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Thinning Operators and Π_4-Reflection

-2008-

Mathematische Logik und Grundlagenforschung

Thinning Operators and Π_4-Reflection

Inaugural-Dissertation
zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften
durch den Fachbereich Mathematik und Informatik
der Westfälischen Wilhelms-Universität Münster

vorgelegt von
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-2008-

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Tag der mündlichen Prüfung:	25.04.2008
Tag der Promotion:	

Contents

Introduction **iii**

1. Preliminaries **1**
1.1. Ordinal Proof Theory 1
1.2. Theories of Reflection 4
1.3. Methods 7

I. The Ordinal Notation System **9**

2. Collapsing Functions **11**
2.1. Basic Definitions 11
2.2. Thinning Hierarchies and Collapsing Functions 14

3. Structure Theory **21**
3.1. Structure Theory for $\mathcal{K}$ 21
3.2. Structure Theory for κ 25
3.3. Structure Theory for π 30
3.4. Comparison of Collapsing Functions 35
3.5. Checking Indescribability 40
3.6. Checking Stationarity 41

4. The Ordinal Notation System **45**

II. Collapsing **49**

5. Operator-controlled derivations **51**
5.1. The Language and Rules of $RS(\mathcal{K})$ 51
5.2. The infinitary calculus 55
5.3. Predicative Cut Elimination 60

6. Embeddings **63**
6.1. The intermediate proof system 63
6.2. Embedding into the main system 65

7. Preparations 67
7.1. The Operators $\mathcal{H}_\gamma$ . . . 67
7.2. Preparations for the First Collapsing Theorem . . . 68
7.3. Preparations for the Second Collapsing Theorem . . . 69
7.4. Preparations for the Third Collapsing Theorem . . . 72

8. Collapsing 77
8.1. The First Collapsing Theorem . . . 77
8.2. The Second Collapsing Theorem . . . 81
8.3. The Third Collapsing Theorem . . . 85

Appendix: An Alternative Collapsing Theorem 95

Introduction

This thesis belongs to the area of (ordinal) proof-theory, which is a subarea of mathematical logic. It deals with assigning certain ordinal numbers to axiom systems by analyzing formal proofs, thereby measuring the strength of these systems and gaining further insight into them. The hour of birth of this fascinating field was GENTZEN's analysis of PEANO-Arithmetic, **PA**, i.e. first order number theory. Since then, much stronger systems of first and second order number theory have been treated proof-theoretically.[1] All these theories also have counterparts in set theory, i.e. (very weak) subsystems of ZERMELO-FRAENKEL set theory that prove the same relevant sentences about the natural numbers. It turned out that such systems are technically much easier to handle.[2] This thesis deals with such a theory, $\mathbf{\Pi_4{-}Ref}$, which axiomatizes a universe that allows reflection of Π_4-formulas. It uses the elegant (new) technique of thinning hierarchies induced by thinning operators, that can implicitly be found in RATHJEN's analysis of Π_3-reflection, [Rat94b], and was in its general, explicit form taught to me by MÖLLERFELD. Although RATHJEN has successfully analyzed much stronger systems, this is the first detailed treatment of $\mathbf{\Pi_4{-}Ref}$. We think that the new technique we use gives some additional insight. We also think that our approach can easily be generalized to arbitrary $\mathbf{\Pi_n{-}Ref}$.

Ideas Recall that an ordinal α is called $\mathcal{F}$-reflecting on a set Y of ordinals $< \alpha$ iff for every formula $F \in \mathcal{F}$ (which may contain parameters) we have

$$L_\alpha \models F \Rightarrow (\exists \beta \in Y)\, L_\beta \models F.$$

In order to analyze a system **T** of set theory, one devises an infinitary calculus which is strong enough to derive the axioms of **T**. If **T** contains reflection axioms, one way to achieve this is to equip the calculus with an appropriate reflection rule. The crucial connection between reflection rules and thinning hierarchies is provided by the

Reflection Lemma. *Let $n \geq 2$, $\Delta \subseteq \Pi_n$ and*

$$\mathcal{A}^Y(X) = \{\alpha \mid \alpha \textit{ is } \Sigma_n\textit{-reflecting on } X \cap Y\}.$$

If

$$(\forall \alpha \in X)\, L_\alpha \models \bigvee\{\Delta, F\},$$

[1] Here we would like to mention the works of SCHÜTTE, FEFERMAN, BUCHHOLZ, POHLERS, RATHJEN, and others.

[2] The first step into this direction was taken by JÄGER.

then

$$(\forall \alpha \in \mathcal{A}^Y(X))\, L_\alpha \models \bigvee\{\Delta, (\exists \beta \in Y) L_\beta \models F\}.$$

If we regard X as a set of model candidates for Δ, F it shows how X has to be *thinned out* in order to get a set of model candidates for $\Delta, (\exists \beta \in Y) L_\beta \models F$. As α is Σ_{n+1}-reflecting on Y iff it is Π_n-reflecting on Y, this indicates how complicated (Π_n-reflecting) ordinals can be replaced by easier (Σ_n-reflecting) ordinals. Of course when we consider infinitary proofs, this process has to be iterated reasonably.

The Reflection Lemma leads to an elegant analysis of the system $\mathbf{\Pi_2}{-}\mathbf{Ref}$ of Π_2-reflection, which is proof-theoretically equivalent to the well-known theory $\mathbf{KP}\boldsymbol{\omega}$. Here, one only needs to compute the ordinal

$$|\mathbf{\Pi_2}{-}\mathbf{Ref}|_{\Sigma_1} = \mu\alpha.\, (\forall F \in \Sigma_1)(\mathbf{\Pi_2}{-}\mathbf{Ref} \vdash F \Rightarrow L_\alpha \models F),$$

because in this special case we have

$$\omega_1^{\mathrm{CK}} = |\mathbf{KP}\boldsymbol{\omega}|_\infty = |\mathbf{\Pi_2}{-}\mathbf{Ref}|_\infty = \mu\alpha.\, L_\alpha \models \mathbf{\Pi_2}{-}\mathbf{Ref}.$$

This fact makes things considerably easier. As indicated, the theory can be embedded into a semiformal system which contains a (Π_2-) reflection rule. Knowing that ρ is Π_1-reflecting on Y iff it is a limit point of Y, one defines

$$\mathcal{A}_\alpha = \{\rho \in \mathrm{Eps} \cap \omega_1^{\mathrm{CK}} \mid (\forall \alpha_0 <_\rho^* \alpha)(\rho \in \mathrm{Lim}(\mathcal{A}_{\alpha_0}))\},$$

where $\{\alpha_0 \mid \alpha_0 <_\rho^* \alpha\}$ is both large enough for proof-theoretical purposes and, as it contains only "simple" α_0, small enough so that $\mathcal{A}_\alpha$ stays nonempty. Following the pattern that an application of the critical reflection rule ($\Pi_2{-}\mathrm{Ref}$) in the semiformal system corresponds to an application of the thinning operator $\mathcal{A}$ (= Lim in the above definition),

$$(\Pi_2{-}\mathrm{Ref})\,\frac{\vdash \Delta, F}{\vdash \Delta, (\exists z)\, z \models F} \quad\cong\quad (\mathcal{A})\,\frac{X \models \bigvee\{\Delta, F\}}{\mathcal{A}(X) \models \bigvee\{\Delta, (\exists z)\, z \models F\}}$$

we get the translation from derivability in the semiformal system to truth in L by

$$\vdash^{\alpha}_{0} \Delta \subseteq \Pi_2 \Rightarrow \mathcal{A}_\alpha \models \bigvee \Delta.$$

Note that, like in the Reflection Lemma, here it is crucial that the complexity of the formulas in Δ does not exceed Π_2.

As the embedding and cut-elimination procedures only require "simple" ordinals, this results in

$$(\forall \alpha \in \mathcal{A}_{\varepsilon_{\omega_1^{\mathrm{CK}}+1}})(\forall F \in \Sigma_1)(\mathbf{\Pi_2}{-}\mathbf{Ref} \vdash F \Rightarrow L_\alpha \models F),$$

or more catchy

$$\mathcal{A}_{\varepsilon_{\omega_1^{\mathrm{CK}}+1}} \models \Sigma_1\text{-consequences}(\mathbf{\Pi_2}{-}\mathbf{Ref}).$$

In particular we get

$$|\mathbf{\Pi_2{-}Ref}|_{\Sigma_1} \leq \mu\mathcal{A}_{\varepsilon_{\omega_1^{\mathrm{CK}}+1}},$$

and indeed this bound can be proved to be sharp. (All this is worked out in [Duc01].)

The advantage of this approach is that it quite semantic — one gets a direct translation from derivability to models in L, and the connection between the elimination of the reflection-rule and the thinning process which consists of taking limits is apparent — and thus makes the ordinal analysis easily accessible.

It was the idea of this thesis to transfer as much of this technique as possible to stronger systems of reflection. As we will explain in a second, we consider $\mathbf{\Pi_4{-}Ref}$ the appropriate choice.

When turning to stronger theories $\mathbf{T}$, the ordinal $|\mathbf{T}|_{\Sigma_1}$ becomes much too large. The Spector-Gandy-Theorem yields the correct definition: One defines $\mathbf{T}$'s proof-theoretical ordinal as

$$|\mathbf{T}| = \mu\alpha.\,(\forall F \in \Sigma_1)(\mathbf{T} \vdash F^{L_{\omega_1^{\mathrm{CK}}}} \Rightarrow L_\alpha \models F)(= |\mathbf{T}|_{\Sigma_1^{\omega_1^{\mathrm{CK}}}}),$$

i.e. the least α such that in L_α all those Σ_1-sentences become true that $\mathbf{T}$ knows to hold at the smallest admissible.

Consequently, it no longer suffices to collapse a derivation below $|\mathbf{T}|_\infty$. We need to iterate the technique outlined above in order to collapse it below ω_1^{CK}.

It is known since Rathjen's [Rat94b] how, under suitable assumptions, to collapse derivations of $\Pi_3^{L_\mathcal{K}}$-sentences below $\mathcal{K}$, where $\mathcal{K}$ is the least Π_3-reflecting ordinal. As this technique is fundamental for the understanding of the whole collapsing procedure, we will sketch it here. One proves by induction on α that (under certain assumptions, $\Delta \subseteq \Pi_3^{L_\mathcal{K}}$ being one of them)

$$\vdash^{\alpha}_{\mathcal{K}+1} \Delta$$

implies

$$\vdash^{\Psi^\mathcal{K}(f(\alpha),\rho)}_{\Psi^\mathcal{K}(f(\alpha),\rho)} \Delta(L_\mathcal{K} \mapsto L_\rho),$$

for all $\rho \in \mathcal{A}^\mathcal{K}_{f(\alpha)}$, where f is some function, in $\Delta(L_\mathcal{K} \mapsto L_\rho)$ every occurrence of $L_\mathcal{K}$ is replaced by L_ρ, $\mathcal{A}^\mathcal{K}_{f(\alpha)}$ is a nonempty set of ordinals $< \mathcal{K}$ and $\Psi^\mathcal{K}(\cdot,\cdot)$ is a function that maps ordinals (in particular those $\geq \mathcal{K}$) below $\mathcal{K}$. The main idea of the proof is as follows: first endow elements of $\mathcal{A}^\mathcal{K}_{f(\alpha)}$ with certain hyper-Σ_3-reflection rules (i.e. rules that express that these ordinals are Σ_3-reflecting *on some set* $\mathcal{A}^\mathcal{K}_\xi$). If now the last inference was an application of the Π_3-reflection rule

$$(\Pi_3{-}\mathrm{Ref})\ \frac{\vdash^{\alpha_0}_{\mathcal{K}+1} \Delta', F}{\vdash^{\alpha}_{\mathcal{K}+1} \Delta', (\exists z \in L_\mathcal{K})\, z \models F}$$

we get by induction hypothesis ($\Delta', F \subseteq \Pi_3$ is still true)

$$\left|\frac{\Psi^{\mathcal{K}}(f(\alpha_0),\rho)}{\Psi^{\mathcal{K}}(f(\alpha_0),\rho)}\right. \Delta'(L_{\mathcal{K}} \mapsto L_\rho), F(L_{\mathcal{K}} \mapsto L_\rho)$$

for all $\rho \in \mathcal{A}^{\mathcal{K}}_{f(\alpha_0)}$. Now we fix $\bar\rho \in \mathcal{A}^{\mathcal{K}}_{f(\alpha)}$. Then $\mathcal{A}^{\mathcal{K}}_{f(\alpha_0)} \cap \bar\rho \neq \emptyset$, thus we obtain after some additional inferences, assuming without loss of generality that Δ' consists only of one formula G,

$$\left|\frac{\Psi^{\mathcal{K}}(f(\alpha_0),\bar\rho)'}{\Psi^{\mathcal{K}}(f(\alpha_0),\bar\rho)'}\right. (\forall\rho \in L_{\bar\rho})(\rho \in \mathcal{A}^{\mathcal{K}}_{f(\alpha_0)} \to G(L_{\mathcal{K}} \mapsto L_\rho)), (\exists z \in L_{\bar\rho}) z \models F. \tag{$*$}$$

As one of the above mentioned rules associated with $\bar\rho$ is

$$(\Sigma_3\text{−Ref}(\mathcal{A}^{\mathcal{K}}_{f(\alpha_0)}))\ \frac{\vdash \Delta', \neg G(L_{\mathcal{K}} \mapsto L_{\bar\rho})}{\vdash \Delta', (\exists\rho \in L_{\bar\rho})\,(\rho \in \mathcal{A}^{\mathcal{K}}_{f(\alpha_0)} \wedge \neg G(L_{\mathcal{K}} \mapsto L_\rho))}$$

we easily get

$$\left|\frac{\Psi^{\mathcal{K}}(f(\alpha_0),\bar\rho)'}{\Psi^{\mathcal{K}}(f(\alpha_0),\bar\rho)'}\right. G(L_{\mathcal{K}} \mapsto L_{\bar\rho}), (\exists\rho \in L_{\bar\rho})(\rho \in \mathcal{A}^{\mathcal{K}}_{f(\alpha_0)} \wedge \neg G(L_{\mathcal{K}} \mapsto L_\rho)) \tag{$**$}$$

by an application of this rule applied to a tautology. Thus we can (cut) ($*$) and ($**$) and are done as $\Psi^{\mathcal{K}}(f(\alpha_0),\bar\rho)' < \Psi^{\mathcal{K}}(f(\alpha),\bar\rho)$. We have seen that a single application of (Π_3-Ref) got replaced by a bunch of (hyper-) Σ_3-reflection rules. Note that the above is just a very formalized version of the proof of the Reflection Lemma (which can be found on page 14).

Observe also that many processes took place at the same time:

- Collapsing of the derivation lengths,
- collapsing of the complexity of the involved formulas ($L_{\mathcal{K}} \mapsto L_\rho$)
- elimination of the critical (reflection-) rule (causing the invention of simpler rules)
- partial cut-elimination.

The coaction of these four patterns will recur in this thesis. Thus when talking about "collapsing" or "cut-elimination", we mostly mean the just described interaction.

The new Σ_3-reflection rules can now (again under appropriate assumptions) be eliminated "for free", using a technique called *Local Predicativity*, which was invented by Pohlers. (In fact, this process is technically much more involved.)

Now let's turn to the theory $\mathbf{\Pi_4{-}Ref}$. If now $\mathcal{K}$ denotes the least Π_4-reflecting ordinal, we can collapse derivations of $\Pi_4^{L_{\mathcal{K}}}$-formulas below $\mathcal{K}$ following exactly the above pattern (with "3" replaced by "4"). But the new rules we have to introduce are of the form (Σ_4-Ref($\mathcal{A}^{\mathcal{K}}_\xi$)), hence they are at least as complicated as (Π_3−Ref)! Consequently, they cannot be removed for free, but we need to iterate the outlined technique instead.

Thus here for the first time such a hyper-reflection-rule has to be eliminated by the invention of new rules. But how to deal with this iteration is exactly the technical equipment one needs for an analysis of arbitrary $\boldsymbol{\Pi_n}$**−Ref**, hence we think that from a technical point of view, our analysis is the generic case.

Here is a sketch of how to define the thinning hierarchies in order to get the collapsing procedure going. We set

$$\mathcal{A}^{\mathcal{K}}_{\alpha} \approx \{\kappa < \mathcal{K} \mid (\forall \alpha_0 <^C_\kappa \alpha)(\kappa \text{ is } \Sigma_4\text{-reflecting on } \mathcal{A}^{\mathcal{K}}_{\alpha_0})\},$$

where again

$$\{\alpha_0 \mid \alpha_0 <^C_\kappa \alpha\}$$

is a reasonably small set containing only "simple" ordinals, but all those needed. Then we endow κ's such that κ is Π_3-reflecting on $\mathcal{A}^{\mathcal{K}}_{\xi}$ with a rule expressing that fact.

If κ is Π_3-reflecting we set

$$\mathcal{A}^{\kappa}_{\alpha} \approx \{\pi < \kappa \mid (\forall \alpha_0, \xi <^C_\pi \alpha)(\kappa\ \Pi_3\text{-reflecting on } \mathcal{A}^{\mathcal{K}}_{\xi} \to \pi\ \Sigma_3\text{-reflecting on } \mathcal{A}^{\mathcal{K}}_{\xi} \cap \mathcal{A}^{\kappa}_{\alpha_0})\}$$

and again endow π's that are Π_2-reflecting on $\mathcal{A}^{\mathcal{K}}_{\xi} \cap \mathcal{A}^{\kappa}_{\xi'}$ with an adequate rule. (Here we simplified matters a lot, cf. the discussion and definition in Section 2.2.) For the elimination of these last rules, however, we stick to the methods just described, i.e. we show under certain assumptions that

$$\vdash^{\alpha}_{\mu} \Delta \subseteq \Pi_2^{L_\pi}$$

implies

$$\vdash^{\Psi^\pi(f(\alpha,\mu),\rho)}_{\Psi^\pi(f(\alpha,\mu),\rho)} \Delta(L_\pi \mapsto L_\rho)$$

for all $\rho \in \mathcal{A}^{\pi}_{f(\alpha,\mu)}$, where we analogously define

$$\mathcal{A}^{\pi}_{\alpha} \approx \{\rho < \pi \mid (\forall \alpha_0, \xi, \xi' <^C_\rho \alpha)(\forall \kappa)$$
$$(\pi\ \Pi_2\text{-reflecting on } \mathcal{A}^{\mathcal{K}}_{\xi} \cap \mathcal{A}^{\kappa}_{\xi'} \to \rho \in \mathrm{Lim}(\mathcal{A}^{\mathcal{K}}_{\xi} \cap \mathcal{A}^{\kappa}_{\xi'} \cap \mathcal{A}^{\pi}_{\alpha_0})\}$$

for Π_2-reflecting ordinals π. But this time we can eliminate the critical rules just using limit arguments. They are available in view of the above definition. Here we seem to deviate from the approach using local predicativity. However, we think that both techniques are just two sides of the same coin, cf. the remark on page 88. We just think that our approach is smoother. It also highlights a more fundamental difference: as we define $\mathcal{A}^{\pi}_{\alpha}$ only using the thinning operator Lim, we define its elements "from the outside". The conventional approach would instead define them as least elements that cannot be captured using operations below α (we have to be very sloppy at this point; cf. the remark on page 19), which we consider a more syntactical view "from the inside".

Results The main result of this thesis is a characterization of an upper bound (we will not prove that this bound is sharp, although this is quite clear) of the proof-theoretical ordinal of $\mathbf{\Pi_4}\mathbf{-Ref}$,

$$|\mathbf{\Pi_4}\mathbf{-Ref}|_{\Sigma_1^{\omega_1^{\mathrm{CK}}}} \leq \Psi^{\omega_1}_{\varepsilon_{\mathcal{K}+1}},$$

using the technique of thinning operators. The most important intermediate results can be found in Chapter 3, where we show that the thinning hierarchies and the collapsing functions pertaining to them have all the necessary properties.

Organization This thesis is divided into two parts. The first culminates in the definition of an ordinal notation system, i.e. a recursive set of natural numbers (together with some additional recursive functions) denoting the ordinals we will use for the ordinal analysis. To this end we first define the thinning hierarchies and collapsing functions that will generate these ordinals. These are the most essential definitions of the whole thesis, and they can be found in Chapter 2.

In the following chapter all crucial properties of the thinning hierarchies are proved. The most important fact here is that under minimal assumptions they can be shown to be nonempty. The second half of that chapter shows that when we regard the involved ordinals as terms, all the necessary predicates on them (such as the $<$-relation) can be verified examining only sub"terms". This implies that the ordinal notation system, which we will define in Chapter 4, really is a *recursive* set of natural numbers.

In the second part we turn to the ordinal analysis of $\mathbf{\Pi_4}\mathbf{-Ref}$: we use the concept of operator-controlled derivations in order to define our semiformal calculus in Chapter 5. In the following chapter we indicate how theorems of $\mathbf{\Pi_4}\mathbf{-Ref}$ can be embedded into such a calculus. As we wanted to keep the proofs of the main theorems as concise as possible, we banned most of the side calculations into Chapter 7.

The final chapter deals with the proofs of the collapsing theorems. For the sake of clarity we split them up into three theorems. In the appendix we indicate how one can formulate and prove all these parts in just one theorem.

Acknowledgments It was Professor Wolfram Pohlers who raised my interest in mathematical logic and in particular in proof theory. He also supervised this thesis, and I thank him for his constant interest in the problems I encountered and many inspiring discussions.

Michael Möllerfeld acquainted me with the general idea of using (abstract) thinning operators in proof theory, and the conversations on the topic, which I enjoyed a lot, were a good starting point for my studies.

Finally, I want to thank all members, former members and students of the Institut für Mathematische Logik und Grundlagenforschung not only for many interesting talks about logic, but also for creating a very pleasant atmosphere.

1. Preliminaries

Throughout this thesis we assume familiarity with (first and second order) number theory, some recursion theory and particularly set theory. We will adopt some notational conventions without mentioning, but hope to define everything that is slightly non-standard.

In this first chapter we will try to explain what ordinal proof theory is (or should be), what system this thesis is about and finally introduce some methods in analyzing a system of set theory.

1.1. Ordinal Proof Theory

Ordinal Proof Theory can roughly be described as (the process of and the techniques for) assigning certain ordinals to systems of first or second order number theory or set theory, which are supposed to measure the "strength" of the system under investigation. Interest in these questions started with GENTZEN's consistency proof of first order number theory, **PA**, which apart from "finitistic" means (i.e. for example means available in primitive recursive arithmetic, **PRA**) only used the wellfoundedness of a well-ordering $\prec$ on the natural numbers, the order type of which was ε_0 (and so by GÖDEL's second incompleteness theorem, **PA** cannot prove $\prec$'s wellfoundedness). As on the other hand **PA** is strong enough to prove the wellfoundedness of all initial segments of $\prec$, ε_0 had been established as the "proof theoretical ordinal" of **PA**. Pursuing HILBERT's programme further, it has been the ultimate goal to achieve similar results for stronger systems, eventually culminating in an analysis of full second order number theory — although this seems to be a long way.

In fact, for systems $\mathbf{T}$ of (first or second order) number theory, there are various competing definitions of this measure called "proof theoretic ordinal", for example

$$|\mathbf{T}|_{\mathrm{Cons}} = \text{least } \alpha.\ \mathbf{PRA} + \text{PR-TI}(\alpha) \vdash \mathrm{Con}(\mathbf{T}),$$

where by PR-TI(α) we denote transfinite induction up to α for primitive recursive predicates (but note that this extremely vague definition only makes sense for "natural" theories and "natural" representations of ordinals, see [Rat99]),

$$|\mathbf{T}|_{\sup} = \sup\{\mathrm{otp}(\prec) \mid \prec \text{ is primitive recursive and } \mathbf{T} \vdash \mathrm{TI}(\prec)\}$$

and

$$|\mathbf{T}|_{\Pi^1_1} = \sup\{\mathrm{tc}(F(X)) \mid (\forall X)F(X) \text{ is a } \Pi^1_1\text{-sentence such that } \mathbf{T} \vdash (\forall X)F(X)\},$$

where

$$\mathrm{tc}(A) = \begin{cases} \min\{\alpha \mid \overset{\alpha}{\models} A\} & \text{if } \mathbb{N} \models A \\ \omega_1 & \text{otherwise.} \end{cases}$$

Here we assume that we have a TAIT-language with no free number variables but which may contain free second order variables (so we talk about so-called *pseudo*-Π^1_1-*sentences*). The verification calculus $\overset{\bullet}{\models}$ is then defined via the three clauses

(AxL) If $s^{\mathbb{N}} = t^{\mathbb{N}}$, then $\overset{\alpha}{\models} \Delta, s \notin X, t \in X$ for all α.

($\bigvee$) If F is of $\bigvee$-type and $\overset{\alpha_0}{\models} \Delta, G$ holds for some $G \in CS(F)$ with $\alpha_0 < \alpha$, then $\overset{\alpha}{\models} \Delta, F$.

($\bigwedge$) If F is of $\bigwedge$-type and $\overset{\alpha_G}{\models} \Delta, G$ with $\alpha_G < \alpha$ holds for all $G \in CS(F)$, then $\overset{\alpha}{\models} \Delta, F$,

where false atomic formulas, formulas of the form $F_0 \vee F_1$ and formulas of the form $(\exists x)F_0(x)$ are of $\bigvee$-type and their sets of characteristic subformulas, $CS(F)$, are the empty set, $\{F_0, F_1\}$ and $\{F_0(\underline{n}) \mid n \in \mathbb{N}\}$, respectively, and analogously if F is of $\bigvee$-type, then $\neg F$ is of $\bigwedge$-type and its set of characteristic subformulas, $CS(\neg F)$, is $\{\neg G \mid G \in CS(F)\}$.

In the following, let ω_1^{CK} be the least ordinal that is not the order type of a (primitive) recursive well-ordering on the natural numbers. (In the next section we will get acquainted will other aspects of this ordinal.)

One can verify (for example via SCHÜTTE's search trees), that indeed for Π^1_1-sentences $(\forall X)A(X)$

$$\mathbb{N} \models (\forall X)A(X) \Leftrightarrow (\exists \alpha < \omega_1^{\mathrm{CK}}) \overset{\alpha}{\models} A(X)$$

holds, so the above definition makes sense — at least for Π^1_1-sound theories. In fact, we also have

$$\sup\{\mathrm{tc}(A) \mid (\forall X)A(X) \in \Pi^1_1 \text{ and } \mathbb{N} \models (\forall X)A(X)\} = \omega_1^{\mathrm{CK}},$$

and if $\mathbf{T}$ is recursively enumerable (which is true for all systems we will encounter) and Π^1_1-sound, then the depth of the proof tree of some formula $(\forall X)A(X)$ that $\mathbf{T}$ derives, i.e. $\mathrm{tc}(A)$, can be computed effectively from A, so in that case we also get

$$|\mathbf{T}|_{\Pi^1_1} < \omega_1^{\mathrm{CK}}.$$

Fortunately $|\mathbf{T}|_{\sup}$ and $|\mathbf{T}|_{\Pi^1_1}$ yield the same ordinal if $\mathbf{T}$ is strong enough (for example if it extends $\mathbf{PA}$) and Π^1_1-sound, see [Poh98] for details and proofs of all these facts, and [Rat99] for an example of a strong, primitive recursive theory $\mathbf{T}'$ that is *not* Π^1_1-sound and satisfies $|\mathbf{T}'|_{\sup} = \omega_1^{\mathrm{CK}}$.

But even when $\mathbf{T}$ is Π^1_1-sound, one can object that $|\mathbf{T}|_{\sup}$ does not convey all the information a proof theoretical ordinal should. RATHJEN does so (again in [Rat99]) by presenting an example of theories $\mathbf{T}_1$ and $\mathbf{T}_2$ such that

$$|\mathbf{T}_1|_{\sup} = |\mathbf{T}_2|_{\sup},$$

but $\mathbf{T}_1$ is *proof-theoretically reducible to* $\mathbf{T}_2$ (which means that in a simple base theory every proof in $\mathbf{T}_1$ can recursively be translated into one in $\mathbf{T}_2$) and not vice versa. To remedy this defect, he instead proposes a definition of proof theoretic ordinal relative to a fixed ordinal notation system — indeed, all notations systems appearing in practice are comparable in strength — which seems to evade most problems, but, alas, it is only partially defined, even if the underlying notation system is strong enough. Nevertheless, we will work with $|\mathbf{T}|_{\sup}$ as official definition, being aware that there are cases when this ordinal may not be the appropriate choice and RATHJEN's approach probably is closer to an overall definition.

Formulas of (second order) number theory can be canonically regarded as formulas in the language of set theory by translating "$(\forall x)$" to "$(\forall x \in \omega)$" (and "$(\forall X)$" to "$(\forall x \subseteq \omega)$") and vice versa: we will sloppily call an $\mathcal{L}_\in$-formula of the form $(\forall x \subseteq \omega)A(x)$, where the only quantifiers occurring in A are of the form "$(\forall y \in \omega)$" or "$(\exists y \in \omega)$", a Π^1_1-formula of number theory. Usually systems of number theory correspond to systems of set theory inasmuch they prove (at least) the same Π^1_1-sentences of number theory. (The most prominent example for this fact are probably the theories $\mathbf{KP}\boldsymbol{\omega}$ (which we will soon turn to in detail) and $\mathbf{ID_1}$ (of non-iterated monotone inductive definitions), which are related in an even much stronger sense, see for example [Poh98] and [Tap99]. On the set-theoretic side, all strong systems that are of particular interest extend $\mathbf{KP}\boldsymbol{\omega}$.

As it turned out, for stronger theories it becomes technically easier to handle systems of set theory instead of number theory. The first step into this direction was taken by JÄGER in [Jäg86].

As by the hyperarithmetical quantifier theorem (or SPECTOR-GANDY theorem), Π^1_1-formulas of second order arithmetic correspond to Σ_1-formulas (of set theory) over ω_1^{CK}, for systems $\mathbf{T}$ of set theory, one defines the proof theoretical ordinal as

$$|\mathbf{T}|_{\Sigma_1^{\omega_1^{\mathrm{CK}}}} = \mu\alpha.\ L_\alpha \models \Sigma_1^{\omega_1^{\mathrm{CK}}}\text{-Cons}(\mathbf{T}),$$

where

$$\Sigma_1^{\omega_1^{\mathrm{CK}}}\text{-Cons}(\mathbf{T}) = \{F \in \Sigma_1 \mid \mathbf{T} \vdash \text{"}F^{L_{\omega_1^{\mathrm{CK}}}}\text{"}\}$$

and $\mathbf{T} \vdash \text{"}F^{L_{\omega_1^{\mathrm{CK}}}}\text{"}$ means that $L_{\omega_1^{\mathrm{CK}}}$ can be talked about in $\mathbf{T}$ (e.g. via an appropriate (admissibility-)predicate).

The connection is that for all sufficiently strong systems $\mathbf{T}$ of set theory (these certainly include all $\mathbf{\Pi}_{\boldsymbol{n}}\mathbf{-Ref}$ with $n \geq 3$) we have

$$|\mathbf{T}|_{\sup} = |\mathbf{T}|_{\Pi^1_1} = |\mathbf{T}|_{\Sigma_1^{\omega_1^{\mathrm{CK}}}},$$

so from now on the computation of this ordinal shall constitute our ordinal analysis.

Remark. For all these systems we also have

$$|\mathbf{T}|_{\Sigma_1^{\omega_1^{CK}}} = |\mathbf{T}|_{\Pi_2^{\omega_1^{CK}}},$$

see [Poh98] for details.

That said, we would like to stress that not much is gained by the mere knowledge that "α is the proof theoretical ordinal of $\mathbf{T}$", because

- α is usually defined via some collapsing function, the understanding of which ultimately involves grasping all the patterns used during the cut-elimination procedure, hence some (more or less deep) insight into the theory itself
- as we see it, the (hidden) beauty of proof theory lies in the subtle interaction between the definition of those collapsing functions, their properties (i.e. the structure theory) and the cut-elimination procedure
- in view of the above mentioned various definitions of "proof theoretical ordinal" (and their drawbacks), it does not convey all the information behind it.

1.2. Theories of Reflection

As this thesis deals with the theory $\mathbf{\Pi_4}-\mathbf{Ref}$ we will in this section introduce the systems of $\mathbf{\Pi_n}-\mathbf{Ref}$. They are based on KRIPKE-PLATEK set theory, $\mathbf{KP}\boldsymbol{\omega}$. In the following we assume some knowledge of set theory, set theoretic notations and conventions, as can be found in [Jec97].

The system of $\mathbf{KP}\boldsymbol{\omega}$ was introduced in the 1960's by S. KRIPKE and R. PLATEK. On the one hand, it is a quite weak subsystem of ZERMELO-FRAENKEL set theory, **ZFC**, and thus has many models, most prominently in the constructible universe, but on the other hand it is strong enough to admit some recursion theory (see for example Theorem 1.2.5). In fact, the origins of $\mathbf{KP}\boldsymbol{\omega}$ rather trace back to generalized recursion theory.

Definition. The axioms of $\mathbf{KP}\boldsymbol{\omega}$ are (the universal closures of):

(Ext) $u = v \leftrightarrow u \subseteq v \wedge v \subseteq u$

(Found) $(\forall x)\big((\forall y \in x)F(y) \to F(x)\big) \to (\forall x)F(x)$ for arbitrary F

(Pair) $(\exists z)(z = \{u, v\})$

(Union) $(\exists z)(z = \bigcup u)$

(ω) $(\exists z)\big((\exists y)(y \in z) \wedge (\forall y \in z)(y \cup \{y\} \in z)\big)$

(Δ_0-Sep) $(\exists z)(z = \{x \in u \mid F(x)\})$ for $F \in \Delta_0$

(Δ_0-Coll) $(\forall x \in u)(\exists y)F(x, y) \to (\exists z)(\forall x \in u)(\exists y \in z)F(x, y)$ for $F \in \Delta_0$

To get a feeling for the strength of this theory, we will list some theorems of $\mathbf{KP}\boldsymbol{\omega}$ without proof. For a deeper study of $\mathbf{KP}\boldsymbol{\omega}$ see [Bar75].

Theorem 1.2.1 (Σ-reflection)**.** *For every Σ-formula F we have*

$$F \leftrightarrow (\exists z)F^z.$$

Theorem 1.2.2 (Σ-collection)**.** *For every Σ-formula F we have*

$$(\forall x \in u)(\exists y)F(x, y) \to (\exists z)(\forall x \in u)(\exists y \in z)F(x, y).$$

Theorem 1.2.3 (Δ-separation)**.** *For every Σ-formula F and every Π-formula F' the following holds:*

$$(\forall x \in u)(F(x) \leftrightarrow F'(x)) \to (\exists z)(z = \{x \in u \mid F(x)\}).$$

Theorem 1.2.4 (Existence of Transitive Closure)**.** *There is a Σ function symbol (which therefore we can treat as a function symbol of our language)* TC*, such that for every x,* $\mathrm{TC}(x)$ *is the smallest transitive set that contains x as a subset.*

Theorem 1.2.5 (Σ-recursion)**.** *If G is an $n+2$-ary Σ function symbol, then we can define a new Σ function symbol F such that the following holds in* $\mathbf{KP}\boldsymbol{\omega}$ *(+ the defining axiom for F):*

$$F(x_1, \ldots, x_n, y) = G(x_1, \ldots, x_n, y, \{\langle z, F(x_1, \ldots, x_n, z)\rangle \mid z \in \mathrm{TC}(y)\}).$$

Remark. Definition by Σ-recursion is quite a powerful tool, as it allows us to define for example ordinal addition or the constructible sets.

Definition. Transitive models of $\mathbf{KP}\boldsymbol{\omega}$ are called *admissible*. An ordinal α is called *admissible* if $\alpha = o(M) = \sup\{\xi \mid \xi \in M\}$ for some admissible M.

Theorem 1.2.6. *An ordinal α is admissible iff L_α is admissible.*

Theorem 1.2.7. *If α is admissible, then it is closed under all α-recursive functions, i.e. functions with an Σ-graph on L_α. In fact, there are arbitrarily large closure points of such functions below α.*

Remarks. (i) Admissible ordinals are known from generalized recursion-theory, where they are called *recursively regular*, see [Hin78].

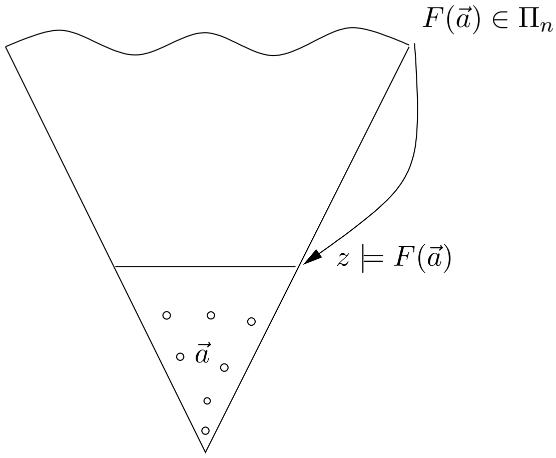
(ii) The least admissible ordinal is ω_1^{CK}.

Now we turn to strenghtenings of $\mathbf{KP}\boldsymbol{\omega}$. We first introduce the axiom scheme of (Π_n−Ref): it states that for every Π_n-formula (which may contain parameters) which is valid, there already exists a reflection point.

$$(\Pi_n\text{−Ref}) \quad F(\vec{a}) \to (\exists z)(z \models F(\vec{a})),$$

where we use $z \models F(\vec{a})$ as an abbreviation for

$$z \neq \emptyset \wedge \mathrm{trans}(z) \wedge \vec{a} \in z \wedge F(\vec{a})^z.$$



Definition. The theory $\boldsymbol{\Pi_n}$**−Ref** is **KP**$\boldsymbol{\omega}$ with (Δ_0-Coll) replaced by (Π_n−Ref). We call, in accord with the above convention, an ordinal α Π_n*-reflecting* if $L_\alpha \models \boldsymbol{\Pi_n}$**−Ref**.

Remark. There is also a recursion-theoretic characterization of the Π_n-reflecting ordinals involving non-monotone inductive definitions: for $n > 0$, the least Π_{n+1}-reflecting ordinal equals $|\Pi^0_n|$, where

$$|\Pi^0_n| = \sup\{|\Gamma| \mid \Gamma \text{ is a } \Pi^0_n\text{-definable operator}\}.$$

Here $|\Gamma|$ denotes the closure ordinal of Γ, i.e. the least ρ such that

$$\Gamma^\rho = \Gamma^{\rho+1},$$

and Γ^ξ is defined via

$$\Gamma^\xi = \bigcup_{\zeta<\xi} \Gamma^\zeta \cup \Gamma(\bigcup_{\zeta<\xi} \Gamma^\zeta).$$

Exact definitions and more details can be found in [RA74].

Now even $\boldsymbol{\Pi_2}$**−Ref** might seem considerably stronger than **KP**$\boldsymbol{\omega}$, because (Δ_0-Coll) easily follows from (Π_2−Ref). However, we have the following

Theorem 1.2.8. *In the constructible hierarchy, the models of* **KP**$\boldsymbol{\omega}$ *and* $\boldsymbol{\Pi_2}$**−Ref** *coincide.*

Thus, the two theories are proof-theoretically equivalent. So it first of all seems to be a natural question to ask for the strength of $\boldsymbol{\Pi_n}$**−Ref** with $n \geq 3$. But apart from that, an analysis of the theory $\boldsymbol{\Pi^1_2}$**−CA**, a subsystem of second order number theory based on comprehension for Π^1_2-formulas, which on the set theoretic side corresponds to the theory $\mathbf{KP}\boldsymbol{\omega} + \mathbf{\Sigma_1} - \mathbf{Sep}$, has for a long time been the ultimate goal in proof theory, and such an analysis necessarily involves the treatment of ordinals which are (much stronger than) Π_n-reflecting for all n (see for example the introduction of [Rat94b]). This can be seen as follows:

- Ordinals κ that satisfy $L_\kappa \models \mathbf{KP}\boldsymbol{\omega} + \mathbf{\Sigma_1} - \mathbf{Sep}$ are *nonprojectible* (see [Bar75] and [Hin78]).
- Nonprojectible ordinals κ are limits of Σ_1-elementary substructures, i.e. satisfy $(\forall \xi < \kappa)(\exists \zeta < \kappa) L_\xi \prec_1 L_\zeta$.
- Even if only $L_\xi \prec_1 L_{\xi+1}$, ξ is Π_n-reflecting for all n, because for any formula φ, if $L_\xi \models \varphi$, then $L_{\xi+1} \models$ "$(\exists z) z \models \varphi$", so $L_\xi \models$ "$(\exists z) z \models \varphi$".

Such an analysis of $\mathbf{\Pi^1_2} - \mathbf{CA}$ has — at least for the parameter-free case — recently been carried out by Rathjen, see [Rat05b] and [Rat05a].

The subsystems of second order arithmetic which correspond to $\mathbf{\Pi_n} - \mathbf{Ref}$ are located strictly between $\mathbf{\Delta^1_2} - \mathbf{CA} + \mathbf{BI}$ (a system based on comprehension for Δ^1_2-formulas plus the scheme of Bar Induction) and $\mathbf{\Pi^1_2} - \mathbf{CA}$, as the following theorem (see also [Rat94b], and for the notion of β-model for example [Sim99]) shows.

Theorem 1.2.9. *For $n \geq 3$, $\mathbf{\Pi_n} - \mathbf{Ref}$ proves the same Π^1_4-sentences of second order arithmetic as $\mathbf{\Delta^1_2} - \mathbf{CA} + \mathbf{BI}$ plus the scheme of β-model reflection for Π^1_{n+1}-formulas.*

The system $\mathbf{\Pi_3} - \mathbf{Ref}$ was analyzed by Rathjen in 1993 (see [Rat94b]), the analysis showing that it was much stronger than every reasonable theory based on (iterated) admissibility alone. His paper will also be the main reference for this thesis.

