicative (since both A and $H^*(K)$ are cocommutative).

To describe our results for $H^*(M_2O)$ we will construct a nontrivial automorphism of a subalgebra of $\underline{A(1)}$, that emulates the changes under Thom isomorphisms: To this end let $A(1)$ denote the subalgebra of A generated by Sq^1 and Sq^2, which indeed is a sub-Hopf-algebra. Similarly, let $\underline{\mathcal{A}(1)}$ denote the subalgebra of $\underline{A}$ generated by $A(1)$ and $H^*(K)$; it is also readily checked to be a sub-Hopf-algebra of $\underline{A}$. Now note that given the spectrum M_2O, we can apply the Thom isomorphism $H^*(M_2O) = H^*(MO) \cong H^*(BO)$. It transforms the $H^*(K)$-action on $H^*(M_2O)$ into the one on $H^*(BO)$ coming from the map $BO \to K$ given by the first and second Stiefel-Whitney class by our convention from right after lemma 3.3.3. The module structure over the Steenrod algebra, however, changes, but the change in the action over $A(1)$ can be described by an automorphism ψ of $\underline{A(1)}$: It is determined by

$$\begin{aligned}\psi(1 \otimes Sq^1) &= 1 \otimes Sq^1 + \iota_1 \otimes 1 \\ \psi(1 \otimes Sq^2) &= 1 \otimes Sq^2 + \iota_1 \otimes Sq^1 + \iota_2 \otimes 1 \\ \psi(k \otimes 1) &= k \otimes 1\end{aligned}$$

and we have $H^*(M_2O) \cong {}_\psi H^*(BO)$, where the lower case ψ denotes pulling back the module structure. For technical reasons it turns out to be more convenient to work with the inverse of ψ for a while, which we denote by φ. We also denote by φ its composition with the inclusion $A(1) \to \underline{A(1)}$. This is given by

$$\begin{aligned}\varphi(Sq^1) &= 1 \otimes Sq^1 + \iota_1 \otimes 1 \\ \varphi(Sq^2) &= 1 \otimes Sq^2 + \iota_1 \otimes Sq^1 + \iota_1^2 \otimes 1 + \iota_2 \otimes 1\end{aligned}$$

The verification that indeed $\varphi \circ \psi = id$ is a little calculation and will be left to the reader.

6.1.5. LEMMA. *The above stipulation indeed defines a unique homomorphism* $\varphi : A(1) \to \underline{A(1)}$ *of algebras. It is a morphism of Hopf algebras.*

Proof. This is just a lengthy computation, since we know that a presentation for $\mathcal{A}(1)$ is given by $(Sq^1)^2 = 0, Sq^1Sq^2Sq^1 = (Sq^2)^2$. We have deferred it to appendix I. □

6.1.6. Lemma. *Extending φ by the identity on $H^*(K)$ produces a Hopf algebra automorphism of $\underline{A(1)}$.*

Proof. To make the extension well-defined, there is again a relation to be checked and we do this in appendix I. The extension is an isomorphism, since it is for example $H^*(K)$-linear and induces the identity on $H^*(K)$-indecomposables. □

6.2. The cohomology of k_2o. We begin with a few simple observations: By the adjointness of Θ and $K \times -$ we find:

$$H^*(E) = H^*(\Theta cE)$$

where the left hand side denotes the twisted cohomology theory represented by $K \times H\mathbb{Z}/2$ and the right hand side the untwisted one represented by $H\mathbb{Z}/2$ (and c is cofibrant resolution). Since we may as well replace all spectra occuring in the following by their cofibrant replacements, we shall suppress it from notation.

6.2.1. Lemma. *The spectrum $\Theta(k_2o)$ is connective with zeroth homotopy group isomorphic to $\mathbb{Z}/2$. The corresponding Postnikov section yields a homotopy class of maps of parametrised spectra*

$$k_2o \longrightarrow K \times H\mathbb{Z}/2$$

This map induces a surjection $\underline{\mathcal{A}} \longrightarrow H^*(k_2o)$ upon passage to cohomology. Furthermore, the Poincaré series of $H^*(k_2o)$ equals the product of that of $H^*(ko)$ and that of $H^*(K)$.

Proof. The statement about the homotopy groups of $\Theta(k_2o)$ follows from the fact that the ABS-orientation $MSpin \to ko$ is an 8-equivalence integrally (and not just 2-locally). By the long exact sequences of the obvious fibration the same holds true for the twisted ABS-orientation and thus for the induced map $MO = \Theta(M_2O) \to \Theta(k_2o)$, where the homotopy groups of the left side are well known.

The second statements follow from the Serre spectral sequences: Stong's calculations of $H^*(BO\langle k\rangle)$ (compare [**St**, Chapter XI, Proposition 6]) show that the map $\mathcal{A} \to H^*(ko)$ induced by 'the' zeroth Postnikov section of ko is a surjection with kernel generated by Sq^1 and Sq^2. The transformation of spectral sequences induced by the Postnikov section $k_2o \longrightarrow K \times H\mathbb{Z}/2$ therefore also is a surjection on the second pages; this is clear once we know that the coefficient system in the spectral sequence for k_2o is constant, which we show below. Since the domain spectral sequence (that of $H^*(K \times H)$) collapses, this implies that we have a surjection on the limit pages and thus a surjection on the abutment, i.e. $\underline{\mathcal{A}} \to H^*(k_2o)$.

Furthermore, the comparison of spectral sequences of k_2o and M_2O shows the final part, since the latter spectral sequence also collapses by the following argument: The Poincaré series of $H^*(M_2O) = H^*(MO)$ is the same as that of the product of $H^*(MSpin)$ with $H^*(K)$, whose tensor product make up the second page of the spectral sequence. Since nontrivial differentials would strictly decrease the Poincaré series, the spectral sequence for M_2O has to collapse and consequently that for k_2o, since it is a summand in that of M_2O. It follows that the Poincaré series of $H^*(k_2o)$ equals that of the second page of its spectral sequence and the proposition follows.

Finally, let us prove that the coefficients in the spectral sequence for M_2O (which in particular implies the same for k_2o) are indeed constant. To this end observe that we have a group homomorphism $\mathbb{Z}/2 \to PG\ell$ that splits the grading map $PG\ell \to \mathbb{Z}/2$ (one such is given by letting the nontrivial element in $\mathbb{Z}/2$ act by swapping the obvious base elements δ_{2n} and δ_{2n+1}). Call the image of the non-trivial element g. Such a split induces an isomorphism on path components and thus an isomorphism on π_1 after taking classifying spaces. Considering the arising diagram

$$\begin{array}{ccc} MSpin'(n) & \longrightarrow & MSpin'(n) \\ \downarrow & & \downarrow \\ E\mathbb{Z}/2 \times_{\mathbb{Z}/2} MSpin'(n) & \longrightarrow & EPG\ell \times_{PG\ell} MSpin'(n) \\ \downarrow & & \downarrow \\ B\mathbb{Z}/2 & \longrightarrow & BPG\ell \end{array}$$

we find that the action of the nontrivial element in $\pi_1(BPG\ell)$ on the cohomology of the fibre is induced by the action of the g on $MSpin'(n)$. By the Thom isomorphism we therefore need to study the action of g on $BSpin'(n)$ and this is given by left multiplication of g on

$$BSpin'(n) = G\ell_n'^{+}/Spin(n) = PG\ell_n'/O(n)$$

This action is covered by a bundle isomorphism of the universal $O(n)$-bundle over $BSpin'(n)$, which will yield the claim since its Stiefel-Whitney classes generate the mod 2 cohomology ring of $BSpin'(n)$. The universal principal $O(n)$-bundle is given by

$$\begin{array}{ccc} G\ell_n'^{+} \times_{Spin(n)} O(n) & \xrightarrow{\cong} & G\ell_n' \times_{Pin(n)} O(n) \\ \downarrow & & \\ BSpin'(n) & & \end{array}$$

and the isomorphism covering the left multiplication by g is again just given by left multiplication with g using the righthandside description of the total space. Note that this does not work for the universal $SO(n)$ or $Spin(n)$ bundles since $Pin(n)$ does not act on these. The coefficient system is indeed non-trivial using integral coefficients (since Pontryagin classes are sensitive to orientations). □

In particular, we find $H^0(k_2o) \cong \mathbb{Z}/2$. Let us denotes by κ the unique non-zero class in that group. Using the splitting of M_2O once more we find:

6.2.2. LEMMA. *We have $\varphi(Sq^1)\kappa = 0$ and $\varphi(Sq^2)\kappa = 0$.*

PROOF. This seems difficult to do directly as we know of no way to track Steenrod operations through the Serre spectral sequence. However, since the ABS-orientation is injective, we can check $\varphi(Sq^{1,2})u = 0$ in $H^*(M_2O) = H^*(MO)$, where the computation reads:

$$\varphi(Sq^1)u = Sq^1(u) + \iota_1 u = w_1 u + w_1 u = 0$$
$$\varphi(Sq^2)u = Sq^2(u) + \iota_1 Sq^1(u) + \iota_1^2 u + \iota_2 u = w_2 u + w_1^2 u + w_1^2 u + w_2 u = 0$$

□

6.2.3. THEOREM. *There is a unique non-trivial class* $\kappa_{8n} \in H^{8n}(k_2o\langle 8n\rangle)$ *and the map* $sh^{-8n}\underline{\mathcal{A}} \to H^*(k_2o\langle 8n\rangle)$ *given by evaluation on* κ_{8n} *induces an isomorphism*

$$sh^{-8n}\underline{\mathcal{A}}_\varphi \otimes_{\mathcal{A}(1)} \mathbb{Z}/2 \to H^*(k_2o\langle 8n\rangle)$$

of left $\underline{\mathcal{A}}$*-modules.*

PROOF. Bott periodicity immediately reduces the claim to the case $n = 0$. By the above lemma the map indeed factors through $\underline{\mathcal{A}}/\varphi(Sq^{1,2})$ and it is surjective by the proposition above that. We will thus be done once we have computed the Poincaré series of both sides. For the right hand side this was also done in the lemma above, and for the left hand side all that remains to be noted is that the Poincaré-series of $\underline{\mathcal{A}}_\varphi \otimes_{\mathcal{A}(1)} \mathbb{Z}/2$ agrees with that of $\underline{\mathcal{A}} \otimes_{\mathcal{A}(1)} \mathbb{Z}/2$ by two applications of the Milnor-Moore theorem and that the latter is indeed the tensor product of $H^*(K)$ and $\mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{Z}/2$ at least as a $\mathbb{Z}/2$ vector space. At this point we are done since $H^*(ko) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{Z}/2$ by Stong's calculation.

□

By similar arguments we can also deal with $k_2o\langle 2\rangle$:

6.2.4. THEOREM. *There is a unique non-trivial class* $\kappa_{8n+2} \in H^{8n+2}(k_2o\langle 8n + 2\rangle)$ *and the map* $sh^{-(8n+2)}\underline{\mathcal{A}} \to H^*(k_2o\langle 8n + 2\rangle)$ *given by evaluation on* κ_{8n+2} *induces an isomorphism*

$$sh^{-(8n+2)}\underline{\mathcal{A}}_\varphi \otimes_{\mathcal{A}(1)} \mathcal{A}(1)/Sq^3 \to H^*(k_2o\langle 8n + 2\rangle)$$

of left $\underline{\mathcal{A}}$*-modules.*

PROOF. By Bott periodicity it suffices again to consider a single value of n, which we choose to be 1, since $k_2o\langle 10\rangle$ 'is a summand of M_2O'. In particular, the Serre spectral sequence of $k_2o\langle 10\rangle$ has constant coefficients (as observed in the proof of 6.2.1), giving us the element κ_{10}. The computation of the Poincaré series is also basically the same as in the previous argument, however, $\varphi(Sq^3)\kappa_{10} = 0$ does not follow directly, since we do not know the image of κ_{10} in $H^{10}(M_2O)$ under a lift of $\overline{\theta}_3 : M_2O \to k_2o\langle 10\rangle$. To proceed we shall decompose $\overline{\theta}_3$ into its components:

$$M_2O \xrightarrow{\Delta} BO_+ \wedge M_2O \xrightarrow{\overline{\theta}_3 \wedge \hat{\alpha}} ko\langle 10\rangle_0 \wedge k_2o \xrightarrow{\mu} k_2o\langle 10\rangle$$

Because $\mu^*\kappa_{10}$ clearly equals $\lambda_{10} \times \kappa$, where λ_{10} refers to the fundamental class in $H^{10}(ko\langle 10\rangle_0)$ (which makes sense since $ko\langle 10\rangle_0$ can be chosen 9-connected with

10th homotopy group isomorphic to $\mathbb{Z}/2$), the following calculation gives the claim:

$$\begin{aligned}
\varphi(Sq^3)(\lambda_{10}\times\kappa) &= \varphi(Sq^1)\varphi(Sq^2)(\lambda_{10}\times\kappa)\\
&= \varphi(Sq^1)\Big((1\otimes Sq^2)(\lambda_{10}\times\kappa) + (\iota_1\otimes Sq^1)(\lambda_{10}\times\kappa)\\
&\quad + (\iota_1^2\otimes 1)(\lambda_{10}\times\kappa) + (\iota_2\otimes 1)(\lambda_{10}\times\kappa)\Big)\\
&= \varphi(Sq^1)\Big(\big[Sq^2(\lambda_{10})\times\iota + \underbrace{Sq^1(\lambda_{10})\times Sq^1(\kappa)}_{=\cancel{Sq^1(\lambda_{10})\times\iota_1\kappa}} + \underbrace{\lambda_{10}\times Sq^2(\kappa)}_{=\cancel{\lambda_{10}\times\iota_2\kappa}}\big]\\
&\quad + \big[\cancel{Sq^1(\lambda_{10})\times\iota_1\kappa} + \underbrace{\lambda_{10}\times\iota_1 Sq^1(\kappa)}_{\cancel{\lambda_{10}\times\iota_1^2\kappa}}\big] + \cancel{\lambda_{10}\times\iota_1^2\kappa} + \cancel{\lambda_{10}\times\iota_2\kappa}\Big)\\
&= (1\otimes Sq^1)(Sq^2(\lambda_{10})\times\kappa) + (\iota_1\otimes 1)(Sq^2(\lambda_{10})\times\kappa)\\
&= \big[\underbrace{Sq^1Sq^2(\lambda_{10})\times\kappa}_{=0} + \underbrace{Sq^2(\lambda_{10})\times Sq^1(\kappa)}_{=Sq^2(\lambda_{10})\times\iota_1\kappa}\big] + Sq^2(\lambda_{10})\times\iota_1\kappa\\
&= 0
\end{aligned}$$

where $Sq^1Sq^2(\lambda_{10}) = 0$ by Stong's calculations of $H^*(BO\langle n\rangle)$. □

With these results it seems natural to guess:

6.2.5. Conjecture. Generalising Stong's computation of $H^*(ko\langle n\rangle)$ we find

$$sh^{-(8n+1)}\underline{\mathcal{A}(1)}/\varphi(Sq^2) \xrightarrow{\cong} H^*(k_2o\langle 8n+1\rangle)$$
$$sh^{-(8n+4)}\underline{\mathcal{A}(1)}/\varphi(Sq^{1,5}) \xrightarrow{\cong} H^*(k_2o\langle 8n+4\rangle)$$

as $\underline{\mathcal{A}(1)}$-modules.

However, the arguments used above break down. Another approach would be to use the fact that $ko\langle n\rangle \simeq X_n \wedge ko$ for some small space X_n and extend the map $X_n \to ko\langle n\rangle$ to an equivalence $k_2o\langle n\rangle \simeq X_n \wedge k_2o$ using the ko-module structure on $k_2o\langle n\rangle$. We have not investigated the details of this though.

