Mathematik

K-theory for Ternary Structures

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Abstract

We define K-theory for different ternary structures, especially for ternary rings of operators and JB^*-triple systems. The latter ones are exactly those Banach spaces whose open unit balls are bounded symmetric domains. Instead of a binary product they rather allow products of three elements. Since the category of JB^*-triple systems has serious limitations, we define as a first step K-theory for ternary rings of operators (or TROs), generalizing the K-theory of C^*-algebras. As an application we give a K-theoretic classification of the inductive limits of finite-dimensional TROs. Next we introduce a functor embedding every JB^*-triple system into a TRO with certain universal properties. We determine these TROs for the building blocks of the finite-dimensional JB^*-triple systems, the Cartan factors. According to this we can define the K-theory of a JB^*-triple system as the K-theory of its corresponding TRO. We give a K-theoretic classification of the finite-dimensional JB^*-triple systems which can be represented faithfully as bounded operators on a Hilbert space using a K-theoretic version of root systems.

Zusammenfassung

Ziel dieser Arbeit ist es K-Theorie für JB^*-Tripelsysteme zu definieren. Dies sind Banachräume, versehen mit einem dreifachen Produkt, die als Kategorie äquivalent zu den beschränkten symmetrischen Gebieten in Banachräumen sind, die einen Basispunkt haben. Da JB^*-Tripelsysteme im Allgemeinen keine eindeutige Operatorraumstruktur besitzen, definieren wir zunächst eine K-Theorie für sogenannte ternäre Ringe von Operatoren (kurz TROs), die diese Einschränkung nicht haben. Als Anwendung klassifizieren wir die induktiven Limiten endlichdimensionaler TROs. Als nächstes betten wir jedes JB^*-Tripelsystem in seinen universellen einhüllenden TRO ein, dessen Existenz wir beweisen. Diese Zuordnung ist funktoriell und erlaubt uns, die K-Theorie eines JB^*-Tripelsystems als K-Theorie seines universellen einhüllenden TROs zu definieren. Nachdem wir die universellen einhüllenden TROs der endlichdimensionalen Cartanfaktoren bestimmt haben, gelingt es uns mit einer K-theoretischen Version der Wurzelsysteme alle endlichdimensionalen, treu darstellbaren JB^*-Tripelsysteme zu klassifizieren.

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Contents

Chapter 1

Introduction

The study of JB^*-triple systems originated in the theory of bounded symmetric domains in Banach spaces. As Kaup showed in [Kau83] every such domain is biholomorphically equivalent to the open unit ball of a JB^*-triple system and according to this, the category of all bounded symmetric domains with base point is equivalent to the category of JB^*-triple systems. Despite their algebraic difficulties, JB^*-triple systems have a lot of appealing properties: The range of a contractive projection on a C^*-algebra is always a JB^*-triple system, although not a C^*-algebra in general. Moreover, the category of JB^*-triple systems is closed under contractive projections (cf. [FR83],[Kau84] and [Sta82]). Another interesting fact is that the isomorphisms of JB^*-triple systems are exactly the surjective isometries. Every C^*-algebra becomes a JB^*-triple system under the ternary product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$.

Since JB^*-triples have on the one hand a strong resemblance to C^*-algebras and on the other hand generalize them in the Jordan theoretic world and include important subcategories such as JB^*-algebras and Hilbert spaces, the idea arose to define K-theory for JB^*-triple systems which is modeled on K-theory for C^*-algebras. Since 'K-theory has revolutionized the study of operator algebras' ([Bla98], Introduction) we are convinced that such a theory is a supplement for the theory of JB^*-triple systems, too. Another motivation is to keep an eye on the connection of K-theory with the classification of finite-dimensional bounded symmetric domains via root systems as developed by É. Cartan ([Car35]).

A first obstruction in the definition of K-theory for JB^*-triple systems was the absence of matrix levels and tensor products, both crucial in K-theory for C^*-algebras. We therefore turned our attention to the study of ternary rings of operators (or TROs). A TRO T is a closed subspace of the space of bounded operators on a Hilbert space such that $xy^*z \in T$ for all $x, y, z \in T$. TROs arise naturally in operator space theory. In contrast to JB^*-triple systems, TROs allow matrix levels and tensor products with C^*-algebras. We define K-theory for TROs with the aid of the linking C^*-algebra which is a generalization of the classical K-theory for C^*-algebras. As a first application we give a classification of inductive limits of finite-dimensional TROs generalizing Elliott's classification ([Ell76]) of AF-algebras. We are also able to generalize Bratteli diagrams to the ternary setting and to study stably isomorphic TROs with K-theoretic means.

The next question is how to apply our K-theory for TROs to JB^*-triple systems, since there is no obvious way to carry over the definitions we made for TROs to the JB^*-triple setting. A way to overcome this obstacle is the introduction of the universal enveloping TRO of a JB^*-triple system. If Z is a JB^*-triple system, then this is a pair $(T^*(Z), \rho_Z)$, where $T^*(Z)$ is a TRO and $\rho_Z : Z \to T^*(Z)$ is a JB^*-triple homomorphism such that $\rho_Z(Z)$ generates $T^*(Z)$ as a TRO and such that for every JB^*-triple homomorphism $\varphi : Z \to T$ to a TRO T, there exists a TRO-homomorphism $T^*(\varphi) : T^*(Z) \to T$ with $T^*(\varphi) \circ \rho_Z = \varphi$. The assignment $Z \to T^*(Z)$ is functorial. It is known that every finite-dimensional JB^*-triple system is the direct sum of so-called Cartan factors. Because we are interested in the classification by finite-dimensional root systems we determine the universal enveloping TROs of the Cartan factors. To compute these TROs we need grids, which are the Jordan theoretic version of root systems.

Now we are able to define the K-theory of a JB^*-triple system to be the K-theory of its universal enveloping TRO. With the help of this K-theory we define a complete isomorphism invariant for the finite-dimensional JB^*-triple systems which can be represented faithfully as operators on a Hilbert space. As invariant we use the order structure and scales defined to classify the inductive limits of finite-dimensional TROs and, as additional data, the elements in the K_0-group of a JB^*-triple system, which stem from the corresponding grid (i.e. root system).

Now we explain the results of the different chapters in more detail:

In **Chapter 2** we illustrate the general concepts of the theory of ternary rings of operators and JB^*-triple systems.

A JB^*-triple system is a Banach space Z combined with a continuous ternary product $(x, y, z) \mapsto \{x, y, z\}$ which is linear in the outer variables and conjugate linear in the inner one. This product is symmetric $\{x, y, z\} = \{z, y, x\}$ and the Jordan triple identity

$$\{x, y, \{a, b, z\}\} - \{a, b, \{x, y, z\}\} = \{\{x, y, a\}, b, z\} - \{a\{y, x, b\}, z\}$$

holds. In addition the following conditions have to be fulfilled for all $x \in Z$: (i) $\|D(x, x)\| = \|x\|^2$; (ii) $D(x, x)$ is hermitian; (iii) $D(x, x)$ has non-negative spectrum as bounded operator on Z.

Every C^*-algebra is a JB^*-triple system under the product

$$\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$$

and more generally every closed subspace of $B(H)$ (the bounded operators on a Hilbert space H), which is closed under the above product, is a JB^*-triple system. Triple systems of this kind are called special. Hilbert spaces, Hilbert C^*-modules and JB^*-algebras are examples of JB^*-triple systems, too.

Another important class of examples is given by the TROs. Every C^*-algebra is a TRO in a natural way and every TRO can be given the structure of a JC^*-triple system. But there are examples of JC^*-triple systems which are not TROs (e.g. the symmetric $n \times n$-matrices with entries in $\mathbb{C}$). It can be shown that a JC^*-triple system is a TRO if and only if its second matrix level is a JC^*-triple system.

Since we want to define K-theory for JB^*-triple systems a first approach is to develop such a theory for ternary rings of operators. Every TRO can be considered, using the product $\langle x, y\rangle = xy^*$, as a Hilbert C^*-module. Thus every TRO can be embedded completely isometrically (what is equivalent to TRO-isomorphy) into its linking C^*-algebra. This C^*-algebra is defined as follows. We consider the two C^*-algebras

$$\mathcal{L}(T) = \overline{\text{lin}}\left\{\sum x_i y_i^* | x_i, y_i \in T\right\} \quad \text{and} \quad \mathcal{R}(T) = \overline{\text{lin}}\left\{\sum x_i^* y_i | x_i, y_i \in T\right\},$$

where we take the closure of the linear span of all finite sums, and define the linking algebra of T to be

$$\mathbb{L}(Z) = \begin{pmatrix} \mathcal{R}(T) & T \\ T^* & \mathcal{L}(T) \end{pmatrix}.$$

This is a C^*-algebra which is Morita equivalent to $\mathcal{L}(T)$ and $\mathcal{R}(T)$. Moreover, $\mathcal{R}$ and $\mathbb{L}$ are covariant functors from the category of TROs to the category of C^*-algebras.

The last section of the Chapter 2 is devoted to the connection between grids and root systems as developed by Neher. Let Z be a (for convenience finite-dimensional) JB^*-triple system. A grid is a certain subset of tripotents (these are elements $e \in Z$ such that $\{e, e, e\} = e$) which carries all relevant information about the JB^*-triple system. It is well known that every finite-dimensional JB^*-triple system is the direct sum of so-called Cartan factors. These are the four classical Cartan factors (with $n, m \in \mathbb{N}$)

$C^1_{n,m}$: the complex $n \times m$-matrices $\mathbb{M}_{n,m}$,
C^2_n : the skew-symmetric, complex $n \times n$-matrices,
C^3_n : the symmetric complex $n \times n$-matrices,
C^4_n : the $n + 1$-dimensional spin factor

and two exceptional Cartan factors in dimensions 16 and 27.

We give the abstract definition of grids and show which grids belong to the classical Cartan factors. Subsequent to this we show how to identify a grid with the defining part of a 3-graded root system.

In **Chapter 3** we define K-theory for TROs and use it to classify AF-TROs, the inductive limits of finite-dimensional TROs. In order to do this we first have to generalize the representation theory of C^*-algebras to TROs. We introduce concepts like unitary equivalence, non-degeneracy and irreducibility of TRO-representation and determine the connection to representations of the linking algebra. By means of this we are able to analyze finite-dimensional TROs and their homomorphisms. Every finite-dimensional TRO is the direct sum of rectangular matrix algebras and, as we show, every homomorphism between finite-dimensional TROs is uniquely, up to unitary equivalence, determined by a rectangular matrix with entries in $\mathbb{N}_0$.

In the next section we take a closer look at the functors $\mathcal{L}$, $\mathcal{R}$ and $\mathbb{L}$ from the category of TROs to the category of C^*-algebras as defined in the prerequisites. We show that all three functors are exact, homotopy invariant, stable and continuous. We can now define the K_0-functor for TROs as the concatenation of the functor $\mathcal{L}$ and the K_0-functor for C^*-algebras. The higher K-groups can be defined as K_0-groups of TRO-suspensions. This definition is equivalent to concatenate $\mathcal{L}$ with the corresponding higher K-functors of C^*-algebras. We obtain functors K_i, $i \in \mathbb{N}_0$ from the category of TROs to the category of Abelian groups which coincide with the usual K-theory functors on the subcategory of C^*-algebras and which are half-exact, split-exact, homotopy invariant, stable and continuous. That we favor the functor $\mathcal{L}$ over the functors $\mathcal{R}$ and $\mathbb{L}$ does not affect the general theory, since, for a TRO T, the canonical embeddings of $\mathcal{L}(T)$ and $\mathcal{R}(T)$ into $\mathbb{L}(T)$ induce isomorphisms of the corresponding K-groups (at least in the separable case). We call the so defined isomorphism from $K_0(\mathcal{R}(T))$ to $K_0(T)$ the Morita isomorphism.

As a first application of our K-theory for TROs we give a complete classification of approximately finite-dimensional TROs in the spirit of Elliott. We equip the K_0-group of an AF-TRO T with an order structure and two positive subsets called the left and right scale. The left scale is the set of elements in $K_0(T)$ which come from the projections in the left C^*-algebra and the right scale are those positive elements which correspond under the Morita isomorphism (which is an order isomorphism) to projections in the right C^*-algebra. They are an instrument to keep track of the dimensions of the left and right C^*-algebra simultaneously. At this point we want to notice that our new K-theory for TROs is more than just the fused K-theory of the left and right C^*-algebras, since there are easy examples of TROs which are not linearly isomorphic but have $*$-isomorphic left and right C^*-algebras. Besides the classification of the AF-TROs, we also give a ternary version of

Bratteli diagrams and an ‘almost’ classification of stably isomorphic TROs with K-theoretic means.

In **Chapter 4** we introduce objects which enable us to apply the K-theory for TROs to JB^*-triple systems. As already mentioned the category of JB^*-triple systems does not allow matrix levels and tensor products. We therefore have to make a construction which assigns an operator space structure to JB^*-triple systems. In many fields of mathematics it is common (as in group theory or JC-algebra theory) to assign a C^*-algebra to the object one wants to study, which fulfills certain universal properties. Since we are not in the binary but in the ternary world we do not assign a C^*-algebra but rather a TRO to every JB^*-triple system, in order to avoid loosing too much information. If Z is a JB^*-triple system the universal enveloping TRO of Z is the pair $(T^*(Z), \rho_Z)$, where $T^*(Z)$ is a TRO and $\rho_Z : Z \to T^*(Z)$ is a JB^*-triple homomorphism such that $\rho_Z(Z)$ generates $T^*(Z)$ as a TRO and such that for every JB^*-triple homomorphism $\varphi : Z \to T$ to a TRO T, there exists a TRO-homomorphism $T^*(\varphi) : T^*(Z) \to T$ with $T^*(\varphi) \circ \rho_Z = \varphi$. The universal enveloping TRO is unique up to TRO-isomorphism and if Z is special ρ_Z becomes injective. As a first application we give a new proof of one of the main theorems of JB^*-triple theory: Every JB^*-triple system has a unique purely exceptional ideal such that the quotient by this ideal is a JC^*-triple system.

The assignment $Z \mapsto T^*(Z)$ defines a functor τ from the category of JB^*-triple systems to the category of ternary rings of operators, if we define for a JB^*-triple homomorphism $\varphi : Z \to W$ the corresponding TRO-homomorphism as $\tau(\varphi) := T^*(\varphi \circ \rho_W) : T^*(Z) \to T^*(W)$. This functor is destined to apply our K-theory for TROs to JB^*-triple systems, but first we have to check its functorial properties. It turns out that the functor τ is exact, continuous and homotopy invariant on the subcategory of JC^*-triple systems. The functor τ does not commute in a reasonable way with the matrix levels of JC^*-triples (i.e. $M_n(\tau(Z)) \ncong \tau(M_n(Z))$, where the right side is not even defined in most cases), as was to be expected.

Working on a replacement for this we were led to the notion of universally reversible JC^*-triple systems, defined in [BFT10]. This allows us to show stability under some assumptions. We are able to generalize an important result of [BFT10].

In the next section we determine the universal enveloping TROs of all finite-dimensional Cartan factors. Because the universal enveloping TROs of the exceptional Cartan factors are necessarily 0, we only have to determine them for the factors of type I–IV. To give an upper bound on the dimensions of the universal enveloping TROs we have to make intensive use of Neher´s grid theory. The universal enveloping TROs are

$$\begin{aligned}
T^*(C^1_{1,n}) &= \bigoplus_{k=1}^{n} \mathbb{M}_{p_k,q_k} \text{ with } p_k = \binom{n}{k} \text{ and } q_k = \binom{n}{k-1},\\
T^*(C^1_{n,m}) &= \mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n} \text{ for } n, m \geq 2,\\
T^*(C^2_n) &= \mathbb{M}_n,\\
T^*(C^3_n) &= \mathbb{M}_n,\\
T^*(C^4_n) &= \mathbb{M}_{2^{k-1}} \oplus \mathbb{M}_{2^{k-1}} \text{ if } n = 2k-1 \text{ and}\\
T^*(C^4_n) &= \mathbb{M}_{2^k} \text{ if } n = 2k.
\end{aligned}$$

In the next section we begin with the study of inductive systems of finite-dimensional JC^*-triple systems. We do the first step in that direction by analyzing the JB^*-triple homomorphisms between Cartan factors with the help of the universal enveloping TRO and the representation theory of finite-dimensional TROs as developed in Chapter 3.

After the preparations made in the last two chapters we are now in the position to define the K-groups of a JB^*-triple system. We deal with this matter in the final **Chapter 5**. If Z is a JB^*-triple system we define the ith K-group of Z to be the ith K-group of the universal enveloping TRO of Z. By our previous results this yields a covariant, half-exact, split-exact, continuous and homotopy invariant functor from the category of JC^*-triple systems to the category of Abelian groups.

On the exceptional Cartan factors the K-functors attain the value 0, because τ does.

Similar to the TRO case, we can introduce an order structure and the left and right scales in the K_0-group of a JB^*-triple system. But this data, contrary to finite-dimensional TROs, is not enough to distinguish between non-isomorphic finite-dimensional JC^*-triple systems. This obstacle can be overcome if we consider the subset of the left scale which stems from the projections that correspond to the grid generating the JB^*-triple system (if possible). We determine this invariant for all finite-dimensional classical Cartan factors (see Section 5.3 for the complete list) and we prove that this defines a complete isomorphism invariant for finite-dimensional JC^*-triple systems.

Chapter 2

Prerequisites

In this chapter we recall some known facts which we need in the rest of the text and that we consider not to be generally known. We assume that the reader is familiar with the concepts of functional analysis and the theory of C^*-algebras.

We first give a brief introduction into the theory of operator spaces and then turn our attention to ternary rings of operators and their equivalent description as Hilbert C^*-modules. We give the definition of the linking algebra and list some of its properties.

Next we introduce JB^*-triple systems, explain some of their properties and give a list of examples including Hilbert spaces, C^*-algebras, TROs and JB^*-algebras. Afterwards we recall the Gelfand-Naimark theorem for JB^*-triple systems which states that every such triple can be embedded isometrically into the direct sum of Cartan factors (these are certain simple JB^*-triple systems, which we describe in detail).

The rest of the chapter is devoted to the study of grids. These are certain subsets of a JB^*-triple system Z, encoding all the relevant information about Z. These grids are in strong correspondence with root systems. We illuminate this connection.

2.1 Notation

At first we fix some notations which will be valid throughout the text unless stated otherwisely. Let X and Y be Banach spaces. We denote by $B(X,Y)$ the Banach space of bounded linear operators from X to Y. We write $B(X) := B(X,X)$, for short. The direct sum $X \oplus Y$ always denotes the l^∞-direct sum of X and Y. By H and K we always denote Hilbert spaces, unless explicitly stated.

The natural numbers are $\mathbb{N} = \{1,2,\ldots\}$ and by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we refer to the set of non-negative integers. The complex $n \times m$-matrices together with the operator norm are called $\mathbb{M}_{n,m}$ and especially $\mathbb{M}_n := \mathbb{M}_{n,n}$ for all $n, m \in \mathbb{N}$.

The symbol $\mathbb{K}$ stands for the compact operators on a separable Hilbert space. If K is a locally compact Hausdorff space the complex valued continuous functions vanishing at infinity are called $C_0(K)$. If $\mathfrak{A}$ is a C^*-algebra we mean by $\mathfrak{A}^+$ the minimal unitization of $\mathfrak{A}$ and $\mathrm{Mult}(\mathfrak{A})$ denotes the multiplier algebra of $\mathfrak{A}$. If $A \subseteq X$ is a subset of a Banach space we denote the closure of the linear span of the elements in A with $\overline{\mathrm{lin}}\, A$.

2.2 Operator spaces and TROs

We take [BLM04] and [Pis03] as references for operator spaces. The book [BLM04] is also a good reference for Hilbert-C^*-modules and TROs. A lot of important results about TROs and their linking algebras are proved in [Ham99].

Definition 2.2.1. *A* ***(concrete) operator space*** *is a closed subspace of $B(H)$.*

Since $B(H,K) \subseteq B(H \oplus^2 K)$, (where $\oplus^2$ denotes the l^2-direct sum of H and K) $B(H,K)$ is always an operator space.

Definition 2.2.2. *Let $E \subseteq B(H_1, K_1)$ and $F \subseteq B(H_2, K_2)$ be operator spaces and*

$$\varphi : E \to F$$

a linear map. For all $n, m \in \mathbb{N}$ we define

$$M_{n,m}(E) := \{(x_{i,j})_{i,j} : x_{i,j} \in E \text{ for all } 1 \le i \le n, 1 \le j \le m\}$$

to be the space of $n \times m$-matrices with entries in E. As usual we set $M_n(E) := M_{n,n}(E)$. By using the identification

$$M_{n,m}(B(H,K)) \simeq B(\underbrace{H \oplus^2 \ldots \oplus^2 H}_{m \text{ times}}, \underbrace{K \oplus^2 \ldots \oplus^2 K}_{n \text{ times}}),$$

via the natural algebraic isomorphism, we can assign a norm $\|\cdot\|_{n,m}$ to $M_{n,m}(B(H,K))$ (resp. $\|\cdot\|_n$ to $M_n(B(H,K))$) and thus to $M_{n,m}(E)$. The sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$ is called ***canonical operator space structure*** *on E. For every $n \in \mathbb{N}$ we can define the* **n*th amplification*** *$\varphi_n : M_n(E) \to M_n(F)$ of φ by*

$$\varphi_n\left((x_{i,j})_{i,j}\right) := (\varphi(x_{i,j}))_{i,j}.$$

In a similar way one can define $\varphi_{n,m} : M_{n,m}(E) \to M_{n,m}(F)$. The map φ is called ***completely bounded (c.b.*** *for short) if*

$$\|\varphi\|_{cb} := \sup_{n\in\mathbb{N}} \|\varphi_n\|_n < \infty.$$

We denote by $CB(E,F)$ the Banach space of all c.b. maps from E to F equipped with the c.b. norm. If all amplifications φ_n are isometries we call φ a

complete isometry. *According to this φ is called a* ***complete contraction*** *if φ_n is a contraction, for all $n \in \mathbb{N}$.*

The theory of operator spaces combines operator algebra with the theory of Banach spaces. Sometimes it is more convenient to think of an operator space as a Banach space with a series of norms attached to its matrix levels.

Definition 2.2.3. *An* ***(abstract) operator space*** *is a pair $\big(X, (\|\cdot\|_n)_{n\in\mathbb{N}}\big)$ consisting of a vector space X and a norm $\|\cdot\|_n$ on $M_n(X)$, for all $n \in \mathbb{N}$, such that there exist Hilbert spaces H and K and a complete isometry $\varphi : X \to B(H,K)$. In this case we call the sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$ an* ***operator space structure*** *on X.*

Next we study an important subclass of operator spaces: the TROs. These spaces appear naturally in the operator space theory as corners $p\mathfrak{A}(1-p)$ of C^*-algebras, where p is a projection. The injectives in the category of operator spaces are TROs as shown by Ruan in [Rua89].

Definition 2.2.4. *A* ***ternary ring of operators****, or* ***TRO****, is a closed subspace $T \subseteq B(H,K)$, such that $xy^*z \in T$ for all $x, y, z \in T$.*

As a closed subspace of $B(H,K)$ a TRO T always carries an operator space structure. It is easy to see that every matrix level $M_n(T) \subseteq M_n(B(H,K))$ is a TRO by itself, if considered with the TRO-product induced by the matrix products $(A,B,C) \mapsto AB^*C$ for $A, B, C \in M_n(T)$. The **TRO-homomorphisms** between TROs T_1 and T_2 are the linear mappings $\varphi : T_1 \to T_2$ with

$$\varphi(xy^*z) = \varphi(x)\varphi(y)^*\varphi(z)$$

for all $x, y, z \in T_1$. The TRO-morphisms respect the canonical operator space structure.

Theorem 2.2.5 ([Ham99], Proposition 2.1)**.** *A TRO-homomorphism $\varphi : T_1 \to T_2$ between TROs T_1 and T_2 is always completely contractive. The mapping φ is bijective if and only if φ is a surjective complete isometry.*

The next definition is due to Zettl (cf. [Zet83]) and plays a central role in this work.

Definition 2.2.6. *Let T be a TRO. Then we call*

$$\mathcal{L}(T) := TT^* := \overline{\operatorname{lin}}\{xy^* : x, y \in T\}$$

the ***left*** $\mathbf{C}^*$***-algebra*** *of T and similar*

$$\mathcal{R}(T) := T^*T := \overline{\operatorname{lin}}\{x^*y : x, y \in T\}$$

the ***right*** $\mathbf{C}^*$***-algebra*** *of T.*

If $\varphi : T_1 \to T_2$ is a homomorphism of TROs T_1 and T_2, then we can define a mapping $\mathcal{L}(\varphi) : \mathcal{L}(T_1) \to \mathcal{L}(T_2)$ on the generators of $\mathcal{L}(T_1)$ by $\mathcal{L}(\varphi)(xy^*) = \varphi(x)\varphi(y)^*$ for all $x, y \in T_1$. The analog mapping $\mathcal{R}(\varphi) : \mathcal{R}(T_1) \to \mathcal{R}(T_2)$ is given by $\mathcal{R}(\varphi)(x^*y) = \varphi(x)^*\varphi(y)$ for all $x, y \in T_1$.

The spaces $\mathcal{L}(T)$ and $\mathcal{R}(T)$ are C^*-algebras and the mappings $\mathcal{L}(\varphi)$ and $\mathcal{R}(\varphi)$ are $*$-homomorphisms with $\mathcal{L}(\varphi \circ \psi) = \mathcal{L}(\varphi) \circ \mathcal{L}(\psi)$ and $\mathcal{R}(\varphi \circ \psi) = \mathcal{R}(\varphi) \circ \mathcal{R}(\psi)$, for compatible TRO-homomorphisms φ and ψ. This makes $\mathcal{L}$ and $\mathcal{R}$ covariant functors from the category of TROs with TRO-homomorphisms to the category of C^*-algebras with $*$-homomorphisms. The TRO-homomorphism φ is injective (resp. surjective) if and only if $\mathcal{L}(\varphi)$ and $\mathcal{R}(\varphi)$ are injective (resp. surjective) (cf. [BLM04], proof of 8.3.5).

Definition 2.2.7. *Let T be a TRO. A closed subspace R of T is called a **subTRO** of T, when it is closed under the TRO-product*

$$(x, y, z) \mapsto xy^*z$$

*for all $x, y, z \in R$. A closed subspace $I \subseteq T$ is called **TRO-ideal** of T if*

$$TT^*I + TI^*T + IT^*T \subseteq I.$$

TRO-ideals and subTROs are obviously TROs themselves.

TROs stand in close connection to an important class of modules of C^*-algebras:

Definition 2.2.8. *Let $\mathfrak{A}$ be a C^*-algebra. A **(right) Hilbert $\mathbf{C}^*$-module** is a right $\mathfrak{A}$-module $\mathcal{H}$ together with a map $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathfrak{A}$, which is linear in the second variable, such that:*

(a) $\langle y, y\rangle \geq 0$ *for all* $y \in \mathcal{H}$,

(b) $\langle y, y\rangle = 0$ *if and only if* $y = 0$,

(c) $\langle y, z.a\rangle = \langle y, z\rangle.a$ *for all* $y, z \in \mathcal{H}, a \in \mathfrak{A}$,

(d) $\langle y, z\rangle^* = \langle z, y\rangle$ *for all* $y, z \in \mathcal{H}$,

(e) $\mathcal{H}$ *is complete in the Norm* $\|y\| = \|\langle y, y\rangle\|^{\frac{1}{2}}$.

*The product $\langle\cdot,\cdot\rangle$ is called **$\mathfrak{A}$-valued inner product** on $\mathcal{H}$. Left Hilbert C^*-modules are defined analogously. A Hilbert C^*-module is called **full** when $\langle\mathcal{H}, \mathcal{H}\rangle$ is dense in $\mathfrak{A}$.*

If $\mathcal{H}$ is a full right Hilbert C^*-module over $\mathfrak{A}$ and a full left Hilbert C^*-module over $\mathfrak{B}$ and the two inner products are compatible in the sense that

$$x.\langle y, z\rangle_1 = \langle x, y\rangle_2.z,$$

*for all $x, y, z \in \mathcal{H}$, we call $\mathcal{H}$ an $\mathfrak{A}$-$\mathfrak{B}$ **equivalence bimodule**. In that case we call $\mathfrak{A}$ and $\mathfrak{B}$ **strongly Morita equivalent**.*

Every TRO T becomes a full right Hilbert C^*-module over $\mathcal{R}(T)$ when we equip it with the inner product

$$\langle x, y\rangle_1 := x^* y \tag{2.1}$$

for all $x, y \in \mathcal{H}$. It becomes a full left Hilbert C^*-module over $\mathcal{L}(T)$ together with the product

$$\langle x, y\rangle_2 := xy^* \tag{2.2}$$

for all $x, y \in \mathcal{H}$. Since these products are obviously compatible T is an equivalence bimodule and $\mathcal{L}(T)$ and $\mathcal{R}(T)$ are strongly Morita equivalent.

In particular every C^*-algebra $\mathfrak{A}$ is an $\mathfrak{A}$-$\mathfrak{A}$-equivalence bimodule.

Definition 2.2.9. *Let Y, Z be Hilbert C^*-modules over the C^*-algebra $\mathfrak{A}$, then we write $\mathbb{B}_{\mathfrak{A}}(Y, Z)$ for the set of **adjointable maps** from Y to Z, where a map from Y to Z is called adjointable if there exists a map $S : Z \to Y$ with*

$$\langle T(y), z\rangle = \langle y, S(z)\rangle$$

for all $y \in Y, z \in Z$. Such a S is unique and is denoted by T^*.

We collect some properties of adjointable maps $T, T_1, T_2 \in \mathbb{B}_{\mathfrak{A}}(Y, Z)$ from [BLM04], p. 299 ff.:

(a) T is a bounded right $\mathfrak{A}$-module map.

(b) T^* is a bounded right $\mathfrak{A}$-module map.

(c) $\mathbb{B}_{\mathfrak{A}}(Y) := \mathbb{B}_{\mathfrak{A}}(Y, Y)$ is a C^*-algebra with respect to the operator norm.

(d) The closure of the linear span of the operators $|z\rangle\langle y| : Y \to Y$, $|z\rangle\langle y|(x) = z\langle y, x\rangle$ for all $y, z \in Y$ is a sub C^*-algebra of $\mathbb{B}_{\mathfrak{A}}(Y)$ denoted by $\mathbb{K}_{\mathfrak{A}}(Y)$.

(e) The right Hilbert C^*-module Y over $\mathfrak{A}$ is also a left Hilbert C^*-module over $\mathbb{K}_{\mathfrak{A}}(Y)$, using $|\cdot\rangle\langle\cdot|$ as inner product.

Definition 2.2.10. *Let Y be a right Hilbert C^*-module over $\mathfrak{A}$. We define $\overline{Y}$ to be the canonical left Hilbert C^*-module over $\mathfrak{A}$, which is the conjugate vector space of Y with inner product $\langle \overline{y}, \overline{z}\rangle := \langle y, z\rangle$ and left action $a\overline{y} := \overline{ya^*}$ for $y, z \in Y, a \in \mathfrak{A}$. The set of 2×2 matrices*

$$\mathbb{L}(Y) = \begin{pmatrix} \mathbb{K}_{\mathfrak{A}}(Y) & Y \\ \overline{Y} & \mathfrak{A} \end{pmatrix}$$

*is called the **linking algebra** of Y.*

The linking algebra becomes an algebra using the usual matrix product and using the inner products and module actions. The algebra $\mathbb{L}(Y)$ carries as a set of 2×2 matrices an involution in an obvious way. We can define an action of $\mathbb{L}(Y)$ on the direct sum module $Y \oplus \mathfrak{A}$ by viewing an element of $Y \oplus \mathfrak{A}$ as a column vector and then formally multiply a matrix in $\mathbb{L}(Y)$ with this vector. One can show that this action defines a $*$-homomorphism from $\mathbb{L}(Y)$ to $\mathbb{B}_{\mathfrak{A}}(Y \oplus \mathfrak{A})$ which is injective. By pulling back the norm from $\mathbb{B}_{\mathfrak{A}}(Y \oplus \mathfrak{A})$ with the aid of this morphism $\mathbb{L}(Y)$ becomes a C^*-algebra. Moreover one can show that the image of $\mathbb{L}(Y)$ in $\mathbb{B}_{\mathfrak{A}}(Y \oplus \mathfrak{A})$ is exactly $\mathbb{K}_{\mathfrak{A}}(Y \oplus \mathfrak{A})$. Therefore one can define a unital C^*-algebra $\mathbb{L}^1(Y)$ within $M(\mathbb{L}(Y))$ (the multiplier operator algebra of $\mathbb{L}(Y)$ (cf. [BLM04]), 2.6.7) as the linear span of $\mathbb{K}_{\mathfrak{A}}(Y \oplus \mathfrak{A})$ and the two diagonal matrices $p := 1 \oplus 0$ and $q := 0 \oplus 1$. Taking the unitization of a unital C^*-algebra to be the algebra itself, we can view the two 1s as the identities of the unitizations of $\mathfrak{A}$ and $\mathbb{K}_{\mathfrak{A}}(Y)$. The unital C^*-algebra $\mathbb{L}^1(Y)$ has the identity $1 = p + q$ and Y becomes the 1-2-corner of both $\mathbb{L}(Y)$ and $\mathbb{L}^1(Z)$, especially

$$Y \simeq p\mathbb{L}(Y)(1-p).$$

Thus every full Hilbert C^*-module can be identified with a TRO.

If on the contrary T is a ternary ring of operators it is a full right Hilbert C^*-module over $\mathcal{R}(T)$ and a full left Hilbert C^*-module over $\mathcal{L}(T)$, as mentioned above. Using [BLM04], 8.1.15 one can show that $\mathcal{R}(T) \simeq \mathbb{K}_{\mathcal{L}(T)}(T)$ and

$$\mathbb{L}(T) \simeq \begin{pmatrix} \mathcal{R}(T) & T \\ T^* & \mathcal{L}(T) \end{pmatrix} \tag{2.3}$$

as C^*-algebras. Since [BLM04], 8.1.18 states that the linking algebra of T is strongly Morita equivalent to $\mathcal{R}(T)$ via the equivalence bimodule $T \oplus \mathcal{R}(T)$, we get the following corollary.

Corollary 2.2.11. *Let T be a ternary ring of operators. The C^*-algebras $\mathcal{L}(T)$, $\mathcal{R}(T)$ and $\mathbb{L}(T)$ are pairwise strongly Morita equivalent.*

The linking algebra of a TRO was studied intensively in [Ham99]. We collect some of the results obtained there, which we will be extensively used in the following.

Theorem 2.2.12 ([Ham99])**.** *Let $T \subseteq B(H)$ and $U \subseteq B(K)$ be ternary rings of operators.*

(a) *If T contains the identity, then T is a C^*-algebra.*

(b) *Every TRO-homomorphism is completely contractive and its kernel is a TRO-ideal.*

(c) *If $\varphi : T \to U$ is a TRO-homomorphism, then $\varphi(T)$ is a subTRO of U.*

(d) *Let $p = 1 \oplus 0 \in \mathbb{L}^1(T)$. The mapping $\iota : T \to \mathbb{L}(T)$, $\iota(x) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ defines a TRO-isomorphism onto the subTRO $p\mathbb{L}(T)(1-p) \subseteq \mathbb{L}(T)$.*

(e) *If I is a TRO-ideal in T, then the C^*-algebra $\mathbb{L}(I)$ can be identified with a closed two-sided ideal in $\mathbb{L}(T)$.*

(f) *A TRO-homomorphism $\varphi : T \to U$ induces a $*$-homomorphism $\mathbb{L}(\varphi) : \mathbb{L}(T) \to \mathbb{L}(U)$: Let $\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \in \begin{pmatrix} \mathcal{R}(T) & T \\ T^* & \mathcal{L}(T) \end{pmatrix}$ and define*

$$\mathbb{L}(\varphi)\left(\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}\right) := \begin{pmatrix} \mathcal{R}(\varphi)(x_{1,1}) & \varphi(x_{1,2}) \\ \varphi^*(x_{2,1}) & \mathcal{L}(\varphi)(x_{2,2}) \end{pmatrix} \in \begin{pmatrix} \mathcal{R}(U) & U \\ U^* & \mathcal{L}(U) \end{pmatrix}$$

where $\varphi^* : T^* \to U^*$ is defined by $\varphi^*(x) = \varphi(x^*)^*$ for all $x \in T^*$.

2.3 JB*-triple systems and related structures

A good overview of the basics of the theory of JB^*-triple systems can be found in [Isi89] and [Upm85]. The book [McC04] is a good reference for the general theory of Jordan algebras and for JB^*-algebras we recommend [HOS84]. The theory of grids was developed in the monograph [Neh87] (see also [FR86], [NR03]).

Definition 2.3.1. *A Banach space Z together with a sesquilinear mapping*

$$Z \times Z \ni (x, y) \mapsto x \,\square\, y \in B(Z)$$

is called a ***JB**-triple system***, if for the triple product*

$$\{x, y, z\} := (x \,\square\, y)(z)$$

and all $a, b, x, y, z \in Z$ the following conditions are fulfilled:

(a) *The triple product $\{x, y, z\}$ is continuous in (x, y, z).*

(b) *It is symmetric in the outer variables: $\{x, y, z\} = \{z, y, x\}$.*

(c) *The C^*-condition is fulfilled: $\|\{x, x, x\}\| = \|x\|^3$.*

(d) *The* ***Jordan triple identity*** *holds:*

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.$$

(e) *The operator $x \,\square\, x$ has non-negative spectrum in the Banach algebra $B(Z)$.*

(f) *The operator $x \,\square\, x$ is hermitian (i.e. $\exp(it(x \,\square\, x))$ is isometric for all $t \in \mathbb{R}$).*

A closed subspace W of a JB^*-triple system Z which is invariant under the triple product, and therefore is a JB^*-triple system itself, is called a **$\mathbf{JB}^*$*-subtriple*** *(or subtriple for short) of Z.*

Let Z and W be JB^*-triple systems. If $\varphi : Z \to W$ is a linear mapping which satisfies

$$\varphi(\{x, y, z\}) = \{\varphi(x), \varphi(y), \varphi(z)\}$$

for all $x, y, z \in Z$, then φ is called a **$\mathbf{JB}^*$*-triple homomorphism***.*

The norm and the triple product of a JB^*-triple system determine each other and we have

$$\|\{x, y, z\}\| \le \|x\|\|y\|\|z\| \text{ for all } x, y, z \in Z.$$

We do not require JB^*-triple homomorphisms to be continuous because they already are.

Theorem 2.3.2 ([Kau83], Theorem 5.5)**.** *Let Z and W be JB^*-triple systems and $\varphi : Z \to W$ linear. Then the following assertions hold:*

(a) *If φ is a JB^*-triple homomorphism, then φ is contractive and in particular continuous.*

(b) *The mapping φ is JB^*-triple isomorphism if and only if φ is a surjective isometry.*

Another important tool for simplifying equations in a JB^*-triple system are the so-called **polarization formulas**:

$$\{x, y, x\} = \frac{1}{4}\sum_{k=0}^{3}(-1)^k\{y + i^k x, y + i^k x, y + i^k x\}$$
$$\{x, y, z\} = \{x + z, y, x + z\} - \{x, y, x\} - \{z, y, z\}.$$

An easy consequence of these formulas is for example, that a linear mapping $\varphi : Z \to W$ between JB^*-triple systems is already a JB^*-triple homomorphism if it satisfies

$$\varphi(\{x, x, x\}) = \{\varphi(x), \varphi(x), \varphi(x)\}$$

for all $x \in Z$.

If $\mathcal{I}$ is an arbitrary index set and $(E_i)_{i\in\mathcal{I}}$ is a family of JB^*-triple systems, then the **direct sum**

$$\bigoplus_{i\in\mathcal{I}} E_i := \left\{(x_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} E_i : \sup_{i\in\mathcal{I}} \|x_i\| < \infty\right\}$$

becomes a JB^*-triple system in the supremum norm and under the pointwise triple product

$$\{(x_i), (y_i), (z_i)\} := (\{x_i, y_i, z_i\})_{i\in\mathcal{I}}$$

for all $(x_i), (y_i), (z_i) \in \bigoplus_{i\in\mathcal{I}} E_i$.

A closed subspace I of a JB^*-triple system Z is called a **JB*-triple ideal**, if

$$\{Z, I, Z\} + \{I, Z, Z\} \subseteq I.$$

JB^*-triple ideals of Z are JB^*-subtriples of Z and the kernel of a JB^*-triple homomorphism is always a JB^*-triple ideal.

Moreover, if $I \subseteq Z$ is a JB^*-triple ideal the **quotient** Z/I becomes a JB^*-triple system in the quotient norm with the triple product

$$\{x + I, y + I, z + I\} := \{x, y, z\} + I,$$

for all $x, y, z \in Z$.

2.3.1 Examples of JB*-triple systems

A lot of important mathematical structures are examples of JB^*-triple systems.

(a) If H is a **Hilbert space** with scaler product $\langle\cdot,\cdot\rangle$ H becomes a JB^*-triple system under the product

$$\{x, y, z\} := \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x)$$

for all $x, y, z \in H$.

(b) A C^*-algebra $\mathfrak{A}$ becomes a JB^*-triple system under the product

$$\{a, b, c\} := \frac{1}{2}(ab^*c + cb^*a) \tag{2.4}$$

for all $a, b, c \in \mathfrak{A}$.

(c) Every **TRO** becomes a JB^*-triple system equipped with the symmetrized product (2.4).

(d) A **Hilbert-C*-module** becomes a JB^*-triple system if endowed with the TRO structure induced by its linking algebra.

(e) A **JC*-triple system** is a closed subspace Z of the space of bounded linear operators on a Hilbert space H such that $aa^*a \in Z$ for all $a \in Z$. By polarization this is equivalent to Z being a JB^*-triple system under the product

$$\{x, y, z\} := \frac{1}{2}(xy^*z + zy^*x)$$

for all $x, y, z \in Z$. JC^*-triple systems were first studied by Harris in [Har74] under the name **J*-algebras**. Obviously C^*-algebras and TROs, with the above products, are JC^*-triple systems although there are examples of JC^*-triple systems which are neither C^*-algebras nor TROs (e.g. the hermitian $n \times n$-matrices). We call a JB^*-triple system **special** if it is JB^*-triple isomorphic to a JC^*-triple system. A JB^*-triple system Z which is not isomorphic to a JC^*-triple system is called **exceptional** and if every homomorphism from Z to a JC^*-triple system is the 0-mapping we call Z **purely exceptional**.

(f) A **Jordan algebra** is a commutative (not necessarily associative) algebra A over $\mathbb{R}$ or $\mathbb{C}$, with a product denoted by $\circ$, satisfying the **Jordan identity**

$$(z \circ z) \circ (z \circ w) = z \circ ((z \circ z) \circ w)$$

for all $z, w \in A$. The Jordan identity expresses a weak form of associativity since A is commutative.

If A is an associative algebra then A becomes a Jordan algebra under the **anti-commutator product**

$$z \circ w = \frac{1}{2}(zw + wz)$$

for all $z, w \in A$.

A **JB*-algebra** is a complex Jordan algebra A with product $\circ$, unit element e, conjugate linear involution * and complete norm such that

(i) $\|e\| = 1$,

(ii) $\|z \circ w\| \leq \|z\| \|w\|$ and

(iii) $\|\{z, z, z\}\| = \|z\|^3$

holds for every $z, w \in A$, where

$$\{x, y, z\} := (x \circ y^*) \circ z + (z \circ y^*) \circ x - (z \circ x) \circ y^* \tag{2.5}$$

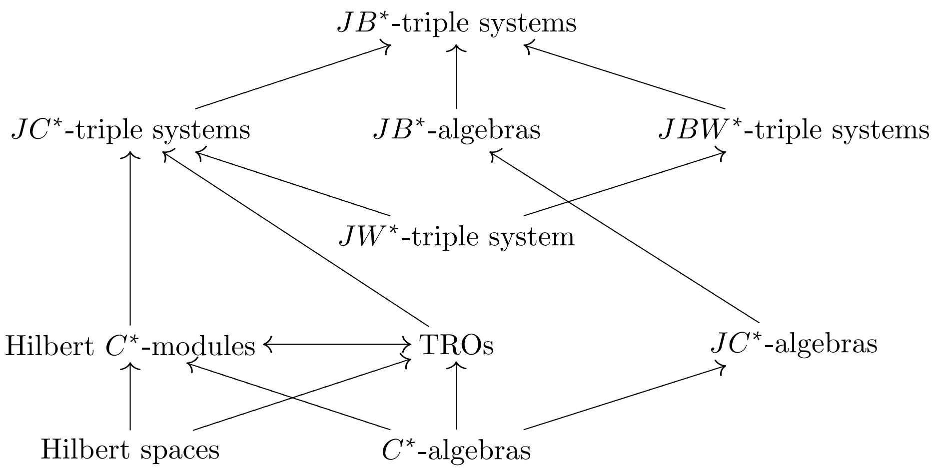
for all $x, y, z \in A$. With this triple product A becomes a JB^*-triple system. If a JB^*-algebra can be represented isomorphically on a Hilbert space it is called a **JC*-algebra**. As above a JB^*-algebra Z is called **exceptional** (resp. **purely exceptional**) if it can not be represented as a JC^*-algebra (resp. if every homomorphism in a JC^*-algebra is the 0-mapping). An example of a purely exceptional JB^*-algebra is the JB^*-algebra $\mathcal{H}_3(\mathbb{O})^{\mathbb{C}}$, which is the complexification of the real (27 dimensional) Jordan algebra $\mathcal{H}_3(\mathbb{O})$, the set of all self-adjoint 3×3-matrices of the octonions under the anti-commutator product. With the triple product (2.5) $\mathcal{H}_3(\mathbb{O})^{\mathbb{C}}$ becomes an example for a purely exceptional JB^*-triple system, too. Every C^*-algebra becomes a JB^*-algebra together with the anti-commutator product, but the TRO consisting of $n \times m$-matrices with $m \neq n$ and canonical product cannot be given the structure of a JB^*-algebra. So there are JB^*-triple systems which are no JB^*-algebras.

(g) A JB^*-triple Z system which is a dual Banach space is called a **JBW*-triple system**. Its predual is usually denoted by Z_*. The triple product of a JBW^*-triple is separately $\sigma(Z, Z_*)$-continuous and its predual is unique. The bidual Z'' of a JB^*-triple system Z is a JBW^*-triple system which contains Z via the canonical injection $Z \hookrightarrow Z''$ as a w^*-dense subtriple.

If a JBW^*-triple system is a w^*-closed subtriple of $B(H)$ we call it **JW*-triple system**.

Since every finite-dimensional vector space is reflexive, every finite-dimensional JB^*-triple system is a JBW^*-triple system.

To guide the reader through this vast amount of examples we give the following diagram, where an arrow from knot A to knot B should be interpreted as "A can be given the structure of B". We do not care about morphisms here.



2.3.2 Tripotents

In the analysis of JB^*-triple systems the so-called tripotents play the important role that projections play in C^*-algebras and von Neumann-algebras and idempotents in rings and algebras.

Definition 2.3.3. *Let Z be a JB^*-triple system. An element $e \in Z$ is called* ***tripotent*** *if $\{e, e, e\} = e$. The collection of all tripotents in Z is denoted by* $\mathrm{Tri}(Z)$.

If $Z \subseteq B(H)$ is a JC^*-triple system (resp TRO, C^*-algebra), then $e \in Z$ is a tripotent if and only if $ee^*e = e$ if and only if e is a partial isometry in $B(H)$. The projections in $B(H)$ which are elements of Z are therefore examples of tripotents.

There are JB^*-triple systems which do not contain any non-zero tripotents. If we denote by $C_0(X)$ the space of all complex-valued continuous functions vanishing at infinity on a locally compact Hausdorff space X it becomes a JB^*-triple system, since it is known to be a C^*-algebra. In the case that X is connected and not compact, $C_0(X)$ does not contain non-zero tripotents.