1.3. Methods

In this section we want to outline how to compute upper bounds for the proof-theoretical ordinal of some system $\mathbf{T}$ of set theory extending $\mathbf{KP}\boldsymbol{\omega}$. First we note that

$$|\mathbf{T}|_\infty = \min\{\alpha \mid L_\alpha \models \mathbf{T}\} \geq \omega_1^{\mathrm{CK}},$$

where $|\mathbf{T}|_\infty > \omega_1^{\mathrm{CK}}$ if $\mathbf{T} = \mathbf{\Pi_n} - \mathbf{Ref}$ and $n > 2$, although by Löwenheim-Skolem and the Condensation Lemma, we know that it is a countable ordinal. (Here we see that $\mathbf{KP}\boldsymbol{\omega}$ itself is some kind of exception, as computing $|\mathbf{KP}\boldsymbol{\omega}|_{\Sigma_1^{\omega_1^{\mathrm{CK}}}}$ is the same as computing

$$|\mathbf{KP}\boldsymbol{\omega}|_{\Sigma_1} = \mu\alpha.\, L_\alpha \models \Sigma_1\text{-Cons}(\mathbf{KP}\boldsymbol{\omega}),$$

because $|\mathbf{KP}\boldsymbol{\omega}|_\infty = \omega_1^{\mathrm{CK}}$, whereas for $n > 2$, $|\mathbf{\Pi_n} - \mathbf{Ref}|_{\Sigma_1}$ is, although easier to compute, still much larger than ω_1^{CK}.) If we have

$$\mathbf{T} \vdash F,$$

then there are axioms $Ax_1, \ldots, Ax_n$ of $\mathbf{T}$ such that

$$\vdash \neg Ax_1, \ldots, \neg Ax_n, F$$

holds in pure logic. The idea is now to employ an infinitary calculus that is strong enough to derive all axioms of $\mathbf{T}$ and to embed pure logic, so that we can link the above derivation with those of

$$\vdash Ax_i$$

by means of a (cut)-rule

$$\frac{\vdash \Gamma, C \quad \vdash \Gamma, \neg C}{\vdash \Gamma}$$

Unfortunately, the resulting derivation of F does not offer much information as in general both its derivation length and its cut cut rank are about as big as $|\mathbf{T}|_\infty$ itself. Up to now, however, we have not yet used the fact that we are only interested in very special formulas F, i.e. $\Sigma_1^{\omega_1^{\mathrm{CK}}}$-formulas. This fact allows us to devise a collapsing procedure, which consists of "pruning, grafting, and relabeling" the proof tree ([Rat94b]). In doing so one constantly talks about very special derivations only (cf. the pancake conditions in Chapter 8). This procedure "collapses" the derivation length (and the complexity of the formulas involved), ultimately even below ω_1^{CK}, thus it must necessarily be accompanied by the elimination of large (cut)s and complicated reflection rules. (In the end it will be crucial to have a cut- and reflection-free derivation, see Theorem 8.3.2.)

Part I.

The Ordinal Notation System

2. Collapsing Functions

We cannot but agree with RATHJEN when he writes in [Rat05b]:

> It makes little sense to present an ordinal notation system without giving some kind of semantic interpretation. For ordinal representation systems in impredicative proof theory it is essential to understand the so-called collapsing functions on which they are built.

So after introducing some basic concepts of ordinals and ordinal functions in the first section, we will present the most crucial definitions of this thesis, including those of the collapsing functions, in the second section. We think that this will also be the right time to explain the idea of using thinning hierarchies for the analysis of $\mathbf{\Pi_4}\text{−}\mathbf{Ref}$.

2.1. Basic Definitions

First we introduce some basic properties of and functions on the ordinals. (For further background see [Poh89].)

Ordinals γ which are *additively indecomposable*, i.e. satisfy

$$(\forall \alpha, \beta < \gamma)(\alpha + \beta < \gamma),$$

are precisely those of the form $\gamma = \omega^{\gamma_0}$. The class of additively indecomposable ordinals is commonly denoted by $\mathbb{H}$.

Every ordinal α can be uniquely written as

$$\alpha =_{CNF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$$

with $\alpha \geq \alpha_1 \geq \cdots \geq \alpha_n$. This is called the CANTOR-*normalform* of α. Based upon this representation, we define the natural sum of two ordinals,

$$\alpha \oplus \beta,$$

as follows: if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ are the respective CANTOR-normalforms, then

$$\alpha \oplus \beta = \omega^{\gamma_1} + \cdots \omega^{\gamma_{m+n}},$$

where $\langle \gamma_1, \ldots, \gamma_{m+n} \rangle$ is a permutation of $\langle \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \rangle$ such that $\gamma_1 \geq \cdots \geq \gamma_{m+n}$.

Ordinals ρ that satisfy

$$\omega^\rho = \rho$$

are called *epsilon numbers*. Their enumeration function is denoted by $\varepsilon.$.

Let φ be the VEBLEN-function, which is defined by recursion on its first argument as follows: $\varphi\alpha\cdot$ is the enumerating function of the class

$$\{\omega^\gamma \mid \gamma \in \mathrm{On} \wedge (\forall\xi < \alpha)\varphi\xi(\omega^\gamma) = \omega^\gamma\}.$$

This immediately implies:

Lemma 2.1.1. *φ has the following properties:*

(i) $\varphi 0\alpha = \omega^\alpha$

(ii) *$\varphi\alpha\beta$ is closed under $+$ and $\oplus$*

(iii) *If $\beta < \beta'$, then $\varphi\alpha\beta < \varphi\alpha\beta'$*

(iv) *If $\alpha < \alpha'$, then $\varphi\alpha(\varphi\alpha'\beta) = \varphi\alpha'\beta$.*

Ordinals ρ which satisfy

$$(\forall\alpha, \beta < \rho)\varphi\alpha\beta < \rho$$

are called *strongly critical*; by SC we denote the class of strongly critical ordinals. Note that every cardinal π is both the πth epsilon number and the πth strongly critical ordinal. We define the VEBLEN-*normalform* by

$$\gamma =_{VNF} \varphi\alpha\beta \Leftrightarrow \gamma = \varphi\alpha\beta \wedge \alpha, \beta < \gamma.$$

We define the *strongly critical parts* of an ordinal ρ, $SCP(\rho)$, as follows:

$$SCP(\rho) = \begin{cases} \emptyset & \text{if } \rho = 0 \\ \{\rho\} & \text{if } \rho \in \mathrm{SC} \\ SCP(\rho_1) \cup \cdots \cup SCP(\rho_n) & \text{if } n > 0, \rho =_{CNF} \rho_0 + \cdots + \rho_n \\ SCP(\rho_1) \cup SCP(\rho_2) & \text{if } \rho =_{VNF} \varphi\rho_1\rho_2 \end{cases}$$

If X is a class of ordinals, we define

$$X[\gamma] = \{\alpha \in X \mid \alpha > \gamma\}$$

and

$$\mu\, X = \text{the least } \alpha \in X.$$

As usual in proof theory, we will base the ordinal notation system on large cardinals rather than (large) admissible ordinals. The advantage of this approach becomes more obvious the more complicated the system under investigation gets: instead of toiling with horrible complexity considerations one can just use (more or less) simple cardinality arguments in order to prove the crucial facts about the collapsing functions. (This will be done in Chapter 3.) However, it should be noted that in principle large cardinals could be dispensed with. We will stick mostly to these cardinal notations (we will for example consider $\Sigma_1^{\omega_1}$-formulas, although the reader should bear in mind that we are still officially interested in $\Sigma_1^{\omega_1^{\mathrm{CK}}}$-formulas).

We will use certain cardinals called *indescribable*, so let's finally introduce them.

Definition. First we expand the language of set theory by second order free variables, X_i and allow for second order quantification. A formula of the resulting language is called Π_0^1, if no second order quantifier occurs in it. A formula is called Π_n^1, if it consists of n alternating blocks of second order quantifiers, starting with a universal quantifier, followed by a Π_0^1 matrix, i.e. if it has the form

$$(\forall X_1)\cdots(\forall X_{m_1})(\exists X_{m_1+1})\cdots(\exists X_{m_2})\cdots(\mathsf{Q}X_{m_{n-1}+1})\cdots(\mathsf{Q}X_{m_n})F(\vec{X}),$$

where F is Π_0^1 and Q is either $\exists$ (if n is even) or $\forall$ (if n is odd).

Definition. A cardinal κ is called Π_n^1*-indescribable*, if whenever $U_1,\dots,U_m \subseteq V_\kappa$ and F is a Π_n^1-sentence in the language of $(V_\kappa, \in, U_1, \dots, U_m)$ such that

$$(V_\kappa, \in, U_1, \dots, U_m) \models F,$$

then there is a $\rho < \kappa$ such that

$$(V_\rho, \in, U_1 \cap V_\rho, \dots, U_m \cap V_\rho) \models F.$$

κ is called Π_1^1*-indescribable on* Y, $\kappa \in \Pi_1^1[Y]$, if in the above situation, a $\rho \in \kappa \cap Y$ can always be found.

As we will later see, we will have to work inside of L instead of V. This does not collide with the use of Π_n^1-indescribable cardinals in view of the following theorem (proved by R. B. Jensen, lecture notes taken by G. Fuchs).

Theorem 2.1.2. *For all n, if κ is Π_n^1-indescribable, then (κ is Π_n^1indescribable)^L.*

There is a reason for using precisely these cardinals. Regular cardinals correspond to admissible ordinals in the following sense: as seen in Theorem 1.2.7, admissible α are closed under α-recursive functions. But regular cardinals π are just closed under *all* functions $f\colon \pi \to \pi$! (Here for example, the gap between all functions and those functions with Σ_1-graph leads to the above mentioned complexity considerations.) So in the case of $\mathbf{\Pi_4}\mathbf{-Ref}$ we get the correspondence

- ω_1 is closed under all functions $f\colon \omega_1 \to \omega_1$
- ω_1^{CK} is closed under all ω_1^{CK}-recursive functions

- $|\mathbf{\Pi_4}{-}\mathbf{Ref}|_{\Sigma_1^{\omega_1^{\mathrm{CK}}}}$ is closed under all ω_1^{CK}-recursive functions the recursivity of which can be proved in $\mathbf{\Pi_4}{-}\mathbf{Ref}$

See also [Poh96].

This analogy can be carried further (this is done in [RA74]) and results in the correspondence of Π_{n+3}-reflecting ordinals and Π^1_{n+1}-indescribable cardinals.

For the rest of the thesis we will stick to the following conventions:

- $\mathcal{K}$ denotes the least Π^1_2-indescribable cardinal,
- ordinals $\kappa, \kappa', \bar{\kappa}, \kappa_0$ etc. are always Π^1_1-indescribable cardinals,
- ordinals $\pi, \pi', \bar{\pi}, \pi_0$ etc. are always regular cardinals.

We might from time to time for explanatory reasons drop back to talking about recursive ordinals, when these conventions have to be translated accordingly.

2.2. Thinning Hierarchies and Collapsing Functions

We think that this the most convenient time for a digression into a general explanation of the relationship between thinning hierarchies and reflection rules, because the following definitions are rather obscure even if you know the ideas behind them.

The quite simple fact underlying the whole approach is the following Lemma, which can implicitly be found in Rathjen's papers, but was in this generality taught to me by Möllerfeld.

Lemma (Reflection Lemma)**.** *Let $\Delta \subseteq \Pi_n$, $X, Y \subseteq \mathrm{On}$ and*

$$\mathcal{A}^Y(X) = \{\rho \mid \rho \text{ is } \Sigma_n\text{-reflecting on } X \cap Y\}.$$

Then

$$X \models \Delta, F,$$

i.e. $(\forall \gamma \in X)(L_\gamma \models \bigvee\{\Delta, F\})$*, implies*

$$\mathcal{A}^Y(X) \models \Delta, (\exists \delta \in Y)(L_\delta \models F),$$

i.e. $(\forall \rho \in \mathcal{A}^Y(X))(L_\rho \models \bigvee\{\Delta, (\exists \delta \in Y)(L_\delta \models F)\})$.

Proof. Let $\rho \in \mathcal{A}^Y(X)$ and assume

$$L_\rho \models \bigwedge \neg\Delta.$$

As $\bigwedge \neg\Delta$ is equivalent to a Σ_n-formula, we find a $\gamma \in \rho \cap X \cap Y$ such that

$$L_\gamma \models \bigwedge \neg\Delta.$$

But $\gamma \in X$ implies

$$L_\gamma \models F,$$

which together with $\gamma \in Y$ yields

$$L_\rho \models (\exists \delta \in Y)(L_\delta \models F). \qquad \square$$

As we always consider reflection *with parameters*, the schemes of Σ_{n+1}-reflection and Π_n-reflection have the same strength. This at least indicates how to resolve the situation that π is Π_{n+1}-reflecting on Y, from now on $\pi \in \Pi_{n+1}[Y]$ for short: In the calculus, we will endow π with a scheme of rules of the form

$$(\mathfrak{R}^Y_{n+1})\ \frac{\Gamma, F}{\Gamma, (\exists \gamma \in \pi \cap Y)(L_\gamma \models F)} \text{ if } F \text{ is } \Pi_{n+1} \text{ on } L_\pi$$

(and sometimes refer to this fact by saying that π *carries* the rule $(\mathfrak{R}^Y_{n+1})$). (Here a formula F is called Π_m *on* L_π if it has the form

$$(\forall x_1 \in L_\pi)(\exists x_2 \in L_\pi) \cdots (\mathsf{Q} x_m \in L_\pi) F_0(\vec{a}, \vec{x}),$$

where all parameters $\vec{a}$ are elements of L_π; this implies that all quantifiers occurring in F_0 are bounded by some element of L_π.) Later it will become clear that we need only consider side formulas (Γ in this case) which are themselves Π_{n+1} on L_π, so if we define

$$\mathcal{A}^Y_\alpha = \{\rho < \pi \mid \rho \text{ is } \alpha\text{-hyper-}\Pi_n\text{-reflecting on } Y\},$$

we can eliminate all applications of $(\mathfrak{R}^Y_{n+1})$ following the pattern

$$(\mathfrak{R}^Y_{n+1})\ \frac{\vdash \Gamma, F}{\vdash \Gamma, (\exists \gamma \in \pi \cap Y)(L_\gamma \models F)} \quad \cong \quad (\mathcal{A}^Y)\ \frac{\mathcal{A}^Y_\alpha \models \Gamma, F}{\mathcal{A}^Y_{\alpha+1} \models \Gamma, (\exists \gamma \in \pi \cap Y)(L_\gamma \models F)}$$

("An application of a reflection rule corresponds to an application of the thinning operator.")

However, here we simplified things in many respects. First of all, this semantic idea has to be translated into a (syntactic) proof system — after all, we want to obtain a modified (and smaller) proof tree, and so we have to endow the ordinals produced according to the above pattern (here: $\mathcal{A}^Y_{\alpha+1}$) with the appropriate (simpler) reflection rules that make these modifications possible (here for example $(\mathfrak{R}^{\mathcal{A}^Y_\alpha}_n)$); secondly, usually π carries many reflection rules, i.e. rules corresponding to different Y's, some π may even be schizophrenic and carry (intrinsically incomparable) Π_m- and Π_n-reflection rules where $m \neq n$. This of course has to be taken care of in the definition of the thinning hierarchy pertaining to π. (So we are in fact defining hierarchies pertaining rather to π than to rules.)

Nevertheless, we think that the patterns we just described give some insight into our proof theoretic treatment of $\mathbf{\Pi_4}$$-$**Ref**, because their traces can quite easily be found in the definitions to follow (they offer an outline for the definition of the thinning hierarchies) and the cut-elimination procedure (one just formalizes the proof of the Reflection Lemma when eliminating a reflection rule).

These explanations lead us to the question in which aspects $\mathbf{\Pi_4}-\mathbf{Ref}$ differs from $\mathbf{\Pi_3}-\mathbf{Ref}$. First of all, we think that there is no fundamental difference — the pattern for eliminating reflection rules remains unchanged. On the other hand, we think that from a technical point of view, the theory $\mathbf{\Pi_4}-\mathbf{Ref}$ is closer to being the generic case for arbitrary $\mathbf{\Pi_n}-\mathbf{Ref}$, for here (for the first time) one has to *iterate* what RATHJEN calls *stationary collapsing* — after eliminating the $(\Pi_4-\mathrm{Ref})$-rule, one must get rid of the newly-introduced hyper-Π_3-reflection-rules, in turn causing the invention of hyper-Π_2-reflection-rules. Thus here we have to resolve rules which express that some κ is Π_3-reflecting on some class Y by inventing a bunch of new (and simpler) reflection rules, which necessarily are connected with Y. This inevitably leads to schizophrenic ordinals, which seemingly do not allow for an elimination of all rules they carry at the same time. (This is somewhat reminiscent of the analysis of $\mathbf{KP}\boldsymbol{i}$ in [Poh98], where the thinning hierarchy pertaining to the least $\mathbf{KP}\boldsymbol{i}$-model (which is both Π_2 and $\Pi_1[\Pi_2]$) consists of $\mathbf{KP}\boldsymbol{\ell}$-models (which still are $\Pi_1[\Pi_2]$).)

Let's try to make things clearer by presenting the easiest example: Let $\kappa \in \Pi_3[\Pi_3]$. Then the thinning hierarchy $(\mathcal{A}^\kappa_\xi)_\xi$ pertaining to κ has to contain, say at stage α, ordinals π which are strong enough to eliminate the rule expressing that $\kappa \in \Pi_3[\Pi_3]$, i.e. ordinals which are $\Pi_2[\Pi_3 \cap \mathcal{A}^\kappa_{\alpha_0}]$ for certain $\alpha_0 < \alpha$. But this again requires talking about ordinals $\in \Pi_3 \cap \mathcal{A}^\kappa_{\alpha_0}$, i.e. a mixing of hierarchies.

The above demonstrations suggest that it is reasonable to split the following definitions into three parts: a hierarchy for $\mathcal{K}$ (which is Π_4-reflecting), one for κ's, which carry at least one Π_3-reflection rule, and one for π's, that don't carry any such.

Notice that later on, we will not introduce any Π_1-reflection rules but eliminate Π_2-reflection rules by certain limit processes. That is why we need collapsing functions of the form $\Psi^{\cdot}_{\cdot}[\bullet]$.

For any ordinal $\rho < \mathcal{K}$, let

$$p(\rho) = \begin{cases} \text{the largest cardinal} < \rho & \text{if such exists,} \\ 0 & \text{otherwise.} \end{cases}$$

(Notice that $p(\rho)$ is not necessarily regular.) For convenience, for $\xi > \mathcal{K}$, we will in the following definition only allow application of the VEBLEN-function φ in the form $\varphi 0 \xi$. (This of course makes no big difference, but allows us to talk only about ordinals less than $\varepsilon_{\mathcal{K}+1}$ rather than ordinals less than $\mathcal{K}^\Gamma$, the least strongly critical ordinal above $\mathcal{K}$.)

By recursion on α we now simultaneously define the sets $C(\alpha, \beta)$ (the αth iterated Skolem-hull of β), the thinning hierarchies $\mathcal{A}^{\cdot}_{\cdot}$ and the collapsing functions $\Psi^{\cdot}_{\cdot}$ pertaining to them.

Definition. Let $\alpha < \varepsilon_{\mathcal{K}+1}$ and $\beta < \mathcal{K}$. $C(\alpha, \beta)$ is the closure of $\beta \cup \{0, \mathcal{K}\}$ under $+, \varphi, p$ and the following collapsing functions:

$$
\begin{aligned}
(\alpha_0, \gamma) &\mapsto \Psi^{\mathcal{K}}_{\alpha_0}[\gamma] \text{ if } \alpha_0 < \alpha \text{ and } \gamma < \mathcal{K} \\
(\kappa, \alpha_0, \xi, \gamma) &\mapsto \Psi^{\kappa}_{\alpha_0,\xi}[\gamma] \text{ if } \xi < \alpha_0 < \alpha,\ \gamma < \kappa \text{ and } \kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi}] \\
(\pi, \kappa, \alpha_0, \alpha_1, \xi, \gamma) &\mapsto \Psi^{\pi,\kappa}_{\alpha_0,\alpha_1,\xi}[\gamma] \text{ if } \xi, \alpha_1 < \alpha_0 < \alpha, (\xi < \alpha_1 \vee \xi = \alpha_1 = 0), \gamma < \pi, \\
&\qquad \kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi}] \text{ and } \mathcal{A}^{\kappa}_{\alpha_1} \cap \mathcal{A}^{\mathcal{K}}_{\xi} \text{ stationary in } \pi
\end{aligned}
$$

Note (see below), that these are partial functions, i.e. might be undefined for some arguments (more about that can be found in Chapter 3). In this case, no new point enters $C(\alpha, \beta)$.

In the following we will use

$$\gamma_0 <^C_\rho \gamma_1 \Leftrightarrow \gamma_0 \in C(\gamma_1, \rho) \cap \gamma_1$$

as an abbreviation.

Definition. Let

$$
\begin{aligned}
\mathcal{A}^{\mathcal{K}}_0 &= \mathrm{SC} \cap \mathcal{K} \\
\mathcal{A}^{\mathcal{K}}_1 &= \mathrm{Reg} \cap \mathcal{K}
\end{aligned}
$$

and for $\alpha > 1$

$$\mathcal{A}^{\mathcal{K}}_\alpha = \big\{\kappa < \mathcal{K} \mid \alpha \in C(\alpha, \kappa) \wedge (\forall \alpha_0 <^C_\kappa \alpha)(\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\alpha_0}])\big\}.$$

The collapsing function $\Psi^{\mathcal{K}}_{\cdot}$ is defined by

$$\Psi^{\mathcal{K}}_\alpha[\gamma] \simeq \mu\, \mathcal{A}^{\mathcal{K}}_\alpha[\gamma].$$

Remarks. (i) The definition of $\mathcal{A}^{\mathcal{K}}_1$ is a little unexpected, but we will need to denote the next regular cardinal above γ anyway, and it really makes no difference that the thinning process only starts with $\mathcal{A}^{\mathcal{K}}_2$.

(ii) $\mathcal{A}^{\mathcal{K}}_2$ consists of the Π^1_1-indescribable cardinals, $\mathcal{A}^{\mathcal{K}}_3$ of those Π^1_1-indescribable on the Π^1_1-indescribables and so on. But $\mathcal{A}^{\mathcal{K}}_{\mathcal{K}}$ is already quite thin — it is contained in the diagonal intersection of the $\mathcal{A}^{\mathcal{K}}_\alpha$ with $\alpha < \mathcal{K}$.

Definition. Let

$$\mathcal{A}^{\kappa}_0 = \mathrm{SC} \cap \kappa$$

and for $\alpha > 0$

$$\begin{aligned}
\mathcal{A}^\kappa_\alpha = \big\{\pi < \kappa \mid\ & \alpha, \kappa \in C(\alpha,\pi) \wedge \\
& (\forall \pi' \in C(\alpha,\pi) \cap \kappa)(\pi' < \pi) \wedge \\
& (\forall \alpha_0, \xi <^C_\pi \alpha)\big(\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\xi] \wedge \kappa, \alpha_0 \in C(\alpha_0,\kappa) \rightarrow \\
& \qquad \mathcal{A}^\kappa_{\alpha_0} \cap \mathcal{A}^{\mathcal{K}}_\xi \text{ stationary in } \pi\big) \wedge \\
& (\forall \alpha_0', \xi' <^C_\pi \alpha)(\forall \kappa' \in C(\alpha,\pi))\big(\kappa' \geq \kappa \rightarrow \\
& \qquad (\mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \text{ stationary in } \kappa \rightarrow \mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \text{ stationary in } \pi) \wedge \\
& \qquad (\mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \text{ stationary in } \pi \rightarrow^* \mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \text{ stationary in } \kappa)\big)\big\},
\end{aligned}$$

where the statement

"X stationary in $\pi \rightarrow^* X$ stationary in κ"

means the following: if X is not stationary in κ, then the ($<_L$-) least club $C \subseteq \kappa$ such that $X \cap C = \emptyset$ is also club in π (and hence a witness for the non-stationarity of X in π).

$\Psi^\kappa_\cdot$ is defined via

$$\Psi^\kappa_{\alpha,\xi}[\gamma] \simeq \mu\,(\mathcal{A}^\kappa_\alpha \cap \mathcal{A}^{\mathcal{K}}_\xi[\gamma]).$$

Remarks. (i) This is exactly the point where $V = L$ comes into play. Notice that the only point in including the condition

"X stationary in $\pi \rightarrow^* X$ stationary in κ"

is to make Section 3.6 work. As Lemma 3.2.2 is crucial, we need to fix *one* club in the above definition. And then $V = L$ helps in showing that $\mathcal{A}^\kappa_\alpha$ is nonempty (Lemma 3.2.4).

(ii) Consequently, for the cut elimination only the conditions

"$(\forall \alpha_0, \xi <^C_\pi \alpha)\big(\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\xi] \wedge \kappa, \alpha_0 \in C(\alpha_0,\kappa) \rightarrow \mathcal{A}^\kappa_{\alpha_0} \cap \mathcal{A}^{\mathcal{K}}_\xi \text{ stationary in } \pi\big)$"

and

"$(\forall \alpha_0', \xi' <^C_\pi \alpha)(\forall \kappa' \in C(\alpha,\pi))$
$(\kappa' \geq \kappa \wedge \mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \text{ stationary in } \kappa \rightarrow \mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \text{ stationary in } \pi)$"

are relevant — they are needed to deal with the rules κ carries. The Π_3-reflection rules are eliminated while the Π_2-reflection rules are maintained.

Definition. Let $\pi \notin \Pi^1_1[\mathrm{On}]$. Define

$$\mathcal{A}^\pi_0 = \mathrm{SC} \cap \pi$$

and for $\alpha > 0$

$$
\begin{aligned}
\mathcal{A}^{\pi}_{\alpha} = \big\{\rho \in \mathrm{SC} \cap (p(\pi), \pi) \mid\ & \alpha, \pi \in C(\alpha, \rho) \wedge \\
& (\forall \pi' \in C(\alpha,\rho) \cap \pi)(\pi' < \rho) \wedge \\
& (\forall \alpha_0, \alpha_0', \xi' <^{C}_{\rho} \alpha)(\forall \kappa' \in C(\alpha,\rho))\big[\kappa' \geq \pi \rightarrow \\
& \quad \big(\kappa' \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}] \wedge \mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \text{ stationary in } \pi \wedge \\
& \quad \alpha_0, \pi \in C(\alpha_0, \pi) \rightarrow \rho \in \mathrm{Lim}(\mathcal{A}^{\pi}_{\alpha_0} \cap \mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'})\big) \wedge \\
& \quad \big(\kappa' \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}] \wedge \mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \text{ stationary in } \rho \rightarrow^{*} \\
& \quad \mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \text{ stationary in } \pi\big)\big]\big\}
\end{aligned}
$$

Unsurprisingly, $\Psi^{\pi}_{\cdot}$ is defined by

$$\Psi^{\pi,\kappa}_{\alpha,\alpha',\xi}[\gamma] \simeq \mu\,(\mathcal{A}^{\pi}_{\alpha} \cap \mathcal{A}^{\kappa}_{\alpha'} \cap \mathcal{A}^{\mathcal{K}}_{\xi}[\gamma]).$$

Remarks. (i) In the definition of $\mathcal{A}^{\pi}_{\alpha}$, the condition "$\rho \in (p(\pi), \pi)$" ensures that the thinning hierarchy pertaining to π stays as close to π as possible. Notice that in general it is not covered by "$(\forall \pi' \in C(\alpha,\rho) \cap \pi)(\pi' < \rho)$", because $p(\pi)$ may be singular!

(ii) Note that here the $\rightarrow^{*}$-part might become quite meaningless: if ρ is not regular, then in general being club in ρ just means being unbounded in ρ. But that is no problem as we will need this condition only in case ρ is regular (see Section 3.6).

(iii) Here, too, this part is only necessary to avoid unwanted exceptions, i.e. sets being stationary in ρ without being so in π. The cut elimination process again only uses that

$$
\begin{aligned}
"&(\forall \alpha_0, \alpha_0', \xi' <^{C}_{\rho} \alpha)(\forall \kappa' \in C(\alpha,\rho))\big[\kappa' \geq \pi \wedge \kappa' \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}] \wedge \\
&\mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \text{ stationary in } \pi \wedge \alpha_0, \pi \in C(\alpha_0, \pi) \rightarrow \rho \in \mathrm{Lim}(\mathcal{A}^{\pi}_{\alpha_0} \cap \mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'})\big]",
\end{aligned}
$$

enabling us to eliminate Π_2-reflection rules by means of limit processes.

Remark. We consider the definitions of the thinning hierarchies, at least those for π's, more "semantic" than the usual definitions, where one would define

$$\mathcal{A}^{\pi}_{\alpha} \approx \{\rho \in \mathrm{SC} \mid \alpha, \pi \in C(\alpha,\rho) \wedge C(\alpha,\rho) \cap \pi = \rho\}.$$

Thus its elements get their strength by the clause

$$C(\alpha,\rho) \cap \pi = \rho,$$

which can be seen as a rather syntactical property "from the inside": forgetting about the normal form condition for a second, you just take the least ordinal you could not name so far as Ψ^{π}_{α}. In contrast, our approach takes — again disregarding the normal form condition — "from the outside" (although in its definition the set $C(\alpha,\rho)$ still figures prominently) the first ordinal that satisfies the respective limit-conditions.

Remark. $C(\alpha, \beta)$ is in fact an inductive definition, which is completed after ω steps. So its stages look like

$$\begin{aligned} C_0(\alpha, \beta) &= \beta \cup \{0, \mathcal{K}\} \text{ and} \\ C_{n+1}(\alpha, \beta) &= C_n(\alpha, \beta) \cup \\ &\quad \{\xi \mid \xi \text{ is constructed with the above functions restricted to } C_n(\alpha, \beta)\} \end{aligned}$$

Later on we will frequently talk about the *maximally simple* element ξ of $C(\alpha, \beta)$ with property P. By this we mean that ξ is least such that

- $\xi \in C_n(\alpha, \beta)$ and ξ has the property P and
- for all $m < n$, no element of $C_m(\alpha, \beta)$ has the property P.

We can state the first (trivial) consequences of these definitions:

Lemma 2.2.1. (i) $\alpha \leq \alpha' \wedge \beta \leq \beta' \Rightarrow C(\alpha, \beta) \subseteq C(\alpha', \beta')$

(ii) $\overline{\overline{C(\alpha, \beta)}} = \max\{\overline{\overline{\beta}}, \aleph_0\}$

(iii) $\lambda \in \mathrm{Lim} \Rightarrow C(\lambda, \beta) = \bigcup_{\alpha<\lambda} C(\alpha, \beta) \wedge C(\alpha, \lambda) = \bigcup_{\beta<\lambda} C(\alpha, \beta)$

(iv) $\rho \in C(\alpha, \beta) \Leftrightarrow \mathit{SCP}(\rho) \subseteq C(\alpha, \beta)$

(v) $C(\alpha, \beta)$ *is closed under* $\oplus$

Proof. As (i) to (iii) are easy and (v) follows from (iv), we will only show (iv). So assume that η is maximally simple such that $\eta \in C(\alpha, \beta)$ and $\mathit{SCP}(\eta) \not\subseteq C(\alpha, \beta)$. If $\eta = \eta_1 + \eta_2$, then by hypothesis $\mathit{SCP}(\eta_1) \cup \mathit{SCP}(\eta_2) \subseteq C(\alpha, \beta)$, but on the other hand $\mathit{SCP}(\eta) \subseteq \mathit{SCP}(\eta_1) \cup \mathit{SCP}(\eta_2)$, a contradiction. If $\eta = \varphi\eta_1\eta_2$, then it is either in normal form (and we get a contradiction again) or it is strongly critical, so that there is nothing to be shown. ☐

Let's introduce some notational simplifications we will use later:

- $C^{\mathcal{K}}(\alpha) = C(\alpha, \Psi^{\mathcal{K}}_{\alpha})$
- $C^{\kappa}_{\xi}(\alpha) = C(\alpha, \Psi^{\kappa}_{\alpha,\xi})$
- $C^{\pi,\kappa}_{\xi,\xi'}(\alpha) = C(\alpha, \Psi^{\pi,\kappa}_{\alpha,\xi,\xi'})$

Notice that $C^{\pi,\kappa}_{0,0}(\alpha) = C^{\pi,\kappa'}_{0,0}(\alpha)$ for arbitrary $\kappa, \kappa' > \pi$; so as a further simplification, we may write

- $C^{\pi}(\alpha)$

instead of $C^{\pi}_{0}(\alpha)$ (if π is Π^1_1-indescribable) or $C^{\pi,\kappa}_{0,0}(\alpha)$ for arbitrary $\kappa > \pi$ (if π is not Π^1_1-indescribable), respectively.

3. Structure Theory

This is the most important (and also the longest) chapter of the first part — all relevant properties of the thinning hierarchies and the collapsing functions pertaining to them will be proved. Note that here we really make use of large cardinal notions, which makes life much easier — those who doubt this might for deterrence want to take a look at [Sch93].

3.1. Structure Theory for $\mathcal{K}$

In this section we prove some essential properties of the thinning hierarchy $\mathcal{A}^{\mathcal{K}}_{\cdot}$ pertaining to $\mathcal{K}$ (in fact, the first such property is that it really deserves the name "thinning" hierarchy), the most important one being Theorem 3.1.5, which implies that $\Psi^{\mathcal{K}}_{\cdot}$ is total.

Lemma 3.1.1. *For all α: If $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\alpha]$, then $\kappa \in \mathcal{A}^{\mathcal{K}}_\alpha$.*

Proof. Induction on α. There is nothing to do if $\alpha = 0$ or $\alpha = 1$. So let $\alpha > 1$. As $\kappa \in \mathrm{Lim}(\mathcal{A}^{\mathcal{K}}_\alpha)$, $\alpha \in C(\alpha, \kappa)$. Let

$$\alpha_0 <^C_\kappa \alpha,$$

$U_1, \ldots, U_n \subseteq V_\kappa$ and $\varphi \in \Pi^1_1$ such that

$$(V_\kappa, \in, U_1, \ldots, U_n) \models \varphi.$$

Pick $\rho < \kappa$ such that

$$\alpha_0 <^C_\rho \alpha$$

and put $\varphi_\rho \equiv \varphi \wedge \rho = \rho$. As $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\alpha]$ there is $\pi \in \mathcal{A}^{\mathcal{K}}_\alpha$ such that

$$(V_\pi, \in, U_1 \cap V_\pi, \ldots, U_n \cap V_\pi) \models \varphi_\rho,$$

in particular $\pi > \rho$, which implies

$$\alpha_0 <^C_\pi \alpha.$$

But then, by definition, $\pi \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\alpha_0}]$, and by induction hypothesis $\pi \in \mathcal{A}^{\mathcal{K}}_{\alpha_0}$. □

Lemma 3.1.2. *For all α and $\rho \in \mathcal{A}^{\mathcal{K}}_\alpha$,*

$$C(\alpha, \rho) \cap \mathcal{K} = \rho.$$

Proof. Induction on α. If $\alpha = 0$ or $\alpha = 1$, the claim is trivial as ρ is closed under $+$ and φ, and $\beta = p(\beta_0)$ cannot be a maximally simple counterexample because $\beta_0 < \rho$ implies $\beta < \rho$. If ρ is regular, then it is also closed under

$$\Psi_0^{\cdot}[\gamma],$$

as this only denotes the next strongly critical above γ. So let $\alpha > 1$. Assume that there is a maximally simple counterexample β, i.e. $\beta \in C(\alpha, \rho) \cap \mathcal{K}$, but $\beta \geq \rho$. As above, $\beta = \beta_0 + \beta_1$, $\beta = \varphi\beta_0\beta_1$ and $\beta = p(\beta_0)$ are clearly impossible. So assume

$$\beta = \Psi_{\alpha_0}^{\mathcal{K}}[\gamma],$$

where $\gamma < \rho$ by choice of β, and $\alpha_0 < \alpha$. But then, by definition of $\mathcal{A}_\alpha^{\mathcal{K}}$, $\rho \in \Pi_1^1[\mathcal{A}_{\alpha_0}^{\mathcal{K}}]$, so $\beta < \rho$. Assume

$$\beta = \Psi_{\alpha_0,\xi}^{\kappa}[\gamma].$$

But then $\beta < \kappa < \rho$ by hypothesis. If finally

$$\beta = \Psi_{\alpha_0,\alpha_1,\xi}^{\pi',\kappa}[\gamma],$$

then again $\beta < \pi' < \rho$. □

The following lemma is an easy corollary.

Lemma 3.1.3. *Let $\kappa \in C(\xi, \kappa)$. Then $\kappa \notin \Pi_1^1[\mathcal{A}_{\xi'}^{\mathcal{K}}]$ for all $\xi' \geq \xi$.*

Proof. Assume $\kappa \in \Pi_1^1[\mathcal{A}_{\xi'}^{\mathcal{K}}]$ for a $\xi' \geq \xi$. Then $\kappa \in \mathcal{A}_{\xi'}^{\mathcal{K}}$ by Lemma 3.1.1. But then

$$\kappa \in C(\xi, \kappa) \cap \mathcal{K} \subseteq C(\xi', \kappa) \cap \mathcal{K} = \kappa$$

by Lemma 3.1.2, a contradiction. □

Now we turn to showing that all $\mathcal{A}_\alpha^{\mathcal{K}}$ are nonempty. First we need the following folklore observation.

Theorem 3.1.4. *There is a Π_2^1-statement $\varphi_{\Pi_1^1}(Y)$ (in the parameter Y), such that we have for all π*

$$V_\pi \models \varphi_{\Pi_1^1}(Y) \Leftrightarrow \pi \text{ is } \Pi_1^1\text{-indescribable on } Y.$$

Now we can prove the main result of this section. Note that there is no big difference to Rathjen's proof in [Rat94b].

Theorem 3.1.5. *For all α, $\mathcal{K} \in \Pi_2^1[\mathcal{A}_\alpha^{\mathcal{K}}]$. In particular, $\mathcal{A}_\alpha^{\mathcal{K}} \neq \emptyset$.*

Proof. Every $\beta \in (\mathcal{K}, \varepsilon_{\mathcal{K}+1})$ has a unique representation of the form

$$\beta =_{CNF} \omega^{\beta_1} + \cdots + \omega^{\beta_{n+1}}$$

where $\beta > \beta_1 \geq \cdots \geq \beta_k$. Therefore

$$f(\beta) = \begin{cases} \beta & \text{if } \beta < \mathcal{K} \\ \{1\} & \text{if } \beta = \mathcal{K} \\ \langle 2, f(\beta_1), \ldots, f(\beta_{n+1}) \rangle & \text{if } \mathcal{K} < \beta \text{ and } \beta =_{CNF} \omega^{\beta_1} + \cdots + \omega^{\beta_{n+1}} \end{cases}$$

is one-one. Put

$$f(\beta) \lhd f(\beta') \Leftrightarrow \beta < \beta'.$$

Then $\lhd$ is an well-ordering of order-type $\varepsilon_{\mathcal{K}+1}$.

Now we prove the theorem by induction on α. If $\alpha = 0$ or $\alpha = 1$, then it is sufficient to express by a Π^1_1 sentence φ_{r} that $\mathcal{K}$ is regular. Take for example

$$\forall F \forall \gamma \big((\mathrm{fct}(F) \wedge \mathrm{dom}(F) = \gamma \wedge \mathrm{rng}(F) \subseteq \mathrm{On}) \rightarrow \exists \delta (F\text{"}\gamma \subseteq \delta) \big).$$

Now if $U_1, \ldots, U_k \subseteq V_{\mathcal{K}}$ and $\varphi \in \Pi^1_2$ are such that

$$(V_{\mathcal{K}}, \in, U_1, \ldots, U_k) \models \varphi,$$

then also

$$(V_{\mathcal{K}}, \in, U_1, \ldots, U_k) \models \varphi \wedge \varphi_{\mathrm{r}},$$

so by the Π^1_2-indescribability of $\mathcal{K}$ there exists a regular $\pi < \mathcal{K}$ such that

$$(V_\pi, \in, U_1 \cap V_\pi, \ldots, U_k \cap V_\pi) \models \varphi.$$

If $\alpha > 1$ again let $U_1, \ldots, U_k \subseteq V_{\mathcal{K}}$ and φ a Π^1_2-sentence such that

$$(V_{\mathcal{K}}, \in, U_1, \ldots, U_k) \models \varphi.$$

Define

$$U_\alpha = \{f(\alpha)\},$$
$$U_\lhd = \{\langle f(\beta), f(\beta') \rangle \mid f(\beta) \lhd f(\beta')\} \text{ and}$$
$$U_{\mathcal{A}} = \bigcup \{\{f(\beta)\} \times \mathcal{A}^{\mathcal{K}}_\beta \mid \beta < \alpha\},$$

so $U_\alpha, U_\lhd, U_{\mathcal{A}} \subseteq V_{\mathcal{K}}$.

Now $(V_{\mathcal{K}}, \in, U_1, \ldots, U_k, U_\alpha, U_\lhd, U_{\mathcal{A}})$ satisfies the following sentences (part (iv) by induction hypothesis):

(i) φ

(ii) φ_{r}

(iii) $U_\alpha \neq \emptyset$

(iv) $\forall x \forall y \big(y \in U_\alpha \wedge \langle x,y\rangle \in U_\triangleleft \rightarrow \varphi_{\Pi^1_1}(\{z \mid \langle x,z\rangle \in U_\mathcal{A}\})\big)$

By Theorem 3.1.4, this is equivalent to a Π^1_2-sentence, so the Π^1_2-indescribability of $\mathcal{K}$ yields the existence of a $\pi < \mathcal{K}$ such that $(V_\pi, \in, U_1 \cap V_\pi, \ldots, U_k \cap V_\pi, U_\alpha \cap V_\pi, U_\triangleleft \cap V_\pi, U_\mathcal{A} \cap V_\pi)$ satisfies

(i) φ

(ii) φ_r

(iii) $U_\alpha \cap V_\pi \neq \emptyset$

(iv) $\forall x \forall y \big(y \in U_\alpha \cap V_\pi \wedge \langle x,y\rangle \in U_\triangleleft \cap V_\pi \rightarrow \varphi_{\Pi^1_1}(\{z \mid \langle x,z\rangle \in U_\mathcal{A} \cap V_\pi\})$

Thus π is regular and $\alpha \in C(\alpha,\pi)$. By (iv) we get

$$(\forall \beta < \alpha)(f(\beta) \in V_\pi \rightarrow \pi \text{ is } \Pi^1_1\text{-indescribable on } \mathcal{A}^\mathcal{K}_\beta). \qquad (*)$$

So the proof is complete if we can show

$$\beta \in C(\alpha,\pi) \rightarrow f(\beta) \in V_\pi.$$

Thus, define

$$X = \{\beta \in C(\alpha,\pi) \mid f(\beta) \in V_\pi\}.$$

X contains $\pi \cup \{0,\mathcal{K}\}$ and is closed under $+$, φ and p because so is π and because V_π is closed under $\langle\cdot,\ldots,\cdot\rangle$. If $\beta_0 \in X \cap \alpha$ and $\gamma < \pi$, then by $(*)$, $\Psi^\mathcal{K}_\beta[\gamma] < \pi$. Finally, if for example $\kappa, \beta_0, \xi, \gamma \in X$ such that $\xi, \beta_0 < \alpha$ and $\gamma < \kappa$, then $\kappa < \pi$ and so $\Psi^\kappa_{\beta_0}[\gamma] < \pi$. The $\Psi^{\pi'}_\cdot$-case runs identically. Thus X has the same closure properties as $C(\alpha,\pi)$, hence $X = C(\alpha,\pi)$. ☐

Corollary 3.1.6. *For all α and all $\gamma < \mathcal{K}$, $\Psi^\mathcal{K}_\alpha[\gamma]$ is defined, so it really is a "collapsing"-function.*

We finally turn to the most basic case of $<$-comparison between ordinals of the shape $\Psi^\mathcal{K}_\cdot$ - in fact, it is the only one we will constantly need.

Lemma 3.1.7. *We have for all α and β:*

(i) $\mathcal{A}^\mathcal{K}_{\alpha\oplus\beta} \subseteq \mathcal{A}^\mathcal{K}_\alpha$

(ii) *If $\beta > 0$ and $\pi \in \mathcal{A}^\mathcal{K}_{\alpha\oplus\beta}$, then $\Psi^\mathcal{K}_\alpha[\pi] < \Psi^\mathcal{K}_{\alpha\oplus\beta}[\pi]$*

Proof. In case of (i), the claim is trivial if $\alpha = 0$ or $\beta = 0$. Otherwise we have

$$\alpha <^C_{\Psi^\mathcal{K}_{\alpha\oplus\beta}} \alpha \oplus \beta$$

because by definition of $\mathcal{A}^\mathcal{K}_{\alpha\oplus\beta}$, $\alpha \oplus \beta \in C^\mathcal{K}(\alpha\oplus\beta)$, hence also $\alpha \in C^\mathcal{K}(\alpha\oplus\beta)$. So if $\pi \in \mathcal{A}^\mathcal{K}_{\alpha\oplus\beta}$, then

$$\pi \in \Pi^1_1[\mathcal{A}^\mathcal{K}_\alpha]$$

and by Lemma 3.1.1, $\pi \in \mathcal{A}^\mathcal{K}_\alpha$.

