Münster J. of Math. **1** (2008), 267–278
urn:nbn:de:hbz:6-43529451393

Münster Journal of Mathematics

A note on Banach $\mathcal{C}_0(X)$-modules

Walther Paravicini

(Communicated by Siegfried Echterhoff)

Abstract. The projective tensor product over $\mathcal{C}_0(X)$ of locally $\mathcal{C}_0(X)$-convex non-degenerate Banach $\mathcal{C}_0(X)$-modules is again locally $\mathcal{C}_0(X)$-convex.

Let X be a locally compact Hausdorff space. A *Banach* $\mathcal{C}_0(X)$*-module* is a Banach space E which is at the same time a (left) $\mathcal{C}_0(X)$-module such that $\|\chi \cdot e\| \leq \|\chi\|_\infty \|e\|$ for all $\chi \in \mathcal{C}_0(X)$ and $e \in E$; such an E is called *non-degenerate* if $\mathcal{C}_0(X)E$ is dense in E. We suggest to call a non-degenerate Banach $\mathcal{C}_0(X)$-module a $\mathcal{C}_0(X)$*-Banach space*. This naming is justified by the fact that $\mathcal{C}_0(X)$-Banach algebras are, in particular, $\mathcal{C}_0(X)$-Banach spaces; more precisely: a $\mathcal{C}_0(X)$-Banach algebra is a Banach algebra which is at the same time a $\mathcal{C}_0(X)$-Banach space such that the product of the algebra is compatible with the $\mathcal{C}_0(X)$-module structure. If E is a Banach space, then we write EX for the $\mathcal{C}_0(X)$-Banach space $\mathcal{C}_0(X, E)$.

The theorem about tensor products of locally $\mathcal{C}_0(X)$-convex spaces that we prove in this note makes it easier to compare the $\mathrm{KK}^{\mathrm{ban}}$-theories for $\mathcal{C}_0(X)$-Banach algebras and for upper semi-continuous fields of Banach algebras over X, see Section 1.3 of [12]. We give two additional applications of the theorem at the end of the first section. In this first section, we explain what locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces are. In the second section, some alternative characterizations of local $\mathcal{C}_0(X)$-convexity are given to facilitate the proof of the main theorem, which is carried out in the third section.

The results of this note are contained in the doctoral thesis [13]; I thank my supervisor Siegfried Echterhoff for his numerous suggestions for improvement. This research has been supported by the Deutsche Forschungsgemeinschaft (SFB 478).

1. Locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces

The notion of local $\mathcal{C}_0(X)$-convexity is crucial if one wants to compare the concept of a $\mathcal{C}_0(X)$-Banach space to the concepts of bundles or sheaves or (upper semi-continuous) fields of Banach spaces over X (see [7]). The same notion appears under different names in the literature (for compact X, say): Hofmann calls it "local $\mathcal{C}(X)$-convexity" in [6] and so does Gierz in his extensive discussion [4]; the same name is used in [7], but there also the abbreviated form "local convexity" is proposed. Dupré and Gillette use "$\mathcal{C}(X)$-convexity" in [2]. Kitchen and Robbins prefer "$\mathcal{C}(X)$-local convexity" in [9] which is in line with similar concepts in [11]. Finally, the same notion appears as "regularity" of $\mathcal{C}_0(X)$-Banach algebras in the preprint [10]. For now, we stick to the lengthy but accurate name "local $\mathcal{C}(X)$-convexity":

Definition 1.1. A $\mathcal{C}_0(X)$-Banach space is called *locally $\mathcal{C}_0(X)$-convex* if

$$\|\chi_1 e_1 + \chi_2 e_2\| \leq \max\{\|e_1\|, \|e_2\|\}$$

holds for all $\chi_1, \chi_2 \in \mathcal{C}_0(X)$ with $\chi_1, \chi_2 \geq 0$ and $\chi_1 + \chi_2 \leq 1$ and for all $e_1, e_2 \in E$.

Closed subspaces, *quotients* and *finite products* of locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces are again locally $\mathcal{C}_0(X)$-convex. These and other properties of the category of locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces were studied in [9]. In the present article, we show the following additional result:

Theorem 1.2. *Let E and F be locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces. Then their $\mathcal{C}_0(X)$-tensor product $E \otimes_{\mathcal{C}_0(X)} F$ is locally $\mathcal{C}_0(X)$-convex.*

By $\mathcal{C}_0(X)$-tensor product we mean the following:

Definition 1.3. Let E and F be $\mathcal{C}_0(X)$-Banach spaces. Then the *projective tensor product over $\mathcal{C}_0(X)$* or *$\mathcal{C}_0(X)$-tensor product* $E \otimes_{\mathcal{C}_0(X)} F$ of E and F is defined to be the quotient of the complete projective tensor product $E \otimes^\pi F$ by the closed subspace generated by elements of the form $\chi e \otimes f - e \otimes \chi f$, where $\chi \in \mathcal{C}_0(X)$, $e \in E$ and $f \in F$.

Note that the $\mathcal{C}_0(X)$-tensor product has the universal property for $\mathcal{C}_0(X)$-*balanced* continuous bilinear maps on $E \times F$; a bilinear map β from $E \times F$ to some Banach space G is called $\mathcal{C}_0(X)$-balanced if $\beta(\chi e, f) = \beta(e, \chi f)$ for all $\chi \in \mathcal{C}_0(X)$, $e \in E$ and $f \in F$.

In [9], the tensor product of locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces was defined to be the so-called Gelfand transform of our $\mathcal{C}_0(X)$-tensor product to make sure that it was also locally $\mathcal{C}_0(X)$-convex. Theorem 1.2 shows that this extra step is not necessary.

One reason to consider locally $\mathcal{C}_0(X)$-convex $\mathcal{C}_0(X)$-Banach spaces is that they are determined by their fibers: If E is a $\mathcal{C}_0(X)$-Banach space, then the subspace $\mathcal{C}_0(U)E$ is closed in E for every open subset $U \subseteq X$. If $x \in X$, then the *fibre* E_x of E is defined as the quotient

$$E_x := E / \left(\mathcal{C}_0(X \setminus \{x\})E\right).$$

For all $e \in E$, we will denote by e_x the corresponding element of the fibre E_x. The canonical projection map from E onto E_x will be denoted by π_x^E. For all $e \in E$, the function $x \mapsto \|e_x\|_{E_x}$ is upper semi-continuous and vanishes at infinity.

