Fach: Mathematik

Double Complexes and Hodge Structures as Vector Bundles

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Chapter 0

Introduction

In this thesis, we study the cohomology of compact complex manifolds and complex algebraic varieties from two less familiar points of view. These are:

- For compact complex manifolds: Consider the double complex of $\mathbb{C}$-valued differential forms instead of cohomology.
- For complex algebraic varieties: Consider equivariant vector bundles instead of Mixed Hodge Structures.

In the following, these points of view and the contributions and applications deduced in this thesis will be explained in more detail.

To a compact complex manifold X, one can attach several finite-dimensional cohomology groups: The de-Rham cohomology $H_{dR}^{\bullet}(X,\mathbb{C})$, which is equipped with a bounded descending filtration F and an antilinear involution σ, and the bigraded Dolbeault, Bott-Chern and Aeppli cohomologies[1]

$$H_{\bar{\partial}}^{\bullet,\bullet}(X),\ H_{BC}^{\bullet,\bullet}(X),\ H_{A}^{\bullet,\bullet}(X).$$

If X is in addition Kähler, all of these spaces coincide with a certain subspace of de-Rham cohomology. Namely, denoting $H^{p,q} := (F^p \cap \sigma F^q)H_{dR}^{p+q}(X,\mathbb{C})$ the space of classes representable by forms of pure type (p,q), there are natural identifications

$$H^{p,q} = H_{BC}^{p,q}(X) = H_{\bar{\partial}}^{p,q}(X) = H_{A}^{p,q}(X)$$

and $H_{dR}(X,\mathbb{C})$ carries a Hodge structure of weight k, i.e. there is the famous 'Hodge decomposition':[2]

$$H_{dR}^{k}(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}$$

This decomposition is known to pose strong restrictions on the topology of compact Kähler manifolds and maps between them: For example, the Betti-numbers b_k for odd k have to be even and any filtered map φ between de-Rham cohomology groups respects the filtration F strictly (i.e. $\varphi F^p = \operatorname{im}\varphi \cap F^p$).

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[1] see e.g. [Sch07].

[2] see e.g. [Voi02, part II].

For general X, the Hodge decomposition fails and all of the above cohomology groups can be different. They are related by maps and the Frölicher spectral sequence, but the latter can be 'arbitrarily nondegenerate'.[3] Even the statement about maps is false in the following strong sense:

Counterexample A (section 1.6)**.** *There is a compact complex 3-fold X and a holomorphic map $f : X \longrightarrow X$ s.t. $f^* : H^1_{dR}(X, \mathbb{C}) \longrightarrow H^1_{dR}(X, \mathbb{C})$ is not strict.*

All of the above cohomology groups can be computed from the Dolbeault double complex, i.e. the double complex of $\mathbb{C}$-valued forms on X:

$$(\mathcal{A}_X^{\bullet,\bullet}, \partial, \bar{\partial})$$

The point of view taken in the first part of the thesis is that one should study this double complex directly by means of the following folklore result, of which to my best knowledge there existed so far no proof in the literature and, apart from [Ang15], almost no study of its consequences:

Theorem B (theorem 1.3)**.** *For any bounded double complex A over a field K there is an isomorphism $A \cong \bigoplus_E E^{\oplus \operatorname{mult}_E(A)}$, where E runs over*

- *Squares*

$$\begin{tikzcd}
K \arrow[r, "-\mathrm{Id}"] & K \\
K \arrow[u, "\mathrm{Id}"] \arrow[r, "\mathrm{Id}"] & K \arrow[u, "\mathrm{Id}"']
\end{tikzcd}$$

- *and zigzags*

$$\begin{tikzcd}
 & & K & & & & & K & & & \\
K & , & & , & K \arrow[r, "\mathrm{Id}"] & K & , & & & , & \cdots \\
 & & K \arrow[uu, "\mathrm{Id}"] & & & & & K \arrow[uu, "\mathrm{Id}"] \arrow[r, "\mathrm{Id}"] & K & &
\end{tikzcd}$$

For any such isomorphism, the (cardinal) numbers $\operatorname{mult}_E(A)$ coincide.

An early step in this direction was made in [Del+75], where it was proved that a complex satisfies the $\partial_1\partial_2$-lemma (which is the case e.g. for $\mathcal{A}_X$ for X compact Kähler) if and only if it is a direct sum of squares and zigzags of length 1 (the length being the number of nonzero components).

Squares do not contribute to any of the above cohomologies. Even length zigzags do not contribute to the total cohomology but account for differentials in the 'Frölicher spectral sequence(s)', which compute the total cohomology by row and column cohomology. For example, one has the following result:

Proposition C (proposition 1.6)**.** *Let A be a bounded double complex over a field. The two Frölicher spectral sequences degenerate on the r-th page if and only if the length of all even zigzags appearing in some (any) decomposition as above is smaller than $2r$.*

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[3] see [BR14].

Odd length zigzags determine and are determined by the (bifiltered) total cohomology. For example, one can characterise in terms of the occuring zigzags when there is a 'Hodge-decomposition' of weight not necesarily equal the cohomological degree.

Proposition D (propositions 1.15 and 1.18)**.** *Let A be a bounded double complex over a field. There is a Hodge decomposition of weight k on the total cohomology in degree d*

$$H_{dR}^d(A) = \bigoplus_{p+q=k} (F_1^p \cap F_2^q) H_{dR}^d(A)$$

if and only if all zigzags contributing to $H_{dR}^d(A)$ have length $2|d-k|+1$ and are concentrated in total degree d and $d + \operatorname{sgn}(d-k)$.
The spaces in the decomposition consist of classes representable by pure elements of $k-d+1$ different types (if $k \geq d$), respectively classes that admit a representative which is a sum of $d-k+1$ elements of different types (if $k \leq d$).

Together with proposition C, this generalises the abovementioned result of [Del+75]. All zigzags together also determine Bott-Chern and Aeppli cohomology.

Writing $A \simeq_1 B$ if two double complexes A, B have 'the same' zigzags, one obtains an equivalence relation fine enough to see all cohomological information. The fruitfulness of this approach will be illustrated by the following result:

Theorem E (theorems 1.36 and 1.39)**.** *Let X be a compact complex manifold*

- *Let $\mathcal{V}$ be a complex vector bundle of rank $n+1$ over X and $\mathbb{P}(\mathcal{V})$ the associated projective bundle. Then:*

$$\mathcal{A}_{\mathbb{P}(V)} \simeq_1 \mathcal{A}_X \otimes \mathcal{A}_{\mathbb{P}_{\mathbb{C}}^n}$$

- *Denoting by $\operatorname{Bl}_Z X$ the blow up of X along a closed compact submanifold $Z \subset X$ of codimension ≥ 2, and by E the exceptional divisor, then*

$$\mathcal{A}_{\operatorname{Bl}_Z X} \oplus \mathcal{A}_Z \simeq_1 \mathcal{A}_X \oplus \mathcal{A}_E$$

This has many applications to bimeromorphic geometry. For example, the multiplicity of certain zigzags is a bimeromorphic invariant. Or, calling a compact complex manifold X a $\partial\bar{\partial}$-manifold if $\mathcal{A}_X$ satisfies the $\partial\bar{\partial}$-lemma (equivalently: if it consists only of squares and zigzags of length 1), one obtains:

Corollary F (corollary 1.40)**.** *The property of being a $\partial\bar{\partial}$-manifold is a bimeromorphic invariant of compact complex manifolds if and only if every submanifold of a $\partial\bar{\partial}$-manifold is again a $\partial\bar{\partial}$-manifold.*

Theorem E and corollary F strengthen and unify recent partial results in [Ang+17], [RYY17] and [YY17], where many other corollaries can also be found.

Considering two compact complex manifolds X and Y to be equivalent if their Dolbeault double complexes are (i.e. $\mathcal{A}_X \simeq_1 \mathcal{A}_Y$) yields an interesting quotient of the Grothendieck ring of compact complex manifolds which generalises the

Hodge ring of Kähler manifolds studied in [KS13].

Smooth projective algebraic varieties over the complex numbers yield important examples of compact Kähler manifolds. Deligne, motivated by analogies with the case of varieties over a finite field ([Del71a], [Del71b], [Del74]) found a different generalisation of the Hodge decomposition for these. In fact, he showed that the functor of singular cohomology with complex coefficients factors through the category of (complex) Mixed Hodge Structures $\mathrm{MHS}_{\mathbb{C}}$, i.e., $H^d(X,\mathbb{C})$ carries a bounded ascending filtration $W_\bullet$, called the weight filtration, and two bounded descending filtrations F_1, F_2 which induce a Hodge decomposition of weight k on each graded piece $\mathrm{gr}_k^W H^d(X,\mathbb{C})$.[4]

A conjectural picture by C. Deninger suggests a more geometric perspective on Mixed Hodge Structures. We summarise the idea here in a very idealised form: To any number field K, it should be possible to associate a compact complex 3-fold X_K, equipped with a foliation by Riemann surfaces and an almost everywhere transversal flow mapping leaves into leaves. The geometry of X_K should encode information about K. For example, there should be a correspondence between the set of closed orbits of positive length of the flow and the set of finite places (i.e., equivalence classes of non-archimedean valuations on K). The 'infinite primes', i.e., equivalence classes of archimedean absolute values, should correspond to closed orbits of length 0 (fixed points) which have to lie in a leaf fixed by the flow. In this picture, a motive over K would correspond to a vector bundle with certain extra structure on M_K and the different realisations to restrictions to appropriate neigbourhoods of the closed orbits. In particular, a Mixed Hodge structure should be realised as a vector bundle with additional structure on a Riemann surface.

In fact, several categories of vector bundles with extra structure are known to be equivalent to $\mathrm{MHS}_{\mathbb{C}}$:[5]

- The category of $\mathbb{G}_m$-equivariant algebraic vector bundles $\mathcal{V}$ over $\mathbb{P}^1_{\mathbb{C}}$ with a bounded ascending filtration $\mathcal{W}_\bullet$ by subbundles s.t. $\mathrm{gr}_k^{\mathcal{W}} \mathcal{V}$ is semistable of slope k.
- The category of $\mathbb{G}_m^2$-equivariant algebraic vector bundles on $\mathbb{P}^2_{\mathbb{C}}$ which are trivial on $\{X_0 = 0\}$.
- The category of $\mathbb{G}_m^2$-equivariant algebraic vector bundles on $\mathbb{A}^2_{\mathbb{C}}$ with an equivariant connection.

The equivalence of $\mathrm{MHS}_{\mathbb{C}}$ to the first category was pointed out by Simpson in [Sim97a]. If one omits the $\mathbb{G}_m$-equivariance condition, one obtains the notion of Mixed Twistor Structures. This point of view is known to align well with

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[4] Actually, one even gets a rational Hodge structure, i.e. the filtrations F_1, F_2 are conjugate to each other and the filtrations $W_\bullet$ can already be defined on the singular cohomology with rational coeficients, but we will stick to this simplified version for the purpose of this introduction.

[5] There are also 'real' versions of these categories, where the dimension of the base space is lower, thus making the later two cases fit better with Deninger's picture.

the more recent development in p-adic Hodge theory,[6] where one has an equivalence of categories of p-adic Galois representations and bundles on the so-called Fargues-Fontaine curve.

The equivalence to the bundles on $\mathbb{P}^2_{\mathbb{C}}$ was described by Penacchio, a student of Simpson, and used, for example, to give a new proof of the fact that $\mathrm{MHS}_{\mathbb{C}}$ is abelian by semistability arguments.[7]

The equivalence to the category of equivariant connections was proved by Kapranov in [Kap12], building on the work of Penacchio and using a non-trivial geometric procedure called the Radon-Penrose transform. There is also an independent proof by Goncharov in [Gon08], using a more direct approach wich has the advantage to work for variations of Hodge Structures as well.

The second part of the thesis (i.e. everything except the first chapter) is devoted to studying the methods used in proving, applications derived from and relations between these equivalences.

A first result is the following:

Theorem G (corollaries 2.27 and 2.28)**.** *For any smooth toric variety X over $\mathbb{C}$, analytification yields equivalences of categories*

$$\left\{\begin{matrix}\textit{equivariant algebraic vector}\\ \textit{bundles on } X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{equivariant holomorphic}\\ \textit{vector bundles on } X^{an}\end{matrix}\right\}$$

and

$$\left\{\begin{matrix}\textit{equivariant algebraic vector}\\ \textit{bundles with an equivariant}\\ \textit{connection on } X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{equivariant holomorphic}\\ \textit{vector bundles with an}\\ \textit{equivariant connection}\\ \textit{on } X^{an}\end{matrix}\right\}.$$

This implies in particular that in all three cases we could replace ‘algebraic’ by ‘holomorphic’ and still obtain equivalent categories, which fits well with Deninger’s picture.

In [Kat13], Kato proposed a definition of height functions for mixed motives over number fields and made a finiteness conjecture for motives with bounded height. The height function has a certain contribution depending only on the MHS on the Betti-realisation of a motive associated with any infinite place: The archimedean height. A different definition for the archimedean height was given in [Kat18]. In terms of the associated bundle with connection $(\mathcal{V}, \nabla)$, the second definition is essentially ‘the norm of the holonomy of ∇ along a certain path’. In the same spirit, one can give a third definition as ‘the norm of the curvature of ∇ at $(1,1)$’.

Theorem H (theorem 4.21)**.** *All three definitions of the archimedean height are equivalent.*

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[6] [FF]

[7] see [Pen03], [Pen11].

Here, the notion of equivalence well-adapted to finiteness considerations.

Attached to a mixed motive M over a number field K, there should be an L-series, defined as a product over 'Euler factors' $L_{\mathfrak{p}}(M,s)$ for each place of K. For the infinite places $\mathfrak{p}$, a definition of the Euler factors in terms of the associated Mixed Hodge Structure $M_{\mathfrak{p}}$ on the Betti-realisation was given in [Ser70] in the pure case and in [FP92]. In [Den91] and [Den01], it was shown how to produce a certain equivariant vector bundle $\xi(M_{\mathfrak{p}})$ s.t. $L_{\mathfrak{p}}(M,s)$ appears as the zeta-regularised determinant of an endomorphism derived from the action. However, the bundle $\xi(M_{\mathfrak{p}})$ carries less information than the whole Mixed Hodge structure. Based on an idea from [Pri16], it will be shown how to produce this bundle from the bundles on $\mathbb{P}^2$ and $\mathbb{P}^1$ associated with $M_{\mathfrak{p}}$ and a related one from the bundle on $\mathbb{A}^2$ in the pure case (section 4.4).

The category $\mathrm{MHS}_{\mathbb{C}}$ is tannakian and hence equivalent to the category of representations of a (pro-)algebraic group $\mathfrak{G}$. This group has been determined by Deligne via a computational approach to be the semi-direct product of $\mathbb{G}_m^2$ with a pro-unipotent group $\mathfrak{U}$. Based on a remark by Kapranov, one obtains the following geometric interpretation of (an uncompleted version of) $\mathfrak{G}$, using holonomy-representations:

Proposition I (proposition 4.26)**.** *There is a $(\mathbb{C}^\times)^2$-invariant subgroup $\mathfrak{L}^{Gon}$ of the group of piecewise smooth loops in $\mathbb{C}^2$ (up to reparametrisation and cancellation) s.t. the category $MHS_{\mathbb{C}}$ is equivalent to the category of representations of $\mathfrak{L}^{Gon} \rtimes (\mathbb{C}^\times)^2$ which are holomorphic in the second factor and continuous (for a natural topology) in the first.*

By the very definition of Mixed Hodge Structures, they come equipped with a filtration by subobjects: the weight filtration. However, in the last two categories such a filtration is not part of the datum. Instead, it appears as an intrinsic feature:

Proposition J (section 4.2)**.** *The weight filtration defined via the equivalence with* $\mathrm{MHS}_{\mathbb{C}}$ *on equivariant bundles with connection on $\mathbb{A}^2_{\mathbb{C}}$, resp. on equivariant $\{X_0 = 0\}$-trivial bundles on $\mathbb{P}^2_{\mathbb{C}}$, coincides with a Harder-Narasimhan-style slope filtration.*

In the pure case, based on a minor modification of an idea by Simpson ([Sim97a], [Sim97b]) one can give a direct construction of Simpson's bundle as the pushforward of a deformed de-Rham complex.

Proposition K (section 4.1)**.** *Let X be a compact complex manifold satisfying the $\partial\overline{\partial}$-lemma (e.g. X compact Kähler) and $A' := \mathbb{C}^2 - \{(0,0)\}$. Denote by*

$$\pi : A' \times X \longrightarrow \mathbb{P}^1_{\mathbb{C}}$$

the projection map, by t_1, t_2 coordinates on $\mathbb{C}^2$ and by $\widetilde{\mathcal{A}}^p_{A'\times X/A'}$ the $\mathbb{C}$-valued relative smooth p-forms which are holomorphic in direction t_1, t_2. Then there is a canonical identification

$$\pi_*(\widetilde{\mathcal{A}}^\bullet_{A'\times X/A'}, t_1\partial + t_2\overline{\partial})^{\Delta(\mathbb{G}_m)} \cong \xi_{\mathbb{P}^1}(H^k_{dR}(X,\mathbb{C}), F, \sigma F),$$

where the right hand side denotes the equivariant semistable bundle of slope k on $\mathbb{P}^1$ associated with $H^k_{dR}(X,\mathbb{C})$ by Simpson's construction. A similar construction also produces the bundle attached to $H^k_{dR}(X,\mathbb{C})$ by Goncharov's construction.

Let us now give a brief overview over the structure of each chapter:

The first three sections of the first chapter deal with the general theory of bounded double complexes over a field, assuming theorem B. How to recognise the multiplicities of each zigzag given a double complex, the relation to the various cohomological invariants and criteria for degeneration of the Frölicher spectral sequences (proposition C) are treated in section 1.1, the generalised Hodge decomposition (proposition D) and criteria for strictness of induced maps on de-Rham cohomology in section 1.2 and the structure of several Grothendieck rings of double complexes, i.e., the behaviour of squares and zigzags under tensor product, in section 1.3.
In the next four sections, the Dolbeault double complex of a compact complex manifold is the central topic. Restrictions on the isomorphism type of the Dolbeault complex of a compact complex manifold are collected and several questions are raised in section 1.4. Then theorem E is proven (sec. 1.5) and the counterexample A is constructed (sec. 1.6). The theory is complemented by two examples in section 1.7, where the Dolbeault double complexes of the counterexample from section 1.6 and of many Calabi-Eckmann manifolds are computed.
The chapter ends with a proof of theorem B in section 1.8, which is postponed since it uses the language of quivers which is not necessary in the rest of the chapter.

The second chapter first (in section 2.1) collects known results from the theory of toric vector bundles on smooth toric varieties. Such bundles can be described completely by linear algebraic data (certain vector spaces of invariant sections and filtrations by pole order along invariant divisors), which will provide the basis for the geometric perspective on Mixed Hodge Structures. In section 2.2, several comparisons are made with the continuous, smooth and holomorphic case and all the results are extended to the holomorphic setting by proving theorem G.

In the third chapter, the known equivalent descriptions of the category of (real or complex) Mixed Hodge structures are recalled. Emphasis is put on the geometric cases, in particular that by Kapranov and Goncharov, where full proofs are given (section 3.6) and some details, such as the computation of the curvature, a comparison between Kapranov and Goncharov's approaches and a variant of Kapranov's proof are added. In the final section 3.7, it is explained how polarisations can be translated to the geometric settings.

The final chapter is devoted to new results related to the geometric points of view on Mixed Hodge Structures: The direct construction (proposition K) is given in section 4.1 and the incarnation of the weight filtration as a slope filtration in section 4.2. After a short reminder on motives and Kato's finiteness conjecture, the equivalence of three definitions of archimedean heights (theorem H) is proven in section 4.3 and the computation of Deninger's bundle controlling the Gamma-factors is made in section 4.4. The chapter ends with a proof of the equivalence of the category of Mixed Hodge Structures to representations of a group of loops in section 4.5.

Three appendices complement the main text: Appendix A gives basic definitions and results on equivariant sheaves and connections and explain how these simplify in the particular case of $\mathbb{G}_m^n$-actions. Appendix B.1 contains a detailed proof of the equivariant Radon-Penrose transform following the (non-equivariant) sketch in [Man97]. Appendix C.2 is a detour into 19^{th} century mathematics: Two functional equations are solved, which were originally thought to be useful in the proof of theorem G, but might be of independent interest.

Chapter 1

Double Complexes and Complex Manifolds

Notations and conventions: By a double complex over a field K, we mean a bigraded K-vector space $A = \bigoplus_{p,q\in\mathbb{Z}} A^{p,q}$ with two endomorphisms ∂_1, ∂_2 of bidegree $(1,0)$ and $(0,1)$ that satisfy the 'boundary condition' $\partial_i \circ \partial_i = 0$ for $i = 1,2$ and anticommute,[1] i.e., $\partial_1 \circ \partial_2 + \partial_2 \circ \partial_1 = 0$. We write $\partial_1^{p,q}$ for the map from $A^{p,q}$ to $A^{p+1,q}$ induced by restriction and similarly for $\partial_2^{p,q}$. We always assume double complexes to be bounded, i.e., $A^{p,q} = 0$ for almost all $(p,q) \in \mathbb{Z}^2$ and denote by DC_K^b the category of bounded double complexes over K and K-linear maps respecting the grading and the ∂_i. If no confusion is likely to result, we say *complex* instead of *double complex.*

The principal example we have in mind is the **Dolbeault double complex**, by which we mean the double complex of $\mathbb{C}$-valued smooth forms on a (compact) complex manifold. In particular, we stress that we do not assume our complexes to be finite dimensional.

1.1 The Isomorphism Type of a Double Complex

Definition 1.1. *An **elementary complex** is a nonzero double complex A s.t.*

1. *Every $\partial_i^{p,q}$ is either zero or an isomorphism and*
2. *Every two nonzero components $A^{p,q}$, $A^{p',q'}$ are connected by a chain of $\partial_i^{r,s}$ which are isomorphisms.*

*The **shape** of an elementary complex A is the set*

$$S(A) := \{(p,q) \in \mathbb{Z}^2 \mid A^{p,q} \neq 0\}.$$

*The **rank** of an elementary complex is* $\mathrm{rk}(A) := \dim A^{p,q}$ *for some $(p,q) \in S(A)$.*

In the following, if we write 'shape' without further specification, we usually mean 'shape of some elementary double complex'.

Example 1.2. 1. *For any $p,q \in \mathbb{Z}$, a complex A of the form*

$$\begin{tikzcd} A^{p-1,q} \arrow[r, "\sim"] & A^{p,q} \\ A^{p-1,q-1} \arrow[u, "\wr"] \arrow[r, "\sim"] & A^{p,q-1} \arrow[u, "\wr"] \end{tikzcd}$$

*is elementary. Such a complex will be called a **square**.*

2. *An elementary complex A supported on at most two antidiagonals (i.e., there is a $d \in \mathbb{Z}$ s.t. $A^{p,q} = 0$ if $p + q \notin \{d, d+1\}$) is called a **zigzag**. We say that it is of length l if its shape consists of l elements.*[2] *E.g., if $V = A^{p,q}$ for some $(p,q) \in S(A)$ and we omit the bigrading in the notation:*

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[1] The *anti-* not essential. In fact, replacing ∂_1 by ∂_1' defined by $(\partial_1')^{p,q} := (-1)^p \partial_1^{p,q}$ we can pass to a commutative double complex (satisfying $\partial_1 \circ \partial_2 = \partial_2 \circ \partial_1$) and vice versa.

[2] One might argue that it would be more natural to count the number of nonzero arrows instead. However, this convention agrees with other sources on the subject and will find a further justification by the behaviour under tensor product, investigated in section 1.3.

$$\begin{tikzcd} V \end{tikzcd} \qquad \begin{tikzcd} V \\ V \arrow[u] \end{tikzcd}, \quad \begin{tikzcd} V \arrow[r] & V \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r] & V \\ & V \arrow[u] \end{tikzcd}, \quad \begin{tikzcd} V & \\ V \arrow[u] \arrow[r] & V \end{tikzcd}$$

length 1 *length* 2 *length* 3

One verifies that every elementary complex is isomorphic to a square or a zigzag and that two elementary complexes are isomorphic iff they have the same shape and rank. In particular, a shape of an elementary complex also determines which arrows $\partial_i^{p,q}$ are nonzero and we will draw shapes as a set of points and arrows.

The following seems to be a folklore result. I became aware of it through a MO-post by Greg Kuperberg[3], who attributes it to Mikhail Khovanov. However, I do not know a reference containing a proof and it seems that several experts do not know one either. For example it is stated without proof and asking for a reference in [Még14]. D. Angella states it (and some of the consequences discussed below) in [Ang15] and [Ang18] as well, referring to the abovementioned MO-post, but phrasing all arguments that use it as heuristics. In the case of a complex satisfying the $\partial_1\partial_2$-lemma (c.f. lemma 1.31), the result is included already in [Del+75]. For finite dimensional complexes, a similar result is contained in [Rin75]. In section 1.8, we give an elementary but nontrivial proof without restrictions on the dimension using the language of quivers. Before that, we will discuss some of its consequences.

Theorem 1.3. *For any double complex $A \in \mathrm{DC}_K^b$, there is a finite collection of elementary complexes $T_1, ..., T_n$ with pairwise distinct shapes and a (noncanonical) isomorphism*

$$A \cong \bigoplus_{i=1}^{n} T_i.$$

Up to reordering and isomorphism, the T_i are unique.

Convention: Given a shape S and a double complex A and some decomposition $A \cong \bigoplus T_i$ as above, we define the multiplicity of S in A as

$$\operatorname{mult}_S(A) := \begin{cases} \operatorname{rk}(T_i) & \text{if } S(T_i) = S \\ 0 & \text{else.} \end{cases}$$

This notion is independent of the chosen decomposition and knowing which factors appear in a decomposition is equivalent to knowing the multiplicity of all shapes S.

Fix a double complex A. A (or the) basic question to ask about it is:

What shapes occur with which multiplicity in A?

In the next paragraphs, we will give an answer to this question for zigzag-shapes in terms of various cohomological invariants associated with A. In order to formulate it, we need to label the various possible shapes. The chosen conventions

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[3] `https://mathoverflow.net/questions/25723/`, see also `https://mathoverflow.net/questions/86947/`.

might seem complicated at first glance, but are motivated by the formulas in proposition 1.4, proposition 1.26 and corollary 1.27.

The shape of an even length zigzag is determined by one endpoint, the length and the direction of the first arrow. Given $p, q \in \mathbb{Z}$, $l \in \mathbb{Z}_{>0}$ and $i \in \{1, 2\}$, denote by $S^{p,q}_{i,l}$ the unique zigzag shape of length $2l$ containing (p, q) and $(p+l, q-l+1)$ if $i = 1$ or $(p - l + 1, q + l)$ if $i = 2$. For example:

$$\begin{tikzcd}
\bullet^{0,1} \arrow[r] & \bullet^{1,1} & \\
& \bullet^{1,0} \arrow[u] \arrow[r] & \bullet^{2,0}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet^{0,1} \\
\bullet^{0,0} \arrow[u]
\end{tikzcd}$$

The shape $S^{0,1}_{1,2}$ The shape $S^{0,0}_{2,1}$

Given $p, q, d \in \mathbb{Z}$, denote by $S^{p,q}_{d}$ the unique zigzag shape of length $2|p+q-d|+1$ containing $(p, d-p)$ and $(d-q, q)$ and satisfying $r+s \in \{d, d+\mathrm{sgn}(d-p-q)\}$ for all $(r, s) \in S^{p,q}_{d}$. Here, d is the total degree the antidiagonal with the most nonzero components and the integers p, q simultaneously determine the endpoints and the second antidiagonal. For example:

$$\begin{tikzcd}
\bullet^{0,1} & \\
\bullet^{0,0} \arrow[u] \arrow[r] & \bullet^{1,0}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet^{1,1}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet^{0,2} \arrow[r] & \bullet^{1,2} \\
& \bullet^{1,1} \arrow[u]
\end{tikzcd}$$

The shape $S^{1,1}_{1}$ The shape $S^{1,1}_{2}$ The shape $S^{0,1}_{2}$

Finally, a square shape is determined by the position of any corner. We choose the top right one and define $S^{p,q} := \{(p - 1, q - 1), (p, q - 1), (p - 1, q), (p, q)\}$.

Recall that a double complex can be equipped with several filtrations in a functorial manner. In particular, there are the so-called 'stupid' filtrations by rows and columns

$$F_1^p A = \bigoplus_{r \geq p} A^{r,s},$$

$$F_2^p A = \bigoplus_{s \geq p} A^{r,s}.$$

These induce filtrations on the total complex A_{tot} given by

$$A^k_{tot} := \bigoplus_{p+q=k} A^{p,q}$$

with differential $d = \partial_1 + \partial_2$ and on the total (or de-Rham) cohomology $H^{\bullet}_{dR}(A) := H^{\bullet}(A_{tot}, d)$. We will still denote by F_i these last filtrations and call them Hodge filtrations. If not explicitely mentioned otherwise, in the following we will always mean these if we write F_i.

The 'stupid' filtrations also induce the converging **Frölicher spectral sequences**, which compute the induced filtrations on the total cohomology from the column or row cohomology of the double complex:

$$S_1: \qquad {}_1E_1^{p,q} = H^q(A^{p,\bullet}, \partial_2) \Longrightarrow \left(H_{dR}^{p+q}(A), F_1\right)$$

$$S_2: \qquad {}_2E_1^{p,q} = H^p(A^{\bullet,q}, \partial_1) \Longrightarrow \left(H_{dR}^{p+q}(A), F_2\right)$$

Now we can state how the multiplicities of shapes are detected:

Proposition 1.4. *With notations as in the preceding discussion:*

1. ***Even length zigzags:*** *Let ${}_id_r^{p,q}$ be the differential on $E_r^{p,q}$ in the spectral sequence S_i. There is an equality*

$$\operatorname{mult}_{S_{i,r}^{p,q}}(A) = \dim \operatorname{im} {}_id_r^{p,q}.$$

2. ***Odd length zigzags:*** *Given $\varphi : \bigoplus T_i \xrightarrow{\sim} A$ a decomposition into elementary complexes with distinct shapes, denote $H_{\varphi,d}^{p,q} := H_{dR}^d(\varphi T_i) \subseteq H_{dR}^d(A)$ if $S(T_i) = S_d^{p,q}$. These spaces split the filtrations F_1 and F_2, i.e.*

$$F_1^p H_{dR}^d(A) = \bigoplus_{r \geq p} H_{\varphi,d}^{r,s}, \qquad F_2^q H_{dR}^d(A) = \bigoplus_{s \geq q} H_{\varphi,d}^{r,s}.$$

In particular,

$$\operatorname{mult}_{S_d^{p,q}}(A) = \dim \operatorname{gr}_{F_1}^p \operatorname{gr}_{F_2}^q H_{dR}^d(A).$$

3. ***Squares:*** *There is an equality*

$$\operatorname{mult}_{S^{p,q}}(A) = \dim(\operatorname{im} \partial_1 \circ \partial_2) \cap A^{p,q}.$$

Proof. Since cohomology commutes with direct sums, we may assume that A is an elementary complex (and φ is the identity in 1.) by theorem 1.3. We invite the reader to draw the first page of the spectral sequences in each case.

If A is a square, formula 3. is clear and the first page of both Frölicher spectral sequences is 0, so the first two points are also satisfied.

If A is a zigzag, $\partial_1 \circ \partial_2$ is always zero, so 3. holds. If A is of even length, it does not contribute to the total cohomology. On the other hand, if it has shape $S_{i,r}^{p,q}$, there are two nonzero spaces of dimension $\operatorname{rk}(A)$ on ${}_iE_1$, namely ${}_iE_1^{p,q}$ and ${}_1E_1^{p+l,q-l+1}$ if $i = 1$ or ${}_2E_1^{p-l+1,q}$ if $i = 2$. All other terms in the first page of both spectral sequences vanish. Therefore, the only possible nonzero differential has to be an isomorphism, which implies 2 in that case.

If $S(A) = S_d^{p,q}$ is an odd length zigzag, the first page of each spectral sequence has just one nonzero entry of dimension $\operatorname{rk}(A)$, namely ${}_1E_1^{p,d-p}$ and ${}_2E_1^{d-q,q}$ if $p + q \leq d$ or ${}_1E_1^{d-q,q}$ and ${}_2E_1^{p,d-p}$ if $p + q \geq d$. In particular, all differentials in the spectral sequences are zero and both filtrations on $H_{dR}^d(A)$ have just one jump in the claimed position, which shows 1. and 2.. □

Remark 1.5. *It has been described before that nonvanishing differentials correspond to subcomplexes with even length zigzag shape, see e.g. [BT95], p. 161 ff. Statements similar to the one of this proposition are also found as a heuristic in [Ang18], who refers to a MO post by David Speyer.*[4] *However, the splitting of the Hodge filtrations does not seem to have been noted.*[5]

Corollary 1.6. *Let $r \in \mathbb{Z}_{>0}$. For a double complex, the following statements are equivalent:*

1. *Both Frölicher spectral sequences degenerate at stage r.*
2. *All even length zigzags of length greater or equal to $2r$ have multiplicity zero.*

We briefly recall two further cohomological invariants. The **Bott-Chern cohomology** is defined as

$$H^{p,q}_{BC}(A) := \frac{\ker \partial_1^{p,q} \cap \ker \partial_2^{p,q}}{\operatorname{im} \partial_1^{p-1,q} \circ \partial_2^{p-1,q-1}}$$

and the **Aeppli cohomology** as

$$H^{p,q}_{A}(A) := \frac{\ker \partial_1^{p,q+1} \circ \partial_2^{p,q}}{\operatorname{im} \partial_1^{p-1,q} + \operatorname{im} \partial_2^{p,q-1}}.$$

As explained in [Ang15], intuitively, $H^{p,q}_{BC}$ counts 'corners' with possibly incoming edges at (p,q) but not belonging to a square

$$\begin{tikzcd} {} \arrow[r, dashed] & \bullet^{p,q} \\ & {} \arrow[u, dashed] \end{tikzcd}$$

and $H^{p,q}_{A}$ counts 'corners' with possibly outgoing edges passing through $A^{p,q}$ and not belonging to a square:

$$\begin{tikzcd} {} & \\ \bullet^{p,q} \arrow[u, dashed] \arrow[r, dashed] & {} \end{tikzcd}$$

This can be made precise in the following way:

Proposition 1.7. *For any decomposition $A = \bigoplus_{i=1}^{n} T_i$ into elementary complexes there are decompositions*

$$H^{p,q}_{BC}(A) = \bigoplus_{\substack{T_i \text{ zigzag} \\ (p,q) \in S(T_i) \\ (p+1,q),(p,q+1) \notin S(T_i)}} H^{p,q}_{BC}(T_i)$$

and

$$H^{p,q}_{A}(A) = \sum_{\substack{T_i \text{ zigzag} \\ (p,q) \in S(T_i) \\ (p-1,q),(p,q-1) \notin S(T_i)}} H^{p,q}_{A}(T_i).$$

---

[4] `https://mathoverflow.net/questions/86947/`.

[5] E.g. on p. 4 of [Ang15] it is stated that via de-Rham cohomology, 'symmetric zigzags [...] can not be detected'.

Proof. One again reduces to the case of A elementary and checks each possible shape separately. $\square$

In particular, we emphasise that the dimensions of $H^{p,q}_{BC}(A)$ and $H^{p,q}_{A}(A)$ are entirely determined by the mutliplicities of all zigzag shapes and thus by the Frölicher spectral sequences and the bigraded space associated with the bifiltered de-Rham cohomology. So one gets for example the following two results:

Lemma 1.8. *$H^{p,q}_{BC}(A)$ and $H^{p,q}_{A}(A)$ are finite dimensional for all $p, q \in \mathbb{Z}$, iff $H^q(A^{p,\bullet}, \partial_2)$ and $H^p(A^{\bullet,q}, \partial_1)$ are finite dimensional for all $p, q \in \mathbb{Z}$ iff all zigzag shapes appear with at most finite multiplicity.*

Remark 1.9. *This shows, for example, that the finite dimensionality for Bott-Chern and Aeppli cohomology of compact complex manifolds follows formally from the finite dimensionality of the Dolbeault cohomology. The former seems to be generally seen as a corollary to the fact that there are modified Laplace-operators s.t. every class in Bott-Chern or Aeppli cohomology has a unique harmonic representative. See e.g. the recent [YY17].*

Lemma 1.10. *If a morphism of double complexes induces an isomorphism on the first page of both Frölicher spectral sequences, it induces an isomorphism in Bott-Chern and Aeppli cohomology.*

From the maps (or spectral sequences) between the various cohomology groups introduced so far, one can derive several 'combinatorial' inequalities and statements which have particularly nice proofs using theorem 1.3. We refer to [Ang15] for a beautiful survey giving such proofs as heuristics. As an example, we invite the reader to figure out a proof of the following using theorem 1.3.[6]

Lemma 1.11. *The following are equivalent:*

1. *The natural map $H^{p,q}_{BC}(A) \longrightarrow H^{p,q}_{A}(A)$ is an isomorphism for all $p, q \in \mathbb{Z}$.*
2. *A zigzag shape with nonzero multiplicity in A has to have length 1.*
3. *If $\alpha = \partial_1\beta_1 + \partial_2\beta_2$ with $\alpha \in A^{p,q}$, $\beta_1 \in A^{p-1,q}$, $\beta_2 \in A^{p,q-1}$, then there exists a $\beta \in A^{p-1,q-1}$ such that $\alpha = \partial_1\partial_2\beta$.*

If A satisfies the equivalent conditions of this lemma, it is said to **satisfy the $\partial_1\partial_2$-lemma**.

1.2 Purity and Strictness

In the following, we prove slightly more general versions of several statements in Hodge Theory for arbitrary double complexes and relate them to theorem 1.3.

We will be considering vector spaces H over a field K that are equipped with two descending filtrations F_1, F_2, satisfying $F_i^n = H$ for some $n \in \mathbb{Z}$ and $F_i^m = 0$ for some $m \in \mathbb{Z}$. Let us denote the category of these and linear, filtration preserving maps by Fil^2_K. The following lemma is well-known. It can be verified by choosing appropriate bases.

---

[6] see [Ang15, thm. 2.3.] for the solution. The equivalence of 2. and 3. was shown before in [Del+75].

Lemma 1.12. *For an object* (H, F_1, F_2) *of* Fil_K^2 *there always exists a splitting, i.e., a bigraduation* $H = \bigoplus_{p,q\in\mathbb{Z}} H^{p,q}$ *such that*

$$F_1^p = \bigoplus_{r\geq p} H^{r,s} \qquad \text{and} \qquad F_2^q = \bigoplus_{s\geq q} H^{r,s}$$

Remark 1.13. *We will sometimes need a variant where* $K|K'$ *is an extension of degree two,* σ *is the only nontrivial* K'*-automorphism of* K *and* H *is equipped with a* σ*-linear involution* τ *satisfying* $\tau^2 = \mathrm{Id}$ *and* $\tau F_1^p = F_2^p$ *for all* $p \in \mathbb{Z}$*. In this case one can choose the decomposition* $H^{p,q}$ *in such a way that* $\tau H^{p,q} = H^{q,p}$*. This can be checked e.g. using Maschke's theorem.*[7]

In general, there is no functorial way to choose such a splitting. However, we have the following functorial construction, which can be viewed as a substitute. The statements can be verified using any splitting.

Lemma 1.14. *Let* (H, F_1, F_2) *be an object in* Fil_K^2 *and denote by* $F^\bullet$ *the* ***total filtration*** *on* H *given by*

$$F^\bullet := \sum_{p+q=\bullet} F_1^p \cap F_2^q.$$

For $k \in \mathbb{Z}$ *let* $\varphi^k : F^k \longrightarrow \mathrm{gr}_F^k H$ *be the projection to the associated graded and for* $p, q \in \mathbb{Z}$ *with* $p + q = k$ *set* $F^{p,q} := \varphi^k(F^p \cap F^q) \subseteq \mathrm{gr}_F^k H$*. Then we have*

1. *The natural inclusion* $F^{p,q} \subseteq \varphi^k(F_1^p) \cap \varphi^k(F_2^q)$ *is an equality.*
2. *The natural maps*

$$\begin{tikzcd} & F^{p,q} \arrow[dl] \arrow[dr] & \\ \mathrm{gr}_{F_1}^p \mathrm{gr}_{F_2}^q H & & \mathrm{gr}_{F_2}^q \mathrm{gr}_{F_1}^p H \end{tikzcd}$$

   are isomorphisms.

3. *The* $F^{p,q}$ *are a splitting for the filtrations on* gr_F^k *induced by the* F_i*.*

We note that the functor $H \mapsto \mathrm{gr}_F^\bullet H$ is in general not exact. We will adress this issue later. First, we interpret point 3. in case H is the de-Rham cohomology of a double complex. This can be viewed as a generalisation of the 'Hodge decomposition'.

Proposition 1.15. *Let* A *be an object of* DC_K^b *and* $d \in \mathbb{Z}$*. Consider the vector space* $H := H_{dR}^d(A)$ *equipped with the two Hodge filtrations* F_1, F_2 *and let* F *be the total filtration. Then there is a canonical decomposition*

$$\mathrm{gr}_F^k H = \bigoplus_{p+q=k} F^{p,q} \qquad \text{where } F^{p,q} := F_1^p \cap F_2^q \text{ mod } F^{p+q+1}.$$

Moreover, the spaces $F^p \cap F^q$ *have the following alternative description:*
If $p + q \geq d$*:*

$$F^p \cap F^q = \left\{ \begin{matrix} \text{classes that admit a representative } \omega \in \\ A^{r,s} \text{ for } (r,s) \in \{(d-q,q), ..., (p,d-p)\} \end{matrix} \right\}$$

---

[7] Thanks to Martin Lüdtke for pointing this out to me.

If $p + q \leq d$*:*

$$F^p \cap F^q = \begin{Bmatrix} \textit{classes that admit a representative} \\ \omega = \sum_{j=p}^{d-q} \omega_{j,d-j} \textit{ with } \omega_{r,s} \in A^{r,s}. \end{Bmatrix}$$

Remark 1.16. *In the situation of remark 1.13,* F *is by definition respected by* τ *and* $\tau F^{p,q} = F^{q,p}$*.*

Proof. By proposition 1.4 and theorem 1.3, one can again proof this easily by reducing to elementary A and investigating the total cohomology of odd length zigzags. To compare the effort, we give a direct proof below.

The statement about the decomposition is just lemma 1.14, so it remains to treat the description of the spaces $F_1^p \cap F_2^q$.

For an element $a \in A$, let us denote by $a^{r,s}$ its component in bidegree (r, s). By definition, a class $\mathfrak{c} \in H^d_{dR}(A)$ is in $F_1^p \cap F_2^q$ if it has a representative $\omega = \sum_{r+s=d} \omega^{r,s}$ with $\omega^{r,s} = 0$ for $r < p$ and another one $\omega' = \sum_{r+s=d} {\omega'}^{r,s}$ with ${\omega'}^{r,s} = 0$ for $s < q$. So the inclusions from right to left are immediate and it remains to show the converse.

Let ω, ω' be two representatives of a class $\mathfrak{c} \in F_1^p \cap F_2^q$ as above and let $\eta = \sum_{r,s\in\mathbb{Z}} \eta^{r,s}$ be a of total degree $k - 1$ with $\omega = \omega' + \eta$. This gives us a sequence of equations

$$\omega^{r,s} = {\omega'}^{r,s} + \partial_1 \eta^{r-1,s} + \partial_2 \eta^{r,s-1} \qquad (*)$$

Set $\tilde{p} := \max\{p, d - q\}$ and $\tilde{q} := \max\{q, d - p\}$. Replacing ω by the cohomologous $\omega - \sum_{i\geq\tilde{p}} d\eta^{i,d-i-1}$ and ω' by $\omega' - \sum_{i\geq\tilde{q}} d\eta^{d-i-1,i}$, we may assume that $\eta^{r,s} = 0$ for $r \geq \tilde{p}$ or $s \geq \tilde{q}$, $\omega^{r,s} = 0$ for $r \notin [p, \tilde{p}]$ and ${\omega'}^{r,s} = 0$ for $r \notin [q, \tilde{q}]$.

Now we distinguish the two cases: If $\tilde{p} = d - q$, i.e. $d \geq k$, we are done. Otherwise, i.e., if $p = \tilde{p}$ and $k > d$, the element $\omega = \omega^{p,d-p}$ is pure of bidegree $(p, d - p)$ and $\omega' = \omega^{d-q,q}$ pure of bidegree $(d - q, q)$. If $p = d - q$, necessarily $\omega' = \omega$, but if $p > d - q$ then by $(*)$, we obtain $\omega = \partial_1 \eta^{p-1,d-p}$ which is cohomologous to the pure element $-\partial_2 \eta^{p-1,d-p}$. Applying the same reasoning over and over again, we obtain representatives for $\mathfrak{c}$ that are pure in degrees $(d - q, q), ..., (p, d - p)$. $\square$

Definition 1.17. 1. *An object* (H, F_1, F_2) *in* Fil^2_K *is called* ***pure (of weight*** k***)*** *if any of the following (equivalent) conditions hold:*

- *The total filtration* F *has only one jump, lying in degree* k*, i.e.* $F^k = H$*,* $F^{k+1} = 0$*.*
- *There is a direct sum decomposition* $H = \bigoplus_{p+q=k} F_1^p \cap F_2^q$
- *The implication* $\mathrm{gr}^p_{F_1} \mathrm{gr}^q_{F_2} H \neq 0 \Rightarrow p + q = k$ *holds.*

2. *A double complex* A *is called* ***(cohomologically) pure of weight*** k ***in degree*** d *if* H^d_{dR} *with the two Hodge filtrations is pure of weight* k*.*

From proposition 1.4 and proposition 1.15 we obtain directly:

Proposition 1.18. *Let A be a double complex. The following are equivalent:*

1. *A is cohomologically pure of weight k in degree d.*
2. *All odd zigzag shapes of the form $S_d^{p,q}$ with nonzero multiplicity satisfy $p+q=k$.*

Recall that a map of filtered vector spaces $\varphi : (H,F) \longrightarrow (H',F')$ is called strict if $\varphi F^p = F'^p \cap \operatorname{im}\varphi$. We will give a sufficient condition when every map of double complexes induces a map on cohomology which is strict with respect to both filtrations F_i.

The following is essentially the well-known lemma that a map between two (pure) Hodge Structures is automatically strict:

Lemma 1.19. *Let (H,F_1,F_2), (H',F_1',F_2') be two bifiltered vector spaces and set*

$$F^\bullet := \sum_{p+q=\bullet} F_1^p \cap F_2^q \text{ and } {F'}^\bullet := \sum_{p+q=\bullet} F_1'^p \cap F_2'^q.$$

Assume there exists an integer d such that $F^m = H$ for $m \leq d$ and $F'^m = 0$ for $m > d$. Then, any bifiltered map between H and H' is strict with respect to both filtrations.