As above, $\Psi^\mathcal{K}_{\alpha\oplus\beta}[\pi] \in \Pi^1_1[\mathcal{A}^\mathcal{K}_\alpha]$ in case of (ii), so in particular $\Psi^\mathcal{K}_{\alpha\oplus\beta}[\pi] \in \mathrm{Lim}(\mathcal{A}^\mathcal{K}_\alpha)$. ☐

3.2. Structure Theory for κ

This time we state the basic properties of the thinning hierarchies $\mathcal{A}^\kappa_\cdot$ pertaining to Π^1_1-indescribable cardinals κ. Again, our first aim is to show that the name "thinning" hierarchy is justified. Later, we turn to the most important property: under the least possible assumptions ($\alpha, \kappa \in C(\alpha, \kappa)$) the value Ψ^κ_α exists and thus is $< \kappa$.

First, however, we need a preparatory lemma.

Lemma 3.2.1. *If X is club in π, then*

$$X' = \{\rho < \pi \mid X \cap \rho \text{ club in } \rho\}$$

is club in π.

Proof. For closedness, assume $\rho_\iota \in X'$ for all $\iota \in I$ and let $\rho = \sup_{\iota \in I} \rho_\iota < \pi$. Then $X \cap \rho$ is unbounded in ρ as $X \cap \rho_\iota$ is unbounded in all the ρ_ι. It is also closed in ρ because X is closed in π.

For unboundedness, let $\gamma < \pi$ be arbitrary. Pick the next ω elements $(\rho_i)_{i \in \omega}$ of X above γ, and let ρ be their supremum. As π is regular and $> \omega$, we get $\rho < \pi$. But trivially, $X \cap \rho$ is club in ρ, so $\gamma < \rho \in X'$. □

Remark. Here we used the expression "club" also in the context of ordinals that are not necessarily regular cardinals, when it loses much of its meaning. For example the intersection of two such clubs may be empty. But whenever we use properties of club-sets (like in the following lemma), they are clubs on regular cardinals.

Lemma 3.2.2. *Let α be arbitrary. If $\mathcal{A}^\kappa_\alpha$ is stationary in $\pi < \kappa$, then $\pi \in \mathcal{A}^\kappa_\alpha$.*

Proof. Induction on α. If $\alpha = 0$, then the claim is trivial, as every limit of strongly critical ordinals is strongly critical itself. So let $\alpha \neq 0$. Let $(\pi_\iota)_\iota$ be an enumeration of $\mathcal{A}^\kappa_\alpha \cap \pi$. Obviously

$$\alpha, \kappa \in C(\alpha, \pi),$$

since $\alpha, \kappa \in C(\alpha, \pi_0)$. If $\pi' \in C(\alpha, \pi) \cap \kappa$, then there is a ι such that

$$\pi' \in C(\alpha, \pi_\iota) \cap \kappa,$$

hence $\pi' < \pi_\iota$. Let now $\alpha_0, \xi <^C_\pi \alpha$ such that $\alpha_0, \kappa \in C(\alpha_0, \kappa)$ and $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\xi]$, and let X be club in π. By Lemma 3.2.1, we can pick

$$\pi_\iota \in \mathcal{A}^\kappa_\alpha \cap \pi$$

such that $\alpha_0, \xi <^C_{\pi_\iota} \alpha$ and X is club in π_ι. Then, by definition, $\mathcal{A}^\kappa_{\alpha_0} \cap \mathcal{A}^{\mathcal{K}}_\xi$ is stationary in π_ι, hence

$$\mathcal{A}^\kappa_{\alpha_0} \cap \mathcal{A}^{\mathcal{K}}_\xi \cap X \neq \emptyset.$$

Finally, let $\alpha_0', \xi' <_\pi^C \alpha$ and $\kappa' \in C(\alpha, \pi)$ such that $\kappa' \geq \kappa$. First let $\mathcal{A}_{\alpha_0'}^{\kappa'} \cap \mathcal{A}_{\xi'}^{\mathcal{K}}$ be stationary in κ and let X be club in π. Again by Lemma 3.2.1 we find $\pi_\iota \in \mathcal{A}_\alpha^\kappa \cap \pi$ such that $\alpha_0', \xi', \kappa' \in C(\alpha, \pi_\iota)$ and X club in π_ι. So again, $\mathcal{A}_{\alpha_0'}^{\kappa'} \cap \mathcal{A}_{\xi'}^{\mathcal{K}} \cap X \neq \emptyset$. Now assume $\mathcal{A}_{\alpha_0'}^{\kappa'} \cap \mathcal{A}_{\xi'}^{\mathcal{K}}$ is not stationary in κ, and let C be the ($<_L$-) least club in κ such that

$$\mathcal{A}_{\alpha_0'}^{\kappa'} \cap \mathcal{A}_{\xi'}^{\mathcal{K}} \cap C = \emptyset.$$

But C is unbounded in π (as it is club in all the $\pi_\iota \in \mathcal{A}_\alpha^\kappa$ by definition) and also closed in π (as it is closed in κ), so $\mathcal{A}_{\alpha_0'}^{\kappa'} \cap \mathcal{A}_{\xi'}^{\mathcal{K}}$ is not stationary in π. □

Again, for $\rho \in \mathcal{A}_\alpha^\kappa$, the set $C(\alpha, \rho)$ is empty in the interval $[\rho, \kappa)$:

Lemma 3.2.3. *If $\rho \in \mathcal{A}_\alpha^\kappa$, then*

$$C(\alpha, \rho) \cap \kappa = \rho.$$

Proof. If $\alpha = 0$, then already $C(0, \rho) \cap \mathcal{K} = \rho$ because $\rho \in \mathrm{SC}$. If $\alpha > 0$, then assume for a contradiction that there is a maximally simple counterexample $\beta \in C(\alpha, \rho) \cap [\rho, \kappa)$. $\beta = \beta_0 + \beta_1$ and $\beta = \varphi\beta_0\beta_1$ are clearly impossible. If

$$\beta = p(\beta_0),$$

then $\beta_0 < \rho$ implies $\beta < \rho$, if $\beta_0 = \kappa$, then $\beta = 0$, because κ is a limit cardinal, and if $\beta_0 > \kappa$, then either $\beta = 0$ or $\beta \geq \kappa$. Now assume

$$\beta = \Psi_{\beta_0}^{\mathcal{K}}[\gamma]$$

where by induction hypothesis $\gamma < \rho$. But if $\beta_0 = 0$, then $\beta < \rho$ because β is only the next strongly critical above γ, and ρ is a cardinal; if on the other hand $\beta_0 > 0$, then β is regular, hence $< \rho$. If

$$\beta = \Psi_{\beta_0,\xi}^{\kappa'}[\gamma],$$

then again $\gamma < \rho$ by induction hypothesis. If $\xi \neq 0$ or $\beta_0 \neq 0$, then β is again regular, hence $< \rho$. But if $\xi = \beta_0 = 0$, then again $\beta = \mu\ \mathrm{SC}[\gamma] < \rho$. So finally assume

$$\beta = \Psi_{\beta_0,\beta_1,\xi}^{\pi',\kappa'}[\gamma]$$

with $\gamma < \rho$. Again, only the case $\xi = \beta_1 = 0$ is interesting. If $\pi' < \kappa$, then by induction hypothesis $\pi' < \rho$, so there is nothing to do. If on the other hand $\pi' > \kappa$, then again either $\beta_0 = 0$, and we are again done, or $\alpha \geq 2$. But then, as κ is a limit cardinal, we get

$$\Psi_1^{\mathcal{K}}[\beta] < \kappa,$$

and thus also $\beta < \Psi_1^{\mathcal{K}}[\beta] < \rho$. □

The following theorem shows that, in a strong sense, the collapses $\Psi^{\kappa}_{\alpha,\xi}$ exist whenever possible.

Theorem 3.2.4. *If $\alpha, \kappa \in C(\alpha,\kappa)$, then $\kappa \in \Pi^1_1[\mathcal{A}^{\kappa}_{\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi}]$ for all ξ such that $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi}]$. If additionally $\pi \in \mathcal{A}^{\kappa}_{\alpha} \cap \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}]$ for some ξ' (or $\pi \in \mathcal{A}^{\pi'}_{\alpha'} \cap \mathcal{A}^{\kappa}_{\alpha} \cap \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}]$ for some π', α', ξ'), then $\mathcal{A}^{\kappa}_{\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$ is stationary in π (or $\mathcal{A}^{\pi'}_{\alpha'} \cap \mathcal{A}^{\kappa}_{\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$ is stationary in π, respectively).*

Proof. Fix κ. The proof runs by induction on α.

In this process, for a given α such that $\alpha, \kappa \in C(\alpha,\kappa)$, we would like to define a coding function mapping elements $\beta \in C(\alpha,\kappa)$ to elements $f(\beta) \in V_\kappa$. Unfortunately, we seemingly cannot avoid the use of multiple codes here. Thus we rather recursively define sets $Cd(\beta)$ of codes for β, the elements of which will be denoted by $\ulcorner\beta\urcorner$. So for $\beta \in C(\alpha,\kappa)$ we define

$$Cd(\beta) \ni \begin{cases} \beta & \text{if } \beta < \kappa \\ \{1\} & \text{if } \beta = \mathcal{K} \\ \langle 2, \ulcorner\beta_0\urcorner, \dots, \ulcorner\beta_n\urcorner\rangle & \text{if } \beta =_{CNF} \beta_0 + \dots + \beta_n \\ \langle 3, \ulcorner\beta_0\urcorner, \ulcorner\beta_1\urcorner\rangle & \text{if } \beta =_{VNF} \varphi\beta_0\beta_1 \\ \langle 4, \ulcorner\beta_0\urcorner\rangle & \text{if } \beta = p(\beta_0) \neq 0 \\ \langle 5, \ulcorner\beta_0\urcorner, \ulcorner\beta_1\urcorner\rangle & \text{if } \beta = \Psi^{\mathcal{K}}_{\beta_0}[\beta_1] \\ \langle 6, \ulcorner\beta_0\urcorner, \dots, \ulcorner\beta_3\urcorner\rangle & \text{if } \beta = \Psi^{\beta_0}_{\beta_1,\beta_2}[\beta_3] \\ \langle 7, \ulcorner\beta_0\urcorner, \dots, \ulcorner\beta_5\urcorner\rangle & \text{if } \beta = \Psi^{\beta_0,\beta_1}_{\beta_2,\beta_3,\beta_4}[\beta_5] \end{cases}$$

Finally define the following subsets of κ:

$$\begin{aligned} U_\alpha &= Cd(\alpha) \quad \text{and} \quad U_\kappa = Cd(\kappa) \\ U_\triangleleft &= \{\langle\ulcorner\beta\urcorner, \ulcorner\beta'\urcorner\rangle \mid \beta < \beta'\} \\ U_{\doteq} &= \{\langle\ulcorner\beta\urcorner, \ulcorner\beta'\urcorner\rangle \mid \beta = \beta'\} \\ U_{\mathrm{Reg}} &= \{\ulcorner\beta\urcorner \mid \beta \text{ is regular}\} \\ U_{\mathrm{Ind}} &= \bigcup\{Cd(\xi) \times \mathcal{A}^{\mathcal{K}}_{\xi} \cap \kappa \mid \kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi}]\} \\ U_{\mathrm{Stat}} &= \bigcup\{Cd(\kappa') \times Cd(\beta) \times Cd(\xi) \times \mathcal{A}^{\kappa'}_{\beta} \cap \mathcal{A}^{\mathcal{K}}_{\xi} \cap \kappa \mid \mathcal{A}^{\kappa'}_{\beta} \cap \mathcal{A}^{\mathcal{K}}_{\xi} \text{ stationary in } \kappa\} \\ U_{\mathrm{NonStat}} &= \bigcup\{Cd(\kappa') \times Cd(\beta) \times Cd(\xi) \times \mathcal{A}^{\kappa'}_{\beta} \cap \mathcal{A}^{\mathcal{K}}_{\xi} \cap \kappa \times C^{\kappa'}_{\beta,\xi} \mid \kappa \geq \kappa \wedge \\ &\qquad C^{\kappa'}_{\beta,\xi} \text{ is the } (<_L\text{-}) \text{ least club } C \text{ in } \kappa \text{ such that } \mathcal{A}^{\kappa'}_{\beta} \cap \mathcal{A}^{\mathcal{K}}_{\xi} \cap \kappa \cap C = \emptyset\} \\ U_{\mathcal{A}} &= \bigcup\{Cd(\beta) \times \mathcal{A}^{\kappa}_{\beta} \mid \beta < \alpha \text{ and } \kappa, \beta \in C(\beta,\kappa)\} \end{aligned}$$

(Here we tacitly assume that whenever we write "$\ulcorner\gamma\urcorner$" or "$Cd(\gamma)$", then $\gamma \in C(\alpha,\kappa)$.)

Now let α be minimal such that $\kappa, \alpha \in C(\alpha,\kappa)$. Let $U_1, \dots, U_k \subseteq V_\kappa$ and $\varphi \in \Pi^1_1$ such that $(V_\kappa, \in, U_1, \dots, U_k) \models \varphi$ and ξ be arbitrary such that $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi}]$. Let φ_{r} be the Π^1_1-sentence from Theorem 3.1.5 expressing regularity and

$$\varphi_{\mathrm{cl}}(u, U_{\mathrm{Reg}}, U_\triangleleft, U_{\doteq}) \equiv (\forall x \in U_{\mathrm{Reg}})\big(\langle x, u\rangle \in U_\triangleleft \to (\exists x')(\langle x, x'\rangle \in U_{\doteq} \wedge x' \in \mathrm{On})\big).$$

Then

$$(V_\kappa, \in, U_1, \ldots, U_k, U_\alpha, U_\kappa, U_{\mathrm{Reg}}, U_\lhd, U_{\doteq}) \models \\ \varphi \wedge \varphi_{\mathrm{r}} \wedge (\exists y_0)(\exists y_1)(y_0 \in U_\alpha \wedge y_1 \in U_\kappa \wedge \varphi_{\mathrm{cl}}(y_1, U_{\mathrm{Reg}}, U_\lhd, U_{\doteq})),$$

whence there exists a $\pi \in \mathcal{A}_\xi^{\mathcal{K}}$ such that

$$(V_\pi, \in, U_1 \cap V_\pi, \ldots, U_k \cap V_\pi, \\ U_\alpha \cap V_\pi, U_\kappa \cap V_\pi, U_{\mathrm{Reg}} \cap V_\pi, U_\lhd \cap V_\pi, U_{\doteq} \cap V_\pi) \models \varphi \wedge \varphi_{\mathrm{r}} \wedge (\exists y_0)(\exists y_1)\big($$

(i) $y_0 \in U_\alpha \cap V_\pi \wedge$

(ii) $y_1 \in U_\kappa \cap V_\pi \wedge$

(iii) $\varphi_{\mathrm{cl}}(y_1, U_{\mathrm{Reg}} \cap V_\pi, U_\lhd \cap V_\pi, U_{\doteq} \cap V_\pi)\big)$.

But then, π is regular, and $\alpha, \kappa \in C(\alpha, \pi)$. To see that

$$(\forall \pi' \in C(\alpha, \pi) \cap \kappa)(\pi' < \pi),$$

which finishes the proof of $\pi \in \mathcal{A}_\alpha^\kappa$, we must check that

$$\beta \in C(\alpha, \pi) \to (\exists x \in Cd(\beta))(x \in V_\pi).$$

But this is trivial, as we allowed for multiple codes.

If α is not minimal, let again $U_1, \ldots, U_k \subseteq V_\kappa$, $\varphi \in \Pi_1^1$ such that $(V_\kappa, \in, U_1, \ldots, U_k) \models \varphi$ and ξ be arbitrary such that $\kappa \in \Pi_1^1[\mathcal{A}_\xi^{\mathcal{K}}]$. Then also (by induction hypothesis)

$$(V_\kappa, \in, U_1, \ldots, U_k, U_\alpha, U_\kappa, U_{\mathrm{Reg}}, U_\lhd, U_{\doteq}, U_{\mathrm{Ind}}, U_{\mathrm{Stat}}, U_{\mathrm{NonStat}}, U_{\mathcal{A}}) \models \varphi \wedge \varphi_{\mathrm{r}} \wedge (\exists y_0)(\exists y_1)\big[$$

(i) $y_0 \in U_\alpha \wedge$

(ii) $y_1 \in U_\kappa \wedge$

(iii) $\varphi_{\mathrm{cl}}(y_1, U_{\mathrm{Reg}}, U_\lhd, U_{\doteq}) \wedge$

(iv) $(\forall z_0 \in (U_{\mathrm{Ind}})_0)(\forall y_2)(\langle y_2, y_0 \rangle \in U_\lhd \to$
$\{z_1 \mid \langle z_0, z_1 \rangle \in U_{\mathrm{Ind}}\} \cap \{y_3 \mid \langle y_2, y_3 \rangle \in U_{\mathcal{A}}\}$ is stationary$) \wedge$

(v) $(\forall z_0 \in (U_{\mathrm{Stat}})_0)(\forall z_1 \in (U_{\mathrm{Stat}})_1)(\forall z_2 \in (U_{\mathrm{Stat}})_2)$
$(\{z_3 \mid \langle z_0, \ldots, z_3 \rangle \in U_{\mathrm{Stat}}\}$ is stationary$) \wedge$

(vi) $(\forall z_0 \in (U_{\mathrm{NonStat}})_0)(\forall z_1 \in (U_{\mathrm{NonStat}})_1)(\forall z_2 \in (U_{\mathrm{NonStat}})_2)(\forall z_3, z_4)$
$\big((\langle z_0, \ldots, z_4 \rangle \in U_{\mathrm{NonStat}} \to z_3 \neq z_4) \wedge$
$(\{z_4' \mid \langle z_0, \ldots, z_4' \rangle \in U_{\mathrm{NonStat}}\}$ is club$)\big)\big]$

By assumption there exists a $\pi \in \mathcal{A}_\xi^{\mathcal{K}}$ such that

$$(V_\pi, \in, U_1 \cap V_\pi, \ldots, U_k \cap V_\pi, U_\alpha \cap V_\pi, U_\kappa \cap V_\pi, \\ U_{\mathrm{Reg}} \cap V_\pi, U_\lhd \cap V_\pi, U_{\mathrm{Ind}} \cap V_\pi, U_{\mathrm{Stat}} \cap V_\pi, U_{\mathcal{A}} \cap V_\pi) \models \varphi \wedge \varphi_{\mathrm{r}} \wedge (\exists y_0)(\exists y_1)\big[$$

(i) $y_0 \in U_\alpha \cap V_\pi \wedge$

(ii) $y_1 \in U_\kappa \cap V_\pi \wedge$

(iii) $\varphi_{\text{cl}}(y_1, U_{\text{Reg}} \cap V_\pi, U_\triangleleft \cap V_\pi, U_{\dot{=}} \cap V_\pi) \wedge$

(iv) $(\forall z_0 \in (U_{\text{Ind}} \cap V_\pi)_0)(\forall y_2)(\langle y_2, y_0\rangle \in U_\triangleleft \cap V_\pi \rightarrow$
$\{z_1 \mid \langle z_0, z_1\rangle \in U_{\text{Ind}} \cap V_\pi\} \cap \{y_3 \mid \langle y_2, y_3\rangle \in U_{\mathcal{A}} \cap V_\pi\}$ is stationary$) \wedge$

(v) $(\forall z_0 \in (U_{\text{Stat}} \cap V_\pi)_0)(\forall z_1 \in (U_{\text{Stat}} \cap V_\pi)_1)(\forall z_2 \in (U_{\text{Stat}} \cap V_\pi)_2)$
$(\{z_3 \mid \langle z_0, \ldots, z_3\rangle \in U_{\text{Stat}} \cap V_\pi\}$ is stationary$)\wedge$

(vi) $(\forall z_0 \in (U_{\text{NonStat}} \cap V_\pi)_0)(\forall z_1 \in (U_{\text{NonStat}} \cap V_\pi)_1)(\forall z_2 \in (U_{\text{NonStat}} \cap V_\pi)_2)(\forall z_3, z_4)$
$\big((\langle z_0, \ldots, z_4\rangle \in U_{\text{NonStat}} \cap V_\pi \rightarrow z_3 \neq z_4) \wedge$
$(\{z_4' \mid \langle z_0, \ldots, z_4'\rangle \in U_{\text{NonStat}} \cap V_\pi\}$ is club$)\big)\big]$

This, however, guarantees that $\pi \in \mathcal{A}^\kappa_\alpha \cap \mathcal{A}^{\mathcal{K}}_\xi$. (Notice that here the claim

$$\xi \in C(\alpha, \pi) \rightarrow (\exists x \in Cd(\xi))(x \in V_\pi)$$

is trivial, as we allowed for multiple codes, in particular for any $\xi \in C(\alpha, \pi) \cap [\pi, \kappa)$.)

The second assertion is easy, as for $\pi \in \mathcal{A}^\kappa_\alpha$ we have a Π^1_1-sentence (with class parameters) saying "I am $\in \mathcal{A}^\kappa_\alpha$". Similarly, there is a Π^1_1-sentence (with class parameters) saying "I am $\in \mathcal{A}^{\pi'}_{\alpha'}$". ☐

Corollary 3.2.5. *If $\alpha, \kappa \in C(\alpha, \kappa)$ and $\gamma < \kappa$, then $\Psi^\kappa_\alpha[\gamma]$ exists. If moreover $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\xi]$, then $\Psi^\kappa_{\alpha,\xi}[\gamma] < \kappa$.*

Although we will study the relationships of different $\mathcal{A}^\kappa_\cdot$ and the collapsing functions pertaining to them in greater generality in sections 3.4 and 3.6, we will close this section by listing two very simple facts, because they are basically all we need to know in the cut elimination process.

Lemma 3.2.6. *The following hold:*

(i) *If $\mathcal{A}^\kappa_\alpha \neq \emptyset \neq \mathcal{A}^\kappa_{\alpha\oplus\beta}$, then $\mathcal{A}^\kappa_{\alpha\oplus\beta} \subseteq \mathcal{A}^\kappa_\alpha$.*

(ii) *If $\beta > 0$, $\pi \in \mathcal{A}^\kappa_{\alpha\oplus\beta}$ and $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\xi]$ then $\Psi^\kappa_{\alpha,\xi}[\pi] < \Psi^\kappa_{\alpha\oplus\beta,\xi}[\pi]$.*

Proof. (i) If $\alpha = 0$ and $\beta \neq 0$, then $\mathcal{A}^\kappa_{\alpha\oplus\beta} \subseteq$ Reg. If $\alpha > 0$, pick $\pi \in \mathcal{A}^\kappa_{\alpha\oplus\beta}$. Because of $\alpha \oplus \beta \in C(\alpha \oplus \beta, \pi)$ we also get $\alpha \in C(\alpha \oplus \beta, \pi)$ and hence $\alpha <^C_\pi \alpha \oplus \beta$. As $\mathcal{A}^\kappa_\alpha \neq \emptyset$ implies $\kappa, \alpha \in C(\alpha, \kappa)$, we get by definition

$\mathcal{A}^\kappa_\alpha$ stationary in π,

so by Lemma 3.2.2, $\pi \in \mathcal{A}^\kappa_\alpha$.

(ii) Again, we get by the definition of "$\Psi^\kappa_{\alpha\oplus\beta,\xi}[\pi] \in \mathcal{A}^\kappa_{\alpha\oplus\beta}$" that $\mathcal{A}^\kappa_\alpha \cap \mathcal{A}^{\mathcal{K}}_\xi$ is stationary in $\Psi^\kappa_{\alpha\oplus\beta,\xi}[\pi]$, so in particular the claim holds. ☐

3.3. Structure Theory for π

As in the previous two sections we first show that the hierarchies $\mathcal{A}^{\pi}_{\cdot}$ pertaining to π's which are not Π^1_1-indescribable have the important thinning property

$$\mathrm{Lim}(\mathcal{A}^{\pi}_{\alpha}) \subseteq \mathcal{A}^{\pi}_{\alpha}.$$

Then we prove that (almost always) $\rho \in \mathcal{A}^{\pi}_{\alpha}$ implies

$$C(\alpha, \rho) \cap \pi = \rho,$$

which turns out to be more involved than in the previous sections. The last — and most important — result once more states that under minimal assumptions the values $\Psi^{\pi}_{\cdot}$ exists; the proof again looks different and only uses the fact that π is regular and thus closed under certain limit processes. We think that these additional difficulties we will encounter reflect the different approach to define the $\mathcal{A}^{\pi}_{\alpha}$'s from the outside, cf. the remark on page 19.

Theorem 3.3.1. *If* $\rho \in \mathrm{Lim}(\mathcal{A}^{\pi}_{\alpha}) \cap \pi$*, then* $\rho \in \mathcal{A}^{\pi}_{\alpha}$*.*

Proof. Induction on α. If $\alpha = 0$, then the claim is trivial because limits of strongly criticals are strongly critical themselves.

So let $\alpha > 0$. Pick a sequence $(\rho_\iota)_\iota$ in $\mathcal{A}^{\pi}_{\alpha}$ with supremum ρ. Clearly, $\alpha, \pi \in C(\alpha, \rho)$, as $\alpha, \pi \in C(\alpha, \rho_0)$. If $\pi' \in C(\alpha, \rho) \cap \pi$, then $\pi' \in C(\alpha, \rho_\iota) \cap \pi$ for sufficiently large ι, hence $\pi' < \rho_\iota < \rho$. For the final condition, take

$$\alpha_0, \alpha_0', \xi' <^C_\rho \alpha \quad \text{and} \quad \kappa' \in C(\alpha, \rho)$$

such that $\kappa' \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}]$. First assume that $\mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$ is stationary in π and $\alpha_0, \pi \in C(\alpha_0, \pi)$. Then there is a ι_0 such that for all $\iota \geq \iota_0$

$$\alpha_0, \alpha_0', \xi', \kappa' \in C(\alpha, \rho_\iota).$$

By definition we get for these ι:

$$\rho_\iota \in \mathrm{Lim}(\mathcal{A}^{\pi}_{\alpha_0} \cap \mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}),$$

and so also

$$\rho \in \mathrm{Lim}(\mathcal{A}^{\pi}_{\alpha_0} \cap \mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}),$$

hence $\rho \in \mathcal{A}^{\pi}_{\alpha}$.

Now let C be the $<_L$-minimal club in π that witnesses that

$$\mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \text{ is not stationary in } \pi.$$

Then by definition C is unbounded in all ρ_ι, hence in ρ itself. Further, it is closed in ρ as it is closed in π, so it is club in ρ. ◻

Lemma 3.3.2. *If $\rho \in \mathcal{A}^\pi_\alpha$ is a limit of strongly critical ordinals, then*

$$C(\alpha, \rho) \cap \pi = \rho.$$

Proof. Assume not. Let β be a maximally simple counterexample, i.e. $\beta \in C(\alpha, \rho) \cap [\rho, \pi)$. As ρ is strongly critical, it is closed under $+$ and φ. If

$$\beta = p(\beta_0) > \rho,$$

then $\beta_0 < \rho$ is impossible, $\beta_0 > \pi$ implies $\beta = 0$ or $\beta \geq \pi$, so the critical case is $\beta = p(\pi)$, but by definition we have $\rho \in (p(\pi), \pi)$!

If $\beta = \Psi^{\mathcal{K}}_0[\gamma]$, $\beta = \Psi^{\kappa}_{0,0}[\gamma]$ or $\beta = \Psi^{\pi',\kappa'}_{0,0,0}[\gamma]$ with $\gamma < \rho$, then $\beta < \rho$, because by assumption, ρ is a limit of strongly critical ordinals. On the other hand, $\beta = \Psi^{\mathcal{K}}_{\beta_0}[\gamma]$, $\beta = \Psi^{\kappa}_{\beta_0,\beta_1}[\gamma]$ or $\beta = \Psi^{\pi',\kappa'}_{\beta_2,\beta_0,\beta_1}[\gamma]$ with $\gamma < \rho$ and $\beta_0 \neq 0$ or $\beta_1 \neq 0$ is again impossible because then β would be regular.

Thus the only critical case is

$$\beta = \Psi^{\pi'}_{\beta_0}[\gamma] (= \Psi^{\pi',\kappa'}_{\beta_0,0,0}[\gamma])$$

with $\gamma < \rho$ and $\beta_0 > 0$. In fact, $\beta_0 = 1$ is also impossible, because $\pi' \in C(1, \beta)$ does not work for $\beta \leq \pi'$. Now we have to examine π' closer.

- If $\pi' < \pi$, then by induction hypothesis $\pi' < \rho$, hence also $\beta < \rho$.
- If $\pi' = \pi$, then by definition $\beta_0, \pi \in C(\beta_0, \pi)$, and because of $\beta_0 <^C_\rho \alpha$, we get

  $$\rho \in \mathrm{Lim}(\mathcal{A}^\pi_{\beta_0}),$$

  which together with $\gamma < \rho$ implies $\beta < \rho$.
- $\pi' > \pi$ and π' is a successor-cardinal. But then $\beta \in (p(\pi'), \pi')$ and $\rho < \pi \leq p(\pi')$.
- $\pi' > \pi$ and π' is a limit cardinal. Then we must have

  $$\pi = \Psi^{\mathcal{K}}_1[\beta],$$

  because $\Psi^{\mathcal{K}}_1[\beta] \in C(\alpha, \rho)$ is regular. Were the cardinals not unbounded in β, we could pick the maximal one below β, call it $\tilde{\pi}$. Because of $\beta_0 > 1$ we would also get $\Psi^{\mathcal{K}}_1[\tilde{\pi}] \in C(\beta_0, \beta)$, which is also regular, $< \pi'$ but $\geq \beta$, a contradiction. So β is a limit of cardinals, hence a cardinal itself, so

  $$\beta = p(\pi),$$

  a contradiction to $\rho < \beta$! □

Theorem 3.3.3. *If $\alpha, \pi \in C(\alpha, \pi)$, then $\mathcal{A}^\pi_\alpha$ is club in π. In particular, if X is stationary in π, then $\mathcal{A}^\pi_\alpha \cap X$ is unbounded in π.*

Proof. The second assertion follows immediately from the first, which is proved by induction on α. If α is minimal such that $\alpha, \pi \in C(\alpha, \pi)$, then

$$\mathcal{A}^\pi_\alpha = \{\rho \in \mathrm{SC} \cap (p(\pi), \pi) \mid \alpha, \pi \in C(\alpha, \rho) \wedge (\forall \pi' \in C(\alpha, \rho) \cap \pi)(\pi' < \rho)\},$$

and one easily verifies that the regularity of π implies that this set is club in π. Therefore, now assume that α is not minimal. First, we show that the set

$$\begin{aligned}\tilde{\mathcal{A}}^\pi_\alpha = \big\{\rho \in \mathrm{SC} \cap (p(\pi), \pi) \mid \alpha, \pi \in C(\alpha, \rho) \wedge \\ (\forall \pi' \in C(\alpha, \rho) \cap \pi)(\pi' < \rho) \wedge \\ (\forall \alpha_0, \alpha_0', \xi' <^C_\rho \alpha)(\forall \kappa' \in C(\alpha, \rho))\big[\kappa' \geq \pi \rightarrow \\ \big(\kappa' \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}] \wedge \mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \text{ stationary in } \pi \wedge \\ \alpha_0, \pi \in C(\alpha_0, \pi) \rightarrow \rho \in \mathrm{Lim}(\mathcal{A}^\pi_{\alpha_0} \cap \mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'})\big)\big]\big\}\end{aligned}$$

is club in π.

For unboundedness, let $\gamma < \pi$ be given. Choose ρ^0 such that $\rho^0 > \gamma$, $\rho^0 > p(\pi)$ and $\alpha, \pi \in C(\alpha, \rho^0)$. Then $\overline{\overline{C(\alpha, \rho^0)}} < \pi$. Let

$$R^0 = \{\pi' \in C(\alpha, \rho^0) \mid \pi' < \pi\}.$$

Then $\overline{\overline{R^0}} =: \lambda_{R^0} < \pi$. Let $(X^0_\iota)_{0<\iota<\lambda_{K^0}}$ be an enumeration of all sets

$$\mathcal{A}^\pi_{\alpha_0} \cap \mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$$

with

- $\alpha_0, \alpha_0', \xi', \kappa' \in C(\alpha, \rho^0)$ and $\alpha_0, \alpha_0', \xi' < \alpha$
- $\alpha_0, \pi \in C(\alpha_0, \pi)$
- $\kappa' \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}]$ and $\mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$ stationary in π.

Note that by induction hypothesis, all X_ι are unbounded in π. Because of $\overline{\overline{C(\alpha, \rho^0)}} < \pi$, $\lambda_{K^0} < \pi$, too. Observe the following race:

$$\begin{aligned}\rho^0_{0,0} &= \mu\, \mathrm{SC}[\max\{\rho^0, \lambda_{R^0}\}] < \pi \\ \rho^0_{0,1} &= \mu\, X^0_1[\rho^0_{0,0}] \\ \rho^0_{0,2} &= \mu\, X^0_2[\rho^0_{0,1}] \ldots\end{aligned}$$

and so on, and at limit ordinals $\lambda < \lambda_{K^0}$

$$\rho^0_{0,\lambda} = \mu\, X^0_\lambda[\sup_{\eta<\lambda}\{\rho^0_{0,\eta}\}],$$

when regularity of π implies that $\rho^0_{0,\eta} < \pi$ for all $\eta < \lambda_{K^0}$. Hence

$$\rho^0_{0,\infty} = \sup_{\eta < \lambda_{K^0}} \{\rho^0_{0,\eta}\} < \pi.$$

Repeating the same process with

$$\rho^0_{1,1} = \mu\, X^0_1[\rho^0_{0,\infty}]$$

leads to a $\rho^0_{1,\infty}$, and iterating this race one eventually gets

$$\rho^1 = \sup_{n\in\omega}\{\rho^0_{n,1}\} < \pi.$$

ρ^1 has the following properties:

- $\rho^1 > p(\pi) \wedge \rho^1 \in \mathrm{SC} \cap \pi$ (because it is a limit of strongly criticals)
- $(\forall \pi' \in C(\alpha, \rho^0) \cap \pi)(\rho^1 > \pi')$ (because $\rho^1 > \lambda_{R^0}$)
- $(\forall \eta < \lambda_{K^0})(\rho^1 \in \mathrm{Lim}(X^0_\eta))$ (because $\rho^1 = \sup_{n\in\omega}\{\rho^0_{n,\eta}\}$)

Now repeat the same procedure ω-times (starting with ρ^1 instead of ρ^0 and defining λ_{R^1} and λ_{K^1} analogously) and get $\rho^2 < \pi$, $\rho^3 < \pi$ and so on. Finally define

$$\rho^\infty = \sup_{n\in\omega}\{\rho^n\}.$$

Then still $\rho^\infty < \pi$. We claim that $\rho^\infty \in \tilde{\mathcal{A}}^\pi_\alpha$. $\alpha, \pi \in C(\alpha, \rho^0) \subseteq C(\alpha, \rho^\infty)$ is easy. If $\pi' \in C(\alpha, \rho^\infty) \cap \pi$, then there exists an $n \in \omega$ such that $\pi' \in C(\lambda, \rho^n) \cap \pi$, hence $\pi' < \rho^{n+1} < \rho^\infty$. So pick $\alpha_0, \alpha_0', \xi', \kappa' \in C(\alpha, \rho^\infty)$ such that

- $\alpha_0, \alpha_0', \xi', \kappa' \in C(\alpha, \rho^\infty)$ and $\alpha_0, \alpha_0', \xi' < \alpha$,
- $\alpha_0, \pi \in C(\alpha_0, \pi)$,
- $\kappa' \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}]$ and $\mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$ stationary in π.

Then there is an $n \in \omega$ such that for all $m \geq n$

$$\alpha_0, \alpha_0', \xi', \kappa' \in C(\alpha, \rho^m).$$

Thus for all $m \geq n$,

$$\rho^{m+1} \in \mathrm{Lim}(\mathcal{A}^\pi_{\alpha_0} \cap \mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'})$$

and so $\rho^\infty \in \mathrm{Lim}(\mathcal{A}^\pi_{\alpha_0} \cap \mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'})$.

Closedness of $\tilde{\mathcal{A}}^\pi_\alpha$ is easily proved by similar arguments.

Now, let $(Y_\iota)_{\iota \in I^\pi}$ be an enumeration of the sets $\mathcal{A}^{\kappa'}_{\alpha_0'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$ such that

- $\alpha_0', \xi', \kappa' \in C(\alpha, \pi) \wedge \alpha_0', \xi' < \alpha \wedge \pi \leq \kappa'$
- $\kappa' \in \Pi_1^1[\mathcal{A}_{\xi'}^{\mathcal{K}}] \wedge \mathcal{A}_{\alpha_0'}^{\kappa'} \cap \mathcal{A}_{\xi'}^{\mathcal{K}}$ not stationary in π

For every such Y_ι let $C_\iota \subseteq \pi$ be the club from the definition of $\mathcal{A}_\alpha^\pi$ that witnesses the non-stationarity of Y_ι. If $\rho \in \tilde{\mathcal{A}}_\alpha^\pi$, then $\overline{\overline{C(\alpha, \rho)}} < \pi$ implies that

$$I^\rho = \{\iota \in I^\pi \mid \text{ the respective } \alpha_0', \xi', \kappa' \text{ satisfy } \alpha_0', \xi', \kappa' \in C(\alpha, \rho)\}$$

has cardinality $< \pi$. Thus

$$C^\rho = \bigcap_{\iota \in I^\rho} (C_\iota)'$$

is still club in π, where

$$(C_\iota)' = \{\delta < \pi \mid C_\iota \text{ is club in } \delta\}.$$

Now let

$$\rho \in \mathrm{Lim}(\tilde{\mathcal{A}}_\alpha^\pi) \cap \mathop{\Delta}_{\bar{\rho} < \pi} C^{\bar{\rho}},$$

where $\mathop{\Delta}_{\bar{\rho}<\pi} C^{\bar{\rho}} = \{\rho' < \pi \mid (\forall \rho'' < \rho')(\rho' \in C^{\rho''})\}$ is the diagonal intersection of the $C^{\bar{\rho}}$. Let $\alpha_0', \xi', \kappa' \in C(\alpha, \rho)$ with $\alpha_0' < \alpha$, $\pi \leq \kappa'$, $\kappa' \in \Pi_1^1[\mathcal{A}_{\xi'}^{\mathcal{K}}]$ and $\mathcal{A}_{\alpha_0'}^{\kappa'} \cap \mathcal{A}_{\xi'}^{\mathcal{K}}$ not stationary in π. Then (as $\rho \in \mathrm{Lim}(\tilde{\mathcal{A}}_\alpha^\pi)$) there is $\bar{\rho} \in \rho \cap \tilde{\mathcal{A}}_\alpha^\pi$ such that $\alpha_0', \xi', \kappa' \in C(\alpha, \bar{\rho})$. As $\bar{\rho} < \rho$, we have $\rho \in C^{\bar{\rho}}$, i.e. C_ι is club in ρ, where ι belongs to α_0', ξ', κ'. As $\mathrm{Lim}(\tilde{\mathcal{A}}_\alpha^\pi) \cap \mathop{\Delta}_{\bar{\rho}<\pi} C^{\bar{\rho}}$ is club in π, we have shown that $\mathcal{A}_\alpha^\pi$ contains a club-set, hence is unbounded. But it is also closed: Let $(\rho_\iota)_\iota$ be unbounded in $\rho < \pi$, and in order to check the last condition, pick $\alpha_0', \xi', \kappa' \in C(\alpha, \rho)$ as above such that $\mathcal{A}_{\alpha_0'}^{\kappa'} \cap \mathcal{A}_{\xi'}^{\mathcal{K}}$ is not stationary in π and let C_{ι_0} be the witness as above. Then $C_{\iota_0} \cap \rho$ is club in ρ: It is unbounded, because $\alpha_0', \xi', \kappa' \in C(\alpha, \rho_\iota)$ for all ι above some ι' and thus C_{ι_0} is unbounded in all these ρ_ι. But it is also closed in ρ, as it is already closed in π. $\square$

The whole procedure only used the fact that π is regular and has all the properties that elements of $\mathcal{A}_\alpha^\pi$ have. So we get the following

Corollary 3.3.4. *If $\pi' \in \mathcal{A}_\alpha^\pi$ is regular, then $\mathcal{A}_\alpha^\pi \cap \pi'$ is club in π'.*

Proof. The proof is more or less literally the same as in the Theorem. If α is minimal, then the claim is trivial (it only requires regularity). If α is not minimal, we only have to check unboundedness (as closedness of $\mathcal{A}_\alpha^\pi \cap \pi'$ in π' follows from closedness of $\mathcal{A}_\alpha^\pi$ in π). To see the unboundedness of $\tilde{\mathcal{A}}_\alpha^\pi$ in π' we just repeat the same race as in the proof above, this time below π', and using the fact that $\pi' \in \mathcal{A}_\alpha^\pi$ implies

$$\mathcal{A}_{\alpha_0}^\pi \cap \mathcal{A}_{\alpha_0'}^{\kappa'} \cap \mathcal{A}_{\xi'}^{\mathcal{K}} \text{ unbounded in } \pi'$$

for all relevant sets $\mathcal{A}_{\alpha_0}^\pi \cap \mathcal{A}_{\alpha_0'}^{\kappa'} \cap \mathcal{A}_{\xi'}^{\mathcal{K}}$. The last argument, showing that even $\mathcal{A}_\alpha^\pi \cap \pi'$ is unbounded in π' can again be adopted literally. $\square$

Corollary 3.3.5. *If $\alpha, \pi \in C(\alpha, \pi)$, $\gamma < \pi$, $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\xi]$ and $\mathcal{A}^\kappa_\beta \cap \mathcal{A}^{\mathcal{K}}_\xi$ stationary in π, then $\Psi^{\pi,\kappa}_{\alpha,\beta,\xi}[\gamma] < \pi$.*

Again we state the most obvious facts about $\mathcal{A}^\pi_\cdot$ and $\Psi^\pi_\cdot$, which we will frequently use later.

Lemma 3.3.6. *The following hold:*

(i) *If $\mathcal{A}^\pi_\alpha \neq \emptyset \neq \mathcal{A}^\pi_{\alpha\oplus\beta}$, then $\mathcal{A}^\pi_{\alpha\oplus\beta} \subseteq \mathcal{A}^\pi_\alpha$.*

(ii) *If $\beta > 0$, $\rho \in \mathcal{A}^\pi_{\alpha\oplus\beta}$, $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\xi]$ and $\mathcal{A}^\kappa_{\alpha'} \cap \mathcal{A}^{\mathcal{K}}_\xi$ stationary in π, then*

$$\Psi^{\pi,\kappa}_{\alpha,\alpha',\xi}[\rho] < \Psi^{\pi,\kappa}_{\alpha\oplus\alpha',\beta,\xi}[\rho].$$

Proof. As this is very similar to the proof of Lemma 3.2.6, we will omit it.