6.3. The cohomology of M_2O. Our generalised Anderson-Brown-Peterson splitting allows us to compute the cohomology of M_2O as a module over the twisted Steenrod algebra. The isomorphism, however, we cannot uniquely specify just as Anderson, Brown and Peterson could not determine theirs due to the non-uniqueness of lifts in theorem 5.1.2.

6.3.1. Corollary. *Any choice of splitting from 5.2.4 determines an isomorphism*

$$H^*(M_2O) \cong \bigoplus_{\substack{J,n(J)\\ even}} sh^{-n_J}\underline{\mathcal{A}}/\varphi(Sq^{1,2}) \oplus \bigoplus_{\substack{J,n(J)\\ odd}} sh^{-n_J}\underline{\mathcal{A}}/\varphi(Sq^3) \oplus \bigoplus_i sh^{-n_i}\underline{\mathcal{A}}$$

of modules over the twisted Steenrod algebra.

In particular, we see that $H^*(M_2O) \cong \underline{\mathcal{A}}_\varphi \otimes_{\mathcal{A}(1)} M$ for the $\mathcal{A}(1)$-module M given as a direct sum of $\mathbb{Z}/2$'s, jokers and free modules according to the above decomposition. Since Anderson, Brown and Peterson showed $H^*(MSpin) \cong \mathcal{A}_\varphi \otimes_{\mathcal{A}(1)} M$ for that very same M, we will call it the ABP-module in the second chapter for easier reference.

This explicitly given structure as an extended module over the twisted Steenrod algebra can be used to get a hand on the second page of a $K \times H\mathbb{Z}/2$-based Adams spectral sequence. We hope to take advantage of this to approach conjecture 4.3.2 along the lines of Stolz' arguments in [**St 94**], which uses the Adams spectral sequence as a principal tool. In the second chapter we start with some preliminary investigations.

6.4. A final observation. We saw in remark3.3.7 that $H_*(M_2O)$ (and similarly $H_*(k_2o)$) carries a multiplicative structure, agreeing with that of $H_*(MO)$. Dualising our cohomological calculations from above one can perform some low-dimensional calculations and find:

6.4.1. PROPOSITION. *The homology of k_2o contains an element $0 \neq y \in H_2(k_2o)$ with $y^4 = 0$. In particular, there can be no injective ring homomorphism $H_*(k_2o) \to H_*(M_2O)$.*

PROOF. In order to carry out this proof we use the description of $H_*(K)$ from proposition 4.1.1. This does not create any circular argument, since the present proposition is only ever used to explain why certain approaches cannot work.
From 4.1.1 we know that $H_*(K) \cong \mathbb{Z}/2[x_{2^k} : k \in \mathbb{N}, x_{2^k+2^l} : k > l]/(x_{2^k}^4, x_{2^k+2^l}^2)$. From this description it is clear that the element $y_2 \in H_2(K)$ dual to ι_1^2 satisfies $x_2^4 = 0$ and so does the element $x_1 \in H_1(K)$, which is the dual of ι_1. We proceed by noting that the element $x_1 \otimes \zeta_1 + x_2 \otimes 1 \in \underline{\mathcal{A}}_2^*$ lies in the image of $H_2(k_2o)$ under the Postnikov section, where ζ_1 is dual to Sq^1: All we need to verify is that $\langle a, x_1 \otimes \zeta_1 + x_2 \otimes 1\rangle = 0$ for $a = (\iota_1 \otimes 1)\varphi(Sq^1), (1 \otimes Sq^1)\varphi(Sq^1)$ and $\varphi(Sq^2)$, which form a generating system for $(\underline{\mathcal{A}} \cdot Sq^{1,2})_2$. Since $(\iota_1 \otimes 1)\varphi(Sq^1) = \iota_1^2 \otimes 1 + \iota_1 \otimes Sq^1 = (1 \otimes Sq^1)\varphi(Sq^1)$ the computations

$$\langle \iota_1^2 \otimes 1 + \iota_1 \otimes Sq^1, x_1 \otimes \zeta_1 + x_2 \otimes 1\rangle = 1 + 1 = 0$$
$$\langle 1 \otimes Sq^2 + \iota_1 \otimes Sq^1 + \iota_1^2 + \iota_2, x_1 \otimes \zeta_1 + x_2 \otimes 1\rangle = 0 + 1 + 1 + 0 = 0$$

suffice for this. The preimage y of $x_1 \otimes \zeta_1 + y_2 \otimes 1$ (the map induced in cohomology is surjective by 6.2.1, so in homology it is injective) then meets our requirements, since the multiplication in $\underline{\mathcal{A}}^*$ is commutative. □

This observation is in stark contrast to the situation in the untwisted case: In [**St 92**] Stolz used the observation of Pengelley, that indeed there is a ring split of the Atiyah-Bott-Shapiro orientation in homology, to produce a structure theorem for the homology of $MSpin$-module spectra. In fact, in [**St 94**] he even produced a map of spectra inducing such a split.

CHAPTER 2

The twisted $\mathbb{H}P^2$-transfer

1. Introduction

1.1. History and motivation. As described in the introduction of the first chapter our interest in twisted *Spin*-cobordism groups arises from their connection to the existence of metrics of positive scalar curvature. The case of spin manifolds in conjecture 4.3.2 from the first chapter was proven by Führing and Stolz. To state their results we need to rephrase the conjecture as the degree-greater-than-4 case of

$$ker\big(\hat{\alpha} : \Omega_*^{Spin}(B\pi, w) \to ko_*(B\pi, w)\big) \subseteq \Omega_*^{Spin}(B\pi, w)^+$$

for every finitely presented group π, and every $H^{1,2}(-, \mathbb{Z}/2)$-twist w on $B\pi$, where $\Omega_*^{Spin}(-)^+ \subseteq \Omega_*^{Spin}(-)$ denotes those elements which can be represented by manifolds admitting positive scalar curvature metrics. In this rather algebraic formulation we can split up the statement by localising at and inverting 2, respectively and Stolz proved the 2-primary and Führing the odd-primary part.
We now set out to explain the method Stolz employed and indicate how one might hope it generalises to include the case of twisted manifolds. One should also be able to adapt Führing's arguments, but we will not discuss that here as his proof is by different, far more geometric methods.
Since explicit generators even for the *Spin*-cobordism ring are hard to come by (and to our best knowledge still are not entirely known), Stolz' idea was to consider total spaces of fibre bundles with fixed fibre F (and indeed structure group G) over varying base spaces. For appropriately chosen fibre these will always admit metrics of positive scalar curvature. The entirety of such bundles can generically be described as the image of the following transfer map

$$\begin{aligned} T_{G,F} : \Omega_*(X \times BG) &\longrightarrow \Omega_{*+dim(F)}(X) \\ [M, (f, g) : M \to X \times BG] &\longmapsto [g^*(EF), f \circ p : g^*(EF) \to M \to X] \end{aligned}$$

where $EF = EG \times_G F$ denotes the universal bundle over BG with fibre F. To impose decorations like twists and *Spin*-structures one needs additional assumptions on G and F of course. For the problem at hand Stolz chooses F as $\mathbb{H}P^2$ and $G = Isom(\mathbb{H}P^2)$, where the isometry group (which in fact is $PSp(3)$) is taken with respect to the Fubini-Study metric on $\mathbb{H}P^2$, which has positive scalar curvature. He then goes on to show that the vertical tangent bundle of $EF \to BG$ admits a *Spin*-structure, wherefore the above map makes sense with *Spin*-decorations. We will review this in a moment. The 2-primary part of 4.3.2 for spin manifolds now follows from

Theorem (Stolz 1994). *We have*

$$ker\big(\alpha : \Omega_*^{Spin}(X) \to ko_*(X)\big)_{(2)} \subseteq im\big(T : \Omega_{*-8}^{Spin}(X \times BPSp(3)) \to \Omega_*^{Spin}(X)\big)_{(2)}$$

for every space X.

Following some preliminaries Stolz' proof of this result consists of two main steps. The preliminiaries are that the transfer map is induced by an explicitly given map of spectra

$$MSpin \wedge S^8 \wedge BG_+ \longrightarrow MSpin$$

and that the composition

$$MSpin \wedge S^8 \wedge BG_+ \longrightarrow MSpin \xrightarrow{\alpha} ko$$

is nullhomotopic by the Atiyah-Singer index theorem (compare [**St 92**, Section 1 & 2]). Therefore, one can pick a lift $t : MSpin \wedge S^8 \wedge BG_+ \to hofib(\alpha)$ of the transfer map. Stolz then proceeds with the following two results:

Theorem (Proposition 1.3 [**St 92**]). *Any such t induces a injection in mod 2-cohomology, which splits over $\mathcal{A}$.*

Theorem (Proposition 8.3 [**St 94**]). *Any split from the previous theorem is realised by a map of (2-localised) spectra.*

From these two results it follows immediately that t induces a split surjection of homology theories, which has the desired statement as a corollary.

1.2. Statement of results and organisation of the chapter. In the second part of this thesis we concern ourselves with the generalisation of the results stated above. It is easy to set up a twisted transfer map

$$M_2O \wedge S^8 \wedge BG_+ \longrightarrow M_2O$$

with the analogous geometric interpretation as in the *Spin*-case. We do this in the second section. In the third we review Stolz' approach to the first step and reduce its generalisation to a certain conjecture about the $\mathcal{A}(1)$-module structure of $\overline{H^*(BSO)}$ (a certain submodule of $H^*(BSO)$), which the main results of the first chapter provide strong evidence for. Along the way we close a small gap that was left in [**St 92**]. In the fifth section we discuss a simplification of Stolz' arguments in [**St 94**] leading to his proof of the second step. We reinterpret several prerequisites from it and thereby decrease the overall complexity of the argumentation. For a generalisation of Stolz' proof, these simplifications should ultimately prove essential; we show, that his arguments can entirely be reduced to algebra once basic facts about the Adams spectral sequence are taken into account. We also discuss how a verification of the conjecture from the second part would lead to an entryway for such a generalisation.
However, in the fourth section, we unfortunately and very surprisingly disprove our conjecture by an explicit computation, showing that the gap we closed for the untwisted case in the third section is indeed a chasm in the general case.
All in all this leaves part of the analysis of the twisted transfer map to future work.

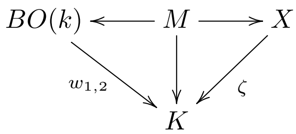
2. The twisted transfer

In this very short section we will sketch the construction of the twisted transfer map, we refer the reader to the exposition in [**St 92**] and the references therein for details and proofs, which work equally well in the parametrised setting.
Recall that given a smooth manifold bundle $p : E \to B$ with compact k-dimensional fibres there is an associated stable Pontryagin-Thom map $S^k \wedge B_+ \to M(-T_vp)$

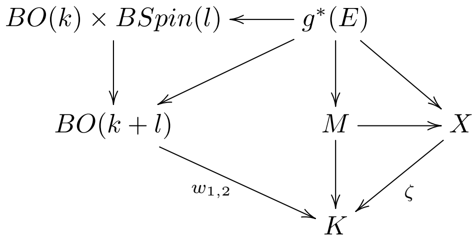
for some choice of inverse of the vertical tangent bundle T_vp of p. If this inverse of the vertical tangent bundle comes equipped with a spin structure one obtains a homotopy class $M(-T_vp) \to MSpin$. Using these two maps we can form the following composition:

$$M_2O \wedge S^k \wedge B_+ \longrightarrow M_2O \wedge M(-T_vp) \longrightarrow M_2O \wedge MSpin \longrightarrow M_2O$$

Interjecting this with the Thom diagonal $M(-T_vp) \to M(-T_vp) \wedge E_+$ produces the usual integration along the fibres, but we shall not need that here. Just as in the unparametrised situation the geometric interpretation of the above map is simple enough: Suppressing the additional bundle data for legibility, it sends a cycle



together with a map $g : M \to B$ to the diamond shaped part in



where the map $g^*(E) \to BO(k) \times BSpin(l)$ is given by $g^*(E) \to M \to BO(k)$ in the first coordinate and a classifying map for some destabilisation of $-T_vp$ over $g^*(E)$ in the second. Note that for some large enough l all such destabilisations produce the same fibre-homotopy class of maps $g^*(E) \to BO(k+l)$, so the class represented by this cycle does not depend on the choice made. Furthermore, by our construction of the $BO(i)$ in the first chapter the diagram still commutes strictly. We want to apply this to the bundle mentioned in the introduction: Let $G = PSp(3) = Isom(\mathbb{H}P^2)$ and consider $EG \times_G \mathbb{H}P^2 \to BG$. We obtain our transfer map

$$T : M_2O \wedge S^8 \wedge BG_+ \longrightarrow M_2O$$

The image of this transfer in $\Omega_*^{Spin}(X, \zeta)$ will consist entirely of manifolds admitting metrics of positive scalar curvature, since they are total spaces of bundles over compact manifolds, whose fibres $\mathbb{H}P^2$ admit such metrics. Finally, we have the following evident generalisation:

PROPOSITION. *The composition*

$$M_2O \wedge S^8 \wedge BG_+ \xrightarrow{T} M_2O \xrightarrow{\hat{\alpha}} k_2o$$

is nullhomotopic.

PROOF. Since $\hat{\alpha}$ is an $MSpin$-module map, we find the above map equal to

$$M_2O \wedge S^k \wedge BG_+ \longrightarrow M_2O \wedge M(-T_vp) \longrightarrow M_2O \wedge MSpin \xrightarrow{\hat{\alpha}\wedge\alpha} k_2o \wedge ko \longrightarrow k_2o$$

However, by the Atiyah-Singer index theorem for families the composition $S^k \wedge BG_+ \to M(-T_vp) \to MSpin \to ko$ is null, since it represents the family index of a bundle with uniformly positive scalar curvature in the fibres, as explained in [**St 92**, Section 2]. □

This concludes our preliminaries on the twisted transfer.

3. Splitting the transfer

In order to utilise the structural results obtained in the first chapter for an analysis of the transfer, it is necessary to produce a version of the isomorphism $H^*(M_2O) \cong \underline{\mathcal{A}}_\varphi \otimes_{\mathcal{A}(1)} M$, where M is a direct sum of $Z/2$'s, Jokers and a free module, that is respected by the transfer map. This is unclear for any isomorphism coming from a choice of KO-theory Pontryagin classes, as was already observed by Stolz in the untwisted case. He resolved this problem by introducing such an isomorphism for $MSpin$-module spectra, that is natural in $MSpin$-module maps. Since the transfer is $MSpin$-linear he was able to proceed. Proposition 6.4.1 of the last chapter, however, prohibits us from directly expanding his methods.
What we obtain from his results is an isomorphism

$$H^*(M_2O) \cong \underline{\mathcal{A}} \otimes_{\underline{\mathcal{A}(1)}} \overline{H^*(M_2O)}$$

for some $\mathcal{A}(1)$-module $\overline{H^*(M_2O)}$ functorially associated to $H^*(M_2O)$, that extends to a natural isomorphism. We shall see that one way of interpreting the results from the previous section is that

$$H^*(M_2O) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} (H^*(MV) \otimes M)$$

for some $\underline{\mathcal{A}(1)}$-module $H^*(MV)$. Therefore, it is very natural to guess that

$$\overline{H^*(M_2O)} \cong H^*(MV) \otimes M$$

with diagonal action. We had originally hoped to formally derive this conjecture from our version of the Anderson-Brown-Peterson-splitting above, just as Stolz seems to deduce the analogous result in the untwisted case ([**St 92**, Corollary 6.4]). However, it became apparent, that this deduction constitutes a small gap in [**St 92**], which we set out to fill. Unfortunately, it turns out that the fix we establish does not extend to the twisted case and, using the classical technique of Q_0 and Q_1-homology, we then analyse $\overline{H^*(M_2O)}$ far enough to ultimately disprove our conjecture, after it passes several consistency checks. This surprising fact leaves part of the analysis of the twisted transfer map to future work.