For example, if $E = E_0X = \mathcal{C}_0(X, E_0)$ for some Banach space E_0, then $E_x \cong E_0$ for all $x \in X$ and the map π_x^E can be identified with evaluation at x.

In [2], Theorem 2.5., it is shown that E is locally $\mathcal{C}_0(X)$-convex if and only if

$$\|e\| = \sup_{x\in X} \|e_x\| \quad \text{for all } e \in E,$$

i.e., the Gelfand representation in the sense of [9] is isometric.

There is an important consequence of local $\mathcal{C}_0(X)$-convexity which concerns linear operators between $\mathcal{C}_0(X)$-Banach spaces. Note that $\mathcal{C}_0(X)$-Banach spaces form a category: If E and F are $\mathcal{C}_0(X)$-Banach spaces, then we take the bounded linear $\mathcal{C}_0(X)$-linear maps from E to F as morphisms and denote the set of these by $\mathrm{L}_{\mathcal{C}_0(X)}(E, F)$.

Let E and F be $\mathcal{C}_0(X)$-Banach spaces and let $x \in X$. Let $T \in \mathrm{L}_{\mathcal{C}_0(X)}(E, F)$. Then there is a unique linear map $T_x \colon E_x \to F_x$ such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ {\scriptstyle \pi_x^E}\downarrow & & \downarrow{\scriptstyle \pi_x^F} \\ E_x & \xrightarrow{T_x} & F_x \end{array}$$

It satisfies $\|T_x\| \le \|T\|$. If T is isometric, so is T_x: This can be read off the following formula of Varela (compare [14], Lemma 1.2) which also follows from Lemme 1.10 of [1]: For every $x \in X$ and every $e \in E$, we have

(1)
$$\|e_x\| = \inf\{\|\varphi e\| : \ \varphi \in \mathcal{C}_c(X), x \in U \subseteq X \text{ open}, \varphi|_U = 1,\ 0 \le \varphi(x) \le 1\}.$$

Also other properties of T are inherited by the fibres, e.g. if T has dense image, is a quotient map or an isometric isomorphism, then the same is true for T_x. By a quotient map we mean not only a surjective map, but what is also called a metric surjection, i.e., we say that $T \in \mathrm{L}(E, F)$ is a quotient map if $\|T\| \le 1$ and if for all $f \in F$ and all $\varepsilon > 0$ there is an $e \in E$ such that

$$T(e) = f \quad \text{and} \quad \|e\| \le \|f\| + \varepsilon,$$

or, alternatively,

$$\|f - T(e)\| \le \varepsilon \quad \text{and} \quad \|e\| \le \|f\|. \tag{2}$$

Now local $\mathcal{C}_0(X)$-convexity comes into play if one wants to prove results which are converse to the above observations:

Proposition 1.4. *Let E and F be $\mathcal{C}_0(X)$-Banach spaces and $T \in \mathrm{L}_{\mathcal{C}_0(X)}(E, F)$ such that $\|T\| \le 1$.*

1) *If E is locally $\mathcal{C}_0(X)$-convex, then T is isometric if and only if the operator $T_x \colon E_x \to F_x$ is isometric for all $x \in X$.*
2) *If F is locally $\mathcal{C}_0(X)$-convex, then T has dense image if and only if the operator $T_x \colon E_x \to F_x$ has dense image for all $x \in X$.*

3) *If E and F are locally $\mathcal{C}_0(X)$-convex, then T is surjective and a quotient map if and only if the operator $T_x\colon E_x \to F_x$ is surjective and a quotient map for all $x \in X$.*
4) *If E and F are locally $\mathcal{C}_0(X)$-convex, then T is an isometric isomorphism if and only if the operator $T_x\colon E_x \to F_x$ is an isometric isomorphism for all $x \in X$.*

Proof. In all four cases, we only prove that T inherits the property in question from its fibres T_x under the respective convexity condition.
1) Let $e \in E$. We have

$$\|e\| = \sup_{x\in X} \|e_x\| = \sup_{x\in X} \|T_x(e_x)\| \leq \|T(e)\| \leq \|e\|,$$

so we have equality throughout, and hence T is isometric.

2) The image of T is fiberwise dense. Since F is locally $\mathcal{C}_0(X)$-convex, a compactness argument shows that the image of T, being a $\mathcal{C}_0(X)$-invariant subspace, is dense in F.

3) We use Equation (2): let $f \in F$ and $\varepsilon > 0$. For every $x \in X$, we pick some $e^x \in E$ such that $\|(T(e^x) - f)_x\| \leq \varepsilon/2$, $\|e^x\| \leq \|f_x\|$ (this is possible since $T_x \circ \pi_x$ is a quotient map for all $x \in X$). Find a compact subset K of X such that $\|f_x\| \leq \varepsilon$ for all $x \subseteq X \setminus K$. Since for all $x \in X$ the function $|T(e^x) - f|$ is upper semi-continuous, the sets $U_x := \{y \in X : \ \|(T(e^x) - f)_y\| < \varepsilon\}$ are open (and contain x). So the set $\{U_x : \ x \in K\}$ forms an open cover of K. Let $S \subseteq K$ be a finite set such that $\{U_s : \ s \in S\}$ is a cover of K. Find a continuous partition of unity on K subordinate to this cover, i.e., a family $(\varphi_s)_{s\in S}$ of elements of $\mathcal{C}_0(X)$ such that $0 \leq \varphi_s \leq 1$, $\operatorname{supp}\varphi_s \subseteq U_s$ and $\sum_{s\in S}\varphi_s(k) = 1$ for all $k \in K$ as well as $\sum_{s\in S}\varphi_s \leq 1$ on the whole of X. Define

$$e := \sum_{s\in S} \varphi_s e^s \in E.$$

Since E is locally $\mathcal{C}_0(X)$-convex, we conclude that $\|e\| \leq \sup_{s\in S}\|f_s\| \leq \|f\|$. Let $\psi := 1 - \sum_{s\in S}\varphi_s$. Note that $f = \sum_{s\in S}\varphi_s f + \psi f$. Let $x \in X$ and $s \in S$. If $x \in U^s$, then $\|T(e^s)_x - f_x\| \leq \varepsilon$, so $\|T(\varphi_s e^s)_x - (\varphi_s f)_x\| \leq \varphi_s(x)\varepsilon$. If $x \notin U_s$, then $\|T(\varphi_s e^s)_x - (\varphi_s f)_x\| = 0 \leq \varphi_s(x)\varepsilon$. So

$$\left\|T(e)_x - \sum_{s\in S}\varphi_s f\right\| \leq \sum_{s\in S}\varphi_s(x)\varepsilon \leq \varepsilon.$$