3.4. Comparison of Collapsing Functions

In order to obtain a *recursive* ordinal notation system (see the next chapter), we need to decide properties of the ordinal notations involved in a syntactic way. This is exactly the point where the additional conditions on elements of $\mathcal{A}^\kappa_\alpha$ and $\mathcal{A}^\pi_\alpha$, which made the definitions too intricate on first sight, come into play.

In this section we show that deciding whether $\Psi < \Psi'$ or not (where Ψ, Ψ' are collapses) can — under good conditions — be decided talking only about sub"terms" of Ψ and Ψ'.

Notice that all of the following arguments use the closedness of the collapsed points (i.e. Lemmas 3.1.2, 3.2.3 and 3.3.2).

The first lemma is still literally the same as in [Rat94b].

Lemma 3.4.1. *Let $\Psi = \Psi^{\mathcal{K}}_\alpha[\gamma]$ and $\Psi' = \Psi^{\mathcal{K}}_{\alpha'}[\gamma']$. If $\gamma, \gamma' < \min\{\Psi, \Psi'\}$, then*

$$\Psi < \Psi'$$

iff one of the following holds:

(i) $\alpha < \alpha' \wedge \alpha, \gamma \in C(\alpha', \Psi')$

(ii) $\alpha' \leq \alpha \wedge \{\alpha', \gamma'\} \not\subseteq C(\alpha, \Psi)$

Proof. We first show "$\Leftarrow$". If (i) holds, then $\Psi \in C(\alpha', \Psi') \cap \mathcal{K} = \Psi'$. If (ii) holds, then $\Psi' \leq \Psi$ contradicts $\alpha', \gamma' \in C(\alpha', \Psi')$.

To prove "$\Rightarrow$", first assume $\alpha < \alpha'$. Then $\{\alpha, \gamma\} \not\subseteq C(\alpha', \Psi')$ contradicts $\alpha, \gamma \in C(\alpha, \Psi)$. If $\alpha' < \alpha$, $\alpha', \gamma' \in C(\alpha, \Psi)$ would imply $\Psi' \in C(\alpha, \Psi) \cap \mathcal{K} = \Psi$. Finally, $\alpha = \alpha'$ is impossible because of $\gamma, \gamma' < \min\{\Psi, \Psi'\}$. □

Lemma 3.4.2. *Let $\kappa^* \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi^*}]$, $\alpha^*, \kappa^* \in C(\alpha^*, \kappa^*)$ and $\gamma^* < \kappa^*$, where $^* \in \{\ ,'\}$. Let*

$$\Psi = \Psi^{\kappa}_{\alpha,\xi}[\gamma] \quad \textit{and} \quad \Psi' = \Psi^{\kappa'}_{\alpha',\xi'}[\gamma']$$

and assume that $\gamma, \gamma' < \min\{\Psi, \Psi'\}$ and $\Psi, \Psi' < \min\{\kappa, \kappa'\}$. Then

$$\Psi < \Psi'$$

iff one of the following holds:

(i) $\alpha < \alpha' \wedge \kappa, \alpha, \xi \in C(\alpha', \Psi')$

(ii) $\alpha' \leq \alpha \wedge \{\kappa', \alpha', \xi'\} \not\subseteq C(\alpha, \Psi)$

(iii) $\alpha = \alpha' \wedge \kappa = \kappa' \wedge \xi <^C_{\Psi'} \xi'$.

Proof. We start with "$\Leftarrow$". If (i) holds, then $\Psi \in C(\alpha', \Psi') \cap \kappa'$. In case of (ii), $\kappa', \alpha', \xi' \in C(\alpha', \Psi')$ forces $\Psi < \Psi'$. If finally (iii) holds, then $\Psi' \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi}]$, so Theorem 3.2.4 implies $\Psi < \Psi'$.

To prove "$\Rightarrow$", first assume $\alpha < \alpha'$. Then immediately (i) follows. If $\alpha' < \alpha$, then (ii) must hold, as otherwise we had $\Psi' \in C(\alpha, \Psi) \cap \kappa = \Psi$. So let's assume $\alpha = \alpha'$. Now $\kappa < \kappa'$ leads to the contradiction $\kappa \in C(\alpha, \Psi) \cap \kappa' \subseteq C(\alpha', \Psi') \cap \kappa' = \Psi'$, whereas in case of $\kappa' < \kappa$, $\kappa' \in C(\alpha, \Psi)$ is impossible, so (ii) must hold. So now assume additionally $\kappa = \kappa'$. $\xi = \xi'$ is obviously impossible. If $\xi < \xi'$, then (iii) follows from $\xi \in C(\xi, \Psi)$. If finally $\xi' < \xi$, then we get $\Psi \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}]$, leading (again using Theorem 3.2.4) to the contradiction $\Psi' < \Psi$. □

The next lemma is more tricky — and one of the reasons for the intricacy of the definition of $\mathcal{A}^{\pi}_{\alpha}$. Notice that in case of (iii) of the "$\Leftarrow$"-direction of the proof, we cannot reason "as usual", as in general $\kappa, \sigma, \xi, \gamma \in C(\sigma', \Psi') \not\Rightarrow \Psi \in C(\alpha', \Psi)$, but have to take a detour via κ'.

Lemma 3.4.3. *Let $\kappa^* \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi^*}]$, $\mathcal{A}^{\kappa^*}_{\sigma^*} \cap \mathcal{A}^{\mathcal{K}}_{\xi^*}$ stationary in π^*, $\alpha^*, \pi^* \in C(\alpha^*, \pi^*)$ and $\gamma^* < \pi^*$, where $^* \in \{\ ,'\}$. Let*

$$\Psi = \Psi^{\pi,\kappa}_{\alpha,\sigma,\xi}[\gamma] \quad \textit{and} \quad \Psi' = \Psi^{\pi',\kappa'}_{\alpha',\sigma',\xi'}[\gamma']$$

and assume that $\gamma, \gamma' < \min\{\Psi, \Psi'\}$ and $\Psi, \Psi' < \min\{\pi, \pi'\}$. Then

$$\Psi < \Psi'$$

iff one of the following holds:

(i) $\alpha < \alpha' \wedge \pi, \kappa, \alpha, \sigma, \xi, \gamma \in C(\alpha', \Psi')$

(ii) $\alpha' \leq \alpha \wedge \{\pi', \kappa', \alpha', \sigma', \xi', \gamma'\} \not\subseteq C(\alpha, \Psi)$

(iii) $\alpha = \alpha' \wedge \pi = \pi' \wedge \sigma < \sigma' \wedge \kappa, \sigma, \xi, \gamma \in C(\sigma', \Psi')$

(iv) $\alpha = \alpha' \wedge \pi = \pi' \wedge \sigma' \leq \sigma \wedge \{\kappa', \sigma', \xi', \gamma'\} \nsubseteq C(\sigma, \Psi)$

(v) $\alpha = \alpha' \wedge \pi = \pi' \wedge \sigma = \sigma' \wedge \kappa = \kappa' \wedge \xi \in C(\xi', \Psi') \cap \xi' \wedge \xi' \neq 1$

(vi) $\alpha = \alpha' \wedge \pi = \pi' \wedge \sigma = \sigma' \wedge \kappa = \kappa' \wedge \xi' < \xi \neq 1 \wedge \xi' \notin C(\xi, \Psi))$

Proof. First we show "$\Leftarrow$".

If we assume (i), then also $\Psi \in C(\alpha', \Psi') \cap \pi'$.

If (ii) holds, then the assumption $\Psi' \leq \Psi$ would lead to the contradiction

$$\pi', \kappa', \alpha', \sigma', \xi', \gamma' \in C(\alpha', \Psi') \subseteq C(\alpha, \Psi).$$

So now assume (iii). In particular, we have $\sigma <_\pi^C \sigma'$, and as

$$\pi = \pi' \in \mathcal{A}_{\sigma'}^{\kappa'}$$

(by Lemma 3.2.2), the fact that $\mathcal{A}_\sigma^\kappa \cap \mathcal{A}_\xi^\mathcal{K}$ is stationary in π implies that

$$\mathcal{A}_\sigma^\kappa \cap \mathcal{A}_\xi^\mathcal{K} \text{ is stationary in } \kappa'.$$

(Here we used the fact that $\kappa < \kappa'$ is impossible, because otherwise we would have

$$\Psi' < \pi < \kappa \in C(\sigma', \Psi') \cap \kappa' = \Psi'.)$$

But now $\Psi' \in \mathcal{A}_{\sigma'}^{\kappa'}$ plus the assumptions yield

$$\mathcal{A}_\sigma^\kappa \cap \mathcal{A}_\xi^\mathcal{K} \text{ is stationary in } \Psi'.$$

As Ψ' must be regular ($\sigma' \neq 0$), Corollary 3.3.4 shows that $\Psi < \Psi'$.

(iv) is again trivial, as $\{\kappa', \sigma', \xi', \gamma'\} \subseteq C(\sigma', \Psi')$.

In the case of (v), $\xi \in C(\xi', \Psi') \cap \xi' \wedge \xi' \neq 1$ implies, as $\Psi' \in \mathcal{A}_{\sigma'}^{\kappa'} = \mathcal{A}_\sigma^\kappa$, that

$$\mathcal{A}_\xi^\mathcal{K} \cap \mathcal{A}_\sigma^\kappa \text{ is stationary in } \Psi'$$

(by Theorem 3.2.4). But this implies, as $\Psi' \in \mathcal{A}_{\alpha'}^{\pi'} = \mathcal{A}_\alpha^\pi$, $\Psi < \Psi'$ by Theorem 3.3.3.

Finally, assuming (vi), $\Psi' \leq \Psi$ would contradict $\xi' \in C(\xi', \Psi')$.

Now we turn to "$\Rightarrow$".

First assume $\alpha < \alpha'$. Then $\{\pi, \kappa, \alpha, \sigma, \xi, \gamma\} \subseteq C(\alpha, \Psi) \subseteq C(\alpha', \Psi')$, so (i) holds. But if $\alpha' < \alpha$, then $\{\kappa', \pi', \alpha', \sigma', \xi', \gamma'\} \subseteq C(\alpha, \Psi)$ is impossible, as this would imply $\Psi' \in C(\alpha, \Psi) \cap \pi = \Psi$, so in this case, (ii) must be true.

So from now on let $\boxed{\alpha = \alpha'}$.

$\pi < \pi'$ is now impossible in view of

$$\pi \in C(\alpha, \Psi) \cap \pi' \subseteq C(\alpha', \Psi') \cap \pi' = \Psi'.$$

But $\pi' < \pi$ implies (ii), because $\pi' \in C(\alpha, \Psi) \cap \pi$ would yield

$$\Psi > \pi' \geq \Psi,$$

a contradiction.

So from now on we may safely assume $\boxed{\pi = \pi'}$ as well.

Then $\sigma < \sigma'$ easily leads to (iii), and if $\sigma' < \sigma \wedge \{\kappa', \sigma', \xi', \gamma'\} \subseteq C(\sigma, \Psi)$, then the same argument of "$\Leftarrow$" (iii) would show $\Psi' < \Psi$.

Now let $\boxed{\sigma = \sigma'}$.

In this situation, $\kappa < \kappa'$ is impossible, because then

$$\kappa \in C(\sigma, \Psi) \cap \kappa' \subseteq C(\sigma', \Psi') \cap \kappa' = \Psi',$$

contradicting the assumption. And if $\kappa' < \kappa$, then either $\kappa' \notin C(\sigma, \Psi)$, hence (iv) is true, or we get the contradiction

$$\kappa' \in C(\sigma, \Psi) \cap \kappa = \Psi.$$

So now let $\boxed{\kappa = \kappa'}$.

If $\xi < \xi'$, then in case of $\xi = 0 \wedge \xi' = 1$, $\sigma = \sigma' > 1$, so the elements of $\mathcal{A}^\kappa_\sigma$ are regular anyway, and hence $\Psi = \Psi'$. Otherwise, $\xi \in C(\xi', \Psi')$ follows directly from $\xi \in C(\xi, \Psi)$. If on the other hand $\xi' < \xi \neq 1$, then the same argument as in "$\Leftarrow$" (v) shows that $\Psi' < \Psi$. But $\xi = \xi'$ together with the hypothesis $\gamma, \gamma' < \min\{\Psi, \Psi'\}$ implies $\Psi = \Psi'$. □

Lemma 3.4.4. *Let $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\xi]$, $\mathcal{A}^\kappa_\sigma \cap \mathcal{A}^{\mathcal{K}}_\xi$ stationary in π, $\alpha, \pi \in C(\alpha, \pi)$ and $\gamma < \pi$. Let further $\gamma' < \mathcal{K}$ and*

$$\Psi = \Psi^{\pi,\kappa}_{\alpha,\sigma,\xi}[\gamma] \quad \textit{and} \quad \Psi' = \Psi^{\mathcal{K}}_{\xi'}[\gamma'].$$

If $\gamma, \gamma' < \min\{\Psi, \Psi'\}$ and $\Psi' < \pi$, then

$$\Psi < \Psi'$$

iff one of the following holds:

(i) $\alpha < \xi' \wedge \{\pi, \kappa, \alpha, \sigma, \xi, \gamma\} \subseteq C(\xi', \Psi')$

(ii) $\xi' < \alpha \wedge \{\xi', \gamma'\} \not\subseteq C(\alpha, \Psi)$

Additionally,

$$\Psi' \neq \Psi.$$

Proof. For the first assertion, we first prove "$\Leftarrow$". If (i) holds, then $\Psi \in C(\xi', \Psi') \cap \mathcal{K} = \Psi'$. If (ii) is true, the assumption $\Psi' \leq \Psi$ would contradict $\xi', \gamma' \in C(\xi', \Psi')$.

Now we turn to "$\Rightarrow$". If $\alpha < \xi'$, $\{\pi, \kappa, \alpha, \sigma, \xi, \gamma\} \subseteq C(\alpha, \Psi)$ implies $\{\pi, \kappa, \alpha, \sigma, \xi, \gamma\} \subseteq C(\xi', \Psi')$, so (i) holds. If $\xi' < \alpha$, then $\xi', \gamma' \in C(\alpha, \Psi)$ is impossible, as it would imply $\Psi' \in C(\alpha, \Psi) \cap \pi = \Psi$. Finally $\alpha = \xi'$ leads to the contradiction $\pi \in C(\alpha, \Psi) \cap \mathcal{K} \subseteq C(\xi', \Psi') \cap \mathcal{K} = \Psi'$.

For the second assertion assume $\Psi = \Psi'$. If $\alpha \leq \xi'$, then $\pi \in C(\alpha, \Psi) \cap \mathcal{K} \subseteq C(\xi', \Psi') \cap \mathcal{K} = \Psi'$ is a contradiction. But $\xi' < \alpha$ is impossible, too, because then $\Psi' \in C(\alpha, \Psi) \cap \pi = \Psi$. □

The proof of the following lemma is completely analogous, so we will omit it.

Lemma 3.4.5. *Let* $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\xi]$*,* $\alpha, \kappa \in C(\alpha, \kappa)$ *and* $\gamma < \kappa$*. Let further* $\gamma' < \mathcal{K}$ *and*

$$\Psi = \Psi^{\kappa}_{\alpha,\xi}[\gamma] \quad \textit{and} \quad \Psi' = \Psi^{\mathcal{K}}_{\xi'}[\gamma'].$$

If $\gamma, \gamma' < \min\{\Psi, \Psi'\}$ *and* $\Psi' < \kappa$*, then*

$$\Psi < \Psi'$$

iff one of the following holds:

(i) $\alpha < \xi' \wedge \{\kappa, \alpha, \xi, \gamma\} \subseteq C(\xi', \Psi')$

(ii) $\xi' < \alpha \wedge \{\xi', \gamma'\} \not\subseteq C(\alpha, \Psi)$

Additionally,

$$\Psi' \neq \Psi.$$

Lemma 3.4.6. *Let* $\kappa \in \mathcal{A}^{\mathcal{K}}_\xi$*,* $\mathcal{A}^{\kappa}_\sigma \cap \mathcal{A}^{\mathcal{K}}_\xi$ *stationary in* π*,* $\alpha, \pi \in C(\alpha, \pi)$*,* $\gamma < \pi$*,* $\kappa' \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}]$*,* $\sigma', \kappa' \in C(\sigma', \kappa')$ *and* $\gamma' < \kappa'$*. If*

$$\Psi = \Psi^{\pi,\kappa}_{\alpha,\sigma,\xi}[\gamma] \quad \textit{and} \quad \Psi' = \Psi^{\kappa'}_{\sigma',\xi'}[\gamma']$$

and $\gamma, \gamma' < \min\{\Psi, \Psi'\}$*,* $\Psi < \kappa'$ *and* $\Psi' < \pi$*, then*

$$\Psi < \Psi'$$

holds iff one of the following is true:

(i) $\alpha < \sigma' \wedge \{\pi, \kappa, \alpha, \sigma, \xi, \gamma\} \subseteq C(\sigma', \Psi')$

(ii) $\sigma' \leq \alpha \wedge \{\kappa', \sigma', \xi', \gamma'\} \not\subseteq C(\alpha, \Psi)$

Proof. We begin with "$\Leftarrow$". (i) implies that $\Psi \in C(\sigma', \Psi') \cap \kappa' = \Psi'$. As $\{\kappa', \sigma', \xi', \gamma'\} \subseteq C(\sigma', \Psi')$, $\Psi < \Psi'$ follows from (ii).

Now we turn to "$\Rightarrow$". If $\alpha < \sigma'$, then we easily get (i). If $\sigma' < \alpha$, $\{\kappa', \sigma', \xi', \gamma'\} \subseteq C(\alpha, \Psi)$ would imply $\Psi' \in C(\alpha, \Psi) \cap \pi = \Psi$, so (ii) must be true. If finally $\alpha = \sigma'$, then neither $\pi < \kappa'$ (because then we would have $\pi \in C(\alpha, \Psi) \cap \kappa' \subseteq C(\sigma', \Psi') \cap \kappa' = \Psi'$) nor $\kappa' < \pi \wedge \kappa' \in C(\alpha, \Psi)$ (because then $\kappa' < \Psi$) is possible. □

Lemma 3.4.7. *Let* $\kappa \in \mathcal{A}^{\mathcal{K}}_\xi$*,* $\mathcal{A}^{\kappa}_\sigma \cap \mathcal{A}^{\mathcal{K}}_\xi$ *stationary in* π*,* $\alpha, \pi \in C(\alpha, \pi)$*,* $\gamma < \pi$*,* $\kappa' \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}]$*,* $\sigma', \kappa' \in C(\sigma', \kappa')$ *and* $\gamma' < \kappa'$*. If*

$$\Psi = \Psi^{\pi,\kappa}_{\alpha,\sigma,\xi}[\gamma] \quad \textit{and} \quad \Psi' = \Psi^{\kappa'}_{\sigma',\xi'}[\gamma']$$

and $\gamma, \gamma' < \min\{\Psi, \Psi'\}$*,* $\Psi < \kappa'$ *and* $\Psi' < \pi$*, then*

$$\Psi' < \Psi$$

iff one of the following holds:

(i) $\sigma' < \alpha \wedge \{\kappa', \sigma', \xi', \gamma'\} \subseteq C(\alpha, \Psi)$

(ii) $\alpha \leq \sigma' \wedge \{\pi, \kappa, \alpha, \sigma, \xi, \gamma\} \nsubseteq C(\sigma', \Psi')$

Proof. "$\Leftarrow$". (i) yields $\Psi' \in C(\alpha, \Psi) \cap \pi$. If (ii) holds, then $\Psi \leq \Psi'$ contradicts $\{\pi, \kappa, \alpha, \sigma, \xi, \gamma\} \subseteq C(\alpha, \Psi)$.

"$\Rightarrow$". If $\sigma' < \alpha$, then (i) holds. In case of $\alpha < \sigma'$, $\{\pi, \kappa, \alpha, \sigma, \xi, \gamma\} \subseteq C(\sigma', \Psi)$ is not possible, as this would also mean $\Psi \in C(\sigma', \Psi') \cap \kappa'$. So finally assume $\alpha = \sigma'$. Now $\kappa' < \pi$ is impossible in view of $\kappa' \in C(\sigma', \Psi') \cap \pi \subseteq C(\alpha, \Psi) \cap \pi = \Psi$. And if $\pi < \kappa'$, then (ii) must be true, as $\pi \in C(\sigma', \Psi')$ would yield $\Psi' > \pi > \Psi'$. □

3.5. Checking Indescribability

In order to decide if $\Psi^{\kappa}_{\alpha_0, \xi} \in C(\alpha, \beta)$, we have to decide in a recursive way whether $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi}]$ or not, and this is what we are going to do in this section. Notice that at that point, the arguments are the same as in [Rat94b].

Definition. For Π^1_1-indescribable κ we define $ind(\kappa)$ as follows:

$$ind(\kappa) = \begin{cases} \alpha & \text{if } \kappa = \Psi^{\mathcal{K}}_{\alpha}[\gamma] \\ \alpha & \text{if } \kappa = \Psi^{\bar{\kappa}}_{\alpha', \alpha}[\gamma] \\ \alpha & \text{if } \kappa = \Psi^{\bar{\pi}, \bar{\kappa}}_{\alpha'', \alpha', \alpha}[\gamma] \end{cases}$$

Lemma 3.5.1. *We have*

(i) $ind(\kappa) = \sup\{\xi \mid \kappa \in \mathcal{A}^{\mathcal{K}}_{\xi}\}$

(ii) $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi}] \Leftrightarrow \xi \in C(ind(\kappa), \kappa) \cap ind(\kappa)$

Proof. (i). $\kappa \in \mathcal{A}^{\mathcal{K}}_{ind(\kappa)}$, so we only have to show that $\kappa \in \mathcal{A}^{\mathcal{K}}_{\xi}$ is impossible for $\xi > ind(\kappa)$. But in that case we would have

$$\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{ind(\kappa)}],$$

as $ind(\kappa) \in C(ind(\kappa), \kappa)$. If $\kappa = \Psi^{\bar{\kappa}}_{\alpha', ind(\kappa)}[\gamma]$, Theorem 3.2.4 then yields that $\mathcal{A}^{\bar{\kappa}}_{\alpha'} \cap \mathcal{A}^{\mathcal{K}}_{ind(\kappa)}$ is stationary in κ, a contradiction. If on the other hand $\kappa = \Psi^{\bar{\pi}, \bar{\kappa}}_{\alpha'', \alpha', ind(\kappa)}[\gamma]$, then we first get (again by Theorem 3.2.4) that $\mathcal{A}^{\bar{\kappa}}_{\alpha'} \cap \mathcal{A}^{\mathcal{K}}_{ind(\kappa)}$ is stationary in κ. But in view of Corollary 3.3.4, $\mathcal{A}^{\bar{\pi}}_{\alpha''}$ is club in κ, hence $\mathcal{A}^{\bar{\pi}}_{\alpha''} \cap \mathcal{A}^{\bar{\kappa}}_{\alpha'} \cap \mathcal{A}^{\mathcal{K}}_{ind(\kappa)}$ is unbounded in κ, again a contradiction.

(ii). As "$\Leftarrow$" is trivial, assume $\xi \geq ind(\kappa)$ exists such that $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi}]$. But then we get $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{ind(\kappa)}]$ (either directly or via the detour $\kappa \in \mathcal{A}^{\mathcal{K}}_{\xi} \wedge ind(\kappa) <^C_{\kappa} \xi$), again leading to a contradiction. So $\xi < ind(\kappa)$ must hold. But this also yields $\xi \in C(ind(\kappa), \kappa)$. □

3.6. Checking Stationarity

Things are getting much more involved when we want to know whether "$\mathcal{A}^{\kappa}_{\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi}$ stationary in π" is true or not — this will be important in order to decide whether or not to include a "term" denoting $\Psi^{\pi,\kappa}_{\beta,\alpha,\xi}[\gamma]$ in the notation system.

Lemma 3.6.1. *Let $\gamma' < \mathcal{K}$, $\pi = \Psi^{\mathcal{K}}_{\alpha'}[\gamma']$, $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi}]$ and $\alpha, \kappa \in C(\alpha, \kappa)$. Then*

$\mathcal{A}^{\kappa}_{\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi}$ *is stationary in* π

iff

$\pi = \kappa.$

Proof. If $\pi = \kappa$, then Theorem 3.2.4 even shows that $\pi \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi} \cap \mathcal{A}^{\kappa}_{\alpha}]$, so "$\Leftarrow$" is trivial.

To prove "$\Rightarrow$", first note that $\kappa < \pi$ is obviously impossible. So we only have to treat the case that $\pi < \kappa$. If $\alpha' < \alpha$, then $\alpha', \gamma' \in C(\alpha', \pi)$ implies that there is a $\pi_0 \in \mathcal{A}^{\kappa}_{\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi} \cap \pi$, such that $\alpha', \gamma' \in C(\alpha, \pi_0)$ (in fact, there are stationary many), and hence $\pi \in C(\alpha, \pi_0) \cap \kappa = \pi_0$, a contradiction. If on the other hand $\alpha \leq \alpha'$, then $\kappa \in C(\alpha', \pi)$, contradicting $C(\alpha', \pi) \cap \mathcal{K} = \pi$. □

In the following lemma, we heavily use the last part of the definition of $\mathcal{A}^{\kappa'}_{\cdot}$.

Lemma 3.6.2. *Let $\kappa' \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}]$, $\gamma' < \kappa'$, $\alpha', \kappa' \in C(\alpha', \kappa')$ and $\pi = \Psi^{\kappa'}_{\alpha',\xi'}[\gamma']$. If $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi}]$ and $\alpha, \kappa \in C(\alpha, \kappa)$, then*

$\mathcal{A}^{\kappa}_{\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi}$ *is stationary in* π

iff

(i) $\kappa = \pi$

(ii) $\kappa' < \kappa \wedge \alpha, \xi <^C_\pi \alpha' \wedge \kappa \in C(\alpha', \pi) \wedge \mathcal{A}^{\kappa}_{\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi}$ *stationary in* κ'

(iii) $\kappa = \kappa' \wedge \alpha, \xi <^C_\pi \alpha'$

(iv) $\kappa = \kappa' \wedge \alpha = \alpha' \wedge \xi <^C_\pi \xi'$

Proof. We start with "$\Leftarrow$". In case of $\kappa = \pi$, we can just apply Theorem 3.2.4. If (ii) or (iii) holds, the claim follows directly from the definitions. In case of (iv), we get $\pi \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi}]$ and are done in view of Theorem 3.2.4.

So let's turn to "$\Rightarrow$". We assume $\pi < \kappa$. First, we exclude $\kappa < \kappa'$. In that case, $\alpha \leq \alpha'$ is impossible, as then

$$\kappa \in C(\alpha, \pi) \cap \kappa' \subseteq C(\alpha', \pi) \cap \kappa' = \pi$$

is an obvious contradiction. But neither is $\alpha' < \alpha$ possible: Then $\kappa', \alpha', \xi' \in C(\alpha', \pi) \wedge \gamma' < \pi$ implies

$$\kappa', \alpha', \xi' \in C(\alpha, \pi_0) \wedge \gamma' < \pi_0$$

for sufficiently large $\pi_0 \in \mathcal{A}^\kappa_\alpha \cap \mathcal{A}^\mathcal{K}_\xi \cap \pi$, so we get

$$\pi \in C(\alpha, \pi_0) \cap \kappa = \pi_0,$$

a contradiction.

So now assume $\kappa' < \kappa$. Now we get $\alpha < \alpha'$, because $\alpha' \leq \alpha$ would lead to the contradiction

$$\kappa' \in C(\alpha, \pi) \cap \kappa \Rightarrow \kappa' \in C(\alpha, \pi_0) \cap \kappa = \pi_0$$

for sufficiently large $\pi_0 \in \mathcal{A}^\kappa_\alpha \cap \mathcal{A}^\mathcal{K}_\xi \cap \pi$. But $\alpha < \alpha'$ again has

$$\kappa, \alpha, \xi \in C(\alpha, \pi) \subseteq C(\alpha', \pi)$$

as a consequence. Finally, as $\pi \in \mathcal{A}^{\kappa'}_{\alpha'}$, we obtain

$$\mathcal{A}^\kappa_\alpha \cap \mathcal{A}^\mathcal{K}_\xi \text{ stationary in } \kappa',$$

so (ii) holds.

The most interesting case is $\kappa = \kappa'$. Here, $\alpha' < \alpha$ is again impossible, as we would get

$$\kappa, \alpha', \xi' \in C(\alpha, \pi_0) \wedge \gamma' < \pi_0$$

for sufficiently large $\pi_0 \in \mathcal{A}^\kappa_\alpha \cap \mathcal{A}^\mathcal{K}_\xi \cap \pi$, and hence

$$\pi \in C(\alpha, \pi_0) \cap \kappa = \pi_0,$$

an obvious contradiction.

If $\alpha < \alpha'$, then $\alpha, \xi \in C(\alpha, \pi) \subseteq C(\alpha', \pi)$ implies (iii).

In case of $\alpha = \alpha'$, $\xi' < \xi$ would imply (in view of $\pi \in \mathcal{A}^\mathcal{K}_{\xi'}$)

$$\xi' \in C(\xi, \pi_0) \cap \xi \wedge \gamma' < \pi_0$$

for sufficiently large $\pi_0 \in \mathcal{A}^\kappa_\alpha \cap \mathcal{A}^\mathcal{K}_\xi \cap \pi$, so $\pi_0 \in \Pi^1_1[\mathcal{A}^\mathcal{K}_{\xi'}]$, and by Theorem 3.2.4 we would get

$$\pi = \Psi^\kappa_{\alpha,\xi'}[\gamma'] < \pi_0,$$

but $\pi_0 < \pi$. If $\xi = \xi'$, then obviously $\mathcal{A}^\kappa_\alpha \cap \mathcal{A}^\mathcal{K}_\xi$ cannot be unbounded in π. So finally assume $\xi < \xi'$. But then $\xi \in C(\xi, \pi_0)$ for all $\pi_0 \in \mathcal{A}^\mathcal{K}_\xi \cap \pi$ easily implies $\xi \in C(\xi', \pi)$. $\square$

The last lemma of this section is the trickiest one. Notice the surprising fact (and its funny proof) that in the "$\Rightarrow$"-direction, the case $\beta' < \alpha$ just never occurs.

Lemma 3.6.3. *Let $\kappa \in \Pi^1_1[\mathcal{A}^\mathcal{K}_{\xi'}]$, $\mathcal{A}^{\kappa'}_{\beta'} \cap \mathcal{A}^\mathcal{K}_{\xi'}$ stationary in π', $\gamma' < \pi'$, $\alpha', \pi' \in C(\alpha', \pi')$ and $\pi = \Psi^{\pi',\kappa'}_{\alpha',\beta',\xi'}[\gamma']$. If $\kappa \in \Pi^1_1[\mathcal{A}^\mathcal{K}_\xi]$ and $\alpha, \kappa \in C(\alpha, \kappa)$, then*

$$\mathcal{A}^\kappa_\alpha \cap \mathcal{A}^\mathcal{K}_\xi \textit{ is stationary in } \pi$$

iff $\pi' < \kappa$, $\alpha < \alpha'$ and one of the following holds:

(i) $\alpha < \beta' \wedge \kappa' < \kappa \wedge \kappa, \alpha, \xi \in C(\beta', \pi) \wedge \mathcal{A}^{\kappa}_{\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi}$ *stationary in* κ'

(ii) $\alpha < \beta' \wedge \kappa = \kappa' \wedge \alpha, \xi \in C(\beta', \pi)$

(iii) $\alpha = \beta' \wedge \kappa = \kappa' \wedge \xi <^{C}_{\pi} \xi'$

Proof. "$\Leftarrow$". If (i) or (ii) is true, then the claim just follows from the definition of "$\pi \in \mathcal{A}^{\kappa'}_{\beta'}$". In case of (iii), π is Π^1_1-reflecting on $\mathcal{A}^{\mathcal{K}}_{\xi}$, so by Theorem 3.2.4 we get that $\mathcal{A}^{\kappa}_{\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi}$ is stationary in π.

"$\Rightarrow$". First, we verify that $\kappa < \pi'$ is impossible, because if then $\alpha \leq \alpha'$, then

$$\kappa \in C(\alpha, \pi) \cap \pi' \subseteq C(\alpha', \pi) \cap \pi' = \pi,$$

and if $\alpha' < \alpha$, then

$$\pi', \kappa', \alpha', \beta', \xi' \in C(\alpha, \pi_0) \wedge \gamma' < \pi_0$$

for sufficiently large $\pi_0 \in \mathcal{A}^{\kappa}_{\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi} \cap \pi$ yields $\pi \in C(\alpha, \pi_0) \cap \kappa = \pi_0$, so in both cases we get a contradiction.

So from now on let $\boxed{\pi' < \kappa}$.

$\alpha' \leq \alpha$ would now imply

$$\pi' \in C(\alpha', \pi_0) \cap \kappa \subseteq C(\alpha, \pi_0) \cap \kappa = \pi_0$$

for sufficiently large $\pi_0 \in \mathcal{A}^{\kappa}_{\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi} \cap \pi$, so we can additionally assume $\boxed{\alpha < \alpha'}$.

Now $\beta' < \alpha$ leads to contradictions, no matter how κ and κ' are arranged:

- $\kappa' < \kappa$ runs into

  $$\kappa' \in C(\beta', \pi_0) \cap \kappa \subseteq C(\alpha, \pi_0) \cap \kappa = \pi_0$$

  for sufficiently large $\pi_0 \in \mathcal{A}^{\kappa}_{\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi} \cap \pi$.

- So now assume $\kappa \leq \kappa'$. In any case, as $\pi' < \kappa$, we have by definition

  $$\mathcal{A}^{\kappa}_{\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi} \text{ stationary in } \pi',$$

  because $\pi \in \mathcal{A}^{\pi'}_{\alpha'}$ and $\alpha < \alpha'$. Thus we get $\pi, \pi' \in \mathcal{A}^{\kappa}_{\alpha}$.

  Is $\mathcal{A}^{\kappa'}_{\beta'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$ stationary in κ?

  If "yes", then $\pi \in \mathcal{A}^{\kappa}_{\alpha}$ (and $\kappa \leq \kappa'$) implies

  $$\mathcal{A}^{\kappa'}_{\beta'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \text{ stationary in } \pi,$$

  too, but this contradicts Corollary 3.3.4. But if the answer is "no", then $\pi' \in \mathcal{A}^{\kappa}_{\alpha}$ (and $\kappa \leq \kappa'$) yields that

  $$\mathcal{A}^{\kappa'}_{\beta'} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \text{ is not stationary in } \pi',$$

  which in turn contradicts the assumptions.

So assume $\alpha < \beta'$. Now, $\kappa < \kappa'$ leads to the contradiction

$$\kappa \in C(\alpha, \pi) \cap \kappa' \subseteq C(\beta', \pi') \cap \kappa' = \pi'.$$

If $\kappa' \leq \kappa$, then obviously $\kappa, \alpha, \xi \in C(\beta', \pi)$, and because of $\pi \in \mathcal{A}_{\beta'}^{\kappa'}$, $\alpha < \beta'$ and $\kappa' \leq \kappa$, we get

$$\mathcal{A}_\alpha^\kappa \cap \mathcal{A}_\xi^\mathcal{K} \text{ stationary in } \kappa'$$

by definition, so (i) or (ii) holds.

Thus assume $\boxed{\alpha = \beta'}$.

$\kappa < \kappa'$ is impossible by the same argument as just seen. Here, $\kappa' < \kappa$ is also not possible, this time because of

$$\kappa' \in C(\beta', \pi_0) \cap \kappa = C(\alpha, \pi_0) \cap \kappa = \pi_0$$

for sufficiently large $\pi_0 \in \mathcal{A}_\alpha^\kappa \cap \mathcal{A}_\xi^\mathcal{K} \cap \pi$.

So we are left with $\boxed{\kappa = \kappa'}$.

First, $\xi = \xi'$ is clearly contradicting Theorem 3.3.3. If $\xi' < \xi$, then $\xi' \in C(\xi', \pi)$ implies that

$$\xi' \in C(\xi', \pi_0) \cap \xi$$

for sufficiently large $\pi_0 \in \mathcal{A}_\alpha^\kappa \cap \mathcal{A}_\xi^\mathcal{K} \cap \pi$, which shows that these π_0 are Π_1^1-reflecting on $\mathcal{A}_{\xi'}^\mathcal{K}$, and thus elements of $\mathcal{A}_{\xi'}^\mathcal{K}$. So $\mathcal{A}_\alpha^\kappa \cap \mathcal{A}_{\xi'}^\mathcal{K}$ is stationary in π, again contradicting Corollary 3.3.4. But if $\xi < \xi'$, then trivially $\xi \in C(\xi', \pi)$, so (iii) holds. $\square$

4. The Ordinal Notation System

The aim of this short chapter is to introduce a set of ordinal terms, $\mathcal{OT} \subseteq C(\varepsilon_{\mathcal{K}+1}, 0)$, together with a relation $\prec$, which both can be regarded as recursive sets of natural numbers. In order to have good control of the ordinals involved, we already incorporated certain normal form conditions in the definition of $C(\alpha, \beta)$, we are for example only interested in collapses of the form

$$\Psi^{\kappa}_{\alpha,\xi}$$

if we know that $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi}]$ and $\kappa, \alpha \in C(\alpha, \kappa)$. These conditions enabled us to prove (in Section 3.4) the relevant lemmas concerning $<$-relationships between collapses only talking about sub"terms" of these ordinals. This now pays off, as for the recursivity of $\prec$ we can just refer to that section. Likewise, Sections 3.5 and 3.6 imply that the predicates answering questions like

"Is κ Π^1_1-indescribable on $\mathcal{A}^{\mathcal{K}}_{\xi}$?"

or

"Is $\mathcal{A}^{\kappa}_{\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi}$ stationary in π?",

where all ordinals involved satisfy all normal form conditions, are recursive.

Nevertheless, we will have to define recursively a whole bunch of functions and predicates, as for example up to now we tacitly assumed that we can decide whether π is regular or not. The most prominent will be

$$K_{\delta}(\alpha),$$

taking care of the question

"Is $\alpha \in C(\gamma, \delta)$?"

by satisfying

$$\alpha \in C(\gamma, \delta) \Leftrightarrow K_{\delta}(\alpha) < \gamma.$$

So the definitions of this chapter have to be read simultaneously.

Definition. Inductively we define the set of ordinal terms, $\mathcal{OT}$, as follows:

- $0, \mathcal{K} \in \mathcal{OT}$
- If $\alpha_1, \ldots, \alpha_n \in \mathcal{OT}$, then also $\alpha =_{CNF} \alpha_1 + \cdots + \alpha_n \in \mathcal{OT}$.

- If $\alpha_1, \alpha_2 \in \mathcal{OT}$, then also $\alpha =_{VNF} \varphi\alpha_1\alpha_2 \in \mathcal{OT}$.
- If $\pi \in \mathcal{OT}$ and $\pi \notin \mathrm{LimCard}$, then also $p(\pi) \in \mathcal{OT}$.
- If $\alpha, \gamma \in \mathcal{OT}$ and $\gamma < \mathcal{K}$, then also $\Psi^{\mathcal{K}}_{\alpha}[\gamma] \in \mathcal{OT}$.
- If $\kappa, \alpha, \xi, \gamma \in \mathcal{OT}$, $\xi < \alpha$, $\gamma < \kappa$, $\alpha, \kappa \in C(\alpha, \kappa)$ and $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi}]$, then also $\Psi^{\kappa}_{\alpha,\xi}[\gamma] \in \mathcal{OT}$.
- If $\pi, \kappa, \alpha, \beta, \xi, \gamma \in \mathcal{OT}$, $(\xi < \beta < \alpha) \vee (0 = \xi = \beta < \alpha)$, $\gamma < \pi$, $\alpha, \pi \in C(\alpha, \pi)$, $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi}]$ and $\mathcal{A}^{\kappa}_{\beta} \cap \mathcal{A}^{\mathcal{K}}_{\xi}$ stationary in π, then also $\Psi^{\pi,\kappa}_{\alpha,\beta,\xi}[\gamma] \in \mathcal{OT}$.

In order to prove Lemma 4.2, which is essential for the whole procedure, we first show the following

Lemma 4.1. *For every α, $C(\alpha, 0) \cap \omega_1$ is a segment of the ordinals.*

Proof. As $C(0, 0) \cap \omega_1$ is obviously a segment and as unions of segments are again segments, we only have to consider the case $\alpha \rightsquigarrow \alpha + 1$. So assume $C(\alpha, 0) \cap \omega_1$ is a segment, but $C(\alpha + 1, 0) \cap \omega_1$ is not. Let

$$\zeta_{\alpha+1} = \mu\xi.\, \xi \notin C(\alpha + 1, 0),$$

so $C(\alpha + 1, 0) = C(\alpha + 1, \zeta_{\alpha+1})$, and

$$\eta = \mu\xi \in (\zeta_{\alpha+1}, \omega_1).\, \xi \in C(\alpha + 1, 0).$$

Then the only possible case is

$$\eta = \Psi^{\omega_1}_{\beta}[\gamma],$$

where $\beta \in C(\alpha + 1, 0) \cap (\alpha + 1)$ and $\gamma < \zeta_{\alpha+1}$. By the normal form condition we also know $\beta \in C(\beta, \eta)$. Now we show $\zeta_{\alpha+1} \in \mathcal{A}^{\omega_1}_{\beta}$.

- Assume that there is a maximally simple $\xi \in C(\beta, \eta)$ such that $\xi \in C(\alpha+1, \zeta_{\alpha+1}) \setminus C(\beta, \zeta_{\alpha+1})$. Then $\xi \in [\zeta_{\alpha+1}, \eta)$ is impossible, as by choice of η, $C(\alpha + 1, \zeta_{\alpha+1}) \cap [\zeta_{\alpha+1}, \eta) = \emptyset$. But apart from that $C(\beta, \zeta_{\alpha+1})$ has the same closure properties as $C(\beta, \eta)$. So in particular we have $\beta \in C(\beta, \zeta_{\alpha+1})$, i.e. the normal form condition is fulfilled.
- Now let $\beta' \in C(\beta, \zeta_{\alpha+1}) \cap C(\beta', \omega_1) \cap \beta$. We have to show $\zeta_{\alpha+1} \in \mathrm{Lim}(\mathcal{A}^{\omega_1}_{\beta'})$, so pick $\delta < \zeta_{\alpha+1}$. But as $\Psi^{\omega_1}_{\beta'}[\delta] \in C(\alpha + 1, \zeta_{\alpha+1}) \cap \eta$ (obviously, $\eta \in \mathrm{Lim}(\mathcal{A}^{\omega_1}_{\beta'})$), $\Psi^{\omega_1}_{\beta'}[\delta] \geq \zeta_{\alpha+1}$ would contradict the minimality of η.

But then $\Psi^{\omega_1}_{\beta}[\gamma] \leq \zeta_{\alpha+1}$, contradicting $\zeta_{\alpha+1} < \eta$. ∎

In the following let $\rho_0 = 1$ and $\rho_{n+1} = \mathcal{K}^{\rho_n}$.

