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Summary

A *branching within branching process*, introduced in [27], is a process that consists of two branching components, where one (often called parasites) proliferates inside the individuals (cells) of the other. It can be used to describe host-parasite populations. In the present thesis, we extend this model of a *branching within branching process* by including a *generational random environment* that governs the parasite reproduction inside the cells.

We show that many results from the standard model also hold in our extended model. These results include *extinction-explosion principles* for the process of parasites $(\mathcal{Z}_n)_{n\geq 0}$ as well as for the process $(\mathcal{T}_n^*)_{n\geq 0}$ counting the number of infected cells. Moreover, we show a Kesten-Stigum-type result for the nondegeneracy of the limit of the normalised process of parasites. Further results concern the long-term behaviour of the process $(F_n(k))_{\geq 0}$ describing the number of infected cells with a given number $k \in \mathbb{N}$ of parasites. Finally, we take a look at two examples, where the conditional parasite offspring

distribution is linear fractional.

The techniques used here are close to the ones used by Gröttrup in [27]. Among others, we use the construction of certain size-biased processes, which, for example, enable us to connect the long-term behaviour of the processes $(\mathcal{T}_n^*)_{n\geq 0}$ and $(F_n(k))_{\geq 0}$ to the long-term behaviour of a certain branching process in random environment.

Contents

Introduction

The topic of this thesis falls in the scope of branching processes. In its simplest form, a branching process is used to describe the evolution of a population of individuals or particles, which reproduce independently and in an identical manner. The original model was introduced by Francis Galton and Henry William Watson in 1875 [56] and is therefore called the Galton-Watson process (GWP) or sometimes Bienaymé-Galton-Watson process, as Irénée-Jules Bienaymé derived a similar model independently of and earlier than the aforementioned. Galton and Watson developed their model to analyse the extinction of aristocratic family names. However, the possible fields of application of their model are numerous, e.g. populations of living organisms, electron multipliers, the spreading of viruses and many more. Hence, branching processes have been extensively studied in the last century, and, as this simple model has its limitations, many extensions to the original model have been made. An overview of some of these extensions and further references can be found in the book of Asmussen and Hering [7] as well as in the book of Athreya and Ney [10].

One extension of particular interest is the consideration of a random environment. Here, the i.i.d. reproduction of different individuals, crucial in the standard GWP setting, is relaxed in the following way: First, in each generation the reproduction law of the individuals is randomly chosen from a set of probability laws on $\mathbb{N}_0$, i.e. in general, individuals in different generations do not reproduce in an identical manner. Second, the multiplication of individuals is only independent given the environment. This model has been introduced by Smith and Wilkinson in [49, 50]. Generalisations and further results can be found in the articles of Athreya and Karlin [8, 9], Tanny [51, 52, 53], Geiger et al. [1, 2, 22, 23] and Liu et al. [25, 32, 30, 31, 42, 43, 54, 55].

In the present thesis, we extend a discrete time model suggested by Bansaye in [11]. Bansaye's model describes the evolution of parasites in dividing cells and is a discrete-time version of a model introduced by Kimmel in [37]. To be more precise, Bansaye's model comprises two branching components. The first one describes the deterministic cell division, i.e. in each generation cells split into two daughter cells. The second component describes an organism living inside these cells, e.g. parasites, as in [11] and in most parts of the following thesis, or some cellular components like organelles. After one unit of time each organism inside a cell produces offspring and shares these into the two daughter cells independently of all the other organisms.

Recently, this model has been extended in a couple of ways. In [12], Bansaye considered the model with a random environment, where in each cell the parasite multiplication is

governed by the random environment, and he also added an immigration component. This means that in each generation new parasites from outside the population contaminate the cells. In his PhD thesis [27], Gröttrup analysed a different extension of the original model. Instead of a deterministic cell division he allowed for a general random cell tree. Moreover, he included the possibility that the cell reproduction can influence the multiplication of the parasites and their sharing into the daughter cells. He called the corresponding process, describing the alive cells and the number of parasites they contain, a *branching within branching process*.
Here, we want to consider a simple *branching within branching process in random environment*. To be more precise, the cell tree is again random but cells can have at most two daughter cells and the cell reproduction does not influence the multiplication of the parasites. Moreover, we consider an environmental sequence that governs the parasite reproduction of an entire generation. Note that in [12] the random environment determined the parasite reproduction for each cell separately.

The thesis is structured as follows. In the first chapter we introduce the model of a branching within branching process including our extension to a generational random environment and give a precise definition of a *branching within branching process in random environment*.
In the following two chapters, we analyse two interesting processes arising from our model. The first one is the *process of contaminated (or infected) cells* $(\mathcal{T}_n^*)_{n\geq 0}$, describing the number of cells that are infected by parasites. The second one is the *process of parasites* $(\mathcal{Z}_n)_{n\geq 0}$, where $\mathcal{Z}_n$ denotes the number of parasites in generation n. Chapter 2 contains results for the process $(\mathcal{T}_n^*)_{n\geq 0}$, including the a.s. convergence after a proper normalisation, and an extinction-explosion principle for the process of parasites. It turns out, that the asymptotic behaviour of the process $(\mathcal{T}_n^*)_{n\geq 0}$ is closely connected to the behaviour of the process of the number of parasites along a randomly chosen cell line $(Z_{V_n})_{n\geq 0}$. Therefore, we introduce this process using the method of *size-biasing* and prove the crucial connection to the process of contaminated cells. Additionally, this chapter includes some results about the process $(F_n(k))_{n\geq 0}$ of the number of cells with exactly k parasites. The third chapter is devoted to the process $(\mathcal{Z}_n)_{n\geq 0}$. We show that after a suitable normalisation this process converges a.s. to a random variable W, say. The second part of this chapter introduces the construction of the *size-biased branching within branching process in random environment*, where this time, the spine is picked along the parasites and not along the cells as in Chapter 2. This construction enables us to prove a dichotomy on the asymptotic behaviour of the normalised process of $(\mathcal{Z}_n)_{n\geq 0}$. In the final section of this chapter, we are going to show that under this spinal construction the process of parasites along the spine forms a branching process in random environment with immigration.
The first part of Chapter 4 is devoted to results about the limit W of the normalised process of parasites in the case $\mathbb{P}(Surv) > 0$. In particular, we give equivalent conditions for the nondegeneracy of W. It turns out, that the *modified* $(Z \log Z)$-condition, known

from branching processes in random environment (BPRE), together with some technical assumptions, is not enough for the nondegeneracy of W. A second condition regarding the aforementioned process $(Z_{V_n})_{n\geq 0}$ is also needed. In the second part of Chapter 4, we show that under certain conditions the number of parasites in strongly infected cells is negligible compared to the total number of parasites, and that strongly infected cells of a generation $n \in \mathbb{N}$ do not have a big impact on the number of infected cells in the following generations.
In Chapter 5, we take a look at two examples of a BwBPRE, where the conditional parasite distribution is linear fractional.

Since this is an extension of the model introduced by Gröttrup, the structure (and approach) of our work is close to that in [27].

1. The model

In this chapter, we are going to extend the model of a branching within branching process, introduced by Gröttrup in [27], by adding a random environment. For easy comparisons, we use a similar notation. As mentioned in the introduction, our random environment solely affects the multiplication of parasites and their sharing into the daughter cells and does not influence the evolution of the cell tree. Before giving a description and a definition of our model, we recall the notion of the well-known Ulam-Harris tree.

1.1. The Ulam-Harris tree

Let

$$\mathbb{V} := \bigcup_{n\in\mathbb{N}_0} \{0,1\}^n,$$

denote the *infinite binary Ulam-Harris tree* where $\{0,1\}^0 := \{\emptyset\}$ and $\emptyset$ is the root. For convenience we write $v_1...v_n$ for the vector $(v_1,\dots,v_n) \in \mathbb{V}$, $n \in \mathbb{N}$, and $v = v_1...v_n$ describes the unique path

$$\emptyset \to v_1 \to \dots \to v_1...v_n = v$$

from the root $\emptyset$ to v. Further let $|v|$ denote the length of this path, so that $|v| = n$ for $v \in \{0,1\}^n$ and in this case we say that v is a member of the n-th generation. Moreover for $v = v_1...v_n$ we use the shorthand notation $v|k$ for the predecessor of v in generation $k \le n$, viz $v|0 := \emptyset$ and $v|k := v_1...v_k$, $k \ge 1$, and we write $u < v$ if v is a descendant of u, so that $u = v|k$ for a $k < n$. Finally, the concatenation $uv = u_1...u_m v_1...v_n$ is the vertex $v = v_1...v_n$ in the tree rooted in $u = u_1...u_m$.

1.2. Informal description of the model

Next, we informally describe the process of proliferating parasites in a cell tree in random environment, where the random environment $\mathbf{e} = (\mathbf{e}_n)_{n\ge 0}$ governs the reproduction and the sharing of the parasites. We assume that cells can die, produce one daughter cell or split into two daughter cells.

The process starts with a single cell contaminated with a single parasite. Given a parasite distribution $\mathbf{e}_0 = u \in \mathbb{M}(\mathbb{N}_0^2)$ in generation 0, the parasite gives birth to $x = (x_1, x_2) \in \mathbb{N}_0^2$ offspring according to this distribution, and the cell produces $i \in \{0, 1, 2\}$ daughter cells, independently of the environment and of the parasite reproduction. In the case $i = 0$ all descendants of the parasite die, in the case $i = 1$ all $x_1 + x_2$ descendants of the parasite go into the one living daughter cell and in the case $i = 2$, x_1 parasites go into the first daughter cell and x_2 parasites go into the second daughter cell. Now, all the cells of the first generation exhibit the same behaviour as before, but the parasites act according to the distribution given by the environmental component $\mathbf{e}_1$. Given the environment, all parasites of the first generation behave independently of one another as well as of the parasites of the previous generation, and the cells reproduce independently and independent of the environment as well as of the parasites. This mechanism goes on indefinitely.

The following picture shows a realization of the above described mechanism. Cells are denoted by ◯ and parasites are denoted by ■ and ●, respectively, where ■ indicates the parasites going to the first daughter cell (if it exists) and ● indicates the parasites going to the second daughter cell (if it exits) or to the first daughter cell (if only one daughter cell exists). Apart from that, the shape of a parasite does not matter and is solely for visual purposes. Moreover, $\longrightarrow$ indicates the parasite multiplication and —— indicates the cell division.

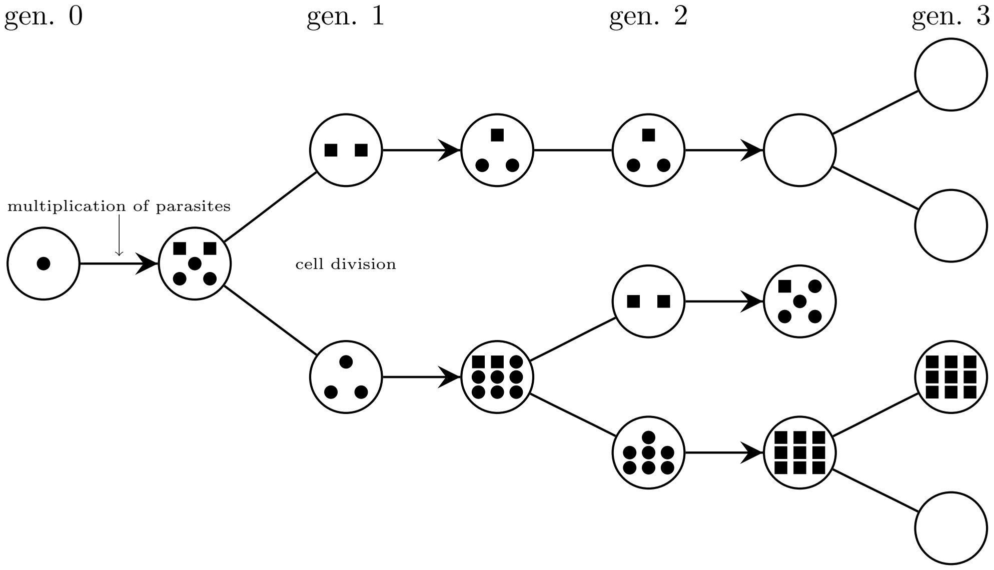

Figure 1.1.: Proliferating parasites in dividing cells up to generation three

1.3. Formal definition of the model

To give a formal representation of this process, let $(\Omega, \mathfrak{A}, \mathbb{P})$ be a probability space large enough to carry all random variables defined hereafter.

1.3.1. The cell population

We consider the cell population first. Therefore, let $(T_v)_{v\in\mathbb{V}}$ be a family of i.i.d. random variables taking values in $\{0,1,2\}$, and set

$$p_k := \mathbb{P}(T_\emptyset = k), \quad k \in \{0,1,2\},$$

as well as $p_k = 0$, $k \geq 3$. We consider the random subtree $\mathbb{T} := \bigcup_{n\in\mathbb{N}_0} \mathbb{T}_n$ of $\mathbb{V}$, where $\mathbb{T}_0 := \{\emptyset\}$ and

$$\mathbb{T}_n := \{v_1...v_n \in \mathbb{V} : v_1...v_{n-1} \in \mathbb{T}_{n-1}, 0 \leq v_n \leq T_{v_1...v_{n-1}} - 1\}$$

is the *set of cells in the n-th generation.* Since $(T_v)_{v\in\mathbb{V}}$ is an i.i.d. family, $\mathbb{T}$ forms a Galton-Watson tree with offspring distribution $(p_k)_{k\geq 0}$ and reproduction mean

$$\mu = p_1 + 2p_2 = 1 - p_0 + p_2.$$

Furthermore let $(\mathsf{A}_v)_{v\in\mathbb{V}}$ be a family of indicators describing which vertices of $\mathbb{V}$ belong to $\mathbb{T}$, viz.

$$\mathsf{A}_v = \begin{cases} 1, & \text{if } v \in \mathbb{T}_n, \\ 0, & \text{if } v \notin \mathbb{T}_n, \end{cases}$$

for $n \in \mathbb{N}_0$ and $v \in \mathbb{V}$ with $|v| = n$. In particular, $\mathsf{A}_\emptyset = 1$ $\mathbb{P}$-a.s. We say that cell v is alive if $\mathsf{A}_v = 1$, otherwise we call the cell dead. For a cell $v = v_1...v_n \in \mathbb{V}$, $n \geq 1$, we have

$$\{\mathsf{A}_v = 1\} = \{v \in \mathbb{T}_n\} = \{\mathsf{A}_{v|n-1} = 1, T_{v|n-1} - 1 \geq v_n\} \quad \mathbb{P}\text{-a.s.}$$

and therefore we get by iteration

$$\mathsf{A}_v = \mathsf{A}_{v|n-1}\mathbb{1}_{\{T_{v|n-1}-1\geq v_n\}} = \mathsf{A}_\emptyset \prod_{i=0}^{n-1} \mathbb{1}_{\{T_{v|i}-1\geq v_{i+1}\}} = \prod_{i=0}^{n-1} \mathbb{1}_{\{T_{v|i}-1\geq v_{i+1}\}} \quad \mathbb{P}\text{-a.s.}$$

Moreover, we have

$$\mathbb{P}\left(\left(\sum_{u=0}^{1}\mathsf{A}_{vu}\right)_{|v|=n}=(k_v)_{|v|=n}\,\middle|\,(\mathsf{A}_v)_{|v|=n}=(t_v)_{|v|=n}\right)=\mathbb{P}\left((t_vT_v)_{|v|=n}=(k_v)_{|v|=n}\right)$$
$$=\prod_{|v|=n,t_v=1}p_{k_v}\prod_{|v|=n,t_v=0}\delta_{0k_v},$$

where $k_v\in\{0,1,2\}$ and $t_v\in\{0,1\}$ for $|v|=n$, and δ_{ij} denotes the Kronecker delta. For $n\in\mathbb{N}_0$ we set

$$\mathcal{T}_n:=\#\mathbb{T}_n=\sum_{|v|=n}\mathsf{A}_v,$$

that is the *number of living cells in the n-th generation.* Clearly, the process $(\mathcal{T}_n)_{n\geq 0}$ is a standard Galton-Watson process with offspring distribution $(p_k)_{k\geq 0}$ and mean number of offspring μ.
In the next section, we are going to introduce the second branching component - the parasites - but it is convenient to give the definition of some related quantities already here. To describe the evolution of parasites in this random cell tree, we denote by Z_v the *number of parasites in a cell* $v\in\mathbb{V}$, and for $n\in\mathbb{N}_0$ we write $\mathbb{T}_n^*$ for the *set of infected cells in generation* n as well as $\mathcal{T}_n^*$ for the *number of infected cells in generation* n, so that

$$\mathbb{T}_n^*=\{v\in\mathbb{T}_n : Z_v>0\}\quad\text{and}\quad\mathcal{T}_n^*=\#\mathbb{T}_n^*.$$

1.3.2. The multiplication of parasites in a generational random environment

Next we focus on the parasite reproduction, and here the random environment comes into play. To define the random environment, set

$$\mathbb{M}:=\mathbb{M}(\mathbb{N}_0^2):=\left\{(u_{nm})_{n,m\in\mathbb{N}_0} : u_{nm}\geq 0\text{ for }n,m\in\mathbb{N}_0,\sum_{n,m\in\mathbb{N}_0}u_{nm}=1\right\},\qquad(1.1)$$

equipped with the trace σ-field $\mathfrak{M}$ induced by $\mathfrak{B}_{[0,1]}^{\infty}$. Let $\mathcal{U}$ be a random variable taking values in $\mathbb{M}$. Furthermore, let $\left(X^{(0)},X^{(1)}\right)$ be a pair of random variables taking values in $\mathbb{N}_0^2$ with

$$\mathbb{P}\left(\left(X^{(0)},X^{(1)}\right)\in\cdot\,\middle|\,\mathcal{U}\right)=\mathcal{U}.$$

The random environment is a sequence $\mathbf{e} = (\mathbf{e}_n)_{n\geq 0}$ of i.i.d. copies of $\mathcal{U}$ and governs the multiplication and sharing mechanism of the parasites. Further, let $\left(X_{k,v}^{(0)}, X_{k,v}^{(1)}\right)_{k\geq 1, v\in\mathbb{V}}$ be a family of conditionally independent random variables given $\mathbf{e}$, with

$$\mathbb{P}\left(\left(X_{k,v}^{(0)}, X_{k,v}^{(1)}\right) \in \cdot \middle| \mathbf{e}\right) = \mathbf{e}_{|v|}, \quad k \geq 1, v \in \mathbb{V}.$$

Additionally, we assume $(T_v)_{v\in\mathbb{V}}$ and $\left(\left(X_{k,v}^{(0)}, X_{k,v}^{(1)}\right)_{k\geq 1, v\in\mathbb{V}}, \mathbf{e}\right)$ to be independent.

More precisely, for a cell $v \in \mathbb{T}$ with at least one parasite and $i \in \{0, 1, 2\}$ daughter cells, $X_{k,v}^{(0)} + X_{k,v}^{(1)}$ is the number of offspring of the k-th parasite in the cell v, whereby all offspring die in the case $i = 0$, all offspring go into the only daughter cell in the case $i = 1$, and in the case $i = 2$, $X_{k,v}^{(0)}$ offspring go into the first daughter cell and $X_{k,v}^{(1)}$ offspring go into the second daughter cell. Thus, the number of parasites in the cells is recursively defined by putting $Z_\emptyset = 1$ and for $v \in \mathbb{V}$

$$\begin{aligned} Z_{v0} &= \mathbb{1}_{\{T_v=1\}} \sum_{k=1}^{Z_v} \left(X_{k,v}^{(0)} + X_{k,v}^{(1)}\right) + \mathbb{1}_{\{T_v=2\}} \sum_{k=1}^{Z_v} X_{k,v}^{(0)}, \\ Z_{v1} &= \mathbb{1}_{\{T_v=2\}} \sum_{k=1}^{Z_v} X_{k,v}^{(1)}. \end{aligned}$$

By definition we have $\{\mathsf{A}_v = 0\} \subseteq \{Z_v = 0\}$.

Note that the i.i.d. structure of the random environment ensures that all parasites show the same biological behaviour, and that parasites in different generations behave independently.

For later usage, let us introduce the random variables

$$\begin{gathered} X^{(0,1)} := X^{(0)} + X^{(1)}, \quad X^{(0,2)} := X^{(0)}, \quad X^{(1,2)} := X^{(1)}, \\ X_{k,v}^{(0,1)} := X_{k,v}^{(0)} + X_{k,v}^{(1)}, \quad X_{k,v}^{(0,2)} := X_{k,v}^{(0)}, \quad X_{k,v}^{(1,2)} := X_{k,v}^{(1)}, \\ X_{k,v}^{(0,0)} := X_{k,v}^{(1,0)} := X_{k,v}^{(1,1)} := 0, \quad k \geq 1, v \in \mathbb{V}, \end{gathered}$$

and define

$$\begin{gathered} \mathcal{U}^{(0,1)} := \mathbb{P}\left(X^{(0,1)} \in \cdot \middle| \mathcal{U}\right), \quad \mathcal{U}^{(0,2)} := \mathbb{P}\left(X^{(0,2)} \in \cdot \middle| \mathcal{U}\right), \\ \mathcal{U}^{(1,2)} := \mathbb{P}\left(X^{(1,2)} \in \cdot \middle| \mathcal{U}\right), \end{gathered}$$

as well as

$$\begin{gathered} \mathbf{e}_n^{(0,1)} := \mathbb{P}\left(X_{1,0^{*n}}^{(0,1)} \in \cdot \middle| \mathbf{e}\right), \quad \mathbf{e}_n^{(0,2)} := \mathbb{P}\left(X_{1,0^{*n}}^{(0,2)} \in \cdot \middle| \mathbf{e}\right), \\ \mathbf{e}_n^{(1,2)} := \mathbb{P}\left(X_{1,0^{*n}}^{(1,2)} \in \cdot \middle| \mathbf{e}\right), n \geq 0, \end{gathered}$$

where $0^{*0} = \emptyset$ and $0^{*n} = 0\dots0$ (n-times), $n \geq 1$. In particular, with these newly defined random variables, the number of parasites in cell $v = ui$, $u \in \mathbb{V}$, $i \in \{0,1\}$, is given by

$$Z_v = \sum_{k=1}^{Z_u} X_{k,u}^{(i,T_u)}.$$

1.3.3. The branching within branching process in random environment

Now we have all we need to define the process of proliferating parasites in a cell tree in random environment. Unless stated otherwise, we assume that we start with a single cell infected by a single parasite, i.e.

$$\mathcal{T}_0 = 1 \quad \text{and} \quad Z_\emptyset = 1 \quad \text{a.s.}$$

Definition 1.1. Given the i.i.d. environmental sequence $\mathbf{e}$ and all above introduced random variables, we call $\mathsf{BP} := (\mathsf{BP}_n)_{n\geq 0}$ with $\mathsf{BP}_n := (\mathsf{A}_v, Z_v)_{|v|=n}$ the associated *branching within branching process in random environment (BwBPRE)* and $\mathsf{BT} := (\mathsf{BT}_n)_{n\geq 0}$ with $\mathsf{BT}_n := (\mathsf{A}_v, Z_v)_{|v|\leq n}$ the associated *branching within branching tree in random environment.*

It will be useful to be able to analyse a BwBPRE with multiple root parasites, e.g. to make assertions about the subprocess rooted in a cell of generation $n \geq 1$. Therefore we introduce the family $(\mathbb{P}_z)_{z\geq 0}$ of probability measures, defined on $(\Omega, \mathfrak{A})$ and satisfying

$$\mathbb{P}_z(\mathcal{T}_0 = 1, Z_\emptyset = z) = 1,$$

such that $(T_v)_{v\in\mathbb{V}}$ is still an i.i.d. family of random variables with law $(p_k)_{k\geq 0}$ under each $\mathbb{P}_z$. Moreover, the law of the environment $\mathbf{e}$ is the same under every $\mathbb{P}_z$, and $\big(X_{k,v}^{(0)}, X_{k,v}^{(1)}\big)_{k\geq 1, v\in\mathbb{V}}$ is a family of conditionally independent random variables given $\mathbf{e}$ under $\mathbb{P}_z$, with

$$\mathbb{P}_z\left(\left(X_{k,v}^{(0)}, X_{k,v}^{(1)}\right) \in \cdot \,\middle|\, \mathbf{e}\right) = \mathbf{e}_{|v|}, \quad k \geq 1, v \in \mathbb{V}.$$

In addition, we assume $(T_v)_{v\in\mathbb{V}}$ and $\left(\left(X_{k,v}^{(0)}, X_{k,v}^{(1)}\right)_{k\geq 1, v\in\mathbb{V}}, \mathbf{e}\right)$ to be independent with respect to $\mathbb{P}_z$. Therefore, the evolution mechanism of the BwBPRE is the same under every $\mathbb{P}_z$. The expectation corresponding to $\mathbb{P}_z$ will be denoted by $\mathbb{E}_z$, and in accordance with the previously used notation we set $\mathbb{P} = \mathbb{P}_1$ and $\mathbb{E} = \mathbb{E}_1$. Sometimes, if convenient, we are going to write $\mathbb{P}_{(1,z)}$ for $\mathbb{P}_z$. Finally, we introduce a measure $\mathbb{P}_{(0,0)}$ for a process that never *lived*, i.e.

$$\mathbb{P}_{(0,0)}(\mathcal{T}_0 = 0, Z_\emptyset = 0) = 1$$

and thus $\mathbb{P}_{(0,0)}(\mathbf{BT} = (0,0)_{v \in \mathbb{V}}) = 1$.

1.3.4. The process of parasites and further notation

Next, we define the *process of parasites* $(\mathcal{Z}_n)_{n \geq 0}$ via

$$\mathcal{Z}_n := \sum_{v \in \mathbb{T}_n} Z_v, \quad n \in \mathbb{N}_0,$$

i.e. $\mathcal{Z}_n$ is the *number of parasites in the n-th generation.* We set

$$\begin{aligned}
\nu(\mathbf{e}_0) := \mathbb{E}\left[\mathcal{Z}_1 \,|\mathbf{e}\right] &= \mathbb{E}\left[\sum_{v \in \mathbb{T}_1} Z_v \middle| \mathbf{e}\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{T_\emptyset = 1\}} X_{1,\emptyset}^{(0,1)} + 1_{\{T_\emptyset = 2\}} \left(X_{1,\emptyset}^{(0,2)} + X_{1,\emptyset}^{(1,2)}\right) \middle| \mathbf{e}\right] \\
&= \mathbb{P}(T_\emptyset \neq 0)\, \mathbb{E}\left[X_{1,\emptyset}^{(0)} + X_{1,\emptyset}^{(1)} \middle| \mathbf{e}\right] \\
&= (p_1 + p_2)\, \mathbb{E}\left[X_{1,\emptyset}^{(0)} + X_{1,\emptyset}^{(1)} \middle| \mathbf{e}\right]
\end{aligned}$$

and

$$\nu := \mathbb{E}[\mathcal{Z}_1] = \mathbb{E}[\nu(\mathbf{e}_0)] = (p_1 + p_2)\mathbb{E}\left[X_{1,\emptyset}^{(0)} + X_{1,\emptyset}^{(1)}\right]. \tag{1.2}$$

We will assume that

$$0 < \nu < \infty \quad \text{and} \quad \mathbb{E}\left|\log \mathbb{E}\left[X_{1,\emptyset}^{(0)} + X_{1,\emptyset}^{(1)} \middle| \mathbf{e}\right]\right| < \infty. \tag{1.3}$$

Finally, we make the assumptions that

$$p_1 < 1 \quad \text{and} \quad \mathbb{P}(\mathcal{Z}_1 = 1) < 1. \tag{1.4}$$

This ensures that neither the number of cells nor the number of parasites stays constant for every generation. An easy consequence of the last assumption, regarding the case of multiple parasites in the root cell, is stated in the next

Lemma 1.2. *If* $\mathbb{P}(\mathcal{Z}_1 = 0) = 0$*, then* $\mathbb{P}(\mathcal{Z}_1 = 1) < 1$ *ensures*

$$\mathbb{P}_z(\mathcal{Z}_1 = z) < 1 \quad \textit{for all } z \in \mathbb{N}.$$

Proof. First note that $\mathbb{P}(\mathcal{Z}_1 = 0) = 0$ obviously yields $p_0 = 0$ and also $\mathbf{e}_0^{(0,1)}(\{0\}) = 0$ $\mathbb{P}$-a.s., since

$$0 = \mathbb{P}(\mathcal{Z}_1 = 0) = \mathbb{P}\left(X_{1,\emptyset}^{(0)} + X_{1,\emptyset}^{(1)} = 0\right) = \mathbb{E}\left[\mathbf{e}_0^{(0,1)}(\{0\})\right].$$

Further, observe that $\mathbf{e}_0^{(0,1)}(\{1\}) < 1$ with positive probability, because

$$1 > \mathbb{P}(\mathcal{Z}_1 = 1) = \mathbb{E}\left[\mathbf{e}_0^{(0,1)}(\{1\})\right].$$

For $z \in \mathbb{N}$ these two observations lead to

$$\begin{aligned}
\mathbb{P}_z(\mathcal{Z}_1 = z) &= \mathbb{P}\left(\sum_{i=1}^{z}\left(X_{i,\emptyset}^{(0)} + X_{i,\emptyset}^{(1)}\right) = z\right)\\
&= \sum_{\substack{x_1,\dots,x_z \in \{0,\dots,z\}\\ \sum_{i=1}^z x_i = z}} \mathbb{P}\left(X_{1,\emptyset}^{(0)} + X_{1,\emptyset}^{(1)} = x_1, \dots, X_{z,\emptyset}^{(0)} + X_{z,\emptyset}^{(1)} = x_z\right)\\
&= \sum_{\substack{x_1,\dots,x_z \in \{0,\dots,z\}\\ \sum_{i=1}^z x_i = z}} \mathbb{E}\left[\mathbb{P}\left(X_{1,\emptyset}^{(0)} + X_{1,\emptyset}^{(1)} = x_1, \dots, X_{z,\emptyset}^{(0)} + X_{z,\emptyset}^{(1)} = x_z \,\middle|\, \mathbf{e}\right)\right]\\
&= \sum_{\substack{x_1,\dots,x_z \in \{0,\dots,z\}\\ \sum_{i=1}^z x_i = z}} \mathbb{E}\left[\prod_{i=1}^{z} \mathbf{e}_0^{(0,1)}(\{x_i\})\right]\\
&= \mathbb{E}\left[\prod_{i=1}^{z} \mathbf{e}_0^{(0,1)}(\{1\})\right]\\
&= \mathbb{E}\left[\mathbf{e}_0^{(0,1)}(\{1\})^z\right]\\
&< 1,
\end{aligned}$$

where we used the conditional independence of $\left(X_{i,\emptyset}^{(0)}, X_{i,\emptyset}^{(1)}\right)$, $1 \le i \le z$, given $\mathbf{e}$, and the Lemma is proved. □

For later usage, we define the two filtrations $\mathcal{F} = (\mathcal{F}_n)_{n \ge 0}$ and $\mathcal{G} = (\mathcal{G}_n)_{n \ge 0}$ by $\mathcal{F}_0 := \sigma(Z_\emptyset)$, $\mathcal{G}_0 := \sigma(\mathbf{e}, Z_\emptyset)$ and

$$\begin{aligned}
\mathcal{F}_n &:= \sigma\left(Z_\emptyset, T_v, X_{k,v}^{(0)}, X_{k,v}^{(1)} : |v| \le n-1, k \ge 1\right), \quad n \ge 1,\\
\mathcal{G}_n &:= \sigma\left(\mathbf{e}, Z_\emptyset, T_v, X_{k,v}^{(0)}, X_{k,v}^{(1)} : |v| \le n-1, k \ge 1\right) = \sigma(\mathbf{e}, \mathcal{F}_n), \quad n \ge 1.
\end{aligned}$$

Put also

$$\mathcal{F}_\infty := \sigma\left(\bigcup_{n\geq 0}\mathcal{F}_n\right) \quad \text{as well as} \quad \mathcal{G}_\infty := \sigma\left(\bigcup_{n\geq 0}\mathcal{G}_n\right).$$

It is clear from the definition that **BP** and **BT** are adapted with respect to $\mathcal{F}$ as well as $\mathcal{G}$, and that they are $\mathcal{F}_\infty$-measurable as well as $\mathcal{G}_\infty$-measurable if seen as a vector.

1.4. The space of host-parasite trees

In this section, we give the construction of the measurable space our BwBPRE lives in. The corresponding notation will become handy in Section 3.3. In the process, we adopt the concept and notation of the space of host-parasite trees used by Gröttrup in [27], to describe the underlying tree structure of our model. Similar approaches can be found in [17, 40, 48].

Set $\mathsf{S} := \{(0,0)\} \cup (\{1\} \times \mathbb{N}_0)$ and denote the *set of host-parasite trees* by

$$\mathbb{S} := \mathsf{S}^{\mathbb{V}} = (\{(0,0)\} \cup (\{1\} \times \mathbb{N}_0))^{\mathbb{V}}.$$

Each element $(s_v, x_v)_{v\in\mathbb{V}}$ of $\mathbb{S}$ represents a host-parasite tree. For $w \in \mathbb{V}$, define the mappings

$$\mathsf{t}_w : \mathbb{S} \to \{0,1\}, \quad (s_v, x_v)_{v\in\mathbb{V}} \mapsto s_w \quad \text{and} \quad \mathfrak{z}_w : \mathbb{S} \to \mathbb{N}_0, \quad (s_v, x_v)_{v\in\mathbb{V}} \mapsto x_w.$$

Next define the sequence $(\mathfrak{S}_n)_{n\geq 0}$ of σ-fields on $\mathbb{S}$ generated by these projections, that is

$$\mathfrak{S}_n := \sigma(\mathsf{t}_v, \mathfrak{z}_v, |v| \leq n)$$

and set $\mathfrak{S} := \sigma(\bigcup_{n\geq 1}\mathfrak{S}_n)$. Clearly, $(\mathfrak{S}_n)_{n\geq 0}$ is a filtration of the measurable space $(\mathbb{S}, \mathfrak{S})$, and the random host-parasite tree $\mathbf{BP} = \mathbf{BT} = (\mathsf{A}_v, Z_v)_{v\in\mathbb{V}}$ is $\mathbb{S}$-valued and $\mathfrak{A}$-$\mathfrak{S}$-measurable, since every (A_v, Z_v) is a random vector taking values in S. Note that $(\mathbb{S}, \mathfrak{S})$ is a Polish space as a countable product of discrete spaces (see [16], Chapter IX §6), and its open subsets generate the σ-field $\mathfrak{S}$.

For $n \in \mathbb{N}_0$, let tr_n and $tr_{|n}$ denote the restrictions of a host-parasite tree to the n-th generation and the first n generations, respectively. To be more precise, set $\mathbb{S}_n := \mathsf{S}^{|v|\leq n}$, $n \in \mathbb{N}_0$, equipped with the canonical σ-field $\mathfrak{S}_{|n}$ and

$$tr_n : \mathbb{S} \to \mathsf{S}^{|v|=n}, \quad (s_v, x_v)_{v\in\mathbb{V}} \mapsto (s_v, x_v)_{|v|=n}$$

as well as

$$tr_{|n} : \mathbb{S} \to \mathbb{S}_n, \quad (s_v, x_v)_{v\in\mathbb{V}} \mapsto (s_v, x_v)_{|v|\leq n},$$

which are surjective mappings and $tr_{|n}$ is $\mathfrak{S}$-$\mathfrak{S}_{|n}$-measurable. With these definitions in mind, we can express the n-th generation and the first n generations of the BwBPRE as follows:

$$\mathbf{BP}_n = tr_n(\mathbf{BT}) \quad \text{and} \quad \mathbf{BT}_n = tr_{|n}(\mathbf{BT}).$$

Obviously, $\mathbf{BT}_n$ is $\mathfrak{A}$-$\mathfrak{S}_{|n}$-measurable and for every $A \in \mathfrak{S}_n$ there exists a set $B \in \mathfrak{S}_{|n}$, such that $tr_{|n}(A) = B$ and $\{\mathbf{BT} \in A\} = \{\mathbf{BT}_n \in B\}$.

Finally, consider the mappings

$$\tau_n : (\mathbb{S}, \mathfrak{S}) \to (\mathbb{N}_0, \mathcal{P}(\mathbb{N}_0)), \quad (s_v, x_v)_{v\in\mathbb{V}} \mapsto \sum_{|w|=n} \mathsf{t}_w((s_v, x_v)_{v\in\mathbb{V}}) = \sum_{|w|=n} s_w,$$
$$\tau_n^* : (\mathbb{S}, \mathfrak{S}) \to (\mathbb{N}_0, \mathcal{P}(\mathbb{N}_0)), \quad (s_v, x_v)_{v\in\mathbb{V}} \mapsto \sum_{|w|=n} s_w(1 - \delta_{0x_w})$$

and

$$\mathbf{z}_n : (\mathbb{S}, \mathfrak{S}) \to (\mathbb{N}_0, \mathcal{P}(\mathbb{N}_0)), \quad (s_v, x_v)_{v\in\mathbb{V}} \mapsto \sum_{|w|=n} s_w x_w.$$

Hence, the number of cells, the number of infected cells and the number of parasites in the n-th generation can be represented by

$$\mathcal{T}_n = \tau_n(\mathbf{BT}), \quad \mathcal{T}_n^* = \tau_n^*(\mathbf{BT}) \quad \text{and} \quad \mathcal{Z}_n = \mathbf{z}_n(\mathbf{BT}),$$

respectively.

1.5. The process started in a cell $|v| \geq 1$ and the process started in the n-th generation

In this short section, we take a look at the evolution of the process started in the n-th generation of the BwBPRE conditional on the random environment $\mathbf{e}$. Let $\mathbf{BT}^{(v)} := (\mathsf{A}_{vu}, Z_{vu})_{u\in\mathbb{V}}$ denote the BwBPRE on the subtree rooted in the cell $v \in \mathbb{V}$. It is clear from our model, that given the history $\mathbf{BT}_n$ of the process the evolution of $\mathbf{BT}^{(v)}$ for a cell v in generation n only depends on (A_v, Z_v). Moreover, given the cell-parasite configuration $\mathbf{BP}_n$ of the n-th generation, the processes starting in the different cells v of generation n behave independently conditional on the random environment $\mathbf{e}$, due to our assumptions. However, it needs to be taken into account, that the behaviour of each of these subprocesses starting in generation n is governed by the shifted environment $[\mathbf{e}]_n := (\mathbf{e}_m)_{m\geq n}$. This leads to the following

Proposition 1.3 (Conditional branching property)**.** *We have* $\mathbb{P}_{(t,z)}$*-a.s.*

$$\mathbb{P}_{(t,z)}\left((\mathsf{BT}^{(v)})_{|v|=n} \in \cdot \,\middle|\, \mathsf{BT}_n = (s_w, x_w)_{|w|\leq n}, \mathbf{e}\right) = \left(\bigotimes_{|v|=n} \mathbb{P}^{\mathsf{BT}|[\mathbf{e}]_n}_{(s_v,x_v)}\right)(\cdot)$$

for $n \in \mathbb{N}_0$*,* $(t,z) \in \mathsf{S}$ *and* $(s_w, x_w)_{|w|\leq n} \in \mathbb{S}_n$*.*

From this point of view, it is useful to allow for the BwBPRE to start with multiple root cells. To this end, let

$$R := \{(0,0)\} \cup \bigcup_{n\in\mathbb{N}} (\{n\} \times \mathbb{N}_0^n)$$

be the *set of possible root configurations*, and consider a family of probability measures $(\mathbb{P}_{(t,z)})_{(t,z)\in R}$ on the space $(\Omega, \mathfrak{A})$, such that under $\mathbb{P}_{(t,z)}$, $t \geq 1$, $z = (z_1, \ldots, z_t) \in \mathbb{N}_0^t$, the BwBPRE starts with t cells and z_i parasites in cell i, $1 \leq i \leq t$, i.e.

$$\mathbb{P}_{(t,z)}\left(\mathcal{T}_0 = t, Z_\emptyset(1) = z_1, \ldots, Z_\emptyset(t) = z_t\right) = 1,$$

where $Z_\emptyset(i)$ denotes the number of parasites in root cell i, $1 \leq i \leq t$. Furthermore, the measure $\mathbb{P}_{(0,0)}$ is the same as before and the environment $\mathbf{e}$ is still a sequence of i.i.d. random variables having the same law under any $\mathbb{P}_{(t,z)}$. To account for multiple root cells, the random variables describing the cell and the parasite evolution, respectively, need to be adjusted. Therefore, let $(T_v(i))_{v\in\mathbb{V},i\geq 1}$ be a family of independent random variables with law $(p_k)_{k\geq 0}$ under $\mathbb{P}_{(t,z)}$, and let $\left(X^{(0)}_{k,v}(i), X^{(1)}_{k,v}(i)\right)_{k\geq 1, v\in\mathbb{V}, i\geq 1}$ be a family of conditionally independent random variables given $\mathbf{e}$ under $\mathbb{P}_{(t,z)}$, with

$$\mathbb{P}_{(t,z)}\left(\left(X^{(0)}_{k,v}(i), X^{(1)}_{k,v}(i)\right) \in \cdot \,\middle|\, \mathbf{e}\right) = \mathbf{e}_{|v|}, \quad k \geq 1, v \in \mathbb{V}, i \geq 1.$$

Moreover, we assume $(T_v(i))_{v\in\mathbb{V},i\geq 1}$ and $\left(\left(X^{(0)}_{k,v}(i), X^{(1)}_{k,v}(i)\right)_{k\geq 1, v\in\mathbb{V}, i\geq 1}, \mathbf{e}\right)$ to be independent under $\mathbb{P}_{(t,z)}$. Recalling the definition of our initial model, the random variables $(T_v(i))_{v\in\mathbb{V}}$ and $\left(X^{(0)}_{k,v}(i), X^{(1)}_{k,v}(i)\right)_{k\geq 1, v\in\mathbb{V}}$ govern the evolution of the cells and parasites of the process starting in root cell i in the obvious way. Furthermore, we denote by $\mathsf{BP}(i)$ the BwBPRE starting in cell i, where $\mathsf{A}_v(i)$, $\mathbb{T}(i)$, $\mathbb{T}_n(i)$, $\mathcal{T}_n(i)$, $\mathbb{T}^*_n(i)$, $\mathcal{T}^*_n(i)$, $Z_v(i)$ and $\mathcal{Z}_n(i)$ are the corresponding random variables describing the alive cells, the cell tree, etc., of the process starting in the i-th ancestor cell. In particular,

$$\mathcal{T}_n(i) = \tau_n(\mathsf{BP}(i)), \quad \mathcal{T}^*_n(i) = \tau^*_n(\mathsf{BP}(i)) \quad \text{and} \quad \mathcal{Z}_n(i) = \mathbf{z}_n(\mathsf{BP}(i)).$$

Then, the total number of living cells, the total number of infected cells and the total number of parasites in the n-th generation of the BwBPRE is given by

$$\mathcal{T}_n = \sum_{i=1}^{t} \mathcal{T}_n(i), \quad \mathcal{T}_n^* = \sum_{i=1}^{t} \mathcal{T}_n^*(i) \quad \text{and} \quad \mathcal{Z}_n = \sum_{i=1}^{t} \mathcal{Z}_n(i) \quad \mathbb{P}_{(t,z)}\text{-a.s.},$$

respectively.