Proof. If $\varphi : H \longrightarrow H'$ is a bifiltered map, it satisfies in particular $\varphi F^m \subseteq F'^m$ for all m. So $\varphi F^m = 0$ for all $m > 0$ and $\varphi F^d \subseteq \bigoplus_{r+s=d} F_1'^r \cap F_2'^s$.

We have to show for all $p \in \mathbb{Z}$ that $F_i'^p \cap \operatorname{im}\varphi \subseteq F_i^p$. So given $\alpha \in F_1'^p \cap \operatorname{im}\varphi$, write $\alpha = \sum_{r+s=d} \alpha^{r,s}$ with unique $\alpha^{r,s} \in F_1'^r \cap F_2'^s$ and $\alpha^{r,s} = 0$ for $r < p$. Now choose some $\beta \in H$ with $\varphi(\beta) = \alpha$. Because $F^d = H$, we can write $\beta = \sum_{r+s=d} \beta^{r,s} + \beta'$ with $\beta^{r,s} \in F_1^r \cap F_2^s$ and $\beta' \in F^{d+1}$. But then $\gamma := \sum_{r \geq p} \beta^{r,s}$ has the same image as β and lies in F_1^p. The proof for F_2 is analogous. ☐

Using the splitting in proposition 1.4, one verifies that the hypotheses of lemma 1.19 are met in the following situation:

Corollary 1.20. *Let $k,d \in \mathbb{Z}$ be integers and A,B two double complexes such that all odd length zigzag shapes $S_k^{p,q}$ have zero multiplicity in A for $p+q < d$ and zero multiplicity in B for $p+q > d$. Then for every map of double complexes between A,B the induced map*

$$H_{dR}^k(A) \longrightarrow H_{dR}^k(B)$$

is strict with respect to the Hodge filtrations.

Corollary 1.21. *Every bifiltered map $(H,F_1,F_2) \longrightarrow (H,F_1,F_2)$ of pure bifiltered vector spaces of weights d,d' with $d \geq d'$ is strict.*

In particular, this corollary applies when $H = H_{dR}^k(A)$, $H' = H_{dR}^k(B)$ for A,B complexes cohomologically pure of weight d,d' in degree k.

We end with a well-known summary of the consequences of satisfying the equivalent conditions of lemma 1.11. The statements follow from what we have seen so far.

Proposition 1.22. *Let A, B be double complexes satisfying the $\partial_1\partial_2$-lemma. Then:*

1. ***Hodge Decomposition:*** *A is pure of weight k in degree k, i.e., there is a decomposition*
$$H^k_{dR}(A) = \bigoplus_{p+q=k} H^{p,q}$$
and $H^{p,q} := F_1^p \cap F_2^q$ consists of classes representable by elements of pure degree (p, q).

2. ***Dolbeault isomorphism:*** *The natural maps*
$$\begin{tikzcd} & H^{p,q}_{BC}(A) \arrow[dl] \arrow[dr] & \\ H^q(A^{p,\bullet}) & & H^{p,q} \end{tikzcd}$$
are isomorphisms. The analogous statement with the left hand side replaced by $H^p(A^{\bullet,q})$ also holds.

3. ***Strictness:*** *Any bifiltered map*
$$(H^k_{dR}(A), F_1, F_2) \longrightarrow (H^k_{dR}(B), F_1, F_2)$$
is strict with respect to both filtrations.

1.3 Rings of Double Complexes

We will be interested in the ring of all finite dimensional double complexes (and formal inverses) up to isomorphism and some variants.

Definition 1.23. *A morphism of double complexes is called an E_r**-quasi-isomorphism** ($r \in \mathbb{Z}_{>0} \cup \{\infty\}$) if it induces an isomorphism on the r-th page (and hence all later pages) of both Frölicher spectral sequences.*

For example, an E_∞-quasi isomorphism is one which induces a bifiltered isomorphism on the total cohomology.

As a direct consequence of theorem 1.3 and proposition 1.4, we obtain:

Lemma 1.24. *For two double complexes A, B, the following are equivalent:*

- *There exists an E_r-quasi-isomorphism $A \longrightarrow B$.*
- *If S is an odd length zigzag shape or an even length zigzag shape of length $\geq 2r$, there is an equality*
$$\operatorname{mult}_S(A) = \operatorname{mult}_S(B).$$

Note that this implies that the relation

$$A \simeq_r B :\Leftrightarrow \text{there is an } E_r\text{-quasi-isomorphism } A \longrightarrow B$$

is an equivalence relation. Without using a decomposition statement, it seems to be nonobvious that this should be symmetric.

The following lemma is a consequence of the Künneth formula and the compatibility of (co)homology with direct sums.

Lemma 1.25. *Let A, B be double complexes. For every $r \geq 1$, there are functorial isomorphisms of bigraded differential algebras*

$$
\begin{aligned}
{}_iE_r(A \oplus B) &\cong {}_iE_r(A) \oplus {}_iE_r(B)\\
{}_iE_r(A \otimes B) &\cong {}_iE_r(A) \otimes {}_iE_r(B).
\end{aligned}
$$

Consider the following categories:

$\mathrm{DC}_K^{b,fin}$	finite dimensional bounded double complexes.
$\mathrm{DC}_K^{b,E_r-fin}$	bounded double complex s.t. the E_r-page of both Frölicher spectral sequences is finite dimensional, localised at E_r-quasi-isomorphisms.
$\mathrm{DC}_K^{b,\partial_1\partial_2-fin}$	bounded double complex satisfying the $\partial_1\partial_2$-lemma s.t. the E_∞ page of both Frölicher spectral sequences is finite dimensional, localised at E_∞-quasi-isomorphisms.

Note that in the last case the Frölicher spectral sequences degenerate. In particular, we could replace E_∞ in the definition by E_r for any r and get the same category.

By lemma 1.25, direct sum and tensor product are well-defined on these categories and we can define the following Grothendieck rings:

$$
\begin{aligned}
R_{dc} &:= K_0(\mathrm{DC}_K^{b,fin})\\
R_{zig} &:= K_0(\mathrm{DC}_K^{b,E_1-fin})\\
R_{odd} &:= K_0(\mathrm{DC}_K^{b,E_\infty-fin})\\
R_{dot} &:= K_0(\mathrm{DC}_K^{b,\partial_1,\partial_2-fin})
\end{aligned}
$$

Given a double complex A with suitable finiteness conditions, write $[A]$ for its class in one of these rings. Abusing notation slightly, given a shape S we write $[S]$ for the class of some elementary complex of rank 1 with shape S. In particular, equations of the form

$$[A] = \sum_{S \text{ shape}} \mathrm{mult}_S(A)[S]$$

hold (if $\mathrm{mult}_S(A)$ is infinite, by construction $[S] = 0$ so one defines the corresponding summand to be 0).

So, as an abelian group, $R_{dc}, R_{zig}, R_{odd}, R_{dot}$ are free with basis given by all shapes, resp. zigzag shapes, resp. odd length zigzag shapes, resp. zigzag shapes of length 1, which also explains the notation. Multiplication is given as follows:

Proposition 1.26. *For a square shape $S^{p,q}$ and any other shape S of an elementary complex, there is an equality in R_{dc}:*

$$[S^{p,q}] \cdot [S] = \sum_{(r,s)\in S} [S^{p+r,q+s}].$$

In particular, the subgroup generated by square shapes is an ideal I_{Sq} in R_{dc}. There are equalities in $R_{zig} = R_{dc}/I_{Sq}$:

$$\begin{aligned}
\left[S_{d}^{p,q}\right] \cdot \left[S_{d'}^{p',q'}\right] &= \left[S_{d+d'}^{p+p',q+q'}\right] \\
\left[S_{1,l}^{p,q}\right] \cdot \left[S_{1,l'}^{p',q'}\right] &= \left[S_{1,\min(l,l')}^{p+p',q+q'}\right] + \left[S_{1,\min(l,l')}^{p+p'+\max(l,l'),q+q'-\max(l,l')+1}\right] \\
\left[S_{2,l}^{p,q}\right] \cdot \left[S_{2,l'}^{p',q'}\right] &= \left[S_{2,\min(l,l')}^{p+p',q+q'}\right] + \left[S_{2,\min(l,l')}^{p+p'-\max(l,l')+1,q+q'+\max(l,l')}\right] \\
\left[S_{1,l}^{p,q}\right] \cdot \left[S_{2,l'}^{p',q'}\right] &= 0 \\
\left[S_{d}^{p,q}\right] \cdot \left[S_{1,l}^{r,s}\right] &= \left[S_{1,l}^{p+p',q+k-p'}\right] \\
\left[S_{d}^{p,q}\right] \cdot \left[S_{2,l}^{r,s}\right] &= \left[S_{2,l}^{p+k'-q',q+q'}\right]
\end{aligned}$$

Proof. To see the equation for squares, let Z be an elementary complex of rank 1 with shape $S^{p,q}$ and Z' an elementary complex of rank 1 with shape S. Choose a basis element $s \in Z^{p-1,q-1}$. In particular, $\partial_1\partial_2 s \neq 0$. Given a basis element $\alpha^{r,s}$ of any nonzero component ${Z'}^{r,s}$, the element

$$\partial_1\partial_2(s\otimes\alpha^{r,s}) = \partial_1\partial_2 s\otimes\alpha^{r,s} + s\otimes\partial_1\partial_2\alpha^{r,s} \in Z^{p,q}\otimes {Z'}^{r,s}\oplus Z^{p-1,q-1}\otimes {Z'}^{r+1,s+1}$$

is not zero, so one obtains

$$\operatorname{mult}_{S^{p+r,q+s}}(Z\otimes Z') \geq 1$$

whenever ${Z'}^{r,s} \neq 0$ and so one has

$$\begin{aligned}
\dim(Z\otimes Z') &\geq 4\cdot \sum_{(r,s)\in S} \operatorname{mult}_{S^{p+r,q+s}}(Z\otimes Z') \\
&\geq 4\cdot \dim Z' \\
&\geq \dim(Z\otimes Z')
\end{aligned}$$

and hence equality, which implies the formula.

The other equations all follow from a consideration of the Frölicher spectral sequences for two elementary double complexes of rank one with the given shapes and using lemma 1.4. In each case there are only very few (i.e. ≤ 2) nonzero entries on each page. We only do this for the most terrible looking formula, the others follow similarly:

Let $l \leq l'$ and Z, Z' elementary double complexes of rank one with shapes $S_{1,l}^{p,q}$ and $S_{1,l'}^{p',q'}$. Then, ${}_2E_r(Z) = {}_2E_r(Z') = 0$ for all r. Therefore, ${}_2E_r(Z\otimes Z') = 0$ for all r and so

$$\operatorname{mult}_{S_{2,d}^{r,s}}(Z\otimes Z') = 0$$

for all $r, s \in \mathbb{Z}, d \in \mathbb{Z}_{>0}$ and

$$\mathrm{mult}_{S_d^{p,q}}(Z \otimes Z') = 0$$

for all odd length zigzag shapes, since the total cohomology has to vanish.

Considering the other spectral sequence, one has ${}_1E_r(Z) = 0$ for $r > l$ and if $r \leq l$ it is nonzero only in bidegrees (p, q) and $(p + l, q - l + 1)$, where it has dimension 1. Similarly, ${}_1E_r(Z') = 0$ for $r > l'$ and if $r \leq l'$, it is nonzero only in bidegrees (p, q) and $(p + l', q - l' + 1)$, where it has dimension 1.

In summary, ${}_1E_r(Z \otimes Z') = 0$ for all $r > l = \min(l, l')$ and nonzero in bidegrees $(p+p', q+q'), (p+p'+l, q+q'-l+1), (p+p'+l', q+q'-l'+1), (p+p'+l+l', q+q'-l-l'+2)$, so the two necesary nonzero differentials of bidegree $(l, -l+1)$ on page ${}_1E_l$ have to start at bidegrees $(p+p', q+q')$ and $(p+p'+l', q+q'-l'+1)$ and all other differentials vanish. ☐

The last equalities can be memorised by the following rules, alluding to the parity of the zigzag length:

even·even = even odd·odd=odd odd·even=even

In particular, two of these rings are just polynomial rings:

Corollary 1.27. *The map*

$$\begin{aligned} R_{odd} &\xrightarrow{\sim} \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, h^{\pm 1}] \\ [S_d^{p,q}] &\longmapsto x^p y^q h^d \end{aligned}$$

is an isomorphism of rings. R_{dot} is identified with the subring $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$.

Remark 1.28. *In contrast, R_{zig} is not finitely generated as a $\mathbb{Z}$-algebra. In fact, for any hypothetical finite set of generators there would be a natural number r s.t. the length of all even length zigzag occuring as summands in the generators is bounded by r. By the multiplication rules, this would also be true for all sums of products of these hypothetical generators.*

1.4 The Dolbeault Complex

In this section, let X be a connected compact complex manifold of dimension n and $\mathcal{A} := \mathcal{A}_X$ its Dolbeault complex. We will be concerned with the question

What restrictions are there on the isomorphism type of $\mathcal{A}$?

First, let us survey the well-known restrictions, also summarised briefly in [Ang15]:

- **Dimension:** As X is of complex dimension n, the complex $\mathcal{A}$ is concentrated in degrees (p, q) with $n \geq p, q \geq 0$. In particular, only shapes that lie in that region can have nonzero multiplicity in $\mathcal{A}$.

- **Real structure:** Consider the following involution on the category $\mathrm{DC}^b_{\mathbb{C}}$:

  $$\sigma : \left(\bigoplus_{p,q\in\mathbb{Z}} A^{p,q}, \partial_1, \partial_2\right) \mapsto \left(\bigoplus_{p,q\in\mathbb{Z}} \overline{A^{q,p}}, \partial_2, \partial_1\right)$$

  There is a canonical isomorphism $\mathcal{A} \cong \sigma\mathcal{A}$, coming from the real structure of $\mathcal{A}$ In particular, for every shape S occuring with multiplicity m in $\mathcal{A}$, its reflection along the diagonal occurs with the same multiplicity. Moreover, all cohomology vector spaces are equipped with a conjugation antilinear involution interchanging the two filtrations, i.e., we are in the situation of remark 1.13.

- **Finite dimensional cohomology:** All zigzag shapes have finite multiplicity because Dolbeault cohomology (or alternatively Bott-Chern and Aeppli cohomology) can be shown to be finite dimensional by elliptic theory (see Lemma 1.8).

- **Duality:** Let $\mathcal{DA}$ denote the 'dual complex' of $\mathcal{A}$, given by $\mathcal{DA}^{p,q} := (\mathcal{A}^{n-p,n-q})^\vee := \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}^{n-p,n-q}, \mathbb{C})$ with differentials $\partial^\vee, \overline{\partial}^\vee$, defined by $(\partial^\vee)^{p,q} := (\varphi \mapsto (-1)^{p+q+1}\varphi \circ \partial^{n-p-1,n-q})$ and similarly for $\overline{\partial}^\vee$.

  By construction and as we know that all zigzag shapes have finite multiplicity, for a zigzag shape occuring with a certain multiplicity in $\mathcal{A}$, the shape obtained by reflection at the antidiagonal $p+q=n$ occurs with the same multiplicity in $\mathcal{DA}$.

  As X is a complex manifold, it is automatically oriented. In particular, integration yields a nondegenerate pairing:

  $$\begin{aligned} \mathcal{A}^{p,q} \otimes \mathcal{A}^{n-p,n-q} &\longrightarrow \mathcal{A}^{n,n} \cong \mathbb{C} \\ (\alpha, \beta) &\longmapsto \int_X \alpha \wedge \beta \end{aligned}$$

  This induces maps $\Phi^{p,q} : \mathcal{A}^{p,q} \longrightarrow \mathcal{DA}^{p,q}$ and the signs are set up so that this yields a morphism of complexes $\Phi : \mathcal{A} \longrightarrow \mathcal{DA}$.

  In particular, we get a morphism between the Frölicher spectral sequence(s) of $\mathcal{A}$ and $\mathcal{DA}$. Serre duality (théorème 4. in [Ser55]) implies that this map is an E_1-quasi-isomorphism. Thus, by proposition 1.4, every zigzag shape occurs with the same multiplicity in $\mathcal{A}$ and $\mathcal{DA}$.

- **Conectedness:** The shape $\{(0,0)\}$ (and therefore, by duality, also the shape $\{(n,n)\}$) has multiplicity 1, as the only functions satisfying $df = 0$ are the constants.

The next two statements are also certainly known to the experts, although maybe not from this point of view. The first is the statement that (pluri-)harmonic functions are constant on compact manifolds and the second is an application of Stokes' theorem.[8] From our point of view, they can be summarised by the phrase: **Only dots in the corners.**

---

[8] For full proofs see e.g. [McH17], p. $7f$.

Proposition 1.29. *Let X be a compact complex manifold. There is no non-constant function f on X satisfying $\partial\overline{\partial}f = 0$, i.e., the shapes*

$$\begin{tikzcd}
 & & \bullet^{0,1} & \bullet^{0,1} & \\
\bullet^{0,0} \arrow[r] & \bullet^{1,0} & & & \\
 & & \bullet^{0,0} \arrow[uu] & \bullet^{0,0} \arrow[uu] \arrow[r] & \bullet^{1,0}
\end{tikzcd}$$

have multiplicity 0 *in* $\mathcal{A}$.

Proposition 1.30. *Let ϕ be a $(n-1,0)$ form on a compact complex manifold of dimension n. If $\partial\phi$ is holomorphic, then $\partial\phi = 0$. In other words, all zigzag shapes containing $(n-1,0)$ and $(n,0)$*

$$\begin{tikzcd}
\cdots \arrow[d, dashed] & \\
\bullet^{n-1,0} \arrow[r] & \bullet^{n,0}
\end{tikzcd}$$

have multiplicity zero in $\mathcal{A}$.

In special situations, there are more restrictions: For example, there is the following famous result (see e.g. [Voi02, prop. 6.3.]):

Proposition 1.31. *If X is a compact Kähler manifold, $\mathcal{A}_X$ satisfies the $\partial\overline{\partial}$-lemma.*

In particular, $\mathcal{A}_X$ satisfies all the consequences listed in proposition 1.22.
Also in small dimensions one has further known restrictions:

Theorem 1.32. *For a compact complex manifold of complex dimension ≤ 2, the Frölicher spectral sequence degenerates on the first page. I.e., all even length zigzags have multiplicity zero in its Dolbeault complex.*

For dimension one this follows already from the previous results (e.g., by proposition 1.30 or, in this case equivalently, by proposition 1.29 or by theorem 1.31 since every Riemann surface admits a Kähler metric). For dimension 2, this result is shown for example in [BPV84, thm IV.2.7] using the Riemann-Roch and Hirzebruch signature theorems. In addition to the restrictions imposed by propositions 1.29 and 1.30, the shape $\{(1,0),(1,1)\}$ and those related by real structure and duality are ruled out. In contrast, for higher dimensional manifolds, 'the Frölicher spectral sequence can be arbitrarily degenerate', see [BR14].

Certain linear combinations of zigzag-multiplicities are known to behave semicontinuously in families:[9]

Proposition 1.33. *Let X be a compact complex manifold, $p : X \longrightarrow B$ a holomorphic submersion and denote by $X_b := p^{-1}(b)$ the fibre over a point $b \in B$. Then*

- *For any $d \in \mathbb{Z}$, the functions $B \ni b \mapsto \dim H^d_{dR}(X,\mathbb{C})$ are continuous (i.e. locally constant).*

---

[9] see e.g. [Voi02] and [Sch07].

- *For any $p, q \in \mathbb{Z}$ the functions $B \ni b \mapsto \dim H_*^{p,q}(X, \mathbb{C})$ are upper-semicontinuous, where H_* denotes Bott-Chern, Aeppli or (conjugate) Dolbeault cohomology.*

From this one obtains with theorem 1.32 and combinatorial arguments the following finer result:

Corollary 1.34. *If in the situation of 1.33 the fibres of p have dimension ≤ 2, the maps*

$$B \ni b \longmapsto \operatorname{mult}_S(\mathcal{A}_{X_b})$$

are continuous (i.e. locally constant).

The results in this chapter suggest several questions:

1. Are there any other restrictions on the isomorphism type of $\mathcal{A}_X$ for a general compact complex manifold?
2. Are there other 'metric' conditions that pose restrictions on the isomorphism type of the Dolbeault complex?
3. Are there other situations where morphisms on cohomology induced by geometric morphisms respect the Hodge filtration strictly?
4. Do the multiplicities of even length (resp. odd) zigzags vary continuously (resp. upper-semicontinuously) in families in general?

Let us sketch an 'algebraic' variant of question 1: Consider the Grothendieck-ring of isomorphism classes of compact complex manifolds with disjoint union as addition and cartesian product as product:

$$R_{cplx} := K_0(\text{compact complex manifolds}, \sqcup, \times)$$

There are maps

$$\begin{aligned} \Phi_{Dol} : R_{cplx} &\longrightarrow R_{zig}[z] \\ \Phi_{dR} : R_{cplx} &\longrightarrow R_{ev}[z] \end{aligned}$$

both given by $X \mapsto [\mathcal{A}_X] z^{\dim X}$. In fact, one has:

Proposition 1.35. *Let X, Y be complex manifolds.*

- *The natural map*

$$\begin{aligned} \mathcal{A}_{X \sqcup Y} &\longrightarrow \mathcal{A}_X \oplus \mathcal{A}_Y \\ \omega &\longmapsto \omega|_X + \omega|_Y \end{aligned}$$

  is an isomorphism of double complexes

- *If X, Y are compact, the natural map*

$$\begin{aligned} \varphi : \mathcal{A}_X \otimes_{\mathbb{C}} \mathcal{A}_Y &\longrightarrow \mathcal{A}_{X \times Y} \\ \omega \otimes \eta &\longmapsto \operatorname{pr}_X^* \omega \wedge \operatorname{pr}_Y^* \eta \end{aligned}$$

  is an E_1-quasi-isomorphism

In particular, the maps Φ_{Dol} and Φ_{dR} are well-defined ring homomorphisms.

Proof. The first statement is elementary, since the the cotangent bundle of the disjoint union is canonically identified with the disjoint union of the cotangent bundles.

For the direct product, this is the known Künneth-theorem for Dolbeault cohomology. We briefly recall the strategy of proof in [GH78]: Choose hermitian metrics on X and Y, consider the product metric on $X \times Y$, let $\Delta_X, \Delta_Y, \Delta_{X\times Y}$ be the $\bar{\partial}$-Laplace operators for these metrics and $\mathcal{H}_X^{p,q}, \mathcal{H}_Y^{p,q}, \mathcal{H}_{X\times Y}^{p,q}$ the spaces of harmonic forms of type (p,q) with respect to these Laplacians. Hodge theory then implies that the projection map $\mathcal{H}_X^{p,q} \to H_{\bar{\partial}}^{p,q}(\mathcal{A}_X)$ is an isomorphism and similarly for Y and $X \times Y$. On the other hand, one shows that the image of φ is in a suitable sense dense and because the metric on $X \times Y$ is the product metric, there is an equality $\Delta_{X\times Y} = \Delta_X + \Delta_Y$ on the image of φ. Because the eigenvalues of the Laplacians are nonnegative, one deduces that the restriction of φ to harmonic forms gives an isomorphism

$$\bigoplus_{\substack{r+r'=p\\ s+s'=q}} \mathcal{H}_X^{r,s} \otimes \mathcal{H}_Y^{r',s'} \cong \mathcal{H}_{X\times Y}^{p,q}$$

from which the statement follows. □

Thus, another version of the question 1. about restrictions to the isomorphism type is:

> 1'. Is there a 'simple' description of the image of Φ_{Dol} and Φ_{dR}?

In more down-to-earth terms, the image of Φ_{dR} can be identified with a subring of the polynomial ring $R_{ev}[z] \cong \mathbb{Z}[x^{\pm}, y^{\pm}, h^{\pm}, z]$ generated by the polynomials

$$\sum_{p,q,d\geq 0} \dim \operatorname{gr}_F^p \operatorname{gr}_{\bar{F}}^q H_{dR}^d(X, \mathbb{C}) x^p y^q h^d z^{\dim X}$$

for all compact complex manifolds X. The restrictions on the isomorphism type of the Dolbeault complex in the beginning of this section can be translated to restrictions on the coefficients of the polynomials that lie in the image. At the present state, it is unclear if these are all restrictions and even if the image is finitely generated. I expect to investigate this further in future work.

If one restricts the map Φ_{dR} (or in this case equivalently Φ_{Dol}) to the subring generated by manifolds satisfying the $\partial\bar{\partial}$-lemma, the image has been described in [KS13]. It is isomorphic to a polynomial ring in three variable and the generators can be chosen to be associated with algebraic varieties (in fact, one- and two-dimensional projective space and an elliptic curve).

As noted in corollary 1.20, question 3 (about strictness of induced maps) has a affirmative answer by linear algebraic reasons for certain kinds of complexes. I do not know of geometric conditions (other than being Kähler or some other cases implying the $\partial\bar{\partial}$-lemma) imposing this kind of shape on the Dolbeault complex. In general, the answer to question 3 is no. If one is willing to also consider noncompact manifolds, an easy counterexample is already given by an inclusion $U \hookrightarrow X$, where X is a Riemann surface of genus $g \geq 1$ and U is the complement of a point.[10] However, the Dolbeault complex might not be 'the right' invariant in that case and as is 'well-known' from the theory of Mixed Hodge Structures, for open complex varieties one does indeed obtain strictness if one redefines the Hodge filtration by using forms with logarithmic singularities. For compact complex manifolds, we give a counterexample in section 1.6.

1.5 Blow-up Formulas

In this section, we compute the Dolbeault complex of projective bundles and blow-ups up to E_1-quasi-isomorphism.

Projective bundles behave as if they were globally trivial.

Proposition 1.36. *Let $\widetilde{\pi} : E \longrightarrow X$ be a complex vector bundle of rank n over a compact complex manifold, $\pi : \mathbb{P}(E) \longrightarrow X$ the associated projective bundle. There is a double complex K and two E_1-quasi-isomorphisms*

$$\begin{tikzcd} & K \arrow[dl] \arrow[dr] & \\ \mathcal{A}_X \otimes \mathcal{A}_{\mathbb{P}^{n-1}} & & \mathcal{A}_{\mathbb{P}(E)}. \end{tikzcd} \qquad (*)$$

In particular, there is an equality

$$[\mathcal{A}_{\mathbb{P}(E)}] = [\mathcal{A}_X] \cdot [\mathcal{A}_{\mathbb{P}^{n-1}}] \in R_{zig}$$

Proof. Let

$$T := \{(e,p) \in E \times \mathbb{P}(E) \mid e \in p\} \subseteq \pi^* E$$

denote the tautological bundle on $\mathbb{P}(E)$. For any fibre $F_x := \pi^{-1}(x) \cong \mathbb{P}^{n-1}$, there is an identification $T|_{F_x} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$.
Choose some hermitian metric g on T and let $\theta \in \mathcal{A}^2_{\mathbb{P}(E)}$ be the curvature of the Chern connection defined by g, s.t.

$$c_1(T) = \left[\frac{1}{2\pi i}\theta\right] \in H^2(\mathbb{P}(E)).$$

It is known that θ is a closed $(1,1)$-form and because $0 \neq c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(-1)) = c_1(T)|_{F_x}$, it is not trivial and not exact. Denote $\theta_x := \theta|_{F_x}$ and by $\mathcal{A}(\theta)$, resp. $\mathcal{A}(\theta_x)$ the finite dimensional (as $\mathbb{C}$-vector space) subcomplexes of $\mathcal{A}_{\mathbb{P}(E)}$, resp. $\mathcal{A}_{F_x}$ with basis $\{1, \theta, \theta^2, ..., \theta^{n-1}\}$ resp. $\{1, \theta_x, \theta_x^2, ..., \theta_x^{n-1}\}$. With this, we can define $K := \mathcal{A}_X \otimes \mathcal{A}(\theta)$. The bigraded Dolbeault cohomology algebra of $\mathbb{P}^{n-1}$ is given by $H(\mathbb{P}^{n-1}) = \mathbb{C}[t]/(t^{n-1})$ with $t = c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(-1))$. In particular, restriction and projection to cohomology (all forms in $\mathcal{A}(\theta_x)$ are closed) yield isomorphisms of differential bigraded bidifferential algebras $\mathcal{A}(\theta) \cong \mathcal{A}(\theta_x) \cong H(\mathbb{P}^{n-1})$ and the inclusion

$$\mathcal{A}(\theta_x) \longrightarrow \mathcal{A}_{F_x}$$

---

[10] This uses that $F^1H(U, \mathbb{C}) = H^1(U, \mathbb{C})$ as U is affine, see e.g. p. 207 in [Voi02]. A similar example was pointed out by the user HYL at `https://mathoverflow.net/questions/249693/`.

is an isomorphism on the first page of the Frölicher spectral sequence. Thus, we can define the left hand map in $(*)$ as the composite

$$\mathcal{A}_X \otimes \mathcal{A}(\theta) \longrightarrow \mathcal{A}_X \otimes \mathcal{A}(\theta_x) \longrightarrow \mathcal{A}_X \otimes \mathcal{A}_{\mathbb{P}^{n-1}},$$

where the maps are the identity on the first factor and restriction and inclusion on the second factor.

The right hand map in $(*)$ is given by π^* on the first factor and the inclusion on the second. It is an isomorphism on the first page of the Frölicher spectral sequence by the Hirsch Lemma for Dolbeault cohomology.[11] $\square$

Definition 1.37. *For a map $p : X \longrightarrow Y$ between compact complex manifolds the **pushforward** p_* is, with the notation of section 1.4, the composite*

$$\mathcal{A}_X \xrightarrow{\Phi} \mathcal{D}\mathcal{A}_X \xrightarrow{\mathcal{D}p^*} \mathcal{D}\mathcal{A}_Y.$$

Lemma 1.38. *For a surjective holomorphic map $f : Y \longrightarrow X$ of compact complex manifolds of the same dimension, the map*

$$\mathcal{A}_Y \xrightarrow{(f_*, \mathrm{pr})} \mathcal{D}\mathcal{A}_X \oplus \mathcal{A}_Y / f^*\mathcal{A}_X$$

is an E_1-quasi-isomorphism.

Proof. By restricting to a connected component, we can assume that Y is connected. Since f is a finite covering with $\deg(f)$ sheets when restricted to appropriate dense open subsets of X and Y, one obtains an exact sequence

$$0 \longrightarrow \mathcal{A}_X \xrightarrow{f^*} \mathcal{A}_Y \longrightarrow \mathcal{A}_Y / f^*\mathcal{A}_X \longrightarrow 0. \qquad (*)$$

As noted in [Wel74], one has $\int_Y f^*\omega = \deg(f) \int_X \omega$ for any form ω on X. In particular, the diagram

$$\begin{tikzcd}
\mathcal{A}_X \arrow[r, "f^*"] \arrow[d, "\deg(f)\cdot\Phi"'] & \mathcal{A}_Y \arrow[d, "\Phi"] \\
\mathcal{D}\mathcal{A}_X & \mathcal{D}\mathcal{A}_Y \arrow[l, "\mathcal{D}f^*"]
\end{tikzcd}$$

commutes. Since Φ induces an isomorphism on the first page of the Frölicher spectral sequence, for every $p \in \mathbb{Z}$, in the long exact sequence of terms on the first page of the Frölicher spectral sequence induced by $(*)$

$$\ldots \xrightarrow{\delta} H^{p,q}_{\bar\partial}(\mathcal{A}_X) \xrightarrow{f^*} H^{p,q}_{\bar\partial}(\mathcal{A}_Y) \xrightarrow{\mathrm{pr}} H^{p,q}_{\bar\partial}(\mathcal{A}_Y / f^*\mathcal{A}_X) \xrightarrow{\delta} \ldots$$

the map f^* is a split injection (with left inverse $\frac{1}{\deg(f)} f_*$) and hence pr is surjective. This implies that the morphism (f_*, pr) in the statement induces an isomorphism on the first page of the Frölicher spectral sequence. $\square$

---

[11] Proven in [Cor+00, lem. 18], as a consequence of a spectral sequence introduced by Borel in the appendix to [Hir78].
Beware of the following potentially confusing notation in [Cor+00]: In their terminology, they assume $H^{*,*}(F)$ to be a *free algebra* with transgressive basis. It suffices however, to have a graded vector space basis for $H^{*,*}(F)$ which consists of transgressive elements as their proof does not make use of the internal multiplicative structure of $H^{*,*}(F)$. It seems this is also what is meant, as otherwise in their corollary 19 the hypotheses of the Hirsch Lemma do not seem to be satisfied.

Theorem 1.39. *Let X be a compact complex manifold, $Z \subset X$ a closed submanifold and $\widetilde{X}$ the blow-up of X at Z and $E \subseteq \widetilde{X}$ the exceptional divisor, so that the following diagram is cartesian:*

$$\begin{tikzcd}
E \arrow[r, "j"] \arrow[d, "\pi_E"] & \widetilde{X} \arrow[d, "\pi"] \\
Z \arrow[r, "i"] & X
\end{tikzcd}$$

The map

$$\mathcal{A}_{\widetilde{X}} \overset{(\pi_*, j^*)}{\longrightarrow} \mathcal{D}\mathcal{A}_X \oplus \mathcal{A}_E / \pi_E^* \mathcal{A}_Z$$

induces an isomorphism on the first page of the Frölicher spectral sequence. In particular, there is an equality in R_{zig}:

$$[\mathcal{A}_{\widetilde{X}}] = [\mathcal{A}_X] + [\mathcal{A}_E] - [\mathcal{A}_Z]$$

Proof. Consider the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] & \mathcal{A}_{\widetilde{X}} \arrow[r, "j^*"] & \mathcal{A}_E \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[u, "\pi^*", "\wr"'] & \mathcal{A}_X \arrow[r, "i^*"] \arrow[u, hook, "\pi^*"] & \mathcal{A}_Z \arrow[r] \arrow[u, hook, "\pi_E^*"] & 0
\end{tikzcd}$$

where K, K' are defined as kernels of i^*, j^*. Because π is an isomorphism on the dense subset $\widetilde{X} \backslash E \cong X \backslash Z$, π^* is injective and induces an isomorphism $K \cong K'$.[12] π_E^* is injective because E is a bundle over Z. From this, one obtains via a diagram chase (or as a special case of the snake lemma) that j^* induces an isomorphism

$$\mathcal{A}_{\widetilde{X}} / \pi^* \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_E / \pi_E^* \mathcal{A}_Z$$

and one concludes with the previous lemma. The additional statement about the decomposition in R_Z comes from the description of $\mathcal{A}_E$ in proposition 1.36. ☐

As corollaries, one obtains formulas for the Dolbeault, de-Rham and Bott-Chern cohomology of projective bundles and blow-ups along complex submanifolds for arbitrary compact complex manifolds. This strenghthens very recent results by [Ang+17], [RYY17], [YY17], where such formulas are proved (some in special cases only) and several corollaries are described. As one example, one obtains:

Corollary 1.40. *A blow-up of a compact complex manifold X along a closed submanifold Z satisfies the $\partial\overline{\partial}$-lemma if and only if X and Z do.*

In particular, since every bimeromorphic map between compact complex manifolds factors as a composition of blow-ups and blow-downs by the weak factorisation theorem ([Abr+02], [Wło03]), the question if satisfying the $\partial\overline{\partial}$-lemma is a bimeromorphic invariant can be reduced to the question wether submanifolds of $\partial\overline{\partial}$-manifolds are again $\partial\overline{\partial}$-manifolds.

Because the complex $\mathcal{A}_E / \mathcal{A}_Z$ in theorem 1.39 is 0 in degrees $(0, q)$ or $(p, 0)$, one obtains, again by the weak factorisation theorem:

---

[12] As noted after print, an isomorphism $K \cong K'$ might be too optimistic. A modified version of the argument will appear on the arXiv.

Corollary 1.41. *Let X be a compact complex manifold of dimension n. For any zigzag-shape S s.t. $S \cap \{(0,p),(p,0),(n,p),(p,n)\} \neq \emptyset$ for some $p \in \mathbb{Z}_{\geq 0}$, the number* $\operatorname{mult}_S(\mathcal{A}_X)$ *is a bimeromorphic invariant.*

This holds in particular for the dimension of $H^{0,q}_{BC}(X)$ and $H^{0,q}_{\bar\partial}(X)$ (both of which are sums of multiplicities as in the above corollary). For the Dolbeault cohomology, this was also deduced in [RYY17] from the formulas proven there.

1.6 A Counterexample

In the following, we give an example of a complex manifold and a holomorphic endomorphism that does not respect the Hodge filtration strictly. This will be done by constructing a morphism of a certain nilpotent Lie algebra, which respects a lattice and as such integrates to a morphism of the corresponding Nilmanifold.

The Lie Algebra

Let $\mathfrak{g}$ be the six dimensional nilpotent Lie algebra $\mathfrak{h}_9$, i.e., a 6-dimensional real vector space with basis $e_1, ..., e_6$ and Lie bracket given by:[13]

$$[e_1, e_2] = e_4, \qquad [e_1, e_3] = [e_2, e_4] = -e_6$$

where the nonmentioned brackets are defined by antisymmetry or 0. We endow this with an endomorphism $J : \mathfrak{g} \longrightarrow \mathfrak{g}$ of complex structure (i.e. $J^2 = -1$) defined by:

$$Je_1 = e_2, \qquad Je_3 = e_4, \qquad Je_5 = e_6$$

A basis of $\mathfrak{g}_{1,0}$, the i eigenspace of J in the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$, is obtained by setting:

$$\omega_1 := e_1 - ie_2, \qquad \omega_2 := e_3 - ie_4, \qquad \omega_3 := e_5 - ie_6$$

The exterior algebra of the dual, $\bigwedge^* \mathfrak{g}^\vee$ comes equipped with the Chevalley-Eilenberg differential d, given on $\bigwedge^1 \mathfrak{g}^\vee$ by

$$\mathfrak{g}^\vee \ni \eta \longmapsto d\eta(\cdot,\cdot) := -\eta([\cdot,\cdot])$$

and extended to satisfy the Leibniz rule. On the complexification $\bigwedge^* \mathfrak{g}^\vee_{\mathbb{C}}$, we have a bigraduation induced by $\mathfrak{g}^\vee_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$, where these are the i and $-i$ eigenspaces for J. The differential d decomposes as $d = \partial + \overline{\partial}$, with ∂ the component of bidegree $(1,0)$ and $\overline{\partial}$ that of bidegree $(0,1)$. Thus, we obtain a double complex equipped with two conjugated 'stupid' filtrations $F, \overline{F}$, just like the Dolbeault complex of a complex manifold.

Let us denote by ω^i the dual basis elements to ω_i. Plugging in the definitions, one checks that the differential is given on $\mathfrak{g}^{1,0}$ by the following rules:

$$d\omega^1 = 0, \qquad d\omega^2 = \frac{1}{2}\bar\omega^1 \wedge \omega^1, \qquad d\omega^3 = \frac{i}{2}(\omega^1 \wedge \bar\omega^2 + \bar\omega^1 \wedge \omega^2)$$

---

[13] This Lie algebra and background information can be found e.g. in [ABD11].

From this, one calculates that the classes

$$[\omega^1], [\bar{\omega}^1], [\omega^2 + \bar{\omega}^2], [\omega^3 + \bar{\omega}^3]$$

form a basis for the first cohomology $H^1 := H^1(\bigwedge^* \mathfrak{g}_{\mathbb{C}}^{\vee}, d)$. The Hodge filtration is given by:

$$F^2 = \{0\} \qquad F^1 = \langle [\omega^1] \rangle \qquad F^0 = H^1.$$

The Morphism

Now, we define a Lie algebra homomorphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{g}$ by:

$$e_i \longmapsto \begin{cases} e_3 - e_6 & i = 1 \\ e_4 + e_5 & i = 2 \\ 0 & \text{else.} \end{cases}$$

This is compatible with the complex structure and the dual morphism $\varphi^{\vee}$ is described by its values on $\mathfrak{g}^{1,0}$, namely:

$$\begin{array}{ccc} \omega^1 & \longmapsto & 0 \\ \omega^2 & \longmapsto & \omega^1 \\ \omega^3 & \longmapsto & i\omega^1 \end{array}$$

Denoting by φ^* the induced morphism on cohomology, one checks

$$\varphi^* F^1 = \{0\} \neq \langle [\omega^1] \rangle = F^1 \cap \varphi^* H^1,$$

i.e., φ^* is not strict.

Integrating to Geometry

The material in this section is 'standard'. We refer to chapter 1.7.2. and chapter 3 of [Ang13] for a survey, including references to the original works containing the results used.

By the correspondence between Lie groups and Lie algebras, the morphism φ integrates to a unique (a priori real analytic) Lie group endomorphism, still denoted φ, of the unique simply connected Lie group G belonging to $\mathfrak{g}$. Further, one checks that the complex structure above is integrable, i.e. $[Je_i, Je_j] = J[Je_i, e_j] + J[e_i, Je_j] + [e_i, e_j] \; \forall i, j \in \{1, ..., 6\}$. As such, it induces a left invariant complex structure on G (as a manifold; multiplication need not be holomorphic). As φ is compatible with the complex structure, its geometric version is holomorphic with respect to this induced left invariant complex structure.

Let $\Gamma \subseteq G$ be a lattice, i.e., a cocompact discrete subgroup. It is a theorem by Malcev that, for a nilpotent Lie group, these exist iff the Lie algebra admits a basis with respect to which it has rational structure constants, as is the case for the basis of $\mathfrak{g}$ given above. More precisely, they correspond, via the exponential map, to lattices in the Lie algebra. Then $X := G/\Gamma$ is a compact complex manifold (a so-called nilmanifold). Note that we can choose Γ to be respected by φ (as this respects e.g. the lattice in $\mathfrak{g}$ spanned by the e_i). So, φ induces a holomorphic map $X \longrightarrow X$.

The complex $\left(\bigwedge^* \mathfrak{g}_{\mathbb{C}}^\vee, d\right)$ can be identified with the subcomplex of the $\mathcal{C}^\infty$-forms on X given by G-invariant differential forms and due to a theorem by Nomizu, this inclusion is a quasi isomorphism. Under certain conditions, one of which is satisfied in this example (namely, the complex structure is abelian, i.e. $[Je_i, e_j] = [e_i, Je_j]$), this inclusion already induces an isomorphism on the first page of the Frölicher spectral sequence. This, however, implies that the isomorphism by Nomizu on de-Rham cohomology is filtered, i.e., the Hodge filtration can also be computed using the smaller complex. Therefore, the geometric map induced by φ is a holomorphic map that induces a non-strict map on cohomology!

1.7 Examples of Decompositions of Dolbeault Complexes

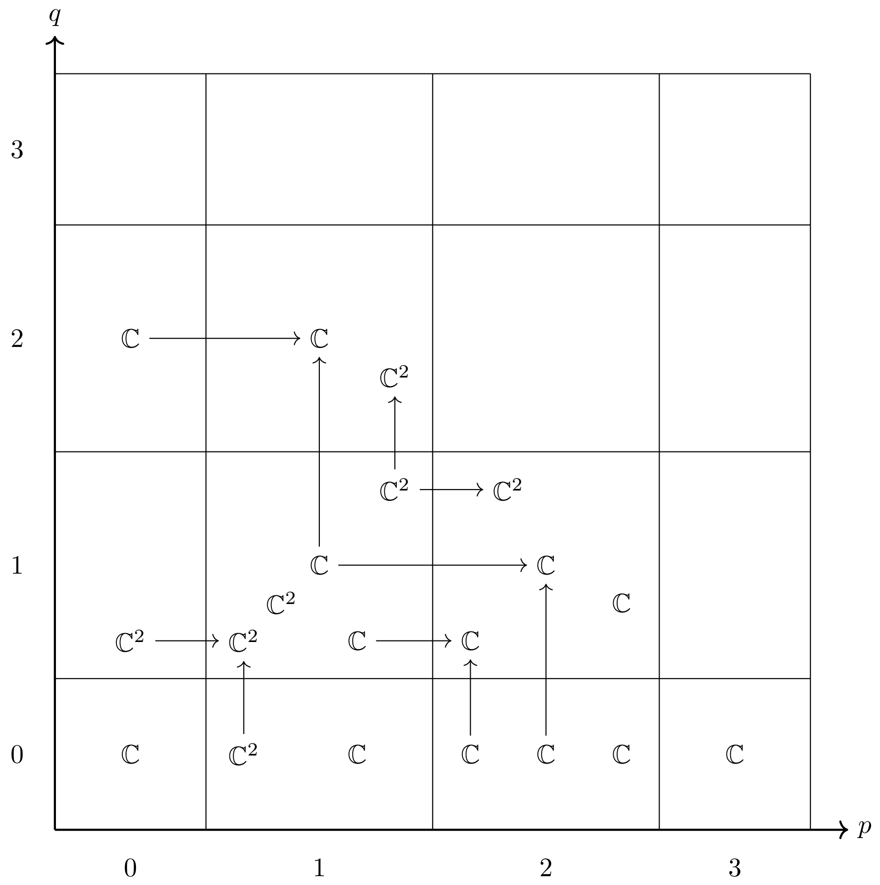
We collect examples of the decomposition of Dolbeault complexes of some non-Kähler manifolds. Similar pictures for the Iwasawa manifold and the Dolbeault complex of a hypothetical complex structure of S^6 can be be found on p. 112 of [Ang13] and in [Ang18].

The nilmanifold from section 1.6

In the following table one finds information on occuring zigzags and their generators in the double complex of the example from section 1.6. We use the notation introduced there. All zigzag shapes whose multiplicity is not determined via duality and real structure from the ones listed here have multiplicity zero.

Shape of zigzag	Generators
$S_0^{0,0}$	1
$S_1^{1,0}$	ω^1
$S_1^{0,0}$	$\omega^2, \omega^3, \overline{\omega}^2, \overline{\omega}^3$
$S_2^{2,0}$	$\omega^1 \wedge \omega^2$
$S_2^{1,0}$	$\omega^1 \wedge \omega^3, \omega^1 \wedge \overline{\omega}^3$
$S_2^{1,1}$	$\omega^2 \wedge \overline{\omega}^1 - \omega^1 \wedge \overline{\omega}^2, i\omega^2 \wedge \overline{\omega}^2 + \omega^1 \wedge \overline{\omega}^3 - \omega^3 \wedge \overline{\omega}^1$
$S_2^{0,0}$	$\omega^2 \wedge \omega^3, \omega^3 \wedge \overline{\omega}^2 + \omega^2 \wedge \overline{\omega}^3, \overline{\omega}^2 \wedge \overline{\omega}^3$
$S_3^{3,0}$	$\omega^1 \wedge \omega^2 \wedge \omega^3$
$S_3^{2,1}$	$\omega^1 \wedge \omega^2 \wedge \overline{\omega}^3$
$S_3^{2,2}$	$\omega^3 \wedge \overline{\omega}^3, \omega^2 \wedge \overline{\omega}^3 - \omega^3 \wedge \overline{\omega}^2$

The following encodes this information (without explicit generators) in a diagram. Again, zigzags determined by duality and real structure are not drawn.



