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Alexandrov meets Lott-Villani-Sturm

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Abstract. Here I show the compatibility of two definitions of generalized curvature bounds: the lower bound for sectional curvature in the sense of Alexandrov and the lower bound for Ricci curvature in the sense of Lott-Villani-Sturm.

Introduction

Let me denote by $\mathrm{CD}[m,\kappa]$ the class of metric-measure spaces which satisfy a weak curvature-dimension condition for dimension m and curvature κ (see preliminaries). By $\mathrm{Alex}^m[\kappa]$, I will denote the class of all m-dimensional Alexandrov spaces with curvature $\geqslant \kappa$ equipped with the volume-measure (so $\mathrm{Alex}^m[\kappa]$ is a class of metric-measure spaces).

Main theorem. $\mathrm{Alex}^m[0] \subset \mathrm{CD}[m,0]$.

The question was first asked by Lott and Villani in [6, Rem. 7.48]. In [12], Villani formulates it more generally:

$$\mathrm{Alex}^m[\kappa] \subset \mathrm{CD}[m,(m-1)\cdot\kappa].$$

The latter statement can be proved along the same lines, but I do not write it down.

About the proof. The idea of the proof is the same as in the Riemannian case (see [4, Thm. 6.2] or [6, Thm. 7.3]). One only needs to extend certain calculus to Alexandrov spaces. To do this, I use the same technique as in [10]. I will illustrate the idea on a very simple problem.

Let M be a 2-dimensional nonnegatively curved Riemannian manifold and $\gamma_\tau : [0,1] \to M$ be a continuous family of unit-speed geodesics such that

$$|\gamma_{\tau_0}(t_0)\,\gamma_{\tau_1}(t_1)| \geqslant |t_1 - t_0|. \tag{1}$$

Set $\ell(t)$ to be the total length of the curve $\sigma_t : \tau \mapsto \gamma_\tau(t)$. Then $\ell(t)$ is a concave function; that is easy to prove.

Now, assume you have $A \in \mathrm{Alex}^2[0]$ instead of M and a noncontinuous family of unit-speed geodesics $\gamma_\tau(t)$ which satisfies (1). Define $\ell(t)$ as the 1-dimensional Hausdorff measure of the image of σ_t. Then ℓ is also concave.

Here is an idea how one can proceed; it is not the simplest one but the one which admits a proper generalization. Consider two functions $\psi = \mathrm{dist}_{\mathrm{Im}\,\sigma_0}$ and $\varphi = \mathrm{dist}_{\mathrm{Im}\,\sigma_1}$. Note that geodesics $\gamma_\tau(t)$ are also gradient curves of ψ and φ. This implies that $\Delta\varphi + \Delta\psi$ vanishes almost everywhere on the image of the map $(\tau, t) \to \gamma_\tau(t)$ (the Laplacians $\Delta\varphi$ and $\Delta\psi$ are Radon sign-measures). Then the result follows from my second variation formula from [9] and calculus on Alexandrov spaces developed by Perelman in [8].

Remark. Although $\mathrm{CD}[m, \kappa]$ is a very natural class of metric-measure spaces, some basic tools in Ricci comparison cannot work there in principle. For instance, *there are* $\mathrm{CD}[m, 0]$*-spaces which do not satisfy the Abresch-Gromoll inequality*, (see [1]). Thus, one has to modify the definition of the class $\mathrm{CD}[m, \kappa]$ to make it suitable for substantial applications in Riemannian geometry.

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1. Preliminaries

Prerequisite. The reader is expected to be familiar with basic definitions and notions of optimal transport theory as in Villani's book [12], measure theory on Alexandrov spaces from paper of Burago, Gromov and Perelman [3], DC-structure on Alexandrov's spaces from Perelman's [8] and technique and notations of gradient flow as my survey [11].

What needs to be proved. Let me recall the definition of the class $\mathrm{CD}[m, 0]$ only; it is sufficient for understanding this paper. The definition of $\mathrm{CD}[m, \kappa]$ can be found in [12, Def. 29.8].

Similar definitions were given by Lott and Villani in [6] and by Sturm in [13]. The idea behind these definitions, convexity of certain functionals in the Wasserstein space over a Riemannian manifold, appears in [7] by Otto an Vilani, [4] by Cordero-Erausquin, McCann and M. Schmuckenschläger, [14] by von Renesse and Sturm. In the Euclidean context, this notion of convexity goes back to [5] by McCann. More on the history of the subject can be found in the Villani's book [12].