Definition 2.3.4. *Let Z be a JB^*-triple system. If e is a non-zero tripotent, then e induces a decomposition of Z into the eigenspaces of $e \,\square\, e$, the **Peirce decomposition***

$$Z = P_0^e(Z) \oplus P_1^e(Z) \oplus P_2^e(Z),$$

*where $P_k^e(Z) := \{z \in Z : \{e,e,z\} = \frac{k}{2}z\}$ is the $\frac{k}{2}$-eigenspace, the **Peirce-k-space**, of $e \,\square\, e$, for $k = 0,1,2$.*

Each Peirce-k-space, $k = 0,1,2$, is again a JB^*-triple system. For $i,j,k \in \{0,1,2\}$ they obey the **Peirce calculus**

$$\left\{P_i^e(Z), P_j^e(Z), P_k^e(Z)\right\} \subseteq P_{i-j+k}^e(Z),$$

where $P_l^e(Z) := 0$ for $l \notin \{0,1,2\}$. The Peirce space $P_2^e(Z)$ has not only the structure of a JB^*-triple system but it becomes a (unital) JB^*-algebra under the product

$$(a,b) \mapsto a \circ b := \{a,e,b\},$$

involution

$$a \mapsto a^\dagger := \{e,a,e\}$$

and unit e.

If T is a TRO equipped with its canonical JB^*-triple structure (2.4), then the Peirce-2-space for a non-zero tripotent $e \in T$ is a unital C^*-algebra with product

$$(a,b) \mapsto a \cdot b := ae^*b$$

and involution

$$a \mapsto a^\dagger := ea^*e.$$

2.3.3 The Cartan factors

A tripotent $e \in Z$ is called **Abelian tripotent** if the Peirce-2-space $P_2^e(Z)$ is an **Abelian JB*-triple system**, i.e.

$$\{\{a,b,c\},d,e\} = \{a,\{b,c,d\},e\} = \{a,b,\{c,d,e\}\}$$

for all $a,b,c,d,e \in Z$.

If a JBW^*-triple system contains an Abelian tripotent it is called **type 1 JBW*-triple system**.

A type 1 JBW^*-triple system which does not contain any non-trivial weak*-closed triple ideals is called **Cartan factor**.

Horn showed in [Hor87b] that there are exactly six different types of Cartan factors. Let H and K be Hilbert spaces of (possible infinite) dimensions n and m. Let J be a conjugation on the Hilbert space H, i.e. a conjugate linear isometry of order 2.

We distinguish between the four **classical Cartan factors** (cf. [Har74], §2) which can be represented as bounded operators on a Hilbert space

$\mathbf{C^1_{n,m}}$: The **rectangular Cartan factor**, or **Cartan factor of type I** is the space $B(H,K)$ with the canonical symmetrized triple product. In finite dimensions the rectangular Cartan factor is the space of complex rectangular $n \times m$-matrices $\mathbb{M}_{n,m}$.

$\mathbf{C^2_n}$: A **symplectic Cartan factor**, or **Cartan factor of type II** is the space

$$C^2_n := \{x \in B(H) : Jx^*J = -x\}.$$

If n is finite this is the space of skew-symmetric $n \times n$-matrices

$$\{A \in \mathbb{M}_n : A^t = -A\}.$$

$\mathbf{C^3_n}$: The **hermitian Cartan factor**, or **Cartan factor of type III**, is the JBW^*-triple system

$$C^3_n := \{x \in B(H) : Jx^*J = x\}.$$

If $2 \le n < \infty$ this becomes

$$\{A \in \mathbb{M}_n : A^t = A\},$$

the symmetric $n \times n$-matrices.

$\mathbf{C^4_\lambda}$: The so-called **spin factor**, or **Cartan factor of type IV**, is the $(\lambda+1)$-dimensional closed linear span of a **spin system** which is a subset of $B(H)$ of arbitrary cardinality $\{\mathrm{id}_H, s_1, \dots, s_\lambda\}$, where $s_1, \dots, s_\lambda$ are self-adjoint elements satisfying

$$\frac{1}{2}(s_i s_j + s_j s_i) = \delta_{i,j}\,\mathrm{id}_H$$

for all $i, j \in \{1, \dots, \lambda\}$ and where $\delta_{i,j}$ denotes the Kronecker delta.

and the two (purely) **exceptional Cartan factors**

$\mathbf{M_{1,2}}(\mathbb{O})$: This Cartan factor is of dimension 16 and is the space of 1×2-matrices over the complex Cayley algebra $\mathbb{O}$ endowed with the triple product

$$\{x, y, z\} := \frac{1}{2}(x(y^*z) + z(y^*x)) \quad \text{where} \quad (x_1, x_2)^* := \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$$

for all $x = (x_1, x_2), y, z \in M_{1,2}(\mathbb{O})$. It becomes a JB^*-triple system endowed with its unique spectral norm (cf. [KU77], 3.17).

$\mathbf{H_3}(\mathbb{O})$: This is the 27-dimensional subspace of $M_3(\mathbb{O})$ consisting of hermitian 3×3-matrices over $\mathbb{O}$ endowed with the spectral norm (cf. [KU77], 3.17).

Definition 2.3.5. *Let Z be a JB^*-triple system. An element $e \in \mathrm{Tri}(Z)$ is called **minimal** if*

$$\{e, Z, e\} = \mathbb{C}e.$$

*A JBW^*-triple system is called **atomic** if it is the w^*-closed linear span of its minimal tripotents.*

Part (b) of the following theorem is a celebrated result of Friedman and Russo known as the Gelfand-Naimark Theorem for JB^*-triple systems. It is a generalization of a representation theorem for JB-algebras given by Alfsen, Shultz and Størmer in [ASS78] and both are generalizations of the classical Gelfand-Naimark Theorem for C^*-algebras.

Theorem 2.3.6 ([FR86], Prop. 1, Prop. 2, Thm. 1)**. (a)** *Every atomic JBW^*-triple system is a l^∞-direct sum of Cartan factors.*

(b) *Every JB^*-triple system is isometrically isomorphic to a subtriple of an atomic JBW^*-triple system and thus to a subtriple of a l^∞-direct sum of Cartan factors.*

Especially every finite-dimensional JB^*-triple system is the direct sum of Cartan factors. One can show that the subtriple of an arbitrary JB^*-triple system generated by a single element is an Abelian C^*-algebra. The commutative version of the Gelfand-Naimark Theorem (cf. [Wer00], Theorem IX.3.4) allows us to take cubic roots:

Lemma 2.3.7. *Let Z be a JB^*-triple system. For every $x \in Z$ there exists an element $y \in Z$ with*

$$\{y, y, y\} = x.$$

2.3.4 Grids and roots

Grids are special families of tripotents which were used by Neher and others to analyze the structure of Jordan triple systems, especially JBW^*-triple systems. In Chapter 4 grid theory helps us to determine the universal enveloping TROs of the classical Cartan factors and in Chapter 5 grids are the special ingredient in the K-theoretic classification of finite-dimensional JC^*-triple systems. Since grids are that important for our work we illuminate the close connection between root systems and grids.

Let Z be a JB^*-triple system and $\mathcal{C} \subseteq \mathrm{Tri}(Z) \setminus \{0\}$. We call $\mathcal{C}$ a **cog** if for each pair $e, f \in \mathcal{C}$ exactly one of the following four relations holds:

$$\begin{aligned} e \perp f &:\Leftrightarrow e \text{ is } \textbf{orthogonal} \text{ to } f \\ &:\Leftrightarrow e \in P_0^f(Z) \text{ and } f \in P_0^e(Z) \\ &\Leftrightarrow \{e,e,f\} = 0 = \{f,f,e\}, \end{aligned}$$

$$\begin{aligned} e \top f &:\Leftrightarrow e \text{ is } \textbf{collinear} \text{ to } f \\ &:\Leftrightarrow e \in P_1^f(Z) \text{ and } f \in P_1^e(Z) \\ &\Leftrightarrow \{e,e,f\} = \frac{1}{2}f \text{ and } \{f,f,e\} = \frac{1}{2}e, \end{aligned}$$

$$\begin{aligned} e \vdash f &:\Leftrightarrow e \textbf{ governs } f \\ &:\Leftrightarrow e \in P_1^f(Z) \text{ and } f \in P_2^e(Z) \\ &\Leftrightarrow \{e,e,f\} = f \text{ and } \{f,f,e\} = \frac{1}{2}e, \end{aligned}$$

$$\begin{aligned} e \dashv f &:\Leftrightarrow f \textbf{ governs } e \\ &:\Leftrightarrow f \vdash e \end{aligned}$$

The relations $\perp$ and $\top$ are symmetric and $e \vdash f$ if and only if $f \dashv e$ for all non-zero tripotents $e, f \in Z$. The abbreviation cog stands for the relations 'collinear', 'orthogonal' and 'govern'. By expressions like for example $e_1 \perp e_2 \vdash e_3 \top e_4$ we mean the obvious.

Let Z be a JB^*-triple system. The maximal cardinality of a system of pairwise orthogonal non-zero tripotents in Z is called the **rank** of Z. We call two tripotents $e, f \in \mathrm{Tri}(Z)$ **associated**, $e \approx f$ if they have the same Peirce spaces.

A cog $\mathcal{C}$ is called **closed** if

(a) for each pair $e_0, e_1 \in \mathcal{C}$ with $e_0 \vdash e_1$ there exists $f \in \mathcal{C}$ satisfying $f \approx \{e_0, e_1, e_0\}$ and

(b) for each family $(e_1, e_2, e_3) \subseteq \mathcal{C}$ with $e_1 \top e_2 \top e_3 \perp e_1$ or $e_1 \vdash e_2 \dashv e_3 \top e_1$ there exists $g \in \mathcal{C}$ satisfying $\{e_1, e_2, e_3\} \approx g$.

A closed cog $\mathcal{G}$ is called a **grid** if the following two conditions are satisfied:

(a) For each family $(e_1, e_2, e_3) \subseteq \mathcal{G}$ with $e_1 \top e_2 \top e_3 \top e_1$ and $\{e_1, e_2, e_3\} \neq 0$ there exists $c \in \mathcal{G}$ such that $e_1 \vdash c \dashv e_3$ and $c \perp e_2$.

(b) If $e_1 \dashv e_2 \vdash e_3 \top e_1$ then $\{e_1, e_2, e_3\} = 0$.

We call two grids $\mathcal{G}_1, \mathcal{G}_2$ **associated** (denoted by $\mathcal{G}_1 \approx \mathcal{G}_2$) if there is a bijection $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ with $\varphi(e) \approx e$ for all $e \in \mathcal{G}_1$.

Two tripotents e, f in grid $\mathcal{G}$ are called **connected tripotents** if there exists a finite sequence $(f_1, \dots, f_n) \subseteq \mathcal{G}$, $n \in \mathbb{N}$, such that

$$f_1 = e, f_n = f \text{ and } f_i \not\perp f_{i+1} \text{ for all } 1 \leq i \leq n-1.$$

Connectedness defines an equivalence relation on $\mathcal{G}$. The equivalence classes of this relation are called **connected components** of $\mathcal{G}$ and $\mathcal{G}$ is called **connected** if all tripotents in $\mathcal{G}$ are connected pairwisely.

There are 6 different types of connected grids (the standard grids I–IV are defined below):

Theorem 2.3.8 ([Neh87], grid classification theorem)**.** *Every connected grid $\mathcal{G}$ is associated to one of the following standard grids, where I and J are index sets:*

(I) *Rectangular grid $\mathcal{R}(I, J)$, $|I|, |J| \geq 1$.*

(II) *Symplectic grid $S(I)$, $|I| \geq 5$.*

(III) *Hermitian grid $H(I)$, $|I| \geq 2$.*

(IV) *Spin grid $Sp(I)$, $|I| \geq 1$.*

And the two exceptional grids

(V) *Bi-Caley grid.*

(VI) *Albert grid.*

We remark that the numeration of the standard grids is not chosen by chance but reflects the numeration of the Cartan factors which are generated by them, as we will see.

We give the abstract definition and standard examples of the grids of type I–IV both playing an essential role in this work. For the two exceptional grids we refer the reader to [Neh87], Chapter II.

We first need a useful definition.

Definition 2.3.9. *Let Z be a JB^*-triple system. A quadruple*

$$(e_1, e_2, e_3, e_4)$$

of tripotents in Z is called a ***quadrangle*** *if*

$$e_i \top e_{i+1}, e_i \perp e_{i+2} \text{ and } \{e_i, e_{i+1}, e_{i+2}\} = \frac{1}{2} e_{i+3}$$

for all $i = 1, 2, 3, 4$ (indices mod 4).

Let I and J be totally ordered sets and Z a JB^*-triple system.

$\mathcal{R}(I,J)$**:** A family $(e_{i,j})_{i\in I,j\in J}$ of non-zero tripotents is called **rectangular grid** if

(i) $e_{i,j} \top e_{i,k}$, $e_{j,i} \top e_{k,i}$ and $e_{i,j} \perp e_{l,k}$ for $i \neq l$, $j \neq k$.

(ii) $(e_{i,j}, e_{i,l}, e_{k,l}, e_{k,j})$ is a quadrangle for $i \neq k$, $j \neq l$.

(iii) $\{e_{i,j}, e_{k,l}, e_{r,s}\} = 0$ for $(k,l) \neq (r,j)$ or $(k,l) \neq (i,s)$.

The standard example for a rectangular grid is the **rectangular matrix system**. This is the set

$$\{E_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\} \subseteq \mathbb{M}_{n,m},$$

spanning the finite-dimensional Cartan factor $C^1_{m,n}$.

$S(I)$**:** Let $|I| \geq 4$. A family $(e_{i,j})_{i,j\in I}$ of non-zero tripotents is called a **symplectic grid** if for all $i,j \in I, i \neq j$:

(i) $e_{i,j} = -e_{j,i}$.

(ii) $P_2^{e_{i,j}}(Z) \subseteq P_1^{e_{k,l}}(Z)$ and $P_2^{e_{k,l}}(Z) \subseteq P_1^{e_{i,j}}(Z)$ if $\{i,j\} \cap \{k,l\} \neq \varnothing$.

(iii) $e_{i,j} \perp e_{k,l}$ for $\{i,j\} \cap \{k,l\} = \varnothing$.

(iv) $(e_{i,j}, e_{k,j}, e_{k,l}, e_{i,l})$ is a quadrangle for pairwise distinct i,j,k,l.

(v) $\{e_{i,j}, e_{k,l}, e_{r,s}\}$ for $(k,l),(l,k) \notin \{i,j\} \times \{r,s\}$.

The finite-dimensional standard example is the **symplectic matrix system** which is the symplectic grid

$$\{E_{i,j} - E_{j,i} : 1 \leq i,j \leq n, i \neq j\} \subseteq \mathbb{M}_n$$

spanning the Cartan factor C^2_n.

$H(I)$**:** A family $(e_{i,j})_{i,j\in J}$ of non-zero tripotents is a **hermitian grid** if the following rules for pairwise distinct $i,j,k,l \in I$ hold:

(i) $e_{i,j} = e_{j,i}$.

(ii) $e_{j,j} \perp e_{i,i} \perp e_{j,k}$, $e_{i,i} \dashv e_{i,j} \top e_{i,k}$ and $e_{i,j} \perp e_{k,l}$.

(iii) $\{e_{i,j}, e_{i,i}, e_{i,j}\} = e_{j,j}$.

(iv) $\{e_{i,m}, e_{m,n}, e_{n,j}\} = e_{i,j}$ (m arbitrary).

(v) $\{e_{i,m}, e_{m,n}, e_{n,i}\} = 2e_{i,i}$ (m,n arbitrary).

(vi) $\{e_{m,n}, e_{p,q}, e_{r,s}\} = 0$ if the indices cannot be linked.

In finite dimensions the standard example of a hermitian grid is given by the hermitian matrix system

$$\{E_{i,j} + E_{j,i} : 1 \leq i,j \leq n, i \neq j\} \cup \{E_{i,i} : 1 \leq i \leq n\}$$

$Sp(I)$**:** A family $\{e_i, \widetilde{e}_i : i \in I\}$, or $\{e_i, \widetilde{e}_i : i \in I\} \cup \{e_0\}$ in finite odd dimensions, of non-zero tripotents is called **spin grid** if for $i, j \in I$:

(i) e_i and $\widetilde{e}_i$ are minimal (but not e_0).

(ii) $e_i \top e_j \top \widetilde{e}_i \top \widetilde{e}_j$ for $i \neq j$.

(iii) $\{e_j, e_i, \widetilde{e}_j\} = -\frac{1}{2}\widetilde{e}_i$, $\{e_i, \widetilde{e}_j, \widetilde{e}_i\} = -\frac{1}{2}\widetilde{e}_j$.

And in the case that e_0 is present:

(iv) $e_i \dashv e_0 \vdash \widetilde{e}_i$ for $i \neq 0$.

(v) $\{e_0, e_i, e_0\} = -\widetilde{e}_i$, $\{e_0, \widetilde{e}_i, e_0\} = -e_i$ for $i \neq 0$.

(vi) All other triple products are 0.

A spin grid generates a Cartan factor of type IV.

Examples of spin grids are given in [Neh87] Ch. I Ex. 1.6 under the name of even and odd dimensional quadratic form grids. In particular they are not represented as matrices. We do not repeat this here but present later (cf. Proposition 5.2.6) a way to construct a spin grid out of a spin system instead. This is a subset of a matrix space which gives us an example of a finite-dimensional spin grid.

The following two theorems are essential in the theory of grids. The first one is called the **structure theorem** and the second one the **isomorphism theorem**.

Theorem 2.3.10 ([Neh87], Theorem 3.14)**.** *Let Z be an atomic JBW^*-triple system and $Z = \bigoplus_i Z_i$ be the decomposition from Theorem 2.3.6 into a l^∞-direct sum of Cartan factors. Then Z contains a grid $\mathcal{G}$, where $\mathcal{G}$ is a union of standard grids $\mathcal{G}_i$, $i \in I$, such that*

$$\mathcal{G} = \bigcup_{i \in I} \mathcal{G}_i, \quad \textit{with } \mathcal{G}_i = \mathcal{G} \cap U_i \textit{ for all } i \in I.$$

Z is triple isomorphic to a Cartan factor if and only if $\mathcal{G}$ is a standard grid.

Theorem 2.3.11 ([Neh87], Theorem 3.18)**.** *Two atomic JBW^*-triple systems Z_k, $k = 1, 2$ with grids $\mathcal{G}_k$ as described in the Structure Theorem 2.3.10 are isometrically isomorphic if $\mathcal{G}_1$ and $\mathcal{G}_2$ are the same union of the same standard grids.*

One can specialize these two results to

Theorem 2.3.12. *Every atomic JW^*-triple system Z is the ℓ^∞-direct sum $Z = \oplus_i Z_i$ of weak*-closed irreducible triple ideals Z_i, where every summand Z_i is the weak*-closure of the complex linear span of a grid of type I–IV. Hence every summand is JB^*-triple isomorphic to a Cartan factor of type I–IV.*

As already noted, grids stand in close connection to root systems. But in contrast to semisimple Lie algebras, root systems are not sufficient to classify the finite-dimensional JB^*-triple systems (i.e. the finite-dimensional bounded symmetric domains). As additional data we need a certain grading of the root system, a so-called 3-grading. Our reference for 3-graded root systems as well as for 3-graded Lie algebras is the work of Neher, especially [Neh90], [Neh91] and [Neh96].

Definition 2.3.13. *Let X be a real, finite-dimensional vector space with scalar product $\langle\cdot,\cdot\rangle$. A subset $R \subseteq X$ is called a **root system** iff*

(a) *R is finite, generates X and does not contain 0.*

(b) *For every root $\alpha \in R$ we have $s_\alpha(R) = R$, where $s_\alpha(x) = x - 2\frac{\langle x,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha$ is the reflection in α.*

(c) *$(\alpha,\beta) := 2\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}$ for all $\alpha,\beta \in R$.*

(d) *For every $\alpha \in R$, we have $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$.*

*Two root systems $R \subseteq X$, $\widetilde{R} \subseteq \widetilde{X}$ are called **isomorphic**, if there exists a vector space isomorphism $\varphi : X \to \tilde{X}$ with $\langle\varphi(\alpha),\varphi(\beta)\rangle = \langle\alpha,\beta\rangle$ for all $\alpha,\beta \in R$.*

*We call a root system $R \neq \emptyset$ **irreducible** if R cannot be decomposed into two orthogonal non-empty subsets.*

*The root system R is called **3-graded root system** if additionally there exist $R_{-1}, R_0, R_1 \subseteq R$ with*

(d) *$R = R_1 \dot\cup R_0 \dot\cup R_{-1}$ (disjoint union).*

(e) *$R_{-1} = -R_1$.*

(f) *$R_0 = \{\alpha-\beta; \alpha,\beta \in R_1, \alpha \neq \beta, \langle\alpha,\beta\rangle = 0\}$.*

(g) *If $\alpha,\beta \in R_1$ then $\alpha+\beta \notin R$.*

(h) *If $\alpha \in R_0, \beta \in R_1$ and $\alpha+\beta \in R$ then $\alpha+\beta \in R_1$.*

*If a disjoint decomposition of R that fulfills (d)–(h) exists it is called a **3-grading** of R.*

Due to the conditions (e) and (f) the grading of a root system is completely determined by its (R_1) part. We thus write (R,R_1) for a 3-graded root system with grading induced by R_1.

*We call two 3-graded root systems $R \subseteq X$, $\tilde{R} \subseteq \tilde{X}$ **isomorphic** if there exists an isomorphism of root systems $\varphi : R \to \tilde{R}$ that maps R_1 onto $\tilde{R}_1$.*

The next results are classical (cf. [Hum78]).

Theorem 2.3.14. *Every nonempty root system $R \subseteq X$ decomposes uniquely as the union of irreducible root systems R^i (in subspaces $X^i \subseteq X$) and $X = \bigoplus X^i$ (orthogonal direct sum).*

Now one can restrict the analysis of root systems to irreducible root systems.

As it turns out there are 9 different types of irreducible root systems. The four classical types $\mathscr{A}_n, \mathscr{B}_n, \mathscr{C}_n, \mathscr{D}_n$ and the 5 exceptional types $\mathscr{E}_6, \mathscr{E}_7, \mathscr{E}_8, \mathscr{F}_4$ and $\mathscr{G}_2$.

Since only the classical root systems play a role in this work we present them in detail and refer the reader to [Hel78] or [Hum78] for a complete list.

$\mathscr{A}_n :$ $(n \geq 1)$ This root system is given by

$$\mathscr{A}_n = \{e_i - e_j : 1 \leq i \neq j \leq n+1\}.$$

The vector space spanned by $\mathscr{A}_n$ is the n-dimensional subspace

$$X = \left\{ \sum_{i=1}^{n+1} x_i e_i : \sum_{i=1}^{n} x_i = 0 \right\}$$

of the vector space $\bigoplus_{i=1}^{n+1} \mathbb{R} e_i$.

$\mathscr{B}_n :$ $(n \geq 2)$ Here the root system is

$$\mathscr{B}_n = \{\pm e_i : 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j : 1 \leq i \neq j \leq n, \pm \text{ independent}\}$$

and spans the vector space $X = \bigoplus_{i=1}^{n} \mathbb{R} e_i$.

$\mathscr{C}_n :$ $(n \geq 3)$ The root system of class $\mathscr{C}_n$ is given by

$$\mathscr{C}_n = \{\pm 2e_i : 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j : 1 \leq i \neq j \leq n, \pm \text{ independent}\}$$

spanning $X = \bigoplus_{i=1}^{n} \mathbb{R} e_i$.

$\mathscr{D}_n :$ $(n \geq 4)$ The root system for $\mathscr{D}_n$ is

$$R = \{\pm e_i \pm e_j : 1 \leq i \neq j \leq n, \pm \text{ independent}\}$$

spanning the vector space $X = \bigoplus_{i=1}^{n} \mathbb{R} e_i$.

The restrictions on n for the types $\mathscr{A}_n$–$\mathscr{D}_n$ are imposed in order to avoid duplication.

Neher described in [Neh91] how to construct a 3-graded root system from what he called an abstract cog, a structure which is very similar to a grid. We adapt this construction to the above defined standard grids.

We associate a 3-graded root system to every classical Cartan factor. The procedure to identify a grid with the R_1 part of a 3-graded root system was developed by Neher in [Neh91].

For a root system one can define similar relations as for tripotents:

Definition 2.3.15. *Let R be a root system and $\alpha, \beta \in R$. We say that the roots*

$$\begin{aligned} \alpha \text{ and } \beta \text{ are } \textbf{orthogonal} &:\Leftrightarrow \alpha \perp \beta \\ &:\Leftrightarrow (\alpha, \beta) = 0 \\ &\Leftrightarrow \langle \alpha, \beta \rangle = 0, \end{aligned}$$

$$\begin{aligned} \alpha \text{ and } \beta \text{ are } \textbf{collinear} &:\Leftrightarrow \alpha \top \beta \\ &:\Leftrightarrow \langle \alpha, \alpha \rangle = 2\langle \alpha, \beta \rangle = \langle \beta, \beta \rangle \\ &\Leftrightarrow (\alpha, \beta) = 1 = (\beta, \alpha), \end{aligned}$$

$$\begin{aligned} \alpha \ \textbf{governs} \ \beta &:\Leftrightarrow \alpha \vdash \beta \text{ or } \beta \dashv \alpha \\ &:\Leftrightarrow 2\langle \alpha, \alpha \rangle = 2\langle \alpha, \beta \rangle = \langle \beta, \beta \rangle \\ &\Leftrightarrow (\alpha, \beta) = 1 \text{ and } (\beta, \alpha) = 2. \end{aligned}$$

Theorem 2.3.16 ([Neh91], Theorem A)**.** *For every closed cog $\mathcal{E}$ there exists a 3-graded root system (R, R_1) and a bijection $\mathcal{E} \to R_1$, $e \mapsto e'$ such that for all $e, f \in \mathcal{E}$ we have*

$$e \,\mathfrak{R}\, f \Leftrightarrow e' \,\mathfrak{R}\, f',$$

where $\mathfrak{R}$ stands for $\perp$, $\top$, $\vdash$ or $\dashv$.

Theorem 2.3.17 ([Neh91], Theorem 3.4)**.** *Two closed cogs are isomorphic if and only if their associated 3-graded root systems are isomorphic (as 3-graded root systems). A grid is connected if and only if its associated 3-graded root systems is irreducible.*

Let $\mathcal{G}$ be a grid spanning an atomic JBW^*-triple system. For every $e \in \mathcal{G}$ we can define an element e' in the dual space Z' of Z such that for every element $f \in \mathcal{G}$ we have

$$e'(f) := \begin{cases} 0 & \text{if } e \perp f, \\ 1 & \text{if } e \top f \text{ or } e \vdash f, \\ 2 & \text{if } e = f \text{ or } e \dashv f. \end{cases}$$

The mapping

$$e \mapsto e'$$

maps $\mathcal{G}$ bijectively to the R_1-part of a 3-graded root system (R, R_1) realizing the mapping in Theorem 2.3.16.

We know that every finite-dimensional JC^*-triple system is the direct sum of Cartan factors of type I–IV. Each Cartan factor is up to JB^*-triple isomorphism, uniquely determined by a grid which is of rectangular, symplectic, hermitian or spin type. These grids are in one to one correspondence with irreducible 3-graded root systems.

We will now give a list (cf. [Neh96], §3) of the 3-graded root systems corresponding to the standard grids and thus to the classical Cartan factors. Let I and J be two finite index sets and $K := I \dot\cup J$.

Type I: A grid of type $\mathcal{R}(I, J)$ is given by the usual matrix units $\{E_{i,j} : i \in I, j \in J\}$ and the corresponding root system R in the vector space X is

$$R = \mathscr{A}_{|K|-1} = \{e_k - e_l : k, l \in K, k \neq l\}$$

with grading induced by

$$R_1 = \{e_i - e_j : i \in I, j \in J\},$$

where $\{e_k : k \in K\}$ is the canonical orthogonal basis of $\widetilde{X} = \bigoplus_{k \in K} \mathbb{R} e_k$ and $X = \left\{\sum_{k \in K} s_k e_k \in \widetilde{X} : \sum_{k \in K} s_k = 0\right\}$.

Type II: The standard example of the symplectic grid $S(I)$, $|I| \geq 5$, is $\{E_{i,j} - E_{j,i} : i, j \in I, i < j\}$ the 3-graded root system is given by

$$R = \mathscr{D}_{|I|} = \{\pm e_i \pm e_j : i, j \in I, i \neq j\},$$

where the grading comes from

$$R_1 = \{e_i + e_j : i, j \in I, i \neq j\}$$

in the real vector space $X = \bigoplus_{i \in I} \mathbb{R} e_i$.

Type III: For a hermitian grid $H(I)$, $I \geq 2$, the standard example is $\{E_{i,j} + E_{j,i} : i, j \in I, i \neq j\} \cup \{E_{i,i} : i \in I\}$ the associated root system is

$$R = \mathscr{C}_{|I|} = \{\pm 2e_i : i \in I\} \cup \{\pm e_i \pm e_j : i, j \in I, i \neq j\}$$

with 1-part

$$R_1 = \{e_i + e_j : i, j \in I, i \neq j\}$$

in $X = \bigoplus_{i \in I} \mathbb{R} e_i$.

Type IV: **(a)** The 3-graded root system associated to a spin grid $Sp(I)$ with $|Sp(I)|$ even is a root system of type $\mathscr{D}_{|I|+1}$ in the vector space $X = \bigoplus_{i \in I \cup \{\infty\}} \mathbb{R} e_i$ with an element $\infty \notin I$.

$$R = \mathscr{D}_{|I|+1} = \{\pm e_i \pm e_j : i, j \in I \cup \{\infty\}, i \neq j\},$$

where the grading comes from

$$R_1 = \{e_\infty \pm e_i : i \in I\}.$$

(b) For a spin grid $Sp(I)$ with $|Sp(I)|$ odd the associated root system R is of type $\mathscr{B}_{|I|+1}$. Let again ∞ be an element not contained in I, then

$$R = \mathscr{B}_{|I|+1} = \{\pm e_i \pm e_j : i,j \in I \cup \{\infty\}, i \neq j\} \cup \{\pm e_i : i \in I \cup \{\infty\}\}$$

is 3-graded with 1-part

$$R_1 = \{e_\infty \pm e_i : i \in I\} \cup \{e_\infty\}.$$

Note that for $i \geq 4$ the gradings for type II and type IV (a) are not isomorphic.

Next we describe the Tits-Kantor-Koecher Lie algebra of a JB^*-triple system. It is not only interesting for us but it is also of historical interest since it is the origin of Jordan triple theory itself. Analyzing the Jacobi identity of the Tits-Koecher-Kantor Lie algebra Meyberg discovered in 1972 (cf. [Mey72]) the outer symmetry and the Jordan triple identity of the Jordan triple product, which led him to the definition of a Jordan triple system (i.e. a real predecessor of our JB^*-triple systems with no norm).

Definition 2.3.18. *Let Z be a JB^*-triple system. We define the* ***box algebra*** *of Z to be the closure of the real linear span of all box operators*

$$Z \square Z := \overline{\text{lin}}\{x \square y : x,y \in Z\}.$$

The Jordan triple identity is equivalent to the operator identity

$$[x \square y, u \square v] = \{x,y,u\} \square v - u \square \{y,x,v\},$$

where $[\cdot,\cdot]$ denotes the commutator product. Equipped with this product $Z \square Z$ is a (real) Lie-algebra. Let

$$\mathfrak{g} := \mathcal{TKK}(Z) := Z_{+1} \oplus Z \square Z \oplus Z_{-1}$$

be the so-called ***Tits-Kantor-Koecher Lie algebra*** *(or TKK-algebra), where $Z_{+1} := Z =: Z_{-1}$. The TKK-algebra becomes a real Lie algebra, when equipped with the following product: Let $x,y \in Z$, $T,T_1,T_2 \in Z \square Z$ and put*

$$\begin{aligned}
&[(0,T,0),(x,0,0)] := (T(x),0,0),\\
&[(0,T,0),(0,0,y)] := (0,0,-T^*(y)),\\
&[(x,0,0),(y,0,0)] := [(0,0,x),(0,0,y)] = 0,\\
&[(x,0,0),(0,0,y)] := (0,x \square y,0) \text{ and}\\
&[(0,T_1,0),(0,T_2,0)] := (0,T_1T_2 - T_2T_1,0).
\end{aligned}$$

Let Z be a Cartan factor of type I–IV. The 3-graded root system associated with Z via its grid is exactly the root systems of its TKK-algebra, as shown in [Neh96], Chapter 3. We refer the reader to [Hum78], Section 1.2 for a detailed description of the classical Lie algebras.

Chapter 3

K-theory for ternary rings of operators

We generalize the representation theory of C^*-algebras to ternary rings of operators and afterwards use this theory to analyze finite-dimensional TROs and their homomorphisms. As it turns out every finite-dimensional TRO is the finite direct sum of rectangular matrix algebras and every TRO-homomorphism is up to unitary equivalence (as defined below) uniquely determined by a rectangular matrix with entries in $\mathbb{N}_0$.

A very powerful tool in the analysis of C^*-algebras is the so-called K-theory which is more or less two covariant functors K_0 and K_1 from the category of C^*-algebras to the category of Abelian groups that are homotopy invariant, stable, half exact and continuous. The functor K_0 attains the values $\mathbb{Z}$ and 0 on the C^*-algebras $\mathbb{C}$ and $C_0(\mathbb{R})$ respectively. Having this in mind, we make a close investigation of three functors from the category of TROs to the category of C^*-algebras of whom the best known is the linking algebra functor $\mathbb{L}$ (cf. Chapter 2.3). The other two are given on the TRO objects by

$$\mathcal{L}(T) = \overline{\operatorname{lin}\{xy^* : x, y \in T\}} \quad \text{and} \quad \mathcal{R}(T) = \overline{\operatorname{lin}\{x^*y : x, y \in T\}}.$$

Since T can be given the structure of a left Hilbert $\mathcal{L}(T)$-module and of a right Hilbert $\mathcal{R}(T)$-module, the C^*-algebras $\mathcal{L}(T)$ and $\mathcal{R}(T)$ are called the left and right C^*-algebras of T. We show that these three functors are also homotopy invariant, stable and continuous. Additionally they are not only half exact but exact. This brings us in the position to define the functor K_0 on the category of TROs as the concatenation of the functor $\mathcal{L}$ with the C^*-algebra K_0-functor. We give an intrinsic definition of K_1 on the category of TROs which coincides with the consecutive application of the functors $\mathcal{L}$ and the C^*-algebra K_1-functor.

The definition of these functors is independent of the choice we made by preferring the functor $\mathcal{L}$ over the functor $\mathcal{R}$: We show on a large subcategory that the embeddings of $\mathcal{L}(T)$ and $\mathcal{R}(T)$ as corners of the linking algebra induce isomorphisms on the K_0-level of C^*-algebras which yields an isomorphism of $K_0(\mathcal{R}(T))$ to $K_0(T)$, the so-called Morita isomorphism. This isomorphism becomes very useful in the examination of the additional structures we attach the K_0-group with. We first define (if possible) an order structure in the K_0-group and afterwards introduce what we call a double-scale. The K_0-group together with this data is called a double-scaled ordered group, a generalization to the ternary context of the concept of a dimension group developed by Elliot. Our concept of double-scales enables us to not only keep track of the dimension of the left C^*-algebra but also of the right C^*-algebra, with the aid of the Morita isomorphism. This will allow us to give a (semi-)classification of stably isomorphic TROs and a complete classification of AF-TROs, the inductive limits of finite-dimensional TROs.

3.1 Representation theory for TROs

In this section we develop a representation theory for ternary rings of operators. We define a natural terminology for these representations such as non-degeneracy, irreducibility and unitary equivalence. It will form a powerful tool in the analysis of TROs to investigate the strong connections between TRO-representations and the representations of the linking algebra. We use ideas from the representation theory of C^*-algebras (cf. [Tak02], [Dav96]) and Hilbert C^*-modules (cf. [BLM04], [BG02] and especially the detailed outline [Ara05]).

Definition 3.1.1. *Let T be a TRO. A TRO-homomorphism $\varphi : T \to B(H,K)$ is called **representation** of T. If φ is injective it is called a **faithful representation** of T.*

If $\varphi : T \to B(H,K)$ is a (faithful) representation of the TRO T, then $\mathcal{L}(\varphi) : \mathcal{L}(T) \to B(H)$ and $\mathcal{R}(\varphi) : \mathcal{R}(T) \to B(K)$ are (faithful) $*$-representations of $\mathcal{L}(T)$ and $\mathcal{R}(T)$.

Let σ be the $*$-isomorphism from $\mathbb{L}(B(H,K)) = \begin{pmatrix} B(K) & B(H,K) \\ B(K,H) & B(H) \end{pmatrix}$ to $B(K \oplus H)$. We obtain a $*$-representation of the linking algebra of T by defining

$$\pi_\varphi := \sigma \circ \mathbb{L}(\varphi) : \mathbb{L}(T) \to B(K \oplus H). \tag{3.1}$$

In particular we have $\pi_\varphi|_{\mathcal{L}(T)} = \sigma \circ \mathcal{L}(\varphi)$, $\pi_\varphi|_{\mathcal{R}(T)} = \sigma \circ \mathcal{R}(\varphi)$ and $\pi_\varphi|_T = \sigma \circ \varphi$ (dropping the canonical inclusions into $\mathbb{L}(T)$). So every TRO-representation of T induces a representation of $\mathbb{L}(T)$.

Proposition 3.1.2. *Let T be a TRO and $\pi : \mathbb{L}(T) \to B(H)$ be a $*$-representation of its linking algebra, then there exist Hilbert spaces H_1, K_1 such that $H = K_1 \oplus H_1$ and there is a TRO-representation $\varphi : T \to B(H_1, K_1)$ with $\pi_\varphi = \pi$.*

Proof. We identify $\mathcal{L}(T), \mathcal{R}(T)$ and T with their images in $\mathbb{L}(T)$ and put $H_1 := \overline{\pi(\mathcal{L}(T))H}$ and $K_1 := H_1^\perp$. There exists a canonical isomorphism ρ between $B(H)$ and $\mathbb{L}(B(H_1, K_1))$. Let $\xi := \rho \circ \pi$, then $\xi(\mathcal{L}(T)) \subseteq \begin{pmatrix} 0 & 0 \\ 0 & B(H_1) \end{pmatrix}$ and $\xi(\mathcal{R}(T)) \subseteq \begin{pmatrix} B(K_1) & 0 \\ 0 & 0 \end{pmatrix}$. Let $x \in B(H_1, K_1)$ and write $\xi := \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix}$, then (identifying x with $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ freely)

$$\begin{aligned}
\xi\begin{pmatrix} xx^* & 0 \\ 0 & 0 \end{pmatrix} &= \xi\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \xi\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}^* \\
&= \begin{pmatrix} \xi_1(x)\xi_1(x)^* + \xi_2(x)\xi_2(x)^* & \xi_1(x)\xi_3(x)^* + \xi_2(x)\xi_4(x)^* \\ \xi_3(x)\xi_1(x)^* + \xi_4(x)\xi_2(x)^* & \xi_3(x)\xi_3(x)^* + \xi_4(x)\xi_4(x)^* \end{pmatrix}
\end{aligned}$$

and therefore $\xi_3 = 0$ and $\xi_4 = 0$. Similarly we get

$$\begin{aligned}
\xi\begin{pmatrix} 0 & 0 \\ 0 & x^*x \end{pmatrix} &= \begin{pmatrix} \xi_1(x)^* & 0 \\ \xi_2(x)^* & 0 \end{pmatrix} \begin{pmatrix} \xi_1(x) & \xi_2(x) \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \xi_1(x)^*\xi_1(x) & \xi_1(x)^*\xi_2(x) \\ \xi_2(x)^*\xi_1(x) & \xi_2(x)^*\xi_2(x) \end{pmatrix}
\end{aligned}$$

and thus $\xi_1 = 0$. Therefore we get a TRO-representation ξ_3 of T with $\mathbb{L}(\xi_2) = \xi$. Now put $\varphi := \rho^{-1} \circ \xi_3$. □

Proposition 3.1.2 and (3.1) yield a 1–1-connection between representations of the TRO T and of its linking algebra.

Note that even though every TRO-homomorphism $\varphi : T \to U$ between TROs T and U induces a $*$-homomorphism $\mathbb{L}(\varphi) : \mathbb{L}(T) \to \mathbb{L}(U)$ of their linking algebras, not every $*$-homomorphism $\phi : \mathbb{L}(T) \to \mathbb{L}(U)$ induces a TRO-homomorphism from T to U.

Definition 3.1.3. *A TRO-representation $\varphi : T \to B(H, K)$ is called **non-degenerate** if $\overline{\varphi(T)H} = K$ and $\overline{\varphi(T)^*K} = H$. (This is equivalent to: If $h \in H, k \in K$ with $\varphi(T)h = 0$ and $\varphi(T)^*k = 0$, then $h = 0$ and $k = 0$.)*

*Let $H_1 \subseteq H$ and $K_1 \subseteq K$ be closed subspaces. We call the pair (H_1, K_1) φ-**invariant** if $\varphi(T)H_1 \subseteq K_1$ and $\varphi(T)^*K_1 \subseteq H_1$. A representation φ is called **irreducible** if $(0,0)$ and (H, K) are the only φ-invariant pairs.*

One can check easily that if $\alpha : \mathfrak{A} \to B(H)$ is a $*$-representation of a C^*-algebra, then α is non-degenerate (irreducible) in the sense of C^*-theory if and only if it is non-degenerate (irreducible) as a TRO-representation.

Lemma 3.1.4. *Let $\varphi : T \to B(H,K)$ be a TRO-representation. Then the following assertions are equivalent:*

(a) *$\varphi : T \to B(H,K)$ is non-degenerate.*

(b) *$\mathcal{L}(\varphi) : \mathcal{L}(T) \to B(K)$ and $\mathcal{R}(\varphi) : \mathcal{R}(T) \to B(H)$ are non-degenerate.*

(c) *$\pi_\varphi : \mathbb{L}(T) \to B(K \oplus H)$ is non-degenerate.*

If $\varphi \neq 0$ is irreducible, then it is non-degenerate.

Proof. $(a) \Rightarrow (b)$: Assume $\varphi : T \to B(H,K)$ is non-degenerate. Take $h \in H$ with $\mathcal{R}(\varphi)(\mathcal{R}(T))h = 0$, then we get for all $x, y, z \in T$ that

$$0 = \varphi(x)\mathcal{R}(\varphi)(y^*z)h = \varphi(xy^*z)h.$$

By Lemma 2.3.7 we obtain for every $x \in T$ an element $y \in T$ with $yy^*y = x$. Thus $\varphi(T)h = 0$ and by assumption $h = 0$. One can similarly show that $\mathcal{L}(\varphi)$ is non-degenerate.

$(b) \Rightarrow (a)$: Let $h \in H$ such that $\varphi(T)h = 0$, then

$$\mathcal{R}(\varphi)(e^*f)h = \varphi(e)^*\varphi(f)h = 0$$

for all $e, f \in T$. It follows $h = 0$. The case that $k = 0$ whenever $\varphi(T)^*k = 0$ is proved similar.

$(c) \Rightarrow (b)$: Let π_φ be non-degenerate and $h \in H$ such that $\mathcal{L}(\varphi)(\mathcal{L}(T))h = 0$ holds. As above we find for every $x \in T$ an element $y \in T$ with $yy^*y = x$. Thus $\varphi(x)h = \mathcal{L}(\varphi(y))\varphi(y)h = 0$. It follows that

$$\pi_\varphi(\mathbb{L}(T))(0 \oplus h) = \begin{pmatrix} \varphi(T)h \\ \mathcal{L}(\varphi)(T)h \end{pmatrix} = 0.$$

Since π_φ is by assumption non-degenerate we get $h = 0$. It can be proved analogously that $\mathcal{R}(\varphi)$ is non-degenerate

$(b) \Rightarrow (c)$: If both $\mathcal{L}(\varphi)$ and $\mathcal{R}(\varphi)$ are non-degenerate, then $H = \mathcal{R}(\varphi)(\mathcal{R}(T))H$ and $K = \mathcal{L}(\varphi)(\mathcal{L}(T))K$. If we take w.l.o.g. $k \oplus h \in K \oplus H$ we can find $a \in \mathcal{R}(T), b \in \mathcal{L}(T), \tilde{h} \in H, \tilde{k} \in K$ with $k = \mathcal{R}(\varphi)(a)\tilde{k}$ and $h = \mathcal{L}(\varphi)(b)\tilde{h}$. It follows that

$$k \oplus h = \pi_\varphi \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} (\tilde{k} \oplus \tilde{h}) \in \pi_\varphi(\mathbb{L}(T))(K \oplus H).$$

Finally, assume that $\varphi \neq 0$ is irreducible and that $h \in H, k \in K$ with $\varphi(T)h = 0$ and $\varphi(T)^*k = 0$. We define H_1 and K_1 to be the closed linear subspaces spanned by h and k. The pairs $(H_1, 0)$ and $(0, K_1)$ are φ-invariant and therefore, since φ is irreducible, both are equal to $(0,0)$. □

Lemma 3.1.5. *Let $\varphi : T \to B(H,K)$ be a TRO-representation. Then the following are equivalent:*

(a) *φ is irreducible.*

(b) *$\mathcal{R}(\varphi)$ and $\mathcal{L}(\varphi)$ are irreducible.*

(c) *π_φ is irreducible.*

Proof. $(a) \Rightarrow (b)$: Let $\varphi \neq 0$ be irreducible and $K_1 \subseteq K$ be a subspace that is $\mathcal{L}(\varphi)$-invariant. If we define $H_1 := \overline{\varphi(T)^* K_1}$the pair (H_1, K_1) is φ-invariant and has to be equal to $(0,0)$ or (H,K). That the mapping $\mathcal{R}(\varphi)$ is irreducible can be shown similarly, under the same assumptions.

$(b) \Rightarrow (a)$: If $\mathcal{L}(\varphi) \neq 0$ and $\mathcal{R}(\varphi) \neq 0$ are irreducible and (H_1, K_1) is φ-invariant, then H_1 is $\mathcal{R}(\varphi)$-invariant and K_1 is $\mathcal{L}(\varphi)$-invariant, thus $(H_1, K_1) = (0,0)$ or $(H_1, K_1) = (H,K)$.