3.1. Stolz' decomposition of the homology of $MSpin$-module spectra. Building on work of Pengelley Stolz constructs a map $s : H^*(MSpin) \to H^*(ko)$ in [**St 92**], which is $\mathcal{A}$-linear, comultiplicative and a one-sided invers of the ABS-orientation. This makes the cohomology of any $MSpin$-module spectrum X into a comodule over the coalgebra $H^*(ko)$. He then shows:

3.1.1. THEOREM (Section 5 [**St 92**]). *The $H^*(ko)$-primitives $\overline{H^*(X)}$ form an $\mathcal{A}(1)$-submodule of $H^*(X)$ and the inclusion $\overline{H^*(X)} \to H^*(X)$ induces an isomorphism*

$$\mathcal{A} \otimes_{\mathcal{A}(1)} \overline{H^*(X)} \to H^*(X)$$

if the cohomology of X is bounded below and locally finite.

Since $MSpin$-module maps are in particular $H^*(ko)$-(co)linear, the naturality of this isomorphism is clear. Stolz now concludes $\overline{H^*(MSpin)} \cong M$ from $\mathcal{A} \otimes_{\mathcal{A}(1)} M \cong H^*(MSpin) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} \overline{H^*(MSpin)}$, where the first isomorphism comes from the work of Anderson, Brown and Peterson. However, a priori this is unclear (as S. Stolz agreed in private conversation), since the combined isomorphism has no reason to come from a map between $\overline{H^*(MSpin)}$ and M (which would suffice since extensions of Hopf-algebras are one-sided free and hence faithfully flat). Due to the special nature of the modules that arise in M (whose structure is known), we show, that this can be corrected, and the original isomorphism deformed into one of the form $id \otimes f$, to which we can apply the bracketed reasoning. The method used however, is easily seen to break down in the case of $\overline{H^*(M_2O)}$ and $H^*(MV) \otimes M$. Let us remark here, that a direct analysis of $\overline{H^*(MSpin)}$ or rather its dual is also feasible, and well within the methods of the later parts of [**St 92**], however, it is not carried out. It is such a direct (and somewhat painful) analysis that will lead to the disprove of our conjecture for the twisted case.

Let us now fill this gap:

3.1.2. PROPOSITION. *Let A be a connected Hopf algebra of finite type over a field k, $B \subseteq A$ a sub-Hopf-algebra, M, N be two graded connected modules of finite type over B and F a free B-module of finite type. If now*

$$g : A \otimes_B N \longrightarrow A \otimes_B (M \oplus F)$$

is an isomorphism of A-modules, then N contains a direct summand N' with complement isomorphic to F such that $A \otimes_B M \cong A \otimes_B N'$.

PROOF. Working from the lowest nontrivial degree of F upwards, it obviously suffices to consider the case with F of rank one. So let f be a basis element for F, $\{1\} \cup \{b_i : i \in I\}$ a right basis of A over B and consider

$$g^{-1}(1 \otimes f) = 1 \otimes n + \sum_i b_i \otimes n_i$$

Here we have $n \neq 0$, since otherwise the right hand side would be decomposable over A. Put $X = B \cdot n \subseteq N$; we claim that X is free with a complement we can choose as N'. To see this consider the following B-linear map

$$N \xrightarrow{g} A \otimes_B (M \oplus F) \xrightarrow{pr} A \otimes_B F = sh^{|f|}A \longrightarrow sh^{|f|}B$$

where the final arrow depends on a choice of left (!) B-complement of B in A. It maps n first to $g(1 \otimes n) = 1 \otimes f + \sum_i b_i \varphi(1 \otimes n_i)$. Now since every b_i is in strictly positive degree, the elements $\varphi(1 \otimes n_i)$ lie in strictly lower degree than $1 \otimes f$ and, therefore, in $A \otimes_B M$. So all in all n is mapped to 1 and thus forms a basis of X. The kernel of this composition is then a complement to X and we choose it as N'. Finally, in this decomposition of N the map induced by g on indecomposables

$$Q_B(N') \oplus k \cdot n = Q_A(A \otimes_B (N' \oplus X)) \xrightarrow{Q_A(g)} Q_A(A \otimes_B (M \oplus F)) = Q_B(M) \oplus k \cdot f$$

is of the form

$$\begin{pmatrix} * & 0 \\ * & 1 \end{pmatrix}$$

so the composition

$$A \otimes_B N' \longrightarrow A \otimes_B (N' \oplus X) \longrightarrow A \otimes_B (M \oplus F) \longrightarrow A \otimes M$$

(which makes up the upper left corner) induces an isomorphism on indecomposables, hence is an isomorphism because the Poincaré series of both sides agree by the Milnor-Moore theorem. □

3.1.3. Remark. We used the freeness of F only two times in the proof: The first time via the existence of a B-linear map $A \otimes_B F \to B \otimes_B F$, which in the above proof is given by a choice of left complement of B in A, say L. If F was only cyclic with annihilator I, such a map does exist if and only if $L \cdot I \subseteq L + I$, a nontrivial condition. The second time is the fact that the arising surjection onto F is automatically an isomorphism. This can be generalised by requiring I to be minimal among all proper annihilators of elements in $M \oplus F$. So in general given a cyclic summand F with annihilator I and an isomorphism

$$A \otimes_B (M \oplus F) \cong A \otimes_B N$$

we can split a copy of F off N if I is minimal among all proper annihilators and we can find a left B-complement L of B in A with $LI \subseteq L + I$.

With this remark the above proposition suffices to close the gap in Stolz' argument, since first cancelling free summands leaves us with jokers and $\mathbb{Z}/2$'s. Then the remark applies to cancel jokers, since a left complement of $\mathcal{A}(1)$ in A starts in degree four, so $L \cdot (Sq^3)$ starts in degree 7 and $L_{\geq 7} = \mathcal{A}_{\geq 7}$. This leaves us with $\mathbb{Z}/2$-summands only, for which computation shows the remark to apply as well. However, once the free summands are cancelled one can argue a little more cleanly:

3.1.4. Proposition. *Let A, B and M, N be as above and $g : A \otimes_B N \to A \otimes_B M$ be an A-linear map. Let furthermore $R \subseteq A$ be a right sub-B-module of A, that is a complement of $B \subseteq A$ and $\pi : A \to B$ the associated right-B-linear projection (these exist by the Milnor-Moore theorem). If now $B \cdot R \subseteq ann(M)$, the map $h : N \to M$ given by*

$$N \longrightarrow A \otimes_B N \xrightarrow{g} A \otimes_B M \xrightarrow{\pi \otimes id} B \otimes_B M \cong N$$

is B-linear and the diagram

$$\begin{array}{ccc} Q_A(A \otimes_B N) & \xrightarrow{\overline{g}} & Q_A(A \otimes_B M) \\ \uparrow \cong & & \uparrow \cong \\ Q_B(N) & \xrightarrow{\overline{h}} & Q_B(M) \end{array}$$

commutes, where Q denotes indecomposoables. In particular, if g is an isomorphism, so is h.

Proof. To verify B-linearity write $g(1 \otimes n) = 1 \otimes m + \sum_i r_i \otimes n_i$ with $b_i \in R$. Then $bg(1 \otimes n) = 1 \otimes bm$ whereas $g(1 \otimes bn) = 1 \otimes bm + \sum_i br_i \otimes n_i$ which projects to the same thing precisely under the above assumption.
The other two statements are hopefully obvious. □

3.1.5. Remark. Note the curious similarity/difference between the two conditions $LI \subseteq L + I$ and $BR \subseteq R + I$. It may however well happen, that in case both remark 3.1.3 and proposition 3.1.4 apply they produce different isomorphisms, as the first construction changes the given map on indecomposables, whereas the second does not!

We can apply these to $\overline{H^*(MSpin)}$ and the Anderson-Brown-Peterson module M, to first take out all the free summands of the right and split off corresponding free summands on the left. The remainder on the right side is then a direct sum of $\mathbb{Z}/2$'s and jokers. In this situation the second proposition applies: The Milnor-Moore theorem tells us that any set of representatives for the (right) $\mathcal{A}(1)$-indecomposables consitutes a right basis for $\mathcal{A}$, so in low degrees one can choose the set $\{1, Sq^4, Sq^6, Sq^7 ...\}$ and, therefore, let L be generated by $\{Sq^4, Sq^6, Sq^7, ...\}$. The set $\pi(\mathcal{A}(1) \cdot L)$ is nontrivial in degree five for the first time, the nontrivial element being $Sq^2Sq^1Sq^2 = Sq^4Sq^1 + Sq^5$, since $Sq^1Sq^4 = Sq^5 = Sq^2Sq^1Sq^2 + Sq^4Sq^1$ (its other nontrivial element is $Sq^2Sq^2Sq^2 = Sq^5Sq^1$ in degree six). However, all elements of degree greater than four obviously act trivially on a sum of jokers and $\mathbb{Z}/2$'s.
It is a rather painful but in the end trivial calculation that a similar argument cannot work after replacing everything in sight by its twisted versions.

3.2. The twisted transfer in cohomology. Proposition 6.4.1 of chapter one prevents us from reducing questions about the twisted transfer to questions about $\mathcal{A}(1)$-modules using the same ideas as above, namely the $H^*(k_2o)$-primitives of an M_2O-module spectrum (note that at present it formally does not even make sense to talk about M_2O-module spectra, since we have not constructed M_2O as a ring spectrum). One therefore has to make do with the $H^*(ko)$-colinearity of the transfer.

3.2.1. PROPOSITION. *For any parametrised $MSpin$-module spectrum $\underline{X \text{ over } K}$, the primitives form an $\underline{\mathcal{A}(1)}$-submodule of $H^*(X)$ and the inclusion map $\overline{H^*(X)} \to H^*(X)$ induces an isomorphism*

$$\underline{\mathcal{A}} \otimes_{\underline{\mathcal{A}(1)}} \overline{H^*(X)} \to H^*(X)$$

of $\underline{\mathcal{A}}$-modules.

PROOF. One can either refer back to Stolz' general statement [**St 92**, (5.2) & Proposition 5.4], which applies immediately, or one can reduce to the untwisted case by the following argument: Since the homology-ko-module structure of X is given by a map over K the comultiplication Δ is $H^*(K)$-linear, so we obtain

$$\Delta(ku) = k\Delta(u) = k(1 \times u) = 1 \times ku$$

For an $\underline{\mathcal{A}(1)}$-module N we in general have that the canonical map

$$\mathcal{A} \otimes_{\mathcal{A}(1)} N \to \underline{\mathcal{A}} \otimes_{\underline{\mathcal{A}(1)}} N$$

is an isomorphism: Both sides have equal Poincaré series and the map is surjective, since any element in $k \otimes \alpha \in \underline{\mathcal{A}}$ can be written as a sum of products $(1 \otimes \alpha_i)(k_i \otimes 1)$ using the antipode. □

Now consider the twisted transfer and the twisted Atiyah-Bott-Shapiro orientation

$$M_2O \wedge BG_+ \wedge S^8 \longrightarrow M_2O \xrightarrow{\alpha} k_2o$$

By the Atiyah-Singer index theorem for families this composition is nullhomotopic (compare [**St 92**, Section 2 & Proposition 1.1]) and we can pick a lift

$$T : M_2O \wedge BG_+ \wedge S^8 \longrightarrow hofib(\underline{\alpha})$$

Since $\underline{\alpha}$ induces an injection on cohomology we find

$$H^*(hofib(\underline{\alpha})) = coker\ \hat{\alpha}^* = \underline{\mathcal{A}} \otimes_{\underline{\mathcal{A}(1)}} coker\ \overline{\hat{\alpha}^*}$$

In order to show that the twisted transfer splits over $\underline{\mathcal{A}}$ in homology it thus suffices to demonstrate that the induced map $coker\ \overline{\hat{\alpha}^*} \to \overline{H^{*-8}(M_2O \wedge BG_+)} \cong \overline{H^*(M_2O)} \otimes H^{*-8}(BG)$ splits over $\underline{\mathcal{A}(1)}$. This map fits into the following diagram:

$$\begin{array}{ccc} & \overset{?}{\dashleftarrow} & \\ coker\ \overline{\hat{\alpha}^*} & \longrightarrow & \overline{H^*(M_2O)} \otimes H^{*-8}(BG) \\ \downarrow & \overset{s}{\longleftarrow} & \downarrow \\ coker\ \overline{\alpha^*} & \longrightarrow & \overline{H^*(MSpin)} \otimes H^{*-8}(BG) \end{array}$$

where s is the split Stolz constructed in [**St 92**, Proposition 7.5] and the question mark indicates the desired generalised split.

As a next step we reinterpret the computation of $H^*(M_2O)$ from section 6 of chapter 1 to fit this program: To this end first of all recall the Hopf algebra maps $\varphi, \psi \mathcal{A}(1) :\to \underline{\mathcal{A}}$ and define

$$H^*(MV) := {}_{\psi}H^*(K)$$

where $H^*(K)$ carries the obvious $\underline{\mathcal{A}(1)}$-module structure.

3.2.2. Remark. This new module structure on $H^*(K)$ comes from the following gedankenexperiment: Suppose there was a vector bundle $V \to K$ with $w_1(V) = \iota_1$ and $w_2(V) = \iota_2$. Then the $\underline{\mathcal{A}(1)}$-module structure of the Thom space MV of V would be fixed and given exactly as above by the very definition of ψ. In this case one might hope

$$MV \wedge MSpin \to MO \wedge MO \to MO$$

to be a 2-local equivalence, since it might just induce an isomorphism on $\mathbb{Z}/2$-cohomology. Of course such a bundle cannot exist, indeed there is no map $H^*(MO) \to H^*(MV)$ that is $\underline{\mathcal{A}(1)}$-linear and an isomorphism in degree 0, as one can easily verify by low dimensional computations. Note that the next proposition tells us, that to some extent this idea may be salvaged (whereas theorem 4.2.1 limits this extent once more).

Following the remark, we shall denote elements in $H^*(MV)$ by ku instead of just k.

3.2.3. Lemma. *Given a left $\mathcal{A}(1)$-module M the inclusion*

$$M \longrightarrow H^*(MV) \otimes M, \qquad m \longmapsto u \otimes m$$

(which is not $\mathcal{A}(1)$-linear) extends to an isomorphism

$$\underline{\mathcal{A}(1)}_{\varphi} \otimes_{\mathcal{A}(1)} M \cong H^*(MV) \otimes M$$

of $\underline{\mathcal{A}(1)}$-modules, where the right hand side is endowed with the diagonal action by $\underline{\mathcal{A}(1)}$ (with trivial $H^*(K)$-action on M).