For a metric-measure space X, I will denote by $|xy|$ the distance between points $x, y \in X$ and by $\mathrm{vol}\, E$ the distinguished measure of Borel subset $E \subset X$ (I will call it *volume*). Let us denote by $\mathrm{P}_2 X$ the set of all probability measures with compact support in X equipped with Wasserstein distance of order 2, see [12, Def. 6.1].

Further, we assume X is a proper geodesic space; in this case $\mathrm{P}_2 X$ is geodesic.

Let μ be a probability measure on X. Denote by μ^r the absolutely continuous part of μ with respect to volume. I.e. μ^r coincides with μ outside a Borel subset of volume zero and there is a Borel function $\varrho : X \to \mathbb{R}$ such that

$\mu^r = \varrho\cdot \mathrm{vol}$. Define

$$U_m\mu \stackrel{def}{=\!=} \int\limits_X \varrho^{1-\frac{1}{m}}\cdot d\,\mathrm{vol} = \int\limits_X \frac{1}{\sqrt[m]{\varrho}}\cdot d\mu^r.$$

Then $X \in \mathrm{CD}[m,0]$ if the functional U_m is concave on P_2X; i.e. for any two measures $\mu_0, \mu_1 \in \mathrm{P}_2X$, there is a geodesic path μ_t, in P_2X, $t \in [0,1]$, such that the real function $t \mapsto U_m\mu_t$ is concave.

Calculus in Alexandrov spaces. Let $A \in \mathrm{Alex}^m[\kappa]$ and $S \subset A$ be the subset of singular points; i.e. $x \in S$ if and only if its tangent space T_x is not isometric to Euclidean m-space $\mathbb{E}^m$. The set S has zero volume ([3, Thm. 10.6]). The set of regular points $A \setminus S$ is convex ([9]); i.e. any geodesic connecting two regular points consists only of regular points.

According to [8], if $f : A \to \mathbb{R}$ is a semiconcave function and $\Omega \subset A$ is an image of a DC_0-chart, then $\partial_k f$ and the components of the metric tensor g^{ij} are functions of locally bounded variation which are continuous in $\Omega \setminus S$.

Further, for almost all $x \in A$ the Hessian of f is well-defined. I.e. there is a subset of full measure $\mathrm{Reg}\, f \subset A \setminus S$ such that for any $p \in \mathrm{Reg}\, f$ there is a bilinear form[1] $\mathrm{Hess}_p\, f$ on T_p such that

$$f(q) = f(p) + d_p f(v) + \mathrm{Hess}_p\, f(v,v) + o(|v|^2),$$

where $v = \log_p q$. Moreover, the Hessian can be found using standard calculus in the DC_0-chart. In particular,

$$\mathrm{Trace}\,\mathrm{Hess}\, f \stackrel{a.e.}{=\!=} \frac{\partial_i(\det g\cdot g^{ij}\cdot \partial_j f)}{\det g}.$$

The following is an extract from the second variation formula [9, Thm. 1.1B] reformulated with formalism of ultrafilters. Let ϖ be a nonprincipal ultrafilter on the natural numbers, $A \in \mathrm{Alex}^m[0]$ and $[pq]$ be a minimizing geodesic in A which is extendable beyond p and q. Assume further that one of (and therefore each) of the points p and q is regular. Then there is a model configuration $\tilde p, \tilde q \in \mathbb{E}^m$ and isometries $\imath_p : \mathrm{T}_pA \to \mathrm{T}_{\tilde p}\mathbb{E}^m$, $\imath_q : \mathrm{T}_qA \to \mathrm{T}_{\tilde q}\mathbb{E}^m$ such that for any fixed $v \in \mathrm{T}_p$ and $w \in \mathrm{T}_q$ we have

$$\Big|\exp_p\big(\tfrac{1}{n}\cdot v\big)\ \exp_q\big(\tfrac{1}{n}\cdot w\big)\Big| \leqslant \Big|\exp_{\tilde p}\circ\imath_p\big(\tfrac{1}{n}\cdot v\big)\ \exp_{\tilde q}\circ\imath_q\big(\tfrac{1}{n}\cdot w\big)\Big| + o(n^2)$$

for ϖ-almost all n (once the left-hand side is well-defined).

If $\tilde\tau : \mathrm{T}_{\tilde p} \to \mathrm{T}_{\tilde q}$ is the parallel translation in $\mathbb{E}^m$, then the isometry $\tau : \mathrm{T}_p \to \mathrm{T}_q$ which satisfies the identity $\imath_q \circ \tau = \tilde\tau \circ \imath_p$ will be called the "parallel transportation" from p to q.