The equivalence $(b) \Leftrightarrow (c)$ is also easily shown. □

Definition 3.1.6. *Let T be a TRO and $\varphi_i : T \to B(H_i, K_i)$, $i \in I$, a family of TRO-representations. The* ***sum representation*** *$\varphi : T \to B(H,K)$ is the TRO-homomorphism with $H := \bigoplus H_i$, $K := \bigoplus K_i$ and $\varphi((h_i)_i) := (\varphi_i(h_i))_i$, for $(h_i)_i \in H$.*

Theorem 3.1.7. *Every non-degenerate representation of a finite-dimensional TRO is the direct sum of irreducible ones.*

Proof. Let T be a finite-dimensional TRO and $\varphi : T \to B(H,K)$ be a non-degenerate representation of T. The induced $*$-representation $\pi_\varphi : \mathbb{L}(T) \to B(K \oplus H)$ of the linking algebra of T splits by [Dav96], Theorem I.10.7 into a direct sum of irreducible $*$-representations $\pi_\varphi = \oplus \pi_i$. For every $*$-representation π_i there exits by Lemma 3.1.2 a TRO-representation $\varphi_i : T \to B(H,K)$ with $\pi_{\varphi_i} = \pi_i$, which is irreducible by Lemma 3.1.5. We get $\varphi = \oplus \varphi_i$. □

Lemma 3.1.8. *Let $I \subseteq T$ be a TRO-ideal in the TRO T and $\varphi : I \to B(H,K)$ a non-degenerate representation of I. There exists a unique extension $\widetilde{\varphi}$: $T \to B(H,K)$ of φ which is a TRO-representation of T. The representation $\widetilde{\varphi}$ is irreducible if and only if φ is irreducible.*

Proof. The induced representation $\pi_\varphi : \mathbb{L}(I) \to B(H \oplus K)$ of the linking algebra of I is by Lemma 3.1.4 non-degenerate and the irreducibility of π_φ is equivalent to the irreducibility of φ by Lemma 3.1.5. Using Theorem 2.2.12 (e) the C^*-algebra $\mathbb{L}(I)$ can be identified with a two-sided ideal inside $\mathbb{L}(T)$. With the aid of [Dav96], Lemma I.9.14 we obtain a unique $*$-representation $\widetilde{\pi}_\varphi : \mathbb{L}(T) \to B(K \oplus H)$ of $\mathbb{L}(T)$ extending π_φ which is irreducible if and only if π_φ is irreducible. By Lemma 3.1.2 we get a TRO-representation $\widetilde{\varphi} : T \to B(H,K)$ such that $\pi_{\widetilde{\varphi}} = \widetilde{\pi}_\varphi$. The TRO-representation

$\widetilde{\varphi}$ is irreducible if and only if $\pi_{\widetilde{\varphi}}$ is irreducible, again by Lemma 3.1.5. The uniqueness of $\widetilde{\varphi}$ follows from the uniqueness of $\widetilde{\pi}_\varphi$: If $\hat{\varphi}$ is an extension of φ with $\hat{\varphi} \neq \widetilde{\varphi}$, then $\pi_{\hat{\varphi}}$ is an extension of of π_φ with $\pi_{\hat{\varphi}} \neq \pi_{\widetilde{\varphi}} = \widetilde{\pi}_\varphi$. ☐

Definition 3.1.9. *Suppose T is a TRO and $\varphi_i : T \to B(H_i, K_i)$, $i = 1, 2$, are TRO-representations. The representations φ_1 and φ_2 are called **unitarily equivalent** if there exist unitary operators $U_1 : H_1 \to H_2$ and $U_2 : K_1 \to K_2$, such that $\varphi_1(x) = U_2^* \varphi_2(x) U_1$ holds for all $x \in T$.*

Proposition 3.1.10. *Assume that $T \subseteq B(H, K)$ is a ternary ring of operators and consider two non-degenerate TRO-representations $\varphi_i : T \to B(H_i, K_i)$, $i = 1, 2$, of T.*

(a) *If φ_1 and φ_2 are unitarily equivalent as TRO-representations, then $\mathcal{L}(\varphi_1)$ and $\mathcal{L}(\varphi_2)$, $\mathcal{R}(\varphi_1)$ and $\mathcal{R}(\varphi_2)$ as well as π_{φ_1} and π_{φ_2} are unitarily equivalent as $*$-representations.*

(b) *If π_{φ_1} and π_{φ_2} are unitarily equivalent $*$-representations, then φ_1 and φ_2 are unitarily equivalent TRO-representations.*

Proof. To prove (a) let $U_1 : H_1 \to H_2$ and $U_2 : K_1 \to K_2$ be unitary operators such that $\varphi_1(x) = U_2^* \varphi_2(x) U_1$ holds for all $x \in T$. If $x = x_1 x_2^* \in TT^*$, then

$$\begin{aligned} \mathcal{L}(\varphi_1)(x) &= \varphi_1(x_1)\varphi_1(x_2)^* \\ &= U_2^* \varphi_2(x_1) U_1 \left(U_2^* \varphi_2(x_2) U_1\right)^* \\ &= U_2^* \varphi_2(x_1) \varphi_2(x_2)^* U_2 \\ &= U_2^* \mathcal{L}(\varphi_2)(x) U_2. \end{aligned}$$

This extends to all of TT^* via continuity and linearity. A similar proof shows that $\mathcal{R}(\varphi_1)(x) = U_1^* \mathcal{R}(\varphi_2)(x) U_1$ for all $x \in \mathcal{R}(T)$. It is also easy to see that $U := \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix}$ is unitary and satisfies $\pi_{\varphi_1} = U^* \pi_{\varphi_2} U$.

To prove (b) let $U : K_1 \oplus H_1 \to K_2 \oplus H_2$ be a unitary operator such that $\pi_{\varphi_1}(x) = U^* \pi_{\varphi_2}(x) U$ holds for all $x \in \mathbb{L}(T)$. Let p be the projection from Theorem 2.2.12 (d) with $\varphi_2(T) \simeq p\mathbb{L}(\varphi_2(T))(1-p)$ (in this case p is just the projection of $K_2 \oplus H_2$ onto K_2 and $1 = \mathrm{id}_{K_2 \oplus H_2}$) and define $U_1 := pUp$ and $U_2 := (1-p)Up$, $U_3 = pU(1-p)$ and $U_4 = (1-p)U(1-p)$. We thus obtain a decomposition $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$. Since $UU^* = \mathrm{id}_{K_2 \oplus H_2}$, we know that

$$\begin{pmatrix} U_1 U_1^* + U_2 U_2^* & U_1 U_3^* + U_2 U_4^* \\ U_3 U_1^* + U_4 U_2^* & U_3 U_3^* + U_4 U_4^* \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{K_2} & 0 \\ 0 & \mathrm{id}_{H_2} \end{pmatrix}$$

and therefore

$$U_1 U_1^* = pUpU^*p = p \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \begin{pmatrix} U_1^* & 0 \\ U_2^* & 0 \end{pmatrix} p = p.$$

Similarly we can deduce from $U^*U = \mathrm{id}_{K_1 \oplus H_1}$ that $U_1^* U_1 = p$. In a similar way one can show that U_4 is a unitary operator. This implies $U_2 = 0$ and $U_3 = 0$ and thus $\varphi_1(x) = U_4^* \varphi_2(x) U_1$ for all $x \in T$. □

3.2 Finite-dimensional TROs

We use the representation theory developed in the last chapter to analyze the structure of TRO-homomorphisms between finite-dimensional TROs. A homomorphism of this type is, up to unitary equivalence, uniquely determined by a rectangular matrix with entries in $\mathbb{N}_0$.

Roger Smith showed that every injective finite-dimensional operator space is the direct sum of rectangular matrix algebras. Since these operator spaces are exactly the finite-dimensional TROs, we can reformulate his result to:

Theorem 3.2.1 ([Smi00])**.** *If T is a finite-dimensional TRO, then T is TRO-isomorphic to a direct sum of rectangular matrix-algebras:*

$$T \simeq \bigoplus_{i=1}^{k} \mathbb{M}_{n_i, m_i}.$$

Definition 3.2.2. *Let $\varphi : \mathbb{M}_{n,m} \to \mathbb{M}_{k,l}$ be a non-degenerate TRO-representation. We know by Theorem 3.1.7 that there exists $\alpha \in \mathbb{N}_0$ such that φ is the direct sum of α irreducible non-zero representations. The non-negative number $M(\varphi) := \alpha$ is called the* ***multiplicity*** *of φ.*

The multiplicity of $*$-homomorphisms is defined analogously (cf. [Tak02], Lemma 11.5). Since the irreducibility of a TRO-homomorphism φ is equivalent to the irreducibility of $\mathbb{L}(\varphi)$ by Lemma 3.1.5 we have that

$$M(\varphi) = M(\pi_\varphi).$$

This gives us the opportunity to compute the multiplicity of φ. Let $e \in \mathbb{M}_{n,m}$ be a non-zero partial isometry, i.e. an element with $ee^*e = e$ (the class of these elements, the so-called tripotents, play a tremendous role in this work), then

$$M(\varphi) = \mathrm{tr}\left(\varphi(e)\varphi(e)^*\right) / \mathrm{tr}(ee^*),$$

where tr is the usual trace mapping.

We will prove two Lemmata on the structure of TRO-representations of finite-dimensional TROs to give a full description of them in Proposition 3.2.5.

Lemma 3.2.3. *Let $\varphi : \mathbb{M}_{n,m} \to \mathbb{M}_{k,l}$ be a non-degenerate TRO-representation with multiplicity α. Then φ is unitarily equivalent to* $\mathrm{id}^\alpha := \oplus_{k=1}^{\alpha} \mathrm{id}$, *where* id *is the identity representation of* $\mathbb{M}_{n,m}$.

Proof. Let $\varphi = \oplus_{i=1}^{\alpha}\varphi_i$ be the direct sum decomposition from Theorem 3.1.7 into irreducible TRO-representations. The induced representation π_φ of $\mathbb{L}(\mathbb{M}_{n,m}) \simeq \mathbb{M}_{n+m}$ is by [Dav96], Corollary III.1.2 unitarily equivalent to $\oplus_{k=1}^{\alpha}\pi_{\mathrm{id}}$. This already implies the statement by Lemma 3.1.2 and Lemma 3.1.10. □

Lemma 3.2.4. *Let $\varphi : T := \bigoplus_{i=1}^{p} \mathbb{M}_{n_i,m_i} \to \mathbb{M}_{k,l}$ be a non-degenerate TRO-representation. There are $\alpha_1, \dots, \alpha_p \in \mathbb{N}_0$ such that φ is unitarily equivalent to $\mathrm{id}_1^{\alpha_1} \oplus \ldots \oplus \mathrm{id}_p^{\alpha_p}$, with id_i denoting the identity representation of $\mathbb{M}_{n_i,m_i}$ for $i = 1, \dots, p$.*

Proof. For $i = 1, \dots, p$ let φ_i be the representation φ restricted to the TRO-ideal $\mathbb{M}_{n_i,m_i} \subseteq T$. Every φ_i splits by Theorem 3.1.7 into a direct sum of non-zero, irreducible representations $\varphi_i = \oplus_{j=1}^{a_i}\varphi_{i,j} : \mathbb{M}_{n_i,m_i} \to \mathbb{M}_{k,l}$. The representations $\varphi_{i,j}$ are non-degenerate by Lemma 3.1.4, so we can extend every $\varphi_{i,j}$ by Lemma 3.1.8 to a unique irreducible representation $\widetilde{\varphi}_{i,j} : T \to \mathbb{M}_{k,l}$. We have $\varphi = \oplus_{i=1}^{p} \oplus_{j=1}^{a_i} \widetilde{\varphi}_{i,j}$. Let $\alpha_i := M(\varphi_i)$, then Lemma 3.2.3 implies that φ is unitarily equivalent to $\oplus_{i=1}^{p} \mathrm{id}^{\alpha_i}$, since φ_i is unitarily equivalent to id^{α_i} for all $1 \le i \le p$. □

Proposition 3.2.5. *Let $\varphi : T := \bigoplus_{i=1}^{p} \mathbb{M}_{n_i,m_i} \to \bigoplus_{j=1}^{q} \mathbb{M}_{k_j,l_j}$ be a non-degenerate TRO-representation. The representation φ is, up to unitary equivalence, uniquely determined by a $q \times p$-matrix $(\alpha_{i,j})$ with entries in $\mathbb{N}_0$. Furthermore we have*

$$\sum_{j=1}^{p} \alpha_{i,j} n_j \le k_i \quad \text{and} \quad \sum_{j=1}^{p} \alpha_{i,j} m_j \le l_i \tag{3.2}$$

for all $i \in \{1, \dots, q\}$.

Proof. Let $\varphi^j : T \to M_{k_j,l_j}$ be the restriction of φ to M_{k_j,l_j} for $j \in \{1, \dots, q\}$, then $\varphi = \varphi^1 \oplus \ldots \oplus \varphi^q$ and every φ^j is a representation of T. Now Lemma 3.2.4 implies, that $\alpha_{1,j}, \dots, \alpha_{q,j} \in \mathbb{N}_0$, which determine φ^j uniquely up to unitary equivalence. A dimension count gives (3.2). □

3.3 Functors

In this section we show that $\mathcal{L}, \mathcal{R}$ and $\mathbb{L}$ are covariant functors from the category of TROs with TRO-homomorphisms to the category of C^*-algebras with $*$-homomorphisms. We determine explicitly the properties of these functors. All three functors are exact, homotopy invariant, stable and continuous and therefore especially additive and split exact which makes them all excellent candidates to define a functor K_0 from the category of TROs to the category of Abelian groups. We choose the functor $\mathcal{L}$, but this does not affect the general theory since all resulting K_0-groups are isomorphic.

The next proposition states that $\mathcal{L}$, $\mathcal{R}$ and $\mathbb{L}$ are covariant functors. The proofs are obvious or known, thus omitted.

Proposition 3.3.1. *Let T, U and V be TROs.*

(a) $\mathcal{L}(\mathrm{id}_T) = \mathrm{id}_{\mathcal{L}(T)}$ $\mathcal{R}(\mathrm{id}_T) = \mathrm{id}_{\mathcal{R}(T)}$, *and* $\mathbb{L}(\mathrm{id}_T) = \mathrm{id}_{\mathbb{L}(T)}$.

(b) *If $\varphi : T \to U$ and $\psi : U \to V$ are TRO-homomorphisms we have*

$$\mathcal{L}(\psi \circ \varphi) = \mathcal{L}(\psi) \circ \mathcal{L}(\varphi),$$
$$\mathcal{R}(\psi \circ \varphi) = \mathcal{R}(\psi) \circ \mathcal{R}(\varphi) \text{ and}$$
$$\mathbb{L}(\psi \circ \varphi) = \mathbb{L}(\psi) \circ \mathbb{L}(\varphi).$$

(c) *Let 0_{TRO} and 0_{C^*} denote the 0-objects in the categories of TROs and C^*-algebras respectively, then*

$$\mathcal{L}(0_{TRO}) = \mathcal{R}(0_{TRO}) = \mathbb{L}(0_{TRO}) = 0_{C^*}.$$

(d) *For the 0-mapping $0_{T,U} : T \to U$ we have*

$$\mathcal{L}(0_{T,U}) = 0_{\mathcal{L}(T),\mathcal{L}(U)},$$
$$\mathcal{R}(0_{T,U}) = 0_{\mathcal{R}(T),\mathcal{R}(U)} \text{ and}$$
$$\mathbb{L}(0_{T,U}) = 0_{\mathbb{L}(T),\mathbb{L}(U)}.$$

Definition 3.3.2. *Let T and U be TROs. Two TRO-homomorphisms $\varphi, \psi : T \to U$ are called **homotopic** (denoted by $\varphi \sim_h \psi$), when there exists a path of TRO-homomorphisms $\gamma_t : T \to W$, $t \in [0,1]$, such that $t \mapsto \gamma_t(x)$ is a continuous map from $[0,1]$ to U for all $x \in T$, satisfying $\gamma_0 = \varphi$ and $\gamma_1 = \psi$. The TROs T and U are called **homotopy equivalent** if there are TRO-homomorphisms $\varphi : T \to U$ and $\psi : U \to T$ with $\varphi \circ \psi \sim_h \mathrm{id}_U$ and $\psi \circ \varphi \sim_h \mathrm{id}_T$.*

Proposition 3.3.3. *Let T and U be TROs.*

(a) *If $\varphi, \psi : T \to U$ are homotopic TRO-homomorphisms, then $\mathcal{L}(\varphi)$ and $\mathcal{L}(\psi)$, $\mathcal{R}(\varphi)$ and $\mathcal{R}(\psi)$ as well as $\mathbb{L}(\varphi)$ and $\mathbb{L}(\psi)$ are homotopic $*$-homomorphisms.*

(b) *If T and U are homotopic TROs, then $\mathcal{L}(T)$ and $\mathcal{L}(U)$, $\mathcal{R}(T)$ and $\mathcal{R}(U)$ as well as $\mathbb{L}(T)$ and $\mathbb{L}(U)$ are homotopic C^*-algebras.*

Proof. To prove (a) assume that the TRO-homomorphisms φ and ψ are homotopic, so there exists a pointwise continuous path of TRO-homomorphisms $t \mapsto \gamma_t$, connecting φ and ψ. If we consider the $*$-homomorphisms

$\mathcal{L}(\varphi), \mathcal{L}(\psi)$ and $\mathcal{L}(\gamma_t) : T \to U$, $t \in [0,1]$, then $\mathcal{L}(\gamma_0) = \mathcal{L}(\varphi)$ and $\mathcal{L}(\gamma_1) = \mathcal{L}(\psi)$. Let (t_n) be a sequence in $[0,1]$ converging towards $t_0 \in [0,1]$ and assume that $a = x_1 x_2^*$ with $x_1, x_2 \in T$, then

$$\lim_{n\to\infty} \mathcal{L}(\gamma_{t_n})(a) = \lim_{n\to\infty} \gamma_{t_n}(x_1)\gamma_{t_n}(x_2)^* = \mathcal{L}(\gamma_{t_0})(a).$$

By linearity the same argument shows that $\lim_{n\to\infty} \mathcal{L}(\gamma_{t_n})(a) = \mathcal{L}(\gamma_{t_0})(a)$ for all $a \in \mathrm{lin}\{xy^* : x, y \in T\} =: A$. Now let $a \in \mathcal{L}(T)$ and (x_n) be a sequence in A converging towards a. Then

$$\begin{aligned}\lim_{n\to\infty} \mathcal{L}(\gamma_{t_n})(a) &= \lim_{n\to\infty}\lim_{m\to\infty} \mathcal{L}(\gamma_{t_n})(x_m)\\ &= \lim_{m\to\infty}\lim_{n\to\infty} \mathcal{L}(\gamma_{t_n})(x_m)\\ &= \mathcal{L}(\gamma_{t_0})(a),\end{aligned}$$

since $[0,1]$ is compact. The other statements of (a) can be proved similarly.

To prove (b) let $\varphi : T \to U$ and $\psi : U \to T$ such that $\psi \circ \varphi \sim_h \mathrm{id}_T$ and $\varphi \circ \psi \sim_h \mathrm{id}_U$. By (a) and functoriality we have that $\mathcal{L}(\psi) \circ \mathcal{L}(\varphi) = \mathcal{L}(\psi \circ \varphi) \sim_h \mathcal{L}(\mathrm{id}_T) = \mathrm{id}_{\mathcal{L}(T)}$ and $\mathcal{L}(\varphi) \circ \mathcal{L}(\psi) \sim_h \mathrm{id}_{\mathcal{L}(U)}$. □

Corollary 3.3.4. *Two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are homotopic as C^*-algebras if and only if they are homotopic as TROs.*

Proof. If $\mathfrak{A}$ and $\mathfrak{B}$ are homotopic as C^*-algebras they are homotopic as TROs, since every $*$-homomorphism is a TRO-homomorphism.

Using Proposition 3.3.3 (b) we get that $\mathcal{L}(\mathfrak{A}) = \mathfrak{A}$ and $\mathcal{L}(\mathfrak{B}) = \mathfrak{B}$ are homotopic as C^*-algebras. □

Remark 3.3.5 (Quotients of TROs by ideals)**.** We briefly explain how to define a norm and a ternary product on the quotient space T/I of a TRO T and a TRO-ideal $I \subseteq T$. This is just a reformulation of the Rieffel quotient equivalence (cf. [BLM04], 8.2.25). We first note that $\mathbb{L}(I)$ can be identified with a closed two-sided ideal in $\mathbb{L}(T)$ by Theorem 2.2.12. Let $\pi : \mathbb{L}(T) \to \mathbb{L}(T)/\mathbb{L}(I)$ be the quotient map. Then the canonical four corners $\begin{pmatrix} \mathcal{L}(T) & T \\ T^* & \mathcal{R}(T) \end{pmatrix}$ induce by [BLM04], 2.6.15 four corners of $\mathbb{L}(T)/\mathbb{L}(I)$, say $\begin{pmatrix} A & U \\ V & D \end{pmatrix}$. The mapping π is corner preserving and it is straightforward to check that U is a TRO with $\mathcal{L}(U) \simeq D$, $\mathcal{R}(U) \simeq A$ and $U^* \simeq V$. Write $\pi = (\pi_{i,j})$. The corner map $\pi_{1,2} : T \to W$ is a complete quotient map with kernel I. Thus the TRO U is completely isometrically isomorphic to T/I where T/I carries the matrix norms $\|(x_{i,j} + I)\|_n = \inf\{\|(x_{i,j} + y_{i,j})\|_n : y_{i,j} \in I\}$ for all $n \in \mathbb{N}$ and $(x_{i,j}) \in \mathbb{M}_n(T)$. One can also show that

$$\mathcal{L}(T/I) \simeq \mathcal{L}(U) \simeq \mathcal{L}(T)/\mathcal{L}(I) \text{ and } \mathcal{R}(T/I) \simeq \mathcal{R}(U) \simeq \mathcal{R}(T)/\mathcal{R}(I) \qquad (3.3)$$

as C^*-algebras. One can define a TRO product on T/I by

$$(x+I)(y+I)^*(z+I) = xy^*z + I$$

for all $x, y, z \in T$. The canonical TRO-structure on T equals the quotient operator space structure.

Proposition 3.3.6. *Let T be a TRO and $I \subseteq T$ a TRO-ideal in T. The exact sequence*

$$0 \longrightarrow I \xrightarrow{\iota} T \xrightarrow{\pi} T/I \longrightarrow 0$$

induces three exact sequences of C^*-algebras

$$0 \longrightarrow \mathcal{L}(I) \xrightarrow{\mathcal{L}(\iota)} \mathcal{L}(T) \xrightarrow{\mathcal{L}(\pi)} \mathcal{L}(T/I) \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{R}(I) \xrightarrow{\mathcal{R}(\iota)} \mathcal{R}(T) \xrightarrow{\mathcal{R}(\pi)} \mathcal{R}(T/I) \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{L}(I) \xrightarrow{\mathbb{L}(\iota)} \mathbb{L}(T) \xrightarrow{\mathbb{L}(\pi)} \mathbb{L}(T/I) \longrightarrow 0.$$

Proof. We give the proof only for the first sequence. Knowing that $\mathcal{L}(\iota)$ is injective and that $\mathcal{L}(\pi)$ is surjective, we have to show exactness at $\mathcal{L}(T)$. We first identify $\mathcal{L}(I)$ with a closed two-sided ideal in $\mathcal{L}(T)$. Since

$$0 \longrightarrow \mathcal{L}(I) \xrightarrow{\mathcal{L}(\iota)} \mathcal{L}(T) \xrightarrow{\widehat{\pi}} \mathcal{L}(T)/\mathcal{L}(I) \longrightarrow 0$$

is obviously an exact sequence where $\widehat{\pi}$ is the quotient homomorphism, the exactness of our sequence follows from (3.3). ☐

Corollary 3.3.7. *Every split exact sequence of TROs*

$$0 \longrightarrow I \longrightarrow T \underset{\lambda}{\overset{\psi}{\rightleftarrows}} U \longrightarrow 0$$

induces split exact sequences of C^*-algebras

$$0 \longrightarrow \mathcal{L}(I) \longrightarrow \mathcal{L}(T) \underset{\mathcal{L}(\lambda)}{\overset{\mathcal{L}(\psi)}{\rightleftarrows}} \mathcal{L}(U) \longrightarrow 0\ ,$$

$$0 \longrightarrow \mathcal{R}(I) \longrightarrow \mathcal{R}(T) \underset{\mathcal{R}(\lambda)}{\overset{\mathcal{R}(\psi)}{\rightleftarrows}} \mathcal{R}(U) \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{L}(I) \longrightarrow \mathbb{L}(T) \underset{\mathbb{L}(\lambda)}{\overset{\mathbb{L}(\psi)}{\rightleftarrows}} \mathbb{L}(U) \longrightarrow 0.$$

Proof. This follows directly from exactness and functoriality of $\mathcal{L}$, $\mathcal{R}$ and $\mathbb{L}$, since for example $\mathbb{L}(\psi) \circ \mathbb{L}(\lambda) = \mathbb{L}(\psi \circ \lambda) = \mathbb{L}(\mathrm{id}_T) = \mathrm{id}_{\mathbb{L}(T)}$. □

Proposition 3.3.8. *For all TROs T and U we have*

$$\mathcal{L}(T \oplus U) = \mathcal{L}(T) \oplus \mathcal{L}(U), \quad \mathcal{R}(T \oplus U) = \mathcal{R}(T) \oplus \mathcal{R}(U) \text{ and}$$

$$\mathbb{L}(T \oplus U) = \mathbb{L}(T) \oplus \mathbb{L}(U).$$

Proof. This follows from Proposition 3.3.6. □

Let $\mathcal{C}$ be a category. We usually write

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \ldots$$

for an inductive sequence in $\mathcal{C}$, where $(A_n)_{n \in \mathbb{N}}$ is a sequence of objects and $\varphi_n : A_n \to A_{n+1}$ is a sequence of morphisms in $\mathcal{C}$ for all $n \in \mathbb{N}$. For $m > n$ we also consider the composed morphisms

$$\varphi_{m,n} = \varphi_{m-1} \circ \varphi_{m-2} \circ \ldots \circ \varphi_n : A_n \to A_m.$$

For $m < n$ we define $\varphi_{m,n} = 0$ (we only consider categories with 0 object) and for $m = n$ we let $\varphi_{n,n} = \mathrm{id}_{A_n}$ and call the set of all of the above mappings the connecting morphisms of the inductive sequence $((A_n), (\varphi_n))$. A system $(A_\infty, (\mu_n))$ is called inductive limit in $\mathcal{C}$ if A_∞ is an object in $\mathcal{C}$ and $\mu_n : A_n \to A_\infty$ are morphisms in $\mathcal{C}$ for all $n \geq 1$ and:

(a) For all $n \in \mathbb{N}$ the diagram

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi_n} & A_{n+1} \\ & {\scriptstyle \mu_n} \searrow \quad \swarrow {\scriptstyle \mu_{n+1}} & \\ & A_\infty & \end{array} \tag{3.4}$$

commutes.

(b) If $(B, (\lambda_n))$ is another system with $\lambda_n = \lambda_{n+1} \circ \varphi_n$ for all $n \in \mathbb{N}$, then there exists a unique morphism λ making

$$\begin{array}{ccc} & A_n & \\ & {\scriptstyle \mu_n} \swarrow \quad \searrow {\scriptstyle \lambda_n} & \\ A_\infty & \xrightarrow{\lambda} & B \end{array} \tag{3.5}$$

commute for all $n \in \mathbb{N}$.

One easily deduces from (b) above, that an inductive limit is unique up to isomorphism.

To show that inductive limits exist in the category of TROs and that the linking algebra of an inductive limit of TROs equals the inductive limit of the corresponding linking algebras, we first have to prove the following special case.

Lemma 3.3.9. *Let T be a TRO and (T_n) an increasing sequence of subTROs of T. Denote by $\varphi_n : T_n \to T_{n+1}$ the inclusion mapping and put*

$$T_\infty := \overline{\bigcup_{n=1}^{\infty} T_n}.$$

Then $(T_\infty, (\iota_n))$ is the inductive limit of $((T_n), (\varphi_n))$, where ι_n denotes the inclusion mapping of T_n into T_∞ for all $n \in \mathbb{N}$.

Proof. For every $n \in \mathbb{N}$, we obviously have the following commutative diagram

$$\begin{array}{ccc} T_n & \xrightarrow{\varphi_n} & T_{n+1} \\ & {\scriptstyle \iota_n}\searrow \quad \swarrow{\scriptstyle \iota_{n+1}} & \\ & T_\infty & \end{array}$$

and if $(U, (\lambda_n))$ is another system consisting of a TRO U and TRO-homomorphisms $\lambda_n : T_n \to U$ such that $\lambda_n = \lambda_{n+1} \circ \varphi_n$ for all $n \in \mathbb{N}$, we can define a map $\lambda : T_\infty \to U$ in the following way: If $x \in T_\infty$ we can find a sequence (x_m) in $\bigcup_{n=1}^{\infty} T_n$ converging to x. Thus we can find for every $m \in \mathbb{N}$ an index $n_m \in \mathbb{N}$ such that $x_m \in T_{n_m}$. Now put $\lambda(x) := \lim_{m\to\infty} \lambda_{n_m}(x_m)$. This is a well-defined TRO-homomorphism and the only possible choice. □

Next we show the existence of inductive limits in the category of TROs.

Proposition 3.3.10. *Let $((T_n), (\varphi_n))$ be an inductive system in the category of TROs, then $((\mathcal{L}(T_n)), (\mathcal{L}(\varphi_n)))$, $((\mathcal{R}(T_n)), (\mathcal{R}(\varphi_n)))$ and $((\mathbb{L}(T_n)), (\mathbb{L}(\varphi_n)))$ are inductive systems of C^*-algebras, and there exists an inductive limit $(T_\infty, (\mu_n))$ of $((T_n), (\varphi_n))$ in the category of ternary rings of operators.*

Proof. That the induced sequences are inductive sequences of C^*-algebras is straightforward. It is well known (see for example [RLL00], 6.2.4), that inductive limits exist in the category of C^*-algebras. Let $(\mathbb{L}_\infty, (\lambda_n))$ be the inductive limit of the sequence of linking algebras then by [RLL00], 6.2.4 (i),

$$\mathbb{L}_\infty = \overline{\bigcup_{n=1}^{\infty} \lambda_n\left(\mathbb{L}(T_n)\right)}.$$

Let $\iota_n : T_n \to \mathbb{L}(T_n)$ be the canonical corner embedding and

$$T_\infty := \overline{\bigcup_{n=1}^{\infty} \lambda_n\left(\iota_n(T_n)\right)} \subseteq \mathbb{L}_\infty.$$

Define $\mu_n := \lambda_n \circ \iota_n : T_n \to T_\infty$, then

$$\begin{aligned}\mu_{n+1} \circ \varphi_n(x) &= \lambda_{n+1} \circ \iota_{n+1} \circ \varphi_n(x) \\ &= \lambda_{n+1} \circ \mathbb{L}(\varphi_n) \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \\ &= \lambda_n \circ \iota_n(x) \\ &= \mu_n(x)\end{aligned}$$

for all $x \in T_n$, $n \in \mathbb{N}$. All μ_n are TRO-homomorphisms and T_∞ is a subTRO of $\mathbb{L}_\infty$.

To prove (3.5) let $(U, (\beta_n))$ be another system satisfying $\beta_{n+1} \circ \varphi_n = \beta_n$, where $\beta_n : T_n \to U$ is a TRO-homomorphism for all $n \in \mathbb{N}$. Since $(\mathbb{L}_\infty, (\lambda_n))$ is the inductive limit of $(\mathbb{L}(T_n), (\mathbb{L}(\varphi_n)))$ and $\mathbb{L}(\beta_{n+1}) \circ \mathbb{L}(\varphi_n) = \mathbb{L}(\beta_{n+1} \circ \varphi_n) = \mathbb{L}(\beta_n)$, there exists one and only one $*$-homomorphism λ making the diagram

$$\begin{array}{ccc} & \mathbb{L}(T_n) & \\ {}^{\lambda_n}\swarrow & & \searrow^{\mathbb{L}(\beta_n)} \\ \mathbb{L}_\infty & \xrightarrow{\quad\lambda\quad} & \mathbb{L}(U) \end{array}$$

commutative. The restriction of λ to T_∞ gives the desired TRO-homomorphism from T_∞ to U. □

An immediate consequence of the proof of Proposition 3.3.10 is the following corollary.

Corollary 3.3.11. *If $(T, (\mu_n))$ is the inductive limit of the inductive sequence of TROs $((T_n), (\varphi_n))$, then*

$$T = \overline{\bigcup_{n=1}^{\infty} \mu_n(T_n)}.$$

To prove that our functors are continuous we need the following lemma.

Lemma 3.3.12. *Let $((T_n), (\varphi_n))$ be an inductive sequence of TROs with inductive limit $(T_\infty, (\mu_n))$, then*

$$\ker(\mu_n) = \{x \in T_n : \lim_{m\to\infty} \|\varphi_{m,n}(x)\| = 0\}$$

for all $n \in \mathbb{N}$.

Proof. Let $(\mathbb{L}_\infty, (\lambda_n))$ be the inductive limit of the sequence of linking algebras $(\mathbb{L}(T_n), (\varphi_n))$. We know from the proof of Proposition 3.3.10 that $\mathbb{L}_\infty = \overline{\bigcup_{n=1}^{\infty} \lambda_n(\mathbb{L}(T_n))}$,

$$T_\infty = \overline{\bigcup_{n=1}^{\infty} \lambda_n(\iota_{T_n}(T_n))} \subseteq \mathcal{L}_\infty$$

and $\mu_n = \lambda_n \circ \iota_{T_n}$ for all $n \in \mathbb{N}$. Since $(\mathbb{L}_\infty, (\lambda_n))$ is the C^*-direct limit we know from [RLL00], Proposition 6.2.4 that

$$\|\lambda_n(x)\| = \lim_{m\to\infty} \|\mathbb{L}(\varphi_{m,n})(x)\|$$

for all $n \in \mathbb{N}$. Especially for all $y \in T_n$ we get

$$\begin{aligned}\|\mu_n(y)\| &= \|\lambda_n(\iota_{T_n}(y))\| \\ &= \lim_{m\to\infty} \|\mathbb{L}(\varphi_{m,n})(\iota_{T_n}(y))\| \\ &= \lim_{m\to\infty} \left\| \begin{pmatrix} 0 & \varphi_{m,n}(y) \\ 0 & 0 \end{pmatrix} \right\| \\ &= \lim_{m\to\infty} \|\varphi_{m,n}(y)\| .\end{aligned}$$

□

We can now show that the functors $\mathcal{L}$, $\mathcal{R}$ and $\mathbb{L}$ are continuous.

Proposition 3.3.13. *If $((T_n), (\varphi_n))$ is an inductive sequence of TROs with inductive limit $(T_\infty, (\mu_n))$, then*

$$\lim_{n\to\infty} \mathcal{L}(T_n) = \mathcal{L}(T_\infty), \quad \lim_{n\to\infty} \mathcal{R}(T_n) = \mathcal{R}(T_\infty) \ \text{and} \ \lim_{n\to\infty} \mathbb{L}(T_n) = \mathbb{L}(T_\infty).$$

Proof. First recall from the proof of Proposition 3.3.10 that the inductive limit of the inductive sequence $((\mathbb{L}(T_n), (\mathbb{L}(\varphi_n)))$ is $\mathbb{L}_\infty = \overline{\bigcup_{n=1}^\infty \lambda_n(\mathbb{L}(T_n))}$ and that $T_\infty = \overline{\bigcup_{n=1}^\infty \mu_n(T_n)}$, with $\mu_n := \lambda_n \circ \iota_{T_n} : T_n \to T_\infty$. If we put

$$\mathcal{L}_\infty := \overline{\bigcup_{n=1}^\infty \lambda_n(\iota_{\mathcal{L}(T_n)}(\mathcal{L}(T_n))} \subseteq \mathbb{L}_\infty$$

we see that this is the inductive limit of the sequence $((\mathcal{L}(T_n)), \mathcal{L}(\varphi_n))$ and

$$\begin{aligned}\mathcal{L}(T_\infty) &= \left(\overline{\bigcup_{n=1}^\infty \lambda_n(\iota_{T_n}(T_n))}\right)\left(\overline{\bigcup_{n=1}^\infty \lambda_n(\iota_{T_n}(T_n))}\right)^* \\ &= \overline{\bigcup_{n=1}^\infty (\lambda_n(\iota_{T_n}(T_n)))\,(\lambda_n(\iota_{T_n}(T_n)))^*} \\ &= \overline{\bigcup_{n=1}^\infty \lambda_n(\iota_{\mathcal{L}(T_n)}(\mathcal{L}(T_n))} \\ &= \mathcal{L}_\infty,\end{aligned}$$

which shows that $\lim_{n\to\infty} \mathcal{L}(T_n) = \mathcal{L}(T_\infty)$. The homomorphisms $\eta_n : \mathcal{L}(T_n) \to \mathcal{L}(T_\infty)$ are given by $\eta_n := \mathcal{L}(\mu_n) = \mathcal{L}(\lambda_n \circ \iota_{T_n}) = \lambda_n \circ \iota_{\mathcal{L}(T_n)}$ for all $n \in \mathbb{N}$. A similar proof shows that $(\mathcal{R}(T_\infty), (\mathcal{R}(\mu_n)))$ is the inductive limit of $((\mathcal{R}(T_n)), (\mathcal{R}(\varphi_n)))$.

Finally we show that $\mathbb{L}_\infty = \lim_{n\to\infty} \mathbb{L}(T_n) = \mathbb{L}(T_\infty)$ and that $\lambda_n = \mathbb{L}(\mu_n)$ for all $n \in \mathbb{N}$. Since $(T_\infty, (\mu_n))$ is the inductive limit of $((T_n), (\varphi_n))$ we notice that $\mu_{n+1} \circ \varphi_n = \mu_n$ and thus

$$\mathbb{L}(\mu_{n+1}) \circ \mathbb{L}(\varphi_n) = \mathbb{L}(\mu_n)$$

for all $n \in \mathbb{N}$.

We get by the universal property of the inductive limit a unique $*$-homomorphism $\lambda : \mathbb{L}_\infty \to \mathbb{L}(T_\infty)$ and for all $n \in \mathbb{N}$ a commutative diagram

$$\begin{array}{ccc} & \mathbb{L}(T_n) & \\ {}^{\lambda_n}\swarrow & & \searrow^{\mathbb{L}(\mu_n)} \\ \mathbb{L}_\infty & \xrightarrow{\quad\lambda\quad} & \mathbb{L}(T_\infty) \end{array}$$

It is well known for inductive limits of C^*-algebras (cf. [RLL00], Proposition 6.2.4), that

$$\lambda \text{ is surjective } \iff \mathbb{L}(T_\infty) = \overline{\bigcup_{n=1}^{\infty} \mathbb{L}(\mu_n)\mathbb{L}(T_n)},$$

which is the case, and that

$$\lambda \text{ is injective } \iff \ker(\mathbb{L}(\mu_n)) \subseteq \ker(\lambda_n)$$

for all $n \in \mathbb{N}$. It follows from [RLL00], Proposition 6.2.4 and the above that

$$\ker(\lambda_n) = \{x \in \mathbb{L}(T_n) : \lim_{m\to\infty} \|\mathbb{L}(\varphi_{m,n})(x)\| = 0\},$$
$$\ker(\mathcal{R}(\varphi_n)) = \{x \in \mathcal{R}(T_n) : \lim_{m\to\infty} \|\mathcal{R}(\varphi_{m,n})(x)\| = 0\}$$

and

$$\ker(\mathcal{L}(\varphi_n)) = \{x \in \mathcal{L}(T_n) : \lim_{m\to\infty} \|\mathcal{L}(\varphi_{m,n})(x)\| = 0\}$$

for all $n \in \mathbb{N}$. Since $(T_\infty, (\mu_n))$ is the inductive limit of $((T_n), (\varphi_n))$ we get with Lemma 3.3.12 that

$$\ker(\mu_n) = \{y \in T_n : \lim_{m\to\infty} \|\varphi_{m,n}(y)\| = 0\},$$

for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \ker(\mathbb{L}(\mu_n)) \subseteq \mathbb{L}(T_n)$, then

$$\lim_{m\to\infty} \|\varphi_{m,n}(x_2)\| = 0$$

and by C^*-theory

$$\lim_{m\to\infty} \|\mathcal{L}(\varphi_{m,n})(x_4)\| = 0 = \lim_{m\to\infty} \|\mathcal{R}(\varphi_{m,n})(x_1)\|.$$

Thus we get

$$\begin{aligned}\lim_{m\to\infty}\|\mathbb{L}(\varphi_{m,n})(x)\| &= \lim_{m\to\infty}\left\|\begin{pmatrix}\mathcal{R}(\varphi_{m,n})(x_1) & \varphi_{m,n}(x_2)\\ \varphi^*_{m,n}(x_3) & \mathcal{L}(\varphi_{m,n})(x_4)\end{pmatrix}\right\|\\ &\leq \lim_{m\to\infty}(\|\mathcal{R}(\varphi_{m,n})(x_1)\| + \|\varphi_{m,n}(x_2)\|\\ &+ \|\varphi^*_{m,n}(x_3)\| + \|\mathcal{L}(\varphi_{m,n})(x_4)\|)\\ &= 0.\end{aligned}$$

Therefore we can conclude that $x \in \ker \lambda_n$ and that λ is an isomorphism. □

Proposition 3.3.14. *If T is a TRO also $M_n(T)$ is a TRO and*

$$\begin{aligned}&\mathcal{L}(M_n(T)) \simeq M_n(\mathcal{L}(T)),\\ &\mathcal{R}(M_n(T)) \simeq M_n(\mathcal{R}(T)) \text{ and}\\ &\mathbb{L}(M_n(T)) \simeq M_n(\mathbb{L}(T))\end{aligned}$$

are isomorphic as C^*-algebras.

Proof. First we prove that $\mathcal{L}(M_n(T)) \simeq M_n(\mathcal{L}(T))$. Since $M_n(T)$ is a TRO, we know by (2.3) that $\mathcal{L}(M_n(T)) \simeq M_n(T)M_n(T)^*$ and we conclude that

$$\begin{aligned}\mathcal{L}(M_n(T)) &= \overline{\text{lin}}\,\{AB^* : A, B \in M_n(T)\}\\ &= M_n\left(\overline{\text{lin}}\{xy^* : x, y \in T\}\right)\\ &= M_n(TT^*)\\ &\simeq M_n(\mathcal{L}(T)).\end{aligned}$$

The obvious analogue gives us $M_n(\mathcal{R}(T)) \simeq M_n(T^*T) \simeq M_n(T)^*M_n(T) \simeq \mathcal{R}(M_n(T))$.

Finally we prove that $\mathbb{L}(M_n(T)) \simeq M_n(\mathbb{L}(T))$. With the above isomorphisms we get

$$\begin{aligned}M_n(\mathbb{L}(T)) &= \begin{pmatrix}\mathbb{L}(T) & \cdots & \mathbb{L}(T)\\ \vdots & \ddots & \vdots\\ \mathbb{L}(T) & \cdots & \mathbb{L}(T)\end{pmatrix}\\ &= \begin{pmatrix}\mathcal{R}(T) & T & \mathcal{R}(T) & \cdots & \mathcal{R}(T) & T\\ T^* & \mathcal{L}(T) & T^* & \cdots & T^* & \mathcal{L}(T)\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \mathcal{R}(T) & T & \mathcal{R}(T) & \cdots & \mathcal{R}(T) & T\\ T^* & \mathcal{L}(T) & T^* & \cdots & T^* & \mathcal{L}(T)\end{pmatrix}\\ &\simeq \begin{pmatrix}M_n(\mathcal{R}(T)) & M_n(T)\\ M_n(T^*) & M_n(\mathcal{L}(T))\end{pmatrix}\\ &\simeq \begin{pmatrix}\mathcal{R}(M_n(T)) & M_n(T)\\ M_n(T)^* & \mathcal{L}(M_n(T))\end{pmatrix}\end{aligned}$$

$$= \mathbb{L}(M_n(T)).$$

We used here (2.3) again and the fact that the permutation of rows and columns yields complete isometries and therefore $*$-isomorphisms. □

Remark 3.3.15 (K-theory for C^*-algebras)**.** We briefly recall the basic definitions and properties of K-theory for C^*-algebras. For further details and proofs see for example [Bla98], [RLL00] or [WO93].

For a C^*-algebra $\mathfrak{A}$ let $\mathcal{P}(\mathfrak{A})$ be the set of projections in $\mathfrak{A}$ and put

$$\mathcal{P}_n(\mathfrak{A}) := \mathcal{P}(M_n(\mathfrak{A})) \text{ and } \mathcal{P}_\infty(\mathfrak{A}) := \bigcup_{n=1}^{\infty} \mathcal{P}_n(\mathfrak{A}).$$

We can define an equivalence relation ~ (often called Murray-von Neumann equivalence) on $\mathcal{P}_\infty(\mathfrak{A})$ as follows: If $p \in \mathcal{P}_n(\mathfrak{A})$ and $q \in \mathcal{P}_m(\mathfrak{A})$ for $n, m \in \mathbb{N}$, we write $p \sim q$ if there exists an element $v \in M_{n,m}(\mathfrak{A})$ such that $vv^* = p$ and $v^*v = q$. We can define a binary relation on $\mathcal{P}_\infty(\mathfrak{A})$ via

$$p \oplus q := \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in \mathcal{P}_\infty(\mathfrak{A})$$

and if we put $\mathcal{D}(\mathfrak{A}) := \mathcal{P}_\infty(\mathfrak{A})/\sim$, then $(\mathcal{D}(\mathfrak{A}), +)$ becomes an Abelian semigroup, where $[p] + [q] = [p \oplus q]$ for equivalence classes $[p], [q] \in \mathcal{D}(\mathfrak{A})$. Similarly to the construction of the integers from the natural numbers, we can construct the so-called Grothendieck group of the Abelian semigroup $(\mathcal{D}(\mathfrak{A}), +)$, this can be viewed as (formal) differences $[p] - [q]$ of elements in $\mathcal{D}(\mathfrak{A})$. This group is denoted by $K_{00}^{C^*}(\mathfrak{A})$ and the canonical homomorphism from $\mathcal{D}(\mathfrak{A})$ to $K_{00}^{C^*}(\mathfrak{B})$ is

$$\iota_{\mathfrak{A}}([p]) = [p] - [0].$$

If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a $*$-homomorphism φ induces a map $\varphi_* : \mathcal{D}(\mathfrak{A}) \to \mathcal{D}(\mathfrak{B})$ by $\varphi_*([\alpha_{i,j}]) = [(\varphi(\alpha_{i,j}))]$, since all amplifications of φ map projections to projections. The universal property of the Grothendieck construction allows us to extend φ_* to a group homomorphism $K_{00}^{C^*}(\varphi)$ from $K_{00}^{C^*}(\mathfrak{A})$ to $K_{00}^{C^*}(\mathfrak{B})$, such that

$$\begin{array}{ccc} \mathcal{D}(\mathfrak{A}) & \xrightarrow{\varphi_*} & \mathcal{D}(\mathfrak{B}) \\ \downarrow{\scriptstyle \iota_{\mathfrak{A}}} & & \downarrow{\scriptstyle \iota_{\mathfrak{B}}} \\ K_{00}^{C^*}(\mathfrak{A}) & \xrightarrow[K_{00}^{C^*}(\varphi)]{} & K_{00}^{C^*}(\mathfrak{B}) \end{array}$$

commutes. Let $\mathfrak{A}^+$ be the unitization of $\mathfrak{A}$ if $\mathfrak{A}$ is not unital and equal to $\mathfrak{A} \oplus \mathbb{C}$ if a is unital. Let $\pi : \mathfrak{A}^+ \to \mathfrak{A}^+/\mathfrak{A} \simeq \mathbb{C}$ be the canonical quotient homomorphism. Finally we can define

$$K_0^{C^*}(\mathfrak{A}) := \ker(K_{00}^{C^*}(\pi) : K_{00}^{C^*}(\mathfrak{A}^+) \to K_{00}^{C^*}(\mathbb{C})).$$

As in the case of $K_{00}^{C^*}$ *we have a canonical map* $\iota_{\mathfrak{A}} : \mathcal{D}(\mathfrak{A}) \to K_0^{C^*}(\mathfrak{A})$ *and any* $*$*-homomorphism* $\varphi : \mathfrak{A} \to \mathfrak{B}$ *induces a homomorphism* $K_0^{C^*}(\varphi) : K_0^{C^*}(\mathfrak{A}) \to K_0^{C^*}(\mathfrak{B})$ *of Abelian groups,*

$$K_0^{C^*}(\varphi)\left([(x_{i,j})] - [(y_{i,j})]\right) = [(\varphi^+ x_{i,j})] - [(\varphi^+ y_{i,j})],$$

where $[(x_{i,j})], [(y_{i,j})] \in \mathcal{D}(\mathfrak{A}^+)$ *and* $\varphi^+ : \mathfrak{A}^+ \to \mathfrak{B}^+$ *is the unital morphism* $\varphi^+(a + \lambda) = \varphi(a) + \lambda$*. Then* $K_0^{C^*}$ *is a covariant functor from the category of* C^**-algebras to the category of Abelian groups. As a functor* $K_0^{C^*}$ *is continuous, half exact, split exact, homotopy invariant, stable and additive.*

The suspension of a C^**-algebra* $\mathfrak{A}$ *is the* C^**-algebra*

$$S\mathfrak{A} = \{f \in C([0,1] \to \mathfrak{A}) : f(0) = f(1) = 0\} \simeq \mathfrak{A} \otimes C_0(\mathbb{R}).$$

The $K_1^{C^*}$*-group of* $\mathfrak{A}$ *can now be defined as*

$$K_1^{C^*}(\mathfrak{A}) := K_0^{C^*}(S\mathfrak{A}).$$

If we put $S^k\mathfrak{A} = SS^{k-1}\mathfrak{A}$ *for* $k \geq 2$ *we can define the higher* K*-groups of* $\mathfrak{A}$ *as* $K_n^{C^*}(\mathfrak{A}) := K_0^{C^*}(S^{n-1}\mathfrak{A})$ *for* $n \geq 2$*. A remarkable fact about* K*-theory for* C^**-algebras is the so-called Bott-periodicity:*

$$K_n^{C^*}(\mathfrak{A}) \simeq K_{n+2}^{C^*}(\mathfrak{A}) \text{ for all } n \geq 0.$$

To define the suspension of a TRO, we first have to state some facts about tensor products of TROs with C^*-algebras.