If we define

$$\Psi^{\omega_1}_{\varepsilon_{\mathcal{K}+1}} = \sup\{\Psi^{\omega_1}_{\rho_n} \mid n \in \omega\},$$

then we get

Lemma 4.2. (i) $\mathcal{OT} = C(\varepsilon_{\mathcal{K}+1}, 0)$ *and*

(ii) $C(\varepsilon_{\mathcal{K}+1}, 0) \cap \omega_1 = \Psi^{\omega_1}_{\varepsilon_{\mathcal{K}+1}}$.

Proof. As (i) is pretty obvious, we will only prove (ii). For "$\subseteq$" pick $\xi \in C(\varepsilon_{\mathcal{K}+1}, 0) \cap \omega_1$. But then there is an n such that $\xi \in C(\rho_n, 0) \cap \omega_1 \subseteq C(\rho_n, \Psi^{\omega_1}_{\rho_n}) \cap \omega_1 = \Psi^{\omega_1}_{\rho_n} < \Psi^{\omega_1}_{\varepsilon_{\mathcal{K}+1}}$. For the other direction note that the ρ_n are "parameter-free", so $\Psi^{\omega_1}_{\rho_n} \in C(\rho_{n+1}, 0) \cap \omega_1$, and as the latter is a segment, also $\Psi^{\omega_1}_{\rho_n} \subseteq C(\rho_{n+1}, 0) \cap \omega_1$. □

Definition. We define sets $K_\delta(\alpha)$ as follows:

$$K_\delta(\alpha) = \begin{cases} \bigcup\{K_\delta(\beta) \mid \beta \in SCP(\alpha)\} & \text{if } \alpha \notin \mathrm{SC} \\ \emptyset & \text{if } \alpha \in \delta \cup \{0, \mathcal{K}\} \\ K_\delta(\beta) \cup K_\delta(\gamma) \cup \{\beta\} & \text{if } \delta \leq \alpha = \Psi^{\mathcal{K}}_{\beta}[\gamma] \\ K_\delta(\kappa) \cup K_\delta(\alpha_0) \cup K_\delta(\xi) \cup K_\delta(\gamma) \cup \{\alpha_0, \xi\} & \text{if } \delta \leq \alpha = \Psi^{\kappa}_{\alpha_0,\xi}[\gamma] \\ K_\delta(\pi) \cup K_\delta(\kappa) \cup K_\delta(\alpha_0) \cup K_\delta(\beta) \cup & \\ \qquad K_\delta(\xi) \cup K_\delta(\gamma) \cup \{\alpha_0, \beta, \xi\} & \text{if } \delta \leq \alpha = \Psi^{\pi,\kappa}_{\alpha_0,\beta,\xi}[\gamma] \end{cases}$$

If we regard "$K_\delta(\alpha) < \gamma$" as an abbreviation for

$$(\forall \zeta \in K_\delta(\alpha))(\zeta < \gamma),$$

we more or less straightforwardly obtain the following lemma:

Lemma 4.3. *For ordinals $\alpha, \delta, \gamma \in \mathcal{OT}$ we have*

$$\alpha \in C(\gamma, \delta) \Leftrightarrow K_\delta(\alpha) < \gamma.$$

Definition. Together with $\mathcal{OT}$, $\prec$ and the above K, we define

- the function *ind* from Section 3.5,
- a predicate $\mathcal{ST}$ such that

  $$(\kappa, \alpha, \xi, \pi) \in \mathcal{ST} \Leftrightarrow \mathcal{A}^{\kappa}_{\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi} \text{ stationary in } \pi$$

  (see Section 3.6),
- predicates $\mathcal{LIM}$ (denoting the limit ordinals), $\mathcal{P}$ (additively indecomposable ordinals), $\mathcal{SC}$ (strongly critical ordinals), $\mathcal{CARD}$ (cardinals), $\mathcal{LIMCARD}$ (limit cardinals) and $\mathcal{REG}$ (regular cardinals).

Henceforth we will restrict ourselves to ordinals from $\mathcal{OT}$.

Remark. This small assertion has far-reaching consequences. It will render the infinitary calculus which we will devise in Section 5.2 unsound "almost everywhere"; as we will only be able to talk about few terms, not every derivable formula of the form

$$(\forall x \in L_{\mathcal{K}})F(x)$$

will be true in L. We will only have soundness for very particular derivations — those of $\Sigma_1^{\omega_1}$-formulas the derivation length and cut rank of which are both below $\Psi^{\omega_1}_{\varepsilon_{\mathcal{K}+1}}$, because as we have seen in Lemma 4.2, all ordinals below $\Psi^{\omega_1}_{\varepsilon_{\mathcal{K}+1}}$ get a name in $\mathcal{OT}$ — but this is just enough. We want to stress a funny symmetry here (illuminated by Prof. Pohlers): one could also approach the problem from the other side (one should probably also do so both for heuristic and historic reasons) — allow all ordinals in the calculus (which leaves it sound everywhere), but then for almost no endsequent the set of ordinals needed for its derivation will be recursive, just in the the above mentioned case. We chose the first option since it is certainly more fun to work with a completely unsound system.

Remark. The recursiveness of $\mathcal{OT}$ implies that $\Psi^{\omega_1}_{\varepsilon_{\mathcal{K}+1}}$ is indeed $< \omega_1^{\mathrm{CK}}$, so that in principle we could replace ω_1 by ω_1^{CK}. It is, however, much more involved to understand that *all* cardinals which appeared can be replaced by their recursive counterparts — here one would have to substitute all cardinality and indescribability arguments by daunting complexity considerations in order to prove all the structure theory, see [Sch93].

Part II.

Collapsing

5. Operator-controlled derivations

In this chapter we will introduce the language of ramified set theory and the infinitary calculus that we will utilize for the analysis of $\mathbf{\Pi_4-Ref}$. Here, the key technical tool are so-called "operator-controlled derivations" which were invented by BUCHHOLZ in [Buc92]. One of the advantages of this concept is its clarity — it bans all mentions of collapsing functions from the definition of the semiformal calculus. They only show up when they are really indispensable, i.e. in the collapsing theorems. The final section treats predicative cut-elimination, which can be proved using arbitrary good operators.

5.1. The Language and Rules of $RS(\mathcal{K})$

Guided by SCHÜTTE's analysis of **PA**, where he introduced the ω-rule

$$\frac{\big(\Gamma, A(\underline{n})\big)_{n\in\mathbb{N}}}{\Gamma, (\forall x)A(x)}$$

in order to be able to derive all axioms of **PA** (in this case, only instances of mathematical induction were critical), we aim for an analogous approach, which will indeed enable us to derive all axioms of $\mathbf{\Pi_4-Ref}$ but the critical scheme (Π_4-Ref) of reflection itself. When talking about systems of set theory it is, however, not at all clear what their canonical models are and how to name their elements. Therefore it is quite helpful that we only need to talk about models within the constructible hierarchy: it is so "thin" that we can denominate all its elements once we have names for the ordinals. So in this section we turn to defining the language and calculus of ramified set theory. Syntactical equality will be denoted by "$\equiv$".

Definition. We augment the language $\mathcal{L}_\in$ of set theory by new unary predicate symbols 3-refl$_\xi$ and 2-refl$^\kappa_{\xi,\xi'}$. (Their intended meaning is, of course, $\{L_\rho \mid \rho \in \mathcal{A}^{\mathcal{K}}_\xi\}$ and $\{L_\rho \mid \rho \in \mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'}\}$, respectively.) Slightly abusing notation, we will call the resulting language $\mathcal{L}_{Ad}$. The *atomic formulas* of $\mathcal{L}_{Ad}$ are those of the form $a \in b$, $\neg(a \in b)$, 3-refl$_\xi(a)$, $\neg$(3-refl$_\xi(a)$), 2-refl$^\kappa_{\xi,\xi'}(a)$ and $\neg$(2-refl$^\kappa_{\xi,\xi'}(a)$). Closure under $\wedge$, $\vee$, $(\exists x \in a)$, $(\forall x \in a)$, $(\exists x)$ and $(\forall x)$ generates all *$\mathcal{L}_{Ad}$-formulas.* (Note that we do not count equality as a symbol of the language but consider it defined via

$$a = b \Leftrightarrow a \subseteq b \wedge b \subseteq a.)$$

Definition. We define *$\mathcal{L}_{RS(\mathcal{K})}$-terms* and their *levels* as follows:

- For every α, L_α is an $\mathcal{L}_{RS(\mathcal{K})}$-term of level $|\mathrm{L}_\alpha| = \alpha$.

- If F is an $\mathcal{L}_{Ad}$-formula and $\vec{a}$ are $\mathcal{L}_{RS(\mathcal{K})}$-terms of levels $< \alpha$, then $[x \in \mathrm{L}_\alpha \mid F^{\mathrm{L}_\alpha}(x, \vec{a})]$ is also an $\mathcal{L}_{RS(\mathcal{K})}$-term; its level is α.

By T_ρ we denote the $\mathcal{L}_{RS(\mathcal{K})}$-terms of level $< \rho$; we will identify $T_\mathcal{K}$ and T.

Definition. Finally we define $\mathcal{L}_{RS(\mathcal{K})}$-formulas as expressions of the form

$$F[t_1, \ldots, t_n]^{\mathrm{L}_\rho},$$

i.e. all unbounded variables are bounded by L_ρ, where $F[a_1, \ldots, a_n]$ is an $\mathcal{L}_{Ad}$-formula (with free variables among $\{a_1, \ldots, a_n\}$), $t_1, \ldots, t_n \in T_\rho$ and $\rho \leq \mathcal{K}$.

Notice that $\mathcal{L}_{RS(\mathcal{K})}$-formulas contain no free variables and no unbounded quantifiers. To stress the fact that some $F \in \mathcal{L}_{RS(\mathcal{K})}$ is of the above mentioned form (i.e. its "widest" quantifiers range over L_ρ and all its parameters are $\in T_\rho$), we will often write $F^{(\mathrm{L}_\rho)}$, or even shorter

$$F^{(\rho)}.$$

If in this situation ρ gets replaced by ρ', which is still larger than all other parameters in F, we will write

$$F^{(\rho,\rho')}.$$

These conventions also extend to sets of formulas.

Let $\mathfrak{X}$ be a set of $\mathcal{L}_{RS(\mathcal{K})}$-terms and -formulas. In $\mathrm{par}(\mathfrak{X})$ we gather all the ordinals needed to build $\mathfrak{X}$: $\alpha \in \mathrm{par}(\mathfrak{X})$ if either L_α occurs in one of the elements of $\mathfrak{X}$ or a formula of the shape "$(\neg)$3-refl$_{\xi_0}(a)$" or "$(\neg)$2-refl$^{\xi_1}_{\xi_2,\xi_3}(a)$" is in $\mathfrak{X}$ and $\alpha = \xi_i$ for some $i < 4$.

From now on we will use the expressions "Σ_n-formula" and "Π_n-formula" in a strict sense, so we define

Definition. Formulas of set theory which contain no unbounded quantifiers are called Δ_0 (or Σ_0 or Π_0); formulas of the shape $(\forall x)F_0(x)$ with $F_0 \in \Sigma_n$ are called Π_{n+1}; analogously, formulas of the shape $(\exists x)F_0(x)$ with $F_0 \in \Pi_n$ are called Σ_{n+1}.

We will need to talk about local Levy-hierarchies, which leads us to the following

Definition. An $\mathcal{L}_{RS(\mathcal{K})}$-formula G is said to be $\Sigma_n(\mathrm{L}_\rho)$ ($\Pi_n(\mathrm{L}_\rho)$, respectively, or even shorter $\Sigma_n(\rho)$ or $\Pi_n(\rho)$, respectively) iff there are a $\Sigma_n(\mathcal{L}_{Ad})$-formula ($\Pi_n(\mathcal{L}_{Ad})$-formula, respectively)

$$F[a_1, \ldots, a_n]$$

and T_ρ-terms $t_1, \ldots, t_n$ such that

$$G \equiv F[t_1, \ldots, t_n]^{(\rho)}.$$

The standard interpretation $\cdot^L$ of $\mathcal{L}_{RS(\mathcal{K})}$-terms is of course

- $(\mathrm{L}_\alpha)^L = L_\alpha$
- $([x \in \mathrm{L}_\alpha \mid F^{\mathrm{L}_\alpha}(x, \vec{a})])^L = \{x \in L_\alpha \mid L_\alpha \models F(x, \vec{a}^L)\}$

Note that under this interpretation there are many terms denoting the same element of L. In particular there are terms with large levels denoting elements of small L-rank. But as for such elements a small term can always be found as well, we do not need to talk about all terms when we are only interested in L_α (for example when asking whether or not $(\exists x \in L_\alpha)F(x)$; see the respective cases in the definition of the following infinitary calculi). This motivates the following

Definition. Let $s, t \in T$ such that $|s| < |t| = \alpha$. We define

$$s \mathrel{\overset{\circ}{\in}} t \equiv \begin{cases} F(s, \vec{a}) & \text{if } t \equiv [x \in \mathrm{L}_\alpha \mid F^{\mathrm{L}_\alpha}(x, \vec{a})] \\ s \notin \mathrm{L}_0 & \text{if } t \equiv \mathrm{L}_\alpha \end{cases}$$

Remark. "$\overset{\circ}{\in}$" and "$\in$" have the same meaning under the standard interpretation.

Definition. The *rules* of $RS(\mathcal{K})$ are the following:

$$(\wedge) \qquad \frac{\Gamma, A_0 \quad \Gamma, A_1}{\Gamma, A_0 \wedge A_1}$$

$$(\vee) \qquad \frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1} \quad \text{if } i \in \{0,1\}$$

$$(\forall) \qquad \frac{\big(\Gamma, s \overset{\circ}{\in} t \to F(s)\big)_{s \in T_{|t|}}}{\Gamma, (\forall x \in t)F(x)}$$

$$(\exists) \qquad \frac{\Gamma, s \overset{\circ}{\in} t \wedge F(s)}{\Gamma, (\exists x \in t)F(x)} \quad \text{if } s \in T_{|t|}$$

$$(\notin) \qquad \frac{\big(\Gamma, s \overset{\circ}{\in} t \to r \neq s\big)_{s \in T_{|t|}}}{\Gamma, r \notin t}$$

$$(\in) \qquad \frac{\Gamma, s \overset{\circ}{\in} t \wedge r = s}{\Gamma, r \in t} \quad \text{if } s \in T_{|t|}$$

$$(\neg\text{3-refl}_\xi) \qquad \frac{\big(\Gamma, \mathrm{L}_\rho \neq t\big)_{\rho \in \mathcal{A}^{\mathcal{K}}_\xi, \rho \leq |t|}}{\Gamma, \neg\text{3-refl}_\xi(t)}$$

$$(\text{3-refl}_\xi) \qquad \frac{\Gamma, \mathrm{L}_\rho = t}{\Gamma, \text{3-refl}_\xi(t)} \quad \text{if } \rho \in \mathcal{A}^{\mathcal{K}}_\xi \text{ and } \rho \leq |t|$$

$$(\neg\text{2-refl}^\kappa_{\xi,\xi'}) \qquad \frac{\big(\Gamma, \mathrm{L}_\rho \neq t\big)_{\rho \in \mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'}, \rho \leq |t|}}{\Gamma, \neg\text{2-refl}^\kappa_{\xi,\xi'}(t)}$$

$$(\text{2-refl}^\kappa_{\xi,\xi'}) \qquad \frac{\Gamma, \mathrm{L}_\rho = t}{\Gamma, \text{2-refl}^\kappa_{\xi,\xi'}(t)} \quad \text{if } \rho \in \mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \text{ and } \rho \leq |t|$$

$$(\text{cut}) \qquad \frac{\Gamma, C \quad \Gamma, \neg C}{\Gamma}$$

$$(\text{4-Ref}^{\mathcal{K}}) \qquad \frac{\Gamma, F}{\Gamma, (\exists z^{\mathcal{K}})(z \models F)} \quad \text{if } F \in \Pi_4(\mathcal{K})$$

$$(\text{3-Ref}^\kappa_\xi) \qquad \frac{\Gamma, F}{\Gamma, (\exists z^\kappa)(\text{3-refl}_\xi(z) \wedge z \models F)} \quad \text{if } F \in \Pi_3(\kappa) \text{ and } \kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\xi]$$

$$(\text{2-Ref}^{\pi,\kappa}_{\xi,\xi'}) \qquad \frac{\Gamma, F}{\Gamma, (\exists z^\pi)(\text{2-refl}^\kappa_{\xi,\xi'}(z) \wedge z \models F)} \quad \text{if } \begin{cases} F \in \Pi_2(\pi), \kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}] \\ \text{and } \mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \text{ stationary in } \pi \end{cases}$$

5.2. The infinitary calculus

A closer look shows that the $\mathcal{L}_{RS(\mathcal{K})}$-sentences can be divided into two groups: those of $\bigwedge$-type and those of $\bigvee$-type. This allows for a concise (re-)formulation of the infinitary calculus.

Definition. We assign a conjunction or disjunction (with index set J) to every $\mathcal{L}_{RS(\mathcal{K})}$-sentence:

- $A_0 \vee A_1 \simeq \bigvee(A_t)_{t\in J}$ with $J = \{0,1\}$
- $a \in b \simeq \bigvee(t \mathrel{\mathring{\in}} b \wedge t = a)_{t\in J}$ with $J = T_{|b|}$
- $(\exists x \in b)F(x) \simeq \bigvee(t \mathrel{\mathring{\in}} b \wedge F(t))_{t\in J}$ with $J = T_{|b|}$
- 3-refl$_\xi(a) \simeq \bigvee(t = a)_{t\in J}$ with $J = \{\mathrm{L}_\rho \mid \rho \in \mathcal{A}^{\mathcal{K}}_{\xi} \wedge \rho \leq |a|\}$
- 2-refl$^{\kappa}_{\xi,\xi'}(a) \simeq \bigvee(t = a)_{t\in J}$ with $J = \{\mathrm{L}_\rho \mid \rho \in \mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \wedge \rho \leq |a|\}$

These formulas will be called *of* $\bigvee$*-type.* Dually we set

- $\neg F \simeq \bigwedge(\neg F_t)_{t\in J}$ if $F \simeq \bigvee(F_t)_{t\in J}$

and call such formulas *of* $\bigwedge$*-type.*

So every $\mathcal{L}_{RS(\mathcal{K})}$-sentence has a set of *characteristic subformulas* (determined by the J's above), which we will refer to as $CS(F)$. The term t figuring prominently in every characteristic subformula will be called *characteristic term.*

The following definition of a rank of an $\mathcal{L}_{RS(\mathcal{K})}$-sentence is technically somewhat involved. Its only purpose, however, is to make Lemma 5.2.1 work.

Definition. We define the *rank* of $\mathcal{L}_{RS(\mathcal{K})}$-sentences as follows

- $\mathrm{rk}(A_0 \vee A_1) = \max\{\mathrm{rk}(A_0), \mathrm{rk}(A_1)\} + 1$
- $\mathrm{rk}((\exists x \in a)A_0(x)) = \begin{cases} \max\{\omega \cdot (3 \cdot |a|), \mathrm{rk}(A_0(\mathrm{L}_0)) + 2\} & \text{if } a \equiv \mathrm{L}_\alpha \\ \max\{\omega \cdot (3 \cdot |a| + 1), \mathrm{rk}(A_0(\mathrm{L}_0))\} & \text{else} \end{cases}$
- $\mathrm{rk}(a \in b) = \max\{\omega \cdot (3 \cdot |a| + 2), \omega \cdot (3 \cdot |b| + 1)\}$
- $\mathrm{rk}(\text{3-refl}_\xi(a)) = \mathrm{rk}(\text{2-refl}^{\kappa}_{\xi,\xi'}(a)) = \omega \cdot (3 \cdot |a| + 2)$
- $\mathrm{rk}(A) = \mathrm{rk}(\neg A)$

Now we can state the intended lemma.

Lemma 5.2.1. *Let F be an $\mathcal{L}_{RS(\mathcal{K})}$-sentence.*

(i) *If $F \simeq \bigvee(F_t)_{t\in J}$ or $F \simeq \bigwedge(F_t)_{t\in J}$, then* $\mathrm{rk}(F_t) < \mathrm{rk}(F)$ *for all $t \in J$.*

(ii) *If* $\gamma \in \text{Eps}$*, then*

$$\text{rk}(F) = \gamma \Leftrightarrow F \equiv (\exists x \in \text{L}_\gamma)F_0(x) \textit{ or } F \equiv (\forall x \in \text{L}_\gamma)F_0(x)$$

Proof. See for example [Bla97]; notice that although our language is richer, there was no need to really modify the definitions, so the proof is literally the same. □

We pause for a moment for a digression. The following considerations are from [Poh98]. Let's drop the limitation to ordinals from $\mathcal{OT}$ for a moment. Then we get

$$L \models F \Leftrightarrow L \models \bigvee_{G \in CS(F)} G$$

for F in $\bigvee$-type and dually

$$L \models F \Leftrightarrow L \models \bigwedge_{G \in CS(F)} G$$

for F in $\bigwedge$-type. Thus defining a verification calculus $\overset{\alpha}{\models}$ by

($\bigvee$) If F is of $\bigvee$-type, $\overset{\alpha_0}{\models} \Delta, G_t$ where $G_t \in CS(F)$, t is its characteristic term and $\alpha_0, |t| < \alpha$, then $\overset{\alpha}{\models} \Delta, F$

($\bigwedge$) If F is of $\bigwedge$-type and $\overset{\alpha_G}{\models} \Delta, G$ with $\alpha_G < \alpha$ holds for all $G \in CS(F)$, then $\overset{\alpha}{\models} \Delta, F$

would imply

$$L \models F \Leftrightarrow (\exists \alpha) \overset{\alpha}{\models} F.$$

So if we only consider $\Sigma_1(\leq \mathcal{K})$-sentences, we also get

$$\overset{\alpha}{\models} F \Rightarrow L_\alpha \models F.$$

Putting

$$\text{tc}(F) = \begin{cases} \min\{\alpha \mid \overset{\alpha}{\models} F\} & \text{if } L \models F \\ \infty & \text{otherwise} \end{cases}$$

then immediately implies

$$\min\{\alpha \mid L_\alpha \models F\} = |F|_{\Sigma_1} \leq \text{tc}(F).$$

As by Lemma 5.2.1 we also get

$$L \models F \Rightarrow \overset{\alpha}{\models} F$$

for $\alpha = \text{rk}(F)$, we can see that the rank of F is an upper bound for the above defined truth complexity of F. Alas, not much is gained by this, as for example the rank of (strict) $\Sigma_1^{\omega_1^{\text{CK}}}$-sentences is not less than ω_1^{CK}. Anyway, the above shows that any rule valid in L (such as the cut rule

$$\models^{\alpha_0} \Delta, C \text{ and } \models^{\alpha_1} \Delta, \neg C \Rightarrow \models^{\alpha} \Delta)$$

are valid in this verification calculus (although this fact gives no hint at how to compute α from α_0 and α_1) and thus cut-elimination alone cannot be the main problem of an ordinal analysis. We also need to collapse derivations. Therefore we will introduce a calculus which allows only particular derivations, but is strong enough to embed $\mathbf{\Pi_4}$**−Ref** into it and admits both collapsing and cut-elimination. How can this technically be achieved?

> Fortunately, Buchholz has provided us with a very elegant and flexible setting for describing uniformity in infinite proofs, called *operator controlled derivations*. (Rathjen in [Rat99].)

The key technical tool thus are *controlling operators*.

Definition. Let $\text{Pow}(\text{On}) = \{X \subseteq \text{On} \mid X \text{ is a set}\}$. A class-function

$$\mathcal{H}\colon \text{Pow}(\text{On}) \to \text{Pow}(\text{On})$$

will be called *operator*. By $\mathcal{H}[\mathfrak{X}]$ we denote the operator

$$X \mapsto \mathcal{H}(\mathfrak{X} \cup X),$$

where $\mathfrak{X}$ is a set of ordinals. (Sometimes we will slightly abuse this notation and also allow $\mathfrak{X}$ to contain $\mathcal{L}_{Ad}$-terms, when we mean the set of parameters occurring in these terms.) Simplifying notation we will often abbreviate $\mathcal{H}(\emptyset)$ by $\mathcal{H}$ and

$$(\forall X \in \text{Pow}(\text{On}))(\mathcal{H}(X) \subseteq \mathcal{H}'(X))$$

by $\mathcal{H} \subseteq \mathcal{H}'$. An operator $\mathcal{H}$ will be called *good*, if it fulfills the following conditions:

($H0$) $\quad 0 \in \mathcal{H}(= \mathcal{H}(\emptyset))$

($H1$) $\quad \mathcal{H}$ is Cantor*ian-closed*, i.e.

$$(\forall X)(\forall \alpha_1) \cdots (\forall \alpha_n)[\alpha_1, \ldots, \alpha_n \in \mathcal{H}(X) \Leftrightarrow \omega^{\alpha_1} \oplus \cdots \oplus \omega^{\alpha_n} \in \mathcal{H}(X)]$$

($H2$) $\quad (\forall X)[X \subseteq \mathcal{H}(X)]$

($H3$) $\quad (\forall X)(\forall Y)[X \subseteq \mathcal{H}(Y) \Rightarrow \mathcal{H}(X) \subseteq \mathcal{H}(Y)]$

Remarks. (i) Good operators are in particular closed under $+$ and $\omega^{\cdot}$.

(ii) Good operators are monotone, i.e. $X \subseteq Y \Rightarrow \mathcal{H}(X) \subseteq \mathcal{H}(Y)$.

Now we can define the final semiformal calculus and with it the notion of "operator-controlled derivability".

Definition. Let $\mathcal{H}$ be a good operator. In the following (and for the rest of this thesis), writing

$$\mathcal{H}\overset{\beta}{\underset{\rho}{\vdash}} \Delta$$

comes with the proviso that $\{\beta\} \cup \mathrm{par}(\Delta) \subseteq \mathcal{H}$. Then we define by induction on α:

$$\mathcal{H}\overset{\alpha}{\underset{\rho}{\vdash}} \Gamma$$

holds iff one of the following cases occurs:

($\bigvee$) $\bigvee(F_t)_{t\in J} \in \Gamma$ and $\mathcal{H}\overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, F_{t_0}$ with $\alpha_0 < \alpha$ for some $t_0 \in J$

($\bigwedge$) $\bigwedge(F^1_{t_1})_{t_1\in J_1}, \ldots, \bigwedge(F^n_{t_n})_{t_n\in J_n} \in \Gamma$ and $\mathcal{H}[\vec{t}]\overset{\alpha_{\vec{t}}}{\underset{\rho}{\vdash}} \Gamma, F^1_{t_1}, \ldots, F^n_{t_n}$ with $|t_1|, \ldots, |t_n| \leq \alpha_{\vec{t}} < \alpha$ holds for all $t_1, \ldots, t_n = \vec{t} \in J = J_1 \times \cdots \times J_n$

(cut) $\mathrm{rk}(C) < \rho$ and both $\mathcal{H}\overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, C$ and $\mathcal{H}\overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, \neg C$ with $\alpha_0 < \alpha$

(4-Ref$^{\mathcal{K}}$) $F \in \Pi_4(\mathcal{K})$, $(\exists z^{\mathcal{K}})(z \models F) \in \Gamma$ and $\mathcal{H}\overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, F$ with $\alpha_0, \mathcal{K} < \alpha$

(3-Ref$^{\kappa}_{\xi}$) $F_i(t_i) \in \Pi_3(\kappa)$ for all $i \leq k$, $(\exists z^{\kappa})\big(\text{3-refl}_{\xi}(z) \wedge \bigwedge_{i\leq k} (\exists x \in z)(F_i(x))^{(\kappa,z)}\big) \in \Gamma$, $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi}]$, $\kappa, \xi \in \mathcal{H}$ and $\mathcal{H}\overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, \bigwedge_{i\leq k} F_i(t_i)$ with $\kappa, \alpha_0 < \alpha$

(2-Ref$^{\pi,\kappa}_{\xi,\xi'}$) $F_i(t_i) \in \Pi_2(\pi)$ for all $i \leq k$, $(\exists z^{\pi})\big(\text{2-refl}^{\kappa}_{\xi,\xi'}(z) \wedge \bigwedge_{i\leq k} (\exists x \in z)(F_i(x))^{(\pi,z)}\big) \in \Gamma$, $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}]$, $\mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$ stationary in π, $\kappa, \xi, \xi' \in \mathcal{H}$ and $\mathcal{H}\overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, \bigwedge_{i\leq k} F_i(t_i)$ with $\pi, \alpha_0 < \alpha$

Remarks. (i) Notice that existential witnesses t_0 in ($\bigvee$), parameters in (cut)-formulas C and the t_i occurring in the (3-Ref$^{\kappa}_{\xi}$)- and (2-Ref$^{\pi,\kappa}_{\xi,\xi'}$)-rules have to be "controlled", i.e. they must have been in $\mathcal{H}$.

(ii) We opted for a slightly unusual formulation of ($\bigwedge$) and the reflection rules analogously to [Bla97] in order to avoid as much reasoning in intermediate calculi as possible. For a more conventional formulation see [Rat94b].

(iii) From now on we will abbreviate

"$\mathcal{H}\overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, C$ and $\mathcal{H}\overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, \neg C$"

by $\mathcal{H}\overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, (\neg)C$.

Remark. Admitting all ordinals in the above calculus, we get the transition to the verification calculus introduced on page 56 (and hence to truth in L) by

$$\mathcal{H}\vert\!\frac{\alpha}{\rho}\ \Delta\ \Rightarrow \models^{\alpha} \Delta.$$

(See [Poh98] for details; prove

$$\mathcal{H}\vert\!\frac{\alpha}{\rho}\ \Delta, \Gamma \text{ and } L \not\models \bigvee \Gamma\ \Rightarrow \models^{\alpha} \Delta;$$

if the last inference was ($\bigwedge$) with main formula $F \in \Gamma$, it is crucial that *every L-term*, i.e. every ordinal, was allowed, so we can use an induction hypothesis like

$$\mathcal{H}\vert\!\frac{\alpha_G}{\rho}\ \Delta, \Gamma, G,$$

where $G \in CS(F)$ and $L \not\models G$.)

However, the proof also shows that this fact is locally true, too. If all ordinals below α are allowed in the calculus and if $\mu, \rho \leq \alpha$, then

$$\mathcal{H}\vert\!\frac{\alpha}{\mu}\ \Delta^{(\rho)}$$

implies $\models^{\alpha} \Delta^{(\rho)}$. This will be crucial in the final theorem, where we have to translate derivability into truth in L — we know from Lemma 4.2 that all ordinals below $\Psi^{\omega_1}_{\varepsilon_{\mathcal{K}+1}}$ are in $\mathcal{OT}$!

Some easy consequences of the above definition are collected in the following

Lemma 5.2.2. *We have:*

(i) *If* $\mathcal{H}\vert\!\frac{\alpha}{\rho}\ \Gamma$, $\alpha \leq \alpha' \in \mathcal{H}$, $\rho \leq \rho'$ *and* $\mathrm{par}(\Delta) \subseteq \mathcal{H}$, *then also* $\mathcal{H}\vert\!\frac{\alpha'}{\rho'}\ \Gamma, \Delta$.

(ii) $\mathcal{H}\vert\!\frac{\alpha}{\rho}\ \Gamma, \bigwedge(F_t)_{t\in J}$ *implies* $\mathcal{H}[t]\vert\!\frac{\alpha}{\rho}\ \Gamma, F_t$ *for all* $t \in J$.

(iii) *If* $\xi \in \mathcal{H}$, $\xi' \leq \xi$ *and* $\mathcal{H}\vert\!\frac{\alpha}{\rho}\ \Gamma, (\exists x^{\xi'})F(x)$, *then* $\mathcal{H}\vert\!\frac{\alpha}{\rho}\ \Gamma, (\exists x^{\xi})F(x)$.

(iv) *If* $\mathcal{H}\vert\!\frac{\alpha}{\rho}\ \Gamma, A_0 \vee A_1$, *then also* $\mathcal{H}\vert\!\frac{\alpha}{\rho}\ \Gamma, A_0, A_1$.

Proof. The proofs are easy inductions on α. We will sketch the proof of (ii) because of the unusual definition of ($\bigwedge$). If the last inference was not ($\bigwedge$) or if $\bigwedge(F_t)_{t\in J}$ was not one of the main formulas of the last ($\bigwedge$)-inference, the claim follows immediately by induction hypothesis. So let us assume that we had

$$\mathcal{H}[\vec{t}]\vert\!\frac{\alpha_{\vec{t}}}{\rho}\ \Gamma, F^1_{t_1}, \ldots, F^n_{t_n}$$

with $|t_1|, \ldots, |t_n| \leq \alpha_{\vec{t}} < \alpha$ for all $t_1, \ldots, t_n \in \bar{J}$ and

$$\bigwedge(F^1_{t_1})_{t_1\in \bar{J}_1}, \ldots, \bigwedge(F^n_{t_n})_{t_n\in \bar{J}_n} \in \Gamma.$$

We may further assume that $F = F^n$, so that $\bar{J} = \bar{J}_1 \times \cdots \times \bar{J}_{n-1} \times J$. In particular, for fixed $t_n \in J$ and for all $t_1, \ldots, t_{n-1} \in \bar{J}_1 \times \cdots \times \bar{J}_{n-1}$ we have

$$\mathcal{H}[\vec{t}] \left|\frac{\alpha_{\vec{t}}}{\rho}\right. \Gamma, F^n_{t_n}, F^1_{t_1}, \ldots, F^{n-1}_{t_{n-1}}.$$

Thus an application of $(\bigwedge)$ yields

$$\mathcal{H}[t_n] \left|\frac{\alpha}{\rho}\right. \underbrace{\Gamma, \bigwedge(F^1_{t_1})_{t_1 \in \bar{J}_1}, \ldots, \bigwedge(F^{n-1}_{t_{n-1}})_{t_{n-1} \in \bar{J}_{n-1}}}_{=\Gamma}, F^n_{t_n}$$

for all $t_n \in J$. ☐

5.3. Predicative Cut Elimination

From now on we will additionally assume that the controlling operator is closed under φ, i.e. satisfies

$$(\forall X)(\forall \alpha_1)(\forall \alpha_2)[\alpha_1, \alpha_2 \in \mathcal{H}(X) \Leftrightarrow \varphi\alpha_1\alpha_2 \in \mathcal{H}(X)].$$

Then we can eliminate all cuts involving formulas which cannot be the main formulas of reflection rules, i.e. formulas the rank of which is not regular. The well-known idea is that in this case the inferences are (more or less) symmetrical, allowing cuts to be eliminated at an earlier stage. The key lemma is the following:

Lemma 5.3.1 (Reduction). *If $A \cong \bigvee(A_t)_{t \in J}$, $\mathrm{rk}(A) \leq \rho$ and $\mathrm{rk}(A)$ is not regular, then*

$$\mathcal{H}[\mathfrak{X}] \left|\frac{\alpha}{\rho}\right. \Gamma, \neg A \quad \textit{and} \quad \mathcal{H}[\mathfrak{X}] \left|\frac{\beta}{\rho}\right. \Delta, A$$

imply

$$\mathcal{H}[\mathfrak{X}] \left|\frac{\alpha + \beta}{\rho}\right. \Gamma, \Delta.$$

Proof. Induction on β. If A was not the main formula of the last inference $(\mathcal{R})$, we can just apply the induction hypothesis and use $(\mathcal{R})$ again. So let's assume that A was the main formula of the last inference, which then must have been $(\bigvee)$ as otherwise $\mathrm{rk}(A)$ would be regular. So we had the hypothesis

$$\mathcal{H}[\mathfrak{X}] \left|\frac{\beta_0}{\rho}\right. \Delta, A, A_{t_0} \tag{$*$}$$

for some $t_0 \in J$. Thus we get by induction hypothesis

$$\mathcal{H}[\mathfrak{X}] \left|\frac{\alpha + \beta_0}{\rho}\right. \Gamma, \Delta, A_{t_0}. \tag{$**$}$$

On the other hand, inversion (Lemma 5.2.2 (ii)) gives us a derivation

$$\mathcal{H}[\mathfrak{X}, t_0] \left|\frac{\alpha}{\rho}\right. \Gamma, \neg A_{t_0},$$

and as $(*)$ implies that $t_0 \in \mathcal{H}[\mathfrak{X}]$ and $(**)$ implies $\alpha + \beta_0 \in \mathcal{H}[\mathfrak{X}]$, we can use a (cut) ($\mathrm{rk}(A_{t_0}) = \mathrm{rk}(\neg A_{t_0}) < \mathrm{rk}(A) \leq \rho$) with $(**)$ and finally get

$$\mathcal{H}[\mathfrak{X}]\overset{\alpha+\beta}{\underset{\rho}{\vdash}} \Gamma, \Delta. \qquad \square$$

Theorem 5.3.2 (Predicative Cut Elimination)**.** *If* $[\rho, \rho+\omega^\alpha) \cap (\mathrm{Reg} \cup \{\mathcal{K}\}) = \emptyset$, $\alpha \in \mathcal{H}$ *and*

$$\mathcal{H}\overset{\beta}{\underset{\rho+\omega^\alpha}{\vdash}} \Gamma,$$

then also

$$\mathcal{H}\overset{\varphi\alpha\beta}{\underset{\rho}{\vdash}} \Gamma.$$

Proof. A proof can for example be found in [Poh98]. $\square$

6. Embeddings

In the first section we introduce an intermediate proof system which is just strong enough to derive all axioms of $\mathbf{\Pi_4}-\mathbf{Ref}$. In the second section, it will be embedded into our main proof system. This detour has some practical advantages — we don't have to bother about derivation lengths, because the derived set of $\mathcal{L}_{RS(\mathcal{K})}$-sentences gives a sufficiently good upper bound. All the methods and results are from BUCHHOLZ' [Buc92]. Other references are [Rat94b] and [Bla97]. We omit all proofs as they can be found there.

6.1. The intermediate proof system

First we introduce a simple hull of a set of ordinals:

Definition. For $X \subseteq \text{On}$ we define

$$X^* = X \cup \{\omega\} \cup \{\xi + 1 \mid \xi \in X\}.$$

Definition. The intermediate $*$-calculus contains only the rules

$$(\bigwedge^*) \qquad \frac{(\Gamma, A_t)_{t \in J}}{\Gamma, \bigwedge(A_t)_{t \in J}}$$

$$(\bigvee^*) \qquad \frac{\Gamma, A_{t_0}}{\Gamma, \bigvee(A_t)_{t \in J}} \quad \text{if } t_0 \in J \text{ and } \text{par}(t_0) \subseteq \text{par}(\Gamma, \bigvee(A_t)_{t \in J})^*,$$

where J is one of the index sets we encountered in the previous chapter.

Derivability in this system will be denoted by "$\overset{*}{\vdash}$"; if we additionally allow cuts, we will refer to the resulting calculus as "$\overset{*}{\underset{\rho}{\vdash}}$".

Now we can state some trivial, but nonetheless important derivability results in this calculus, which will later be needed for the embedding.

Lemma 6.1.1. (i) $\overset{*}{\vdash} A, \neg A$

(ii) $\overset{*}{\vdash} a \notin a$

(iii) $\overset{*}{\vdash} a \subseteq a$

(iv) $\overset{*}{\vdash} a = a$

(v) $\overset{*}{\vdash} \mathrm{L}_\gamma \neq \emptyset$ *if* $\gamma > 0$

(vi) $\overset{*}{\vdash} a \in \mathrm{L}_\gamma$ *and* $a \mathrel{\overset{\circ}{\in}} \mathrm{L}_\gamma$ *if* $|a| < \gamma$

(vii) $\overset{*}{\vdash} \mathrm{trans}(\mathrm{L}_\gamma)$

(viii) $\overset{*}{\vdash} (\exists x \in \mathrm{L}_\gamma)\mathrm{infinite}(x)$ *if* $\gamma > \omega$

(ix) $\overset{*}{\vdash} 3\text{-}\mathit{refl}_\xi(\mathrm{L}_\gamma)$ *if* $\gamma \in \mathcal{A}_\xi^{\mathcal{K}}$

(x) $\overset{*}{\vdash} 2\text{-}\mathit{refl}^{\kappa}_{\xi,\xi'}(\mathrm{L}_\gamma)$ *if* $\gamma \in \mathcal{A}_\xi^{\kappa} \cap \mathcal{A}_{\xi'}^{\mathcal{K}}$

(xi) $\overset{*}{\vdash} [s_1 \neq t_1], \ldots, [s_n \neq t_n], \neg A(\vec{s}), A(\vec{t})$,
where every x_i *may occur at most once in* $A(\vec{x})$. *(Here* $[s \neq t]$ *is an abbreviation for* $\neg s \subseteq t, \neg t \subseteq s$.*)*

As a corollary to (xi) we obtain

Lemma 6.1.2 (Equality Lemma)**.** *We have*

$$\overset{*}{\vdash} s \neq t, \neg A(s), A(t).$$

Lemma 6.1.3. *We have*

$$\overset{*}{\vdash} s \notin t, s \mathrel{\overset{\circ}{\in}} t.$$

Lemma 6.1.4. (i) *For every limit ordinal* λ *we have*

$$\overset{*}{\vdash} (\mathrm{Ext})^\lambda \wedge (\mathrm{Found})^\lambda \wedge (\mathrm{Pair})^\lambda \wedge (\mathrm{Union})^\lambda \wedge (\Delta_0\text{−}\mathrm{Sep})^\lambda$$

(ii) *Now we finally add the rule*

$$(4\text{−Ref}^*) \quad \frac{\Gamma, F}{\Gamma, (\exists^{\mathcal{K}} z)(z \models F)}$$

for $F \in \Pi_4(\mathcal{K})$ *to the intermediate calculus.*

Then we have

$$\overset{*}{\vdash} A^{\mathcal{K}}$$

for every instance A *of* $(\Pi_4\text{−Ref})$.

6.2. Embedding into the main system

This short section shows that theorems of $\mathbf{\Pi_4}-\mathbf{Ref}$ (which thus contain no parameters) are derivable in a very controlled way: all we need to know is that $\mathcal{K} \in \mathcal{H}$.

Definition. For a set $\Gamma = \{A_1, \dots, A_n\}$ of $\mathcal{L}_{RS(\mathcal{K})}$-formulas we define its *norm* $||\Gamma||$ as

$$||\Gamma|| = \omega^{\mathrm{rk}(A_1)} \oplus \dots \oplus \omega^{\mathrm{rk}(A_n)}.$$

In the following, let $\mathcal{H}$ always be a good operator. First we get a connection between $\overset{*}{\underset{\cdot}{\vdash}}$ and $\mathcal{H}\overset{\cdot}{\underset{\cdot}{\vdash}}$ as follows:

Lemma 6.2.1. *If* $\overset{*}{\underset{\rho}{\vdash}} \Gamma$*, then* $\mathcal{H}[\Gamma]\overset{||\Gamma||}{\underset{\rho}{\vdash}} \Gamma$.