Note that the definition of these probability measures is in accordance with our previous notation. The expectation corresponding to $\mathbb{P}_{(t,z)}$ is denoted by $\mathbb{E}_{(t,z)}$.

2. The process of infected cells

This chapter comprises results about the process of infected cells $(\mathcal{T}_n^*)_{n\geq 0}$ and about the process $(F_n(k))_{n\geq 0}$ of the number of cells with exactly k parasites, defined later. As it is going to turn out, the asymptotic behaviour of these processes is closely related to the asymptotics of the process of parasites along a randomly chosen cell line. This random path in the cell tree is picked by using the method of *size-biasing*, an approach that will be introduced in the following section.

2.1. Size-biasing I

In [46] Lyons, Pemantle and Peres presented a tool to give probabilistic proofs of some classical limit theorems for GWP's. The original, more analytic proofs of these results relied heavily on the analysis of generating functions. The central idea of the method of Lyons et al. is the construction of a *size-biased measure* on the space of marked trees, using the notion of *size-biased distributions.* At this point, we only need the concept of size-biased distributions to construct a marked cell tree, but we will use their method in Chapter 3 in greater generality to construct a size-biased BwBPRE. The definition of a sized-biased distribution is given in Section A.3.1 of the appendix.

In the years following the article of Lyons, Pemantle and Peres, many authors have used similar approaches to give probabilistic proofs in a wide variety of different branching models. For example, in [45] Lyons used this approach to give a new proof of Biggins' martingale convergence theorem for branching random walks and in [41] Kyprianou and Rahimzadeh Sani did the same for multi-type branching random walks. Moreover, Kuhlbusch proved limit theorems for weighted branching processes in random environment in [40] and Gröttrup proved results for the branching within branching process in [27], both using the size-biased construction.

Now, we turn to the actual construction. Let $(\check{T}_n, C_n)_{n\geq 0}$ be a sequence of i.i.d. random variables on $(\Omega, \mathfrak{A})$ that is also independent of $(T_v)_{v\in\mathbb{V}}$ as well as independent of $\left(\left(X_{k,v}^{(0)}, X_{k,v}^{(1)}\right)_{k\geq 1, v\in\mathbb{V}}, \mathbf{e}\right)$, and satisfies for $n \in \mathbb{N}_0$

$$\mathbb{P}\left(\check{T}_n = k\right) = \frac{kp_k}{\mu}, \quad k \geq 0,$$

and in the case $p_k > 0$

$$\mathbb{P}\left(C_n = l \middle| \check{T}_n = k\right) = \frac{1}{k}$$

for $0 \le l \le k-1$. In particular, we have in the case $p_k > 0$

$$\mathbb{P}\left(\check{T}_n = k, C_n = l\right) = \mathbb{P}\left(C_n = l \middle| \check{T}_n = k\right)\mathbb{P}\left(\check{T}_n = k\right) = \frac{p_k}{\mu}$$

for $0 \le l \le k-1$, and this is even true in the case $p_k = 0$. Note that in our model $p_k = 0$ for $k \ge 3$. The *spine* $(V_n)_{n\ge 0}$ is now recursively defined by putting $V_0 := \emptyset$ and

$$V_n := V_{n-1}C_{n-1}, \quad n \ge 1.$$

Thus

$$\emptyset = V_0 \to V_1 \to V_2 \to \ldots \to V_n \to \ldots$$

is a random cell line in $\mathbb{V}$.

Plainly speaking, in each generation n the spinal cell produces offspring according to the size-biased distribution of the original cell offspring distribution and among these offspring the new spinal cell is chosen uniformly via C_n. The parasite reproduction is not influenced by this mechanism.

The following figure illustrates this procedure. Here, $\square$ denotes the cells along the randomly chosen cell line, i.e. the spine, and $\bigcirc$ denotes all other cells. Parasites are indicated by $\bullet$.

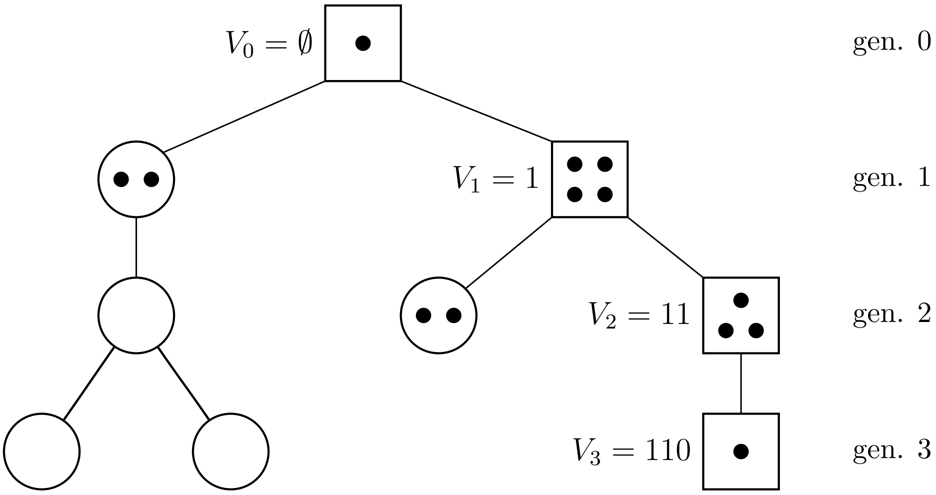

Figure 2.1.: Realisation of a random cell line up to generation three

As a final note, this structure including the sequence $(\check{T}_n, C_n)_{n\geq 0}$ shall be the same under any $\mathbb{P}_z$, $z \geq 0$.

2.2. The number of parasites along the spine I

Now, we consider the number of parasites along the spine $(V_n)_{n\geq 0}$. Put $Z_{V_0} = Z_\emptyset$ and for $n \geq 0$

$$\begin{aligned} Z_{V_{n+1}} &= \mathbb{1}_{\{\check{T}_n=1\}} \sum_{k=1}^{Z_{V_n}} \left(X^{(0)}_{k,V_n} + X^{(1)}_{k,V_n} \right) + \mathbb{1}_{\{\check{T}_n=2\}} \sum_{u=0}^{1} \mathbb{1}_{\{C_n=u\}} \sum_{k=1}^{Z_{V_n}} X^{(u)}_{k,V_n} \\ &= \mathbb{1}_{\{\check{T}_n=1\}} \sum_{k=1}^{Z_{V_n}} \left(X^{(0)}_{k,V_n} + X^{(1)}_{k,V_n} \right) + \mathbb{1}_{\{\check{T}_n=2\}} \sum_{k=1}^{Z_{V_n}} X^{(C_n)}_{k,V_n} \\ &= \sum_{k=1}^{Z_{V_n}} X^{(C_n,\check{T}_n)}_{k,V_n}. \end{aligned}$$

Furthermore set $\xi_n := (\check{T}_n, C_n, \mathbf{e}_n)$ as well as

$$f_{\xi_n}(s) := \mathbb{E}\left[s^{X^{(C_n,\check{T}_n)}_{k,V_n}} \,\middle|\, \xi_n \right]$$

for $n \geq 0, k \geq 1$ and $s \in [0,1]$.

The first important observation is that the process $(Z_{V_n})_{n\geq 0}$ forms a BPRE. This is stated in the following

Proposition 2.1. *The process of the number of parasites along the spine $(Z_{V_n})_{n\geq 0}$ is a BPRE in i.i.d. random environment $\xi := (\xi_n)_{n\geq 0}$.*

Since the above statement is obvious from our model assumptions, we omit the tedious calculations. Note that the corresponding conditional offspring distribution of the process $(Z_{V_n})_{n\geq 0}$ is given by

$$\begin{aligned} \mathbb{P}\left(X^{(C_n,\check{T}_n)}_{k,V_n} \in \cdot \,\middle|\, \xi \right) &= \sum_{v\in\{0,1\}^n} \sum_{t=1}^{2} \sum_{l=0}^{t-1} \mathbb{1}_{\{V_n=v,\check{T}_n=t,C_n=l\}} \mathbb{P}\left(X^{(l,t)}_{k,v} \in \cdot \,\middle|\, \xi \right) \\ &= \sum_{v\in\{0,1\}^n} \sum_{t=1}^{2} \sum_{l=0}^{t-1} \mathbb{1}_{\{V_n=v,\check{T}_n=t,C_n=l\}} \cdot \mathbf{e}^{(l,t)}_n \\ &= \mathbf{e}^{(C_n,\check{T}_n)}_n \quad \mathbb{P}\text{-a.s. for all } k \geq 1,\ n \geq 0. \end{aligned}$$

Here we used the independence of $(\check{T}_n, C_n)_{n\geq 0}$ and $\left(\left(X^{(0)}_{k,v}, X^{(1)}_{k,v} \right)_{k\geq 1, v\in\mathbb{V}}, \mathbf{e} \right)$.

Proposition 2.1 enables us to use the many results in the area of BPRE to determine the asymptotic behaviour of the process $(Z_{V_n})_{n\geq 0}$ depending on its criticality, see e.g. Afanasyev, Geiger, Kersting, Vatutin [2], Athreya and Karlin [8, 9], Geiger and Kersting [22], Geiger, Kersting and Vatutin [23], Smith and Wilkinson [50] as well as Tanny [53]. We formulate these results in our situation and postpone their proofs to the end of this section.

Proposition 2.2. *Let*

$$\mathbb{E}\left|\log \mathbb{E}[X_{1,\emptyset}^{(0)}|\mathbf{e}]\right|, \mathbb{E}\left|\log \mathbb{E}[X_{1,\emptyset}^{(1)}|\mathbf{e}]\right| < \infty. \tag{2.1}$$

For the process of parasites along the spine $(Z_{V_n})_{n\geq 0}$*, the following assertions are equivalent:*

(a) $\mathbb{E}[\log f'_{\xi_0}(1)] > 0$ *and* $\mathbb{E}[\log^-(1-f_{\xi_0}(0))] < \infty$*.*

(b) $\lim_{n\to\infty} \mathbb{P}(Z_{V_n} > 0) = c > 0$*, for some* $c \in (0,1]$*.*

Remark 2.3. Since this will be needed later on, note that the above result is also valid in the case of $z \geq 2$ root parasites, i.e. $\lim_{n\to\infty} \mathbb{P}_z(Z_{V_n} > 0) > 0$ iff

$$\mathbb{E}[\log f'_{\xi_0}(1)] > 0 \quad \text{and} \quad \mathbb{E}[\log^-(1-f_{\xi_0}(0))] < \infty.$$

Proposition 2.4 (Supercritical case)**.** *Let*

$$\mathbb{E}[\log f'_{\xi_0}(1)] > 0 \quad \textit{and} \quad \mathbb{E}[\log^-(1-f_{\xi_0}(0))] < \infty.$$

The process $\widetilde{W}_n := \pi_n^{-1} Z_{V_n}$*,* $n \geq 0$*, where* $\pi_0 := 1$ *and* $\pi_n := \prod_{i=0}^{n-1} f'_{\xi_i}(1)$*,* $n \geq 1$*, forms a nonnegative martingale with respect to the filtration*

$$(\sigma(Z_{V_0}, \ldots, Z_{V_n}, \xi_0, \xi_1, \xi_2, \ldots))_{n\geq 0},$$

and therefore $\widetilde{W} := \lim_{n\to\infty} \widetilde{W}_n$ *exists a.s. Moreover,* $\widetilde{W}$ *is nondegenerate iff*

$$\mathbb{E}\left[\frac{Z_{V_1}\log^+ Z_{V_1}}{f'_{\xi_0}(1)}\right] < \infty,$$

and in this case we have

$$\mathbb{E}[\widetilde{W}|(\xi_n)_{n\geq 0}] = 1 \quad \mathbb{P}\text{-}a.s. \quad \textit{and} \quad \{\widetilde{W} = 0\} = \{Z_{V_n} \to 0\} \quad \mathbb{P}\text{-}a.s.$$

Proposition 2.5 (Critical case)**.** *Let* $\mathbb{E}[\log f'_{\xi_0}(1)] = 0$ *and assume*

$$\mathbb{E}\left[\log^2 f'_{\xi_0}(1)\right] \in (0,\infty) \quad \textit{and} \quad \mathbb{E}\left[\frac{(1+\log^+ f'_{\xi_0}(1))f''_{\xi_0}(1)}{f'_{\xi_0}(1)^2}\right] < \infty.$$

Then there exists a constant $0 < c < \infty$*, such that*

$$\mathbb{P}(Z_{V_n} > 0) \sim cn^{-1/2}, \quad n \to \infty.$$

Proposition 2.6 (Strongly subcritical case)**.** *Let*

$$\mathbb{E}[\log f'_{\xi_0}(1)] < 0, \quad \mathbb{E}[f'_{\xi_0}(1) \log f'_{\xi_0}(1)] < 0$$

and

$$\mathbb{E}\left[\left(X^{(0)} + X^{(1)}\right) \log^+ \left(X^{(0)} + X^{(1)}\right)\right] < \infty$$

be valid. Then we have for some $0 < c_1 \le 1$ *that*

$$\mathbb{P}(Z_{V_n} > 0) \sim c_1 \left(\frac{\nu}{\mu}\right)^n, \quad n \to \infty.$$

Moreover,

$$\lim_{n\to\infty} \mathbb{P}(Z_{V_n} = k | Z_{V_n} > 0) = q_1(k), \quad k \ge 1,$$

where

$$\sum_{k=1}^{\infty} q_1(k) = 1 \quad \textit{and} \quad \sum_{k=1}^{\infty} kq_1(k) = c_1^{-1} < \infty,$$

and Z_{V_n} *is conditioned on* $\{Z_{V_n} > 0\}$ *uniformly integrable, i.e.*

$$\lim_{a\to\infty} \sup_{n\in\mathbb{N}} \mathbb{E}\left[Z_{V_n} \mathbb{1}_{\{Z_{V_n} > a\}} \middle| Z_{V_n} > 0\right] = 0.$$

Proposition 2.7 (Intermediately subcritical case)**.** *Let*

$$\mathbb{E}[\log f'_{\xi_0}(1)] < 0, \quad \mathbb{E}[f'_{\xi_0}(1) \log f'_{\xi_0}(1)] = 0$$

and assume

$$\mathbb{E}[f'_{\xi_0}(1) \log^2 f'_{\xi_0}(1)] < \infty \quad \textit{and} \quad \mathbb{E}[(1 + \log^- f'_{\xi_0}(1))f''_{\xi_0}(1)] < \infty.$$

Then there exists a constant $0 < c_2 < \infty$*, such that*

$$\mathbb{P}(Z_{V_n} > 0) \sim c_2 n^{-1/2} \left(\frac{\nu}{\mu}\right)^n, \quad n \to \infty.$$

Moreover,

$$\lim_{n\to\infty} \mathbb{P}(Z_{V_n} = k | Z_{V_n} > 0) = q_2(k), \quad k \geq 1,$$

where $\sum_{k\geq 1} q_2(k) = 1$.

For the weakly subcritical case set

$$\beta := \inf_{\vartheta\in[0,1]} \mathbb{E} f'_{\xi_0}(1)^{\vartheta}$$

and choose $\alpha \in [0,1]$, such that $\beta = \mathbb{E} f'_{\xi_0}(1)^{\alpha}$.

Proposition 2.8 (Weakly subcritical case)**.** *Let*

$$\mathbb{E}[\log f'_{\xi_0}(1)] < 0 \quad \textit{and} \quad 0 < \mathbb{E}[f'_{\xi_0}(1) \log f'_{\xi_0}(1)] < \infty$$

and assume

$$\mathbb{E}\left[\frac{f''_{\xi_0}(1)}{f'_{\xi_0}(1)^{1-\alpha}}\right] < \infty \quad \textit{and} \quad \mathbb{E}\left[\frac{f''_{\xi_0}(1)}{f'_{\xi_0}(1)^{2-\alpha}}\right] < \infty.$$

Then there exists a constant $0 < c_3 < \infty$*, such that*

$$\mathbb{P}(Z_{V_n} > 0) \sim c_3 n^{-3/2} \beta^n, \quad n \to \infty.$$

Moreover,

$$\lim_{n\to\infty} \mathbb{P}(Z_{V_n} = k | Z_{V_n} > 0) = q_3(k), \quad k \geq 1,$$

where $\sum_{k\geq 1} q_3(k) = 1$.

Now, we introduce some further common notation for BPRE. Recall that

$$f_{\xi_n}(s) = \mathbb{E}\left[s^{X^{(C_n,\check{T}_n)}_{k,V_n}} \middle| \xi \right], \quad s \in [0,1],$$

is the probability generating function (pgf) of $X^{(C_n,\check{T}_n)}_{k,V_n}$ under the measure $\mathbb{P}(\cdot|\xi)$. Theorem A.1 gives us

$$f_{n,\xi}(s) := \mathbb{E}\left[s^{Z_{V_n}} \,\middle|\, \xi\right] = f_{\xi_0} \circ \ldots \circ f_{\xi_{n-1}}(s)$$

and

$$f_n(s) := \mathbb{E}\left[s^{Z_{V_n}}\right] = \mathbb{E}\left[f_{\xi_0} \circ \ldots \circ f_{\xi_{n-1}}(s)\right]$$

for every $n \in \mathbb{N}$ and $s \in [0,1]$. Note that $(f_{\xi_n})_{n\geq 0}$ is an i.i.d. sequence with

$$\begin{aligned}
\mathbb{E}[f'_{\xi_0}(1)] = \mathbb{E}\left[X_{1,\emptyset}^{(C_0,\check{T}_0)}\right] &= \sum_{k=1}^{2}\sum_{l=0}^{k-1} \mathbb{P}(\check{T}_0 = k, C_0 = l) \cdot \mathbb{E}\left[X_{1,\emptyset}^{(l,k)}\right] \\
&= \sum_{k=1}^{2}\sum_{l=0}^{k-1} \frac{p_k}{\mu} \cdot \mathbb{E}\left[X_{1,\emptyset}^{(l,k)}\right] \\
&= \frac{p_1 + p_2}{\mu} \cdot \mathbb{E}\left[X_{1,\emptyset}^{(0)} + X_{1,\emptyset}^{(1)}\right] \\
&= \frac{\nu}{\mu},
\end{aligned}$$

where we used (1.2) for the last equation. Moreover, this yields

$$\mathbb{E}[Z_{V_n}] = f'_n(1) = \prod_{i=0}^{n-1} \mathbb{E}[f'_{\xi_i}(1)] = \left(\frac{\nu}{\mu}\right)^n .$$

Furthermore, if $\mathbb{E}|\log f'_{\xi_0}(1)| < \infty$, the theory of BPRE ([50], Theorem 3.1 and [8], Section 1) yields that $(Z_{V_n})_{n\geq 0}$ survives with positive probability iff

$$\mathbb{E}\left[\log f'_{\xi_0}(1)\right] > 0 \quad \text{and} \quad \mathbb{E}\left[\log^-\left(1 - f_{\xi_0}(0)\right)\right] < \infty. \tag{2.2}$$

Finally, we define for $s \in [0,1]$ the random pgf

$$g_{\mathbf{e}_n^{(l,t)}}(s) := \mathbb{E}\left[s^{X_{1,0*n}^{(l,t)}} \,\middle|\, \mathbf{e}\right], \quad n \geq 0, 0 \leq l \leq t-1 \leq 1.$$

We want to use the process $(Z_{V_n})_{n\geq 0}$ to prove assertions about the limiting behaviour of the process of the number of infected cells $(\mathcal{T}_n^*)_{n\geq 0}$ and other related quantities. The required connection is subject of the following

Proposition 2.9. *For every $m, n, z \in \mathbb{N}_0$ we have*

$$\mathbb{P}_z(Z_{V_n} = m|\mathbf{e}) = \mu^{-n} \cdot \mathbb{E}_z[\#\{v \in \mathbb{T}_n : Z_v = m\}|\mathbf{e}] \quad \mathbb{P}_z\text{-a.s.},$$

and thus

$$\mathbb{P}_z(Z_{V_n} > 0|\mathbf{e}) = \frac{\mathbb{E}_z[\mathcal{T}_n^*|\mathbf{e}]}{\mu^n} \quad \mathbb{P}_z\text{-}a.s.$$

In particular, this gives

$$\mathbb{P}_z(Z_{V_n} = m) = \mu^{-n} \cdot \mathbb{E}_z[\#\{v \in \mathbb{T}_n : Z_v = m\}],$$

and therefore

$$\mathbb{P}_z(Z_{V_n} > 0) = \frac{\mathbb{E}_z[\mathcal{T}_n^*]}{\mu^n}.$$

The proof of this proposition will be given after two easy consequences.

Corollary 2.10. *In the subcritical case, the expected number of infected cells* $(\mathcal{T}_n^*)_{n\geq 0}$ *satisfies*

$$\mathbb{E}\mathcal{T}_n^* \sim \begin{cases} c_1\nu^n, & \text{if the assumptions of Proposition 2.6 are fullfilled,} \\ c_2 n^{-1/2}\nu^n, & \text{if the assumptions of Proposition 2.7 are fullfilled,} \\ c_3 n^{-3/2}\mu^n\beta^n, & \text{if the assumptions of Proposition 2.8 are fullfilled,} \end{cases}$$

as $n \to \infty$, *with* c_1, c_2, c_3 *from the respective propositions.*

Proof. We know from Proposition 2.9 that

$$\mathbb{E}\mathcal{T}_n^* = \mu^n \mathbb{P}(Z_{V_n} > 0).$$

for all $n \geq 0$. Then, using the asymptotics for $\mathbb{P}(Z_{V_n} > 0)$ shown in Propositions 2.6, 2.7 and 2.8, respectively, we get the desired result. ☐

Next, we can state a first assertion about the number of cells with a given number of parasites and, in the strongly subcritical case, about the expected number of parasites within all these cells. To that end, define the *number of cells in the n-th generation having exactly k parasites*, i.e.

$$F_n(k) := \#\{v \in \mathbb{T}_n : Z_v = k\}, \quad n, k \geq 0. \tag{2.3}$$

Corollary 2.11. *We have for every* $k, z \geq 1$

$$\lim_{n\to\infty} \frac{F_n(k)}{\mu^n} = 0, \quad \text{in probability and in } L^1 \text{ under } \mathbb{P}_z.$$

Furthermore, under the assumptions of Proposition 2.6 we have for $k \geq 1$

$$\lim_{n\to\infty} \frac{\mathbb{E}[F_n(k)]}{\nu^n} = \frac{q_1(k)}{c_1^{-1}},$$

i.e. $(k\nu^{-n}\mathbb{E}[F_n(k)])_{k\geq 1}$ *converges weakly to the size-biasing of* $(q_1(k))_{k\geq 1}$.

Proof. Starting with $F_n(\cdot)$ and using once again Proposition 2.9 as well as the extinction-explosion principle for BPRE (see e.g. [8]), we have

$$\lim_{n\to\infty} \mathbb{E}_z\left[\frac{F_n(k)}{\mu^n}\right] = \lim_{n\to\infty} \mathbb{P}_z(Z_{V_n} = k) = 0$$

for $k, z \geq 1$. Hence, $F_n(k)/\mu^n$ converges in L^1 to 0 and therefore converges in particular in probability to 0 under $\mathbb{P}_z$ for $n \to \infty$.

Turning to the second assertion, we get

$$\begin{aligned}\lim_{n\to\infty} \frac{\mathbb{E}[F_n(k)]}{\nu^n} &= \lim_{n\to\infty}\left[\left(\frac{\mu}{\nu}\right)^n \mathbb{P}(Z_{V_n} = k)\right] \\ &= \lim_{n\to\infty}\left[\left(\frac{\mu}{\nu}\right)^n \mathbb{P}(Z_{V_n} > 0)\cdot \mathbb{P}(Z_{V_n} = k | Z_{V_n} > 0)\right] = \frac{q_1(k)}{c_1^{-1}},\end{aligned}$$

for $k \geq 1$, where we used Proposition 2.9 and Proposition 2.6. □

Now, we prove Proposition 2.9.

Proof of Proposition 2.9. For $n \in \mathbb{N}$, $v = v_1 ... v_n \in \mathbb{V}$, $u := v|n-1$ and $t_0, \dots, t_{n-1} \in \{1, 2\}$ with $v_1 \leq t_0 - 1, \dots, v_n \leq t_{n-1} - 1$, we get for $s \in [0, 1]$

$$\begin{aligned}&\mathbb{E}_z\left[\prod_{k=1}^{Z_u} s^{X_{k,u}^{(v_n,t_{n-1})}} \,\middle|\, (T_{v|j})_{0\leq j\leq n-2}, \left(X_{k',v'}^{(0)}, X_{k',v'}^{(1)}\right)_{k'\geq 1}^{|v'|\leq n-2}, \mathbf{e}\right] \\ =\ &\sum_{j\geq 0} \mathbb{1}_{\{Z_u=j\}}\cdot \mathbb{E}\left[\prod_{k=1}^{j} s^{X_{k,u}^{(v_n,t_{n-1})}} \,\middle|\, \left(X_{k',v'}^{(0)}, X_{k',v'}^{(1)}\right)_{k'\geq 1}^{|v'|\leq n-2}, \mathbf{e}\right] \\ =\ &\sum_{j\geq 0} \mathbb{1}_{\{Z_u=j\}}\cdot \mathbb{E}\left[\prod_{k=1}^{j} s^{X_{k,u}^{(v_n,t_{n-1})}} \,\middle|\, \mathbf{e}\right] \qquad (2.4) \\ =\ &\prod_{k=1}^{Z_u} \mathbb{E}\left[s^{X_{k,u}^{(v_n,t_{n-1})}} \,\middle|\, \mathbf{e}\right] \\ =\ &g_{\mathbf{e}_{n-1}^{(v_n,t_{n-1})}}(s)^{Z_u} \quad \mathbb{P}_z\text{-a.s.},\end{aligned}$$

where we used the independence of $(T_w)_{w\in\mathbb{V}}$ and $\left(\left(X^{(0)}_{k,w}, X^{(1)}_{k,w}\right)_{k\geq 1, w\in\mathbb{V}}, \mathbf{e}\right)$ in the first equation and the conditional independence of $\left(X^{(0)}_{k,w}, X^{(1)}_{k,w}\right)_{k\geq 1, w\in\mathbb{V}}$ given $\mathbf{e}$ in the third equation. For the second equality note that for $A \in \mathfrak{M}^{\mathbb{N}_0}$ and $B \in \bigotimes_{|w|\leq n-2}\mathcal{P}(\mathbb{N}_0^2)^{\mathbb{N}}$

$$
\begin{aligned}
&\int_{\left\{\mathbf{e}\in A, \left(X^{(0)}_{k',v'}, X^{(1)}_{k',v'}\right)_{k'\geq 1, |v'|\leq n-2}\in B\right\}} \prod_{k=1}^{j} s^{X^{(v_n,t_{n-1})}_{k,u}}\, d\mathbb{P} \\
=\ & \int_{\{\mathbf{e}\in A\}} \mathbb{E}\left[\mathbb{1}_{\left\{\left(X^{(0)}_{k',v'}, X^{(1)}_{k',v'}\right)_{k'\geq 1, |v'|\leq n-2}\in B\right\}} \prod_{k=1}^{j} s^{X^{(v_n,t_{n-1})}_{k,u}} \,\middle|\, \mathbf{e}\right] d\mathbb{P} \\
=\ & \int_{\{\mathbf{e}\in A\}} \mathbb{P}\left(\left(X^{(0)}_{k',v'}, X^{(1)}_{k',v'}\right)_{k'\geq 1, |v'|\leq n-2} \in B \,\middle|\, \mathbf{e}\right) \mathbb{E}\left[\prod_{k=1}^{j} s^{X^{(v_n,t_{n-1})}_{k,u}} \,\middle|\, \mathbf{e}\right] d\mathbb{P} \qquad (2.5) \\
=\ & \int_{\{\mathbf{e}\in A\}} \mathbb{E}\left[\mathbb{1}_{\left\{\left(X^{(0)}_{k',v'}, X^{(1)}_{k',v'}\right)_{k'\geq 1, |v'|\leq n-2}\in B\right\}} \mathbb{E}\left[\prod_{k=1}^{j} s^{X^{(v_n,t_{n-1})}_{k,u}} \,\middle|\, \mathbf{e}\right] \,\middle|\, \mathbf{e}\right] d\mathbb{P} \\
=\ & \int_{\left\{\mathbf{e}\in A, \left(X^{(0)}_{k',v'}, X^{(1)}_{k',v'}\right)_{k'\geq 1, |v'|\leq n-2}\in B\right\}} \mathbb{E}\left[\prod_{k=1}^{j} s^{X^{(v_n,t_{n-1})}_{k,u}} \,\middle|\, \mathbf{e}\right] d\mathbb{P},
\end{aligned}
$$

since $\left(X^{(0)}_{k,v'}, X^{(1)}_{k,v'}\right)_{k\geq 1, |v'|\leq n-2}$ and $\left(X^{(0)}_{k,u}, X^{(1)}_{k,u}\right)_{k\geq 1}$ are conditionally independent given $\mathbf{e}$.

From (2.4), it follows that

$$
\begin{aligned}
& \mathbb{E}_z\left[s^{Z_v} \,\middle|\, T_{v|j} = t_j, 0\leq j\leq n-1, \mathbf{e}\right] \\
=\ & \mathbb{E}_z\left[\prod_{k=1}^{Z_u} s^{X^{(v_n,t_{n-1})}_{k,u}} \,\middle|\, T_{v|j} = t_j, 0\leq j\leq n-2, \mathbf{e}\right] \\
=\ & \mathbb{E}_z\left[g_{\mathbf{e}_{n-1}^{(v_n,t_{n-1})}}(s)^{Z_u} \,\middle|\, T_{v|j} = t_j, 0\leq j\leq n-2, \mathbf{e}\right] \quad \mathbb{P}_z\text{-a.s.,}
\end{aligned}
$$

and hence iteration yields

$$
\mathbb{E}_z\left[s^{Z_v} \,\middle|\, T_{v|j} = t_j, 0\leq j\leq n-1, \mathbf{e}\right] = \left(g_{\mathbf{e}_0^{(v_1,t_0)}} \circ \ldots \circ g_{\mathbf{e}_{n-1}^{(v_n,t_{n-1})}}(s)\right)^z \quad \mathbb{P}_z\text{-a.s.}
$$

Analogously, we get for $s\in[0,1]$

$$
\mathbb{E}_z\big[s^{Z_{V_n}} | (\check{T}_j, C_j) = (t_j, v_{j+1}), 0\leq j\leq n-1, \mathbf{e}\big] = \left(g_{\mathbf{e}_0^{(v_1,t_0)}} \circ \ldots \circ g_{\mathbf{e}_{n-1}^{(v_n,t_{n-1})}}(s)\right)^z
$$

$\mathbb{P}_z$-a.s., i.e.

$$\mathbb{E}_z\left[s^{Z_v}\middle|T_{v|j}=t_j, 0\le j\le n-1, \mathbf{e}\right]=\mathbb{E}_z\left[s^{Z_{V_n}}\middle|(\check{T}_j, C_j)=(t_j, v_{j+1}), 0\le j\le n-1, \mathbf{e}\right]$$

$\mathbb{P}_z$-a.s. This implies for $m\in\mathbb{N}_0$

$$\begin{aligned}
&\mathbb{P}_z(Z_v=m, \mathsf{A}_v=1|\mathbf{e})\\
=&\sum_{t_0=v_1+1}^{2}\cdots\sum_{t_{n-1}=v_n+1}^{2}\mathbb{P}_z(Z_v=m, T_{v|j}=t_j, 0\le j\le n-1|\mathbf{e})\\
=&\sum_{t_0=v_1+1}^{2}\cdots\sum_{t_{n-1}=v_n+1}^{2}\mathbb{P}(T_{v|j}=t_j, 0\le j\le n-1|\mathbf{e})\\
&\cdot\mathbb{P}_z(Z_v=m|T_{v|j}=t_j, 0\le j\le n-1, \mathbf{e})\\
=&\sum_{t_0=v_1+1}^{2}\cdots\sum_{t_{n-1}=v_n+1}^{2}\prod_{k=0}^{n-1}p_{t_k}\mathbb{P}_z\big(Z_{V_n}=m\big|(\check{T}_j, C_j)=(t_j, v_{j+1}), 0\le j\le n-1, \mathbf{e}\big)\\
=&\sum_{t_0=v_1+1}^{2}\cdots\sum_{t_{n-1}=v_n+1}^{2}\prod_{k=0}^{n-1}p_{t_k}\mathbb{P}\big((\check{T}_j, C_j)=(t_j, v_{j+1}), 0\le j\le n-1\big|\mathbf{e}\big)^{-1}\\
&\cdot\mathbb{P}_z\big(Z_{V_n}=m, (\check{T}_j, C_j)=(t_j, v_{j+1}), 0\le j\le n-1|\mathbf{e}\big)\\
=&\ \mu^n\sum_{t_0=v_1+1}^{2}\cdots\sum_{t_{n-1}=v_n+1}^{2}\mathbb{P}_z\big(Z_{V_n}=m, (\check{T}_j, C_j)=(t_j, v_{j+1}), 0\le j\le n-1|\mathbf{e}\big)
\end{aligned}$$

$\mathbb{P}_z$-a.s., and therefore

$$\begin{aligned}
&\mu^{-n}\cdot\mathbb{E}_z[\#\{v\in\mathbb{T}_n: Z_v=m\}|\mathbf{e}]\\
=&\ \mu^{-n}\sum_{|v|=n}\mathbb{P}_z(Z_v=m, \mathsf{A}_v=1|\mathbf{e})\\
=&\sum_{|v|=n}\sum_{t_0=v_1+1}^{2}\cdots\sum_{t_{n-1}=v_n+1}^{2}\mathbb{P}_z\big(Z_{V_n}=m, (\check{T}_j, C_j)=(t_j, v_{j+1}), 0\le j\le n-1|\mathbf{e}\big)\\
=&\sum_{t_0=1}^{2}\sum_{v_1=0}^{t_0-1}\cdots\sum_{t_{n-1}=1}^{2}\sum_{v_n=0}^{t_{n-1}-1}\mathbb{P}_z\big(Z_{V_n}=m, (\check{T}_j, C_j)=(t_j, v_{j+1}), 0\le j\le n-1|\mathbf{e}\big)\\
=&\ \mathbb{P}_z(Z_{V_n}=m|\mathbf{e})\quad\mathbb{P}_z\text{-a.s.}
\end{aligned}$$

From this, the other three assertions follow easily and the proposition is proved. □

Finally, we provide the remaining proofs of the Propositions 2.2 to 2.8. Since $(Z_{V_n})_{n\geq 0}$ is a BPRE, the results can be concluded more or less directly from the theory of BPRE.

Proof of Proposition 2.2. The proposition is a consequence of results in [50] as well as [8] and has been stated in (2.2), where we note that (2.1) ensures $\mathbb{E}|\log f'_{\xi_0}(1)| < \infty$. □

Proof of Proposition 2.4. The assertions are consequences of Theorem 1 in [9] and Theorem 2 in [53]. □

Proof of Proposition 2.5. This follows from Theorem 1.1 in [22]. □

Proof of Proposition 2.6. Note that

$$\begin{aligned}\mathbb{E}[Z_{V_1}\log^+ Z_{V_1}] &= \mathbb{E}\left[X_{1,\emptyset}^{(C_0,\check{T}_0)}\log^+ X_{1,\emptyset}^{(C_0,\check{T}_0)}\right]\\ &= \sum_{k=1}^{2}\sum_{l=0}^{k-1}\mathbb{E}\left[\mathbb{1}_{\{\check{T}_0=k,C_0=l\}}X_{1,\emptyset}^{(l,k)}\log^+ X_{1,\emptyset}^{(l,k)}\right]\\ &= \sum_{k=1}^{2}\sum_{l=0}^{k-1}\frac{p_k}{\mu}\mathbb{E}\left[X^{(l,k)}\log^+ X^{(l,k)}\right]\\ &\leq \mathbb{E}\left[\left(X^{(0)}+X^{(1)}\right)\log^+\left(X^{(0)}+X^{(1)}\right)\right] < \infty\end{aligned}$$

and $\mathbb{E}[f'_{\xi_0}(1)] = \nu/\mu$. Thus the assertions follow from Theorem 1.1 in [23] and Corollary 2.3 in [2]. □

Proof of Proposition 2.7. This is Theorem 1.2 in [23] with the assumptions transferred to our notation and observing $\mathbb{E}[f'_{\xi_0}(1)] = \nu/\mu$. □

Proof of Proposition 2.8. This is Theorem 1.3 in [23] with the assumptions transferred to our notation. □

2.3. A Markov chain and the extinction-explosion principle for the process of parasites

This part follows along the lines of Section 1.2.2 in [27]. Apart from some minor changes, the results shown there are directly transferable to our situation. First, we need some further notation. Let

$$R^{\uparrow} = \{(s,(z_1,\dots,z_s)) \in R : 1 \leq z_1 \leq z_2 \leq \dots \leq z_s\}$$

be the *set of ordered configurations of infected cells in a generation* and set $R_0^{\uparrow} := \{(0,0)\} \cup R^{\uparrow}$.

For a vector $(\tau^{(1)},\dots,\tau^{(k)})$, $k\in\mathbb{N}$, of host-parasite trees, we want to get a vector representing the total number of infected cells in a given generation and the number of parasites in these cells in ascending order. That is, for $n\ge 0$, setting $t(k):=\sum_{i=1}^{k}\boldsymbol{\tau}_n^*(\tau^{(i)})$ for a vector $(\tau^{(1)},\dots,\tau^{(k)})$, we define

$$\begin{aligned}\chi_n:\left(\bigcup_{l\ge1}\mathbb{S}^l,\sigma\left(\bigcup_{l\ge1}\mathfrak{S}^l\right)\right)&\to(R_0^{\uparrow},\mathcal{P}(R_0^{\uparrow})),\\ \left(\tau^{(i)}\right)_{1\le i\le k}&\mapsto\begin{cases}(t,z), & \text{if } t:=t(k)>0,\\ (0,0), & \text{if } t=0,\end{cases}\end{aligned}$$

where $z=(z_1,\dots,z_t)$ is the t-dimensional vector of increasing entries $z_j=\mathfrak{z}_{w_j}(\tau^{(i_j)})$, $1\le j\le t$, for distinct tuples $(i_1,w_1),\dots,(i_t,w_t)\in\{1,\dots,k\}\times\{|v|=n\}$, denoting the number of parasites in the infected cells over all trees in generation n, i.e. $\mathsf{t}_{w_j}(\tau^{(i_j)})\mathfrak{z}_{w_j}(\tau^{(i_j)})>0$ for every $1\le j\le t$, and $z_1\le z_2\le\ldots\le z_t$.

We define the process **BPG** $=(\textbf{BPG}_n)_{n\ge0}$ by

$$\textbf{BPG}_n:=\chi_n(\textbf{BP}),\quad n\in\mathbb{N}_0.$$

Thus, $\textbf{BPG}_n=(s,(z_1,\dots,z_s))$ means that the n-th generation of the BwBPRE **BP** has s infected cells containing $z_1,\dots,z_s$ parasites, respectively.

Given the environment $\mathbf{e}$, all cells and their parasites multiply independently of all other cells and their parasites in the same generation. Thus given $\mathbf{e}$, the exact positions of the infected cells in a generation are not important for the number of infected cells and the parasites they contain in the next generation. Therefore, by standard but rather tedious calculations, one can show that the process **BPG** is a Markov chain with state space $R_0^{\uparrow}$ and transition probabilities

$$p((s,x),(t,z)):=\mathbb{P}_{(s,x)}(\textbf{BPG}_1=(t,z))=\mathbb{P}_{(s,x)}(\textbf{BP}\in\chi_1^{-1}(t,z))\tag{2.6}$$

for $(s,x),(t,z)\in R_0^{\uparrow}$. We omit the details. This leads to the following

Proposition 2.12. *The process* **BPG** *is a homogeneous Markov chain with state space* $R_0^{\uparrow}$ *and transition probabilities given in* (2.6). *Moreover, all states in* $R^{\uparrow}$ *are transient.*

Before we prove this statement, we need a

Lemma 2.13. *In the case* $\mathbb{P}(\mathcal{Z}_1=0)=0$*, we have* $\mathbb{P}_z(\mathcal{Z}_1<z)=0$ *for every* $z\in\mathbb{N}$.