Remark 3.3.16 (Tensor product of TROs with C^*-algebras)**.** In [KR02] Kaur and Ruan developed a theory of tensor products between C^*-algebras and TROs. In general, like in the C^*-case, there are a lot of different tensor norms on the algebraic tensor product $T \otimes \mathfrak{A}$ of a TRO T and a C^*-algebra $\mathfrak{A}$, turning this product into a TRO. Recall that a C^*-algebra $\mathfrak{A}$ is called nuclear if there is only one C^*-tensor norm on $\mathfrak{A} \otimes \mathfrak{B}$ for every C^*-algebra $\mathfrak{B}$. A TRO T is accordingly called **nuclear** if for every C^*-algebra $\mathfrak{B}$ there is a unique tensor norm which turns $T \otimes \mathfrak{B}$ into a TRO. It turns out that a TRO T is nuclear as a TRO if and only if its linking algebra is nuclear as a C^*-algebra. If T is a nuclear TRO, then

$$\mathcal{L}(T \otimes \mathfrak{A}) \simeq \mathcal{L}(T) \otimes \mathfrak{A}, \quad \mathcal{R}(T \otimes \mathfrak{A}) \simeq \mathcal{R}(T) \otimes \mathfrak{A} \tag{3.6}$$

and

$$\mathbb{L}(T \otimes \mathfrak{A}) \simeq \mathbb{L}(T) \otimes \mathfrak{A}$$

for every C^*-algebra $\mathfrak{A}$.

Corollary 3.3.17. *If $\mathfrak{A}$ is a nuclear C^*-algebra, then for every TRO T there exists a unique norm on $T \otimes \mathfrak{A}$ making it a TRO.*

Proof. This follows directly from [KR02], Theorem 5.5. □

Since we only consider tensor products of TROs with nuclear C^*-algebras we do not bother about the possible tensor norms. It is known that the C^*-algebra $\mathbb{K}$ of compact operators on a separable Hilbert space is nuclear as well as every commutative and every finite-dimensional C^*-algebra.

Definition 3.3.18. *Let T be a TRO. The **suspension** of T is the TRO*

$$ST := T \otimes C_0(\mathbb{R}) \simeq \{f \in C([0,1], T) : f(0) = f(1) = 0\}.$$

*The **i-fold suspension** of T is recursively defined by*

$$S^i T = S(S^{i-1}T)$$

for $i \geq 2$ and $S^1 T = ST$.

One can associate to every TRO-homomorphism $\varphi : T \to U$ an TRO-homomorphism of the corresponding suspensions $S\varphi : ST \to SU$ by

$$S\varphi(f)(t) = \varphi(f(t))$$

for all $t \in [0,1]$. It is not hard to check that S becomes in this way a covariant functor from the category of TROs to itself.

Definition 3.3.19. *Let T be TRO. We define the **(ternary) K-groups** of T to be*

$$K_0^{TRO}(T) := K_0^{C^*}(\mathcal{L}(T)) \text{ and}$$
$$K_i^{TRO}(T) := K_0^{TRO}(S^i T) \text{ for } i \geq 1,$$

where $S^i T$ is the i-fold suspension of T. Every TRO-homomorphism $\varphi : T \to U$ induces a homomorphism of Abelian groups

$$K_0^{TRO}(\varphi) : K_0^{TRO}(T) \to K_0^{TRO}(U),$$

when we put $K_0^{TRO}(\varphi) := K_0^{C^*}(\mathcal{L}(\varphi))$.

The ternary K_0-groups of a C^*-algebra $\mathfrak{A}$ are exactly the K_0-groups in the C^*-algebra sense, since $\mathfrak{A}\mathfrak{A}^* = \mathfrak{A}$. Therefore it poses no danger of confusion to write $K_0(\mathfrak{A})$ for the ternary K_0-group of $\mathfrak{A}$. If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a $*$-homomorphism we know that $\mathcal{L}(\varphi) = \varphi$ and thus $K_0^{\text{TRO}}(\varphi) = K_0^{\text{C}^*}(\mathcal{L}(\varphi)) = K_0^{\text{C}^*}(\varphi)$. We therefore drop the superscripts for K_0.

Next we want to show that the functor K_1^{TRO} restricted to the category of C^*-algebras coincides with $K_1^{\text{C}^*}$. Since we defined the higher K-groups on the category of TROs itself, we first have to prove that the suspension functor commutes with the functors $\mathcal{L}$, $\mathcal{R}$ and $\mathbb{L}$.

Lemma 3.3.20. *If T is a TRO, then*

$$S\mathcal{L}(T) \simeq \mathcal{L}(ST),\ S\mathcal{R}(T) \simeq \mathcal{R}(ST) \text{ and } S\mathbb{L}(T) \simeq \mathbb{L}(ST).$$

Proof. This follows from the theory of tensor products, since for example

$$\mathbb{L}(ST) \simeq \mathbb{L}(T \otimes C_0(\mathbb{R})) \simeq \mathbb{L}(T) \otimes C_0(\mathbb{R}) \simeq S\mathbb{L}(T).$$

□

As a consequence of Lemma 3.3.20 we conclude that for a TRO T we have

$$K_1^{\mathrm{TRO}}(T) = K_0(ST) = K_0(\mathcal{L}(ST)) \simeq K_0(S\mathcal{L}(T)) = K_1^{\mathrm{C}^*}(\mathcal{L}(T))$$

in correspondence with C^*-theory (and similar for morphisms). We therefore drop the superscripts on K_1, too.

Before we state the next proposition about the functorial properties of ternary K-theory we note that we can deduce from Lemma 3.3.20, that Bott periodicity is still valid, i.e. for all $n \geq 0$

$$K_n(T) \simeq K_{n+2}(T)$$

for every TRO T.

Proposition 3.3.21. *Let S, T and U be TROs and $i \in \mathbb{N}_0$.*

(a) *Every short exact sequence of TROs*

$$0 \longrightarrow S \xrightarrow{\ \iota\ } T \underset{\psi}{\overset{\pi}{\rightleftarrows}} U \longrightarrow 0$$

induces a half-exact sequence

$$K_i(S) \xrightarrow{\ K_i(\iota)\ } K_i(T) \xrightarrow{\ K_i(\pi)\ } K_i(U)$$

of Abelian groups. If there exists a homomorphism $\psi : U \to T$ such that

$$0 \longrightarrow S \xrightarrow{\ \iota\ } T \underset{\psi}{\overset{\pi}{\rightleftarrows}} U \longrightarrow 0$$

is split exact also

$$0 \longrightarrow K_i(S) \xrightarrow{\ K_i(\iota)\ } K_i(T) \underset{K_i(\psi)}{\overset{K_i(\pi)}{\rightleftarrows}} K_i(U) \longrightarrow 0$$

is split exact.

(b) *If $\varphi, \psi : S \to T$ are homotopic TRO-morphisms, then $K_i(\varphi) = K_i(\psi)$ for the induced group homomorphisms $K_i(\varphi), K_i(\psi) : K_i(S) \to K_i(T)$.*

(c) *If $(T, (\mu_n))$ is the direct limit of the inductive sequence $(T_n, (\varphi_n))$ of TROs it follows that $(K_i(T), (K_i(\mu_n))$ is the direct limit of the inductive sequence $(K_i(T_n), K_i(\varphi_n))$.*

(d) $K_i(S \oplus T) \simeq K_i(S) \oplus K_i(T)$.

Proof. For K_0 this is just basic K-theory combined with our functorial results for $\mathcal{L}$ from this section. For the functor K_1 we know from Lemma 3.3.20 and the definition of K_1 that

$$K_1 = K_0 \circ S \circ \mathcal{L}$$

and on the category of C^*-algebras, the stated characteristics for $K_1^{C^*} = K_0 \circ S$ are well known. For higher indices the results are obtained by Bott periodicity. □

The next proposition shows that the choice we made by preferring the functor $\mathcal{L}$ over the functors $\mathcal{R}$ and $\mathbb{L}$ does not affect the theory. To prove this theorem we make use of the theory of stably isomorphic C^*-algebras developed by Brown. Since this theory is intended for separable C^*-algebras, we restrict our attention to separable TROs.

Proposition 3.3.22. *Let T be a separable TRO, then*

$$K_0(T) = K_0(\mathcal{L}(T)) \simeq K_0(\mathcal{R}(T)) \simeq K_0(\mathbb{L}(T)).$$

Proof. It follows from [Bro77], Lemma 2.5 and Lemma 2.6 that, if $\mathfrak{A}$ is a C^*-algebra containing a strictly positive element and $\mathfrak{B}$ is a full corner of $\mathfrak{A}$ (i.e. there exists a projection p in the multiplier algebra $\mathrm{Mult}(\mathfrak{A})$ of $\mathfrak{A}$ such that $\mathfrak{B} = p\mathfrak{A}p$ is not contained in any proper closed two-sided ideal of $\mathfrak{A}$), then there exists a tripotent v in the multiplier algebra of $\mathfrak{A} \otimes \mathbb{K}$ such that $v^*v = \mathrm{id}$ and $vv^* = p \otimes \mathrm{id}$. Moreover, $\mathfrak{A}$ and $\mathfrak{B}$ are stably isomorphic and the isomorphism is induced by the partial isometry v, here p is the projection with $\mathfrak{B} = p\mathfrak{A}p$ and $\mathfrak{B}$ can be identified with $(p \otimes \mathrm{id})(\mathfrak{A} \otimes \mathbb{K})(p \otimes \mathrm{id})$.

We first notice that $\mathcal{L}(T)$, $\mathcal{R}(T)$ and $\mathbb{L}(T)$ are separable, thus containing a positive element. Both $\mathcal{L}(T)$ and $\mathcal{R}(T)$ are full corners of $\mathbb{L}(T)$. Let p be the projection such that $p\mathbb{L}(T)p = \mathcal{L}(T)$ and v the tripotent from above with $vv^* = p \otimes \mathrm{id}$ and $v^*v = \mathrm{id}$. We can identify $\mathcal{L}(T) \otimes \mathbb{K}$ with $(p \otimes \mathrm{id})\mathbb{L}(T)(p \otimes \mathrm{id})$. Let

$$\phi : \mathbb{L}(T) \otimes \mathbb{K} \to \mathbb{L}(T) \otimes \mathbb{K}, \quad x \mapsto vxv^*$$

and notice that $\mathrm{im}(\phi) \subseteq \mathcal{L}(T) \otimes \mathbb{K}$ since $(p \otimes \mathrm{id})(\phi(x))(p \otimes \mathrm{id}) = vv^*vxv^*vv^* = \phi(x)$ for all $x \in \mathbb{L}(T) \otimes \mathbb{K}$. The linear mapping ϕ is bijective with inverse $\phi^{-1} : \mathbb{L}(T) \otimes \mathbb{K} \to \mathbb{L}(T) \otimes \mathbb{K}$, $x \mapsto v^*xv$ and ϕ is an algebra homomorphism since $\phi(xy) = vxyv^* = vxv^*vyv^* = \phi(x)\phi(y)$ for all $x, y \in \mathbb{L}(T) \otimes \mathbb{K}$. Since K_0 is stable this gives us an isomorphism from $K_0(\mathcal{L}(T))$ to $K_0(\mathbb{L}(T))$. The isomorphism from $K_0(\mathcal{R}(T))$ to $K_0(\mathbb{L}(T))$ is constructed in a similar way. □

We discovered different ways to prove the above lemma. Using the theory of stably isomorphic C^*-algebras developed by Brown, as used above, was the proof which needed the least mathematical machinery. The two others involve the use of KK-theory, one even ideas worked out in the context of KK-theory for Banach algebras developed in [Par09]. Both build on the fact that T can be interpreted as a Morita equivalence between $\mathcal{L}(T)$ and $\mathcal{R}(T)$, and thus T induces an isomorphism in KK-theory.

Corollary 3.3.23. *Let T be a separable TRO. The canonical embeddings $\iota_{\mathcal{L}} : \mathcal{L}(T) \to \mathbb{L}(T)$ and $\iota_{\mathcal{R}} : \mathcal{R}(T) \to \mathbb{L}(T)$ induce isomorphisms*

$$K_0(\iota_{\mathcal{L}}) : K_0(T) \to K_0(\mathbb{L}(T))$$

and

$$K_0(\iota_{\mathcal{R}}) : K_0(\mathcal{R}(T)) \to K_0(\mathbb{L}(T)).$$

Proof. Choose a system $\{e_{i,j} : i, j \in \mathbb{N}\}$ of matrix-units of $\mathbb{K}$ and consider the commutative diagram

$$\begin{array}{ccc} \mathcal{L}(T) & \xrightarrow{\iota_{\mathcal{L}}} & \mathbb{L}(T) \\ \downarrow & & \downarrow \\ \mathcal{L}(T) \otimes \mathbb{K} & \xrightarrow[\iota_{\mathcal{L}} \otimes \mathrm{id}]{} & \mathbb{L}(T) \otimes \mathbb{K} \end{array}$$

where the vertical maps are given by $a \mapsto a \otimes e_{1,1}$. On the level of K_0, the vertical maps become isomorphisms (this is just the stability of K_0) so we only have to prove that $K_0(\iota_{\mathcal{L}} \otimes \mathrm{id})$ is an isomorphism, but this follows from the proof of 3.3.22 since $\iota_{\mathcal{L}} \otimes \mathrm{id}$ induces the same map on the K_0-level as ϕ^{-1}, by [Bro77], Lemma 2.7. □

Definition 3.3.24. *For a separable TRO T the isomorphism*

$$\eta_T := K_0(\iota_{\mathcal{L}})^{-1} \circ K_0(\iota_{\mathcal{R}}) : K_0(\mathcal{R}(T)) \to K_0(T)$$

is said to be the ***Morita isomorphism*** *of the left and right K_0-groups of T.*

The next Lemma shows that the Morita isomorphism respects the group homomorphisms induced by TRO-homomorphisms. This naturality becomes important in the next chapter.

Lemma 3.3.25. *Let $\varphi : T \to U$ be a TRO-homomorphism of separable TROs, then the diagram*

$$\begin{array}{ccc} K_0(\mathcal{R}(T)) & \xrightarrow{\eta_T} & K_0(T) \\ {\scriptstyle K_0(\mathcal{R}(\varphi))}\downarrow & & \downarrow{\scriptstyle K_0(\varphi)} \\ K_0(\mathcal{R}(U)) & \xrightarrow[\eta_U]{} & K_0(U) \end{array}$$

commutes.

Proof. This is just part of the commuting diagram

$$\begin{array}{ccccc} \mathcal{R}(T) & \xrightarrow{\iota_{\mathcal{R}(T)}} & \mathbb{L}(T) & \xleftarrow{\iota_{\mathcal{L}(T)}} & \mathcal{L}(T) \\ {\scriptstyle \mathcal{R}(\varphi)}\downarrow & & \downarrow{\scriptstyle \mathbb{L}(\varphi)} & & \downarrow{\scriptstyle \mathcal{L}(\varphi)} \\ \mathcal{R}(U) & \xrightarrow[\iota_{\mathcal{R}(U)}]{} & \mathbb{L}(U) & \xleftarrow[\iota_{\mathcal{L}(U)}]{} & \mathcal{L}(U) \end{array}$$

under the functor K_0, where the horizontal arrows become isomorphisms due to Corollary 3.3.23. □

3.4 Classification of AF-TROs

As an application of our new K-theory for TROs we generalize the classic theory of dimension groups and AF-algebras developed by Elliott (cf. [Ell76]) to our ternary case. To give a classification of - how we call them - AF-TROs (inductive limits of finite-dimensional TROs) we will endow the K_0-group of these TROs with the additional data of an order structure and two positive subsets: the left and right scale. These scales originate from the projections in the left and right C^*-algebras and serve as a tool to recreate the dimension of the TRO. We also define a ternary version of Bratteli diagrams and give an 'almost' classification of stably isomorphic TROs.

Our definition of an AF-TRO is not the first time that inductive limits of ternary structures appear in the literature. In [Rua04] Ruan developed, studying the analogue of hyperfiniteness of von Neumann algebras for W^*-TROs, the notion of the rectangular approximately finite-dimensional property (AFD) for W^*-TROs which is a weak* version of our AF-property.

A lot has been written about AF-algebras and ordered groups. The references we used are [Bla98], Chapter 7, [RLL00], Chapter 5 and 6, [WO93] Chapter 12 and [Dav96], Chapter 3.

Definition 3.4.1. *A pair (G, G_+) is called* ***ordered Abelian group*** *if G is an Abelian group and $G_+ \subseteq G$ is a subset such that*

$$G_+ + G_+ \subseteq G_+, \quad G_+ - G_+ = G, \quad G_+ \cap (-G_+) = \{0\}.$$

We can define a partial ordering on G via

$$g \leq h :\Leftrightarrow h - g \in G_+.$$

Definition 3.4.2. *For a TRO T the* ***positive cone*** *of* $K_0(T)$ *is the set*

$$K_0(T)_+ := \{[p] \in K_0(T) : p \in \mathcal{P}_\infty(\mathcal{L}(T))\}.$$

The positive cone of $K_0(T)$ is just the image of $\mathcal{P}_\infty(\mathcal{L}(T))$ under the canonical mapping $\iota_{\mathcal{L}(T)}$ from Remark 3.3.15.

The positive cone of $K_0(T \oplus U)$ of the K_0-group of two TROs T and U is given by $K_0(T \oplus U)_+ = K_0(T)_+ \oplus K_0(U)_+$, which follows from [RLL00], Proposition 5.1.9 and our Proposition 3.3.8.

In general the pair $(K_0(T), K_0(T)_+)$ is not an ordered Abelian group.

Definition 3.4.3. *Let* (G, G_+) *and* (H, H_+) *be ordered Abelian groups. We call a group homomorphism* $\varphi : G \to H$ *an* ***order homomorphism*** *or* ***positive homomorphism*** *if it maps* G_+ *to* H_+.

If φ *is an isomorphism of groups with* $\varphi(G_+) = H_+$*, then* φ *is called* ***order isomorphism****.*

If $\varphi : T \to U$ is a TRO-homomorphism, then

$$K_0(\varphi)(K_0(T)_+) \subseteq K_0(U)_+,$$

since $\mathcal{L}(\varphi)$ maps projections to projections. If φ is bijective in addition $K_0(\varphi)$ becomes an isomorphism. In abuse of language (since in general $(K_0(T), K_0(T)_+)$ is not an ordered group) we call group homomorphisms $\psi : (K_0(T), K_0(U)_+) \to (K_0(U), K_0(U)_+)$ which preserve the positive cones **order homomorphisms** or **positive homomorphisms**.

Recall that a C^*-algebra $\mathfrak{A}$ is said to have cancellation if the semigroup $\mathcal{D}(\mathfrak{A})$ has cancellation, which means that if $x, y, z \in \mathcal{D}(\mathfrak{A})$ with $x + z = y + z$, then $x = y$.

Definition 3.4.4. *A TRO T is called* ***pre-unital*** *if* $\mathbb{L}(T)$ *is unital and we say that T has* ***cancellation*** *if* $\mathbb{L}(T)$ *has cancellation.*

If a TRO T has cancellation, then $(K_0(T), K_0(T)_+)$ is an ordered Abelian group ([Bla98], Proposition 6.3.3 and Proposition 6.4.1).

Note that $\mathbb{L}(T)$ is unital if and only if $\mathcal{L}(T)$ and $\mathcal{R}(T)$ are unital. The C^*-algebra $\mathbb{L}(T)$ has cancellation if and only if $\mathcal{L}(T)$ has cancellation or if and only if $\mathcal{R}(T)$ has cancellation. Since the linking algebra of every finite-dimensional TRO is also finite-dimensional and therefore has the cancellation property by [RLL00], 7.3.1, so does the TRO. Every finite-dimensional C^*-algebra is unital by the Artin-Wedderburn theorem for C^*-algebras. Therefore every finite-dimensional TRO is pre-unital.

Recall that any TRO is a (left) Hilbert C^*-module over $\mathcal{L}(T)$ and $\mathcal{R}(T)$ is $\star$-isomorphic to the C^*-algebra of compact adjointable operators on T. With that in mind we can deduce the next lemma from [WO93], 15.4.2.

Lemma 3.4.5. *Let T be a TRO. Then the following are equivalent:*

(a) *T is pre-unital.*

(b) *$\mathcal{L}(T)$ and $\mathcal{R}(T)$ are unital.*

(c) *$\mathcal{L}(T)$ is unital and T is finitely generated as left Hilbert $\mathcal{L}(T)$-module.*

(d) *$\mathcal{R}(T)$ is unital and T is finitely generated as right Hilbert $\mathcal{R}(T)$-module.*

Lemma 3.4.6. *Let T be a separable TRO. Then the Morita isomorphism $\eta_T : K_0(\mathcal{R}(T)) \to K_0(T)$ is an order isomorphism.*

Proof. It suffices to show that the canonical embedding $\iota_{\mathcal{L}} : \mathcal{L}(T) \to \mathbb{L}(T)$ induces an order isomorphism on the K_0-level. Recall the commutative diagram from the proof of Corollary 3.3.23

$$\begin{array}{ccc} \mathcal{L}(T) & \xrightarrow{\ \iota_{\mathcal{L}}\ } & \mathbb{L}(T) \\ \downarrow & & \downarrow \\ \mathcal{L}(T) \otimes \mathbb{K} & \xrightarrow[\iota_{\mathcal{L}} \otimes \mathrm{id}]{} & \mathbb{L}(T) \otimes \mathbb{K} \end{array}$$

where the vertical maps are given by $a \mapsto a \otimes e_{1,1}$ for a system $\{e_{i,j} : i, j \in \mathbb{N}\}$ of matrix-units of $\mathbb{K}$. Now $K_0(\iota_{\mathcal{L}} \otimes \mathrm{id})$ is an order isomorphism since it is induced by an isomorphism of C^*-algebras. Because the compact operators are isomorphic to the inductive limit of the matrix algebras over $\mathbb{C}$ with left upper corner embeddings as connecting morphisms, we see that for every projection in $p \in \mathcal{L}(T) \otimes \mathbb{K}$ we find a $n \in \mathbb{N}$, such that $p \in \mathbb{M}_n(\mathcal{L}(T))$. Therefore, as $\mathcal{L}(T) \otimes \mathbb{K}$ is stable, we get $P_\infty(\mathcal{L}(T)) = P(\mathcal{L}(T) \otimes \mathbb{K}) = P_\infty(\mathcal{L}(T) \otimes \mathbb{K})$. This shows that the left vertical map (and analogously the right vertical map) is an order isomorphism. □

We introduce another notion which is a ternary generalization of a concept which has proven itself to be useful in C^*-theory.

Definition 3.4.7. *Let (G, G_+) be an ordered Abelian group. A **scale** of G is a subset $S \subseteq G_+$ which is*

(a) ***hereditary**: if $g \in G$ and $s \in S$ with $0 \le g \le s$ and $s \in S$, then $g \in S$,*

(b) ***directed**: if $s_1, s_2 \in S$ then there is a $s \in S$ so that $s_1 \le s$ and $s_2 \le s$,*

(c) ***generating**: every $g \in G_+$ is the sum of finitely many elements of S.*

*A pair $(S_1, S_2) \subseteq G_+ \times G_+$ is called **double-scale** if both S_1 and S_2 are scales in G. The tuple*

$$(G, G_+, S_1, S_2)$$

is called ***double-scaled ordered group****. A* ***homomorphism of double-scaled ordered groups***

$$\varphi : (G, G_+, S_1, S_2) \to (H, H_+, R_1, R_2)$$

is an order homomorphism which maps S_1 into R_1 and S_2 into R_2. The homomorphism φ is called ***isomorphism of double-scaled ordered groups*** *if it is an order isomorphism with*

$$\varphi(S_1) = R_1 \text{ and } \varphi(S_2) = R_2.$$

In C^*-algebra theory the scale, defined as the set containing all Murray-von Neumann equivalence classes of projections in the original C^*-algebra, is a tool to control the dimension of the C^*-algebra (at least in some important cases). We introduced the notion of a double-scale, since a single scale is not sufficient for our intent. A very good example is the TRO $\mathbb{M}_{n,m}$, determined not only by one dimension but by the pair $(n, m) \in \mathbb{N} \times \mathbb{N}$ or equivalently by the dimensions of $\mathcal{L}(\mathbb{M}_{n,m}) = \mathbb{M}_n$ and $\mathcal{R}(\mathbb{M}_{n,m}) = \mathbb{M}_m$. We therefore consider the scales in the left and right C^*-algebra simultaneously, transporting the scale of the right C^*-algebra with the Morita isomorphism to the K_0-group of the TRO.

Definition 3.4.8. *Let T be a separable TRO with K_0-group $K_0(T)$ and positive cone $K_0(T)_+$. The* ***left scale*** *of $K_0(T)$ is the set*

$$\Sigma^{\mathcal{L}}(T) := \{[p] \in K_0(T) : p \text{ is a projection in } \mathcal{L}(T)\} \subseteq K_0(T)_+.$$

Let η_T be the Morita isomorphism of T. We define the ***right scale*** *of $K_0(T)$ to be*

$$\Sigma^{\mathcal{R}}(T) := \eta_T\left(\{[p] \in K_0(\mathcal{R}(T)) : p \text{ is a projection in } \mathcal{R}(T)\}\right) \subseteq K_0(T)_+.$$

The quadruple

$$\left(K_0(T), K_0(T)_+, \Sigma^{\mathcal{L}}(T), \Sigma^{\mathcal{R}}(T)\right)$$

is called ***double-scaled ordered $\mathbf{K_0}$-group*** *of T.*

The homomorphisms of double-scaled ordered K_0-groups are those positive group homomorphisms which map the left scale into the left scale and the right scale into the right scale.

For a separable C^*-algebra $\mathfrak{A}$ we get that $\Sigma^{\mathcal{L}}(\mathfrak{A}) = \Sigma^{\mathcal{R}}(\mathfrak{A}) =: \Sigma(\mathfrak{A})$, since $\mathcal{L}(\mathfrak{A}) = \mathcal{R}(\mathfrak{A}) = \mathfrak{A}$. We thus write

$$(K_0(\mathfrak{A}), K_0(\mathfrak{A})_+, \Sigma(\mathfrak{A}))$$

for our doubled-scaled ordered group, consistent with C^*-theory.

If $\varphi : T \to U$ is a TRO-homomorphism $K_0(\varphi) : K_0(T) \to K_0(U)$ is positive with $K_0(\varphi)(\Sigma^{\mathcal{L}}(T)) \subseteq \Sigma^{\mathcal{L}}(U)$, since $\mathcal{L}(\varphi)$ maps projections to projections. In addition $K_0(\varphi)(\Sigma^{\mathcal{R}}(T)) \subseteq \Sigma^{\mathcal{R}}(U)$ because $K_0(\varphi) \circ \eta_T = \eta_U \circ K_0(\mathcal{R}(\varphi))$ holds by Lemma 3.3.25.

The pair $\big(\Sigma^{\mathcal{L}}(T), \Sigma^{\mathcal{R}}(T)\big)$ is not a double-scale for every TRO T in the sense of Definition 3.4.7 in general.

Lemma 3.4.9. *Let T be a separable, pre-unital TRO with cancellation, then $\Sigma^{\mathcal{L}}(T)$ and $\Sigma^{\mathcal{R}}(T)$ are hereditary subsets of $K_0(T)_+$. Moreover*

$$\Sigma^{\mathcal{L}}(T) = \left\{g \in K_0(T)_+ : g \leq [\mathrm{id}_{\mathcal{L}(T)}]\right\}$$

and

$$\Sigma^{\mathcal{R}}(T) = \left\{g \in K_0(T)_+ : g \leq \eta_T\left([\mathrm{id}_{\mathcal{R}(T)}]\right)\right\}.$$

Proof. The C^*-algebras $\mathcal{L}(T)$ and $\mathcal{R}(T)$ are both unital and have the cancellation property. By [RLL00], Proposition 5.1.7 $[\mathrm{id}_{\mathcal{L}(T)}]$ and $[\mathrm{id}_{\mathcal{R}(T)}]$ are order units for $K_0(T)$ and $K_0(\mathcal{R}(T))$, respectively. The beginning of IV.3 in [Dav96] shows that $\Sigma^{\mathcal{L}}(T)$ is a hereditary subset of $K_0(T)$ as well as $\Sigma(\mathcal{R}(T)) = \{[p] \in K_0(\mathcal{R}(T)) : p \text{ is a projection in } \mathcal{R}(T)\}$ is a hereditary subset of $K_0(\mathcal{R}(T))$. It follows from [Bla98], 6.6 that

$$\Sigma^{\mathcal{L}}(T) = \left\{g \in K_0(T)_+ : g \leq [\mathrm{id}_{\mathcal{L}(T)}]\right\}$$

and

$$\Sigma(\mathcal{R}(T)) = \left\{g \in K_0(\mathcal{R}(T))_+ : g \leq [\mathrm{id}_{\mathcal{R}(T)}]\right\}.$$

Since the Morita morphism is an order isomorphism of the ordered Abelian groups $(K_0(\mathcal{R}(T)), K_0(\mathcal{R}(T))_+)$ and $(K_0(T), K_0(T)_+)$ it is easy to see that $\Sigma^{\mathcal{R}}(T)$ has the desired form. □

The above lemma shows that it sometimes is not necessary to know the whole double-scale of an ordered Abelian group but rather the upper bounds of the scales, which motivates the next definition.

Definition 3.4.10. *Let (G, G_+) be an ordered Abelian group. An element $u \in G_+$ is called an* ***order unit*** *in G if for every $g \in G$ there exists a $n \in \mathbb{N}$ such that $-nu \leq g \leq nu$. A pair $(u_{\mathcal{L}}, u_{\mathcal{R}})$ is called* ***pair of order units****, if $u_{\mathcal{L}}$ as well as $u_{\mathcal{R}}$ are order units. We refer to $u_{\mathcal{L}}$ as* ***left order unit*** *and to $u_{\mathcal{R}}$ as* ***right order unit****.*

Remark 3.4.11. Suppose that T is a separable pre-unital TRO with cancellation. We can use Lemma 3.4.9 to obtain a canonical pair of order units for $K_0(T)$. The left order unit $u_{\mathcal{L}}(T)$ is just the equivalence class in $K_0(T)$ of the unit of $\mathcal{L}(T)$. The right order unit $u_{\mathcal{R}}(T)$ is the image of $[\mathrm{id}_{\mathcal{R}(T)}] \in K_0(\mathcal{R}(T))$ under the Morita morphism η_T, which is an order isomorphism by Lemma 3.4.6. We call the tuple

$$\left(K_0(T), K_0(T)_+, u_{\mathcal{L}(T)}, u_{\mathcal{R}(T)}\right) \tag{3.7}$$

the **ordered $\mathbf{K_0}$-group of T with canonical order units**. If U is a pre-unital TRO with cancellation such that there exists a TRO-isomorphism φ from T to U, then $\mathcal{L}(T)$ and $\mathcal{L}(U)$ as well as $\mathcal{R}(T)$ and $\mathcal{R}(U)$ are $*$-isomorphic. These $*$-isomorphisms induce order isomorphisms on the K_0-level and we get a commuting diagram of order isomorphisms

$$\begin{array}{ccc}(K_0(T), K_0(T)_+) & \xleftarrow{\ \eta_T\ } & (K_0(\mathcal{R}(T)), K_0(\mathcal{R}(T))_+) \\ \Big\downarrow{\scriptstyle K_0(\varphi)} & & \Big\downarrow{\scriptstyle K_0(\mathcal{R}(\varphi))} \\ (K_0(U), K_0(U)_+) & \xleftarrow[\ \eta_U\ ]{} & (K_0(\mathcal{R}(U)), K_0(\mathcal{R}(U))_+)\end{array}$$

and moreover

$$\begin{aligned} K_0(\varphi)(u_{\mathcal{R}(T)}) &= K_0(\varphi) \circ \eta_T \left([\mathrm{id}_{\mathcal{R}(T)}]\right) \\ &= \eta_U \circ K_0(\mathcal{R}(\varphi)) \left([\mathrm{id}_{\mathcal{R}(T)}]\right) \\ &= \eta_U \left([\mathrm{id}_{\mathcal{R}(U)}]\right) \\ &= u_{\mathcal{R}(U)} \end{aligned}$$

and obviously $K_0(\varphi)(u_{\mathcal{L}(T)}) = u_{\mathcal{L}(U)}$. This proves that (3.7) is an isomorphism invariant for T. In particular, if T and U are two separable pre-unital TROs which have cancellation and for whom $(K_0(T), K_0(T)_+, u_{\mathcal{L}(T)}, u_{\mathcal{R}(T)})$ is not isomorphic to $(K_0(U), K_0(U)_+, u_{\mathcal{L}(U)}, u_{\mathcal{R}(U)})$, then T and U can not be completely isometric. For example if $n \neq m$ (for details see Example 3.4.12)

$$\begin{aligned} (K_0(\mathbb{M}_{n,m}), K_0(\mathbb{M}_{n,m})_+, u_{\mathbb{M}_n}, u_{\mathbb{M}_m}) &\simeq (\mathbb{Z}, \mathbb{N}_0, n, m) \\ &\not\simeq (\mathbb{Z}, \mathbb{N}_0, m, n) \\ &= (K_0(\mathbb{M}_{m,n}), K_0(\mathbb{M}_{m,n})_+, u_{\mathbb{M}_m}, u_{\mathbb{M}_n}) \end{aligned}$$

yields that $\mathbb{M}_{n,m}$ can not be TRO-isomorphic to $\mathbb{M}_{m,n}$.

Example 3.4.12. Let T be the TRO $\mathbb{M}_{n,m}$. Observe that

$$\begin{aligned} \mathcal{L}(T) &= \overline{\mathrm{lin}}\{AB^* : A, B \in \mathbb{M}_{n,m}\} = \mathbb{M}_n, \\ \mathcal{R}(T) &= \overline{\mathrm{lin}}\{A^*B : A, B \in \mathbb{M}_{n,m}\} = \mathbb{M}_m \end{aligned}$$

and

$$\mathbb{L}(T) = \begin{pmatrix} \mathbb{M}_m & \mathbb{M}_{n,m} \\ \mathbb{M}_{m,n} & \mathbb{M}_n \end{pmatrix} = \mathbb{M}_{n+m}.$$

Recall from [RLL00] that the group $K_0(\mathbb{M}_n)$ is isomorphic to $\mathbb{Z}$ for every $n \in \mathbb{N}$. The isomorphism from $K_0(\mathbb{M}_n)$ to $\mathbb{Z}$ is induced by the standard trace and the cyclic group $K_0(\mathbb{M}_n)$ is generated by $[e]$, where e is any one-dimensional projection in $\mathbb{M}_n$. As a TRO, $\mathbb{M}_{n,m}$ is obviously pre-unital and has cancellation, since

$$K_0(\mathrm{tr})\left(K_0(T)_+\right) = \mathbb{N}_0.$$

Thus we get a left order unit $u_{\mathcal{L}} = [\mathrm{id}_{\mathbb{M}_n}]$ dominating the left scale and a right order unit $u_{\mathcal{R}} = [\eta_T(\mathrm{id}_{\mathbb{M}_m})]$ dominating the right scale. Under the trace map we get $u_{\mathcal{L}} = n$, $u_{\mathcal{R}} = m$ and therefore

$$\Sigma^{\mathcal{L}}(T) = \{0, \ldots, n\}$$

and

$$\Sigma^{\mathcal{R}}(T) = \{0, \ldots, m\}.$$

Let more generally U be a finite-dimensional TRO. By Theorem 3.2.1 we can assume that there exists a $k \in \mathbb{N}$ such that

$$U = \mathbb{M}_{n_1,m_1} \oplus \ldots \oplus \mathbb{M}_{n_k,m_k}.$$

We get with the above and Proposition 3.3.21 that

$$K_0(U) = \mathbb{Z}^k \text{ and } K_0(U)_+ = \mathbb{N}_0^k.$$

Every finite-dimensional TRO is pre-unital and has the cancellation property. The canonical order units of U are

$$u_{\mathcal{L}} = (n_1, \ldots, n_k) \text{ and } u_{\mathcal{R}} = (m_1, \ldots, m_k)$$

with corresponding scales

$$\Sigma^{\mathcal{L}}(U) = \{(\alpha_1, \ldots, \alpha_k) \in \mathbb{N}_0^k : \alpha_i \leq n_i \text{ for } 1 \leq i \leq k\}$$

and

$$\Sigma^{\mathcal{R}}(U) = \{(\beta_1, \ldots, \beta_k) \in \mathbb{N}_0^k : \beta_i \leq m_i \text{ for } 1 \leq i \leq k\}.$$

Definition 3.4.13. *Let T and U be TROs. We call T and U* ***stably isomorphic*** *if $T \otimes \mathbb{K}$ and $U \otimes \mathbb{K}$ are isomorphic as TROs.*

A consequence of (3.6) is that if T and U are stably isomorphic TROs, then their linking algebras are stably isomorphic as C^*-algebras

$$\mathbb{L}(T) \otimes \mathbb{K} \simeq \mathbb{L}(T \otimes \mathbb{K}) \simeq \mathbb{L}(U \otimes \mathbb{K}) \simeq \mathbb{L}(U) \otimes \mathbb{K}.$$

Moreover we have

$$\mathcal{L}(T) \otimes \mathbb{K} \simeq \mathcal{L}(U) \otimes \mathbb{K} \quad \text{and} \quad \mathcal{L}(T) \otimes \mathbb{K} \simeq \mathcal{L}(U) \otimes \mathbb{K}.$$

In the next remark we discuss the close relationship between stable isomorphy of TROs and the isomorphy of their double-scaled ordered K_0-groups. A full classification is not even possible for C^*-algebras.

Remark 3.4.14 (‘Classification’ of stably isomorphic TROs)**.** Let T be a separable pre-unital TRO with cancellation and

$$(K_0(T), K_0(T)_+, u_{\mathcal{L}(T)}, u_{\mathcal{R}(T)})$$

its ordered K_0-group with canonical order units. The left and right scales of T are by Lemma 3.4.9 given by

$$\Sigma^{\mathcal{L}}(T) = \left\{g \in K_0(T)_+ : g \leq [\mathrm{id}_{\mathcal{L}(T)}]\right\} \text{ and}$$
$$\Sigma^{\mathcal{R}}(T) = \left\{g \in K_0(T)_+ : g \leq \eta_T\left([\mathrm{id}_{\mathcal{R}(T)}]\right)\right\}.$$

If U is a TRO which is stably isomorphic to T, then $(K_0(T), K_0(T)_+)$ is order isomorphic to $(K_0(U), K_0(U)_+)$ and the image of $\Sigma^{\mathcal{L}}(T)$ is a closed interval $[0, u_1] \subseteq K_0(U)_+$ for some order unit u_1. We have that $\mathcal{R}(T) \otimes \mathbb{K} \simeq \mathcal{R}(U) \otimes \mathbb{K}$ as C^*-algebras which gives us by stability of K_0 that

$$K_0(\mathcal{R}(T)) \simeq K_0(\mathcal{R}(T) \otimes \mathbb{K}) \simeq K_0(\mathcal{R}(U) \otimes \mathbb{K}) \simeq K_0(\mathcal{R}(U)),$$

where the isomorphisms are order preserving. The image of $\Sigma(\mathcal{R}(T)) \subseteq K_0(\mathcal{R}(T))$ therefore is a closed interval $[0, \widetilde{u}_2] \subseteq K_0(\mathcal{R}(U))_+$. Under the Morita morphism η_U this becomes a closed interval $[0, u_2] \subseteq K_0(U)_+$. Let conversely u_1 and u_2 be two order units of $K_0(T)$. We construct a pre-unital TRO U such that $K_0(U) = K_0(T)$,

$$\Sigma^{\mathcal{L}}(U) = \{g \in K_0(U)_+ : g \leq u_1\} \text{ and}$$
$$\Sigma^{\mathcal{R}}(U) = \{g \in K_0(U)_+ : g \leq u_2\}.$$

Let $n_1, n_2 \in \mathbb{N}$ and take first $p \in M_{n_1}(\mathcal{L}(T))$ with $[p] = u_1$ and $q \in M_{n_2}(\mathcal{R}(T))$ with $[q] = \eta_T^{-1}(u_2)$. Now assume w.l.o.g. that $n_1 = n_2 =: n$. One can see that $pM_n(\mathcal{L}(T))p$ is strongly Morita equivalent to $qM_n(\mathcal{R}(T))q$ (first notice that $pM_n(\mathcal{L}(T))p$ is Morita equivalent to $M_n(\mathcal{L}(T))$, which is Morita equivalent to $\mathcal{L}(T)$. Now we can apply (2.1) and (2.2) to see that T is an equivalence bimodule between $\mathcal{L}(T)$ and $\mathcal{R}(T)$). Let U be the TRO with $\mathcal{L}(U) = pM_n(\mathcal{L}(T))p$ and $\mathcal{R}(U) = qM_n(\mathcal{R}(T))q$. Then we have

$$\mathbb{L}(T) \otimes \mathbb{K} \simeq \mathcal{L}(T) \otimes \mathbb{K} \simeq M_n(\mathcal{L}(T)) \otimes \mathbb{K} \simeq pM_n(\mathcal{L}(T))p \otimes \mathbb{K},$$

which yields

$$\begin{aligned} T \otimes \mathbb{K} &\simeq \mathcal{L}(T) \otimes \mathbb{K} \\ &= \mathbb{L}(T) \otimes \mathbb{K} \\ &\simeq pM_n(\mathcal{L}(T))p \otimes \mathbb{K} \\ &= \mathcal{L}(U) \otimes \mathbb{K} \\ &\simeq U \otimes \mathbb{K}, \end{aligned}$$

where the first and last isomorphism follow from the Brown-Kasparov stabilization theorem (cf. [BLM04], Corollary 8.2.6 (5)). By construction we get

$$\Sigma^{\mathcal{L}}(U) = \{g \in K_0(U)_+ : g \le u_1\} \text{ and } \Sigma^{\mathcal{R}}(U) = \{g \in K_0(U)_+ : g \le u_2\}.$$

So we obtained almost a classification of all pre-unital TROs stably isomorphic to T by pairs of order units in $K_0(T)$. Unfortunately the correspondence is not 1–1 in general. It is known (cf. [Bla98], 6.6) that if T is a C^*-algebra the C^*-algebras corresponding to pairs of order units (u,u) and (v,v) may be isomorphic if there exists an order automorphism of $K_0(T)$ mapping u to v, but the existence of such an order automorphism does not assure in general that the C^*-algebras are stably isomorphic (not every order isomorphism is induced by an isomorphism on the TRO-level).

Proposition 3.4.15. *Suppose*

$$\varphi : T = \bigoplus_{i=1}^{p} \mathbb{M}_{n_i,m_i} \to U = \bigoplus_{j=1}^{q} \mathbb{M}_{l_j,k_j}$$

is a TRO-homomorphism and let

$$\varphi_j : \bigoplus_{i=1}^{p} \mathbb{M}_{n_i,m_i} \to \mathbb{M}_{l_j,k_j}$$

be the restrictions of φ for $j = 1,\dots,q$ with $\varphi = \varphi_1 + \dots + \varphi_q$. For every $i \in \{1,\dots,p\}$ let ι_i be the embedding of $\mathbb{M}_{n_i,m_i}$ into $\bigoplus_{i=1}^{p} \mathbb{M}_{n_i,m_i}$ and $\varphi_{i,j} := \varphi_j \circ \iota_i$. The induced group homomorphism $K_0(\varphi) : \mathbb{Z}^p \to \mathbb{Z}^q$ is given by the $q \times p$ matrix

$$K_0(\varphi) = (\alpha_{i,j})_{i,j},$$

where $\alpha_{i,j}$ is the multiplicity $M(\varphi_{i,j})$ for $1 \le i \le p$, $1 \le j \le q$.

Proof. This follows from our results about homomorphisms of finite-dimensional TROs. □

Proposition 3.4.16. *Let $T = \bigoplus_{i=1}^{p} \mathbb{M}_{n_i,m_i}$ and $U = \bigoplus_{j=1}^{q} \mathbb{M}_{k_j,l_j}$ be finite-dimensional ternary rings of operators and*

$$\left(K_0(T), K_0(T)_+, \Sigma^{\mathcal{L}}(T), \Sigma^{\mathcal{R}}(T)\right) = \left(\mathbb{Z}^p, \mathbb{N}_0^p, \prod_{i=1}^{p}\{0,\dots,n_i\}, \prod_{i=1}^{p}\{0,\dots,m_i\}\right)$$

and

$$\left(K_0(U), K_0(U)_+, \Sigma^{\mathcal{L}}(U), \Sigma^{\mathcal{R}}(U)\right) = \left(\mathbb{Z}^q, \mathbb{N}_0^q, \prod_{j=1}^{q}\{0,\dots,k_j\}, \prod_{j=1}^{q}\{0,\dots,l_j\}\right)$$

their double-scaled ordered K_0-groups. Let $\alpha : K_0(T) \to K_0(U)$ be a homomorphism of double-scaled groups.

(a) *The homomorphism α can be represented as a $q \times p$-matrix $(a_{i,j})_{i,j}$ with entries $a_{i,j} \in \mathbb{N}_0$.*

(b) *For all $(z_1, \dots z_p) \in \Sigma^{\mathcal{L}}(T)$ we have $\sum_{j=1}^{p} a_{i,j} z_j \leq k_i$ for all $i = 1, \dots, q$.*

(c) *For all $(z_1, \dots z_p) \in \Sigma^{\mathcal{R}}(T)$ we have $\sum_{j=1}^{p} a_{i,j} z_j \leq l_i$ for all $i = 1, \dots, q$.*

(d) *There exists a TRO-homomorphism $\varphi : T \to U$ with $K_0(\varphi) = (a_{i,j})_{i,j}$.*

Proof. All group homomorphisms from $\mathbb{Z}^p$ to $\mathbb{Z}^q$ can be viewed as $q \times p$-matrices with entries in $\mathbb{Z}$. The homomorphism α is a homomorphism of ordered groups and therefore maps $K_0(T)_+ = \mathbb{N}_0^p$ to $K_0(U)_+ = \mathbb{N}_0^q$, thus all entries in the matrix have to be positive or 0.

To prove (b) let $x = (z_1, \dots z_p) \in \Sigma^{\mathcal{L}}(T)$. Since $\alpha(\Sigma^{\mathcal{L}}(T)) \subseteq \Sigma^{\mathcal{L}}(U)$ holds, we see that

$$\alpha(x) = \left(\sum a_{1,i} z_i, \dots, \sum a_{q,i} z_i\right) \leq (k_1, \dots, k_q).$$

An analogous argument shows (c).