Laplacians of semiconcave functions. Here are some facts from my paper on harmonic functions [10].

[1] Note that $p \in A \setminus S$, thus T_p is isometric to Euclidean m-space.

Given a function $f : A \to \mathbb{R}$, define its *Laplacian* Δf to be a Radon sign-measure which satisfies the following identity

$$\int\limits_A u\cdot d\Delta f = -\int\limits_A \langle \nabla u, \nabla f\rangle\cdot d\,\mathrm{vol}$$

for any Lipschitz function $u : A \to \mathbb{R}$.

1.1. Claim. *Let $A \in \mathrm{Alex}^m[\kappa]$ and $f : A \to \mathbb{R}$ be λ-concave Lipschitz function. Then the Laplacian Δf is well-defined and*

$$\Delta f \leqslant m\cdot\lambda\cdot \mathrm{vol}\,.$$

In particular, $\Delta^s f$—the singular part of Δf—is negative.

Moreover,

$$\Delta f = \mathrm{Trace\,Hess}\, f\cdot \mathrm{vol} + \Delta^s f.$$

Proof. Let us denote by $F_t : A \to A$ the f-gradient flow for the time t.

Given a Lipschitz function $u : A \to \mathbb{R}$, consider the family $u_t(x) = u \circ F_t(x)$. Clearly, $u_0 \equiv u$ and u_t is Lipschitz for any $t \geqslant 0$. Further, for any $x \in A$ we have $\left|\tfrac{d^+}{dt} u_t(x)|_{t=0}\right| \leqslant \mathrm{Const}$. Moreover

$$\tfrac{d^+}{dt} u_t(x)|_{t=0} \overset{a.e.}{=\!=} d_x u(\nabla_x f) \overset{a.e.}{=\!=} \langle \nabla_x u, \nabla_x f\rangle.$$

Further,

$$\int\limits_A u_t\cdot d\,\mathrm{vol} = \int\limits_A u\cdot d(F_t\#\,\mathrm{vol}),$$

where $\#$ stands for push-forward. Since $|F_t(x)F_t(y)| \leqslant e^{\lambda t}\cdot|xy|$ (see [11, 2.1.4(i)]) for any $x, y \in A$ we have

$$F_t\#\,\mathrm{vol} \geqslant \exp(-m\cdot\lambda\cdot t)\cdot \mathrm{vol}\,.$$

Therefore, for any nonnegative Lipschitz function $u : A \to \mathbb{R}$,

$$\int\limits_A u_t\cdot d\,\mathrm{vol} = \int\limits_A u\cdot d(F_t\#\,\mathrm{vol}) \geqslant \exp(-m\cdot\lambda\cdot t)\cdot \int\limits_A u\cdot d\,\mathrm{vol}\,.$$

Therefore

$$\int\limits_A \langle \nabla u, \nabla f\rangle\cdot d\,\mathrm{vol} = \tfrac{d^+}{dt}\int\limits_A u_t\cdot d\,\mathrm{vol}\Big|_{t=0} \geqslant -m\cdot\lambda\cdot \int\limits_A u\cdot d\,\mathrm{vol}\,.$$

I.e. there is a Radon measure χ on A such that

$$\int\limits_A u\cdot d\chi = \int\limits_A [\langle \nabla u, \nabla f\rangle + m\cdot\lambda\cdot u]\cdot d\,\mathrm{vol}$$

Set $\Delta f = -\chi + m\cdot\lambda$, this is a Radon sign-measure and $\chi = -\Delta f + m\cdot\lambda \geqslant 0$.

To prove the second part of the theorem, assume u is a nonnegative Lipschitz function with support in a DC_0-chart $U \to A$, where $U \subset \mathbb{R}^m$ is an open subset. Then

$$\int_A \langle \nabla u, \nabla f \rangle = \int_U \det g \cdot g^{ij} \cdot \partial_i u \cdot \partial_j f \cdot dx^1 \cdot dx^2 \cdots dx^m$$
$$= - \int_U u \cdot \partial_i (\det g \cdot g^{ij} \cdot \partial_j f) \cdot dx^1 \cdot dx^2 \cdots dx^m.$$

Thus

$$\Delta f = \partial_i (\det g \cdot g^{ij} \cdot \partial_j f) \cdot dx^1 \cdot dx^2 \cdots dx^m \overset{a.e.}{=\!=} \operatorname{Trace} \operatorname{Hess} f.$$

□

Gradient curves. Here I extend the notion of gradient curves to families of functions, see [11] for all necessary definitions.