On the other hand, $\|(\psi f)_x\| \leq \varepsilon$, so

$$\left\|T(e)_x - f_x\right\| \leq \left\|T(e)_x - \sum_{s\in S}\varphi_s f\right\| + \|(\psi f)_x\| \leq 2\varepsilon.$$

This is true for all $x \in X$, so $\|T(e) - f\| = \sup_{x\in X}\|T(e) - f\| \leq 2\varepsilon$ because F is locally $\mathcal{C}_0(X)$-convex. So T is surjective and a quotient map.

4) This follows from 1. and 2. (or 3.). □

We now turn to the fibres of tensor products of $\mathcal{C}_0(X)$-Banach spaces.

Proposition 1.5. *If E and F are $\mathcal{C}_0(X)$-Banach spaces and $x \in X$, then there is an isometric isomorphism*

$$(E \otimes_{\mathcal{C}_0(X)} F)_x \quad \cong \quad E_x \otimes^\pi F_x.$$

Proof. We define maps in both directions which are linear and of norm less than or equal to one and which are inverse to each other; in both cases, we use suitable universal properties.

Firstly, we can regard E_x as a $\mathcal{C}_0(X)$-Banach space by defining $\chi e := \chi(x)e$ for $\chi \in \mathcal{C}_0(X)$ and $e \in E_x$; we can do the same with F_x. Then the map $(e, f) \mapsto e_x \otimes f_x$ is a continuous $\mathcal{C}_0(X)$-balanced bilinear map from $E \times F$ to $E_x \otimes^\pi F_x$ which hence gives rise to a $\mathcal{C}_0(X)$-linear map from $E \otimes_{\mathcal{C}_0(X)} F$ to $E_x \otimes^\pi F_x$ of norm less than or equal to one. Its kernel contains the kernel of $\pi_x^{E \otimes_{\mathcal{C}_0(X)} F}$, so it induces a linear map Φ from $(E \otimes_{\mathcal{C}_0(X)} F)_x$ to $E_x \otimes^\pi F_x$. If $e \in E$ and $f \in F$, then $\Phi((e \otimes f)_x) = e_x \otimes f_x$.

To define an inverse map, we give a bilinear continuous map μ of norm less than or equal to one from $E_x \times F_x$ to $(E \otimes_{\mathcal{C}_0(X)} F)_x$. Let $e' \in E_x$ and $f' \in F_x$. Let $e \in E$ and $f \in F$ such that $e_x = e'$ and $f_x = f'$. Define $\mu(e', f') := (e \otimes f)_x$. A short calculation shows that $\mu(e', f')$ does not depend on the choice of e and f and, moreover, one can choose e and f so that $\|e\|$ and $\|f\|$ are as close as desired to $\|e'\|$ and $\|f'\|$, respectively, showing that $\|\mu\| \leq 1$. It follows that there is a continuous linear map Ψ from $E_x \otimes^\pi F_x$ to $(E \otimes_{\mathcal{C}_0(X)} F)_x$ sending $e_x \otimes f_x$ to $(e \otimes f)_x$ for every $e \in E$ and $f \in F$. This precisely says that Ψ is an inverse to Φ and that Φ and Ψ are isometric. □

We now give two corollaries of Theorem 1.2:

Corollary 1.6. *Let E and F be Banach spaces. Then*

$$EX \otimes_{\mathcal{C}_0(X)} FX \cong (E \otimes^\pi F)\, X.$$

Proof. Define

$$\begin{aligned} \Phi\colon EX \otimes_{\mathcal{C}_0(X)} FX &\to (E \otimes^\pi F)\, X, \\ e \otimes f &\mapsto (x \mapsto e(x) \otimes f(x)). \end{aligned}$$

This map is $\mathcal{C}_0(X)$-linear and of norm less than or equal to one. Let $x \in X$. If we identify the fibre at x on both sides with $E \otimes^\pi F$, then Φ_x is simply the identity and hence an isometric isomorphism.

From Theorem 1.2 and Proposition 1.4, 4., we can deduce that Φ is an isometric isomorphism. □

Recall that a Banach algebra B is called self-induced (in the sense of [5]) if the canonical map from $B \otimes_B B$ to B is an isometric isomorphism.

Corollary 1.7. *Let B be a $\mathcal{C}_0(X)$-Banach algebra, i.e., a Banach algebra which is also a $\mathcal{C}_0(X)$-Banach space such that the product is $\mathcal{C}_0(X)$-bilinear. Assume that B is locally $\mathcal{C}_0(X)$-convex. Then B is self-induced if and only if B_x is self-induced for all $x \in X$.*

Proof. First observe that the tensor product $B \otimes_B B$ is $\mathcal{C}_0(X)$-balanced, so it is indeed a quotient of $B \otimes_{\mathcal{C}_0(X)} B$. In particular, it is locally $\mathcal{C}_0(X)$-convex by Theorem 1.2 as a quotient of a locally $\mathcal{C}_0(X)$-convex space. The canonical linear map T from $B \otimes_B B$ to B is $\mathcal{C}_0(X)$-linear (where we put the obvious $\mathcal{C}_0(X)$-Banach space structure on $B \otimes_B B$). Let $x \in X$. Note that $(B \otimes_B B)_x$ is canonically isomorphic to $B_x \otimes_{B_x} B_x$. Moreover, $T_x\colon (B \otimes_B B)_x \to B_x$ can be identified with the canonical map from $B_x \otimes_{B_x} B_x$ to B_x. Now apply Proposition 1.4 to see that T is an isometric isomorphism if and only if T_x is an isometric isomorphism for all $x \in X$. □

2. ALTERNATIVE CHARACTERIZATIONS OF LOCAL $\mathcal{C}_0(X)$-CONVEXITY

In this section, let E be a $\mathcal{C}_0(X)$-Banach space.