The Calabi-Eckmann manifolds

In [CE53], for any $\alpha \in \mathbb{C}\backslash\mathbb{R}$ and $u, v \in \mathbb{Z}_{\geq 0}$ a manifold $M^{\alpha}_{u,v}$ was defined by putting a complex structure on the product $S^{2u+1} \times S^{2v+1}$ such that the projection

$$S^{2u+1} \times S^{2v+1} \longrightarrow \mathbb{P}^u_{\mathbb{C}} \times \mathbb{P}^v_{\mathbb{C}}$$

is a holomorphic fibre bundle with fibre $T = \mathbb{C}/(\mathbb{Z} + \alpha\mathbb{Z})$. Explicitly, they can be realised as the quotient

$$M^{\alpha}_{u,v} = ((\mathbb{C}^{u+1}\backslash\{0\}) \times (\mathbb{C}^{v+1}\backslash\{0\}))/\sim$$

where

$$(x, y) \sim (e^t x, e^{\alpha t} y) \qquad \text{for any } t \in \mathbb{C}.$$

Since the following discussion does not depend on the choice of α, we write $M_{u,v}$ for the product $S^{2u+1} \times S^{2v+1}$ equipped with any of these complex structures.

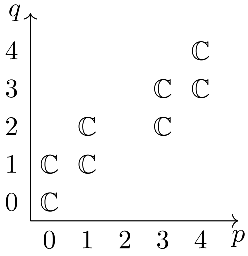
Example 1.42. *$M_{0,0}$ is a complex torus and $M_{0,v}$ or $M_{u,0}$ are Hopf manifolds.*

In the following, we assume $u < v$ for simplicity.

In [Hir78], Borel computed the first page of the Frölicher spectral sequence for $M_{u,v}$. Numerically, the result reads:

$$h^{p,q}_{M_{u,v}} = \begin{cases} 1 & \text{if } p \leq u \text{ and } q = p, p+1 \\ 1 & \text{if } p > v \text{ and } q = p, p-1 \\ 0 & \text{else.} \end{cases}$$

The following is a picture of the E_1-page (without differentials) for $u = 1, v = 2$:



Since the underlying topological space of $M_{u,v}$ is a product of spheres, the Betti-numbers are given by

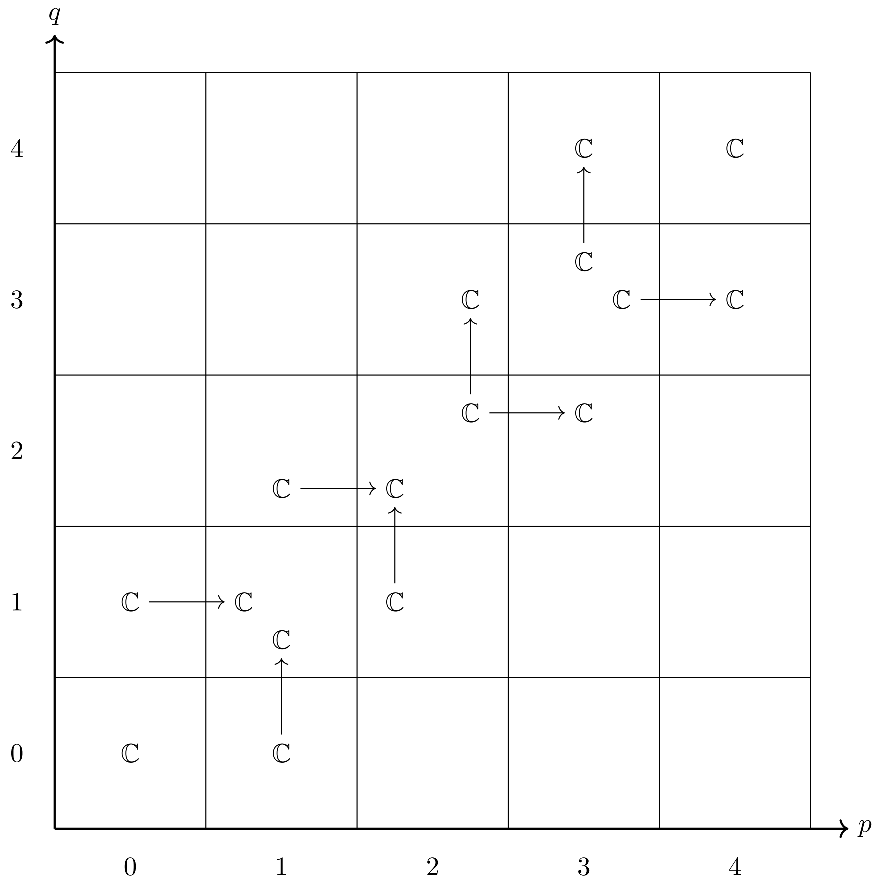
$$b^{i}_{M_{u,v}} = \begin{cases} 1 & \text{if } i = 0, 2u+1, 2v+1, 2(u+v)+2 \\ 0 & \text{else.} \end{cases}$$

In particular, the Frölicher spectral sequence degenerates at the second stage and the multiplicities of all zigzags can be determined combinatorially. (For the odd-length zigzags, use that the de-Rham cohomology is one-dimensional, so knowing the breakpoints of each filtration individually determines the nonzero bidegree of the associated bigraded.) This results in:

$$\operatorname{mult}_S(\mathcal{A}_{M_{u,v}}) = \begin{cases} 1 & \text{if } S = S^{0,0}_0, S^{u,u}_{2u+1} \\ 1 & \text{if } S = S^{p,p+1}_{1,1} \text{ for } 0 \leq p < u \\ 0 & \text{in all other cases not determined by duality and real structure.} \end{cases}$$

Continuing with the case $u = 1$, $v = 2$, a picture of all the zigzags in $\mathcal{A}_{M_{1,2}}$

looks as follows:



Remark 1.43. *For the cases u or v equal to 0, this coincides with a description of the zigzags in the Dolbeault double complex obtained (under a conjecture) by G. Kuperberg on MO.*[14]

Remark 1.44. *For $u = v > 1$, it seems one cannot just argue numerically since in this case $b^{2u+1}_{M_{u,u}} = 2$ and so it does not suffice to know in which degree the Hodge filtration and its conjugate jump individually. More precisely, the methods used here only allow to say that there are either two zigzags with shapes $S^{u,u+1}_{2u+1}, S^{u+1,u}_{2u+1}$ or two zigzags with shapes $S^{u,u}_{2u+1}, S^{u+1,u+1}_{2u+1}$.*

1.8 A Proof of Theorem 1.3

In order to prove theorem 1.3, we will use the language of quivers, which we briefly recall. For details, we refer to [Gab72].

A **quiver** is a directed graph with multiple edges between two points and edges from one point to itself allowed. More precisely, a quiver Q consists of a two sets $A(Q)$ ('arrows') and $P(Q)$ ('points') and two maps of sets $h_Q, t_Q : A(Q) \longrightarrow$

---

[14] `https://mathoverflow.net/questions/25723/`

$P(Q)$ that associate to each arrow α its head $h_Q(\alpha)$ and its tail $t_Q(\alpha)$. There are 'obvious' notions of morphisms of quivers, subquivers, disjoint union of quivers etc.

Example 1.45. *Let $n \in \mathbb{N}$. We can obtain a quiver from a diagram*

$$1 \text{ ——— } 2 \text{ ——— } ... \text{ ——— } n$$

by assigning a direction to each dash. A quiver isomorphic to one obtained in this way is said to be of ***type*** $\mathbb{A}_n$.

Example 1.46. *Let D be the quiver given by $P(D) = \mathbb{Z}^2$,*

$$A(D) = \{((p,q),(p+1,q))\} \cup \{((p,q),(p,q+1))\} \subseteq \mathbb{Z}^2 \times \mathbb{Z}^2,$$

with t_D coming from projection from $\mathbb{Z}^2 \times \mathbb{Z}^2$ to the first factor and h_D from projection to the second factor. We also write $\partial_1^{p,q}$ for the element $((p,q),(p+1,q))$ in the first set of the union and $\partial_2^{p,q}$ for the element $((p,q),(p,q+1))$ in the second set of the union defining $A(D)$. The quiver D can be visualised as the infinite grid:

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\dots \arrow[r] & (p,q+1) \arrow[r] \arrow[u] & (p+1,q+1) \arrow[r] \arrow[u] & \dots \\
\dots \arrow[r] & (p,q) \arrow[r] \arrow[u] & (p+1,q) \arrow[r] \arrow[u] & \dots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

A **representation** V **of a quiver** Q is the datum of a vector space $V(p)$ for every $p \in P(Q)$ and a linear map $V(\alpha) : V(t_Q(\alpha)) \longrightarrow V(h_Q(\alpha))$ for every $\alpha \in A(Q)$. Again there are 'obvious' notions of morphisms of representations, direct sum, subrepresentation, restriction to a subquiver, etc.

The following notions might be nonstandard:
The **support** of a representation V of a quiver Q is the subquiver $\mathrm{supp}_V \hookrightarrow Q$ consisting of those $p \in P(Q)$ and $\alpha \in A(Q)$ s.t. $V(p) \neq \{0\}$ or $V(\alpha) \neq 0$, respectively. Two points $p, q \in P(Q)$ of a quiver are called adjacent if there is an arrow $\alpha \in A(Q)$ with $t_Q(\alpha) = p$ and $h_Q(\alpha) = q$ or $t_Q(\alpha) = q$ and $h_Q(\alpha) = p$. A quiver is said to be **connected** if there is just one equivalence class of points with respect to the equivalence relation generated by adjacency. Finally, a representation of quivers is called **elementary**, if it is nonzero, its support is connected and all nonzero maps are isomorphisms.

Example 1.47. *A representation of a quiver of type $\mathbb{A}_n$ is elementary if all nonzero maps are isomorphisms and the support is again of type $\mathbb{A}_m$ for some $m \leq n$.*

Example 1.48. *Let A be a representation of the quiver D from example 1.46. Abusing notation, we also write $\partial_i^{p,q}$ for $A(\partial_i^{p,q})$. A is called a double complex if it has finite support (meaning both defining sets of the support quiver are finite) and satisfies in addition $\partial_1^{p,q} \circ \partial_1^{p-1,q} = 0$, $\partial_2^{p,q} \circ \partial_2^{p,q-1} = 0$ and $\partial_1^{p,q+1} \circ \partial_2^{p,q} + \partial_2^{p+1,q} \circ \partial_1^{p,q} = 0$ for all $p, q \in \mathbb{Z}$.*

Remark 1.49. *The notion of elementary and general double complexes agrees with our conventions from the first section. A zigzag in that language is a double complex (in the above sense) s.t. all arrows are zero or isomorphisms and the support is of type $\mathbb{A}_n$. The shape in the sense of definition 1.1 entirely determines the support of an elementary double complex.*

The proof of the following is a minor modification (in order to include the infinite dimensional case) of arguments I learned from the lecture notes [Rin12].

Proposition 1.50. *Let Q be a quiver of type $\mathbb{A}_n$. Given a (not necessarily finite-dimensional) representation A of Q, there are finitely many elementary representations $T_1, ..., T_r$ of Q with pairwise distinct support and a (noncanonical) isomorphism*

$$A \cong \bigoplus_{i_1}^{r} T_i.$$

Up to reordering and isomorphism, the T_i are unique.

To prove uniqueness, we will use the following elementary

Lemma 1.51. *Given a filtered isomorphism between two filtered representations of a quiver, it induces an isomorphism between the associated graded pieces.*

Proof of proposition 1.50: We first proof this for $n = 1, 2, 3$. The general case will then follow by induction. The case $n = 1$ is clear, as each representation is elementary itself. In the case $n = 2$, up to isomorphism Q is of the form

$$1 \longrightarrow 2$$

and the possible elementary representations are of the form

$$W \longrightarrow \{0\} \qquad W \xrightarrow{\sim} W \qquad \{0\} \longrightarrow W$$

for some vector space W. Given any representation $V = \big(V(1) \xrightarrow{\alpha} V(2)\big)$, consider the filtration

$$\Big(\{0\} \longrightarrow \{0\}\Big) \subseteq \Big(\ker\alpha \longrightarrow \{0\}\Big) \subseteq \Big(V(1) \xrightarrow{\alpha} \operatorname{im}\alpha\Big) \subseteq \Big(V(1) \longrightarrow V(2)\Big).$$

This filtration is functorial in the representation and given any decomposition of V into elementary representations, it splits the filtration (after possibly reordering). In particular, the nonzero graded pieces are isomorphic to the different elementary representations and hence, by lemma 1.51, the uniqueness statement follows. On the other hand, by choosing bases successively, one verifies that this filtration splits,[15] which gives us existence.

---

[15] Note that there really is something to show: It is not true that any filtration of (double) complexes splits, as already the example $(0 \longrightarrow k) \subseteq (k \xrightarrow{\sim} k)$ shows.

The case $n = 3$ is treated with the same method. Up to isomorphism, there are three possiblities for Q, namely:

$$\begin{tikzcd} 1 & 2 \arrow[l] \arrow[r] & 3 & \text{”source”} \\ 1 \arrow[r] & 2 & 3 \arrow[l] & \text{”sink”} \\ 1 \arrow[r] & 2 \arrow[r] & 3 & \text{”river”} \end{tikzcd}$$

In each case, there are six possibilities for the support of an elementary representation which we encode as $\bullet 00, 0 \bullet 0, 00\bullet, \bullet \bullet 0, 0 \bullet \bullet, \bullet \bullet \bullet$, where a 0 in position i means $V(i) = \{0\}$. We will write down a filtration on representations of a quiver of type $\mathbb{A}_3$ in each of the three cases that is functorial and s.t. a decomposition into elementary representations in appropriate order provides a splitting.

Let $A \xleftarrow{\alpha} C \xrightarrow{\beta} B$ be a representation of the ‘source’ quiver. The first column of the following table is the promised filtration, whereas the second indicates the support of the associated graded in that step.

$$\begin{tikzcd} 0 & 0 \arrow[l] \arrow[r] & 0 & 000 \\ 0 & \ker\alpha \cap \ker\beta \arrow[l] \arrow[r] & 0 & 0 \bullet 0 \\ 0 & \ker\alpha \arrow[l] \arrow[r] & \beta(\ker\alpha) & 0 \bullet \bullet \\ \alpha(\ker\beta) & \ker\alpha + \ker\beta \arrow[l] \arrow[r] & \beta(\ker\alpha) & \bullet \bullet 0 \\ \operatorname{im}\alpha & C \arrow[l] \arrow[r] & \operatorname{im}\beta & \bullet \bullet \bullet \\ \operatorname{im}\alpha & C \arrow[l] \arrow[r] & B & 00\bullet \\ A & C \arrow[l] \arrow[r] & B & \bullet 00 \end{tikzcd}$$

If $A \xrightarrow{\alpha} C \xleftarrow{\beta} B$ is a representation of the ‘sink’ quiver, the corresponding table looks like:

$$\begin{tikzcd}
0 \arrow[r] & 0 & 0 \arrow[l] & 000 \\
\ker\alpha \arrow[r] & 0 & 0 \arrow[l] & \bullet 00 \\
\ker\alpha \arrow[r] & 0 & \ker\beta \arrow[l] & 00\bullet \\
\alpha^{-1}(\operatorname{im}\beta) \arrow[r] & \operatorname{im}\beta\cap\operatorname{im}\alpha & \beta^{-1}(\operatorname{im}\alpha) \arrow[l] & \bullet\bullet\bullet \\
A \arrow[r] & \operatorname{im}\alpha & \beta^{-1}(\operatorname{im}\alpha) \arrow[l] & \bullet\bullet 0 \\
A \arrow[r] & \operatorname{im}\alpha+\operatorname{im}\beta & B \arrow[l] & 0\bullet\bullet \\
A \arrow[r] & C & B \arrow[l] & 0\bullet 0
\end{tikzcd}$$

Finally, if $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is a representation of the ‘river’ quiver:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 & 000 \\
\ker\alpha \arrow[r] & 0 \arrow[r] & 0 & \bullet 00 \\
\alpha^{-1}(\ker\beta) \arrow[r] & \ker\beta\cap\operatorname{im}\alpha \arrow[r] & 0 & \bullet\bullet 0 \\
\alpha^{-1}(\ker\beta) \arrow[r] & \ker\beta \arrow[r] & 0 & 0\bullet 0 \\
A \arrow[r] & \ker\beta+\operatorname{im}\alpha \arrow[r] & \beta(\operatorname{im}\alpha) & \bullet\bullet\bullet \\
A \arrow[r] & B \arrow[r] & \operatorname{im}\beta & 0\bullet\bullet \\
A \arrow[r] & B \arrow[r] & C & 00\bullet
\end{tikzcd}$$

Each of these filtrations splits, as one verifies by choosing appropriate bases. This ends the proof for $n = 3$.

For general $n > 3$, let V be a representation of a quiver of type $\mathbb{A}_n$. Denote by $V|_{\{1,\dots,n-1\}}$ its restriction to the first $n-1$ nodes. Inductively, we can assume $V|_{\{1,\dots,n-1\}} \cong \bigoplus_{i=1}^{r} T_i$ for some (essentially unique) elementary representations T_i. Grouping together those T_i with $T_i(n-1) = 0$ and those with $T_i(n-1) \neq 0$, we obtain a decomposition

$$V = V^{-} \oplus V',$$

where $V^-(n-1) = 0 = V^{-1} = 0$ and V' is **increasing** up to degree $n-1$, i.e., if an arrow goes from $V'(i-1)$ to $V'(i)$ with $1 < i \leq n-1$, it is injective, whereas it is surjective if if goes from $V'(i)$ to $V'(i-1)$. Both summands are unique up to isomorphism.

Similarly, denote $V'|_{\{2,\ldots,n\}}$ the restriction of V' to the subquiver given by the last $n-1$ nodes. Again, this splits by induction as a sum of elementary representations and we obtain a splitting with summands unique up to isomorphism

$$V' = V'' \oplus V^+,$$

where $V^+(1) = 0 = V^+(2)$ and V'' is **decreasing** from degrees 2 to n, i.e., for $2 \leq i < n$, if a morphism goes from $V''(i)$ to $V''(i+1)$ it is surjective, and if it goes from $V''(i+1)$ to $V''(i)$, it is injective. But it is also increasing, hence all morphisms between $V''(2)$ and $V''(n-1)$ are isomorphisms and we may contract V'' to a representation of a quiver of type $\mathbb{A}_3$, where we know the statement. $\square$

With this statement at hand, we can now prove theorem 1.3.

Proof of theorem 1.3: Consider the functorial ascending filtration $W_\bullet$ on A given in degree k by the subcomplex generated by all components in (total) degree $\leq k$, i.e.,

$$(W_kA)^{p,q} = \begin{cases} A^{p,q} & \text{if } p+q \leq k \\ (\operatorname{im}\partial_1 + \operatorname{im}\partial_2)^{p,q} & \text{if } p+q = k+1 \\ (\operatorname{im}\partial_1 \circ \partial_2)^{p,q} & \text{if } p+q = k+2 \\ \{0\} & \text{else.} \end{cases}$$

Note that given any decomposition into elementary complexes $A \cong \bigoplus T_i$, it induces an isomorphism

$$\operatorname{gr}_k^W A \cong \bigoplus_{\substack{T_i \text{ generated} \\ \text{in degree } k}} T_i.$$

Using lemma 1.51, we can thus reduce the question of uniqueness to the case that A is generated in a single total degree k.

Given such A, for all p,q with $p+q=k$, set $K^{p,q} := (\ker\partial_1 \circ \partial_2)^{p,q}$ and define K to be the subcomplex generated by the $K^{p,q}$. Thus, we have a two-step filtration W':

$$W_0' = \{0\} \subseteq K \subseteq A = W_2'$$

Given any decomposition as in the statement, there are isomorphisms

$$\operatorname{gr}_1^{W'} \cong \bigoplus_{T_i \text{ zigzag}} T_i \qquad \operatorname{gr}_2^{W'} \cong \bigoplus_{T_i \text{ square}} T_i.$$

Thus, by lemma 1.51, we are reduced to check uniqueness in the two cases that the complex is generated in degree k and either all the T_i are zigzags or all the T_i are squares. In this last case, the T_i with base in (p,q) has the intrinsic definition as the subcomplex generated by $A^{p,q}$, so the decomposition is unique.

The case of zigzags follows from proposition 1.50 above.

Regarding existence of the decomposition, we first note that the filtration $W_\bullet$ splits: Let A be an arbitrary bounded double complex. Because A is assumed to be bounded, by choosing bases succesively, we can find for all $p, q \in \mathbb{Z}$ subspaces $B_r^{p,q} \subseteq A^{p,q}$ that split the filtrations W_k on $A^{p,q}$ in such a way that $\partial_1 B_r^{p,q} \subseteq B_r^{p+1,q}$ and $\partial_2 B_r^{p,q} \subseteq B_r^{p,q+1}$.

Similarly, for A generated in total degree k, $W'_\bullet$ splits: In fact, for every $p, q \in \mathbb{Z}$ with $p+q=k$, choose a complement $S^{p,q}$ s.t. $A^{p,q} = K^{p,q} \oplus S^{p,q}$. Let S denote the subcomplex generated by the $S^{p,q}$. One verifies that by construction S splits uniquely as a direct sum of squares generated in degree k (the subcomplexes generated by the $S^{p,q}$) and we have a direct sum decomposition

$$A = S \oplus K.$$

It thus remains to show uniqueness and existence of the decomposition in the case of a double complex generated in one total degree k and satisfying $\partial_1 \circ \partial_2 = 0$. But this is nothing more than a (special) representation of a quiver of type $\mathbb{A}_n$ and thus, the statement follows from proposition 1.50. ☐

Remark 1.52.

- *The consideration of cohomological invariants in the first section of this chapter (proposition 1.4) yields another proof for uniqueness.*
- *Essentially the only thing used in the proof is that one can choose complements of subvectorspaces. Therefore, one can adapt theorem 1.3 to double complexes in other abelian categories which are semisimple in a suitable sense, e.g., representations of a finite group G. The definition of elementary double complexes stays the same, but their classification can of course become more complicated (e.g., they are classified by shape and isomorphism type of the representation on a nonzero component in the case of representations).*
- *After replacing ∂_1 by $(-1)\partial_1$, a double complex is a complex of complexes. So one might hope to get a similar statement for complexes of complexes of complexes and so on. In particular, this would also treat maps between double complexes.*

  However, as was pointed out to me by L. Hille, there is no longer a discrete classification of elementary complexes. In fact, for any isomorphism $\alpha : K \cong K$ consider the following complex, where all arrows except α are the identity:

$$\begin{tikzcd}
 & K & & \\
K \arrow[ur] \arrow[rr] & & K & \\
 & K \arrow[uu, crossing over] \arrow[rr] & & K \\
 & & K \arrow[uu, crossing over] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

These are pairwise nonisomorphic for different α*.*

Even worse, the direct analogue of theorem 1.3 fails in this situation. For example, a triple complex of the following form, where only nonzero arrows are drawn, is not elementary and cannot be decomposed into elementary complexes:

$$\begin{tikzcd}
K \arrow[r] & K & & \\
 & & K \arrow[r] & K \\
K \arrow[uu] \arrow[r] \arrow[urr] & K^2 \arrow[uu] \arrow[r] \arrow[urr] & K & \\
 & & & K \arrow[uu] \\
 & K \arrow[uu] \arrow[r] \arrow[urr] & K \arrow[uu] &
\end{tikzcd}$$

Chapter 2

Rees-bundles

Two well-known theorems state that every holomorphic or algebraic vector bundle on $\mathbb{C}^n$ is trivial.[1] In a more technical terms, this could be stated as follows:

Theorem 2.1. *For every point $x \in \mathbb{C}^n$, the association*

$$\mathcal{V} \longmapsto \mathcal{V}(x),$$

where $\mathcal{V}(x)$ denotes the fibre at x, is a functor which induces a bijection on isomorphism classes:

$$\left\{ \begin{matrix} \textit{vector bundles on} \\ \mathbb{C}^n \end{matrix} \right\}_{/\sim} \longleftrightarrow \left\{ \begin{matrix} \textit{finite dimensional complex} \\ \textit{vector spaces} \end{matrix} \right\}_{/\sim}$$

In this chapter, we review a construction studied by C. Deninger, C. Simpson, O. Penacchio, A. Klyachko and others that yields an even stronger statement for equivariant bundles and filtered vector spaces. The first section deals with toric algebraic vector bundles on $\mathbb{A}^n$ and $\mathbb{P}^n$. This is a review of known facts,[2] but for the convenience of the reader and to establish some techniques, we give proofs of all the results. In the second section, we prove variants of the previous statements and constructions for holomorphic vector bundles and vector bundles with a connection. We also prove GAGA-style results.

Notational conventions: We fix a field k of characteristic zero. The letter n will always denote some natural number and Fil_k^n the category of finite dimensional k-vector spaces with n separated and exhaustive filtrations. (Note that this is a slightly different convention from chapter 1.2, where we did not assume finite dimensionality.) Given two filtered vector spaces $(V, F), (V', F')$, the tensor product $V \otimes_k V'$ is equipped with the filtration $(F \otimes F')^\bullet = \sum_{p+q=\bullet} F^p \otimes_k F'^q$ and this induces a tensor product on Fil_k^n. We often write simply V instead of $(V, F_1, ..., F_n)$ for an object in Fil_k^n.
We write $A := k[z_1, ..., z_n]$ and $B := k[z_1^{\pm 1}, ..., z_n^{\pm 1}]$. We view these as equipped with the standard n-grading, i.e., z_i^p has degree p. For a multiindex $p = (p_1, ..., p_n) \in \mathbb{Z}^n$, we write $|p| := \sum_{i=1}^n p_i$ and $z^p := z_1^{p_1} \cdot ... \cdot z_n^{p_n} \in B$. For n filtrations F_i on some vector space, we set $F^p := F_1^{p_1} \cap ... \cap F_n^{p_n}$. Given some $\lambda \in \mathbb{Z}$, we use the notation $\lambda p := (\lambda p_1, ..., \lambda p_n)$ and $p \pm_i \lambda := (p_1, ..., p_{i-1}, p_i \pm \lambda, p_{i+1}, ..., p_n) \in \mathbb{Z}^n$ and similarly with $\geq_i$ instead of $\pm_i$. Also, for $p, r \in \mathbb{Z}^n$, we write $p \geq r :\Leftrightarrow p_i \geq r_i \; \forall i \in \{1, ..., n\}$.

We refer to appendix A for background on equivariant sheaves and connections.

2.1 Algebraic Rees-bundles

Consider the affine space $\mathbb{A}^n := \mathbb{A}_k^n := \operatorname{Spec} A$. Via the inclusion

$$j : \mathbb{G}_m^n \longrightarrow \mathbb{A}^n$$

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[1] see e.g. [Qui76] for the algebraic case. The holomorphic case follows e.g. from [Gra58] and the fact that every topological bundle is trivial as $\mathbb{C}^n$ is contractible.

[2] The results are essentially all contained in much greater generality in [Kly91]. At the time of writing, I was not aware of this but followed [Pen03], which also contains most of them. We only add details as e.g. the notion of r-strictness in remark 2.15.

it is equipped with a $\mathbb{G}_m^n = \operatorname{Spec} B$-action by multiplication. For every $i \in \{1, ..., n\}$, there is a torus-invariant divisor Z_i, defined by the equation $z_i = 0$. We denote

$$U_i := \mathbb{A}^n \backslash \bigcup_{j \neq i} Z_i.$$

For example, one has $U_1 = \mathbb{A}^1 \times \mathbb{G}_m^{n-1}$.
The ring $B = \Gamma(\mathbb{G}_m^n, \mathcal{O}_{\mathbb{G}_m^n})$ is equipped with two structures: an n-grading, by degree of the z_i, corresponding to the action of $\mathbb{G}_m^n$ on itself and n filtrations according to pole order along the Z_i. Explicitly, if

$$B_i := \Gamma(U_i, \mathcal{O}_{U_i}) = k[z_1^{\pm 1}, ..., z_{i-1}^{\pm 1}, z_i, z_{i+1}^{\pm 1}, ..., z_n^{\pm 1}],$$

for every $i \in \{1, ..., n\}$, the ring B is equipped with the separated and exhaustive filtration of B_i-submodules given by

$$F_i^p B := z_i^p B_i.$$

Let $\mathcal{V}$ be a coherent $\mathbb{G}_m^n$-equivariant sheaf on $\mathbb{A}^n$. For every $i \in \{1, ..., n\}$, the sections

$$\Gamma(\mathbb{G}_m^n, \mathcal{V}) = \Gamma(\mathbb{A}^n, \mathcal{V}) \otimes_A B$$

are thus equipped with the n separated and exhaustive filtrations of B_i-submodules

$$F_i^p \Gamma(\mathbb{G}_m^n, \mathcal{V}) = \Gamma(\mathbb{A}^n, \mathcal{V}) \otimes_A F_i^p B$$

Hence, the k-vector space of invariant sections $\mathcal{V}^{inv} := \Gamma(\mathbb{G}_m^n, \mathcal{V})^{\mathbb{G}_m^n}$, which can be calculated as the degree $(0, ..., 0)$ elements, is equipped with n separated and exhaustive descending filtrations by k-vector spaces, given explicitly as $v \in F_i^p \mathcal{V}^{inv} :\Leftrightarrow \operatorname{ord}_{z_i}(v z_i^{-p}) \geq 0$. Thus, this procedure yields a functor

$$\begin{aligned} \operatorname{Coh}(\mathbb{A}^n, \mathbb{G}_m^n) &\longrightarrow \operatorname{Fil}_k^n \\ \mathcal{V} &\longmapsto \eta_{\mathbb{A}^n} := (\mathcal{V}^{inv}, F_1, ..., F_n). \end{aligned}$$

To see that this is well-defined, note that we have the following

Lemma 2.2. *Every quasi-coherent $\mathbb{G}_m^n$-equivariant sheaf $\mathcal{F}$ over $\mathbb{G}_m^n$ is canonically isomorphic to the trivial vector bundle $\mathcal{F}^{inv} \otimes_k \mathcal{O}_{\mathbb{G}_m^n}$.*
Equivalently, every n-graded B-module M is canonically isomorphic to the free module $M^{(0,...,0)} \otimes_k B$, where $M^{(0,...,0)}$ denotes the elements of degree $(0, ..., 0)$.
In particular, $\mathcal{F}^{inv} = \Gamma(\mathbb{G}_m^n, \mathcal{F})^{\mathbb{G}_m^n}$ is finite-dimensional iff $\mathcal{F}$ is coherent.

Conversely, given $(V, F_1, ..., F_n) \in \operatorname{Fil}_k^n$, the space $V \otimes_k B$ is equipped with a grading via the right factor and n filtrations by B_i-submodules defined by:

$$F_i^p (V \otimes B) := \sum_{r+s=p} F_i^r V \otimes_k F_i^s B$$

In particular, one can define the **Rees-module**

$$\operatorname{Rs}^n(V) := F^{(0,...,0)}(V \otimes_k B) = \sum_{p \in \mathbb{Z}^n} F^p V \otimes_k z^{-p} A \subseteq V \otimes_k B.$$

The (algebraic) **Rees-sheaf** $\xi_{\mathbb{A}^n}(V)$ is defined to be the coherent sheaf associated to this module, with $\mathbb{G}_m^n$-equivariant structure corresponding to the grading.

Theorem 2.3. *The functors $\eta_{\mathbb{A}^n}, \xi_{\mathbb{A}^n}$ induce an equivalence of categories*

$$\mathrm{Coh}^{t.f.}(\mathbb{A}^n, \mathbb{G}_m^n) \longleftrightarrow \mathrm{Fil}_k^n,$$

where the left hand side means torsion free $\mathbb{G}_m^n$-equivariant sheaves on $\mathbb{A}^n$. This equivalence is compatible with direct sums and tensor products.

Proof. Let $\mathcal{M}_B^n$ denote the category of finitely generated n-graded B-modules with n separated and exhaustive filtrations of B_i-submodules satisfying $z_i^p F_i^q = F_i^{p+q}$ and let $\mathcal{M}_A^n$ denote the category of finitely generated n-graded A-modules. Note that $\mathrm{Coh}(\mathbb{A}^n, \mathbb{G}_m^n)$ and $\mathcal{M}_A^n$ are equivalent via the global sections functor.

Taking the degree $(0, ..., 0)$-component and tensoring $\_ \otimes_k B$ yield an adjunction

$$\mathcal{M}_B^n \underset{\longleftarrow}{\overset{\longrightarrow}{}} \mathrm{Fil}_k^n$$

and using lemma 2.2, one verifies that these functors are actually quasi-inverse.

On the other hand, there is an adjunction

$$\mathcal{M}_B^n \underset{\longleftarrow}{\overset{\longrightarrow}{}} \mathcal{M}_A^n$$

given by taking $F^{(0,\dots,0)}$ and tensoring $\_ \otimes_A B$. This is not an equivalence. However, observe that for $M \in \mathcal{M}_{\mathcal{B}}$, we always have $F^{(0,\dots,0)}(M) \otimes_A B = M$ as the filtrations are supposed to be exhaustive. If $N \in \mathcal{M}_A$, we always have an identification $i(N) = F^{(0,\dots,0)}(N \otimes_A B)$ for $i : N \longrightarrow N \otimes_A B$ the natural map. Therefore we get an equivalence if we restrict to such N with i injective. This means that there is no $z_1 \cdot \ldots \cdot z_n$-torsion, which in turn implies that there can also be no other torsion. In fact, for any $f \in N$ the map $N \longrightarrow N_{f \cdot z_1 \cdot \ldots \cdot z_n}$ is injective as it is the composite of $N \longrightarrow N_{z_1 \cdot \ldots \cdot z_n}$ and $N_{z_1 \cdot \ldots \cdot z_n} \longrightarrow N_{f \cdot z_1 \cdot \ldots \cdot z_n}$, which is injective as $N_{z_1 \cdot \ldots \cdot z_n}$ is free by lemma 2.2. So, a fortiori $N \longrightarrow N_f$ is injective.

Compatibility with direct sums and tensor products is straightforward and documented, for example, in [Pen03, §2.2], so we omit verification. □

Remark 2.4. *O. Penacchio has shown that the Rees-sheaf of some object in* Fil_k^n *is a reflexive sheaf (i.e., isomorphic to its bidual).*[3] *Hence, by the above equivalence, the sheaf underlying an object in* $\mathrm{Coh}^{t.f.}(\mathbb{A}^n, \mathbb{G}_m^n)$ *is even a reflexive sheaf, which is a priori a stronger condition than torsion-freeness.*

We now investigate under what conditions $\xi_{\mathbb{A}^n}(V)$ is a vector bundle (in which case we also call it the **Rees-bundle**).
Recall that a splitting of n filtrations $F_1, ..., F_n$ on a vector space V is the datum of a direct sum decomposition

$$V = \bigoplus_{p \in \mathbb{Z}^n} V^p$$

such that

$$F_i^r = \bigoplus_{\substack{p \in \mathbb{Z} \\ p \geq_i r}} V^p.$$

If several filtrations admit a splitting can be tested numerically:

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[3] [Pen03, §2.4.2]

Lemma 2.5. *Let* $\underline{V} = (V, F_1^\bullet, ..., F_n^\bullet)$ *be an object in* Fil_k^n*. For every* $p \in \mathbb{Z}^n$*, denote*

$$D^p := \left(\frac{F^p}{\sum_{i=1}^n F^{p+_i 1}} \right)$$

and $d^p := \dim(D^p)$*. The filtrations of* $\underline{V}$ *admit a splitting if and only if the equality*

$$\dim V = \sum_{p \in \mathbb{Z}^n} d^p$$

holds.

Proof. Considering the 'only if' part, note that if we are given a splitting

$$V = \bigoplus_{p \in \mathbb{Z}^n} V^p,$$

then d_p is just the dimension of the space V^p and hence, the equality follows. For the other direction, for every $p \in \mathbb{Z}^n$, choose subspaces $V^p \subseteq F^p$ s.t. V^p projects isomorphically onto D^p. By construction, we have

$$F^p = \sum_{\substack{r \in \mathbb{Z}^n \\ r \geq p}} V^r.$$

In particular,

$$\dim V \leq \sum_{p \in \mathbb{Z}^n} \dim V^p = \sum_{p \in \mathbb{Z}^n} \dim D^p = \dim V.$$

Therefore, there has to be an equality and the sum of the V^p is direct. □

Let $\mathrm{Fil}_k^{n,splittable}$ denote the full subcategory of Fil_k^n which has as objects finite dimensional vector spaces with n filtrations that admit a splitting.

Theorem 2.6. *The functors* $\eta_{\mathbb{A}^n}$ *and* $\xi_{\mathbb{A}^n}$ *restrict to an equivalence*

$$\mathrm{Bun}(\mathbb{A}^n, \mathbb{G}_m^n) \longleftrightarrow \mathrm{Fil}_k^{n,splittable},$$

where $\mathrm{Bun}(\mathbb{A}^n, \mathbb{G}_m^n)$ *denotes the full subcategory of* $\mathrm{Coh}(\mathbb{A}^n, \mathbb{G}_m^n)$ *of locally free equivariant sheaves.*

Proof. Let V be an object of Fil_k^n. We have to show that the filtrations F_i admit a splitting if and only if $\xi_{\mathbb{A}^n}(V)$ is a vector bundle. As the base is a connected and reduced noetherian scheme, $\xi_{\mathbb{A}^n}(V)$ is a vector bundle iff all the fibres have the same dimension. Furthermore, as the fibre dimension is constant along each orbit of the $\mathbb{G}_m^n$-action, by semicontinuity of the rank it suffices to check wether the fibre at $(0, ..., 0)$ has the expected dimension.[4] One computes this fibre to be canonically isomorphic to $\bigoplus_{p \in \mathbb{Z}^n} D^p$ in the notation of lemma 2.5, using the explicit description of the Rees-module,[5] hence the result follows from lemma 2.5. □

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[4] see e.g. [Har77, p. 125, ex. 5.8.] for the statements used here.

[5] cf. the proof of proposition 2.11 for a similar computation

Corollary 2.7. *For $n = 1, 2$, the functors $\eta_{\mathbb{A}^n}$ and $\xi_{\mathbb{A}^n}$ yield an equivalence of categories:*

$$\mathrm{Bun}(\mathbb{A}^n, \mathbb{G}_m^n) \longleftrightarrow \mathrm{Fil}_k^n$$

Proof. This follows from the fact that for one and two filtrations, there always exists a splitting (see lemma 1.12). ☐

Remark 2.8. *Three or more filtrations do not necessarily admit a splitting. (Examples can be constructed considering* 3 *distinct lines in* k^2*.) Via the above equivalence, this can be seen as an incarnation of the fact that not every reflexive sheaf is a vector bundle if* $n \geq 3$*.*

There is again a variant if $k|k'$ is an extension of degree 2 with only nontrivial k'-automorphism σ. In this case, let $\mathrm{Fil}^2_{k|k'}$ denote the category of k-vector spaces with a σ-antilinear involution τ and two filtrations F_1, F_2 satisfying $\tau F_1^p = F_2^p$. Further, denote by $(\mathbb{G}_m)_{k'} := Res_{k|k'}\mathbb{G}_m$ the Weil restriction.

Corollary 2.9. *The functors $\eta_{\mathbb{A}^n}$ and $\xi_{\mathbb{A}^n}$ induce an equivalence of categories*

$$\mathrm{Bun}(\mathbb{A}^2_{k'}, (\mathbb{G}_m)_{k'}) \longleftrightarrow \mathrm{Fil}^2_{k|k'}\,.$$

We now classify the equivariant line bundles and show that every equivariant bundle on $\mathbb{A}^n$ splits equivariantly as a direct sum of such:

For $p \in \mathbb{Z}^n$, let $\mathcal{O}_{\mathbb{A}^n}(p)$ be the equivariant line bundle on $\mathbb{A}^n$ associated with the free A-module of rank one with generator in degree $-p$. Equivalently, it is the Rees-bundle associated to a one-dimensional vector space with filtrations F_i having the only jump in p_i. Because the equivalence is compatible with direct sums and tensor products, we get:

Corollary 2.10. *Let $(V, F_1^\bullet, ..., F_n^\bullet)$ be an object of Fil_k^n. A splitting*

$$V = \bigoplus_{p \in \mathbb{Z}^n} V^p$$

of the F_i induces an equivariant splitting

$$\xi_{\mathbb{A}^n}(V) \cong \bigoplus_{p \in \mathbb{Z}^n} V^p \otimes_k \mathcal{O}_{\mathbb{A}^n}(p).$$

In particular, there is an isomorphism of abelian groups

$$\begin{aligned} \mathbb{Z}^n &\cong Pic(\mathrm{Coh}(\mathbb{A}^n, \mathbb{G}_m^n)) \\ p &\mapsto \mathcal{O}_{\mathbb{A}^n}(p). \end{aligned}$$

For later use, let us record several restriction and compatibility properties of the Rees-bundle.

Proposition 2.11. *The functors $\xi_{\mathbb{A}^n}(\_)$ behave as follows when restricted to torus-invariant subsets:*

1. *There is a canonical, $\mathbb{G}_m^n$-equivariant isomorphism*

$$\xi_{\mathbb{A}^n}(V, F_1, ..., F_n)|_{\mathbb{G}_m \times \mathbb{A}^{n-1}} \cong pr^*_{\mathbb{A}^{n-1}} \xi_{\mathbb{A}^{n-1}}(V, F_2, ..., F_n)$$

   and similarly for other subvarieties of the form $\mathbb{A}^r \times \mathbb{G}_m^s \times \mathbb{A}^t$.

2. *If $(V, F_1, ..., F_n)$ is an object in $\mathrm{Fil}_k^{n,splittable}$, there is a canonical $\mathbb{G}_m^n$-equivariant isomorphism*

$$\xi_{\mathbb{A}^n}(V)|_{Z_1} \cong \bigoplus_{p\in\mathbb{Z}} \xi_{\mathbb{A}^{n-1}}(\mathrm{gr}_{F_1}^{-p} V, F_2, ..., F_n),$$

where the first component of $\mathbb{G}_m^n$ on the right hand side acts trivially on the base and according to the grading on the bundle. There is the analogous isomorphism for restrictions to other invariant divisors Z_i. In particular, for any permutation σ of $\{1, ..., n\}$, there is a canonical isomorphism

$$\xi_{\mathbb{A}^n}(V)|_{\{(0,...,0)\}} \cong \bigoplus_{(p_1,...,p_n)\in\mathbb{Z}^n} \mathrm{gr}_{F_{\sigma(1)}}^{-p_{\sigma(1)}} ... \mathrm{gr}_{F_{\sigma(n)}}^{-p_{\sigma(n)}} V.$$

Proof. As all involved bases are affine, we can compute on global sections. Here, the restriction in 1. just corresponds to inverting z_1 and the isomorphism follows immediately. For 2. note that we always have a map

$$\frac{F_1^{p_1} \cap ... \cap F_n^{p_n}}{F_1^{p_1+1} \cap F_2^{p_2} ... \cap F_n^{p_n}} \longrightarrow F_2^{p_2} \cap ... \cap F_n^{p_n}(\mathrm{gr}_{F_1}^{p_1}),$$

which is an isomorphism if the filtrations are splittable. The claimed isomorphism (as k-modules) is then given by

$$\begin{aligned} \frac{\mathrm{Rs}^n(\underline{V})}{z_1 \,\mathrm{Rs}^n(\underline{V})} &\cong \bigoplus_{p_1\in\mathbb{Z}} \left(\bigoplus_{p_2,...,p_n\in\mathbb{Z}} \frac{F_1^{-p_1} \cap ... \cap F_n^{-p_n}}{F_1^{-p_1+1} \cap F_2^{-p_2} ... \cap F_n^{-p_n}} z_2^{p_2} ... z_n^{p_n} \right) z_1^{p_1} \\ &\cong \bigoplus_{p_1\in\mathbb{Z}} \left(\bigoplus_{p_2,...,p_n\in\mathbb{Z}} F_2^{-p_2} \cap ... \cap F_n^{-p_n}(\mathrm{gr}_{F_1}^{-p_1}) z_2^{p_2} ... z_n^{p_n} \right) z_1^{p_1}. \end{aligned}$$

The right hand sides carries the structure of a $k[z_2, ...z_n]$-$\mathbb{G}_m^n$-module[6] suggested by the appearance of the z_i and one checks the isomorphism to be compatible with this structure. □

Remark 2.12. *As* 2. *shows, if we start with a graded vector space $V = \bigoplus_{p\in\mathbb{Z}} V^p$, equip it with the filtration $F^p = \bigoplus_{i\geq p} V^i$ and build the Rees-bundle, there is an isomorphism $\xi_{\mathbb{A}^1}(V, F)|_{\{0\}} \cong \bigoplus V^{-p}$. I.e., we get the opposite of the grading we started with.*

Proposition 2.13. *Let $f : V \longrightarrow W$ be a map in Fil_k^n and $\xi_{\mathbb{A}^n}(f)$ the associated map of Rees-sheaves.*

- *The morphism induced by the inclusion $\ker f \hookrightarrow \underline{V}$ gives a canonical identification*

$$\xi_{\mathbb{A}^n}(\ker f) \cong \ker \xi_{\mathbb{A}^n}(f).$$

- *There is an exact sequence*

$$0 \longrightarrow T \longrightarrow \mathrm{coker}\, \xi_{\mathbb{A}^n}(f) \xrightarrow{\varphi} \xi_{\mathbb{A}^n}(\mathrm{coker}\, f) \longrightarrow 0,$$

where $\mathrm{coker}\, \xi_{\mathbb{A}^n}(f)$ is the sheaf-theoretic cokernel and T is the torsion subsheaf. Further, $\mathrm{codim}(\mathrm{supp}(T)) \geq 2$ if and only if f is strict with respect to all filtrations.

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[6] I.e. a graded $k[z_2, ..., z_n]$-module, see appendix A for details

Proof. It suffices to check this on global sections. For the first point, note that for any $p \in \mathbb{Z}^n$, a section $v_p \otimes z^p$ with $v_p \in F_V^p$ is mapped to zero iff $v_p \in \ker f$.

For the second point, let $\pi : W \to \operatorname{coker} f$ denote the projection. The global sections of the two right members of the claimed sequence are then given (where the sum is direct as k-vector spaces) by

$$\Gamma(\mathbb{A}^n, \operatorname{coker} \xi_{\mathbb{A}^n}(f)) = \bigoplus_{p \in \mathbb{Z}^n} \frac{F_W^p}{f(F_V^p)} z^{-p}$$

and

$$\Gamma(\mathbb{A}^n, \xi_{\mathbb{A}^n}(\operatorname{coker} f)) = \bigoplus_{p \in \mathbb{Z}^n} \pi F_W^p z^{-p}.$$

Denoting by $\varphi^p : \frac{F_W^p}{f(F_V^p)} \longrightarrow \pi F_W^p$ the natural map, we can define the map φ to be induced by the direct sum of the φ^p. Using

$$\ker \varphi^p = \{w \in F_W^p \mid \exists q \in \mathbb{N}^n \text{ s.t. } w \in f(F_W^{p-q})\} \bmod f(F_V^p),$$

one verifies that $ker\varphi = T$ coincides with the torsion subsheaf.