PROOF. In general the map $\varphi \otimes id :_{\varphi^{-1}} S \otimes_R M \cong S_\varphi \otimes_R M$ is an isomorphism of S-modules for any (not necessarily R-algebra-) automorphism ψ of an R-algebra S. Applying this to our situation we find

$$\underline{\mathcal{A}(1)}_\varphi \otimes_{\mathcal{A}(1)} M \cong_\psi \underline{\mathcal{A}(1)} \otimes_{\mathcal{A}(1)} M \cong_\psi (H^*(K) \otimes M) = ({}_\psi H^*(K)) \otimes M$$

where the second isomorphism is the obvious map striking out the middle factor of $\mathcal{A}(1)$ and the last identity holds since image of $\psi - id$ consists of $H^*(K)$-decomposable elements, which act trivially on M. □

Therefore, we can rewrite corollary 6.3.1 from chapter 1 as:

$$H^*(M_2O) \cong \underline{\mathcal{A}}_\varphi \otimes_{\mathcal{A}(1)} M \cong \underline{\mathcal{A}} \otimes_{\underline{\mathcal{A}(1)}} \underline{\mathcal{A}(1)}_\varphi \otimes_{\mathcal{A}(1)} M \cong \underline{\mathcal{A}} \otimes_{\underline{\mathcal{A}(1)}} (M \otimes H^*(MV))$$

(where we are allowed to switch the factors by the cocommutativity of $\underline{\mathcal{A}}$). Putting this together with the isomorphism from theorem 3.1.1 and our fix from above we obtain:

$$\begin{aligned}\underline{\mathcal{A}} \otimes_{\underline{\mathcal{A}(1)}} \overline{H^*(M_2O)} &\cong H^*(M_2O)\\ &\cong \underline{\mathcal{A}} \otimes_{\underline{\mathcal{A}(1)}} (M \otimes H^*(MV))\\ &\cong \underline{\mathcal{A}} \otimes_{\underline{\mathcal{A}(1)}} (\overline{H^*(MSpin)} \otimes H^*(MV))\end{aligned}$$

Therefore, it seems natural to guess, that indeed

$$\overline{H^*(M_2O)} \cong \overline{H^*(MSpin)} \otimes H^*(MV)$$

with similar formulas following for *coker* $\overline{\hat{\alpha}^*}$ and $\overline{H^*(M_2O)} \otimes H^{*-8}(BG)$.

3.2.4. PROPOSITION. *Any isomorphism* $\overline{H^*(M_2O)} \cong \overline{H^*(MSpin)} \otimes H^*(MV)$ *determines and is determined by a* φ*-linear map* $\overline{H^*(MSpin)} \to \overline{H^*(M_2O)}$*, via restriction and* $H^*(K)$*-linear expansion, respectively. Such a map determines an isomorphism if and only if it induces an isomorphism on* $H^*(K)$*-indecomposable quotients.*

PROOF. There is an evident general statement to this end for semidirect products of Hopf algebras, but we stick to our concrete case: It is trivial to verify that for an $\mathcal{A}(1)$-module M and an $\underline{\mathcal{A}(1)}$-module N the $\underline{\mathcal{A}(1)}$-linear maps $H^*(K) \otimes M \to N$ correspond exactly to $\overline{\mathcal{A}(1)}$-linear maps $M \to N$. Applying this to $N = {}_\varphi\overline{H^*(M_2O)}$ yields that maps as described in the proposition correspond to maps ${}_\psi(H^*(K) \otimes \overline{H^*(MSpin)}) \to \overline{H^*(M_2O)}$. It remains to verify that ${}_\psi(H^*(K) \otimes \overline{H^*(MSpin)})$ and $H^*(MV) \otimes \overline{H^*(MSpin)} = {}_\psi H^*(K) \otimes \overline{H^*(MSpin)}$ carry the same $\underline{\mathcal{A}(1)}$-action: However, this immediately follows from the $H^*(K)$-decomposability of $\psi - id$, since $H^*(K)$-decomposable elements act trivially on M by definition. It is trivial to note that the process described in the one direction is indeed given by restriction, however, the expansion proceeds by mapping a φ-linear map $r : \overline{H^*(MSpin)} \to \overline{H^*(M_2O)}$ to the map $H^*(MV) \otimes \overline{H^*(MSpin)} \longrightarrow \overline{H^*(M_2O)}$ sending $ku \otimes m$ to $\varphi(k) \cdot r(m)$. To see that this is indeed the $H^*(K)$-linear expansion of r one has to note that φ restricts to the identity on $H^*(K)$.
The final statement is trivial upon noting that the respective Poincaré series coincide, since they do after inducing up to $\underline{\mathcal{A}}$. □

Now suppose we had such a φ-linear map $r : \overline{H^*(MSpin)} \to \overline{H^*(M_2O)}$. Using the corresponding isomorphisms a map $\overline{H^*(M_2O)} \otimes H^{*-8}(BG) \longrightarrow$ *coker* $\overline{\hat{\alpha}^*}$ is then determined by a φ-linear map $\overline{H^*(MSpin)} \otimes H^{*-8}(BG) \longrightarrow coker\ \overline{\hat{\alpha}^*}$. For such a map there is the obvious candidate $r \circ s$, since r automatically factor through the cokernels of the various incarnations of α. All in all from a choice of r we can construct a dotted map, say S, in the diagram above. By construction it makes the diagram

$$\begin{array}{ccc} coker\ \overline{\hat{\alpha}^*} & \xleftarrow{\ S\ } & \overline{H^*(M_2O)} \otimes H^{*+8}(BG) \\ r \uparrow & & \uparrow r\otimes id \\ coker\ \overline{\alpha^*} & \xleftarrow{\ s\ } & \overline{H^*(MSpin)} \otimes H^{*+8}(BG) \end{array}$$

commute.
It remains to check that S indeed gives a split of the transfer. To ensure this we need one more assumption on r, namely that the composition

$$\overline{H^*(MSpin)} \xrightarrow{r} \overline{H^*(M_2O)} \longrightarrow \overline{H^*(MSpin)}$$

is the identity (it is certainly linear, since $\varphi - id$ consists of $H^*(K)$-decomposable elements, which are killed by the projection map). If this condition is satisfied, then taking $H^*(K)$-indecomposable quotients in the diagrams above makes the vertical maps inverse isomorphisms and thereby shows that the twisted transfer composed with S induces the identity on $Q_{H^*(K)}(\overline{H^*(M_2O)})$. This composition is therefore an isomorphism and, in particular, the twisted transfer splits.

In order to study the existence of such a map r, we first 'dethomify' the statement: By the Thom isomorphism ${}_{\varphi}\overline{H^*(M_2O)} \cong \overline{H^*(BO)}$ as $\mathcal{A}(1)$-modules, where $H^*(ko)$ acts on $H^*(BO)$, since the Thom isomorphism is comultiplicative. Since the Thom isomorphism for *Spin*-bundles is $\mathcal{A}(1)$-linear, we similarly have $\overline{H^*(MSpin)} \cong \overline{H^*(BSpin)}$. Thus, the existence of r is equivalent to the existence of an $\mathcal{A}(1)$-linear split of the projection

$$\overline{H^*(BO)} \longrightarrow \overline{H^*(BSpin)}$$

and any such induces an isomorphism $\overline{H^*(BO)} \cong \overline{H^*(BSpin)} \otimes H^*(K)$ as $\mathcal{A}(1)$-modules. In order to facilitate computation we find it convenient to switch to homology.

Before doing so let us, however, observe one final simplification: Since $BO \simeq BSO \times K_1$, we can try to split off the $H^*(K_1)$-factor on both sides of our conjectured $\mathcal{A}(1)$-isomorphism $\overline{H^*(BO)} \cong \overline{H^*(BSpin)} \otimes H^*(K)$; first of all it is clear that the splitting of BO induces an $\mathcal{A}(1)$-isomorphism $\overline{H^*(BO)} \cong \overline{H^*(BSO)} \otimes H^*(K_1)$, since the map $BSpin \to BO$ factors through BSO under the homotopy equivalence above. We now claim that an $\mathcal{A}(1)$-isomorphism

$$\overline{H^*(BSO)} \otimes H^*(K_1) \cong \overline{H^*(BSpin)} \otimes H^*(K_2) \otimes H^*(K_1)$$

induces an $\mathcal{A}(1)$-isomorphism of these terms with the $H^*(K_1)$-factors cancelled: First of all from any such isomorphism i we obtain an $\mathcal{A}(1)$-linear map

$$\begin{aligned}\overline{H^*(BSpin)} \otimes H^*(K_2) &\xrightarrow{id\otimes 1} \overline{H^*(BSpin)} \otimes H^*(K_2) \otimes H^*(K_1)\\ &\xrightarrow{i} \overline{H^*(BSO)} \otimes H^*(K_1)\\ &\xrightarrow{id\otimes\pi} \overline{H^*(BSO)}\end{aligned}$$

The composition of the first two maps sends an element $m \otimes k$ to $\sum_j ka_j \otimes a'_j$, where $r(a) = \sum_j a_j \otimes a'_j \in \overline{H^*(BSO)} \otimes H^*(K_1) \cong \overline{H^*(BO)}$. The third then picks out ka_j for the summand with $a'_j \in H^0(K_1)$, but by assumption on r the element a_j reduces back to a when sent to $\overline{H^*(BSpin)}$. So in order for ka_j to be zero, either k or a_j (and thus a) have to be zero (since the action of $H^*(K_2)$ on $H^*(BSO)$ is torsionfree), proving that this map is injective and thus an isomorphism.

Switching to homology now, one considers modules over $H_*(ko)$ and we let $\overline{M}$ denote the $H_*(ko)$-indecomposable quotient of M. However, we do not switch to $\mathcal{A}(1)$-comodules, but rather consider homology as a right $\mathcal{A}(1)$-module via the dual Steenrod operations. By the same argument as above (only replacing $H^*(K)$ by $H^*(K_2)$ where necessary), we see that the existence of our desired map r is equivalent to:

3.2.5. Guess. The obvious map $\overline{H_*(BSpin)} \to \overline{H_*(BSO)}$ is the inclusion of a direct right $\mathcal{A}(1)$-summand.

As before a choice of complement determines a (dual) map r and any such map gives rise to an isomorphism

$$\overline{H_*(BSO)} \cong \overline{H_*(BSpin)} \otimes H_*(K_2)$$

as right $\mathcal{A}(1)$-modules, say. In the next section we analyse both sides far enough, to show that no such isomorphism can exist.

4. The structure of $\overline{H_*(BSO)}$ as an $\mathcal{A}(1)$-module

4.1. A presentation of $H_*(BSO)$. To analyse $\overline{H_*(BSO)}$ we use the description of $H_*(BSO)$ given by Giambalvo, Pengelley and Ravenel in [**GiPeRa 88**]. They construct classes $x_i \in H_i(BO)$, such that $H_*(BO) = \mathbb{Z}/2[x_i : i \in \mathbb{N}]$. They also describe the action of the Steenrod algebra in terms of these classes. Since the procedure is somewhat complicated we briefly describe it here: Consider the polynomial ring $\mathbb{Z}[y_i : i \in \mathbb{N}]$, where again y_i has degree i, and then consider the classes d_i given by writing $i = j2^k$ with odd j and putting

$$d_i = \sum_{l=0}^{j} 2^l y_{j2^l}^{2^{k-l}}$$

There is a unique set of linear selfmaps Sq_i on this ring obeying

$$Sq_j d_i = \frac{i}{i-j}\binom{i-j}{j} d_{i-j}$$

and the usual Cartan formula. Explicitly for Sq_1 and Sq_2 we have

$$Sq_1 d_i = i d_{i-1}$$
$$Sq_2 d_i = \frac{i(i-3)}{2} d_{i-2}$$

These two rules together allow algorithmic computations of $Sq_j y_i$ in terms of the y_k, which upon replacing y_i by x_i and reduction *mod* 2 yield the correct formulas for $Sq_j x_i$, see [**GiPeRa 88**, Section 2] for details.

We can now identify several relevant homology rings in terms of this description of $H_*(BO)$.

4.1.1. PROPOSITION. *We have*

$$\begin{aligned} H_*(BSO) &= \mathbb{Z}/2[x_{2^k}^2 : k \in \mathbb{N}, x_i : \alpha(i) \geq 2] \\ H_*(BSpin) &= \mathbb{Z}/2[x_{2^k}^4 : k \in \mathbb{N}, x_{2^k+2^l}^2 : k > l, x_i : \alpha(i) \geq 3] \\ H_*(ko) &= \mathbb{Z}/2[x_1^4, x_3^2, x_{2^k-1} : k \geq 3] \end{aligned}$$

where we view $H_*(ko)$ as embedded in $H_*(BSpin)$ via Stolz' multiplicative splitting of the Atiyah-Bott-Shapiro orientation

$$\begin{aligned} \overline{H_*(BO)} &= \mathbb{Z}/2[x_i : i \neq 2^k - 1][x_1, x_3]/(x_1^4, x_3^2) \\ \overline{H_*(BSO)} &= \mathbb{Z}/2[x_{2^k}^2 : k \geq 1, x_i : \alpha(i) \geq 2 \ \&\ i \neq 2^k - 1][x_1^2, x_3]/(x_1^4, x_3^2) \\ \overline{H_*(BSpin)} &= \mathbb{Z}/2[x_{2^k}^4 : k \geq 1, x_{2^k+2^l}^2 : k > l \ \&\ (k,l) \neq (1,0), \\ &\qquad x_i : \alpha(i) \geq 3 \ \&\ i \neq 2^k - 1] \\ H_*(K) &= \mathbb{Z}/2[x_{2^k} : k \in \mathbb{N}, x_{2^k+2^l} : k > l]/(x_{2^k}^4, x_{2^k+2^l}^2) \\ H_*(K_1) &= \Lambda[x_{2^k} : k \in \mathbb{N}] \\ H_*(K_2) &= \Lambda[x_{2^k}^2 : k \in \mathbb{N}, x_{2^k+2^l} : k > l] \end{aligned}$$

Furthermore, all the obvious maps connecting these send all generators which are not explicitely mentioned in the target to 0.

PROOF. The first two statements follow from [**GiPeRa 88**, Lemma 2.2] describing the images of the injective maps $H_*(BSpin) \to H_*(BSO) \to H_*(BO)$. The third is [**St 92**, Corollary 4.7]. The fourth, fifth and sixth isomorphism should then be clear. For the seventh, note that the map $H_*(BO) \to H_*(K)$ is multiplicative by the choice of multiplication on K and is surjective, for example by [**St**, Chapter IX, Corollary to Lemma 7]. Since the composition

$$BSpin \to BO \to K$$

is nullhomotopic $\mathbb{Z}/2[x_{2^k} : k \in \mathbb{N}, x_{2^k+2^l} : k > l]/(x_{2^k}^4, x_{2^k+2^l}^2)]$ maps onto $H_*(K)$. However, once more both sides have equal Poincaré series, as can be seen from $H^*(K) = \mathbb{Z}/2[\iota_1, \iota_2, Sq^1(\iota_2), Sq^2Sq^1(\iota_2), ...]$, which has generators in degrees 1 and $2^i + 1$: The polynomial ring formed by ι_1 and ι_2 has the same Poincaré-series as $\mathbb{Z}/2[x_{2^k} : k \in \mathbb{N}]/(x_{2^k}^4)$ and a polynomial ring on a generator in degree $2^i + 1$ for $i \geq 1$ corresponds to $\Lambda[x_{2^i+1}, x_{2^{i+1}+2}, x_{2^{i+2}+4}, ...]$. The same argument applied to the other combinations of $BSO, BSpin$ and BO gives the last two claims. □

4.2. The Q_i-structure of $\overline{H_*(BSO)}$. The ultimate goal of this section will be to prove:

4.2.1. THEOREM. *It is not true that*

$$\overline{H_*(BSO)} \cong \overline{H_*(BSpin)} \otimes H_*(K)$$

as $\mathcal{A}(1)$-modules. Indeed even as an $\mathcal{A}(1)$-module $\overline{H_*(BSO)}$ is not induced along the inclusion $\mathcal{A}(1) \subseteq \underline{\mathcal{A}(1)}$.