Let $\mathbb{I}$ be an open real interval and $\lambda : \mathbb{I} \to \mathbb{R}$ be a continuous function. A one parameter family of functions $f_t : A \to \mathbb{R}$, $t \in \mathbb{I}$, will be called $\lambda(t)$-*concave* if the function $(t, x) \mapsto f_t(x)$ is locally Lipschitz and f_t is $\lambda(t)$-concave for each $t \in \mathbb{I}$.

We will write $\alpha^{\pm}(t) = \nabla f_t$ if for any $t \in \mathbb{I}$, the right/left tangent vector $\alpha^{\pm}(t)$ is well-defined and $\alpha^{\pm}(t) = \nabla_{\alpha(t)} f_t$. The solutions of $\alpha^{+}(t) = \nabla f_t$ will be also called f_t-gradient curves.

The following is a slight generalization of [11, 2.1.2 and 2.2(2)]; it can be proved along the same lines.

1.2. Proposition-Definition. *Let $A \in \mathrm{Alex}^m[\kappa]$, let $\mathbb{I}$ be an open real interval, let $\lambda : \mathbb{I} \to \mathbb{R}$ be a continuous function and let $f_t : A \to \mathbb{R}$, $t \in \mathbb{I}$ be a $\lambda(t)$-concave family.*

Then for any $x \in A$ and $t_0 \in \mathbb{I}$ there exists an f_t-gradient curve α which is defined in a neighborhood of t_0 and such that $\alpha(t_0) = x$.

Moreover, if $\alpha, \beta : \mathbb{I} \to A$ are f_t-gradient then for any $t_0, t_1 \in \mathbb{I}$, $t_0 \leqslant t_1$,

$$|\alpha(t_1)\beta(t_1)| \leqslant L \cdot |\alpha(t_0)\beta(t_0)|,$$

where $L = \exp\left(\int_{t_0}^{t_1} \lambda(t) \cdot dt\right)$.

Note that the above proposition implies that the value $\alpha(t_0)$ of an f_t-gradient curve $\alpha(t)$ uniquely determines it for all $t \geqslant t_0$ in $\mathbb{I}$. Thus we can define the f_t-*gradient flow* as a family of maps $F_{t_0,t_1} : A \to A$ such that

$$F_{t_0,t_1}(\alpha(t_0)) = \alpha(t_1) \quad \text{if} \quad \alpha^{+}(t) = \nabla f_t.$$

1.3. Claim. *Let $f_t : A \to \mathbb{R}$ be a $\lambda(t)$-concave family and F_{t_0,t_1} be f_t-gradient flow. Let $E \subset A$ be a bounded Borel set. Fix t_1 and consider the function*

$v(t) = \operatorname{vol} F_{t,t_1}^{-1}(E)$. *Then*

$$v\big|_t^{t_1} = \int_t^{t_1} \Delta f_{\tau}\big[F_{\tau,t_1}^{-1}(E)\big] \cdot d\tau$$

Proof. Let $u : A \to \mathbb{R}$ be a Lipschitz function with compact support. Set $u_t = u \circ F_{t,t_1}$. Clearly every $(x,t) \mapsto u_t(x)$ is locally Lipschitz. Thus, the function

$$w_u : t \mapsto \int_A u_t \cdot d\operatorname{vol}$$

is locally Lipschitz. Further

$$w_u'(t) \stackrel{a.e.}{=\!=} -\int_A \langle \nabla u_t, \nabla f_t \rangle \cdot d\operatorname{vol} = \int_A u_t \cdot d\Delta f_t.$$

Therefore

$$w_u\big|_t^{t_1} = \int_t^{t_1} d\tau \cdot \int_A u_\tau \cdot d\Delta f_\tau.$$

The last formula extends to an arbitrary Borel function $u : A \to \mathbb{R}$ with bounded support. Applying it to the characteristic function of E we get the result. □

2. Games with Hamilton-Jacobi shifts.

Let $A \in \operatorname{Alex}^m[0]$. For a function $f : A \to \mathbb{R} \cup \{+\infty\}$, let us define its Hamilton-Jacobi shift[2] $\mathcal{H}_t f : A \to \mathbb{R}$ for the time $t > 0$ as follows

$$(\mathcal{H}_t f)(x) \stackrel{def}{=\!=} \inf_{y \in A} \big\{ f(y) + \tfrac{1}{2t} \cdot |xy|^2 \big\}.$$

We say that $\mathcal{H}_t f$ is well-defined if the above infimum is $> -\infty$ everywhere in A. Note that

$$\mathcal{H}_{t_0+t_1} f = \mathcal{H}_{t_1} \mathcal{H}_{t_0} f \qquad (2)$$

for any $t_0, t_1 > 0$. (The inequality $\mathcal{H}_{t_0+t_1} f \leqslant \mathcal{H}_{t_1} \mathcal{H}_{t_0} f$ is a direct consequence of the triangle inequality and it is actually equality for any intrinsic metric, in particular for Alexandrov space.)