For the proof of (d) let φ be the direct sum $\varphi := \varphi_1 \oplus \dots \oplus \varphi_q$, where $\varphi_j : T \to \mathbb{M}_{k_j, l_j}$ is defined via

$$\varphi_j(x_1 \oplus \dots \oplus x_p) := \operatorname{diag}(\underbrace{x_1, \dots, x_1}_{a_{1,j} times}, \dots, \underbrace{x_p, \dots x_p}_{a_{p,j} times}, 0, \dots, 0),$$

for $j = 1, \dots, q$. These TRO-homomorphisms are well-defined by (b) and (c) and using Proposition 3.4.15 we get $K_0(\varphi) = \alpha$. □

Proposition 3.4.17. *Two finite-dimensional TROs are isomorphic if and only if their double-scaled ordered groups are isomorphic.*

Proof. Let T and U be finite-dimensional TROs. If $\varphi : T \to U$ is a TRO-isomorphism, then $K_0(\varphi)$ becomes an isomorphism of double-scaled ordered groups.

If on the other hand $\left(K_0(T), K_0(T)_+, \Sigma^{\mathcal{L}}(T), \Sigma^{\mathcal{R}}(T)\right)$ is the scaled ordered group of T, then we know from Example 3.4.12 that there exist natural numbers $k, n_1, \dots, n_k, m_1, \dots m_k$ such that $K_0(T) \simeq \mathbb{Z}^k$, $K_0(T)_+ \simeq \mathbb{N}_0^k$, $\Sigma^{\mathcal{L}}(T) \simeq \prod_{i=1}^{k} \{0, \dots, n_i\}$ and $\Sigma^{\mathcal{R}}(T) \simeq \prod_{i=1}^{k} \{0, \dots, m_i\}$. Since every finite-dimensional TRO is isomorphic to the direct sum of rectangular matrix algebras, the double-scaled ordered K_0-group of T carries all the necessary data to recover T up to isomorphism. If $T \simeq \bigoplus_{i=1}^{l} \mathbb{M}_{r_i, s_i}$, then $l = k$ and (after maybe changing the summation order) $(r_i, s_i) = (m_i, n_i)$ for $i = 1, \dots, k$. □

Remark 3.4.18. One can easily get the impression that ternary K-theory is rather the doubled K-theory of two Morita equivalent C^*-algebras than the K-theory of the TRO itself. But this is not the case as the following simple but illuminating example shows. The TROs

$$T = \mathbb{M}_{1,2} \oplus \mathbb{M}_{2,1} \quad \text{and} \quad U = \mathbb{M}_{1,1} \oplus \mathbb{M}_{2,2}$$

are non-isomorphic (not even linear isomorphic) with

$$\mathcal{L}(T) = \mathbb{M}_1 \oplus \mathbb{M}_2 = \mathcal{L}(U)$$

and

$$\mathcal{R}(T) = \mathbb{M}_2 \oplus \mathbb{M}_1 \simeq \mathbb{M}_1 \oplus \mathbb{M}_2 = \mathcal{R}(U).$$

This yields the two non-isomorphic double-scaled ordered groups

$$\begin{aligned} K_0(T) = (\mathbb{Z}^2, \mathbb{N}_0^2, &\{(0,0),(0,1),(0,2),(1,1),(1,2)\},\\ &\{(0,0),(1,0),(1,1),(2,0),(2,1)\}) \end{aligned}$$

and

$$\begin{aligned} K_0(U) = (\mathbb{Z}^2, \mathbb{N}_0^2, &\{(0,0),(0,1),(0,2),(1,1),(1,2)\},\\ &\{(0,0),(0,1),(0,2),(1,1),(1,2)\}). \end{aligned}$$

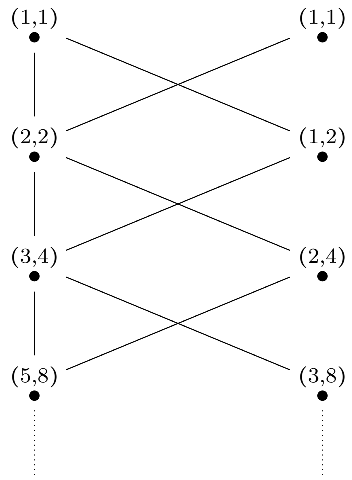
The example shows that the K-theory of TROs is not a ‘fused’ K-theory of two Morita equivalent C^*-algebras, but can distinguish between different TROs, even if they have isomorphic left and right C^*-algebras.

Definition 3.4.19. *An **AF-TRO** (approximately finite-dimensional TRO) is a TRO which is (TRO-isomorphic to) the inductive limit of an inductive sequence of finite-dimensional TROs.*

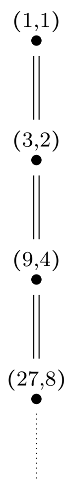
Note that if T is an AF-TRO, then $\mathcal{L}(T)$ is an AF-algebra and thus T has cancellation by [Dav96], Theorem IV.1.6. Especially $(K_0(T), K_0(T)_+)$ is an ordered Abelian group. Moreover it is possible to deduce from [Dav96], IV.3 that $\Sigma^{\mathcal{L}}(T)$ and $\Sigma^{\mathcal{R}}(T)$ are scales in the sense of Definition 3.4.7.

Definition 3.4.20. *Let $((T_\lambda), (\varphi_\lambda))$ be an inductive sequence of finite-dimensional TROs. We can view this inductive sequence as a so-called **ternary Bratteli diagram**. This is a weighted graph, consisting of rows of a finite number of vertices and a number of edges connecting these vertices to the vertices in the next row. Each vertex has an attached pair of positive integers. The vertices in the λth row represent the direct summands in the decomposition of T_λ into rectangular matrix algebras. Assume $T_\lambda \simeq \bigoplus_{i=1}^p \mathbb{M}_{n_i,m_i}$, then the λth row of the Bratteli diagram has p vertices and the ith vertex has the pair (n_i, m_i) attached, representing the number of rows and columns of $\mathbb{M}_{n_i,m_i}$. If $T_{\lambda+1} = \bigoplus_{j=1}^q \mathbb{M}_{l_j,k_j}$, then the number of edges between the ith vertex in the λth row and the jth vertex in the $(\lambda+1)$th row is given by the multiplicity of the mapping $\mathbb{M}_{n_i,m_i} \xrightarrow{\iota} T_\lambda \xrightarrow{\varphi} T_{\lambda+1} \xrightarrow{\pi} \mathbb{M}_{l_j,k_j}$, where ι and π denote the canonical injection and projection.*

Example 3.4.21. Let (f_n) be the Fibonacci sequence, recursively defined via $f_0 := f_1 := 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ and (a_n) be the sequence given by $a_n = 2^{n-1}$ for all $n \in \mathbb{N}$. We define an inductive sequence of TROs by $T_n := \mathbb{M}_{f_n,a_n} \oplus \mathbb{M}_{f_{n-1},a_n}$, and $\varphi_n : T_n \to T_{n+1}$, $\varphi_n((x,y)) = \left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}\right)$. The corresponding ternary Bratteli diagram is given by



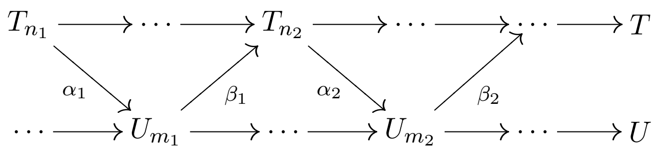
For the inductive sequence $((T_n),(\varphi_n))$, with $T_n = \mathbb{M}_{3^{n-1},2^{n-1}}$ and $\varphi_n(x) := \begin{pmatrix} x & 0 \\ 0 & x \\ 0 & 0 \end{pmatrix}$ we have the ternary Bratteli diagram



Lemma 3.4.22. *Let $(T,(\mu_n))$ and $(U,(\nu_n))$ be the inductive limits of the inductive sequences of TROs $((T_n),(\varphi_n))$ and $((U_n),(\psi_n))$. Furthermore let there be indices*

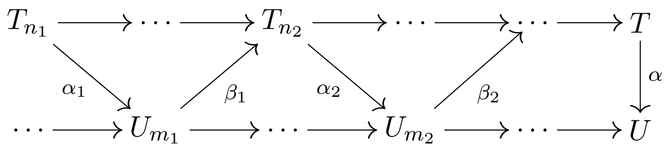
$$n_1 \leq m_1 < n_2 \leq m_2 < n_3 \leq m_3 < \dots$$

and TRO-homomorphisms $\alpha_k : T_{n_k} \to U_{m_k}$, $\beta_k : U_{m_k} \to T_{n_{k+1}}$, *for* $k \in \mathbb{N}$ *such that*



is commutative.

Then there exists a unique TRO-isomorphism $\alpha : T \to U$, *such that*



commutes.

Proof. Let $k \in \mathbb{N}$ and $a_{n_k} \in T_{n_k}$. We define $\alpha : T \to U$ by $\alpha(\mu_{n_k}(a_{n_k})) := \nu_{m_k}(\alpha_k(a_{n_k}))$. Then α is, by the properties of the inductive limit, a well-defined TRO-homomorphism with inverse $\beta : U \to T$ given by

$$\beta(\nu_{m_k}(b_{m_k})) := \mu_{n_{k+1}}(\beta_k(b_{m_k})),$$

for $b_{m_k} \in U_{m_k}$. Uniqueness follows from the properties of the inductive limit. □

Proposition 3.4.23. *Let* (T_n) *and* (U_n) *be increasing sequences of finite-dimensional TROs. If* $T = \overline{\bigcup_{n=1}^{\infty} T_n}$ *and* $U = \overline{\bigcup_{n=1}^{\infty} U_n}$ *have the same ternary Bratteli diagram, then they are isomorphic.*

Proof. We denote by $\varphi_\lambda : U_\lambda \to U_{\lambda+1}$ and $\psi_\lambda : U_\lambda \to U_{\lambda+1}$ the connecting homomorphisms for all $\lambda \in \mathbb{N}$. Since T and U have the same ternary Bratteli diagram we know from its λth row (which for example consists of p vertices attached with the numbers $(n_1, m_1), \ldots, (n_p, m_p)$) that both T_λ and U_λ are TRO-isomorphic to $\bigoplus_{i=1}^{p} \mathbb{M}_{n_i,m_i}$, which gives us a TRO-isomorphism τ_λ from T_λ to U_λ. This can be done for every $\lambda \in \mathbb{N}$ and we obtain two injections $\tau_{\lambda+1} \circ \varphi_\lambda$ and $\psi_\lambda \circ \tau_\lambda$, both having the same multiplicities (in the obvious meaning), since by assumption φ_λ and ψ_λ have the same multiplicities. We can now use Lemma 3.1.10 and Lemma 3.2.3 and obtain unitaries $U'_{\lambda+1} \in \mathcal{R}(U_{\lambda+1})$ and $K_{\lambda+1} \in \mathcal{L}(U_{\lambda+1})$ such that

$$\psi_\lambda \circ \tau_\lambda = K^*_{\lambda+1} \, (\tau_{\lambda+1} \circ \varphi_\lambda) \, U'_{\lambda+1}.$$

We define recursively two sequences of unitaries: Let

$$A_1 = \mathrm{id}_{\mathcal{R}(U_1)} \text{ and } A_{\lambda+1} = U'_{\lambda+1} \, (\mathcal{R}(\psi_\lambda)(A_\lambda))$$

as well as

$$B_1 = \mathrm{id}_{\mathcal{L}(U_1)} \text{ and } B_{\lambda+1} = K_{\lambda+1}\left(\mathcal{L}(\psi_\lambda)(B_\lambda)\right)$$

for $\lambda \geq 1$. If we put

$$\alpha_\lambda := B_\lambda^* \tau_\lambda A_\lambda,$$

then

$$\begin{aligned}
\psi_\lambda \circ \alpha_\lambda &= \psi_\lambda \circ B_\lambda^* \tau_\lambda A_\lambda \\
&= \left(\mathcal{L}(\psi_\lambda)(B_\lambda)\right)^* \left(\psi_\lambda \circ \tau_\lambda\right) \left(\mathcal{R}(\psi_\lambda)(A_\lambda)\right) \\
&= \left(\mathcal{L}(\psi_\lambda)(B_\lambda)\right)^* K_{\lambda+1}^* \left(\tau_{\lambda+1} \circ \varphi_\lambda\right) U_{\lambda+1} \left(\mathcal{R}(\psi_\lambda)(A_\lambda)\right) \\
&= B_{\lambda+1}^* \left(\tau_{\lambda+1} \circ \varphi_\lambda\right) A_{\lambda+1} \\
&= B_{\lambda+1}^* \tau_{\lambda+1} A_{\lambda+1} \circ \varphi_\lambda \\
&= \alpha_{\lambda+1} \circ \varphi_\lambda
\end{aligned}$$

for all $\lambda \geq 1$. We therefore obtain a commuting diagram

$$\begin{array}{ccccccccccc}
T_1 & \xrightarrow{\varphi_1} & T_2 & \xrightarrow{\varphi_2} & T_3 & \xrightarrow{\varphi_3} & T_4 & \xrightarrow{\varphi_4} & \cdots & \longrightarrow & T \\
{\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\alpha_2}\downarrow & & {\scriptstyle\alpha_3}\downarrow & & {\scriptstyle\alpha_3}\downarrow & & & & \\
U_1 & \xrightarrow[\psi_1]{} & U_2 & \xrightarrow[\psi_2]{} & U_3 & \xrightarrow[\psi_3]{} & U_4 & \xrightarrow[\psi_4]{} & \cdots & \longrightarrow & U
\end{array}$$

which induces a TRO-isomorphism of the inductive limits T and U by Lemma 3.4.22. □

Notice that the ternary Bratteli diagram is not unique since two isomorphic AF-TROs can be the inductive limits of quite different inductive sequences with different ternary Bratteli diagrams. Another problem is, that it is not known (even for C^*-algebras) how to determine if two diagrams yield isomorphic AF-TROs and there does not exist a reasonable algorithm to construct a ternary Bratteli diagram from a given AF-TRO.

Theorem 3.4.24 ([RLL00], Proposition 6.2.5, Proposition 6.2.6, [Dav96], IV.3.3)**.** *Each inductive sequence $((G_n),(\varphi_n))$ of Abelian groups has an inductive limit (G,μ_n) and $G = \bigcup_{n=1}^\infty \mu_n(G_n)$. Assume in addition that all of the G_n are ordered and that all of the connecting morphisms are positive. If we put*

$$G_+ := \bigcup_{n=1}^{\infty} \mu_n((G_n)_+),$$

then (G,G_+) becomes an ordered Abelian group, the μ_n are positive group homomorphism for all $n \in \mathbb{N}$ and $((G,G_+),(\mu_n))$ is the corresponding inductive limit in the category of ordered Abelian groups.

Moreover if for every $n \in \mathbb{N}$ there is a scale $\Sigma_n \subseteq (G_n)_+$ and each φ_n maps Σ_n into Σ_{n+1}, then

$$\Sigma := \bigcup_{n \in \mathbb{N}} \mu_n(\Sigma_n)$$

is a scale in G.

Lemma 3.4.25. *Let $(T, (\mu_n))$ be the inductive limit of the inductive sequence of finite-dimensional TROs $((T_n), (\varphi_n))$ and suppose that all connecting homomorphisms preserve the double-scales, then*

$$\Sigma^{\mathcal{L}}(T) = \bigcup_{n \in \mathbb{N}} K_0(\mu_n) \left(\Sigma^{\mathcal{L}}(T_n)\right)$$

and

$$\Sigma^{\mathcal{R}}(T) = \bigcup_{n \in \mathbb{N}} K_0(\mu_n) \left(\Sigma^{\mathcal{R}}(T_n)\right)$$

are scales in the sense of Definition 3.4.7.

Proof. It follows from Theorem 3.4.24 and Proposition 3.3.21 that $(K_0(T), K_0(T)_+, \Sigma^{\mathcal{L}}(T))$ is the limit of $\left(K_0(T_n), K_0(T_n)_+, \Sigma^{\mathcal{L}}(T_n)\right)$ and especially $\Sigma^{\mathcal{L}}(T) = \bigcup_{n \in \mathbb{N}} K_0(\mu_n) \left(\Sigma^{\mathcal{L}}(T_n)\right)$. We can use the same assertions to conclude that $(K_0(\mathcal{R}(T)), K_0(\mathcal{R}(T))_+, \Sigma(\mathcal{R}(T)))$ is the inductive limit of $(K_0(\mathcal{R}(T_n)), K_0(\mathcal{R}(T_n))_+, \Sigma(\mathcal{R}(T_n)))$ with scale

$$\Sigma(\mathcal{R}(T)) = \bigcup_{n \in \mathbb{N}} K_0(\mathcal{R}(\mu_n))(\Sigma(\mathcal{R}(T_n))) \subseteq K_0(\mathcal{R}(T))$$

and we only have to prove that

$$\eta_T \left(\bigcup_{n \in \mathbb{N}} K_0(\mathcal{R}(\mu_n)) \left(\Sigma(\mathcal{R}(T_n))\right) \right) = \bigcup_{n \in \mathbb{N}} K_0(\mu_n) \left(\Sigma^{\mathcal{R}}(T_n)\right). \tag{3.8}$$

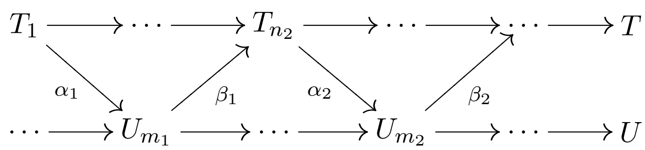
Let $n \in \mathbb{N}$ and $p \in \Sigma(\mathcal{R}(T_n))$, then $\eta_T(K_0(\mathcal{R}(\mu_n))(p)) = K_0(\mu_n)(\eta_{T_n}(p))$, which proves (3.8) since by definition $\Sigma^{\mathcal{R}}(T_n) = \eta_{T_n}(\Sigma(\mathcal{R}(T_n)))$. That $\Sigma^{\mathcal{L}}(T)$ and $\Sigma^{\mathcal{R}}(T)$ are scales now follows from Theorem 3.4.24. ☐

Lemma 3.4.26. *Let T and U be finite-dimensional TROs. If $\varphi, \psi : T \to U$ are TRO-homomorphisms with $K_0(\varphi) = K_0(\psi)$, then φ and ψ are unitarily equivalent as TRO-homomorphisms.*

Proof. We first extend φ and ϕ to mappings of the linking algebras $\mathbb{L}(\varphi)$, $\mathbb{L}(\psi) : \mathbb{L}(T) \to \mathbb{L}(U)$ and then it follows from [Bla98], Proposition 7.3.1 (b) that there exists a unitary $u \in \mathbb{L}(U)$ such that $\mathbb{L}(\varphi)(x) = u\mathbb{L}(\psi)(x)u^*$ for all $x \in \mathbb{L}(T)$. Now apply Proposition 3.1.10 (b). ☐

Theorem 3.4.27. *If T and U are AF-TROs and $\sigma : K_0(T) \to K_0(W)$ is an isomorphism of double-scaled ordered groups, then there exists a TRO-isomorphism $\varphi : T \to U$ with $K_0(\varphi) = \sigma$.*

Proof. Assume that $(T, (\tau_n))$ is the inductive limit of $(T_n, (\varphi_n))$ and $(U, (\mu_n))$ is the inductive limit of $(U_n, (\psi_n))$. We construct TRO-homomorphisms $\alpha_k : T_{n_k} \to U_{m_k}$ and $\beta_k : U_{m_k} \to T_{n_{k+1}}$, $k \in \mathbb{N}$, such that



commutes for indices $1 = n_1 \leq m_1 < n_2 \leq m_2 < n_3 \leq m_3 < \ldots$. Lemma 3.4.22 then provides us with a TRO isomorphism $\varphi : T \to U$ such that

$$\begin{array}{ccccccccc}
T_1 & \longrightarrow \cdots \longrightarrow & T_{n_2} & \longrightarrow \cdots \longrightarrow & \cdots & \longrightarrow & T \\
& \searrow^{\alpha_1} \quad \nearrow_{\beta_1} & & \searrow^{\alpha_2} \quad \nearrow_{\beta_2} & & & \downarrow \varphi \\
\cdots \longrightarrow & U_{m_1} & \longrightarrow \cdots \longrightarrow & U_{m_2} & \longrightarrow & \cdots \longrightarrow & U
\end{array}$$

commutes. We also require that $K_0(\varphi) = \sigma$ and therefore

$$\begin{array}{ccccccccc}
K_0(T_1) & \longrightarrow \cdots \longrightarrow & K_0(T_{n_2}) & \longrightarrow \cdots \longrightarrow & \cdots & \longrightarrow & K_0(T) \\
& \searrow^{K_0(\alpha_1)} \quad \nearrow_{K_0(\beta_1)} & & \searrow^{K_0(\alpha_2)} \quad \nearrow_{K_0(\beta_2)} & & & \downarrow \sigma \\
\cdots \longrightarrow & K_0(U_{m_1}) & \longrightarrow \cdots \longrightarrow & K_0(U_{m_2}) & \longrightarrow & \cdots \longrightarrow & K_0(U)
\end{array}$$

has to be a commuting diagram, too. We first construct the group homomorphisms $K_0(\alpha_k)$ and $K_0(\beta_k)$ for all $k \in \mathbb{N}$ and afterwards lift them to TRO-homomorphisms with the aid of Proposition 3.4.16 (d). We know by Example 3.4.12 that $K_0(T_1)$ is isomorphic to $\mathbb{Z}^l$ for some $l \in \mathbb{N}$. Let $e_1 := (1, 0, \ldots, 0), \ldots, e_l := (0, \ldots, 0, 1)$ be the generators of $\mathbb{Z}^l$. The group homomorphism σ preserves double-scales by assumption, the mapping $\sigma \circ K_0(\tau_1) : K_0(T_1) \to K_0(U)$ does it alike and we know that $\sigma \circ K_0(\tau_1)(e_j) \in \Sigma^{\mathcal{L}}(U)$, since $e_j \in \Sigma^{\mathcal{L}}(T_1)$ for all $j \in \{1, \ldots, l\}$. Since $\Sigma^{\mathcal{L}}(U) = \bigcup_{n=1}^{\infty} K_0(\mu_n)\Sigma^{\mathcal{L}}(U_n)$ by Lemma 3.4.25 we can find a natural number $m_1 \geq 1$ and positive elements $x_1, \ldots, x_n \in K_0(U_{m_1})_+$ with

$$\sigma \circ K_0(\tau_1)(e_j) = K_0(\mu_{m_1})(x_j) \in \Sigma^{\mathcal{L}}(U)$$

for all $j = 1, \ldots, n$. If we focus our attention on the right scale we see that also

$$\sigma \circ K_0(\tau_1)(e_j) = K_0(\mu_{m_1})(x_j) \in \Sigma^{\mathcal{R}}(U)$$

for all $j = 1, \ldots, l$ holds. By Theorem 3.2.1 we can write the finite-dimensional TRO $T_1 = \mathbb{M}_{\zeta_1,\xi_1} \oplus \ldots \oplus \mathbb{M}_{\zeta_l,\xi_l}$, with $\zeta_i, \xi_i \in \mathbb{N}$ for $i = 1, \ldots, l$. Therefore

$$\sigma \circ K_0(\tau_1)(\zeta_1 e_1 + \ldots + \zeta_l e_l) \in \Sigma^{\mathcal{L}}(U)$$

and

$$\sigma \circ K_0(\tau_1)(\xi_1 e_1 + \ldots + \xi_l e_l) \in \Sigma^{\mathcal{R}}(U).$$

By increasing m_1, if necessary, we can assume that

$$\zeta_1 x_1 + \ldots + \zeta_l x_l \in \Sigma^L(U_{m_1}) \text{ and } \xi_1 x_1 + \ldots + \xi_l x_l \in \Sigma^R(U_{m_1}).$$

We can now define a group homomorphism $\phi_1 : K_0(T_1) \to K_0(U_{m_1})$, via $\phi_1(e_j) := x_j$ for $j = 1, \ldots, l$, which is a homomorphism of double-scaled ordered groups satisfying

$$\sigma \circ K_0(\tau_1) = K_0(\mu_{m_1}) \circ \phi_1.$$

Similar to the above construction we can find an index n_2 and a double-scale preserving group homomorphism $\gamma_1 : K_0(U_{m_1}) \to K_0(T_{n_2})$ with

$$\sigma^{-1} \circ K_0(\mu_{m_1}) = K_0(\tau_{n_2}) \circ \gamma_1,$$

or equivalently $K_0(\mu_{m_1}) = K_0(\tau_{n_2}) \circ \gamma_1 \circ \sigma$. By construction we have

$$K_0(\tau_{n_2}) \circ \gamma_1 \circ \phi_1 = K_0(\tau_1).$$

By increasing n_2, if necessary, we may assume that $\gamma_1 \circ \phi_1 = K_0(\varphi_{n_2,1})$ (recall that $\varphi_{n_2,1} = \varphi_{n_2-1} \circ \ldots \circ \varphi_1$). Now, applying Proposition 3.4.16, we can lift our homomorphisms ϕ_1 and γ_1 to TRO-homomorphisms $\alpha_1 : T_1 \to U_{m_1}$ and $\omega : U_{m_1} \to T_{n_2}$ with $K_0(\alpha_1) = \phi_1$ and $K_0(\omega) = \gamma_1$. Since $K_0(\omega \circ \alpha_1) = K_0(\varphi_{n_2,1})$ we know by Proposition 3.1.10 (b) that there exist unitaries u and k such that $u^*(\omega \circ \alpha_1)(x)k = \varphi_{n_2,1}(x)$ for all $x \in T_1$. If we put $\beta_1(y) := u^*\omega(y)k$ for all $y \in U_{m_1}$, then $K_0(\beta_1) = \gamma_1$. All α_n and β_n, $n \geq 2$ can be constructed analogously. □

We can rephrase Theorem 3.4.27 to

Theorem 3.4.28. *Two AF-TROs are isomorphic if and only if their double-scaled ordered groups are isomorphic.*

Corollary 3.4.29. *Two AF-TROs are stably isomorphic if and only if their K_0-groups are isomorphic as ordered Abelian groups.*

Proof. Let T and U be stably isomorphic AF-TROs. Then $K_0(T)$ is order isomorphic to $K_0(T \otimes \mathbb{K})$ which is order isomorphic to $K_0(U \otimes \mathbb{K})$, which is order isomorphic to $K_0(U)$.

Let T and U be AF-TROs such that $(K_0(T), K_0(T)_+)$ is order isomorphic to $(K_0(U), K_0(U)_+)$. The left and right scale of a stable TRO is the whole positive cone, which yields

$$\Sigma^{\mathcal{L}}(T \otimes \mathbb{K}) = \Sigma^{\mathcal{R}}(T \otimes \mathbb{K}) = K_0(T \otimes \mathbb{K})_+ \simeq K_0(U \otimes \mathbb{K})_+ = \Sigma^{\mathcal{L}}(U \otimes \mathbb{K}) = \Sigma^{\mathcal{R}}(U \otimes \mathbb{K}).$$

Therefore the K_0-groups of the AF-TROs $T \otimes \mathbb{K}$ and $U \otimes \mathbb{K}$ are isomorphic as double-scaled ordered groups and Theorem 3.4.27 gives us an isomorphism of $T \otimes \mathbb{K}$ and $U \otimes \mathbb{K}$. □

Chapter 4

The universal enveloping TRO

In this chapter we associate to every JB^*-triple system Z an, up to TRO-isomorphism, unique pair $(T^*(Z), \rho_Z)$, where $T^*(Z)$ is a TRO and $\rho_Z : Z \to T^*(Z)$ is a JB^*-triple homomorphism with the following two universal properties: (i) for every JB^*-triple homomorphism $\varphi : Z \to T$ to a TRO T there exists a TRO-homomorphism $T^*(\varphi) : T^*(Z) \to T$ with $T^*(\varphi) \circ \rho_Z = \varphi$; (ii) $\rho_Z(Z)$ generates $T^*(Z)$ as a TRO. As a first application we are able to give an alternative proof for one of the main theorems of JB^*-triple theory: Every JB^*-triple system contains a unique purely exceptional ideal such that the quotient of the triple by this ideal is special.

The universal properties of T^* yield a functor τ from the category of JB^*-triple homomorphisms to the category of TROs. We establish some functorial properties of this mapping, respectively its restriction to the category of JC^*-triple systems. As it turns out τ is homotopy invariant, continuous, additive and exact. But the functors lack to be stable (the matrix levels of JC^*-triples are not even JC^*-triples themselves). Building on the theory of reversibility we overcome this obstacle in a large number of examples. We compute the universal enveloping TROs of Abelian JB^*-triple systems and generalize an important result from [BFT10].

Next we use the theory of grids to compute the universal enveloping TROs of the finite-dimensional Cartan factors and thus, since every finite-dimensional JB^*-triple system is the direct sum of Cartan factors, the universal enveloping TROs of all finite-dimensional JB^*-triple systems.

As another application we use the universal enveloping TRO and its properties to analyze how the Cartan factors can be represented on each other.

After working out the details of the first and the forth section of this chapter, we learned that Bunce, Feely and Timoney independently and at the same time studied the operator structure of JC^*-triple systems and developed a theory of universal enveloping TROs of JC^*-triple systems. They calculated the universal enveloping TROs of the special Cartan factors. Their methods differ completely from ours, where the most striking difference lies in our usage of grid theory (cf. [BFT10]).

4.1 Universal objects

We prove the existence of the universal enveloping TRO and the universal enveloping C^*-algebra of a JB^*-triple system. As a corollary we obtain a new proof of one of the main theorems of JB^*-triple theory.

The following Lemma and theorem are generalizations of classical results for real JB-algebras (cf. [HOS84], Theorem 7.1.3 and [AS03], Theorem 4.36).

Lemma 4.1.1. *Let Z be a JB^*-triple system. Then there exists a Hilbert space H such that for every JB^*-triple homomorphism $\varphi : Z \to B(K)$ the C^*-algebra $\mathfrak{A}_\varphi$ generated by $\varphi(Z)$ can be embedded $*$-isomorphically into $B(H)$.*

Proof. The cardinality of $\varphi(Z)$ is less or equal to the cardinality of Z. Therefore we get an upper bound for the cardinality of the $*$-algebra generated by $\varphi(Z)$, independent of φ. It follows that there exists an, again independent of φ, upper bound on the cardinality of the set of all Cauchy sequences in this $*$-algebra. This is also an upper bound for $\mathfrak{A}_\varphi$. Hence there is an upper bound for the cardinality of the state space and thus for the cardinality of every GNS-representation of $\mathfrak{A}_\varphi$. We get an upper bound κ on the dimension of the universal representation of $\mathfrak{A}_\varphi$. Let H be a Hilbert space with $\dim H = \kappa$. $\square$

Theorem 4.1.2. *Let Z be a JB^*-triple system.*

(a) *There exist up to $*$-isomorphism a unique C^*-algebra $C^*(Z)$ and a JB^*-triple homomorphism $\psi_Z : Z \to C^*(Z)$ such that*

- **(i)** *For every JB^*-triple homomorphism $\varphi : Z \to \mathfrak{A}$, where $\mathfrak{A}$ is an arbitrary C^*-algebra, exists a $*$-homomorphism $C^*(\varphi) : C^*(Z) \to \mathfrak{A}$ with $C^*(\varphi) \circ \psi_Z = \varphi$.*
- **(ii)** *$C^*(Z)$ is generated as a C^*-algebra by $\psi_Z(Z)$.*

(b) *There exists up to TRO-isomorphism a unique TRO $T^*(Z)$ and a JB^*-triple homomorphism $\rho_Z : Z \to T^*(Z)$ such that*

- **(i)** *For every JB^*-triple homomorphism $\alpha : Z \to T$, where T is an arbitrary TRO, exists a TRO-homomorphism $T^*(\alpha) : T^*(Z) \to T$ with $T^*(\alpha) \circ \rho_Z = \alpha$.*
- **(ii)** *$T^*(Z)$ is generated as a TRO by $\rho_Z(Z)$.*

Proof. Let H be the Hilbert space from Lemma 4.1.1 and I the family of JB^*-triple homomorphisms from Z to $B(H)$. Let $\psi_Z := \rho_Z := \bigoplus_{\psi\in I}\psi$ and $\hat{H} := \bigoplus_{\psi\in I} H_\psi$ be l^2-direct sums with $H_\psi := H$. Then ψ_Z and ρ_Z are JB^*-triple homomorphisms from Z to $B(\hat{H})$. Let $C^*(Z)$ be the C^*-algebra and $T^*(Z)$ the TRO generated by $\rho(Z)$ in $B(\hat{H})$. If $\mathfrak{A}$ is a C^*-algebra and $\varphi : Z \to \mathfrak{A}$ is a JB^*-triple homomorphism, where $\varphi(Z)$ w.l.o.g. generates $\mathfrak{A}$ as a C^*-algebra, then we can suppose that $\mathfrak{A}$ is a subset of $B(H)$. Therefore φ can be regarded as an element of I. Let $\pi_\varphi : \bigoplus_{\psi\in I} B(H_\psi) \to B(H_\varphi)$ be the projection onto the φ-component, then $\pi_\varphi(\psi_Z(z)) = \pi_\varphi(\rho_Z(z)) = \varphi(z)$ for all $z \in Z$. We define $C^*(\varphi)$ resp. $T^*(\varphi)$ to be the restrictions of π_φ to $C^*(Z)$ resp. $T^*(Z)$. Uniqueness is proved in the usual way using the universal properties. □

We call $(T^*(Z), \rho_Z)$ the **universal enveloping TRO** and $(C^*(Z), \psi_Z)$ the **universal enveloping C*-algebra** of Z respectively. Most of the time we only use $T^*(Z)$ and $C^*(Z)$ as shorter versions.

Similar to the classical case there exists a TRO-antiautomorphism on $T^*(Z)$:

Proposition 4.1.3. *Let Z be a JB^*-triple system. There exists a TRO-antiautomorphism θ (i.e. a linear, bijective mapping from $T^*(Z)$ to $T^*(Z)$ such $\theta(xy^*z) = \theta(z)\theta(y)^*\theta(x)$ for all $x, y, z \in T^*(Z)$) of $T^*(Z)$ of order 2 such that $\theta \circ \rho_Z = \rho_Z$.*

Proof. Denote by $T^*(Z)^{op}$ the opposite TRO of $T^*(Z)$, i.e. the TRO that coincides with $T^*(Z)$ as a set and is equipped with the same norm, if $\gamma : T^*(Z) \to T^*(Z)^{op}$, $\gamma(a) = a^{op}$ is the identity mapping then $(xy^*z)^{op} = z^{op}(y^{op})^* x^{op}$ for all $x, y, z \in T^*(Z)$.

The composed mapping $\gamma \circ \rho_Z : Z \to T^*(Z)^{op}$ is a JB^*-triple homomorphism and thus lifts to a TRO-homomorphism $T^*(\gamma\circ\rho_Z) : T^*(Z) \to T^*(Z)^{op}$. We put

$$\theta := \gamma^{-1} \circ T^*(\gamma \circ \rho_Z) : T^*(Z) \to T^*(Z).$$

Easily it can be seen (since θ fixes by construction $\rho_Z(Z)$ which generates $T^*(Z)$ as a TRO) using the universal properties of $T^*(Z)$ that θ is a TRO-antiautomorphism of order 2. □

We refer to θ as the **canonical TRO-antiautomorphism of order 2** on $T^*(Z)$.

Corollary 4.1.4. *If the JB^*-triple system Z in Theorem 4.1.2 is a JC^*-triple then the mappings ψ_Z and ρ_Z are injective.*

Obviously ψ_Z and ρ_Z are the 0 mappings if Z is purely exceptional.

We obtain a new proof of an important theorem of Friedman and Russo (cf. [FR86], Theorem 2):

Corollary 4.1.5. *Any JB^*-triple system Z contains a unique purely exceptional ideal J such that Z/J is JB^*-triple isomorphic to a JC^*-triple system.*

Proof. Let J be the kernel of the mapping $\rho_Z : Z \to T^*(Z)$, which is a JB^*-triple ideal. We know that Z/J is a JB^*-triple system which is JB^*-triple isomorphic to the JB^*-triple system $\rho_Z(Z) \subseteq T^*(Z)$ and hence to a JC^*-triple system.

Let us assume that J is not purely exceptional which means that there exists a non-zero JB^*-triple homomorphism φ from J into some $B(H)$. This JB^*-triple homomorphism extends by next section's Lemma 4.2.3 to a JB^*-triple homomorphism $\phi : Z \to B(H)$. Since $\phi = T^*(\phi) \circ \rho_Z$ holds, ϕ vanishes on J, which is a contradiction.

Now let I be another purely exceptional ideal such that Z/I is JB^*-triple isomorphic to a JC^*-triple system. On the one hand we have $I \subseteq \ker(\rho_Z) = J$. On the other hand let $\varphi : Z \to B(H)$ be a JB^*-triple homomorphism with kernel I. Then φ has to vanish on J and therefore $J \subseteq I$. □

4.2 Functorial properties

The universal properties in Theorem 4.1.2 give us a functor τ from the category of JB^*-triple systems with JB^*-triple homomorphisms to the category of TROs with TRO-homomorphisms, if we map a JB^*-triple system Z to its universal enveloping TRO $\tau(Z) := T^*(Z)$ and JB^*-triple homomorphisms $\varphi : Z_1 \to Z_2$ to TRO-homomorphisms $\tau(\varphi) := T^*(\rho_{Z_2} \circ \varphi) : \tau(Z_1) \to \tau(Z_2)$.

Proposition 4.2.1. *If X and Y are JB^*-triple systems, then*

$$\tau(X \oplus Y) = \tau(X) \oplus \tau(Y).$$

Proof. Let $\iota_X : X \to X \oplus Y$, $\iota_{\tau(X)} : \tau(X) \to \tau(X) \oplus \tau(Y)$, $\pi_X : X \oplus Y \to X$, and $\pi_Y : X \oplus Y \to Y$ be the canonical injections and projections respectively. Let $\Phi : \tau(X \oplus Y) \to \tau(X) \oplus \tau(Y)$, $\Phi(z) = (\tau(\pi_X)(z), \tau(\pi_Y)(z))$, then Φ is a TRO-homomorphism and we have by the five lemma: If in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tau(X) & \xrightarrow{\tau(\iota_X)} & \tau(X \oplus Y) & \xrightarrow{\tau(\pi_Y)} & \tau(Y) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \Phi} & & \| & & \\
0 & \longrightarrow & \tau(X) & \xrightarrow[\iota_{\tau(X)}]{} & \tau(X) \oplus \tau(Y) & \xrightarrow[\pi_{\tau(Y)}]{} & \tau(Y) & \longrightarrow & 0
\end{array}$$

the small squares commute, then Φ is an isomorphism. Obviously the right square commutes. Let $x \in \tau(X)$, then we have

$$\Phi \circ \tau(\iota_X)(x) = (\tau(\pi_X)(\tau(\iota_X)(x)), \tau(\pi_Y)(\tau(\iota_X)(x)))$$

$$= (\tau(\rho_X)(x), 0)$$
$$= \big(\iota_{\tau(X)}(x), 0\big),$$

since $\tau(\rho_X) = id_{\tau(X)}$ by the universal property of $T^*(X)$. □

We need the following two lemmas to prove exactness of τ.

Lemma 4.2.2. *Let Z be a JC^*-triple system and T a TRO such that Z generates T as a TRO. If $I \subseteq Z$ is a JB^*-triple ideal, then the TRO $[I]$ generated by I in T is a TRO-ideal in T.*

Proof. We have to show that $[I]T^*T + T[I]^*T + TT^*[I] \subseteq [I]$. Since I generates $[I]$ and Z generates T it suffices to show that for all $i \in I$ and for all $x, y \in Z$: ix^*y, $xi^*y, xy^*i \in [I]$. To prove this we first show that for all $i_1, i_2 \in I$, $z \in Z$: $i_1 i_2^* z$, $z i_2^* i_1 \in [I]$. By Lemma 2.3.7 there exists $j \in I$ with $\{j, j, j\} = jj^*j = i_2$. Thus

$$\begin{aligned} i_1 i_2^* z = i_1 j^* j j^* z &= 2 i_1 j^* \{j, j, z\} - i_1 j^* z j^* j \\ &= 2 i_1 j^* \{j, j, z\} - i_1 \{j, z, j\}^* j \in [I] \end{aligned}$$

and similarly $z i_2^* i_1 \in [I]$. Now let $x, y \in Z$, $i \in I$ and, as above, $j \in I$ with $jj^*j = i$. We have

$$\begin{aligned} x i^* y = x j^* j j^* y &= 2\{x, j, j\} j^* y - j j^* x j^* y \\ &= 2\{x, j, j\} j^* y - j\{j, x, j\}^* y \in [I], \end{aligned}$$

and similarly we get $ix^*y \in [I]$ and $xy^*i \in [I]$. □

Lemma 4.2.3. *For every JB^*-triple ideal I in a JB^*-triple system Z and every JB^*-triple homomorphism $\varphi : I \to W$, where W is a JBW^*-triple system, there exists a JB^*-triple homomorphism $\Phi : Z \to W$ which extends φ.*

Proof. We know by [Din86] that the second dual Z'' of Z is a JBW^*-triple system and the canonical embedding $\iota : Z \to Z''$ is an isometric JB^*-triple isomorphism onto a norm closed w^*-dense subtriple of Z''. By [BC92], Remark 1.1 and since W is a JBW^*-triple system there exists a unique, w^*-continuous extension $\overline{\varphi} : I'' \to W$ of φ with $\overline{\varphi}(I'') = \overline{\varphi(I)}^{w^*}$. Let

$$I^\perp := \{x \in Z'' : y \mapsto \{x, i, y\} \text{ is the 0 mapping for all } i \in I''\}$$

be the w^*-closed orthogonal complement of I'' with $Z'' = I'' \oplus I^\perp$ (cf. [Hor87a], Theorem 4.2 (4)). If we denote the projection of Z'' onto I'' with π we get the desired extension of φ by defining $\Phi := \overline{\varphi} \circ \pi \circ \iota$. □

Likewise to the case of JC-algebras (cf. [HO83]) the functor τ is exact.

Theorem 4.2.4. *Every exact sequence of JC^*-triple systems*

$$0 \to I \to Z \to Z/I \to 0,$$

where I is a JB^*-triple ideal of Z, induces an exact sequence of the corresponding universal enveloping TROs:

$$0 \to \tau(I) \to \tau(Z) \to \tau(Z/I) \to 0.$$

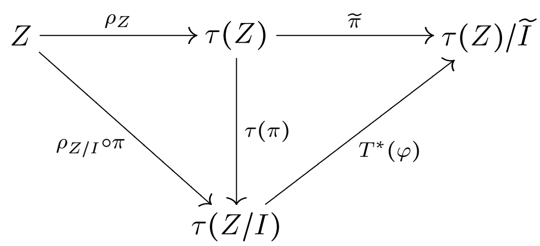
Proof. Let $\iota : I \to Z$ and $\pi : Z \to Z/I$ be the canonical injection and quotient homomorphism, respectively.

We first show exactness at $\tau(Z/I)$: The TRO $\tau(Z)$ is generated by $\rho_Z(Z)$ and the TRO $\tau(Z/I)$ by

$$\begin{aligned} \rho_{Z/I}(Z/I) &= \rho_{Z/I}(\pi(Z)) \\ &= T^*(\rho_{Z/I} \circ \pi)(\rho_Z(Z)) \\ &= \tau(\pi)(\rho_Z(Z)). \end{aligned}$$

Next we show exactness at $\tau(Z)$: We have $\tau(\pi) \circ \tau(\iota) = \tau(\pi \circ \iota) = 0$ by functoriality. Let $\widetilde{I} := \tau(\iota)(\tau(I))$, then $\widetilde{I}$ is a TRO (cf. Theorem 2.2.12 (c)) and the JB^*-triple ideal $\rho_Z(\iota(I)) \subseteq \rho_Z(Z)$ generates $\widetilde{I}$ as a TRO. By Lemma 4.2.2 the subTRO $\widetilde{I}$ is a TRO-ideal of $\tau(Z)$.

Let $\widetilde{\pi} : \tau(Z) \to \tau(Z)/\widetilde{I}$ be the quotient homomorphism onto the TRO $\tau(Z)/\widetilde{I}$ (cf. Remark 3.3.5), then $\widetilde{\pi} \circ \rho_Z \circ \iota = 0$, since $\widetilde{I}$ is generated by $\rho_Z(\iota(I))$. Therefore the JB^*-triple homomorphism $\widetilde{\pi} \circ \rho_Z$ induces a JB^*-triple homomorphism $\varphi : Z/I \to \tau(Z)/\widetilde{I}$, which induces the TRO-homomorphism $T^*(\varphi) : \tau(Z/I) \to \tau(Z)/\widetilde{I}$. In the diagram



the left triangle trivially commutes and the outer triangle commutes by the definition of φ. If $z \in \rho_Z(Z) \subseteq \tau(Z)$, we find an element $x \in Z$ with $\rho_Z(x) = z$. Since the left and the outer triangle commute we have

$$\begin{aligned} \widetilde{\pi}(z) &= \widetilde{\pi}(\rho_Z(x)) \\ &= T^*(\varphi) \circ \rho_{Z/I} \circ \pi(x) \\ &= T^*(\varphi) \circ \tau(\pi)(\rho_Z(x)) \\ &= T^*(\varphi) \circ \tau(\pi)(z). \end{aligned}$$

Now $\rho_Z(Z)$ generates $\tau(Z)$ and thus the right triangle commutes. We obtain the desired inclusion $\ker(\tau(\pi)) \subseteq \widetilde{I} = \tau(\iota)(\tau(I))$.

Finally we show exactness at $\tau(I)$: Let H be a Hilbert space and $\alpha : \tau(I) \to B(H)$ an injective TRO-homomorphism. Then $\alpha \circ \rho_I : I \to B(H)$ is an injective JB^*-homomorphism and we get by Lemma 4.2.3 an extension $\overline{\alpha} : Z \to B(H)$ of $\alpha \circ \rho_I$. This JB^*-homomorphism lifts to a TRO-homomorphism $T^*(\overline{\alpha}) : \tau(Z) \to B(H)$. With the universal property we get $T^*(\overline{\alpha}) \circ \tau(\iota) = \alpha$. Since α is injective, so is $\tau(\iota) : \tau(I) \to \tau(Z)$. □

Definition 4.2.5. *Let Z_1 and Z_2 be JB^*-triple systems and $\alpha, \beta : Z_1 \to Z_2$ JB^*-triple homomorphisms. The mappings α and β are called* ***homotopic****, denoted $\alpha \sim_h^{JB^*} \beta$, when there is a path $(\gamma_t)_{t\in[0,1]}$ of JB^*-triple homomorphisms $\gamma_t : Z_1 \to Z_2$ such that $t \mapsto \gamma_t(z)$ is a norm continuous path in Z_2 for every $z \in Z_1$ with $\gamma_0 = \alpha$, $\gamma_1 = \beta$.*

A JB^*-triple homomorphism $\alpha : Z_1 \to Z_2$ is called a ***homotopic equivalence*** *when there is a JB^*-triple homomorphism $\beta : Z_2 \to Z_1$ such that $\alpha \circ \beta$ and $\beta \circ \alpha$ both are homotopic to the identity.*

Proposition 4.2.6. *Let $\alpha, \beta : Z_1 \to Z_2$ be homotopic JB^*-triple homomorphisms. If $(T^*(Z_1), \rho_{Z_1})$ and $(T^*(Z_2), \rho_{Z_2})$ are the corresponding universal enveloping TROs, then the functor τ induces a TRO homotopy between $\tau(\alpha)$ and $\tau(\beta)$.*

Proof. Let $(\gamma_t)_{t\in[0,1]}$ be a pointwise continuous path in Z_2 which connects α and β and $(\widetilde{\gamma}_t)_{t\in[0,1]}$ be the path defined by $\widetilde{\gamma}_t := \tau(\gamma_t) : \tau(Z_1) \to \tau(Z_2)$. Obviously $(\widetilde{\gamma}_t)$ connects $\tau(\alpha)$ and $\tau(\beta)$, so the only thing to show is that $t \mapsto \widetilde{\gamma}_t(z)$ defines a norm continuous path in $\tau(Z_2)$ for every $z \in \tau(Z_1)$. Since $\tau(Z_1)$ is generated by $\rho_{Z_1}(Z_1)$ we can assume w.l.o.g. that

$$z = \rho_{Z_1}(z_1)\rho_{Z_1}(z_2)^*\rho_{Z_1}(z_3)\ldots\rho_{Z_1}(z_{2n})^*\rho_{Z_1}(z_{2n+1})$$

with $z_i \in Z_1$. Then

$$\begin{aligned}\widetilde{\gamma}_t(z) &= \widetilde{\gamma}_t(\rho_{Z_1}(z_1))\widetilde{\gamma}_t(\rho_{Z_1}(z_2))^*\widetilde{\gamma}_t(\rho_{Z_1}(z_3))\ldots\widetilde{\gamma}_t(\rho_{Z_1}(z_{2n}))^*\widetilde{\gamma}_t(\rho_{Z_1}(z_{2n+1}))\\ &= \rho_{Z_2}(\gamma_t(z_1))\rho_{Z_2}(\gamma_t(z_2))^*\rho_{Z_2}(\gamma_t(z_3))\ldots\rho_{Z_2}(\gamma_t(z_{2n}))^*\rho_{Z_2}(\gamma_t(z_{2n+1})),\end{aligned}$$

which is norm continuous in t. □

Recall the definition of an inductive sequence and an inductive limit from Section 3.3. With the help of our functor τ we are able to show that inductive limits exist in the category of JC^*-triple systems.