Proof. Similar to the proof of Lemma 1.2 we get

$$\mathbb{P}_z(\mathcal{Z}_1<z)=\sum_{\substack{x_1,\dots,x_z\in\{0,\dots,z-1\}\\ \sum_{i=1}^{z}x_i<z}}\mathbb{E}\left[\prod_{i=1}^{z}\mathbf{e}_0^{(0,1)}(\{x_i\})\right]=0,$$

since at least one x_i has to be zero and $\mathbf{e}_0^{(0,1)}(\{0\}) = 0$ $\mathbb{P}$-a.s. □

Proof (Proposition 2.12). The proof follows along similar lines as the proof of Proposition 1.5 in [27], but some changes occur due to the random environment.

Since the Markov property has already been stated prior to the proposition, it is only left to prove that all states in $R^{\uparrow}$ are transient. First note that

$$\{\mathcal{Z}_1 = 0\} = \{T_\emptyset = 0\} \cup \left\{ T_\emptyset \neq 0, \sum_{i=1}^{z} \left(X_{i,\emptyset}^{(0)} + X_{i,\emptyset}^{(1)} \right) = 0 \right\} \quad \mathbb{P}_z\text{-a.s.}$$

for all $z \in \mathbb{N}_0$. Therefore $\mathbb{P}(\mathcal{Z}_1 = 0) > 0$ implies

$$\mathbb{P}(T_\emptyset = 0) > 0 \quad \text{or} \quad \mathbb{E}\left[\mathbf{e}_0^{(0,1)}(\{0\})\right] = \mathbb{P}\left(X_{1,\emptyset}^{(0)} + X_{1,\emptyset}^{(1)} = 0 \right) > 0.$$

Hence, together with

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{z} \left(X_{i,\emptyset}^{(0)} + X_{i,\emptyset}^{(1)} \right) = 0 \right) &= \mathbb{P}\left(X_{1,\emptyset}^{(0)} + X_{1,\emptyset}^{(1)} = 0, \ldots, X_{z,\emptyset}^{(0)} + X_{z,\emptyset}^{(1)} = 0 \right) \\ &= \mathbb{E}\left[\mathbb{P}\left(X_{1,\emptyset}^{(0)} + X_{1,\emptyset}^{(1)} = 0 \middle| \mathbf{e} \right)^{z} \right] \\ &= \mathbb{E}\left[\mathbf{e}_0^{(0,1)}(\{0\})^{z} \right], \end{aligned}$$

we have in the case $\mathbb{P}(\mathcal{Z}_1 = 0) > 0$

$$\mathbb{P}_z(\mathcal{Z}_1 = 0) \geq \left\{ \begin{matrix} \mathbb{P}(T_\emptyset = 0), & \text{if } \mathbb{P}(T_\emptyset = 0) > 0, \\ \mathbb{E}\left[\mathbf{e}_0^{(0,1)}(\{0\})^{z} \right], & \text{if } \mathbb{P}(T_\emptyset = 0) = 0, \end{matrix} \right\} > 0.$$

Thus, for $(s,x) \in R^{\uparrow}$ with $x = (x_1 \ldots, x_s)$ we get

$$\begin{aligned} &\mathbb{P}_{(s,x)}\left(\mathbf{BPG}_n \neq (s,x) \text{ for all } n \geq 1 \right) \\ &\geq \left\{ \begin{matrix} \mathbb{P}_{(s,x)}(\mathcal{Z}_1 = 0), & \text{if } \mathbb{P}(\mathcal{Z}_1 = 0) > 0, \\ 1 - \mathbb{P}_{(s,x)}\left(\mathcal{Z}_1 = \sum_{i=1}^{s} x_i \right), & \text{if } \mathbb{P}(\mathcal{Z}_1 = 0) = 0, \end{matrix} \right\} \\ &\geq \left\{ \begin{matrix} \mathbb{P}(T_\emptyset = 0)^{s}, & \text{if } \mathbb{P}(T_\emptyset = 0) > 0, \\ \mathbb{P}_{\sum_{i=1}^{s} x_i}(\mathcal{Z}_1 = 0), & \text{if } \mathbb{P}(\mathcal{Z}_1 = 0) > 0, \mathbb{P}(T_\emptyset = 0) = 0, \\ 1 - \mathbb{P}_{\sum_{i=1}^{s} x_i}\left(\mathcal{Z}_1 = \sum_{i=1}^{s} x_i \right), & \text{if } \mathbb{P}(\mathcal{Z}_1 = 0) = 0, \end{matrix} \right\} \\ &> 0, \end{aligned}$$

where we used Lemma 2.13 for the first and the second step and Lemma 1.2 for the last step. This proves the transience of (s,x) and finishes the proof. □

Now, the extinction-explosion principle for the process of parasites is an easy

Corollary 2.14 (Extinction-explosion principle)**.** *For the process of parasites* $(\mathcal{Z}_n)_{n\geq 0}$ *we have*

$$\mathbb{P}_{(t,z)}(\mathcal{Z}_n \to 0) + \mathbb{P}_{(t,z)}(\mathcal{Z}_n \to \infty) = 1$$

for any $(t,z) \in R$.

Proof. First note that root cells with no parasites have no effect on the survival of the parasite population. Therefore we can assume $(t,z) \in R^{\uparrow}$, and then the transience of elements in $R^{\uparrow}$ for the process **BPG** yields

$$\lim_{n\to\infty} \mathbb{P}_{(t,z)}(1 \leq \mathcal{Z}_n \leq K) \leq \lim_{n\to\infty} \sum_{s=1}^{K} \sum_{1\leq x_1\leq \ldots \leq x_s \leq K} \mathbb{P}_{(t,z)}(\mathbf{BPG}_n = (s,x)) = 0$$

for any $K \in \mathbb{N}$, where $x = (x_1, \ldots, x_s)$. This proves the assertion. □

For the process of parasites, we define the *set of extinction* and the *set of survival* through

$$Ext := \{\mathcal{Z}_n \to 0\} \quad \text{and} \quad Surv := Ext^c,$$

respectively. In particular, the previous corollary implies that $Surv = \{\mathcal{Z}_n \to \infty\}$ $\mathbb{P}_{(t,z)}$-a.s. for every $(t,z) \in R$. Since the processes starting from different ancestor cells are independent given $\mathbf{e}$, we have for $(t,z) \in R^{\uparrow}$

$$\mathbb{P}_{(t,z)}(\mathcal{Z}_n = 0|\mathbf{e}) = \prod_{i=1}^{t} \mathbb{P}_{z_i}(\mathcal{Z}_n = 0|\mathbf{e}) \quad \mathbb{P}\text{-a.s.}$$

Thus

$$\mathbb{P}_{(t,z)}(Ext\,|\mathbf{e}) = \prod_{i=1}^{t} \mathbb{P}_{z_i}(Ext\,|\mathbf{e}) \quad \mathbb{P}\text{-a.s.},$$

which in turn implies $\mathbb{P}$-a.s. $\mathbb{P}_{(t,z)}(Ext\,|\mathbf{e}) = 1$ iff $\mathbb{P}_{z_i}(Ext\,|\mathbf{e}) = 1$ for every $1 \leq i \leq t$.

Furthermore, for $z' \in \mathbb{N}$, $\mathbb{P}_{z'}(Ext\,|\mathbf{e}) = 1$ implies $\mathbb{P}(Ext|\mathbf{e}) = 1$. If on the other hand $\mathbb{P}(Ext|\mathbf{e})(\omega) = 1$, it follows from the independence of $\left(\left(X^{(0)}_{k,v}, X^{(1)}_{k,v}\right)_{k\geq 1, v\in\mathbb{V}}, \mathbf{e}\right)$ and

$\mathbf{T} := (T_v)_{v\in\mathbb{V}}$ that

$$1 = \mathbb{P}(\mathit{Ext}|\mathbf{e})(\omega) = \int \mathbb{P}(\mathit{Ext}\ |\mathbf{T} = \tau, \mathbf{e})(\omega)\ \mathbb{P}^{\mathbf{T}}(\mathrm{d}\tau) = \int \mathbb{P}(\mathit{Ext}(\tau)|\mathbf{e})(\omega)\ \mathbb{P}^{\mathbf{T}}(\mathrm{d}\tau).$$

Therefore, $\mathbb{P}(\mathit{Ext}(\tau)|\mathbf{e})(\omega) = 1$ $\mathbb{P}^{\mathbf{T}}$-a.s., where

$$\mathbb{P}(\mathit{Ext}(\tau)|\mathbf{e}) = \lim_{n\to\infty} \mathbb{P}(\mathcal{Z}_n(\tau) = 0|\mathbf{e})$$

and $\mathcal{Z}_n(\tau)$ is the number of parasites in the n-th generation of a BwBPRE when the cell tree is spanned by τ. Furthermore, if we consider a single ancestor cell with $z' \in \mathbb{N}$ parasites, let $\mathcal{Z}_{n,i}$ denote the number of progeny in the n-th generation of the i-th ancestor parasite and let $\mathcal{Z}_{n,i}(\tau)$ denote the same when the cell tree is spanned by τ, $1 \le i \le z'$, $\tau \in \{0,1,2\}^{\mathbb{V}}$. We then get, using once again the independence of $\mathbf{T}$ and $\left(\left(X^{(0)}_{k,v}, X^{(1)}_{k,v}\right)_{k\ge1, v\in\mathbb{V}}, \mathbf{e}\right)$,

$$\begin{aligned}
\mathbb{P}_{z'}(\mathcal{Z}_n = 0|\mathbf{e}) &= \mathbb{P}(\mathcal{Z}_{n,1} = 0, \dots, \mathcal{Z}_{n,z'} = 0|\mathbf{e}) \\
&= \int \mathbb{P}(\mathcal{Z}_{n,1} = 0, \dots, \mathcal{Z}_{n,z'} = 0|\mathbf{T} = \tau, \mathbf{e})\ \mathbb{P}^{\mathbf{T}}(\mathrm{d}\tau) \\
&= \int \mathbb{P}(\mathcal{Z}_{n,1}(\tau) = 0, \dots, \mathcal{Z}_{n,z'}(\tau) = 0|\mathbf{e})\ \mathbb{P}^{\mathbf{T}}(\mathrm{d}\tau) \\
&= \int \prod_{i=1}^{z'} \mathbb{P}(\mathcal{Z}_{n,i}(\tau) = 0|\mathbf{e})\ \mathbb{P}^{\mathbf{T}}(\mathrm{d}\tau) \\
&= \int \mathbb{P}(\mathcal{Z}_n(\tau) = 0|\mathbf{e})^{z'}\ \mathbb{P}^{\mathbf{T}}(\mathrm{d}\tau) \quad \mathbb{P}\text{-a.s.},
\end{aligned}$$

where we used the conditional independence of $\mathcal{Z}_{n,1}(\tau), \dots, \mathcal{Z}_{n,z'}(\tau)$ given $\mathbf{e}$. Thus we have for almost all ω with $\mathbb{P}(\mathit{Ext}|\mathbf{e})(\omega) = 1$

$$\mathbb{P}_{z'}(\mathit{Ext}\ |\mathbf{e})(\omega) = \int (\mathbb{P}(\mathit{Ext}(\tau)|\mathbf{e})(\omega))^{z'}\ \mathbb{P}^{\mathbf{T}}(\mathrm{d}\tau) = 1.$$

Therefore we have $\mathbb{P}$-a.s. $\mathbb{P}_{z'}(\mathit{Ext}\ |\mathbf{e}) = 1$ iff $\mathbb{P}(\mathit{Ext}|\mathbf{e}) = 1$. Summarising, this shows that $\mathbb{P}$-a.s.

$$\mathbb{P}_{(t,z)}(\mathit{Ext}\ |\mathbf{e}) = 1 \quad \text{iff} \quad \mathbb{P}(\mathit{Ext}|\mathbf{e}) = 1. \tag{2.7}$$

In particular $\mathbb{P}_{(t,z)}(\mathit{Ext}) = 1$ iff $\mathbb{P}(\mathit{Ext}) = 1$.

Next we are going to show, that the set $\{\mathbb{P}(\mathit{Surv}\ |\mathbf{e}) > 0\}$ has either probability 0 or 1, i.e. extinction is certain under almost every environment or survival is possible under almost every environment. The same phenomenon is known for BPRE, see Proposition 1 in [8]. Recall that $[\mathbf{e}]_m = (\mathbf{e}_n)_{n\geq m}$.

Proposition 2.15. *We have* $\{\mathbb{P}(\mathit{Surv}\,|\mathbf{e}) > 0\} = \{\mathbb{P}(\mathit{Surv}\,|[\mathbf{e}]_1) > 0\}$ $\mathbb{P}$*-a.s., and hence* $\mathbb{P}(\mathbb{P}(\mathit{Surv}\,|\mathbf{e}) > 0) \in \{0, 1\}$.

Proof. We start by showing $\{\mathbb{P}(\mathit{Surv}\,|\mathbf{e}) > 0\} \subseteq \{\mathbb{P}(\mathit{Surv}\,|[\mathbf{e}]_1) > 0\}$ $\mathbb{P}$-a.s. Choose $\omega \in \{\mathbb{P}(\mathit{Surv}\,|\mathbf{e}) > 0\}$ and suppose $\mathbb{P}(\mathit{Surv}\,|[\mathbf{e}]_1)(\omega) = 0$. Then, according to what has been shown prior to the proposition, we have $\mathbb{P}_{(t,z)}(\mathit{Surv}\,|[\mathbf{e}]_1)(\omega) = 0$ for all $(t, z) \in R^{\uparrow}$, except for ω from a set of measure 0, and thus

$$\begin{aligned}
0 &< \mathbb{P}(\mathit{Surv}\,|\mathbf{e})(\omega) = \mathbb{P}(\mathbf{BPG}_m \in R^{\uparrow} \text{ for all } m \geq 1|\mathbf{e})(\omega) \\
&= \sum_{(t,z)\in R^{\uparrow}} \mathbb{P}(\mathbf{BPG}_m \in R^{\uparrow} \text{ for all } m \geq 2|\mathbf{BPG}_1 = (t,z), \mathbf{e})(\omega) \cdot \mathbb{P}(\mathbf{BPG}_1 = (t,z)|\mathbf{e})(\omega) \\
&= \sum_{(t,z)\in R^{\uparrow}} \mathbb{P}_{(t,z)}(\mathbf{BPG}_m \in R^{\uparrow} \text{ for all } m \geq 1|[\mathbf{e}]_1)(\omega) \cdot \mathbb{P}(\mathbf{BPG}_1 = (t,z)|\mathbf{e})(\omega) \\
&= \sum_{(t,z)\in R^{\uparrow}} \mathbb{P}_{(t,z)}(\mathit{Surv}\,|[\mathbf{e}]_1)(\omega) \cdot \mathbb{P}(\mathbf{BPG}_1 = (t,z)|\mathbf{e})(\omega) = 0.
\end{aligned}$$

This is a contradiction and we must have

$$\{\mathbb{P}(\mathit{Surv}\,|\mathbf{e} > 0\} \subseteq \{\mathbb{P}(\mathit{Surv}\,|[\mathbf{e}]_1) > 0\} \quad \mathbb{P}\text{-a.s.} \tag{2.8}$$

Now, since the random environment consists of i.i.d. random variables, we further have

$$\mathbb{P}(\mathbb{P}(\mathit{Surv}\,|\mathbf{e}) > 0) = \mathbb{P}(\mathbb{P}(\mathit{Surv}\,|[\mathbf{e}]_1) > 0) \tag{2.9}$$

Combining (2.8) and (2.9), we arrive at the desired result. The second assertion of the proposition follows from the ergodicity of the shift operator. □

Remark 2.16. In other words, Proposition 2.15 tells us

$$\mathbb{P}(\mathbb{P}(\mathit{Ext}\,|\mathbf{e}) = 1) \in \{0, 1\}.$$

2.4. An extinction-explosion principle and the normalised process of infected cells

With the help of the Markov chain introduced in the previous section, we can prove that the process of contaminated cells satisfies an extinction-explosion principle, apart from some degenerate cases. This is stated in the following

Theorem 2.17. *Let* $\mathbb{P}(\mathit{Surv}) > 0$ *and* $z \in \mathbb{N}$.

(a) If $\mathbb{P}_2(\mathcal{T}_1^* = 2) = 0$*, then*

$$\mathbb{P}_z(\mathcal{T}_n^* = 1 \ \forall\ n \geq 0 | \mathit{Surv}) = 1.$$

(b) If $\mathbb{P}_2(\mathcal{T}_1^* = 2) > 0$*, then*

$$\mathbb{P}_z(\mathcal{T}_n^* \to \infty | \mathit{Surv}) = 1.$$

Proof. The proof follows the general idea of the proof of Theorem 1.7 in [27], but some adaptations are necessary due to the random environment. Let $z \in \mathbb{N}$.

(a) Note that $\mathbb{P}(\mathit{Surv}) > 0$ implies $p_2 > 0$, for otherwise, since $p_1 < 1$, the cell tree would die out $\mathbb{P}$-a.s., and thus in particular $\mathbb{P}(\mathit{Surv}) = 0$. Furthermore, the independence assumptions of our model imply

$$0 = \mathbb{P}_2(\mathcal{T}_1^* = 2) \geq p_2 \ \mathbb{P}\left(X_{i,\emptyset}^{(0)} > 0, X_{j,\emptyset}^{(1)} > 0\right)$$

for all $i, j \in \{1, 2\}$, and thus $\mathbb{P}\left(X_{i,\emptyset}^{(0)} > 0, X_{j,\emptyset}^{(1)} > 0\right) = 0$ for all $i, j \in \mathbb{N}$. Then we get

$$\begin{aligned}
\mathbb{P}_{z'}(\mathcal{T}_1^* = 2) &= p_2 \ \mathbb{P}\left(\sum_{i=1}^{z'} X_{i,\emptyset}^{(0)} > 0, \sum_{j=1}^{z'} X_{j,\emptyset}^{(1)} > 0\right) \\
&= p_2 \ \mathbb{P}\left(\exists (i,j) \in \{1, \ldots, z'\}^2 : X_{i,\emptyset}^{(0)} > 0, X_{j,\emptyset}^{(1)} > 0\right) \\
&\leq p_2 \sum_{i,j=1}^{z'} \mathbb{P}\left(X_{i,\emptyset}^{(0)} > 0, X_{j,\emptyset}^{(1)} > 0\right) \\
&= 0
\end{aligned}$$

for all $z' \in \mathbb{N}$. Therefore, there is a.s. at most one infected cell in every generation, and since $\mathit{Surv} = \{\mathcal{T}_n^* \geq 1 \ \forall\ n \geq 0\}$ $\mathbb{P}_z$-a.s. and $\mathbb{P}_z(\mathit{Surv}) > 0$, we get

$$\mathbb{P}_z(\mathcal{T}_n^* = 1 \ \forall\ n \geq 0 | \mathit{Surv}) = 1.$$

(b) For $t \geq 1$ we are going to show that

$$\mathbb{P}_z(\mathcal{T}_n^* = t \text{ i.o.}) = 0. \tag{2.10}$$

This yields

$$\mathbb{P}_z(1 \leq \mathcal{T}_n^* \leq t \text{ i.o.}) = 0$$

for all $t \geq 1$, i.e. the extinction-explosion principle for $(\mathcal{T}_n^*)_{n\geq 0}$. Since $Ext = \{\mathcal{T}_n^* \to 0\}$ $\mathbb{P}_z$-a.s. and thus $Surv = \{\mathcal{T}_n^* \to \infty\}$ $\mathbb{P}_z$-a.s., (b) can be concluded.

To show (2.10) for every $t \geq 1$, we use the Markov chain **BPG** introduced in the previous section. To this end, set

$$A_t := \{(t, x_1, \ldots, x_t) \in R^{\uparrow} : x_t \geq 2\} \subset \mathbb{N}^{t+1},$$

for $t \geq 1$, and note that

$$\{\mathcal{T}_n^* = t\} = \{\mathbf{BPG}_n \in A_t\} \cup \{\mathbf{BPG}_n = (t, 1 \ldots, 1)\} \quad \mathbb{P}_z\text{-a.s.}$$

Furthermore, we have

$$\mathbb{P}_z(\mathcal{T}_n^* = t \text{ i.o.}) = \mathbb{P}_z(\mathbf{BPG}_n \in A_t \text{ i.o.}),$$

since $(t, 1, \ldots, 1) \in R^{\uparrow}$ is a transient state, see Proposition 2.12.

Fix $t \geq 1$ and use the notation $\mathcal{T}_{n,i}^*$ for the number of infected cells in the n-th generation stemming from ancestor cell i, $1 \leq i \leq t$. Now, we get by the conditional independence of the processes starting from different ancestor cells

$$\begin{aligned}
& \mathbb{P}_{(t,x)}(\mathbf{BPG}_n \notin A_t \text{ for all } n \geq 1) \\
\geq\ & \mathbb{P}_{(t,x)}(\mathcal{T}_n^* > t \text{ for all } n \geq 1) \\
\geq\ & \mathbb{P}_{(t,x)}(\mathcal{T}_{n,1}^* \geq 1, \ldots, \mathcal{T}_{n,t-1}^* \geq 1, \mathcal{T}_{n,t}^* \geq 2 \text{ for all } n \geq 1) \\
=\ & \mathbb{E}_{(t,x)}[\mathbb{P}_{(t,x)}(\mathcal{T}_{n,1}^* \geq 1, \ldots, \mathcal{T}_{n,t-1}^* \geq 1, \mathcal{T}_{n,t}^* \geq 2 \text{ for all } n \geq 1|\mathbf{e})] \\
=\ & \mathbb{E}[\mathbb{P}_{x_1}(\mathcal{T}_n^* \geq 1 \text{ for all } n \geq 1|\mathbf{e}) \cdots \mathbb{P}_{x_{t-1}}(\mathcal{T}_n^* \geq 1 \text{ for all } n \geq 1|\mathbf{e}) \\
& \quad \cdot \mathbb{P}_{x_t}(\mathcal{T}_n^* \geq 2 \text{ for all } n \geq 1|\mathbf{e})],
\end{aligned}$$

for $(t, x) \in A_t$ and $x = (x_1, \ldots, x_t)$. One possibility for the event in the last factor to occur is: There are two infected cells in the first generation and in both of the resulting branches the parasites survive. Note that both of these branches are conditionally independent as well as independent of the first generation given the number of parasites in the two starting cells. Moreover, the survival probability gets smaller if these cells are only inhabited by one parasite. This leads to

$$\begin{aligned}
& \mathbb{P}_{(t,x)}(\mathbf{BPG}_n \notin A_t \text{ for all } n \geq 1) \\
\geq\ & \mathbb{E}[\mathbb{P}_{x_1}(Surv\,|\mathbf{e}) \cdots \mathbb{P}_{x_{t-1}}(Surv\,|\mathbf{e})\mathbb{P}_{x_t}(\mathcal{T}_1^* \geq 2|\mathbf{e})\mathbb{P}(Surv\,|[\mathbf{e}]_1)^2] \\
\geq\ & \mathbb{E}[\mathbb{P}(Surv\,|\mathbf{e})^{t-1}\mathbb{P}_2(\mathcal{T}_1^* = 2|\mathbf{e})\mathbb{P}(Surv\,|[\mathbf{e}]_1)^2] =: \tilde{c} > 0,
\end{aligned}$$

where we used $x_t \geq 2$, Proposition 2.15 and the assumptions for the final line. In particular, this lower bound does not depend on x. Let $\tau_0 := 0$ and

$$\tau_{n+1} := \inf\{k > \tau_n : \mathsf{\mathbf{BPG}}_k \in A_t\}, \quad n \geq 0,$$

where, as usual, $\inf \emptyset := \infty$. The strong Markov property yields

$$\begin{aligned} &\mathbb{P}_z(\tau_{n+1} - \tau_n < \infty | \mathsf{\mathbf{BPG}}_{\tau_n} = (t,x), \tau_n < \infty) \\ &\quad = \mathbb{P}_{(t,x)}(\tau_1 < \infty) = 1 - \mathbb{P}_{(t,x)}(\mathsf{\mathbf{BPG}}_n \notin A_t \text{ for all } n \geq 1) \leq 1 - \tilde{c} =: c < 1 \end{aligned}$$

for all $n \geq 1$ and $(t,x) \in A_t$. By iteration we get for $n \geq 2$

$$\begin{aligned} \mathbb{P}_z(\tau_n < \infty) &= \sum_{(t,x)\in A_t} \mathbb{P}_z(\mathsf{\mathbf{BPG}}_{\tau_{n-1}} = (t,x), \tau_n - \tau_{n-1} < \infty, \tau_{n-1} < \infty) \\ &= \sum_{(t,x)\in A_t} \mathbb{P}_z(\tau_n - \tau_{n-1} < \infty | \mathsf{\mathbf{BPG}}_{\tau_{n-1}} = (t,x), \tau_{n-1} < \infty) \\ &\qquad \cdot \mathbb{P}_z(\mathsf{\mathbf{BPG}}_{\tau_{n-1}} = (t,x), \tau_{n-1} < \infty) \\ &\leq c \; \mathbb{P}_z(\tau_{n-1} < \infty) \\ &\leq c^{n-1} \; \mathbb{P}_z(\tau_1 < \infty) \\ &\leq c^{n-1} \end{aligned}$$

and thus

$$\begin{aligned} \mathbb{P}_z(\mathsf{\mathbf{BPG}}_n \in A_t \text{ i.o.}) &= \mathbb{P}_z(\tau_n < \infty \text{ for all } n \geq 1) \\ &= \lim_{n\to\infty} \mathbb{P}_z(\tau_n < \infty) \leq \lim_{n\to\infty} c^{n-1} = 0. \end{aligned}$$

This shows $\mathbb{P}_z(\mathcal{T}_n^* = t \text{ i.o.}) = 0$ and completes the proof of (b). □

Now we prove a limit theorem for the process of infected cells $(\mathcal{T}_n^*)_{n\geq 0}$. This result is an easy consequence of the martingale convergence theorem applied to the supermartingale $(\mu^{-n}\mathcal{T}_n^*)_{n\geq 0}$. Moreover, we can show a dichotomy known from the theory of GWP's, namely either this supermartingale converges to 0 $\mathbb{P}_z$-a.s. or μ^{-n} gives the right normalisation, that is

$$\left\{\lim_{n\to\infty} \mu^{-n}\mathcal{T}_n^* > 0\right\} = \{\mathcal{T}_n^* \to \infty\} \quad \mathbb{P}_z\text{-a.s.}$$

Proposition 2.18. *Let $z \geq 1$. Under $\mathbb{P}_z$ the process $(\mu^{-n}\mathcal{T}_n^*)_{n\geq 0}$ forms a nonnegative supermartingale with respect to $(\mathcal{G}_n)_{n\geq 0}$ and therefore converges $\mathbb{P}_z$-a.s. to an integrable random variable L for $n \to \infty$. Moreover we have:*

(a) $L = 0$ $\mathbb{P}_z$-a.s. iff one of the following conditions holds

(i) $\mu \leq 1$,

(ii) $\mathbb{E}\log f'_{\xi_0}(1) \leq 0$ *or* $\mathbb{E}\left[\log^-(1-f_{\xi_0}(0))\right] = \infty$.

(b) $\mathbb{P}_z(\mathbb{P}_z(L=0|\mathbf{e})=1) \in \{0,1\}$.

(c) $\{L=0\} = Ext$ $\mathbb{P}_z$*-a.s. if* $\mathbb{P}_z(L=0)<1$.

Proof. The first part of the proof is similar to the one of Theorem 1.8 in [27].

Since integrability and measurability are obvious, we are left to prove the supermartingale property. For $n \geq 1$ we have

$$\begin{aligned}\mathbb{E}_z\left[\mathcal{T}^*_{n+1}\middle|\,\mathcal{G}_n\right] &= \sum_{v\in\mathbb{T}^*_n}\mathbb{E}_z\left[\sum_{u=0}^{T_v-1}\mathbb{1}_{\{Z_{vu}>0\}}\middle|\,\mathcal{G}_n\right]\\ &\leq \sum_{v\in\mathbb{T}^*_n}\mathbb{E}_z\left[T_v\middle|\,\mathcal{G}_n\right]\\ &= \sum_{v\in\mathbb{T}^*_n}\mathbb{E}\left[T_v\right]\\ &= \mu\mathcal{T}^*_n \quad \mathbb{P}_z\text{-a.s.,}\end{aligned}$$

where we have used the independence of $(T_v)_{|v|=n}$ and $\mathcal{G}_n$ for the second equality. Thus $(\mu^{-n}\mathcal{T}^*_n)_{n\geq 0}$ is a nonnegative supermartingale and the martingale convergence theorem yields the $\mathbb{P}_z$-a.s. convergence to an integrable random variable L. This finishes the proof of the first part of the proposition.

(a) Note that in the case $\mu > 1$ the process $(\mu^{-n}\mathcal{T}_n)_{n\geq 0}$ forms a normalised supercritical GWP satisfying the $(Z\log Z)$-condition and is therefore uniformly integrable. Since

$$\mu^{-n}\mathcal{T}^*_n \leq \mu^{-n}\mathcal{T}_n, \quad n \geq 1,$$

the process $(\mu^{-n}\mathcal{T}^*_n)_{n\geq 0}$ is also uniformly integrable, and Proposition 2.9 yields

$$\mathbb{E}_z L = \lim_{n\to\infty}\mathbb{E}_z\left[\frac{\mathcal{T}^*_n}{\mu^n}\right] = \lim_{n\to\infty}\mathbb{P}_z\left(Z_{V_n}>0\right).$$

Therefore, Proposition 2.2 and Remark 2.3 imply that $L=0$ $\mathbb{P}_z$-a.s. iff condition *(ii)* is fulfilled. On the contrary, if $\mu \leq 1$, then the GWP $(\mathcal{T}_n)_{n\geq 0}$ is critical or subcritical and as a consequence dies out with probability one. So for almost every ω there exists a $n_0(\omega) \in \mathbb{N}_0$, such that $\mathcal{T}^*_n(\omega) \leq \mathcal{T}_n(\omega) = 0$ for all $n \geq n_0(\omega)$, and thus $L=0$ $\mathbb{P}_z$-a.s.

(b) W.l.o.g. suppose $\mathbb{P}_z(L=0)<1$, for otherwise $\mathbb{P}_z(L=0|\mathbf{e})=1$ $\mathbb{P}_z$-a.s. Then, part (a) ensures $\mu>1$ and, as before, $(\mu^{-n}\mathcal{T}_n)_{n\geq 0}$ forms a normalised supercritical GWP satisfying the $(Z\log Z)$-condition. Therefore, the independence of $(T_v)_{v\in\mathbb{V}}$ and $\mathbf{e}$ yields

$$\lim_{a\to\infty}\sup_{n\in\mathbb{N}_0}\mathbb{E}_z\left[\frac{\mathcal{T}_n}{\mu^n}\mathbb{1}_{\left\{\frac{\mathcal{T}_n}{\mu^n}>a\right\}}\middle|\,\mathbf{e}\right] = \lim_{a\to\infty}\sup_{n\in\mathbb{N}_0}\mathbb{E}\left[\frac{\mathcal{T}_n}{\mu^n}\mathbb{1}_{\left\{\frac{\mathcal{T}_n}{\mu^n}>a\right\}}\right] = 0 \quad \mathbb{P}_z\text{-a.s.,}$$

i.e. $(\mu^{-n}\mathcal{T}_n)_{n\geq 0}$ is uniformly integrable under $\mathbb{P}_z(\cdot|\mathbf{e})$ $\mathbb{P}_z$-a.s., and so the same holds true for $(\mu^{-n}\mathcal{T}_n^*)_{n\geq 0}$. Thus, with the help of Proposition 2.9 we infer

$$\mathbb{E}_z[L|\mathbf{e}] = \lim_{n\to\infty} \mathbb{E}_z\left[\frac{\mathcal{T}_n^*}{\mu^n}\middle|\,\mathbf{e}\right] = \lim_{n\to\infty} \mathbb{P}_z(Z_{V_n} > 0|\mathbf{e}) = \mathbb{E}_z\left[\lim_{n\to\infty} \mathbb{P}_z(Z_{V_n} > 0|\xi)\middle|\,\mathbf{e}\right]$$

$\mathbb{P}_z$-a.s. Setting $q_z(\xi) := \lim_{n\to\infty} \mathbb{P}_z(Z_{V_n} = 0|\xi)$, we have $q_z(\xi) = q_1(\xi)^z$. Note that $\mathbb{P}(q_1(\xi) = 1) \in \{0, 1\}$ according to Proposition 1 in [8], since $(Z_{V_n})_{n\geq 0}$ is a BPRE. Therefore, $\mathbb{P}_z(q_z(\xi) = 1|\mathbf{e}) = 1$ $\mathbb{P}_z$-a.s. or $\mathbb{P}_z(q_z(\xi) = 1|\mathbf{e}) = 0$ $\mathbb{P}_z$-a.s., and thus $\mathbb{E}_z[L|\mathbf{e}] = 0$ $\mathbb{P}_z$-a.s. or $\mathbb{E}_z[L|\mathbf{e}] > 0$ $\mathbb{P}_z$-a.s. Since $\mathbb{P}_z(L = 0) < 1$, this gives us $\mathbb{P}_z(\mathbb{P}_z(L = 0|\mathbf{e}) = 1) = 0$.
Altogether, we have $\mathbb{P}_z(\mathbb{P}_z(L = 0|\mathbf{e}) = 1) \in \{0, 1\}$, as desired.

(c) Note first, that *Ext* $\subseteq \{L = 0\}$ $\mathbb{P}_z$-a.s. Now, consider for every $n \in \mathbb{N}$ the first time the number of contaminated cells is greater than or equal to n, i.e.

$$\tau_n = \inf\{m \in \mathbb{N} : \mathcal{T}_m^* \geq n\}.$$

Observe that for $m \in \mathbb{N}$, we have

$$\mathcal{T}_{m+l}^* = \sum_{u\in\mathbb{T}_m^*} \mathcal{T}_l^*(u), \quad l \geq 0,$$

and thus

$$L = \lim_{l\to\infty} \frac{\mathcal{T}_{m+l}^*}{\mu^{m+l}} = \frac{1}{\mu^m} \sum_{u\in\mathbb{T}_m^*} \lim_{l\to\infty} \frac{\mathcal{T}_l^*(u)}{\mu^l},$$

where $\mathcal{T}_l^*(u)$ is the number of infected cells in the l-th generation of the BwBPRE with root cell u and Z_u root parasites. According to our model assumptions, these processes starting in the cells $u \in \mathbb{T}_m^*$ are conditionally independent given $\mathcal{G}_m$, and are evolving as the original process under $\mathbb{P}$ with the shifted environment $[\mathbf{e}]_m$, apart from the fact that there might be more than one parasite in the starting generation. Therefore, we get for $n \in \mathbb{N}$

$$\begin{aligned}
\mathbb{P}_z(L = 0) &\leq \sum_{m\geq 1} \mathbb{P}_z(\tau_n = m, L = 0) + \mathbb{P}_z(\tau_n = \infty) \\
&= \sum_{m\geq 1} \mathbb{P}_z\left(\{\tau_n = m\} \cap \bigcap_{u\in\mathbb{T}_m^*} \left\{\lim_{l\to\infty} \frac{\mathcal{T}_l^*(u)}{\mu^l} = 0\right\}\right) + \mathbb{P}_z(\tau_n = \infty) \\
&= \sum_{m\geq 1} \mathbb{E}_z\left[\mathbb{1}_{\{\tau_n = m\}} \cdot \mathbb{P}_z\left(\bigcap_{u\in\mathbb{T}_m^*} \left\{\lim_{l\to\infty} \frac{\mathcal{T}_l^*(u)}{\mu^l} = 0\right\}\middle|\,\mathcal{G}_m\right)\right] + \mathbb{P}_z(\tau_n = \infty)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m\geq 1} \mathbb{E}_z \left[\mathbb{1}_{\{\tau_n=m\}} \prod_{u\in\mathbb{T}^*_m} \mathbb{P}_z \left(\lim_{l\to\infty} \frac{\mathcal{T}^*_l(u)}{\mu^l} = 0 \middle| \mathcal{G}_m \right) \right] + \mathbb{P}_z(\tau_n = \infty) \\
&\leq \sum_{m\geq 1} \mathbb{E}_z \left[\mathbb{1}_{\{\tau_n=m\}} \cdot \mathbb{P}(L = 0 | [\mathbf{e}]_m)^{\mathcal{T}^*_m} \right] + \mathbb{P}_z(\tau_n = \infty) \\
&\leq \sum_{m\geq 1} \mathbb{E}_z \left[\mathbb{P}_z(\tau_n = m | \mathbf{e}) \cdot \mathbb{P}(L = 0 | [\mathbf{e}]_m)^n \right] + \mathbb{P}_z(\tau_n = \infty) \\
&= \sum_{m\geq 1} \mathbb{P}_z(\tau_n = m) \cdot \mathbb{E}_z \left[\mathbb{P}(L = 0 | [\mathbf{e}]_m)^n \right] + \mathbb{P}_z(\tau_n = \infty) \\
&= \sum_{m\geq 1} \mathbb{P}_z(\tau_n = m) \cdot \mathbb{E}_z \left[\mathbb{P}(L = 0 | \mathbf{e})^n \right] + \mathbb{P}_z(\tau_n = \infty) \\
&\leq \mathbb{E}_z \left[\mathbb{P}(L = 0 | \mathbf{e})^n \right] + \mathbb{P}_z(\tau_n = \infty),
\end{aligned}
$$

where we used the i.i.d. property of the random environment in the antepenultimate and in the penultimate line. If $\mathbb{P}_z(L = 0) < 1$, we have $\mathbb{P}(L = 0) < 1$ and $\mathbb{P}(L = 0|\mathbf{e}) < 1$ a.s., according to part (a) and (b). Hence, the extinction-explosion principle for the process of contaminated cells yields

$$
\begin{aligned}
\mathbb{P}_z(L = 0) &\leq \lim_{n\to\infty} \mathbb{P}_z(\tau_n = \infty) \\
&= \lim_{n\to\infty} \mathbb{P}_z \left(\sup_{m\geq 1} \mathcal{T}^*_m < n \right) \\
&= \mathbb{P}_z \left(\sup_{m\geq 1} \mathcal{T}^*_m < \infty \right) \\
&= \mathbb{P}_z(\mathit{Ext}).
\end{aligned}
$$

Combining this with $\mathit{Ext} \subseteq \{L = 0\}$ $\mathbb{P}_z$-a.s., completes the proof. ☐

With the previous proposition at hand, we can take a look at the notion of a *recovering organism.* In [11] Bansaye called an organism *recovering*, if the number of infected cells becomes negligible compared to the total number of cells as $n \to \infty$. Therefore, we analyse the asymptotic behaviour of $\frac{\mathcal{T}^*_n}{\mathcal{T}_n}$. Note that in our model the cell process is a GWP that may die out, and obviously, the notion of a recovering organism only makes sense if the cell population survives. Thus, we let $\mathit{Surv}_{\text{cell}}$ denote the survival set of the cell population $(\mathcal{T}_n)_{n\geq 0}$, and we say that an organism recovers if the number of infected cells becomes asymptotically negligible compared to the total number of cells on $\mathit{Surv}_{\text{cell}}$. An immediate consequence of Proposition 2.18 is the following

Corollary 2.19. *Let $\mu > 1$ (or equivalently $\mathbb{P}(\mathit{Surv}_{\mathit{cell}}) > 0$). The organism is a.s. re-*

covering, i.e.

$$\frac{\mathcal{T}_n^*}{\mathcal{T}_n} \to 0 \quad a.s.\ on\ Surv_{cell},$$

as $n \to \infty$, *iff*

$$\mathbb{E} \log f'_{\xi_0}(1) \le 0 \quad or \quad \mathbb{E}[\log^-(1 - f_{\xi_0}(0))] = \infty.$$

Moreover, if the organism is not a.s. recovering, then the parasites need to die out for the organism to recover.

This completes our analysis of the long-term behaviour of the process of infected cells.

2.5. The number of cells with a given number of parasites

In the final section, we come back to the process $(F_n(k))_{n\ge 0}$ of the number of cells with exactly k parasites defined in (2.3). In Corollary 2.11 we compared this quantity to the expected number of cells. Here we compare it to the number of cells $(\mathcal{T}_n)_{n\ge 0}$ and the number of infected cells $(\mathcal{T}_n^*)_{n\ge 0}$. This is done in the next

Theorem 2.20. *Let* $z \ge 1$ *and* $\mathbb{P}(Surv) > 0$. *The following assertions hold for* $k \ge 1$*:*

(a) $\lim\limits_{n\to\infty} \mathbb{P}_z\left(\frac{F_n(k)}{\mathcal{T}_n} > \varepsilon \middle| Surv\right) = 0$ *for all* $\varepsilon > 0$.