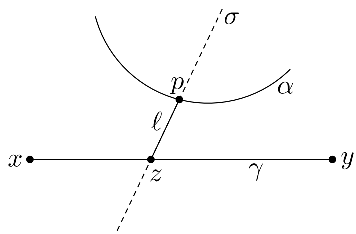
Note that for $t > 0$, $f_t = \mathcal{H}_t f$ forms a $\frac{1}{t}$-concave family, thus, we can apply Proposition-Definition 1.2 and Claim 1.3. The next theorem gives a more delicate property of the gradient flow for such families; it is an analog of [11, 3.3.6].

2.1. Claim. *Let $A \in \operatorname{Alex}^m[0]$, $f_0 : A \to \mathbb{R}$ be a function and let $f_t = \mathcal{H}_t f_0$ be well-defined for $t \in (0,1)$. Assume that $\gamma : [0,1] \to A$ is a geodesic path which is an f_t-gradient curve for $t \in (0,1)$ and that $\alpha : (0,1) \to A$ is another*

[2] There is a lot of similarity between the Hamilton-Jacobi shift of a function and an equidistant for a hypersurface.

f_t-gradient curve. Then if for some $t_0 \in (0,1)$, $\alpha(t_0)=\gamma(t_0)$, then $\alpha(t)=\gamma(t)$ for all $t\in(0,1)$.

Proof. Set $\ell=\ell(t)=|\alpha(t)\gamma(t)|$; this is a locally Lipschitz function defined on $(0,1)$. We have to show that if $\ell(t_0)=0$ for some t_0, then $\ell(t)=0$ for all t.



According to Proposition-Definition 1.2, $\ell(t)=0$ for all $t\geqslant t_0$. In order to prove that $\ell(t)=0$ for all $t\leqslant t_0$, it is sufficient to show that

$$\ell'\geqslant -[\tfrac{1}{t}+\tfrac{2}{1-t}]\cdot\ell$$

for almost all t.

Since α is locally Lipschitz, for almost all t, $\alpha^+(t)$ and $\alpha^-(t)$ are well-defined and *opposite*[3] to each other.

Fix such t and set $x=\gamma(0)$, $z=\gamma(t)$, $y=\gamma(1)$, $p=\alpha(t)$, so $\ell(t)=|pz|$. Note that the function

$$f_t+\tfrac{1}{2(1-t)}\cdot\operatorname{dist}_y^2 \tag{3}$$

has a minimum at z. Extend a geodesic $[zp]$ by a both-sides infinite unit-speed quasigeodesic[4] $\sigma:\mathbb{R}\to A$, so $\sigma(0)=z$ and $\sigma^+(0)=\Uparrow_{[zp]}$. The function $f_t\circ\sigma:\mathbb{R}\to\mathbb{R}$ is $\frac{1}{t}$-concave and from (3),

$$f_t\circ\sigma(s)\geqslant f_t(z)+\langle\gamma^+(t),\Uparrow_{[zp]}\rangle\cdot s-\tfrac{1}{2(1-t)}\cdot s^2.$$

It follows that

$$\begin{aligned}\langle\nabla_p f_t,\sigma^+(\ell)\rangle&\geqslant d_p f_t(\sigma^+(\ell))\\&=(f_t\circ\sigma)^+(\ell)\\&\geqslant\langle\gamma^+(t),\Uparrow_{[zp]}\rangle-[\tfrac{1}{t}+\tfrac{2}{1-t}]\cdot\ell.\end{aligned}$$

Now,

(a) the vectors $\sigma^\pm(\ell)$ are polar, thus $\langle\alpha^\pm(t),\sigma^+(\ell)\rangle+\langle\alpha^\pm(t),\sigma^-(\ell)\rangle\geqslant 0$,
(b) the vectors $\alpha^\pm(t)$ are opposite, thus $\langle\alpha^+(t),\sigma^\pm(\ell)\rangle+\langle\alpha^-(t),\sigma^\pm(\ell)\rangle=0$,
(c) $\alpha^+(t)=\nabla_p f_t$ and $\sigma^-(\ell)=\Uparrow_{[pz]}$.