Proposition 2.1. *The following are equivalent:*

1) *E is locally $\mathcal{C}_0(X)$-convex.*
2) $\|(\varphi_1 + \varphi_2)e\| = \max\{\|\varphi_1 e\|, \|\varphi_2 e\|\}$ *holds for all* $e \in E$, $\varphi_1, \varphi_2 \in \mathcal{C}_b(X)$ *with* $\varphi_1\varphi_2 = 0$.
3) $\|(\varphi_1 + \varphi_2)e\| = \max\{\|\varphi_1 e\|, \|\varphi_2 e\|\}$ *holds for all* $e \in E$, $\varphi_1, \varphi_2 \in \mathcal{C}_0(X)$ *with* $\varphi_1\varphi_2 = 0$.
4) $\|(\varphi_1 + \varphi_2)e\| = \max\{\|\varphi_1 e\|, \|\varphi_2 e\|\}$ *holds for all* $e \in E$, $\varphi_1, \varphi_2 \in \mathcal{C}_c(X)$ *with* $\varphi_1\varphi_2 = 0$.

Proof. 1. $\Leftrightarrow$ 2.: This is part of proposition 7.14 of [4].
The implications 2. $\Rightarrow$ 3. and 3. $\Rightarrow$ 4. are trivial.
4. $\Rightarrow$ 2.: Take a bounded approximate unit $(\chi_\lambda)_{\lambda\in\Lambda}$ of $\mathcal{C}_0(X)$ which is contained in $\mathcal{C}_c(X)$. Let $e \in E$ and $\varphi_1, \varphi_2 \in \mathcal{C}_b(X)$ such that $\varphi_1\varphi_2 = 0$. Then

$$\|(\varphi_1 + \varphi_2)e\| = \lim_\lambda \|(\chi_\lambda\varphi_1 + \chi_\lambda\varphi_2)e\| = \\ = \lim_\lambda \max\{\|\chi_\lambda\varphi_1 e\|, \|\chi_\lambda\varphi_2 e\|\} = \max\{\|\varphi_1 e\|, \|\varphi_2 e\|\}$$

since $(\chi_\lambda\varphi_1)(\chi_\lambda\varphi_2) = 0$ for every $\lambda \in \Lambda$ (allowing us to apply 4.). □

For technical reasons, we want to refine this proposition a tiny bit. The condition $\varphi_1\varphi_2 = 0$ says that the sets $U_{\varphi_i} := \{x \in X : \varphi_i(x) \neq 0\}$, $i = 1, 2$, are disjoint. We can impose the slightly stronger condition that the supports, being the closures of these sets, do not intersect either. This is an easy consequence of the following trivial observation:

Lemma 2.2. *Let φ be an element of $\mathcal{C}_0(X)$ and $\varepsilon > 0$. Let $U_\varphi := \{x \in X : \varphi(x) \neq 0\}$. Then there is a function $\varphi^\varepsilon \in \mathcal{C}_c(X)$ of compact support contained in U_φ such that $\|\varphi - \varphi^\varepsilon\| \leq \varepsilon$.*

From this it follows:

Lemma 2.3. *The $\mathcal{C}_0(X)$-Banach space E is locally $\mathcal{C}_0(X)$-convex if and only if it has the following property:*

4') $\|(\varphi_1 + \varphi_2)e\| = \max\{\|\varphi_1 e\|, \|\varphi_2 e\|\}$ *holds for all* $e \in E$, $\varphi_1, \varphi_2 \in \mathcal{C}_c(X)$ *with* $\operatorname{supp}\varphi_1 \cap \operatorname{supp}\varphi_2 = \varnothing$.

Proof. It is clear that 4) ⇒ 4'). For the opposite direction, let $e \in E$ and $\varphi_1, \varphi_2 \in \mathcal{C}_0(X)$ such that $\varphi_1\varphi_2 = 0$. Let $\varepsilon > 0$. Find functions φ_1^ε and φ_2^ε in $\mathcal{C}_c(X)$ such that the support of φ_i^ε is contained in $U_{\varphi_i} := \{x \in X: \ \varphi_i(x) \neq 0\}$ and such that $\|\varphi_i - \varphi_i^\varepsilon\| \leq \varepsilon$. Note that the supports of these two functions are separated by the open sets U_{φ_i}. We can hence apply 4') to get

$$\begin{aligned}
\|(\varphi_1 + \varphi_2)e\| &= \|((\varphi_1 - \varphi_1^\varepsilon) + \varphi_1^\varepsilon + (\varphi_2 - \varphi_2^\varepsilon) + \varphi_2^\varepsilon)e\| \\
&\leq \|(\varphi_1^\varepsilon + \varphi_2^\varepsilon)e\| + \|(\varphi_1 - \varphi_1^\varepsilon)e\| + \|(\varphi_2 - \varphi_2^\varepsilon)e\| \\
&\leq \|(\varphi_1^\varepsilon + \varphi_2^\varepsilon)e\| + 2\varepsilon\,\|e\| \\
&\overset{4')}{=} \max\{\|\varphi_1^\varepsilon e\|, \|\varphi_2^\varepsilon e\|\} + 2\varepsilon\,\|e\| \\
&\leq \max\{\|\varphi_1 e\| + \varepsilon\,\|e\|, \|\varphi_2 e\| + \varepsilon\,\|e\|\} + 2\varepsilon\,\|e\| \\
&= \max\{\|\varphi_1 e\|, \|\varphi_2 e\|\} + 3\varepsilon\,\|e\|.
\end{aligned}$$

Since ε was arbitrary, we get the desired result. □

Definition 2.4. Let $e \in E$. The support $\operatorname{supp} e$ of e is defined as

$$\operatorname{supp} e := X \setminus \{x \in X: \text{ there exists } U \subseteq X \text{open with } x \in U, \text{ and } \varphi e = 0 \text{ for all } \varphi \in \mathcal{C}_0(U)\}.$$

Define

$$E_c := \{e \in E: \ \operatorname{supp} e \text{ is compact}\}.$$

Lemma 2.5. *Let $e \in E$. If $\varphi \in \mathcal{C}_c(X)$ such that $\operatorname{supp}\varphi \cap \operatorname{supp} e = \varnothing$, then $\varphi e = 0$.*