(b) If $\mathbb{E}[\log f'_{\xi_0}(1)] > 0$ *and* $\mathbb{E}[\log^-(1 - f_{\xi_0}(0))] < \infty$, *then*

$$\lim_{n\to\infty} \frac{F_n(k)}{\mathbb{E}_z[\mathcal{T}_n^*|\mathbf{e}]} = 0, \quad in\ probability\ and\ in\ L^1\ under\ \mathbb{P}_z.$$

(c) If $\mathbb{E}[\log f'_{\xi_0}(1)] > 0$ *and* $\mathbb{E}[\log^-(1 - f_{\xi_0}(0))] < \infty$, *then*

$$\lim_{n\to\infty} \mathbb{P}_z\left(\frac{F_n(k)}{\mathcal{T}_n^*} > \varepsilon \middle| Surv\right) = 0$$

for all $\varepsilon > 0$.

Proof. (a) First note that $(\mathcal{T}_n)_{n\ge 0}$ is a supercritical GWP satisfying the $(Z \log Z)$-condition. Therefore, $(\mathcal{T}_n/\mu^n)_{n\ge 0}$ converges a.s. to an integrable random variable, say $\mathcal{T}$, satisfying $\{\mathcal{T} > 0\} \supseteq Surv$. Hence, we get with the help of Corollary 2.11

$$\frac{F_n(k)}{\mathcal{T}_n}\mathbb{1}_{Surv} = \frac{F_n(k)/\mu^n}{\mathcal{T}_n/\mu^n}\mathbb{1}_{Surv} \xrightarrow{\mathbb{P}_z} 0,$$

i.e. $\lim_{n\to\infty} \mathbb{P}_z\left(\frac{F_n(k)}{\mathcal{T}_n} > \varepsilon \,\middle|\, Surv\right) = 0$ for all $\varepsilon > 0$.

(b) In the given situation, Theorem 5.3 in [51] tells us

$$\mathbb{P}_z(Z_{V_n} \to 0 \text{ or } Z_{V_n} \to \infty \text{ as } n \to \infty | \xi) = 1 \quad \text{a.s.}$$

Therefore, the dominated convergence theorem ensures

$$\lim_{n\to\infty} \mathbb{P}_z(Z_{V_n} = k|\mathbf{e}) = \mathbb{E}_z\left[\lim_{n\to\infty} \mathbb{P}(Z_{V_n} = k|\xi) \,\middle|\, \mathbf{e}\right] = 0 \quad \text{a.s.}$$

Furthermore, since $\lim_{n\to\infty} \mathbb{P}_z(Z_{V_n} > 0) > 0$ according to Remark 2.3, a similar reasoning as in the proof of Proposition 2.18 part (b) yields

$$\lim_{n\to\infty} \mathbb{P}_z(Z_{V_n} > 0|\mathbf{e}) = \lim_{n\to\infty} (1 - \mathbb{P}_z(Z_{V_n} = 0|\mathbf{e})) > 0 \quad \text{a.s.}$$

So, another appeal to the dominated convergence theorem gives us in combination with Proposition 2.9

$$\mathbb{E}_z\left[\frac{F_n(k)}{\mathbb{E}_z[\mathcal{T}_n^*|\mathbf{e}]}\right] = \mathbb{E}_z[\mathbb{P}_z(Z_{V_n} = k|Z_{V_n} > 0, \mathbf{e})] \xrightarrow{n\to\infty} 0,$$

which is the desired result.

(c) Using Proposition 2.18, we have $\mathbb{P}_z(L = 0) < 1$ and $\{L > 0\} = Surv$ $\mathbb{P}_z$-a.s. Therefore, Corollary 2.11 yields

$$\frac{F_n(k)}{\mathcal{T}_n^*}\mathbb{1}_{Surv} = \frac{F_n(k)/\mu^n}{\mathcal{T}_n^*/\mu^n}\mathbb{1}_{Surv} \xrightarrow{\mathbb{P}_z} 0,$$

i.e. $\lim_{n\to\infty} \mathbb{P}_z\left(\frac{F_n(k)}{\mathcal{T}_n^*} > \varepsilon \,\middle|\, Surv\right) = 0$ for all $\varepsilon > 0$. □

3. The process of parasites

Now, we are turning to the process of parasites $(\mathcal{Z}_n)_{n\geq 0}$. In Corollary 2.14 of the previous chapter, we have already seen that this process satisfies an extinction-explosion principle.

In the following section, we will take a closer look at the $\mathbb{P}$-a.s. extinction of the process of parasites. Afterwards, after a proper normalisation, we establish a convergence result for the process $(\mathcal{Z}_n)_{n\geq 0}$ by means of the martingale convergence theorem. Furthermore, we give conditions for L^2-boundedness of this normalised process. Then, we come back to the earlier mentioned method of size-biasing and construct a *size-biased BwBPRE*. This enables us to prove a dichotomy on the asymptotic behaviour of the normalised process of $(\mathcal{Z}_n)_{n\geq 0}$. The final section of this chapter deals with the process of parasites along the spine in the here given spinal construction.

Similar results can be found in [27].

3.1. The a.s. extinction of the process of parasites

Again, we have to distinguish the cases $\mathbb{P}_2(\mathcal{T}_1^* = 2) > 0$ and $\mathbb{P}_2(\mathcal{T}_1^* = 2) = 0$. Let us start with the latter one.

Proposition 3.1. *If* $\mathbb{P}_2(\mathcal{T}_1^* = 2) = 0$*, then* $\mathbb{P}(\mathit{Ext}) = 1$ *iff*

$$\mathbb{E}\left[\log \mathbb{E}\left[\mathcal{Z}_1 | T_\emptyset, \mathbf{e}_0\right]\right] \leq 0 \quad \mathit{or} \quad \mathbb{E}[\log^-(1 - \mathbb{P}(\mathcal{Z}_1 = 0 | T_\emptyset, \mathbf{e}_0))] = \infty.$$

Proof. Let $\mathbb{P}_2(\mathcal{T}_1^* = 2) = 0$ and $p_2 > 0$. As seen in the proof of Theorem 2.17, we get

$$\mathbb{P}_z(\mathcal{T}_n^* \leq 1 \ \forall \ n \geq 0) = 1,$$

for any $z \in \mathbb{N}_0$. Obviously, this is also true in the case $p_2 = 0$. Hence, the process $(\mathcal{Z}_n)_{n\geq 0}$ is equal in law to the BPRE $(\tilde{\mathcal{Z}}_n)_{n\geq 0}$ in i.i.d. random environment $(T_{0^{*n}}, \mathbf{e}_n)_{n\geq 0}$, where $\tilde{\mathcal{Z}}_0 = 1$ and

$$\tilde{\mathcal{Z}}_{n+1} = \sum_{i=1}^{\tilde{\mathcal{Z}}_n} \left(X_{i,0^{*n}}^{(0,T_{0^{*n}})} + X_{i,0^{*n}}^{(1,T_{0^{*n}})} \right), \quad n \geq 1.$$

If $p_0 = 0$, we see with the help of (1.3) that this BPRE satisfies $\mathbb{E}|\log \mathbb{E}[\mathcal{Z}_1 | T_\emptyset, \mathbf{e}_0]| < \infty$.

Therefore, according to results in [8] and [50] we get $\mathbb{P}(\mathit{Ext}) = 1$ iff

$$\mathbb{E}[\log \mathbb{E}[\mathcal{Z}_1 | T_\emptyset, \mathbf{e}_0]] \leq 0 \quad \text{or} \quad \mathbb{E}[\log^-(1 - \mathbb{P}(\mathcal{Z}_1 = 0 | T_\emptyset, \mathbf{e}_0))] = \infty.$$

If $p_0 > 0$, we have $\mathbb{P}(\mathit{Ext}) = 1$ as well as $\mathbb{E}[\log \mathbb{E}[\mathcal{Z}_1 | T_\emptyset, \mathbf{e}_0]] \leq 0$. ☐

Unfortunately, we cannot give a full characterisation of $\mathbb{P}$-a.s. extinction in the case $\mathbb{P}_2(\mathcal{T}_1^* = 2) > 0$. At least, we can state a necessary condition and a different condition which is sufficient. We start with the necessary condition.

Proposition 3.2. *Let* $\mathbb{P}_2(\mathcal{T}_1^* = 2) > 0$. *If* $\mathbb{P}(\mathit{Ext}) = 1$, *then we have*

$$\mathbb{E}[\log \mathbb{E}[\mathcal{T}_n^* | \mathbf{e}]] \leq 0 \quad \textit{or} \quad \mathbb{E}[\log^-(1 - \mathbb{P}(\mathcal{T}_n^* = 0 | \mathbf{e}))] = \infty$$

for all $n \in \mathbb{N}_0$.

Proof. Suppose that there exists an $m \in \mathbb{N}_0$, such that

$$\mathbb{E}[\log \mathbb{E}[\mathcal{T}_m^* | \mathbf{e}]] > 0 \quad \text{and} \quad \mathbb{E}[\log^-(1 - \mathbb{P}(\mathcal{T}_m^* = 0 | \mathbf{e}))] < \infty.$$

Then one can consider the supercritical BPRE $(S_n)_{n \geq 0}$ with $S_0 = 1$ and generic random generating function $\mathbb{E}[s^{\mathcal{T}_m^*} | \mathbf{e}]$. Note that

$$\mathbb{P}(S_n > k | \mathbf{e}) \leq \mathbb{P}(\mathcal{T}_{nm}^* > k | \mathbf{e}), \quad k \in \mathbb{N}_0,$$

since the presence of more than one parasite in an infected cell increases the probability to get more infected cells. Moreover, the BPRE $(S_n)_{n \geq 0}$ is supercritical and satisfies

$$\mathbb{E}[\log^-(1 - \mathbb{P}(S_1 = 0 | \mathbf{e}))] = \mathbb{E}[\log^-(1 - \mathbb{P}(\mathcal{T}_m^* = 0 | \mathbf{e}))] < \infty.$$

Thus, Theorem 3 in [8] or Theorem 3.1 in [50] yields

$$\lim_{n \to \infty} \mathbb{P}(\mathcal{T}_{nm}^* > 0) \geq \lim_{n \to \infty} \mathbb{P}(S_n > 0) > 0,$$

i.e. parasites survive with positive probability. This is a contradiction. ☐

The sufficient condition for a.s. extinction is content of the next

Proposition 3.3. *Let* $\mathbb{P}_2(\mathcal{T}_1^* = 2) > 0$. *If* $\sup_{n \in \mathbb{N}_0} \mathbb{E}[\log^+ \mathbb{E}[\mathcal{T}_n^* | \mathbf{e}]] < \infty$, *then we have* $\mathbb{P}(\mathit{Ext}) = 1$.

Proof. Assume $\mathbb{P}(\mathit{Surv}) > 0$. We have

$$\mathbb{P}(\mathcal{T}_n^* \to \infty | \mathit{Surv}) = 1$$

according to Theorem 2.17, and therefore

$$\mathbb{P}(\mathcal{T}_n^* \to \infty, \mathit{Surv}\,|\mathbf{e}) > 0$$

with positive probability. Using Fatou's lemma, we get

$$\liminf_{n\to\infty} \mathbb{E}[\mathcal{T}_n^*|\mathbf{e}] \geq \mathbb{E}\left[\mathbb{1}_{\mathit{Surv}} \liminf_{n\to\infty} \mathcal{T}_n^* \middle| \mathbf{e}\right] = \infty$$

with positive probability and thus, using Fatou's lemma once again,

$$\infty > \sup_{n\in\mathbb{N}_0} \mathbb{E}[\log^+ \mathbb{E}[\mathcal{T}_n^*|\mathbf{e}]] \geq \liminf_{n\to\infty} \mathbb{E}[\log^+ \mathbb{E}[\mathcal{T}_n^*|\mathbf{e}]] \geq \mathbb{E}\left[\liminf_{n\to\infty} \log^+ \mathbb{E}[\mathcal{T}_n^*|\mathbf{e}]\right] = \infty.$$

This is a contradiction. □

3.2. The process of parasites and two martingales

Note that the number of parasites $\mathcal{Z}_n$ in the n-th generation, $n \geq 1$, can be written as

$$\mathcal{Z}_n = \sum_{v\in\mathbb{T}_{n-1}^*} \sum_{i=1}^{Z_v} \left(X_{i,v}^{(0,T_v)} + X_{i,v}^{(1,T_v)}\right).$$

Let us introduce

$$\nu(\mathbf{e}_n) := \mathbb{E}\left[X_{1,v}^{(0,T_v)} + X_{1,v}^{(1,T_v)} \middle| \mathbf{e}\right], \quad |v| = n,$$

for $n \in \mathbb{N}_0$, and

$$\nu_n(\mathbf{e}) := \prod_{i=0}^{n-1} \nu(\mathbf{e}_i),$$

where $\nu_0(\mathbf{e}) := 1$. Note that assumption (1.3) ensures

$$0 < \nu(\mathbf{e}_0) < \infty \quad \mathbb{P}\text{-a.s.}$$

Using $(\nu_n(\mathbf{e}))_{n\geq 0}$ as a norming sequence, we can state a first convergence result for the process of parasites $(\mathcal{Z}_n)_{n\geq 0}$. The corresponding process $(W_n)_{n\geq 0}$ given by

$$W_n := \frac{\mathcal{Z}_n}{\nu_n(\mathbf{e})}$$

will be analysed in greater detail here and in the following chapter.

Proposition 3.4. *Under each* $\mathbb{P}_z$, $z \in \mathbb{N}_0$, *the process* $(W_n)_{n\geq 0}$ *forms a nonnegative martingale with respect to* $(\mathcal{G}_n)_{n\geq 0}$ *and therefore converges* $\mathbb{P}_z$*-a.s. to an integrable random variable* W *satisfying* $\mathbb{E}_z W \leq z$. *Moreover:*

(a) $\{\mathbb{P}(W > 0|\mathbf{e}) > 0\} = \{\mathbb{P}(W > 0|[\mathbf{e}]_1) > 0\}$ $\mathbb{P}$*-a.s., and therefore*

$$\mathbb{P}(\mathbb{P}(W = 0|\mathbf{e}) = 1) \in \{0, 1\}.$$

(b) If $\mathbb{P}(W = 0) < 1$, *then* $\{W = 0\} = \mathit{Ext}$ $\mathbb{P}$*-a.s.*

Remark 3.5. Another natural norming sequence would be $(\nu^n)_{n\geq 0}$, where ν is the annealed mean of $\mathcal{Z}_1$. However, as we will see later in Proposition 3.13, it turns out that this norming sequence is in general not suitable.

Proof. It is clear from the definition of $\mathcal{G}_n$ that W_n is $\mathcal{G}_n$-measurable, $n \geq 0$. Moreover, for $n \geq 0$

$$\begin{aligned}
\mathbb{E}_z[W_{n+1}|\mathcal{G}_n] &= \nu_{n+1}(\mathbf{e})^{-1} \sum_{v\in\mathbb{T}_n} \sum_{i=1}^{Z_v} \mathbb{E}_z \left[X_{i,v}^{(0,T_v)} + X_{i,v}^{(1,T_v)} \middle| \mathcal{G}_n \right] \\
&= \nu_{n+1}(\mathbf{e})^{-1} \sum_{v\in\mathbb{T}_n} \sum_{i=1}^{Z_v} \mathbb{E} \left[\mathbb{1}_{\{T_v\neq 0\}} \mathbb{E} \left[X_{i,v}^{(0)} + X_{i,v}^{(1)} \middle| \mathcal{G}_n, T_v \right] \middle| \mathcal{G}_n \right] \\
&= \nu_{n+1}(\mathbf{e})^{-1} \sum_{v\in\mathbb{T}_n} \sum_{i=1}^{Z_v} \mathbb{E} \left[\mathbb{1}_{\{T_v\neq 0\}} \mathbb{E} \left[X_{i,v}^{(0)} + X_{i,v}^{(1)} \middle| \mathbf{e} \right] \middle| \mathcal{G}_n \right] \\
&= \nu_{n+1}(\mathbf{e})^{-1} \sum_{v\in\mathbb{T}_n} \sum_{i=1}^{Z_v} \mathbb{E} \left[X_{i,v}^{(0)} + X_{i,v}^{(1)} \middle| \mathbf{e} \right] \mathbb{P}(T_v \neq 0) \\
&= \nu_{n+1}(\mathbf{e})^{-1} \sum_{v\in\mathbb{T}_n} \sum_{i=1}^{Z_v} \nu(\mathbf{e}_n) \\
&= \nu_{n+1}(\mathbf{e})^{-1} \nu(\mathbf{e}_n) \cdot \sum_{v\in\mathbb{T}_n} Z_v \\
&= \nu_n(\mathbf{e})^{-1} \cdot \mathcal{Z}_n \\
&= W_n \quad \mathbb{P}_z\text{-a.s.},
\end{aligned}$$

where we used the independence of $(T_v)_{|v|=n}$ and $\mathcal{G}_n$ for the fifth equation. Regarding the fourth equation note that

$$\begin{aligned}
\mathbb{E} \left[X_{i,v}^{(0)} + X_{i,v}^{(1)} \middle| \mathcal{G}_n, T_v \right] &= \mathbb{E} \left[X_{i,v}^{(0)} + X_{i,v}^{(1)} \middle| \mathbf{e}, X_{k,w}^{(0)}, X_{k,w}^{(1)}, k \geq 1, |w| \leq n-1 \right] \\
&= \mathbb{E} \left[X_{i,v}^{(0)} + X_{i,v}^{(1)} \middle| \mathbf{e} \right],
\end{aligned}$$

where we used the independence of $\left(\left(X_{k,w}^{(0)}, X_{k,w}^{(1)}\right)_{k\geq 1, w\in\mathbb{V}}, \mathbf{e}\right)$ and $(T_w)_{w\in\mathbb{V}}$, and a similar calculation as in (2.5). Since the integrability follows from the above by iteration, $(W_n)_{n\geq 0}$ forms a nonnegative martingale with respect to $(\mathcal{G}_n)_{n\geq 0}$ and the martingale convergence theorem ensures the $\mathbb{P}_z$-a.s. convergence to an integrable random variable W. Finally, $\mathbb{E}_z W_0 = z$ in combination with Fatou's lemma yields

$$\mathbb{E}_z W \leq \liminf_{n\to\infty} \mathbb{E}_z W_n = z.$$

(a) First, we are going to show $\{\mathbb{P}(W>0|\mathbf{e})>0\} \subseteq \{\mathbb{P}(W>0|[\mathbf{e}]_1)>0\}$ $\mathbb{P}$-a.s. To that end, we choose $\omega \in \{\mathbb{P}(W>0|\mathbf{e})>0\}$ and suppose $\mathbb{P}(W>0|[\mathbf{e}]_1)(\omega)=0$. Similar to (2.7), one can show that this entails

$$\mathbb{P}_{(t,z)}(W>0|[\mathbf{e}]_1)(\omega) = 0$$

for every $(t,z)\in R^{\uparrow}$ and almost all such ω. Then

$$\begin{aligned} 0 <\; & \mathbb{P}(W>0|\mathbf{e})(\omega) \\ =\; & \sum_{z\geq 1} \mathbb{P}(T_\emptyset = 1, Z_0 = z|\mathbf{e})(\omega)\mathbb{P}_z(W>0|[\mathbf{e}]_1)(\omega) \\ & + \sum_{\substack{z_0,z_1\geq 0 \\ z_0+z_1\geq 1}} \mathbb{P}(T_\emptyset = 2, Z_0 = z_0, Z_1 = z_1|\mathbf{e})(\omega)\mathbb{P}_{(2,(z_0,z_1))}(W>0|[\mathbf{e}]_1)(\omega) \\ =\; & 0. \end{aligned}$$

This is a contradiction and so $\{\mathbb{P}(W>0|\mathbf{e})>0\} \subseteq \{\mathbb{P}(W>0|[\mathbf{e}]_1)>0\}$ $\mathbb{P}$-a.s. Since the environmental sequence $\mathbf{e}$ is i.i.d., we also get

$$\mathbb{P}(\mathbb{P}(W>0|\mathbf{e})>0) = \mathbb{P}(\mathbb{P}(W>0|[\mathbf{e}]_1)>0),$$

and therefore $\{\mathbb{P}(W>0|\mathbf{e})>0\} = \{\mathbb{P}(W>0|[\mathbf{e}]_1)>0\}$ $\mathbb{P}$-a.s. The second assertion follows from the ergodicity of the shift operator.

(b) This part follows similar to part (c) of Proposition 2.18. First note that obviously *Ext* $\subseteq \{W=0\}$ $\mathbb{P}$-a.s.
If $\mathbb{P}_2(\mathcal{T}_1^* = 2) = 0$ and $\mathbb{P}(W=0)<1$, then the law of the process $(\mathcal{Z}_n)_{n\geq 0}$ equals the law of a BPRE, as seen in the proof of Proposition 3.1. Therefore, Theorem 1 in [9] gives us $\{W=0\} =$ *Ext* $\mathbb{P}$-a.s.
Let $\mathbb{P}_2(\mathcal{T}_1^* \geq 2) > 0$. Again, as in the proof of Proposition 2.18 (c), consider the first time the number of infected cells is greater than or equal to $n\in\mathbb{N}$, i.e.

$$\tau_n = \inf\{m\in\mathbb{N} : \mathcal{T}_m^* \geq n\}.$$

Now, for $m \in \mathbb{N}$ we have

$$\mathcal{Z}_{m+l} = \sum_{u \in \mathbb{T}_m^*} \mathcal{Z}_l(u), \quad l \geq 0,$$

and thus

$$W = \lim_{l\to\infty} \frac{\mathcal{Z}_{m+l}}{\nu_{m+l}(\mathbf{e})} = \frac{1}{\nu_m(\mathbf{e})} \sum_{u \in \mathbb{T}_m^*} \lim_{l\to\infty} \frac{\mathcal{Z}_l(u)}{\nu_{m+l}(\mathbf{e})/\nu_m(\mathbf{e})},$$

where $\mathcal{Z}_l(u)$ denotes the number of parasites in the l-th generation of the BwBPRE starting in the cell u with Z_u root parasites. These processes, starting in the cells $u \in \mathbb{T}_m^*$, are conditionally independent given $\mathcal{G}_m$, and evolve like the original process under $\mathbb{P}$ with the shifted environment $[\mathbf{e}]_m$, except for the fact that there might be more than one parasite in the root cell u. This gives us for $n \in \mathbb{N}$

$$\begin{aligned}
\mathbb{P}(W=0) \leq\ & \sum_{m\geq 1} \mathbb{P}(\tau_n = m, W=0) + \mathbb{P}(\tau_n=\infty) \\
=\ & \sum_{m\geq 1} \mathbb{P}\left(\{\tau_n = m\} \cap \bigcap_{u\in\mathbb{T}_m^*} \left\{ \lim_{l\to\infty} \frac{\mathcal{Z}_l(u)}{\frac{\nu_{m+l}(\mathbf{e})}{\nu_m(\mathbf{e})}} = 0 \right\}\right) + \mathbb{P}(\tau_n=\infty) \\
=\ & \sum_{m\geq 1} \mathbb{E}\left[\mathbb{1}_{\{\tau_n=m\}} \mathbb{P}\left(\bigcap_{u\in\mathbb{T}_m^*} \left\{ \lim_{l\to\infty} \frac{\mathcal{Z}_l(u)}{\frac{\nu_{m+l}(\mathbf{e})}{\nu_m(\mathbf{e})}} = 0 \right\} \middle| \mathcal{G}_m \right)\right] \\
& + \mathbb{P}(\tau_n=\infty) \\
=\ & \sum_{m\geq 1} \mathbb{E}\left[\mathbb{1}_{\{\tau_n=m\}} \prod_{u\in\mathbb{T}_m^*} \mathbb{P}\left(\lim_{l\to\infty} \frac{\mathcal{Z}_l(u)}{\frac{\nu_{m+l}(\mathbf{e})}{\nu_m(\mathbf{e})}} = 0 \middle| \mathcal{G}_m \right)\right] + \mathbb{P}(\tau_n=\infty) \\
\leq\ & \sum_{m\geq 1} \mathbb{E}\left[\mathbb{1}_{\{\tau_n=m\}} \cdot \mathbb{P}(W=0|[\mathbf{e}]_m)^{\mathcal{T}_m^*} \right] + \mathbb{P}(\tau_n=\infty) \\
\leq\ & \sum_{m\geq 1} \mathbb{E}\left[\mathbb{P}(\tau_n = m|\mathbf{e}) \cdot \mathbb{P}(W=0|[\mathbf{e}]_m)^n \right] + \mathbb{P}(\tau_n=\infty) \\
=\ & \sum_{m\geq 1} \mathbb{P}(\tau_n = m) \cdot \mathbb{E}\left[\mathbb{P}(W=0|[\mathbf{e}]_m)^n \right] + \mathbb{P}(\tau_n=\infty) \\
=\ & \sum_{m\geq 1} \mathbb{P}(\tau_n = m) \cdot \mathbb{E}\left[\mathbb{P}(W=0|\mathbf{e})^n \right] + \mathbb{P}(\tau_n=\infty) \\
\leq\ & \mathbb{E}\left[\mathbb{P}(W=0|\mathbf{e})^n \right] + \mathbb{P}(\tau_n=\infty),
\end{aligned}$$

where we used the i.i.d. property of the random environment. If $\mathbb{P}(W=0)<1$, we have $\mathbb{P}(W=0|\mathbf{e})<1$ a.s. according to part (a), and the extinction-explosion principle for the process of infected cells (Theorem 2.17 (b)) yields

$$\mathbb{P}(W=0)\leq \lim_{n\to\infty}\mathbb{P}(\tau_n=\infty)=\mathbb{P}\left(\sup_{m\geq 1}\mathcal{T}_m^*<\infty\right)=\mathbb{P}(Ext),$$

where we recall that we are in the case $\mathbb{P}_2(\mathcal{T}_1^*\geq 2)>0$. Together with $Ext\subseteq\{W=0\}$ $\mathbb{P}$-a.s. this gives the desired result. □

Sometimes it is convenient to have W defined on the entire probability space $(\Omega,\mathfrak{A},\mathbb{P})$. Therefore we put $W:=\limsup_{n\to\infty}W_n$.

Now, we turn to the problem of finding conditions that ensure $\mathbb{P}(W>0)>0$. An obvious sufficient condition is uniform integrability (u.i.) and a fortiori L^2-boundedness of $(W_n)_{n\geq 0}$. Later, in Theorem 4.5, we give a necessary and sufficient condition for u.i., but here we start with the L^2-setting. First, we compute the quenched and annealed second moment of our martingale $(W_n)_{n\geq 0}$ in order to make assertions about the L^2-boundedness under $\mathbb{P}(\cdot|\mathbf{e})$ as well as under $\mathbb{P}$. To give the following expressions in a more compact form, let us introduce the notation

$$\nu^{(p)}(\mathbf{e}_n)=\mathbb{E}\left[\left(X_{1,v}^{(0,T_v)}+X_{1,v}^{(1,T_v)}\right)^p\middle|\,\mathbf{e}\right],\quad |v|=n,$$

for $n\in\mathbb{N}_0$ and $p>0$, and note that $\nu^{(1)}(\mathbf{e}_n)=\nu(\mathbf{e}_n)$. Moreover, define

$$\beta(\mathbf{e}_n):=\mu\cdot\mathbb{E}[f''_{\xi_n}(1)|\mathbf{e}],\quad \alpha(\mathbf{e}_n):=\mu\cdot\mathbb{E}[f'_{\xi_n}(1)^2|\mathbf{e}],\quad n\geq 0,$$

and

$$\alpha_0(\mathbf{e}):=1,\quad \alpha_n(\mathbf{e}):=\prod_{m=1}^n\alpha(\mathbf{e}_m),\quad n\geq 1,$$

and put

$$\sigma^2:=\mathbb{V}\mathrm{ar}\left(\frac{\mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right),\quad \tilde{\nu}:=\mathbb{E}[\nu^{-1}(\mathbf{e}_0)],\quad \beta:=\mu\cdot\mathbb{E}\left[\frac{f''_{\xi_0}(1)}{\nu^2(\mathbf{e}_0)}\right]$$
$$\text{and}\quad \alpha:=\mu\cdot\mathbb{E}\left[\left(\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right)^2\right].$$

Then we get the following

Proposition 3.6. *Suppose* $(W_n)_{n\geq 0}$ *is square-integrable. For any* $n \geq 1$ *we have*

$$\begin{aligned}\mathbb{E}[W_n^2|\mathbf{e}] = 1 &+ \sum_{k=0}^{n-1} \frac{1}{\nu_k(\mathbf{e})}\left(\frac{\nu^{(2)}(\mathbf{e}_k)}{\nu^2(\mathbf{e}_k)} - 1\right)\\ &+ \frac{p_0}{p_1+p_2}\sum_{k=0}^{n-1}\frac{1}{\nu_k^2(\mathbf{e})}\sum_{j=0}^{k-1}\frac{\alpha_{k-1}(\mathbf{e})}{\alpha_j(\mathbf{e})}\beta(\mathbf{e}_j)\nu_j(\mathbf{e}) \quad \mathbb{P}\text{-}a.s.\end{aligned}$$

and

$$\mathbb{E}[W_n^2] = 1 + \sigma^2\sum_{k=0}^{n-1}\tilde{\nu}^k + \frac{p_0}{p_1+p_2}\beta\sum_{k=0}^{n-1}\alpha^{k-1}\sum_{j=0}^{k-1}\left(\frac{\tilde{\nu}}{\alpha}\right)^j. \tag{3.1}$$

In particular

$$\mathbb{V}\mathrm{ar}\, W_n = \sigma^2\sum_{k=0}^{n-1}\tilde{\nu}^k + \frac{p_0}{p_1+p_2}\beta\sum_{k=0}^{n-1}\alpha^{k-1}\sum_{j=0}^{k-1}\left(\frac{\tilde{\nu}}{\alpha}\right)^j. \tag{3.2}$$

Proof. Note that

$$W_{n+1} - W_n = \frac{1}{\nu_n(\mathbf{e})}\sum_{v\in\mathbb{T}_n}\sum_{i=1}^{Z_v}\left(\frac{X_{i,v}^{(0,T_v)} + X_{i,v}^{(1,T_v)}}{\nu(\mathbf{e}_n)} - 1\right), \tag{3.3}$$

and that $(W_n)_{n\geq 0}$ forms a martingale under $\mathbb{P}(\cdot|\mathbf{e})$ with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$. The latter is immediate from the proof of Proposition 3.4, and by our assumption this martingale is $\mathbb{P}$-a.s. square-integrable under $\mathbb{P}(\cdot|\mathbf{e})$. Therefore, the orthogonality of martingale increments yields

$$\begin{aligned}\mathbb{E}[W_n^2|\mathbf{e}] &= 1 + \sum_{k=0}^{n-1}\mathbb{E}\left[(W_{k+1}-W_k)^2\middle|\, \mathbf{e}\right]\\ &= 1 + \sum_{k=0}^{n-1}\frac{1}{\nu_k^2(\mathbf{e})}\cdot\mathbb{E}\left[\left(\sum_{v\in\mathbb{T}_k}\sum_{i=1}^{Z_v}\left(\frac{X_{i,v}^{(0,T_v)}+X_{i,v}^{(1,T_v)}}{\nu(\mathbf{e}_k)} - 1\right)\right)^2\middle|\,\mathbf{e}\right] \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

Denoting the quenched expectation on the right hand side of the previous equation by I_k, we get

$$I_k = I_{k,1} + I_{k,2} + I_{k,3},$$

with

$$\begin{aligned}
I_{k,1} &:= \mathbb{E}\left[\sum_{v\in\mathbb{T}_k}\sum_{i=1}^{Z_v}\left(\frac{X_{i,v}^{(0,T_v)}+X_{i,v}^{(1,T_v)}}{\nu(\mathbf{e}_k)}-1\right)^2\middle|\,\mathbf{e}\right]\\
&= \mathbb{E}\left[\sum_{v\in\mathbb{T}_k}\sum_{i=1}^{Z_v}\mathbb{E}\left[\left(\frac{X_{i,v}^{(0,T_v)}+X_{i,v}^{(1,T_v)}}{\nu(\mathbf{e}_k)}-1\right)^2\middle|\,\mathcal{F}_k,\mathbf{e}\right]\middle|\,\mathbf{e}\right]\\
&= \mathbb{E}\left[\sum_{v\in\mathbb{T}_k}\sum_{i=1}^{Z_v}\left(\mathbb{E}\left[\left(\frac{X_{i,v}^{(0,T_v)}+X_{i,v}^{(1,T_v)}}{\nu(\mathbf{e}_k)}\right)^2\middle|\,\mathbf{e}\right]-1\right)\middle|\,\mathbf{e}\right]\\
&= \left(\frac{\nu^{(2)}(\mathbf{e}_k)}{\nu^2(\mathbf{e}_k)}-1\right)\mathbb{E}[\mathcal{Z}_k|\mathbf{e}]\\
&= \left(\frac{\nu^{(2)}(\mathbf{e}_k)}{\nu^2(\mathbf{e}_k)}-1\right)\nu_k(\mathbf{e})\quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

and

$$\begin{aligned}
I_{k,2} &:= \mathbb{E}\left[\sum_{v\in\mathbb{T}_k}\sum_{\substack{i,j=1\\ i\neq j}}^{Z_v}\left(\frac{X_{i,v}^{(0,T_v)}+X_{i,v}^{(1,T_v)}}{\nu(\mathbf{e}_k)}-1\right)\left(\frac{X_{j,v}^{(0,T_v)}+X_{j,v}^{(1,T_v)}}{\nu(\mathbf{e}_k)}-1\right)\middle|\,\mathbf{e}\right]\\
&= \mathbb{E}\left[\sum_{v\in\mathbb{T}_k}\sum_{\substack{i,j=1\\ i\neq j}}^{Z_v}\mathbb{E}\left[\left(\frac{X_{i,v}^{(0,T_v)}+X_{i,v}^{(1,T_v)}}{\nu(\mathbf{e}_k)}-1\right)\left(\frac{X_{j,v}^{(0,T_v)}+X_{j,v}^{(1,T_v)}}{\nu(\mathbf{e}_k)}-1\right)\middle|\,\mathcal{F}_k,\mathbf{e}\right]\middle|\,\mathbf{e}\right]\\
&= \mathbb{E}\left[\sum_{v\in\mathbb{T}_k}\sum_{\substack{i,j=1\\ i\neq j}}^{Z_v}\mathbb{E}\left[\left(\frac{X_{i,v}^{(0,T_v)}+X_{i,v}^{(1,T_v)}}{\nu(\mathbf{e}_k)}-1\right)\left(\frac{X_{j,v}^{(0,T_v)}+X_{j,v}^{(1,T_v)}}{\nu(\mathbf{e}_k)}-1\right)\middle|\,\mathbf{e}\right]\middle|\,\mathbf{e}\right]\\
&= \mathbb{E}\left[\sum_{v\in\mathbb{T}_k}\sum_{\substack{i,j=1\\ i\neq j}}^{Z_v}\left(\frac{p_1+p_2}{\nu^2(\mathbf{e}_k)}\cdot\mathbb{E}\left[X_{i,v}^{(0)}+X_{i,v}^{(1)}\middle|\,\mathbf{e}\right]\mathbb{E}\left[X_{j,v}^{(0)}+X_{j,v}^{(1)}\middle|\,\mathbf{e}\right]-1\right)\middle|\,\mathbf{e}\right]\\
&= \mathbb{E}\left[\sum_{v\in\mathbb{T}_k}\sum_{\substack{i,j=1\\ i\neq j}}^{Z_v}\left(\frac{1}{p_1+p_2}-1\right)\middle|\,\mathbf{e}\right]
\end{aligned}$$

$$= \frac{p_0}{p_1+p_2} \cdot \mathbb{E}\left[\sum_{v\in\mathbb{T}_k} Z_v^2 - \mathcal{Z}_k \middle| \mathbf{e}\right] \quad \mathbb{P}\text{-a.s.}$$

as well as

$$\begin{aligned}
I_{k,3} &:= \mathbb{E}\left[\sum_{\substack{v,w\in\mathbb{T}_k\\ v\neq w}} \left(\sum_{i=1}^{Z_v}\left(\frac{X_{i,v}^{(0,T_v)}+X_{i,v}^{(1,T_v)}}{\nu(\mathbf{e}_k)}-1\right)\right)\left(\sum_{j=1}^{Z_w}\left(\frac{X_{j,w}^{(0,T_w)}+X_{j,w}^{(1,T_w)}}{\nu(\mathbf{e}_k)}-1\right)\right)\middle| \mathbf{e}\right]\\
&= \mathbb{E}\left[\sum_{\substack{v,w\in\mathbb{T}_k\\ v\neq w}} \sum_{i=1}^{Z_v}\sum_{j=1}^{Z_w}\mathbb{E}\left[\left(\frac{X_{i,v}^{(0,T_v)}+X_{i,v}^{(1,T_v)}}{\nu(\mathbf{e}_k)}-1\right)\left(\frac{X_{j,w}^{(0,T_w)}+X_{j,w}^{(1,T_w)}}{\nu(\mathbf{e}_k)}-1\right)\middle| \mathcal{F}_k,\mathbf{e}\right]\middle| \mathbf{e}\right]\\
&= \mathbb{E}\left[\sum_{\substack{v,w\in\mathbb{T}_k\\ v\neq w}} \sum_{i=1}^{Z_v}\sum_{j=1}^{Z_w}\mathbb{E}\left[\frac{X_{i,v}^{(0,T_v)}+X_{i,v}^{(1,T_v)}}{\nu(\mathbf{e}_k)}-1\middle| \mathbf{e}\right]\mathbb{E}\left[\frac{X_{j,w}^{(0,T_w)}+X_{j,w}^{(1,T_w)}}{\nu(\mathbf{e}_k)}-1\middle| \mathbf{e}\right]\middle| \mathbf{e}\right]\\
&= 0 \quad \mathbb{P}\text{-a.s.},
\end{aligned}$$

where we used the conditional independence of $(X_{i,v}^{(0,T_v)}, X_{i,v}^{(1,T_v)})_{i\geq 1,|v|=k}$ and $\mathcal{F}_k$ as well as the conditional independence of $(X_{i,v}^{(0)}, X_{i,v}^{(1)})$ and $(X_{j,v}^{(0)}, X_{j,v}^{(1)})$ if $i \neq j$ and the conditional independence of $(X_{i,v}^{(0,T_v)}, X_{i,v}^{(1,T_v)})$ and $(X_{j,w}^{(0,T_w)}, X_{j,w}^{(T_w)})$ if $v \neq w$. Further, using Proposition 2.9, we can conclude

$$\begin{aligned}
\mathbb{E}\left[\sum_{v\in\mathbb{T}_k} Z_v^2 - \mathcal{Z}_k \middle| \mathbf{e}\right] &= \mu^k \mathbb{E}[Z_{V_k}^2 - Z_{V_k}|\mathbf{e}] = \mu^k \mathbb{E}[f''_{k,\xi}(1)|\mathbf{e}]\\
&= \mu^k\left(\mathbb{E}[f''_{k-1,\xi}(1) f'_{\xi_{k-1}}(1)^2|\mathbf{e}] + \mathbb{E}[f'_{k-1,\xi}(1) f''_{\xi_{k-1}}(1)|\mathbf{e}]\right)\\
&= \mu^k\left(\mathbb{E}[f''_{k-1,\xi}(1)|\mathbf{e}]\cdot\mathbb{E}[f'_{\xi_{k-1}}(1)^2|\mathbf{e}] + \mathbb{E}[Z_{V_{k-1}}|\mathbf{e}]\cdot\mathbb{E}[f''_{\xi_{k-1}}(1)|\mathbf{e}]\right)\\
&= \mu^k\left(\mathbb{E}[f''_{k-1,\xi}(1)|\mathbf{e}]\cdot\frac{\alpha(\mathbf{e}_{k-1})}{\mu} + \frac{\nu_{k-1}(\mathbf{e})}{\mu^{k-1}}\cdot\frac{\beta(\mathbf{e}_{k-1})}{\mu}\right)\\
&= \ldots = \sum_{j=0}^{k-1}\frac{\alpha_{k-1}(\mathbf{e})}{\alpha_j(\mathbf{e})}\beta(\mathbf{e}_j)\nu_j(\mathbf{e}) \quad \mathbb{P}\text{-a.s.},
\end{aligned}$$

where we used the definitions made beforehand. Altogether, this yields

$$\mathbb{E}[W_n^2|\mathbf{e}] = 1 + \sum_{k=0}^{n-1}\frac{1}{\nu_k(\mathbf{e})}\left(\frac{\nu^{(2)}(\mathbf{e}_k)}{\nu^2(\mathbf{e}_k)}-1\right)$$

$$+\frac{p_0}{p_1+p_2}\sum_{k=0}^{n-1}\frac{1}{\nu_k^2(\mathbf{e})}\sum_{j=0}^{k-1}\frac{\alpha_{k-1}(\mathbf{e})}{\alpha_j(\mathbf{e})}\beta(\mathbf{e}_j)\nu_j(\mathbf{e}) \quad \mathbb{P}\text{-a.s.}$$