To verify the statement on the dimension of the support of T, we can, using proposition 2.11, restrict to the case where $\underline{V} = (V, F_V), \underline{W} = (W, F_W) \in \mathrm{Fil}_k^1$. Then, we have to show that φ is an isomorphism. But if we only consider one filtration, strictness is equivalent to the condition that all the φ^p are isomorphisms, which is in turn equivalent to φ being an isomorphism. □

Remark 2.14. *This means that the categorical cokernel of a morphism in* $\mathrm{Coh}^{t.f.}(\mathbb{A}^n, \mathbb{G}_m^n)$ *does not coincide with the cokernel in the category of sheaves, but rather is a quotient of it. Alternatively, the categorical cokernel can be described as the double dual (which is reflexive) of the sheaf-theoretic cokernel.*

Remark 2.15. *One can generalise the condition on the cokernel as follows: Let us say a map*

$$f : (V, F_1, ..., F_n) \longrightarrow (W, G_1, ..., G_n)$$

in Fil_k^n *is* r*-strict if for every collection* $\{i_1, ..., i_r\}$ *of indices in* $\{1, ..., n\}$ *and all* r*-tuples of integers* $(p_1, ..., p_r) \in \mathbb{Z}^r$*,*

$$f(F_{i_1}^{p_1} \cap ... \cap F_{i_r}^{p_r}) = G_{i_1}^{p_r} \cap ... \cap G_{i_r}^{p_r} \cap \mathrm{im}(f).$$

E.g., 1*-strictness coincides with usual strictness for all filtrations and if there exist splittings for the* F_i *and* G_i *respected by* f*, then* f *is* n*-strict.*
Then we have, in the terminology of the proposition,

$$f \text{ is } r\text{-strict} \Leftrightarrow \operatorname{codim}(\operatorname{supp}(T)) \geq r + 1.$$

Next, we consider a variant of the Rees-construction with projective base. Let $\mathbb{P}^n := \mathbb{P}_k^n = \operatorname{Proj}(k[X_0, ..., X_n])$ be projective n-space over k. It is obtained as the quotient of punctured affine space by a diagonally embedded multiplicative group, i.e. $(\mathbb{A}^{n+1} \backslash \{0\})/\Delta(\mathbb{G}_m)$, and carries an induced group action of $\mathbb{G}_m^{n+1}/\Delta(\mathbb{G}_m)$. It is covered by the standard open sets

$$U_i := \{X_i \neq 0\} \cong \mathbb{A}^n = \operatorname{Spec}\left(k\left[\frac{X_0}{X_i}, ..., \frac{X_n}{X_i}\right]\right).$$

On these sets the action of $\mathbb{G}_m^{n+1}/\Delta(\mathbb{G}_m) \cong \mathbb{G}_m^n$ (the isomorphism being division by the i-th coordinate) is the standard action by multiplication, i.e., corresponding to the n-grading, where the variables have degree 1 (note that this differs from our earlier use of the letters U_i).

Given $(V, F_0, ..., F_n)$ in Fil_k^{n+1}, let us define a sheaf $\xi_{\mathbb{P}^n}(V)$ on $\mathbb{P}^n$ as follows: On each open set U_i, consider the sheaf $\xi_{\mathbb{A}^n}(\delta_i V)$, where δ_i means forgetting about the i-th filtration. By proposition 2.11, these coincide on the intersections and hence glue to a sheaf on $\mathbb{P}^n$. This defines a functor $\xi_{\mathbb{P}^n}$ from vector spaces with $n+1$ filtrations to $\mathbb{G}_m^n$-equivariant sheaves on $\mathbb{P}^n$

$$\xi_{\mathbb{P}^n} : \mathrm{Fil}_k^{n+1} \longrightarrow \mathrm{Coh}(\mathbb{P}^n, \mathbb{G}_m^n).$$

An alternative description is as the composition

$$\xi_{\mathbb{P}^n} := p_*^{\Delta(\mathbb{G}_m)} \circ i^* \circ \xi_{\mathbb{A}^{n+1}},$$

where $i : \mathbb{A}^{n+1}\backslash\{0\} \hookrightarrow \mathbb{A}^{n+1}$ is the inclusion, $p : \mathbb{A}^{n+1}\backslash\{0\} \longrightarrow \mathbb{P}^n$ is the projection and $p_*^{\Delta(\mathbb{G}_m)}$ means pushforward of sheaves followed by taking invariants with respect to the diagonal action of $\mathbb{G}_m$ (which now acts trivially on the base). Similarly, applying $\eta_{\mathbb{A}^n}$ on every U_i yields a functor

$$\eta_{\mathbb{P}^n} : \mathrm{Coh}(\mathbb{P}^n, \mathbb{G}_m^n) \longrightarrow \mathrm{Fil}_k^{n+1},$$

which can equivalently be described as $\eta_{\mathbb{A}^{n+1}} \circ i_* \circ p^*$.

Let us say an object $\underline{V} = (V, F_1, ..., F_n)$ of Fil_k^n is r-splittable if every tuple of r filtrations among the F_i admits a splitting. For example, usual splittability coincides with n-splittability and $\underline{V}$ is always 1- and 2-splittable. Denote the category of vector spaces with n r-splittable filtrations by $\mathrm{Fil}_k^{n,r\text{-}splittable}$. By theorem 2.6, we obtain:

Proposition 2.16. *The functors $\xi_{\mathbb{P}^n}$ and $\eta_{\mathbb{P}^n}$ are compatible with direct sum and tensor product and yield equivalences of categories*

$$\mathrm{Fil}_k^{n+1,n\text{-}splittable} \longleftrightarrow \mathrm{Bun}(\mathbb{P}^n, \mathbb{G}_m^n).$$

Corollary 2.17. *For $n = 1, 2$ the functors $\xi_{\mathbb{P}^n}$ and $\eta_{\mathbb{P}^n}$ yield an equivalence of categories*

$$\mathrm{Fil}_k^{n+1} \longleftrightarrow \mathrm{Bun}(\mathbb{P}^n, \mathbb{G}_m^n).$$

The restriction property 2. in proposition 2.11 translates to:

Proposition 2.18. *Let $\underline{V} = (V, F_0, ..., F_n)$ be in $\mathrm{Fil}_k^{n+1,n\text{-}splittable}$. There is a canonical equivariant isomorphism*

$$\xi_{\mathbb{P}^n}(\underline{V})|_{\{X_0=0\}} \cong \bigoplus_{p\in\mathbb{Z}} \xi_{\mathbb{P}^{n-1}}(\mathrm{gr}_{F_0}^{-p} V, F_1, ..., F_n) \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(-p)$$

and similarly for other invariant divisors $\{X_i = 0\}$.

For a multiindex $p \in \mathbb{Z}^{n+1}$, let $\mathcal{O}_{\mathbb{P}^n}(p)$ be the equivariant bundle on $\mathbb{P}^n$ obtained by glueing the bundles $\mathcal{O}_{\mathbb{A}^n}(\delta_i p)$ on the charts U_i, where $\delta_i p$ denotes forgetting the i-th entry. This can alternatively be described as the equivariant pushforward to $\mathbb{P}^n$ of $\mathcal{O}_{\mathbb{A}^{n+1}}(p)|_{\mathbb{A}^{n+1}\setminus\{0\}}$. The underlying line bundle (without equivariant structure) is $\mathcal{O}_{\mathbb{P}^n}(|p|)$.

The splitting property in corollary 2.10 translates as

Proposition 2.19. *Let V be an object in Fil_k^{n+1}. A splitting*

$$V = \bigoplus_{p \in \mathbb{Z}^{n+1}} V^p$$

of the filtrations induces an equivariant splitting

$$\xi_{\mathbb{P}^n}(V) \cong \bigoplus_{p \in \mathbb{Z}^n} V^p \otimes_k \mathcal{O}_{\mathbb{P}^n}(p).$$

As a corollary, we obtain an equivariant version of the classification of vector bundles on $\mathbb{P}^1$:

Corollary 2.20. *For every equivariant vector bundle $\mathcal{V}$ of rank r on $\mathbb{P}^1$, there is a unique nonincreasing sequences of pairs in $\mathbb{Z}^2$ (with lexicographic order)*

$$(p_1, q_1) \geq (p_2, q_2) \geq ... \geq (p_r, q_r)$$

and an isomorphism

$$\mathcal{V} \cong \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^1}(p_i, q_i).$$

Recall that the classical theorem, in this form due to Grothendieck in [Gro57], although in linear algebraic forms going back at least to [DW82], says:

Theorem 2.21. *For a vector bundle $\mathcal{V}$ of rank r on $\mathbb{P}^1$, there is a unique nonincreasing sequence of integers $a_1 \geq a_2 ... \geq a_r$ and an isomorphism*

$$\mathcal{V} \cong \bigoplus_{i=1}^{r} \mathcal{O}(a_i).$$

2.2 Variants

In this section, we are interested in the case $k = \mathbb{C}$ and we only treat vector bundles, not more general equivariant sheaves. We consider equivariant connections and weaker (than being algebraic) regularity conditions on transition functions and group actions.

Notations and conventions: Let $\mathrm{Gra}_{\mathbb{C}}^n$ denote the category of finite dimensional n-graded $\mathbb{C}$-vector spaces. For $G = (\mathbb{C}^\times)^n$ or $G = \mathbb{T}^n := (S^1)^n$, by $\mathrm{Bun}^\omega(\mathbb{C}^n, G)$ with $\omega \in \{\mathrm{alg}, \mathrm{hol}, \mathrm{sm}, \mathrm{cont}\}$, we mean algebraic, holomorphic, smooth or continuous G-equivariant bundles on $\mathbb{C}^n$ when meaningful. We identify $\mathrm{Bun}(\mathbb{A}^n, \mathbb{G}_m^n)$ from the previous section with $\mathrm{Bun}^{\mathrm{alg}}(\mathbb{C}^n, (\mathbb{C}^\times)^n)$ (by considering the complex valued points) and we switch freely between geometric vector bundles and locally free sheaves over the corresponding structure sheaf. By $\mathrm{Rep}^{\omega}(G)$, we denote algebraic, holomorphic, smooth or continuous representations of G. Those cases of interest to us are related to each other by the following diagram:

$$\begin{tikzcd}
\mathrm{Bun}^{\mathrm{alg}}(\mathbb{C}^n, (\mathbb{C}^\times)^n) \arrow[r] \arrow[d] & \mathrm{Rep}^{\mathrm{alg}}((\mathbb{C}^\times)^n) \arrow[d] \arrow[ddr] & \\
\mathrm{Bun}^{\mathrm{hol}}(\mathbb{C}^n, (\mathbb{C}^\times)^n) \arrow[r] \arrow[dd] & \mathrm{Rep}^{\mathrm{hol}}((\mathbb{C}^\times)^n) \arrow[dd] \arrow[dr] & \\
 & & \mathrm{Gra}^n_{\mathbb{C}} \\
\mathrm{Bun}^{\mathrm{sm}}(\mathbb{C}^n, \mathbb{T}^n) \arrow[r] \arrow[d] & \mathrm{Rep}^{\mathrm{sm}}(\mathbb{T}^n) \arrow[d] \arrow[ur] & \\
\mathrm{Bun}^{\mathrm{cont}}(\mathbb{C}^n, \mathbb{T}^n) \arrow[r] & \mathrm{Rep}^{\mathrm{cont}}(\mathbb{T}^n) \arrow[uur] &
\end{tikzcd} \qquad (*)$$

Here the vertical arrows forget some structure (or in sheaf-theoretic terms, tensor by a bigger structure sheaf and restrict the action), the horizontal ones are restriction to the fibre at 0 and the diagonal ones are given by the rule $(V, \rho) \mapsto V = \bigoplus_{p \in \mathbb{Z}^n} V^p$ where V^p with $p = (p_1, ..., p_n)$ is the eigenspace of the character

$$\chi_p : (\lambda_1, ..., \lambda_n) \mapsto \lambda^{-p} := \lambda_1^{-p_1} \cdot ... \cdot \lambda_n^{-p_n}.$$

All diagonal arrows are equivalences of categories, hence so are all arrows in the middle column. To a G-representation (V, ρ) in $\mathrm{Rep}^{\omega}(G)$, one can associate the trivial (geometric) vector bundle $\widetilde{V} := \mathbb{C}^n \times V$ with product action

$$\lambda.(x, v) = (\lambda.x, \rho(\lambda)x).$$

We also denote this $\widetilde{V}^{\omega}$ if we want to emphasize that we consider it as a bundle in $\mathrm{Bun}^{\omega}(\mathbb{C}^n, G)$. E.g., in the notation of the previous section, $\widetilde{V}^{\mathrm{alg}} = \bigoplus_{p \in \mathbb{Z}^n} V^p \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^n}$, where as above V^p is the eigenspace of χ_p, but equipped with the trivial action.

The invariant global sections of $\widetilde{V}^{\omega}$ can be identified with equivariant (algebraic, holomorphic, smooth or continuous) maps $\mathbb{C}^n \longrightarrow V$ and the restriction of $\widetilde{V}$ to $(0, ..., 0)$ is canonically isomorphic to (V, ρ). In particular, all horizontal arrows in $(*)$ are essentially surjective.

Because $\mathbb{C}^n$ is $\mathbb{T}^n$-equivariantly contractible to the point $(0, ..., 0)$, the restriction to the fibre

$$\mathrm{Bun}^{\omega}(\mathbb{C}^n, \mathbb{T}^n) \longrightarrow \mathrm{Rep}^{\omega}(\mathbb{T}^n)$$

induces a bijection on isomorphism classes[7] for $\omega \in \{\mathrm{sm}, \mathrm{cont}\}$ and by the results of the previous section (corollary 2.10) so does

$$\mathrm{Bun}^{\mathrm{alg}}(\mathbb{C}^n, (\mathbb{C}^\times)^n) \longrightarrow \mathrm{Gra}^n_{\mathbb{C}} .$$

Finally, by a general theorem of Heinzner and Kutzschebauch on equivariant bundles on Stein spaces[8] the forgetful functor

$$\mathrm{Bun}^{\mathrm{hol}}(\mathbb{C}^n, (\mathbb{C}^\times)^n) \longrightarrow \mathrm{Bun}^{\mathrm{cont}}(\mathbb{C}^n, \mathbb{T}^n)$$

induces a bijection on isomorphism classes. Summing up the statements in the previous paragraphs, one obtains:

Proposition 2.22. *In the diagram* $(*)$*, all arrows induce bijections on isomorphism classes. The sides of the triangles on the right are equivalences of categories.*

For the functor from algebraic to holomorphic bundles, even more is true:

Theorem 2.23. *The functor*

$$\mathrm{Bun}^{\mathrm{alg}}(\mathbb{C}^n, (\mathbb{C}^\times)^n) \longrightarrow \mathrm{Bun}^{\mathrm{hol}}(\mathbb{C}^n, (\mathbb{C}^\times)^n)$$

is an equivalence of categories.

Proof. We already know that it is essentially surjective by the previous considerations. It is also obviously faithful, so what remains to be checked is that it is full. Consider a map of equivariant holomorphic bundles $\mathcal{V} \longrightarrow \mathcal{W}$. Without loss of generality, we may assume $\mathcal{V} = \widetilde{V}, \mathcal{W} = \widetilde{W}$ for some $(\mathbb{C}^\times)^n$-representations V, W. In this case, the statement follows from the following lemma applied to the bundle $\mathrm{Hom}(\mathcal{V}, \mathcal{W})$.

Lemma 2.24. *Let V be an object of* $\mathrm{Rep}^{\mathrm{hol}}((\mathbb{C}^\times)^n)$*. For any equivariant section $z \mapsto (z, s(z))$ of the bundle $\widetilde{V}^{\mathrm{hol}}$, there are a finite subset $I \subseteq \mathbb{Z}^n_{\geq 0}$ and elements $v_p \in V^{-p}$ for all $p \in I$ s.t.*

$$s = \sum_{p \in I} z^p v_p$$

It suffices to check this equality on the dense open subset $U = (\mathbb{C}^\times)^n \subseteq \mathbb{C}^n$. Since $(\mathbb{C}^\times)^n$ acts simply transitively on U, a section is determined by its value $s(1, ..., 1) = \sum_{p \in I} v_p$. Because the section extends to the whole of $\mathbb{C}^n$, necessarily $I \subseteq \mathbb{Z}^n_{\geq 0}$. This implies the lemma and consequently the theorem. ☐

Remark 2.25. *The other two forgetful functors are still faithful and essentially surjective, but no longer full.*

---

[7] This follows from the 'homotopy invariance of isomorphism classes of equivariant vector bundles': Given a compact topological group G and two homotopic equivariant maps $f_0, f_1 : X \longrightarrow Y$ between G-spaces with X paracompact. Then for any vector bundle $\mathcal{V}$ on Y the pullbacks $f_0^*\mathcal{V}$ and $f_1^*\mathcal{V}$ are isomorphic.
This is essentially proven in [Ati67], p. 40f. However, there only the case of a compact hausdorff base and a finite group is treated. See [Seg68] for the case of a compact group (but still compact base) and, e.g., [Zoi10, p. 21] for the case of non-equivariant bundles over a paracompact base, which can be adapted to the equivariant case by a standard averaging trick.

[8] [HK95, p. 341], see also [Hei91, par. 1.4.] for the notions used in the statement.

Remark 2.26. *Using the whole diagram* $(*)$ *and in particular [HK95] to show essential surjectivity of analytification seems to be quite an overkill. It would be interesting to see a direct derivation of the classification of isomorphism classes in* $\mathrm{Bun}^{\mathrm{hol}}(\mathbb{C}^n, (\mathbb{C}^\times)^n)$. *In a (failed) attempt to produce one, I came across some nice functional equations for certain classes of matrices of holomorphic functions which can be found in appendix C.*

Corollary 2.27. *For any smooth toric variety* X*, analytification yields an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{equivariant algebraic vector} \\ \textit{bundles on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{equivariant holomorphic} \\ \textit{vector bundles on } X^{\mathrm{an}} \end{array} \right\}$$

Proof. For any groups G, G' and G-space X, one obtains by formal nonsense an identification of the $G \times G'$-equivariant bundles over $X \times G'$ and the G-equivariant bundles over X, regardless wether one considers this in the algebraic or holomorphic category. In particular, there is an equivalence

$$\mathrm{Bun}^{\omega}(\mathbb{C}^n \times (\mathbb{C}^\times)^m, (\mathbb{C}^\times)^{n+m}) \longrightarrow \mathrm{Bun}^{\omega}(\mathbb{C}^n, (\mathbb{C}^\times)^n)$$

for $\omega = \mathrm{alg}, \mathrm{hol}$ and analytification gives an equivalence of algebraic and holomorphic equivariant bundles over $\mathbb{C}^n \times (\mathbb{C}^\times)^m$. One concludes by noting that every toric variety is covered by open sets equivariantly isomorphic to $\mathbb{C}^n \times (\mathbb{C}^\times)^m$ for some $n, m \in \mathbb{Z}_{\geq 0}$.[9]. ☐

Corollary 2.28. *Let* X *be a smooth toric variety. Analytification yields an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{equivariant algebraic vector} \\ \textit{bundles with an equivariant} \\ \textit{connection on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{equivariant holomorphic} \\ \textit{vector bundles with an} \\ \textit{equivariant connection} \\ \textit{on } X^{an} \end{array} \right\}.$$

It restricts to an equivalence of the subcategories of flat connections.

Proof. That the functor is fully faithful follows from the corresponding statement for bundles without a connection. In fact, a map between algebraic vector bundles with connection is compatible with the connections if and only if its analytification is. Similarly a connection is flat if and only if its analytification is. For essential surjectivity, we can assume that $X = \mathbb{C}^n \times (\mathbb{C}^\times)^m$ with action by multiplication of $(\mathbb{C}^\times)^{n+m}$ and that a holomorphic bundle $\mathcal{V}$ is given in trivialised form, i.e., as $\mathcal{V} = \mathcal{V}\widetilde{(0, ..., 0)} \boxtimes \mathcal{O}_{(\mathbb{C}^\times)^m}$. One checks that on these (trivial) bundles the canonical connection d is equivariant, so any connection is given as

$$\nabla = d + \Omega,$$

where Ω is a global invariant holomorphic 1-form on $\mathbb{C}^n \times (\mathbb{C}^\times)^m$ with values in the vector space $\mathrm{End}(\mathcal{V}(0, ..., 0))$. Arguing as in the proof of lemma 2.24, such a form can be written as

$$\Omega = \sum_{p \in \mathbb{Z}^n_{\geq 0}} \sum_{i=1}^{n+m} A_{p,i} z_1^{p_1} \cdot ... \cdot z_n^{p_n} \frac{dz_i}{z_i}$$

---

[9] See [CLS11, thm. 3.1.19, thm. 1.3.12, ex. 1.2.21]

where the $A_{p,i}$ are endomorphisms of $\mathcal{V}(0,...,0)$ of multidegree $-p$ (which are taken to be zero if they would cause a pole, i.e., if $p_i = 0$). In particular, the connection is algebraic. $\square$

For flat connections, there is also a nicer comparison to representations, which also accounts for the smooth case. Let $\mathrm{Bun}^{\omega}_{\nabla}(\mathbb{C}^n, (\mathbb{C}^{\times})^n)$ with $\omega \in \{\mathrm{alg}, \mathrm{hol}\}$ denote the category of $(\mathbb{C}^{\times})^n$ algebraic or holomorphic equivariant vector bundles with an equivariant connection and denote by $\mathrm{Bun}^{\omega}_{\nabla^{\flat}}(\mathbb{C}^n, (\mathbb{C}^{\times})^n)$ the respective subcategories of flat connections. Let $\mathrm{Bun}^{\mathrm{sm}}_{\nabla^{\flat}}(\mathbb{C}^n, \mathbb{T}^n)$ be the category of smooth equivariant vector bundles with flat equivariant connections. As above, there is a commutative diagram:

$$\begin{tikzcd}
\mathrm{Bun}^{\mathrm{alg}}_{\nabla^{\flat}}(\mathbb{C}^n, (\mathbb{C}^{\times})^n) \arrow[r] \arrow[d] & \mathrm{Rep}^{\mathrm{alg}}((\mathbb{C}^{\times})^n) \arrow[d] \\
\mathrm{Bun}^{\mathrm{hol}}_{\nabla^{\flat}}(\mathbb{C}^n, (\mathbb{C}^{\times})^n) \arrow[r] \arrow[d] & \mathrm{Rep}^{\mathrm{hol}}((\mathbb{C}^{\times})^n) \arrow[d] \\
\mathrm{Bun}^{\mathrm{sm}}_{\nabla^{\flat}}(\mathbb{C}^n, \mathbb{T}^n) \arrow[r] & \mathrm{Rep}^{\mathrm{sm}}(\mathbb{T}^n)
\end{tikzcd} \tag{$**$}$$

Here, horizontal arrows are again restriction to $(0,...,0)$ and vertical ones forget about the stronger regularity conditions imposed.

Proposition 2.29. *In the diagram* $(**)$*, all arrows are equivalences of categories.*

Proof. Since we already know the functors in the right column and the one from algebraic to holomorphic bundles to be an equivalence, it suffices to show that restriction to the invariant point is an equivalence in the holomorphic and smooth cases. We do this in the holomorphic case, the smooth case works the same way.

Sending (V, ∇) to $\ker \nabla$ is an equivalence of categories between flat equivariant connections and equivariant local systems on $\mathbb{C}^n$. Since $\mathbb{C}^n$ is contractible, any local system is necessarily trivial. So restriction from global sections to any point is an isomorphism. In particular, restriction to the fixed point $(0,...,0)$ induces an equivalence of categories with $\mathrm{Rep}^{\mathrm{hol}}((\mathbb{C}^{\times})^n)$.

For completeness, here is an explicit description of the pseudo-inverse to the restriction functor in $(**)$: For any representation (V, ρ) in $\mathrm{Rep}^{\mathrm{hol}}((\mathbb{C}^{\times})^n)$, as before, consider the bundle $\widetilde{V}$ with product action. Its sheaf of sections $V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n}$ is equipped with the canonical equivariant connection d given by

$$d(v \otimes f) = v \otimes df$$

and sending a map $f : (V, \rho_V) \longrightarrow (W, \rho_W)$ of representations to $f \otimes \mathrm{Id}$ this defines the pseudo-inverse

$$\mathrm{Rep}^{\mathrm{hol}}((\mathbb{C}^{\times})^n) \longrightarrow \mathrm{Bun}_{\nabla^{\flat}}(\mathbb{C}^n, (\mathbb{C}^{\times})^n).$$

$\square$

Chapter 3

The Many Faces of Mixed Hodge Structures

The aim of this chapter is to introduce and compare various equivalent descriptions of the category of (Real or Complex) Mixed Hodge Structures. This is a review of known facts and definitions. Therefore, we allow ourselves to be brief and provide references to the literature if the full details of the proofs can be found therein. Emphasis is put on the last example, by A. Goncharov and M. Kapranov, where we add some details compared to the original sources and give a slightly different proof of the main result of [Kap12] using the toric GAGA principle from the previous chapter.

We will always describe categories with an involution. Forgetting the involution, this will yield the category of Complex Mixed Hodge Structures. Taking the category of fixed points, this will yield the category of Real Mixed Hodge Structures. Therefore, we recall the concept of an involution on a category and the category of fixed points.

Definition 3.1. *An* ***involution on a category*** $\mathcal{C}$ *is an endofunctor* φ *of* $\mathcal{C}$ *together with a natural isomorphism* $\varphi \circ \varphi \Rightarrow id_{\mathcal{C}}$.

For example, the category $Vect_{\mathbb{C}}$ of finite dimensional $\mathbb{C}$-vector spaces is equipped with the involution

$$\varphi : V \mapsto V \otimes_{\mathbb{C},\sigma} \mathbb{C},$$

where $\sigma : \lambda \mapsto \overline{\lambda}$ denotes complex conjugation and we consider the right hand side as a $\mathbb{C}$-vector space via multiplication on the right side of the tensor product.

Remark 3.2. *We will usually suppress the natural isomorphism in the notation and simply write* $A = \varphi^2 A$ *for objects* A *of* $\mathcal{C}$ *and similarly for morphisms. This is justified by the fact that in all the examples we will consider, we could actually find a* φ'*, isomorphic to* φ*, such that there really is an equality* $\varphi'^2 = id_{\mathcal{C}}$.

In the example of $Vect_{\mathbb{C}}$, we could replace φ by φ', sending a vector space V to $\overline{V}$, which is the same as V as an abelian group, but with scalar multiplication given by $(\lambda, v) \mapsto \bar{\lambda} v$, where $\lambda \in \mathbb{C}, v \in \bar{V}$ and scalar multiplication on the right is taken in V.

Definition 3.3. *Let* $(\mathcal{C}, \varphi)$ *be a category with involution. The* ***category*** $\mathcal{C}^{\varphi}$ ***of fixed points*** *is defined in the following way:*

As objects, it has pairs (A, τ) *with* $A \in \mathcal{C}$ *and* $\tau : \varphi A \to A$ *an isomorphism such that the composition*

$$A = \varphi^2 A \xrightarrow{\varphi\tau} \varphi A \xrightarrow{\tau} A$$

equals the identity.

Morphisms $(A, \tau_A) \longrightarrow (B, \tau_B)$ *are given by morphisms* $f : A \longrightarrow B$ *in* $\mathcal{C}$ *such that the diagram*

$$\begin{tikzcd}
\varphi A \arrow[r, "\tau_A"] \arrow[d, "\varphi f"] & A \arrow[d, "f"] \\
\varphi B \arrow[r, "\tau_B"] & B
\end{tikzcd}$$

commutes.

In the example of $Vect_{\mathbb{C}}$, fixed points are complex vector spaces with a conjugation-antilinear involution. This can be identified with the category $Vect_{\mathbb{R}}$ of finite dimensional real vector spaces ('Galois descent').

Remark 3.4. *For a category with an endofunctor φ without any further conditions, there is another notion of category of fixed points, which is the subcategory of all objects and morphisms on which φ acts as the identity. This notion is not well-suited for our purposes since it is too strict. In the example of $(Vect_{\mathbb{C}}, \varphi')$, the category of fixed points in this sense consists of zero-dimensional vector spaces.*

In the rest of this chapter, we adopt the notations and conventions of chapter 2.

3.1 Triples of Opposite Filtrations

References for this section are [Del71a], [Del71b] and [Del94]. The following definition has essentially been given by Deligne, although in its present form, it is found in [Sim97a].

Definition 3.5. *A* ***Complex Mixed Hodge Structure*** *is a tuple (V, W, F_1, F_2), where*

- *V is a finite dimensional complex vector space,*
- *W is a separated and exhaustive increasing filtration on V, called the* ***weight filtration****,*
- *F_1, F_2 are separated and exhaustive decreasing filtrations on V, called the* ***Hodge filtrations****,*
- *for every $n \in \mathbb{Z}$, the space $\operatorname{gr}_n^W V$ with filtrations induced by F_1, F_2 is pure of weight n in the sense of definition 1.17, i.e., the implication*

  $$\operatorname{gr}_{F_1}^p \operatorname{gr}_{F_2}^q \operatorname{gr}_n^W V \neq 0 \Rightarrow p + q = n$$

  holds.

Together with linear maps respecting all three filtrations, these objects form the ***category of Complex Mixed Hodge Structures****. It is denoted by* $\mathrm{MHS}_{\mathbb{C}}$.

The last condition is also referred to as 'W, F_1, F_2 form a triple of opposite filtrations'.

There is an involution on $\mathrm{MHS}_{\mathbb{C}}$ given in the following way:

$$(V, W, F_1, F_2) \longmapsto (\overline{V}, \overline{W}, \overline{F}_2, \overline{F}_1)$$

The category of fixed points of this involution is called the category of **Real Mixed Hodge Structures** and is denoted $\mathrm{MHS}_{\mathbb{R}}$. An object in $\mathrm{MHS}_{\mathbb{R}}$ is given by an object (V, W, F_1, F_2) of $\mathrm{MHS}_{\mathbb{C}}$ with a conjugation antilinear involution fixing W and interchanging F_1 with F_2. Thus, $\mathrm{MHS}_{\mathbb{R}}$ can be identified with the category of real vector spaces V together with an ascending filtration W on V

and a descending filtration F on $V_{\mathbb{C}} := V \otimes \mathbb{C}$ such that $(V_{\mathbb{C}}, W_{\mathbb{C}}, F, (\mathrm{Id} \otimes \sigma)F)$, where σ denotes complex conjugation, form a Complex Mixed Hodge Structure.

In general, the three filtrations of a Complex Mixed Hodge Structure do not split. However, there is a functorial splitting of the weight filtration and either one of the Hodge filtrations.

Proposition 3.6. *Let $(V, W_., F_1^., F_2^.)$ be a complex MHS. For $p, q \in \mathbb{Z}$ define subspaces of V by*

$$\begin{aligned} I_{F_1}^{p,q} &:= (W_{p+q} \cap F_1^p) \cap \left((W_{p+q} \cap F_2^q) + \sum_{i \geq 0} (W_{p+q-i} \cap F_2^{q-i+1}) \right), \\ I_{F_2}^{p,q} &:= (W_{p+q} \cap F_2^q) \cap \left((W_{p+q} \cap F_1^p) + \sum_{i \geq 0} (W_{p+q-i} \cap F_1^{p-i+1}) \right). \end{aligned}$$

Then, for $i = 1$ or $i = 2$, we have

$$V = \bigoplus_{p,q} I_{F_i}^{p,q}$$

and $I_{F_i}^{\bullet,\bullet}$ is a splitting for (W, F_i) in the following sense:

$$\begin{aligned} W_n V &= \bigoplus_{p+q \leq n} I_{F_i}^{p,q} \qquad \text{for } i = 1 \text{ or } 2 \\ F_1^p V &= \bigoplus_{p' \geq p} I_{F_1}^{p',q} \\ F_2^q V &= \bigoplus_{q' \geq q} I_{F_2}^{p,q'} \end{aligned}$$

These splittings are (by definition) functorial with respect to morphisms in $\mathrm{MHS}_{\mathbb{C}}$.

An easy consequence of the functoriality of these splittings is the fact that morphisms in $\mathrm{MHS}_{\mathbb{C}}$ are automatically strict.[1] This, in turn, implies that the category is abelian. Recall that a **neutral tannakian category**[2] (over a field k) is a k-linear rigid[3] abelian tensor category s.t. the endomorphisms of the unit object are identified with k. A k-linear faithful exact tensor functor from a neutral tannakian category to k-vector spaces is called a **fibre functor**.

Theorem 3.7. *With the tensor product of 3-filtered vector spaces, the categories* $\mathrm{MHS}_{\mathbb{C}}$ *and* $\mathrm{MHS}_{\mathbb{R}}$ *are neutral tannakian categories. The functors*

$$(V, W, F_1, F_2) \longmapsto V \qquad \text{'underlying vector space'}$$

---

[1] They are even 2-strict with respect to the weight filtrations and either one of the Hodge-filtrations. However, they are **not** 2-strict with respect to the two Hodge-filtrations.

[2] See [DM82].

[3] I.e. internal hom's exist, are well behaved with respect to the tensor product and every object is reflexive, i.e., isomorphic to its double dual.

and

$$(V, W, F_1, F_2) \longmapsto \bigoplus_{k\in\mathbb{Z}} \operatorname{gr}_k^W V \qquad\qquad \text{'associated graded'}$$

are fibre functors.

Let us give a description of the structure of $\mathrm{MHS}_{\mathbb{C}}$ and $\mathrm{MHS}_{\mathbb{R}}$. This is not completely included in the references given, but elementary:

For any integer $k \in \mathbb{Z}$, $\mathrm{MHS}_{\mathbb{C}}$ (resp. $\mathrm{MHS}_{\mathbb{R}}$) has the full abelian subcategories $\mathrm{HS}_{\mathbb{C}}^k$ (resp. $\mathrm{HS}_{\mathbb{R}}^k$) of objects such that the weight filtration has a single jump in degree k. Such objects are called (complex or real) **Pure Hodge Structures of weight** k. In fact, the bifiltered vector space (H, F_1, F_2) is pure for an object (H, W, F_1, F_2) in $\mathrm{HS}_{\mathbb{C}}^k$. Thus, we will identify $\mathrm{HS}_{\mathbb{C}}^k$ with the category of bigraded vector spaces $H^{\bullet,\bullet}$ s.t. $p + q = k$ for all nonzero $H^{p,q}$. Similarly, $\mathrm{HS}_{\mathbb{R}}^k$ can be identified with the category of real vector spaces H with a bigrading on $H_{\mathbb{C}} := H \otimes \mathbb{C}$ s.t. $H_{\mathbb{C}}$ is in $\mathrm{HS}_{\mathbb{C}}^k$ and $(\mathrm{Id} \otimes \sigma)H^{p,q} = H^{q,p}$, but we will usually use the description as objects in $\mathrm{HS}_{\mathbb{C}}^k$ with an antilinear involution switching the bigrading. We denote by $\mathrm{HS}_{\mathbb{C}}$ (resp. $\mathrm{HS}_{\mathbb{R}}$) the smallest subcategory of $\mathrm{MHS}_{\mathbb{C}}$ (resp. $\mathrm{MHS}_{\mathbb{R}}$) which is closed under direct sums and contains the subcategories $\mathrm{HS}_{\mathbb{C}}^k$ (resp. $\mathrm{HS}_{\mathbb{R}}^k$) for all $k \in \mathbb{Z}$.

Example 3.8. *Let $p, q \in \mathbb{Z}$ be integers.*

- *The pair (p, q) defines a one-dimensional object $H(p, q)$ in $\mathrm{HS}_{\mathbb{C}}^{p+q}$: The one-dimensional vector space $\mathbb{C}$ sitting in bidegree (p, q).*
- *The object $H(-p, -p)$ considered with conjugation as an involution defines an object $T(2p)$ in $\mathrm{HS}_{\mathbb{R}}^{-2p}$. This is called the* ***Tate-structure****.*[4]
- *The space $H(p, q) \oplus H(q, p)$ equipped with the antilinear involution exchanging the two factors defines a 2-dimensional object $\tilde{H}(p, q)$ in $\mathrm{HS}_{\mathbb{R}}^{p+q}$. One has $\tilde{H}(p, q) \cong \tilde{H}(r, s)$ iff $\{p, q\} = \{r, s\}$.*

In fact, these examples are essentially all:

Proposition 3.9. *Let $k \in \mathbb{Z}$ be an integer.*

1. *The categories $\mathrm{HS}_{\mathbb{C}}^k$ and $\mathrm{HS}_{\mathbb{R}}^k$ are semisimple.*
2. *An object in $\mathrm{HS}_{\mathbb{C}}^k$ is simple iff it is isomorphic to some $H(p, q)$ with $p+q = k$.*
3. *An object in $\mathrm{HS}_{\mathbb{R}}^k$ is simple iff it is isomorphic to $T(-k)$ or $\tilde{H}(p, q)$ for some $p, q \in \mathbb{Z}$ with $p + q = k$ and $p \neq q$.*

It is an elementary observation that *the weight filtration is a filtration by subobjects*, i.e., for any (real or complex) Mixed Hodge Structure (V, W, F_1, F_2) and any $n \in \mathbb{Z}$, the subspace $W_n V$ with induced filtrations is again a Mixed Hodge Structure. In particular, a simple object has to lie in one of the $\mathrm{HS}_{\mathbb{C}}^k$ (resp. $\mathrm{HS}_{\mathbb{R}}^k$) and an object is simple there if and only if it is simple in $\mathrm{MHS}_{\mathbb{C}}$ (resp. $\mathrm{MHS}_{\mathbb{R}}$). In summary:

---

[4] That $T(k)$ has weight $-k$ is not a typo, but standard convention.

Proposition 3.10. *Let $K = \mathbb{R}$ or $K = \mathbb{C}$. An object in MHS_K is simple iff it is simple in HS_K^k for some $k \in \mathbb{Z}$, i.e., HS_K is the subcategory consisting of all semisimple objects. Every object in MHS_K has a finite composition series refining the weight filtration.*

In fact, the extensions between two Mixed Hodge Structures can also be described explicitely and higher Ext-groups vanish (see [Car79] and $I.3.5$ of [PS08]).

3.2 Vector Spaces with an Endomorphism

This section gives a review of parts of [Del94]. Recall from the previous section that we identify the category of Complex Pure Hodge Structures $\mathrm{HS}_{\mathbb{C}}$ with the category of finite-dimensional bigraded complex vector spaces.

Given a Complex Mixed Hodge Structure (V, W, F_1, F_2) recall that $V_W := \bigoplus_{k\in\mathbb{Z}} \mathrm{gr}_k^W V$ lies in $\mathrm{HS}_{\mathbb{C}}$. Let $\alpha_{F_i}^{p,q}$ denote the projections $I_{F_i}^{p,q} \longrightarrow V_W^{p,q}$, where we use the notations from proposition 3.6. Because the $I_{F_i}^{\bullet,\bullet}$ are a splitting, it follows that the $\alpha_{F_i}^{p,q}$ are isomorphisms. Hence, so are the direct sums $\alpha_i := \bigoplus_{p,q\in\mathbb{Z}} \alpha_{F_i}^{p,q}$. As a result, the composition

$$d_V := \alpha_2\alpha_1^{-1}$$

is an automorphism of V_W (not respecting the grading), which we call the **Deligne operator** (of V). This construction is functorial and can be shown to take values in category $\mathcal{DO}_{\mathbb{C}}^{exp}$ defined below:

Definition 3.11. 1. *For V an object in $\mathrm{HS}_{\mathbb{C}}$, denote*

$$\mathrm{End}^{\mathrm{Del}}(V) := \left\{ D \in \mathrm{End}_{\mathbb{C}\text{-}vs}(V) \;\middle|\; \forall p,q \in \mathbb{Z} : DV^{p,q} \subseteq \bigoplus_{\substack{p'<p \\ q'<q}} V^{p',q'} \right\}.$$

2. *Let $\mathcal{DO}_{\mathbb{C}}$ be the category of pairs (V, D), where $V \in \mathrm{HS}_{\mathbb{C}}$ and $D \in \mathrm{End}^{\mathrm{Del}}(V)$. Morphisms $(V, D_V) \longrightarrow (W, D_W)$ are linear maps $f : V \longrightarrow W$ s.t. $f \circ D_V = D_W \circ f$.*

3. *Let $\mathcal{DO}_{\mathbb{C}}^{\mathrm{exp}}$ be the category of pairs (V, d) s.t. $V \in \mathrm{HS}_{\mathbb{C}}$ and $d = \exp D$ for some $D \in \mathrm{End}^{\mathrm{Del}}(V)$. Morphisms are as in 2.*

Note that endomorphisms $D \in \mathrm{End}^{\mathrm{Del}}(V)$ as above are in particular nilpotent. Hence, their exponential is unipotent and one has the easier criterion

$$d = \exp D \text{ for some } D \in \mathrm{End}^{\mathrm{Del}}(V) \Longleftrightarrow (d-1) \in \mathrm{End}^{\mathrm{Del}}(V).$$

$\mathcal{DO}_{\mathbb{C}}$ has the structure of a tensor category via

$$(V, D) \otimes (V', D') := (V \otimes V', \mathrm{Id}_V \otimes D' + D \otimes \mathrm{Id}_{V'})$$

and similarly for $\mathcal{DO}_{\mathbb{C}}^{\mathrm{exp}}$

$$(V, d) \otimes (V', d') := (V \otimes V', d \otimes d').$$

Recall the involution on $\mathrm{HS}_{\mathbb{C}}$ is given by

$$V = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q} \longmapsto \overline{V} := \bigoplus_{p,q\in\mathbb{Z}} \overline{V}^{p,q} \text{ where } \overline{V}^{p,q} := \overline{V^{q,p}}.$$

The category $\mathcal{DO}_{\mathbb{C}}$ is equipped with the involution:

$$(V, D) \longmapsto (\overline{V}, -D)$$

and the category $\mathcal{DO}_{\mathbb{C}}^{\exp}$ with the involution

$$(V, d) \longmapsto (\overline{V}, d^{-1}).$$

We denote the category of fixed points by $\mathcal{DO}_{\mathbb{R}}$, (resp. $\mathcal{DO}_{\mathbb{R}}^{\exp}$). $\mathcal{DO}_{\mathbb{R}}$ can be identified with the category of pairs (V, D), where V is real vector space with a bigrading on the complexification $V_{\mathbb{C}}$ and $D \in \mathrm{End}^{\mathrm{Del}}(V_{\mathbb{C}})$ such that

$$(\mathrm{Id}\otimes\sigma)V_{\mathbb{C}}^{p,q} = V_{\mathbb{C}}^{q,p} \text{ and } (\mathrm{Id}\otimes\sigma)D(\mathrm{Id}\otimes\sigma) = -D,$$

where σ is complex conjugation. There is a similar description for $\mathcal{DO}_{\mathbb{R}}^{\exp}$.

Theorem 3.12. *The categories $\mathcal{DO}_{\mathbb{C}}$, $\mathcal{DO}_{\mathbb{C}}^{\exp}$ and $\mathrm{MHS}_{\mathbb{C}}$ are equivalent as $\mathbb{C}$-linear tensor categories with involution. In particular, the categories $\mathcal{DO}_{\mathbb{R}}$, $\mathcal{DO}_{\mathbb{R}}^{\exp}$ and $\mathrm{MHS}_{\mathbb{R}}$ are equivalent.*

Proof. The equivalence between $\mathcal{DO}_{\mathbb{C}}$ and $\mathcal{DO}_{\mathbb{C}}^{\exp}$ is given by sending (V, D) to $(V, \exp D)$.
We merely recall here how to get back from $\mathcal{DO}_{\mathbb{C}}^{\exp}$ to $\mathrm{MHS}_{\mathbb{C}}$: Given (V, d), set

$$W_{\bullet} := \bigoplus_{p+q\leq\bullet} V^{p,q},$$

$$F_W^{\bullet} := \bigoplus_{p\geq\bullet} V^{p,q}$$

and

$$\overline{F}_W^{\bullet} := \bigoplus_{q\geq\bullet} V^{p,q}.$$

Then,

$$(V, W, d^{-\frac{1}{2}}(F_W), d^{\frac{1}{2}}(\overline{F}_W))$$

is a Complex Mixed Hodge Structure. The roots ensure that this also respects real structures. For a full proof, see [Del94]. ☐

Note that under these equivalences, direct sums of Pure Hodge Structures correspond to pairs $(V, 0)$. The notions of opposed filtrations and Deligne operator can thus be seen as two different (although equivalent) ways of embedding the category $\mathrm{HS}_{\mathbb{C}}$ or $\mathrm{HS}_{\mathbb{R}}$ into an abelian category which admits interesting extensions.

3.3 Representations of a Pro-algebraic Group

By general theory of tannakian categories, $\mathrm{MHS}_{\mathbb{C}}$ and $\mathrm{MHS}_{\mathbb{R}}$ are equivalent to categories of representations of pro-algebraic groups. These groups were calculated by Deligne in [Del94] and we briefly recall here the result, following the original source almost verbatim.

Let $\mathfrak{L}_n$ be the free complex Lie algebra on generators $D^{i,j}$ where $i, j < 0$ with grading s.t. $D^{i,j}$ has degree (i, j). Denote by $W_\bullet$ the ascending filtration by total degree., i.e.,

$$W_n\mathfrak{L} := \bigoplus_{p+q\leq n} \mathfrak{L}^{p,q}.$$

Define a pro-algebraic group $\mathfrak{U}$ by

$$\mathfrak{U} := \varprojlim_{n\in\mathbb{Z}} \exp(\mathfrak{L}/W_n\mathfrak{L}).$$

The grading induces an action of $\mathbb{G}_m^2$ on $\mathfrak{L}$ which respects $W_\bullet$ and hence induces a grading on $\mathfrak{U}$. Thus, one can define the semidirect product

$$\mathfrak{G} := \mathfrak{U} \rtimes \mathbb{G}_m^2.$$

There is an involution on $\mathfrak{G}$ defined as follows:

- On $\mathbb{G}_m^2(\mathbb{C})$ it is given as $(\lambda, \mu) \mapsto (\overline{\mu}, \overline{\lambda})$
- On $\mathfrak{L}$ it is determined by conjugation-antilinearity and $D^{i,j} \mapsto -D^{j,i}$ for all $i, j \in \mathbb{Z}_{<0}$.

This determines a real form $\mathfrak{M}$ of $\mathfrak{G}$.