Proof. Let K be the support of φ. For all $k \in K \subseteq X \setminus \operatorname{supp} e$, there is an open neighborhood U_k of k such that $\psi e = 0$ for all $\psi \in \mathcal{C}_0(U_k)$. Now $\{U_k: \ k \in K\}$ is an open covering of K, so we can find a finite set $S \subseteq K$ such that $\{U_s: \ s \in S\}$ covers K. Find a continuous partition of unity $(\chi_s)_{s\in S}$ on K subordinate to $(U_s)_{s\in S}$ (with $\chi_s \in \mathcal{C}_c(X)$). Then $\chi_s\varphi$ is in $\mathcal{C}_0(U_s)$ so $\chi_s\varphi e = 0$. But $\sum_{s\in S}\chi_s\varphi = \varphi$, so $\varphi e = 0$. □

Lemma 2.6. *If $e \in E$ and $\varphi \in \mathcal{C}_0(X)$, then $\operatorname{supp}(\varphi e) \subseteq \operatorname{supp}\varphi \cap \operatorname{supp} e$.*

Proof. Let $x \in X$ such that $x \notin (\operatorname{supp}\varphi \cap \operatorname{supp} e)$. If $x \notin \operatorname{supp}\varphi$, then $U := X \setminus \operatorname{supp}\varphi$ is a neighborhood of x. Let $\psi \in \mathcal{C}_0(U)$. Then $\psi(\varphi e) = (\psi\varphi)e = 0e = 0$, so $x \notin \operatorname{supp}(\varphi e)$. If $x \notin \operatorname{supp} e$, then $U := X \setminus \operatorname{supp} e$ is a neighborhood of x. Let $\psi \in \mathcal{C}_0(U)$. Then $\psi(\varphi e) = \varphi(\psi e) = \varphi 0 = 0$, so $x \notin \operatorname{supp}(\varphi e)$. □

Lemma 2.7. *Let $e \in E$. Then $e \in E_c$ if and only if there is a $\varphi \in \mathcal{C}_c(X)$ such that $\varphi e = e$. If $e \in E_c$, then φ can be chosen to be supported in any given compact neighborhood of* $\operatorname{supp} e$ *and such that* $0 \leq \varphi \leq 1$.

Proof. If $\varphi e = e$ for some $\varphi \in \mathcal{C}_c(X)$, then this means $\operatorname{supp} e \subseteq \operatorname{supp}\varphi$, so the support of e is compact.

If $K := \operatorname{supp} e$ is compact and L is a compact neighborhood of K, then we can find a function $\varphi \in \mathcal{C}_c(X)$ such that $\varphi|_L = 1$ and $0 \leq \varphi \leq 1$. Let M be a compact set containing the support of φ and χ_M be a function in $\mathcal{C}_c(X)$ such that $\chi_M|_M = 1$ and $0 \leq \chi_M \leq 1$. Then $\chi_M\varphi = \varphi$ and hence $\chi_M\varphi e = \varphi e$. On the other hand we have $\operatorname{supp}(\chi_M - \varphi) \subseteq X \setminus K$ and hence $(\chi_M - \varphi)e = 0$, i.e., $\chi_M e = \varphi e$. If M gets larger and larger, then $\chi_M e$ approaches e, so $e = \varphi e$. □

Lemma 2.8. *The $\mathcal{C}_0(X)$-Banach space E is locally $\mathcal{C}_0(X)$-convex if and only if it has the following property:*

5) $\|e_1 + e_2\| = \max\{\|e_1\|, \|e_2\|\}$ *holds for all* $e_1, e_2 \in E_c$ *with the property that* $\operatorname{supp} e_1 \cap \operatorname{supp} e_2 = \varnothing$.

Proof. Assume that 5) is satisfied. We show 4'). Let $e \in E$ and $\varphi_1, \varphi_2 \in \mathcal{C}_c(X)$ such that $\operatorname{supp}\varphi_1 \cap \operatorname{supp}\varphi_2 = \varnothing$. Let $e_i := \varphi_i e$ for $i = 1, 2$. Then $\operatorname{supp} e_i \subseteq \operatorname{supp}\varphi_i$ so $\operatorname{supp} e_1 \cap \operatorname{supp} e_2 = \varnothing$. An application of 5) now gives 4').

Assume now that 4') holds. Let $e_1, e_2 \in E_c$ such that $\operatorname{supp} e_1 \cap \operatorname{supp} e_2 = \varnothing$. Find $\varphi_1, \varphi_2 \in \mathcal{C}_c(X)$ such that $\varphi_i e_i = e_i$ for $i = 1, 2$ and $\operatorname{supp}\varphi_1 \cap \operatorname{supp}\varphi_2 = \varnothing$. Define $e := e_1 + e_2$. Note that $\varphi_2 e_1 = 0 = \varphi_1 e_2$, so $\varphi_i e = e_i$. An application of 4') now gives 5). □

3. The proof of the theorem

Lemma 3.1. *Let E and F be $\mathcal{C}_0(X)$-Banach spaces and $e \in E_c$, $f \in F_c$ such that* $\operatorname{supp} e \cap \operatorname{supp} f = \varnothing$. *Then $e \otimes f = 0 \in E \otimes_{\mathcal{C}_0(X)} F$.*

Proof. Let K be a compact neighborhood of $\operatorname{supp} e$ and let L be a compact neighborhood of $\operatorname{supp} f$ such that $K \cap L = \varnothing$. Find functions φ and ψ in $\mathcal{C}_c(X)$ such that $\operatorname{supp}\varphi \subseteq K$ and $\varphi e = e$ and $\operatorname{supp}\psi \subseteq L$ and $\psi f = f$. Now $e \otimes f = (\varphi e) \otimes (\psi f) = e \otimes (\varphi\psi f) = e \otimes 0 = 0$. □