Lemma 6.2.2. *Let* $\lambda \in \mathrm{Lim}$ *such that* $\lambda \in \mathcal{H}$*. If* $\Gamma[\vec{u}]$ *is a logically valid set of* $\mathcal{L}_{Ad}$-*formulas, then there is an* $m \in \omega$ *such that for all* $\vec{s} \in T_\lambda$

$$\mathcal{H}[\vec{s}]\overset{\omega^{\omega\cdot\lambda+m}}{\underset{\omega\cdot\lambda}{\vdash}} \Gamma[\vec{s}]^\lambda.$$

As we have seen in the previous section that we can derive all axioms of $\mathbf{\Pi_4}-\mathbf{Ref}$ relativized to $\mathcal{K}$, this result immediately implies the embedding theorem we were aiming for:

Theorem 6.2.3. *Let* $\mathcal{K} \in \mathcal{H}$*. For each theorem* A *of* $\mathbf{\Pi_4}-\mathbf{Ref}$ *there is an* $m \in \omega$ *such that*

$$\mathcal{H}\overset{\mathcal{K}\cdot\omega^m}{\underset{\mathcal{K}+m}{\vdash}} A^{\mathcal{K}}.$$

7. Preparations

In order to disburden the proofs of the collapsing theorems of the next chapter, which constitute the central results of this thesis, we try to shift some technical side-calculations into this preparatory chapter.

The collapsing theorems will be proved inductively. In that process the controlling operators will have to grow stronger, so no longer one single operator suffices. Hence in the first section we define a hierarchy of good operators, $(\mathcal{H}_\gamma)_\gamma$. The remaining sections deal with the "pancake" conditions, which are needed to make the inductive proofs of the collapsing theorems work. They describe certain derivations which can be collapsed into derivations with much smaller ordinal labels.

7.1. The Operators $\mathcal{H}_\gamma$

We want the operators to be both monotone and as close to C-sets as possible. This leads to the following

Definition. Set

$$\mathcal{H}_\gamma(X) = \bigcap_{\alpha>\gamma} \{C(\alpha,\beta) \mid X \subseteq C(\alpha,\beta)\}.$$

We immediately see the following consequences:

Lemma 7.1.1. *For arbitrary $\mathfrak{X}$ we have:*

(i) *$\mathcal{H}_\gamma$ is a good operator and is closed under φ.*

(ii) *If $\gamma \leq \gamma'$, then $\mathcal{H}_\gamma[\mathfrak{X}] \subseteq \mathcal{H}_{\gamma'}[\mathfrak{X}]$.*

(iii) *If $\alpha \leq \gamma$, $\rho < \mathcal{K}$ and $\alpha, \rho \in \mathcal{H}_\gamma[\mathfrak{X}]$, then $\Psi^{\mathcal{K}}_\alpha[\rho] \in \mathcal{H}_\gamma[\mathfrak{X}]$.*

(iv) *If $\xi < \alpha \leq \gamma$, $\rho < \kappa$, $\kappa, \alpha, \xi, \rho \in \mathcal{H}_\gamma[\mathfrak{X}]$ and $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\xi]$, then $\Psi^\kappa_{\alpha,\xi}[\rho] \in \mathcal{H}_\gamma[\mathfrak{X}]$.*

(v) *If $(\xi' < \xi < \alpha \leq \gamma \vee \xi' = \xi = 0 < \alpha \leq \gamma)$, $\rho < \pi$, $\kappa, \pi, \alpha, \xi, \xi', \rho \in \mathcal{H}_\gamma[\mathfrak{X}]$, $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}]$ and $\mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$ is stationary in π, then also $\Psi^{\pi,\kappa}_{\alpha,\xi,\xi'}[\rho] \in \mathcal{H}_\gamma[\mathfrak{X}]$.*

We omit the easy *Proof* here.

7.2. Preparations for the First Collapsing Theorem

Now we turn to the conditions we need to prove the first collapsing theorem, which is connected to eliminating the Π_4-reflection rule. They just express that the index of the operator controlling the existing derivation, γ, has to be nice and that the parameters $\mathfrak{X}$ adjoined to $\mathcal{H}_\gamma$ have to be controlled in a suitable way — conditions which are preserved by the cut-elimination procedure.

Definition. Let $\mathfrak{A}^4(\mathfrak{X};\gamma) :\Leftrightarrow \gamma \in \mathcal{H}_\gamma[\mathfrak{X}] \wedge \mathrm{par}(\mathfrak{X}) \subseteq C^{\mathcal{K}}(\gamma+1)$.

Lemma 7.2.1. *Let $\mathfrak{A}^4(\mathfrak{X};\gamma), \alpha \in \mathcal{H}_\gamma[\mathfrak{X}]$ and $\kappa \in \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}}$, where $\hat{\alpha} = \gamma \oplus \omega^{\omega^{\mathcal{K}\oplus\alpha}}$. Then:*

(i) $\mathcal{H}_\gamma[\mathfrak{X}] \subseteq C^{\mathcal{K}}(\gamma+1)$ *and thus* $\mathcal{H}_\gamma[\mathfrak{X}] \cap \mathcal{K} \subseteq \Psi^{\mathcal{K}}_{\gamma+1}$.

(ii) $\Psi^{\mathcal{K}}_{\hat{\alpha}}[\kappa] \in \mathcal{H}_{\hat{\alpha}}[\mathfrak{X},\kappa] \cap \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}}$.

(iii) $\Psi^{\mathcal{K}}_{\gamma+1} < \Psi^{\mathcal{K}}_{\hat{\alpha}}$ *and hence* $\mathrm{par}(\mathfrak{X}) \cap \mathcal{K} \subseteq \Psi^{\mathcal{K}}_{\hat{\alpha}}$.

(iv) *If* $\alpha_0 \in \mathcal{H}_\gamma[\mathfrak{X}]$ *and* $\alpha_0 < \alpha$, *then* $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\hat{\alpha}_0}]$, $\kappa \in \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}_0}$ *and* $\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\kappa] < \Psi^{\mathcal{K}}_{\hat{\alpha}}[\kappa]$.

(v) *Let* $t_1,\ldots,t_n \in T_{\mathcal{K}}$, $|t_1|,\ldots,|t_n| \le \alpha_{\vec{t}} < \alpha$, $\alpha_{\vec{t}} \in \mathcal{H}_\gamma[\mathfrak{X},t_1,\ldots,t_n]$ *and define*

$$\gamma_{\vec{t}} = \gamma \oplus \omega^{\omega^{\mathcal{K}\oplus\alpha_{\vec{t}}\oplus|t_1|\oplus\cdots\oplus|t_n|}}$$

and $\beta_{\vec{t}} = \gamma_{\vec{t}} \oplus \omega^{\omega^{\mathcal{K}\oplus\alpha_{\vec{t}}}}$. *Then*

$$\mathfrak{A}^4(\mathfrak{X}\cup\{t_1,\ldots,t_n\};\gamma_{\vec{t}})$$

and $\beta_{\vec{t}} \in \mathcal{H}_{\gamma_{\vec{t}}}[\mathfrak{X},t_1,\ldots,t_n]$. *If additionally* $t_1,\ldots,t_n \in T_\kappa$, *then*

$$\Psi^{\mathcal{K}}_{\beta_{\vec{t}}}[\kappa] < \Psi^{\mathcal{K}}_{\hat{\alpha}}[\kappa]$$

and $\kappa \in \mathcal{A}^{\mathcal{K}}_{\beta_{\vec{t}}}$.

Proof. Most parts are trivial:

(i) $\mathcal{H}_\gamma[\mathfrak{X}] \subseteq C^{\mathcal{K}}(\gamma+1)$ by definition and $C^{\mathcal{K}}(\gamma+1) \cap \mathcal{K} = \Psi^{\mathcal{K}}_{\gamma+1}$ by Lemma 3.1.2.

(ii) Follows immediately from the definition and Theorem 3.1.5.

(iii) As $\gamma+1 < \hat{\alpha}$, the normal form condition implies $\Psi^{\mathcal{K}}_{\gamma+1} < \Psi^{\mathcal{K}}_{\hat{\alpha}}$, and thus the second claim follows from Lemma 3.1.2 again.

(iv) As $\hat{\alpha}_0 < \hat{\alpha}$, it suffices to show

$$\alpha_0, \gamma \in C^{\mathcal{K}}(\hat{\alpha}). \tag{$*$}$$

But $\alpha_0,\gamma \in C^{\mathcal{K}}(\gamma+1) \subseteq C^{\mathcal{K}}(\hat{\alpha})$ as $\Psi^{\mathcal{K}}_{\gamma+1} < \Psi^{\mathcal{K}}_{\hat{\alpha}}$ by (iii). Thus $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\hat{\alpha}_0}]$ by definition, and $\Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\hat{\alpha}_0}] \subseteq \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}_0}$ follows from Theorem 3.1.1. But $(*)$ also implies $\alpha_0,\gamma \in C(\hat{\alpha},\Psi^{\mathcal{K}}_{\hat{\alpha}}[\kappa])$, hence $\Psi^{\mathcal{K}}_{\hat{\alpha}}[\kappa] \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\hat{\alpha}_0}]$ follows immediately, and therefore $\Psi^{\mathcal{K}}_{\hat{\alpha}}[\kappa] > \Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\kappa]$ holds.

(v) Because of $|t_1|, \dots, |t_n|, \alpha_{\vec{t}}, \gamma \in \mathcal{H}_\gamma[\mathfrak{X}, t_1, \dots, t_n] \subseteq \mathcal{H}_{\gamma_{\vec{t}}}[\mathfrak{X}, t_1, \dots, t_n]$ we obtain $\gamma_{\vec{t}} \in \mathcal{H}_{\gamma_{\vec{t}}}[\mathfrak{X}, t_1, \dots, t_n]$. The normal form condition yields $|t_1|, \dots, |t_n|, \gamma + 1 \in C^{\mathcal{K}}(\gamma_{\vec{t}} + 1)$, which implies

$$\Psi^{\mathcal{K}}_{\gamma+1} \leq \Psi^{\mathcal{K}}_{\gamma_{\vec{t}}+1},$$

and thus also

$$\mathrm{par}(\mathfrak{X}) \subseteq C^{\mathcal{K}}(\gamma + 1) \subseteq C^{\mathcal{K}}(\gamma_{\vec{t}} + 1),$$

hence $\mathfrak{A}^4(\mathfrak{X} \cup \{t_1, \dots, t_n\}; \gamma_{\vec{t}})$ is proved. Obviously, $\beta_{\vec{t}} \in \mathcal{H}_{\gamma_{\vec{t}}}[\mathfrak{X}, t_1, \dots, t_n]$ because $\gamma_{\vec{t}}, \alpha_{\vec{t}} \in \mathcal{H}_{\gamma_{\vec{t}}}[\mathfrak{X}, t_1, \dots, t_n]$. Let now $t_1, \dots, t_n \in T_\kappa$. We first show

$$\beta_{\vec{t}} <^C_\kappa \hat{\alpha}.$$

Clearly, $\beta_{\vec{t}} < \hat{\alpha}$ because

$$\omega^{\mathcal{K} \oplus \alpha_{\vec{t}}}, |t_1|, \dots, |t_n| < \omega^{\mathcal{K} \oplus \alpha}$$

implies

$$\omega^{\omega^{\mathcal{K} \oplus \alpha_{\vec{t}}} \oplus |t_1| \oplus \dots \oplus |t_n|} \oplus \omega^{\omega^{\mathcal{K} \oplus \alpha_{\vec{t}}}} < \omega^{\omega^{\mathcal{K} \oplus \alpha}}.$$

Further we have $|t_1|, \dots, |t_n| < \kappa$, and as $\kappa \in \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}}$ implies $\Psi^{\mathcal{K}}_{\gamma+1} \leq \Psi^{\mathcal{K}}_{\hat{\alpha}} \leq \kappa$, we get

$$|t_1|, \dots, |t_n|, \gamma, \alpha_{\vec{t}} \in \mathcal{H}_\gamma[\mathfrak{X}, t_1, \dots, t_n] \subseteq C(\gamma + 1, \kappa) \subseteq C(\hat{\alpha}, \kappa).$$

Hence $\beta_{\vec{t}} \in C(\hat{\alpha}, \kappa)$. Therefore

$$\kappa, \Psi^{\mathcal{K}}_{\hat{\alpha}}[\kappa] \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\beta_{\vec{t}}}]$$

and hence $\kappa \in \mathcal{A}^{\mathcal{K}}_{\beta_{\vec{t}}}$ and $\Psi^{\mathcal{K}}_{\beta_{\vec{t}}}[\kappa] < \Psi^{\mathcal{K}}_{\hat{\alpha}}[\kappa]$. □

7.3. Preparations for the Second Collapsing Theorem

Again the pancake conditions associated with collapsing a derivation of $\Pi_3(\kappa)$-formulas contain niceness conditions on κ, the index γ of the controlling operator of the old derivation, $\mathcal{H}_\gamma[\mathfrak{X}]$ and a ξ such that $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\xi]$.

Definition. Let $\mathfrak{A}^3(\mathfrak{X}; \gamma, \kappa, \xi) :\Leftrightarrow$

- $\gamma, \kappa, \xi \in \mathcal{H}_\gamma[\mathfrak{X}]$
- $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\xi]$
- $\mathrm{par}(\mathfrak{X}) \subseteq C^\kappa(\gamma + 1)$
- $\xi \leq \gamma$

Remarks. (i) The condition

$$\mathrm{par}(\mathfrak{X}) \subseteq C^\kappa(\gamma+1)$$

is supposed to mean that $\Psi^\kappa_{\gamma+1} = \Psi^\kappa_{\gamma+1,0}$ *is defined* and $\mathrm{par}(\mathfrak{X}) \subseteq C^\kappa(\gamma+1)$.

(ii) Whenever $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\sigma]$ such that $\sigma, \gamma < \alpha'$ and $\alpha' \in \mathcal{H}_\gamma[\mathfrak{X}]$, then $\Psi^\kappa_{\alpha',\sigma}$ is defined, as

$$\alpha', \kappa \in C^\kappa(\gamma+1) \subseteq C(\alpha', \kappa).$$

Lemma 7.3.1. *Assume* $\mathfrak{A}^3(\mathfrak{X}; \gamma, \kappa, \xi), \alpha \in \mathcal{H}_\gamma[\mathfrak{X}]$ *and* $\pi \in \mathcal{A}^\kappa_{\hat\alpha} \cap \mathcal{A}^{\mathcal{K}}_\xi$*, where* $\hat\alpha = \gamma \oplus \omega^{\omega^{\kappa\oplus\alpha}}$*. Then:*

(i) $\mathcal{H}_\gamma[\mathfrak{X}] \subseteq C^\kappa(\gamma+1)$ *and thus* $\mathcal{H}_\gamma[\mathfrak{X}] \cap \kappa \subseteq \Psi^\kappa_{\gamma+1}$.

(ii) $\Psi^\kappa_{\hat\alpha,\xi}[\pi] \in \mathcal{H}_{\hat\alpha}[\mathfrak{X}, \pi] \cap \mathcal{A}^\kappa_{\hat\alpha} \cap \mathcal{A}^{\mathcal{K}}_\xi$.

(iii) $\Psi^\kappa_{\gamma+1} < \Psi^\kappa_{\hat\alpha,\xi}$*, and so* $\mathrm{par}(\mathfrak{X}) \cap \kappa \subseteq \Psi^\kappa_{\hat\alpha,\xi}$.

(iv) *If* $\alpha_0 \in \mathcal{H}_\gamma[\mathfrak{X}]$ *and* $\alpha_0 < \alpha$*, then* $\mathcal{A}^\kappa_{\hat\alpha_0} \cap \mathcal{A}^{\mathcal{K}}_\xi$ *is stationary in* π*,* $\pi \in \mathcal{A}^\kappa_{\hat\alpha_0} \cap \mathcal{A}^{\mathcal{K}}_\xi$ *and* $\Psi^\kappa_{\hat\alpha_0,\xi}[\pi] < \Psi^\kappa_{\hat\alpha,\xi}[\pi]$.

(v) *Let* $t_1, \ldots, t_n \in T_\kappa$*,* $|t_1|, \ldots, |t_n| \le \alpha_{\vec t} < \alpha$*,* $\alpha_{\vec t} \in \mathcal{H}_\gamma[\mathfrak{X}, t_1, \ldots, t_n]$ *and define*

$$\gamma_{\vec t} = \gamma \oplus \omega^{\omega^{\kappa\oplus\alpha_{\vec t}\oplus|t_1|\oplus\cdots\oplus|t_n|}}$$

and $\beta_{\vec t} = \gamma_{\vec t} \oplus \omega^{\omega^{\kappa\oplus\alpha_{\vec t}}}$*. Then*

$$\mathfrak{A}^3(\mathfrak{X} \cup \{t_1, \ldots, t_n\}; \gamma_{\vec t}, \kappa, \xi)$$

and $\beta_{\vec t} \in \mathcal{H}_{\gamma_{\vec t}}[\mathfrak{X}, t_1, \ldots, t_n]$*. If further* $|t_1|, \ldots, |t_n| \in T_\pi$*, then*

$$\Psi^\kappa_{\beta_{\vec t},\xi}[\pi] < \Psi^\kappa_{\hat\alpha,\xi}[\pi]$$

and $\pi \in \mathcal{A}^\kappa_{\beta_{\vec t}}$.

Proof. (i) follows directly from the definition, the above remark and Lemma 3.2.3,

(ii) again follows immediately from the above remark.

(iii) We only have to show

$$\gamma + 1 <^C_{\Psi^\kappa_{\hat\alpha,\xi}} \hat\alpha,$$

but this follows immediately from the normal form condition on $\Psi^\kappa_{\hat\alpha,\xi}$.

(iv) $\alpha_0, \gamma, \xi, \mu \in \mathcal{H}_\gamma[\mathfrak{X}] \subseteq C^\kappa(\gamma+1) \subseteq C^\kappa_\xi(\hat{\alpha})$ (see (iii)) implies

$$\hat{\alpha}_0, \xi <^C_\pi \hat{\alpha}.$$

By the same argument, $\kappa, \hat{\alpha}_0 \in C(\hat{\alpha}_0, \kappa)$, so by definition we get

$$\mathcal{A}^\kappa_{\hat{\alpha}_0} \cap \mathcal{A}^{\mathcal{K}}_\xi \text{ stationary in both } \pi \text{ and } \Psi^\kappa_{\hat{\alpha},\xi}[\pi],$$

and hence $\pi \in \mathcal{A}^\kappa_{\hat{\alpha}_0}$ (by Lemma 3.2.2) and $\Psi^\kappa_{\hat{\alpha}_0,\xi}[\pi] < \Psi^\kappa_{\hat{\alpha},\xi}[\pi]$.

(v) Like in Lemma 7.2.1 we have

$$|t_1|, \ldots, |t_n|, \alpha_{\vec{t}}, \gamma \in \mathcal{H}_\gamma[\mathfrak{X}, t_1, \ldots, t_n] \subseteq \mathcal{H}_{\gamma_{\vec{t}}}[\mathfrak{X}, t_1, \ldots, t_n],$$

and hence $\gamma_{\vec{t}} \in \mathcal{H}_{\gamma_{\vec{t}}}[\mathfrak{X}, t_1, \ldots, t_n]$. Further we know that $\Psi^\kappa_{\gamma_{\vec{t}}+1}$ is defined, because $|t_1|, \ldots, |t_n| < \kappa$ together with the assumption $\mathrm{par}(\mathfrak{X}) \subseteq C^\kappa(\gamma+1)$ implies that

$$\mathcal{H}_\gamma[\mathfrak{X}, t_1, \ldots, t_n] \subseteq C(\gamma+1, \kappa),$$

which proves that

$$\kappa, \gamma, \alpha_{\vec{t}}, |t_1|, \ldots, |t_n| \in \mathcal{H}_\gamma[\mathfrak{X}, t_1, \ldots, t_n] \subseteq C(\gamma+1, \kappa) \subseteq C(\gamma_{\vec{t}}, \kappa), \qquad (*)$$

and thus also $\kappa, \gamma_{\vec{t}} \in C(\gamma_{\vec{t}}, \kappa)$. The normal form condition again yields

$$|t_1|, \ldots, |t_n|, \gamma+1 \in C^\kappa(\gamma_{\vec{t}}+1),$$

which implies

$$\Psi^\kappa_{\gamma+1} \leq \Psi^\kappa_{\gamma_{\vec{t}}+1},$$

and thus also

$$\mathrm{par}(\mathfrak{X}) \subseteq C^\kappa(\gamma+1) \subseteq C^\kappa(\gamma_{\vec{t}}+1).$$

Hence, $\mathfrak{A}^3(\mathfrak{X} \cup \{t_1, \ldots, t_n\}; \gamma_{\vec{t}}, \kappa, \xi)$ is proved. Now let $t_1, \ldots, t_n \in T_\pi$. Again we get

$$\beta_{\vec{t}} <^C_\pi \hat{\alpha}$$

analogously to Lemma 7.2.1, and again is

$$\mathcal{A}^\kappa_{\beta_{\vec{t}}} \cap \mathcal{A}^{\mathcal{K}}_\xi \text{ stationary in } \pi, \Psi^\kappa_{\hat{\alpha},\xi}[\pi],$$

this time using the additional fact that

$$\kappa, \beta_{\vec{t}} \in C(\beta_{\vec{t}}, \kappa),$$

which easily follows from $(*)$. This implies $\pi \in \mathcal{A}^\kappa_{\beta_{\vec{t}}} \cap \mathcal{A}^{\mathcal{K}}_\xi$ (by Lemma 3.2.2), and $\Psi^\kappa_{\beta_{\vec{t}},\xi}[\pi] < \Psi^\kappa_{\hat{\alpha},\xi}[\pi]$. □

7.4. Preparations for the Third Collapsing Theorem

The preparations for the final theorem, which treats collapsing of derivations of $\Pi_2(\pi)$-formulas where π is not Π^1_1-indescribable, become somewhat more involved, because in the course of its proof we have to examine arbitrary cut-ranks. (That's why we need the additional parameter μ in $\mathfrak{A}^2(\ldots,\mu)$.) In order to apply the induction hypothesis we thus need to check both conditions of the form $\mathfrak{A}^2(\ldots)$ and $\mathfrak{A}^3(\ldots)$. This will be done in parts (vi) to (viii) of the lemma. Proving these parts requires the additional technical conditions in $\mathfrak{A}^2(\ldots)$.

Definition. Let $\mathfrak{A}^2(\mathfrak{X};\gamma,\pi,\kappa,\xi,\xi',\mu) :\Leftrightarrow$

- $\gamma,\pi,\kappa,\xi,\xi',\mu \in \mathcal{H}_\gamma[\mathfrak{X}]$
- $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}]$ and $\mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$ stationary in π
- $\mathrm{par}(\mathfrak{X}) \subseteq C^\pi(\gamma+1)$
- $\pi \in \bigcap\{C^{\pi'}(\delta) \mid \pi' > \pi \wedge \delta > \gamma\}$
- $(\xi' < \xi \leq \gamma \vee \xi' = \xi = 0 \leq \gamma)$ and $\pi \leq \mu$
- $\pi \leq \Psi^{\mathcal{K}}_{\gamma+1}$

Remarks. (i) The meaning of the condition

$$\mathrm{par}(\mathfrak{X}) \subseteq C^\pi(\gamma+1)$$

should be clear by now: we require $\Psi^\pi_{\gamma+1}$ to be defined. Analogously,

$$\pi \in \bigcap\{C^{\pi'}(\delta) \mid \pi' > \pi \wedge \delta > \gamma\}$$

is supposed to mean: if $\pi' > \pi$, $\delta > \gamma$ and $\Psi^{\pi'}_\delta$ *is defined*, then $\pi \in C^{\pi'}(\delta)$. But this is fulfilled in all relevant cases, for example if $\pi < \pi' \in \mathcal{H}_\gamma[\mathfrak{X}]$ and $\gamma < \delta \in \mathcal{H}_\gamma[\mathfrak{X}]$, then by assumption

$$\delta, \pi' \in C^\pi(\gamma+1) \subseteq C(\delta,\pi'),$$

and so $\Psi^{\pi'}_\delta$ is defined. In particular we get $\pi \leq \Psi^\kappa_{\gamma+1}$.

(ii) As in the previous section we get that

$$\Psi^{\pi,\kappa'}_{\alpha',\sigma',\sigma''} \text{ is defined}$$

whenever $\kappa' \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\sigma''}]$, $\mathcal{A}^{\kappa'}_{\sigma'} \cap \mathcal{A}^{\mathcal{K}}_{\sigma''}$ stationary in π (such that $\sigma'' < \sigma' < \alpha'$ or $\sigma'' = \sigma' < \alpha'$) and $\gamma < \alpha' \in \mathcal{H}_\gamma[\mathfrak{X}]$.

Lemma 7.4.1. *Let $\mathfrak{A}^2(\mathfrak{X};\gamma,\pi,\kappa,\xi,\xi',\mu)$, $\alpha \in \mathcal{H}_\gamma[\mathfrak{X}]$ and $\rho \in \mathcal{A}^\pi_{\hat\alpha} \cap \mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$, where $\hat\alpha = \gamma \oplus \omega^{\omega^{\mu\oplus\alpha}}$. Then the following hold:*

(i) $\mathcal{H}_\gamma[\mathfrak{X}] \subseteq C^\pi(\gamma+1)$, *and so* $\mathcal{H}_\gamma[\mathfrak{X}] \cap \pi \subseteq \Psi^\pi_{\gamma+1}$ *if* $\Psi^\pi_{\gamma+1} \in \mathrm{Lim(SC)}$.

(ii) $\Psi^{\pi,\kappa}_{\hat\alpha,\xi,\xi'}[\rho] \in \mathcal{H}_{\hat\alpha}[\mathfrak{X},\rho] \cap \mathcal{A}^\pi_{\hat\alpha} \cap \mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$.

(iii) $\Psi^\pi_{\gamma+1} < \Psi^{\pi,\kappa}_{\hat\alpha,\xi,\xi'}$ *and hence* $\mathrm{par}(\mathfrak{X}) \cap \pi \subseteq \Psi^{\pi,\kappa}_{\hat\alpha,\xi,\xi'}$.

(iv) *If* $\alpha_0 \in \mathcal{H}_\gamma[\mathfrak{X}] \cap \alpha$, *then* $\rho \in \mathcal{A}^\pi_{\hat\alpha_0} \cap \mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$ *and* $\Psi^{\pi,\kappa}_{\hat\alpha_0,\xi,\xi'}[\rho] < \Psi^{\pi,\kappa}_{\hat\alpha,\xi,\xi'}[\rho]$.

(v) *If* $\alpha_{\vec t} \in \mathcal{H}_\gamma[\mathfrak{X}] \cap \alpha$, $t_1,\dots,t_n \in T_\pi$ *and* $\gamma_{\vec t} = \gamma \oplus \omega^{\omega^{\mu\oplus\alpha_{\vec t}\oplus|t_1|\oplus\cdots\oplus|t_n|}}$, *then*

$$\mathfrak{A}^2(\mathfrak{X}\cup\{t_1,\dots,t_n\};\gamma_{\vec t},\pi,\kappa,\xi,\xi',\mu).$$

If further $t_1,\dots,t_n \in T_\rho$, *then*

$$\rho \in \mathcal{A}^\pi_{\beta_{\vec t}} \cap \mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \quad \text{and} \quad \Psi^{\pi,\kappa}_{\beta_{\vec t},\xi,\xi'}[\rho] < \Psi^{\pi,\kappa}_{\hat\alpha,\xi,\xi'}[\rho],$$

where $\beta_{\vec t} = \gamma_{\vec t} \oplus \omega^{\omega^{\mu\oplus\alpha_{\vec t}}}$.

(vi) *If* $\alpha_0 \in \mathcal{H}_\gamma[\mathfrak{X}]$, $\bar\kappa,\bar\pi \in \mathcal{H}_\gamma[\mathfrak{X}] \cap (\pi,\mu]$, $\bar\kappa$ *is* Π^1_1*-indescribable and* $\bar\pi$ *is not, then*

$$\mathfrak{A}^2(\mathfrak{X};\hat\alpha_0,\pi,\kappa,\xi,\xi',\Psi^{\mathcal{K}}_{\hat\alpha_0}[\Psi^{\mathcal{K}}_{\hat\alpha_0}])$$
$$\mathfrak{A}^2(\mathfrak{X};\hat\alpha_0,\pi,\kappa,\xi,\xi',\Psi^{\bar\kappa}_{\hat\alpha_0}[\Psi^{\bar\kappa}_{\hat\alpha_0}])$$

and

$$\mathfrak{A}^2(\mathfrak{X};\hat\alpha_0,\pi,\kappa,\xi,\xi',p^*(\Psi^{\bar\pi}_{\hat\alpha_0}[\Psi^{\bar\pi}_{\hat\alpha_0}])),$$

where p^* *is defined via*

$$p^*(\beta) = \begin{cases} \beta & \text{if it is a cardinal,} \\ p(\beta) & \text{if not.} \end{cases}$$

(vii) *If* $\pi < \bar\kappa \in \mathcal{H}_\gamma[\mathfrak{X}]$, $\bar\kappa \le \mu$ *and* $\bar\kappa$ *is* Π^1_1*-indescribable, then*

$$\mathfrak{A}^3(\mathfrak{X};\gamma,\bar\kappa,0).$$

(viii) *If* $\gamma \ge 2$, $\pi < \bar\pi \in \mathcal{H}_\gamma[\mathfrak{X}]$, $\bar\pi \le \mu$ *and* $\bar\pi$ *is not* Π^1_1*-indescribable, then*

$$\mathfrak{A}^2(\mathfrak{X};\gamma,\bar\pi,\Psi^{\mathcal{K}}_2[\bar\pi],0,0,\mu).$$

Proof. The first parts are routine by now.

(i) is again clear in view of the above remark and Lemma 3.3.2;

(ii) holds as we already noted that $\Psi^{\pi,\kappa}_{\hat\alpha,\xi,\xi'}[\rho]$ is defined.

(iii) $\gamma+1 <^C_{\Psi^{\pi,\kappa}_{\hat\alpha,\xi,\xi'}} \hat\alpha$ follows immediately from the normal form condition.

(iv) First we have

$$\hat{\alpha}_0, \pi \in C(\hat{\alpha}_0, \Psi^{\pi,\kappa}_{\hat{\alpha},\xi,\xi'}) \tag{$*$}$$

because of $\gamma, \mu, \alpha_0, \pi \in \mathcal{H}_\gamma[\mathfrak{X}]$ and $\Psi^\pi_{\gamma+1} < \Psi^{\pi,\kappa}_{\hat{\alpha},\xi,\xi'}$ (see (iii)). But as obviously $\hat{\alpha}_0 < \hat{\alpha}$, $(*)$ yields both

$$\hat{\alpha}_0 <^C_{\Psi^{\pi,\kappa}_{\hat{\alpha},\xi,\xi'}} \hat{\alpha}$$

and

$$\pi, \hat{\alpha}_0 \in C(\hat{\alpha}_0, \pi),$$

hence by definition and Theorem 3.3.1 we are done.

(v) We easily get $\gamma_{\vec{t}} \in \mathcal{H}_{\gamma_{\vec{t}}}[\mathfrak{X}, t_1, \ldots, t_n]$. We also know that

$$\Psi^\pi_{\gamma_{\vec{t}}+1} \text{ is defined,}$$

as $|t_1|, \ldots, |t_n| < \pi$ and $\mathrm{par}(\mathfrak{X}) \subseteq C^\pi(\gamma+1)$ together imply

$$\mathcal{H}_\gamma[\mathfrak{X}, t_1, \ldots, t_n] \subseteq C(\gamma+1, \pi),$$

and thus we get

$$\pi, \gamma_{\vec{t}} \in C(\gamma+1, \pi) \subseteq C(\gamma_{\vec{t}}+1, \pi).$$

So we get

$$\Psi^\pi_{\gamma+1} \leq \Psi^\pi_{\gamma_{\vec{t}}+1}$$

because of the normal form condition. Hence $\mathrm{par}(\mathfrak{X}) \subseteq C^\pi(\gamma_{\vec{t}}+1)$, and also by the normal form condition, every $|t_i| \in C^\pi(\gamma_{\vec{t}}+1)$. Additionally,

$$\pi \in \bigcap\{C^{\pi'}(\delta) \mid \pi' > \pi \wedge \delta > \gamma_{\vec{t}}\}$$

follows directly from the hypothesis, and $\pi \leq \Psi^{\mathcal{K}}_{\gamma+1}$ implies $\pi \leq \Psi^{\mathcal{K}}_{\gamma_{\vec{t}}+1}$, so we have already successfully proved

$$\mathfrak{A}^2(\mathfrak{X} \cup \{t_1, \ldots, t_n\}; \gamma_{\vec{t}}, \pi, \kappa, \xi, \xi', \mu).$$

If $t_1, \ldots, t_n \in T_\rho$, then first we get

$$\beta_{\vec{t}} < \hat{\alpha},$$

because $\alpha_{\vec{t}} < \alpha$ implies $\omega^{\mu \oplus \alpha_{\vec{t}}} < \omega^{\mu \oplus \alpha}$, and $|t_1|, \ldots, |t_n| < \omega^{\mu \oplus \alpha}$ holds because of $|t_1|, \ldots, |t_n| < \rho < \pi \leq \mu$. Now, $|t_1|, \ldots, |t_n| < \rho$, $\mathrm{par}(\mathfrak{X}) \subseteq C^\pi(\gamma+1)$ and $\Psi^\pi_{\gamma+1} \leq \Psi^\pi_{\hat{\alpha}} \leq \rho$ imply

$$\mathcal{H}_\gamma[\mathfrak{X}, t_1, \ldots, t_n] \subseteq C(\gamma+1, \rho),$$

hence we obtain $\gamma, \mu, \alpha_{\vec{t}}, |t_1|, \dots, |t_n| \in \mathcal{H}_\gamma[\mathfrak{X}, t_1, \dots, t_n] \subseteq C(\gamma+1, \rho)$, which shows both

$$\beta_{\vec{t}} \in C(\hat{\alpha}, \rho) \text{ and } \beta_{\vec{t}} \in C(\beta_{\vec{t}}, \pi).$$

Thus we get

$$\rho, \Psi^{\pi,\kappa}_{\hat{\alpha},\xi,\xi'}[\rho] \in \mathrm{Lim}(\mathcal{A}^{\pi}_{\beta_{\vec{t}}} \cap \mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}),$$

by definition, which yields $\Psi^{\pi,\kappa}_{\beta_{\vec{t}},\xi,\xi'}[\rho] < \Psi^{\pi,\kappa}_{\hat{\alpha},\xi,\xi'}[\rho]$ and $\rho \in \mathcal{A}^{\pi}_{\beta_{\vec{t}}} \cap \mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$.

(vi) First, $\alpha_0, \bar{\kappa}, \bar{\pi} \in \mathcal{H}_\gamma[\mathfrak{X}]$ implies

$$\hat{\alpha}_0, \Psi^{\mathcal{K}}_{\hat{\alpha}_0}, \Psi^{\bar{\kappa}}_{\hat{\alpha}_0}, \Psi^{\bar{\pi}}_{\hat{\alpha}_0} \in \mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}],$$

thus also $p^*(\Psi^{\bar{\pi}}_{\hat{\alpha}_0}[\Psi^{\bar{\pi}}_{\hat{\alpha}_0}]) \in \mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}]$. The normal form condition yields $\Psi^{\pi}_{\gamma+1} \leq \Psi^{\pi}_{\hat{\alpha}_0+1}$, so we also get

$$\mathrm{par}(\mathfrak{X}) \subseteq C^{\pi}(\hat{\alpha}_0 + 1).$$

As $\gamma \leq \hat{\alpha}_0$ we trivially have

$$\bigcap\{C^{\pi'}(\delta) \mid \pi' > \pi \wedge \delta > \gamma\} \subseteq \bigcap\{C^{\pi'}(\delta) \mid \pi' > \pi \wedge \delta > \hat{\alpha}_0\}.$$

Again the normal form condition shows

$$\Psi^{\mathcal{K}}_{\gamma+1} \leq \Psi^{\mathcal{K}}_{\hat{\alpha}_0} \leq \Psi^{\mathcal{K}}_{\hat{\alpha}_0+1},$$

so the only thing left to prove is

$$\pi \leq \Psi^{\bar{\kappa}}_{\hat{\alpha}_0}[\Psi^{\bar{\kappa}}_{\hat{\alpha}_0}], p^*(\Psi^{\bar{\pi}}_{\hat{\alpha}_0}[\Psi^{\bar{\pi}}_{\hat{\alpha}_0}]).$$

But as $\bar{\kappa}, \bar{\pi} \in \mathcal{H}_\gamma[\mathfrak{X}]$, we get

$$\hat{\alpha}_0, \bar{\kappa}, \bar{\pi} \in C^{\pi}(\gamma + 1) \subseteq C(\hat{\alpha}_0, \pi),$$

and because of $\pi < \bar{\kappa}, \bar{\pi}$, both $\Psi^{\bar{\kappa}}_{\hat{\alpha}_0}$ and $\Psi^{\bar{\pi}}_{\hat{\alpha}_0}$ are defined, which by the assumption $\pi \in \bigcap\{C^{\pi'}(\delta) \mid \pi' > \pi \wedge \delta > \gamma\}$ implies

$$\pi < \Psi^{\bar{\kappa}}_{\hat{\alpha}_0}, \Psi^{\bar{\pi}}_{\hat{\alpha}_0},$$

and hence also $\pi \leq p^*(\Psi^{\bar{\pi}}_{\hat{\alpha}_0}[\Psi^{\bar{\pi}}_{\hat{\alpha}_0}])$.

(vii) From $\pi < \bar{\kappa} \in \mathcal{H}_\gamma[\mathfrak{X}]$ we can conclude

$$\gamma, \bar{\kappa} \in C(\gamma + 1, \pi) \subseteq C(\gamma + 1, \bar{\kappa}),$$

so $\Psi^{\bar{\kappa}}_{\gamma+1}$ is defined. As a consequence, $\pi < \Psi^{\bar{\kappa}}_{\gamma+1}$ and hence

$$\mathrm{par}(\mathfrak{X}) \subseteq C^{\pi}(\gamma + 1) \subseteq C^{\bar{\kappa}}(\gamma + 1).$$

(viii) Using the by now familiar trick we get that $\Psi^{\bar{\pi}}_{\gamma+1}$ is defined and hence

$$\pi < \Psi^{\bar{\pi}}_{\gamma+1}.$$

This again implies

$$\mathrm{par}(\mathfrak{X}) \subseteq C^{\pi}(\gamma+1) \subseteq C^{\bar{\pi}}(\gamma+1).$$

By definition we also have

$$\pi < \Psi^{\pi'}_{\delta}$$

whenever $\pi' > \pi$, $\delta > \gamma$ and $\Psi^{\pi'}_{\delta}$ is defined. Thus in these cases we obtain

$$\bar{\pi} \in C^{\pi}(\gamma+1) \subseteq C^{\pi'}(\delta).$$

Finally, $\pi \leq \Psi^{\mathcal{K}}_{\gamma+1}$ again implies $\bar{\pi} \leq \Psi^{\mathcal{K}}_{\gamma+1}$. □

8. Collapsing

In the presence of reflection rules, cut elimination (and hence also collapsing) becomes more involved. The reason for this is that while in the Predicative Cut Elimination (Theorem 5.3.2), only symmetrical inferences were allowed (and so we could just make the necessary (cut) earlier in the derivation), we now have to face an asymmetrical scenario like:

$$\dfrac{\dfrac{\vdash \Gamma, F}{\vdash \Gamma, (\exists x \in \mathrm{L}_{\mathcal{K}})(x \models F)}\ (\text{4-Ref}^{\mathcal{K}}) \qquad \dfrac{(\vdash \Gamma, \neg(t \models F))_{t \in T_{\mathcal{K}}}}{\vdash \Gamma, (\forall x \in \mathrm{L}_{\mathcal{K}})\neg(x \models F)}\ (\bigwedge)}{\vdash \Gamma}\ (\text{cut})$$

The solution to this problem is of course known since Rathjen's [Rat94b]: With the proviso that the side formulas (Γ in this case) are $\Pi_4(\mathcal{K})$, an application of an (4-Ref$^{\mathcal{K}}$)-rule can be replaced by a bunch of applications of (3-Ref$^{\kappa}_{\xi}$)-rules (this will be done in section 8.1), such rules can again be replaced by (2-Ref$^{\kappa,\pi}_{\xi,\xi'}$)-rules, using the same technique. To get rid of those last rules, one simply uses a limit process, exploiting the fact that then it is sufficient to consider only side formulas of complexity Π_2, which admits an upward persistency argument. Therefore, "collapsing" is supposed to describe the whole process of cut-elimination, rule-elimination and collapsing of the proof-tree.

A short remark on the arrangement of this chapter (or equivalently on the arrangement of the collapsing theorems). We opted for (hopefully) much clarity but little uniformity, which means that we split up the collapsing procedure in as many parts as possible (three in this case). As one cannot allow oneself such laxity anymore when moving to stronger theories, we will indicate how to compress all parts of the collapsing procedure into just one theorem. Then it will also become clear that one could in fact prove slightly more than what is actually needed. However, we hope that the advantage of the main part approach — seeing clearly the problems that can arise at every stage of the collapsing process — outweighs its drawbacks.

8.1. The First Collapsing Theorem

As we already mentioned, there will be no surprises in this section. Technically we stick closer to [Bla97] than the original [Rat94b].

Theorem 8.1.1. *Assume* $\mathfrak{A}^4(\mathfrak{X};\gamma)$ *and* $\Delta^{(\mathcal{K})} \subseteq \Pi_4(\mathcal{K})$*. If*

$$\mathcal{H}_\gamma[\mathfrak{X}] \left|\frac{\alpha}{\mathcal{K}+1}\right. \Delta^{(\mathcal{K})},$$

then

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}, \kappa] \left|\frac{\Psi^{\mathcal{K}}_{\hat{\alpha}}[\kappa]}{\bullet}\right. \Delta^{(\mathcal{K},\kappa)}$$

holds for all $\kappa \in \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}}$*, where* $\hat{\alpha} = \gamma \oplus \omega^{\omega^{\mathcal{K}\oplus\alpha}}$.

Proof. The proof is by induction on α. We examine the last inference.