Lemma 4.2.7. *Let $((Z_n), (\varphi_n))$ be an inductive sequence in the category of JC^*-triple systems, then $((\tau(Z_n)), (\tau(\varphi_n)))$ is an inductive sequence of TROs. Moreover, if $(T_\infty, (\mu_n))$ is the inductive limit of $((\tau(Z_n)), (\tau(\varphi_n)))$*

in the category of TROs, then $(Z_\infty, (\nu_n))$ *is the inductive limit in the category of* JC^**-triple systems, where*

$$Z_\infty := \overline{\bigcup_{n=1}^{\infty} \mu_n(\rho_{Z_n}(Z_n))}$$

with homomorphisms $\nu_n := \mu_n \circ \rho_{Z_n} : Z_n \to Z_\infty$ *for all* $n \in \mathbb{N}$.

Proof. It is straightforward to check that $((\tau(Z_n)), (\tau(\varphi_n)))$ is an inductive sequence. This sequence of TROs has an inductive limit by Proposition 3.3.10 and an argument similar to the proof of Proposition 3.3.10 shows the rest of the Lemma. □

Now that we have proved the existence of inductive limits in the category of JC^*-triples we can prove the continuity of the functor τ.

Proposition 4.2.8. *Let* $((Z_n), (\varphi_n))$ *be an inductive sequence in the category of* JC^**-triple systems. If the pair* $(Z_\infty, (\mu_n))$ *is the inductive limit of* $((Z_n), (\varphi_n))$*, then* $(\tau(Z_\infty), (\tau(\mu_n)))$ *is the inductive limit of the induced sequence of TROs.*

Proof. We know by functoriality of τ that

$$\tau(\mu_{n+1}) \circ \tau(\varphi_n) = \tau(\mu_{n+1} \circ \varphi_n) = \tau(\mu_n)$$

for all $n \in \mathbb{N}$, so (3.4) is fulfilled. To show (3.5) let $(T_\infty, (\lambda_n))$ be another system such that $\lambda_{n+1} \circ \tau(\mu_n) = \lambda_n$ for all $n \in \mathbb{N}$. We first notice that the commutative diagram

$$\begin{array}{ccccc}
Z_n & & \xrightarrow{\mu_n} & & Z_{n+1} \\
\downarrow{\scriptstyle \rho_{Z_n}} & & & & \downarrow{\scriptstyle \rho_{Z_{n+1}}} \\
T^*(Z_n) & & \xrightarrow{\tau(\mu_n)} & & T^*(Z_{n+1}) \\
& {\scriptstyle \lambda_n}\searrow & & \swarrow{\scriptstyle \lambda_{n+1}} & \\
& & T_\infty & &
\end{array}$$

induces by the universal property of the inductive limit a unique JB^*-triple homomorphism $\lambda : Z_\infty \to T_\infty$ such that

$$\begin{array}{ccc}
Z_n & \xrightarrow{\rho_{Z_n}} & T^*(Z_n) \\
\downarrow{\scriptstyle \mu_n} & & \downarrow{\scriptstyle \lambda_n} \\
Z_\infty & \xrightarrow{\lambda} & T_\infty
\end{array}$$

commutes for all $n \in \mathbb{N}$. This induces a unique TRO-homomorphism $T^*(\lambda) : T^*(Z_\infty) \to T_\infty$ such that

$$\begin{array}{ccc} & T^*(Z_n) & \\ {}^{\tau(\mu_n)}\swarrow & & \searrow^{\lambda_n} \\ T^*(Z_\infty) & \xrightarrow[T^*(\lambda)]{} & T_\infty \end{array}$$

commutes for all $n \in \mathbb{N}$, which shows that $(\tau(Z_\infty), (\tau(\mu_n)))$ is the inductive limit $(\tau(Z_n), (\tau(\varphi_n)))$. □

4.3 Universally reversible JC^*-triple systems

We use the theory of reversibility developed in [BFT10] to make comments on how the functor τ operates on matrix levels. In order to do this we first have to introduce a notion for the matrix levels of JC^*-triple systems using the universal enveloping TRO since the higher matrix levels of JC^*-triple systems are not JC^*-triple systems themselves in general. We show that, if $T \subseteq T^*(T) \simeq T \oplus \theta(T)$ is a universally reversible TRO (see below for the definition) that does not contain an ideal of codimension 1, then

$$M_n(T^*(T)) \simeq M_n(T) \oplus \theta_n\left(M_n(T)\right) \simeq M_n(T^*(T)),$$

where θ is the canonical TRO-antiautomorphism of $T^*(T)$, constructed in Proposition 4.1.3, and θ_n is its transposed amplification for all $n \in \mathbb{N}$. A similar result is obtained for JC^*-triple systems. Thus the phenomenon of duplication which we were able to witness in the case of finite-dimensional Cartan factors of type I and rank ≥ 2 (which are exactly the simple, finite-dimensional TROs which are universally reversible) carries over to the matrix levels of TROs.

We also consider the case in which a universally reversible TRO T contains an ideal of codimension 1 which is not covered in [BFT10]. We show here that there exists an ideal $\mathbf{R}(T)$ in T which is universally reversible and which does not contain an ideal of codimension 1 itself, such that $T/\mathbf{R}(T)$ is an Abelian JB^*-triple system. We obtain an exact sequence

$$0 \longrightarrow \mathbf{R}(T) \oplus \theta(\mathbf{R}(T)) \longrightarrow T^*(T) \longrightarrow C_0^{\mathbb{T}}(\mathrm{Epi}(T/\mathbf{R}(T), \mathbb{C})) \longrightarrow 0,$$

where the notation is taken from (4.1) ff.

Finally we define the reversible hull of a JC^*-triple system and analyze its universal properties.

4.3.1 Matrix levels

We adopt the following definition from [BFT10]. It is the generalization of reversibility of JC^*-algebras.

Definition 4.3.1. *A JC^*-triple system $Z \subseteq B(H)$ is said to be **reversible** if*

$$\frac{1}{2}(x_1 x_2^* x_3 \dots x_{2n}^* x_{2n+1} + x_{2n+1} x_{2n}^* \dots x_3 x_2^* x_1) \in Z$$

*for all $x_1, \dots, x_n \in Z$ and $n \in \mathbb{N}$. We call a JC^*-triple system **universally reversible** if it is reversible in every representation.*

Obviously every TRO, and therefore every C^*-algebra, is reversible (but not necessarily universally reversible, since we have to cope with JB^*-triple homomorphisms).

A JC^*-algebra is universally reversible if its canonical embedding is reversible in its universal enveloping C^*-algebra. The analogue for JC^*-triple system is easily seen: A JC^*-triple system is universally reversible if and only if it is reversible when embedded in its universal enveloping TRO.

In general the nth matrix level of a JC^*-triple system is not a JC^*-triple system itself for $n \geq 2$. A result of Hamana in [Ham99] even states that a JC^*-triple system is a TRO if and only if its second matrix level is a JC^*-triple system. Therefore we do not consider the nth matrix level of a JC^*-triple system but rather the JC^*-triple generated by this set. This definition becomes independent of the embedding, if we endow the JC^*-triple system with the operator space structure inherited from its universal enveloping TRO.

Definition 4.3.2. *Let $Z \subseteq B(H)$ be a JC^*-triple system and $n \in \mathbb{N}$. We denote by $J_n^*(Z)$ the JC^*-triple system generated by $M_n(\rho_Z(Z))$ in the TRO $M_n(T^*(Z))$ and call it the **nth matrix level** of Z.*

To determine the universal enveloping TRO of the nth matrix levels of TROs and thus be able to make some comments on the stability of the functor τ, we have to recall some results from [BFT10]. We state in Theorem 4.3.13 a generalization of Lemma 4.3.4, dropping the assumption that the TRO has no ideals of codimension 1.

Lemma 4.3.3 ([BFT10], Theorem 4.4)**.** *Let Z be a universally reversible JC^*-triple system and let $\varphi : Z \to B(H)$ be an injective triple homomorphism. Suppose there exists a TRO antiautomorphism Ψ of the TRO-span $\mathrm{TRO}(\varphi(Z))$ such that $\Psi \circ \varphi = \varphi$, then $T^*(\varphi) : T^*(Z) \to \mathrm{TRO}(\varphi(Z))$ is a TRO-isomorphism.*

Lemma 4.3.4 ([BFT10], Corollary 4.5)**.** *Let T be a universally reversible TRO in a C^*-algebra $\mathfrak{A}$. Suppose T has no TRO-ideals of codimension 1 and there is a TRO antiautomorphism $\theta : \mathfrak{A} \to \mathfrak{A}$ of order 2. Then $T^*(T) \simeq T \oplus \theta(T)$ with universal embedding $a \mapsto (a, \theta(a))$.*

The universally reversible TROs can be characterized: Corollary 11.26 in [BFT10] states, that a TRO T is universally reversible if and only if it has no

TRO representations onto a TRO isometric to a Hilbert space of dimension strictly greater than 2.

Theorem 4.3.5. *Let T be a universally reversible TRO which does not contain a TRO-ideal of codimension* 1*. If we assume $T \subseteq T^*(T)$, then*

$$T^*(M_n(T)) \simeq M_n(T) \oplus \theta_n(M_n(T)) \simeq M_n(T \oplus \theta(T)) \simeq M_n(T^*(T)),$$

for all $n \in \mathbb{N}$, where θ is the canonical involutive antiautomorphism of $T^*(T)$ and $\theta_n((x_{i,j})) = (\theta(x_{j,i}))$.

Proof. Let $n \in \mathbb{N}$. Since every TRO-ideal in $M_n(T)$ is of the form $M_n(I)$, for a TRO-ideal $I \subseteq T$, the TRO $M_n(T)$ does not contain a one codimensional ideal for all $n \in \mathbb{N}$. We first show that $T^*(M_n(T)) \simeq M_n(T) \oplus \theta_n(M_n(T))$. The mapping θ_n is an involutive antiautomorphism of $M_n(C^*(T))$ and $M_n(T)$ is universally reversible, since T is universally reversible. Thus we can use Lemma 4.3.4.

Next we to prove that $M_n(T \oplus \theta(T)) \simeq M_n(T^*(T))$. Since T has no ideal of codimension 1, this follows directly from Lemma 4.3.4, because $T^*(T) \simeq T \oplus \theta(T)$.

Finally we show that $M_n(T) \oplus \theta_n(M_n(T)) \simeq M_n(T \oplus \theta(T))$. Let $\varphi : M_n(T) \to M_n(T \oplus \theta(T))$ be given by

$$\varphi((x_{i,j})) = (x_{i,j}, \theta(x_{j,i}))_{i,j}.$$

We first show that the TRO generated by the image of φ equals $M_n(T \oplus \theta(T))$. To prove this we show that the elements $(x, \theta(y))^{i,j} \in M_n(T \oplus \theta(T))$ which are 0 everywhere except $(x, \theta(y))$ in the ith row and jth column are contained in the TRO-span of the image of φ, for all $1 \leq i, j \leq n$, $x, y \in T$. Let $(x)^{i,j}$ be the matrix in $M_n(T)$ which is 0 everywhere except x in the ith row and the jth column. Since for every element $x \in T$ there exists by Lemma 2.3.7 an element $z \in T$ with $zz^*z = x$ we have for $i \neq k$ or $j \neq l$

$$\begin{aligned}
&\varphi\left((z)^{i,j}\right)\left(\varphi\left((z)^{k,j}\right)\right)^* \varphi\left((z)^{k,l}\right) \\
&= \left((z,0)^{i,j} + (0,\theta(z))^{j,i}\right)\left((z,0)^{k,j} + (0,\theta(z))^{j,k}\right)^* \left((z,0)^{k,l} + (0,\theta(z))^{l,k}\right) \\
&= \left((z,0)^{i,j} + (0,\theta(z))^{j,i}\right)\left((z^*,0)^{j,k} + (0,\left(\theta(z)^*\right)^{k,j})\right)\left((z,0)^{k,l} + (0,\theta(z))^{l,k}\right) \\
&= (z,0)^{i,j}(z^*,0)^{j,k}(z,0)^{k,l} \\
&= (x,0)^{i,l},
\end{aligned}$$

for all $1 \leq i, j, k, l \leq n$, and analogously for the other component using that θ is a TRO-antiautomorphism. Once we have shown that the mapping Ω of $M_n(T \oplus \theta(T))$ which is given by

$$\Omega((x_{i,j}, \theta(y_{i,j}))) = (y_{j,i}, \theta(x_{j,i}))$$

is an involutive antiautomorphism. Then, since Ω leaves $\varphi(M_n(T \oplus \theta(T)))$ fixed and since $M_n(T) \oplus \theta_n(M_n(T))$ is the universal enveloping TRO of $M_n(T)$, we get with Lemma 4.3.3 that $T^*(\varphi)$ is a TRO-isomorphism. So let $A = (\alpha^1_{i,j}, \theta(\alpha^2_{i,j}))$, $B = (\beta^1_{i,j}, \theta(\beta^2_{i,j}))$ and $C = (\gamma^1_{i,j}, \theta(\gamma^2_{i,j})) \in M_n(T \oplus \theta(T))$, then

$$\begin{aligned}
\Omega(AB^*C) &= \Omega\Big(\left(\alpha^1_{i,j}, \theta\left(\alpha^2_{i,j}\right)\right)_{i,j} \left(\beta^1_{i,j}, \theta\left(\beta^2_{i,j}\right)\right)^*_{i,j} \left(\gamma^1_{i,j}, \theta\left(\gamma^2_{i,j}\right)\right)_{i,j}\Big) \\
&= \Omega\Big(\left(\alpha^1_{i,j}, \theta\left(\alpha^2_{i,j}\right)\right)_{i,j} \left(\left(\beta^1_{j,i}\right)^*, \theta\left(\beta^2_{j,i}\right)^*\right)_{i,j} \left(\gamma_{i,j}, \theta\left(\gamma^2_{i,j}\right)\right)_{i,j}\Big) \\
&= \Omega\left(\left(\sum_{l=1}^n \sum_{k=1}^n \alpha^1_{i,k}\left(\beta^1_{l,k}\right)^* \gamma^1_{l,j}, \sum_{l=1}^n \sum_{k=1}^n \theta\left(\alpha^2_{i,k}\right)\theta\left(\beta^2_{l,k}\right)^* \theta\left(\gamma^2_{l,j}\right)\right)_{i,j}\right) \\
&= \Omega\left(\left(\sum_{l=1}^n \sum_{k=1}^n \alpha^1_{i,k}\left(\beta^1_{l,k}\right)^* \gamma^1_{l,j}, \theta\left(\sum_{l=1}^n \sum_{k=1}^n \gamma^2_{l,j}\left(\beta^2_{l,k}\right)^* \alpha^2_{i,k}\right)\right)_{i,j}\right) \\
&= \left(\sum_{l=1}^n \sum_{k=1}^n \gamma^2_{l,i}\left(\beta^2_{l,k}\right)^* \alpha^2_{j,k}, \theta\left(\sum_{l=1}^n \sum_{k=1}^n \alpha^1_{j,k}\left(\beta^1_{l,k}\right)^* \gamma^1_{l,i}\right)\right)_{i,j} \\
&= \left(\sum_{l=1}^n \sum_{k=1}^n \gamma^2_{l,i}\left(\beta^2_{l,k}\right)^* \alpha^2_{j,k}, \sum_{l=1}^n \sum_{k=1}^n \theta\left(\gamma^1_{l,i}\right)\theta\left(\beta^1_{l,k}\right)^* \theta\left(\alpha^1_{j,k}\right)\right)_{i,j} \\
&= \left(\sum_{k=1}^n \sum_{l=1}^n \gamma^2_{l,i}\left(\beta^2_{l,k}\right)^* \alpha^2_{j,k}, \sum_{k=1}^n \sum_{l=1}^n \theta\left(\gamma^1_{l,i}\right)\theta\left(\beta^1_{l,k}\right)^* \theta\left(\alpha^1_{j,k}\right)\right)_{i,j} \\
&= \left(\gamma^2_{j,i}, \theta\left(\gamma^1_{j,i}\right)\right)_{i,j} \left(\left(\beta^2_{i,j}\right)^*, \left(\theta\left(\beta^1_{i,j}\right)\right)^*\right)_{i,j} \left(\alpha^2_{j,i}, \theta\left(\alpha^1_{j,i}\right)\right)_{i,j} \\
&= \left(\gamma^2_{j,i}, \theta\left(\gamma^1_{j,i}\right)\right)_{i,j} \left(\beta^2_{j,i}, \theta\left(\beta^1_{j,i}\right)\right)^*_{i,j} \left(\alpha^2_{j,i}, \theta\left(\alpha^1_{j,i}\right)\right)_{i,j} \\
&= \Omega\left(\left(\gamma^1_{i,j}, \theta\left(\gamma^2_{i,j}\right)\right)_{i,j}\right) \Omega\left(\left(\beta^1_{i,j}, \theta\left(\beta^2_{i,j}\right)\right)_{i,j}\right)^* \Omega\left(\left(\alpha^1_{i,j}, \theta\left(\alpha^2_{i,j}\right)\right)_{i,j}\right) \\
&= \Omega(C)\,\Omega(B)^*\,\Omega(A).
\end{aligned}$$

□

Proposition 4.3.6. *Let Z be a universally reversible JC^*-triple system such that $T^*(Z)$ is universally reversible and does not contain a TRO-ideal of codimension 1. If $J^*_n(Z)$ is universally reversible for all $n \in \mathbb{N}$, then*

$$T^*(J^*_n(Z)) \simeq M_n(T^*(Z)) \oplus \theta_n(M_n(T^*(Z)),$$

for all $n \in N$.

Proof. The mapping

$$\begin{aligned}
\varphi : & J^*_n(Z) \to M_n(T^*(Z)) \oplus \theta_n(M_n(T^*(Z))), \\
& (x_{i,j})_{i,j} \mapsto ((x_{i,j})_{i,j}, \theta_n((x_{i,j})_{i,j}))\,.
\end{aligned}$$

is a JB^*-triple homomorphism and its image is fixed under the involutive TRO-antiautomorphism Ω on $M_n(T^*(Z)) \oplus \theta_n(M_n(T^*(Z)))$ given by

$$\Omega\left(\left((x_{i,j})_{i,j}, \theta_n\left((y_{i,j})_{i,j}\right)\right)\right) = \left((y_{i,j})_{i,j}, \theta_n\left((x_{i,j})_{i,j}\right)\right).$$

The image of φ generates $M_n(T^*(Z)) \oplus \theta_n(M_n(T^*(Z)))$ as a TRO (here one needs that for every element x there exists an element y with $yy^*y = x$, which is true by Lemma 2.3.7). Thus

$$T^*(\varphi) : T^*(J_n^*(Z)) \to M_n(T^*(Z)) \oplus \theta_n(M_n(T^*(Z)))$$

is a TRO-isomorphism by Lemma 4.3.3. □

4.3.2 The radical

We now establish the announced generalization of Lemma 4.3.4. In order to do so we define an ideal in a universal TRO T such that the quotient of T by this ideal is Abelian. We are first recalling some known facts about Abelian JB^*-triple systems which allow us to compute the universal enveloping TRO of a general Abelian triple. Afterwards we show that every ideal of a universal reversible JC^*-triple system is universally reversible.

Recall that a JB^*-triple system Z is called Abelian, if

$$\{\{a,b,c\},d,e\} = \{a,\{b,c,d\},e\} = \{a,b,\{c,d,e\}\}$$

for all $a,b,c,d,e \in Z$. The importance of Abelian JB^*-triple systems derives from the fact that every JB^*-triple system is locally Abelian, which means that every element in a JB^*-triple system generates an Abelian subtriple. Every commutative C^*-algebra is an Abelian JB^*-triple system with the product $\{a,b,c\} = ab^*c$. We call the elements of

$$\mathrm{Epi}(Z,\mathbb{C}) := \{\varphi : Z \to \mathbb{C} : \varphi \neq 0 \text{ is a triple homomorphism}\}$$

the **characters** of Z. Following [Kau83], §1 we consider $\mathrm{Epi}(Z,\mathbb{C})$ as a subspace of $Z' = B(Z,\mathbb{C})$ and endow it with the $\sigma(Z^*, Z)$ topology. Then $\mathrm{Epi}(Z,\mathbb{C})$ becomes a locally compact space and a principal $\mathbb{T}$-bundle for the group

$$\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}.$$

The base space $\mathrm{Epi}(Z,\mathbb{C})/\mathbb{T}$ can be identified with the set of all JB^*-triple ideals $I \subseteq Z$ such that Z/I is isometric to $\mathbb{C}$. The space

$$C_0^{\mathbb{T}}(\mathrm{Epi}(Z,\mathbb{C})) := \{f \in C_0(\mathrm{Epi}(Z,\mathbb{C})) | \forall t \in \mathbb{T}\ \forall \lambda \in \mathrm{Epi}(Z,\mathbb{C}) : f(t\lambda) = tf(\lambda)\}$$

is a subtriple of the Abelian C^*-algebra $C_0(\mathrm{Epi}(Z,\mathbb{C}))$, the continuous functions on $\mathrm{Epi}(Z,\mathbb{C})$ vanishing at infinity. The mapping

$$\hat{\ } : Z \to C_0^{\mathbb{T}}(\mathrm{Epi}(Z,\mathbb{C})) \tag{4.1}$$

defined by $\hat{x}(\lambda) = \lambda(x)$ for all $x \in Z$ and $\lambda \in \mathrm{Epi}(Z,\mathbb{C})$ is called the **Gelfand transform** of Z.

Theorem 4.3.7 ([Kau95], Theorem 6.2)**.** *For every JB^*-triple system Z the following assertions are equivalent:*

(a) *Z is Abelian.*

(b) *Z is a subtriple of a commutative C^*-algebra.*

(c) *The Gelfand transform of Z is a surjective isometry onto $C_0^{\mathbb{T}}(\mathrm{Epi}(Z,\mathbb{C}))$.*

Especially every Abelian JB^*-triple system is a TRO.

Lemma 4.3.8. *Let Z be an Abelian JC^*-triple. Then Z is a universally reversible TRO.*

Proof. We only have to show that every Abelian JC^*-triple system is already a TRO since every TRO is already reversible, but by Theorem 4.3.7 we know that Z is a subtriple of an Abelian C^*-algebra and therefore a TRO. □

Proposition 4.3.9. *Let Z be an Abelian JC^*-triple system, then*

$$T^*(Z) \simeq C_0^{\mathbb{T}}(\mathrm{Epi}(Z,\mathbb{C}))$$

and the universal embedding $\rho_Z : Z \to C_0^{\mathbb{T}}(\mathrm{Epi}(Z,\mathbb{C}))$ is given by the Gelfand transform of Z.

Proof. The Abelian JC^*-triple system Z is by Lemma 4.3.8 a universally reversible TRO. Let $\hat{}: Z \to C_0^{\mathbb{T}}(\mathrm{Epi}(Z,\mathbb{C}))$ be the Gelfand transform, which is by Theorem 4.3.7 a JB^*-triple isomorphism. The identity mapping id on $C_0^{\mathbb{T}}(\mathrm{Epi}(Z,\mathbb{C}))$ is, since we are in the Abelian world, also an antiautomorphism, satisfying $\mathrm{id}\circ\hat{} = \hat{}$. Since $\hat{Z}$ generates $C_0^{\mathbb{T}}(\mathrm{Epi}(Z,\mathbb{C}))$ as a TRO we obtain the statement from Lemma 4.3.3. □

Definition 4.3.10. *Let Z be an universally reversible JC^*-triple system. Define the* ***radical*** *of Z to be the set*

$$\mathbf{R}(Z) := \bigcap_{\varphi\in \mathrm{Epi}(Z,\mathbb{C})\cup\{0\}} \ker(\varphi).$$

In the case that $\mathrm{Epi}(Z,\mathbb{C}) = \varnothing$ we have $\mathbf{R}(Z) = Z$.

The next proposition helps us to show that the radical of a universal reversible JC^*-triple system is universally reversible.

Proposition 4.3.11. *Let Z be a universally reversible JC^*-triple system and $I \subseteq Z$ a JB^*-triple ideal, then I is also universally reversible.*

Proof. We assume that $T^*(I) \subseteq T^*(T)$. It suffices to show that $\rho_Z(I) \subseteq T^*(Z)$ is reversible. Since $T^*(I)$ is a TRO-ideal and $\rho_Z(Z)$ is reversible by definition, we know that $\rho_Z(I)$ is reversible, if

$$\rho_Z(I) = T^*(I) \cap \rho_Z(Z).$$

Let $x \in T^*(I) \cap \rho_Z(Z)$ and $\pi : \rho_Z(Z) \to \rho_Z(Z)/\rho_Z(I)$ be the JB^*-quotient homomorphism. It follows from Theorem 4.2.4 that

$$T^*(Z)/T^*(I) \simeq T^*(\rho_Z(Z)/\rho_Z(I))$$

and therefore $\pi(x) = \tau(\pi)(x) = 0$, which yields $x \in \rho_Z(I)$. □

Since the radical is always a JB^*-triple ideal the next corollary follows immediately.

Corollary 4.3.12. *Let Z be a universally reversible JC^*-triple system, then $\mathbf{R}(Z)$ is universally reversible.*

Theorem 4.3.13. *Let T be a universally reversible TRO embedded in a C^*-algebra $\mathfrak{A}$ such that there exists a TRO antiautomorphism $\theta : \mathfrak{A} \to \mathfrak{A}$ of order 2. Then we have an exact sequence of TROs*

$$0 \longrightarrow \mathbf{R}(T) \oplus \theta(\mathbf{R}(T)) \longrightarrow T^*(T) \longrightarrow C_0^{\mathbb{T}}(\mathrm{Epi}(T/\mathbf{R}(T), \mathbb{C})) \longrightarrow 0. \quad (4.2)$$

Proof. By Corollary 4.3.12 we know that the radical $\mathbf{R}(T)$ is universally reversible and does not contain a TRO-ideal of codimension 1 by construction. Using Lemma 4.3.4 we get

$$T^*(\mathbf{R}(T)) = \mathbf{R}(T) \oplus \theta(\mathbf{R}(T)).$$

The quotient $T/\mathbf{R}(T)$ is an Abelian JB^*-triple system and we get with Proposition 4.3.9 that

$$T^*(T/(\mathbf{R}(T))) = C_0^{\mathbb{T}}(\mathrm{Epi}(T/\mathbf{R}(T), \mathbb{C})).$$

The exactness of (4.2) follows now from the exactness of

$$0 \longrightarrow \mathbf{R}(T) \longrightarrow T \longrightarrow T/\mathbf{R}(T) \longrightarrow 0,$$

and Theorem 4.2.4. □

Theorem 4.3.13 is a generalization of Lemma 4.3.4. If we add the additional assumption that T does not contain a one codimensional TRO-ideal, then $\mathbf{R}(T) = T$ and thus (4.2) becomes

$$0 \longrightarrow T \oplus \theta(T) \longrightarrow T^*(T) \longrightarrow 0 \longrightarrow 0.$$

4.3.3 The reversible hull

We now define another universal object, the reversible hull of a JC^*-triple system, and show some equivalent characterizations of it.

Definition 4.3.14. *Let Z be a JC^*-triple system with universal enveloping TRO $(T^*(Z), \rho_Z)$. We call*

$$\mathrm{Rev}(Z) := \overline{\mathrm{lin}}\{x_1 x_2^* x_3 \dots x_{2n}^* x_{2n+1} + x_{2n+1} x_{2n}^* x_{2n-1} \dots x_2^* x_1 : x_1, \dots, x_{2n+1} \in \rho_Z(Z), n \in \mathbb{N}\}$$

the ***reversible hull*** *of Z.*

Proposition 4.3.15. *Let Z be a JC^*-triple system. The reversible hull of Z has the following universal properties:* $\mathrm{Rev}(Z)$ *is the unique (up to reversible triple isomorphisms) reversible JC^*-triple system such that there exists a triple homomorphism $\xi_Z : Z \to \mathrm{Rev}(Z)$ with*

(i) $\mathrm{Rev}(Z)$ *is generated as a reversible JC^*-triple system by $\xi_Z(Z)$.*

(ii) *Whenever $\varphi : Z \to W$ is a homomorphism from Z to a reversible JC^*-triple system W there exists a JC^*-triple homomorphism* $\mathrm{Rev}(\varphi) : \mathrm{Rev}(Z) \to W$, *respecting the generalized triple product, with* $\mathrm{Rev}(\varphi) \circ \xi_Z = \varphi$.

Proof. $\mathrm{Rev}(Z)$ is a reversible JC^*-triple system by definition. Let $\xi_Z := \rho_Z$ and $\mathrm{Rev}(\varphi) := T^*(\varphi)|_{\mathrm{Rev}(Z)}$, then it is easy to check that (i) and (ii) are fulfilled. Uniqueness is proved as usual. □

Proposition 4.3.16. *Let Z be JC^*-triple system and θ be the canonical involutive antiautomorphism of $T^*(Z)$, then*

$$\mathrm{Rev}(Z) = \{x \in T^*(Z) : \theta(x) = x\}.$$

Proof. Let $A := \{x \in T^*(Z) : \theta(x) = x\}$. Then A is the image of $T^*(Z)$ under the projection $\frac{1}{2}(\theta + \mathrm{id})$, therefore a JC^*-triple system by [FR83]. Since θ fixes $\rho_Z(Z)$ and reverses the order of products A is a reversible JC^*-triple system containing Z, thus $\mathrm{Rev}(Z) \subseteq A$.

On the other hand let $x \in A$. Then w.l.o.g. $x = x_1 x_2^* x_3 \dots x_{2n+1}$, with $x_1, x_2, \dots, x_{2n+1} \in Z$, $n \in \mathbb{N}$. Therefore

$$\begin{aligned} 2x &= x + \theta(x) \\ &= x_1 x_2^* x_3 \dots x_{2n+1} + \theta(x_{2n+1})\theta(x_{2n})^*\theta(x_{2n-1}) \dots \theta(x_1) \\ &= x_1 x_2^* x_3 \dots x_{2n+1} + x_{2n+1} x_{2n}^* x_{2n-1} \dots x_1 \in \mathrm{Rev}(Z). \end{aligned}$$

□

Proposition 4.3.16 shows in particular that Z is universally reversible if and only if $\rho_Z(Z) = \mathrm{Rev}(Z)$.

Unfortunately the reversible hull is not universally reversible in general:

Let $C^1_{1,3} = \mathbb{M}_{1,3}$ be the 3-dimensional type 1 rank 1 Cartan factor. We see in the next section that

$$T^*(Z) = \mathbb{M}_{1,3} \oplus \mathbb{M}_{3,3} \oplus \mathbb{M}_{3,1}.$$

Let π be the projection of $T^*(Z)$ onto the first summand. From the construction of the space H^1_3 (cf. 4.4.4 for details) we see that $\pi(\rho_Z(Z)) = \mathbb{M}_{1,3}$, thus $\mathrm{Rev}(Z)$ has to contain $\mathbb{M}_{1,3}$ as direct summand and therefore has a representation on a Hilbert space.

4.4 Cartan factors

In this section we compute the universal enveloping TROs of the finite-dimensional Cartan factors. Since the universal enveloping TROs of the two exceptional factors are 0, we have to compute the factors of type I–IV. We do so by using the grids spanning these factors (cf. [DF87] and Chapter 2).

4.4.1 Factors of type IV

A spin system is a subset $S = \{\mathrm{id}, s_1, \dots, s_n\}$, $n \geq 2$, of self-adjoint elements of $B(H)$ which satisfy the anti-commutator relation $s_i s_j + s_j s_i = 2\delta_{i,j}$ for all $i, j \in \{1, \dots, n\}$. The complex linear span of S is a JC^*-algebra of dimension $n + 1$ (cf. [HOS84]). Every JC^*-triple system which is JB^*-isomorphic to such a JC^*-algebra is called a spin factor. We now recall the definition of a spin grid: A spin grid is a collection $\{u_j, \widetilde{u}_j | j \in J\}$ (or $\{u_j, \widetilde{u}_j | j \in J\} \cup \{u_0\}$ in finite odd dimensions), where J is an index set with $0 \notin J$, for $j \in J$, $u_j, \widetilde{u}_j$ are minimal tripotents and, if we let $i, j \in J$, $i \neq j$, then

(SPG1) $\{u_i, u_i, \widetilde{u}_j\} = \frac{1}{2}\widetilde{u}_j$, $\{\widetilde{u}_j, \widetilde{u}_j, u_i\} = \frac{1}{2}u_i$,

(SPG2) $\{u_i, u_i, u_j\} = \frac{1}{2}u_j$, $\{u_j, u_j, u_i\} = \frac{1}{2}u_i$,

(SPG3) $\{\widetilde{u}_i, \widetilde{u}_i, \widetilde{u}_j\} = \frac{1}{2}\widetilde{u}_j$, $\{\widetilde{u}_j, \widetilde{u}_j, \widetilde{u}_i\} = \frac{1}{2}\widetilde{u}_i$,

(SPG4) $\{u_i, u_j, \widetilde{u}_i\} = -\frac{1}{2}\widetilde{u}_j$,

(SPG5) $\{u_j, \widetilde{u}_i, \widetilde{u}_j\} = -\frac{1}{2}u_i$,

(SPG6) All other products of elements from the spin grid are 0.

In the case of finite odd dimensions (where u_0 is present) we have, for all $i \in J$, the additional conditions (as exceptions of (SPG6))

(SPG7) $\{u_0, u_0, u_i\} = u_i$, $\{u_i, u_i, u_0\} = \frac{1}{2}u_0$,

(SPG8) $\{u_0, u_0, \widetilde{u}_i\} = \widetilde{u}_i$, $\{\widetilde{u}_i, \widetilde{u}_i, u_0\} = \frac{1}{2}u_0$,

(SPG9) $\{u_0, u_i, u_0\} = -\widetilde{u}_i, \{u_0, \widetilde{u}_i, u_0\} = -u_i$.

It is known (cf. [DF87]) that every finite-dimensional spin factor is linearly spanned by a spin grid (but not necessarily by a spin system).

Let $\mathfrak{G} := \{u_i, \widetilde{u}_i : i \in I\}$ (resp. $\widetilde{\mathfrak{G}} := \mathfrak{G} \cup \{u_0\}$) be a spin grid which spans the JC^*-triple Z and $1 \in I$ an arbitrary index. If we define a tripotent $v := i(u_1 + \widetilde{u}_1)$, Neal and Russo gave a method how to construct from $\mathfrak{G}$ (resp. $\widetilde{\mathfrak{G}}$) and v a JC^*-triple system in [NR03], which is JB^*-triple isomorphic to Z and contains a spin system. First they have shown for the Peirce-2-space $P_2^v(Z)$ of v that $P_2^v(Z) = Z$ and that, if $\mathfrak{A}$ is any von Neumann algebra containing Z, then $P_2^v(\mathfrak{A})^{(v)}$ is a C^*-algebra TRO-isomorphic to $P_2^v(\mathfrak{A})$ (the isomorphism is the identity mapping). Moreover, they proved:

Theorem 4.4.1 ([NR03], 3.1)**.** *The space $P_2^v(Z)^{(v)}$ is the linear span of a spin grid. More precisely, let $s_j = u_j + \widetilde{u}_j, j \in I \setminus \{1\}$; $t_j := i(u_j - \widetilde{u}_j), j \in I$. Then a spin system in the unital C^*-algebra $\mathfrak{A}_2(v)^{(v)}$, which linearly spans $P_2^v(Z)^{(v)}$, is given by*

$$\{s_j, t_k, v : j \in I \setminus \{1\}, k \in I\}$$

or, if the spin factor is of odd finite dimension

$$\{s_j, t_k, v, u_0 : j \in I \setminus \{1\}, k \in I\}.$$

Lemma 4.4.2. *Let T be a TRO and $v \in \mathrm{Tri}(T)$.*

(a) *We have $P_2^v(T) = \{z \in T : v\,(vz^*v)^*\,v = z\}$.*

(b) *Let $Z \subseteq B(H)$ be a JC^*-triple system and T the TRO generated by Z. If $Z = P_2^v(Z)$, then $T = P_2^v(T)$.*

(c) *If v is a tripotent in the TRO T, then the Peirce-2-space $P_2^v(T)$ is a subTRO of T.*

Proof. **(a)** Let $z \in T$ with $vv^*z + zv^*v = 2z$. Then vv^* and v^*v are projections with $vv^*zv^*v + zv^*v = 2zv^*v$ and $vv^*zv^*v + vv^*z = 2vv^*z$. Thus we have $vv^*zv^*v = zv^*v = vv^*z$ and therefore $vv^*zv^*v = \frac{1}{2}(vv^*z + zv^*v) = z$.

If $z \in Z$ with $vv^*zv^*v = z$, then $vv^*zv^*v = zv^*v$ and $vv^*zv^*v = vv^*z$. We get $\frac{1}{2}(vv^*z + zv^*v) = vv^*zv^*v = z$.

(b) Let $x = z_1 z_2^* z_3 \dots z_{2n} z_{2n+1} \in T$, with $z_j \in Z = P_2^v(Z)$. By (a) we get $vv^*z_jv^*v = z_j$ and $z_j = vv^*z_j = z_jvv^*$. Thus $vv^*xv^*v = (vv^*z_1)z_2^*z_3 \dots z_{2n}^*(z_{2n+1}v^*v) = z_1z_2^*z_3 \dots z_{2n}^*z_{2n+1} = x$ and it follows that $x \in P_2^v(T)$.

(c) Let $a, b, c \in P_2^v(T)$, then $vv^*ab^*cv^*v = vv^*a(vv^*bv^*v)^*cv^*v = (vv^*av^*v)b^*(vv^*cv^*v) = ab^*c$.

□

As a first result we get an upper bound for the dimension of the universal enveloping TRO of a spin system:

Proposition 4.4.3. *Let Z be a spin factor of dimension $k+1<\infty$. Then*

$$\dim T^*(Z) \le 2^k.$$

Proof. For $k = 2n$ let $\mathfrak{G} = \{u_1, \widetilde{u}_1, \dots, u_n, \widetilde{u}_n\}$ (resp. $\mathfrak{G} = \{u_1, \widetilde{u}_1, \dots, u_n, \widetilde{u}_n\} \cup \{u_0\}$ for $k = 2n+1$) be a spin grid generating Z. Then $\rho_Z(\mathfrak{G})$ is a spin grid in $\rho_Z(Z) \subseteq T^*(Z)$. By Lemma 4.4.2 we have for $v := i(u_1 + \widetilde{u}_1)$ that $P_2^v(T^*(Z)) = T^*(Z)$, which is TRO-isomorphic to $P_2^v(T^*(Z))^{(v)}$. The unital C^*-algebra $P_2^v(T^*(Z))^{(v)}$ contains by Theorem 4.4.1 a spin system $\{\mathrm{id}, s_1, \dots, s_k\}$, which generates it as a C^*-algebra. It is easy to see (cf. [HOS84], Remark 7.1.12) that $P_2^v(T^*(Z))^{(v)}$ is linearly spanned by the 2^k elements $s_{i_1} \dots s_{i_j}$, where $1 \le i_1 < i_2 < \dots < i_j$ and $0 \le j \le k$. □

From the proof of Proposition 4.4.3 we can deduce that the universal enveloping TRO of a spin factor is TRO-isomorphic to its universal enveloping C^*-algebra, once we have shown that $\dim T^*(Z) = 2^k$.

In Jordan-C^*-theory the following famous spin system appears (cf. [HOS84], 6.2.1):

Let

$$\sigma_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

be the Pauli spin matrices.

For matrices $a = (\alpha_{i,j}) \in \mathbb{M}_k$ and $b \in \mathbb{M}_l$ we define $a \otimes b := (\alpha_{i,j} b) \in M_k(\mathbb{M}_l) = \mathbb{M}_{kl}$.

The so-called standard spin system, which is linearly generating a $(k+1)$-dimensional spin factor in $\mathbb{M}_{2^n}$, when $k \le 2n$, is given via $\{\mathrm{id}, s_1, \dots, s_k\}$ with

$$s_1 := \sigma_1 \otimes \underbrace{\mathrm{id} \otimes \dots \otimes \mathrm{id}}_{n-1 \text{ times}},$$

$$s_2 := \sigma_2 \otimes \underbrace{\mathrm{id} \otimes \dots \otimes \mathrm{id}}_{n-1 \text{ times}},$$

$$s_3 := \sigma_3 \otimes \sigma_1 \otimes \underbrace{\mathrm{id} \otimes \dots \otimes \mathrm{id}}_{n-2 \text{ times}},$$

$$s_4 := \sigma_3 \otimes \sigma_2 \otimes \underbrace{\mathrm{id} \otimes \dots \otimes \mathrm{id}}_{n-2 \text{ times}},$$

$$s_{2l+1} := \underbrace{\sigma_3 \otimes \dots \otimes \sigma_3}_{l \text{ times}} \otimes \sigma_1 \otimes \underbrace{\mathrm{id} \otimes \dots \otimes \mathrm{id}}_{n-l-1 \text{ times}},$$

$$s_{2l+2} := \underbrace{\sigma_3 \otimes \ldots \otimes \sigma_3}_{l \text{ times}} \otimes \sigma_2 \otimes \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{n-l-1 \text{ times}} \quad \text{for } 1 \le l \le n-1.$$

Lemma 4.4.4. *Let $S = \{\mathrm{id}, s_1, \ldots, s_k\}$ be the standard spin system. If $k = 2n$, then the TRO generated by S in $\mathbb{M}_{2^n}$ is $\mathbb{M}_{2^n}$. If $k = 2n-1$ then the generated TRO is TRO-isomorphic to $\mathbb{M}_{2^{n-1}} \oplus \mathbb{M}_{2^{n-1}}$.*

Proof. Let T be the TRO generated by S.

Let $k = 2n$. It suffices to show that the $3k$ elements

$$a_j := \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{j-1 \text{ times}} \otimes \sigma_1 \otimes \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{n-j times},$$

$$b_j := \mathrm{id} \otimes \ldots \otimes \mathrm{id} \otimes \sigma_2 \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id},$$

$$c_j := \mathrm{id} \otimes \ldots \otimes \mathrm{id} \otimes \sigma_3 \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}$$

for every $j = 1, \ldots, k$ are elements of T, since a_j, b_j, c_j and $\mathrm{id} \otimes \ldots \otimes \mathrm{id}$ span $\mathbb{C} \otimes \ldots \otimes \mathbb{C} \otimes \mathbb{M}_2 \otimes \mathbb{C} \otimes \ldots \otimes \mathbb{C}$.

Obviously $a_1 = s_1 \in T$. Suppose we have shown $a_j \in T$ for a fixed $j \ge 1$, then

$$\begin{aligned} s_{2j} s_{2j+1}^* a_j &= (\underbrace{\sigma_3 \otimes \ldots \otimes \sigma_3}_{j-1 \text{ times}} \otimes \sigma_2 \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}) \\ &\quad (\underbrace{\sigma_3 \otimes \ldots \otimes \sigma_3}_{j \text{ times}} \otimes \sigma_1 \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id})^* \\ &\quad (\underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{j-1 \text{ times}} \otimes \sigma_1 \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}) \\ &= \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{j-1 \text{ times}} \otimes \sigma_2 \sigma_3 \sigma_1 \otimes \sigma_1\, \mathrm{id} \otimes \ldots \otimes \mathrm{id} \\ &= i a_{j+1}. \end{aligned}$$

Similarly we have $b_1 = s_2 \in T$. If we have shown for a fixed $j \ge 1$ that $b_j \in T$, then

$$\begin{aligned} s_{2j} s_{2j+2}^* a_j &= (\underbrace{\sigma_3 \otimes \ldots \otimes \sigma_3}_{j-1 \text{ times}} \otimes \sigma_2 \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}) \\ &\quad (\underbrace{\sigma_3 \otimes \ldots \otimes \sigma_3}_{j \text{ times}} \otimes \sigma_2 \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id})^* \\ &\quad (\underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{j-1 \text{ times}} \otimes \sigma_1 \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}) \\ &= \underbrace{\mathrm{id} \otimes \ldots \otimes \mathrm{id}}_{j-1 \text{ times}} \otimes \sigma_2 \sigma_3 \sigma_1 \otimes \sigma_2 \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id} \end{aligned}$$

$$= ib_{j+1}.$$

Another easy induction shows that $c_j \in T$ for all $j = 1, \dots, n$.

If $k = 2n-1$ we have $a_n \in T$, $b_n, c_n \notin T$. Since σ_1 and $\mathrm{id} \otimes \ldots \otimes \mathrm{id}$ generate the diagonal matrices, the statement is clear.

Alternatively we could argue that T contains the identity so T has to be a C^*-algebra. Then the statement follows from [HOS84], Theorem 6.2.2. □

Theorem 4.4.5. *For the universal enveloping TRO of a spin factor Z with $\dim Z = k+1$ we have*

$$T^*(Z) = \begin{cases} \mathbb{M}_{2^{n-1}} \oplus \mathbb{M}_{2^{n-1}} & \text{if } k = 2n-1, \\ \mathbb{M}_{2^n} & \text{if } k = 2n. \end{cases}$$

Proof. The JC^*-triple system Z is JB^*-isomorphic to the JC^*-algebra J linearly generated by the standard spin system $\{1, s_1, \dots, s_k\}$. By the universal property of $T^*(Z)$ we get, since J generates $\mathbb{M}_{2^{n-1}} \oplus \mathbb{M}_{2^{n-1}}$ if $k = 2n-1$ (respectively $\mathbb{M}_{2^n}$ if $k = 2n$) as a TRO, a surjective TRO-homomorphism from $T^*(Z)$ onto $\mathbb{M}_{2^{n-1}} \oplus \mathbb{M}_{2^{n-1}}$ if $k = 2n-1$ (respectively $\mathbb{M}_{2^n}$ if $k = 2n$). By Proposition 4.4.3 this has to be an isomorphism. □

4.4.2 Factors of type III

A hermitian grid is a family $\{u_{ij} : i, j \in I\}$ of tripotents in Z such that for all $i, j, k, l \in I$:

(HG1) $u_{ij} = u_{ji}$ for all $i, j \in I$.

(HG2) $\{u_{kl}, u_{kl}, u_{ij}\} = 0$ if $\{i, j\} \cap \{k, l\} = \varnothing$.