Thus, $\langle\nabla_p f_t,\sigma^+(\ell)\rangle+\langle\alpha^+(t),\Uparrow_{[pz]}\rangle=0$. Therefore

$$\ell'=-\langle\alpha^+(t),\Uparrow_{[pz]}\rangle-\langle\gamma^+(t),\Uparrow_{[zp]}\rangle\geqslant-[\tfrac{1}{t}+\tfrac{2}{1-t}]\cdot\ell.$$

□

2.2. Proposition. *Let $A\in\mathrm{Alex}^m[0]$, let $f:A\to\mathbb{R}$ be a bounded continuous function and let $f_t=\mathcal{H}_t f$. Assume that $\gamma:(0,a)\to A$ is an f_t-gradient curve which is also a constant-speed geodesic. Assume that the function*

$$h(t)\overset{def}{=\!=}\operatorname{Trace}\operatorname{Hess}_{\gamma(t)} f_t$$

[3]I.e. $|\alpha^+(t)|=|\alpha^-(t)|$ and $\measuredangle(\alpha^+(t),\alpha^-(t))=\pi$

[4]A careful proof of existence of quasigeodesics can be found in [11].

is defined for almost all $t \in (0,a)$*. Then*

$$h' \leqslant -\tfrac{1}{m}\cdot h^2$$

in the sense of distributions; i.e. for any nonnegative Lipschitz function $u : (0,a) \to \mathbb{R}$ *with compact support,*

$$\int_0^a \left(\tfrac{1}{m}\cdot h^2\cdot u - h\cdot u'\right)\cdot dt \geqslant 0.$$

Proof. Since h is defined a.e., all $\mathrm{T}_{\gamma(t)}$ for $t \in (0,a)$ are isometric to Euclidean m-space. From (2),

$$f_{t_1}(x) = \inf_{y\in A}\left\{f_{t_0}(y) + \frac{|xy|^2}{2\cdot(t_1 - t_0)}\right\}.$$

Thus, for a parallel transportation $\tau : \mathrm{T}_{\gamma(t_0)} \to \mathrm{T}_{\gamma(t_1)}$ along γ, we have

$$\mathrm{Hess}_{\gamma(t_1)} f_{t_1}(y) \leqslant \mathrm{Hess}_{\gamma(t_0)} f_{t_0}(x) + \frac{|\tau(x)\,y|^2}{2\cdot(t_1 - t_0)}$$

for any $x \in \mathrm{T}_{\gamma(t_0)}$ and $y \in \mathrm{T}_{\gamma(t_1)}$. Taking trace leads to the result. □

3. Proof of the main theorem

Let $A \in \mathrm{Alex}^m[0]$; in particular A is a proper geodesic space. Let μ_t be a family of probability measures on A for $t \in [0,1]$ which forms a *geodesic path*[5] in $\mathrm{P}_2 A$ and both μ_0 and μ_1 are absolutely continuous with respect to volume on A.

It is sufficient[6] to show that the function

$$\Theta : t \mapsto U_m \mu_t$$

is concave.

According to [12, 7.22], there is a probability measure п on the space of all geodesic paths in A which satisfies the following: If $\Gamma = \operatorname{supp}$ п and $e_t : \Gamma \to A$ is evaluation map $e_t : \gamma \mapsto \gamma(t)$ then $\mu_t = e_t\#$п.

The measure п is called the *dynamical optimal coupling* for μ_t and the measure $\pi = (e_0, e_1)\#$п is the corresponding *optimal transference plan.* The space Γ will be considered further equipped with the metric

$$|\gamma\,\gamma'| = \max_{t\in[0,1]} |\gamma(t)\gamma'(t)|.$$

First we present μ_t *as the push-forward for gradient flows of two opposite families of functions.* According to [12, 5.10], there are optimal price functions $\varphi, \psi : A \to \mathbb{R}$ such that

$$\varphi(y) - \psi(x) \leqslant \tfrac{1}{2}\cdot|xy|^2$$

[5] i.e. a constant-speed minimizing geodesic defined on $[0,1]$.

[6] It follows from [12, 30.32] since Alexandrov's spaces are nonbranching.

for any $x, y \in A$ and equality holds for any $(x, y) \in \operatorname{supp} \pi$. We can assume that $\psi(x) = +\infty$ for $x \notin \operatorname{supp} \mu_0$ and $\varphi(y) = -\infty$ for $y \notin \operatorname{supp} \mu_1$.