To see this we calculate that on the one hand the class $x_{18} \in H_{18}(K)$ is a Q_1-cycle but not a Q_1-boundary and has trivial Sq^2 but nontrivial Sq^1 (namely x_{17}), which is again not a Q_1-boundary. The class $1 \otimes x_{18}$ then has those same properties and we show that no such class can exist in $\overline{H_{18}(BSO)}$.

To this end we will calculate the Q_0- and Q_1-cohomology of $H_*(K_2)$ and $\overline{H_*(BSO)}$ using the spectral sequence associated to the image/pullback of the augmentation filtration on $H_*(BO)$. The following lemma will determine the first differential:

4.2.2. LEMMA. *Let I denote the augmentation ideal in $H_*(BO)$. Then*

$$Q_0(x_i) \equiv \begin{cases} x_{i-1} & i \text{ even} \\ 0 & i \text{ odd} \end{cases} \quad mod\ I^2$$

$$Q_1(x_i) \equiv \begin{cases} x_{i-3} & i \text{ even} \\ 0 & i \text{ odd} \end{cases} \quad mod\ I^2$$

$$Sq_2(x_i) \equiv \begin{cases} x_{i-2} & i \equiv 0,1 \ mod\ 4 \\ 0 & i \equiv 2,3 \ mod\ 4 \end{cases} \quad mod\ I^2$$

PROOF. Write $i = 2^k j$ with j odd and $i-1 = 2^l h$ with h odd. Then

$$\begin{aligned} 2^k j(x_h^{2^l} + 2x_{2h}^{2^{l-1}} + \dots + 2^{l-1}x_{2^{l-1}h}^2 + 2^l x_{2^l h}) \\ &= i d_{i-1} \\ &= Sq_1(d_i) \\ &= Sq_1(x_j^{2^k} + 2x_{2j}^{2^{k-1}} + \dots + 2^{k-1}x_{2^{k-1}j}^2 + 2^k x_{2^k j}) \end{aligned}$$

Disregarding all terms with more than one factor, we obtain

$$2^l j x_{i-1} \equiv Sq_1(x_i)$$

which yields the first claim. For Sq_2 write $i = 2^k j$, $i-2 = 2^l h$ and $i-3 = 2^m f$ with j, h, f odd. We find

$$\begin{aligned} 2^{k+m-1} jf(x_h^{2^l} + 2x_{2h}^{2^{l-1}} + \dots + 2^{l-1}x_{2^{l-1}h}^2 + 2^l x_{2^l h}) \\ &= \frac{i(i-3)}{2} d_{i-2} \\ &= Sq_2(d_i) \\ &= Sq_2(x_j^{2^k} + 2x_{2j}^{2^{k-1}} + \dots + 2^{k-1}x_{2^{k-1}j}^2 + 2^k x_{2^k j}) \end{aligned}$$

Again, disregarding all terms that go to zero we have:

$$2^{l+m-1} jf x_{i-2} \equiv Sq_2(x_i)$$

which gives the third claim, the second following by putting the others together. □

While this gets us started, we have relegated all further explicit computations to the second appendix.

4.2.3. Proposition.

$$H_*(K_2, Q_0) = \Lambda[x_5, x_{2^k}^2 : k \geq 1]$$

Proof. We have $H_*(K_2) \cong \Lambda[x_{2^k}^2, x_{2^k+2^l} : k > l]$. Consider the spectral sequence coming from the filtration by powers of the images of I. The differentials on the first page then connect even, non-square to odd, non-square generators one degree lower. Therefore, as a differential, graded algebra the first page decomposes into a tensor product of $\Lambda[x_{2^k+1}, x_{2^k+2}]$ for $k \geq 2$ with differential given by $x_{2^k+2} \mapsto x_{2^k+1}$ and an exterior algebra on all other generators with trivial differential. By the Künneth-formula we find the second page given by

$$\Lambda[x_{2^k}^2, x_3, x_{2^k+2^l} : k > l \geq 2, x_{2^k+2}x_{2^k+1} : k \geq 2]$$

The next differentials are computed by $Sq_1(x_3) = x_1^2$ and

$$Sq_1(x_{2^k+2^l}) \equiv x_{2^{k-1}+2^{l-1}} \cdot x_{2^{k-2}+2^{l-2}} \cdots x_{2^{k-l+1}+2} \cdot x_{2^{k-l+1}+1} \quad mod\ I^{l+1}$$

for $l \geq 2$, which we cite from lemma 2 in the appendix. In particular

$$\begin{aligned} d_2(x_3) &= x_1^2 \\ d_2(x_{2^{k+1}+4}) &= x_{2^k+2}x_{2^k+1} && \text{for } k \geq 2 \\ d_2(x_{2^k+2^l}) &= 0 && \text{for } k > l \geq 3 \end{aligned}$$

Again, by Künneth we find the third page given by

$$\Lambda[x_{2^k}^2 : k \geq 1, x_1^2 x_3, x_{2^k+2^l} : k > l \geq 3, x_{2^{k+1}+4}x_{2^k+2}x_{2^k+1} : k \geq 2]$$

Since $Sq_1(x_1^2 x_3) = 0$ we find

$$\begin{aligned} d_3(x_1^2 x_3) &= 0 \\ d_3(x_{2^{k+2}+8}) &= x_{2^{k+1}+4}x_{2^k+2}x_{2^k+1} && \text{for } k \geq 2 \\ d_3(x_{2^k+2^l}) &= 0 && \text{for } k > l \geq 4 \end{aligned}$$

Therefore, the fourth page is

$$\Lambda[x_{2^k}^2 : k \geq 1, x_1^2 x_3, x_{2^k+2^l} : k > l \geq 4, x_{2^{k+2}+8}x_{2^{k+1}+4}x_{2^k+2}x_{2^k+1} : k \geq 2]$$

From here on we inductively have

$$E_r = \Lambda[x_{2^k}^2 : k \geq 1, x_1^2 x_3, x_{2^k+2^l} : k > l \geq r, x_{2^{k+r-2}+2^{r-1}} \cdots x_{2^k+2}x_{2^k+1} : k \geq 2]$$

with differential given by

$$\begin{aligned} d_r(x_1^2 x_3) &= 0 \\ d_r(x_{2^{k+r-1}+2^r}) &= x_{2^{k+r-2}+2^{r-1}} \cdots x_{2^k+2}x_{2^k+1} && \text{for } k \geq 2 \\ d_r(x_{2^k+2^l}) &= 0 && \text{for } k > l \geq r+1 \end{aligned}$$

We thus find

$$\Lambda[x_{2^k}^2 : k \geq 1, x_1^2 x_3]$$

as the limiting page. Finally, we note $Sq_1(x_6) = x_5 + x_1^2 x_3$ (lemma 1 of appendix II) giving the result. □

Similarly:

4.2.4. Proposition.

$$H_*(K_2, Q_1) = \Lambda[x_9, x_{17}, x_{2^k}^2 : k \geq 1, x_{2^k+2} : k \geq 4]$$

PROOF. Again the first page is given by $H_*(K_2) = \Lambda[x_{2^k}^2, x_{2^k+2^l} : k > l]$ with differentials connecting even, non-square generators to odd, non-square generators three degrees lower. So the first page breaks up into $\Lambda[x_{2^k+4}, x_{2^k+1}]$ for $k \geq 3$, $\Lambda[x_6, x_3]$ and an exterior algebra with trivial differential on all other generators. The second page is thus given by

$$\Lambda[x_{2^k}^2, x_3x_6, x_5, x_{2^k+2} : k \geq 3, x_{2^k+2^l} : k > l \geq 3, x_{2^k+4}x_{2^k+1} : k \geq 3]$$

Here we have $Q_1(x_3x_6) = 0, Q_1(x_5) = x_1^2, Q_1(x_{10}) = x_1^2x_5, Q_1(x_{2^k+2}) = 0$ for $k \geq 4$ and

$$Q_1(x_{2^k+2^l}) \equiv x_{2^{k-1}+2^{l-1}} \cdot x_{2^{k-2}+2^{l-2}} \cdots x_{2^{k-l+2}+4} \cdot x_{2^{k-l+2}+1} \quad mod\ I^l$$

Therefore, the second differential is given by

$$\begin{array}{rcll} d_2(x_3x_6) & = & 0 & \\ d_2(x_5) & = & x_1^2 & \\ d_2(x_{2^k+2}) & = & 0 & \text{for } k \geq 3 \\ d_2(x_{2^{k+1}+8}) & = & x_{2^k+4}x_{2^k+1} & \text{for } k \geq 3 \\ d_2(x_{2^k+2^l}) & = & 0 & \text{for } k > l \geq 4 \end{array}$$

The third page:

$$\Lambda[x_{2^k}^2 : k \geq 1, x_3x_6, x_1^2x_5, x_{2^k+2} : k \geq 3, x_{2^k+2^l} : k > l \geq 4, x_{2^{k+1}+8}x_{2^k+4}x_{2^k+1} : k \geq 3]$$

And the third differentials:

$$\begin{array}{rcll} d_3(x_3x_6) & = & 0 & \\ d_3(x_{10}) & = & x_1^2x_5 & \\ d_3(x_{2^k+2}) & = & 0 & \text{for } k \geq 4 \\ d_3(x_{2^{k+2}+16}) & = & x_{2^{k+1}+8}x_{2^k+4}x_{2^k+1} & \text{for } k \geq 3 \\ d_3(x_{2^k+2^l}) & = & 0 & \text{for } k > l \geq 5 \end{array}$$

The fourth page is:

$$\begin{array}{l} \Lambda[x_{2^k}^2 : k \geq 1, x_3x_6, x_1^2x_5x_{10}, x_{2^k+2} : k \geq 4, \\ \quad x_{2^k+2^l} : k > l \geq 5, x_{2^{k+2}+16}x_{2^{k+1}+8}x_{2^k+4}x_{2^k+1} : k \geq 3] \end{array}$$

Since $Q_1(x_1^2x_5x_{10}) = 0$ (lemma 1 of the appendix) we inductively have

$$\begin{array}{l} E_r = \Lambda[x_{2^k}^2 : k \geq 1, x_3x_6, x_1^2x_5x_{10}, x_{2^k+2} : k \geq 4, \\ \quad x_{2^k+2^l} : k > l \geq r+1, x_{2^{k+r-2}+2^r} \cdots x_{2^k+4}x_{2^k+1} : k \geq 3] \end{array}$$

with differentials

$$\begin{array}{rcll} d_r(x_3x_6) & = & 0 & \\ d_r(x_1^2x_5x_{10}) & = & 0 & \\ d_r(x_{2^k+2}) & = & 0 & \text{for } k \geq 4 \\ d_r(x_{2^{k+r-1}+2^{r+1}}) & = & x_{2^{k+r-2}+2^r} \cdots x_{2^k+4}x_{2^k+1} & \text{for } k \geq 3 \\ d_r(x_{2^k+2^l}) & = & 0 & \text{for } k > l \geq r+2 \end{array}$$

Thus we find

$$\Lambda[x_{2^k}^2 : k \geq 1, x_3x_6, x_1^2x_5x_{10}, x_{2^k+2} : k \geq 4]$$

as the limiting page. Since $Q_1(x_{12}) = x_3x_6 + x_9$ and $Q_1(x_{20}) = x_{17} + x_{10}x_5x_1^2$ (again by appendix II, lemma 1) we have the result. □

Now consider the element $x_{18} \in H_{18}(K)$. We have $Sq_1(x_{18}) = x_{17}$ and $Sq_2(x_{18}) = 0$. By the above computation we see that it has the desired properties, namely that neither lie in the image of Q_1.

Now for $\overline{H_*(BSO)}$. Let us compute the Q_i-homologies.

4.2.5. LEMMA. *The associated graded of the filtration on $H_*(\overline{BSO}, Q_1)$ is given by*

$$\mathbb{Z}/2[x_{2i}^2 : i \geq 1] \otimes \Lambda[x_5]$$

.

PROOF. We have $H_*(\overline{BSO}) = P[x_{2^k}^2, x_i : i \neq 2^l - 1]/(x_1^4, x_3^2)$ and the first differential is again given by $d_1(x_i) = ix_{i-1}$. Therefore, the first page breaks up into $P[x_{2i}, x_{2i-1}]$ where $i \neq 2^l$ and $i \geq 3$. The second page then is:

$$P[x_{2i}^2 : i \geq 1] \otimes \Lambda[x_1^2, x_3]$$

Since $Q_0(x_3) = x_1^2$ the next page is

$$P[x_{2i}^2 : i \geq 1] \otimes \Lambda[x_1^2 x_3]$$

All of these are permanent cycles, so $Sq_1(x_6) = x_5 + x_1^2 x_3$ gives the claim. □

4.2.6. LEMMA. *We have the associated graded of $H_*(\overline{BSO}, Q_1)$ isomorphic to*

$$P[x_2^2, x_4^2, x_6^2, x_{10}^2, x_{2i}^2 : i \neq 2^l + 1, x_{2^k+2} : k \geq 4] \otimes \Lambda[x_9, x_{17}]$$

.

PROOF. Again, the first page is

$$\overline{H_*(BSO)} = P[x_{2^k}^2, x_i : \alpha(i) \geq 2 \ \&\ i \neq 2^l - 1]/(x_1^4, x_3^2)$$

and the differential is given by $d_1(x_i) = ix_{i-3}$. The second page thus looks as follows:

$$P[x_2^2, x_4^2, x_6^2, x_{2^k+2} : k \geq 3, x_{2^k-3} : k \geq 3, x_{2i}^2 : i \neq 2^l + 1] \otimes \Lambda[x_3 x_6]$$

Now $Q_1(x_{10}) = x_1^2 x_5$, $Q_1(x_{2^k+2}) = 0$ for $k \geq 4$ and $Q_1(x_{2^k-3}) = x_{2^{k-1}-3}^2$ once $k \geq 3$ and finally $Q_1(x_3 x_6) = 0$. The third page:

$$P[x_2^2, x_4^2, x_6^2, x_{2^k+2} : k \geq 3, x_{2i}^2 : i \neq 2^l + 1] \otimes \Lambda[x_1^2 x_5, x_3 x_6]$$

which has $d_3(x_{10}) = x_1^2 x_5$. Thus the fourth page is

$$P[x_2^2, x_4^2, x_6^2, x_{10}^2, x_{2^k+2} : k \geq 3, x_{2i}^2 : i \neq 2^l + 1] \otimes \Lambda[x_1^2 x_5 x_1 0, x_3 x_6]$$

and everything here is a permanent cycle. Finally, noting that $Q_1(x_{20} + x_1^2 x_5 x_{13}) = x_{17} + x_1^2 x_5 x_{10}$ and $Q_1(x_{12}) = x_9 + x_3 x_6$ we have the result. □

Similarly one computes:

4.2.7. LEMMA. *We have the associated graded of $H_*(\overline{BSpin}, Q_i)$ for both $i = 0, 1$ is given by*

$$P[x_{2^k}^4 : k \geq 2, x_{2i} : \alpha(i) \geq 2]$$

.

PROOF. The statement about Q_0-homology is [**St 92**, Lemma 8.4]. Instead of giving a direct computation Stolz uses the fact that $\overline{H^*(MSpin)}$ is isomorphic to the Anderson-Brown-Peterson module, which he in turn does not prove (and for which we gave an indirect argument above). The claim about Q_1-homology follows since M consists of a direct sum of Jokers, $\mathbb{Z}/2$'s and free summands, all of whose Q_0 and Q_1 homology is carried by the same classes.
A simple calculation using the spectral sequences above is of course also possible. This can then be used to give a more direct proof, that indeed $\overline{H_*(MSpin)} \cong M$. □

These computations immediately show that as expected the map

$$\overline{H_*(BSpin)} \longrightarrow \overline{H_*(BSO)}$$

is injective in Q_i-homology, supporting our guess (i.e. 3.2.5) that its image is a direct summand. Furthermore, at the level of Q_i-homology our conjecture also holds true: The Poincaré series of $\overline{H_*(BSO, Q_i)}$ and $\overline{H_*(BSpin, Q_i)} \otimes \overline{H_*(K_2, Q_i)}$ agree, as should be the case by the Künneth theorem if our conjecture held: This is a relatively easy counting argument, which we leave to the reader as it is of no further use. All of this makes theorem 4.2.1 all the more surprising (and disappointing).