Since the environmental sequence is i.i.d., we obtain for the annealed second moment

$$\begin{aligned}
\mathbb{E}[W_n^2] &= 1+\sum_{k=0}^{n-1}\mathbb{E}[\nu_k^{-1}(\mathbf{e})]\cdot\mathbb{E}\left[\frac{\nu^{(2)}(\mathbf{e}_k)}{\nu^2(\mathbf{e}_k)}-1\right] \\
&\quad+\frac{p_0}{p_1+p_2}\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}\mathbb{E}\left[\beta(\mathbf{e}_j)\frac{\alpha(\mathbf{e}_{j+1})\cdots\alpha(\mathbf{e}_{k-1})}{\nu(\mathbf{e}_0)\cdots\nu(\mathbf{e}_{j-1})\nu^2(\mathbf{e}_j)\cdots\nu^2(\mathbf{e}_{k-1})}\right] \\
&= 1+\sum_{k=0}^{n-1}\tilde{\nu}^k\cdot\mathbb{E}\left[\frac{\mathcal{Z}_1^2}{\nu^2(\mathbf{e}_0)}-1\right] \\
&\quad+\frac{p_0}{p_1+p_2}\cdot\mathbb{E}\left[\frac{\beta(\mathbf{e}_0)}{\nu^2(\mathbf{e}_0)}\right]\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}\left(\mathbb{E}[\nu^{-1}(\mathbf{e}_0)]\right)^j\cdot\left(\mathbb{E}\left[\frac{\alpha(\mathbf{e}_0)}{\nu^2(\mathbf{e}_0)}\right]\right)^{k-1-j} \\
&= 1+\mathbb{V}\mathrm{ar}\left(\frac{\mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right)\sum_{k=0}^{n-1}\tilde{\nu}^k \\
&\quad+\frac{p_0}{p_1+p_2}\cdot\mu\mathbb{E}\left[\frac{f_{\xi_0}''(1)}{\nu^2(\mathbf{e}_0)}\right]\sum_{k=0}^{n-1}\left(\mu\mathbb{E}\left[\left(\frac{f_{\xi_0}'(1)}{\nu(\mathbf{e}_0)}\right)^2\right]\right)^{k-1}\sum_{j=0}^{k-1}\left(\frac{\tilde{\nu}}{\mu\mathbb{E}\left[\left(\frac{f_{\xi_0}'(1)}{\nu(\mathbf{e}_0)}\right)^2\right]}\right)^j \\
&= 1+\sigma^2\sum_{k=0}^{n-1}\tilde{\nu}^k+\frac{p_0}{p_1+p_2}\beta\sum_{k=0}^{n-1}\alpha^{k-1}\sum_{j=0}^{k-1}\left(\frac{\tilde{\nu}}{\alpha}\right)^j.
\end{aligned}$$

Now, the formula for the variance is a consequence of $\mathbb{E}[W_n]=1$. □

We can make a case analysis to give more explicit formulas for the variance. Put

$$c=\frac{p_0}{p_1+p_2}.$$

The results are summarised in the next

Corollary 3.7. *For $n \geq 0$, the variance of W_n is given by*

$$\mathbb{V}\mathrm{ar} W_n = \begin{cases} n\sigma^2 + c\beta \dfrac{n(n-1)}{2}, & \tilde{\nu} = 1, \alpha = 1, \\ n\sigma^2 + \dfrac{c\beta}{1-\alpha}\left(n - \dfrac{1-\alpha^n}{1-\alpha}\right), & \tilde{\nu} = 1, \alpha \neq 1, \\ \sigma^2 \dfrac{1-\tilde{\nu}^n}{1-\tilde{\nu}} + \dfrac{c\beta}{1-\tilde{\nu}}\left(n - \dfrac{1-\tilde{\nu}^n}{1-\tilde{\nu}}\right), & \tilde{\nu} \neq 1, \alpha = 1, \\ \sigma^2 \dfrac{1-\tilde{\nu}^n}{1-\tilde{\nu}} + c\beta \dfrac{1 - n\alpha^{n-1} + (n-1)\alpha^n}{(1-\alpha)^2}, & \tilde{\nu} \neq 1, \alpha \neq 1, \tilde{\nu} = \alpha, \\ \sigma^2 \dfrac{1-\tilde{\nu}^n}{1-\tilde{\nu}} + \dfrac{c\beta}{\alpha - \tilde{\nu}}\left(\dfrac{1-\alpha^n}{1-\alpha} - \dfrac{1-\tilde{\nu}^n}{1-\tilde{\nu}}\right), & \tilde{\nu} \neq 1, \alpha \neq 1, \tilde{\nu} \neq \alpha. \end{cases}$$

Proof. Using (3.2) from Proposition 3.6 and the formula for the partial sum of the geometric series, the above expressions in the different cases are readily obtained. □

With the help of Proposition 3.6, we can give conditions for the L^2-boundedness of $(W_n)_{n\geq 0}$ under $\mathbb{P}(\cdot|\mathbf{e})$ as well as under $\mathbb{P}$. We need the following

Lemma 3.8 ([32], Lemma 3.1, [26], Theorem 1)**.** *Let $(\alpha_n, \beta_n)_{n\geq 0}$ be a stationary and ergodic sequence of nonnegative random variables. If $\mathbb{E}[\log \alpha_0] < 0$ and $\mathbb{E}[\log^+ \beta_0] < \infty$, then*

$$\sum_{n\geq 0} \alpha_0 \cdots \alpha_{n-1}\beta_n < \infty \quad \mathbb{P}\text{-}a.s.$$

The previous proposition and the cited lemma entail

Theorem 3.9. *Let $\mathbb{E}[\log \nu(\mathbf{e}_0)] > 0$, $p_0 > 0$ and suppose $(W_n)_{n\geq 0}$ is square-integrable.*

(a) For $\mathbb{P}$-almost all $\mathbf{e}$, the martingale $(W_n)_{n\geq 0}$ is L^2-bounded under $\mathbb{P}(\cdot|\mathbf{e})$ iff

$$\sum_{n\geq 0} \frac{1}{\nu_n(\mathbf{e})}\left(\frac{\nu^{(2)}(\mathbf{e}_n)}{\nu^2(\mathbf{e}_n)} - 1\right) < \infty$$

and

$$\sum_{n\geq 0} \frac{1}{\nu_n^2(\mathbf{e})} \sum_{j=0}^{n-1} \frac{\alpha_{n-1}(\mathbf{e})}{\alpha_j(\mathbf{e})} \beta(\mathbf{e}_j)\nu_j(\mathbf{e}) < \infty.$$

These sums are $\mathbb{P}$-a.s. finite if

$$\mathbb{E}\left[\log \mathbb{E}\left[\left(\frac{\mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right)^2 \middle| \mathbf{e}\right]\right] < \infty, \quad \mathbb{E}\left[\log^+ \frac{\mathbb{E}[f''_{\xi_0}(1)|\mathbf{e}]}{\mathbb{E}[f'_{\xi_0}(1)^2|\mathbf{e}]}\right] < \infty$$

$$\textit{and}\quad \mathbb{E}\left[\log \mathbb{E}\left[\left(\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right)^2\middle|\,\mathbf{e}\right]\right] < -\log\mu < \mathbb{E}\left[\log \mathbb{E}\left[\frac{f'_{\xi_0}(1)^2}{\nu(\mathbf{e}_0)}\middle|\,\mathbf{e}\right]\right].$$

(b) The martingale $(W_n)_{n\geq 0}$ *is* L^2*-bounded under* $\mathbb{P}$ *iff*

$$\mathbb{V}\mathrm{ar}\left(\frac{\mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right) < \infty, \quad \mathbb{E}[\nu^{-1}(\mathbf{e}_0)] < 1 \quad \textit{and} \quad \mathbb{E}\left[\left(\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right)^2\right] < \frac{1}{\mu}.$$

Proof. (a) The first assertion is a direct consequence of Proposition 3.6 because all summands are nonnegative. For the second assertion we use Lemma 3.8. Regarding the first sum, note that

$$\mathbb{E}[\log \nu^{-1}(\mathbf{e}_0)] = -\mathbb{E}[\log \nu(\mathbf{e}_0)] < 0$$

and

$$\mathbb{E}\left[\log^+\left(\frac{\nu^{(2)}(\mathbf{e}_0)}{\nu^2(\mathbf{e}_0)} - 1\right)\right] \leq \mathbb{E}\left[\log^+ \mathbb{E}\left[\left(\frac{\mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right)^2\middle|\,\mathbf{e}\right]\right] < \infty.$$

Thus, the $\mathbb{P}$-a.s. finiteness of the first sum is a consequence of Lemma 3.8.

For the second sum, note that

$$\begin{aligned}&\sum_{n\geq 0}\frac{1}{\nu_n^2(\mathbf{e})}\sum_{j=0}^{n-1}\frac{\alpha_{n-1}(\mathbf{e})}{\alpha_j(\mathbf{e})}\beta(\mathbf{e}_j)\nu_j(\mathbf{e})\\ \leq\ & \frac{1}{\nu(\mathbf{e}_0)}\sum_{n\geq 1}\frac{\alpha(\mathbf{e}_1)}{\nu^2(\mathbf{e}_1)}\cdots\frac{\alpha(\mathbf{e}_{n-1})}{\nu^2(\mathbf{e}_{n-1})}\sum_{j\geq 0}\frac{\nu(\mathbf{e}_1)}{\alpha(\mathbf{e}_1)}\cdots\frac{\nu(\mathbf{e}_{j-1})}{\alpha(\mathbf{e}_{j-1})}\frac{\beta(\mathbf{e}_j)}{\alpha(\mathbf{e}_j)}.\end{aligned}$$

Furthermore, the assumptions ensure

$$\begin{aligned}\mathbb{E}\left[\log\frac{\nu(\mathbf{e}_0)}{\alpha(\mathbf{e}_0)}\right] = -\mathbb{E}\left[\log\frac{\alpha(\mathbf{e}_0)}{\nu(\mathbf{e}_0)}\right] &= -\log\mu - \mathbb{E}\left[\log \mathbb{E}\left[\frac{f'_{\xi_0}(1)^2}{\nu(\mathbf{e}_0)}\middle|\,\mathbf{e}\right]\right] < 0,\\ \mathbb{E}\left[\log^+\frac{\beta(\mathbf{e}_0)}{\alpha(\mathbf{e}_0)}\right] &= \mathbb{E}\left[\log^+\frac{\mathbb{E}[f''_{\xi_0}(1)|\mathbf{e}]}{\mathbb{E}[f'_{\xi_0}(1)^2|\mathbf{e}]}\right] < \infty\end{aligned}$$

as well as

$$\mathbb{E}\left[\log\frac{\alpha(\mathbf{e}_0)}{\nu^2(\mathbf{e}_0)}\right] = \log\mu + \mathbb{E}\left[\log \mathbb{E}\left[\left(\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right)^2\middle|\,\mathbf{e}\right]\right] < 0.$$

Hence, Lemma 3.8 gives us the $\mathbb{P}$-a.s. finiteness of the second sum.

(b) Recalling $\mathbb{E}[W_n^2]$ from (3.1) in Proposition 3.6 and using the definitions made prior to Proposition 3.6, we see that $(W_n)_{n\geq 0}$ is L^2-bounded under $\mathbb{P}$ iff

$$\sigma^2 < \infty, \quad \sum_{n\geq 0} \mathbb{E}[\nu^{-1}(\mathbf{e}_0)]^n < \infty, \quad \mathbb{E}\left[\frac{f''_{\xi_0}(1)}{\nu^2(\mathbf{e}_0)}\right] < \infty$$

$$\text{and} \quad \sum_{n\geq 0}\sum_{j=0}^{n-1}\left(\mu\mathbb{E}\left[\left(\frac{f_{\xi_0'(1)}}{\nu(\mathbf{e}_0)}\right)^2\right]\right)^{n-1-j}\left(\mathbb{E}[\nu^{-1}(\mathbf{e}_0)]\right)^j < \infty.$$

First note that $\mathbb{V}\mathrm{ar}(\mathcal{Z}_1/\nu(\mathbf{e}_0)) < \infty$ implies

$$\mathbb{E}\left[\frac{f''_{\xi_0}(1)}{\nu^2(\mathbf{e}_0)}\right] \leq \mathbb{E}\left[\mathbb{E}\left[\frac{Z_{V_1}^2}{\nu^2(\mathbf{e}_0)}\,\middle|\,\xi\right]\right] \leq \mathbb{E}\left[\left(\frac{\mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right)^2\right] < \infty.$$

Furthermore

$$\begin{aligned}
&\sum_{n\geq 0}\sum_{j=0}^{n-1}\left(\mu\mathbb{E}\left[\left(\frac{f_{\xi_0'(1)}}{\nu(\mathbf{e}_0)}\right)^2\right]\right)^{n-1-j}\left(\mathbb{E}[\nu^{-1}(\mathbf{e}_0)]\right)^j \\
= &\sum_{j\geq 0}\sum_{n\geq 0}\left(\mu\mathbb{E}\left[\left(\frac{f_{\xi_0'(1)}}{\nu(\mathbf{e}_0)}\right)^2\right]\right)^{n}\left(\mathbb{E}[\nu^{-1}(\mathbf{e}_0)]\right)^j,
\end{aligned}$$

and this double series is obviously finite iff

$$\mathbb{E}[\nu^{-1}(\mathbf{e}_0)] < 1 \quad \text{and} \quad \mathbb{E}\left[\left(\frac{f_{\xi_0'(1)}}{\nu(\mathbf{e}_0)}\right)^2\right] < \frac{1}{\mu}.$$

This completes the proof. □

Remark 3.10. Obviously, one can prove part (b) of the previous proposition by considering the different cases according to Corollary 3.7.

With the previous proposition at hand, we can give sufficient conditions for L^2-convergence of the martingale $(W_n)_{n\geq 0}$, which in turn also ensure L^1-convergence of the martingale, and thus the nondegeneracy of the martingale limit W. This is stated in the next

Corollary 3.11. *If $\sigma^2 < \infty$, $\tilde{\nu} < 1$ and $\alpha < 1$, then*

$$\lim_{n\to\infty} \mathbb{E}(W_n - W)^2 = 0,$$

and hence

$$\mathbb{E}W = 1, \quad \mathbb{V}\mathrm{ar}W = \frac{\sigma^2}{1-\tilde{\nu}} + \frac{c\beta}{(1-\alpha)(1-\tilde{\nu})} \quad \textit{and} \quad \mathbb{P}(W=0) = \mathbb{P}(\mathit{Ext}).$$

Since the conditions stated above are not necessary for L^1-convergence, this is not the end of the story. Later in Theorem 4.5 we give a necessary and sufficient condition, using the size-biased construction of the next section.

We close this section by giving two consequences from Proposition 3.4. The first one is an easy consequence of the SLLN and yields the convergence of $\mathcal{Z}_n^{1/n}$ to a finite and positive constant on the event $\{W > 0\}$. Note that $\mathbb{E}|\log \nu(\mathbf{e}_0)| < \infty$ by (1.3).

Corollary 3.12. *On the event* $\{W > 0\}$ *we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathcal{Z}_n = \mathbb{E}[\log \nu(\mathbf{e}_0)] \quad \mathbb{P}\text{-}a.s.$$

Proof. For any ω with $W_n(\omega) \to W(\omega) \in (0,\infty)$, we obviously have $\lim_{n\to\infty} \frac{1}{n} \log W_n(\omega) = 0$. Furthermore, by the SLLN

$$\lim_{n\to\infty} \frac{1}{n} \log \nu_n(\mathbf{e}) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \nu(\mathbf{e}_i) = \mathbb{E}[\log \nu(\mathbf{e}_0)] \quad \mathbb{P}\text{-a.s.},$$

and thus

$$\lim_{n\to\infty} \frac{1}{n} \log \mathcal{Z}_n = \lim_{n\to\infty} \frac{1}{n} \log W_n + \lim_{n\to\infty} \frac{1}{n} \log \nu_n(\mathbf{e}) = \mathbb{E}[\log \nu(\mathbf{e}_0)] \quad \mathbb{P}\text{-a.s.}$$

on the event $\{W > 0\}$. ☐

Finally, we take a look at another possible norming sequence for $(\mathcal{Z}_n)_{n\geq 0}$, namely $(\nu^n)_{n\geq 0}$, where ν is the annealed mean of $\mathcal{Z}_1$. It is easy to see that $(\mathcal{Z}_n/\nu^n)$ forms again a martingale, and we can further deduce that $(\nu^n)_{n\geq 0}$ is not a suitable norming sequence for the process of parasites, unless $\nu(\mathbf{e}_0)$ is deterministic. This is stated in the next

Proposition 3.13. *The process* $\tilde{W}_n := \nu^{-n}\mathcal{Z}_n$, $n \geq 0$, *forms a nonnegative martingale with respect to* $(\mathcal{F}_n)_{n\geq 0}$ *and thus converges* $\mathbb{P}$*-a.s. to an integrable random variable* $\tilde{W}$*. If* $\nu(\mathbf{e}_0)$ *is nondeterministic, i.e.* $\mathbb{V}\mathrm{ar}(\nu(\mathbf{e}_0)) > 0$, *we have* $\tilde{W} = 0$ $\mathbb{P}$*-a.s.*

Proof. Obviously, $\tilde{W}_n$ is $\mathcal{F}_n$-measurable, $n \geq 0$. Furthermore for $n \geq 0$ we have

$$\mathbb{E}[\mathcal{Z}_{n+1}|\mathcal{F}_n] = \sum_{v\in\mathbb{T}_n} \sum_{i=1}^{Z_v} \mathbb{E}\left[X_{i,v}^{(0,T_v)} + X_{i,v}^{(1,T_v)} \middle| \mathcal{F}_n \right]$$

$$\begin{aligned}
&= \sum_{v\in\mathbb{T}_n}\sum_{i=1}^{Z_v}\mathbb{E}\left[X_{i,v}^{(0,T_v)}+X_{i,v}^{(1,T_v)}\right]\\
&= \sum_{v\in\mathbb{T}_n}\sum_{i=1}^{Z_v}\mathbb{E}\mathcal{Z}_1\\
&= \nu\sum_{v\in\mathbb{T}_n} Z_v\\
&= \nu\mathcal{Z}_n \quad \mathbb{P}\text{-a.s.},
\end{aligned}$$

where we used the independence of $X_{i,v}^{(0,T_v)}+X_{i,v}^{(1,T_v)}$ and $\mathcal{F}_n$ for the second equality. The integrability follows from the above by induction. Hence $(\tilde{W}_n)_{n\geq 0}$ forms a nonnegative martingale and the martingale convergence theorem ensures the $\mathbb{P}$-a.s. convergence to an integrable random variable $\tilde{W}$.

For the second part note that Jensen's inequality in combination with the fact that $\nu(\mathbf{e}_0)$ is nondeterministic implies

$$\mathbb{E}[\log\nu(\mathbf{e}_0)] < \log\mathbb{E}\nu(\mathbf{e}_0) = \log\mathbb{E}\mathcal{Z}_1 = \log\nu.$$

Therefore, by the SLLN

$$\begin{aligned}
\left(\frac{\nu_n(\mathbf{e})}{\nu^n}\right)^{1/n} &= \exp\left(\frac{1}{n}\sum_{i=0}^{n-1}\log\nu(\mathbf{e}_i)-\log\nu\right)\\
&\to \exp\left(\mathbb{E}[\log\nu(\mathbf{e}_0)]-\log\nu\right) < 1 \quad \mathbb{P}\text{-a.s.},
\end{aligned}$$

and together with Proposition 3.4 we get

$$\tilde{W}_n = W_n\cdot\frac{\nu_n(\mathbf{e})}{\nu^n}\to 0 \quad \mathbb{P}\text{-a.s.},$$

that is $\tilde{W}=0$ $\mathbb{P}$-a.s. This completes the proof. □

Remark 3.14. The second part of the previous proposition can also be proved by means of Kakutani's product martingale theorem. In fact, the process $(\nu_n(\mathbf{e})/\nu^n)_{n\geq 0}$ forms a nonnegative product martingale, and since $\nu(\mathbf{e}_0)$ is nondeterministic, Jensen's inequality implies

$$q := \mathbb{E}\left[\left(\frac{\nu(\mathbf{e}_n)}{\nu}\right)^{1/2}\right] < \left(\mathbb{E}\left[\frac{\nu(\mathbf{e}_n)}{\nu}\right]\right)^{1/2} = 1, \quad n\geq 0.$$

Therefore

$$\prod_{k=0}^{n} \mathbb{E}\left[\left(\frac{\nu(\mathbf{e}_k)}{\nu}\right)^{1/2}\right] = q^{n+1} \to 0, \quad \text{as } n \to \infty.$$

This yields $\nu_n(\mathbf{e})/\nu^n \to 0$ $\mathbb{P}$-a.s., as $n \to \infty$, and we can proceed as before.

3.3. Size-biasing II

This section provides us with the tools to give the desired necessary and sufficient conditions for the nondegeneracy of the martingale limit W in Chapter 4.

We come back to the method of size-biasing discussed in Section 2.1. References and a short description of the method can be found there. This time we are interested in the construction of a size-biased tree, not just in picking a random cell line. Furthermore, we pick the spine along the parasites rather than the cells.

Before formally constructing the size-biased process, we give an intuitive description of the method, which is also depicted in Figure 3.1.
Given the random environment $\mathbf{e}$, we want to construct a spine in our cell tree. Therefore, every cell along the spine needs to have at least one daughter cell. For convenience, we let spinal cells produce one daughter cell with probability $p_1/(p_1+p_2)$ and two daughter cells with probability $p_2/(p_1+p_2)$. Note that this is not the size-biased cell distribution. Now, we describe how the spinal cells are chosen.
At generation 0, we start with a single cell and a single parasite, called the spinal cell and the spinal parasite of generation 0. This cell reproduces according to the cell distribution specified above, and the parasite produces offspring and shares these into the daughter cell(s) according to a *size-biased parasite distribution* specified later. Among these offspring, the spinal parasite of the first generation is chosen uniformly, and the cell containing this parasite becomes the spinal cell of the first generation. Now, the same procedure applies to the spinal cell and the spinal parasite of the first generation, while the possible other cell and the other parasites multiply in the original way. In particular, the possible other cell of the first generation starts a usual BwBPRE - potentially having more than one starting parasite - in the shifted environment $[\mathbf{e}]_1$. This goes on indefinitely.
As already mentioned, the above described evolution of the size-biased process is visualized later in Figure 3.1 and includes the notation we introduce next.

Let $(\hat{T}_n, \hat{C}_n, (\hat{X}_n^{(0)}, \hat{X}_n^{(1)}))_{n\geq 0}$ be a sequence of random variables satisfying the following properties:

- $(\hat{T}_n)_{n\geq 0}$ is an independent sequence and independent of

$$\left((\hat{C}_n, (\hat{X}_n^{(0)}, \hat{X}_n^{(1)}))_{n\geq 0}, (T_v)_{v\in\mathbb{V}}, (X_{k,v}^{(0)}, X_{k,v}^{(1)})_{k\geq 1, v\in\mathbb{V}}, \mathbf{e}\right).$$

- Given $\mathbf{e}$, the random variables $\big(\hat{C}_n, (\hat{X}_n^{(0)}, \hat{X}_n^{(1)})\big)_{n\geq 0}$ are independent and independent of $\big((T_v)_{v\in\mathbb{V}}, (X_{k,v}^{(0)}, X_{k,v}^{(1)})_{k\geq 1, v\in\mathbb{V}}\big)$.
- The distributions are defined by

$$\begin{aligned}&\mathbb{P}\left(\hat{T}_n = t, \left(\hat{X}_n^{(0)}, \hat{X}_n^{(1)}\right) \in A, \hat{C}_n = l \,\middle|\, \mathbf{e}\right)\\ =\ & \frac{p_t}{\nu(\mathbf{e}_n)}\, \mathbb{P}\left(\left(X_{1,0^{*n}}^{(0)}, X_{1,0^{*n}}^{(1)}\right) \in A, X_{1,0^{*n}}^{(0)} + X_{1,0^{*n}}^{(1)} \geq l \,\middle|\, \mathbf{e}\right) \quad \mathbb{P}\text{-a.s.,}\end{aligned}$$

  for $t \in \{1,2\}$, $A \subseteq \mathbb{N}_0^2$, $l \geq 1$, and 0 otherwise.

This yields for the marginal distributions

$$\mathbb{P}(\hat{T}_n = t) = \begin{cases} \dfrac{p_t}{p_1 + p_2}, & \text{if } t \in \{1,2\},\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\mathbb{P}\left(\left(\hat{X}_n^{(0)}, \hat{X}_n^{(1)}\right) \in A \,\middle|\, \mathbf{e}\right) = \frac{\mathbb{E}\left[\mathbb{1}_{\left\{\left(X_{1,0^{*n}}^{(0)}, X_{1,0^{*n}}^{(1)}\right)\in A\right\}}\left(X_{1,0^{*n}}^{(0)} + X_{1,0^{*n}}^{(1)}\right) \,\middle|\, \mathbf{e}\right]}{\mathbb{E}\left[X_{1,0^{*n}}^{(0)} + X_{1,0^{*n}}^{(1)} \,\middle|\, \mathbf{e}\right]} \quad \mathbb{P}\text{-a.s.,}$$

for $A \subseteq \mathbb{N}_0^2$, as well as

$$\mathbb{P}(\hat{C}_n = l|\mathbf{e}) = \frac{\mathbb{P}\left(X_{1,0^{*n}}^{(0)} + X_{1,0^{*n}}^{(1)} \geq l \,\middle|\, \mathbf{e}\right)}{\mathbb{E}\left[X_{1,0^{*n}}^{(0)} + X_{1,0^{*n}}^{(1)} \,\middle|\, \mathbf{e}\right]} \quad \mathbb{P}\text{-a.s.,} \quad l \geq 1.$$

Here, the sequence $(\hat{T}_n)_{n\geq 0}$ describes the multiplication of the spinal cells, the sequence $(\hat{X}_n^{(0)}, \hat{X}_n^{(1)})_{n\geq 0}$ describes the multiplication of the spinal parasites and the sequence $(\hat{C}_n)_{n\geq 0}$ describes the choosing of the spinal parasites.
Note that for $x_0, x_1 \in \mathbb{N}_0$, with $x_0 + x_1 \geq 1$ and $1 \leq l \leq x_0 + x_1$

$$\begin{aligned}\mathbb{P}(\hat{C}_n = l|\hat{X}_n^{(0)} = x_0, \hat{X}_n^{(1)} = x_1, \mathbf{e}) &= \frac{\mathbb{P}\left(\hat{X}_n^{(0)} = x_0, \hat{X}_n^{(1)} = x_1, \hat{C}_n = l|\mathbf{e}\right)}{\mathbb{P}\left(\hat{X}_n^{(0)} = x_0, \hat{X}_n^{(1)} = x_1|\mathbf{e}\right)}\\ &= \frac{1}{x_0 + x_1}.\end{aligned}$$

This means that $\hat{C}_n$ picks uniformly at random one of the offspring of the spinal parasite of the n-th generation to be the spinal parasite of the $(n+1)$-st generation.

Note that this structure including the sequence $(\hat{T}_n, \hat{C}_n, (\hat{X}_n^{(0)}, \hat{X}_n^{(1)}))_{n\geq 0}$ shall be the same under any $\mathbb{P}_z$, $z \geq 0$, where the spinal parasite of the root cell is chosen uniformly among the root parasites.

Next, we define the random variables $\hat{V}_0 := \emptyset$ and $\hat{V}_{n+1} := \hat{V}_n \hat{U}_n$, $n \geq 0$, where

$$\hat{U}_n := \mathbb{1}_{\left\{\hat{C}_n > \hat{X}_n^{(0,\hat{T}_n)}\right\}}, \quad n \geq 0.$$

Here the notation $\hat{X}_n^{(k,l)}$ is the same as for the nonhatted version. Thus, $(\hat{V}_n)_{n\geq 0}$ is a cell line in the cell tree and $\hat{V}_n$ gives the cell the spinal parasite of the n-th generation is living in. Furthermore, put $\hat{\mathsf{A}}_\emptyset := 1$ and

$$\hat{\mathsf{A}}_{vu} := \begin{cases} \mathbb{1}_{\left\{u \leq \hat{T}_n - 1\right\}}, & \text{if } v = \hat{V}_n \text{ for some } n \geq 0, \\ \mathbb{1}_{\{u \leq T_v - 1\}} \hat{\mathsf{A}}_v, & \text{otherwise,} \end{cases}$$

for all $v \in \mathbb{V}$ and $u \in \{0,1\}$, as well as $\hat{Z}_\emptyset := Z_\emptyset$ and

$$\hat{Z}_{vu} := \begin{cases} \displaystyle\sum_{i=1}^{\hat{Z}_{\hat{V}_n}-1} X_{i,\hat{V}_n}^{(u,\hat{T}_n)} + \hat{X}_n^{(u,\hat{T}_n)}, & \text{if } v = \hat{V}_n \text{ for some } n \geq 0, \\ \displaystyle\sum_{i=1}^{\hat{Z}_v} X_{i,v}^{(u,T_v)}, & \text{otherwise,} \end{cases}$$

for all $v \in \mathbb{V}$ and $u \in \{0,1\}$. Finally set

$$\bar{T}_v := \begin{cases} \hat{T}_n, & \text{if } v = \hat{V}_n \text{ for some } n \geq 0, \\ T_v, & \text{otherwise,} \end{cases}$$

for all $v \in \mathbb{V}$.

Now, the corresponding *size-biased branching within branching process in random environment*, $\widehat{\mathsf{\mathbf{BP}}} := (\widehat{\mathsf{\mathbf{BP}}}_n)_{n\geq 0}$, is given by $\widehat{\mathsf{\mathbf{BP}}}_n := (\hat{\mathsf{A}}_v, \hat{Z}_v)_{|v|=n}$, and the *size-biased branching within branching tree in random environment*, $\widehat{\mathsf{\mathbf{BT}}} := (\widehat{\mathsf{\mathbf{BT}}}_n)_{n\geq 0}$, is given by $\widehat{\mathsf{\mathbf{BT}}}_n := (\hat{\mathsf{A}}_v, \hat{Z}_v)_{|v|\leq n}$.

Now, the quantities $\hat{\mathbb{T}}, \hat{\mathbb{T}}_n, \hat{\mathcal{T}}_n, \hat{\mathbb{T}}_n^*, \hat{\mathcal{T}}_n^*$ and $\hat{\mathcal{Z}}_n$, $n \geq 0$, are defined in an obvious way for the size-biased BwBPRE .

In the following figure, a realization of the first three generations of the size-biased BwBPRE is shown. The spinal parasites and the cells they live in are represented by ■ and □, respectively, whereas normal cells and normal parasites are represented by the corresponding circular versions. Here, the nonspinal cell 0 starts a BwBPRE in the shifted environment $[\mathbf{e}]_1$ with two ancestor parasites, and the cell 11 starts a BwBPRE in the shifted environment $[\mathbf{e}]_2$ with one ancestor parasite. The spinal cells up to generation three are given by $\hat{V}_0 = \emptyset$, $\hat{V}_1 = 1$, $\hat{V}_2 = 10$ and $\hat{V}_3 = 100$, and the corresponding parasite counts are $\hat{Z}_{\hat{V}_0} = 1$, $\hat{Z}_{\hat{V}_1} = 4$, $\hat{Z}_{\hat{V}_2} = 3$ and $\hat{Z}_{\hat{V}_3} = 5$.

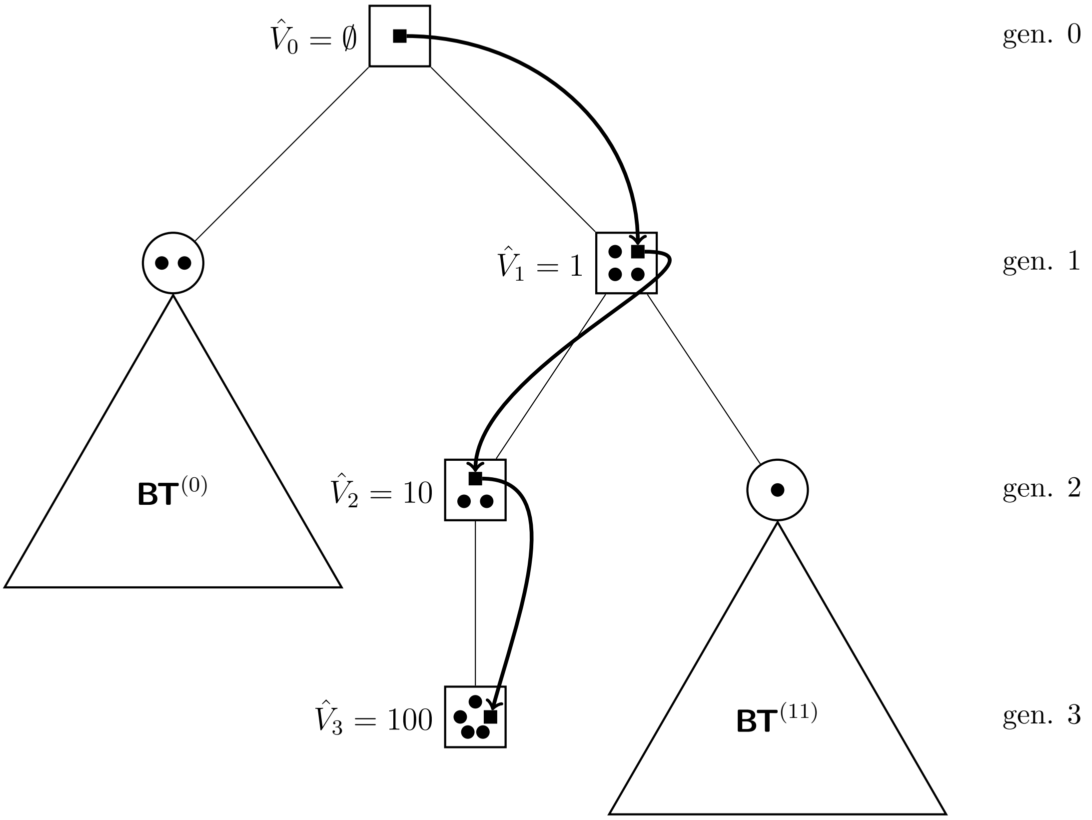

Figure 3.1.: A realization of the size-biased BwBPRE up to generation three

To derive the relation between the law of $(\widehat{\mathbf{BT}}, \mathbf{e})$ and the law of $(\mathbf{BT}, \mathbf{e})$ we need a

Lemma 3.15. *Let* $n \in \mathbb{N}_0$.

(a) For all $x_0, x_1 \in \mathbb{N}_0$, $t \in \{1, 2\}$ *and* $0 \le u \le t-1$ *we have*

$$\begin{aligned} &\mathbb{P}\left(\hat{X}_n^{(0)} = x_0, \hat{X}_n^{(1)} = x_1, \hat{T}_n = t, \hat{U}_n = u \,\middle|\, \mathbf{e}\right) \\ = \; &\frac{p_t x^{(u,t)}}{\nu(\mathbf{e}_n)} \, \mathbb{P}\left(X_{1,0*n}^{(0)} = x_0, X_{1,0*n}^{(1)} = x_1 \,\middle|\, \mathbf{e}\right) \quad \mathbb{P}\text{-}a.s., \end{aligned}$$

where $x^{(0,1)} = x_0 + x_1$, $x^{(0,2)} = x_0$ *and* $x^{(1,2)} = x_1$.

(b) *For all* $t \in \{1,2\}$, $0 \leq u \leq t-1$, $(z_v)_{0 \leq v \leq t-1} \in \mathbb{N}_0^t$ *and* $z \in \mathbb{N}$

$$\begin{aligned} & \mathbb{P}_z \left((\hat{Z}_v)_{0 \leq v \leq t-1} = (z_v)_{0 \leq v \leq t-1}, \hat{T}_0 = t, \hat{U}_0 = u \middle| \mathbf{e} \right) \\ = \ & \frac{1}{\nu(\mathbf{e}_0)} \frac{z_u}{z} \ \mathbb{P}_z \left((Z_v)_{0 \leq v \leq t-1} = (z_v)_{0 \leq v \leq t-1}, T_\emptyset = t \middle| \mathbf{e} \right) \quad \mathbb{P}_z\text{-}a.s. \end{aligned}$$

In particular,

$$\begin{aligned} & \mathbb{P}_z \left((\hat{Z}_v)_{0 \leq v \leq t-1} = (z_v)_{0 \leq v \leq t-1}, \hat{T}_0 = t \middle| \mathbf{e} \right) \\ = \ & \frac{1}{\nu(\mathbf{e}_0)} \frac{\sum_{u=0}^{t-1} z_u}{z} \ \mathbb{P}_z \left((Z_v)_{0 \leq v \leq t-1} = (z_v)_{0 \leq v \leq t-1}, T_\emptyset = t \middle| \mathbf{e} \right) \quad \mathbb{P}_z\text{-}a.s. \end{aligned}$$

Proof. (a) We establish the assertion just for $t = 2$ and $u = 0$, as the other two cases can be proved in a similar fashion. For $x_0, x_1 \in \mathbb{N}_0$, $t = 2$ and $u = 0$ we get

$$\begin{aligned} & \mathbb{P} \left(\hat{X}_n^{(0)} = x_0, \hat{X}_n^{(1)} = x_1, \hat{T}_n = 2, \hat{U}_n = 0 \middle| \mathbf{e} \right) \\ = \ & \sum_{l=1}^{x_0} \mathbb{P} \left(\hat{X}_n^{(0)} = x_0, \hat{X}_n^{(1)} = x_1, \hat{T}_n = 2, \hat{C}_n = l \middle| \mathbf{e} \right) \\ = \ & \frac{p_2}{\nu(\mathbf{e}_n)} \sum_{l=1}^{x_0} \mathbb{P} \left(X_{1,0*n}^{(0)} = x_0, X_{1,0*n}^{(1)} = x_1 \middle| \mathbf{e} \right) \\ = \ & \frac{p_2 x_0}{\nu(\mathbf{e}_n)} \ \mathbb{P} \left(X_{1,0*n}^{(0)} = x_0, X_{1,0*n}^{(1)} = x_1 \middle| \mathbf{e} \right) \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

(b) For $t \in \{1,2\}$, $0 \leq u \leq t-1$, $(z_v)_{0 \leq v \leq t-1} \in \mathbb{N}_0^t$ and $z \in \mathbb{N}$ we get

$$\begin{aligned} & \mathbb{P}_z \left((\hat{Z}_v)_{0 \leq v \leq t-1} = (z_v)_{0 \leq v \leq t-1}, \hat{T}_0 = t, \hat{U}_0 = u \middle| \mathbf{e} \right) \\ = \ & \sum_{\substack{x_0, x_1 \geq 0 \\ x^{(v,t)} \leq z_v, 0 \leq v \leq t-1}} \mathbb{P} \left(\hat{X}_0^{(0)} = x_0, \hat{X}_0^{(1)} = x_1, \hat{T}_0 = t, \hat{U}_0 = u \middle| \mathbf{e} \right) \\ & \qquad \cdot \mathbb{P} \left(\left(\sum_{i=1}^{z-1} X_{i,\emptyset}^{(v,t)} \right)_{0 \leq v \leq t-1} = (z_v - x^{(v,t)})_{0 \leq v \leq t-1} \middle| \mathbf{e} \right) \\ = \ & \frac{p_t}{\nu(\mathbf{e}_0)} \sum_{\substack{x_0, x_1 \geq 0 \\ x^{(v,t)} \leq z_v, 0 \leq v \leq t-1}} x^{(u,t)} \ \mathbb{P} \left(X_{z,\emptyset}^{(0)} = x_0, X_{z,\emptyset}^{(1)} = x_1 \middle| \mathbf{e} \right) \\ & \qquad \cdot \mathbb{P} \left(\left(\sum_{i=1}^{z-1} X_{i,\emptyset}^{(v,t)} \right)_{0 \leq v \leq t-1} = (z_v - x^{(v,t)})_{0 \leq v \leq t-1} \middle| \mathbf{e} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{p_t}{\nu(\mathbf{e}_0)} \sum_{\substack{x_0,x_1\ge 0\\ x^{(v,t)}\le z_v, 0\le v\le t-1}} x^{(u,t)}\ \mathbb{P}\left(\left(\sum_{i=1}^{z} X^{(v,t)}_{i,\emptyset}\right)_{0\le v\le t-1} = (z_v)_{0\le v\le t-1},\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. X^{(0)}_{z,\emptyset} = x_0, X^{(1)}_{z,\emptyset} = x_1 \,\middle|\, \mathbf{e}\right)\\
&= \frac{p_t}{\nu(\mathbf{e}_0)}\ \mathbb{E}\left[X^{(u,t)}_{z,\emptyset}\,\middle|\,\left(\sum_{i=1}^{z} X^{(v,t)}_{i,\emptyset}\right)_{0\le v\le t-1} = (z_v)_{0\le v\le t-1}, \mathbf{e}\right]\\
&\quad\ \cdot \mathbb{P}\left(\left(\sum_{i=1}^{z} X^{(v,t)}_{i,\emptyset}\right)_{0\le v\le t-1} = (z_v)_{0\le v\le t-1}\,\middle|\,\mathbf{e}\right)\\
&= \frac{1}{\nu(\mathbf{e}_0)}\frac{z_u}{z}\ \mathbb{P}\left(\left(\sum_{i=1}^{z} X^{(v,T_\emptyset)}_{i,\emptyset}\right)_{0\le v\le t-1} = (z_v)_{0\le v\le t-1}, T_\emptyset = t\,\middle|\,\mathbf{e}\right)\\
&= \frac{1}{\nu(\mathbf{e}_0)}\frac{z_u}{z}\ \mathbb{P}_z\left((Z_v)_{0\le v\le t-1} = (z_v)_{0\le v\le t-1}, T_\emptyset = t\,\middle|\,\mathbf{e}\right) \quad \mathbb{P}_z\text{-a.s.},
\end{aligned}$$

where we used the independence assumptions for the first equality and part (a) for the second equality. Due to the conditional independence, the penultimate equality follows from the fact that $\mathbb{E}[X_1|S_n] = \frac{S_n}{n}$ $\mathbb{P}$-a.s. for any random walk $(S_n)_{n\ge 0}$ with $S_0 = 0$ and i.i.d. increments $(X_n)_{n\ge 1}$.
The second part of (b) follows by summation over $u \in \{0,1\}$. □

With the help of the previous lemma, we are now able to prove the crucial connection between the law of $(\widehat{\mathbf{BT}}, \mathbf{e})$ and the law of $(\mathbf{BT}, \mathbf{e})$. But beforehand, we need some further notation. For $z \in \mathbb{N}_0$ let

$$\hat{Q}_z(\cdot) := \mathbb{P}_z((\widehat{\mathbf{BT}}, \mathbf{e}) \in \cdot) \quad \text{and} \quad Q_z(\cdot) := \mathbb{P}_z\left((\mathbf{BT}, \mathbf{e}) \in \cdot\right),$$

and put

$$\begin{gathered}
w_n : (\mathbb{S} \times \mathbb{M}^{\mathbb{N}_0}, \mathfrak{S} \otimes \mathfrak{M}^{\mathbb{N}_0}) \to ([0,\infty), \mathscr{B}([0,\infty))), \quad (s,u) \mapsto \frac{\mathbf{z}_n(s)}{\nu_n(u)},\\
w := \limsup_{n\to\infty} w_n, \quad \hat{\mathcal{Z}}_n := \mathbf{z}_n \circ \widehat{\mathbf{BT}}, \quad \widehat{W}_n := w_n \circ (\widehat{\mathbf{BT}}, \mathbf{e}),\\
\mathfrak{S}'_n := \mathfrak{S}_n \otimes \mathfrak{M}^{\mathbb{N}_0}.
\end{gathered} \tag{3.4}$$

Note that w_n is $\mathfrak{S}'_n$-measurable. In particular, this yields the following representations

$$W_n = w_n \circ (\mathbf{BT}, \mathbf{e}) \quad \text{and} \quad W = w \circ (\mathbf{BT}, \mathbf{e}).$$

The aforementioned connection between the law of $(\widehat{\mathbf{BT}}, \mathbf{e})$ and the law of $(\mathbf{BT}, \mathbf{e})$ is stated in the next

Lemma 3.16. *Let $n \geq 0$.*

(a) For all $(t_v, z_v)_{|v|\leq n} \in \mathbb{S}_n$, $u \in \mathbb{V}$ with $|u| = n$ and $z \in \mathbb{N}$ we have

$$\mathbb{P}_z(\widehat{\mathbf{BT}}_n = (t_v, z_v)_{|v|\leq n}, \hat{V}_n = u|\mathbf{e}) = \frac{1}{\nu_n(\mathbf{e})}\frac{z_u}{z}\ \mathbb{P}_z(\mathbf{BT}_n = (t_v, z_v)_{|v|\leq n}|\mathbf{e}) \quad \mathbb{P}_z\text{-}a.s.$$

In particular,

$$\mathbb{P}_z(\widehat{\mathbf{BT}}_n = (t_v, z_v)_{|v|\leq n}|\mathbf{e}) = \frac{\sum_{|u|=n} z_u}{\nu_n(\mathbf{e})z}\ \mathbb{P}_z(\mathbf{BT}_n = (t_v, z_v)_{|v|\leq n}|\mathbf{e}) \quad \mathbb{P}_z\text{-}a.s.,$$

as well as

$$\mathbb{P}_z(\widehat{\mathbf{BT}}_n \in A|\mathbf{e}) = \frac{1}{z}\mathbb{E}_z[W_n \mathbb{1}_{\{\mathbf{BT}_n \in A\}}|\mathbf{e}] \quad \mathbb{P}_z\text{-}a.s.$$

for all $A \in \mathfrak{S}_{|n}$.