Consider two families of functions

$$\psi_t = \mathcal{H}_t \psi \quad \text{and} \quad \varphi_t = \mathcal{H}_{1-t}(-\varphi).$$

Clearly, ψ_t forms a $\frac{1}{t}$-concave family for $t \in (0, 1]$ and φ_t forms[7] a $\frac{1}{1-t}$-concave family for $t \in [0, 1)$.

It is straightforward to check that for any $\gamma \in \Gamma$ and $t \in (0, 1)$

$$\pm\langle\gamma^{\pm}(t), v\rangle = d_{\gamma(t)}\psi_t(v) = -d_{\gamma(t)}\varphi_t(v);$$

in particular,

$$\gamma^+(t) = \nabla\psi_t \quad \text{and} \quad \gamma^-(t) = \nabla\varphi_t. \tag{4}$$

For $0 < t_0 \leqslant t_1 \leqslant 1$, let us consider the maps $\Psi_{t_0,t_1} : A \to A$, the gradient flow of ψ_t, defined by

$$\Psi_{t_0,t_1}\alpha(t_0) = \alpha(t_1) \quad \text{if} \quad \alpha^+(t) = \nabla\psi_t.$$

Similarly, $0 \leqslant t_0 \leqslant t_1 < 1$, define map $\Phi_{t_1,t_0} : A \to A$

$$\Phi_{t_1,t_0}\beta(t_1) = \beta(t_0) \quad \text{if} \quad \beta^-(t) = \nabla\varphi_t.$$

According to Proposition-definition 1.2,

$$\Psi_{t_0,t_1} \text{ is } \tfrac{t_1}{t_0}\text{-Lipschitz} \quad \text{and} \quad \Phi_{t_1,t_0} \text{ is } \tfrac{1-t_0}{1-t_1}\text{-Lipschitz.} \tag{5}$$

From (4), $e_{t_1} = \Psi_{t_0,t_1} \circ e_{t_0}$ and $e_{t_0} = \Phi_{t_1,t_0} \circ e_{t_1}$. Thus, for any $t \in (0, 1)$, the map $e_t : \Gamma \to A$ is bi-Lipschitz. In particular, for any measure χ on A, there is a uniquely determined one-parameter family of "pull-back" measures χ_t^* on Γ, i.e. such that $\chi_t^* E = \chi(e_t E)$ for any Borel subset $E \subset \Gamma$.

Fix some $z_0 \in (0, 1)$ (one can take $z_0 = \frac{1}{2}$) and equip Γ with the measure $\nu = \operatorname{vol}_{z_0}^*$. Thus, from now on "almost everywhere" has sense in Γ, $\Gamma \times (0, 1)$ and so on.

Now we will represent Θ in terms of families of functions on Γ. Note that $\mu_1 = \Psi_{t,1}\#\mu_t$ and $\Psi_{t,1}$ is $\frac{1}{t}$-Lipschitz. Since μ_1 is absolutely continuous, so is μ_t for all t. Set $\mu_t = \varrho_t \cdot \operatorname{vol}$. Note that from (5), we get that

$$\left(\frac{1-t_1}{1-t_0}\right)^m \leqslant \frac{\varrho_{t_1}(\gamma(t_1))}{\varrho_{t_0}(\gamma(t_0))} \leqslant \left(\frac{t_1}{t_0}\right)^m$$

for almost all $\gamma \in \Gamma$ and $0 < t_0 < t_1 < 1$. For $\gamma \in \Gamma$ set $r_t(\gamma) = \varrho_t(\gamma(t))$. Then

$$\Theta(t) = \int_A \varrho_t^{-\frac{1}{m}} \cdot d\mu_t = \int_\Gamma r_t^{-\frac{1}{m}} \cdot d\Pi. \tag{6}$$

In particular, Θ is locally Lipschitz in $(0, 1)$.

[7] Note that usually φ_t is defined with opposite sign, but I wanted to work with semiconcave functions only.

Next we show that the measure $\Delta\varphi_t$ is absolutely continuous on $e_t\Gamma$ and that $r_t(\gamma(t)) = \varrho_t(\gamma(t))\cdot\Delta\varphi_t$ in some weak sense. From (5), $\mathrm{vol}^*_t = e^{w_t}\cdot\nu$ for some Borel function $w_t : \Gamma \to \mathbb{R}$. Thus

$$\mathrm{vol}\, e_t E = \int\limits_E e^{w_t}\cdot d\nu$$

for any Borel subset $E \subset \Gamma$. Moreover, for almost all $\gamma \in \Gamma$, we have that the function $t \mapsto w_t(\gamma)$ is locally Lipschitz in $(0,1)$ (more precisely, $t \mapsto w_t(\gamma)$ coincides with a Lipschitz function outside of a set of zero measure). In particular $\frac{\partial w_t}{\partial t}$ is well-defined a.e. in $\Gamma \times (0,1)$ and moreover

$$w_t \overset{a.e.}{=\!=} \int\limits_{z_0}^{t} \frac{\partial w_\xi}{\partial \xi}\cdot d\xi.$$