A representation of a semidirect product is determined by representations of both factors satisfying a certain compatibility condition. Using this, one obtains:

Theorem 3.13. *The category of representations of* $\mathfrak{G}$ *(resp.* $\mathfrak{M}$*) is equivalent to* $\mathcal{DO}_{\mathbb{C}}^{\exp}$ *(resp.* $\mathcal{DO}_{\mathbb{R}}^{\exp}$*).*

The category $\mathrm{HS}_{\mathbb{C}}$ (resp. $HS_{\mathbb{R}}$) of pure Hodge structures corresponds to the subcategory of those representations that factor over $\mathbb{G}_m^2$ (resp. the Weil-restriction $\mathrm{Res}_{\mathbb{C}|\mathbb{R}}\,\mathbb{G}_m$, which is the real form of $\mathbb{G}_m^2$ defined by the above involution).

3.4 Filtered Equivariant Bundles on $\mathbb{P}^1_{\mathbb{C}}$

The reference for this chapter is [Sim97a].

To a Complex Mixed Hodge Structure (V, W, F_1, F_2), we can associate the nested sequence of Rees-bundles on $\mathbb{P}^1 := \mathbb{P}^1_{\mathbb{C}}$:

$$\ldots \hookrightarrow \xi_{\mathbb{P}^1}(W_nV, F_1, F_2) \hookrightarrow \xi_{\mathbb{P}^1}(W_{n+1}V, F_1, F_2) \hookrightarrow \ldots$$

This can also be considered as a single equivariant filtered vector bundle. Furthermore, by lemma 2.13, since the inclusions $W_nV \subseteq W_{n+1}V$ are strict as maps of filtered vector spaces, we have

$$\frac{\xi_{\mathbb{P}^1}(W_nV, F_1, F_2)}{\xi_{\mathbb{P}^1}(W_{n-1}V, F_1, F_2)} \cong \xi_{\mathbb{P}^1}(\operatorname{gr}_n^W V, F_1, F_2).$$

In particular, this is again a vector bundle and so this process yields an embedding of $\mathrm{MHS}_{\mathbb{C}}$ into the category of equivariant vector bundles on $\mathbb{P}^1_{\mathbb{C}}$ with a separated and exhaustive filtration by strict subbundles. The essential image of this has been described in [Sim97a]:

Definition 3.14. *We call* ***Simpson-bundles*** *and denote by* $\mathrm{Simp}_{\mathbb{C}}(\mathbb{P}^1, \mathbb{G}_m)$ *the category which has*

- *as* ***objects*** *equivariant vector bundles* $\mathcal{V}$ *on* $\mathbb{P}^1$*, together with a separated and exhaustive ascending filtration* $\mathcal{W}$ *by strict equivariant subbundles such that the associated graded bundles* $\operatorname{gr}_n^{\mathcal{W}} \mathcal{V}$ *are semistable of slope n,*
- *as* ***morphisms*** *morphisms of equivariant sheaves that respect the filtration.*

The tensor product of vector bundles induces the structure of a tensor category on $\mathrm{Simp}_{\mathbb{C}}(\mathbb{P}^1, \mathbb{G}_m)$.

There is a $\mathbb{C}$-antilinear involution on $\mathbb{P}^1$, which is given on points by

$$\tau_{\mathbb{P}^1} : [z_1 : z_2] \longmapsto [\overline{z_2} : \overline{z_1}].$$

On $\mathbb{G}_m(\mathbb{C})$, this acts as $\lambda \longmapsto \overline{\lambda}^{-1}$. By pullback, this also acts on bundles over $\mathbb{P}^1$ and hence on $\mathrm{Simp}_{\mathbb{C}}(\mathbb{P}^1, \mathbb{G}_m)$. We denote the category of fixed points by $\mathrm{Simp}_{\mathbb{R}}(\mathbb{P}^1, \mathbb{G}_m)$.

Remark 3.15. *Simpson also considers the different involution*

$$\sigma_{\mathbb{P}^1} : \lambda \mapsto -\overline{\lambda}^{-1}.$$

Since the subgroups of the automorphism group of $\mathbb{P}^1(\mathbb{C})$ *generated by* $\mathbb{C}^\times$ *and either* $\sigma_{\mathbb{P}^1}$ *or* $\tau_{\mathbb{P}^1}$ *are the same, the distinction is not important for our purposes. It does, however, make a difference if one considers filtered vector bundles with the same semistability conditions without* $\mathbb{G}_m$*-equivariance condition, so-called Mixed Twistor Structures. These are the main topic in [Sim97a] and have applications, e.g., in the theory of representations of the fundamental group of complex algebraic varieties. We do not consider them any further.*

Theorem 3.16. *The categories* $\mathrm{MHS}_{\mathbb{C}}$ *and* $\mathrm{Simp}_{\mathbb{C}}(\mathbb{P}^1, \mathbb{G}_m)$ *are equivalent via the above Rees-bundle construction as* $\mathbb{C}$*-linear tensor categories equipped with an involution. The fibre functor 'underlying vector space' on* $\mathrm{MHS}_{\mathbb{C}}$ *corresponds to taking invariants over* $\mathbb{G}_m$*:*

$$\mathcal{V} \longmapsto H^0(\mathbb{G}_m, \mathcal{V})^{\mathbb{G}_m}$$

Proof. The compatibility with the involution is straightforward and compatibility with the tensor product is a general fact for the Rees-bundle construction. It remains to show that the condition of opposedness on the filtrations translates as stated. This is the subject of the following

Lemma 3.17. *For a vector space H with two filtrations $F^\bullet, G^\bullet$ the following conditions are equivalent:*

1. *(H, F, G) is pure of weight n.*
2. *There is an isomorphism as vector bundles*
$$\xi_{\mathbb{P}^1}(H, F^\bullet, G^\bullet) \cong H \otimes \mathcal{O}_{\mathbb{P}^1}(n).$$
3. *$\xi_{\mathbb{P}^1}(H, F^\bullet, G^\bullet)$ is semistable of slope n.*

Pick a splitting $H = \bigoplus_{p,q\in\mathbb{Z}} H^{p,q}$ for the two filtrations. By proposition 2.19, this induces an isomorphism

$$\xi_{\mathbb{P}^1}(H, F^\bullet, G^\bullet) \cong \bigoplus_{p,q\in\mathbb{Z}} H^{p,q} \otimes \mathcal{O}_{\mathbb{P}^1}(p,q).$$

Since $\mathcal{O}_{\mathbb{P}^1}(p,q) \cong \mathcal{O}_{\mathbb{P}^1}(p+q)$ as vector bundles, this implies the result of the lemma and hence the theorem. ☐

3.5 Equivariant Bundles on $\mathbb{P}^2_{\mathbb{C}}$

References for this chapter are [Pen03] and [Kap12].

Starting from a Mixed Hodge Structure (V, W, F_1, F_2), one can consider the projective Rees-bundle $\xi_{\mathbb{P}^2}(V)$ on $\mathbb{P}^2 := \mathbb{P}^2_{\mathbb{C}} = \mathrm{Proj}(\mathbb{C}[X_0, X_1, X_2])$ considering W as a decreasing filtration via $W^\bullet := W_{-\bullet}$. According to corollary 2.17, this yields an embedding of $\mathrm{MHS}_{\mathbb{C}}$ into the category $\mathrm{Bun}(\mathbb{P}^2, \mathbb{G}_m^2)$. O. Penacchio, a student of C. Simpson, studied this in his thesis [Pen03] and determined the essential image to be the following category:

Definition 3.18. *We call* ***Penacchio-bundles*** *and denote by* $\mathrm{Penac}_{\mathbb{C}}(\mathbb{P}^2, \mathbb{G}_m^2)$ *the category which has*

- *as* ***objects*** *equivariant vector bundles $\mathcal{V}$ on $\mathbb{P}^2$ that are (non-equivariantly) trivial on $Z_0 := \{X_0 = 0\}$*
- *as* ***morphisms*** *morphisms of equivariant sheaves.*

Again, we view this as a tensor category with the tensor product induced by that of vector bundles.

There is an involution on this category induced by the involution of $\mathbb{P}^2$ given on points by $[z_0 : z_1 : z_2] \longmapsto [\overline{z_0} : \overline{z_2} : \overline{z_1}]$. We denote the category of fixed points by $\mathrm{Penac}_{\mathbb{R}}(\mathbb{P}^2, \mathbb{G}_m^2)$.

Theorem 3.19. *The categories* $\mathrm{MHS}_{\mathbb{C}}$ *and* $\mathrm{Penac}_{\mathbb{C}}(\mathbb{P}^2, \mathbb{G}_m^2)$ *are equivalent as $\mathbb{C}$-linear tensor categories with involution. In particular,* $\mathrm{MHS}_{\mathbb{R}}$ *and* $\mathrm{Penac}_{\mathbb{R}}(\mathbb{P}^2, \mathbb{G}_m^2)$ *are equivalent. The fibre functor 'underlying vector space' is identified with taking invariants over the open $\mathbb{G}_m^2$-orbit:*

$$\mathcal{V} \longmapsto H^0(\mathbb{G}_m^2, \mathcal{V})^{\mathbb{G}_m^2}$$

The fibre functor 'associated graded' is identified with global sections of the restriction to Z_0:

$$\mathcal{V} \longmapsto H^0(Z_0, \mathcal{V})$$

Proof. The key point is to show how the condition of opposed filtrations translates to bundles under the Rees-equivalence. Let V be a vector space with one increasing filtration $W_\bullet$ and two decreasing filtrations $F_1^\bullet, F_2^\bullet$. By the restriction properties from proposition 2.18, we get

$$\xi_{\mathbb{P}^2}(V)|_{Z_0} \cong \bigoplus_{k\in\mathbb{Z}} \xi_{\mathbb{P}^1}(\mathrm{gr}_{-k}^W V, F_1, F_2) \otimes \mathcal{O}_{\mathbb{P}^1}(k).$$

Now one concludes using lemma 3.17. □

For later use, we record in more detail the description of the fibre functor 'associated graded', which follows from the isomorphism given in the proof:

Corollary 3.20. *Given a Mixed Hodge Structure* (V, W, F_1, F_2)*, there is a canonical equivariant identification*

$$H^0(Z_0, \xi_{\mathbb{P}^2}(V)) \cong V_W,$$

where $V_W := \bigoplus \mathrm{gr}_k^W V$ *is equipped with the* $\mathbb{G}_m^2 = \mathrm{Spec}(\mathbb{C}[\frac{X_1}{X_0}, \frac{X_2}{X_0}])$*-action given on* $v \in V_W^{p;q}$ *by the coaction*

$$v \mapsto v \otimes \left(\frac{X_1}{X_0}\right)^{-p} \left(\frac{X_2}{X_0}\right)^{-q}.$$

The following two observations are due to Kapranov [Kap12], the second one being implicit in [Pen03] as well:

Lemma 3.21. *An equivariant bundle on* $\mathbb{P}^2$ *that is trivial on* Z_0 *is also trivial restricted to each line not meeting* $[1:0:0]$.

Proof. Let $\check{\mathbb{P}}^2 = \mathrm{Proj}(\mathbb{C}[\check{X}_0, \check{X}_1, \check{X}_2])$ denote the dual projective space, i.e., the space of lines in $\mathbb{P}^2$. It is again a projective 2-space. A point $[a:b:c] \in \check{\mathbb{P}}^2(\mathbb{C})$ corresponds to the line $\ell_{[a:b:c]} = \{aX_0 + bX_1 + cX_2 = 0\}$ in the original $\mathbb{P}^2$ and vice versa. The dual projective space inherits a $\mathbb{G}_m^3/\Delta(\mathbb{G}_m)$-action from the original one given on points by

$$[\lambda_1 : \lambda_2 : \lambda_3].[a:b:c] := [\lambda_1^{-1}a : \lambda_2^{-1}b : \lambda_3^{-1}c].$$

It is shown in [OSS11, lemma 3.2.2] that for a vector bundle $\mathcal{V}$ on $\mathbb{P}^2$ of rank n the function

$$a_{\mathcal{V}} : \check{\mathbb{P}}^2_{\mathbb{C}} \longrightarrow \mathbb{Z}^n \qquad \ell \longmapsto \text{splitting type of } \mathcal{V}|_\ell$$

is upper semicontinuous. The splitting type is the unique tuple $a = (a_1, ..., a_n)$ with $a_i \geq a_{i+1}$ s.t. $\mathcal{V}|_\ell \cong \bigoplus_{i=1}^n \mathcal{O}(a_i)$ given by theorem 2.21. Upper semicontinuity means that for every $b = (b_1, ..., b_n)$ the sets $A(\mathcal{V}, b) := \{\ell \mid a_{\mathcal{V}}(\ell) \geq (b_1, ..., b_n)\}$ are closed, where $\mathbb{Z}^n$ is equipped with the lexicographic order (contrary to our overall assumptions on multiindices, this will be the case in the whole proof). Equivalently, $U(\mathcal{V}, b) = A(\mathcal{V}, b)^c = \{\ell \mid a_{\mathcal{V}}(\ell) \leq (b_1, ..., b_n)\}$ is open.

For vector bundles restriction to a subvariety commutes with dualising. Hence we have for fixed $\mathcal{V}$, ℓ as above with $a_{\mathcal{V}}(\ell) = (a_1, ..., a_n)$ the equality $a_{\mathcal{V}^*}(\ell) = (-a_n, ..., -a_1)$. In particular, the (open) intersection $U_{triv} := U(\mathcal{V}, (0, ..., 0)) \cap$

$U(\mathcal{V}^*, (0, ..., 0))$ consists of those lines on which the bundle is trivial: For ℓ in the intersection one has $0 \leq a_n \leq a_i \leq a_1 \leq 0$ and therefore, equality.

In general on any G-variety X the restriction of a G-equivariant bundle to a subvariety is isomorphic to the restriction of any of its G-translates. In particular, U_{triv} is a $G_m^3/\Delta(\mathbb{G}_m)$-invariant set. But then it has to contain $\{\check{X}_0 \neq 0\}$ as this is the smallest invariant set containing the point $[1:0:0] \in \check{\mathbb{P}}^2(\mathbb{C})$ corresponding to Z_0. This is what was claimed since a line in $\mathbb{P}^2$ meeting the point $[1:0:0]$ corresponds to a point $[0:b:c] \in \check{\mathbb{P}}^2(\mathbb{C})$. □

Lemma 3.22. *Morphisms in* $\mathrm{Penac}_{\mathbb{C}}(\mathbb{P}^2, \mathbb{G}_m^2)$ *have constant rank in every point except possibly in* $[1:0:0]$.

Proof. Let $\varphi : \mathcal{V} \longrightarrow \mathcal{W}$ be a morphism in $\mathrm{Penac}_{\mathbb{C}}(\mathbb{P}^2, \mathbb{G}_m^2)$. We give two variants to prove the statement, the second one being due to Kapranov.

For the first, use theorem 3.19, i.e., that φ comes from a morphism of Mixed Hodge structures $f : (H, W, F_1, F_2) \longrightarrow (H', W', F_1', F_2')$. By functoriality of the Deligne splittings (proposition 3.6), f is 2-strict for the pairs $(W, F_i), (W', F_i')$. As noted in remark 2.15, this means that the cokernel is locally free on the sets U_1 and U_2, i.e., everywhere except possibly $[1:0:0]$.

Alternatively, note that a morphism between trivial bundles on $\mathbb{P}^n$ always has constant rank as it can be described by a matrix with constant entries. In particular, as seen in lemma 3.21, φ has constant rank on every line in $\mathbb{P}^2 \backslash \{[1:0:0]\}$. But any two points in $\mathbb{P}^2 \backslash \{[1:0:0]\}$ can be connected by finitely many such lines, so the rank is constant everywhere except possibly in $[1:0:0]$. □

Summarising these two lemmas yields:

Proposition 3.23. *The category* $\mathrm{Penac}_{\mathbb{C}}(\mathbb{P}^2, \mathbb{G}_m^2)$ *is equivalent as a* $\mathbb{C}$*-linear tensor category with involution to the category* $\mathrm{Penac}_{\mathbb{C}}(\mathbb{P}_0^2, \mathbb{G}_m^2)$ *of equivariant vector bundles on* $\mathbb{P}_0^2 := \mathbb{P}^2 \backslash \{[1:0:0]\}$ *that are trivial on each line, with maps of equivariant vector bundles.*

Finally, let us describe how to obtain the Deligne operator directly from Penacchio's description. This has been shown in [Kap12]. Namely, given an object $\mathcal{V}$ of $\mathrm{Penac}(\mathbb{P}^2, \mathbb{G}_m^2)$, define $H_{\mathcal{V}} := \Gamma(Z_0, \mathcal{V})$ with bigrading induced by the $\mathbb{G}_m^2$-action as in corollary 3.20. Further define an endomorphism $d_{\mathcal{V}}$ of $H_{\mathcal{V}}$ by enforcing the commutativity of the following diagram:

$$\begin{tikzcd}
H_{\mathcal{V}} \arrow[r, "res"] \arrow[dd, "d_{\mathcal{V}}"] & \Gamma([0:0:1], \mathcal{V}) \arrow[r, "res^{-1}"] & \Gamma(\mathbb{P}^1_{(1,-1,0)}, \mathcal{V}) \arrow[d, "res"] \\
 & & \Gamma([1:1:1], \mathcal{V}) \arrow[d, "res^{-1}"] \\
H_{\mathcal{V}} & \Gamma([0:1:0], \mathcal{V}) \arrow[l, "res^{-1}"'] & \Gamma(\mathbb{P}^1_{(1,0,-1)}, \mathcal{V}) \arrow[l, "res"']
\end{tikzcd} \qquad (*)$$

Note that all restriction maps are isomorphisms by lemma 3.21. Let $\Phi_{\mathrm{MHS}}^{\mathrm{Del}}$ and $\Phi_{\mathrm{MHS}}^{\mathrm{Penac}}$ be the functors from $\mathrm{MHS}_{\mathbb{C}}$ to $\mathcal{DO}_{\mathbb{C}}^{\exp}$ and $\mathrm{Penac}_{\mathbb{C}}(\mathbb{P}^2, \mathbb{G}_m^2)$ previously constructed.

Proposition 3.24. *The assignment $\mathcal{V} \mapsto (H_{\mathcal{V}}, d_{\mathcal{V}})$ defines a functor*

$$\Phi^{\mathrm{Del}}_{\mathrm{Penac}} : \mathrm{Penac}_{\mathbb{C}}(\mathbb{P}^2, \mathbb{G}_m^2) \longrightarrow \mathcal{DO}^{\exp}_{\mathbb{C}}$$

and there is a natural isomorphism $\Phi^{\mathrm{Del}}_{\mathrm{Penac}} \circ \Phi^{\mathrm{Penac}}_{\mathrm{MHS}} \cong \Phi^{\mathrm{Del}}_{\mathrm{MHS}}$, compatible with the involutions. In particular, $\Phi^{\mathrm{Del}}_{\mathrm{Penac}}$ is an equivalence of categories with involutions.

Proof. $\Phi^{\mathrm{Del}}_{\mathrm{Penac}}$ becomes a functor by sending maps to their restriction to the sections above Z_0. That it lands in the correct category is a byproduct of the compatibility statement $\Phi^{\mathrm{Del}}_{\mathrm{Penac}} \circ \Phi^{\mathrm{Penac}}_{\mathrm{MHS}} \cong \Phi^{\mathrm{Del}}_{\mathrm{MHS}}$, so let us prove this.

Consider the open sets $U_1 := \mathrm{Spec}(\mathbb{C}[r_0, r_2])$ and $U_2 := \mathrm{Spec}(\mathbb{C}[s_0, s_1])$, where $r_i := \frac{X_i}{X_1}$ and $s_i := \frac{X_i}{X_2}$. Let (V, W, F_1, F_2) be a Mixed Hodge Structure and denote by $\mathcal{V} := \xi_{\mathbb{P}^2}(\mathcal{DO}^{\exp}_{\mathbb{C}})$ the associated Rees-bundle on $\mathbb{P}^2$. The Deligne splittings $I^{p,q}_{F_j}$ $(j = 1, 2)$ from proposition 3.6 yield trivialisations with fibre $V_W := \mathrm{gr}^W_\bullet V$ over U_i for $i = 2, 1$, given on global sections

$$\Phi_{U_1} : \Gamma(U_1, \mathcal{V}) = \bigoplus_{p,q\in\mathbb{Z}} I^{p,q}_{F_2} \otimes r_0^{p+q} r_2^{-q} \mathbb{C}[r_0, r_2] \longrightarrow \bigoplus_{p,q\in\mathbb{Z}} V_W^{p,q} \otimes \mathbb{C}[r_0, r_2]$$

by applying $\alpha_2^{p,q}$ followed by multiplication with $r_0^{-(p+q)} r_2^q$ in each summand. Similarly,

$$\Phi_{U_2} : \Gamma(U_2, \mathcal{V}) = \bigoplus_{p,q\in\mathbb{Z}} I^{p,q}_{F_1} \otimes s_0^{p+q} s_1^{-q} \mathbb{C}[s_0, s_1] \longrightarrow \bigoplus_{p,q\in\mathbb{Z}} V_W^{p,q} \otimes \mathbb{C}[s_0, s_1]$$

is given by applying $\alpha_1^{p,q}$ followed by multiplication by $s_0^{-(p+q)} s_1^p$ in each summand. Computing with these explicit expressions, one can identify the top and bottom rows of the diagram $(*)$ with V_W and horizontal maps with the identity. On the other hand, over the open $\mathbb{G}_m^2$-orbit containing $[1 : 1 : 1]$, there is the canonical trivialisation with fibre V. Using this to describe $\Gamma([1 : 1 : 1], \mathcal{V})$, the vertical restriction maps are given by α_1^{-1} and α. ☐

3.6 Equivariant Bundles with Connection on $\mathbb{A}^2_{\mathbb{C}}$

This section revisits [Gon08] and [Kap12].

There is a further geometric description of Mixed Hodge Structures as a category of equivariant vector bundles with an equivariant connection on affine space. We refer to Appendix A for basics on equivariant connections. Let $\mathbb{A}^2 := \mathbb{A}^2_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[t_1, t_2])$ denote affine space equipped with the standard action by multiplication coming from the inclusion $\mathbb{G}_m^2 := \mathbb{G}_{m,\mathbb{C}}^2 \subseteq \mathbb{A}^2$.

Definition 3.25. *Let $\mathrm{Bun}_\nabla(\mathbb{A}^2, \mathbb{G}_m^2)$ denote the category which has as*

- ***objects**: $\mathbb{G}_m^2$-equivariant vector bundles $(\mathcal{V}, \nabla)$ equipped with an equivariant connection,*
- ***morphisms** from $(\mathcal{V}, \nabla_{\mathcal{V}})$ to $(\mathcal{W}, \nabla_{\mathcal{W}})$: morphisms of equivariant sheaves $\mathcal{V} \longrightarrow \mathcal{W}$ that are flat.*

This inherits the structure of a tensor category from the tensor product of connections, i.e., for two bundles $(\mathcal{V}, \nabla_{\mathcal{V}}), (\mathcal{W}, \nabla_{\mathcal{W}})$ one equips $\mathcal{V} \otimes \mathcal{W}$ with the unique connection $\nabla_{\mathcal{V}\otimes\mathcal{W}}$ satisfying for all local sections s of $\mathcal{V}$ and s' of $\mathcal{W}$

$$\nabla_{\mathcal{V}\otimes\mathcal{W}}(s \otimes s') = \nabla_{\mathcal{V}}(s) \otimes s' + s \otimes \nabla_{\mathcal{W}}(s').$$

There is an involution on $\mathbb{A}^2$ given on complex points by $(a, b) \mapsto (\bar{b}, \overline{a})$ and this induces, by pullback, an involution on the category $\mathrm{Bun}_\nabla(\mathbb{A}^2, \mathbb{G}_m^2)$. We denote the category of fixed points by $\mathrm{Bun}_\nabla^{\mathbb{R}}(\mathbb{A}^2, \mathbb{G}_m^2)$. It can be identified with the category $\mathrm{Bun}_\nabla(\mathbb{A}_{\mathbb{R}}^2, \mathbb{S})$ of $\mathbb{S} := Res_{\mathbb{C}|\mathbb{R}}\mathbb{G}_m$-equivariant bundles over $\mathbb{A}_{\mathbb{R}}^2 = \mathrm{Res}_{\mathbb{C}|\mathbb{R}} \mathbb{A}^1$ with an equivariant connection.

Theorem 3.26. *The categories* $\mathrm{MHS}_{\mathbb{C}}$ *and* $\mathrm{Bun}_\nabla(\mathbb{A}^2, \mathbb{G}_m^2)$ *are equivalent as* $\mathbb{C}$*-linear tensor categories with an involution. The subcategory* $\mathrm{HS}_{\mathbb{C}}$ *of* $\mathrm{MHS}_{\mathbb{C}}$ *corresponds to the subcategory of flat connections.*

Remark 3.27. *Since sheaf-theoretic kernel and cokernel of vector bundles with connections are again canonically equipped with a connection and a coherent sheaf with connection is automatically locally free,*[5] *this also yields a different proof that the category of Mixed Hodge Structures is abelian.*

There are several ways to prove this, all somewhat more involved than the former equivalences. We will review two of them, one due to A. Goncharov and one due to M. Kapranov and will fill in some details when compared to the original sources. The first uses the equivalence $\mathrm{MHS}_{\mathbb{C}} \leftrightarrow \mathcal{DO}_{\mathbb{C}}^{\exp}$ as a starting point and the latter the equivalence $\mathrm{MHS}_{\mathbb{C}} \leftrightarrow \mathrm{Penac}_{\mathbb{C}}(\mathbb{P}^2, \mathbb{G}_m^2)$.

3.6.1 Goncharov's Approach

Recall from chapter 2 that every $\mathbb{G}_m^2$-equivariant bundle $\mathcal{V}$ on $\mathbb{A}^2$ has a (non-canonical) trivialisation $\mathcal{V} \cong \mathcal{V}(0, 0) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^2}$ such that the $\mathbb{G}_m^2$-action is induced by the action on $\mathcal{V}(0, 0)$. Further, given such a trivialisation, we get a canonical flat equiviarant connection d that acts as $v \otimes f \mapsto v \otimes df$. In $\mathrm{Bun}_\nabla(\mathbb{A}^2, \mathbb{G}_m^2)$, objects even have a 'canonical' trivialisation:

Proposition 3.28. *Let* $(\mathcal{V}, \nabla)$ *be an object of* $\mathrm{Bun}_\nabla(\mathbb{A}^2, \mathbb{G}_m^2)$ *and* $V_0 := \mathcal{V}(0, 0)$ *the fibre at zero. There is a unique isomorphism*

$$\varphi : \mathcal{V} \longrightarrow V_0 \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^2}$$

such that

- φ *induces the identity on* V_0 *and*
- *the induced connection* $\nabla_\varphi := (\varphi \otimes \mathrm{Id})\nabla\varphi^{-1}$ *takes the form*

  $$\nabla_\varphi = d + \sum_{p,q>0} A_{p,q}(t_1^{p-1}t_2^q dt_1 - t_1^p t_2^{q-1} dt_2) \text{ with } A_{p,q} \in \mathrm{End}(V_0)^{-p,-q},$$

  where $\mathrm{End}(V_0)$ *inherits a bigrading from the one on* V_0 *defined by the action.*

---

[5] Intuitively, because by parallel transport all fibres must have the same dimension. For a formal proof in an algebraic setting see [And01, cor. 2.5.2.2.].

This construction is functorial in the following sense: For every map $f : \mathcal{V} \longrightarrow \mathcal{W}$ *in* $\mathrm{Bun}_\nabla(\mathbb{A}^2, \mathbb{G}_m^2)$*, denote by* $f_0 : V_0 \longrightarrow W_0$ *its restriction to the fibre at* $(0,0)$*. If* $\varphi_\mathcal{V}, \varphi_\mathcal{W}$ *are the canonical trivialisations of* $\mathcal{V}$ *and* $\mathcal{W}$*, there is a commutative diagram*

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "\varphi_\mathcal{V}"] \arrow[d, "f"] & V_0 \otimes \mathcal{O}_{\mathbb{A}^2} \arrow[d, "f_0 \otimes \mathrm{Id}"] \\
\mathcal{W} \arrow[r, "\varphi_\mathcal{W}"] & W_0 \otimes \mathcal{O}_{\mathbb{A}^2}
\end{tikzcd}$$

and $f_0 \circ A_{p,q}^V = A_{p,q}^W \circ f_0$*.*

We call this the **canonical form of** $(\mathcal{V}, \nabla)$. This proposition (in a different form and without the statement about maps) is found without proof in [Kap12, prop. 2.4.4] and with almost no proof in [Gon08, lem. 5.17].

Proof. We will work on global sections: By the previous remarks, we already know there is *some* isomorphism

$$\Gamma(\mathbb{A}^2, \mathcal{V}) \cong V_0 \otimes \mathbb{C}[t_1, t_2]$$

s.t. the action on the right hand side is induced from the tensor product bigrading. We can always assume this isomorphism to be the identity on V_0 and so we are reduced to finding an automorphism of $V_0 \otimes \mathbb{C}[t_1, t_2]$ with the desired properties.

A general connection on $V_0 \otimes_\mathbb{C} \mathbb{C}[t_1, t_2]$ has the form $\nabla = d + \Omega$, where Ω is a one-form with values in $\mathrm{End}(V_0)$ of degree 0, i.e.

$$\Omega = \sum_{p,q \geq 0} A_{p,q} t_1^{p-1} t_2^q dt_1 + B_{p,q} t_1^p t_2^{q-1} dt_2$$

with $A_{p,q}, B_{p,q} \in \mathrm{End}(V_0)$ of degree $(-p, -q)$ and $A_{0,q} = B_{p,0} = 0$ for all $p, q \in \mathbb{Z}$. We want to achieve $A_{p,q} + B_{p,q} = 0$ for all $(p,q) \in \mathbb{Z}^2$.

Automorphisms that induce the identity on V_0 have the form

$$g = \sum_{p,q \geq 0} C_{p,q} t_1^p t_2^q,$$

where $C_{0,0} = \mathrm{Id}_{V_0}$ and $C_{p,q} \in \mathrm{End}(V_0)^{-p,-q}$. They transform ∇ via

$$\Omega \longmapsto g^{-1} dg + g^{-1} \Omega g.$$

Let $I_l \subseteq \mathbb{C}[t_1, t_2]$ be the ideal generated by homogeneous polynomials of degree l. We show by induction on l that we can bring Ω to the desired form modulo I_l for all l.

The initial step $l = 0$ follows because $\Omega \equiv 0$ modulo I_0.

For the induction step, $l \rightsquigarrow l + 1$, suppose we can write Ω as

$$\begin{aligned}
\Omega \equiv & \sum_{\substack{p,q \geq 1 \\ p+q \leq l}} A_{p,q} (t_1^{p-1} t_2^q dt_1 - t_1^p t_2^{q-1} dt_2) \\
& + \sum_{p+q=l+1} A_{p,q} t_1^{p-1} t_2^q dt_1 + B_{p,q} t_1^p t_2^{q-1} dt_2 \qquad \text{mod } I_{l+1}.
\end{aligned}$$

Under a transformation of the form

$$g \equiv Id + \sum_{p+q=l+1} \tilde{C}_{p,q} t_1^p t_2^q \qquad \bmod I_{l+2},$$

the first sum does not change and the second one becomes (modulo I_{l+1}):

$$\sum_{p+q=l+1} (A_{p,q} + p\tilde{C}_{p,q}) t_1^{p-1} t_2^q dt_1 + (B_{p,q} + q\tilde{C}_{p,q}) t_1^p t_2^{q-1} dt_2.$$

Hence, to reach the desired normal form modulo I_{l+1}, it is necessary and sufficient to define

$$\tilde{C}_{p,q} := -\frac{A_{p,q} + B_{p,q}}{p+q}. \tag{$*$}$$

Therefore, we find a sequence of automorphisms of $V_0 \otimes \mathbb{C}[t_1, t_2]$ that, if successively applied, transform Ω to the desired normal form modulo I_l for arbitrarily high l. But the vector space V_0 is finite dimensional and so this process terminates (the $\tilde{C}_{p,q}$ are necessarily 0 for p, q big enough). The same calculations also show uniqueness because if we have two representations both satisfying $A_{p,q} + B_{p,q} = 0$ for all p, q and an automorphism g as above that transports one to the other, then by equation $(*)$ it successively follows that g has to be the identity.

Let us now show the statement about maps. Consider $\mathcal{V}, \mathcal{W}$ with fibres at zero V_0, W_0 to be in trivialised form with connections $\nabla_{\mathcal{V}}, \nabla_{\mathcal{W}}$ as above. A morphism f can then be written (on global sections) as:

$$f = \sum_{p,q \geq 0} C_{p,q} t_1^p t_2^q \qquad \text{with } C_{p,q} \in Hom^{-p,-q}(V_0, W_0)$$

Writing out the relation $\nabla_{\mathcal{W}} \circ f = (f \otimes \mathrm{Id}) \circ \nabla_{\mathcal{V}}$, one sees that this is equivalent to an equality

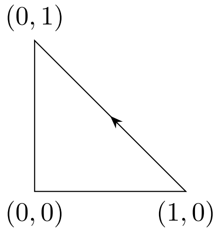
$$C_{a,b} d(t_1^a t_2^b) = \sum_{\substack{a=p+r \\ b=q+s}} (C_{p,q} \circ A_{r,s}^V - A_{r,s}^W \circ C_{p,q})(t_1^{a-1} t_2^b dt_1 - t_1^a t_2^{b-1} dt_2)$$

for all (a, b) with $a, b \in \mathbb{Z}_{\geq 0}$. By comparing coefficients of dt_i this implies $C_{a,b} = 0$ if a or b are greater than 0 and $C_{0,0} A_{r,s}^V = A_{r,s}^W C_{0,0}$ for all $r, s \in \mathbb{Z}$. □

Remark 3.29. *Given an object $(\mathcal{V}, \nabla)$, we can produce a flat connection $\nabla^\flat := (\varphi \otimes \mathrm{Id})^{-1} \circ d \circ \varphi$ on $\mathcal{V}$ this way. This will correspond to the operation of sending a mixed Hodge structure to its associated graded (a direct sum of pure Hodge Structures). From the explicit expression, one checks that ∇ coincides with $\nabla^\flat$ when restricted to lines through the origin.*

Remark 3.30. *This states that an object in $\mathrm{Bun}_\nabla(\mathbb{A}^2, \mathbb{G}_m^2)$ is actually uniquely determined by a bigraded vector space V (the fibre at 0) and an endomorphism of strictly negative degree (the sum of the $A_{p,q}$). Together with the functoriality statement, this yields an equivalence of $\mathrm{Bun}_\nabla(\mathbb{A}^2, \mathbb{G}_m^2)$ to $\mathcal{DO}_{\mathbb{C}}$. However, this is not the one Kapranov and Goncharov describe. Instead, they consider the holonomy of the connection along a certain path.*

Let γ be a path in $\mathbb{A}^2(\mathbb{C}) = \mathbb{C}^2$ going along straight lines from $(0,0)$ to $(1,0)$ to $(0,1)$ and back to $(0,0)$:



Proposition 3.31. *Let $(\mathcal{V}, \nabla)$ be an object in $\mathrm{Bun}_\nabla(\mathbb{A}^2, \mathbb{G}_m^2)$ and use the terminology of proposition 3.28. Assuming $\mathcal{V}$ is in canonical form, i.e., φ is the identity, we have:*

1. *The curvature is given by*
$$\nabla^2 = -\sum_{p,q>0} (p+q) A_{p,q} t_1^{p-1} t_2^{q-1} dt_1 \wedge dt_2$$
2. *For any $r, s > 0$ define a function on $\mathbb{R}$ by*
$$\varphi_{r,s}(t) := (1-t)^{r-1} t^{s-1}.$$
If $g_{p,q}$ denotes the components of bidegree $(-p,-q)$ of $\mathrm{Hol}_\gamma^\nabla$, there are equalities $g_{0,0} = \mathrm{Id}$, for $p, q > 0$
$$g_{p,q} = \sum_{r \in \mathbb{N}} \sum_{\substack{p_1+\ldots+p_r=p \\ q_1+\ldots+q_r=q \\ p_i,q_i>0}} \int_0^1 \varphi_{p_1,q_1}(t_1) \ldots \int_0^{t_{r-1}} \varphi_{p_r,q_r}(t_r) dt_r \ldots dt_1 A_{p_1,q_1} \ldots A_{p_n,q_n}$$
(note that this is a finite sum because $A_{p,q} = 0$ for p, q big enough) and $g_{p,q} = 0$ otherwise.

Proof. 1. To compute the curvature, it suffices to compute how it acts on a pair of vector fields and a section v of $\mathcal{V}$. Let $X := \frac{\partial}{\partial t_1}$, $Y := \frac{\partial}{\partial t_2}$ and $v \in \Gamma(\mathbb{A}^2, \mathcal{V})$. Note that X, Y form a basis of the tangent space of $\mathbb{A}^2$ at every point and so it suffices to compute the curvature for them. Further, the Lie-bracket $[X,Y] = 0$ vanishes. So we have, using a standard formula for the curvature,[6] where $\nabla_X(v)$ means 'apply ∇ to v and then plug X into the differential forms':

$$\begin{aligned}
\nabla^2(v)(X,Y) &= \nabla_X \nabla_Y v - \nabla_Y \nabla_X v - \nabla_{[X,Y]} v \\
&= -\nabla_X \left(\sum_{p,q>0} A_{p,q} v t_1^p t_2^{q-1} \right) - \nabla_Y \left(\sum_{p,q>0} A_{p,q} v t_1^{p-1} t_2^q \right) \\
&= -\sum_{p,q>0} p A_{p,q>} v t_1^{p-1} t_2^{q-1} - \sum_{\substack{p',q'>0 \\ p,q>0}} A_{p',q'} \circ A_{p,q} v t_1^{p'-1+p} t_2^{q'+q-1} \\
&\quad -\sum_{p,q>0} q A_{p,q} v t_1^{p-1} t_2^{q-1} + \sum_{\substack{p',q'>0 \\ p,q>0}} A_{p',q'} \circ A_{p,q} v t_1^{p'+p-1} t_2^{q'-1+q} \\
&= -\sum_{p,q>0} (p+q) A_{p,q} v t_1^{p-1} t_2^{p-1}.
\end{aligned}$$

---

[6] see e.g. [Del70, eq. 2.13.2].

This proves 1.

In order to prove 2, first note that the connection is trivial if restricted to the coordinate axes $t_i = 0,\ i = 1, 2$. Therefore, it suffices to compute the holonomy along the line segment from $(1, 0)$ to $(0, 1)$. Let $\gamma : [0, 1] \longrightarrow \mathbb{C}^2$ be the path given by $t \longmapsto (1 - t, t)$ so that $\gamma(0) = (1, 0)$ and $\gamma(1) = (0, 1)$. Then $\gamma^*\nabla$ is of the form

$$\begin{aligned}\gamma^*\nabla &= d + \sum_{p,q>0} A_{p,q}\left((1-t)^{p-1}t^q d(1-t) - (1-t)^p t^{q-1} dt\right) \\ &= d - \sum_{p,q>0} A_{p,q}\varphi_{p,q}(t)dt.\end{aligned}$$

In order to compute the holonomy, we have to solve the (matrix valued) differential equation:

$$G'(t) = \sum_{p,q>0} \varphi_{p,q}(t)A_{p,q}G(t), \qquad G(0) = \operatorname{Id}.$$

We do this by induction on the bidegree. First, the $(0, 0)$ component of the equation reads:

$$G'(t)_{0,0} = 0$$

It follows that $G(t)_{0,0} = \operatorname{Id}$ and similarly the components of bidegree (p, q) vanish for $(p, q) \neq \{(0, 0)\} \in \mathbb{Z}^2_{\geq 0}$. Now, fix $p, q > 0$ and consider the components $G_{p',q'}$ of bidegree $(-p', -q')$ for $p' < p, q' < q$ known. Then, we have:

$$G'(t)_{p,q} = \sum_{\substack{p \geq p' > 0 \\ q \geq q' > 0}} \varphi_{p',q'}(t)A_{p',q'}G(t)_{p-p',q-q'}$$

But the right hand side is a known function by assumption. So, a solution for the homogeneous bidegree $(-p, -q)$ part is given as:

$$G(t)_{p,q} = \int_0^t \sum_{\substack{p \geq p' > 0 \\ q \geq q' > 0}} \varphi_{p',q'}(s)A_{p',q'}G(s)_{p-p',q-q'}ds$$

From this the statement follows, since $g_{p,q} = G(1)_{p,q}$. □

Corollary 3.32. *The holonomy satisfies:*

$$(\operatorname{Hol}^{\nabla}_{\gamma} - 1)V_0^{p,q} \subseteq \bigoplus_{\substack{p' < p \\ q' < q}} V_0^{p',q'},$$

i.e., $(V_0, \operatorname{Hol}^{\nabla}_{\gamma})$ *is an object of* $\mathcal{DO}^{exp}_{\mathbb{C}}$.

Corollary 3.33. *The transformation of* $\operatorname{End}^{Del}(V_0)$ *that sends* $A := \sum_{p,q>0} A_{p,q}$ *to* $\log \operatorname{Hol}^{\nabla}_{\gamma}$ *is a homeomorphism.*

This will follow from the following elementary lemma, that I do not know a reference for. It says that operators with a 'triangular shape' are invertible. We make a slightly more general statement than needed at the moment. Recall that for two multiindices $I = (a_1, ..., a_n), J = (b_1, ..., b_n) \in \mathbb{Z}^n$, we write $I < J$ if $a_i < b_i$ for all $i \in \{1, ..., n\}$.

Lemma 3.34. *Let* W *be a* n*-graded topological vector space s.t. there are only finitely many* $I \in \mathbb{Z}^n$ *with* $W^I \neq 0$ *and s.t.* W *is the topological product of the* W^I*. In particular, the projections* $W \longrightarrow W^I$ *are continuous. Let* $\alpha : W \longrightarrow W$ *be a (not necessarily linear) map s.t. the components* $\alpha^I : W \longrightarrow W^I$ *can be written as* $\alpha^I = M^I + \alpha^{<I}$ *where*

- M^I *factors over the projection to* W^I *and the resulting map* $\tilde{M}^I : W^I \longrightarrow W^I$ *is a homeomorphism.*
- $\alpha^{<I}$ *factors over the projection to* $W^{<I} := \bigoplus_{I'<I} W^I$ *and the resulting map* $\tilde{\alpha}^{<I} : W^{<I} \longrightarrow W^I$ *is continuous.*

Then α *is a homeomorphism of* W*.*

Proof. By assumption, the projection maps to the components W^I are continuous, so M^I and $\alpha^{<I}$ are continuous and because addition is continuous α^I is and so is α, since all of its components are.

Let $\tilde{N}^I$ denote the inverse of $\tilde{M}^I$ and N^I the map $W \longrightarrow W^I$ resulting by precomposition with the projection $W \longrightarrow W^I$.
We construct the components $\beta^I : W \longrightarrow W^I$ of an inverse map β recursively. Set $\beta^I := N^I - \tilde{N}^I \circ \beta^{<I}$ where $\beta^{<I}$ is the composition

$$W \overset{\Pi_{J<I}\beta^J}{\longrightarrow} W^{<J} \overset{\tilde{\alpha}^{<I}}{\longrightarrow} W^I.$$

(Note that by our assumption, the direct sum $W^{<J}$ is also the topological direct product of its components.) To define this, we only needed β^J for $J < I$. For sufficiently small J, we can have $W^J = 0$ and hence necessarily $\beta^J = 0$. The so-constructed map β is continuous because all of its components are.

Given $w \in W$ write w^J (resp. $w^{<J}$) for the projection to W^J (resp. $W^{<J}$). Using the equation

$$\begin{aligned}
\beta^I \alpha(w) &= \beta^I \left(\sum_{J \in \mathbb{Z}^n} \alpha^J(w) \right) \\
&= \beta^I \left(\sum_{J \in \mathbb{Z}^n} \tilde{M}^J(w^J) + \alpha^{<J}(w^{<J}) \right) \\
&= \underbrace{\tilde{N}^I \tilde{M}^I(w^J)}_{w^J} + \tilde{N}^I \alpha^{<I} \left(\sum_{J<I} w^J \right) - \tilde{N}^I \alpha^{<I} \left(\sum_{J<I} \beta^J \alpha(w) \right)
\end{aligned}$$

one (again recursively) verifies that $\beta \circ \alpha = \mathrm{Id}_W$ and a similar calculation shows $\alpha \circ \beta = \mathrm{Id}_W$. ☐

Proof of corollary 3.33. Note that the proposition 3.31 yields:

$$\begin{aligned}
(\log \mathrm{Hol}^{\nabla}_{\gamma})_{p,q} = &\int_0^1 \varphi_{p,q}(t) dt A_{p,q} \\
&+ \text{ noncom. polynomial in the } A_{p',q'} \text{ with } p' < p, q' < q
\end{aligned}$$

So by lemma 3.34, it suffices to show that

$$\int_0^1 \varphi_{p,q}(t)dt \neq 0.$$

The integral can be evaluated using integration by parts. For any $p, q \geq 0$, we have

$$\begin{aligned}\int_0^1 (1-t)^p t^q dt &= -\frac{p}{q+1}\int_0^1 (1-t)^{p-1}t^{q+1}dt \\ &= \frac{p(p-1)}{(q+1)(q+2)}\int_0^1 (1-t)^{p-2}t^{q+2} \\ &\cdots \\ &= (-1)^p \binom{p+q}{p}^{-1}\frac{1}{p+q+1} \neq 0.\end{aligned}$$

□

Now we are ready to prove the main theorem of this section.

Theorem 3.35. *The functor*

$$\mathrm{Bun}_\nabla(\mathbb{A}^2, \mathbb{G}_m^2) \longrightarrow \mathcal{DO}_{\mathbb{C}}^{\exp}$$

given on objects by

$$(\mathcal{V}, \nabla) \longmapsto (V_0, \mathrm{Hol}_\gamma^\nabla)$$

and on maps by restriction to the fibre at $(0,0)$ *is essentially surjective and fully faithful. Hence, it induces an equivalence of categories. It is also compatible with the involutions and tensor products on both sides.*

Proof. Essentially surjective: Given an object (V, D) of $\mathcal{DO}_{\mathbb{C}}$ with $D = \sum_{p,q>0} D_{p,q}$, where $D_{p,q}$ are of bidegree $(-p,-q)$, then by corollary 3.33 there are unique $A_{p,q} \in \mathrm{End}(V)^{-p,-q}$ s.t. the connection

$$d + \sum_{p,q>0} A_{p,q}(t_1^{p-1}t_2^q dt_1 - t_1^p t_2^{q-1}dt_2)$$

on the equivariant bundle with global sections $V \otimes_{\mathbb{C}} \mathbb{C}[t_1, t_2]$ has holonomy $\exp D$ along γ.

Faithful: This follows from the general fact that two flat morphisms between vector bundles with connections (on a connected base) that agree in the fibre over one point x_0 have to agree everywhere. In fact, for any other point x take any path δ from x_0 to x. Then, we have $\mathrm{Par}_\delta^\nabla f_{x_0} = f_x \mathrm{Par}_\delta^\nabla$.