PROOF OF THEOREM 4.2.1. We are looking for a class in $\overline{H_{18}(BSO)}$ that has Q_1 and Sq_2 trivial but whose Sq_1 is not a Q_1-boundary: Call such a class a. By the above calculations we can write $a = p + ux_{18} + Q_1(b)$ as a sum of some polynomial in $x_2^2, x_4^2, x_6^2, x_8^2, x_9^2$ and x_{18}, which we seperate as $p + ux_{18}$ with p a polynomial in all the squares, $u \in \mathbb{Z}/2$ and a Q_1-boundary $Q_1(b)$. In order to have $Sq_1(a) = Sq_1(p + ux_{18} + Q_1(b)) = ux_{17} + Q_1(Sq_1 b)$ not be a Q_1-boundary we had better have $u = 1$. In particular, a has a nontrivial x_{18}-term in the obvious monomial basis of H_{18}, since p does not have one and $Q_1(b)$ also cannot have one, as $Q_1(H_{21}) \subseteq I^2$. However, no such element of $\overline{H_{18}(BSO)}$ can have simultaneously vanishing Sq^2 and Q_1 for the following reason:
$Sq_2(x_{18}) = x_2^8$ which in turn only occurs as a term in the expansion of Sq_2 of two other monomials, namely $x_2^4 x_{10}$ and $x_2^6 x_6$, this can easily be seen by the Cartan formula and the table in the second appendix: Any monomial m having Sq^2 contain an x_2^8 summand can be written as $m = x_2^k a$ with a not divisible by x_2. Now a has to lie in I^2, since otherwise applying Sq^2 via the Cartan formula to it always leaves one factor $\neq x_2$ unchanged. Therefore, either a is a generator itself and $Sq_2(a)$ having x_2-power arising or $a = bc$ with b, c generators and $Sq_1(a), Sq_1(b)$ having a power of x_2 in it. Looking at the table in the sppendix reveals $x_2^4 x_{10}$ and $x_2^6 x_6$ as the only possibilities. Hence exactly one of these terms has to appear in the expansion of a in addition to x_{18}. Now $Q_1(x_2^4 x_{10}) = x_1^2 x_2^4 x_5$ and $Q_1(x_2^6 x_6) = x_2^6 x_3$, which both do not occur in the expansion of Q_1 of any other monomial as is easily checked using the table once more.

□

5. The Adams spectral sequence for homology-ko-module spectra

This last part is concerned with a somewhat more conceptual reinterpretation of the paper [**St 94**]. We shall see, that, what Stolz in effect does, is the following: Given homology-ko-module spectra X and Y he computes a certain part of the second page of the Adams spectral sequence comptuing $[X, Y]$; namely the part, in whose subquotients the differentials of any $H^*(ko)$-linear map have to lie. With this reinterpretation we change the role that finite spectra, whose wedge-sum realises $\overline{H^*(X)}$, play in Stolz' paper.This has the following major advantage: It seperates the topological from the algebraic considerations, which are very much tied together in Stolz' approach, and thereby allows for a generalisation to the twisted context. The main idea is simple: Given a homology-ko-module spectrum Y each spectrum in the canonical Adams tower of Y

$$\begin{array}{ccc} \downarrow & & \\ Y_2 & \longrightarrow & Y_2 \wedge H\mathbb{Z}/2 \\ \downarrow & & \\ Y_1 & \longrightarrow & Y_1 \wedge H\mathbb{Z}/2 \\ \downarrow & & \\ Y & \longrightarrow & Y \wedge H\mathbb{Z}/2 \end{array}$$

can be given a homology-ko-module structure, such that all structure maps are $H^*(ko)$-colinear. Given now an $H^*(ko)$-colinear and $\mathcal{A}$-linear map $g : H^*(Y) \longrightarrow H^*(X)$, one obtains an homology-ko-module map $f : X \to Y \wedge H\mathbb{Z}/2$, which represents it on the second page. Since all differentials are induced by maps in the above tower, one may hope these differentials carry f into something $H^*(ko)$-colinear. Stolz' structural computation of the cohomology of homology-ko-module spectra shows that on the second page, the homology-ko-module maps are identified with the image of

$$\begin{aligned} Ext^*_{\mathcal{A}(1)}(\overline{H^*(Y)}, \overline{H^*(X)}) \longrightarrow\ & Ext^*_{\mathcal{A}(1)}(\overline{H^*(Y)}, H^*(X)) \\ \cong\ & Ext^*_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{A}(1)} \overline{H^*(Y)}, H^*(X)) \\ \cong\ & Ext^*_{\mathcal{A}}(H^*(Y), H^*(X)) \end{aligned}$$

It is the former groups that Stolz shows do vanish in all relevant cases, thereby establishing the realisability of a specific g, namely a split of the $\mathbb{H}P^2$-transfer map.

5.1. Differentials on linear maps. We start by constructing a homology-ko-module structure on the Adams tower of any homology-ko-module spectrum Y. The only nontrivial part is giving one to the homotopy cofibre, call it Z for now, of the obvious map $Y_i \to Y_i \wedge H\mathbb{Z}/2$, whose desuspension is Y_{i+1}. Since the diagram

$$\begin{array}{ccc} ko \wedge Y_i & \xrightarrow{id \wedge i} & ko \wedge Y_i \wedge H\mathbb{Z}/2 \\ \downarrow & & \downarrow \\ Y_i & \xrightarrow{\quad i \quad} & Y_i \wedge H\mathbb{Z}/2 \end{array}$$

commutes strictly, the composition

$$ko \wedge Y_i \xrightarrow{id \wedge i} ko \wedge Y_i \wedge H\mathbb{Z}/2 \longrightarrow Y_i \wedge h\mathbb{Z}/2 \longrightarrow Z$$

is canonically nullhomotopic, giving a candidate for a map $ko \wedge Z \to Z$. To verify that this indeed gives a homology-*ko*-module structure to Z, note that the map $H_*(Y_i \wedge H\mathbb{Z}/2) \to H_*(Z)$ is surjective (since $H_*(Y_i) \to H_*(Y_i \wedge H\mathbb{Z}/2)$ is injective). Hence the relevant diagrams are even more commutative than they are in $H_*(Y_i \wedge H\mathbb{Z}/2)$ and the map $Y_i \wedge H\mathbb{Z}/2 \to Z$ is even strictly multiplicative (i.e. the diagram analagous to the one above commutes on the nose). Finally, note that the connecting map $Z \to shY$ is homology-*ko*-linear, since it induces the zero map in homology, though this is a fact we will not need.

Given now a $H^*(ko)$-colinear, $\mathcal{A}$-linear map $g : H^*(Y) \longrightarrow H^*(X)$, consider the Adams spectral sequence for maps $X \to Y$. Suppressing the internal grading its first page E_1^i is given by $[X, Y_i \wedge H\mathbb{Z}/2] \cong Hom_{\mathcal{A}}(H^*(Y_i) \otimes \mathcal{A}, H^*(X))$ and the first differential is induced by

$$Y_i \wedge H\mathbb{Z}/2 \longrightarrow sh^1 Y_{i+1} \longrightarrow sh^1 Y_{i+1} \wedge H\mathbb{Z}/2$$

On E_1 the map g canonically corresponds to a map $f : X \to Y \wedge H\mathbb{Z}/2$, which obviously is a homology-*ko*-module map. By the construction the first differential of f will also be represented by a homology-*ko*-module map. Now a map $p : M \to N$ between two homology-*ko*-module spectra is a homology-*ko*-module map if and only if $p^*(\overline{N}) \subseteq \overline{M}$. Therefore, by Stolz' theorem 3.1.1 f and its differential is represented by an element in

$$Hom_{\mathcal{A}(1)}(\overline{H^*(Y_i)} \otimes \mathcal{A}, \overline{H^*(X)}) \subseteq Hom_{\mathcal{A}(1)}(\overline{H^*(Y_i)} \otimes \mathcal{A}, H^*(X))$$

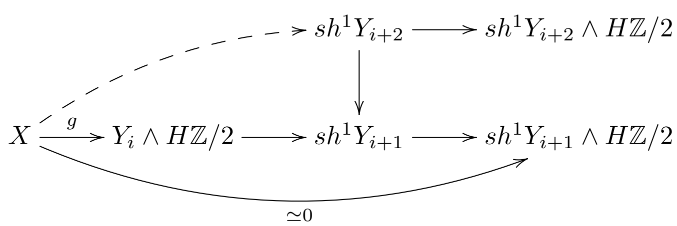
Now the sequence

$$\cdots \to H^{*+i+1}(Y_{i+1} \wedge H\mathbb{Z}/2) \to H^{*+i}(Y_i \wedge H\mathbb{Z}/2) \to \cdots$$

forms a free $\mathcal{A}$-resolution of $H^*(Y)$ and the second page of the spectral sequence has $E_2^i \cong Hom_{\mathcal{A}}(H^*(Y_i) \otimes \mathcal{A}, H^*(X))$. As the sequence consists of $H^*(ko)$-comodules and $H^*(ko)$-colinear maps and by Stolz' theorem taking *ko*-primitives preserves exactness, we find that

$$\cdots \to \overline{H^{*+i+1}(Y_{i+1} \wedge H\mathbb{Z}/2)} \to \overline{H^{*+i}(Y_i \wedge H\mathbb{Z}/2)} \to \cdots$$

is a resolution of $\overline{H^*(Y)}$. By proposition 3.1.2 it is indeed a free resolution and, therefore, the 'image' of $Hom_{\mathcal{A}(1)}(\overline{H^*(Y_i)} \otimes \mathcal{A}, \overline{H^*(X)})$ on the second page is precisely the image of $Ext^*_{\mathcal{A}(1)}(\overline{H^*(Y)}, \overline{H^*(X)}) \longrightarrow Ext^*_{\mathcal{A}}(H^*(Y), H^*(X))$.
In order to conclude that higher differentials of g (supossing it survives the first one) can also be represented by homology-*ko*-module maps, we seem to need an additional assumption on the source spectrum: Given a 1-cycle g its second differential is represented by the upper composition using the dotted lift



However, without further arguments it is unclear whether this lift can be chosen in a $H^*(ko)$-colinear fashion. It is certainly sufficient to have $X \simeq ko \wedge F$ as homology-ko-module spectra for some arbitrary spectrum F. This discussion directly carries over to higher differentials which are given by similar lifting prodcedures.

5.2. Conclusion. We can now try to rewrite parts of [**St 92**]: The first step is to show that the map $s : H^*(MSpin) \to H^*(ko)$ from 3.1, which is a comultiplicative split of the ABS-orientation, is in fact given by a map of (2-completed) spectra. To establish this Stolz essentially considers the Adams spectral sequence with second page isomorphic to

$$Ext^*_{\mathcal{A}}(H^*(MSpin), H^*(ko)) = Ext_{\mathcal{A}(1)}(M, H^*(K))$$

with M the Anderson-Brown-Peterson module. Stolz computes these Ext-groups far enough to prove that no nontrivial differential can start in position $(0,0)$ and therefore the element s survives the spectral sequence (compare [**St 94**, Examples 3.6 & 3.7]). We now conclude that indeed s has to come from a map of spectra by convergence of the Adams spectral sequence, see [**Ad**, Part III, Theorem 15.1].
The second step is to show that the split $H^*(MSpin) \otimes H^*(BG) \to H^{*+8}(hofib(\alpha))$ of the transfer is also realised by a map of spectra. Since both sides acquire homology-ko-module structures by the previous step and the split by construction is linear for these, it remains to verify that the groups

$$Ext^*_{\mathcal{A}(1)}(M \otimes H^*(BG), M_{>0})$$

vanish along the line the differentials from source $(8,0)$ might hit and that the spectrum $MSpin$ is of the form $ko \wedge X$. The first is precisely what Stolz does in [**St 94**, Proposition 8.5] and the second follows from the Anderson-Brown-Peterson splitting, as $H\mathbb{Z}/2$ and all connective covers of ko are of the required form.
In the twisted context, the same argument shows that a $H^*(ko)$-colinear split of the twisted transfer should have differentials represented in

$$Ext^*_{\underline{\mathcal{A}(1)}}(\overline{H^*(M_2O)} \otimes H^*(BG), N)$$

where N denotes the cokernel of

$$\overline{H^*(k_2o)} \xrightarrow{\hat{\alpha}^*} \overline{H^*(M_2O)}$$

Had our conjecture in the second part been correct, this group would have been isomorphic to

$$Ext^*_{\mathcal{A}(1)}(M \otimes H^*(BG), M_{>0} \otimes H^*(K))$$

and these groups seemed very much approachable.

Appendix I

We here finish the purely calculational proofs of lemmas 6.1.5 and 6.1.6 of the first chapter. The former of which read as follows:

Lemma (6.1.5). *The stipulation*

$$
\begin{aligned}
Sq^1 &\longmapsto 1 \otimes Sq^1 + \iota_1 \otimes 1 \\
Sq^2 &\longmapsto 1 \otimes Sq^2 + \iota_1 \otimes Sq^1 + \iota_1^2 \otimes 1 + \iota_2 \otimes 1
\end{aligned}
$$

specifies a unique morphism of Hopf algebras $\varphi : \mathcal{A}(1) \longrightarrow \underline{\mathcal{A}}$.