(b) For every $z \in \mathbb{N}$ and $B \in \mathfrak{S}'_n$

$$\hat{Q}_z(B) = \frac{1}{z}\ \mathbb{E}_z[W_n \mathbb{1}_{\{(\mathbf{BT},\mathbf{e})\in B\}}] = \int_B \frac{w_n}{z}\ \mathrm{d}Q_z,$$

i.e. $\hat{Q}_{z|\mathfrak{S}'_n} \ll Q_{z|\mathfrak{S}'_n}$ with $\frac{\mathrm{d}\hat{Q}_{z|\mathfrak{S}'_n}}{\mathrm{d}Q_{z|\mathfrak{S}'_n}} = \frac{w_n}{z}$.

(c) For every $z \in \mathbb{N}$ we have the dichotomy

(i) $\hat{Q}_z(w < \infty) = 1 \iff \mathbb{E}_z W = z$,

(ii) $\hat{Q}_z(w = \infty) = 1 \iff Q_z(w = 0) = 1$.

Proof. Let $z \in \mathbb{N}$.

(a) The assertion is obviously true for $n = 0$. Let $n \in \mathbb{N}$, $(t_v, z_v)_{|v|\leq n} \in \mathbb{S}_n$ and $u = u'u_n \in \mathbb{V}$, where $|u'| = n-1$ and $u_n \in \{0,1\}$. W.l.o.g. we can assume that $t_{w1} = 0$ whenever $t_{w0} = 0$, $|w| \leq n-1$, and $t_v = 0$ whenever $t_{v|k} = 0$ for a $0 \leq k \leq |v|$, $|v| \leq n$. Moreover, $t_{u|k} = 1$ for all $k \leq n$, as well as $z_u > 0$ (and hence $z_{u|k} > 0$, $k \leq n$). Otherwise both sides of the assertion are equal to 0. Finally set $\hat{t} := \max\{v' \in \{0,1\} : t_{u'v'} = 1\} + 1$ and

$$E_n := \{\widehat{\mathbf{BT}}_{n-1} = (t_v, z_v)_{|v|\leq n-1}, \hat{V}_{n-1} = u'\}.$$

Note that

$$\mathbb{P}_z(\widehat{\mathbf{BT}}_n = (t_v, z_v)_{|v|\leq n}, \hat{V}_n = u|\mathbf{e}) = I_1 \cdot \mathbb{P}_z(E_n|\mathbf{e}) \quad \mathbb{P}_z\text{-a.s.}, \tag{3.5}$$

with

$$\begin{aligned}
I_1 = {} & \mathbb{P}_z(\widehat{\mathsf{BT}}_n = (t_v, z_v)_{|v|\leq n}, \hat{V}_n = u | E_n, \mathbf{e}) \\
= {} & \mathbb{P}_z((\hat{\mathsf{A}}_v, \hat{Z}_v)_{|v|=n} = (t_v, z_v)_{|v|=n}, \hat{U}_{n-1} = u_n | E_n, \mathbf{e}) \\
= {} & \mathbb{P}\Bigg(\sum_{i=1}^{z_{v|n-1}} X_{i,v|n-1}^{(v_n, T_{v|n-1})} = z_v, |v| = n, v|n-1 \neq u', \\
& \quad \sum_{i=1}^{z_{u'}-1} X_{i,u'}^{(v', \hat{T}_{n-1})} + \hat{X}_{n-1}^{(v', \hat{T}_{n-1})} = z_{u'v'}, |v'| = 1, \\
& \quad (t_{v|n-1} \mathbb{1}_{\{v_n \leq \bar{T}_{v|n-1} - 1\}})_{|v|=n} = (t_v)_{|v|=n}, \hat{U}_{n-1} = u_n \Bigg| E_n, \mathbf{e} \Bigg) \\
= {} & \mathbb{P}\Bigg(\sum_{i=1}^{z_{v|n-1}} X_{i,v|n-1}^{(v_n, T_{v|n-1})} = z_v, t_{v|n-1} \mathbb{1}_{\{v_n \leq T_{v|n-1} - 1\}} = t_v, |v| = n, v|n-1 \neq u', \\
& \quad \sum_{i=1}^{z_{u'}-1} X_{i,u'}^{(v', \hat{T}_{n-1})} + \hat{X}_{n-1}^{(v', \hat{T}_{n-1})} = z_{u'v'}, \mathbb{1}_{\{v' \leq \hat{T}_{n-1} - 1\}} = t_{u'v'}, |v'| = 1, \\
& \quad \hat{U}_{n-1} = u_n \Bigg| E_n, \mathbf{e} \Bigg) \\
= {} & \mathbb{P}\Bigg(\sum_{i=1}^{z_{v|n-1}} X_{i,v|n-1}^{(v_n, T_{v|n-1})} = z_v, t_{v|n-1} \mathbb{1}_{\{v_n \leq T_{v|n-1} - 1\}} = t_v, |v| = n, v|n-1 \neq u', \\
& \quad \sum_{i=1}^{z_{u'}-1} X_{i,u'}^{(v', \hat{T}_{n-1})} + \hat{X}_{n-1}^{(v', \hat{T}_{n-1})} = z_{u'v'}, |v'| = 1, \hat{T}_{n-1} = \hat{t}, \hat{U}_{n-1} = u_n \Bigg| \mathbf{e} \Bigg) \\
= {} & \prod_{\substack{|v|=n-1 \\ v \neq u'}} \mathbb{P}\left(\sum_{i=1}^{z_v} X_{i,v}^{(v', T_v)} = z_{vv'}, t_v \mathbb{1}_{\{v' \leq T_v - 1\}} = t_{vv'}, |v'| = 1 \middle| \mathbf{e} \right) \\
& \cdot \mathbb{P}\Bigg(\sum_{i=1}^{z_{u'}-1} X_{i,u'}^{(v', \hat{T}_{n-1})} + \hat{X}_{n-1}^{(v', \hat{T}_{n-1})} = z_{u'v'}, |v'| = 1, \hat{T}_{n-1} = \hat{t}, \hat{U}_{n-1} = u_n \Bigg| \mathbf{e} \Bigg) \\
= {} & \prod_{\substack{|v|=n-1 \\ v \neq u'}} \mathbb{P}\left(\sum_{i=1}^{z_v} X_{i,v}^{(v', T_v)} = z_{vv'}, t_v \mathbb{1}_{\{v' \leq T_v - 1\}} = t_{vv'}, |v'| = 1 \middle| \mathbf{e} \right) \\
& \cdot \mathbb{P}_{z_{u'}}\Bigg((\hat{Z}_{v'})_{0 \leq v' \leq \hat{t}-1} = (z_{u'v'})_{0 \leq v' \leq \hat{t}-1}, \hat{T}_0 = \hat{t}, \hat{U}_0 = u_n \Bigg| (\mathbf{e}_m)_{m \geq n-1} \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{z_u}{\nu(\mathbf{e}_{n-1}) z_{u'}} \mathbb{P}\left(\sum_{i=1}^{z_{v|n-1}} X_{i,v|n-1}^{(v_n, T_{v|n-1})} = z_v, t_{v|n-1} \mathbb{1}_{\{v_n \le T_{v|n-1}-1\}} = t_v, |v| = n \,\middle|\, \mathbf{e} \right) \\
&= \frac{1}{\nu(\mathbf{e}_{n-1})} \frac{z_u}{z_{u'}}\ \mathbb{P}_z \left(\mathsf{BT}_n = (t_v, z_v)_{|v| \le n} \middle| \, \mathsf{BT}_{n-1} = (t_v, z_v)_{|v| \le n-1}, \mathbf{e} \right) \quad \mathbb{P}_z\text{-a.s.},
\end{aligned}
$$

where we used the independence and conditional independence assumptions for the fifth, sixth, eighth and ninth equality, and also Lemma 3.15 for the eighth equality. Hence by induction we get from equation (3.5)

$$
\begin{aligned}
&\mathbb{P}_z(\widehat{\mathsf{BT}}_n = (t_v, z_v)_{|v| \le n}, \hat{V}_n = u | \mathbf{e}) \\
= {}& \frac{1}{\nu(\mathbf{e}_{n-1})} \frac{z_u}{z_{u'}}\ \mathbb{P}_z \left(\mathsf{BT}_n = (t_v, z_v)_{|v| = n} \middle| \, \mathsf{BT}_{n-1} = (t_v, z_v)_{|v| \le n-1}, \mathbf{e} \right) \\
&\cdot \frac{1}{\nu_{n-1}(\mathbf{e})} \frac{z_{u'}}{z}\ \mathbb{P}_z(\mathsf{BT}_{n-1} = (t_v, z_v)_{|v| \le n-1} | \mathbf{e}) \\
= {}& \frac{1}{\nu_n(\mathbf{e})} \frac{z_u}{z}\ \mathbb{P}_z(\mathsf{BT}_n = (t_v, z_v)_{|v| \le n} | \mathbf{e}) \quad \mathbb{P}_z\text{-a.s.}
\end{aligned}
$$

The second part of (a) follows by summation over $|u| = n$, and we then get for $A \in \mathfrak{S}_{|n}$

$$
\begin{aligned}
\mathbb{P}_z(\widehat{\mathsf{BT}}_n \in A | \mathbf{e}) &= \int_A \mathbb{P}_z(\widehat{\mathsf{BT}}_n \in \mathrm{d}(t_v, z_v)_{|v| \le n} | \mathbf{e}) \\
&= \int_A \frac{\sum_{|u|=n} z_u}{\nu_n(\mathbf{e}) z} \mathbb{P}_z(\mathsf{BT}_n \in \mathrm{d}(t_v, z_v)_{|v| \le n} | \mathbf{e}) \\
&= \frac{1}{z} \mathbb{E}_z \left[\frac{\mathcal{Z}_n}{\nu_n(\mathbf{e})} \mathbb{1}_{\{\mathsf{BT}_n \in A\}} \middle| \, \mathbf{e} \right] \\
&= \frac{1}{z} \mathbb{E}_z \left[W_n \mathbb{1}_{\{\mathsf{BT}_n \in A\}} \middle| \, \mathbf{e} \right] \quad \mathbb{P}_z\text{-a.s.}
\end{aligned}
$$

This completes the proof of part (a).

(b) With part (a) in mind, we get for $B = C \times D$ with $C \in \mathfrak{S}_n$ and $D \in \mathfrak{M}^{\mathbb{N}_0}$

$$
\begin{aligned}
\hat{Q}_z(B) &= \int_{\{\mathbf{e} \in D\}} \mathbb{P}_z(\widehat{\mathsf{BT}} \in C | \mathbf{e})\ \mathrm{d}\mathbb{P}_z \\
&= \int_{\{\mathbf{e} \in D\}} \mathbb{P}_z(\widehat{\mathsf{BT}}_n \in C' | \mathbf{e})\ \mathrm{d}\mathbb{P}_z \\
&= \int_{\{\mathbf{e} \in D\}} \frac{1}{z} \mathbb{E}_z[W_n \mathbb{1}_{\{\mathsf{BT}_n \in C'\}} | \mathbf{e}]\ \mathrm{d}\mathbb{P}_z \\
&= \frac{1}{z} \mathbb{E}_z[W_n \mathbb{1}_{\{(\mathsf{BT}, \mathbf{e}) \in B\}}],
\end{aligned}
$$

where we used the existence of a $C' \in \mathfrak{S}_{|n}$ with $\{\widehat{\mathbf{BT}} \in C\} = \{\widehat{\mathbf{BT}}_n \in C'\}$ and $\{\mathbf{BT} \in C\} = \{\mathbf{BT}_n \in C'\}$, see Section 1.4 for further details. Since the sets of the above product form are a π-system and generate the product σ-field $\mathfrak{S}'_n$, part (b) is proved.

(c) With the help of part (b) and Theorem A.5 we infer for every $B \in \mathfrak{S} \otimes \mathfrak{M}^{\mathbb{N}_0}$

$$\hat{Q}_z(B) = \int_B \frac{w}{z} \, \mathrm{d}Q_z + \hat{Q}_z(B \cap \{w = \infty\})$$

and thus

$$\frac{1}{z}\mathbb{E}_z W = 1 - \hat{Q}_z(w = \infty),$$

i.e. the asserted dichotomy. □

The previous lemma also yields the following connection between an integrability condition for the size-biased process of parasites and the modified $(Z \log Z)$-condition for the process of parasites.

Lemma 3.17. *One has*

$$\mathbb{E}[\log \hat{\mathcal{Z}}_1] = \mathbb{E}\left[\frac{\mathcal{Z}_1 \log \mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right].$$

Proof. First, note that

$$\mathbb{E}[\log \hat{\mathcal{Z}}_1] = \int_{(0,\infty)} \mathbb{P}(\log \hat{\mathcal{Z}}_1 > t) \; \lambda\!\!\lambda(\mathrm{d}t).$$

Furthermore, with the help of Lemma 3.16 (b) we can infer that

$$\mathbb{P}(\log \hat{\mathcal{Z}}_1 > t) = \int w_1(s,u) \mathbb{1}_{\{\log \mathbf{z}_1(s) > t\}} \; Q(\mathrm{d}(s,u)) = \mathbb{E}\left[\frac{\mathcal{Z}_1}{\nu(\mathbf{e}_0)} \mathbb{1}_{\{\log \mathcal{Z}_1 > t\}}\right]$$

for $t > 0$, and thus we get with the help of Fubini's theorem

$$\mathbb{E}[\log \hat{\mathcal{Z}}_1] = \mathbb{E}\left[\frac{\mathcal{Z}_1}{\nu(\mathbf{e}_0)} \int_{(0,\infty)} \mathbb{1}_{\{\log \mathcal{Z}_1 > t\}} \; \lambda\!\!\lambda(\mathrm{d}t)\right] = \mathbb{E}\left[\frac{\mathcal{Z}_1 \log \mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right].$$

This concludes the proof. □

Remark 3.18. Part (b) of Lemma 3.16 gives us for any $A \in \mathscr{B}([0,\infty))$ and $n \geq 0$

$$\mathbb{P}_z^{\widehat{W}_n}(A) = \hat{Q}_z(w_n \in A) = \frac{1}{z} \int_{\{w_n \in A\}} w_n \, \mathrm{d}Q_z = \frac{1}{\mathbb{E}_z W_n} \int_A x \; \mathbb{P}_z^{W_n}(\mathrm{d}x),$$

that is $\widehat{W}_n$ is the size-biasing of W_n. This explains the notion of size-biasing from the beginning of this section, since the use of this terminology was not directly evident from the definition of the distribution of the random vector $(\hat{T}_n, \hat{C}_n, (\hat{X}_n^{(0)}, X_n^{(1)}))$.

3.4. The number of parasites along the spine II

Recall that $\hat{Z}_{\hat{V}_0} = Z_\emptyset$ and

$$\hat{Z}_{\hat{V}_{n+1}} = \sum_{i=1}^{\hat{Z}_{\hat{V}_n}-1} X_{i,\hat{V}_n}^{(\hat{U}_n,\hat{T}_n)} + \hat{X}_n^{(\hat{U}_n,\hat{T}_n)}, \quad n \geq 0.$$

Therefore

$$\hat{Z}_{\hat{V}_{n+1}} - 1 = \sum_{i=1}^{\hat{Z}_{\hat{V}_n}-1} X_{i,\hat{V}_n}^{(\hat{U}_n,\hat{T}_n)} + \hat{X}_n^{(\hat{U}_n,\hat{T}_n)} - 1, \quad n \geq 0.$$

Looking at this representation, imagining that the spinal parasite lives outside the population and its offspring immigrate into the spinal cell of the next generation, it should not come as a surprise that this process actually forms a BPRE with immigration (BPREI). This is stated in the next

Proposition 3.19. *The process $(\hat{Z}_{\hat{V}_n} - 1)_{n\geq 0}$ forms a BPREI in i.i.d. random environment $\Xi := (\Xi_n)_{n\geq 0} := (\hat{T}_n, \hat{U}_n, \mathbf{e}_n)_{n\geq 0}$ and with immigration sequence $(\hat{X}_n^{(\hat{U}_n,\hat{T}_n)} - 1)_{n\geq 0}$.*

Since this assertion is fairly obvious from our model assumptions, we omit the tedious calculations.

We are going to use this result in the upcoming chapter when we derive an equivalent condition for the nondegeneracy of the martingale limit W.

4. Main results in the case $\mathbb{P}(\textbf{Surv}) > 0$

In the first part of this chapter we want to give equivalent conditions for the nondegeneracy of the martingale limit W, i.e. conditions that ensure that the sequence of quenched means $(\nu_n(\mathbf{e}))_{n\geq 0}$ is the right norming sequence for the process of parasites $(\mathcal{Z}_n)_{n\geq 0}$. This leads to a $(Z \log Z)$-condition and a Kesten-Stigum type result. To prove this result, we use the dichotomy established in part (c) of Lemma 3.16 of the previous chapter.

The second part of the chapter contains some additional results, e.g. that under certain assumptions the number of parasites in 'strongly' infected cells is negligible compared to the total number of parasites, or that 'strongly' infected cells in a generation have not a big impact on the number of infected cells in the following generations.

4.1. Conditions for the nondegeneracy of W

Part (a) of the following theorem shows that the $(Z \log Z)$-condition known from BPRE in combination with a $(Z \log Z)$-condition regarding the process of parasites along a randomly chosen cell line implies $\mathbb{E}W = 1$. If $p_0 > 0$, we see in part (b) that the converse is also true.

Theorem 4.1. *Suppose* $\mathbb{P}(Surv) > 0$ *and* $\mathbb{E}[\log \nu(\mathbf{e}_0)] > 0$.

(a) *If*

$$\mathbb{E}\left[\frac{\mathcal{Z}_1 \log \mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right] < \infty \quad \textit{and} \quad \mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)} \log \frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right] < 0,$$

then $\mathbb{E}W = 1$.

(b) *If further* $p_0 > 0$, *then the converse of (a) holds true, i.e.*

$$\mathbb{E}\left[\frac{\mathcal{Z}_1 \log \mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right] = \infty \quad \textit{or} \quad \mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)} \log \frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right] \geq 0$$

imply $W = 0$ $\mathbb{P}$-*a.s.*

Proof of Theorem 4.1 (a). We show that $\widehat{W} := \limsup_{n\to\infty} \widehat{W}_n$ is $\mathbb{P}$-a.s. finite and then use Lemma 3.16 to finish the proof. Consider the representation

$$\hat{\mathcal{Z}}_{n+1} = \sum_{v\in\hat{\mathbb{T}}_n}\sum_{u=0}^{\bar{T}_v-1}\hat{Z}_{vu} = \sum_{u=0}^{\hat{T}_n-1}\hat{Z}_{\hat{V}_n u} + \sum_{v\in\hat{\mathbb{T}}_n\setminus\{\hat{V}_n\}}\sum_{u=0}^{T_v-1}\sum_{i=1}^{\hat{Z}_v} X_{i,v}^{(u,T_v)}, \quad n\in\mathbb{N}_0,$$

and define the σ-field

$$\mathcal{H} := \sigma\left((\hat{T}_n)_{n\geq 0}, (\hat{X}_n^{(0)}, \hat{X}_n^{(1)})_{n\geq 0}, (\hat{U}_n)_{n\geq 0}, \mathbf{e}\right).$$

Using the independence assumptions from the beginning of Section 3.3, we infer by iteration that

$$\begin{aligned}
\mathbb{E}[\hat{\mathcal{Z}}_{n+1}|\mathcal{H}] &= \sum_{u=0}^{\hat{T}_n-1}\mathbb{E}\left[\hat{Z}_{\hat{V}_n u}\middle|\mathcal{H}\right] + \mathbb{E}\left[\sum_{v\in\hat{\mathbb{T}}_n\setminus\{\hat{V}_n\}}\sum_{u=0}^{T_v-1}\sum_{i=1}^{\hat{Z}_v} X_{i,v}^{(u,T_v)}\middle|\mathcal{H}\right] \\
&= \sum_{u=0}^{\hat{T}_n-1}\mathbb{E}\left[\hat{Z}_{\hat{V}_n u}\middle|\mathcal{H}\right] + \mathbb{E}\left[\sum_{v\in\hat{\mathbb{T}}_n\setminus\{\hat{V}_n\}}\sum_{i=1}^{\hat{Z}_v}\mathbb{E}\left[X_{i,v}^{(0,T_v)} + X_{i,v}^{(1,T_v)}\middle|\mathbf{e}\right]\middle|\mathcal{H}\right] \\
&= \sum_{u=0}^{\hat{T}_n-1}\mathbb{E}\left[\hat{Z}_{\hat{V}_n u}\middle|\mathcal{H}\right] + \mathbb{E}\left[\sum_{v\in\hat{\mathbb{T}}_n\setminus\{\hat{V}_n\}}\sum_{i=1}^{\hat{Z}_v}\nu(\mathbf{e}_n)\middle|\mathcal{H}\right] \\
&\leq \sum_{u=0}^{\hat{T}_n-1}\mathbb{E}\left[\hat{Z}_{\hat{V}_n u}\middle|\mathcal{H}\right] + \nu(\mathbf{e}_n)\mathbb{E}\left[\hat{\mathcal{Z}}_n\middle|\mathcal{H}\right] \\
&\leq \ldots \leq \sum_{k=0}^{n}\nu(\mathbf{e}_{k+1})\cdots\nu(\mathbf{e}_n)\sum_{u=0}^{\hat{T}_k-1}\mathbb{E}\left[\hat{Z}_{\hat{V}_k u}\middle|\mathcal{H}\right] \\
&= \sum_{k=0}^{n}\frac{\nu_{n+1}(\mathbf{e})}{\nu_{k+1}(\mathbf{e})}\sum_{u=0}^{\hat{T}_k-1}\left(\hat{X}_k^{(u,\hat{T}_k)} + \mathbb{E}\left[\sum_{i=1}^{\hat{Z}_{\hat{V}_k}-1} X_{i,\hat{V}_k}^{(u,\hat{T}_k)}\middle|\mathcal{H}\right]\right) \qquad \mathbb{P}\text{-a.s.},
\end{aligned}$$

where we used the definition of $\hat{Z}_{\hat{V}_k u}$, $k\geq 0$, for the last line. Once again, with the independence assumptions from the beginning of Section 3.3, we can further infer that

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{\hat{Z}_{\hat{V}_k}-1} X_{i,\hat{V}_k}^{(u,\hat{T}_k)}\middle|\mathcal{H}\right] &= \sum_{|v|=k}\sum_{t=1}^{2}\sum_{j\geq 1}\mathbb{1}_{\{\hat{V}_k=v,\hat{T}_k=t\}}\mathbb{E}\left[\mathbb{1}_{\{\hat{Z}_v=j\}}\sum_{i=1}^{j-1}\mathbb{E}\left[X_{i,v}^{(u,t)}\middle|\mathcal{H},\mathcal{F}_k\right]\middle|\mathcal{H}\right] \\
&= \sum_{|v|=k}\sum_{t=1}^{2}\sum_{j\geq 1}\mathbb{1}_{\{\hat{V}_k=v,\hat{T}_k=t\}}\mathbb{E}\left[\mathbb{1}_{\{\hat{Z}_v=j\}}\sum_{i=1}^{j-1}\mathbb{E}\left[X_{1,0^{*k}}^{(u,t)}\middle|\mathbf{e}\right]\middle|\mathcal{H}\right]
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}\left[\hat{Z}_{\hat{V}_k} - 1 \middle| \mathcal{H}\right] \cdot \sum_{t=1}^{2} \mathbb{1}_{\{\hat{T}_k = t\}} \mathbb{E}\left[X^{(u,t)}_{1,0^{*k}} \middle| \mathbf{e}\right] \\
&= \mathbb{E}\left[\hat{Z}_{\hat{V}_k} - 1 \middle| \mathcal{H}\right] \mathbb{E}\left[X^{(u,\hat{T}_k)}_{1,0^{*k}} \middle| \hat{T}_k, \mathbf{e}\right] \quad \mathbb{P}\text{-a.s.,}
\end{aligned}$$

and so we get $\mathbb{P}$-a.s.

$$\begin{aligned}
\mathbb{E}[\hat{\mathcal{Z}}_{n+1}|\mathcal{H}] &\le \sum_{k=0}^{n} \frac{\nu_{n+1}(\mathbf{e})}{\nu_{k+1}(\mathbf{e})} \sum_{u=0}^{\hat{T}_k - 1} \left(\hat{X}^{(u,\hat{T}_k)}_k + \mathbb{E}\left[\hat{Z}_{\hat{V}_k} - 1 \middle| \mathcal{H}\right] \mathbb{E}\left[X^{(u,\hat{T}_k)}_{1,0^{*k}} \middle| \hat{T}_k, \mathbf{e}\right]\right) \\
&= \sum_{k=0}^{n} \frac{\nu_{n+1}(\mathbf{e})}{\nu_{k+1}(\mathbf{e})} \hat{X}^{(0,1)}_k + \sum_{k=0}^{n} \frac{\nu_{n+1}(\mathbf{e})}{\nu_{k+1}(\mathbf{e})} \mathbb{E}\big[\hat{Z}_{\hat{V}_k} - 1 \big| \mathcal{H}\big] \cdot \mathbb{E}\Big[X^{(0,1)}_{1,0^{*k}} \Big| \mathbf{e}\Big].
\end{aligned}$$

Thus, we have $\mathbb{P}$-a.s.

$$\begin{aligned}
\mathbb{E}[\widehat{W}_{n+1}|\mathcal{H}] &\le \sum_{k\ge 0} \frac{1}{\nu_{k+1}(\mathbf{e})} \hat{X}^{(0,1)}_k + \sum_{k\ge 0} \frac{1}{\nu_{k+1}(\mathbf{e})} \mathbb{E}\big[\hat{Z}_{\hat{V}_k} - 1 \big| \mathcal{H}\big] \cdot \mathbb{E}\Big[X^{(0,1)}_{1,0^{*k}} \Big| \mathbf{e}\Big] \\
&= \sum_{k\ge 0} \frac{1}{\nu_{k+1}(\mathbf{e})} \hat{X}^{(0,1)}_k + \frac{1}{p_1 + p_2} \sum_{k\ge 0} \frac{1}{\nu_k(\mathbf{e})} \mathbb{E}\big[\hat{Z}_{\hat{V}_k} - 1 \big| \mathcal{H}\big] =: I_1 + I_2 \qquad (4.1)
\end{aligned}$$

for all $n \ge 0$. Next we show the $\mathbb{P}$-a.s. finiteness of the sums I_1 and I_2.

Let us start with the first one. The definitions from the beginning of Section 3.3 together with the i.i.d. property of the environment $\mathbf{e}$ ensure that under $\mathbb{P}$ the sequence $(\hat{X}^{(0,1)}_k)_{k\ge 0}$ consists of i.i.d. random variables distributed as $\hat{\mathcal{Z}}_1$. Recalling Lemma 3.17, the assumption $\mathbb{E}\left[\frac{\mathcal{Z}_1 \log \mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right] < \infty$ implies $\mathbb{E}[\log^+ \hat{\mathcal{Z}}_1] < \infty$. Furthermore,

$$\mathbb{E}\left[\log \frac{1}{\nu(\mathbf{e}_0)}\right] = -\mathbb{E}[\log \nu(\mathbf{e}_0)] < 0.$$

Therefore, Lemma 3.8 yields

$$I_1 = \sum_{k\ge 0} \nu(\mathbf{e}_0)^{-1} \cdots \nu(\mathbf{e}_k)^{-1} \hat{X}^{(0,1)}_k < \infty \quad \mathbb{P}\text{-a.s.}$$

For the second sum I_2, recall that $(\hat{Z}_{\hat{V}_n} - 1)_{n\ge 0}$ is a BPREI in random environment Ξ and with immigration sequence $(\hat{X}^{(\hat{U}_n,\hat{T}_n)}_n - 1)_{n\ge 0}$. Since $\mathbb{E}[\log \hat{\mathcal{Z}}_1] < \infty$, as was pointed out along the way of showing the $\mathbb{P}$-a.s. finiteness of I_1, the immigration sequence satisfies

$$\mathbb{E}\left[\log^+ \left(\hat{X}^{(\hat{U}_0,\hat{T}_0)}_0 - 1\right)\right] \le \mathbb{E}[\log \hat{\mathcal{Z}}_1] < \infty. \qquad (4.2)$$

Moreover, for the reproduction mean $\mathbb{E}\left[X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)}\middle|\,\Xi\right]$ of the first marginal distribution of the BPREI we have

$$\begin{aligned}\mathbb{E}\left[\log\mathbb{E}\left[X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)}\middle|\,\Xi\right]\right] &= \sum_{t=1}^{2}\sum_{u=0}^{t-1}\mathbb{E}\left[\mathbb{1}_{\{\hat{T}_0=t,\hat{U}_0=u\}}\log\mathbb{E}\left[X_{1,\emptyset}^{(u,t)}\middle|\,\mathbf{e}\right]\right]\\ &= \sum_{t=1}^{2}\sum_{u=0}^{t-1}\mathbb{E}\left[\mathbb{P}(\hat{T}_0=t,\hat{U}_0=u|\mathbf{e})\;\log\mathbb{E}\left[X_{1,\emptyset}^{(u,t)}\middle|\,\mathbf{e}\right]\right]\\ &= \sum_{t=1}^{2}\sum_{u=0}^{t-1}p_t\;\mathbb{E}\left[\frac{\mathbb{E}\left[X_{1,\emptyset}^{(u,t)}\middle|\,\mathbf{e}\right]}{\nu(\mathbf{e}_0)}\log\mathbb{E}\left[X_{1,\emptyset}^{(u,t)}\middle|\,\mathbf{e}\right]\right],\end{aligned}$$

where we used Lemma 3.15 (a) for the last equality. Recalling the definitions from the beginning of Section 2.1, we get

$$\begin{aligned}\mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\log f'_{\xi_0}(1)\right] &= \sum_{t=1}^{2}\sum_{u=0}^{t-1}\mathbb{E}\left[\mathbb{1}_{\{\check{T}_0=t,C_0=u\}}\frac{\mathbb{E}\left[X_{1,\emptyset}^{(u,t)}\middle|\,\mathbf{e}\right]}{\nu(\mathbf{e}_0)}\log\mathbb{E}\left[X_{1,\emptyset}^{(u,t)}\middle|\,\mathbf{e}\right]\right]\\ &= \frac{1}{\mu}\sum_{t=1}^{2}\sum_{u=0}^{t-1}p_t\;\mathbb{E}\left[\frac{\mathbb{E}\left[X_{1,\emptyset}^{(u,t)}\middle|\,\mathbf{e}\right]}{\nu(\mathbf{e}_0)}\log\mathbb{E}\left[X_{1,\emptyset}^{(u,t)}\middle|\,\mathbf{e}\right]\right] \qquad (4.3)\\ &= \frac{1}{\mu}\;\mathbb{E}\left[\log\mathbb{E}\left[X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)}\middle|\,\Xi\right]\right].\end{aligned}$$

Combining this with the second assumption $\mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\log\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right] < 0$ we see that

$$\begin{aligned}\mathbb{E}\left[\log\mathbb{E}\left[X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)}\middle|\,\Xi\right]\right] &< \mu\;\mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\log\nu(\mathbf{e}_0)\right]\\ &= \mu\sum_{t=1}^{2}\sum_{u=0}^{t-1}\mathbb{E}\left[\mathbb{1}_{\{\check{T}_0=t,C_0=u\}}\frac{\mathbb{E}\left[X_{1,\emptyset}^{(u,t)}\middle|\,\mathbf{e}\right]}{\nu(\mathbf{e}_0)}\log\nu(\mathbf{e}_0)\right]\\ &= \sum_{t=1}^{2}\sum_{u=0}^{t-1}p_t\;\mathbb{E}\left[\frac{\mathbb{E}\left[X_{1,\emptyset}^{(u,t)}\middle|\,\mathbf{e}\right]}{\nu(\mathbf{e}_0)}\log\nu(\mathbf{e}_0)\right] \qquad (4.4)\\ &= \mathbb{E}\left[\frac{\mathbb{E}\left[X_{1,\emptyset}^{(0,T_\emptyset)}+X_{1,\emptyset}^{(1,T_\emptyset)}\middle|\,\mathbf{e}\right]}{\nu(\mathbf{e}_0)}\log\nu(\mathbf{e}_0)\right]\\ &= \mathbb{E}\left[\log\nu(\mathbf{e}_0)\right],\end{aligned}$$

using the independence assumptions from the beginning of Section 2.1. Hence, we can choose a $c \in (1, \exp(\mathbb{E}[\log \nu(\mathbf{e}_0)]))$, such that

$$\mathbb{E}\left[\log \mathbb{E}\left[X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)} \middle| \Xi\right]\right] < \log c$$

and together with (4.2), Lemma A.3 implies

$$\hat{Z}_\infty := \sup_{n\geq 0}\left(\frac{1}{c^n}\ \mathbb{E}[\hat{Z}_{\hat{V}_n} - 1|\mathcal{H}]\right) < \infty \quad \mathbb{P}\text{-a.s.}$$

We now arrive at the following estimate for the second sum

$$I_2 \leq \frac{\hat{Z}_\infty}{p_1+p_2}\sum_{k\geq 0}\frac{c^k}{\nu_k(\mathbf{e})} \quad \mathbb{P}\text{-a.s.}$$

Since

$$\mathbb{E}\left[\log\left(\frac{c}{\nu(\mathbf{e}_0)}\right)\right] = \log c - \mathbb{E}[\log \nu(\mathbf{e}_0)] < 0,$$

we can again use Lemma 3.8 to get

$$I_2 \leq \frac{\hat{Z}_\infty}{p_1+p_2}\sum_{k\geq 0}\frac{c}{\nu(\mathbf{e}_0)}\cdots\frac{c}{\nu(\mathbf{e}_k)} < \infty \quad \mathbb{P}\text{-a.s.}$$

Recalling (4.1) and gathering everything that we have shown so far, we arrive at

$$\sup_{n\geq 0}\mathbb{E}[\widehat{W}_n|\mathcal{H}] < \infty \quad \mathbb{P}\text{-a.s.},$$

and with the help of Fatou's lemma, this gives us

$$\hat{Q}\left(\liminf_{n\to\infty} w_n < \infty\right) = \mathbb{P}\left(\liminf_{n\to\infty}\widehat{W}_n < \infty\right) = 1.$$

To finish the proof we need to show that $(w_n)_{n\geq 0}$ converges $\hat{Q}$-a.s. To this end, we show that $(1/w_n)_{n\geq 0}$ forms a $\hat{Q}$-supermartingale with respect to the filtration $(\mathfrak{S}'_n)_{n\geq 0}$, defined in (3.4). Obviously, w_n is $\mathfrak{S}'_n$-measurable and Lemma 3.16 (b) gives us

$$\hat{Q}(w_n = 0) = \int_{\{w_n=0\}} w_n \,\mathrm{d}Q = 0,$$

for all $n \geq 0$. Then for every $A \in \mathfrak{S}'_n \subseteq \mathfrak{S}'_{n+1}$ we get, using once again Lemma 3.16, that

$$\begin{aligned}\int_A \mathbb{E}_{\hat{Q}}\left[\frac{1}{w_{n+1}}\middle|\mathfrak{S}'_n\right]\mathrm{d}\hat{Q} &= \int_{A\cap\{w_{n+1}>0\}}\frac{1}{w_{n+1}}\,\mathrm{d}\hat{Q}\\ &= \int_{A\cap\{w_{n+1}>0\}}\frac{1}{w_{n+1}}\cdot w_{n+1}\,\mathrm{d}Q\\ &= Q(A\cap\{w_{n+1}>0\})\\ &\le Q(A\cap\{w_n>0\})\\ &= \int_A\frac{1}{w_n}\,\mathrm{d}\hat{Q}.\end{aligned}$$

This ensures the supermartingale property and the integrability. Hence, according to the martingale convergence theorem $(1/w_n)_{n\geq 0}$ converges $\hat{Q}$-a.s. and therefore, so does $(w_n)_{n\geq 0}$. Thus we have

$$\hat{Q}(w<\infty) = \hat{Q}(\liminf_{n\to\infty} w_n < \infty) = 1$$

and Lemma 3.16 (c) ensures $\mathbb{E}W = 1$. This completes the proof. □

For the proof of Theorem 4.1 (b), we need the following auxiliary lemma which follows directly by the Borel-Cantelli lemma.

Lemma 4.2. *Let $(X_n)_{n\geq 0}$ be a sequence of identical distributed and nonnegative random variables. If $\mathbb{E}X_0 < \infty$, then*

$$\limsup_{n\to\infty}\frac{X_n}{n} = 0 \quad \mathbb{P}\text{-}a.s.$$

If $\mathbb{E}X_0 = \infty$, and in addition $(X_n)_{n\geq 0}$ is an independent sequence, then

$$\limsup_{n\to\infty}\frac{X_n}{n} = \infty \quad \mathbb{P}\text{-}a.s.$$

Proof of Theorem 4.1 (b). Once again, we want to use Lemma 3.16. For this purpose, we show that $\hat{Q}(w=\infty) = 1$.

Case I: Suppose $\mathbb{E}\left[\frac{\mathcal{Z}_1\log\mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right] = \infty$. Due to Lemma 3.17, this implies $\mathbb{E}[\log\hat{\mathcal{Z}}_1] = \infty$. Moreover, we have for $n \geq 1$

$$\widehat{W}_n = \frac{1}{\nu_n(\mathbf{e})}\sum_{v\in\hat{\mathbb{T}}_n}\hat{Z}_v \geq \frac{1}{\nu_n(\mathbf{e})}\sum_{u=0}^{\hat{T}_{n-1}-1}\hat{Z}_{\hat{V}_{n-1}u} \geq \frac{1}{\nu_n(\mathbf{e})}\hat{X}^{(0,1)}_{n-1} \quad \mathbb{P}\text{-a.s.}$$

As already pointed out in the proof of the previous theorem, the sequence $(\hat{X}^{(0,1)}_{n-1})_{n\geq 1}$

consists of i.i.d. random variables distributed as $\hat{\mathcal{Z}}_1$, and hence

$$\limsup_{n\to\infty} \frac{1}{n} \log \hat{X}_{n-1}^{(0,1)} = \infty \quad \mathbb{P}\text{-a.s.},$$

by Lemma 4.2. Together with the SLLN this yields

$$\limsup_{n\to\infty} \widehat{W}_n \geq \limsup_{n\to\infty} \exp\left(\frac{1}{n}\log \hat{X}_{n-1}^{(0,1)} - \frac{1}{n}\sum_{i=0}^{n-1} \log \nu(\mathbf{e}_i)\right)^n = \infty \quad \mathbb{P}\text{-a.s.},$$

i.e. $\hat{Q}(w=\infty) = \mathbb{P}(\limsup_{n\to\infty} \widehat{W}_n = \infty) = 1$, and part (c) of Lemma 3.16 ensures $\mathbb{P}(W=0)=1$.