Full: Given two bundles $(\mathcal{V}, \nabla_{\mathcal{V}})$ and $(\mathcal{W}, \nabla_{\mathcal{W}})$, for every morphism $f_0 : \mathcal{V}_0 \longrightarrow \mathcal{W}_0$ that commutes with the holonomy along γ we have to produce a flat morphism $f : (\mathcal{V}, \nabla_{\mathcal{V}}) \longrightarrow (\mathcal{W}, \nabla_{\mathcal{W}})$ that restricts to f_0. Assuming w.l.o.g. that $\mathcal{V}$ and $\mathcal{V}$ are in canonical form, we can define $f := f_0 \otimes \mathrm{Id}$ on global sections. This commutes with the connection since each $A_{p,q}$ is a polynomial in the $D_{r,s}$.

Finally, compatibility with the tensor product follows from the formula

$$\mathrm{Hol}_\gamma^{\nabla\otimes\nabla'} = \mathrm{Hol}_\gamma^\nabla \otimes \mathrm{Hol}_\gamma^{\nabla'},$$

which holds in general for vector bundles with connection and compatibility with the involution since γ is mapped to its inverse path under $(a, b) \mapsto (\overline{b}, \overline{a})$. □

3.6.2 Kapranov's Approach

The Radon-Penrose transform is a geometric construction that, under certain circumstances, relates vector bundles on one space to vector bundles with a connection on another one. Since a reference containing a full proof appears to be unavailable, we included an essentially self-contained review in appendix B. The main idea of [Kap12] is to apply this construction to Penacchio's equivariant bundles on projective 2-space.

Let $\mathbb{P}^2 = \mathrm{Proj}(\mathbb{C}[X_0, X_1, X_2])$. A line in $\mathbb{P}^2$ is given by an equation of the form $aX_0 + bX_1 + cX_2$, i.e., it is uniquely determined by a tripel (a, b, c), defined up to a scalar multiple. So lines in $\mathbb{P}^2$ form another projective 2-space, the 'dual projective space' $\check{\mathbb{P}}^2$. The two are related by the incidence variety

$$\mathcal{Q}(\mathbb{C}) = \{(x, \ell) \in (\mathbb{P}^2 \times \check{\mathbb{P}}^2)(\mathbb{C}) \mid x \in \ell\}$$

If $\check{\mathbb{P}}^2$ is given as $\check{\mathbb{P}}^2 = \mathrm{Proj}(\mathbb{C}[\check{X}_0, \check{X}_1, \check{X}_2])$, then the incidence variety is given as

$$\mathcal{Q} := \mathrm{Proj}\left(\mathbb{C}[X_0, X_1, X_2, \check{X}_0, \check{X}_1, \check{X}_2]/(\check{X}_0X_0 + \check{X}_1X_1 + \check{X}_2X_2)\right).$$

We equip all three spaces with a $\mathbb{G}_m^2 \cong \mathbb{G}_m^3/\Delta(\mathbb{G}_m)$-action as follows:

- On $\mathbb{P}^2$, the action corresponds to the grading by the degree of the X_i, i.e., on points it is given by:

$$\begin{array}{rcl} (\mathbb{G}_m^3/\Delta(\mathbb{G}_m) \times \mathbb{P}^2)(\mathbb{C}) & \longrightarrow & (\mathbb{P}^2)(\mathbb{C}) \\ ([\lambda_1 : \lambda_2 : \lambda_3], [z_1 : z_2 : z_3]) & \longmapsto & [\lambda_1 z_1 : \lambda_2 z_2 : \lambda_3 z_3] \end{array}$$

- On $\check{\mathbb{P}}^2$, it is given by the inverse grading of the $\check{X}_i$, i.e., on points as

$$\begin{array}{rcl} (\mathbb{G}_m^3/\Delta(\mathbb{G}_m) \times \check{\mathbb{P}}^2)(\mathbb{C}) & \longrightarrow & (\mathbb{P}^2)(\mathbb{C}) \\ ([\lambda_1 : \lambda_2 : \lambda_3], [z_1 : z_2 : z_3]) & \longmapsto & [\lambda_1^{-1} z_1 : \lambda_2^{-1} z_2 : \lambda_3^{-1} z_3]. \end{array}$$

- On $\mathcal{Q}$, it is given by the diagonal action.

In the following, we are interested in certain invariant subvarieties of the above, namely $\mathbb{A}^2 := \mathbb{P}^2 \backslash V(X_0)$ with coordinates $t_1 := \frac{X_1}{X_0}, t_2 := \frac{X_2}{X_0}$ and $\check{\mathbb{P}}^2_0 := \check{\mathbb{P}}^2 \backslash V(\check{X}_1, \check{X}_2)$. We obtain the following commutative diagram:

$$\begin{tikzcd}
\mathbb{A}^2 \arrow[d, hook] & \mathcal{Q}_0 \arrow[l, "q_0"'] \arrow[r, "p_0"] \arrow[d, hook] & \check{\mathbb{P}}^2_0 \arrow[d, hook] \\
\mathbb{P}^2 & \mathcal{Q} \arrow[l, "q"'] \arrow[r, "p"] & \check{\mathbb{P}}^2
\end{tikzcd}$$

Here the vertical arrows are inclusions, the horizontal ones are projections and

$$\mathcal{Q}_0(\mathbb{C}) := \{(x, \ell) \in (\mathbb{A}^2 \times \check{\mathbb{P}}^2_0)(\mathbb{C}) \mid x \in \ell\}.$$

Let us denote by $\mathrm{Penac}_{\mathbb{C}}(\check{\mathbb{P}}^2, \mathbb{G}_m^2)$ the category of equivariant bundles on $\mathbb{P}^2$ which are equivariant on every line, with morphisms of vector bundles. Theorem 3.19 and proposition 3.23 imply that this category is equivalent to $\mathrm{MHS}_{\mathbb{C}}$ as categories with involution (Note that the $\mathbb{G}_m^2$-action is the inverse of the one in theorem 3.19).

Let $\Omega_{\mathcal{Q}_0/\check{\mathbb{P}}^2_0}$ be the sheaf of relative differentials. For a connection ∇ on a vector bundle $\mathcal{V}$ on $\mathcal{Q}_0$, we denote by $\nabla_{\mathcal{Q}_0/\check{\mathbb{P}}^2_0}$ the composite with the projection:

$$\nabla_{\mathcal{Q}_0/\check{\mathbb{P}}^2_0} : \mathcal{V} \xrightarrow{\nabla} \mathcal{V} \otimes \Omega^1_{\mathcal{Q}_0} \longrightarrow \mathcal{V} \otimes \Omega_{\mathcal{Q}_0/\check{\mathbb{P}}^2_0}$$

Theorem 3.36. *For an object* $(\mathcal{V}, \nabla)$ *of* $\mathrm{Bun}_\nabla(\mathbb{A}^2, \mathbb{G}^2_m)$*, the equivariant sheaf*

$$(p_0)_* \ker((q_0^* \nabla)_{\mathcal{Q}_0/\check{\mathbb{P}}^2_0})$$

is a vector bundle on $\check{\mathbb{P}}^2_0$*. Moreover, this construction yields an equivalence of* $\mathbb{C}$*-linear tensor categories with involution*

$$\mathrm{Bun}_\nabla(\mathbb{A}^2, \mathbb{G}^2_m) \longleftrightarrow \mathrm{Penac}(\check{\mathbb{P}}^2_0, \mathbb{G}^2_m).$$

The inverse is given by $\mathcal{W} \longmapsto (q_0)_* p_0^* \mathcal{W}$*, which carries a canonical connection.*

Proof. Let us prove a variant, where $\mathbb{A}^2, \mathcal{Q}_0, \check{\mathbb{P}}^2_0, \mathbb{G}^2_m$ are replaced by their complex points, considered as complex manifolds and we seek to prove the statement for holomorphic equivariant vector bundles. This is suffices because by the GAGA principle in theorem 2.27 and by proposition 2.28, the algebraic and analytic categories on both sides coincide.

In the holomorphic setting, the statement follows from the general theorem on the equivariant Radon-Penrose transform in the Appendix B. It remains to verify the conditions for the Radon-Penrose transform that are stated in B.1 and which we recall here:

The maps q_0 and p_0 are fibre bundles (condition 1. of B.1) with fibre $\mathbb{A}^1$ and $\mathbb{P}^1$, respectively. In fact, the map

$$\begin{array}{rcl} \mathcal{Q}_0(\mathbb{C}) & \xrightarrow{\sim} & (\mathbb{A}^2 \times \mathbb{P}^1)(\mathbb{C}) \\ ([1 : a_1 : a_2], [b_0 : b_1 : b_2]) & \longmapsto & ((a_1, a_2), [b_1 : b_2]) \end{array}$$

identifies q_0 with a trivial $\mathbb{P}^1$-bundle over $\mathbb{A}^2$. On the other hand, over the chart $U_1 := \{\check{X}_1 \neq 0\} \subseteq \check{\mathbb{P}}^2_0$, there is the identification of p_0 with the trivial bundle via

$$\begin{array}{rcl} p_0^{-1} U_1(\mathbb{C}) & \xrightarrow{\sim} & (\mathbb{A}^1 \times U_1)(\mathbb{C}) \\ ([1 : a_1 : a_2], [b_0 : 1 : b_2]) & \longmapsto & (a_2, [b_0 : 1 : b_2]) \end{array}$$

and similarly over $U_2 := \{\check{X}_2 \neq 0\}$. In particular, the fibres of p_0 are (even simply) connected and the fibres of q_0 are compact and connected (conditions 3. and 4. of B.1).

Condition 2. of B.1 says that the map

$$(q_0, p_0) : \mathcal{Q}_0 \longrightarrow \mathbb{A}^2 \times \check{\mathbb{P}}^2_0$$

is a closed immersion, which is true by construction.

Denote $\mathcal{N} := q_0^*\Omega_{\mathbb{A}^2} \cap p_0^*\Omega_{\check{\mathbb{P}}_0^2}$, where the intersection takes place in $\Omega_{\mathcal{Q}_0}$. We want to show that $R^1(q_0)_*\mathcal{N} = 0 = (q_0)_*\mathcal{N}$, which is condition 5. of B.1. Because $\mathbb{A}^2$ is affine, these are the sheaves associated to the $\mathbb{C}[t_1, t_2]$-modules $H^1(\mathcal{Q}_0, \mathcal{N})$ and $H^0(\mathcal{Q}_0, \mathcal{N})$. Therefore, let us compute these, using Čhech cohomology of the cover V_1, V_2, where $V_i := p_0^{-q}U_i$ for $i = 1, 2$.

First, we explicitely describe several modules of differential forms. We do this for the algebraic differential forms. The V_i are affine, in fact $V_i = \operatorname{Spec} A_i$, where

$$A_1 := \mathbb{C}[t_1, t_2, r_0, r_2]/(L_1)$$

with $r_i := \frac{\check{X}_i}{\check{X}_1}$ and $L_1 := r_0 + t_1 + r_2t_2$ and similarly,

$$A_2 := \mathbb{C}[t_1, t_2, s_0, s_1]/(L_2)$$

with $s_i := \frac{\check{X}_i}{\check{X}_2}$ and $L_2 := s_0 + s_1t_1 + t_2$. From this, one computes the differential forms of $\mathcal{Q}_0$ on V_1 as:

$$\Gamma(V_1, \Omega_{\mathcal{Q}_0}) = \left(\bigoplus_{i \in \{1,2\}} A_1 dt_i \oplus \bigoplus_{i \in \{0,2\}} A_1 dr_i \right) \operatorname{mod} dL_1$$

This is freely generated by dt_2, dr_0, dr_2. Similarly,

$$\Gamma(V_2, \Omega_{\mathcal{Q}_0}) = \left(\bigoplus_{i \in \{1,2\}} A_2 dt_i \oplus \bigoplus_{i \in \{0,1\}} A_2 ds_i \right) \operatorname{mod} dL_2,$$

which is free with basis dt_1, ds_0, ds_1. The pullback bundles $q_0^*\Omega_{\mathbb{A}^2}$ and $p_0^*\Omega_{\check{\mathbb{P}}_0^2}$ are given on V_1 as follows:

$$\begin{aligned}
\Gamma(V_1, p_0^*\Omega_{\check{\mathbb{P}}_0^2}) &= A_1 dr_0 \oplus A_1 dr_2 \\
\Gamma(V_2, p_0^*\Omega_{\check{\mathbb{P}}_0^2}) &= A_2 ds_0 \oplus A_2 ds_1 \\
\Gamma(V_1, q_0^*\Omega_{\mathbb{A}^2}) &= A_1 dt_1 \oplus A_1 dt_2 \\
\Gamma(V_2, q_0^*\Omega_{\mathbb{A}^2}) &= A_2 dt_1 \oplus A_2 dt_2
\end{aligned}$$

Using the above identifications, one checks that $\Gamma(V_1, \mathcal{N})$ is the free A_1-submodule of $\Gamma(V_1, \mathcal{Q}_0)$ generated by $\omega_1 := dr_0 + t_2dr_2$ and $\Gamma(V_2, \mathcal{N})$ is the free A_2-submodule of $\Gamma(V_2, \mathcal{Q}_0)$ generated by $\omega_2 := ds_0 + t_1ds_1$. Let A_{12} denote the coordinate ring of the intersection $V_{12} = V_1 \cap V_2$, i.e.,

$$A_{12} = \mathbb{C}[t_1, t_2, s_0, s_1^{\pm 1}]/(L_2).$$

Here we have the relations $-t_1s_1 - t_2 = s_0 = r_0r_2^{-1}$, $s_1 = r_2^{-1}$ and in $\Gamma(V_{12}, \Omega_{\mathcal{Q}_0})$ the relations $dr_2 = -s_1^{-2}ds_1$ and $dr_0 = s_1^{-1}ds_0 - s_0s_1^{-2}ds_1$. In particular,

$$\omega_1 = s_1^{-1}ds_0 - (s_0s_1^{-2} + t_2s_1^{-2})ds_1 = s_1^{-1}\omega_2.$$

Now, suppose we are given

$$f_1(t_1, t_2, r_0, r_2)\omega_1 \in A_1\omega_1 = \Gamma(V_1, \mathcal{N})$$

and

$$g_2(t_1, t_2, s_0, s_1)\omega_2 \in A_2\omega_2 = \Gamma(V_2, \mathcal{N}).$$

Under the identification $A_{12} \cong \mathbb{C}[t_1, t_2, s_1^{\pm 1}]$, the condition for the $f_1\omega_1$ and $f_2\omega_2$ to be restrictions of a single global section reads as

$$f_1(t_1, t_2, -t_1 - t_2 s_1^{-1}, s_1^{-1})s_1^{-1} = f_2(t_1, t_2, -t_1 s_1 - t_2, s_1),$$

which cannot be true unless $f_1 = f_2 = 0$ by comparing the s_1-degrees, so $H^0(\mathcal{Q}_0, \mathcal{N}) = 0$.

Let us make the identifications $A_1 \cong \mathbb{C}[t_1, t_2, s_1^{-1}]$ and $A_2 \cong \mathbb{C}[t_1, t_2, s_1]$. In order to show $H^1(\mathcal{Q}_0, \mathcal{N}) = 0$, we have to verify that every section $s \in \Gamma(V_{12}, \mathcal{N})$ can be written as a difference

$$s = \theta_1|_{V_{12}} - \theta_2|_{V_{12}} \text{ with } \theta_1 \in \Gamma(V_1, \mathcal{N}), \theta_2 \in \Gamma(V_2, \mathcal{N}).$$

Writing $s = a\omega_2$ with $a \in A_{12}$, this amounts to the statement

$$a\omega_2 = (fs^{-1} - g)\omega_2 \qquad \text{with } f \in A_1, g \in A_2,$$

which can always be achieved.

This ends the verification of the conditions for the Radon-Penrose transform. So far, we get an identification

$$\left\{ \begin{array}{c} \mathbb{G}_m^2\text{-equivariant vector} \\ \text{bundles on } \mathbb{A}^2 \text{ with a} \\ \text{connection } (\mathcal{V}, \nabla) \text{ s.t. } q_0^*\nabla \text{ is} \\ \text{flat with trivial monodromy} \\ \text{on every fibre of } p_0 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathbb{G}_m^2\text{-equivariant} \\ \text{vector bundles on } \check{\mathbb{P}}_0^2 \\ \text{trivial on every line} \end{array} \right\}$$

The right hand side is the category $\mathrm{Penac}_{\mathbb{C}}(\check{\mathbb{P}}_0^2, \mathbb{G}_m^2)$. Also, the condition on curvature and monodromy on the left hand side are automatically satisfied: The fibres of p_0 are one dimensional, so any connection is flat and they are simply connected, so no connection can have monodromy.

As for compatibility with the involution, note that also $\mathcal{Q}$ is equipped with an involution given by

$$([a_0 : a_1 : a_2], [b_0 : b_1 : b_2]) \mapsto ([\overline{a_0} : \overline{a_2} : \overline{a_1}], [\overline{b_0}, \overline{b_2}, \overline{b_1}]),$$

which preserves $\mathcal{Q}_0$ and p, q are equivariant with respect to this involution.

Pullback and the Riemann-Hilbert correspondence (i.e., taking the kernel of the connection) are compatible with tensor products and since, as seen in appendix B.1, every bundle $\mathcal{V}$ on $\mathcal{Q}_0$ which is trivial on every fibre of q_0 is of the form $\mathcal{V} \cong q_0^*(q_0)_*\mathcal{V}$, also $(q_0)_*$ is compatible with the tensor product by the projection formula. This finishes the proof. □

Remark 3.37. *This proof is slightly different from the one given in [Kap12], where, after showing the result for holomorphic bundles, it is deduced by explicit considerations that the Radon-Penrose transform maps algebraic bundles to algebraic bundles in this situation. We can omit this since, as was shown in the previous chapter, the holomorphic and algebraic categories on both sides agree.*

It remains to check in how far the approaches of Kapranov and Goncharov coincide. It turns out that they do 'up to signs'. Let us make this more precise:

Let (H, W, F_1, F_2) be a complex Mixed Hodge Structure and $H_W^{\bullet,\bullet}$ its associated graded with Deligne Operator $d_H = \alpha_2\alpha_1^{-1}$ with notations as in section 3.2. Let $\mathcal{W} = \xi_{\check{\mathbb{P}}^2}(H, W, F_1, F_2)|_{\check{\mathbb{P}}^2_0}$ be the associated equivariant bundle on $\check{\mathbb{P}}^2_0$ by Penacchio's construction and $(\mathcal{V}, \nabla)$ be the associated bundle with connection on $\mathbb{A}^2$ obtained from the Radon-Penrose-transformation.

Let $F_{(a,b)}$ be the fibre of q_0 over (a, b). Via p_0, it maps isomorphically onto the subspace $\mathbb{P}^1_{(a,b)} := \{\check{X}_0 + a\check{X}_1 + b\check{X}_2 = 0\} \subseteq \check{\mathbb{P}}^2_0$. By the definition of the Radon-Penrose transform and proper base change, we have isomorphisms

$$\mathcal{V}(a, b) \cong ((q_0)_* p_0^* \mathcal{W})(a, b) \cong H^0(\mathbb{P}^1_{(a,b)}, \mathcal{W})$$

For $(a, b) = (0, 0)$, we already know the right hand side (corollary 3.20), so this yields an equivariant identification $\mathcal{V}(0, 0) \cong H_W$. Note that the $\mathbb{G}_m^2$-action truly gives the original grading on H_W and not its inverse as the action on $\check{\mathbb{P}}^2$ is inverse to the one on $\mathbb{P}^2$.

Now let $\widetilde{\gamma}$ run along straight lines from $(0, 0)$ to $(-1, 0)$ to $(0, -1)$ and back to $(0, 0)$. To compute the holonomy along $\widetilde{\gamma}$, we employ proposition B.10, which tells us that it can be computed via the diagram

$$\begin{tikzcd}
H_W \arrow[r, "res"] \arrow[dd, "\mathrm{Hol}^{\nabla}_{\widetilde{\gamma}}"] & \Gamma([0:0:1], \mathcal{W}) \arrow[r, "res^{-1}"] & \Gamma(\mathbb{P}^1_{(-1,0)}, \mathcal{W}) \arrow[d, "res"] \\
 & & \Gamma([1:1:1], \mathcal{W}) \arrow[d, "res^{-1}"] \\
H_W & \Gamma([0:1:0], \mathcal{W}) \arrow[l, "res^{-1}"'] & \Gamma(\mathbb{P}^1_{(0,-1)}, \mathcal{W}). \arrow[l, "res"']
\end{tikzcd}$$

But as seen in the proof of proposition 3.24 this implies $\mathrm{Hol}^{\nabla}_{\widetilde{\gamma}} = d_H$.

In particular, as $\widetilde{\gamma} = (-1, -1).\gamma$, we obtain that $\mathrm{Hol}^{\nabla}_{\gamma} = (-1, -1).D := \sum_{p,q}(-1)^{p+q}(d_H)_{p,q}$, where $(d_H)_{p,q}$ are the degree $(-p, -q)$-components of d_H. In terms of the bundles, this means:

Proposition 3.38. *Let H be a Mixed Hodge Structure, $(\mathcal{V}, \nabla)$ be the associated bundle with connection via Kapranov's Construction and $(\mathcal{V}', \nabla')$ be the associated bundle with connection via Goncharov's Construction. There is a natural isomorphism of equivariant vector bundles with connection*

$$(-1, -1)^*(\mathcal{V}, \nabla) \cong (\mathcal{V}', \nabla'),$$

where $(-1, -1)^*$ means pullback by the automorphism of $\mathbb{A}^2$ induced by $(-1, -1) \in \mathbb{G}_m^2(\mathbb{C})$ via the action of $\mathbb{G}_m^2$ on $\mathbb{A}^2$.

.

3.7 Polarisations

We recall the definition of (graded) polarised Mixed Hodge Structure (see e.g., [Del71b]) and translate this in terms of equivariant bundles with a conenction. For a pure Hodge Structure H, the Weil-operator C is defined by $C.v := i^{p-q}$ for $v \in H^{p,q}$.

Definition 3.39. *1. Let H be a pure real Hodge structure of weight k. A **polarisation** of H is a morphism of Hodge structures*

$$(\ . \ , \ . \ ) : H \otimes H \longrightarrow T(-k)$$

s.t. $Q := (C\ .\ ,\ .\ )$ is positive definite and symmetric.

2. *Let $H = \bigoplus_{k \in \mathbb{Z}} H_k$ be a direct sum of pure Hodge real structures, where H_k is of weight k. A **polarisation** on H is the datum of a polarisation as in* 1. *on each H_k.*

3. *Let H be a real Mixed Hodge Structure. A **polarisation** is the datum of a polarisation in the sense of* 2. *on $H_W := \bigoplus_{k \in \mathbb{Z}} \mathrm{gr}_k^W H$. A real Mixed Hodge Structure with polarisation is called **(graded) polarised**.*

Note that $(\ . \ , \ . \ )$ is completely determined by Q, but not every positive definite symmetric form gives rise to a polarisation.

A polarisation yields a hermitian form on the complexification $H_{\mathbb{C}}$ and a norm on both H and its complexification $H_{\mathbb{C}}$.

In terms of equivariant bundles with connections, we translate this as follows:

Proposition 3.40. *Let H be a real Mixed Hodge Structure and $\mathcal{V}^{Gon}$ and $\mathcal{V}^{\mathrm{Penac}}$ the associated (real) bundles via Goncarov's and Penacchio's description. Then the following data are equivalent:*

- *A polarisation on H.*
- *A polarisation on $\mathcal{V}^{\mathrm{Penac}}|_{\{X_0=0\}}$.*
- *A bundle metric on $\mathcal{V}^{Gon}$ which is flat for the canonical flat connection and gives rise to a polarisation on the fibre above $(0,0) \in \mathbb{A}^2_{\mathbb{R}}(\mathbb{R}) = \mathbb{R}^2$.*

There is also a translation of polarisations in terms of Simpson's bundles for which we refer to [Sim97a, §3].

Chapter 4

Complements and Applications

Here, we present new results related to the various descriptions of Mixed Hodge Structures in terms of vector bundles from the previous chapter.

4.1 A Direct Construction in the Pure Case

The constructions in this section are a minor modification of ideas already found in [Sim97b] and [Sim97a].

Let X be a compact complex manifold and $A := \mathbb{A}^2_{\mathbb{C}}$ complex affine space ($\mathbb{C}^2$ with Zariski-topology) with coordinates t_1, t_2. Let $\mathcal{O}_A$ be the structure sheaf of A and regard X as a locally ringed space with structure sheaf $\mathcal{A}_X$, the sheaf of complex valued smooth functions. Consider the diagram of topological spaces

$$\begin{tikzcd}
A \times X \arrow[r, "p"] \arrow[d, "\pi"] & X \arrow[d, "q"] \\
A \arrow[r] & \{\mathrm{pt}\},
\end{tikzcd}$$

where $\pi = \mathrm{Id} \times q$ and equip $A \times X$ with the structure sheaf $\mathcal{O}\mathcal{A} := \pi^{-1}\mathcal{O}_A \otimes_{\mathbb{C}} p^{-1}\mathcal{A}_X$ and A, resp. $A \times X$, are equipped with the action of $\mathbb{G}_m^2$ by multiplication.[1] As in chapter 1, write $(\mathcal{A}_X^\bullet, d = \partial + \bar\partial)$ for the complex of complex-valued differential forms (in particular $\mathcal{A}_X^0 = \mathcal{A}_X$). The Rees-bundle construction works just as well for filtered complexes of sheaves and yields a complex of $\mathcal{O}\mathcal{A}$-modules with $\pi^{-1}\mathcal{O}_A$-linear differential:

$$(\xi^\bullet_{A\times X}, d) := \xi((p^*\mathcal{A}_X^\bullet, \partial + \bar\partial), F, \bar F),$$

where $F, \bar F$ are the filtrations by type coming from the 'stupid' filtrations on the double complex $(\mathbb{A}_X^{\bullet,\bullet}, \partial, \bar\partial)$. Let

$$\xi_A^k := \xi(H^k(X, \mathbb{C}), F, \bar F)$$

be the Rees-bundle associated with the Hodge structure $H^k(X, \mathbb{C}) = R^k q_* \mathcal{A}_X$. There is the following 'base-change' property:

Proposition 4.1. *There is a canonical isomorphism*

$$R^k\pi_*(\xi^\bullet_{A\times X}, d) \ mod\ T \cong \xi^k_A,$$

where T is the torsion subsheaf.

Proof. Since $\mathcal{A}_X^{p,q}$ is q_*-acyclic, the sheaves $\xi^k_{A\times X}$ are π_*-acyclic. We can therefore compute the pushforward as the cohomology of the complex

$$(\pi_*\xi^\bullet_{A\times X}, d).$$

The individual terms of this complex are again Rees-sheaves on A. In fact, one has

$$\pi_*\xi^k_{A\times X} = \xi(\mathcal{A}^k_X(X), F, \bar F).$$

---

[1] The reader uncomfortable with this mixing of smooth and algebraic structure may safely replace A by $\mathbb{C}^2$ with the usual topology and structure sheaf of holomorphic functions in view of theorem 2.27.

Recall from proposition 2.13 that for a map of $V \longrightarrow W$ of bifiltered vector spaces one has

$$\xi(\ker(V \longrightarrow W)) \cong \ker(\xi(V) \longrightarrow \xi(W))$$

and

$$\xi(\operatorname{coker}(V \longrightarrow W)) \cong \operatorname{coker}(\xi(V) \longrightarrow \xi(W))/T,$$

where T denotes the torsion. In particular, for a complex of bifiltered vector spaces, the Rees-bundle construction commutes with cohomology up to torsion, which proves the claim. $\square$

Remark 4.2.

- *So far, there is nothing particular about differential forms here and one could make an analogous statement for any complex of q_*-acyclic sheaves on X with multiple filtrations by acyclic subsheaves.*

- *If one views $\mathcal{A}_X = \mathcal{A}_{X,\mathbb{R}} \otimes \mathbb{C}$ as being equipped with the involution given by complex conjugation in the second factor and $\mathbb{C}[t_1, t_2]$ with the antilinear involution exchanging t_1 and t_2, the morphism of the proposition is compatible with the induced involutions on both sides.*

- *Using the fact that $F, \bar{F}$ are split by the sheaves $\mathcal{A}_X^{p,q}$, one checks that there is an isomorphism*

  $$(\xi^\bullet_{A\times X}, d) \cong (p^*\mathcal{A}^\bullet_X, t_1\partial + t_2\bar{\partial}),$$

  where the action of $(\lambda_1, \lambda_2) \in \mathbb{G}_m^2(\mathbb{C})$ on the right hand side is given on $\mathcal{A}^{p,q}$ by multiplication with $\lambda_1^p \cdot \lambda_2^q$. I.e., one is essentially considering the de-Rham complex with deformed differential.

- *For $(x, y) \in A(\mathbb{C})\backslash\{(0,0)\}$ consider the fibre of $R^k\xi^\bullet_{A\times X}$ at (x, y): If $x \neq 0 \neq y$, it is isomorphic to the de-Rham cohomology, if $x = 0$ to Dolbeault cohomology and if $y = 0$ to conjugate Dolbeault cohomology. In particular, the torsion T is supported in $(0,0)$ if and only if the Frölicher spectral sequence degenerates, e.g., for X Kähler.*[2] *By proposition 2.13 or a direct consideration with squares and zigzags, this is the case if and only if d is strict with respect to F (and hence $\bar{F}$).*

- *For a pure complex Hodge Structure $H = \bigoplus H^{p,q}$ of weight k, the equivariant bundle underlying Goncharov's construction is associated with $\mathbb{C}[t_1, t_2] \otimes H$ with tensor product bigrading, while the Rees-bundle $\xi_{\mathbb{A}^2}(H, F, \sigma F)$ is isomorphic to $\mathbb{C}[t_1, t_2] \otimes \widetilde{H}$, where $\widetilde{H} = H$ as vector spaces, but with grading $\widetilde{H}^{p,q} = H^{-p,-q}$, i.e., $\widetilde{H} = \sigma H[k]$, where σ denotes the involution on the category of complex Hodge Structures. In particular, composing the above pushforward construction with a twist and pullback by $(a, b) \mapsto (\overline{b}, \overline{a})$ yields a direct construction of the equivariant bundle underlying Goncharov's construction, starting from a complex manifold s.t. the de-Rham cohomology carries a pure Hodge structure.*

---

[2] In fact, the theory developed in chapter 1 originated in an early attempt to analyse the sheaf T.

- *Even simpler, one also obtains a direct construction of Simpson's twistor bundles from certain complex manifolds: Let* $A' := \mathbb{A}^2 \backslash \{(0,0)\} \subseteq A$ *and* $\mathrm{pr} : A' \longrightarrow \mathbb{P}^1$*. Then,*

  $$(\mathrm{pr}_*(\pi_* \xi^k_{A \times X}|_{A' \times X}))^{\Delta(\mathbb{G}_m)}$$

  is an equivariant sheaf on $\mathbb{P}^1$*. It is a vector bundle if the Frölicher spectral sequence degenerates in total degree d on the first page and then coincides with* $\xi_{\mathbb{P}^1}(H^k(X,\mathbb{C}), F, \overline{F})$*. It is semistable of slope k (i.e., a Simpson-style bundle) if and only if* $(H^k(X,\mathbb{C}), F, \overline{F})$ *is pure of weight k (which can be interpreted as only certain even-length zigzags appearing in* $\mathcal{A}_X^{\bullet,\bullet}$*, see proposition 1.18).*

4.2 The Weight Filtration as a Slope Filtration

Recall the following notions from [And09]: Let $\mathcal{C}$ be an essentially small quasi-abelian category and let $sk\mathcal{C}$ denote the set of isomorphism classes of objects of $\mathcal{C}$. Assume $\mathcal{C}$ is equipped with a rank function, i.e., a function

$$\mathrm{rk} : sk\mathcal{C} \longrightarrow \mathbb{N}$$

which is additive on short exact sequences and takes the value 0 only on the zero-object.

Example 4.3.

- *The category* Fil^n_k *of n-filtered vector spaces is essentially small quasi-abelian. It has a rank function that assigns the dimension of the underlying vector space to an object.*
- *The categories* $\mathrm{MHS}_{\mathbb{C}}$ *and* $\mathrm{MHS}_{\mathbb{R}}$ *are essentially small abelian (hence quasi-abelian). The rank function is given by the dimension of the underlying* $\mathbb{R}$*-vector space. Accordingly, all of the other equivalent descriptions from the previous chapter yield examples. In each case, the rank function is given by the rank of the underlying vector space or bundle.*

Definition 4.4. *A (rational)* ***degree function*** *is a function*

$$\deg : sk\mathcal{C} \longrightarrow \mathbb{Q},$$

which is additive on short exact sequences, takes the value 0 *on the zero object and has an associated* ***slope function***

$$\mu := \frac{\deg}{\mathrm{rk}} : sk\mathcal{C} \backslash \{0\} \longrightarrow \mathbb{Q},$$

which satisfies $\mu(M) \leq \mu(N)$ *for any morphism* $M \longrightarrow N$ *which is epi and monic.*

Given a degree function, an object N *of* $\mathcal{C}$ *is called* ***semistable*** *if for all subobjects* $M \hookrightarrow N$ *one has* $\mu(M) \leq \mu(N)$*.*

Remark 4.5.

- *If $\mathcal{C}$ is abelian, the condition on the slope function μ is empty as any epi-monic is an isomorphism.*
- *Degree and slope function determine each other.*
- *The sum of two degree (slope) functions is again a degree (slope) function and so it is a positive multiple of a degree (slope) function.*

Example 4.6. *For a multifiltered vector space $(V, F_1, ..., F_n)$ in Fil_k^n, set*

$$\deg(V) := \sum_{i=1}^{n}\sum_{p\in\mathbb{Z}} p\cdot\dim\mathrm{gr}_{F_i}^{p} V = \sum_{p_1,\ldots,p_n\in\mathbb{Z}} (p_1+...+p_n)\cdot\dim\mathrm{gr}_{F_1}^{p_1}...\mathrm{gr}_{F_n}^{p_n} V$$

This defines a degree function on Fil_k^n. For a proof of the second equality, see [And10].

- *An object in Fil_k^1 is semistable of slope $d\in\mathbb{Z}$ iff the filtration has its only jump in degree d.*
- *An object in Fil_k^2 is semistable of slope $d\in\mathbb{Z}$ iff it is pure of weight d.*

Remark 4.7. *It would be interesting to interpret the condition for three filtrations to be opposite as some kind of semistability condition, as this could yield a different algebraic proof that the category of Mixed Hodge Structures is abelian. In terms of the associated bundles on $\mathbb{P}^2$, this was done in [Pen03].*

The importance of the notion of degree (slope) function and semistability comes from the following theorem:

Theorem 4.8 ([And09])**.** *Given a slope function μ on $\mathcal{C}$, on every object V of $\mathcal{C}$ there is a uniquely determined functorial descending separated and exhaustive filtration $F_\mu^\bullet$ indexed by rational numbers and with finitely many jumps*

$$0\hookrightarrow F_\mu^{\alpha_1}\hookrightarrow ...\hookrightarrow F_\mu^{\alpha_r}=V$$

*with $\alpha_1 > ... > \alpha_n$ and $\mathrm{gr}^{\alpha_i} V$ is semistable of slope α_i. (Here $\mathrm{gr}_{F_{HN}}^{\alpha} := F_HN^{\alpha}/\bigcup_{\alpha'>\alpha}F_{HN}^{\alpha'}$). It is called the **slope filtration** or **Harder-Narasimhan filtration**.*

Example 4.9. *If $\underline{V} = (V, F)$ is an object in Fil_k^1, the Harder-Narasimhan filtration has integral jump points and coincides with F.*

Remark 4.10. *By the way the formalism is set up, the Harder-Narasimhan filtration is always descending. Since we are interested in the (ascending) weight filtration, we will call **ascending** Harder-Narasimhan (or slope) filtration the one defined by $(F_\mu)_\bullet := F_\mu^{-\bullet}$.*

Recall that for a Mixed Hodge Structure (H, W, F_1, F_2) (real or complex does not matter for this argument, we could also take $\mathbb{Q}$ here), the individual subspaces W_kH (with induced filtrations on $(W_k)_\mathbb{C}$) define a filtration by subobjects. In particular, setting $\deg_{\mathrm{MHS}}(H) := -\sum_{r\in\mathbb{Z}} k\cdot\mathrm{gr}_r^W H$, the corresponding ascending slope filtration is again the weight filtration considered as a filtration by sub

Hodge structures.

A more 'categorical' version of this definition is the following: First, define the degree on one-dimensional objects in $\mathrm{MHS}_{\mathbb{C}}$ as:

$$\begin{aligned} \deg'_{\mathrm{MHS}} : \mathrm{Pic}(\mathrm{MHS}_{\mathbb{C}}) &\longrightarrow \mathbb{Z} \\ H(p,q) &\longmapsto -(p+q). \end{aligned}$$

Then the general degree function is given by $\deg'_{\mathrm{MHS}}(H) := \deg(\det H)$. It is then a basic result about slope functions on filtered vector spaces (see e.g. [And09]) that $\deg_{\mathrm{MHS}} = \deg'_{\mathrm{MHS}}$. Note that, using that $\mathrm{MHS}_{\mathbb{C}}$ is an abelian tensor category with internal hom's and $\mathrm{End}(1) = \mathbb{C}$, the definition of $\dim H$ and $\Lambda^r H$ can be given in purely category-theoretic terms (namely as the trace of the identity, resp. the image of an 'antisymmetrisation'-map, see [Del] for details).

So, because all the equivalences in the previous chapter are compatible with tensor products, we get an intrinsic characterisation of the weight filtration for $\mathcal{DO}_{\mathbb{C}}$, $\mathrm{Simp}_{\mathbb{C}}(\mathbb{P}^1, \mathbb{G}_m)$, $\mathrm{Penac}_{\mathbb{C}}(\mathbb{P}^2, \mathbb{G}_m^2)$ and $\mathrm{Bun}_{\nabla}(\mathbb{A}^2, \mathbb{G}_m^2)$ and similarly for their categories of fixed points. Of course, the categorically defined dimension and determinant are as expected in each case, i.e., the rank and top exterior power of the underlying vector space or bundle with induced extra structures.

As an application, we construct a direct equivalence

$$\Phi^{\mathrm{Simp}}_{\mathrm{Penac}} : \mathrm{Penac}_{\mathbb{C}}(\mathbb{P}^2, \mathbb{G}_m^2) \longrightarrow \mathrm{Simp}_{\mathbb{C}}(\mathbb{P}^1, \mathbb{G}_m)$$

compatible with the involution.

For this, let

$$j : A := \mathbb{P}^2 \backslash (V(X_0) \cup V(X_1, X_2)) \subseteq \mathbb{P}^2$$

denote the inclusion of the punctured affine space stable under $\mathbb{G}_m^2$ and the involution, let

$$A \longrightarrow \mathbb{P}^1$$

be the projection. If HN denotes the functor which sends an equivariant bundle on $\mathbb{P}^2$ to the same bundle equipped with its ascending Harder-Narasimhan filtration, we can define

$$\Phi^{\mathrm{Simp}}_{\mathrm{Penac}} := p_*^{\Delta(\mathbb{G}_m)} \circ j^* \circ HN,$$

where $p_*^{\Delta(\mathbb{G}_m)}$ means equivariant pushforward, i.e., the usual sheaf-theoretic pushforward followed by taking invariants for the $\Delta(\mathbb{G}_m)$-action (which acts trivially on $\mathbb{P}^1$). Let $\Phi^{\mathrm{Penac}}_{\mathrm{MHS}}$ and $\Phi^{\mathrm{Simp}}_{\mathrm{MHS}}$ denote the functors (Rees-bundle construction) from $\mathrm{MHS}_{\mathbb{C}}$ to $\mathrm{Penac}_{\mathbb{C}}(\mathbb{P}^2, \mathbb{G}_m^2)$ and $\mathrm{Simp}_{\mathbb{C}}(\mathbb{P}^1, \mathbb{G}_m)$ defined in the previous chapter.

Proposition 4.11. *The functor $\Phi^{\mathrm{Simp}}_{\mathrm{Penac}}$ is compatible with the involutions and there is a natural isomorphism $\Phi^{\mathrm{Simp}}_{\mathrm{MHS}} \cong \Phi^{\mathrm{Simp}}_{\mathrm{Penac}} \circ \Phi^{\mathrm{Simp}}_{\mathrm{MHS}}$. In particular, $\Phi^{\mathrm{Simp}}_{\mathrm{Penac}}$ is an equivalence of categories.*

Proof. Given a complex Mixed Hodge Structure (H, W, F_1, F_2), by definition of the projective Rees-bundle, there is a natural isomorphism

$$\xi_{\mathbb{P}^2}(H)|_A \cong \xi_{\mathbb{A}^2}(H, F_1, F_2)|_A$$

and so the Harder Narasimhan filtration on it is given by the subbundles $\xi_{\mathbb{A}^2}(W_\bullet H, F_1, F_2)|_A$. But $p_*^{\Delta(\mathbb{G}_m)} \circ \xi_{\mathbb{A}^2}|_A$ is naturally isomorphic to $\xi_{\mathbb{P}^1}$. The involution $[a : b : c] \mapsto [\overline{a} : \overline{c} : \overline{b}]$ on $\mathbb{P}^2(\mathbb{C})$ induces $(x, y) \mapsto (\overline{y}, \overline{x})$ and thus after modding out $\Delta(\mathbb{G}_m)(\mathbb{C})$ becomes $[x : y] \mapsto [\overline{y} : \overline{x}]$. $\square$

4.3 Kato's Local Archimedean Height

4.3.1 Generalities on Motives

This section deals with an invariant of motives. As the theory of motives is still in parts conjectural, we very briefly recall some of its underlying assumptions (possibly in a too idealised form) which are needed for the upcoming constructions. We used [Kat18] and [Jan90] as sources to which we refer for more information.

Let K be a number field. Intuitively, a motive should be something like an essential piece of an algebraic variety over K encoding cohomological information: The category Mot_K of motives over K should be a $\mathbb{Q}$-linear, abelian tensor category and every object should be equipped with an ascending filtration W, called the weight filtration. Furthermore, to a motive M, it should be possible to associate the following 'realisations' in a functorial way:

1. The **de-Rham realisation** M_{dR}, which is an K-vector space with a descending filtration F, the **Hodge filtration**.
2. A free $A^f := \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$-module of finite rank $M_{ét}$ equipped with a continuous action of the absolute Galois-group G_K: the **étale realisation**. In particular, for each prime number p, one has the $\mathbb{Q}_p$-component $M_{ét,\mathbb{Q}_p}$.
3. For every $a : K \longrightarrow \mathbb{C}$ a finite dimensional $\mathbb{Q}$-vector space $M_{a,B}$, the **Betti-realisation**, together with isomorphisms $M_{a,B} \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow M_{dR} \otimes_{a,K} \mathbb{C}$, equipping the right hand-side with a (rational) Mixed Hodge structure M_a (with weight filtration induced by the one on M) and similar comparison isomorphisms with the étale realisation.

In addition, there should be a functor

$$\begin{aligned} \{\text{algebraic varieties over } K\} &\longrightarrow \mathrm{Mot}_K \\ V &\longmapsto H(V) \end{aligned}$$

s.t. composing this with the realisations yields the de-Rham, étale and Betti cohomology theories.

A motive is called **pure of weight** w if $\mathrm{gr}_s^W M = 0 \ \forall s \neq w$. A pure motive is a direct sum of pure motives of (possibly different) given weights. A general motive is also called a mixed motive.

For every $r \in \mathbb{Z}$, there is a special pure motive $\mathbb{Q}(r)$ of weight $-2r$, the r-th **Tate motive**. Its de-Rham realisation is a one-dimensional vector space with unique jump of the filtration in degree $-r$.

A pure motive M of weight w is called **polarised** if there is a morphism

$$M \otimes M \longrightarrow \mathbb{Q}(-w),$$

which induces a polarisation of the pure Hodge-structure M_a for all a as above. A polarisation is a polarisation on every summand of fixed weight and a mixed motive M is called (graded) polarised if there is a polarisation on $\operatorname{gr}_\bullet^W M$ (see definition 3.39 for the definition of polarisations in the Hodge-context).

By a motive with $\mathbb{Z}$-coefficients, one means a motive M together with a G_K-stable $\hat{\mathbb{Z}}$-lattice $T \subseteq H_{ét}(M)$.

We note that at the moment there is, to our knowledge, no satisfying definition of the abelian category of mixed motives. However, in the following, we will only deal with the Hodge structures coming from de-Rham and Betti-realisation, which are so kind as to lie in perfectly existing categories.

4.3.2 Kato's Finiteness Conjecture

In [Kat18], building on earlier work [Kat14] and [Kat13], K. Kato proposes a definition of the height of a mixed motive: To every mixed motive M over a number field, he associates a non-negative real number $\operatorname{ht}(M)$ and conjectures:[3]

Conjecture: Let K be a number field. Assume we are given a polarised pure motive M_w of weight w for each $w \in \mathbb{Z}$ with $\mathbb{Z}$ coefficients over K s.t. $M_w = 0$ for all but finitely many w. Then, for every $C \in \mathbb{R}_{\geq 0}$, there exist only finitely many isomorphism classes of graded-polarised mixed motives M that satisfy $\operatorname{gr}_w^W M = M_w$ and $\operatorname{ht}(M) \leq C$.

The number $\operatorname{ht}(M)$ is defined as a sum

$$\operatorname{ht}(M) := \operatorname{ht}_1(M) + \sum_v \operatorname{ht}_v(M),$$

where $\operatorname{ht}_1(M)$ depends only on the extensions[4]

$$0 \longrightarrow \operatorname{gr}_d^W M \longrightarrow \frac{W_{d+1}M}{W_{d-1}M} \longrightarrow \operatorname{gr}_{d+1}^W \longrightarrow 0$$

and v runs over the places (i.e., equivalence classes of absolute values) of K and $\operatorname{ht}_v(M)$ depends only on the étale realisation for finite v. Recall that an archimedean v can be described by a conjugate pair of embeddings $a, \overline{a} : K \longrightarrow$

---

[3] 2.1.2 in [Kat18]. He also proposes definitions for pure motives and has finiteness conjectures without restricting to fixed associated graded. However, we restrict to this case as it has a particularly nice interpretation in terms of the Kapranov/Goncharov description of Mixed Hodge Structures.

[4] Note that in $\mathrm{MHS}_{\mathbb{C}}$ or $\mathrm{MHS}_{\mathbb{R}}$ there are no extensions between pure HS of weights $k, k+1$ as can be seen easily e.g. by the Deligne-operator description.

$\mathbb{C}$ (which may or may not coincide). The number ht_v for archimedean v depends only on the real Mixed Hodge structure given by M_a.