Proof of Theorem 1.2. We use Lemma 2.8. Let t_1 and t_2 be tensors in $\left(E \otimes_{\mathcal{C}_0(X)} F\right)_c$ such that $\operatorname{supp} t_1 \cap \operatorname{supp} t_2 = \varnothing$. Without loss of generality we assume that both, t_1 and t_2, are non-zero. Let L_1, L_2 be compact neighborhoods of $\operatorname{supp} t_1$ and $\operatorname{supp} t_2$, respectively, such that $L_1 \cap L_2 = \varnothing$. Find functions φ_1 and φ_2 such that $\operatorname{supp}\varphi_i \subseteq L_i$, $0 \leq \varphi_i \leq 1$ and $\varphi_i t_i = t_i$, for $i = 1, 2$. Note that

$$\|t_i\| = \|\varphi_i(t_1 + t_2)\| \leq \|\varphi_i\| \, \|t_1 + t_2\| = \|t_1 + t_2\|$$

for $i = 1, 2$, which shows $\|t_1 + t_2\| \geq \max\{\|t_1\|, \|t_2\|\}$. The other inequality is the non-trivial one. Let $\varepsilon > 0$. Find sequences $(e_n^1)_{n\in\mathbb{N}}$ and $(e_n^2)_{n\in\mathbb{N}}$ in E and $(f_n^1)_{n\in\mathbb{N}}$ and $(f_n^2)_{n\in\mathbb{N}}$ in F such that

$$t_i = \sum_{n\in\mathbb{N}} e_n^i \otimes f_n^i \quad \text{and} \quad \sum_{n\in\mathbb{N}} \|e_n^i\| \, \|f_n^i\| \leq \|t_i\| + \varepsilon \tag{3}$$

for $i = 1, 2$. Without loss of generality we can assume that for all $i \in \{1, 2\}$ and $n \in \mathbb{N}$,

$$\operatorname{supp} e_n^i, \ \operatorname{supp} f_n^i \subseteq L_i, \tag{4}$$

$$\|f_n^i\| = 1, \tag{5}$$

$$(6) \qquad \|e_n^1\| \geq \|e_n^2\| \quad \text{for all } n \in \mathbb{N} \qquad \text{or} \qquad \|e_n^1\| \leq \|e_n^2\| \quad \text{for all } n \in \mathbb{N}.$$

Before justifying these assumptions, we show how to use them to finish the proof. Assume that the first part of (6) holds. From (4), it follows that

$$e_n^1 \otimes f_n^2 = 0 = e_n^2 \otimes f_n^1,$$

for all $n \in \mathbb{N}$ and hence

$$\sum_{n\in\mathbb{N}} (e_n^1 + e_n^2) \otimes (f_n^1 + f_n^2) = \sum_{n\in\mathbb{N}} e_n^1 \otimes f_n^1 + \sum_{n\in\mathbb{N}} e_n^2 \otimes f_n^2 = t_1 + t_2.$$

Moreover, we have

$$\|e_n^1 + e_n^2\| \overset{(4)}{=} \max\{\|e_n^1\|, \|e_n^2\|\} \overset{(6)}{=} \|e_n^1\|$$

and

$$\|f_n^1 + f_n^2\| \overset{(4)}{=} \max\{\|f_n^1\|, \|f_n^2\|\} \overset{(5)}{=} 1 = \|f_n^1\|$$

for all $n \in \mathbb{N}$. It follows that

$$\|t_1 + t_2\| \leq \sum_{n\in\mathbb{N}} \|e_n^1 + e_n^2\| \, \|f_n^1 + f_n^2\| =$$

$$= \sum_{n\in\mathbb{N}} \|e_n^1\| \, \|f_n^1\| \leq \|t_1\| + \varepsilon \leq \max\{\|t_1\|, \|t_2\|\} + \varepsilon.$$

If the second part of (6) holds, then we arrive at the same inequality. Since we have shown this for all $\varepsilon > 0$, it follows that

$$\|t_1 + t_2\| \leq \max\{\|t_1\|, \|t_2\|\}.$$

Now we justify the assumptions (4)-(6), step by step.

1) For (4), consider the sequences $(\varphi_i e_n^i)_{n\in\mathbb{N}}$ and $(\varphi_i f_n^i)_{n\in\mathbb{N}}$ for $i = 1, 2$. They satisfy the conditions $\operatorname{supp} \varphi_i e_n^i \subseteq L_i$ and $\operatorname{supp} \varphi_i f_n^i \subseteq L_i$ for all $n \in \mathbb{N}$, $i = 1, 2$. Moreover,

$$\sum_{n\in\mathbb{N}} \varphi_i e_n^i \otimes \varphi_i f_n^i = \varphi_i^2 \sum_{n\in\mathbb{N}} e_n^i \otimes f_n^i = \varphi_i^2 t_i = t_i$$

for $i = 1, 2$ because $\varphi_i t_i = t_i$. Additionally,

$$\sum_{n\in\mathbb{N}} \|\varphi_i e_n^i\| \, \|\varphi_i f_n^i\| \leq \sum_{n\in\mathbb{N}} \|e_n^i\| \, \|f_n^i\| \leq \|t_i\| + \varepsilon,$$

so substituting e_n^i with $\varphi_i e_n^i$ and f_n^i with $\varphi_i f_n^i$ gives sequences which satisfy (3) as well as (4).

2) We show that we can assume (5). Let $i \in \{1, 2\}$. We can assume that $f_n^i \neq 0$ for all $n \in \mathbb{N}$: Because $t_i \neq 0$ by assumption, there has to exist an $f_0^i \in F$ such that $\operatorname{supp} f_0^i \subseteq L_i$ and $f_0^i \neq 0$. If $n \in \mathbb{N}$ such that $f_n^i = 0$, then substitute e_n^i by zero and f_n^i by f_0^i.

Now consider the sequences $(\|f_n^i\| \, e_n^i)_{n\in\mathbb{N}}$ and $(\frac{1}{\|f_n^i\|} f_n^i)_{n\in\mathbb{N}}$. If we take these sequences instead of $(e_n^i)_{n\in\mathbb{N}}$ and $(f_n^i)_{n\in\mathbb{N}}$, then (3), (4) and (5) are satisfied.