(HG3) $\{u_{ii}, u_{ii}, u_{ij}\} = \frac{1}{2}u_{ij}$, $\{u_{ij}, u_{ij}, u_{ii}\} = u_{ii}$ if $i \neq j$.

(HG4) $\{u_{ij}, u_{ij}, u_{jk}\} = \frac{1}{2}u_{jk}$, $\{u_{jk}, u_{jk}, u_{ij}\} = \frac{1}{2}u_{ij}$ if i, j, k are pairwise distinct.

(HG5) $\{u_{ij}, u_{jk}, u_{kl}\} = \frac{1}{2}u_{il}$ if $i \neq l$.

(HG6) $\{u_{ij}, u_{jk}, u_{ki}\} = u_{ii}$ if at least two of these tripotents are distinct.

(HG7) All other products of elements from the hermitian grid are 0.

Let Z be a finite-dimensional JC^*-triple system spanned by a hermitian grid $\{u_{ij} : 1 \leq i, j \leq n\}$ and T the TRO generated by this grid. Define $e_{ij} := u_{ii}(\sum_{k=1}^n u_{kk})^* u_{ji} \in T$. By [NR03], Lemma 3.2 (b) we get that $u_{ij} = e_{ij} + e_{ji}$. Since, similar to the spin factor case, $P_2^v(T) = T$, for $v = \sum_{i=1}^n u_{ii}$, the set $\{e_{ij}\}$ forms by the previously mentioned lemma a system of C^*-matrix units for $P_2^v(T)$. One can easily show that the TRO product is the same no matter if it is calculated in $P_2^v(T)$ or in T. Thus we see that the TRO T is linearly spanned by the n^2 elements e_{ij} and we conclude:

Lemma 4.4.6. *Let Z be a JC^*-triple system of dimension n spanned by a hermitian grid. For the universal enveloping TRO of Z we have*

$$\dim T^*(Z) \le n^2.$$

Next we recall the standard representation of a finite-dimensional hermitian Cartan factor. Let $H = \left\{A \in \mathbb{M}_n : A^t = A\right\}$ and $U_{i,j}$ be the $n \times n$-matrix, which is 0 everywhere, except for the entries (i,j) and (j,i), which are 1. Then the collection $\{U_{i,j} : 1 \le i,j \le n\}$ is a hermitian grid, that linearly spans H and generates $\mathbb{M}_n$ as a TRO. With Lemma 4.4.6 and the universal property of the universal enveloping TRO we get:

Theorem 4.4.7. *If Z is a hermitian Cartan factor spanned by the hermitian grid $\{u_{ij} : 1 \le i,j \le n\}$ then*

$$T^*(Z) = \mathbb{M}_n.$$

Remark 4.4.8. Let Z be a finite-dimensional TRO. Then the direct sum

$$T = \bigoplus_{\alpha=1}^{r} \mathbb{M}_{n_\alpha,m_\alpha}$$

can be described by so-called **rectangular matrix units**: Let $E(\alpha,i,j) := E_{i,j} \in \mathbb{M}_{n_\alpha,m_\alpha}$ be the matrix in $\mathbb{M}_{n_\alpha,m_\alpha}$ which is 0 everywhere except 1 in the (i,j)-component for all $1 \le i \le n_\alpha$, $1 \le j \le m_\alpha$ and $\alpha \in \{1,\dots,r\}$. Put

$$e_{i,j}^{(\alpha)} := (0,\dots,0,E(\alpha,i,j),0,\dots,0) \in T,$$

where $E(\alpha,i,j)$ is in the αth summand. The rectangular matrix units satisfy

(i) $e_{i,j}^{(\alpha)} \left(e_{l,j}^{(\alpha)}\right)^* e_{l,k}^{(\alpha)} = e_{i,k}^{(\alpha)}$.

(ii) $e_{i,j}^{(\alpha)} \left(e_{n,m}^{(\beta)}\right)^* e_{p,q}^{(\gamma)} = 0$ for $j \ne m$, $n \ne p$, $\alpha \ne \beta$ or $\beta \ne \gamma$.

(iii) $T = \operatorname{lin}\{e_{i,j}^{(\alpha)} : 1 \le \alpha \le r,\ 1 \le i \le n_\alpha,\ 1 \le j \le m_\alpha\}$.

If U is another TRO which contains elements $f_{i,j}^{(\beta)}$ satisfying the analogues of *(i)*–*(iii)* for $1 \le i \le n_\alpha$, $1 \le j \le m_\alpha$ and $\alpha,\beta \in \{1,\dots,r\}$, then it is easy to see that the mapping sending $e_{i,j}^{(\alpha)}$ to $f_{i,j}^{(\alpha)}$ for $1 \le i \le n_\alpha$, $1 \le j \le m_\alpha$ and $\alpha \in \{1,\dots,r\}$ is a TRO-isomorphism. With the help of the rectangular matrix units we could have argued, instead making the above detour to the Peirce space, that if $\{u_{i,j}\}$ is a hermitian grid in $T^*(Z)$ spanning $\rho_Z(Z)$, then

$$e_{ij} := u_{ii}\left(\sum_{k=1}^{n} u_{kk}\right)^* u_{ji},$$

for $1 \le i,j \le n$, forms a system of rectangular matrix units in $T^*(Z)$, thus $T^*(Z) \simeq \mathbb{M}_n$. This gives an interesting connection between the hermitian grid and the rectangular matrix units of the universal enveloping TRO (recall that $u_{ij} = e_{ij} + e_{ji}$ for all $i,j \in \{1,\dots,n\}$)

4.4.3 Factors of type II

A symplectic grid is a family $\{u_{ij} : i, j \in I, i \neq j\}$ of minimal tripotents such that for all $i, j, k, l \in I$

(SYG1) $u_{ij} = -u_{ji}$ for $i \neq j$.

(SYG2) $\{u_{ij}, u_{ij}, u_{kl}\} = \frac{1}{2}u_{kl}$, $\{u_{kl}, u_{kl}, u_{ij}\} = \frac{1}{2}u_{ij}$ for $\{i, j\} \cap \{k, l\} \neq \varnothing$.

(SYG3) $\{u_{kl}, u_{kl}, u_{ij}\} = 0$ if $\{i, j\} \cap \{k, l\} = \varnothing$.

(SYG4) $\{u_{ij}, u_{il}, u_{kl}\} = \frac{1}{2}u_{kj}$ for i, j, k, l pairwise distinct.

(SYG5) All other triple products in the symplectic grid are 0.

The standard example of a finite-dimensional symplectic grid is the collection $\{U_{i,j} : 1 \leq i, j \leq n, i \neq j\} \subseteq \mathbb{M}_n$, where $U_{i,j}$, for $i < j$, is a complex $n \times n$-matrix, which is 0 everywhere except for the (i, j)-entry, which is 1 and the (j, i)-entry, which is -1. This grid spans linearly the JC^*-triple system $\{A \in \mathbb{M}_n : A^t = -A\}$ of skew-symmetric $n \times n$ matrices; its TRO span is $\mathbb{M}_n$.

Let $\mathfrak{G} := \{u_{ij} : i, j \in I, i \neq j\}$ be a symplectic grid, Z the JC^*-triple system spanned by $\mathfrak{G}$ and T the TRO generated by it. Since for $\dim Z = 3$ Z is JB^*-triple isomorphic to a type I Cartan factor and for $\dim Z = 6$ it is JB^*-triple isomorphic to a type IV Cartan factor, both covered in other sections, let $\dim Z \geq 10$.

Lemma 4.4.9 ([NR03], Lemma 4.1 and Lemma 4.3)**.** *For $1 \leq i, j, k, l \leq n$ pairwise distinct let $e_{i,i} := u_{ik}u_{kl}^*u_{il}$ and $e_{ij} := e_{ii}e_{ii}^*u_{ij}e_{jj}^*e_{jj}$. Then we have:*

(a) *The elements e_{ii} and e_{ij} are unambiguously defined.*

(b) *$u_{ij} = e_{ij} - e_{ji}$.*

(c) *For $v := \sum e_{kk}$, $e_{ij} \in P_2^v(T)$ and therefore $u_{ij} \in P_2^v(T)$.*

(d) *$ve_{ij}^*v = e_{ji}$ and $e_{ij}v^*e_{kl} = \delta_{jk}e_{il}$.*

We see with Lemma 4.4.9 that $Z = \text{lin}\{u_{ij} : i \neq j\} \subseteq \text{lin}\{e_{ij} : i \neq j\} \subseteq P_2^v(T)$. Since $P_2^v(T)$ is a TRO and T is generated by Z we see that

$$P_2^v(T) = T \tag{4.3}$$

and we conclude that T is generated as a TRO by $\{e_{ij} : i \neq j\}$. Since by Lemma 4.4.9 (d)

$$\begin{aligned} e_{ij}e_{kl}^*e_{mn} &= e_{ij}(vv^*e_{kl}v^*v)^*e_{mn} \text{ with (4.3) and 4.4.2} \\ &= e_{ij}v^*(ve_{kl}^*v)v^*e_{mn} \\ &= e_{ij}v^*e_{lk}v^*e_{mn} \\ &\in \{e_{ij} : i \neq j\}, \end{aligned}$$

we know that $\dim T \leq n^2$, if the grid has n elements. We summarize:

Theorem 4.4.10. *If Z is a JC^*-triple system spanned by a symplectic grid $\mathfrak{G} := \{u_{ij} : 1 \leq i, j \leq n, i \neq j\}$ with $\dim Z \geq 10$, then*

$$T^*(Z) = \mathbb{M}_n.$$

Similar to the hermitian case the above argumentation gives us a nice connection between the symplectic grid $\{u_{i,j}\}$ and rectangular matrix units as defined in Remark 4.4.8. The calculations show that the system $\{e_{i,j}\}$ given by

$$e_{m,m} := u_{mk} u_{kl}^* u_{ml} \quad \text{and} \quad e_{ij} := e_{ii} e_{ii}^* u_{ij} e_{jj}^* e_{jj}$$

for $1 \leq m \leq n$ and $1 \leq i \neq j \leq n$, is a system of rectangular matrix units in $T^*(Z)$. We can rebuild the grid from the matrix units since

$$u_{ij} = e_{ij} - e_{ji}$$

for $i, j \in \{1, \ldots, n\}$, $i \neq j$.

4.4.4 Factors of type I

Let Δ and Σ be two index sets. A rectangular grid is a family $\{u_{ij} : i \in \Delta, j \in \Sigma\}$ of minimal tripotents such that

(RG1) $\{u_{il}, u_{il}, u_{jk}\} = 0$ if $i \neq j, k \neq l$.

(RG2) $\{u_{il}, u_{il}, u_{jk}\} = \frac{1}{2} u_{jk}$, $\{u_{jk}, u_{jk}, u_{il}\} = \frac{1}{2} u_{il}$ if either $j = i, k \neq l$ or $j \neq i, k = l$.

(RG3) $\{u_{jk}, u_{jl}, u_{il}\} = \frac{1}{2} u_{ik}$ if $j \neq i$ and $k \neq l$.

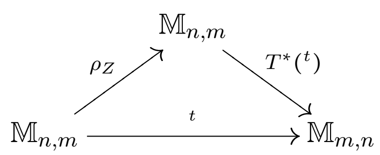
(RG4) All other triple products in the rectangular grid equal 0.

Let Z be the JC^*-triple system generated by a finite rectangular grid. We assume that Z is finite-dimensional and hence JB^*-triple isomorphic to $\mathbb{M}_{n,m}$ with $m = |\Delta|$ and $n = |\Sigma|$.

We first exclude some candidates for $T^*(Z)$:

Lemma 4.4.11. *For the JC^*-triple system $Z = \mathbb{M}_{n,m}$ its universal enveloping TRO $T^*(Z)$ is neither TRO-isomorphic to $\mathbb{M}_{n,m}$ nor to $\mathbb{M}_{m,n}$.*

Proof. Assume that $T^*(Z)$ is TRO-isomorphic to $\mathbb{M}_{n,m}$. Let ${}^t : \mathbb{M}_{n,m} \to \mathbb{M}_{m,n}$ be the transposition mapping. According to the universal property of $T^*(Z)$ there is a mapping $T^*({}^t)$ such that



commutes. Since ρ_Z is bijective there is a TRO-isomorphism $T^*(\rho_Z) : \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ with $T^*(\rho_Z) \circ \rho_Z = \mathrm{id}$. This means $T^*(\rho_Z) = \rho_Z^{-1}$, in particular ρ_Z is a complete isometry.

Since ρ_Z and ^{t} are bijective the same holds for $T^*({}^t)$ and it follows that ^{t} is a complete isometry. We get a contradiction because ^{t} is not even completely bounded. The other statement can be proved analogously. □

Lemma 4.4.12 ([NR03], Lemma 5.1 (b), Lemma 5.2 (b)). *Let $\{u_{ij}\}$ be a rectangular grid spanning Z.*

(a) *If for $i \in \Delta$, $k, l \in \Sigma$, where $k \neq l$, we have $u_{il}u_{ik}^* = 0$ or for $i, j \in \Delta$, $k \in \Sigma$, where $i \neq j$, we have $u_{ik}^* u_{jk} = 0$, then Z is TRO-isomorphic to $\mathbb{M}_{n,m}$.*

(b) *If for $i \in \Delta$, $k, l \in \Sigma$, where $k \neq l$, we have $u_{il}^* u_{ik} = 0$ or for $i, j \in \Delta$, $k \in \Sigma$, where $i \neq j$, we have $u_{ik}u_{jk}^* = 0$, then Z is TRO-isomorphic to $\mathbb{M}_{m,n}$.*

By this we get

Lemma 4.4.13. *Let $\{e_{ij}\}$ be a rectangular grid spanning $\rho_Z(Z) \subseteq T^*(Z)$, then we have*

$$e_{ik}e_{il}^* \neq 0 \text{ and } e_{ik}^* e_{il} \neq 0 \text{ for all } i \in \Delta, k, l \in \Sigma \tag{4.4}$$

as well as

$$e_{ik}e_{jk}^* \neq 0 \text{ and } e_{ik}^* e_{jk} \neq 0 \text{ for all } i, j \in \Delta, k \in \Sigma. \tag{4.5}$$

Proof. If one of these conditions is not fulfilled we get by Lemma 4.4.12 and since $\rho_Z(Z)$ generates $T^*(Z)$ as a TRO, that $\rho_Z(Z) = T^*(Z)$ and hence is isomorphic to $\mathbb{M}_{n,m}$ respectively $\mathbb{M}_{m,n}$. But this is a contradiction to Lemma 4.4.11. □

Lemma 4.4.14. *Let $\operatorname{rank} Z \geq 2$ and $\{e_{ij}\}$ be a rectangular grid spanning $\rho_Z(Z)$, then*

$$p := \sum_{i \in \Delta} \prod_{j \in \Sigma} e_{ij}e_{ij}^* \in C^*(Z)$$

is a sum of non-zero orthogonal projections. We have:

$$pT^*(Z) \subseteq T^*(Z), \ \ (1-p)T^*(Z) \subseteq T^*(Z).$$

Proof. Since (4.4) and (4.5) hold we can use [NR03], Lemma 5.5 and get that $\prod_{j \in \Sigma} e_{ij}e_{ij}^* \neq 0$ are orthogonal projections for all $i \in \Delta$.

The fact that p leaves $T^*(Z)$ invariant is obvious. □

Lemma 4.4.15. *For all $i, k, a \in \Delta$, $j, l, b \in \Sigma$ we have*

$$pe_{ij}(pe_{kl})^* pe_{ab} = pe_{ij}e_{kl}^* pe_{ab} \in \operatorname{lin}\{pe_{ij}\}$$

and for $q := (1-p)$

$$qe_{ij}(qe_{kl})^* qe_{ab} = qe_{ij}e_{kl}^* qe_{ab} \in \operatorname{lin}\{qe_{ij}\}.$$

Proof. Since $\{e_{ij}\}$ is a rectangular grid we know for $i \neq k$ and $j \neq l$ that

$$e_{ij}e_{kl}^* = 0 \qquad \text{and} \qquad e_{ij}^*e_{kl} = 0$$

and therefore, for $i \neq k$ and $j \neq l$,

$$pe_{il}(pe_{kl})^* = pe_{il}e_{kl}^*p = 0 \tag{4.6}$$

as well as

$$\begin{aligned}(pe_{il})^*pe_{kl} &= e_{il}^*pe_{kl}\\ &= e_{il}^*\left(\sum_{\alpha\in\Delta}\prod_{\beta\in\Sigma}e_{\alpha\beta}e_{\alpha\beta}^*\right)e_{kl}\\ &= e_{il}^*e_{i1}e_{i1}^*\dots e_{in}\underbrace{e_{in}^*e_{kl}}_{=0 \text{ if } n\neq l}\\ &= 0,\end{aligned} \tag{4.7}$$

since the range projections of collinear tripotents commute by [NR03], Lemma 5.4.

Equation (4.6) and (4.7) lead us to the fact that we only have to prove (for arbitrary $a \in \Delta, b \in \Sigma$) that

- $pe_{ik}(pe_{il})^*pe_{ab}$ $\quad k \neq l$
- $pe_{il}(pe_{il})^*pe_{ab}$
- $pe_{ab}(pe_{il})^*pe_{jl}$ $\quad i \neq j$
- $pe_{jk}(pe_{ik})^*pe_{ab}$ $\quad i \neq j$
- $pe_{ab}(pe_{il})^*pe_{ik}$ $\quad k \neq l$
- $pe_{ab}(pe_{il})^*pe_{il}$

are elements of $\text{lin}\{pe_{ij}\}$.

Using (4.6) and (4.7) again, we have to prove this in the following cases:

- $pe_{ik}(pe_{il})^*pe_{ib}$ $\quad k \neq l,\ k \neq b \neq l$
- $pe_{ik}(pe_{il})^*pe_{il}$ $\quad k \neq l$
- $pe_{jk}(pe_{ik})^*pe_{ib}$ $\quad b \neq k,\ i \neq j$
- $pe_{jk}(pe_{ik})^*pe_{jk}$ $\quad i \neq j$
- $pe_{il}(pe_{il})^*pe_{ib}$ $\quad b \neq l$
- $pe_{il}(pe_{il})^*pe_{il}$.
- $pe_{ik}(pe_{il})^*pe_{ik}$ $\quad k \neq l$
- $pe_{ik}(pe_{il})^*pe_{al}$ $\quad k \neq l,\ a \neq i$
- $pe_{jk}(pe_{ik})^*pe_{ik}$ $\quad i \neq j$
- $pe_{jk}(pe_{ik})^*pe_{ak}$ $\quad i \neq j,\ a \neq i$
- $pe_{il}(pe_{il})^*pe_{al}$ $\quad a \neq i$

We obtain a similar list for q. Luckily Neal and Russo calculated all these products to show that $\{pe_{ij}\}$ is a rectangular grid (cf. the proof of [NR03], Lemma 5.6) and it is true that all of them are elements of $\{pe_{ij}\}$. One can show by similar methods that all products in the list for q are elements of the rectangular grid $\{(1-p)e_{ij}\}$. □

Proposition 4.4.16. *If* $\operatorname{rank} Z \geq 2$ *we have for the universal enveloping TRO of Z*

$$T^*(Z) = \operatorname{lin}\{pe_{ij}, (1-p)e_{ij} : 1 \leq i \leq n, 1 \leq i \leq m\}$$

especially

$$\dim T^*(Z) \leq 2nm.$$

Proof. The rectangular grid $\{e_{ij}\}$ spans $\rho_Z(Z)$ which generates $T^*(Z)$ as a TRO, so an element $x \in T^*(Z)$ has to be of the form

$$x = \sum_{\alpha=1}^{n} \lambda_\alpha e_1^\alpha (e_2^\alpha)^* e_3^\alpha \dots (e_{2n}^\alpha)^* e_{2k_\alpha+1}^\alpha,$$

with $e_1^\alpha, \dots, e_{2k_\alpha+1}^\alpha \in \{e_{ij}\}$, $\lambda_\alpha \in \mathbb{C}$ and $k_\alpha \in \mathbb{N}$ for all $1 \leq \alpha \leq n$, $n \in \mathbb{N}$. Let $e_1, \dots, e_{2n+1}$ and $e := e_1 e_2^* e_3 \dots e_{2n} e_{2n+1}^* \in T^*(Z)$, then

$$\begin{aligned} e &= (pe_1 + (1-p)e_1)(pe_2 + (1-p)e_2)^* \dots (pe_{2n+1} + (1-p)e_{2n+1}) \\ &= pe_1(pe_2)^* \dots pe_{2n+1} + (1-p)e_1((1-p)e_2)^* \dots (1-p)e_{2n+1} \\ &+ \text{ mixed terms in } p \text{ and } (1-p) \\ &= pe_1(pe_2)^* \dots pe_{2n+1} + (1-p)e_1((1-p)e_2)^* \dots (1-p)e_{2n+1}, \end{aligned}$$

since $\{pe_{ij}\} \perp \{(1-p)e_{ij}\}$ by Lemma [NR03], Lemma 5.6. An inductive use of Lemma 4.4.15 gives us $e \in \{pe_{ij}, (1-p)e_{ij} : 1 \leq i \leq n, 1 \leq i \leq m\}$. □

Theorem 4.4.17. *Let Z be a JC^*-triple system of* rank ≥ 2 *and isomorphic to a finite-dimensional Cartan factor of type I. Let $\{u_{ij}; 1 \leq i \leq n, 1 \leq j \leq m\}$ be a grid spanning Z. Then*

$$T^*(Z) = \mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n}.$$

Proof. We identify Z with $\mathbb{M}_{n,m}$. The mapping $\Phi : \mathbb{M}_{n,m} \to \mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n}$, $A \mapsto (A, A^t)$ is a JB^*-triple isomorphism onto a JB^*-subtriple of $\mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n}$ which generates $\mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n}$ as a TRO. Since by 4.4.16 $\dim T^*(Z) \leq 2nm$ the induced mapping $T^*(\Phi) : T^*(Z) \to \mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n}$ has to be a TRO isomorphism. □

For the rest of this section we assume that $\operatorname{rank} Z = 1$ and Z is of finite dimensions. This implies, that if $\{u_{ij}; 1 \leq i \leq n, 1 \leq j \leq m\}$ is a rectangular grid spanning Z then n or m have to be equal to 1. In this special case the definition of a rectangular grid becomes simpler:

A finite rectangular grid of rank 1 is a set $\{u_1, \dots, u_n\}$ of tripotents with

(RG'1) $\{u_i, u_j, u_i\} = 0$ for $i \neq j$.

(RG'2) $\{u_i, u_i, u_k\} = \frac{1}{2} u_k$ for $i \neq k$.

(RG'3) All other products are 0.

Let Z be a n-dimensional type 1 Cartan factor of rank 1. We fix a finite rectangular grid $\{e_1, \ldots, e_n\}$ of rank 1 spanning $\rho_Z(Z) \subseteq T^*(Z)$.

Lemma 4.4.18. *Let Z be as above, then*

$$\dim T^*(Z) \leq \sum_{k=1}^{n} \binom{n}{k-1} \binom{n}{k}.$$

Proof. Using the grid properties (RG'1),(RG'2),(RG'3) we show that

$$T^*(Z) = \text{lin}\{e_{i_1} e_{i_2}^* e_{i_3} \ldots e_{i_{2k}}^* e_{i_{2k+1}} : i_j < i_{j+2}, 1 \leq j \leq 2k-1, \\ 0 \leq k \leq \frac{1}{2}(n-1)\}.$$

For a fixed k we have $\binom{n}{k-1}\binom{n}{k}$ choices for $e_{i_1} e_{i_2}^* e_{i_3} \ldots e_{i_{2k}}^* e_{i_{2k+1}}$. This is true because $i_j < i_{j+2}$. We have $\binom{n}{k}$ choices for $i_1 < i_3 < \ldots < i_{2k+1}$ and $\binom{n}{k-1}$ choices for $i_2 < i_4 < \ldots < i_{2k}$.

To prove that $T^*(Z)$ is the above mentioned linear span we give an induction which takes $x = e_{i_1} e_{i_2}^* e_{i_3} \ldots e_{i_{2k}}^* e_{i_{2k+1}} \in T^*(Z)$ and rearranges the grid elements such that x is a sum of elements of the form $e_{j_1} e_{j_2}^* e_{j_3} \ldots e_{j_{2k}}^* e_{j_{2k+1}}$ with $j_1 \leq j_3 \leq \ldots \leq j_{2l+1}$ and $j_2 \leq j_4 \leq \ldots \leq j_{2l}$. Since the grid elements are tripotents we can assume that we do not have three equal indices in a row. If we have the case $e_\alpha e_\beta^* e_\alpha$, where $\alpha \neq \beta$ this equals 0 by the minimality of the tripotents (cf. (RG'1)). Therefore $j_a < j_{a+2}$ for all $1 \leq a \leq 2l-1$. Especially $l \leq \frac{1}{2}(n-1)$.

So let $x = e_{i_1} e_{i_2}^* e_{i_3} \ldots e_{i_{2k}}^* e_{i_{2k+1}} \in T^*(Z)$. Since the e_{i_a} are all minimal tripotents we can assume $e_{i_a} \neq e_{i_{a+2}}$.

For $k = 0$ nothing is to prove. Additionally we prove the case when $k = 1$. Let $x = e_{i_1} e_{i_2}^* e_{i_3}$.

If $i_1 < i_3$ we are done.

If $i_1 = i_2 > i_3$ we can use (RG'2) and get $x = e_{i_3} - e_{i_3} e_{i_1}^* e_{i_1}$.

If $i_1 > i_2 = i_3$ we can also use (RG'2) and get $x = e_{i_1} - e_{i_2} e_{i_2}^* e_{i_1}$.

If $i_1 \neq i_2 \neq i_3$:

If $i_1 > i_3$ we can use (RG'3) and we deduce $x = -e_{i_3} e_{i_2}^* e_{i_1}$.

Now we assume that we have shown the statement for $2k+1 \in \mathbb{N}$, $2k+3 \leq n$ and for all lesser indices. If we apply our induction statement to the first $2k+1$ grid elements in the product and then apply the beginning of the induction to all the last three elements of the products in the resulting sum, then one can easily convince himself that in at most three repetitions of this procedure we get the desired form for x. □

Again we have to give a faithful representation T of $T^*(Z)$. This happens to be more complicated than in the other cases. Again we can use the work of Neal and Russo. In [NR03] they showed that a JC^*-triple system, which is linearly spanned by a finite rectangular grid of rank 1 with n elements, has to be completely isometric (especially JB^*-triple isomorphic) to one of the spaces H_n^k, where $k = 1, \dots, n$, that are generalizations of the row and column Hilbert space.

We recall the construction of the spaces H_n^k (cf. [NR03], Section 6 and 7 or [NR06], Section 1). Let $1 \leq k \leq n$ and I, J be subsets of $\{1, \dots, n\}$ such that I has $k-1$ and J has $n-k$ elements. There are $q_k := \binom{n}{k-1}$ choices for I and $p_k := \binom{n}{n-k} = \binom{n}{k}$ choices for J. We assume that the collections $\mathcal{I} := \{I_1, \dots, I_{q_k}\}$ and $\mathcal{J} := \{J_1, \dots, J_{p_k}\}$ of such sets are ordered lexicographically. Let $e_{I_1}, \dots, e_{I_{q_k}}$ and $e_{J_1}, \dots, e_{J_{p_k}}$ be the canonical bases of $\mathbb{C}^{p_k}$ and $\mathbb{C}^{q_k}$. We can define an element in $\mathbb{M}_{p_k,q_k}$ by $E_{I,J} := E_{i,j}$, when $I = I_i \in \mathcal{I}$ and $J = J_j \in \mathcal{J}$. The space H_n^k is the linear span of matrices $b_i^{n,k}$, where $1 \leq i \leq n$, given by

$$b_i^{n,k} := \sum_{I \cap J = \varnothing, (I \cup J)^c = \{i\}} \operatorname{sgn}(I, i, J) E_{J,I}, \tag{4.8}$$

where $\operatorname{sgn}(I, i, J)$ is the signature of the permutation taking $(i_1, \dots, i_{k-1}, i, j_1, \dots, j_{n-k})$ to $(1, \dots, n)$, when $I = \{i_1, \dots, i_{k-1}\}$, where $i_1 < i_2 < \dots < i_{k-1}$, and $J = \{j_1, \dots, j_{n-k}\}$ and where $j_1 < j_2 < \dots < j_{n-k}$.

One can show that the TRO spanned by $b_1^{n,k}, \dots, b_n^{n,k}$ equals $\mathbb{M}_{p_k,q_k}$, so if we represent our JC^*-triple system Z as $\bigoplus_{k=1}^n H_n^k$ we get with Lemma 4.4.18:

Theorem 4.4.19. *If Z is a JC^*-triple system spanned by a finite rectangular grid of rank 1, then*

$$T^*(Z) = \bigoplus_{k=1}^{n} \mathbb{M}_{p_k,q_k},$$

where $p_k = \binom{n}{k}$ and $q_k = \binom{n}{k-1}$ for all $k = 1, \dots, n$.

With this result the list of universal enveloping TROs of the finite-dimensional Cartan factors is complete.

4.5 Homomorphisms of Cartan factors

As an application of our representation theory for ternary rings of operators and the ternary envelope we analyze the JB^*-triple homomorphisms between the finite-dimensional Cartan factors. Let Z and W be Cartan factors and $\varphi : Z \to W$ a JB^*-homomorphism. Via the canonical maps $\rho_Z : Z \to T^*(Z)$ and $\rho_W : W \to T^*(W)$ we can assume that $Z \subseteq T^*(Z)$ and $W \subseteq T^*(W)$. Especially we can think of the homomorphism φ as the restriction of the completely isometric mapping $\tau(\varphi) : T^*(Z) \to T^*(W)$, i.e.

$$\varphi = \tau(\varphi)|_Z.$$

If Z and W are finite-dimensional, then the TRO-homomorphism $\tau(\varphi)$ is by Proposition 3.2.5, up to unitary equivalence, uniquely determined by a matrix with entries in $\mathbb{N}_0$. As additional data we know that $\tau(\varphi)$ maps the image $\rho_Z(Z) \subseteq T^*(Z)$ into $\rho_W(W) \subseteq T^*(W)$.

Recall that the rank of a JB^*-triple system is the maximal cardinality of a system of non-zero tripotents. The rank is an isomorphism invariant and if $\varphi : Z \to W$ is a homomorphism of JB^*-triple systems, then we have necessarily that

$$\operatorname{rank} \varphi(Z) \leq \operatorname{rank} W.$$

The ranks of the different Cartan factors are well known. If $Z \simeq \mathbb{M}_{n,m}$ is a type I Cartan factor, then $\operatorname{rank} Z = \min\{n, m\}$. The symplectic Cartan factors C^2_{2n} and C^2_{2n+1} have rank n as well as the hermitian Cartan factor C^3_n. All Spin factors, independent of their dimension, have the rank 2.

Lemma 4.5.1. *Let $\varphi_1, \varphi_2 : Z \to W$ be JB^*-triple homomorphisms, where Z and W are finite-dimensional JC^*-triple systems, then the following assertions are equivalent:*

(a) *The TRO-homomorphisms $\tau(\varphi_1)$ and $\tau(\varphi_2)$ are unitarily equivalent.*

(b) *There exists a JB^*-triple automorphism ψ of W with $\varphi_1(x) = \psi \circ \varphi_2(x)$ for all $x \in Z$.*

Proof. We know that the universal enveloping TROs of Z and W are finite-dimensional. By the universal properties of T^* we get the following commuting diagram

$$\begin{array}{ccc} T^*(Z) & \xrightarrow{\ \tau(\varphi_i)\ } & T^*(W) \\ {\scriptstyle \rho_Z}\uparrow & & \uparrow{\scriptstyle \rho_W} \\ Z & \xrightarrow{\ \varphi_i\ } & W \end{array}$$

and see that $\tau(\varphi_i)(\rho_Z(Z)) \subseteq \rho_W(W)$ for all $i = 1, 2$.

If we assume that (a) is true, then there exist unitaries U and K such that $\tau(\varphi_1)(x) = U\tau(\varphi_2)(x)K$ for all $x \in T^*(Z)$. Since ρ_W is a JB^*-triple isomorphism onto its image, (b) follows.

Next we assume (b) to be true. If ψ is a JB^*-triple automorphism of W mapping $\varphi_1(Z)$ onto $\varphi_2(Z)$, then $\tau(\psi)$ is a TRO-automorphism of $T^*(W)$ mapping $\rho_W(\varphi_1(Z))$ onto $\rho_W(\varphi_2(Z))$. Since every automorphism of finite-dimensional TROs is inner, we find unitaries U and K such that for all $x \in Z$

$$\begin{aligned}\tau(\varphi_1)(\rho_Z(x)) &= \rho_W(\varphi_1(x))\\ &= \rho_W(\psi \circ \varphi_2(x))\\ &= \tau(\psi \circ \varphi_2)(\rho_Z(x))\\ &= U\tau(\varphi_2)(\rho_Z(x))K.\end{aligned}$$

Since $\rho_Z(Z)$ generates $T^*(Z)$ as a TRO and because $\tau(\varphi_1)$ and $\tau(\varphi_2)$ are TRO-homomorphisms, we obtain (a). □

Let Z and W be type I Cartan factors with $\operatorname{rank} Z \geq 2$. If $\operatorname{rank} W = 1$ an embedding of Z into W does not exist. Otherwise we can say the following.

Proposition 4.5.2. *Let Z and W be type I Cartan factors with $\operatorname{rank} Z$, $\operatorname{rank} W \geq 2$. We can suppose that Z and W are embedded in their universal enveloping TROs*

$$Z = \{(A, A^t) : A \in \mathbb{M}_{n,m}\} \subseteq T^*(Z) = \mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n}$$

and

$$W = \{(B, B^t) : B \in \mathbb{M}_{N,M}\} \subseteq T^*(Z) = \mathbb{M}_{N,M} \oplus \mathbb{M}_{M,N}.$$

If $\varphi : Z \to W$ is a JB^*-triple homomorphism, then φ is uniquely, up to unitary equivalence, determined by a 2×2 matrix $\begin{pmatrix}\alpha & \beta\\ \beta & \alpha\end{pmatrix} \in M_2(\mathbb{N}_0)$ with

$$0 \leq \alpha n + \beta m \leq N \quad \text{and} \quad 0 \leq \beta n + \alpha m \leq M.$$

Proof. We know from Proposition 3.2.5 that the mapping $\tau(\varphi) : T^*(Z) \to T^*(W)$ is uniquely, up to unitary equivalence, determined by a 2×2 matrix $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ with entries in $\mathbb{N}_0$. Moreover, we know from (3.2) that

$$\begin{aligned}\alpha n + \beta m &\leq N, & \gamma n + \delta m &\leq M,\\ \alpha m + \beta n &\leq M, & \gamma m + \delta n &\leq N.\end{aligned}$$

By the construction used to prove Proposition 3.2.5 there exist unitaries U_1, U_2 and K_1, K_2 such that

$$\tau(\varphi) = \left(U_1 \left(\mathrm{id}^{\alpha}_{n,m} \oplus \mathrm{id}^{\beta}_{m,n}\right) K_1, U_2 \left(\mathrm{id}^{\gamma}_{n,m} \oplus \mathrm{id}^{\delta}_{m,n}\right) K_2\right),$$

where $\mathrm{id}^{\gamma}_{j,k}$ denotes the γ-fold identity representation of $\mathbb{M}_{j,k}$ for all $j, k \in \mathbb{N}$. Now $\tau(\varphi)|_Z = \varphi$, thus $\tau(\varphi)(Z) \subseteq W$ and therefore

$$\left(U_1 \left(\mathrm{id}^{\alpha}_{n,m}(A) \oplus \mathrm{id}^{\beta}_{m,n}\left(A^t\right)\right) K_1\right)^t = U_2 \left(\mathrm{id}^{\gamma}_{n,m}(A) \oplus \mathrm{id}^{\delta}_{m,n}\left(A^t\right)\right) K_2.$$

This yields

$$\begin{aligned}\mathrm{id}^{\gamma}_{n,m}(A) \oplus \mathrm{id}^{\delta}_{m,n}\left(A^t\right) &= U_2^* K_1^t \left(\mathrm{id}^{\alpha}_{m,n}\left(A^t\right) \oplus \mathrm{id}^{\beta}_{n,m}(A)\right) U_1^t K_2^* \\ &= \widetilde{U}\left(\mathrm{id}^{\beta}_{n,m}(A) \oplus \mathrm{id}^{\alpha}_{m,n}\left(A^t\right)\right)\widetilde{K},\end{aligned}$$

for suitable unitaries $\widetilde{U}$ and $\widetilde{K}$. But, since Z generates $T^*(Z)$ as a TRO, this can only be true for $\alpha = \delta$ and $\beta = \gamma$. □

We now determine how type I Cartan factors with rank greater or equal to 2 can be represented on type III Cartan factors.

Proposition 4.5.3. *Let Z be a type I Cartan factor with* $\operatorname{rank} Z \geq 2$ *and W a type III Cartan factor. By section 4.4 we can assume that*

$$Z = \{(A, A^t) : A \in \mathbb{M}_{n,m}\} \subseteq T^*(Z) = \mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n},$$

for suitable $n, m \in \mathbb{N}$. Similar we can assume that

$$W = \{A \in \mathbb{M}_N : A^t = A\} \subseteq T^*(W) = \mathbb{M}_N,$$

for a suitable $N \in \mathbb{N}$. If $\varphi : Z \to W$ is a JB^*-triple homomorphism, then φ is, up to (ternary) unitary equivalence, uniquely determined by an 1×2 matrix $\begin{pmatrix}\alpha & \alpha\end{pmatrix}$ where $0 \leq \alpha \leq \frac{N}{n+m}$, $\alpha \in \mathbb{N}_0$.

Proof. We know that the TRO-homomorphism $\tau(\varphi)$ is, up to unitary equivalence, determined uniquely by a 1×2 matrix $\begin{pmatrix}\alpha & \beta\end{pmatrix}$ with entries in $\mathbb{N}_0$. By (3.2) we know that $0 \leq \alpha n + \beta n \leq N$. Assume that $\tau(\varphi)$ is given by the matrix $\begin{pmatrix}\alpha & \beta\end{pmatrix}$. Then there exist unitary matrices U and K such that

$$\tau(\varphi) = U(\mathrm{id}^{\alpha}_{n,m} \oplus \mathrm{id}^{\beta}_{m,n})K,$$

where $\mathrm{id}^{\gamma}_{j,k}$ denotes the γ-fold identity representation of $\mathbb{M}_{j,k}$ for all $j, k \in \mathbb{N}$. The TRO-homomorphism $\tau(\varphi)$ has to coincide with φ on Z. We get that

$$\tau(\varphi)((A, A^t)) = U(A \oplus \ldots \oplus A \oplus A^t \oplus \ldots \oplus A^t \oplus 0 \oplus \ldots \oplus 0)K \in W,$$

as 'diagonal' in $\mathbb{M}_N$ for all $A \in \mathbb{M}_{n,m}$. Especially

$$(\tau(\varphi)((A, A^t)))^t = \tau((A, A^t))$$

and thus

$$\begin{aligned} &U(\underbrace{A \oplus \ldots \oplus A}_{\alpha \text{ times}} \oplus \underbrace{A^t \oplus \ldots \oplus A^t}_{\beta \text{ times}} \oplus 0 \oplus \ldots \oplus 0)K \\ &\qquad = K^t(\underbrace{A^t \oplus \ldots \oplus A^t}_{\alpha \text{ times}} \oplus \underbrace{A \oplus \ldots \oplus A}_{\beta \text{ times}} \oplus 0 \oplus \ldots \oplus 0)U^t \\ &\qquad = \widetilde{U}(\underbrace{A \oplus \ldots \oplus A}_{\beta \text{ times}} \oplus \underbrace{A^t \oplus \ldots \oplus A^t}_{\alpha \text{ times}} \oplus 0 \oplus \ldots \oplus 0)\widetilde{K},\end{aligned}$$

with unitary matrices $\widetilde{U}$ and $\widetilde{K}$. Since $\mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n}$ is generated by $Z = \{(A, A^t) : A \in \mathbb{M}_{n,m}\}$, we see that necessarily $\alpha = \beta$ by unitary equivalence.

A concrete realization of φ for $\begin{pmatrix}\alpha & \alpha\end{pmatrix}$ is given by the mapping

$$(A, A^t) \mapsto \begin{pmatrix} & & & & & & A \\ & & & & & \iddots & \\ & & & & A & & \\ & & & 0 & & & \\ & & A^t & & & & \\ & \iddots & & & & & \\ A^t & & & & & & \end{pmatrix}.$$

□

Proposition 4.5.4. *Let Z be a type I Cartan factor of rank greater or equal to 2 realized as*

$$Z = \{(A, A^t) : A \in \mathbb{M}_{n,m}\} \subseteq T^*(Z) = \mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n},$$

for suitable $n, m \in \mathbb{N}$ and assume that W is a symplectic Cartan factor with

$$W = \{A \in \mathbb{M}_N : A^t = -A\} \subseteq T^*(W) = \mathbb{M}_N.$$

If $\varphi : Z \to W$ is a JB^*-triple homomorphism, then φ is, up to unitary equivalence, uniquely determined by an 1×2 matrix $\begin{pmatrix}\alpha & \alpha\end{pmatrix}$ with $0 \le \alpha \le \frac{N}{n+m}$ and $\alpha \in \mathbb{N}_0$.

Proof. The proof of this lemma follows the lines of the preceding one, where a concrete realization of φ is given by the mapping

$$(A, A^t) \mapsto \begin{pmatrix} & & & & & & A \\ & & & & & \iddots & \\ & & & & A & & \\ & & & 0 & & & \\ & & -A^t & & & & \\ & \iddots & & & & & \\ -A^t & & & & & & \end{pmatrix}.$$

□

A similar argumentation to the proofs of the preceding propositions gives us:

Proposition 4.5.5. *Let Z be the finite-dimensional type II (resp. type III) Cartan factor C_n^2 (resp. C_n^3) and $W = C_{N,M}^1$ be a finite-dimensional rectangular Cartan factor, for $n, N, M \in \mathbb{N}$, $N, M \geq 2$. If $\varphi : Z \to W$ is a JB^*-triple homomorphism, then φ is, up to unitary equivalence, uniquely determined by a 2×1-matrix $\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$ with $0 \leq \alpha \leq \frac{\min\{M,N\}}{n}$.*

Since the rank of the Cartan factor $C_{1,n}^1$ is equal to 1 for all $n \in \mathbb{N}$, there can not be any non-zero JB^*-triple homomorphisms between Cartan factors with rank greater or equal to 2 and $C_{1,n}^1$. A similar statement is true for spin factors since they have constant rank 2: Every JB^*-triple homomorphisms between a Cartan factor with rank greater or equal 3 and a spin factor must be the 0-mapping.

Chapter 5

K-theory for JB^*-triple systems

We use the results of the previous two chapters to define the K-groups of JB^*-triple systems. For a given JB^*-triple system Z we define the ith K-group of Z to be the ith (ternary) K-group of its universal enveloping TRO $T^*(Z)$. By our previous results we obtain covariant, continuous, half-exact, split-exact and homotopy invariant functors on the subcategory of JC^*-triple systems.

Using the theory of grids combined with our ordered K-theory for TROs we are able to define an invariant for atomic JBW^*-triple systems given by a tuple

$$\left(K_0^{\mathrm{JB}^*}(Z), K_0^{\mathrm{JB}^*}(Z)_+, \Sigma_{\mathcal{L}}^{\mathrm{JB}^*}(Z), \Sigma_{\mathcal{R}}^{\mathrm{JB}^*}(Z), \Gamma(Z)\right),$$

where $(K_0^{\mathrm{JB}^*}(Z), K_0^{\mathrm{JB}^*}(Z)_+, \Sigma_{\mathcal{L}}^{\mathrm{JB}^*}(Z), \Sigma_{\mathcal{R}}^{\mathrm{JB}^*}(Z))$ is the double-scaled ordered K_0-group of $T^*(Z)$ and $\Gamma(Z)$ is the set of equivalence classes in $K_0^{\mathrm{JB}^*}(Z)_+$ which stem from a grid spanning Z. We show that this invariant is a well defined, complete isomorphism invariant for the finite-dimensional JC^*-triple systems, which means that two finite-dimensional JC^*-triple systems are isomorphic if and only if they have isomorphic invariants. We prove this by computing the invariants of all finite-dimensional JC^*-triple systems.

5.1 K-groups

Recall from Section 4.2 that the functor τ maps a JB^*-triple system Z to its universal enveloping TRO $T^*(Z)$. If $\varphi : Z_1 \to Z_2$ is a JB^*-triple homomorphism, then $\tau(\varphi) = T^*(\rho_{Z_2} \circ \varphi) : T^*(Z_1) \to T^*(Z_2)$ is a TRO-homomorphism.

Definition 5.1.1. *Let Z be a JB^*-triple system and $T^*(Z)$ its universal enveloping TRO defined in Section 4.1. We define the* ***ith K-group****, $i \in \mathbb{N}_0$, of Z by*

$$K_i^{JB^*}(Z) := K_i(\tau(Z)).$$

If $\varphi : Z \to W$ is a JB^*-triple homomorphism, then φ induces for all $i \in \mathbb{N}_0$ group homomorphisms

$$K_i^{JB^*}(\varphi) : K_i^{JB^*}(T) \to K_i^{JB^*}(W),$$

defined by $K_i^{JB^*}(\varphi) = K_i(\tau(\varphi))$.

The next proposition follows immediately from Section 4.2 and Proposition 3.3.21.

Proposition 5.1.2. *Let Z and W be JB^*-triple systems. Let $i \in \mathbb{N}_0$, then the following is true .*

(a) *$K_i^{JB^*}$ is a covariant functor from the category of JB^*-triple systems to the category of Abelian groups.*

(b) $K_i^{JB^*}(Z \oplus W) \simeq K_i^{JB^*}(Z) \oplus K_i^{JB^*}(W)$.

If Z is a JC^*-triple system, then the following assertions hold:

(c) *Every short exact sequence of JC^*-triple systems*

$$0 \longrightarrow W \xrightarrow{\iota} Z \xrightarrow{\pi} U \longrightarrow 0$$

induces an exact sequence

$$K_i^{JB^*}(W) \xrightarrow{K_i^{JB^*}(\iota)} K_i^{JB^*}(Z) \xrightarrow{K_i^{JB^*}(\pi)} K_i^{JB^*}(U)$$

of Abelian groups. If there exists a homomorphism $\psi : U \to W$ such that

$$0 \longrightarrow W \xrightarrow{\iota} Z \underset{\psi}{\overset{\pi}{\rightleftarrows}} U \longrightarrow 0$$

is split exact, then also

$$0 \longrightarrow K_i^{JB^*}(W) \xrightarrow{K_i^{JB^*}(\iota)} K_i^{JB^*}(Z) \underset{K_i^{JB^*}(\psi)}{\overset{K_i^{JB^*}(\pi)}{\rightleftarrows}} K_i^{JB^*}(U) \longrightarrow 0$$

is split exact.