Proof. As mentioned in 6.1.5 the following two identities

$$
\begin{aligned}
&\varphi(Sq^1)^2 = 0 \\
&\varphi(Sq^2)^2 = \varphi(Sq^1)\varphi(Sq^2)\varphi(Sq^1)
\end{aligned}
$$

will show both multiplicativity of φ and that it is even well defined. Here goes:

$$
\begin{aligned}
\varphi(Sq^1)^2 &= (1 \otimes Sq^1 + \iota_1 \otimes 1)^2 \\
&= (1 \otimes Sq^1)^2 + (1 \otimes Sq^1)(\iota_1 \otimes 1) + (\iota_1 \otimes 1)(1 \otimes Sq^1) + (\iota_1 \otimes 1)^2 \\
&= \underbrace{1 \otimes (Sq^1)^2}_{=0} + \big[\underbrace{Sq^1(\iota_1) \otimes 1}_{=\iota_1^2 \otimes 1} + \iota_1 \otimes Sq^1\big] + \iota_1 \otimes Sq^1 + \iota_1^2 \otimes 1 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
&\varphi(Sq^1)\varphi(Sq^2)\varphi(Sq^1) \\
&= \ (1 \otimes Sq^1 + \iota_1 \otimes 1)(1 \otimes Sq^2 + \iota_1 \otimes Sq^1 + \iota_1^2 \otimes 1 + \iota_2 \otimes 1)(1 \otimes Sq^1 + \iota_1 \otimes 1) \\
&= \Big((1 \otimes Sq^1)(1 \otimes Sq^2) + (1 \otimes Sq^1)(\iota_1 \otimes Sq^1) + (1 \otimes Sq^1)(\iota_1^2 \otimes 1) \\
&\quad + (1 \otimes Sq^1)(\iota_2 \otimes 1) + (\iota_1 \otimes 1)(1 \otimes Sq^2) + (\iota_1 \otimes 1)(\iota_1 \otimes Sq^1) \\
&\quad + (\iota_1 \otimes 1)(\iota_1^2 \otimes 1) + (\iota_1 \otimes 1)(\iota_2 \otimes 1)\Big)(1 \otimes Sq^1 + \iota_1 \otimes 1) \\
&= \Big(1 \otimes Sq^1 Sq^2 + \big[\underbrace{Sq^1(\iota_1) \otimes Sq^1}_{=\cancel{\iota_1^2 \otimes Sq^1}} + \underbrace{\iota_1 \otimes (Sq^1)^2}_{=0}\big] + \big[\underbrace{Sq^1(\iota_1^2) \otimes 1}_{=0} + \cancel{\iota_1^2 \otimes Sq^1}\big] \\
&\quad + \big[Sq^1(\iota_2) \otimes 1 + \iota_2 \otimes Sq^1\big] + \iota_1 \otimes Sq^2 + \iota_1^2 \otimes Sq^1 \\
&\quad + \iota_1^3 \otimes 1 + \iota_1\iota_2 \otimes 1\Big)(1 \otimes Sq^1 + \iota_1 \otimes 1)
\end{aligned}
$$

$$
\begin{aligned}
&= (1\otimes Sq^1Sq^2)(1\otimes Sq^1) + (Sq^1(\iota_2)\otimes 1)(1\otimes Sq^1) + (\iota_2\otimes Sq^1)(1\otimes Sq^1)\\
&\quad + (\iota_1\otimes Sq^2)(1\otimes Sq^1) + (\iota_1^2\otimes Sq^1)(1\otimes Sq^1) + (\iota_1^3\otimes 1)(1\otimes Sq^1)\\
&\quad + (\iota_1\iota_2\otimes 1)(1\otimes Sq^1) + (1\otimes Sq^1Sq^2)(\iota_1\otimes 1) + (Sq^1(\iota_2)\otimes 1)(\iota_1\otimes 1)\\
&\quad + (\iota_2\otimes Sq^1)(\iota_1\otimes 1) + (\iota_1\otimes Sq^2)(\iota_1\otimes 1) + (\iota_1^2\otimes Sq^1)(\iota_1\otimes 1)\\
&\quad + (\iota_1^3\otimes 1)(\iota_1\otimes 1) + (\iota_1\iota_2\otimes 1)(\iota_1\otimes 1)\\
&= \underbrace{1\otimes Sq^1Sq^2Sq^1}_{=1\otimes Sq^3Sq^1} + Sq^1(\iota_2)\otimes Sq^1 + \underbrace{\iota_2\otimes (Sq^1)^2}_{=0} + \iota_1\otimes Sq^2Sq^1 + \underbrace{\iota_1^2\otimes(Sq^1)^2}_{=0}\\
&\quad + \iota_1^3\otimes Sq^1 + \cancel{\iota_1\iota_2\otimes Sq^1} + \big[\underbrace{Sq^1Sq^2(\iota_1)\otimes 1}_{=0} + \underbrace{Sq^1(\iota_1)\otimes Sq^2}_{\cancel{\iota_1^2\otimes Sq^2}} + \underbrace{Sq^2(\iota_1)\otimes Sq^1}_{=0}\\
&\quad + \underbrace{\iota_1\otimes Sq^1Sq^2}_{\iota_1\otimes Sq^3}\big] + Sq^1(\iota_2)\iota_1\otimes 1 + \big[\underbrace{\iota_2Sq^1(\iota_1)\otimes 1}_{\cancel{\iota_1^2\iota_2\otimes 1}} + \cancel{\iota_2\iota_1\otimes Sq^1}\big]\\
&\quad + \big[\underbrace{\iota_1Sq^2(\iota_1)\otimes 1}_{=0} + \underbrace{\iota_1Sq^1(\iota_1)\otimes Sq^1}_{\cancel{\iota_1^3\otimes Sq^1}} + \cancel{\iota_1^2\otimes Sq^2}\big] + \big[\underbrace{\iota_1^2Sq^1(\iota_1)\otimes 1}_{\cancel{\iota_1^4\otimes 1}} + \cancel{\iota_1^3\otimes Sq^1}\big]\\
&\quad + \cancel{\iota_1^4\otimes 1} + \cancel{\iota_1^2\iota_2\otimes 1}\\
&= 1\otimes Sq^3Sq^1 + Sq^1(\iota_2)\otimes Sq^1 + \iota_1\otimes Sq^2Sq^1 + \iota_1^3\otimes Sq^1 + \iota_1\otimes Sq^3\\
&\quad + \iota_1Sq^1(\iota_2)\otimes 1
\end{aligned}
$$

$$
\begin{aligned}
&\varphi(Sq^2)\varphi(Sq^2)\\
&= (1\otimes Sq^2 + \iota_1\otimes Sq^1 + \iota_1^2\otimes 1 + \iota_2\otimes 1)(1\otimes Sq^2 + \iota_1\otimes Sq^1 + \iota_1^2\otimes 1 + \iota_2\otimes 1)\\
&= (1\otimes Sq^2)^2 + (\iota_1\otimes Sq^1)(1\otimes Sq^2) + (\iota_1^2\otimes 1)(1\otimes Sq^2) + (\iota_2\otimes 1)(1\otimes Sq^2)\\
&\quad + (1\otimes Sq^2)(\iota_1\otimes Sq^1) + (\iota_1\otimes Sq^1)^2 + (\iota_1^2\otimes 1)(\iota_1\otimes Sq^1) + (\iota_2\otimes 1)(\iota_1\otimes Sq^1)\\
&\quad + (1\otimes Sq^2)(\iota_1^2\otimes 1) + (\iota_1\otimes Sq^1)(\iota_1^2\otimes 1) + (\iota_1^2\otimes 1)^2 + (\iota_2\otimes 1)(\iota_1^2\otimes 1)\\
&\quad + (1\otimes Sq^2)(\iota_2\otimes 1) + (\iota_1\otimes Sq^1)(\iota_2\otimes 1) + (\iota_1^2\otimes 1)(\iota_2\otimes 1) + (\iota_2\otimes 1)^2\\
&= \underbrace{1\otimes (Sq^2)^2}_{=1\otimes Sq^3Sq^1} + \underbrace{\iota_1\otimes Sq^1Sq^2}_{=\iota_1\otimes Sq^3} + \cancel{\iota_1^2\otimes Sq^2} + \cancel{\iota_2\otimes Sq^2} + \big[\underbrace{Sq^2(\iota_1)\otimes Sq^1}_{=0}\\
&\quad + \underbrace{Sq^1(\iota_1)\otimes (Sq^1)^2}_{=0} + \iota_1\otimes Sq^2Sq^1\big] + \big[\underbrace{\iota_1Sq^1(\iota_1)\otimes Sq^1}_{=\iota_1^3\otimes Sq^1} + \underbrace{\iota_1^2\otimes (Sq^1)^2}_{=0}\big]\\
&\quad + \cancel{\iota_1^3\otimes Sq^1} + \cancel{\iota_2\iota_1\otimes Sq^1} + \big[\underbrace{Sq^2(\iota_1^2)\otimes 1}_{=\cancel{\iota_1^4\otimes 1}} + \underbrace{Sq^1(\iota_1^2)\otimes Sq^1}_{=0} + \cancel{\iota_1^2\otimes Sq^2}\big]\\
&\quad + \big[\underbrace{\iota_1Sq^1(\iota_1^2)\otimes 1}_{=0} + \cancel{\iota_1^3\otimes Sq^1}\big] + \cancel{\iota_1^4\otimes 1} + \cancel{\iota_2\iota_1^2\otimes 1} + \big[\underbrace{Sq^2(\iota_2)\otimes 1}_{=\cancel{\iota_2^2\otimes 1}}\\
&\quad + Sq^1(\iota_2)\otimes Sq^1 + \cancel{\iota_2\otimes Sq^2}\big] + \big[\iota_1Sq^1(\iota_2)\otimes 1 + \cancel{\iota_1\iota_2\otimes Sq^1}\big] + \cancel{\iota_1^2\iota_2\otimes 1}\\
&\quad + \cancel{\iota_2^2\otimes 1}
\end{aligned}
$$

$$= 1 \otimes Sq^3Sq^1 + \iota_1 \otimes Sq^3 + \iota_1 \otimes Sq^2Sq^1 + \iota_1^3 \otimes Sq^1 + Sq^1(\iota_2) \otimes Sq^1 + \iota_1 Sq^1(\iota_2) \otimes 1$$

These two outcomes obviously agree. For the comultiplicativity recall that the inclusion maps induce a coalgebra isomorphism $H^*(K) \otimes \mathcal{A} \to \underline{\mathcal{A}}$. The coalgebra structure of $H^*(K)$ is of course determined by 3.3.2:

$$\Delta(\iota_1) = \iota_1 \otimes 1 + 1 \otimes \iota_1$$
$$\Delta(\iota_2) = \iota_2 \otimes 1 + \iota_1 \otimes \iota_1 + 1 \otimes \iota_2$$

With these data it is trivial to verify $\varphi(\Delta(Sq^{1,2})) = \Delta(\varphi(Sq^{1,2}))$. By multiplicativity of φ and the coproducts we are done. □

LEMMA (6.1.6). *Extending φ by the identity on $H^*(K)$ produces a Hopf algebra automorphism of $\underline{\mathcal{A}(1)}$.*

PROOF. Putting $\varphi(k \otimes \alpha) = (k \otimes 1)\varphi(\alpha)$ just leaves the verificaton of the multiplicativity. To this end calculate

$$\begin{aligned} \varphi(k \otimes \alpha)\varphi(l \otimes \beta) &= (k \otimes 1)\varphi(\alpha)(l \otimes 1)\varphi(\beta) \\ \varphi((k \otimes \alpha)(l \otimes \beta)) &= \sum \varphi(k \cdot \alpha'(l) \otimes \alpha'' \circ \beta) \\ &= \sum (k\alpha'(l) \otimes 1)\varphi(\alpha'' \circ \beta) \\ &= \sum (k \otimes 1)(\alpha'(l) \otimes 1)\varphi(\alpha'')\varphi(\beta) \end{aligned}$$

Note that we would be done if $\varphi(\alpha)(l \otimes 1) = \sum(\alpha'(l) \otimes 1)\varphi(\alpha'')$ for all $\alpha \in \mathcal{A}, l \in H^*(K)$, observe this propagates under multiplying α's and then calculate once more:

$$\begin{aligned} \varphi(Sq^1)(l \otimes 1) &= (\iota_1 \otimes 1 + 1 \otimes Sq^1)(l \otimes 1) \\ &= (\iota_1 l \otimes 1) + (Sq^1(l) \otimes 1) + (l \otimes Sq^1) \\ &= (Sq^1(l) \otimes 1) + (l \otimes 1)(\iota_1 \otimes 1 + 1 \otimes Sq^1) \\ \varphi(Sq^2)(l \otimes 1) &= (\iota_2 \otimes 1 + \iota_1^2 \otimes 1 + \iota_1 \otimes Sq^1 + 1 \otimes Sq^2)(l \otimes 1) \\ &= \iota_2 l \otimes 1 + \iota_1^2 \otimes 1 + \iota_1 Sq^1(l) \otimes 1 + \iota_1 l \otimes Sq^1 + Sq^2(l) \otimes 1 \\ &\quad + Sq^1(l) \otimes Sq^1 + l \otimes Sq^2 \\ &= (Sq^2(l) \otimes 1) + (Sq^1(l) \otimes 1)(\iota_1 \otimes Sq^1) \\ &\quad + (l \otimes 1)(\iota_2 \otimes 1 + \iota_1^2 \otimes 1 + \iota_1 \otimes Sq^1 + 1 \otimes Sq^2) \end{aligned}$$

Of course this is nothing but the verification of compatibility of φ and $id_{H^*(K)}$ in the universal property of a semidirect product of Hopf algebras. □

Appendix II

In this appendix we compile several calculations of Steenrod operations in $\overline{H_*(BSO)}$ that are needed in section 4.2 of chapter 2.

LEMMA 1. *In low degrees of* $\overline{H_*(BO)}$ *we have:*

x	$Sq_1(x) = Q_0(x)$	$Sq_2(x)$	$Q_1(x)$
x_1	0	0	0
x_2	x_1	0	0
x_3	x_1^2	0	0
x_4	$x_1x_2 + x_3$	x_2	x_1
x_5	0	x_3	x_1^2
x_6	$x_5 + x_1^2x_3$	x_2^2	x_3
x_8	$x_1x_2^3 + x_1x_2x_4 + x_3x_4$	$x_6 + x_4x_2$	$x_5 + x_1x_4 + x_1^2x_3 + x_1^3x_2$
x_9	0	0	0
x_{10}	x_9	$x_3x_5 + x_2^4$	$x_1^2x_5$
x_{11}	x_5^2	0	0
x_{12}	$x_{11} + x_6x_5 + x_6x_3x_1^2$	$x_{10} + x_5^2 + x_2^2x_6$	$x_9 + x_3x_6$
x_{13}	0	x_{11}	x_5^2
x_{14}	x_{13}	x_6^2	x_{11}
x_{16}	$-$	$-$	$-$
x_{17}	0	0	0
x_{18}	x_{17}	x_2^8	0
x_{19}	x_9^2	0	0
x_{20}	$x_{19} + x_9x_10$	$x_{18} + x_3x_5^3 +$ $+x_3x_5x_{10} + x_2^4x_{10}$	$x_{17} + x_1^2x_5^3 + x_1^2x_5x_{10}$
x_{21}	x_5^4	x_{19}	x_9^2

where we have not given the values for x_{16} since they are not needed in section 4.2 (as $x_{16}^2 \in H_{32}(BSO)$, but $x_{16} \notin H_{16}(BSO)$) and they are too long to fit into the table.