Further, from Claim 2.1, if $0 < t_0 \leqslant t_1 < 1$ then for any $\gamma \in \Gamma$,

$$\Psi_{t_0,t_1}(x) = \gamma(t_1) \iff x = \gamma(t_0),$$
$$\Phi_{t_1,t_0}(x) = \gamma(t_0) \iff x = \gamma(t_1).$$

Thus, for any Borel subset $E \subset \Gamma$,

$$e_{t_1}E = \Psi_{t_0,t_1} \circ e_{t_0}E = \Phi^{-1}_{t_1,t_0}(e_{t_0}E)\,,$$
$$e_{t_0}E = \Phi_{t_1,t_0} \circ e_{t_1}E = \Psi^{-1}_{t_0,t_1}(e_{t_1}E)\,.$$

Set

$$v(t) \overset{def}{=\!=} \mathrm{vol}\, e_t E = \int\limits_E e^{w_t}\cdot d\nu.$$

From Claim 1.3,

$$v'(t) \overset{a.e.}{=\!=} \Delta\psi_t(e_tE) \overset{a.e.}{=\!=} -\Delta\varphi_t(e_tE).$$

Thus, $\Delta\psi_t + \Delta\varphi_t = 0$ everywhere on $e_t\Gamma$. From Claim 1.1,

$$\Delta\psi_t \leqslant \tfrac{m}{t}\cdot \mathrm{vol}, \qquad \Delta\varphi_t \leqslant \tfrac{m}{1-t}\cdot \mathrm{vol}\,.$$

Thus, both restrictions $\Delta\psi_t|_{e_t\Gamma}$ and $\Delta\varphi_t|_{e_t\Gamma}$ are absolutely continuous with respect to volume. Therefore

$$v'(t) \overset{a.e.}{=\!=} \int\limits_{e_tE} \mathrm{Trace}\,\mathrm{Hess}\,\varphi_t\cdot d\,\mathrm{vol}\,.$$

For the one parameter family of functions $h_t(\gamma) = \mathrm{Trace}\,\mathrm{Hess}_{\gamma(t)}\,\varphi_t$, we have

$$v\big|_{z_0}^{t} = \int\limits_E (e^{w_t} - 1)\cdot d\nu = \int\limits_{z_0}^{t} d\xi\cdot \int\limits_E h_\xi\cdot e^{w_\xi}\cdot d\nu$$

or any Borel set $E \subset \Gamma$. Equivalently,

$$\frac{\partial w_t}{\partial t} \overset{a.e.}{=\!=} h_t$$

From Proposition 2.2,

$$\frac{\partial h_t}{\partial t} \leqslant -\tfrac{1}{m}\cdot h_t^2.$$

Thus, for almost all $\gamma \in \Gamma$, the following inequality holds in the sense of distributions:

$$\frac{\partial^2}{\partial t^2}\exp\left(\frac{w_t(\gamma)}{m}\right) = \left(\tfrac{1}{m^2}\cdot h_t^2 + \tfrac{1}{m}\cdot\frac{\partial h_t}{\partial t}\right)\cdot\exp\left(\frac{w_t(\gamma)}{m}\right) \leqslant 0;$$

i.e. $t \mapsto \exp\left(\frac{w_t(\gamma)}{m}\right)$ is concave. Moreover precisely, $t \mapsto \exp\left(\frac{w_t(\gamma)}{m}\right)$ coincides with a concave function almost everywhere.

Clearly, for any t we have $\mu = r_t\cdot e^{w_t}\cdot\nu$. Thus, for almost all γ there is a nonnegative Borel function $a : \Gamma \to \mathbb{R}_{\geqslant 0}$ such that $r_t \overset{a.e.}{=\!=} a\cdot e^{-w_t}$. Continue (6),

$$\Theta(t) = \int\limits_{\Gamma} r_t^{-\frac{1}{m}}\cdot d\Pi = \int\limits_{\Gamma} e^{\frac{w_t}{m}}\cdot \sqrt[m]{a}\cdot d\Pi$$

I.e. Θ is concave as an average of concave functions. Again, more precisely, Θ coincides with a concave function a.e., but since Θ is locally Lipschitz in $(0, 1)$ we get that Θ is concave. □

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