Case II: Suppose $\mathbb{E}\left[\frac{\mathcal{Z}_1 \log \mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right] < \infty$ but $\mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)} \log \frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right] \geq 0$. For $n \geq 1$ we get

$$\widehat{W}_n = \frac{1}{\nu_n(\mathbf{e})} \sum_{v\in\hat{\mathbb{T}}_n} \hat{Z}_v \geq \frac{1}{\nu_n(\mathbf{e})} \hat{Z}_{\hat{V}_n} \geq \frac{1}{\nu_n(\mathbf{e})}(\hat{Z}_{\hat{V}_n} - 1) \quad \mathbb{P}\text{-a.s.}$$

Now, $(\hat{Z}_{\hat{V}_n} - 1)_{n\geq 0}$ is a BPREI in random environment Ξ and with immigration sequence $(\hat{X}_n^{(\hat{U}_n,\hat{T}_n)} - 1)_{n\geq 0}$. As in (4.2), the immigration sequence satisfies

$$\mathbb{E}\left[\log^+\left(\hat{X}_0^{(\hat{U}_0,\hat{T}_0)} - 1\right)\right] \leq \mathbb{E}[\log \hat{\mathcal{Z}}_1] < \infty,$$

since $\mathbb{E}[\log \hat{\mathcal{Z}}_1] < \infty$ by Lemma 3.17. Furthermore, using (4.3) and a similar reasoning as in (4.4), the assumption

$$\mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)} \log \frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right] \geq 0$$

yields

$$\begin{aligned} \mathbb{E}\left[\log \mathbb{E}\left[X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)} \,\middle|\, \Xi\right]\right] &= \mu\ \mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)} \log f'_{\xi_0}(1)\right] \\ &\geq \mu\ \mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)} \log \nu(\mathbf{e}_0)\right] \\ &= \mathbb{E}[\log \nu(\mathbf{e}_0)] > 0. \end{aligned} \tag{4.5}$$

Thus, the process $(\hat{Z}_{\hat{V}_n} - 1)_{n\geq 0}$ is supercritical and Theorem A.2 (a) gives the existence of an a.s. finite random variable Z_∞, such that

$$\lim_{n\to\infty} \frac{\hat{Z}_{\hat{V}_n} - 1}{\prod_{i=0}^{n-1} \mathbb{E}[X_{1,0^{*i}}^{(\hat{U}_i,\hat{T}_i)} | \Xi]} = Z_\infty \quad \mathbb{P}\text{-a.s.}$$

Note that

$$\begin{aligned}
\mathbb{E}\left[\frac{X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)} \log^+ X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)}}{\mathbb{E}[X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)} | \Xi]}\right] &= \sum_{t=1}^{2} \sum_{u=0}^{t-1} \mathbb{E}\left[\mathbb{1}_{\{\hat{T}_0 = t, \hat{U}_0 = u\}} \frac{X_{1,\emptyset}^{(u,t)} \log^+ X_{1,\emptyset}^{(u,t)}}{\mathbb{E}[X_{1,\emptyset}^{(u,t)} | \mathbf{e}]}\right] \\
&= \sum_{t=1}^{2} \sum_{u=0}^{t-1} \mathbb{E}\left[\mathbb{P}(\hat{T}_0 = t, \hat{U}_0 = u | \mathbf{e}) \frac{\mathbb{E}[X_{1,\emptyset}^{(u,t)} \log^+ X_{1,\emptyset}^{(u,t)} | \mathbf{e}]}{\mathbb{E}[X_{1,\emptyset}^{(u,t)} | \mathbf{e}]}\right] \\
&= \sum_{t=1}^{2} \sum_{u=0}^{t-1} p_t \ \mathbb{E}\left[\frac{X_{1,\emptyset}^{(u,t)} \log^+ X_{1,\emptyset}^{(u,t)}}{\nu(\mathbf{e}_0)}\right] \\
&\le \sum_{t=1}^{2} \sum_{u=0}^{t-1} p_t \ \mathbb{E}\left[\frac{X_{1,\emptyset}^{(u,t)}}{\nu(\mathbf{e}_0)} \log^+ \left(\sum_{u'=0}^{t-1} X_{1,\emptyset}^{(u',t)}\right)\right] \\
&= \sum_{t=1}^{2} \mathbb{E}\left[\mathbb{1}_{\{T_\emptyset = t\}} \frac{X_{1,\emptyset}^{(0)} + X_{1,\emptyset}^{(1)}}{\nu(\mathbf{e}_0)} \log^+ \left(X_{1,\emptyset}^{(0)} + X_{1,\emptyset}^{(1)}\right)\right] \\
&= \mathbb{E}\left[\frac{\mathcal{Z}_1 \log \mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right] < \infty,
\end{aligned}$$

where we used the conditional independence given $\mathbf{e}$ in the second equality and part (a) of Lemma 3.15 for the third equality. Thus, Theorem A.2 (a) ensures that $\mathbb{P}(Z_\infty > 0) = 1$. Taking everything into account we have shown so far, we see that

$$\begin{aligned}
\limsup_{n\to\infty} \widehat{W}_n &\ge Z_\infty \limsup_{n\to\infty} \frac{\prod_{i=0}^{n-1} \mathbb{E}\left[X_{1,0^{*i}}^{(\hat{U}_i,\hat{T}_i)} \middle| \Xi\right]}{\nu_n(\mathbf{e})} \\
&= Z_\infty \exp\left(\limsup_{n\to\infty} \sum_{i=0}^{n-1} \log\left(\frac{\mathbb{E}\left[X_{1,0^{*i}}^{(\hat{U}_i,\hat{T}_i)} \middle| \Xi\right]}{\nu(\mathbf{e}_i)}\right)\right) \qquad \mathbb{P}\text{-a.s.},
\end{aligned}$$

and this expression depends on

$$\mathbb{E}\left[\log\left(\frac{\mathbb{E}\left[X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)} \middle| \Xi\right]}{\nu(\mathbf{e}_0)}\right)\right] = \mathbb{E}\left[\log \mathbb{E}\left[X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)} \middle| \Xi\right]\right] - \mathbb{E}[\log \nu(\mathbf{e}_0)] \ge 0,$$

where we used (4.5). If $\mathbb{E}\left[\log \mathbb{E}\left[X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)}\middle|\,\Xi\right]\right] > \mathbb{E}[\log \nu(\mathbf{e}_0)]$, the SLLN gives us

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\log\left(\frac{\mathbb{E}\left[X_{1,0^{*i}}^{(\hat{U}_i,\hat{T}_i)}\middle|\,\Xi\right]}{\nu(\mathbf{e}_i)}\right) = \infty \quad \mathbb{P}\text{-a.s.}$$

If $\mathbb{E}\left[\log \mathbb{E}\left[X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)}\middle|\,\Xi\right]\right] = \mathbb{E}[\log \nu(\mathbf{e}_0)]$, note that the assumption $p_0 > 0$ ensures

$$\mathbb{P}\left(\log\left(\frac{\mathbb{E}\left[X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)}\middle|\,\Xi\right]}{\nu(\mathbf{e}_0)}\right) = 0\right) = \mathbb{P}\left(\mathbb{E}\left[X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)}\middle|\,\Xi\right] = \nu(\mathbf{e}_0)\right) < 1,$$

according to Lemma A.8, and thus the Chung-Fuchs theorem (the last part in Theorem A.7) yields

$$\limsup_{n\to\infty}\sum_{i=0}^{n-1}\log\left(\frac{\mathbb{E}\left[X_{1,0^{*i}}^{(\hat{U}_i,\hat{T}_i)}\middle|\,\Xi\right]}{\nu(\mathbf{e}_i)}\right) = \infty \quad \mathbb{P}\text{-a.s.}$$

Hence, in both cases we get $\limsup_{n\to\infty}\widehat{W}_n = \infty$ $\mathbb{P}$-a.s., and by Lemma 3.16 (c) this implies $\mathbb{P}(W = 0) = 1$ again. □

Remark 4.3. If

$$\mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\log\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right] < 0,$$

then the supercriticality or criticality of the process $(Z_{V_n})_{n\geq 0}$, i.e. $\mathbb{E}[\log f'_{\xi_0}(1)] \geq 0$, automatically ensures $\mathbb{E}[\log \nu(\mathbf{e}_0)] > 0$. To see this, observe that due to Jensen's inequality

$$\begin{aligned}0 > \mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\log\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right] &\geq \mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right]\log\mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right]\\ &\geq \mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right]\mathbb{E}\left[\log\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right],\end{aligned}$$

and thus $\mathbb{E}\left[\log\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right] < 0$. Since we are in the supercritical or critical case, we further get

$$0 \leq \mathbb{E}[\log f'_{\xi_0}(1)] < \mathbb{E}[\log \nu(\mathbf{e}_0)].$$

Now, we take a look at the the case $p_0 = 0$. In this case $(\mathcal{Z}_n)_{n\geq 0}$ actually is a BPRE with quenched reproduction mean $\mathbb{E}[\mathcal{Z}_1|\mathbf{e}] = \nu(\mathbf{e}_0)$. Therefore, Theorem 2 in [53] yields the following

Theorem 4.4. *Let* $\mathbb{P}(Surv) > 0$, $\mathbb{E}[\log \nu(\mathbf{e}_0)] > 0$ *and* $p_0 = 0$. *We have*

$$\mathbb{E}W = 1 \quad \textit{iff} \quad \mathbb{E}\left[\frac{\mathcal{Z}_1 \log \mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right] < \infty.$$

It is pretty obvious from our model assumptions that $(\mathcal{Z}_n)_{n\geq 0}$ forms a supercritical BPRE with quenched reproduction mean

$$\mathbb{E}[\mathcal{Z}_1|\mathbf{e}] = \nu(\mathbf{e}_0)$$

if $p_0 = 0$. We omit the details.

Let us summarise the previous results in the following

Theorem 4.5. *Let* $\mathbb{P}(Surv) > 0$ *and assume* $\mathbb{E}[\log \nu(\mathbf{e}_0)] > 0$.

(a) If $p_0 > 0$, *then the following conditions are equivalent:*
  - *(i)* $\mathbb{P}(W > 0) > 0$,
  - *(ii)* $\mathbb{E}W = 1$,
  - *(iii)* $(W_n)_{n\geq 0}$ *is uniformly integrable,*
  - *(iv)* $\mathbb{E}\left[\frac{\mathcal{Z}_1 \log \mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right] < \infty$ *and* $\mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)} \log \frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right] < 0$.

(b) If $p_0 = 0$, *then*

$$\mathbb{E}W = 1 \quad \textit{iff} \quad \mathbb{E}\left[\frac{\mathcal{Z}_1 \log \mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right] < \infty.$$

4.2. The impact of strongly infected cells

In this section we give additional results about the long-time behaviour of the number of infected cells and of the number of parasites. More precisely, we are interested in the following question: What is the impact of strongly infected cells on the total number of parasites and on the number of infected cells in future generations? We need the following assumption for the process of parasites along the spine $(Z_{V_n})_{n\geq 0}$:

$$\lim_{K\to\infty} \sup_{n\geq 0} \mathbb{E}\left[\frac{\mu^n}{\nu_n(\mathbf{e})} Z_{V_n} \mathbb{1}_{\{Z_{V_n} > K\}}\right] = 0. \tag{4.6}$$

Remark 4.6. If we could replace $\nu_n(\mathbf{e})$ by ν^n in (4.6), this would turn into an u.i. condition for the process $(Z_{V_n})_{n\geq 0}$. In fact,

$$\mathbb{E}\left[\frac{\mu^n}{\nu^n} Z_{V_n} \mathbb{1}_{\{Z_{V_n}>K\}}\right] \leq \frac{1}{\mathbb{P}(Z_{V_n}>0)} \mathbb{E}[Z_{V_n} \mathbb{1}_{\{Z_{V_n}>K\}}] = \mathbb{E}[Z_{V_n} \mathbb{1}_{\{Z_{V_n}>K\}} | Z_{V_n}>0],$$

and if $(Z_{V_n})_{n\geq 0}$ is a strongly subcritical BPRE satisfying $\mathbb{E}[Z_{V_1} \log Z_{V_1}] < \infty$, we have

$$\lim_{K\to\infty} \sup_{n\geq 0} \mathbb{E}[Z_{V_n} \mathbb{1}_{\{Z_{V_n}>K\}} | Z_{V_n}>0],$$

cf. Proposition 2.6. Since in general $\nu^n/\nu_n(\mathbf{e}) \to \infty$ $\mathbb{P}$-a.s. for $n\to\infty$, as we have seen in the proof of Proposition 3.13, our assumption is stronger. This is probably due to the fluctuations of the random environment. We expect that we could replace (4.6) with the assumptions from Proposition 2.6 and a condition controlling for the fluctuations of the random environment.

Furthermore we assume

$$\mathbb{E}[\log \nu(\mathbf{e}_0)] > 0,\ p_0 > 0,\ \mathbb{E}\left[\frac{\mathcal{Z}_1 \log \mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)} \log \frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right] < 0. \tag{A1}$$

According to Theorem 4.5, these assumptions ensure the nondegeneracy of W if $\mathbb{P}(Surv) > 0$. Next we give a simple example, where the conditions (4.6) and (A1) are satisfied. Recall (1.1) for the definition of $\mathbb{M}$.

Example 4.7. Let $p_0 = 1/4$, $p_2 = 3/4$ and $\mathbb{P}(\mathbf{e}_0 = u) = \frac{1}{2} = \mathbb{P}(\mathbf{e}_0 = \hat{u})$, where $u = (u_{n,m})_{n,m\geq 1}$, $\hat{u} = (\hat{u}_{n,m})_{n,m\geq 1} \in \mathbb{M}$, with

$$u_{1,0} = \frac{1}{2} = u_{1,1} \quad \text{and} \quad \hat{u}_{1,1} = 1.$$

Then we obviously have $Z_{V_n} \leq 1$ $\mathbb{P}$-a.s. for all $n \geq 0$, and therefore

$$\lim_{K\to\infty} \sup_{n\geq 0} \mathbb{E}\left[\frac{\mu^n}{\nu_n(\mathbf{e})} Z_{V_n} \mathbb{1}_{\{Z_{V_n}>K\}}\right] = 0.$$

Moreover, it is easy to verify that (A1) is also satisfied. We omit the details.

We follow the approach of Section 6 in [11] and of Section 4.3 in [27]. We need the following

Lemma 4.8. *Let* $\mathbb{P}(Surv) > 0$ *and assume* (A1). *There exist* $\mathbb{P}$*-a.s. finite random variables* C *and* D*, such that*

$$C \leq \frac{\mathcal{Z}_n}{\nu_n(\mathbf{e})} \leq D \quad \mathbb{P}\text{-}a.s.$$

for any $n \in \mathbb{N}_0$, *and* $\mathbb{P}(C = 0|\mathit{Surv}) = \mathbb{P}(D = 0|\mathit{Surv}) = 0$.

Proof. Using Theorem 4.5 and Proposition 3.4, the assumptions ensure that

$$\mathcal{Z}_n/\nu_n(\mathbf{e}) \to W \ \mathbb{P}\text{-a.s.}, \quad \mathbb{P}(W = 0) < 1 \quad \text{and} \quad \{W = 0\} = \mathit{Ext}\ \mathbb{P}\text{-a.s.}$$

In particular, $\mathbb{P}(W = 0|\mathit{Surv}) = 0$. Since $\mathcal{Z}_n/\nu_n(\mathbf{e}) < \infty$ $\mathbb{P}$-a.s. for every $n \geq 0$ and

$$\limsup_{n\to\infty} \frac{\mathcal{Z}_n}{\nu_n(\mathbf{e})} = W < \infty \quad \mathbb{P}\text{-a.s.},$$

we get

$$C := \inf_{n\geq 0} \frac{\mathcal{Z}_n}{\nu_n(\mathbf{e})} \leq \frac{\mathcal{Z}_n}{\nu_n(\mathbf{e})} \leq \sup_{n\geq 0} \frac{\mathcal{Z}_n}{\nu_n(\mathbf{e})} =: D < \infty \quad \mathbb{P}\text{-a.s.}$$

Moreover, $\mathcal{Z}_n/\nu_n(\mathbf{e}) > 0$ $\mathbb{P}$-a.s. on *Surv* for all $n \geq 0$ and thus

$$\begin{aligned}\mathbb{P}(D = 0|\mathit{Surv}) &\leq \mathbb{P}(C = 0|\mathit{Surv}) \\ &= \mathbb{P}\left(\liminf_{n\to\infty} \frac{\mathcal{Z}_n}{\nu_n(\mathbf{e})} = 0 \middle| \mathit{Surv}\right) = \mathbb{P}(W = 0|\mathit{Surv}) = 0.\end{aligned}$$

This completes the proof. □

With the help of the previous lemma, we can show that the number of parasites in strongly infected cells is negligible compared to the total number of parasites, cf. Lemma 6.1 in [11] and Lemma 4.13 (a) in [27].

Theorem 4.9. *Let* $\mathbb{P}(\mathit{Surv}) > 0$ *and assume* (4.6) *and* (A1). *We have*

$$\lim_{K\to\infty} \sup_{n\geq 0} \mathbb{P}\left(\frac{\sum_{v\in\mathbb{T}_n^*} Z_v \mathbb{1}_{\{Z_v > K\}}}{\mathcal{Z}_n} \geq \eta \middle| \mathit{Surv}\right) = 0$$

for all $\eta > 0$.

Proof. Fix $\eta > 0$ and define

$$A_n(K, \eta) := \left\{\frac{\sum_{v\in\mathbb{T}_n^*} Z_v \mathbb{1}_{\{Z_v > K\}}}{\mathcal{Z}_n} \geq \eta\right\} \cap \mathit{Surv}$$

for $n, K \geq 0$. By Lemma 4.8, there exists a random variable C such that $C \leq \mathcal{Z}_n/\nu_n(\mathbf{e})$

$\mathbb{P}$-a.s. for all $n \geq 0$ and $\mathbb{P}(C = 0|Surv) = 0$. Therefore

$$\mathbb{E}\left[\sum_{v\in\mathbb{T}_n^*} Z_v \mathbb{1}_{\{Z_v>K\}}\middle| \mathbf{e}\right] \geq \eta\mathbb{E}\left[\mathcal{Z}_n \mathbb{1}_{A_n(K,\eta)}\middle| \mathbf{e}\right] \geq \eta\nu_n(\mathbf{e})\mathbb{E}[C\mathbb{1}_{A_n(K,\eta)}|\mathbf{e}] \quad \mathbb{P}\text{-a.s.},$$

and thus

$$\begin{aligned}
\eta\mathbb{E}[C\mathbb{1}_{A_n(K,\eta)}|\mathbf{e}] &\leq \frac{1}{\nu_n(\mathbf{e})}\sum_{k>K}\mathbb{E}\left[\sum_{v\in\mathbb{T}_n^*} k\mathbb{1}_{\{Z_v=k\}}\middle| \mathbf{e}\right] \\
&= \frac{1}{\nu_n(\mathbf{e})}\sum_{k>K} k\mathbb{E}[\#\{v\in\mathbb{T}_n : Z_v = k\}|\mathbf{e}] \\
&= \frac{\mu^n}{\nu_n(\mathbf{e})}\sum_{k>K} k\mathbb{P}(Z_{V_n} = k|\mathbf{e}) \qquad (4.7) \\
&= \frac{\mu^n}{\nu_n(\mathbf{e})}\mathbb{E}[Z_{V_n}\mathbb{1}_{\{Z_{V_n}>K\}}|\mathbf{e}],
\end{aligned}$$

where we used Proposition 2.9 for the second equality. This leads to

$$\lim_{K\to\infty}\sup_{n\geq 0}\eta\mathbb{E}[C\mathbb{1}_{A_n(K,\eta)}] \leq \lim_{K\to\infty}\sup_{n\geq 0}\mathbb{E}\left[\frac{\mu^n}{\nu_n(\mathbf{e})} Z_{V_n}\mathbb{1}_{\{Z_{V_n}>K\}}\right] = 0$$

according to (4.6). Since $C > 0$ $\mathbb{P}$-a.s. on *Surv*, this implies

$$\lim_{K\to\infty}\sup_{n\geq 0}\mathbb{P}\left(\frac{\sum_{v\in\mathbb{T}_n^*} Z_v\mathbb{1}_{\{Z_v>K\}}}{\mathcal{Z}_n} \geq \eta\middle| Surv\right) = 0,$$

and the proof is finished. □

The next result is the analogue of Proposition 6.3 in [11] and of Lemma 4.13 (b) in [27].

Proposition 4.10. *In the situation of Theorem 4.9, for any* $\varepsilon > 0$ *there exist constants* $0 < a < b < \infty$*, such that*

$$\mathbb{P}\left(a \leq \frac{\mathcal{T}_n^*}{\nu_n(\mathbf{e})} \leq b\middle| Surv\right) \geq 1 - \varepsilon$$

for all $n \geq 0$.

Proof. Fix $\varepsilon > 0$ and choose random variables C, D according to Lemma 4.8. Then we get for all $n \geq 0$

$$\frac{\mathcal{T}_n^*}{\nu_n(\mathbf{e})} \leq \frac{\mathcal{Z}_n}{\nu_n(\mathbf{e})} \leq D \quad \mathbb{P}\text{-a.s.}, \tag{4.8}$$

and since D is $\mathbb{P}$-a.s. finite there exists a $b \in (0, \infty)$ with $\mathbb{P}(D \leq b | \mathit{Surv}) \geq 1 - \frac{\varepsilon}{2}$. Furthermore, Lemma 4.8 yields

$$\frac{\mathcal{T}_n^*}{\nu_n(\mathbf{e})} \geq \frac{\sum_{v \in \mathbb{T}_n^*} Z_v \mathbb{1}_{\{Z_v \leq K\}}}{K \nu_n(\mathbf{e})} \geq \frac{C}{K} \cdot \frac{\sum_{v \in \mathbb{T}_n^*} Z_v \mathbb{1}_{\{Z_v \leq K\}}}{\mathcal{Z}_n} \quad \mathbb{P}\text{-a.s.} \tag{4.9}$$

for any $K \geq 1$. Owing to Theorem 4.9, we can find a $K_0 \in \mathbb{N}$, such that

$$\mathbb{P}\left(\frac{\sum_{v \in \mathbb{T}_n^*} Z_v \mathbb{1}_{\{Z_v \leq K\}}}{\mathcal{Z}_n} \geq \frac{1}{2} \middle| \mathit{Surv} \right) \geq 1 - \frac{\varepsilon}{4}$$

for all $K \geq K_0$ and $n \geq 0$. Now, on Surv we have $C > 0$ $\mathbb{P}$-a.s. and thus there exists an $0 < \eta < 2b$, such that

$$\mathbb{P}\left(\frac{C}{K_0} > \eta \middle| \mathit{Surv} \right) \geq 1 - \frac{\varepsilon}{4}.$$

Taking $a := \frac{\eta}{2}$ we get for any $n \geq 0$

$$\begin{aligned} & \mathbb{P}\left(\frac{C}{K_0} \cdot \frac{\sum_{v \in \mathbb{T}_n^*} Z_v \mathbb{1}_{\{Z_v \leq K_0\}}}{\mathcal{Z}_n} \geq a \middle| \mathit{Surv} \right) \\ \geq \; & \mathbb{P}\left(\frac{\sum_{v \in \mathbb{T}_n^*} Z_v \mathbb{1}_{\{Z_v \leq K_0\}}}{\mathcal{Z}_n} \geq \frac{1}{2}, \frac{C}{K_0} > \eta \middle| \mathit{Surv} \right) \\ \geq \; & \mathbb{P}\left(\frac{\sum_{v \in \mathbb{T}_n^*} Z_v \mathbb{1}_{\{Z_v \leq K_0\}}}{\mathcal{Z}_n} \geq \frac{1}{2} \middle| \mathit{Surv} \right) + \mathbb{P}\left(\frac{C}{K_0} > \eta \middle| \mathit{Surv} \right) - 1 \\ \geq \; & 1 - \frac{\varepsilon}{2}. \end{aligned}$$

Using (4.8) and (4.9), this choice of $0 < a < b < \infty$ yields

$$\mathbb{P}\left(a \leq \frac{\mathcal{T}_n^*}{\nu_n(\mathbf{e})} \leq b \middle| \mathit{Surv} \right) \geq 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 1 - \varepsilon$$

for all $n \geq 0$. □

Now, we show that strongly infected cells do not have a big influence on the number of infected cells in future generations. Similar results in the case without a random environment can be found in [27, Lemma 4.13 (c)] and in [11, Lemma 6.5].

Theorem 4.11. *In the situation of Theorem 4.9, for any $\eta > 0$ we have*

$$\lim_{K\to\infty} \sup_{n,m\geq 0} \mathbb{P}\left(\frac{\#\{v \in \mathbb{T}^*_{n+m} : Z_{v|n} > K\}}{\mathcal{T}^*_{n+m}} \geq \eta \,\middle|\, Surv \right) = 0.$$

To prove this result, we use the previous proposition and details from the proof of Theorem 4.9.

Proof. Fix $\eta, \varepsilon > 0$. According to Proposition 4.10, we can find an $a > 0$, such that

$$\mathbb{P}\left(\frac{\mathcal{T}^*_{n+m}}{\nu_{n+m}(\mathbf{e})} \geq a \,\middle|\, Surv \right) \geq 1 - \frac{\varepsilon}{2}$$

for all $n, m \geq 0$. Set

$$A_{n,m}(K,\eta) := \left\{ \frac{\#\{v \in \mathbb{T}^*_{n+m} : Z_{v|n} > K\}}{\mathcal{T}^*_{n+m}} \geq \eta \right\} \cap \left\{ \frac{\mathcal{T}^*_{n+m}}{\nu_{n+m}(\mathbf{e})} \geq a \right\}$$

for $n, m, K \geq 0$. Then we get

$$\#\{v \in \mathbb{T}^*_{n+m} : Z_{v|n} > K\} \geq \eta \mathcal{T}^*_{n+m} \mathbb{1}_{A_{n,m}(K,\eta)} \geq \eta a \nu_{n+m}(\mathbf{e}) \mathbb{1}_{A_{n,m}(K,\eta)} \quad \mathbb{P}\text{-a.s.}$$

Now, let $\mathcal{Z}^{(v)}_m$ denote the number of parasites in the m-th generation of the process starting in cell $v \in \mathbb{T}^*_n$. We get

$$\begin{aligned}
\mathbb{P}(A_{n,m}(K,\eta)) &\leq \frac{1}{\eta a} \mathbb{E}\left[\frac{\#\{v \in \mathbb{T}^*_{n+m} : Z_{v|n} > K\}}{\nu_{n+m}(\mathbf{e})} \right] \\
&\leq \frac{1}{\eta a} \mathbb{E}\left[\frac{\sum_{v\in\mathbb{T}^*_n} \mathcal{Z}^{(v)}_m \mathbb{1}_{\{Z_v > K\}}}{\nu_{n+m}(\mathbf{e})} \right] \\
&= \frac{1}{\eta a} \mathbb{E}\left[\frac{\sum_{v\in\mathbb{T}^*_n} \sum_{k>K} \mathbb{E}[\mathcal{Z}^{(v)}_m \mathbb{1}_{\{Z_v = k\}} | \mathcal{G}_n]}{\nu_{n+m}(\mathbf{e})} \right] \\
&= \frac{1}{\eta a} \mathbb{E}\left[\frac{\sum_{v\in\mathbb{T}^*_n} \sum_{k>K} \mathbb{1}_{\{Z_v = k\}} \mathbb{E}_k[\mathcal{Z}_m | [\mathbf{e}]_n]}{\nu_{n+m}(\mathbf{e})} \right] \\
&= \frac{1}{\eta a} \mathbb{E}\left[\frac{\sum_{v\in\mathbb{T}^*_n} \sum_{k>K} \mathbb{1}_{\{Z_v = k\}} k \frac{\nu_{n+m}(\mathbf{e})}{\nu_n(\mathbf{e})}}{\nu_{n+m}(\mathbf{e})} \right] \\
&= \frac{1}{\eta a} \mathbb{E}\left[\frac{1}{\nu_n(\mathbf{e})} \sum_{k>K} \mathbb{E}\left[\sum_{v\in\mathbb{T}^*_n} k \mathbb{1}_{\{Z_v = k\}} \,\middle|\, \mathbf{e} \right] \right]
\end{aligned}$$

$$= \frac{1}{\eta a}\mathbb{E}\left[\frac{\mu^n}{\nu_n(\mathbf{e})} Z_{V_n} \mathbb{1}_{\{Z_{V_n} > K\}}\right],$$

where the last equality follows as in (4.7). By (4.6), this yields

$$\lim_{K\to\infty} \sup_{m,n\geq 0} \mathbb{P}(A_{n,m}(K,\eta)) = 0.$$

In particular, there exists a $K_0 \geq 0$, such that for each $K \geq K_0$ and $n, m \geq 0$

$$\mathbb{P}(A_{n,m}(K,\eta) \cap \mathit{Surv}) \leq \mathbb{P}(A_{n,m}(K,\eta)) \leq \frac{\varepsilon}{2}\mathbb{P}(\mathit{Surv})$$

and thus

$$\begin{aligned} & \mathbb{P}\left(\frac{\#\{v \in \mathbb{T}^*_{n+m} : Z_{v|n} > K\}}{\mathcal{T}^*_{n+m}} \geq \eta \,\middle|\, \mathit{Surv}\right) \\ \leq\ & \mathbb{P}(A_{n,m}(K,\eta)|\mathit{Surv}) + \mathbb{P}\left(\frac{\mathcal{T}^*_{n+m}}{\nu_{n+m}(\mathbf{e})} < a \,\middle|\, \mathit{Surv}\right) \\ \leq\ & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $K \geq K_0$ and $n, m \geq 0$. This completes the proof. □

5. Linear fractional parasite reproduction

The final chapter comprises two examples where the conditional parasite reproduction is linear fractional. At first we look at the simple example of a complete binary cell tree, i.e. $p_2 = 1$. The second example is in the setting of chapter one, i.e. cells can die, produce one daughter cell or produce two daughter cells. We gathered some facts about the linear fractional distribution in Section A.5 of the appendix.

5.1. The complete binary cell tree

5.1.1. The model and known results about perpetuities

We consider a simple example, where the cell tree is a complete binary tree, i.e. $p_2 = 1$. Moreover, the parasite reproduction given the environment is linear fractional and the environment governs the parameters of these linear fractional distributions. Therefore, let $\mathbf{e} = (A_n, B_n)_{n\geq 0}$ be a sequence of i.i.d. $\mathbb{R}^2_>$-valued random vectors satisfying $A_0 + B_0 \geq 1$ $\mathbb{P}$-a.s. and $\mathbb{E}|\log A_0|, \mathbb{E}|\log B_0| < \infty$. Furthermore, we assume that, given the environment, the sharing of parasites into the cells is independent and symmetric, i.e.

$$\mathbb{P}\left(\left(X^{(0)}_{k,v}, X^{(1)}_{k,v}\right) \in \cdot \,\middle|\, \mathbf{e}\right) = LF(A_{|v|}, B_{|v|}) \otimes LF(A_{|v|}, B_{|v|}), \quad k \in \mathbb{N}, v \in \mathbb{V}.$$

Here, $LF(a, b)$ denotes the linear fractional distribution with parameters a, b.

Remark 5.1. For convenience, we defined our random environment by $\mathbf{e} = (A_n, B_n)_{n\geq 0}$ instead of $\mathbf{e} = (LF(A_n, B_n) \otimes LF(A_n, B_n))_{n\geq 0}$.

The corresponding generating function h of $LF(a, b)$ is given by

$$\frac{1}{1 - h(s)} = \frac{a}{1 - s} + b, \quad s \in [0, 1).$$

If $h_{\mathbf{e}_n}$ denotes the generating function of $X^{(0)}_{1,0*n}$ under $\mathbb{P}(\cdot|\mathbf{e})$ (and also of $X^{(1)}_{1,0*n}$),

$n \geq 0$, it is well known that

$$\frac{1}{1-h_{1:n,\mathbf{e}}(s)} := \frac{1}{1-h_{\mathbf{e}_0} \circ \ldots \circ h_{\mathbf{e}_{n-1}}(s)} = \frac{\mathit{\Pi}_n}{1-s} + S_n, \quad n \geq 1,$$

where $\mathit{\Pi}_0 := 1$, $S_0 := 0$,

$$\mathit{\Pi}_n = \prod_{i=0}^{n-1} A_i \quad \text{and} \quad S_n = \sum_{i=0}^{n-1} \mathit{\Pi}_i B_i.$$

For $n \geq 0$ set $g_{\mathbf{e}_n}(s) := A_n s + B_n$ and

$$g_{1:n,\mathbf{e}} := g_{\mathbf{e}_0} \circ \ldots \circ g_{\mathbf{e}_{n-1}} \quad \text{as well as} \quad g_{n:1,\mathbf{e}} := g_{\mathbf{e}_{n-1}} \circ \ldots \circ g_{\mathbf{e}_0}.$$

Then

$$g_{1:n,\mathbf{e}}(s) = \mathit{\Pi}_n s + S_n \quad \text{and} \quad g_{n:1,\mathbf{e}}(s) = \mathit{\Pi}_n s + \sum_{i=1}^{n} \frac{\mathit{\Pi}_n}{\mathit{\Pi}_i} B_{i-1}.$$

Further, setting $\varphi(s) := (1-s)^{-1}$ for $s \in [0,1)$, the above computations yield

$$\varphi \circ h_{1:n,\mathbf{e}}(s) = g_{1:n,\mathbf{e}} \circ \varphi(s).$$

Next, we gather some known facts about these quantities. The following proposition is due to Goldie and Maller.

Proposition 5.2 (Goldie and Maller [24], Theorem 2.1)**.** *The following assertions are equivalent:*

(a) $\mathit{\Pi}_n \to 0$ $\mathbb{P}$*-a.s.* *and* $I_- := \int_{[0,\infty)} \frac{x}{J^-(x)} \, \mathbb{P}(\log B_0 \in \mathrm{d}x) < \infty$,
where $J^-(x) := \int_0^x \mathbb{P}(-\log A_0 > y) \, \mathrm{d}y = \mathbb{E}[x \wedge \log^- A_0]$ *for* $x > 0$.

(b) The so-called perpetuity

$$S_\infty := \sum_{i \geq 0} \mathit{\Pi}_i B_i$$

is $\mathbb{P}$*-a.s. finite and, for each* $x \in \mathbb{R}$*, the backward iterations* $g_{1:n,\mathbf{e}}(x)$ *converge* $\mathbb{P}$*-a.s. (monotonically) to* S_∞*, while the forward iterations* $g_{n:1,\mathbf{e}}(x)$ *converge in distribution to* S_∞*.*

Conversely, if

$$\mathbb{P}(A_0 x + B_0 = x) < 1 \quad \textit{for all } x \in \mathbb{R}$$

and at least one of the conditions in (a) fails, then $S_\infty = \infty$ $\mathbb{P}$*-a.s.*

Replacing A_n with A_n^{-1} as well as B_n with $A_n^{-1}B_n$, the above proposition immediately yields

Proposition 5.3. *Defining* $g_{\mathbf{e}_n}^{(-1)}(s) := A_n^{-1}s + A_n^{-1}B_n$ *for* $n \geq 0$*, the duality relation*

$$\frac{S_n}{\Pi_n} \;=\; g_{n:1,\mathbf{e}}^{(-1)}(0) \;\overset{d}{=}\; g_{1:n,\mathbf{e}}^{(-1)}(0) \;=\; \sum_{i=0}^{n-1} \Pi_{i+1}^{-1} B_i \;=:\; S_n^{(-1)}$$

holds true for all $n \in \mathbb{N}$*. Moreover, the following assertions are equivalent:*

(a) $\Pi_n \to \infty$ $\mathbb{P}$*-a.s. and* $I_+ := \int_{[0,\infty)} \frac{x}{J^+(x)}\, \mathbb{P}(\log(B_0/A_0) \in \mathrm{d}x) < \infty$*,*
where $J^+(x) := \mathbb{E}[x \wedge \log^+ A_0]$ *for* $x > 0$*.*

(b) $S_\infty^{(-1)} := \sum_{i\geq 0} \Pi_{i+1}^{-1} B_i < \infty$ $\mathbb{P}$*-a.s.*

In particular, (a) and (b) imply

$$\frac{S_n}{\Pi_n} \xrightarrow{d} S_\infty^{(-1)}.$$

5.1.2. The process of parasites along the spine

Next, we compute some of the quantities needed for a deeper analysis of the results from the previous chapters of this thesis, in the given linear fractional situation. First we get

$$\mathbb{E}\left[X_{k,v}^{(0)}\middle|\,\mathbf{e}\right] = \mathbb{E}\left[X_{k,v}^{(1)}\middle|\,\mathbf{e}\right] = \frac{1}{A_{|v|}} \quad \text{and thus} \quad \nu(\mathbf{e}_n) = \frac{2}{A_n}, \quad \nu_n(\mathbf{e}) = \frac{2^n}{\Pi_n},$$
$$\nu = \mathbb{E}[\nu(\mathbf{e}_0)] = 2\cdot\mathbb{E}[A_0^{-1}], \quad \mathbb{E}[\log\nu(\mathbf{e}_0)] = \log 2 - \mathbb{E}[\log A_0].$$

For any $v \in \{0,1\}^n$, $n \geq 1$, the number of parasites in cell v is, conditioned on $\mathbf{e}$, again linear fractional. More precisely

$$\mathcal{L}(Z_v|\mathbf{e}) = LF(\Pi_n, S_n).$$

In particular, the same holds true for the number of parasites along the spine $(Z_{V_n})_{n\geq 0}$, and so we have

$$\mathbb{P}(Z_{V_n} = k|\mathbf{e}) = \begin{cases} 1 - \dfrac{1}{\Pi_n + S_n}, & k = 0, \\[2ex] \dfrac{\Pi_n}{(\Pi_n + S_n)^2}\left(\dfrac{S_n}{\Pi_n + S_n}\right)^{k-1}, & k \geq 1, \end{cases} \tag{5.1}$$

and therefore $\mathbb{P}(Z_{V_n} > 0|\mathbf{e}) = \frac{1}{\Pi_n + S_n}$. In the notation of Chapter 2, we get

$$\begin{aligned} f_{\xi_0}(s) = \mathbb{E}\left[s^{Z_{V_1}} \middle| \xi\right] = 1 - \frac{1-s}{A_0 + B_0(1-s)}, \\ f'_{\xi_0}(s) = \frac{A_0}{(A_0 + B_0(1-s))^2}, \quad f'_{\xi_0}(1) = \frac{1}{A_0}, \quad \mathbb{E}[\log f'_{\xi_0}(1)] = -\mathbb{E}[\log A_0], \\ \mathbb{E}[f'_{\xi_0}(1) \log f'_{\xi_0}(1)] = -\mathbb{E}\left[\frac{\log A_0}{A_0}\right], \\ f''_{\xi_0}(s) = \frac{2A_0B_0}{(A_0 + B_0(1-s))^3}, \quad f''_{\xi_0}(1) = \frac{2B_0}{A_0^2}. \end{aligned}$$

Therefore,

$$\mathbb{E}[\log f'_{\xi_0}(1)] \begin{cases} > 0, & \text{if } \mathbb{E}[\log A_0] < 0, \\ = 0, & \text{if } \mathbb{E}[\log A_0] = 0, \\ < 0, & \text{if } \mathbb{E}[\log A_0] > 0. \end{cases}$$

Moreover, since $\mathbb{E}[\log^+ B_0] < \infty$ and $\mathbb{E}[\log^+(B_0/A_0)] < \infty$, we get from Proposition 5.2 and Proposition 5.3

$$\begin{aligned} S_\infty < \infty = S_\infty^{(-1)} \ \mathbb{P}\text{-a.s.} \quad &\text{iff} \quad \mathbb{E}[\log A_0] < 0, \\ S_\infty = \infty = S_\infty^{(-1)} \ \mathbb{P}\text{-a.s.} \quad &\text{iff} \quad \mathbb{E}[\log A_0] = 0, \\ S_\infty^{(-1)} < \infty = S_\infty \ \mathbb{P}\text{-a.s.} \quad &\text{iff} \quad \mathbb{E}[\log A_0] > 0. \end{aligned} \tag{5.2}$$

Subcritical case: $S_\infty^{(-1)} < \infty = S_\infty$ $\mathbb{P}$-a.s.