- The last inference was $(\bigwedge)$. We only consider the case that the main formulas are

  $$\bigwedge((F^1_t)^{(\mathcal{K})})_{t\in T_{\mathcal{K}}}, \ldots, \bigwedge((F^n_t)^{(\mathcal{K})})_{t\in T_{\mathcal{K}}} \in \Delta^{(\mathcal{K})},$$

  as all other cases run similarly (but easier). Then for all $\vec{t} = t_1, \ldots, t_n \in T_{\mathcal{K}}$ there was an $\alpha_{\vec{t}}$ such that $|t_1|, \ldots, |t_n| \leq \alpha_{\vec{t}} < \alpha$ and

  $$\mathcal{H}_\gamma[\mathfrak{X}, \vec{t}]\Big|\frac{\alpha_{\vec{t}}}{\mathcal{K}+1}\ \Delta^{(\mathcal{K})}, (F^1_{t_1})^{(\mathcal{K})}, \ldots, (F^n_{t_n})^{(\mathcal{K})}.$$

  Setting $\gamma_{\vec{t}} = \gamma \oplus \omega^{\omega^{\mathcal{K}\oplus\alpha_{\vec{t}}\oplus|t_1|\oplus\cdots\oplus|t_n|}}$ and $\beta_{\vec{t}} = \gamma_{\vec{t}} \oplus \omega^{\omega^{\mathcal{K}\oplus\alpha_{\vec{t}}}}$, Lemma 7.2.1(v) guarantees

  $$\mathfrak{A}^4(\mathfrak{X} \cup \{t_1, \ldots, t_n\}; \gamma_{\vec{t}}),$$

  so we can apply the induction hypothesis to get

  $$\mathcal{H}_{\beta_{\vec{t}}}[\mathfrak{X}, \vec{t}, \kappa']\Big|\frac{\Psi^{\mathcal{K}}_{\beta_{\vec{t}}}[\kappa']}{\bullet}\ \Delta^{(\mathcal{K},\kappa')}, (F^1_{t_1})^{(\mathcal{K},\kappa')}, \ldots, (F^n_{t_n})^{(\mathcal{K},\kappa')} \quad \text{for all } \kappa' \in \mathcal{A}^{\mathcal{K}}_{\beta_{\vec{t}}}$$

  for all $t_1, \ldots, t_n \in T_{\mathcal{K}}$. Now fix $\kappa \in \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}}$ and let $t_1, \ldots, t_n \in T_\kappa$. Then again Lemma 7.2.1(v) implies

  $$\kappa \in \mathcal{A}^{\mathcal{K}}_{\beta_{\vec{t}}} \quad \text{and} \quad \Psi^{\mathcal{K}}_{\beta_{\vec{t}}}[\kappa] < \Psi^{\mathcal{K}}_{\hat{\alpha}}[\kappa],$$

  and so we only need to apply $(\bigwedge)$ to get

  $$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}, \kappa]\Big|\frac{\Psi^{\mathcal{K}}_{\hat{\alpha}}[\kappa]}{\bullet}\ \underbrace{\Delta^{(\mathcal{K},\kappa)}, (\forall x^\kappa)(F^1(x))^{(\mathcal{K},\kappa)}, \ldots, (\forall x^\kappa)(F^n(x))^{(\mathcal{K},\kappa)}}_{=\Delta^{(\mathcal{K},\kappa)}}.$$

  Note that here the index set of the inference changed from $T_{\mathcal{K}}$ to T_κ.

- The last inference was $(\bigvee)$, and so we had

  $$\mathcal{H}_\gamma[\mathfrak{X}]\Big|\frac{\alpha_0}{\mathcal{K}+1}\ \Delta^{(\mathcal{K})}, F^{(\mathcal{K})}_{t_0}$$

  with $\bigvee(F^{(\mathcal{K})}_t)_{t\in J} \in \Delta^{(\mathcal{K})}$ and $t_0 \in J$. The induction hypothesis implies

  $$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}, \kappa']\Big|\frac{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\kappa']}{\bullet}\ \Delta^{(\mathcal{K},\kappa')}, F^{(\mathcal{K},\kappa')}_{t_0}$$

  for all $\kappa' \in \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}_0}$. Fix $\kappa \in \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}}$. Then Lemma 7.2.1(iii) yields

  $$\Psi^{\mathcal{K}}_{\gamma+1} \leq \Psi^{\mathcal{K}}_{\hat{\alpha}},$$

and thus also

$$t_0 \in C^{\mathcal{K}}(\hat{\alpha}) \cap \mathcal{K} = \Psi^{\mathcal{K}}_{\hat{\alpha}}.$$

So $t_0 < \kappa$, and as Lemma 7.2.1(iv) implies $\kappa \in \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}_0}$ and $\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\kappa] < \Psi^{\mathcal{K}}_{\hat{\alpha}}[\kappa]$, we can just apply a $(\bigvee)$-inference. Note that the index set J of the inference might shrink here, too — $T_{\mathcal{K}}$ for example becomes T_κ.

- The last inference was a (cut). So first assume that $\mathrm{rk}(C) = \mathrm{rk}(\neg C) = \mathcal{K}$ and we had

$$\mathcal{H}_\gamma[\mathfrak{X}] \Big|\frac{\alpha_0}{\mathcal{K}+1}\ \Delta^{(\mathcal{K})}, (\neg)C^{(\mathcal{K})}.$$

Then for example $C \in \Sigma_1(\mathcal{K})$ and $\neg C \in \Pi_1(\mathcal{K})$ by Lemma 5.2.1, hence we can apply the induction hypothesis and get

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}, \kappa'] \Big|\frac{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\kappa']}{\bullet}\ \Delta^{(\mathcal{K},\kappa')}, (\neg)C^{(\mathcal{K},\kappa')}$$

for all $\kappa' \in \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}_0}$. But if $\kappa \in \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}}$, then $\kappa \in \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}_0}$ and $\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\kappa], \mathrm{rk}(C^{(\mathcal{K},\kappa)}) < \Psi^{\mathcal{K}}_{\hat{\alpha}}[\kappa]$ by Lemma 7.2.1(iv), so a (cut) works here.

If $\mathrm{rk}(C) < \mathcal{K}$, then we can just apply the induction hypothesis, and $\mathrm{rk}(C) < \Psi^{\mathcal{K}}_{\gamma+1}$ and $\Psi^{\mathcal{K}}_{\gamma+1} \leq \Psi^{\mathcal{K}}_{\hat{\alpha}}$ allows us to do the same (cut).

- The last inference was of the form ($3\text{-Ref}^{\kappa}_{\xi}$) or ($2\text{-Ref}^{\pi,\kappa}_{\xi,\xi'}$). Then we can apply first the induction hypothesis and then the same rule, using the fact that

$$\mathrm{par}(\Delta) \cap \mathcal{K} < \Psi^{\mathcal{K}}_{\gamma+1} \leq \Psi^{\mathcal{K}}_{\hat{\alpha}}.$$

- The last inference was ($4\text{-Ref}^{\mathcal{K}}$) and we had

$$\mathcal{H}_\gamma[\mathfrak{X}] \Big|\frac{\alpha_0}{\mathcal{K}+1}\ \Delta^{(\mathcal{K})}, F^{(\mathcal{K})}$$

for some $\alpha_0 < \alpha$ and $F^{(\mathcal{K})} \in \Pi_4(\mathcal{K})$ such that $(\exists z^{\mathcal{K}})(z \models F) \in \Delta^{(\mathcal{K})}$. Using the induction hypothesis we get

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}, \kappa'] \Big|\frac{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\kappa']}{\bullet}\ \Delta^{(\mathcal{K},\kappa')}, F^{(\mathcal{K},\kappa')}$$

for all $\kappa' \in \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}_0}$. Let $\kappa \in \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}}$. Then $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\hat{\alpha}_0}]$ by Lemma 7.2.1 (iv). As we also have

$$\Big|\!\frac{*}{}\ \mathrm{trans}(\mathrm{L}_{\kappa'}) \wedge \mathrm{L}_{\kappa'} \neq \emptyset$$

for all $\kappa' \in \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}_0}$ by Lemma 6.1.1, we get

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}, \kappa, \kappa'] \Big|\frac{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\kappa'] + \omega}{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\kappa']}\ \bigvee \Delta^{(\mathcal{K},\kappa')}, (\exists^{\kappa} z)(z \models F) \qquad (*)$$

for all $\kappa' \in \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}_0} \cap \kappa$ by means of $(\bigvee)$. Now fix $s \in T_\kappa$. As we also obtain

$$\big|\!\frac{\;*\;}{\;} \mathrm{L}_{\kappa'} \neq s, \bigwedge \neg\Delta^{(\mathcal{K},\kappa')}, \bigvee \Delta^{(\mathcal{K},s)}, \tag{$**$}$$

we can combine $(*)$ and $(**)$ and arrive at

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X},\kappa,s,\kappa']\Big|\!\frac{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[|s|]+\omega+1}{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[|s|]}\; \mathrm{L}_{\kappa'} \neq s, \bigvee \Delta^{(\mathcal{K},s)}, (\exists^\kappa z)(z \models F)$$

for all $\kappa' \in \mathcal{A}^{\mathcal{K}}_{\hat{\alpha}_0}$ such that $\kappa' \leq |s|$. Applying $(\bigwedge)$ in the guise of $(\neg 3\text{-refl}_{\hat{\alpha}_0})$ has

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X},\kappa,s]\Big|\!\frac{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[|s|]+\omega+2}{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[|s|]}\; \neg 3\text{-refl}_{\hat{\alpha}_0}(s), \bigvee \Delta^{(\mathcal{K},s)}, (\exists^\kappa z)(z \models F)$$

as a consequence. As $s \in T_\kappa$ was arbitrary, we can use $(\bigvee)$ and then $(\bigwedge)$ and finally get

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X},\kappa]\Big|\!\frac{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\kappa]}{\bullet}\; (\forall v^\kappa)\big(3\text{-refl}_{\hat{\alpha}_0}(v) \to \bigvee \Delta^{(\mathcal{K},v)}\big), (\exists^\kappa z)(z \models F). \tag{✄}$$

(Here we used the fact that for $s \in T_\kappa$, $\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[|s|] < \kappa$.)

On the other hand we also obtain

$$\big|\!\frac{\;*\;}{\;} \Delta_{\mathrm{inv}}(\vec{t})^{(\mathcal{K},\kappa)}, \bigwedge \neg\Delta_{\mathrm{inv}}(\vec{t})^{(\mathcal{K},\kappa)}$$

for all $\vec{t} \in T_\kappa$, when $\Delta^{(\mathcal{K},\kappa)}_{\mathrm{inv}}$ arises from $\Delta^{(\mathcal{K},\kappa)}$ by replacing all formulas of the shape $(\forall x^\kappa)G(x)^{(\mathcal{K},\kappa)} \in \Delta^{(\mathcal{K},\kappa)}$ by $G(u_G)^{(\mathcal{K},\kappa)}$ (where u_G is a (new) free variable), so that $\Delta^{(\mathcal{K},\kappa)}_{\mathrm{inv}}$ becomes a set of $\Sigma_3(\kappa)$-formulas (with additional free variables; those will, however, not occur in the calculus as they are immediately replaced by the $\vec{t}$). Thus we may apply a $(3\text{-Ref}^\kappa_{\hat{\alpha}_0})$-rule and get

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X},\kappa,\vec{t}]\Big|\!\frac{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\kappa]+1}{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\kappa]}\; \Delta_{\mathrm{inv}}(\vec{t})^{(\mathcal{K},\kappa)}, (\exists v^\kappa)\big(3\text{-refl}_{\hat{\alpha}_0}(v) \wedge \bigwedge \neg\Delta^{(\mathcal{K},v)}\big)$$

for all $\vec{t} \in T_\kappa$. Using some $(\bigwedge)$-inferences yields

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X},\kappa]\Big|\!\frac{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\kappa]+\omega}{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\kappa]}\; \Delta^{(\mathcal{K},\kappa)}, (\exists v^\kappa)\big(3\text{-refl}_{\hat{\alpha}_0}(v) \wedge \bigwedge \neg\Delta^{(\mathcal{K},v)}\big). \tag{✃}$$

As we also have $\Psi^{\mathcal{K}}_{\hat{\alpha}}[\kappa] \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\hat{\alpha}_0}]$ (by the same arguments as in Lemma 7.2.1(iv)), we get

$$\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\kappa] + \omega < \Psi^{\mathcal{K}}_{\hat{\alpha}}[\kappa],$$

and so we can apply a (cut) to the derivations (✄) and (✃) and finally obtain

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X},\kappa]\Big|\!\frac{\Psi^{\mathcal{K}}_{\hat{\alpha}}[\kappa]}{\bullet}\; \Delta^{(\mathcal{K},\kappa)}, (\exists^\kappa z)(z \models F).$$

□

Remark. All $(3\text{-Ref}^\kappa_\sigma)$-rules, which we introduced in order to get rid of the $(4\text{-Ref}^{\mathcal{K}})$-rule, satisfied $\sigma < \hat{\alpha}$.

8.2. The Second Collapsing Theorem

In this section we turn to eliminating the (3-Ref$^{\kappa}_{\xi}$)-rules. This can be done exactly the same way as in the previous section by just shifting all complexities down by one. Note however that κ's may be schizophrenic, so the additional case that the last inference was a "small" Π_2-reflection rule living on κ has to be considered, too.

Note also that in the following theorem we only consider derivations with cut-rank $\kappa+1$; this is clearly sufficient for the purpose of our ordinal analysis (see the respective case in the proof of Theorem 8.3.1), one could however bypass this restriction by proving Theorems 8.2.1 and 8.3.1 simultaneously (by induction on the cut-rank and side induction on the derivation length), using the fact that if the last inference was a (cut) of large rank, one has the appropriate induction hypothesis at hand. This will be outlined in the appendix.

Theorem 8.2.1. *Assume* $\mathfrak{A}^3(\mathfrak{X};\gamma,\kappa,\xi)$ *and* $\Delta^{(\kappa)} \subseteq \Pi_3(\kappa)$. *If* $\mathcal{H}_\gamma[\mathfrak{X}] \overset{\alpha}{\underset{\kappa+1}{\vdash}} \Delta^{(\kappa)}$ *and if all* (2-Ref$^{\cdot,\cdot}_{\sigma,\sigma'}$)*- and* (3-Ref$^{\cdot}_{\sigma}$)*-inferences occurring in this derivation satisfy* $\sigma,\sigma' < \gamma$, *then also*

$$\mathcal{H}_{\hat\alpha}[\mathfrak{X},\pi] \overset{\Psi^{\kappa}_{\hat\alpha,\xi}[\pi]}{\underset{\bullet}{\vdash}} \Delta^{(\kappa,\pi)}$$

for all $\pi \in \mathcal{A}^{\kappa}_{\hat\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi}$, *where* $\hat\alpha = \gamma \oplus \omega^{\omega^{\kappa\oplus\alpha}}$.
Furthermore, all (2-Ref$^{\cdot,\cdot}_{\sigma,\sigma'}$)*- and* (3-Ref$^{\cdot}_{\sigma}$)*-inferences occurring in this new derivation satisfy* $\sigma,\sigma' < \hat\alpha$.

Proof. We proceed by induction on α.

- The last inference was ($\bigwedge$). We only consider the case that the main formulas are

  $$\bigwedge((F^1_t)^{(\kappa)})_{t\in T_\kappa}, \dots, \bigwedge((F^n_t)^{(\kappa)})_{t\in T_\kappa} \in \Delta^{(\kappa)}.$$

  For all $\vec{t} = t_1,\dots,t_n \in T_\kappa$ there is an $\alpha_{\vec{t}}$ such that $|t_1|,\dots,|t_n| \le \alpha_{\vec{t}} < \alpha$ and

  $$\mathcal{H}_\gamma[\mathfrak{X},\vec{t}] \overset{\alpha_{\vec{t}}}{\underset{\kappa+1}{\vdash}} \Delta^{(\kappa)}, (F^1_{t_1})^{(\kappa)}, \dots, (F^n_{t_n})^{(\kappa)}.$$

  Defining $\gamma_{\vec{t}} = \gamma \oplus \omega^{\omega^{\kappa\oplus\alpha_{\vec{t}}}\oplus|t_1|\oplus\cdots\oplus|t_n|}$ and $\beta_{\vec{t}} = \gamma_{\vec{t}} \oplus \omega^{\omega^{\kappa\oplus\alpha_{\vec{t}}}}$, Lemma 7.3.1(v) guarantees

  $$\mathfrak{A}^3(\mathfrak{X}\cup\{t_1,\dots,t_n\};\gamma_{\vec{t}},\kappa,\xi),$$

  hence we can apply the induction hypothesis and get

  $$\mathcal{H}_{\beta_{\vec{t}}}[\mathfrak{X},\vec{t},\pi'] \overset{\Psi^{\kappa}_{\beta_{\vec{t}},\xi}[\pi']}{\underset{\bullet}{\vdash}} \Delta^{(\kappa,\pi')}, (F^1_{t_1})^{(\kappa,\pi')}, \dots, (F^n_{t_n})^{(\kappa,\pi')} \quad \text{for all } \pi' \in \mathcal{A}^{\kappa}_{\beta_{\vec{t}}} \cap \mathcal{A}^{\mathcal{K}}_{\xi}$$

  for all $t_1,\dots,t_n \in T_\kappa$. Now fix $\pi \in \mathcal{A}^{\kappa}_{\hat\alpha} \cap \mathcal{A}^{\mathcal{K}}_{\xi}$ and let $t_1,\dots,t_n \in T_\pi$. Then again in Lemma 7.3.1(v) it was proved that

  $$\pi \in \mathcal{A}^{\kappa}_{\beta_{\vec{t}}} \cap \mathcal{A}^{\mathcal{K}}_{\xi} \quad \text{and} \quad \Psi^{\kappa}_{\beta_{\vec{t}},\xi}[\pi] < \Psi^{\kappa}_{\hat\alpha,\xi}[\pi],$$

so we can apply $(\bigwedge)$ to get

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X},\pi]\frac{\Psi^{\kappa}_{\hat{\alpha},\xi}[\pi]}{\bullet}\ \underbrace{\Delta^{(\kappa,\pi)},(\forall x^{\pi})(F^1(x))^{(\kappa,\pi)},\ldots,(\forall x^{\pi})(F^n(x))^{(\kappa,\pi)}}_{=\Delta^{(\kappa,\pi)}}.$$

- The last inference was $(\bigvee)$ and we had

$$\mathcal{H}_{\gamma}[\mathfrak{X}]\frac{\alpha_0}{\kappa+1}\ \Delta^{(\kappa)},F^{(\kappa)}_{t_0},$$

where $t_0\in J$ and $\bigvee(F^{(\kappa)}_t)_{t\in J}\in\Delta^{(\kappa)}$. We can apply the induction hypothesis and obtain

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X},\pi']\frac{\Psi^{\kappa}_{\hat{\alpha}_0,\xi}[\pi']}{\bullet}\ \Delta^{(\kappa,\pi')},F^{(\kappa,\pi')}_{t_0}$$

for all $\pi'\in\mathcal{A}^{\kappa}_{\hat{\alpha}_0}\cap\mathcal{A}^{\mathcal{K}}_{\xi}$. Now fix $\pi\in\mathcal{A}^{\kappa}_{\hat{\alpha}}\cap\mathcal{A}^{\mathcal{K}}_{\xi}$. Then by Lemma 7.3.1(iv) we get $\pi\in\mathcal{A}^{\kappa}_{\hat{\alpha}_0}\cap\mathcal{A}^{\mathcal{K}}_{\xi}$ and $\Psi^{\kappa}_{\hat{\alpha}_0,\xi}[\pi]<\Psi^{\kappa}_{\hat{\alpha},\xi}[\pi]$, and as $|t_0|<\kappa$ together with $\Psi^{\kappa}_{\gamma+1}\le\Psi^{\kappa}_{\hat{\alpha},\xi}$ implies $|t_0|<\Psi^{\kappa}_{\hat{\alpha},\xi}\le\pi$, this yields

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X},\pi]\frac{\Psi^{\kappa}_{\hat{\alpha},\xi}[\pi]}{\bullet}\ \underbrace{\Delta^{(\kappa,\pi)},\bigvee((F_t)^{(\kappa,\pi)})_{t\in\tilde{J}}}_{=\Delta^{(\kappa,\pi)}}$$

by $(\bigvee)$, where $\tilde{J}=J\restriction\pi$.

- The last inference was $(2\text{-Ref}^{\kappa,\bar{\kappa}}_{\sigma,\sigma'})$. Then we have Π_2-formulas F_i and T_κ-terms t_i $(i\le k)$ such that

$$\mathcal{H}_{\gamma}[\mathfrak{X}]\frac{\alpha_0}{\kappa+1}\ \Delta^{(\kappa)},\bigwedge_{i\le k}F_i(t_i)^{(\kappa)}.$$

Applying the induction hypothesis has the consequence

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X},\pi']\frac{\Psi^{\kappa}_{\hat{\alpha}_0,\xi}[\pi']}{\bullet}\ \Delta^{(\kappa,\pi')},\bigwedge_{i\le k}F_i(t_i)^{(\kappa,\pi')}$$

for all $\pi'\in\mathcal{A}^{\kappa}_{\hat{\alpha}_0}\cap\mathcal{A}^{\mathcal{K}}_{\xi}$. (Note that $|t_i|<\pi'$ like in the previous case.) If $\pi\in\mathcal{A}^{\kappa}_{\hat{\alpha}}\cap\mathcal{A}^{\mathcal{K}}_{\xi}$, then

  - $\pi\in\mathcal{A}^{\kappa}_{\hat{\alpha}_0}\cap\mathcal{A}^{\mathcal{K}}_{\xi}$ (by Lemma 7.4.1(iv))
  - $\sigma,\sigma'<^{C}_{\pi}\hat{\alpha}$ (as $\sigma,\sigma'\le\gamma$ and $\sigma,\sigma'\in\mathcal{H}_{\gamma}[\mathfrak{X}]$)
  - $\bar{\kappa}\in C(\hat{\alpha},\pi)$ (as $\bar{\kappa}\in\mathcal{H}_{\gamma}[\mathfrak{X}]$),

and as also $\Psi^{\kappa}_{\hat{\alpha}_0,\xi}[\pi] < \Psi^{\kappa}_{\hat{\alpha},\xi}[\pi]$ holds, again by Lemma 7.3.1 (iv), we get by an application of the same (2-Ref$^{\pi,\bar{\kappa}}_{\sigma,\sigma'}$)-rule

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X},\pi]\overset{\Psi^{\kappa}_{\hat{\alpha},\xi}[\pi]}{\underset{\bullet}{\vdash}} \Delta^{(\kappa,\pi)}, \underbrace{(\exists z^{\pi})\big(\text{2-refl}^{\bar{\kappa}}_{\sigma,\sigma'}(z) \wedge \bigwedge_{i\leq k}(\exists x \in z)(F_i(x))^{(\kappa,z)}\big)}_{\in\Delta^{(\kappa,\pi)}}.$$

- If the last inference was (2-Ref$^{\kappa',\bar{\kappa}}_{\sigma,\sigma'}$) such that $\kappa' < \kappa$, we simply use the induction hypothesis and then apply the same inference, using the fact that all occurring parameters ($\kappa', \bar{\kappa}, \sigma, \sigma'$ in this case) had already been controlled by $\mathcal{H}_{\gamma}[\mathfrak{X}]$.
- The last inference was (3-Ref$^{\kappa}_{\sigma}$). Then we had Π_3-formulas F_i and T_{κ}-terms t_i ($i \leq k$) such that

  $$\mathcal{H}_{\gamma}[\mathfrak{X}]\overset{\alpha_0}{\underset{\kappa+1}{\vdash}} \Delta^{(\kappa)}, \bigwedge_{i\leq k} F_i(t_i)^{(\kappa)}.$$

  As by assumption $\sigma \leq \gamma$ and $\sigma \in \mathcal{H}_{\gamma}[\mathfrak{X}]$, we immediately get

  $$\mathfrak{A}^3(\mathfrak{X};\gamma,\kappa,\sigma),$$

  so we can apply the induction hypothesis and get

  $$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X},\pi']\overset{\Psi^{\kappa}_{\hat{\alpha}_0,\sigma}[\pi']}{\underset{\bullet}{\vdash}} \Delta^{(\kappa,\pi')}, \bigwedge_{i\leq k} F_i(t_i)^{(\kappa,\pi')}$$

  for all $\pi' \in \mathcal{A}^{\kappa}_{\hat{\alpha}_0} \cap \mathcal{A}^{\mathcal{K}}_{\sigma}$. Now fix $\pi \in \mathcal{A}^{\kappa}_{\hat{\alpha}} \cap \mathcal{A}^{\mathcal{K}}_{\xi}$. Then

  $$\mathcal{A}^{\kappa}_{\hat{\alpha}_0} \cap \mathcal{A}^{\mathcal{K}}_{\sigma} \text{ stationary in } \pi$$

  can be verified analogously to Lemma 7.3.1 (iv). (The same argument also shows

  $$\mathcal{A}^{\kappa}_{\hat{\alpha}_0} \cap \mathcal{A}^{\mathcal{K}}_{\sigma} \text{ stationary in } \Psi^{\kappa}_{\hat{\alpha},\xi}[\pi], \tag{†}$$

  which we will need later.) For all $\pi' \in \mathcal{A}^{\kappa}_{\hat{\alpha}_0} \cap \mathcal{A}^{\mathcal{K}}_{\sigma} \cap \pi$ we have

  $$\overset{*}{\vdash} \text{3-refl}_{\sigma}(\mathrm{L}_{\pi'}) \wedge \mathrm{L}_{\pi'} \in \mathrm{L}_{\pi}.$$

  Therefore, for those π' we get

  $$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X},\pi,\pi']\overset{\Psi^{\kappa}_{\hat{\alpha}_0,\sigma}[\pi']+\omega}{\underset{\Psi^{\kappa}_{\hat{\alpha}_0,\sigma}[\pi']}{\vdash}} \bigvee \Delta^{(\kappa,\pi')}, (\exists u^{\pi})\vec{F}^{\kappa}_{\sigma}(u), \tag{$*$}$$

  where $\vec{F}^{\kappa}_{\sigma}(u) \equiv \text{3-refl}_{\sigma}(u) \wedge \bigwedge_{i\leq k}(\exists x \in u)(F_i(x))^{(\kappa,u)}$. Now we also have for all $s \in T_{\pi}$

  $$\overset{*}{\vdash} \mathrm{L}_{\pi'} \neq s, \bigwedge \neg\Delta^{(\kappa,\pi')}, \bigvee \Delta^{(\kappa,s)}. \tag{$**$}$$

Combining $(*)$ and $(**)$ via a (cut) we obtain

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X},\pi,s,\pi'] \Big|\frac{\Psi^{\kappa}_{\hat{\alpha}_0,\sigma}[|s|]+\omega+1}{\Psi^{\kappa}_{\hat{\alpha}_0,\sigma}[|s|]}\ \mathrm{L}_{\pi'} \neq s, \bigvee \Delta^{(\kappa,s)}, (\exists u^{\pi})\vec{F}^{\kappa}_{\sigma}(u)$$

for all $s \in T_\pi$ and all $\pi' \in \mathcal{A}^{\kappa}_{\hat{\alpha}_0} \cap \mathcal{A}^{\mathcal{K}}_{\sigma}$ such that $\pi' \leq |s|$. Here, an application of $(\bigwedge)$, i.e. $(\neg 2\text{-refl}^{\kappa}_{\hat{\alpha}_0,\sigma})$, leads to

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X},\pi,s] \Big|\frac{\Psi^{\kappa}_{\hat{\alpha}_0,\sigma}[|s|]+\omega+2}{\Psi^{\kappa}_{\hat{\alpha}_0,\sigma}[|s|]}\ \neg 2\text{-refl}^{\kappa}_{\hat{\alpha}_0,\sigma}(s), \bigvee \Delta^{(\kappa,s)}, (\exists u^{\pi})\vec{F}^{\kappa}_{\sigma}(u),$$

for all $s \in T_\pi$, and after applying $(\bigvee)$ and $(\bigwedge)$ we arrive at

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X},\pi] \Big|\frac{\Psi^{\kappa}_{\hat{\alpha}_0,\sigma}[\pi]}{\bullet}\ (\forall v^{\pi})\big(2\text{-refl}^{\kappa}_{\hat{\alpha}_0,\sigma}(v) \rightarrow \bigvee \Delta^{(\kappa,v)}\big), (\exists u^{\pi})\vec{F}^{\kappa}_{\sigma}(u). \qquad (✄)$$

(Note that $\pi \in \mathrm{Lim}(\mathcal{A}^{\kappa}_{\hat{\alpha}_0} \cap \mathcal{A}^{\mathcal{K}}_{\sigma})$ implies that all the $\Psi^{\kappa}_{\hat{\alpha}_0,\sigma}[|s|]$ are below $\Psi^{\kappa}_{\hat{\alpha}_0,\sigma}[\pi]$.) As we also have

$$\Big|\frac{*}{}\ \Delta_{\mathrm{inv}}(\vec{t})^{(\kappa,\pi)}, \bigwedge \neg\Delta_{\mathrm{inv}}(\vec{t})^{(\kappa,\pi)},$$

where $\Delta^{(\kappa,\pi)}_{\mathrm{inv}}$ is the result of replacing each formula $(\forall x^{\pi})G(x)^{(\kappa,\pi)} \in \Delta^{(\kappa,\pi)}$ by $G(u_G)^{(\kappa,\pi)}$ (with a fresh free variable u_G), so that $\Delta^{(\kappa,\pi)}_{\mathrm{inv}}$ becomes a set of $\Sigma_2(\pi)$-formulas (with free variables). This implies

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X},\pi,\vec{t}] \Big|\frac{\Psi^{\kappa}_{\hat{\alpha}_0,\sigma}[\pi]}{\bullet}\ \Delta_{\mathrm{inv}}(\vec{t})^{(\kappa,\pi)}, \bigwedge \neg\Delta_{\mathrm{inv}}(\vec{t})^{(\kappa,\pi)}$$

for all $\vec{t} \in T_\pi$. So now we are in the situation to introduce a $(2\text{-Ref}^{\pi,\kappa}_{\hat{\alpha}_0,\sigma})$-rule and get

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X},\pi,\vec{t}] \Big|\frac{\Psi^{\kappa}_{\hat{\alpha}_0,\sigma}[\pi]+1}{\Psi^{\kappa}_{\hat{\alpha}_0,\sigma}[\pi]}\ \Delta_{\mathrm{inv}}(\vec{t})^{(\kappa,\pi)}, (\exists v^{\pi})\big(2\text{-refl}^{\kappa}_{\hat{\alpha}_0,\sigma}(v) \wedge \bigwedge \neg\Delta^{(\kappa,v)}\big)$$

for all $\vec{t} \in T_\pi$. To get rid of the terms, we use some $(\bigwedge)$- inferences and finally arrive at

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X},\pi] \Big|\frac{\Psi^{\kappa}_{\hat{\alpha}_0,\sigma}[\pi]+\omega}{\Psi^{\kappa}_{\hat{\alpha}_0,\sigma}[\pi]}\ \Delta^{(\kappa,\pi)}, (\exists v^{\pi})\big(2\text{-refl}^{\kappa}_{\hat{\alpha}_0,\sigma}(v) \wedge \bigwedge \neg\Delta^{(\kappa,v)}\big). \qquad (✃)$$

Now (cut)ting (✄) and (✃) and verifying $\Psi^{\kappa}_{\hat{\alpha}_0,\sigma}[\pi]+\omega < \Psi^{\kappa}_{\hat{\alpha},\xi}[\pi]$ (which immediately follows from (†)) leads to

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X},\pi] \Big|\frac{\Psi^{\kappa}_{\hat{\alpha},\xi}[\pi]}{\bullet}\ \Delta^{(\kappa,\pi)}, (\exists u^{\pi})\vec{F}^{\kappa}_{\sigma}(u).$$

- The last inference was (3-Ref$_\sigma^{\kappa'}$) with $\kappa' < \kappa$. Then we can, analogously to the small (2-Ref$^{\cdot}_{\cdot}$)-cases, just use the same inference after applying the induction hypothesis.

- The last inference was a (cut). We have the hypothesis

$$\mathcal{H}_\gamma[\mathfrak{X}]\left|\frac{\alpha_0}{\kappa+1}\right. \Delta^{(\kappa)}, (\neg)C^{(\kappa)}$$

with $\mathrm{rk}(C) < \kappa + 1$. In view of Lemma 5.2.1 this implies that $C \in \Sigma_1(\kappa)$ and $\neg C \in \Pi_1(\kappa)$ (or the other way round). Thus we can apply the induction hypothesis and get

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}, \pi']\left|\frac{\Psi^{\kappa}_{\hat{\alpha}_0,\xi}[\pi']}{\bullet}\right. \Delta^{(\kappa,\pi')}, (\neg)C^{(\kappa,\pi')}$$

for all $\pi' \in \mathcal{A}^{\kappa}_{\hat{\alpha}_0} \cap \mathcal{A}^{\mathcal{K}}_{\xi}$. Now if $\pi \in \mathcal{A}^{\kappa}_{\hat{\alpha}} \cap \mathcal{A}^{\mathcal{K}}_{\xi}$, then also $\pi \in \mathcal{A}^{\kappa}_{\hat{\alpha}_0} \cap \mathcal{A}^{\mathcal{K}}_{\xi}$ and $\mathrm{rk}(C^{(\kappa,\pi)}) = \mathrm{rk}(\neg C^{(\kappa,\pi)}) = \pi < \Psi^{\kappa}_{\hat{\alpha},\xi}[\pi]$, so we can just apply a (cut). □

8.3. The Third Collapsing Theorem

This section differs from the two preceding ones in some respects: later we want to use it to collapse a derivation of a $\Sigma_1^{\omega_1^{\mathrm{CK}}}$-formula below ω_1^{CK}, but the cut-rank of the original derivation will in that situation be $\mathcal{K}+1$, so here we finally have to consider arbitrarily large cut-ranks. In view of predicative cut-elimination, only a few of these cut-ranks are really critical. That motivates the following

Definition. Let $\mu \in \mathrm{Card}$. Set

$$\bar{\mu} = \begin{cases} \mu + 1 & \text{iff } \mu \in \mathrm{Reg} \\ \mu & \text{otherwise.} \end{cases}$$

Another point is that, in contrast to Theorem 8.2.1, here we have to eliminate all Π_2-reflection rules, which has to be done "by hand", as we don't want to introduce new "Π_1-reflection rules". Note that here our approach deviates, although marginally, from the original [Rat94b] in that we consider $\Pi_2(\pi)$-formulas instead of $\Sigma_1(\pi)$-formulas and collapse the complexity of the involved formulas in the process (like in the previous theorems), see the discussion on page 88.

Theorem 8.3.1. *Assume* $\mathfrak{A}^2(\mathfrak{X}; \gamma, \pi, \kappa, \xi, \xi', \mu)$ *and* $\Delta^{(\pi)} \subseteq \Pi_2(\pi)$. *If* $\mathcal{H}_\gamma[\mathfrak{X}]\left|\frac{\alpha}{\bar{\mu}}\right. \Delta^{(\pi)}$ *and if all* (2-Ref$^{\cdot,\cdot}_{\sigma,\sigma'}$)*- and* (3-Ref$^{\cdot}_{\sigma}$)*-inferences occurring in this derivation satisfy* $\sigma, \sigma' < \gamma$, *then also*

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}, \rho]\left|\frac{\Psi^{\pi,\kappa}_{\hat{\alpha},\xi,\xi'}[\rho]}{\bullet}\right. \Delta^{(\pi,\rho)}$$

for all $\rho \in \mathcal{A}^{\pi}_{\hat{\alpha}} \cap \mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$, *where* $\hat{\alpha} = \gamma \oplus \omega^{\omega^{\mu \oplus \alpha}}$.
Furthermore, all (2-Ref$^{\cdot,\cdot}_{\sigma,\sigma'}$)*- and* (3-Ref$^{\cdot}_{\sigma}$)*-inferences occurring in this new derivation satisfy* $\sigma, \sigma' < \hat{\alpha}$.

Proof. The proof runs by main induction on μ and side induction on α.

- The last inference was $(\bigwedge)$, so we had the assumptions

$$\mathcal{H}_\gamma[\mathfrak{X},\vec{t}]\Big|\frac{\alpha_{\vec{t}}}{\mu}\ \Delta^{(\pi)},(F^1_{t_1})^{(\pi)},\dots,(F^n_{t_n})^{(\pi)}$$

for all $t_1 \in J_1,\dots,t_n \in J_n$ where $J_i \subseteq T_\pi$ for all $i \le n$. We will again only examine the case $J_1 = \cdots = J_n = T_\pi$ explicitly, as all the other cases can be treated very similarly. We have $|t_1|,\dots,|t_n| \le \alpha_{\vec{t}}$, and setting

$$\gamma_{\vec{t}} = \gamma \oplus \omega^{\omega^{\mu\oplus\alpha_{\vec{t}}}\oplus|t_1|\oplus\cdots\oplus|t_n|},$$

Lemma 7.4.1(v) guarantees

$$\mathfrak{A}^2(\mathfrak{X}\cup\{t_1,\dots,t_n\};\gamma_{\vec{t}},\pi,\kappa,\xi,\xi',\mu).$$

So we can apply the side induction hypothesis and obtain

$$\mathcal{H}_{\beta_{\vec{t}}}[\mathfrak{X},t_1,\dots,t_n,\rho']\Big|\frac{\Psi^{\pi,\kappa}_{\beta_{\vec{t}},\xi,\xi'}[\rho']}{\bullet}\ \Delta^{(\pi,\rho')},(F^1_{t_1})^{(\pi,\rho')},\dots,(F^n_{t_n})^{(\pi,\rho')}$$

for all $\rho' \in \mathcal{A}^\pi_{\beta_{\vec{t}}}\cap\mathcal{A}^\kappa_\xi\cap\mathcal{A}^{\mathcal{K}}_{\xi'}$ for all $t_1,\dots,t_n \in T_\pi$, where $\beta_{\vec{t}} = \gamma_{\vec{t}}\oplus\omega^{\omega^{\mu\oplus\alpha_{\vec{t}}}}$. Now fix $\rho \in \mathcal{A}^\pi_{\hat{\alpha}}\cap\mathcal{A}^\kappa_\xi\cap\mathcal{A}^{\mathcal{K}}_{\xi'}$. Let further $t_1,\dots,t_n \in T_\rho$ be arbitrary. Then Lemma 7.4.1(v) implies

$$\rho \in \mathcal{A}^\pi_{\beta_{\vec{t}}} \quad\text{and}\quad \Psi^{\pi,\kappa}_{\beta_{\vec{t}},\xi,\xi'}[\rho] < \Psi^{\pi,\kappa}_{\hat{\alpha},\xi,\xi'}[\rho],$$

so we can just apply an $(\bigwedge)$-inference (with $J_i = T_\rho$ for all $i \le n$) and are done.

- The last inference was $(\bigvee)$ and we had

$$\mathcal{H}_\gamma[\mathfrak{X}]\Big|\frac{\alpha_0}{\mu}\ \Delta^{(\pi)},F^{(\pi)}_{t_0},$$

where $t_0 \in J$ and $\bigvee((F_t)^{(\pi)})_{t\in J} \in \Delta^{(\pi)}$. We can apply the side induction hypothesis and obtain

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X},\rho']\Big|\frac{\Psi^{\pi,\kappa}_{\hat{\alpha}_0,\xi,\xi'}[\rho']}{\bullet}\ \Delta^{(\pi,\rho')},F^{(\pi,\rho')}_{t_0}$$

for all $\rho' \in \mathcal{A}^\pi_{\hat{\alpha}_0}\cap\mathcal{A}^\kappa_\xi\cap\mathcal{A}^{\mathcal{K}}_{\xi'}$. Now fix $\rho \in \mathcal{A}^\pi_{\hat{\alpha}}\cap\mathcal{A}^\kappa_\xi\cap\mathcal{A}^{\mathcal{K}}_{\xi'}$. Then by Lemma 7.4.1(iv) we get $\rho \in \mathcal{A}^\pi_{\hat{\alpha}_0}\cap\mathcal{A}^\kappa_\xi\cap\mathcal{A}^{\mathcal{K}}_{\xi'}$ and $\Psi^{\pi,\kappa}_{\hat{\alpha}_0,\xi,\xi'}[\rho] < \Psi^{\pi,\kappa}_{\hat{\alpha},\xi,\xi'}[\rho]$. As we also get (like in the previous theorem) that if $t_0 < \pi$, then $t_0 < \Psi^{\pi,\kappa}_{\hat{\alpha},\xi,\xi'} \le \rho$, this implies

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X},\rho]\Big|\frac{\Psi^{\pi,\kappa}_{\hat{\alpha},\xi,\xi'}[\rho]}{\bullet}\ \underbrace{\Delta^{(\pi,\rho)},\bigvee((F_t)^{(\pi,\rho)})_{t\in\tilde{J}}}_{=\Delta^{(\pi,\rho)}}$$

by an application of $(\bigvee)$ where $\tilde{J} = J \restriction \rho$.

- The last inference was $(2\text{-Ref}^{\pi,\kappa'}_{\sigma,\sigma'})$. Before this inference we had T_π-Terms t_i such that

$$\mathcal{H}_\gamma[\mathfrak{X}]\frac{|\alpha_0}{\bar{\mu}}\ \Delta^{(\pi)}, \bigwedge_{i\leq k} F_i(t_i)^{(\pi)},$$

and $(\exists z^\pi)\big(2\text{-refl}^{\kappa'}_{\sigma,\sigma'}(z) \wedge \bigwedge_{i\leq k}(\exists x \in z)(F_i(x))^{(\pi,z)}\big) \in \Delta^{(\pi)}$. All the $F_i^{(\pi)}$ are $\Pi_2(\pi)$. So after verifying

$$\mathfrak{A}^2(\mathfrak{X};\gamma,\pi,\kappa',\sigma,\sigma',\mu),$$

where we need the fact that by hypothesis, $\sigma,\sigma' < \gamma$, we get by side induction hypothesis

$$\mathcal{H}_{\hat\alpha_0}[\mathfrak{X},\rho']\frac{|\Psi^{\pi,\kappa'}_{\hat\alpha_0,\sigma,\sigma'}[\rho']}{\bullet}\ \Delta^{(\pi,\rho')}, \bigwedge_{i\leq k} F_i(t_i)^{(\pi,\rho')} \tag{$\diamond$}$$

for all $\rho' \in \mathcal{A}^\pi_{\hat\alpha_0} \cap \mathcal{A}^{\kappa'}_\sigma \cap \mathcal{A}^{\mathcal{K}}_{\sigma'}$. (Notice here that the t_i were already controlled, i.e. $|t_i| \in C^\pi(\gamma+1)\cap\pi$, and so $|t_i| < \Psi^{\pi,\kappa'}_{\hat\alpha_0,\sigma,\sigma'}$.) Now fix $\rho \in \mathcal{A}^\pi_{\hat\alpha} \cap \mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$. As we have

  - $\hat\alpha_0,\kappa',\sigma,\sigma' \in C(\hat\alpha,\rho)$ (because $\hat\alpha_0,\kappa',\sigma,\sigma' \in \mathcal{H}_\gamma[\mathfrak{X}] \subseteq C^{\pi,\kappa}_{\xi,\xi'}(\hat\alpha)$)
  - $\hat\alpha_0,\pi \in C(\hat\alpha_0,\pi)$ (because $\mathcal{H}_\gamma[\mathfrak{X}] \subseteq C(\hat\alpha_0,\pi)$, too)

we get by definition

$$\rho \in \mathrm{Lim}(\mathcal{A}^\pi_{\hat\alpha_0} \cap \mathcal{A}^{\kappa'}_\sigma \cap \mathcal{A}^{\mathcal{K}}_{\sigma'}).$$

For every $\rho' \in \mathcal{A}^\pi_{\hat\alpha_0} \cap \mathcal{A}^{\kappa'}_\sigma \cap \mathcal{A}^{\mathcal{K}}_{\sigma'}$ we have

$$\left|\frac{*}{}\ 2\text{-refl}^{\kappa'}_{\sigma,\sigma'}(\mathrm{L}_{\rho'}).\right. \tag{$\diamond\diamond$}$$

Thus for all $\rho' \in \mathcal{A}^\pi_{\hat\alpha_0} \cap \mathcal{A}^{\kappa'}_\sigma \cap \mathcal{A}^{\mathcal{K}}_{\sigma'} \cap \rho$ we get

$$\mathcal{H}_{\hat\alpha_0}[\mathfrak{X},\rho,\rho']\frac{|\Psi^{\pi,\kappa'}_{\hat\alpha_0,\sigma,\sigma'}[\rho']+\omega}{\Psi^{\pi,\kappa'}_{\hat\alpha_0,\sigma,\sigma'}[\rho']}\ \Delta^{(\pi,\rho')}, (\exists z^\rho)\underbrace{\big(2\text{-refl}^{\kappa'}_{\sigma,\sigma'}(z) \wedge \bigwedge_{i\leq k}(\exists x\in z)(F_i(x))^{(\pi,z)}\big)}_{=\vec{F}^\pi_{(\kappa';\sigma,\sigma')}(z)}$$

by first applying the Inversion Lemma (5.2.2(ii)) to $(\diamond)$, then using $(\bigvee)$ (with the t_i's as witnesses), combining these derivations with $(\diamond\diamond)$ by use of $(\bigwedge)$ and finally $(\exists)$-quantifying again, this time $L_{\rho'}$ being the witness.