Kato gave two different definitions of ht_v in the archimedean in the archimedean case, which we review below. The new contribution here is to propose a third definition to and show that all three are equivalent in the following sense:

Definition 4.12. *Let S be a set. Two functions $f, g : S \longrightarrow \mathbb{R}_{\geq 0}$ are called **equivalent** if there are monotoneously increasing functions $P, Q : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ s.t.*

$$(P \circ f)(s) \geq g(s) \text{ and } (Q \circ g)(s) \geq f(s) \text{ for all } s \in S$$

This is a good notion if we consider the finiteness conjecture because of:

Lemma 4.13. *Let S be a set and $f, g : S \longrightarrow \mathbb{R}_{\geq 0}$ two equivalent functions. The sets $F_c := \{s \in S \mid f(s) \leq c\}$ are finite for all $c \in \mathbb{R}_{\geq 0}$ iff the sets $G_c := \{s \in S \mid g(s) \leq c\}$ are finite for all $c \in \mathbb{R}_{\geq 0}$.*

Proof. Let P, Q be as in the definition of equivalent functions. As $f(s) \leq c$ implies $g(s) \leq P(f(s)) \leq P(c)$, we have $F_c \subseteq G_{P(c)}$ and similarly $G_c \subseteq F_{Q(c)}$. This implies the claim. □

In order to apply this to the archimedean summand, we note:

Lemma 4.14. *For three functions $f, g, h : S \longrightarrow \mathbb{R}_{\geq 0}$ with g, h equivalent, $f + g$ and $f + h$ are equivalent.*

Proof. Let $P : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ be monotoneously increasing s.t. $P(g(s)) \geq h(s)$. Set $\widetilde{P}(s) := s + P(s)$. Then,

$$\widetilde{P}(f(s) + g(s)) = f(s) + g(s) + P(f(s) + g(s)) \geq f(s) + P(g(s)) \geq f(s) + h(s).$$

The other condition follows by reversing the roles of g and h. □

In particular, replacing ht_v by an equivalent function for archimedean places v should not change the validity of the finiteness conjecture.

4.3.3 Three Definitions of the Archimedean Height

Notational convention: In this section, for a bigraded vector space, we write upper indices $()^{p,q}$ for the projection to the (p, q) component and lower indices $()_{p,q}$ for the projection to the $(-p, -q)$ component. Similarly, for a (bi-)graded vector space $()^d$ means (total-)degree d part, $()_d$ means (total-)degree $-d$.

In the following, we use the description of Mixed Hodge Structures via the Deligne operator (cf. section 3.2). I.e., when we say Mixed Hodge Structure, we mean a tupel (V, D), where $V = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$ is a bigraded finite-dimensional complex vector space with a $\mathbb{C}$-antilinear involution σ satisfying $\sigma V^{p,q} = V^{q,p}$ (i.e., a direct sum of pure Hodge Structures) and $D : V \longrightarrow V$ is an endomorphism satisfying

$$DV^{p,q} \subseteq \bigoplus_{p' < p, q' < q} V^{p',q'} \tag{1}$$

and

$$\sigma D \sigma = -D. \tag{2}$$

Let us denote the set of endomorphisms satisfying (1) and (2) by $\mathrm{End}_{\mathbb{R}}^{\mathrm{Del}}(V)$. It parametrises the set of real Mixed Hodge Structures (in the usual sense) with associated graded V.

Assume we fix the direct sum of pure Hodge structures V and assume it is equipped with a polarisation P_V. Let $G := \mathrm{Aut}(V)$ be the set of degree, involution and polarisation preserving automorphisms of V. The group G acts by conjugation on $\mathrm{End}_{\mathbb{R}}^{\mathrm{Del}}(V)$ and the isomorphism classes of MHS with associated graded equal to V are the orbits of this action. We will thus define the local archimedean heights as functions on $\mathrm{End}_{\mathbb{R}}^{\mathrm{Del}}(V)$, invariant under G.

Let's denote by $\mathrm{End}^-(V) := \bigoplus_{d>0} \mathrm{End}(V)_d$ the endomorphisms of negative total degree and by

$$N : \mathrm{End}^-(V) \longrightarrow \mathbb{R}$$

the operator given on each piece $\mathrm{End}(V)_d$ by

$$v \mapsto \|v\|_d^{\frac{1}{d}},$$

where $\|.\|_d$ is the norm induced on $\mathrm{End}(V)_d$ by the polarisation P_V. Denote the restriction to $\mathrm{End}_{\mathbb{R}}^{\mathrm{Del}}(V)$ by the same letter N. Up to a factor of $\frac{1}{2i}$, which does not change any of our conclusions, the following is the definition in [Kat18]:

Definition 4.15. *The **holonomy height** of a Hodge Structure (V, D) is defined as*

$$\mathrm{ht}^{\mathrm{hol}}(D) := N(D).$$

Remark 4.16. *If $(\mathcal{V}, \nabla)$ is the equivariant bundle with connection associated to (V, D) by Goncharov's or Kapranov's construction, N can naturally be regarded as a function on the fibre at $(0, 0)$ and D is the logarithm of the holonomy along a certain path, hence the name.*

In the earlier paper [Kat13], Kato gives a different definition of the archimedean height contribution, which we briefly recall here:

Associated with D, there is a nilpotent endomorphism $\zeta := \zeta(D)$ coming from the theory of degenerations of Hodge structures as in [CKS86]. We do not give its definition here, but we recall some facts about it. In order to formulate them, let $F_{(p,q)}$ be the free Lie-algebra on generators $X_{r,s}$ with $0 < r \leq p$ and $0 < s \leq q$. It is bigraded by giving $X_{r,s}$ the bidegree $(-r, -s)$. We consider the elements of $F_{(p,q)}$ as (Lie-)polynomials in the $X_{r,s}$. In particular, for every bigraded Lie-algebra A and collection of elements $(a_{r,s})_{0<r\leq p, 0<s\leq q}$, there is an evaluation homomorphism

$$\begin{aligned} F_{(p,q)} &\longrightarrow A \\ S &\longmapsto S((a_{r,s})_{0<r\leq p, 0<s\leq q}), \end{aligned}$$

defined by sending $X_{r,s}$ to $a_{r,s}$. The following is proven in [KNU08]:

Proposition 4.17. *For every $p, q \in \mathbb{Z}$, there is an element $S_{(p,q)} \in F_{(p-1,q-1)}$ s.t. for any Mixed Hodge Structure (V, D), we have*

$$\zeta(D)_{p,q} = c_{p,q} D_{p,q} + S_{(p,q)}((D_{r,s})_{0<r<p, 0<s<q})$$

with

$$c_{p,q} = \frac{1}{2^{p+q}} \left(\sum_{q \leq k < p} \binom{p+q-1}{k} \right).$$

In particular, if $f : (V, D_V) \longrightarrow (W, D_W)$ is a morphism of MHS, we have $f \circ \zeta(D_V)_{p,q} = \zeta(D_W)_{p,q} \circ f$.

Given a MHS (V, D), $\zeta(D)$ yields a new bigrading on V with (p, q) component $\exp(\zeta)V^{p,q}$. Define $\tilde{N}$ just as N, but with respect to this new bigrading. It is related to the old N by $\tilde{N}(D) = N(\exp(-\zeta)D\exp(\zeta))$. The following is the definition given in [Kat13]:[5]

Definition 4.18. *The **degeneration height** of a Hodge Structure (V, D) is defined as*

$$\mathrm{ht}^{dgn}(D) := \tilde{N}(D) = N(\exp(-\zeta)D\exp(\zeta)).$$

Note that by the functoriality statement in proposition 4.17, this indeed yields a well-defined function on isomorphism classes.

A third definition can be given by Goncharov's description of Mixed Hodge structures. Let $(\mathcal{V}, \nabla)$ be the bundle associated to (V, D). In the canonical trivialisation of $\mathcal{V}$ with fibre V (proposition 3.28), the connection takes the form

$$d + \sum_{p,q>0} A_{p,q}(t_1^{p-1}t_2^q dt_1 - t_1^p t_2^{q-1} dt_2),$$

where $A_{p,q} \in \mathrm{End}(V)_{p,q}$. Setting $A := \sum_{p,q} A_{p,q}$, we can define:

Definition 4.19. *The **Goncharov-height** of a Hodge structure (V, D) is defined as*

$$\mathrm{ht}^{\mathrm{Gon}}(D) := N(A).$$

By the functoriality statement in proposition 3.28, this is a well-defined function on isomorphism classes.

Remark 4.20. *If one wants to avoid using the canonical trivialisation in the definition of* $\mathrm{ht}^{\mathrm{Gon}}$*, one could consider the following variant: Let $\nabla^2 \in \mathrm{End}(\mathcal{V}) \otimes \Omega^2_{\mathbb{A}^2}$ be the curvature of the connection. Restricting to the fibre at $(1, 1)$ and plugging in the standard basis of the tangent space yields an endomorphism of the fibre at $(1, 1)$:*

$$\nabla^2_{(1,1)} := \nabla^2_{\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}}(1, 1) \in \mathrm{End}(\mathcal{V}(1, 1))$$

Given a path α along a straight line with $\alpha(0) = (1, 1)$ and $\alpha(1) := (0, 0)$, define

$$\widetilde{A} := \mathrm{Par}^{\nabla}_{\alpha} \circ \nabla^2_{(1,1)} \circ \mathrm{Par}^{\nabla}_{\alpha^{-1}} \in \mathrm{End}(\mathcal{V}(0, 0))$$

---

[5] again we changed the argument of $\tilde{N}$ from $\frac{1}{2i}D$ to D.

and set

$$\mathrm{ht}^{\mathrm{Gon}'} := N(\widetilde{A}).$$

Using the fact that ∇ *agrees with the canonical flat connection on rays through the origin and the explicit expression for the curvature in the canonical trivialisation (proposition 3.31), one can check that* $-(p+q)\widetilde{A}_{p,q} = A_{p,q}$*. In particular, theorem 4.21 below remains valid.*

Theorem 4.21. *Let* V *be a direct sum of pure Hodge Structures. The three functions* $\mathrm{ht}^{\mathrm{hol}}, \mathrm{ht}^{\mathrm{dfm}}, \mathrm{ht}^{\mathrm{Gon}} : \mathrm{End}^{\mathrm{Del}}_{\mathbb{R}}(V) \longrightarrow \mathbb{R}$ *are equivalent.*

For the proof, we use the following

Lemma 4.22. *For homeomorphism* φ *of* $\mathrm{End}^{\mathrm{Del}}_{\mathbb{R}}(V)$ *to itself, the functions* N *and* $N \circ \varphi$ *are equivalent.*

Proof. Define $P : \mathbb{R}_{\geq} 0 \longrightarrow \mathbb{R}_{\geq 0}$ by $P(r) := \max\{N \circ \varphi(N^{-1}[0,r])\}$. Note that the maximum exists because φ is continuous and N is proper (it's essentially a norm). Then we have $P \circ N \geq N \circ \varphi$. Define Q in a similar way, exchanging φ with φ^{-1}. ☐

Proof. All three heights are defined by applying N to certain elements of $\mathrm{End}^{\mathrm{Del}}_{\mathbb{R}}(V)$. By lemma 4.22, it suffices to show that the maps

$$\varphi_1 : D \longmapsto \exp(-\zeta)\frac{1}{2i}D\exp(\zeta)$$

and

$$\varphi_2 : D \longmapsto A = \sum_{p,q>0} A_{p,q}$$

are homeomorphisms.

Let us show this for φ_1. Expanding the exponential series and plugging in the explicit formula for ζ, one sees that

$$(\varphi_1 D)_{p,q} = \frac{1}{2i}D_{p,q} + \text{continuous function in the } D_{r,s} \text{ with } r > p,\ s > q.$$

The conclusion follows from lemma 3.34. For φ_2, this was already shown in corollary 3.33. $\Box_{Prop}$

4.4 Gamma-Factors

Let (H, F_1, F_2, W) be a Complex Mixed Hodge structure and consider the following filtration on H:

$$\gamma^\bullet := F_1^\bullet \cap F_2^\bullet.$$

When H is a real Hodge structure with underlying real vector space $H_{\mathbb{R}}$, this could also be described as

$$\gamma^\bullet := F^\bullet \cap H_{\mathbb{R}}.$$

In [Den01], C. Deninger gives a formula for the Gamma-factors attached to (real) Mixed Hodge structure in terms of the bundle $\xi_{\mathbb{A}^1}(H, \gamma)$. We describe in the following how to obtain this bundle from the various geometric interpretations of Mixed Hodge Structures introduced in the previous chapter. This was inspired by a construction found in [Pri16].

Let k be a field and consider the morphism of varieties over k

$$N : \mathbb{A}^2 \longrightarrow \mathbb{A}^1$$

given on points by $(a, b) \mapsto a \cdot b$, i.e., induced from the map of rings

$$\begin{aligned} N^* : k[t] &\longrightarrow k[z_1, z_2] \\ t &\longmapsto z_1 \cdot z_2. \end{aligned}$$

Note that the restriction of N is part of the following short exact sequence of algebraic groups:

$$1 \longrightarrow \Delta'(\mathbb{G}_m) \longrightarrow \mathbb{G}_m^2 \xrightarrow{N} \mathbb{G}_m \longrightarrow 1$$

Here, the kernel is given on points as $\Delta'(\mathbb{G}_m)(\mathbb{C}) = \{(a, a^{-1}) | a \in k^\times\}$. Note that if $k = \mathbb{C}$, $\mathbb{G}_m(\mathbb{C})$ is equipped with complex conjugation and $\mathbb{G}_m^2(\mathbb{C})$ is equipped with the involution $(a, b) \mapsto (\overline{b}, \overline{a})$, this induces a sequence on the real points (i.e., the fixed points of the involution):

$$1 \longrightarrow S^1 \longrightarrow \mathbb{C}^\times \xrightarrow{z \mapsto z \cdot \overline{z}} \mathbb{R}^\times$$

Lemma 4.23. *Let* (V, F_1, F_2) *be an object in* Fil_k^2 *and* $\gamma^\bullet := F_1^\bullet \cap F_2^\bullet$. *There is a natural* $\mathbb{G}_m$*-equivariant identification*

$$(N_* \xi_{\mathbb{A}^2}(V, F_1, F_2))^{\Delta'(\mathbb{G}_m)} \cong \xi_{\mathbb{A}^1}(V, \gamma).$$

Proof. This can be computed on global sections. We have

$$\Gamma(\mathbb{A}^1, N_* \xi_{\mathbb{A}^2}(V, F_1, F_2)) = \Gamma(\mathbb{A}^2, \xi_{\mathbb{A}^2}(V, F_1, F_2)) = \sum_{p,q \in \mathbb{Z}} F_1^{-p} \cap F_2^{-q} z_1^p z_2^q,$$

where we consider the right hand side as a $k[t]$-module via N^*. The sum is direct if considered as vector spaces (as opposed to $k[t]$-modules). The $\mathbb{G}_m^2$-action in this description is given by the degrees of the z_i. Taking the invariants w.r.t. $\Delta'(\mathbb{G}_m)$ means looking at only those components with $p = q$ and identifiying t with $z_1 z_2$ yields the result. ☐

Using this lemma, one obtains the following result almost by definition of the projective Rees-bundles.

Proposition 4.24. *Consider the following diagram:*

$$\begin{tikzcd}
\mathbb{A}^2 \backslash \{0\} \arrow[r, hook, "j"] \arrow[d, "p"] & \mathbb{A}^2 \arrow[r, hook, "\varphi"] \arrow[d, "N"] & \mathbb{P}^2 \\
\mathbb{P}^1 & \mathbb{A}^1 &
\end{tikzcd}$$

Here, p *is reduction modulo the diagonal* $\mathbb{G}_m$*-action,* φ *is the inverse of the coordinate* $\{X_0 \neq 0\} \xrightarrow{\sim} \mathbb{A}^2$ *and* j *is the inclusion.*

Let (H, W, F_1, F_2) be a finite dimensional vector space with two descending filtrations F_i and one ascending filtration W and let $\gamma^\bullet := F_1^\bullet \cap F_2^\bullet$. There are the following isomorphisms, natural in H:

$$(N_* \varphi^* \xi_{\mathbb{P}^2}(H, W, F_1, F_2))^{\Delta'(\mathbb{G}_m)} \cong \xi_{\mathbb{A}^1}(H, \gamma) \cong ((N \circ j)_* p^* \xi_{\mathbb{P}^1}(H, F_1, F_2))^{\Delta'(\mathbb{G}_m)}$$

In particular, if (H, W, F_1, F_2) is a Mixed Hodge Structure, we get a description of Deninger's bundle in terms of the associated bundles on $\mathbb{P}^1$ and $\mathbb{P}^2$. Note that the underlying bundle in Goncharov's description is isomorphic to $\xi_{\mathbb{A}^2}(\tilde{H}_W, F_1, F_2)$, where $\tilde{H}_W$ as a vector space is $H_W := \bigoplus_{r \in \mathbb{Z}} \mathrm{gr}_r^W H$ but with inverse grading $\tilde{H}_W^{p,q} := H^{-p,-q}$ and F_1, F_2 are the two filtrations given by the grading. So applying N_* in this case yields the bundle governing the Gamma-factors of $\tilde{H}_W$ instead of H itself.

4.5 Hodge Structures and Loops

In [Kap12, rem. 0.4.], it is stated that via the description of Mixed Hodge Structures via equivariant vector bundles with a connection, the pro-unipotent group $\mathfrak{U}$ of [Del94] 'is realised as (the pro-algebraic completion of) the group of piecewise smooth loops in $\mathbb{C}$ considered up to reparametrisation and cancellation (with the group operation being the composition of loops). This group of loops acts in any bundle with connection via the holonomy.' This statement is not proven as such but rather discussed at a Lie-algebra level there. We will present one possibility of making a variant of this statement precise.

4.5.1 Generalities on Groups of Loops

Let X be a differentiable manifold. For two points $x, y \in X$, recall that a path in X from x to y is defined to be a continuous map

$$\gamma : [0, 1] \longrightarrow X$$

s.t. $\gamma(0) = x$, $\gamma(1) = y$. It is called piecewise smooth if there is a subdivision of $[0, 1]$ into closed intervalls, s.t. γ restricted to each of them is smooth.
Concatenation of two paths γ from x to y and γ' from y to z is defined by

$$\gamma' \circ \gamma(s) := \begin{cases} \gamma(2s) & 0 \leq s \leq \frac{1}{2} \\ \gamma'(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Consider the following two types of homotopies:

Cancellation
Consider a path $\gamma : [0, 1] \longrightarrow X$, s.t. there is a subinterval $[a, b] \subseteq [0, 1]$ with $\gamma(a + s) = \gamma(b - s)$ for all $s \in [0, b - a]$. This means, you take some path and then go back the same way with the same speed. The path that for arguments in $[a, b]$ stays constant on $\gamma(a)$ and is equal to γ otherwise is homotopic to this

via the following homotopy:

$$\begin{aligned} C : [0,1]\times[0,1] &\longrightarrow X \\ (s,t) &\longmapsto \begin{cases} \gamma(a+t(s-a)) & \text{if } s\in[a,\frac{a+b}{2}] \\ \gamma(b+t(s-b)) & \text{if } s\in[\frac{a+b}{2},b] \\ \gamma(s) & \text{else} \end{cases} \end{aligned}$$

Reparametrisation
Consider a (piecewise) smooth function $c : [0,1] \longrightarrow [0,1]$ with $c(0) = 0, c(1) = 1$. Then a path γ is homotopic to $\gamma \circ c$ via

$$\begin{aligned} R : [0,1]\times[0,1] &\longrightarrow X \\ (s,t) &\longmapsto \gamma((1-t)s + tc(s)). \end{aligned}$$

A path is called a loop based at $x \in X$ if $\gamma(0) = \gamma(1) = x$. Let us call the set of all piecewise smooth loops based at a fixed point x, modulo the equivalence relation generated by homotopies of these two types, the **group of loops** at x and denote it by $\mathfrak{L}(X, x)$. It is actually a group with unit element $e_x = \mathrm{const}_x : t \mapsto x$, inverse of a path $\gamma(s)$ given by $\gamma(1-s)$ (due to the cancellation homotopies) and associativity ensured by the reparametrisation.[6]

Given a complex vector bundle with connection $(\mathcal{V}, \nabla)$ on X, parallel transport yields a representation

$$\begin{aligned} \mathrm{Hol} : \mathfrak{L}(X,x) &\longrightarrow \mathrm{GL}(\mathcal{V}(x)) \\ \gamma &\mapsto \mathrm{Hol}_\gamma\,. \end{aligned}$$

If we additionally assume that $\mathcal{V}$ is an equivariant vector bundle for an action of a Lie-group G s.t. x is a fixed point, one gets an additional representation

$$\rho_G : G \longrightarrow \mathrm{GL}(\mathcal{V}(x))$$

and a morphism

$$\begin{aligned} G &\longrightarrow \mathrm{Aut}(\mathfrak{L}(X,x)) \\ g &\longmapsto (\gamma \mapsto g.\gamma := L_g \circ \gamma), \end{aligned}$$

where L_g means left translation by g. If ∇ is equivariant, these satisfy the compatibility relation

$$\mathrm{Hol}_{g.\gamma} = \rho(g)\,\mathrm{Hol}_\gamma\,\rho(g^{-1}).$$

So ρ_G and Hol combine into a representation of the semidirect product

$$\rho^\nabla : \mathfrak{L}(X,x) \rtimes G \longrightarrow \mathrm{GL}(\mathcal{V}(x)).$$

Two natural questions arise:

1. To what extent is an equivariant vector bundle with equivariant connection determined by ρ^∇?

---

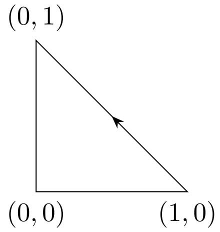
[6] As a more elegant but less explicit variant one could consider the equivalence relation of **thin homotopies**, i.e., those homotopies which sweep out no area meaning they have a differential of rank ≤ 1. Cancellation and Reparametrisation are thin.

2. Can every representation ρ of the semidirect product arise in this way?

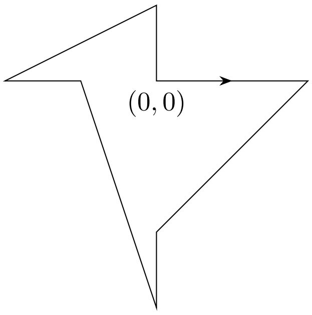
Non-equivariant versions of these questions, i.e., just considering representations of the group of loops, have been studied e.g., in [Bar91], [Lew93], [CP94], [CP99], [SW07], [BH11]. There are many small variants in the precise settings considered, but roughly the answers are that the holonomy representation determines a connection completely but only 'continuous' representations can arise. We will not go into detail about the different approaches.

4.5.2 Hodge Structures as Representations of a Group of Loops

Let us consider holomorphic vector bundles with connection on $X := \mathbb{C}^2$ with the action of $G := (\mathbb{C}^\times)^2$ by multiplication with fixed point $x := (0, 0)$.
Let $\gamma_{Gon} \in \mathfrak{L}(X, x)$ be the path



which we have already seen in section 3.6. Denote by $\mathfrak{L}^{Gon}$ the subgroup generated by the G-orbit of γ_{Gon}, equipped with the subspace topology.
The following would be a (representative of a) typical element in $\mathfrak{L}^{Gon}$ (with image in $R^2 \subseteq \mathbb{C}^2$):



As a group, $\mathfrak{L}^{Gon}$ can be identified with the free group on $(\mathbb{C}^\times)^2$:

$$\begin{aligned} F_{(\mathbb{C}^\times)^2} &\xrightarrow{\sim} \mathfrak{L}^{Gon} \\ (a, b) &\longmapsto (a, b).\gamma_{Gon} \end{aligned}$$

We will be considering finite dimensional representations ρ of $\mathfrak{L}^{Gon} \rtimes G$ on complex vector spaces V. Such a ρ is given equivalently by representations ρ_G and $\rho_{\mathfrak{L}^{Gon}}$ of G and $\mathfrak{L}^{Gon}$ on V, which satisfy

$$\rho_G(\lambda)\rho_{\mathfrak{L}^{Gon}}(\gamma)\rho_G(\lambda^{-1}) = \rho_{\mathfrak{L}^{Gon}}(\lambda.\gamma)$$

for all $\lambda \in G, \gamma \in \mathfrak{L}^{Gon}$.

Definition 4.25. *A representation $\rho : \mathfrak{L}^{Gon} \rtimes G \longrightarrow \mathrm{GL}(V)$ on some complex vector space V is called* ***ad-hoc continuous*** *if the following conditions are satisfied:*

1. *The associated representation ρ_G is holomorphic.*
2. *For every sequence $(\lambda_i)_{i\in\mathbb{N}}$ with $\lambda_i \in G \subseteq \mathbb{C}^2$ converging to $(0,0)$, one has*

$$\rho_{\mathfrak{L}^{Gon}}(\lambda_i.\gamma_{Gon}) \overset{i\to\infty}{\longrightarrow} \mathrm{Id}_V \,.$$

The category of ad-hoc continuous representations is denoted $\mathrm{Rep}^{ahc}(\mathfrak{L}^{Gon} \rtimes G)$*.*

The involution $\tau : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ given by $(a,b) \mapsto (\bar{b}, \bar{a})$ also acts on $\mathfrak{L}^{Gon}$ and G. Together with the involution $\varphi : V \mapsto \overline{V}$, it induces an involution on the category $\mathrm{Rep}^{ahc}(\mathfrak{L}^{Gon} \rtimes G)$ given by

$$\rho \longmapsto \varphi(\rho \circ \tau(\_))\varphi^{-1}.$$

The fixed points can be identified with representations on complex vector spaces of the group $\rho : \mathfrak{L}^{Gon} \rtimes G \rtimes \langle\tau\rangle$ in which the first two factors act complex linearly and τ acts conjugation antilinearly.

Proposition 4.26. *The functor*

$$\begin{aligned} \mathrm{Rep}^{ahc}(\mathfrak{L}^{Gon} \rtimes G) &\longrightarrow \mathcal{DO}^{exp} \\ (\rho, V) &\mapsto (V, \rho(\gamma_{Gon})) \end{aligned}$$

is well-defined and an yields an equivalence of $\mathbb{C}$-linear tensor categories with involution.

Proof. Giving a holomorphic representation of $G = (\mathbb{C}^\times)^2$ on a complex vector space V is equivalent to demanding that there is a decomposition $V = \bigoplus V^{p,q}$, s.t. ρ_G acts on $V^{p,q}$ via the character $(\lambda_1, \lambda_2) \mapsto \lambda_1^{-p}\lambda_2^{-q}$.

Knowing ρ_G, a representation ρ on the LHS is entirely determined by the image of the generating path of $\mathfrak{L}^{Gon}$, $d_\rho := \rho(\gamma_{Gon})$. Writing $d_\rho = \sum_{p,q\in\mathbb{Z}} d^{p,q}$ according to the grading induced by ρ_G, we have for any $(\lambda_1, \lambda_2) \in G$

$$\rho((\lambda_1, \lambda_2).\gamma_{Gon}) = \sum_{p,q\in\mathbb{Z}} \lambda_1^{-p}\lambda_2^{-q} d^{p,q}$$

and one sees that condition 2 in definition 4.25 is equivalent to $d^{p,q} = 0$ for all $p > 0, q > 0$ and $d^{0,0} = \mathrm{Id}_V$, i.e., $d_\rho \in \mathrm{End}^{\mathrm{Del}}(V)$. In particular, the above functor is well-defined and an inverse is given by prescribing ρ_G according to the grading and $\rho(\gamma_{Gon})$ to be the Deligne-operator.

To see compatibility with the involutions, note that $\tau.\gamma_{Gon} = \gamma_{Gon}^{-1}$. □

Corollary 4.27. *The functor*

$$\mathrm{Bun}_\nabla(\mathbb{C}^2, (\mathbb{C}^\times)^2) \longrightarrow \mathrm{Rep}^{ahc}(\mathfrak{L}^{Gon} \rtimes G)$$

given by holonomy and restriction to the fibre at $(0,0)$ is an equivalence of $\mathbb{C}$-linear tensor categories with involution.

Proof. In the following commutative diagram, two of the three arrows are equivalences of categories

$$\begin{tikzcd}
\mathrm{Bun}_{\nabla}(\mathbb{C}^2, (\mathbb{C}^\times)^2) \arrow[rr] \arrow[dr] & & \mathcal{DO}^{exp} \\
& \mathrm{Rep}^{ahc}(\mathfrak{L}^{Gon} \rtimes G) \arrow[ur] &
\end{tikzcd}$$

□

Remark 4.28. *In [Bar91], a topology on* $\mathfrak{L}(X, x)$ *is introduced for general* X *as follows: Consider the space* ΩX *of piecewise smooth loops at* X. *A map* $\psi : U \to \Omega X$ *is called a smooth finite dimensional familiy of loops if there exist an* $n \in \mathbb{N}$ *and real numbers* $0 \leq t_0 < ... < t_n \leq 1$ *s.t. the map*

$$\begin{aligned} U \times I &\longrightarrow X \\ (u, t) &\longmapsto \psi(u)(t) \end{aligned}$$

is continuous and smooth on every piece $U \times [t_i, t_{i+1}]$. *Equip* $\mathfrak{L}(X, x)$ *with the finest topology s.t. for every smooth finite dimensional family of loops* ψ *the map* $\mathrm{pr} \circ \psi$ *is continuous, where* $\mathrm{pr} : \Omega X \to \mathfrak{L}(X, x)$ *is the projection. Any holonomy representation is continuous for this topology and so, by corollary 4.27, if one restricts this topology to* $\mathfrak{L}^{Gon}$, *the ad-hoc continuity is equivalent to actual continuity (plus holomorphicity of* ρ_G*).*

Remark 4.29. *It might be surprising from the point of view of the general discussion on groups of loops that it suffices to know the holonomy representation only on the (comparatively) small subgroup* $\mathfrak{L}^{Gon} \subseteq \mathfrak{L}(X, x)$.

Appendices

Appendix A

Equivariant Sheaves, Connections and Local Systems

A.1 Sheaf Theory

In this section, we collect definitions and elementary properties of equivariant sheaves, relative equivariant connections and relative equivariant local systems. References are [MFK94] for equivariant sheaves and [Del70] for relative connections and local systems. For the discussion of pullbacks and the definition of equivariant (relative) local systems and connections I do not know a reference.

We begin with a motivation for the definition of an equivariant sheaf:
Let X be a G-manifold (or variety), i.e., a complex manifold (or an algebraic variety) equipped with an action of a complex Lie-group (resp. algebraic group) G:[1]

$$\rho_X : G \times X \longrightarrow X$$

An equivariant vector bundle is then a holomorphic (resp. algebraic) vector bundle $p : E \longrightarrow X$ s.t. the total space E is equipped with a (fibrewise linear) action

$$\rho_E : G \times E \longrightarrow E$$

which commutes with the projection, i.e., the diagram

$$\begin{tikzcd}
G \times E \arrow[r, "\rho_E"] \arrow[d, "id\times p"] & E \arrow[d, "p"] \\
G \times X \arrow[r, "\rho_X"] & X
\end{tikzcd}$$

is commutative.

A vector bundle E on X can equivalently be described by its sheaf of holomorphic (resp. algebraic) sections

$$\mathcal{V}_E : U \longmapsto \Gamma(U, E)$$

viewed as a sheaf of $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ is the sheaf of holomorphic (resp. algebraic) functions on X. Let us translate the equivariant structure in terms of sheaves. For this, denote by $\rho_X^* E$ the fibre product of

$$G \times X \xrightarrow{\rho_X} X \xleftarrow{p} E.$$

By the universal property of the fibre product, we obtain the dashed map

$$\begin{tikzcd}
G \times E \arrow[rrd, bend left, "\rho_E"] \arrow[rd, dotted, "\varphi"] \arrow[rdd, bend right, "\mathrm{Id}\times p"'] & & \\
& \rho_X^* E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
& G \times X \arrow[r, "\rho_X"] & X
\end{tikzcd}$$

which one checks to be an isomorphism (in the fibre over $(g, x) \in G \times X$ it is given by multiplication with g). Let $\mathrm{pr}_2 : G \times X \longrightarrow X$ be the projection. The isomorphism induced by φ on the sheaves of sections

$$\Phi : \mathrm{pr}_2^* \mathcal{V}_E \longrightarrow \rho_X^* \mathcal{V}_E$$

---

[1] What follows works just as well in the topological or differentiable setting

satisfies the following cocycle condition:

$$(\mathrm{Id}_G \times \rho_X)^* \Phi \circ p_{23}^* \Phi = (\mu_G \times \mathrm{Id}_X)^* \Phi$$

where

$$\mathrm{pr}_{23} : G \times G \times X \longrightarrow G \times X$$

is the projection to the last two factors and μ_G denotes the multiplication of G. On the level of fibres this means that the diagram

$$\begin{tikzcd} E(x) \arrow[rr, "\sim"] \arrow[dr, "\sim"] & & E(gh.x) \\ & E(h.x) \arrow[ur, "\sim"] & \end{tikzcd}$$

commutes, which follows from the axioms of a group action. In fact, the datum of this isomorphism together with the cocycle condition is an equivalent description of a group action. This motivates the following definition for a general sheaf of $\mathcal{O}_X$-modules, where for technical reasons we replace Φ as above by its inverse.

Definition A.1. *An **equivariant sheaf** (of $\mathcal{O}_X$-modules) on X is an $\mathcal{O}_X$-module $\mathcal{F}$ on X together with an isomorphism*

$$\Phi : \rho_X^* \mathcal{F} \longrightarrow \mathrm{pr}_2^* \mathcal{F}$$

of $\mathcal{O}_{G \times X}$-modules that satisfies the cocycle condition

$$\mathrm{pr}_{23}^* \Phi \circ (\mathrm{Id}_G \times \rho_X)^* \Phi = (\mu_G \times \mathrm{Id}_X)^* \Phi.$$

*If G acts trivially on X, i.e. $\rho_X = \mathrm{pr}_X$, the **sheaf of invariant sections** is defined to be the subsheaf of $\mathcal{F}$ on which the adjoint $\Phi^{ad} : \mathcal{F} \longrightarrow \rho_{X*} \mathrm{pr}^* \mathcal{F}$ coincides with $\mathrm{Id}^{ad} : \mathcal{F} \longrightarrow \mathrm{pr}_{2*} \mathrm{pr}_2^* \mathcal{F}$.*

The category of equivariant (geometric) vector bundles as in the above discussion is equivalent to equivariant locally free sheaves in the sense of this definition. From now on, we will always mean the latter if we talk about (equivariant) vector bundles.

Example A.2.

1. *Every sheaf is equivariant with respect to the trivial action*

   $$\rho_X = \mathrm{pr}_2 : G \times X \longrightarrow X$$

   with $\Phi = \mathrm{Id}$.

2. *Let Ω_X be the sheaf of holomorphic (resp. Kähler) differentials. Recall that for any holomorphic (resp. algebraic) map $f : Y \longrightarrow X$ there is an exact sequence*

   $$f^* \Omega_X \longrightarrow \Omega_Y \xrightarrow{res} \Omega_{Y/X} \longrightarrow 0,$$

   where the right arrows is restriction to vector fields along the fibres of f and the left arrow is pullback via f. It is injective if f is a submersion (resp. smooth). Further on any product $Y = Z \times X$ with $f = \mathrm{pr}_X$ there is *an isomorphism* $\Omega_{Y/X} \cong \operatorname{pr}_Z^* \Omega_Z$ *and pullback via* pr_Z *splits the sequence, so* $\Omega_{X\times Z} \cong \operatorname{pr}_X^* \Omega_X \oplus \operatorname{pr}_Z^* \Omega_Z$*. As a consequence,* Ω_X *is equipped with an action via*

$$\Phi_\Omega : \rho_X^* \Omega_X \longrightarrow \Omega_{G\times X} \cong \operatorname{pr}_1^* \Omega_G \oplus \operatorname{pr}_2^* \Omega_X \longrightarrow \operatorname{pr}_2^* \Omega_X,$$

where the first map is pullback via ρ_X *and the second map is projection.*

From now on, let us fix a submersive (smooth) map of G-manifolds (varieties) $f : X \longrightarrow S$. Denote by $d : \mathcal{O}_X \longrightarrow \Omega_X$ the exterior differential and by $d_{X/S}$ the composition

$$d_{X/S} : \mathcal{O}_X \xrightarrow{d} \Omega_X \xrightarrow{res} \Omega_{X/S}.$$

Definition A.3. *Let* $\mathcal{V}$ *be a sheaf of* $\mathcal{O}_X$*-modules on* X*. A* ***relative connection*** *(with respect to* f*) on* $\mathcal{V}$ *is a* $f^{-1}\mathcal{O}_S$*-linear map*

$$\nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes_{\mathcal{O}_X} \Omega_{X/S}$$

that satisfies the Leibniz-rule, i.e.

$$\nabla(s.v) = v \otimes d_{X/S}s + s.\nabla(v),$$

where s *is a local section of* $\mathcal{O}_X$ *and* v *a local section of* $\mathcal{V}$*.*

A map $f : \mathcal{V} \longrightarrow \mathcal{W}$ *of* $\mathcal{O}_X$*-modules with connections* $\nabla_\mathcal{V}$ *and* $\nabla_\mathcal{W}$ *is called* ***compatible with the connections*** *or* ***flat*** *if* $\nabla_\mathcal{W} \circ f = (f \otimes \operatorname{Id}) \circ \nabla_\mathcal{V}$*.*

Given a vector bundle with relative connection $(\mathcal{V}, \nabla)$, there is a natural way to equip the bundles $\mathcal{V} \otimes_{\mathcal{O}_X} \Omega^i_{X/S}$ with a relative connection, which we denote $\nabla^{(i)}$ or ∇ as well. It turns out that the **curvature** $\nabla^2 := \nabla^{(1)} \circ \nabla$ is $\mathcal{O}_X$-linear, i.e., an element of $\operatorname{End}_{\mathcal{O}_X}(\mathcal{V}) \otimes_{\mathcal{O}_X} \Omega_{X/S}$. If it is zero, the connection is called **flat**.

Definition A.4. *A* ***relative local system*** *(with respect to* f*) is a locally free* $f^{-1}\mathcal{O}_S$*-module.*[2]

For the rest of this section, we work in the analytic topology.

When S is a point, we call a relative local system L just local system or absolute local system. Then for any path $\gamma : [0,1] \longrightarrow X$, with $\gamma(0) =: x$, $\gamma(1) =: y$ the pullback $\gamma^{-1}L$ is necessarily trivial. In particular, we get an isomorphism

$$L_x \cong (\gamma^{-1}L)_0 \xrightarrow{res^{-1}} \Gamma([0,1], \gamma^{-1}L) \xrightarrow{res} (\gamma^{-1}L)_1 \cong L_y.$$

This construction behaves well under concatenation of paths, i.e., gives a representation of the fundamental groupoid of X. In fact one has:

Theorem A.5. *Suppose* X *is connected and locally simply connected and* $x \in X$ *a point. Then sending a local system to the fibre over* x *gives an equivalence of categories*

---

[2] This definition differs from the one given in [Del70], who defines a relative local system as a coherent $f^{-1}\mathcal{O}_S$-module. We do not need this generality, however.

$$\{\textit{absolute local systems on } X\} \longleftrightarrow \{\textit{representations of } \pi(X,x)\}.$$

A well-known theorem on analytic differential equations[3] states:

Theorem A.6. *Taking the kernel of the connection yields a rank-preserving equivalence of categories*

$$\left\{ \begin{array}{c} \textit{vector bundles on } X \textit{ with a flat} \\ \textit{relative connection w.r.t. } f \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{relative local systems} \\ \textit{on } X \textit{ w.r.t. } f \end{array} \right\}.$$

The inverse functor is denoted RH. It is locally (on X) given as

$$f^{-1}\mathcal{O}_S^n \longmapsto (\mathcal{O}_X^n, d_{X/S}).$$

This equivalence is compatible with pullbacks and equivariant structures in an appropriate sense. I do not know a reference for this, so in the following it is going to be spelt out in detail what this means:

Given a map $g : Y \longrightarrow X$, and a vector bundle $\mathcal{V}$ with relative connection ∇ on X define a pullback $g^*\nabla : g^*\mathcal{V} \longrightarrow g^*\mathcal{V} \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$ in several steps:

1. Form the sheaf-theoretic pullback $g^{-1}\nabla : g^{-1}\mathcal{V} \longrightarrow g^{-1}\mathcal{V} \otimes_{g^{-1}\mathcal{O}_X} g^{-1}\Omega_{X/S}$.
2. Compose with the map $g^{-1}\mathcal{V} \otimes g^{-1}\Omega_{X/S} \longrightarrow g^*\mathcal{V} \otimes_{\mathcal{O}_Y} g^*\Omega_{X/S}$ and then with $g^*\Omega_{X/S} \longrightarrow \Omega_{Y/S}$. Denote the result $\overline{g^{-1}\nabla}$.
3. Prolong the so-constructed map to $g^*\mathcal{V} = \mathcal{O}_Y \otimes g^{-1}\mathcal{V}$ by enforcing the Leibniz-rule $s \otimes v \longmapsto s.\overline{g^{-1}\nabla} v + v \otimes ds$ on the level of presheaves and then sheafify.

This $g^*\nabla$ satisfies the Leibniz-rule and is $(f \circ g)^{-1}\mathcal{O}_S$ linear. Note however, that $f \circ g$ does no longer have to be a submersion.

Suppose we are given a cartesian diagram:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \widetilde{S} \arrow[r, "h"] & S \end{tikzcd}$$

Then f' is again a submersion and the composition

$$BC_h\nabla : g^*\mathcal{V} \overset{g^*\nabla}{\longrightarrow} g^*\mathcal{V} \otimes_{\mathcal{O}_Y} \Omega_{Y/S} \overset{\mathrm{Id}\otimes res}{\longrightarrow} g^*\mathcal{V} \otimes_{\mathcal{O}_Y} \Omega_{Y/\widetilde{S}}$$

is called the **base change of ∇ along** h. For fixed h, this is a functor

$$\left\{ \begin{array}{c} \text{vector bundles on } X \text{ with a} \\ \text{relative connection w.r.t. } f \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{vector bundles on } Y \text{ with a} \\ \text{relative connections w.r.t. } f' \end{array} \right\}$$

and for a map $h' : S' \longrightarrow \widetilde{S}$ there is a natural isomorphism $BC_{h \circ h'} \cong BC_{h'} \circ BC_h$.

If one talks about the **restriction** of a relative connection ∇ on X to a fibre $F_s := f^{-1}(s)$ with inclusion $i : F_s \hookrightarrow S$, one actually means the base change $BC_i\nabla$.

---

[3] In the form needed here this can be found in [Del70, ch. 1, thm. 2.23].

Definition A.7. *Suppose f is an equivariant map and let $\mathcal{V}$ be a vector bundle on X that is equipped with an equivariant structure Φ. A relative connection ∇ on $\mathcal{V}$ is called* ***equivariant*** *if the following diagram commutes:*

$$\begin{tikzcd}
\rho_X^*\mathcal{V} \arrow[r, "BC_{\rho_S}\nabla"] \arrow[d, "\Phi"] & \rho_X^*\mathcal{V} \otimes \Omega_{G\times X/G\times S} \arrow[d, "\Phi\otimes\mathrm{Id}"] \\
\mathrm{pr}_X^*\mathcal{V} \arrow[r, "BC_{\mathrm{pr}_S}\nabla"] & \mathrm{pr}_2^*\mathcal{V} \otimes \Omega_{G\times X/G\times S}
\end{tikzcd}$$

Remark A.8. *One might wonder why the action on $\Omega_{X/S}$ does not appear. In fact it does, but this is disguised in the definition of BC_{ρ_S}, as $\Omega_{G\times X/G\times X} \cong \mathrm{pr}_X^*\Omega_{X/S}$. Using this, the above definition is equivalent to the commutativity of the following diagram:*

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "\nabla"] \arrow[d, "\Phi^{ad}"] & \mathcal{V} \otimes \Omega_{X/S} \arrow[d, "\Phi^{ad}\otimes\Phi^{ad}_{\Omega_{X/S}}"] \\
\rho_{X*}\mathrm{pr}_X^*\mathcal{V} \arrow[r, "\rho_{X*}BC_{\mathrm{pr}_S}\nabla"] & \rho_{X*}(\mathrm{pr}_X^*\mathcal{V} \otimes \mathrm{pr}_X^*\Omega_{X/S})
\end{tikzcd}$$

There is an analogous story for relative local systems. Namely, for a cartesian diagram

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
\widetilde{S} \arrow[r, "h"] & S
\end{tikzcd}$$

consider X as a locally ringed space with structure sheaf $f^{-1}\mathcal{O}_S$ and Y as a locally ringed space with structure sheaf $f'^{-1}\mathcal{O}_{\widetilde{S}}$. For a relative local system L on X, the base change along h is defined as the pullback in the sense of locally ringed spaces, i.e.

$$BC_hL := g^*L := g^{-1}L \otimes_{(h\circ f')^{-1}\mathcal{O}_S} f'^{-1}\mathcal{O}_{\widetilde{S}}$$

Again this is a functor

$$\begin{Bmatrix}\text{relative local systems}\\ \text{w.r.t. } f\end{Bmatrix} \longrightarrow \begin{Bmatrix}\text{relative local systems}\\ \text{w.r.t. } f'\end{Bmatrix}$$

and for a map $h' : S' \longrightarrow \widetilde{S}$ there is a natural isomorphism $BC_{h\circ h'} \cong BC_{h'} \circ BC_h$. Because this is really a pullback in the sense of sheaves of modules we will often favor the notation g^* over BC_h.

If one talks about the **restriction** of a relative local system L on X to a fibre $F_s := f^{-1}(s)$ with inclusion $i : F_s \hookrightarrow S$ one actually means the base change BC_iL.

Definition A.9. *An* ***equivariant relative local system*** *on X (with respect to f) is a tuple (L, Φ) s.t. L is a relative local system and Φ an isomorphism of $(Id_G \times f)^{-1}\mathcal{O}_{G\times X}$-modules*

$$\Phi : \rho_X^*L \longrightarrow \mathrm{pr}_X^*L$$

that satisfies the cocycle condition

$$\mathrm{pr}_{23}^* \Phi \circ (\mathrm{Id}_G \times \rho_X)^* \Phi = (\mu_G \times \mathrm{Id}_X)^* \Phi.$$

If one talks about the **restriction** of a relative local system L on X to a fibre $F_s := f^{-1}(s)$ with inclusion $i : F_s \hookrightarrow S$ one actually means the base change $BC_i L$

Theorem A.10. *The equivalence in theorem A.6 is compatible with base changes. Suppose we are given a relative connection* ∇, *a relative local system* L *on* X *and a cartesian diagram:*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \widetilde{S} \arrow[r, "h"] & S \end{tikzcd}$$

Then there are natural identifications

$$\ker(BC_h \nabla) \cong BC_h(\ker \nabla) \qquad RH(BC_h(L)) \cong BC_h(RH(L))$$

This yields the following stronger form of theorem A.6.