(d) *If $\varphi, \psi : Z \to W$ are homotopic JB^*-triple homomorphisms, then*

$$K_i^{JB^*}(\varphi) = K_i^{JB^*}(\psi).$$

(e) *If* $((Z_n),(\varphi_n))$ *is an inductive sequence of* JC^**-triple systems, then* $((K_i^{JB^*}(Z_n)),(K_i^{JB^*}(\varphi_n)))$ *is an inductive sequence of Abelian groups and*

$$\lim_{n\to\infty} K_i^{JB^*}(Z_n) \simeq K_i^{JB^*}(\lim_{n\to\infty} Z_n).$$

(f) *We have*

$$K_i^{JB^*}(Z) \simeq K_{i+2}^{JB^*}(Z).$$

Definition 5.1.3. *Let* Z *be an atomic* JBW^**-triple system spanned by a grid* $\mathcal{G}$. *The* ***K-grid invariant*** *of* Z *is the tuple*

$$\mathcal{KG}(Z) := \left(K_0^{JB^*}(Z), K_0^{JB^*}(Z)_+, \Sigma_{\mathcal{L}}^{JB^*}(Z), \Sigma_{\mathcal{R}}^{JB^*}(Z), \Gamma(Z)\right),$$

where $K_0^{JB^*}(Z)_+ := K_0(T^*(Z))_+$, $\Sigma_{\mathcal{L}}^{JB^*}(Z)$ *and* $\Sigma_{\mathcal{R}}^{JB^*}(Z)$ *are the left and right scale of the TRO* $T^*(Z)$ *and* $\Gamma(Z)$ *is the set of equivalence classes*

$$\Gamma(Z) := \left\{[\rho_Z(g)\rho_Z(g)^*] \in K_0^{JB^*}(Z) : g \in \mathcal{G}\right\} \subseteq \Sigma_{\mathcal{L}}^{JB^*}(Z).$$

Let $\varphi : K_0^{JB^*}(Z_1) \to K_0^{JB^*}(Z_2)$ *be a group homomorphism. We say that* φ *is a* ***K-grid isomorphism of*** $\mathbf{K_0}$ ***-groups*** *if* φ *is a group isomorphism with* $\varphi(K_0^{JB^*}(Z)_+) = K_0^{JB^*}(W)_+$, $\varphi(\Sigma_{\mathcal{L}}^{JB^*}(Z_1)) = \Sigma_{\mathcal{L}}^{JB^*}(Z_2)$, $\varphi(\Sigma_{\mathcal{R}}^{JB^*}(Z_1)) = \Sigma_{\mathcal{R}}^{JB^*}(Z_2)$ *and* $\varphi(\Gamma(Z_1)) = \Gamma(Z_2)$.

Our choice of the equivalence classes of the grid elements as additional classifying data is of course not by chance. The grids, as shown by Neher (cf. for example [Neh90], [Neh91] and [Neh96] or section 2.3.4), are the Jordan analogue of the root systems which were used by É. Cartan to classify the bounded symmetric spaces in finite dimensions.

The notion of the K-grid invariant can be extended to general JB^*-triple systems: We first recall from [FR86], Theorem D that every JBW^*-triple system Z with predual Z_* decomposes into an orthogonal direct sum $Z = A \oplus N$ of w^*-closed ideals A and N, where A is the w^*-closure of the linear span of its minimal tripotents and N does not contain minimal tripotents. The ideal A is called the atomic part of Z. Moreover, they showed in Proposition 1 of the same article that if $\widetilde{A}$ is the atomic part of Z'' and if one composes the canonical embedding $\iota : Z \to Z''$ with the canonical projection $\pi : Z'' \to \widetilde{A}$, then $\pi \circ \iota$ is a JB^*-triple embedding. Therefore we can define the K-grid invariant of a JB^*-triple system Z.

Let Z be a JB^*-triple system. The **JBW*-K-grid invariant** of Z is the K-grid invariant of the atomic part of Z''.

The definition of the K-grid invariant of a JBW^*-triple system and the JB^*-K-grid invariant of a JB^*-triple system coincide in the case that the triple system is reflexive as a Banach space, especially in finite dimensions, where our main interest lies. It is known (cf. [CI90], Theorem 6) that a JB^*-triple system is reflexive if and only if it does not contain a copy of the function space c_0.

However we still have to show that the K-grid invariant is well-defined.

Lemma 5.1.4. *Let Z be an atomic JBW^*-triple system and $\mathcal{G}_1$ and $\mathcal{G}_2$ two grids spanning Z, then there exists a JB^*-triple automorphism of Z mapping $\mathcal{G}_1$ onto $\mathcal{G}_2$.*

Proof. We know by the Isomorphism Theorem and the Structure Theorem for atomic JBW^*-triple systems that $\mathcal{G}_1$ and $\mathcal{G}_2$ are both the union of the same standard grids of type 1–6. So we can w.l.o.g. assume that $\mathcal{G}_1$ and $\mathcal{G}_2$ are connected and of the same type. If one analyzes the proof of the Isomorphism Theorem 3.18 in [Neh87] one sees that there exists a triple isomorphism that maps $\mathcal{G}_1$ onto $\mathcal{G}_2$. □

Lemma 5.1.5. *Let Z_1 and Z_2 be atomic JBW^*-triple systems and $\varphi : Z_1 \to Z_2$ be a JB^*-triple isomorphism. Let $\mathcal{G}_i$ be a grid spanning Z_i, $i = 1,2$. Then there exists a JB^*-triple isomorphism $\psi : Z_1 \to Z_2$ such that the induced map $K_0^{JB^*}(\psi)$ is a K-grid isomorphism of K_0-groups.*

Proof. We assume w.l.o.g. that Z_1 and Z_2 are factors and therefore $\varphi(\mathcal{G}_1)$ and $\varphi(\mathcal{G}_1)$ are grids of the same type. Let $\psi' : Z_2 \to Z_2$ be the JB^*-triple isomorphism from Lemma 5.1.4 which maps $\varphi(\mathcal{G}_1)$ onto $\mathcal{G}_2$, then $\psi := \psi' \circ \varphi$ is a JB^*-triple isomorphism with $\psi(\mathcal{G}_1) = \mathcal{G}_2$. Therefore $K_0^{\mathrm{JB}^*}(\psi) : K_0^{\mathrm{JB}^*}(Z_1) \to K_0^{\mathrm{JB}^*}(Z_2)$ is an isomorphism of double-scaled ordered groups which maps $\Gamma(Z_1)$ onto $\Gamma(Z_2)$. □

Proposition 5.1.6. *Let Z_1 and Z_2 be finite-dimensional JC^*-triple systems. If $\varphi : Z_1 \to Z_2$ is a JB^*-triple isomorphism then*

$$K_0(\varphi)\,(\Gamma(Z_1)) = \Gamma(Z_2).$$

Proof. We can assume that Z_1 and Z_2 are simple and spanned by grids $\mathcal{G}_1 \subseteq Z_1$ and $\mathcal{G}_2 \subseteq Z_2$, which are of the same type. We consider the images of $\mathcal{G}_1$ and $\mathcal{G}_2$ in $T^*(Z_2)$, say $\mathcal{G}_1' := \rho_{Z_2}(\varphi(\mathcal{G}_1))$ and $\mathcal{G}_2' := \rho_{Z_2}(\mathcal{G}_2)$, then there exists by Lemma 5.1.5 a JB^*-triple automorphism ψ mapping $\mathcal{G}_1'$ to $\mathcal{G}_2'$. By the universal property of the universal enveloping TRO, we know that ψ has to be the restriction of a TRO-automorphism $\tau(\psi)$ of $T^*(Z_2)$. It is known that every TRO-automorphism in finite dimensions is inner, thus there exist unitaries U and K such that $\tau(\psi)(z) = UzK$ for all $z \in T^*(Z_2)$. Especially we have

$$U\mathcal{G}_1'K = \mathcal{G}_2'.$$

□

Thus any isomorphism of finite-dimensional JC^*-triple systems yields a K-grid isomorphism of their K_0-groups, independent of the choice of the grid.

By the direct sum of two K-grid invariants we mean the direct sum of all the components.

Proposition 5.1.7. *Let Z_1 and Z_2 be atomic JBW^*-triple systems, then there exists a K-grid isomorphism of K_0-groups*

$$\mathcal{KG}(Z_1 \oplus Z_2) \simeq \mathcal{KG}(Z_1) \oplus \mathcal{KG}(Z_2).$$

Proof. We already know that the functor K_0 from the category of TROs to the category of Abelian groups is additive (cf. Proposition 3.3.21). This also holds for the positive cone and the scales. The functor τ defined in section 4.2 is additive by Proposition 4.2.1. If $\mathcal{G}$ is a grid spanning $Z_1 \oplus Z_2$ and p_i is the projection onto Z_i, then $p_i\mathcal{G}$ is a grid which generates Z_i for $i = 1, 2$. Therefore $\Gamma(Z_1 \oplus Z_2) = \Gamma(Z_1) \oplus \Gamma(Z_2)$. □

Remark 5.1.8. In general the universal enveloping TRO of a C^*-algebra is not the C^*-algebra itself (for example if $n \geq 2$, then $T^*(\mathbb{M}_n) = M_n \oplus \mathbb{M}_n$). Therefore the functor τ is not the identity restricted to the category of C^*-algebras and thus the composed functors $K_i^{\text{JB}^*}$ are not the usual functors of the K-theory for C^*-algebras, for all $i \in \mathbb{N}_0$. This happens because by defining the K-groups of a JB^*-triple system as the K-groups of its universal enveloping TRO, we fix a specific operator space structure on the JB^*-triple system. Bunce, Feely and Timoney developed in [BFT10] a theory of the different operator space structures a JC^*-triple system can carry. An operator space structure of a JC^*-triple system Z is an operator space structure determined by a linear isometry onto a subtriple of $B(H)$. One JC^*-triple system can carry different (i.e not completely isometric) operator space structures. For example the JC^*-triple systems $\mathbb{M}_{n,m}$ and $\mathbb{M}_{m,n}$ are isometric but not completely isometric for $n \neq m$ if endowed with their canonical TRO operator space structures. There is a close connection between operator space structures on a JC^*-triple systems Z and the ideals I in $T^*(Z)$ with $\rho_Z(Z) \cap I = \{0\}$ which are called the operator space ideals of $T^*(Z)$. For every operator space ideal I of $T^*(Z)$ we have an operator space structure Z_I on Z determined by the isometric embedding $Z \to T^*(Z)/I$, $z \mapsto \rho_Z(z) + I$. In [BFT10] it was shown that, if $\varphi : Z \to W$ is a surjective isometry onto a JC^*-triple system $W \subseteq B(K)$ (regarded as an operator subspace of $B(K)$), then there exists an operator space ideal $I \subseteq T^*(Z)$ such that $\pi : Z_I \to W$ is a complete isometry.

In this context we always endow a JC^*-triple system with its unique 'maximal' operator space structure, the one induced by the $\{0\}$ operator space ideal. It would be possible to define K-theory on smaller operator space structures but it is not clear how to find ideals in a canonical way to preserve functoriality. If we only consider universally reversible C^*-algebras which do not contain an ideal of codimension 1 we know by 4.3.5 that $T^*(M_n(\mathfrak{A})) = M_n(\mathfrak{A}) \oplus \theta_n(M_n(\mathfrak{A})) = M_n(T^*(\mathfrak{A}))$ for all $n \in \mathbb{N}$. By always considering the operator space structure induced by the operator space ideal $\{0\} \oplus \theta_n(M_n(\mathfrak{A}))$ we would obtain a 'K-theory' which coincides with C^*-algebraic K-theory.

5.2 A complete isomorphism invariant

We determine the K-grid invariants of all finite-dimensional JC^*-triple systems. We do this by making a case by case study of the K-grid invariants of the finite-dimensional Cartan factors of type I–IV.

5.2.1 Rectangular factors

Recall that a finite-dimensional rectangular Cartan factor is a JC^*-triple system which is isometric to

$$C^1_{n,m} = \mathbb{M}_{n,m}$$

for $n, m \in \mathbb{N}$. The standard example of a rectangular grid spanning $\mathbb{M}_{n,m}$ is

$$\mathcal{G} = \{E_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Let Z be JC^*-triple system which is isomorphic to the finite-dimensional Cartan factor $C^1_{n,m}$. We have to distinguish between the case when Z is a rank 1 JB^*-triple system and the case $1 < n, m < \infty$.

Proposition 5.2.1. *If Z is the finite-dimensional type I Cartan factor $Z = C^1_{n,m}$, with $n, m \geq 2$, then $\mathcal{KG}(C^1_{n,m})$ is given by*

$$\big(\mathbb{Z}^2, \mathbb{N}_0^2, \{1,\dots,n\} \times \{1,\dots,m\}, \{1,\dots,m\} \times \{1,\dots,n\}, \{(1,1)\}\big)$$

Proof. We know by Section 4.4.4 that the universal enveloping TRO of Z is

$$T^*(Z) = \mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n}.$$

If we identify Z with the diagonal

$$\{(A, A^t) : A \in \mathbb{M}_{n,m}\} \subseteq \mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n},$$

then Z is spanned by the rectangular grid

$$\mathcal{G} = \{(E_{i,j}, E_{j,i}) : 1 \leq i \leq n, 1 \leq j \leq m\},$$

thus $\Gamma(Z)$ collapses to

$$\Gamma(Z) = \{(1,1)\}.$$

The equality

$$\big(K_0^{\mathrm{JB}^*}(Z), K_0^{\mathrm{JB}^*}(Z)_+, \Sigma_{\mathcal{L}}^{\mathrm{JB}^*}(Z), \Sigma_{\mathcal{R}}^{\mathrm{JB}^*}(Z)\big) = (\mathbb{Z}^2, \mathbb{N}_0^2, \{1,\dots,n\} \times \{1,\dots,m\}, \{1,\dots,m\} \times \{1,\dots,n\})$$

follows from the K-theory for ternary rings of operators. □

Recall from 4.4.4 that, if Z is a finite-dimensional type I Cartan factor of rank 1, then its universal enveloping TRO is given by

$$T^*(Z) = \bigoplus_{k=1}^{n} \mathbb{M}_{p_k,q_k},$$

where $p_k = \binom{n}{k}$ and $q_k = \binom{n}{k-1}$ for $k = 1,\ldots,n$. The image of Z under the injection into $T^*(Z)$ is located inside the direct sum of the spaces H_n^k,

$$\rho_Z(Z) \subseteq \bigoplus_{k=1}^{n} H_n^k \subseteq \bigoplus_{k=1}^{n} \mathbb{M}_{p_k,q_k}$$

(cf. (4.8) for details on the spaces H_n^k).

Proposition 5.2.2. *If Z is isometric to a finite-dimensional Hilbert space, i.e. $Z = C_{1,n}^1$, $n \in \mathbb{N}$, then $\mathcal{KG}(C_{1,n}^1)$ is given by*

$$\begin{aligned}
K_0^{JB^*}(C_{1,n}^1) &= \mathbb{Z}^n,\\
K_0^{JB^*}(C_{1,n}^1)_+ &= \mathbb{N}_0^n,\\
\Sigma_{\mathcal{L}}^{JB^*}(C_{1,n}^1) &= \prod_{k=1}^{n}\left\{1,\ldots,\binom{n}{k}\right\},\\
\Sigma_{\mathcal{R}}^{JB^*}(C_{1,n}^1) &= \prod_{k=1}^{n}\left\{1,\ldots,\binom{n}{k-1}\right\} \quad \textit{and}\\
\Gamma(C_{1,n}^1) &= \left\{\left(\binom{n-1}{0},\binom{n-1}{1},\ldots,\binom{n-1}{n-1}\right)\right\}.
\end{aligned}$$

Proof. Let $n \in \mathbb{N}$. First we identify Z with its image $\rho_Z(Z) \subseteq \bigoplus_{k=1}^n H_n^k$ in $T^*(Z)$. Recall from (4.8) that for every $k \in \{1,\ldots,n\}$ the space H_n^k is spanned by the matrices

$$b_i^{n,k} = \sum_{I\cap J=\varnothing,(I\cup J)^c=\{i\}} \operatorname{sgn}(I,i,J)E_{J,I},$$

$i = 1,\ldots,n$. To compute $\Gamma(Z)$ we have do determine a grid in $\rho_Z(Z)$, which spans $\rho_Z(Z)$. From [NR06], §1 it is known, that for every $k = 1\ldots,n$ the matrices $b_1^{n,k},\ldots,b_n^{n,k}$ are the isometric image of a rectangular grid. Thus

$$\mathcal{G} = \{g_i := (b_i^{n,1},\ldots,b_i^{n,n}) : i = 1,\ldots,n\}$$

is a rectangular grid spanning Z.

One observes that the matrices $b_i^{n,k}$ can also be written as

$$b_i^{n,k} = \sum_{\substack{I\subseteq\{1,\ldots,n\},\\ |I|=k-1,i\notin I}} \operatorname{sgn}(I,i,(I\cup\{i\})^c)E_{(I\cup\{i\})^c,I},$$

for all $i, k \in \{1, \dots, n\}$. Therefore

$$\begin{aligned}
b_i^{n,k}\left(b_i^{n,k}\right)^* &= \sum_{\substack{I\subseteq\{1,\dots,n\},\\ |I|=k-1, i\notin I}} \sum_{\substack{J\subseteq\{1,\dots,n\},\\ |J|=k-1, i\notin J}} \operatorname{sgn}(I,i,(I\cup\{i\})^c)\operatorname{sgn}(J,i,(J\cup\{i\})^c) \\
&\qquad E_{(I\cup\{i\})^c,I}E_{J,(J\cup\{i\})^c} \\
&= \sum_{\substack{I\subseteq\{1,\dots,n\},\\ |I|=k-1, i\notin I}} \operatorname{sgn}(I,i,(I\cup\{i\})^c)^2 E_{(I\cup\{i\})^c,I}E_{I,(I\cup\{i\})^c} \\
&= \sum_{\substack{I\subseteq\{1,\dots,n\},\\ |I|=k-1, i\notin I}} E_{(I\cup\{i\})^c,(I\cup\{i\})^c},
\end{aligned}$$

which is a matrix of rank $\binom{n-1}{k-1}$, since we have that many choices for $I \subseteq \{1, \dots, n\} \smallsetminus \{i\}$, $|I| = k-1$. We get

$$g_i g_i^* = \left(\sum_{\substack{I\subseteq\{1,\dots,n\},\\ |I|=0, i\notin I}} E_{(I\cup\{i\})^c,(I\cup\{i\})^c}, \dots, \sum_{\substack{I\subseteq\{1,\dots,n\},\\ |I|=n-1, i\notin I}} E_{(I\cup\{i\})^c,(I\cup\{i\})^c} \right)$$

and therefore all elements of $\Gamma(C^1_{1,n})$ lie in the same equivalence class:

$$[g_i g_i^*] = \left(\binom{n-1}{0}, \binom{n-1}{1}, \dots, \binom{n-1}{n-1}\right) \in \mathbb{Z}^n$$

for all $i = 1, \dots, n$. □

5.2.2 Hermitian and symplectic factors

Proposition 5.2.3. *Let Z be isometric to a Cartan factor of type II with* $\dim Z \geq 10$. *Then*

$$\mathcal{KG}(C_n^2) = (\mathbb{Z}, \mathbb{N}_0, \{1, \dots, n\}, \{1, \dots, n\}, \{2\}).$$

Proof. The universal enveloping TRO of C_n^2 is the C^*-algebra $\mathbb{M}_n$ by Section 4.4.3 . A grid spanning the skew-symmetric $n \times n$-matrices is

$$\mathcal{G} = \{g_{i,j} := E_{i,j} - E_{j,i} : 1 \leq i < j \leq n\}.$$

Thus $\Gamma(C_n^2)$ is given by the equivalence classes of

$$\begin{aligned}
g_{i,j}g_{i,j}^* &= (E_{i,j} - E_{j,i})(E_{i,j} - E_{j,i})^* \\
&= E_{i,i} + E_{j,j},
\end{aligned}$$

for $1 \leq i < j \leq n$. These are, independent of i and j, all rank 2 matrices. □

Proposition 5.2.4. *If Z is JB^*-triple isomorphic to the finite-dimensional Cartan factor C_n^3, then*

$$\mathcal{KG}(C_n^3) = (\mathbb{Z}, \mathbb{N}_0, \{1, \ldots, n\}, \{1, \ldots, n\}, \{1, 2\}).$$

Proof. The universal enveloping TRO of Z is by section 4.4.2 completely isometric to $\mathbb{M}_n$, thus $K_0^{\mathrm{JB}^*}(C_n^3) = \mathbb{Z}$ with positive cone $\mathbb{N}_0$ and double-scales $\Sigma_{\mathcal{L}}^{\mathrm{JB}^*}(Z) = \Sigma_{\mathcal{R}}^{\mathrm{JB}^*}(Z) = \{1, \ldots, n\}$. The Cartan factor C_n^3 is spanned by the hermitian grid

$$\{g_{i,j} := E_{i,j} + E_{j,i} : 1 \le i < j \le n\} \cup \{g_{i,i} := E_{i,i} : 1 \le i \le n\}.$$

This leads to

$$\begin{aligned} g_{i,j} g_{i,j}^* &= (E_{i,j} + E_{j,i})^2 \\ &= E_{i,i} + E_{j,j}, \text{ for } 1 \le i < j \le n, \end{aligned}$$

and

$$g_{i,i} g_{i,i}^* = E_{i,i}, \text{ for } 1 \le i \le n.$$

Thus

$$\Gamma(C_n^3) = \{1, 2\}.$$

□

At this point we are already able to see that the mapping $Z \mapsto \mathcal{KG}(Z)$ is not in a natural way a functor from the category of JC^*-triple systems with JB^*-triple homomorphisms to the category of extended Abelian groups with group homomorphisms. If $\varphi : W \to Z$ is a JB^*-triple homomorphism the induced group homomorphism $K_0^{\mathrm{JB}^*}(\varphi) : K_0^{\mathrm{JB}^*}(W) \to K_0^{\mathrm{JB}^*}(Z)$ does not necessarily map $\Gamma(W)$ to $\Gamma(Z)$:

Let Z be the Cartan factor $C_{n,n}^1$, with $n > 1$ and W be the hermitian Cartan factor C_n^3. We identify W with the symmetric $n \times n$-matrices as a subset of $\mathbb{M}_n = Z$. We have $T^*(Z) = \mathbb{M}_n \oplus \mathbb{M}_n$ containing $\rho_Z(Z)$ as diagonal

$$\rho_Z(Z) = \{(A, A^t) : A \in \mathbb{M}_n\}.$$

If we apply the functor τ to the canonical injection $\iota : W \to Z$ we get the TRO-homomorphism

$$\tau(\iota) : T^*(W) = \mathbb{M}_n \to \mathbb{M}_n \oplus \mathbb{M}_n, \ \tau(\iota)(A) = (A, A^t).$$

Thus if $\mathcal{G}$ is the hermitian grid $\{E_{i,j} + E_{j,i} : 1 \le i < j \le n\} \cup \{E_{i,i} : 1 \le i \le n\}$ we have

$$\begin{aligned} \tau(\iota)(\mathcal{G}) = \{&(E_{i,j} + E_{j,i}, E_{i,j} + E_{j,i}) : 1 \le i < j \le n\} \cup \{(E_{i,i}, E_{i,i}) : 1 \le i \le n\} \\ &\subseteq \mathbb{M}_n \oplus \mathbb{M}_n. \end{aligned}$$

It follows that

$$\begin{aligned} K_0^{\mathrm{JB}^*}(\iota)(\Gamma(W)) &= K_0^{\mathrm{TRO}}(\tau(\iota))(\{1,2\}) \\ &= \{1,2\} \times \{1,2\} \\ &\nsubseteq \{1\} \times \{1\} \\ &= \Gamma(Z), \end{aligned}$$

where $K_0^{\mathrm{TRO}}(\tau(\iota))$ is the group homomorphism from the K-theory of TROs.

5.2.3 Spin factors

To determine the K-grid invariant of the finite-dimensional spin factors we need a little preparation. We already know that if Z is a spin factor with $\dim Z = k + 1 \geq 3$, then

$$T^*(Z) = \begin{cases} \mathbb{M}_{2^{n-1}} \oplus \mathbb{M}_{2^{n-1}} & \text{if } k = 2n-1, \\ \mathbb{M}_{2^n} & \text{if } k = 2n. \end{cases}$$

Therefore one can easily conclude that

$$\left(K_0^{\mathrm{JB}^*}(Z), \Sigma_{\mathcal{L}}^{\mathrm{JB}^*}(Z), \Sigma_{\mathcal{R}}^{\mathrm{JB}^*}(Z)\right) = \begin{cases} \left(\mathbb{Z}^2, \{1,\ldots,2^{n-1}\}^2, \{1,\ldots,2^{n-1}\}^2\right) \\ \text{if } k = 2n-1, \\ (\mathbb{Z}, \{1,\ldots,2^n\}, \{1,\ldots,2^n\}) \\ \text{if } k = 2n. \end{cases}$$

To compute $\Gamma(Z)$ we need to determine a spin grid which spans $\rho_Z(Z) \subseteq T^*(Z)$. Obviously we have to distinguish between the even and odd dimensional case. From [HOS84] a spin system is known that linearly spans $\rho_Z(Z) \subseteq \mathbb{M}_{2^n}$ as a JB^*-algebra, in the case that Z is odd dimensional, but it is unfortunately not a spin grid. It is called the standard spin system (using the abbreviation $a^n := \underbrace{a \otimes \ldots \otimes a}_{n \text{ times}}$, $a \in \mathbb{M}_2$):

$$\begin{aligned} s_0^{odd} &:= id^n, \\ s_1^{odd} &:= \sigma_1 \otimes \mathrm{id}^{n-1}, \\ s_2^{odd} &:= \sigma_2 \otimes \mathrm{id}^{n-1}, \\ s_3^{odd} &:= \sigma_3 \otimes \sigma_1 \otimes \mathrm{id}^{n-2}, \\ s_4^{odd} &:= \sigma_3 \otimes \sigma_2 \otimes \mathrm{id}^{n-2}, \\ s_{2l+1}^{odd} &:= \sigma_3^l \otimes \sigma_1 \otimes \mathrm{id}^{n-l-1}, \\ s_{2l+2}^{odd} &:= \sigma_3^l \otimes \sigma_2 \otimes \mathrm{id}^{n-l-1}, \end{aligned}$$

for $1 \leq l \leq n-1$. If we drop the last idempotent s_{2n} we get a spin system which generates an even dimensional spin factor embedded in $\mathbb{M}_{2^n}$. However we are interested in the spin system generating $\rho_Z(Z)$ inside the universal enveloping TRO $T^*(Z) = \mathbb{M}_{2^{n-1}} \oplus \mathbb{M}_{2^{n-1}}$.

Lemma 5.2.5. *Let Z be an even dimensional spin factor with $\dim Z = 2n$ then the following idempotents define a spin system generating $\rho_Z(Z) \subseteq T^*(Z) = \mathbb{M}_{2^{n-1}} \oplus \mathbb{M}_{2^{n-1}}$:*

$$
\begin{aligned}
s_0^{even} &:= \left(\mathrm{id}^{n-1}, \mathrm{id}^{n-1}\right),\\
s_1^{even} &:= \left(\sigma_1 \otimes \mathrm{id}^{n-2}, \sigma_1 \otimes \mathrm{id}^{n-2}\right),\\
s_2^{even} &:= \left(\sigma_2 \otimes \mathrm{id}^{n-2}, \sigma_2 \otimes \mathrm{id}^{n-2}\right),\\
s_3^{even} &:= \left(\sigma_3 \otimes \sigma_1 \otimes \mathrm{id}^{n-3}, \sigma_3 \otimes \sigma_1 \otimes \mathrm{id}^{n-3}\right),\\
s_4^{even} &:= \left(\sigma_3 \otimes \sigma_2 \otimes \mathrm{id}^{n-3}, \sigma_3 \otimes \sigma_2 \otimes \mathrm{id}^{n-3}\right),\\
s_{2l+1}^{even} &:= \left(\sigma_3^l \otimes \sigma_1 \otimes \mathrm{id}^{n-l-2}, \sigma_3^l \otimes \sigma_1 \otimes \mathrm{id}^{n-l-2}\right),\\
s_{2l+2}^{even} &:= \left(\sigma_3^l \otimes \sigma_2 \otimes \mathrm{id}^{n-l-2}, \sigma_3^l \otimes \sigma_2 \otimes \mathrm{id}^{n-l-2}\right),\\
s_{2n-1}^{even} &:= \left(\sigma_3^{n-1}, -\sigma_3^{n-1}\right),
\end{aligned}
$$

for $1 \leq l \leq n-2$.

Proof. This is just the image of the standard spin system under the map $\varphi : \mathbb{M}_{2^{n-1}} \otimes \mathbb{M}_2 \to \mathbb{M}_{2^{n-1}} \oplus \mathbb{M}_{2^{n-1}}$, $\varphi\left(A \otimes \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\right) = (\lambda_1 A, \lambda_2 A)$. Restricted to the TRO-span of the standard spin system in $\mathbb{M}_{2^n}$, which is $\mathbb{M}_{2^{n-1}} \otimes \mathcal{D}$, where $\mathcal{D}$ denotes the diagonal matrices in $\mathbb{M}_2$, this becomes a $*$-isomorphism. □

Next we prove a proposition which enables us to construct a spin grid out of any given spin system. With the help of this proposition we can construct spin grids which generate the even and odd dimensional spin factors embedded in their universal enveloping TROs allowing us to compute their K-grid invariants.

Proposition 5.2.6. *Let $S = \{\mathrm{id}, s_1, \ldots, s_n\}$ be a spin system. If n is odd (i.e. the corresponding spin factor is of even dimension) we can define a spin grid $\mathfrak{G} = \{u_i, \widetilde{u}_i : i = 1, \ldots, n\}$ by*

$$u_1 := \frac{1}{2}(\mathrm{id} - s_1), \quad \widetilde{u}_1 := -\frac{1}{2}(\mathrm{id} + s_1) \text{ and}$$

$$u_{k+1} := \frac{1}{2}(s_{2k} + i s_{2k+1}), \quad \widetilde{u}_{k+1} = \frac{1}{2}(s_{2k} - i s_{2k+1}) \text{ for } k = 1, \ldots, \frac{1}{2}(n-1).$$

In the case that n is even, a spin grid is given by $\mathfrak{G} = \{u_i, \widetilde{u}_i : i = 1, \ldots, n\} \cup \{u_0\}$ with $u_0 := s_n$.

Proof. To prove this proposition we have to verify that all elements of $\mathfrak{G}$ are minimal (except u_0 in the case of odd dimensions) tripotents which satisfy the spin grid axioms (SPG1), ..., (SPG9) from section 4.4.1. As an example we prove (SPG5):

(a) Let $j,k \in \{1,\dots,n-1\}$. Using the anticommutator relations of the spin system we get

$$\begin{aligned}\{u_{j+1},\widetilde{u}_{k+1},\widetilde{u}_{j+1}\} &= \frac{1}{2}(u_{j+1}u_{k+1}\widetilde{u}_{j+1} + \widetilde{u}_{j+1}u_{k+1}u_{j+1}) \\ &= \frac{1}{16}((s_{2j}+is_{2j+1})(s_{2k}+is_{2k+1})(s_{2j}-is_{2j+1}) \\ &\quad + (s_{2j}-is_{2j+1})(s_{2k}+is_{2k+1})(s_{2j}+is_{2j+1})) \\ &= \frac{1}{8}(s_{2j}(s_{2k}+is_{2k+1})s_{2j} - is_{2j}(s_{2j}-is_{2j+1})s_{2j+1} \\ &\quad + is_{2j+1}(s_{2k}+is_{2k+1})s_{2k} + s_{2j+1}(s_{2k}+is_{2k+1})s_{2j+1}) \\ &= -\frac{1}{4}(s_{2k}+is_{2k+1}) \\ &= -\frac{1}{2}u_{k+1}.\end{aligned}$$

(b) For $j \in \{1,\dots,n-1\}$ we have

$$\begin{aligned}\{u_{j+1},\widetilde{u}_1,\widetilde{u}_{j+1}\} &= -\frac{1}{16}((s_{2j}+is_{2j+1})(\mathrm{id}+s_1)(s_{2j}-is_{2j+1}) \\ &\quad + (s_{2j}-is_{2j+1})(\mathrm{id}+s_1)(s_{2j}+is_{2j+1})) \\ &= -\frac{1}{8}\left(s_{2j}^2 + s_{2j}s_1s_{2j} + s_{2j+1}^2 + s_{2j+1}s_1s_{2j+1}\right) \\ &= -\frac{1}{2}u_1.\end{aligned}$$

(c) Similarly we get $\{u_1,\widetilde{u}_j,\widetilde{u}_1\} = -\frac{1}{2}u_j$ for all $j \in \{2,\dots,n\}$.

□

Proposition 5.2.7. *Let Z be a finite-dimensional spin factor with* $\dim Z = k+1$.

If Z is of even dimension, i.e. $k = 2n-1$, $n \geq 2$, then the K-grid invariant of Z is given by

$$\begin{aligned}K_0^{JB^*}(Z) &= \mathbb{Z}^2, \\ K_0^{JB^*}(Z)_+ &= \mathbb{N}_0^2, \\ \Sigma_{\mathcal{L}}^{JB^*}(Z) &= \left\{1,\dots,2^{n-1}\right\}^2, \\ \Sigma_{\mathcal{R}}^{JB^*}(Z) &= \left\{1,\dots,2^{n-1}\right\}^2, \\ \Gamma(Z) &= \left\{\left(2^{n-2},2^{n-2}\right),\left(2^{n-1},2^{n-1}\right)\right\}.\end{aligned}$$

If Z is of odd dimensions, i.e. $k = 2n$, $n \geq 2$, then $\mathcal{KG}(Z)$ has the components

$$\begin{aligned}
K_0^{JB^*}(Z) &= \mathbb{Z}, \\
K_0^{JB^*}(Z)_+ &= \mathbb{N}_0, \\
\Sigma_{\mathcal{L}}^{JB^*}(Z) &= \{1, \dots, 2^n\}, \\
\Sigma_{\mathcal{R}}^{JB^*}(Z) &= \{1, \dots, 2^n\}, \\
\Gamma(Z) &= \left\{2^{n-1}\right\}.
\end{aligned}$$

Proof. We have to prove the statements for $\Gamma(Z)$. Let first $\dim Z$ be odd and $\mathcal{S} = \{\mathrm{id}, s_1^{odd}, \dots, s_{2n-2}^{odd}\}$ be the standard spin system in $\rho_Z(Z)$ defined as above. By Proposition 5.2.6 we can construct a spin grid $\mathcal{G}$ from $\mathcal{S}$ linearly spanning $\rho_Z(Z)$. We get

$$\begin{aligned}
u_1^{odd} &= \frac{1}{2}\left(\mathrm{id}^n - \sigma_1 \otimes \mathrm{id}^{n-1}\right) = \frac{1}{2}\left((\mathrm{id} - \sigma_1) \otimes \mathrm{id}^{n-1}\right) \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathrm{id}^{n-1},
\end{aligned}$$

$$\begin{aligned}
\widetilde{u}_1^{odd} &= -\frac{1}{2}\left(\mathrm{id}^n + \sigma_1 \otimes \mathrm{id}^{n-1}\right) = -\frac{1}{2}\left((\mathrm{id} + \sigma_1) \otimes \mathrm{id}^{n-1}\right) \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathrm{id}^{n-1},
\end{aligned}$$

$$\begin{aligned}
u_{l+1}^{odd} &= \frac{1}{2}\left(s_{2l}^{odd} + i s_{2l+1}^{odd}\right) \\
&= \frac{1}{2}\left(\sigma_3^{l-1} \otimes \sigma_2 \otimes \mathrm{id}^{n-l} + i\sigma_3^l \otimes \sigma_1 \otimes \mathrm{id}^{n-l-1}\right) \\
&= \frac{1}{2}\left(\sigma_3^{l-1} \otimes (\sigma_2 \otimes \mathrm{id} + i\sigma_3 \otimes \sigma_1) \otimes \mathrm{id}^{n-l-1}\right) \\
&= \frac{1}{2}\left(\sigma_3^{l-1} \otimes \left(\begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\right) \otimes \mathrm{id}^{n-l-1}\right) \\
&= \sigma_3^{l-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \mathrm{id}^{n-l-1}
\end{aligned}$$

and similarly

$$\widetilde{u}_{l+1}^{odd} = \sigma_3^{l-1} \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \otimes \mathrm{id}^{n-l-1} .$$

This leads us to

$$u_1^{odd} \left(u_1^{odd}\right)^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathrm{id}^{n-1},$$

$$\widetilde{u}_1^{odd} \left(\widetilde{u}_1^{odd}\right)^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathrm{id}^{n-1},$$

$$u_{l+1}^{odd} \left(u_{l+1}^{odd}\right)^* = \mathrm{id}^{l-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \mathrm{id}^{n-l-1} \quad \text{and}$$

$$\widetilde{u}_{l+1}^{odd} \left(\widetilde{u}_{l+1}^{odd}\right)^* = \mathrm{id}^{l-1} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \mathrm{id}^{n-l-1} .$$

Since $\operatorname{rank}(A \otimes B) = (\operatorname{rank} A)(\operatorname{rank} B)$ we can conclude from the above, for $\dim Z = 2n - 1$ with $n \geq 2$, that

$$\Gamma(Z) = \left\{2^{n-1}\right\}.$$

If $\dim Z$ is even we can deduce from the above results and Lemma 5.2.5 that

$$\begin{aligned} u_1^{even} &= \frac{1}{2}\left(\left(\mathrm{id}^{n-1}, \mathrm{id}^{n-1}\right) - \left(\sigma_1 \otimes \mathrm{id}^{n-2}, \sigma_1 \otimes \mathrm{id}^{n-2}\right)\right) \\ &= \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathrm{id}^{n-2}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathrm{id}^{n-2}\right), \end{aligned}$$

$$\begin{aligned} \widetilde{u}_1^{even} &= -\frac{1}{2}\left(\left(\mathrm{id}^{n-1}, \mathrm{id}^{n-1}\right) + \left(\sigma_1 \otimes \mathrm{id}^{n-2}, \sigma_1 \otimes \mathrm{id}^{n-2}\right)\right) \\ &= \left(\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathrm{id}^{n-2}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathrm{id}^{n-2}\right), \end{aligned}$$

$$\begin{aligned} u_{l+1}^{even} &= \frac{1}{2}\left(s_{2l}^{even} + i s_{2l+1}^{even}\right) \\ &= \left(\sigma_3^{l-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \mathrm{id}^{n-l-2}, \sigma_3^{l-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \mathrm{id}^{n-l-2}\right), \end{aligned}$$

$$\begin{aligned} \widetilde{u}_{l+1}^{even} &= \frac{1}{2}\left(s_{2l}^{even} - i s_{2l+1}^{even}\right) \\ &= \left(\sigma_3^{l-1} \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \otimes \mathrm{id}^{n-l-2}, \sigma_3^{l-1} \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \otimes \mathrm{id}^{n-l-2}\right). \end{aligned}$$

The element u_0^{even} is given by

$$u_0^{even} = s_{2n-1}^{even} = \left(\sigma_3^{n-1}, -\sigma_3^{n-1}\right).$$

The corresponding projections are

$$u_1^{even}\left(u_1^{even}\right)^* = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathrm{id}^{n-2}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathrm{id}^{n-2}\right),$$

$$\widetilde{u}_1^{even}\left(\widetilde{u}_1^{even}\right)^* = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathrm{id}^{n-2}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathrm{id}^{n-2}\right),$$

$$\begin{aligned} u_{l+1}^{even}\left(u_{l+1}^{even}\right)^* = \Bigg(& \mathrm{id}^{l-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \mathrm{id}^{n-l-2}, \\ & \mathrm{id}^{l-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \mathrm{id}^{n-l-2} \Bigg), \end{aligned}$$

$$\widetilde{u}_{l+1}^{even}\left(\widetilde{u}_{l+1}^{even}\right)^* = \left(\mathrm{id}^{l-1}\otimes\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\otimes\mathrm{id}^{n-l-2},\right.$$

$$\left.\mathrm{id}^{l-1}\otimes\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\otimes\mathrm{id}^{n-l-2}\right) \text{ and}$$

$$u_0^{even}\left(u_0^{even}\right)^* = \left(\mathrm{id}^{n-1},\mathrm{id}^{n-1}\right).$$

For an even dimensional spin factor we can conclude that

$$\Gamma(Z) = \left\{\left(2^{n-2},2^{n-2}\right),\left(2^{n-1},2^{n-1}\right)\right\}.$$

□

5.3 Classification

Finally we are in the position to give the announced K-theoretic classification of the finite-dimensional JC^*-triple systems. We first notice that, if Z_1 and Z_2 are two isomorphic finite-dimensional JC^*-triple, the corresponding double-scaled ordered K_0-groups of their universal enveloping TROs are isomorphic. This isomorphism is given by $K_0^{\mathrm{JB}^*}(\varphi)$, which also induces a bijection from $\Gamma(Z_1)$ to $\Gamma(Z_2)$ by Proposition 5.1.6. Thus we can conclude that if two finite-dimensional JC^*-triple systems are isomorphic, then their K-grid invariants are isomorphic.

To prove the opposite direction of the above statement we first collect, for easier accessibility, the results from the previous section in the following list. Let $n \in \mathbb{N}$, $k \geq 2$, then the K-grid invariants of the Cartan factors are

$\mathbf{C^1_{1,n}}$: The invariant $\mathcal{KG}(C^1_{1,n})$ is given by

$$\begin{aligned}
K_0^{\mathrm{JB}^*}(C^1_{1,n}) &= \mathbb{Z}^n,\\
K_0^{\mathrm{JB}^*}(C^1_{1,n})_+ &= \mathbb{N}_0^n,\\
\Sigma_{\mathcal{L}}^{\mathrm{JB}^*}(C^1_{1,n}) &= \prod_{k=1}^{n}\left\{1,\dots,\binom{n}{k}\right\},\\
\Sigma_{\mathcal{R}}^{\mathrm{JB}^*}(C^1_{1,n}) &= \prod_{k=1}^{n}\left\{1,\dots,\binom{n}{k-1}\right\} \text{ and}\\
\Gamma(C^1_{1,n}) &= \left\{\left(\binom{n-1}{0},\binom{n-1}{1},\dots,\binom{n-1}{n-1}\right)\right\}.
\end{aligned}$$

$\mathbf{C^1_{n,m}}$: For the rectangular Cartan factor with $n, m \geq 2$ we have

$$\begin{aligned}
K_0^{\mathrm{JB}^*}(C^1_{n,m}) &= \mathbb{Z}^2,\\
K_0^{\mathrm{JB}^*}(C^1_{n,m})_+ &= \mathbb{N}_0^2,\\
\Sigma_{\mathcal{L}}^{\mathrm{JB}^*}(C^1_{n,m}) &= \{1,\dots,n\} \times \{1,\dots,m\},\\
\Sigma_{\mathcal{R}}^{\mathrm{JB}^*}(C^1_{n,m}) &= \{1,\dots,m\} \times \{1,\dots,n\} \text{ and}\\
\Gamma(C^1_{n,m}) &= \{(1,1)\}.
\end{aligned}$$

$\mathbf{C^2_n}$: For the symplectic Cartan factors with $\dim C^2_n \geq 10$ the K-grid invariant is given by

$$\begin{aligned}
K_0^{\mathrm{JB}^*}(C^2_n) &= \mathbb{Z},\\
K_0^{\mathrm{JB}^*}(C^2_n)_+ &= \mathbb{N}_0,\\
\Sigma_{\mathcal{L}}^{\mathrm{JB}^*}(C^2_n) &= \{1,\dots,n\},\\
\Sigma_{\mathcal{R}}^{\mathrm{JB}^*}(C^2_n) &= \{1,\dots,n\} \text{ and}\\
\Gamma(C^2_n) &= \{2\}.
\end{aligned}$$

$\mathbf{C^3_n}$: For the hermitian Cartan factors we have

$$\begin{aligned}
K_0^{\mathrm{JB}^*}(C^3_n) &= \mathbb{Z},\\
K_0^{\mathrm{JB}^*}(C^3_n)_+ &= \mathbb{N}_0,\\
\Sigma_{\mathcal{L}}^{\mathrm{JB}^*}(C^3_n) &= \{1,\dots,n\},\\
\Sigma_{\mathcal{R}}^{\mathrm{JB}^*}(C^3_n) &= \{1,\dots,n\} \text{ and}\\
\Gamma(C^3_n) &= \{1,2\}.
\end{aligned}$$

$\mathbf{C^4_k}$: Let Z be a finite-dimensional spin factor with $\dim Z = k+1$.

If Z is of even dimension, i.e. $k = 2n-1$, $n \geq 2$, then the K-grid invariant of Z is given by

$$\begin{aligned}
K_0^{\mathrm{JB}^*}(Z) &= \mathbb{Z}^2,\\
K_0^{\mathrm{JB}^*}(Z)_+ &= \mathbb{N}_0^2,\\
\Sigma_{\mathcal{L}}^{\mathrm{JB}^*}(Z) &= \left\{1,\dots,2^{n-1}\right\}^2,\\
\Sigma_{\mathcal{R}}^{\mathrm{JB}^*}(Z) &= \left\{1,\dots,2^{n-1}\right\}^2 \text{ and}\\
\Gamma(Z) &= \left\{\left(2^{n-2},2^{n-2}\right),\left(2^{n-1},2^{n-1}\right)\right\}.
\end{aligned}$$

If Z is of odd dimensions, i.e. $k = 2n$, $n \geq 2$, then $\mathcal{KG}(Z)$ has the components

$$\begin{aligned} K_0^{\mathrm{JB}^*}(Z) &= \mathbb{Z}, \\ K_0^{\mathrm{JB}^*}(Z)_+ &= \mathbb{N}_0, \\ \Sigma_{\mathcal{L}}^{\mathrm{JB}^*}(Z) &= \{1, \dots, 2^n\}, \\ \Sigma_{\mathcal{R}}^{\mathrm{JB}^*}(Z) &= \{1, \dots, 2^n\} \text{ and} \\ \Gamma(Z) &= \left\{2^{n-1}\right\}. \end{aligned}$$

Theorem 5.3.1. *Let Z_1 and Z_2 be finite-dimensional JC^*-triple systems. If $\sigma : K_0^{JB^*}(Z_1) \to K_0^{JB^*}(Z_2)$ is an isomorphism with $\sigma(\mathcal{KG}(Z_1)) = \mathcal{KG}(Z_2)$, then there exists a JB^*-isomorphism $\varphi : Z_1 \to Z_2$ such that $K_0^{JB^*}(\varphi) = \sigma$.*

Proof. Since $\sigma(\mathcal{KG}(Z_1)) = \mathcal{KG}(Z_2)$, we especially know that σ is an isomorphism of the double-scaled ordered K_0-groups of $T^*(Z_1)$ and $T^*(Z_2)$. Using 3.4.27 we can lift σ to a complete isometry $\varphi' : T^*(Z_1) \to T^*(Z_2)$ with $K_0^{\mathrm{JB}^*}(\varphi') = \sigma$. The TRO $T^*(Z_1)$ is by 3.2.1 the finite sum of rectangular matrix algebras $T^*(Z_1) \simeq \bigoplus_{i=1}^p \mathbb{M}_{n_i,m_i}$, determined by the double-scaled ordered K_0-group of $T^*(Z_1)$. Now we can use the information encoded in $\Gamma(Z_1)$ and the above list of the K-grid invariants to recover which summands correspond to which Cartan factor (the list allows no ambiguities). Since the K-grid invariant is additive by Proposition 5.1.7 we can recover the image $\rho_{Z_1}(Z_1) \subseteq T^*(Z_1)$ up to (ternary) unitary equivalence. The same works for $\rho_{Z_2}(Z_2) \subseteq T^*(Z_2)$.

Let $\mathcal{G}_1$ be a grid spanning Z_1 and $\mathcal{G}_2' := \varphi'(\mathcal{G}_1) \subseteq T^*(Z_2)$ its image under φ'. Let $\phi : T^*(Z_2) \to T^*(Z_2)$ be the TRO-isomorphism mapping the linear span of $\mathcal{G}_2'$ to $\rho_{Z_2}(Z_2)$ (we construct ϕ by using the universal property of $T^*(Z_2)$). Since Z_2 is finite-dimensional so is $T^*(Z_2)$ and thus ϕ is automatically inner and unitary equivalent to the identity. If we put

$$\varphi := \rho_{Z_2}^{-1} \circ \phi \circ \varphi' \circ \rho_{Z_1} : Z_1 \to Z_2,$$

where $\rho_{Z_2}^{-1} : \rho_{Z_2}(Z_2) \to Z_2$ is the inverse of ρ_{Z_2} restricted to its image, then φ is a JB^*-isomorphism with $K_0^{\mathrm{JB}^*}(\varphi) = \sigma$. □

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