3) For (6), we have to work a little harder. First note that, because of (5), we have $\sum_{n\in\mathbb{N}} \|e^i_n\| = \sum_{n\in\mathbb{N}} \|e^i_n\|\|f^i_n\| \leq \|t_i\|+\varepsilon < \infty$ for $i = 1, 2$; so $(\|e^i_n\|)_{n\in\mathbb{N}}$ is in $l^1(\mathbb{N})$. We may assume $\sum_{n\in\mathbb{N}} \|e^1_n\| \geq \sum_{n\in\mathbb{N}} \|e^2_n\|$, without loss of generality. We show that in this case we can assume for all $n \in \mathbb{N}$ that $\|e^1_n\| \geq \|e^2_n\|$.

Note that we have the freedom to rearrange the sequences $(e^i_n, f^i_n)_{n\in\mathbb{N}}$ in any order we like and that we can, informally speaking, replace some entry (e^i_n, f^i_n) by the two entries $(\lambda e^i_n, f^i_n)$ and $((1-\lambda)e^i_n, f^i_n)$ for any $\lambda \in [0, 1]$. Both moves will not affect the properties (3), (4) or (5). Our strategy is to take one entry of $(e^2_n)_{n\in\mathbb{N}}$ after the other and split it up into smaller entries which we can match with entries of $(e^1_n)_{n\in\mathbb{N}}$ of the same size. Since $\sum_{n\in\mathbb{N}} \left\|e^1_n\right\| \geq \sum_{n\in\mathbb{N}} \left\|e^2_n\right\|$, it will be possible to match all entries of the sequence $(e^2_n)_{n\in\mathbb{N}}$ with entries of the other sequence. There might still be some bits of $(e^1_n)_{n\in\mathbb{N}}$ which are left over, but these entries will be matched with zero entries.

For technical reasons, we would like to assume that $(e^2_n)_{n\in\mathbb{N}}$ has infinitely many non-zero entries: Because $t_2 \neq 0$ we know that at least one entry is non-zero. Substitute this entry by infinitely many "copies with weight 2^{-n}", where n runs through the natural numbers.

To gain space, we want the sequences to be indexed over a larger set; for notational convenience, we take $\mathbb{Z}$. So define $e^1_k := e^2_k := 0 \in E$ for all $k \in \{0, -1, -2, \ldots\}$ and choose arbitrary f^1_k and f^2_k in F with norm 1 such that $\operatorname{supp} f^i_k \subseteq L_i$. Then the double-sequences $(e^i_k)_{k\in\mathbb{Z}}$ and $(f^i_k)_{k\in\mathbb{Z}}$ satisfy the relations (3), (4) and (5) (with $\mathbb{Z}$ replacing $\mathbb{N}$).

Description of the inductive procedure: We are going to give an inductive definition of a sequence $\left({}_ne^1, {}_nf^1, {}_ne^2, {}_nf^2\right)_{n\in\mathbb{N}_0}$ of four-tuples of double-sequences, starting with the four double-sequences $\left(e^1, f^1, e^2, f^2\right) =: \left({}_0e^1, {}_0f^1, {}_0e^2, {}_0f^2\right)$ we have just defined. In each step, an entry of the sequence corresponding to $(e^2_k)_{k\in\mathbb{N}}$ is set to zero and "moved to the negative part of the double-sequence". Also some (parts of) entries of the sequence corresponding to $(e^1_k)_{k\in\mathbb{N}}$ are moved to the negative part, to ensure that the negative part of the sequences is always "balanced" in the sense that

$$\left\|{}_ne^1_k\right\| = \left\|{}_ne^2_k\right\| \text{ for all } n \in \mathbb{N}_0 \text{ and all } k \in \mathbb{Z}_{\leq 0}. \tag{7}$$

Also, the procedure is designed in a way ensuring that the relations (3), (4) and (5) remain true. In the limit, all positive entries of the sequences corresponding to $(e^2_k)_{k\in\mathbb{N}}$ vanish and we are left with sequences which are "balanced" on the negative side. There might still be some non-vanishing entries of the sequence corresponding to $(e^1_k)_{k\in\mathbb{N}}$, but the sequence corresponding to $(e^2_k)_{k\in\mathbb{N}}$ vanishes, so Condition (6) holds. Also the other relations hold for the limit.

The inductive definition: Let $n \in \mathbb{N}$ and assume that we have already defined the quadruple $\left({}_{n-1}e^1, {}_{n-1}f^1, {}_{n-1}e^2, {}_{n-1}f^2\right)$, satisfying the relations (3), (4) and (5) as well as $\left\|{}_{n-1}e^1_k\right\| = \left\|{}_{n-1}e^2_k\right\|$ for all $k \in \mathbb{Z}_{\leq 0}$ and $\sum_{k\in\mathbb{N}} \left\|{}_{n-1}e^1_k\right\| \geq \sum_{k\in\mathbb{N}} \left\|{}_{n-1}e^2_k\right\|$, and such that the set $\{k \in \mathbb{Z}_{\leq 0} :\ {}_{n-1}e^2_k \neq 0\}$ is finite whereas

$\{k \in \mathbb{N} : {}_{n-1}e^2_k \neq 0\}$ is infinite. Note that

$$\left\|{}_{n-1}e^2_n\right\| < \sum_{m\in\mathbb{N}} \left\|{}_{n-1}e^2_m\right\| \leq \sum_{m\in\mathbb{N}} \left\|{}_{n-1}e^1_m\right\|$$

and so we can find a $p \in \mathbb{N}$ such that $r := \sum_{m=1}^{p-1} \left\|{}_{n-1}e^1_m\right\| < \left\|{}_{n-1}e^2_n\right\|$ and $\sum_{m=1}^{p} \left\|{}_{n-1}e^1_m\right\| \geq \left\|{}_{n-1}e^2_n\right\|$. Find $N \in \mathbb{Z}_{\leq 0}$ such that ${}_{n-1}e^2_k = 0$ for all $k < N$. Define

$${}_ne^1_k := \begin{cases} {}_{n-1}e^1_l & \text{if } k = N - l \text{ for some } l \in \{1,\dots,p-1\} \\ \frac{\left\|{}_{n-1}e^2_n\right\| - r}{\left\|{}_{n-1}e^1_p\right\|}\,{}_{n-1}e^1_p & \text{if } k = N - p \\ 0 & \text{if } k \in \{1,\dots,p-1\} \\ \frac{\left\|{}_{n-1}e^1_n\right\| - (\left\|{}_{n-1}e^2_n\right\| - r)}{\left\|{}_{n-1}e^1_p\right\|}\,{}_{n-1}e^1_p = {}_{n-1}e^1_p - {}_ne^1_{N-p} & \text{if } k = p \\ {}_{n-1}e^1_k & \text{else,} \end{cases}$$