PROOF. This is just a tedious calculation using the algorithm from [**GiPeRa 88**]. □

LEMMA 2. *For $k > l + 1 \geq 3$ we have*

$$Sq_1(x_{2^l+1}) = 0$$
$$Sq_1(x_{2^{l+1}+2}) = x_{2^k+1}$$

in $\overline{H_*(BO)}$. In $H_*(K_2)$ we furthermore have:

$$\begin{aligned} Sq_1(x_{2^{l+1}+2^l}) &= x_{2^l+2^{l-1}} \cdot x_{2^{l-1}+2^{l-2}} \cdot \ldots \cdot x_{12} \cdot x_6 \cdot x_5 \\ &\quad + x_{2^l+2^{l-1}} \cdot x_{2^{l-1}+2^{l-2}} \cdot \ldots \cdot x_{12} \cdot x_6 \cdot x_3 \cdot x_1^2 \\ Sq_1(x_{2^k+2^l}) &= x_{2^{k-1}+2^{l-1}} \cdot x_{2^{k-2}+2^{l-2}} \cdot \ldots \cdot x_{2^{k-l+1}+2} \cdot x_{2^{k-l+1}+1} \end{aligned}$$

Similarly in $\overline{H_*(BO)}$:

$$\begin{aligned} Sq_2(x_{2^{l+1}+1}) &= 0 \\ Sq_2(x_{2^{l+2}+2}) &= 0 \\ Sq_2(x_{2^{l+3}+4}) &= x_{2^{l+3}+2} \end{aligned}$$

And in $H_*(K_2)$ we find:

$$\begin{aligned} Sq_2(x_{2^{l+2}+2^{l+1}}) &= x_{2^{l+1}+2^l} \cdot x_{2^l+2^{l-1}} \cdot \ldots \cdot x_{12} \cdot x_{10} \\ &\quad + x_{2^{l+1}+2^l} \cdot x_{2^l+2^{l-1}} \cdot \ldots \cdot x_{12} \cdot x_6 \cdot x_2^2 \\ Sq_2(x_{2^{l+3}+2^{l+1}}) &= x_{2^{l+2}+2^l} \cdot x_{2^{l+1}+2^{l-1}} \cdot \ldots \cdot x_{20} \cdot x_{18} \\ &\quad + x_{2^{l+2}+2^l} \cdot x_{2^{l+1}+2^{l-1}} \cdot \ldots \cdot x_{20} \cdot x_{10} \cdot x_5 \cdot x_3 \\ Sq_2(x_{2^{k+2}+2^{l+1}}) &= x_{2^{k+1}+2^l} \cdot x_{2^k+2^{l-1}} \cdot \ldots \cdot x_{2^{k-l+3}+4} \cdot x_{2^{k-l+3}+2} \end{aligned}$$

and finally in $H_*(K_2)$ we obtain:

$$Q_1(x_{2^k+2^{l+1}}) \equiv x_{2^{k-1}+2^l} \cdot x_{2^{k-2}+2^{l-1}} \cdot \ldots \cdot x_{2^{k-l+1}+4} \cdot x_{2^{k-l+1}+1} \quad mod\ I^{l+1}$$

Proof. The first case:

$$\begin{aligned} Sq_1(d_{2^n+1}) &= Sq_1(y_{2^n+1}) \\ Sq_1(d_{2^n+1}) &= (2^n+1)d_{2^n} \\ &= (2^n+1)(y_1^{2^n} + 2y_2^{2^{n-1}} + \ldots + 2^{n-1}y_{2^{n-1}}^2 + 2^n y_{2^n}) \end{aligned}$$

Reducing this *mod* 2 gives $Sq_1(x_{2^n+1}) = x_1^{2^n}$ which is zero for $n \geq 2$. The second case:

$$\begin{aligned} Sq_1(d_{2^n+2}) &= Sq_1(y_{2^{n-1}+1}^2 + 2y_{2^n+2}) \\ &= 2y_{2^{n-1}+1}Sq_1(y_{2^{n-1}+1}) + 2Sq_1(y_{2^n+2}) \\ Sq_1(d_{2^n+2}) &= (2^n+2)d_{2^n+1} \\ &= (2^n+2)y_{2^n+1} \end{aligned}$$

Reducing *mod* 2 yields

$$Sq_1(x_{2^n+2}) = x_{2^{n-1}+1}Sq_1(x_{2^{n-1}+1}) + x_{2^n+1} = x_{2^n+1}$$

once $n \geq 3$. The third case:

$$\begin{aligned}
Sq_1(d_{2^n+2^m}) &= Sq_1\left(y^{2^m}_{2^{n-m}+1} + 2y^{2^{m-1}}_{2^{n-m+1}+2} + \ldots + 2^{m-1}y^2_{2^{n-1}+2^{m-1}} + 2^m y_{2^m+2^n}\right) \\
&= 2^m\left(y^{2^m-1}_{2^{n-m}+1}Sq_1(y_{2^{n-m}+1}) + y^{2^{m-1}-1}_{2^{n-m+1}+2}Sq_1(y_{2^{n-m+1}+2})\right. \\
&\quad \left. + \ldots + y_{2^{n-1}+2^{m-1}}Sq_1(y_{2^{n-1}+2^{m-1}}) + Sq_1(y_{2^m+2^n})\right) \\
Sq_1(d_{2^n+2^m}) &= (2^n+2^m)d_{2^n+2^m-1} \\
&= (2^n+2^m)y_{2^n+2^m-1}
\end{aligned}$$

Reducing this *mod* 2 and projecting into $H_*(K)$ yields

$$\begin{aligned}
Sq_1(x_{2^n+2^m}) &= x_{2^n+2^m-1} + x_{2^{n-1}+2^{m-1}}Sq_1(x_{2^{n-1}+2^{m-1}}) \\
&= x_{2^{n-1}+2^{m-1}}Sq_1(x_{2^{n-1}+2^{m-1}})
\end{aligned}$$

once $m \geq 2$, giving a recursion whose initial value can be read off from either 1 for $n = m+1$ (i.e. $Sq_1(x_6)$) or from the previous case when $n > m+1$. This recursion immediately gives the claimed formulas.

The fourth case:

$$\begin{aligned}
Sq_2(d_{2^n+1}) &= Sq_2(y_{2^n+1}) \\
Sq_2(d_{2^n+1}) &= (2^n+1)(2^{n-1}-1)d_{2^n-1} \\
&= (2^n+1)(2^{n-1}-1)y_{2^n-1}
\end{aligned}$$

which yields

$$Sq_2(x_{2^n+1}) = x_{2^n-1} = 0$$

once $n \geq 3$. The fifth case:

$$\begin{aligned}
Sq_2(d_{2^n+2}) &= Sq_2(y^2_{2^{n-1}+1} + 2y_{2^n+2}) \\
&= 2y_{2^{n-1}+1}Sq_2(y_{2^{n-1}+1}) + Sq_1(y_{2^{n-1}+1})^2 + 2Sq_2(y_{2^n+2}) \\
&= 2y_{2^{n-1}+1}Sq_2(y_{2^{n-1}+1}) + 2Sq_2(y_{2^n+2}) \\
&\quad + \left((2^{n-1}+1)(y_1^{2^{n-1}} + 2y_2^{2^{n-2}} + \ldots + 2^{n-2}y^2_{2^{n-2}} + 2^{n-1}y_{2^{n-1}})\right)^2 \\
&\equiv 2y_{2^{n-1}+1}Sq_2(y_{2^{n-1}+1}) + 2Sq_2(y_{2^n+2}) + y_1^{2^n} \quad mod\ 4 \\
Sq_2(d_{2^n+2}) &= (2^{n-1}+1)(2^n-1)d_{2^n} \\
&= (2^{n-1}+1)(2^n-1)(y_1^{2^n} + 2y_2^{2^{n-1}} + \ldots + 2^{n-1}y^2_{2^{n-1}} + 2^n y_{2^n}) \\
&\equiv 3y_1^{2^n} + 2y_2^{2^{n-1}}
\end{aligned}$$

Reducing *mod* 2:

$$Sq_2(x_{2^n+2}) = x_{2^{n-1}+1}Sq_2(x_{2^{n-1}+1}) + x_1^{2^n} + x_2^{2^{n-1}} = 0$$

once $n \geq 4$. The sixth:

$$\begin{aligned}
Sq_2(d_{2^n+4}) &= Sq_2(y_{2^{n-2}+1}^4 + 2y_{2^{n-1}+2}^2 + 4y_{2^n+4}) \\
&= 4y_{2^{n-2}+1}^3 Sq_2(y_{2^{n-2}+1}) + 6y_{2^{n-2}+1}^2 Sq_1(y_{2^{n-2}+1})^2 \\
&\quad + 4y_{2^{n-1}+2} Sq_2(y_{2^{n-1}+2}) + 2Sq_1(y_{2^{n-1}+2})^2 + 4Sq_2(y_{2^n+4}) \\
&= 4y_{2^{n-2}+1}^3 Sq_2(y_{2^{n-2}+1}) + 4y_{2^{n-1}+2} Sq_2(y_{2^{n-1}+2}) + 4Sq_2(y_{2^n+4}) \\
&\quad + 6y_{2^{n-2}+1}^2 \left((2^{n-2}+1)(y_1^{2^{n-2}} + 2y_2^{2^{n-3}} + \ldots + 2^{n-3}y_{2^{n-3}}^2 + 2^{n-2}y_{2^{n-2}})\right)^2 \\
&\quad + 2\left((2^{n-2}+1)y_{2^{n-1}+1} - y_{2^{n-2}+1}Sq_1(y_{2^{n-2}+1})\right)^2 \\
&\equiv 4y_{2^{n-2}+1}^3 Sq_2(y_{2^{n-2}+1}) + 4y_{2^{n-1}+2} Sq_2(y_{2^{n-1}+2}) + 4Sq_2(y_{2^n+4}) \\
&\quad + 6y_{2^{n-2}+1}^2 y_1^{2^{n-1}} + 2y_{2^{n-2}+1}^2 Sq_1(y_{2^{n-2}+1})^2 \\
&\quad + 4y_{2^{n-2}+1} Sq_1(y_{2^{n-2}+1}) y_{2^{n-1}+1} + 2y_{2^{n-1}+1}^2 \quad mod\ 8 \\
&= 4y_{2^{n-2}+1}^3 Sq_2(y_{2^{n-2}+1}) + 4y_{2^{n-1}+2} Sq_2(y_{2^{n-1}+2}) + 4Sq_2(y_{2^n+4}) \\
&\quad + 4y_{2^{n-2}+1} Sq_1(y_{2^{n-2}+1}) y_{2^{n-1}+1} + 2y_{2^{n-1}+1}^2 + 6y_{2^{n-2}+1}^2 y_1^{2^{n-1}} \\
&\quad + 2y_{2^{n-2}+1}^2 \left((2^{n-2}+1)(y_1^{2^{n-2}} + 2y_2^{2^{n-3}} + \ldots + 2^{n-3}y_{2^{n-3}}^2 + 2^{n-2}y_{2^{n-2}})\right)^2 \\
&\equiv 4y_{2^{n-2}+1}^3 Sq_2(y_{2^{n-2}+1}) + 4y_{2^{n-1}+2} Sq_2(y_{2^{n-1}+2}) + 4Sq_2(y_{2^n+4}) \\
&\quad + 4y_{2^{n-2}+1} Sq_1(y_{2^{n-2}+1}) y_{2^{n-1}+1} + 2y_{2^{n-1}+1}^2 \quad mod\ 8 \\
Sq_2(d_{2^n+4}) &= (2^{n-1}+2)(2^n+1)d_{2^n+2} \\
&= (2^{n-1}+2)(2^n+1)(y_{2^{n-1}+1}^2 + 2y_{2^n+2}) \\
&\equiv 2y_{2^{n-1}+1}^2 + 4y_{2^n+2} \quad mod\ 8
\end{aligned}$$

Reducing *mod* 2:

$$\begin{aligned}
Sq_2(x_{2^n+4}) &= x_{2^{n-2}+1}^3 Sq_2(x_{2^{n-2}+1}) + x_{2^{n-1}+2} Sq_2(x_{2^{n-1}+2}) \\
&\quad + x_{2^{n-2}+1} Sq_1(x_{2^{n-2}+1}) x_{2^{n-1}+1} + x_{2^n+2} \\
&= x_{2^n+2}
\end{aligned}$$

for $n \geq 5$. The seventh:

$$\begin{aligned}
Sq_2(d_{2^n+2^m}) &= Sq_2\left(y_{2^{n-m}+1}^{2^m} + 2y_{2^{n-m+1}+2}^{2^{m-1}} + \ldots + 2^{m-1}y_{2^{n-1}+2^{m-1}}^2 + 2^m y_{2^m+2^n}\right) \\
&= 2^m \left(y_{2^{n-m}+1}^{2^m-1} Sq_2(y_{2^{n-m}+1}) + y_{2^{n-m+1}+2}^{2^{m-1}-1} Sq_2(y_{2^{n-m+1}+2})\right. \\
&\quad \left. + \ldots + y_{2^{n-1}+2^{m-1}} Sq_2(y_{2^{n-1}+2^{m-1}}) + Sq_2(y_{2^m+2^n})\right) \\
&\quad + 2^{m-1}\left((2^m-1)y_{2^{n-m}+1}^{2^m-2} Sq_1(y_{2^{n-m}+1})^2\right. \\
&\quad \left. + (2^{m-1}-1)y_{2^{n-m+1}+2}^{2^{m-1}-2} Sq_1(y_{2^{n-m+1}+2})^2 + \ldots + Sq_1(y_{2^{n-1}+2^{m-1}})^2\right)
\end{aligned}$$

To continue without having to expand the various Sq_i's denote by J the ideal in $\mathbb{Z}[y_i : i \in \mathbb{N}]$ formed by those elements all of whose 2-power-'demultiples' (i.e. the element itself, its half, its quarter ..., whenever those exist) vanish when sent to $H_*(K)$. We then find:

$$\begin{aligned}
&\equiv 2^m y_{2^{n-1}+2^{m-1}} Sq_2(y_{2^{n-1}+2^{m-1}}) + 2^m Sq_2(y_{2^m+2^n}) + \\
&\quad + 2^{m-1} Sq_1(y_{2^{n-1}+2^{m-1}})^2 \quad mod\ J \\
&= 2^m y_{2^{n-1}+2^{m-1}} Sq_2(y_{2^{n-1}+2^{m-1}}) + 2^m Sq_2(y_{2^m+2^n}) \\
&\quad + 2^{m-1} \Big((2^{n-1}+1) y_{2^{n-1}+2^{m-1}-1} - y_{2^{n-m}+1}^{2^{m-1}-1} Sq_1(y_{2^{n-m}+1}) \\
&\quad - y_{2^{n-m+1}+2}^{2^{m-2}-1} Sq_1(y_{2^{n-m+1}+2}) - \ldots - y_{2^{n-2}+2^{m-2}} Sq_1(y_{2^{n-2}+2^{m-2}}) \Big)^2 \\
&\equiv 2^m y_{2^{n-1}+2^{m-1}} Sq_2(y_{2^{n-1}+2^{m-1}}) + 2^m Sq_2(y_{2^m+2^n}) \quad mod\ J
\end{aligned}$$

$$\begin{aligned}
Sq_2(d_{2^n+2^m}) &= (2^{n-1}+2^{m-1})(2^n+2^m-3) d_{2^n+2^m-2} \\
&= (2^{n-1}+2^{m-1})(2^n+2^m-3)(y^2_{2^{n-1}+2^{m-1}-1} + 2y_{2^n+2^m-2}) \\
&\equiv 0 \quad mod\ J
\end{aligned}$$

Equating these produces the recursion

$$Sq_2(x_{2^n+2^m}) = x_{2^{n-1}+2^{m-1}} Sq_2(x_{2^{n-1}+2^{m-1}})$$

for $m \geq 3$ whose inital values are again obtained from 1 for $n = m+1, m+2$ (i.e. $Sq_2(x_{12})$ and $Sq_2(x_{20})$) and which are covered by the previous calculation for $n > m+2$.
The final claim now follows by simple inspection. □

Lemma 3. *For $k \geq 3$ we have:*

$$\begin{aligned}
Sq_1(x_{2^k-3}) &= x^4_{2^{k-2}-1} \\
Sq_2(x_{2^k-3}) &= x_{2^k-5} \\
Sq_1(x_{2^k-5}) &= x^2_{2^{k-1}-3} \\
Q_1(x_{2^k-3}) &= x^2_{2^{k-1}-3}
\end{aligned}$$

in $\overline{H_*(BO)}$.

Proof. This is comparatively simple: The equations

$$Sq_1(y_{2^k-3}) = Sq_1(d_{2^k-3}) = (2^k-3)d_{2^k-4} = (2^k-3)(y^4_{2^{k-2}-1} + 2y^2_{2^{k-1}-2} + 4y_{2^k-4})$$
$$Sq_2(y_{2^k-3}) = Sq_2(d_{2^k-3}) = (2^k-3)(2^{k-1}-3)d_{2^k-5} = (2^k-3)(2^{k-1}-3)y_{2^k-5}$$
$$Sq_1(y_{2^k-5}) = Sq_1(d_{2^k-5}) = (2^k-5)d_{2^k-6} = (2^k-5)(y^2_{2^{k-1}-3} + 2y_{2^k-6})$$

immediately reduce to the first three claims, and the fourth follows by putting them together. □

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