Now let without loss of generality

$$\Delta^{(\pi)} = (\forall x_1^\pi)(\exists y_1^\pi)G_1(x_1,y_1),\ldots,(\forall x_n^\pi)(\exists y_n^\pi)G_n(x_n,y_n)$$

with $\Delta_0(\pi)$-formulas G_i. Pick arbitrary $t_1, \ldots, t_n \in T_\rho$. By the Inversion Lemma (5.2.2(ii)) we get

$$\mathcal{H}_{\hat\alpha_0}[\mathfrak{X}, \rho, \rho', t_1, \ldots, t_n] \Big|\frac{\Psi^{\pi,\kappa'}_{\hat\alpha_0,\sigma,\sigma'}[\rho'] + \omega}{\Psi^{\pi,\kappa'}_{\hat\alpha_0,\sigma,\sigma'}[\rho']} \; (\vec{G}_{\text{inv}}(\vec{t}))^{(\rho')}, (\exists z^\rho)\vec{F}^{\pi}_{(\kappa';\sigma,\sigma')}(z)$$

for all $\rho' \in \mathcal{A}^\pi_{\hat\alpha_0} \cap \mathcal{A}^{\kappa'}_\sigma \cap \mathcal{A}^{\mathcal{K}}_{\sigma'} \cap \rho$, where $\vec{G}_{\text{inv}}(\vec{t}) \equiv (\exists y_1)G_1(t_1, y_1), \ldots, (\exists y_n)G_n(t_n, y_n)$. Thus by upward persistency (Lemma 5.2.2(iii)) also

$$\mathcal{H}_{\hat\alpha_0}[\mathfrak{X}, \rho, \rho', t_1, \ldots, t_n] \Big|\frac{\Psi^{\pi,\kappa'}_{\hat\alpha_0,\sigma,\sigma'}[\rho'] + \omega}{\Psi^{\pi,\kappa'}_{\hat\alpha_0,\sigma,\sigma'}[\rho']} \; (\vec{G}_{\text{inv}}(\vec{t}))^{(\rho)}, (\exists z^\rho)\vec{F}^{\pi}_{(\kappa';\sigma,\sigma')}(z)$$

holds for all $\rho' \in \mathcal{A}^\pi_{\hat\alpha_0} \cap \mathcal{A}^{\kappa'}_\sigma \cap \mathcal{A}^{\mathcal{K}}_{\sigma'} \cap \rho$ such that $|t_1|, \ldots, |t_n| < \rho'$. Let

$$\rho_{\vec{t}} = \Psi^{\pi,\kappa'}_{\hat\alpha_0,\sigma,\sigma'}[|t_1| \oplus \cdots \oplus |t_n|].$$

Because of $\rho \in \text{Lim}(\mathcal{A}^\pi_{\hat\alpha_0} \cap \mathcal{A}^{\kappa'}_\sigma \cap \mathcal{A}^{\mathcal{K}}_{\sigma'})$ we have $\rho_{\vec{t}} < \rho$, and by definition is $\rho_{\vec{t}} \in \mathcal{H}_{\hat\alpha_0}[\mathfrak{X}, \rho, t_1, \ldots, t_n]$, so in particular we obtain

$$\mathcal{H}_{\hat\alpha_0}[\mathfrak{X}, \rho, t_1, \ldots, t_n] \Big|\frac{\Psi^{\pi,\kappa'}_{\hat\alpha_0,\sigma,\sigma'}[\rho_{\vec{t}}] + \omega}{\Psi^{\pi,\kappa'}_{\hat\alpha_0,\sigma,\sigma'}[\rho_{\vec{t}}]} \; (\vec{G}_{\text{inv}}(\vec{t}))^{(\rho)}, (\exists z^\rho)\vec{F}^{\pi}_{(\kappa';\sigma,\sigma')}(z).$$

But as clearly

$$|t_1|, \ldots, |t_n| < \Psi^{\pi,\kappa'}_{\hat\alpha_0,\sigma,\sigma'}[\rho_{\vec{t}}] < \Psi^{\pi,\kappa}_{\hat\alpha,\xi,\xi'}[\rho],$$

we finally get by an application of $(\bigwedge)$ the desired

$$\mathcal{H}_{\hat\alpha}[\mathfrak{X}, \rho] \Big|\frac{\Psi^{\pi,\kappa}_{\hat\alpha,\xi,\xi'}[\rho]}{\bullet} \; \underbrace{\Delta^{(\pi,\rho)}, (\exists z^\rho)\vec{F}^{\pi}_{(\kappa';\sigma,\sigma')}(z)}_{=\Delta^{(\pi,\rho)}}.$$

Notice that here we needed collapsing functions of the form $\Psi^{\cdot}_{\cdot}[\bullet]$ and the limit process in order to get rid of the additional parameter ρ' in the operator, thus enabling us to eliminate the rule "for free".

We also want to stress a funny asymmetry here: the customary way to prove collapsing for a set $\Delta^{(\pi)}$ of formulas living on a π that only carries Π_2-reflection rules is as follows:

- only consider $\Sigma_1(\pi)$-formulas
- don't collapse $\Delta^{(\pi)}$ in the process
- if the last inference was a Π_2-reflection rule, use inversion and boundedness in order to derive the conclusion of the rule "by hand"

– in the end apply boundedness to the set of $\Sigma_1(\pi)$-formulas

Here we are doing something different: as we collapse the π in the process, we get this conclusion for free; but we have to work a bit to get Δ collapsed to the correct ρ, which involves (upwards) persistency — quite the opposite of boundedness!

- The last inference was a (cut). We get as hypothesis

$$\mathcal{H}_\gamma[\mathfrak{X}]\frac{\alpha_0}{\bar{\mu}}\ \Delta^{(\pi)}, (\neg)C$$

with $\mathrm{rk}(C) < \bar{\mu}$ and have to examine several cases:

$\boxed{\mathrm{rk}(C) \leq \pi.}$ Then we can simply apply the side induction hypothesis and obtain

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}, \rho]\frac{\Psi^{\pi,\kappa}_{\hat{\alpha}_0,\xi,\xi'}[\rho]}{\bullet}\ \Delta^{(\pi,\rho)}, (\neg)C^{(\pi,\rho)}$$

for all $\rho \in \mathcal{A}^\pi_{\hat{\alpha}_0} \cap \mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$. Note that $\mathrm{par}(\Delta, C) \cap \pi \subseteq \Psi^{\pi,\kappa}_{\hat{\alpha}_0,\xi,\xi'}$ follows from the assumptions. So if $\mathrm{rk}(C) < \pi$, then also $\mathrm{rk}(C) < \Psi^{\pi,\kappa}_{\hat{\alpha}_0,\xi,\xi'}$. If on the other hand $\mathrm{rk}(C) = \pi$, then by definition of the rank-function $\mathrm{rk}(C^{(\pi,\rho)}) = \rho$. As we additionally get

$$\rho \in \mathcal{A}^\pi_{\hat{\alpha}_0} \cap \mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \quad \text{and} \quad \Psi^{\pi,\kappa}_{\hat{\alpha}_0,\xi,\xi'}[\rho] < \Psi^{\pi,\kappa}_{\hat{\alpha},\xi,\xi'}[\rho]$$

for all $\rho \in \mathcal{A}^\pi_{\hat{\alpha}} \cap \mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$ by Lemma 7.4.1(iv), in both cases a (cut) yields the desired result.

$\boxed{\mathrm{rk}(C) = \mathcal{K}}$ and thus $\mu = \mathcal{K}$. Then $C \in \Sigma_1(\mathcal{K})$. From $\pi \leq \Psi^{\mathcal{K}}_{\gamma+1}$ we easily conclude

$$\mathfrak{A}^4(\mathfrak{X}; \gamma),$$

so Theorem 8.1.1 implies

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}]\frac{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\bar{\kappa}]}{\bullet}\ \Delta^{(\pi)}, (\neg)C^{(\mathcal{K},\bar{\kappa})},$$

where $\bar{\kappa} = \Psi^{\mathcal{K}}_{\hat{\alpha}_0}$. (Notice that $\bar{\kappa} \in \mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}]$ and $\bar{\kappa} > \mathrm{par}(\Delta^{(\pi)})$ by Lemma 7.2.1(iii), so $\Delta^{(\pi)}$ remains unchanged.) After a (cut) ($\mathrm{rk}(C^{(\mathcal{K},\bar{\kappa})}) = \bar{\kappa} < \Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\bar{\kappa}]$) we get

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}]\frac{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\bar{\kappa}] + 1}{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\bar{\kappa}]}\ \Delta^{(\pi)}.$$

$\mathfrak{A}^2(\mathfrak{X}; \hat{\alpha}_0, \pi, \kappa, \xi, \xi', \Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\bar{\kappa}])$ was already checked in Lemma 7.4.1(vi), and so we can apply the main induction hypothesis to get

$$\mathcal{H}_\beta[\mathfrak{X}, \rho]\frac{\Psi^{\pi,\kappa}_{\beta,\xi,\xi'}[\rho]}{\bullet}\ \Delta^{(\pi,\rho)}$$

for all $\rho \in \mathcal{A}^\pi_\beta \cap \mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$, where $\beta = \hat{\alpha}_0 \oplus \omega^{\omega^{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\bar{\kappa}]\cdot 2+1}}$. Now we have

– $\beta < \hat{\alpha}$, because $\alpha_0 < \alpha$ implies $\hat{\alpha}_0 < \hat{\alpha}$ and $\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\bar{\kappa}] < \mathcal{K}$ yields $\omega^{\omega^{\Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\bar{\kappa}]\cdot 2+1}} < \hat{\alpha}$, too,

– $\beta \in C^{\pi,\kappa}_{\xi,\xi'}(\hat{\alpha})$, because in view of $\hat{\alpha}_0 < \hat{\alpha}$ with $\alpha_0, \gamma \in \mathcal{H}_\gamma[\mathfrak{X}] \subseteq C^{\pi,\kappa}_{\xi,\xi'}(\hat{\alpha})$ (Lemma 7.4.1(iii)) also $\Psi^{\mathcal{K}}_{\hat{\alpha}_0}, \beta \in C^{\pi,\kappa}_{\xi,\xi'}(\hat{\alpha})$,

– $\beta, \pi \in C(\beta, \pi)$, because $\alpha_0, \gamma, \pi \in \mathcal{H}_\gamma[\mathfrak{X}] \subseteq C(\beta, \pi)$ and $\hat{\alpha}_0 \leq \beta$.

Thus we obtain by the familiar arguments

$$\mathcal{A}^\pi_{\hat{\alpha}} \subseteq \mathcal{A}^\pi_\beta \quad \text{and} \quad \Psi^{\pi,\kappa}_{\beta,\xi,\xi'}[\rho] < \Psi^{\pi,\kappa}_{\hat{\alpha},\xi,\xi'}[\rho] \quad \text{for all } \rho \in \mathcal{A}^\pi_{\hat{\alpha}} \cap \mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'},$$

so we are done.

$\boxed{\mathrm{rk}(C) = \kappa_C > \pi}$ and κ_C is Π^1_1-indescribable. Then $C \in \Sigma_1(\kappa_C)$. First assume that $\kappa_C = \mu$. Then we are in the situation of Theorem 8.2.1, and as Lemma 7.4.1(vii) implies $\mathfrak{A}^3(\mathfrak{X}; \gamma, \kappa_C, 0)$, we can conclude

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}] \overset{\Psi^{\kappa_C}_{\hat{\alpha}_0}[\bar{\pi}]}{\underset{\bullet}{\Big|\!\!\!-\!\!\!-\!\!\!-}} \Delta^{(\pi)}, (\neg)C^{(\kappa_C,\bar{\pi})}$$

where $\bar{\pi} = \Psi^{\kappa_C}_{\hat{\alpha}_0}$. (Notice that $\bar{\pi} \in \mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}]$ and $\bar{\pi} > \mathrm{par}(\Delta^{(\pi)})$, for example by Lemma 7.3.1(iii).) Thus after a (cut) ($\mathrm{rk}(C^{(\kappa_C,\bar{\pi})}) = \bar{\pi} < \Psi^{\kappa_C}_{\hat{\alpha}_0}[\bar{\pi}]$) we are left with

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}] \overset{\Psi^{\kappa_C}_{\hat{\alpha}_0}[\bar{\pi}]+1}{\underset{\Psi^{\kappa_C}_{\hat{\alpha}_0}[\bar{\pi}]}{\Big|\!\!\!-\!\!\!-\!\!\!-}} \Delta^{(\pi)}.$$

Now, $\Psi^{\kappa_C}_{\hat{\alpha}_0}[\bar{\pi}] = \tilde{\pi}$ is regular and $< \kappa_C$, and as $\mathfrak{A}^2(\mathfrak{X}; \hat{\alpha}_0, \pi, \kappa, \xi, \xi', \tilde{\pi})$ has already been checked in Lemma 7.4.1(vi), we can apply the main induction hypothesis:

$$\mathcal{H}_\beta[\mathfrak{X}, \rho] \overset{\Psi^{\pi,\kappa}_{\beta,\xi,\xi'}[\rho]}{\underset{\bullet}{\Big|\!\!\!-\!\!\!-\!\!\!-}} \Delta^{(\pi,\rho)}$$

for all $\rho \in \mathcal{A}^\pi_\beta \cap \mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$, where

$$\beta = \hat{\alpha}_0 \oplus \omega^{\omega^{\tilde{\pi}\cdot 2+1}}.$$

But now we have

– $\beta < \hat{\alpha}$, because one the one hand, $\alpha_0 < \alpha$ implies $\hat{\alpha}_0 < \hat{\alpha}$, and on the other hand $\tilde{\pi} < \kappa_C = \mu$ implies $\omega^{\omega^{\tilde{\pi}\cdot 2+1}} < \hat{\alpha}$,

– $\beta \in C^{\pi,\kappa}_{\xi,\xi'}(\hat{\alpha})$, because $\gamma, \alpha_0, \kappa_C \in \mathcal{H}_\gamma[\mathfrak{X}] \subseteq C^{\pi,\kappa}_{\xi,\xi'}(\hat{\alpha})$ and $\hat{\alpha}_0 < \hat{\alpha}$ imply $\bar{\pi}, \tilde{\pi} \in C^{\pi,\kappa}_{\xi,\xi'}(\hat{\alpha})$ and thus also $\beta \in C^{\pi,\kappa}_{\xi,\xi'}(\hat{\alpha})$,

– $\pi, \beta \in C(\beta, \pi)$ because of $\pi, \gamma, \alpha_0, \kappa_C \in \mathcal{H}_\gamma[\mathfrak{X}] \subseteq C(\beta, \pi)$ and $\hat{\alpha}_0 < \beta$.

So we get

$$\mathcal{A}^{\pi}_{\hat{\alpha}} \cap \mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \subseteq \mathcal{A}^{\pi}_{\beta} \cap \mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$$

and

$$\Psi^{\pi,\kappa}_{\beta,\xi,\xi'}[\rho] < \Psi^{\pi,\kappa}_{\hat{\alpha},\xi,\xi'}[\rho] \quad \text{for all } \rho \in \mathcal{A}^{\pi}_{\hat{\alpha}} \cap \mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'},$$

hence we are done.

Now assume $\kappa_C < \mu$. Then $\Psi^{\mathcal{K}}_1[\kappa_C] = \kappa_C^+ \leq \mu$ and $\Delta^{(\pi)}, C^{(\kappa_C)} \subseteq \Delta_0(\kappa_C^+)$, so we can proceed like in the next case.

$\boxed{\mathrm{rk}(C) = \pi_C > \pi}$ and π_C is not Π^1_1-indescribable. Then $C \in \Sigma_1(\pi_C)$ and again we have

$$\mathfrak{A}^2(\mathfrak{X}; \gamma, \pi_C, \Psi^{\mathcal{K}}_2[\pi_C], 0, 0, \mu)$$

by Lemma 7.4.1(viii). So we can apply the side induction hypothesis resulting in

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}] \Big|\frac{\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0]}{\bullet}\ \Delta^{(\pi)}, (\neg)C^{(\pi_C,\rho_0)}$$

where $\rho_0 = \Psi^{\pi_C}_{\hat{\alpha}_0}$. Notice that as usual $\rho_0 \in \mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}]$ and $\mathrm{par}(\Delta^{(\pi)}) \subseteq \pi + 1 < \rho_0$ by assumption. Now a (cut) yields

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}] \Big|\frac{\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0] + 1}{\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0]}\ \Delta^{(\pi)}.$$

If $\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0]$ is a cardinal, then in view of Corollary 3.3.4, it cannot be regular, so, although not important here, we get $\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0] = \overline{\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0]}$. As Lemma 7.4.1(vi) implies $\mathfrak{A}^2(\mathfrak{X}; \hat{\alpha}_0, \pi, \kappa, \xi, \xi', \Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0])$ and as $\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0] < \pi_C \leq \mu$, we can apply the main induction hypothesis and obtain

$$\mathcal{H}_{\beta}[\mathfrak{X}, \rho] \Big|\frac{\Psi^{\pi,\kappa}_{\beta,\xi,\xi'}[\rho]}{\bullet}\ \Delta^{(\pi,\rho)}$$

for all $\rho \in \mathcal{A}^{\pi}_{\beta} \cap \mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$, where

$$\beta = \hat{\alpha}_0 \oplus \omega^{\omega^{\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0] \cdot 2 + 1}}.$$

By the same arguments as in the previous case we now get

$$\beta <^{C}_{\Psi^{\pi,\kappa}_{\hat{\alpha},\xi,\xi'}} \hat{\alpha} \quad \text{and} \quad \pi, \beta \in C(\beta, \pi)$$

and thus

$$\mathcal{A}^{\pi}_{\hat{\alpha}} \cap \mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} \subseteq \mathcal{A}^{\pi}_{\beta} \cap \mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$$

and

$$\Psi^{\pi,\kappa}_{\beta,\xi,\xi'}[\rho] < \Psi^{\pi,\kappa}_{\hat{\alpha},\xi,\xi'}[\rho] \quad \text{for all } \rho \in \mathcal{A}^{\pi}_{\hat{\alpha}} \cap \mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'},$$

so we are done.

Now assume that $\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0]$ is not a cardinal. Then we first have to use predicative cut elimination before we can apply the main induction hypothesis. As

$$\left[\overline{p(\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0])}, \overline{p(\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0])} + \Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0]\right) \cap \mathrm{Reg} = \emptyset,$$

Theorem 5.3.2 implies

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}] \Big|\frac{\varphi(\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0])(\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0]+1)}{p(\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0])} \; \Delta^{(\pi)}.$$

Notice that $p(\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0])$ is a cardinal $< \pi_C \leq \mu$, and as Lemma 7.4.1(vi) also yields $\mathfrak{A}^2(\mathfrak{X}; \hat{\alpha}_0, \pi, \kappa, \xi, \xi', p(\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0]))$, we get

$$\mathcal{H}_{\beta}[\mathfrak{X}, \rho] \Big|\frac{\Psi^{\pi,\kappa}_{\beta,\xi,\xi'}[\rho]}{\bullet} \; \Delta^{(\pi,\rho)}$$

for all $\rho \in \mathcal{A}^{\pi}_{\beta} \cap \mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$, where

$$\beta = \hat{\alpha}_0 \oplus \omega^{\omega^{p\left(\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0]\right) \oplus \varphi\left(\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0]\right)\left(\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0]+1\right)}}.$$

But if $\rho \in \mathcal{A}^{\pi}_{\hat{\alpha}} \cap \mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$, then $\rho \in \mathcal{A}^{\pi}_{\beta} \cap \mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$ and $\Psi^{\pi,\kappa}_{\beta,\xi,\xi'}[\rho] \leq \Psi^{\pi,\kappa}_{\hat{\alpha},\xi,\xi'}[\rho]$, because

- $\beta < \hat{\alpha}$, because $\hat{\alpha}_0 < \hat{\alpha}$ and $\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0] < \mu$ (and hence also $p(\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0]) < \mu$ and $\varphi(\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0])(\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0]+1) < \mu$),
- $\beta \in C^{\pi,\kappa}_{\xi,\xi'}(\hat{\alpha})$, because $\gamma, \mu, \alpha_0, \pi_C \in C^{\pi}(\gamma+1) \subseteq C^{\pi,\kappa}_{\xi,\xi'}(\hat{\alpha})$ and $\hat{\alpha}_0 < \hat{\alpha}$, so that also $\Psi^{\pi_C}_{\hat{\alpha}_0}[\rho_0] \in C^{\pi,\kappa}_{\xi,\xi'}(\hat{\alpha})$,
- $\beta, \pi \in C(\beta, \pi)$ because $\gamma, \mu, \alpha_0, \pi_C \in \mathcal{H}_{\gamma}[\mathfrak{X}] \subseteq C(\beta, \pi)$ and $\hat{\alpha}_0 < \beta$.

$\boxed{\mathrm{rk}(C) = \rho_C > \pi}$ and ρ_C is not regular. Then again $\Psi^{\mathcal{K}}_1[\rho_C] \in \mathcal{H}_{\gamma}[\mathfrak{X}]$, $\Psi^{\mathcal{K}}_1[\rho_C] \leq \mu$ and $C \in \Delta_0(\Psi^{\mathcal{K}}_1[\rho_C])$, so we can argue exactly as in the previous case. □

In the following let $\rho_0 = 1$ and $\rho_{n+1} = \mathcal{K}^{\rho_n}$.

In order to formulate the concluding theorem, we first have to point out that the property of being admissible is quite simply expressible — there is a parameter-free Π_3-formula σ_0 such that

$$x \models \sigma_0 \Leftrightarrow x \text{ is admissible}$$

holds for all transitive x (see for example [RA74]). Thus we may assume that there is a Δ_0-formula $Ad(x)$ expressing that x is an admissible set containing ω.

Theorem 8.3.2. *If A is a Σ_1-formula such that*

$$\mathbf{\Pi_4{-}Ref} \vdash (\forall x)(Ad(x) \to A^x),$$

then there is an $n \in \omega$ such that

$$\mathcal{H}_{\rho_n} \overset{\Psi^{\omega_1}_{\rho_n}[\rho]}{\underset{\bullet}{\vdash}} A^{(\rho)}$$

holds for all $\rho \in \mathcal{A}^{\omega_1}_{\rho_n}$. In particular, $L_{\Psi^{\omega_1}_{\rho_n}} \models A$.

Hence at $L_{\Psi^{\omega_1}_{\varepsilon_{\mathcal{K}+1}}}$ all $\Sigma_1^{\omega_1^{CK}}$-sentences of $\mathbf{\Pi_4{-}Ref}$ are true, so

$$|\mathbf{\Pi_4{-}Ref}|_{\Sigma_1^{\omega_1^{CK}}} \leq \Psi^{\omega_1}_{\varepsilon_{\mathcal{K}+1}}.$$

Proof. By Theorem 6.2.3 there is an $m \in \omega$ such that

$$\mathcal{H}_0 \overset{\mathcal{K}\cdot\omega^m}{\underset{\mathcal{K}+m}{\vdash}} (\forall x^{\mathcal{K}})(Ad(x) \to A^x),$$

so using inversion (Lemma 5.2.2(ii)), we also get

$$\mathcal{H}_0[\omega_1] \overset{\mathcal{K}\cdot\omega^m}{\underset{\mathcal{K}+m}{\vdash}} \neg Ad(\mathrm{L}_{\omega_1}), A^{\mathrm{L}_{\omega_1}}.$$

Now applying predicate cut-elimination (Theorem 5.3.2) leads to

$$\mathcal{H}_0[\omega_1] \overset{\rho_{m+2}}{\underset{\mathcal{K}+1}{\vdash}} \neg Ad(\mathrm{L}_{\omega_1}), A^{\mathrm{L}_{\omega_1}},$$

so, after verifying $\mathfrak{A}^4(\{\omega_1\};1)$, Theorem 8.1.1 implies

$$\mathcal{H}_{\rho_k} \overset{\Psi^{\mathcal{K}}_{\rho_k}[\Psi^{\mathcal{K}}_{\rho_k}]}{\underset{\bullet}{\vdash}} \neg Ad(\mathrm{L}_{\omega_1}), A^{\mathrm{L}_{\omega_1}},$$

for $k \geq m+3$. Since we also have

$$\mathcal{H}_0 \overset{\omega_1\cdot\omega}{\underset{\omega_1+\omega}{\vdash}} Ad(\mathrm{L}_{\omega_1}),$$

we get

$$\mathcal{H}_{\rho_k} \overset{\Psi^{\mathcal{K}}_{\rho_k}[\Psi^{\mathcal{K}}_{\rho_k}]+1}{\underset{\Psi^{\mathcal{K}}_{\rho_k}[\Psi^{\mathcal{K}}_{\rho_k}]}{\vdash}} A^{\mathrm{L}_{\omega_1}} \qquad (*)$$

by means of a (cut). Now one easily sees that $\omega_1, \Psi^{\mathcal{K}}_{\rho_k} \in C(\rho_k+1, 0)$, which suffices to check

$$\mathfrak{A}^2(\emptyset; \rho_k, \omega_1, \Psi^{\mathcal{K}}_2[\omega_1], 0, 0, \Psi^{\mathcal{K}}_{\rho_k}[\Psi^{\mathcal{K}}_{\rho_k}]).$$

As by the remark following Theorem 8.1.1 all (3-Ref$_{\sigma}^{\kappa}$)-rules occurring in the derivation of ($*$) (there are no 2-Ref$_{\because}^{\because}$-rules in it) satisfy $\sigma < \rho_k$, we can apply Theorem 8.3.1 and obtain

$$\mathcal{H}_\alpha \frac{|\Psi_\alpha^{\omega_1}[\rho]}{|\bullet} A^{(\omega_1,\rho)}$$

for all $\rho \in \mathcal{A}_\alpha^{\omega_1}$, where $\alpha = \rho_k \oplus \omega^{\omega^{\Psi_{\rho_k}^{\mathcal{K}}[\Psi_{\rho_k}^{\mathcal{K}}]\cdot 2+1}}$. Using predicative cut elimination this leads to

$$\mathcal{H}_\alpha \frac{|\varphi\big(\Psi_\alpha^{\omega_1}[\rho]\big)\big(\Psi_\alpha^{\omega_1}[\rho]\big)}{|0} A^{(\omega_1,\rho)}$$

for all $\rho \in \mathcal{A}_\alpha^{\omega_1}$. Finally it is easy to find an $n \in \omega$ such that

$$\alpha <^C_{\Psi_{\rho_n}^{\omega_1}} \rho_n,$$

hence $\mathcal{A}_{\rho_n}^{\omega_1} \subseteq \mathcal{A}_\alpha^{\omega_1}$ and $\Psi_\alpha^{\omega_1}[\rho] < \Psi_{\rho_n}^{\omega_1}[\rho]$ holds for all $\rho \in \mathcal{A}_{\rho_n}^{\omega_1}$, so that we can conclude

$$\mathcal{H}_{\rho_n} \frac{|\Psi_{\rho_n}^{\omega_1}[\rho]}{|0} A^{(\omega_1,\rho)}$$

for all $\rho \in \mathcal{A}_{\rho_n}^{\omega_1}$.

But by the remark on page 59 following the definition of our calculus, if we take $\rho = \Psi_{\rho_n}^{\omega_1}$, the calculus is correct in this situation, so we get also $L_{\Psi_{\rho_n}^{\omega_1}} \models A$.

As Σ_1-formulas are upwards persistent, the second claim now follows immediately. □

Appendix: An Alternative Collapsing Theorem

As promised we want to line out how to formulate and prove the collapsing procedure in just one theorem. Therefore we first define

$$n(\pi) = \begin{cases} 4 & \text{if } \pi = \mathcal{K} \\ 3 & \text{if } \pi \text{ is } \Pi^1_1\text{-indescribable} \\ 2 & \text{otherwise.} \end{cases}$$

As the main result of this chapter is slightly stronger than the ones before, we need to modify the pancake conditions. The uniformity of this approach makes it look more entangled than it really is, but the following only gives a slight foretaste of the technical problems one encounters in [Rat05b]. As we want to talk about all stages of the collapsing process in a uniform way, we introduce the concept of *reflection configurations* R, which are sequences of ordinals, starting with $(R)_0 = s(R) =$ the *sphere* of the reflection configuration, i.e. the ordinal that carries the reflection rule connected with the configuration, followed by additional information about $s(R)$. Further we define the *reflection area* $a(R)$ as

$$a(R) = \begin{cases} \mathrm{On} & \text{if } R = \langle \mathcal{K} \rangle \\ \mathcal{A}^{\mathcal{K}}_{\xi} & \text{if } R = \langle \kappa, \xi \rangle \\ \mathcal{A}^{\kappa}_{\xi} \cap \mathcal{A}^{\mathcal{K}}_{\xi'} & \text{if } R = \langle \pi, \kappa, \xi, \xi' \rangle \end{cases}$$

and for $\pi = s(R)$ the collapsing function pertaining to R via

$$\Psi^{\pi}_{\alpha,R} = \mu\, \mathcal{A}^{\pi}_{\alpha} \cap a(R).$$

To simplify notation, we will again abbreviate $C(\alpha, \Psi^{\pi}_{\alpha,R})$ by $C^{\pi}_{R}(\alpha)$, and further use the notations Ψ^{π}_{α}, Ψ^{κ}_{α}, $C^{\pi}(\alpha)$ or $C^{\kappa}(\alpha)$, respectively, instead of Ψ^{π}_{α,R_0}, $\Psi^{\kappa}_{\alpha,R'_0}$, $C^{\pi}_{R_0}(\alpha)$ or $C^{\kappa}_{R'_0}(\alpha)$, respectively, where $R_0 = \langle \pi, \bar{\kappa}, 0, 0 \rangle$ for some/any $\bar{\kappa} > \pi$ and $R'_0 = \langle \kappa, 0 \rangle$.

We now (re-)define $\mathfrak{A}^4(\mathfrak{X}; \gamma, R, \mu) \Leftrightarrow$

- $R = \langle \mathcal{K} \rangle$
- $\gamma \in \mathcal{H}_{\gamma}[\mathfrak{X}]$
- $\mathrm{par}(\mathfrak{X}) \subseteq C^{\mathcal{K}}(\gamma + 1)$
- $\mu = \mathcal{K}$,

$\mathfrak{A}^3(\mathfrak{X}; \gamma, R, \mu) \Leftrightarrow$

- $R = \langle \kappa, \xi \rangle$
- $\gamma, \kappa, \xi, \mu \in \mathcal{H}_\gamma[\mathfrak{X}]$
- $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_\xi]$
- $\mathrm{par}(\mathfrak{X}) \subseteq C^\kappa(\gamma + 1)$
- $\kappa \in \bigcap\{C^{\pi'}(\delta) \mid \pi' > \kappa \wedge \delta > \gamma\}$
- $\xi \leq \gamma$ and $\kappa \leq \mu$
- $\kappa \leq \Psi^{\mathcal{K}}_{\gamma+1}$,

and finally $\mathfrak{A}^2(\mathfrak{X}; \gamma, R, \mu) \Leftrightarrow$

- $R = \langle \pi, \kappa, \xi, \xi' \rangle$
- $\gamma, \pi, \kappa, \xi, \xi', \mu \in \mathcal{H}_\gamma[\mathfrak{X}]$
- $\kappa \in \Pi^1_1[\mathcal{A}^{\mathcal{K}}_{\xi'}]$ and $\mathcal{A}^\kappa_\xi \cap \mathcal{A}^{\mathcal{K}}_{\xi'}$ stationary in π
- $\mathrm{par}(\mathfrak{X}) \subseteq C^\pi(\gamma + 1)$
- $\pi \in \bigcap\{C^{\pi'}(\delta) \mid \pi' > \pi \wedge \delta > \gamma\}$
- $(\xi' < \xi \leq \gamma \vee \xi' = \xi = 0 \leq \gamma)$ and $\pi \leq \mu$
- $\pi \leq \Psi^{\mathcal{K}}_{\gamma+1}$

Note that the only sensible difference to the definitions in the main part of this thesis consists in allowing the additional parameter μ and postulating the additional condition

$$\kappa \in \bigcap\{C^{\pi'}(\delta) \mid \pi' > \kappa \wedge \delta > \gamma\}$$

in $\mathfrak{A}^3$. Of course, the latter is caused by the former: as we will in the following theorem allow arbitrary cut-ranks $\mu \geq \kappa$ when collapsing a derivation living on a κ, we need to take care that if the last inference was a large cut we still get the induction hypothesis. Thus we add the following items to Lemma 7.3.1:

(vi) If $\alpha_0 \in \mathcal{H}_\gamma[\mathfrak{X}] \cap \alpha$, $\bar{\kappa}, \bar{\pi} \in \mathcal{H}_\gamma[\mathfrak{X}] \cap (\kappa, \mu]$, $\bar{\kappa}$ is Π^1_1-indescribable and $\bar{\pi}$ is not, then

$$\mathfrak{A}^3(\mathfrak{X}; \hat{\alpha}_0, \langle \kappa, \xi \rangle, \Psi^{\mathcal{K}}_{\hat{\alpha}_0}[\Psi^{\mathcal{K}}_{\hat{\alpha}_0}]),$$
$$\mathfrak{A}^3(\mathfrak{X}; \hat{\alpha}_0, \langle \kappa, \xi \rangle, \Psi^{\bar{\kappa}}_{\hat{\alpha}_0}[\Psi^{\bar{\kappa}}_{\hat{\alpha}_0}])$$

and

$$\mathfrak{A}^3(\mathfrak{X}; \hat{\alpha}_0, \langle \kappa, \xi \rangle, p^*(\Psi^{\bar{\pi}}_{\hat{\alpha}_0}[\Psi^{\bar{\pi}}_{\hat{\alpha}_0}])),$$

where

$$p^*(\beta) = \begin{cases} \beta & \text{if it is a cardinal,} \\ p(\beta) & \text{if not.} \end{cases}$$

(vii) If $\kappa < \bar{\kappa} \in \mathcal{H}_\gamma[\mathfrak{X}]$, $\bar{\kappa} \le \mu$ and $\bar{\kappa}$ is Π^1_1-indescribable, then

$$\mathfrak{A}^3(\mathfrak{X}; \gamma, \langle \bar{\kappa}, 0 \rangle, \mu).$$

(viii) If $\gamma \ge 2$, $\pi \in \mathcal{H}_\gamma[\mathfrak{X}]$, $\kappa < \pi \le \mu$ and π is not Π^1_1-indescribable, then

$$\mathfrak{A}^2(\mathfrak{X}; \gamma, \langle \pi, \Psi_2^{\mathcal{K}}[\pi], 0, 0 \rangle, \mu).$$

But (as one can see in the proof of 7.4.1), in order to prove these, the above mentioned additional condition is necessary!

So after proving these extended preparatory lemmas, one can state the main

Theorem. *Assume* $\pi = s(R)$, $\mathfrak{A}^{n(\pi)}(\mathfrak{X}; \gamma, R, \mu)$ *and* $\Delta^{(\pi)} \subseteq \Pi_{n(\pi)}(\pi)$. *If*

$$\mathcal{H}_\gamma[\mathfrak{X}] \overset{\alpha}{\underset{\bar{\mu}}{\vdash}} \Delta^{(\pi)}$$

and if all (2-Ref$^{\because}_{\sigma,\sigma'}$)*- and* (3-Ref$^{\cdot}_{\sigma}$)*-inferences occurring in this derivation satisfy* $\sigma, \sigma' < \gamma$*, then also*

$$\mathcal{H}_{\hat{\alpha}}[\mathfrak{X}, \rho] \overset{\Psi^{\pi}_{\hat{\alpha},R}[\rho]}{\underset{\bullet}{\vdash}} \Delta^{(\pi,\rho)}$$

for all $\rho \in \mathcal{A}^\pi_{\hat{\alpha}} \cap a(R)$*, where* $\hat{\alpha} = \gamma \oplus \omega^{\omega^{\mu \oplus \alpha}}$*.*

Furthermore, all (2-Ref$^{\because}_{\sigma,\sigma'}$)*- and* (3-Ref$^{\cdot}_{\sigma}$)*-inferences occurring in this new derivation satisfy* $\sigma, \sigma' < \hat{\alpha}$*.*

The *Proof* is by main induction on μ and side induction on α. As already mentioned, basically the only situation that we did not yet consider in the main part is that $R = \langle \kappa, \xi \rangle$ and the last inference was a (cut) of rank $\pi_C > \kappa$. We will exemplarily treat the case that π_C is regular, but not Π^1_1-indescribable. Then we had the assumption

$$\mathcal{H}_\gamma[\mathfrak{X}] \overset{\alpha_0}{\underset{\bar{\mu}}{\vdash}} \Delta^{(\kappa)}, (\neg)C^{(\pi_C)}$$

for some $\alpha_0 < \alpha$. Now again $C \in \Sigma_1(\pi_C)$ and because we also get

$$\mathfrak{A}^2(\mathfrak{X}; \gamma, \langle \pi_C, \Psi_2^{\mathcal{K}}[\pi_C], 0, 0 \rangle, \mu)$$

by the above mentioned new part (viii) of Lemma 7.3.1, we can apply the side induction hypothesis (now with $R_C = \langle \pi_C, \Psi_2^{\mathcal{K}}[\pi_C], 0, 0 \rangle$ of course) and get

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}] \overset{\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0]}{\underset{\bullet}{\vdash}} \Delta^{(\kappa)}, (\neg)C^{(\pi_C,\rho_0)}$$

where we set $\rho_0 = \Psi^{\pi_C}_{\hat{\alpha}_0,R_C}$. Notice that $\rho_0 \in \mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}]$ and $\mathrm{par}(\Delta) \subseteq \kappa + 1 < \rho_0$ by assumption. A (cut) yields

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}] \Big|\frac{\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0]+1}{\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0]}\ \Delta^{(\kappa)}.$$

If $\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0]$ is a cardinal (necessarily singular then), Lemma 7.3.1's new part (vi) implies $\mathfrak{A}^3(\mathfrak{X}; \hat{\alpha}_0, \langle\kappa,\xi\rangle, \Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0])$ and as $\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0] < \pi_C \leq \mu$, we can apply the main induction hypothesis and get

$$\mathcal{H}_{\beta}[\mathfrak{X},\pi] \Big|\frac{\Psi^{\kappa}_{\beta,R}[\pi]}{\bullet}\ \Delta^{(\kappa,\pi)}$$

for all $\pi \in \mathcal{A}^{\kappa}_{\beta} \cap a(R)$, where $\beta = \hat{\alpha}_0 \oplus \omega^{\omega^{\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0]\cdot 2+1}}$. Now we can show

$$\beta <^{C}_{\Psi^{\kappa}_{\hat{\alpha},R}} \hat{\alpha} \tag{$*$}$$

and thus

$$\mathcal{A}^{\kappa}_{\beta} \cap a(R) \subseteq \mathcal{A}^{\kappa}_{\hat{\alpha}} \cap a(R) \quad \text{and} \quad \Psi^{\kappa}_{\beta,R}[\pi] < \Psi^{\kappa}_{\hat{\alpha},R}[\pi],$$

for all $\pi \in \mathcal{A}^{\kappa}_{\hat{\alpha}} \cap a(R)$, so we are done.

Now assume that $\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0]$ is not a cardinal. Then we first have to use Predicative Cut Elimination before we can apply the main induction hypothesis. As

$$\left[\overline{p(\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0])}, \overline{p(\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0])} + \Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0]\right) \cap \mathrm{Reg} = \emptyset,$$

Theorem 5.3.2 implies

$$\mathcal{H}_{\hat{\alpha}_0}[\mathfrak{X}] \Big|\frac{\varphi(\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0])(\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0]+1)}{\overline{p(\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0])}}\ \Delta^{(\kappa)}.$$

Notice that $p(\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0])$ is a cardinal $< \pi_C \leq \mu$ and as part (vi) of Lemma 7.3.1 also implies $\mathfrak{A}^3(\mathfrak{X}; \hat{\alpha}_0, \langle\kappa,\xi\rangle, p(\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0]))$, we get by main induction hypothesis

$$\mathcal{H}_{\beta'}[\mathfrak{X},\pi] \Big|\frac{\Psi^{\kappa}_{\beta,R}[\pi]}{\bullet}\ \Delta^{(\kappa,\pi)}$$

for all $\pi \in \mathcal{A}^{\kappa}_{\beta'} \cap a(R)$, where

$$\beta' = \hat{\alpha}_0 \oplus \omega^{\omega^{p\left(\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0]\right) \oplus \varphi\left(\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0]\right)\left(\Psi^{\pi_C}_{\hat{\alpha}_0,R_C}[\rho_0]+1\right)}}.$$

Again we have

$$\beta' <^{C}_{\Psi^{\kappa}_{\hat{\alpha},R}} \hat{\alpha}, \tag{$**$}$$

which yields

$$\mathcal{A}^{\kappa}_{\beta'} \cap a(R) \subseteq \mathcal{A}^{\kappa}_{\hat{\alpha}} \cap a(R) \quad \text{and} \quad \Psi^{\kappa}_{\beta',R}[\pi] < \Psi^{\kappa}_{\hat{\alpha},R}[\pi]$$

for all $\pi \in \mathcal{A}^{\kappa}_{\hat{\alpha}} \cap a(R)$, so we are again done.

In order to accept $(*)$ and $(**)$, we note that

- $\beta, \beta' < \hat{\alpha}$ because
  - $\hat{\alpha}_0 < \hat{\alpha}$ and
  - $\Psi^{\pi_C}_{\hat{\alpha}_0, R_C}[\rho_0] < \pi_C \leq \mu$ implies

    $$p\left(\Psi^{\pi_C}_{\hat{\alpha}_0, R_C}[\rho_0]\right) < \mu \quad \text{and} \quad \varphi\left(\Psi^{\pi_C}_{\hat{\alpha}_0, R_C}[\rho_0]\right)\left(\Psi^{\pi_C}_{\hat{\alpha}_0, R_C}[\rho_0] + 1\right) < \mu,$$

    too.
- $\gamma, \mu, \alpha_0, \pi_C \in C^\kappa(\gamma + 1) \subseteq C^\kappa_R(\hat{\alpha})$ and $\hat{\alpha}_0 < \hat{\alpha}$ implies that

  $$\Psi^{\pi_C}_{\hat{\alpha}_0, R_C}[\rho_0] \in C^\kappa_R(\hat{\alpha}).$$

  But then also

  $$p\left(\Psi^{\pi_C}_{\hat{\alpha}_0, R_C}[\rho_0]\right), \varphi\left(\Psi^{\pi_C}_{\hat{\alpha}_0, R_C}[\rho_0]\right)\left(\Psi^{\pi_C}_{\hat{\alpha}_0, R_C}[\rho_0] + 1\right) \in C^\kappa_R(\hat{\alpha}),$$

  consequently $\beta, \beta' \in C^\kappa_R(\hat{\alpha})$.

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