$${}_nf^1_k := \begin{cases} {}_{n-1}f^1_l & \text{if } k = N - l \text{ for some } l \in \{1,\dots,p\} \\ {}_{n-1}f^1_k & \text{else,} \end{cases}$$

$${}_ne^2_k := \begin{cases} \frac{\left\|{}_{n-1}e^1_l\right\|}{\left\|{}_{n-1}e^2_n\right\|}\,{}_{n-1}e^2_n & \text{if } k = N - l \text{ for some } l \in \{1,\dots,p-1\} \\ \frac{\left\|{}_{n-1}e^2_n\right\| - r}{\left\|{}_{n-1}e^2_n\right\|}\,{}_{n-1}e^2_n & \text{if } k = N - p \\ 0 & \text{if } k = n \\ {}_{n-1}e^2_k & \text{else,} \end{cases}$$

$${}_nf^2_k := \begin{cases} {}_{n-1}f^2_l & \text{if } k = N - l \text{ for some } l \in \{1,\dots,p\} \\ {}_{n-1}f^2_k & \text{else.} \end{cases}$$

The resulting quadruple $\left({}_ne^1, {}_nf^1, {}_ne^2, {}_nf^2\right)$ has all the properties of the original quadruple $\left({}_{n-1}e^1, {}_{n-1}f^1, {}_{n-1}e^2, {}_{n-1}f^2\right)$ listed above, plus it satisfies ${}_ne^2_n = 0$. Note that $\left\|{}_ne^1 - {}_{n-1}e^1\right\|_1 = 2\left\|e^2_n\right\| = \left\|{}_ne^2 - {}_{n-1}e^2\right\|_1$. Hence $({}_ne^1)_{n\in\mathbb{N}}$ and $({}_ne^2)_{n\in\mathbb{N}}$ converge in $l^1(\mathbb{Z})$. The sequences $({}_nf^1)_{n\in\mathbb{N}}$ and $({}_nf^2)_{n\in\mathbb{N}}$ converge pointwise and are uniformly bounded by 1. Let $\left({}_\infty e^1, {}_\infty f^1, {}_\infty e^2, {}_\infty f^2\right)$ denote the limit-quadruple. The recursively defined sequences $\left(({}_ne^1_k \otimes {}_nf^1_k)_{k\in\mathbb{Z}}\right)_{n\in\mathbb{N}}$ and $\left(({}_ne^2_k \otimes {}_nf^2_k)_{k\in\mathbb{Z}}\right)_{n\in\mathbb{N}}$ converge in l^1 (the sums being t_1 and t_2, respectively). Hence the limit-quadruple satisfies (3). The relations (4), and (5) are stable under pointwise convergence of the involved sequences, hence they remain true in the limit as they are true in each step of the induction. The negative part of the sequences are "balanced" in every step of the induction, and ${}_\infty e^2_k = 0$ for all $k \in \mathbb{N}$. Hence (6) is true in the limit. □

References

[1] É. Blanchard, Déformations de C^*-algèbres de Hopf, Bull. Soc. Math. France **124** (1996), no. 1, 141–215. MR1395009 (97f:46092)

[2] M. J. Dupré and R. M. Gillette, *Banach bundles, Banach modules and automorphisms of C^*-algebras*, Pitman, Boston, MA, 1983. MR0721812 (85j:46127)

[3] J. Dixmier, *Les C^*-algèbres et leurs représentations*, Gauthier-Villars & Cie, Éditeur, 1964. MR0171173 (30 #1404)

[4] G. Gierz, *Bundles of topological vector spaces and their duality*, Springer, Berlin, 1982. MR0674650 (84c:46076)

[5] N. Grønbæk, Morita equivalence for self-induced Banach algebras, Houston J. Math. **22** (1996), no. 1, 109–140. MR1434388 (98c:46090)

[6] K. H. Hofmann, Bundles and sheaves are equivalent in the category of Banach spaces, in *K-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, Ga., 1975)*, 53–69. Lecture Notes in Math., 575, Springer, Berlin. MR0487491 (58 #7117)

[7] K. H. Hofmann and K. Keimel, Sheaf-theoretical concepts in analysis: bundles and sheaves of Banach spaces, Banach $C(X)$-modules, in *Applications of sheaves (Proc. Res. Sympos. Appl. Sheaf Theory to Logic, Algebra and Anal., Univ. Durham, Durham, 1977)*, 415–441, Lecture Notes in Math., 753, Springer, Berlin. MR0555553 (81f:46085)

[8] J. W. Kitchen and D. A. Robbins, Errata: "Linear algebra in the category of $C(M)$-locally convex modules", Rocky Mountain J. Math. **19** (1989), no. 4, 1111. MR1039548 (92b:46120b)

[9] J. W. Kitchen and D. A. Robbins, Linear algebra in the category of $C(M)$-locally convex modules, Rocky Mountain J. Math. **19** (1989), no. 2, 433–480. MR1018831 (92b:46120a)

[10] Gennadi G. Kasparov and Guoliang Yu. Groups with coarse positive duality and the Novikov conjecture. Preprint.

[11] L. Nachbin, Algebras of finite differential order and the operational calculus, Ann. of Math. (2) **70** (1959), 413–437. MR0108730 (21 #7444)

[12] Walther Paravicini. A generalised Green-Julg theorem for proper groupoids and Banach algebras. *Preprintreihe SFB 478 - Geometrische Strukturen in der Mathematik*, 480, 2007.

[13] Walther Paravicini. KK-*Theory for Banach Algebras And Proper Groupoids*. PhD thesis, Universität Münster, 2007. Persistent identifier: urn:nbn:de:hbz:6-39599660289.

[14] J. Varela, Sectional representation of Banach modules, Math. Z. **139** (1974), 55–61. MR0352987 (50 #5473)

Received September 19, 2007; accepted May 28, 2008

Walther Paravicini
Westfälische Wilhelms-Universität Münster, Mathematisches Institut
Einsteinstr. 62, D-48149 Münster, Germany
E-mail: w.paravicini@uni-muenster.de