If $S_\infty^{(-1)} = \lim_{n\to\infty} S_n^{(-1)} < \infty$ $\mathbb{P}$-a.s., then

$$\begin{aligned} \mathbb{P}(Z_{V_n} = k | Z_{V_n} > 0, \mathbf{e}) &= \frac{\Pi_n}{\Pi_n + S_n}\left(\frac{S_n}{\Pi_n + S_n}\right)^{k-1} \\ &= \frac{1}{1 + S_n/\Pi_n}\left(\frac{S_n/\Pi_n}{1 + S_n/\Pi_n}\right)^{k-1} \\ &\overset{d}{=} \frac{1}{1 + S_n^{(-1)}}\left(\frac{S_n^{(-1)}}{1 + S_n^{(-1)}}\right)^{k-1} \\ &\xrightarrow{n\to\infty} \frac{1}{1 + S_\infty^{(-1)}}\left(\frac{S_\infty^{(-1)}}{1 + S_\infty^{(-1)}}\right)^{k-1} \qquad \mathbb{P}\text{-a.s.} \end{aligned}$$

for $k \geq 1$, according to Proposition 5.3. In particular

$$\mathcal{L}(Z_{V_n}|Z_{V_n} > 0, \mathbf{e}) = Geom\left(\frac{1}{1 + S_n/\varPi_n}\right).$$

Supercritical case: $S_\infty < \infty = S_\infty^{(-1)}$ $\mathbb{P}$-a.s.

In this case, we have

$$\frac{1}{\varPi_n}\mathbb{P}(Z_{V_n} = k|\mathbf{e}) = \frac{1}{(\varPi_n + S_n)^2}\left(\frac{S_n}{\varPi_n + S_n}\right)^{k-1} \xrightarrow{n\to\infty} \frac{1}{S_\infty^2} \quad \mathbb{P}\text{-a.s.},$$

in other words

$$\mathbb{P}(Z_{V_n} = k|\mathbf{e}) = \frac{\varPi_n}{S_\infty^2} + R_n \quad \mathbb{P}\text{-a.s.},$$

as $n \to \infty$, where R_n is a random variable satisfying $R_n/\varPi_n \to 0$ $\mathbb{P}$-a.s.

5.1.3. The process of infected cells and related quantities

Regarding the process of infected cells $(\mathcal{T}_n^*)_{n\geq 0}$ in the linear fractional situation, Proposition 2.9 and Proposition 2.18 yield the following

Proposition 5.4. *The process $(2^{-n}\mathcal{T}_n^*)_{n\geq 0}$ converges $\mathbb{P}$-a.s. to a random variable L taking values in $[0,1]$ with*

$$\mathbb{E}[L|\mathbf{e}] = \lim_{n\to\infty} \frac{1}{\varPi_n + S_n} \xrightarrow{n\to\infty} \frac{1}{S_\infty} \quad \mathbb{P}\text{-a.s.}$$

and

$$\mathbb{E}L = \lim_{n\to\infty} \mathbb{E}\left[\frac{1}{\varPi_n + S_n}\right] \xrightarrow{n\to\infty} \mathbb{E}\left[\frac{1}{S_\infty}\right].$$

Furthermore,

$$\mathbb{P}(L = 0) = 1 \quad \textit{iff} \quad S_\infty = \infty \ \mathbb{P}\text{-a.s.} \quad \textit{iff} \quad \mathbb{E}[\log A] \geq 0,$$

and if $\mathbb{P}(L = 0) < 1$ then $Ext = \{L = 0\}$ $\mathbb{P}$-a.s.

Note that in the given linear fractional situation

$$\mathbb{P}(\mathbb{P}(L = 0|\mathbf{e}) = 1) \in \{0, 1\}$$

follows from the fact that either $S_\infty < \infty$ $\mathbb{P}$-a.s. or $S_\infty = \infty$ $\mathbb{P}$-a.s.

Additionally, for $k \geq 0$, we want to take a look at the process $(F_n(k))_{n\geq 0}$ of the number of cells with exactly k parasites, where we recall

$$F_n(k) = \#\{v \in \{0,1\}^n : Z_v = k\}.$$

According to Corollary 2.11, we have

$$\frac{F_n(k)}{2^n} \xrightarrow{n\to\infty} 0 \quad \text{in probability and in } L^1,$$

for every $k \geq 1$. To analyse the behaviour of the conditional expectation of the number of cells with exactly k parasites $\mathbb{E}[F_n(k)|\mathbf{e}]$, we again distinguish the three cases according to (5.2).

In the following, fix $k \geq 1$ and recall from Proposition 2.9 and (5.1)

$$\mathbb{E}[F_n(k)|\mathbf{e}] = 2^n \mathbb{P}(Z_{V_n} = k|\mathbf{e}) = \frac{2^n \mathit{\Pi}_n}{(\mathit{\Pi}_n + S_n)^2}\left(\frac{S_n}{\mathit{\Pi}_n + S_n}\right)^{k-1}, \quad n \geq 1.$$

The supercritical case: $S_\infty < \infty = S_\infty^{(-1)}$ $\mathbb{P}$-a.s.

In this case we have $\mathbb{P}(L = 0) < 1$ and hence $\mathit{Ext} = \{L = 0\}$ $\mathbb{P}$-a.s. Since $S_\infty < \infty$ $\mathbb{P}$-a.s., Proposition 5.2 yields $\mathit{\Pi}_n \to 0$ $\mathbb{P}$-a.s. and we get

$$\frac{\mathbb{E}[F_n(k)|\mathbf{e}]}{2^n \mathit{\Pi}_n} = \frac{1}{(\mathit{\Pi}_n + S_n)^2}\left(\frac{S_n}{\mathit{\Pi}_n + S_n}\right)^{k-1} \xrightarrow{n\to\infty} \frac{1}{S_\infty^2} \quad \mathbb{P}\text{-a.s.}$$

In other words

$$\mathbb{E}[F_n(k)|\mathbf{e}] = \frac{2^n \mathit{\Pi}_n}{S_\infty^2} + R_{1,n} \quad \mathbb{P}\text{-a.s.}$$

as $n \to \infty$, where $R_{1,n}$ is a random variable satisfying $R_{1,n}/(2^n \mathit{\Pi}_n) \to 0$ $\mathbb{P}$-a.s.

The critical case: $S_\infty = \infty = S_\infty^{(-1)}$ $\mathbb{P}$-a.s.

Here we have $\mathbb{P}(L = 0) = 1$, and further

$$\frac{(\mathit{\Pi}_n + S_n)^2}{2^n \mathit{\Pi}_n} \cdot \mathbb{E}[F_n(k)|\mathbf{e}] = \left(\frac{S_n/\mathit{\Pi}_n}{1 + S_n/\mathit{\Pi}_n}\right)^{k-1} \overset{d}{=} \left(\frac{S_n^{(-1)}}{1 + S_n^{(-1)}}\right)^{k-1} \xrightarrow{n\to\infty} 1 \quad \mathbb{P}\text{-a.s.}$$

and thus

$$\frac{(\Pi_n + S_n)^2}{2^n \Pi_n} \cdot \mathbb{E}[F_n(k)|\mathbf{e}] \xrightarrow{n\to\infty} 1 \quad \text{in probability.}$$

The subcritical case: $S_\infty^{(-1)} < \infty = S_\infty$ $\mathbb{P}$**-a.s.**

In this case we have again $\mathbb{P}(L = 0) = 1$. Moreover,

$$\begin{aligned}\frac{\mathbb{E}[F_n(k)|\mathbf{e}]}{\nu_n(\mathbf{e})} &= \frac{\Pi_n}{2^n} \cdot \mathbb{E}[F_n(k)|\mathbf{e}] = \frac{1}{(1+S_n/\Pi_n)^2}\left(\frac{S_n/\Pi_n}{1+S_n/\Pi_n}\right)^{k-1} \\ &\xrightarrow[n\to\infty]{d} \frac{1}{\left(1+S_\infty^{(-1)}\right)^2}\left(\frac{S_\infty^{(-1)}}{1+S_\infty^{(-1)}}\right)^{k-1}\end{aligned}$$

according to Proposition 5.3.

5.1.4. The process of parasites

After some easy but tedious computations, we obtain for the distribution of $\mathcal{Z}_1$ given $\mathbf{e}$

$$\mathbb{P}(\mathcal{Z}_1 = k|\mathbf{e}) = \begin{cases} \left(1 - \frac{1}{A_0+B_0}\right)^2, & k = 0, \\ \frac{2A_0(A_0+B_0-1)}{(A_0+B_0)^3}\left(\frac{B_0}{A_0+B_0}\right)^{k-1} & \\ +(k-1)\frac{A_0^2}{(A_0+B_0)^4}\left(\frac{B_0}{A_0+B_0}\right)^{k-2}, & k \geq 1. \end{cases}$$

Note that in the given situation $(\mathcal{Z}_n)_{n\geq 0}$ is even a BPRE and if this process is supercritical, i.e. $\mathbb{E}[\log \nu(\mathbf{e}_0)] > 0$, we have

$$\mathbb{E}W = 1 \quad \text{iff} \quad \mathbb{E}\left[\frac{\mathcal{Z}_1 \log \mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right] < \infty.$$

Moreover, our assumptions from the beginning of this section ensure

$$\begin{aligned}\mathbb{E}[\log^-(1 - \mathbb{P}(\mathcal{Z}_1 = 0|\mathbf{e}))] &= -\mathbb{E}\left[\log\left(\frac{2}{A_0+B_0} - \frac{1}{(A_0+B_0)^2}\right)\right] \\ &= \mathbb{E}\left[\log(A_0+B_0)\right] - \mathbb{E}\left[\log\left(2 - \frac{1}{A_0+B_0}\right)\right] \\ &\leq \mathbb{E}\left[\log(A_0+B_0)\right] < \infty.\end{aligned}$$

Thus, Theorem 3 in [8] and Theorem 3.1 in [50] yield

$$\mathbb{P}(\mathit{Ext}) = 1 \quad \text{iff} \quad \mathbb{E}[\log \nu(\mathbf{e}_0)] \leq 0,$$

i.e.

$$\mathbb{P}(\mathit{Ext}) = 1 \quad \text{iff} \quad \mathbb{E}[\log A_0] \geq \log 2.$$

Next, we compute the quenched and annealed second moment of W_n, $n \geq 0$, using Theorem 3.6. Note that $p_0 = 0$ and

$$\nu^{(2)}(\mathbf{e}_k) = \mathbb{E}\big[(X^{(0)}_{1,0^{*k}})^2\big|\mathbf{e}\big] + \mathbb{E}\big[(X^{(1)}_{1,0^{*k}})^2\big|\mathbf{e}\big] + 2\mathbb{E}\big[X^{(0)}_{1,0^{*k}}\big|\mathbf{e}\big]\mathbb{E}\big[X^{(1)}_{1,0^{*k}}\big|\mathbf{e}\big],$$

where we used the conditional independence of the sharing of the parasites into the cells. Now

$$\begin{aligned}\mathbb{E}\big[(X^{(0)}_{1,0^{*k}})^2\big|\mathbf{e}\big] &= \sum_{j\geq 1} j^2 \frac{A_k}{(A_k+B_k)^2}\left(\frac{B_k}{A_k+B_k}\right)^{j-1} \\ &= \frac{A_k}{(A_k+B_k)^2}\frac{1+B_k/(A_k+B_k)}{(1-B_k/(A_k+B_k))^3} \\ &= \frac{A_k+2B_k}{A_k^2},\end{aligned} \tag{5.3}$$

and therefore

$$\nu^{(2)}(\mathbf{e}_k) = 2\left(\frac{A_k+2B_k}{A_k^2} + \frac{1}{A_k^2}\right) = 2\cdot\frac{A_k+2B_k+1}{A_k^2}.$$

This yields for the quenched second moment

$$\mathbb{E}[W_n^2|\mathbf{e}] = 1 + \sum_{k=0}^{n-1}\frac{\mathit{\Pi}_k}{2^k}\cdot\frac{A_k+2B_k-1}{2},$$

and for the annealed second moment

$$\mathbb{E}[W_n] = 1 + \frac{\mathbb{E}[A_0+2B_0]-1}{2}\cdot\sum_{k=0}^{n-1}\left(\frac{\mathbb{E}[A_0]}{2}\right)^k.$$

Thus, $(W_n)_{n\geq 0}$ is L^2-bounded iff

$$\mathbb{E}[A_0] < 2 \quad \text{and} \quad \mathbb{E}[B_0] < \infty.$$

5.2. The random cell tree

5.2.1. The model

Now, we consider the cell tree as defined in Section 1.3.1 with $p_0 > 0$. As before, let $\mathbf{e} = (A_n, B_n)_{n\geq 0}$ be a sequence of i.i.d. $\mathbb{R}^2_>$-valued random vectors satisfying $A_0 + B_0 \geq 1$ $\mathbb{P}$-a.s. and $\mathbb{E}|\log A_0|, \mathbb{E}|\log B_0| < \infty$. Let $(Y_{k,v})_{k\geq 1, v\in\mathbb{V}}$ be a family of conditionally independent random variables given $\mathbf{e}$, which is independent of $(T_v)_{v\in\mathbb{V}}$, and with conditional distribution

$$\mathbb{P}(Y_{k,v} \in \cdot|\mathbf{e}) = LF(A_{|v|}, B_{|v|}), \quad k \geq 1, v \in \mathbb{V}.$$

Moreover, let $(\alpha_{k,v}(j))_{k,j\geq 1, v\in\mathbb{V}}$ be a family of i.i.d. random variables with distribution $Bern(\vartheta)$, $\vartheta \in (0,1)$, which is independent of $((Y_{k,v})_{k\geq 1, v\in\mathbb{V}}, \mathbf{e}, (T_v)_{v\in\mathbb{V}})$. Now, the offspring vector of the parasites is defined by

$$\left(X^{(0)}_{k,v}, X^{(1)}_{k,v}\right) = \left(\sum_{j=1}^{Y_{k,v}} \alpha_{k,v}(j), \sum_{j=1}^{Y_{k,v}} (1 - \alpha_{k,v}(j))\right), \quad k \geq 1, v \in \mathbb{V}.$$

Let $h_{\mathbf{e}_n}$, $g^{(0)}_{\mathbf{e}_n}$ and $g^{(1)}_{\mathbf{e}_n}$ denote the generating functions of $Y_{1,0^{*n}}$, $X^{(0)}_{1,0^{*n}}$ and $X^{(1)}_{1,0^{*n}}$ under $\mathbb{P}(\cdot|\mathbf{e})$, respectively. Then

$$\frac{1}{1 - h_{\mathbf{e}_n}(s)} = \frac{A_n}{1-s} + B_n,$$

and by the independence of $(\alpha_{k,v}(j))_{k,j\geq 1, v\in\mathbb{V}}$ and $((Y_{k,v})_{k\geq 1, v\in\mathbb{V}}, \mathbf{e})$ this gives us

$$\frac{1}{1 - g^{(0)}_{\mathbf{e}_n}(s)} = \frac{1}{1 - h_{\mathbf{e}_n}(\vartheta s + 1 - \vartheta)} = \frac{A_n}{1 - (\vartheta s + 1 - \vartheta)} + B_n = \frac{A_n/\vartheta}{1-s} + B_n$$

and

$$\frac{1}{1 - g^{(1)}_{\mathbf{e}_n}(s)} = \frac{A_n/(1-\vartheta)}{1-s} + B_n.$$

Therefore, $X^{(0)}_{k,v}$ and $X^{(1)}_{k,v}$ are again linear fractional given $\mathbf{e}$ with

$$\mathbb{P}(X^{(0)}_{k,v} \in \cdot|\mathbf{e}) = LF\left(\frac{A_{|v|}}{\vartheta}, B_{|v|}\right) \quad \text{and} \quad \mathbb{P}(X^{(1)}_{k,v} \in \cdot|\mathbf{e}) = LF\left(\frac{A_{|v|}}{1-\vartheta}, B_{|v|}\right),$$

$k \geq 1, v \in \mathbb{V}$. Set $\mathit{\Pi}_0 := 1$ and $\mathit{\Pi}_n = \prod_{i=0}^{n-1} A_i$. We get

$$\mathbb{E}\left[X_{k,v}^{(0)}\middle|\,\mathbf{e}\right] = \frac{\vartheta}{A_{|v|}}, \quad \mathbb{E}\left[X_{k,v}^{(1)}\middle|\,\mathbf{e}\right] = \frac{1-\vartheta}{A_{|v|}}, \quad \nu(\mathbf{e}_n) = (p_1+p_2)\frac{1}{A_n},$$
$$\nu_n(\mathbf{e}) = \frac{(p_1+p_2)^n}{\mathit{\Pi}_n}, \quad \nu = \mathbb{E}[\nu(\mathbf{e}_0)] = (p_1+p_2)\mathbb{E}[A_0^{-1}],$$
$$\mathbb{E}[\log \nu(\mathbf{e}_0)] = \log(p_1+p_2) - \mathbb{E}[\log A_0].$$

5.2.2. The process of parasites along the spine

Recall the definition of $(Z_{V_n})_{n\geq 0}$ from Section 2.2. In the present model we have

$$\begin{aligned} f_{\xi_0}(s) &= \mathbb{E}\left[s^{Z_{V_1}}\middle|\,\xi\right] \\ &= \mathbb{1}_{\{\check{T}_0=1\}}\mathbb{E}\left[s^{Y_{1,\emptyset}}\middle|\,\mathbf{e}\right] + \mathbb{1}_{\{\check{T}_0=2,C_0=0\}}\mathbb{E}\left[s^{X_{1,\emptyset}^{(0)}}\middle|\,\mathbf{e}\right] + \mathbb{1}_{\{\check{T}_0=2,C_0=1\}}\mathbb{E}\left[s^{X_{1,\emptyset}^{(1)}}\middle|\,\mathbf{e}\right] \\ &= 1 - \Bigg(\mathbb{1}_{\{\check{T}_0=1\}}\frac{1-s}{A_0+B_0(1-s)} + \mathbb{1}_{\{\check{T}_0=2,C_0=0\}}\frac{1-s}{A_0/\vartheta+B_0(1-s)} \\ &\qquad + \mathbb{1}_{\{\check{T}_0=2,C_0=1\}}\frac{1-s}{A_0/(1-\vartheta)+B_0(1-s)}\Bigg). \end{aligned}$$

Therefore

$$f'_{\xi_0}(1) = \mathbb{1}_{\{\check{T}_0=1\}}\frac{1}{A_0} + \mathbb{1}_{\{\check{T}_0=2,C_0=0\}}\frac{\vartheta}{A_0} + \mathbb{1}_{\{\check{T}_0=2,C_0=1\}}\frac{1-\vartheta}{A_0}$$

and

$$\begin{aligned} \mathbb{E}[\log f'_{\xi_0}(1)] &= \frac{-p_1}{p_1+2p_2}\mathbb{E}[\log A_0] + \frac{p_2}{p_1+2p_2}(\log\vartheta - \mathbb{E}[\log A_0] + \log(1-\vartheta) - \mathbb{E}[\log A_0]) \\ &= -\mathbb{E}[\log A_0] + \frac{p_2}{p_1+2p_2}(\log\vartheta + \log(1-\vartheta)). \end{aligned}$$

As a consequence we get

$$\mathbb{E}[\log f'_{\xi_0}(1)]\begin{cases} > 0, & \text{if } \mathbb{E}[\log A_0] < \dfrac{p_2}{p_1+2p_2}(\log\vartheta + \log(1-\vartheta)), \\ = 0, & \text{if } \mathbb{E}[\log A_0] = \dfrac{p_2}{p_1+2p_2}(\log\vartheta + \log(1-\vartheta)), \\ < 0, & \text{if } \mathbb{E}[\log A_0] > \dfrac{p_2}{p_1+2p_2}(\log\vartheta + \log(1-\vartheta)). \end{cases}$$

5.2.3. The process of parasites

Now, we restate the conditions for L^2-boundedness of $(W_n)_{n\geq 0}$ and for the nondegeneracy of W using the parameters of our linear fractional model.

First note that the distribution of $\mathcal{Z}_1$ under $\mathbb{P}(\cdot|\mathbf{e})$ is again linear fractional. In fact

$$\begin{aligned}\mathbb{E}[s^{\mathcal{Z}_1}|\mathbf{e}] = \mathbb{E}[\mathbb{1}_{\{T_\emptyset=0\}} + s^{Y_{1,\emptyset}}\mathbb{1}_{\{T_\emptyset>0\}}] &= p_0 + (p_1+p_2)\left(1 - \frac{1-s}{A_0 + B_0(1-s)}\right)\\ &= 1 - \frac{1-s}{\frac{A_0}{p_1+p_2} + \frac{B_0}{p_1+p_2}(1-s)},\end{aligned}$$

i.e. the conditional distribution of $\mathcal{Z}_1$ given $\mathbf{e}$ is $LF\left(\frac{A_0}{p_1+p_2}, \frac{B_0}{p_1+p_2}\right)$. Unfortunately, $\mathcal{Z}_n$ is not linear fractional under $\mathbb{P}(\cdot|\mathbf{e})$ for generations $n > 1$.

Conditions for the L^2-boundedness of $(W_n)_{n\geq 0}$

According to Theorem 3.9, we need to compute

$$\mathbb{V}\mathrm{ar}\left(\frac{\mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right), \quad \mathbb{E}[\nu^{-1}(\mathbf{e}_0)] \quad \text{and} \quad \mathbb{E}\left[\left(\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right)^2\right].$$

Using (5.3), we get

$$\mathbb{E}[\mathcal{Z}_1^2|\mathbf{e}] = \frac{A_0/(p_1+p_2) + 2B_0/(p_1+p_2)}{A_0^2/(p_1+p_2)^2} = \frac{(p_1+p_2)}{A_0^2}(A_0 + 2B_0)$$

and therefore

$$\mathbb{V}\mathrm{ar}\left(\frac{\mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right) = \mathbb{E}\left[\left(\frac{\mathcal{Z}_1}{\nu(\mathbf{e}_0)}\right)^2\right] - 1 = \frac{1}{p_1+p_2}\mathbb{E}[A_0 + 2B_0] - 1.$$

Moreover, we have

$$\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)} = \frac{1}{p_1+p_2}\mathbb{1}_{\{\check{T}_0=1\}} + \frac{\vartheta}{p_1+p_2}\mathbb{1}_{\{\check{T}_0=2, C_0=0\}} + \frac{1-\vartheta}{p_1+p_2}\mathbb{1}_{\{\check{T}_0=2, C_0=1\}}.$$

This gives us

$$\mathbb{E}\left[\left(\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right)^2\right] = \frac{p_1 + p_2(\vartheta^2 + (1-\vartheta)^2}{(p_1+p_2)^2\mu} = \frac{p_1 + p_2 - 2p_2\vartheta(1-\vartheta)}{(p_1+p_2)^2\mu}.$$

Now, we can conclude from Theorem 3.9 that $(W_n)_{n\geq 0}$ is L^2-bounded iff

$$\mathbb{E}[A_0+2B_0]<\infty,\quad \frac{1}{p_1+p_2}\mathbb{E}[A_0]<1\quad\text{and}\quad \frac{p_1+p_2-2p_2\vartheta(1-\vartheta)}{(p_1+p_2)^2\mu}<\frac{1}{\mu},$$

or, equivalently,

$$\mathbb{E}[B_0]<\infty,\quad \mathbb{E}[A_0]<p_1+p_2\quad\text{and}\quad \frac{(p_1+p_2)p_0}{2p_2}<\vartheta(1-\vartheta)$$

Conditions for the nondegeneracy of W

Here, we can state the second condition in Theorem 4.5 (a) (iv) in terms of p_1, p_2 and ϑ. Recall

$$\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}=\frac{1}{p_1+p_2}\mathbb{1}_{\{\check{T}_0=1\}}+\frac{\vartheta}{p_1+p_2}\mathbb{1}_{\{\check{T}_0=2,C_0=0\}}+\frac{1-\vartheta}{p_1+p_2}\mathbb{1}_{\{\check{T}_0=2,C_0=1\}},$$

which gives us

$$\begin{aligned}\mathbb{E}\left[\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\log\frac{f'_{\xi_0}(1)}{\nu(\mathbf{e}_0)}\right] &= \frac{1}{(p_1+p_2)(p_1+2p_2)}\left(p_1\log\left(\frac{1}{p_1+p_2}\right)\right.\\ &\quad\left.+p_2\left(\vartheta\log\left(\frac{\vartheta}{p_1+p_2}\right)+(1-\vartheta)\log\left(\frac{1-\vartheta}{p_1+p_2}\right)\right)\right)\\ &= -\frac{\log(p_1+p_2)}{p_1+2p_2}+\frac{p_2(\vartheta\log\vartheta+(1-\vartheta)\log(1-\vartheta))}{(p_1+p_2)(p_1+2p_2)}.\end{aligned}$$

Now, the second condition in Theorem 4.5 (a) (iv) is equivalent to

$$\vartheta\log\vartheta+(1-\vartheta)\log(1-\vartheta)<\frac{p_1+p_2}{p_2}\log(p_1+p_2).$$

A. Appendix

A.1. Branching processes in random environment

Theorem A.1. *Let* $(Z_n)_{n\geq 0}$ *be a BPRE in random environment* $\mathcal{U}$*.*

(a) For every $i, n \in \mathbb{N}_0$ *and* $s \in [0,1]$ *we have*

$$\mathbb{E}_i[s^{Z_n}|\mathcal{U}] = \left(\mathbb{E}[s^{Z_n}|\mathcal{U}]\right)^i = (f_{\mathcal{U}_1} \circ \ldots \circ f_{\mathcal{U}_n}(s))^i \quad \mathbb{P}_i\text{-}a.s.$$

and

$$\mathbb{E}_i[s^{Z_n}] = \mathbb{E}\left[(f_{\mathcal{U}_1} \circ \ldots \circ f_{\mathcal{U}_n}(s))^i\right]$$

(b) For every $i, n \in \mathbb{N}_0$ *we have*

$$\mathbb{E}_i[Z_n|\mathcal{U}] = i \prod_{j=1}^{n} \mu_{\mathcal{U}_j} \quad \mathbb{P}_i\text{-}a.s.$$

and

$$\mathbb{E}_i[Z_n] = i\mathbb{E}\left[\prod_{j=1}^{n} \mu_{\mathcal{U}_j}\right].$$

A.2. Branching processes in random environment with immigration

We adopt the notation used in Section 3.1 in [27]. In particular, let $(Z_n)_{n\geq 0}$ be a BPREI in random environment $\mathcal{U} = (\mathcal{U}_n)_{n\geq 0}$ and with immigration sequence $(\xi_n)_{n\geq 0}$. Further, let $\mu_{\mathcal{U}_n}$ denote the mean of the first marginal distribution of $\mathcal{U}_n$.

Theorem A.2 (Gröttrup [27], Theorem 3.5)**.** *Let* $\mathbb{E}\log \mu_{\mathcal{U}_0} > 0$ *(and* $\mu_{\mathcal{U}_0} < \infty$ $\mathbb{P}$*-a.s.).*

(a) If $\mathbb{E}\log^+ \xi_0 < \infty$*, then for every* $x \in \mathbb{N}_0$ *there exists a finite random variable* Z_∞

such that

$$\lim_{n\to\infty} \frac{Z_n}{\prod_{i=0}^{n-1} \mu_{\mathcal{U}_i}} = Z_\infty \quad \mathbb{P}_x\text{-}a.s.$$

Furthermore,

$$\mathbb{P}_x(Z_\infty > 0) = 1 \quad \textit{iff} \quad \mathbb{P}_x(Z_\infty > 0) > 0 \quad \textit{iff} \quad \mathbb{E}[(X_{1,0}\log^+ X_{1,0})/\mu_{\mathcal{U}_0}] < \infty.$$

(b) If $\mathbb{E}\log^+ \xi_0 = \infty$*, then* $\limsup_{n\to\infty} c^{-n} Z_n = \infty$ $\mathbb{P}_x$*-a.s. for every* $x \in \mathbb{N}_0$ *and* $c \in (0, \infty)$*.*

Lemma A.3 (Gröttrup [27], Remark 3.10)**.** *Set* $\mathcal{F}_0 := \sigma((\xi_n)_{n\geq 0}, \mathcal{U})$*. Let* $x \in \mathbb{N}_0$ *and assume that there exists a constant* $c > 1$ *such that* $\mathbb{E}[\log \mu_{\mathcal{U}_n}] < \log c$*. Then*

$$\limsup_{n\to\infty} \left(\frac{1}{c^n} \mathbb{E}_x[Z_n | \mathcal{F}_0] \right) = 0 \quad \mathbb{P}_x\text{-}a.s.$$

A.3. Definitions and results from probability theory

A.3.1. The size-biased distribution

Definition A.4 (Alsmeyer [4], Definition 3.1)**.** Let ν be a distribution on the space $([0,\infty), \mathfrak{B}_{[0,\infty)})$ with finite and positive mean $\gamma = \int_{[0,\infty)} x\nu(dx)$. Then $\hat{\nu}$, given by

$$\hat{\nu}(B) = \frac{1}{\gamma} \int_B x\nu(dx), \quad B \in \mathfrak{B}_{[0,\infty)},$$

is called the *size-biased distribution on* $[0, \infty)$ *corresponding to* ν.

A.3.2. Radon-Nikodym derivatives

Let μ be a finite measure and let ν be a probability measure on $(\Omega, \mathcal{F})$. Furthermore, let $\mathcal{F}_n \uparrow \mathcal{F}$ be σ-fields (i.e., $\sigma(\cup \mathcal{F}_n) = \mathcal{F}$) and let μ_n as well as ν_n be the restrictions of μ and ν to $\mathcal{F}_n$.

Theorem A.5 (Durrett [20], Theorem 5.3.3)**.** *Suppose* $\mu_n \ll \nu_n$ *for all* n*. Let* $X_n = \mathrm{d}\mu_n/\mathrm{d}\nu_n$ *and let* $X = \limsup_{n\to\infty} X_n$*. Then*

$$\mu(A) = \int_A X \,\mathrm{d}\nu + \mu(A \cap \{X = \infty\}).$$

A.3.3. Sums of i.i.d. random variables

Theorem A.6 (Alsmeyer [5], Proposition 7.11)**.** *For an independent sequence* $(X_n)_{n\geq 1}$ *of identical distributed random variables, the corresponding sequence of partial sums* $(S_n)_{n\geq 0}$ *always satisfies one of the following four alternatives:*

(1) $S_n = 0$ $\mathbb{P}$-*a.s. for all* $n \geq 1$.

(2) $\lim_{n\to\infty} S_n = \infty$ $\mathbb{P}$-*a.s.*

(3) $\lim_{n\to\infty} S_n = -\infty$ $\mathbb{P}$-*a.s.*

(4) $\liminf_{n\to\infty} S_n = -\infty$ *and* $\limsup_{n\to\infty} S_n = \infty$ $\mathbb{P}$-*a.s.*

Theorem A.7 (Alsmeyer [5], Proposition 7.12)**.** *If, in addition, we assume that* $\mu := \mathbb{E}X_1$ *exists in the previous theorem, we have*

$$(1) \;\Leftrightarrow\; X_1 = 0 \ \mathbb{P}\text{-a.s.}; \qquad (2) \;\Leftrightarrow\; \mu \in (0,\infty];$$

$$(3) \;\Leftrightarrow\; \mu \in [-\infty, 0); \qquad (4) \;\Leftrightarrow\; \mu = 0 \text{ and } \mathbb{P}(X_1 = 0) < 1.$$

A.4. A lemma for the proof of Theorem 4.1 (b)

Lemma A.8. *Let* $\mathbb{P}(\mathit{Surv}) > 0$ *and* $p_0 > 0$*. Then*

$$\mathbb{P}\left(\mathbb{E}\left[X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)}\middle|\,\Xi\right] = \nu(\mathbf{e}_0)\right) < 1.$$

Proof. If $\mathbb{P}(\mathit{Surv}) > 0$ and $p_0 > 0$, then $p_2 > p_0$. We get

$$\begin{aligned}
\mathbb{P}\left(\mathbb{E}\left[X_{1,\emptyset}^{(\hat{U}_0,\hat{T}_0)}\middle|\,\Xi\right] = \nu(\mathbf{e}_0)\right) &= \mathbb{P}(\hat{T}_0 = 1, \mathbb{E}[X_{1,\emptyset}^{(0,1)}|\mathbf{e}] = (p_1+p_2)\mathbb{E}[X_{1,\emptyset}^{(0,1)}|\mathbf{e}]) \\
&\quad + \mathbb{P}(\hat{T}_0 = 2, \hat{U}_0 = 0, \mathbb{E}[X_{1,\emptyset}^{(0)}|\mathbf{e}] = \nu(\mathbf{e}_0)) \\
&\quad + \mathbb{P}(\hat{T}_0 = 2, \hat{U}_0 = 1, \mathbb{E}[X_{1,\emptyset}^{(1)}|\mathbf{e}] = \nu(\mathbf{e}_0)) \\
&= p_2\left(\mathbb{E}\left[\mathbb{1}_{\left\{\mathbb{E}[X_{1,\emptyset}^{(0)}|\mathbf{e}]=\nu(\mathbf{e}_0)\right\}}\frac{\mathbb{E}[X_{1,\emptyset}^{(0)}|\mathbf{e}]}{\nu(\mathbf{e}_0)}\right]\right. \\
&\quad \left. + \mathbb{E}\left[\mathbb{1}_{\left\{\mathbb{E}[X_{1,\emptyset}^{(1)}|\mathbf{e}]=\nu(\mathbf{e}_0)\right\}}\frac{\mathbb{E}[X_{1,\emptyset}^{(1)}|\mathbf{e}]}{\nu(\mathbf{e}_0)}\right]\right) \\
&= p_2\left(\mathbb{P}\left(\mathbb{E}[X_{1,\emptyset}^{(0)}|\mathbf{e}] = \frac{p_1+p_2}{p_0}\mathbb{E}[X_{1,\emptyset}^{(1)}|\mathbf{e}]\right)\right. \\
&\quad \left. + \mathbb{P}\left(\mathbb{E}[X_{1,\emptyset}^{(0)}|\mathbf{e}] = \frac{p_0}{p_1+p_2}\mathbb{E}[X_{1,\emptyset}^{(1)}|\mathbf{e}]\right)\right) \\
&< 1,
\end{aligned}$$

where we used Lemma 3.15 for the second equality. □

A.5. The linear fractional distribution

This section is meant as a short introduction to the linear fractional distribution. In the following, we give two different representations of the corresponding generating function f. The second one is used in Chapter 5.

The first representation is based on two parameters $b, p \in (0,1)$ satisfying $b + p \leq 1$. Let

$$p_k = bp^{k-1}, k \in \mathbb{N}, \quad p_0 = 1 - \sum_{k \geq 1} p_k = \frac{1-p-b}{1-p}.$$

The generating function of the distribution $(p_k)_{k \geq 0}$ is given by

$$f(s) = 1 - \frac{b}{1-p} + \frac{bs}{1-ps}, \quad s \in [0,1]. \tag{A.1}$$

We then have

$$f'(s) = \frac{b}{(1-ps)^2}$$

and thus

$$\mathsf{m} = f'(1) = \frac{b}{(1-p)^2}.$$

Now we come to the second representation of the generating function of the linear fractional distribution. With this representation, it is very easy to determine all iterations $f_n = f \circ \ldots \circ f$, $n \geq 1$, of f. Let f be the generating function satisfying

$$\frac{1}{1-f(s)} = \frac{\alpha}{1-s} + \beta, \quad s \in [0,1), \tag{A.2}$$

where $\alpha, \beta \in \mathbb{R}_>$ with $\alpha + \beta \geq 1$. In particular

$$f(s) = 1 - \frac{1-s}{\alpha + \beta(1-s)}$$

and thus

$$f'(s) = \frac{\alpha}{(\alpha + \beta(1-s))^2} \quad \text{and} \quad f'(1) = \frac{1}{\alpha}.$$

Define the bijection $\varphi : [0,1) \to [1,\infty)$, $\varphi(s) = (1-s)^{-1}$, and the function $g : \mathbb{R} \to \mathbb{R}$, $g(s) = \alpha s + \beta$, having n-th iteration $g_n(s) = \alpha^n s + \beta(\alpha^{n-1} + \ldots + \alpha + 1)$. Then we can restate (A.2) as

$$\varphi \circ f(s) = g \circ \varphi(s)$$

or, equivalently,

$$f(s) = \varphi^{-1} \circ g \circ \varphi(s)$$

for $s \in [0,1)$. Thus, we get for the n-th iteration of f

$$f_n(s) = \varphi^{-1} \circ g_n \circ \varphi(s), \quad s \in [0,1).$$

This gives us

$$\frac{1}{1-f_n(s)} = \frac{\alpha^n}{1-s} + \beta(\alpha^{n-1} + \ldots + \alpha + 1),$$

i.e. f_n is the generating function of a linear fractional distribution with parameters α^n and $\beta(\alpha^{n-1} + \ldots + \alpha + 1)$.

To conclude this short section, we give the connection between the two representations:

- (A.1)$\rightsquigarrow$(A.2) with $\alpha, \beta \in \mathbb{R}_>$, $\alpha + \beta \geq 1$, by choosing $p = \frac{\beta}{\alpha+\beta}$ and $b = \frac{\alpha}{(\alpha+\beta)^2}$.
- (A.2)$\rightsquigarrow$(A.1) with $b, p \in (0,1)$, $b + p \leq 1$, by choosing $\alpha = \frac{(1-p)^2}{b}$ and $\beta = \frac{p(1-p)}{b}$.

In particular,

$$p_0 = 1 - \frac{1}{\alpha+\beta} \quad \text{and} \quad p_k = \frac{\alpha}{(\alpha+\beta)^2}\left(\frac{\beta}{\alpha+\beta}\right)^{k-1}, k \in \mathbb{N}.$$

Acronyms

BPRE	branching process in random environment
BPREI	branching process in random environment with immigration
BwBPRE	branching within branching process in random environment
CLT	Central limit theorem
GWP	Galton-Watson process
SLLN	strong law of large numbers

List of symbols

$\mathbb{V}$	infinite binary Ulam-Harris tree
$\emptyset$	root cell of $\mathbb{V}$
T_v	number of daughter cells of cell v
$(p_k)_{k\geq 0}$	cell reproduction
μ	reproduction mean of the cells
$\mathbb{T}_n$	set of cells in generation n
$\mathcal{T}_n$	number of living cells in generation n
$\mathbb{T}$	cell tree
A_v	status (alive, dead) of a cell
$\mathbb{T}_n^*$	set of infected cells in generation n
$\mathcal{T}_n^*$	number of infected cells in generation n
$\mathbb{M}(\mathbb{N}_0^2), \mathfrak{M}$	space of probability measures on $\mathbb{N}_0^2$ and corresponding σ-field
$[\mathbf{e}]_m$	$:= (\mathbf{e}_n)_{n\geq m}$
$(X_{k,v}^{(0)}, X_{k,v}^{(1)})$	offspring of the k-th parasite in cell v
$X_{k,v}^{(0,1)}$	$= X_{k,v}^{(0)} + X_{k,v}^{(1)}$
$X_{k,v}^{(0,2)}$	$= X_{k,v}^{(0)}$
$X_{k,v}^{(1,2)}$	$= X_{k,v}^{(1)}$
$X_{k,v}^{(i,j)}$	$= 0$ if $(i,j) \in \{(0,0),(1,0),(1,1)\}$
Z_v	number of parasites in cell v
$\mathbf{e}_n^{(i,j)}$	$= \mathbb{P}(X_{1,0^{*n}}^{(i,j)} \in \cdot \mid \mathbf{e})$

0^{*n}	$= 0\dots 0$ (n-times)
0^{*0}	$= \emptyset$
$\mathbb{P}_z, \mathbb{E}_z$	starting with z parasites in the root cell
$\mathbb{P}_{(t,z)}, \mathbb{E}_{(t,z)}$	starting with t cells hosting $z_1, \dots, z_t$ parasites, respectively, where $z = (z_1, \dots, z_t)$
$\mathbf{\mathsf{BP}}$	branching within branching process in random environment
$\mathbf{\mathsf{BP}}_n$	nth generation of the branching within branching process in random environment
$\mathbf{\mathsf{BT}}$	branching within branching tree in random environment
$\mathbf{\mathsf{BT}}_n$	branching within branching tree in random environment up to generation n
$\mathcal{Z}_n$	number of parasites in generation n
$\nu(\mathbf{e}_n)$	$= \mathbb{E}[\mathcal{Z}_n \vert \mathbf{e}]$
ν	$= \mathbb{E}[\nu(\mathbf{e}_0)]$
$\mathcal{F}$	canonical filtration not including the random environment
$\mathcal{G}$	canonical filtration including the random environment
S	$= \{(0,0)\} \cup (\{1\} \times \mathbb{N}_0)$
$(\mathbb{S}, \mathfrak{S})$	space of host-parasite trees
$\mathfrak{S}_n$	sub σ-field generated by the projections on the first n generations of the host-parasite trees
R	set of possible root configurations
V_n	spinal cell in generation n ('chosen along the cells')
ξ_n	$= (\check{T}_n, C_n, \mathbf{e}_n)$
f_{ξ_n}	$= \mathbb{E}[s^{X_{k,V_n}^{C_n,\check{T}_n}} \vert \xi_n]$
$F_n(k)$	number of cells in generation n containing exactly k parasites
$R^\uparrow$	set of ordered configurations of infected cells
$R_0^\uparrow$	$= \{(0,0)\} \cup R^\uparrow$
$\mathbf{\mathsf{BPG}}$	process denoting the number of infected cells and the number of parasites they contain
$\mathbf{\mathsf{BPG}}_n$	number of infected cells and the number of parasites they contain in generation n
$Ext/Surv$	set of extinction/survival of the parasites
W_n	$= \mathcal{Z}_n / \nu_n(\mathbf{e})$
$\hat{V}_n$	spinal cell in generation n ('chosen along the parasites')
Q_z	$= \mathbb{P}_z((\mathbf{\mathsf{BT}}, \mathbf{e}) \in \cdot)$
$\mathfrak{S}'_n$	$= \mathfrak{S}_n \otimes \mathfrak{M}^{\mathbb{N}_0}$
Ξ_n	$= (\hat{T}_n, \hat{U}_n, \mathbf{e}_n)$

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