Corollary A.11. *The functors of theorem A.6 yield an equivalence*

$$\left\{ \begin{array}{c} \textit{equivariant vector bundles on} \\ X \textit{ with a flat relative} \\ \textit{equivariant connection w.r.t. } f \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{equivariant relative local} \\ \textit{systems on } X \textit{ w.r.t. } f \end{array} \right\}.$$

A.2 Algebraic Description

Let us spell out what equivariant vector bundles and connections mean if we work with algebraic groups and varieties over the spectrum of a field, i.e. $S = \operatorname{Spec} k$. Here, by variety we will always mean a separated and reduced scheme of finite type over a base field k.[4] If not indicated differently, all tensor (resp. fibre) products will be over k (resp. $\operatorname{Spec} k$). Nothing in this section is new although I did not find a reference for all statements. The source for algebraic groups and equivariant bundles was the very well-readable [KR16], although this was certainly not invented there.

Recall that an algebraic group G is a variety s.t. the functor of points, which sends a k-scheme X to the X-valued points $G(X) := \mathrm{Hom}_{\mathrm{Sch}_k}(X, G)$ of G, factors through the category of groups. Equivalently, this is given by morphisms $\mu_G : G \times G \longrightarrow G$ ('multiplication'), $e_G : \operatorname{Spec} k \longrightarrow G$ ('unit') and $i_G : G \longrightarrow G$ ('inverse') satisfying the commutative diagrams obtained from writing the group

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[4] Most of the results in this section hold in far greater generality, but we will not need this.

axioms in terms of maps, i.e.:

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\mu_G \times \mathrm{Id}_G"] \arrow[d, "\mathrm{Id}_G \times \mu_G"'] & G \times G \arrow[d, "\mu_G"] \\
G \times G \arrow[r, "\mu_G"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times G \arrow[r, "\mu_G"] & G \\
G \arrow[u, "{(e_G, \mathrm{Id}_G)}"] \arrow[ur, "\mathrm{Id}_G"] \arrow[r, "{(\mathrm{Id}_G, e_G)}"'] & G \times G \arrow[u, "\mu_G"']
\end{tikzcd}$$

$$\begin{tikzcd}
G \times G \arrow[r, "\mu_G"] & G \\
G \arrow[u, "\mathrm{Id}_G \times i_G"] \arrow[ur, "e_G"'] \arrow[r, "i_G \times \mathrm{Id}_G"'] & G \times G \arrow[u, "\mu_G"']
\end{tikzcd}$$

If $G = \operatorname{Spec} R$ is affine, this is equivalently given by the structure of a Hopf algebra on the k-algebra R, i.e., by a comultiplication $\Delta : R \longrightarrow R \otimes R$, a counit $\varepsilon : R \longrightarrow k$ and a coinverse $\tau : R \longrightarrow R$, satisfying compatibility conditions obtained from dualising the diagrams for algebraic groups.

An action of an algebraic group G on a variety X is then a morphism

$$\rho : G \times X \longrightarrow X$$

which induces a group action on points. Equivalently, the diagrams

$$\begin{tikzcd}
G \times G \times X \arrow[r, "\mathrm{Id}_G \times \rho"] \arrow[d, "\mu_G \times \mathrm{Id}_X"] & G \times X \arrow[d, "\rho"] \\
G \times X \arrow[r, "\rho"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times X \arrow[r, "\rho"] & X \\
\operatorname{Spec} k \times X \arrow[u, "e_G \times \mathrm{Id}_X"] \arrow[ur] &
\end{tikzcd}$$

commute. If $G = \operatorname{Spec} R$ and $X = \operatorname{Spec} A$ are affine, this is equivalent to a coaction, i.e., a map of k-algebras $\rho^* : A \longrightarrow A \otimes R$ which gives A the structure of an R-comodule, i.e., $(\rho^* \otimes \mathrm{Id}) \circ \rho^* = (\mathrm{Id} \otimes \Delta) \circ \rho^*$ and $(\varepsilon \otimes \mathrm{Id}) \circ \rho^* = \mathrm{Id}$.

If $X = \operatorname{Spec} A$ is affine, recall that a quasi-coherent sheaf $\mathcal{F}$ is equivalently given by its A-module of global sections $M = \Gamma(X, \mathcal{F})$. If in addition $G = \operatorname{Spec} R$ is affine we can hence ask for purely module-theoretic descriptions of equivariant sheaves. In the following we give two such descriptions, one which is a direct translation of definition A.1 in terms of modules and one which spells out what the cocycle condition means in the affine case for the morphism $\Phi^{ad} : \mathcal{F} \longrightarrow \rho_* \operatorname{pr}_2^* \mathcal{F}$ obtained by adjunction.

Definition A.12. *Let $G = \operatorname{Spec} R$ be an affine algebraic group and $X = \operatorname{Spec} A$ an affine algebraic G-variety. Denote by $\rho_A : A \longrightarrow A \otimes R$ the corresponding coaction. An **A-G-module** M is an A-module M equipped with a k-linear map $\rho_M : M \longrightarrow M \otimes R$ such that*

1. *(R-comodule structure)*

$$(\rho_M \otimes \mathrm{Id}) \circ \rho_M = (\mathrm{Id} \otimes \Delta) \circ \rho_M$$

and

$$(\varepsilon \otimes \mathrm{Id}) \circ \rho_M = \mathrm{Id}_M \, .$$

2. *(Compatibility with comodule structure of A)*

$$\rho_M(a.m) = \rho_A(a).\rho_M(m).$$

Proposition A.13. *Let $G = \operatorname{Spec} R$ be an affine algebraic group and $X = \operatorname{Spec} A$ be an affine G-variety. The following three categories are equivalent:*

1. *Quasi-coherent equivariant sheaves on X.*
2. *A-modules M with an isomorphism $\varphi : M \otimes_{A,\rho_X^*} (A \otimes R) \xrightarrow{\sim} M \otimes_k R$ satisfying the cocycle condition.*
3. *A-G-modules (M, ρ_M).*

Sketch of proof. The passage from 1. to 2. is a direct translation via the dictionary between quasi-coherent sheaves on affine schemes and modules. To pass from 2. to 3. set $\rho_M(m) := \varphi(m \otimes 1)$ for $m \in M$. From 3. to 2., define $\varphi(m \otimes s) := s.\rho_M(m)$ where $m \in M$, $s \in A \otimes R$.[5] □

For affine varieties, the notion of an absolute connection has the expected module-theoretic interpretation, which is slightly nontrivial because connections are not $\mathcal{O}_X$-linear.

Lemma A.14. *Let $X = \operatorname{Spec} A$ be an affine variety over k and $\Omega_A := \Omega_{A/k}$ the module of Kähler-differentials. The global sections functor induces an equivalence of categories between*

1. *Quasi-coherent sheaves with a connection $(\mathcal{F}, \nabla)$.*
2. *A-modules M with a connection, i.e., a k-linear map $M \longrightarrow M \otimes_A \Omega_A$ satisfying the Leibniz-rule.*

Proof. Because I could not find a direct reference, here is a sketch how to prove this by hand:[6] Define a functor from A-modules M with a connection $\nabla^{alg} : M \longrightarrow M \otimes_A \Omega_{A/k}$ to quasi-coherent sheaves with a connection as follows: Note that the algebraic differential $d : A \longrightarrow \Omega_{A/k}$ behaves well w.r.t. localisation, so it induces the differential on sheaves $d : \mathcal{O}_X \longrightarrow \Omega_X$. So given (M, ∇^{alg}) we can extend ∇^{alg} to a map of sheaves $\nabla : \tilde{M} = \mathcal{O}_X \otimes_A M \longrightarrow \tilde{M} \otimes_{\mathcal{O}_X} \Omega_X$ by enforcing the Leibniz-rule. Using that a connection is determined by its values on generating sections, one checks that this is a pseudoinverse to the global sections functor. □

In terms of modules the pullback of a connection means the following: Given a map of k-algebras $\varphi : A \longrightarrow B$ and an A-module with connection (M, ∇), we get a a map $\varphi^*\nabla : M \otimes_A B \longrightarrow (M \otimes_A B) \otimes_B \Omega_B$ defined via $\nabla(m \otimes b) := m \otimes db + b.\nabla m$
Using lemma A.14, the following is a direct translation of the definition of an equivariant sheaf:

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[5] For the full computations, see [KR16, prop. 3.6]. However, note that in their notation θ has to be replaced by θ^{-1}.

[6] A proof for flat connections using the theory of D_X-modules can be found in [HTT08, prop. 1.4.4.]. It works for general connections word by word if one replaces D_X by the sheaf of noncommutative differential operators, c.f. [Kap12, sec. 1.5.].

Proposition A.15. *Let $G = \operatorname{Spec}(R)$ be an affine algebraic group and $X = \operatorname{Spec}(A)$ an affine G-variety over k. Write ρ_A for the induced coaction on A and $\Omega_A := \Omega_{A/k}$ for the module of Kähler differentials. The following categories are equivalent:*

1. *equivariant quasi-coherent sheaves on X with an equivariant connection.*
2. *A-G-modules (M, ρ_M) with a k-linear map that satisfies the Leibniz-rule*

$$\nabla : M \longrightarrow M \otimes_A \Omega_A$$

such that the diagram

$$\begin{tikzcd}
M \otimes R \arrow[r, "i^*\nabla_A"] & (M \otimes_k R) \otimes_{A\otimes R} (\Omega_A \otimes_k R) \\
M \arrow[u, "\rho_M"] \arrow[r, "\nabla"] & M \otimes_A \Omega_A \arrow[u, "\rho_M \otimes (\rho_A)_*"']
\end{tikzcd}$$

commutes, where $i : A \to A \otimes R$ sends $a \mapsto a \otimes 1$, $(\rho_A)_* : \Omega_A \to \Omega_A \otimes_k R$ is induced by the coaction on A and $i^*\nabla_A$ is $i^*\nabla$ followed by $id \otimes p$ where p is the projection

$$\Omega_{A\otimes_k R} \cong \Omega_A \otimes_A (A \otimes_k R) \oplus \Omega_R \otimes_R (A \otimes_k R) \longrightarrow \Omega_A \otimes_A (A \otimes_k R).$$

A.3 The Case $G = \mathbb{G}_m^n$ and X Affine

In this section, let $R = k[t_1^{\pm 1}, ..., t_n^{\pm 1}]$, $G = \mathbb{G}_m^n = \operatorname{Spec}(R)$ be some power of the multiplicative group and $X = \operatorname{Spec}(A)$ be an affine scheme. We want to spell out explicitly what the equivariant notions introduced above mean in this context. The general slogan is 'An action of the multiplicative group is the same as a grading'. Apart from the last proposition, this material is taken from [KR16], to which we refer for details.

Proposition A.16. *Let M be a k-module (resp. a k-algebra). The following structures are equivalent:*

1. *A k-linear R-comodule structure $\rho : M \longrightarrow M \otimes_k R$ (that is also an algebra-homomorphism).*
2. *A $\mathbb{Z}^n$-grading*

$$M = \bigoplus_{p\in\mathbb{Z}^n} M^p$$

(that is compatible with the multiplication, i.e., $M^p M^q \subseteq M^{p+q}$).

Proof. Let us just sketch how to get from one viewpoint to the other:
Given a k-linear R-comodule structure ρ, define for every $p \in \mathbb{Z}^n$ a k-submodule of M as

$$M^p := \{m \in M \mid \rho(m) = m \otimes t^p\}.$$

This gives the decomposition in 2.[7]
Vice versa, given a decomposition $M = \bigoplus M^p$, define ρ via $\rho(m) := m \otimes t^p$ for $m \in M^p$. ☐

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[7] For a more coneceptual point of view, note that $\mathbb{Z}^n$ is in bijection with the set of characters of $\mathbb{G}_m^n$.

For example the action of $\mathbb{G}_m^n$ on $\mathbb{A}^n = \operatorname{Spec} k[t_1, ..., t_n]$ by multiplication is given on rings by $t_i \mapsto t_i \otimes t_i$ and hence corresponds to the multigrading of $k[t_1, ..., t_n]$ by the degrees of the monomials.

Proposition A.17. *The global sections functor* $\Gamma(X, \ - \ )$ *induces an equivalence of categories*

1. $\mathbb{G}_m^n$*-equivariant quasi-coherent sheaves on an affine* $\mathbb{G}_m^n$*-variety* $X = \operatorname{Spec} A$
2. A*-modules with an* n*-grading compatible with the* n*-grading of* A*.*

Under this correspondence, the invariant sections correspond to elements of multidegree $(0, ..., 0)$*.*

Proof. This follows from proposition A.16 and the equivalence of 1. and 3. in proposition A.13. ☐

So a $\mathbb{G}_m^n$-equivariant sheaf on $\mathbb{A}^n$ is nothing but a $k[t_1, ..., t_n]$-module with a decomposition into k-vectorspaces $M = \bigoplus_{p \in \mathbb{Z}^n} M^p$ such that $t^q M^p \subseteq M^{q+p}$.

Proposition A.18. *Let* $\mathcal{V}$ *be a* $\mathbb{G}_m^n$*-equivariant vector bundle on an affine* $\mathbb{G}_m^n$*-variety* $X = \operatorname{Spec} A$*. By lemma A.14 a connection* ∇' *on* $\mathcal{V}$ *can equivalently be given by its connection on the global sections* $M := \Gamma(X, \mathcal{V})$

$$\nabla : M \longrightarrow M \otimes_A \Omega_A.$$

Then ∇' *is equivariant if and only if* ∇ *respects the multigrading.*

Proof. By proposition A.15 and proposition A.17 ∇' is equivariant iff the diagram

$$\begin{tikzcd}
M \otimes R \arrow[r, "i^*\nabla_A"] & (M \otimes_k R) \otimes_{A \otimes R} (\Omega_A \otimes_k R) \\
M \arrow[u, "\rho_M"] \arrow[r, "\nabla"] & M \otimes_A \Omega_A \arrow[u, "\rho_M \otimes (\rho_A)_*"']
\end{tikzcd}$$

is commutative. Therefore, let $m \in M^p$ be an element of degree $p \in \mathbb{Z}^n$ and $\nabla m = \sum_q \theta_q$ its decomposition according to degree in $M \otimes_A \Omega_A$, s.t. $\rho_M \otimes (\rho_A)_*(\nabla m) = \sum_q \theta_q \otimes t^q$. On the other hand, $i^*\nabla_A \rho_M(m) = i^*\nabla_A(m \otimes t^p) = \sum_q \theta_q \otimes t^p$. The two expressions agree iff $\theta_q = 0$ for $q \neq p$. So, because m and p were arbitrary, the diagram commutes iff ∇ respects the grading. ☐

Appendix B

The Radon-Penrose Transform

B.1 The Setup

This section is an elaboration and expansion to the equivariant case of [Man97, sect. 2.1.1-3].[1]

Let G be a complex Lie-group and consider a diagram of complex G-manifolds

$$A \xleftarrow{q} Q \xrightarrow{p} B$$

such that the following conditions hold:

1. The maps p, q are equivariant holomorphic fibre bundles.
2. At each point $x \in Q$ with $q(x) = a, p(x) = b$ the tangent spaces $T_x q^{-1}(a)$ and $T_x p^{-1}(b)$ have zero intersection.[2]
3. The fibre of p is connected.
4. The fibre of q ist compact and connected.
5. If $\mathcal{N} := p^*\Omega_B \cap q^*\Omega_A$ then the vanishing $q_*\mathcal{N} = 0 = R^1 q_*\mathcal{N}$ holds.

An equivalent, more geometric interpretation of 2. is that the map

$$(q,p) : Q \longrightarrow A \times B$$

is an immersion. In particular, we can consider fibres of p (resp. q) as immersed submanifolds of A (resp. B).

It may also be useful to have the following diagram in mind, where $*$ is a placeholder in order not to give names to too much stuff. One can interpret 2. as the sketched vanishing and 5. as the statement that the last nonzero morphisms in the first row and column become isomorphisms after applying q_*.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{N} \arrow[r] \arrow[d] & p^*\Omega_B \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & q^*\Omega_A \arrow[r] \arrow[d] & \Omega_Q \arrow[r] \arrow[d] & \Omega_{Q/A} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & * \arrow[r] \arrow[d] & \Omega_{Q/B} \arrow[r] \arrow[d] & *_{=0} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

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[1] After writing this up, I became aware of the works [Eas85] and [BE89], in particular chapters 7 and 9, which also treat the relevant sections in [Man97]. However they too are somewhat brief on the basics and do not treat the equivariant case, so I think this elaboration is still useful. The first reinterpretation of condition 2. below is only implicit in Manin and, in a slightly stronger form (embedding instead of immersion), explicated in [Eas85].

[2] This condition seems to be missing in [Man97] or is maybe implicit in the term 'double fibration'.

A vector bundle on B is called **A-trivial**, if its restriction to every fibre of q is isomorphic to a trivial bundle and a vector bundle with connection on A is called **B-trivial** if its restriction to every fibre of p is flat and has vanishing monodromy (i.e., it is trivial as a bundle with connection). The main theorem of this appendix is the following equivalence, called the Radon-Penrose transform:

Theorem B.1. *There is an equivalence of categories*

$$\left\{\begin{array}{c} \textit{equivariant } B\textit{-trivial vector} \\ \textit{bundles on } A \textit{ with an} \\ \textit{equivariant connection} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{c} \textit{equivariant} \\ A\textit{-trivial vector} \\ \textit{bundles on } B \end{array}\right\}.$$

Note that the notions of A- and B-triviality do not invoke the equivariant structure, i.e., we do not require the bundles to be equivariantly trivial on every fibre.

As am immediate corollary of theorem A.6, or rather corollary A.11, we obtain that passing to the kernel of a connection induces an equivalence

$$\left\{\begin{array}{c} \text{equivariant } A\text{- and } B\text{-trivial} \\ \text{vector bundles on } Q \text{ with an} \\ \text{equivariant flat relative} \\ \text{connection w.r.t. } p \end{array}\right\} \longleftrightarrow \left\{\begin{array}{c} \text{equivariant } A\text{- and} \\ B\text{-trivial relative local} \\ \text{systems w.r.t. } p \text{ on } Q \end{array}\right\},$$

where we call a relative local system A-trivial (resp. B-trivial) if its restriction to every fibre of q, (resp p) is isomorphic to a trivial vector bundle (resp. local system). Similarly, an equivariant flat relative connection is called A-trivial if its restriction to every fibre of q is a trivial vector bundle (note that $\Omega_{F/B} = 0$ for every fibre F of Q, so the connection on this bundle is always trivial) and B-trivial if the monodromy of the restriction to every fibre of p is trivial.

Thus, in order to prove theorem B.1, we will describe in the next two sections how the data on A and B can be interpreted via pullback and pushforward as data on Q.

B.2 Equivalences along q

Because a fibre bundle with compact fibre is a proper map, as an immediate consequence of the complex analytic proper base change theorem[3] one obtains:

Proposition B.2. *Let $\mathcal{F}$ be a vector bundle on Q s.t. the restriction to every fibre of q is trivial. Then the following holds:*

1. *The pushforward $q_*\mathcal{F}$ is again a vector bundle.*
2. *For any cartesian diagram*

$$\begin{tikzcd} \widetilde{Q} \arrow[r, "f'"] \arrow[d, "q'"] & Q \arrow[d, "q"] \\ \widetilde{A} \arrow[r, "f"] & A \end{tikzcd}$$

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[3] see e.g. [BPV84] for the statement, who refer to [BS76, ch. III] for a proof.

the base change map

$$f^*q_*\mathcal{F} \longrightarrow q'_*f'^*\mathcal{F}$$

is an isomorphism.

Using this, we obtain:

Proposition B.3. *The adjunction between pullback and pushforward yields an equivalence of categories:*

$$\left\{\begin{matrix}\textit{equivariant vector}\\ \textit{bundles on } A\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{equivariant vector bundles on}\\ Q\textit{, trivial on the fibres of } q\end{matrix}\right\}$$

Proof. Ignoring the equivariant structure for a moment, the adjunction between pushforward and pullback of coherent sheaves gives natural transformations

$$\varepsilon : \mathrm{Id} \Rightarrow q_*q^*$$
$$\eta : q^*q_* \Rightarrow \mathrm{Id}\,.$$

We have to show that both transformations are natural isomorphisms when restricted to the above categories. In order to see this for the first one let $\mathcal{V}$ be a vector bundle on A. Then by proposition B.2 the sheaf $q_*q^*\mathcal{V}$ is again a vector bundle and it suffices to check that the map $\varepsilon_{\mathcal{V}}$ is an isomorphism on every fibre. Let $i : x \longrightarrow A$ be the inclusion of a point and $j : F_x := q^{-1}(x) \longrightarrow Q$ the inclusion of the fibre above x. Then $j^*q^*\mathcal{V} = q^*i^*\mathcal{V} = \underline{\mathcal{V}(x)}$ is a constant sheaf and the composition of $\varepsilon_{\mathcal{V}}(x)$ with the base change isomorphism $i^*q_* \cong q_*j^*$ is given as:

$$\mathcal{V}(x) \longrightarrow H^0(F_x, \underline{\mathcal{V}(x)}),$$

which sends an element on the left hand side to the constant section with this value. Because F_x is connected, this is an isomorphism.

For η, let $\mathcal{F}$ be a fibre-wise trivial vector bundle on Q, $y \in Q$ a point, $x := q(y) \in A$ and $F_x := q^{-1}(x)$ the fibre through y. The map $\eta_{\mathcal{F}}(y)$ factors as:

$$q^*q_*\mathcal{F}(y) = (q_*\mathcal{F})(x) \longrightarrow H^0(F_x, \mathcal{F}) \longrightarrow \mathcal{F}(y)$$

Here the first map is the base change isomorphism and the second one restriction, which is an isomorphism because $\mathcal{F}$ is assumed to be trivial on F_x, which is compact and connected.

Now consider $\mathcal{V}$ as above to be equipped with an equivariant structure $\Phi_{\mathcal{V}}$ and set $q_G := \mathrm{Id}_G \times q$. Denoting by ρ_Q and ρ_A the action maps, the morphism

$$\rho_Q^*q^*\mathcal{V} = q_G^*\rho_A^*\mathcal{V} \xrightarrow{q_G^*\Phi_{\mathcal{V}}} q_G^*\,\mathrm{pr}_2^*\,\mathcal{V} = \mathrm{pr}_2^*\,q^*\mathcal{V}$$

equips $\mathcal{V}$ with an equivariant structure. The cocycle condition follows formally from functoriality of the pullback.

Because the q is supposed to be an equivariant map, the diagram

$$\begin{tikzcd} G \times Q \arrow[r] \arrow[d, "q_G"] & Q \arrow[d, "q"] \\ G \times A \arrow[r] & A \end{tikzcd}$$

is cartesian if the horizontal arrows are either both projections or the action maps ρ_Q and ρ_A. So, for $\mathcal{F}$ as above with equivariant structure $\Phi_{\mathcal{F}}$, the morphism

$$\rho_A^* q_* \mathcal{F} \cong (q_G)_* \rho_Q^* \xrightarrow{(q_G)_* \Phi_{\mathcal{F}}} (q_G)_* \operatorname{pr}_2^* \mathcal{F} \cong \operatorname{pr}_2^* q_* \mathcal{F},$$

where the first and last identifications come from the base change proposition B.2, equips $q_*\mathcal{F}$ with an equivariant structure. In order to check the cocycle conditions note that q_G satisfies the same hypotheses as q so the proposition B.2 also applies. For this same reason, the whole previous discussion applies to q_G as well and these constructions are inverse to each other.

□

So far we only used that $Q \longrightarrow A$ is a fibre bundle with compact, connected fibre. Using properties 2. and 5., we can show the equivalence also holds in the following enriched form:

Proposition B.4. *Pullback and pushforward induce an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{equivariant vector} \\ \textit{bundles on } A \textit{ with an} \\ \textit{equivariant} \\ \textit{connection} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{equivariant vector bundles on } Q, \\ \textit{trivial on every fibre of } q, \textit{ with an} \\ \textit{equivariant relative connection with} \\ \textit{respect to } p \end{array} \right\}.$$

Proof. Let us describe how the connections on the pullback, respectively pushforward bundles are constructed.

For a vector bundle with connection $(\mathcal{V}, \nabla)$ on A endow the pullback-bundle with the operator

$$\nabla_{Q/B} : q^*\mathcal{V} \xrightarrow{q^*\nabla} q^*\mathcal{V} \otimes \Omega_Q \xrightarrow{id \otimes res} q^*\mathcal{V} \otimes \Omega_{Q/B}.$$

This is a relative connection with respect to p: The Leibniz-rule follows from the Leibniz-rule for $q^*\nabla$ and for the $p^{-1}\mathcal{O}_B$-linearity use the again the Leibniz-rule and the fact that $d_{Q/B}$ is $p^{-1}\mathcal{O}_B$-linear.
Starting with a vector bundle $\mathcal{F}$ on Q with relative connection $\nabla_{Q/B}$, consider the pushforward

$$q_*\nabla_{Q/B} : q_*\mathcal{F} \longrightarrow q_*(\mathcal{F} \otimes \Omega_{Q/B}).$$

Using $\mathcal{F} \cong q^*q_*\mathcal{F}$ by the previous proposition and the projection formula, we identify the target with

$$q_*(\mathcal{F} \otimes \Omega_{Q/B}) \cong q_*\mathcal{F} \otimes q_*\Omega_{Q/B}.$$

By the next lemma we can make a further identification, thus formally obtaining the correct target for a connection on A:

Lemma B.5. *The maps* $\Omega_A \longrightarrow q_*q^*\Omega_A \xrightarrow{q_*res} q_*\Omega_{Q/B}$ *are isomorphisms*

Proof. The first isomorphism follows from the previous proposition. The transversality of the fibres implies that the sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow q^*\Omega_A \longrightarrow \Omega_{Q/B} \longrightarrow 0,$$

where the last arrow means restriction to vector fields along the fibres of p, is exact (the nontrivial part being the surjectivity). This together with condition 5. implies the second isomorphism. □

So we have a map

$$q_*\mathcal{F} \longrightarrow q_*(\mathcal{F} \otimes \Omega_{Q/B}) \cong q_*\mathcal{F} \otimes q_*\Omega_{Q/B} \cong q_*V \otimes \Omega_A,$$

and it remains to show that this operator is indeed a connection. By the previous proposition one reduces to the case that $\mathcal{F}$ is trivial. In this case the sections of the tensor product agree with the tensor product of sections (i.e., the first isomorphism is the identity) and it remains to see that the relative differential $d_{Q/B}$ is identified with the differential d_A of Ω_A under the isomorphism of the lemma B.5. This follows from the following commutative diagram:

$$\begin{tikzcd}
q_*q^*\mathcal{O}_A \cong \mathcal{O}_A \arrow[r, "d_A"] \arrow[d] & \Omega_A \cong q_*q^*\Omega_A \arrow[d] \arrow[dr, "\sim"] & \\
q_*\mathcal{O}_Q \arrow[r, "q_*d_Q"] \arrow[rr, bend right, "q_*d_{Q/B}"'] & q_*\Omega_Q \arrow[r, "q_*res"] & q_*\Omega_{Q/B}
\end{tikzcd}$$

Let us check that these constructions are inverse to each other: In one direction, this follows from the adjunction between q^{-1} and q_* by considering the following commutative diagram:

$$\begin{tikzcd}
q^{-1}\mathcal{V} \arrow[r, "q^{-1}\nabla"] \arrow[d] & q^{-1}(\mathcal{V} \otimes \Omega_A) \arrow[d] \arrow[dr] & \\
q^*\mathcal{V} \arrow[r, "q^*\nabla"] & q^*\mathcal{V} \otimes \Omega_Q \arrow[r, "\mathrm{Id} \otimes res"] & q^*\mathcal{V} \otimes \Omega_{Q/B}
\end{tikzcd}$$

Here the vertical maps are the canonical inclusions and the diagonal one is defined as the composition of the other two in the triangle.

Similarly, using the isomorphism of lemma B.5, adjunction yields a commutative diagram

$$\begin{tikzcd}
q^{-1}q_*\mathcal{F} \arrow[r, "q^{-1}q_*\nabla_{Q/B}"] \arrow[d] & q^{-1}q_*\mathcal{F} \otimes q^{-1}\Omega_A \arrow[d] \\
\mathcal{F} \arrow[r, "\nabla_{Q/B}"] & \mathcal{F} \otimes \Omega_{Q/B}.
\end{tikzcd}$$

From this one concludes by going through the definition of the pullback of a connection.

□

B.3 Equivalence along p

In this section, we only use that p is a fibre bundle with connected fibre. The goal is an analogue of proposition B.3 in this setting. However, starting with a vector bundle $\mathcal{W}$ on B, the pullback $p^*\mathcal{W}$ does not carry enough information to recover $\mathcal{W}$. Instead, we are going to consider the subsheaf $p^{-1}\mathcal{W} \subseteq p^*\mathcal{W}$, which is a relative local system.

The following is an analogue of proposition B.2 for relative local systems:

Proposition B.6. *Let L be a relative local system with respect to p s.t. the restriction $L|_F$ to every fibre F of p is trivial.*

1. *The adjunction morphism $\mathcal{W} \longrightarrow p_*p^{-1}\mathcal{W}$ is an isomorphism for all sheaves $\mathcal{W}$ on B (in particular, for the structure sheaf $\mathcal{O}_B$) and the push-forward p_*L is a vector bundle of the same rank as L.*
2. *For any cartesian diagram*

$$\begin{tikzcd} \widetilde{Q} \arrow[r, "f'"] \arrow[d, "p'"] & Q \arrow[d, "p"] \\ \widetilde{B} \arrow[r, "f"] & B \end{tikzcd}$$

   there is a functorial base change map

$$f^*p_*L \longrightarrow p'_*f'^*L,$$

   which is an isomorphism.

For the proof we will need two lemmas. The first is 'well-known', but I did not find a reference containing not only the statement but also a proof:

Lemma B.7. *For a projection $\mathrm{pr}_X : X \times F \longrightarrow X$ with F connected and a sheaf $\mathcal{G}$ on X, the adjunction $\mathcal{G} \longrightarrow (\mathrm{pr}_X)_* \mathrm{pr}_X^{-1} \mathcal{G}$ induces an isomorphism on global sections:*

$$\Gamma(X, \mathcal{G}) \xrightarrow{\sim} \Gamma(X \times F, \mathrm{pr}_X^{-1} \mathcal{G}) \qquad (*)$$

Proof. Denote by $\mathcal{G}'$ the presheaf on $X \times F$ given by $U \mapsto \mathcal{G}(pU)$, so $\mathrm{pr}_X^{-1} \mathcal{G}$ is the sheafification of $\mathcal{G}'$. This is explicitely given as the sheaf of (continuous) sections to the map

$$Et(\mathcal{G}') \longrightarrow X \times F,$$

where $Et(\mathcal{G}')$ means the étale space of $\mathcal{G}'$. Similarly, $\mathcal{G}$ can be considered as the sheaf of sections to

$$\pi : Et(\mathcal{G}) \longrightarrow X.$$

Using the definition of the étale space, one checks that there is a tautological homeomorphism[4] $Et(\mathcal{G}') \longrightarrow Et(\mathcal{G}) \times F$ such that the following diagram of topological spaces commutes:

$$\begin{tikzcd} Et(\mathcal{G}') \arrow[rr, "\sim"] \arrow[dr] & & Et(\mathcal{G}) \times F \arrow[dl, "\pi \times \mathrm{Id}_F"] \\ & X \times F & \end{tikzcd}$$

---

[4] Recall that by definition $Et(\mathcal{G}') = \coprod_{(x,f) \in X \times F} \mathcal{G}'_{(x,v)}$ and a basis for the open sets is given by $[U \times V, s] := \{s_{(x,y)} \mid (x, y) \in U \times V\}$, where $U \subseteq X$, $V \subseteq F$ open and s is a section of $\mathcal{G}'$ over $U \times V$. Note that $\mathcal{G}'_{(x,v)} = \mathcal{G}_x$.

Under this identifications, the map $(*)$ sends a section t of π to $t \times \mathrm{Id}_F$. This is obviously injective, so it remains to show surjectivity. A section to $\pi \times \mathrm{Id}_F$ can equivalently given by a continuous map $s : X \times F \longrightarrow Et(\mathcal{G})$ such that $\pi \circ s = \mathrm{pr}_X$. Over a fibre $\pi^{-1}(x)$ this restricts to $s_x : \{x\} \times F \longrightarrow \mathcal{G}_x$ where the right hand side has the discrete topology, so, as F is connected, the map is constant on every fibre, hence comes from a map $\tilde{s} : X \longrightarrow Et(\mathcal{G})$ that satisfies $\pi \circ \tilde{s} = \mathrm{Id}_X$, i.e., an element of $\Gamma(X, \mathcal{G})$. ☐

Next, the assumption of triviality along the fibres actually implies something (a priori) stronger:

Lemma B.8. *Let L be a relative local system on Q with respect to p. The following statements are equivalent:*

1. *For every point $y \in Q$ there is an open neigbourhood of the form $y \in V = p^{-1}U$ for some open set $U \subseteq B$ s.t. $L|_V \cong p^{-1}\mathcal{O}_B|_V$.*
2. *For every point $y \in Q$, with F_y the fibre of p through y, the restriction $L|_{F_y}$ is a trivial local system.*

Proof. The implication 1. $\Rightarrow$ 2. is clear, so let us suppose L satisfies 2. Without loss of generality, let us assume Q is globally a product $Q = B \times F$ and p the projection. For $b \in B$ write L_b for the local system arising as the restriction of L to $\{b\} \times F$. Pick a connected open set $U \times V$ with $U \subseteq B$ and $V \subseteq F$ such that $L|_{U \times V}$ is trivial and pick a basis $s_1, ..., s_n \in \Gamma(U \times V, L)$, where n denotes the rank of L. We claim that this can be extended to a basis of $\Gamma(U \times F, L)$. To show this, cover $U \times F$ by connected open sets of the form $U_i \times V_i$ with $U_i \subseteq B$ and $V_i \subseteq F$, s.t. $V \cap V_i$ is connected and $L|_{U_i \times V_i}$ is trivial for all indices i. For $b \in U$, let us denote by $s_j(b)$ the image of s_j under the restriction map

$$\Gamma(U \times V, L) \longrightarrow \Gamma(\{b\} \times V, L_b).$$

Now we produce unique sections $s^i_j \in \Gamma(U_i \times V_i, L)$ that satisfy for all $b \in U_i$:

Condition $(*)$: The section $s_j(b)$ is sent to $s^i_j(b)$ under the isomorphism

$$\Gamma(\{b\} \times V, L_b) \xrightarrow{res^{-1}} \Gamma(\{b\} \times F, L_b) \xrightarrow{res} \Gamma(\{b\} \times V_i, L_b).$$

Because by assumption $L|_{U_i \times V_i}$ is trivial, uniqueness follows as functions are determined by their value in each point and this value is determined by condition $(*)$. It remains to show existence. For this, let us first assume that V_i and V hav nonempty intersection. Then by lemma B.7, the composite

$$\Gamma(U_i \times V, L) \xrightarrow{res} \Gamma(U_i \times (V \cap V_i), L) \xrightarrow{res^{-1}} \Gamma(U_i \times V_i, L)$$

is an isomorphism and we define s^i_j as the image of s_j under this map. That it satisfies the condition $(*)$ follows from the commutativity of the following diagram, where all arrows are restriction maps:

$$\begin{tikzcd}
\Gamma(U_i \times V, L) \arrow[r] \arrow[d] & \Gamma(U_i \times (V \cap V_i), L) \arrow[d] & \Gamma(U_i \times V_i, L) \arrow[l] \arrow[d] \\
\Gamma(\{b\} \times V, L_b) \arrow[r] & \Gamma(\{b\} \times (V \cap V_i), L_b) & \Gamma(\{b\} \times V_i, L_b) \arrow[l] \\
 & \Gamma(\{b\} \times F, L_b) \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

For a point (b, f) in a general $U_i \times V_i$, there is a finite sequence $(U_k \times V_k)_{k \in \{1,\dots,r\}}$ such that $b \in \bigcap_{k=1}^{r} U_k$, $V_k \cap V_{k+1} \neq \emptyset$ and $f \in V_r$. Applying the above reasoning succesively and glueing yields s^i_j in this case. By construction the s^i_j agree on intersections and hence glue to global sections which generate L and are linearly independent in every point, i.e., we constructed a trivialisation above $U \times F$. □

proof of proposition B.6: Let $\mathcal{W}$ be any sheaf on B. Let x be a point in B and $F_x = p^{-1}(x)$ be the fibre. The unit of the adjunction on the stalk at x reads as:

$$\varinjlim_{x \in U} \Gamma(U, \mathcal{W}) \longrightarrow \varinjlim_{x \in U} \Gamma(p^{-1}U, p^{-1}\mathcal{W})$$

Because p is a fibre bundle, for small enough U the map p can be assumed to be a projection, hence this is an isomorphism by lemma B.7. This proves the first part of 1. The second is an immediate consequence of lemma B.7 and lemma B.8.

On we go with 2.: We define the base change map by the usual abstract nonsense. Namely, the unit of the adjunction between f'^* and f'_* is a natural transformation

$$\mathrm{Id} \Rightarrow f'_* f'^*$$

which yields a natural transformation

$$p_* \Rightarrow p_* \circ f'_* \circ f'^* = f_* \circ p'_* \circ f'^*$$

and the base change transformation is the adjoint of this. In order to show that for given L it is an isomorphism, note that by lemma B.8, we can assume that $L = p^{-1}\mathcal{O}_B$. But then $f'^* p^{-1}\mathcal{O}_B = p'^{-1}\mathcal{O}_{\widetilde{B}}$ and the assertion follows from 1.

□

Proposition B.9. *The adjunction between sheaf-theoretic pullback and push-forward along p induces the following equivalence of categories:*

$$\left\{ \begin{array}{c} \textit{equivariant relative (with} \\ \textit{respect to p) local systems on} \\ \textit{Q, trivial on every fibre of p} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{equivariant} \\ \textit{vector bundles} \\ \textit{on B} \end{array} \right\}$$

Proof. As in the proof of proposition B.3, first we show that unit and counit

$$\varepsilon : \mathrm{Id} \Rightarrow p_* p^{-1}$$
$$\eta : p^{-1} p_* \Rightarrow \mathrm{Id}\,.$$

of the adjunction are isomorphisms when restricted to these subcategories without equivariant structure. For ε, this is part of proposition B.6.

For the counit, we could argue as in the proof of B.3 once one checks that an morphism of relative local systems is an isomorphism when restricted (in an appropriate sense) to every fibre, or as follows: Let L be a local system on Q, $x \in Q$ and $F = p^{-1}p(x)$. Assume we have open neighbourhoods $x \in U$ and $V = p^{-1}U$ s.t. $L|_V \cong p^{-1}\mathcal{O}_V$. Then, by naturality of the adjunction and the unit-counit equations, we have a commutative diagram:

$$\begin{tikzcd}
p^{-1}p_*L|_V \arrow[r] & L|_V \\
p^{-1}p_*p^{-1}\mathcal{O}_V \arrow[r] \arrow[u] & p^{-1}\mathcal{O}_V \arrow[u] \\
p^{-1}\mathcal{O}_V \arrow[u] \arrow[ur, "\mathrm{Id}_{p^{-1}\mathcal{O}_V}"'] &
\end{tikzcd}$$

Here, the horizontal arrows come from the counit η of the adjunction, the upper vertical ones are (induced by) the trivialisation on V and the lower vertical one comes from the unit ε of the adjunction. But we know all arrows except the horizontal ones to be isomorphisms, so they have to be as well.

Taking into account the equivariant structures works as in proposition B.3, thanks to proposition B.6. $\square$

B.4 Parallel Transport

Finally, we explain how parallel transport along curves in A which are entirely contained in a fibre of p can be interpreted in terms of data on B. This is modelled on a special case stated in [Kap12] without proof. Let us assume a slightly stronger variant of condition 2., namely

$$2.' \text{ The map } (q,p) : Q \longrightarrow A \times B \text{ is an embedding.}$$

Let $(\mathcal{V}, \nabla)$ be a B-trivial bundle with connection on A. Consider a point $b \in B$ with fibre $F_b = p^{-1}(b)$. If we want to understand parralel transport along curves with image in qF_b, we have to study the restriction map

$$H^0(qF_b, \ker\nabla) \longrightarrow \mathcal{V}(a)$$

for any point $a \in qF_b$. We claim that this can be described as the inverse of a restriction map on B. To make this precise, fix such an a, with fibre $F_a = q^{-1}(a)$ and let $\mathcal{W}$ be the bundle corresponding to $\mathcal{V}$ under the Radon-Penrose transform and $(\mathcal{F}, \nabla^{rel})$ the bundle with flat relative connection corresponding to $\mathcal{V}$ and $\mathcal{W}$.

Proposition B.10. *The following diagram commutes and all arrows are isomorphisms:*

$$\begin{tikzcd}
H^0(qF_b, \ker\nabla) \arrow[rr] \arrow[d] & & \mathcal{V}(a) \arrow[d] \\
H^0(F_b, \ker BC_{i_b}\nabla^{rel}) \arrow[r] & H^0(F_a \cap F_b, \mathcal{F}) & H^0(F_a, \mathcal{F}) \arrow[l] \\
\mathcal{W}(b) \arrow[u] & & H^0(pF_a, \mathcal{W}) \arrow[ll] \arrow[u]
\end{tikzcd} \tag{*}$$

Here all horizontal maps are restrictions, the top right and bottom left maps are base change maps on global sections and the top left and bottom right maps are formal identifications expained in the proof.

Proof. The situation is depicted in the following commutative diagram, where all vertical arrows are inclusions and all horizontal or diagonal ones are induced by p or q:

$$\begin{tikzcd}
A & qF_b \arrow[l, "i_b"'] & & \{a\} \arrow[ll] \arrow[r] & A \\
Q \arrow[u] \arrow[d] & F_b \arrow[l, "j_b"] \arrow[u, "q_b"] \arrow[d] \arrow[ul, phantom, "\lrcorner", very near start] & F_a \cap F_b \arrow[l] \arrow[r] \arrow[ur] \arrow[dl] & F_a \arrow[u] \arrow[d] \arrow[r] \arrow[dr, phantom, "\ulcorner", very near start] & Q \arrow[u] \arrow[d] \\
B & \{b\} \arrow[l] \arrow[rr] & & pF_a \arrow[r] & B
\end{tikzcd}$$

Since we assume 2.$'$, the intersection $F_a \cap F_b = \{x\}$ has only one element. In particular, all the restriction maps in $(*)$ are isomorphisms since by assumption $\mathcal{F}$ is trivial on the compact fibres of q and $\ker \nabla^{rel}$ is a trivial local system on every fibre of p. The base change maps are isomorphisms by proposition B.6 and proposition B.2.

Concerning the formal identifications alluded to, note that, since q_b an p_b are isomorphisms, the differentials on F_b and qF_b are identified and one can make identifications of functors $(q_b^{-1})^* = (q_b)_*$ and $(p_a^{-1})^{-1} = (p_a)_*$. So

$$i_b^* \mathcal{V} = (q_b)_* j_b^* \underbrace{q^* \mathcal{V}}_{\cong \mathcal{F}}$$

gives the identification on the top left in the diagram $(*)$ and the analogous computation yields the identification on the bottom right.

That the whole diagram $(*)$ commutes is maybe most obvious if one imagines the global sections as actual sections to the étale spaces: In this case the vertical maps are just pullback by (restricted versions of) p and q. $\square$

Appendix C

Two Functional Equations

Here are two solutions to two functional equations, which originally seemed to be useful in a direct classification of equivariant holomorphic bundles over $\mathbb{C}^n$, but might be of independent interest.

Lemma C.1. *Let $A(z) : \mathbb{C} \longrightarrow \mathrm{Mat}_{n\times n}(\mathbb{C})$ be a holomorphic function with $A(0) \in \mathrm{GL}_n(\mathbb{C})$ that satisfies the functional equation*

$$A(z) = A(-z)A(2z).$$

Then there is a matrix $C \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ s.t.

$$A(z) = \exp(zC).$$

Proof. As A is an entire holomorphic function, it can be represented by an everywhere convergent power series, i.e., we find matrices $A_k \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, such that

$$A(z) = \sum_{k\geq 0} A_k z^k.$$

Writing out the functional equation and comparing coefficients one gets the series of equations

$$A_k = \sum_{i+j=k} (-1)^i 2^j A_i A_j.$$

The first equation reads $A_0 = A_0 A_0$. Since $A(0) \in \mathrm{GL}_n(\mathbb{C})$, it follows that $A_0 = \mathrm{Id}$. Using this, we rewrite the remaining equations as

$$(1 - (-1)^k + 2^k)A_k = \sum_{\substack{i+j=k \\ i,j\neq k}} (-1)^i 2^j A_i A_j.$$

But the factor $1 - (-1)^k + 2^k$ is invertible for $k \geq 2$, so $A(z)$ is uniquely determined by A_1 (i.e., the derivative of $A(z)$ at 0). This implies $A(z) = \exp(zA_1)$, as the right hand side satisfies the functional equation and has the required derivative at 0. $\square$

Lemma C.2. *Let $f : \mathbb{C}^\times \longrightarrow \mathrm{Mat}_{n\times n}(\mathbb{C})$ be a holomorphic function with $f(1) \in \mathrm{GL}_n(\mathbb{C})$ that satisfies the functional equation*

$$f(z) = f(z^{-1})f(z^2).$$

Then there are an invertible matrix $S \in \mathrm{GL}_n(\mathbb{C})$ and integers $\alpha_1, ..., \alpha_n \in \mathbb{Z}$ s.t.

$$S^{-1}f(z)S = \begin{bmatrix} z^{\alpha_1} & & & \\ & z^{\alpha_2} & & \\ & & \ddots & \\ & & & z^{\alpha_n} \end{bmatrix},$$

where all the other entries are zero.

Proof. Let $A(z) := f(\exp(z))$. By the previous Lemma, we find a matrix $C \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ s.t. $A(z) = \exp(zC)$. We investigate what conditions on C are enforced by the relation

$$A(z) = A(z + 2\pi i r) \qquad \forall r \in \mathbb{Z}. \tag{$\star$}$$

To this end, look at the Jordan-Chevalley decomposition of C, i.e., write $C = D + N$ with D diagonalisable, N nilpotent and $DN = ND$. Set $A_D(z) := \exp(zD)$ and $A_N(z) := \exp(zN)$, so that $A(z) = A_D(z)A_N(z)$. By definition, $(\star)$ holds for $A_D(z)$ and $A_N(z)$ as well. But $A_N(z)$ is a polynomial in z (with matrix coefficients). Let $n \in \mathbb{N}$ be the greatest natural number s.t. $N^n \neq 0$. Then by comparing the constant term in $(\star)$ for $A_N(z)$, we get

$$\mathrm{Id} = \sum_{k=0}^{n} \frac{(2\pi i)^k}{k!} N^k.$$

One sees that necessarily $N = 0$ as nonzero powers of a nilpotent matrix are linearly independent.
Now pick an invertible matrix S s.t. $S^{-1}DS =: \tilde{D}$ is a diagonal matrix with (a priori complex valued) entries $\alpha_1, ..., \alpha_n$. Then $\exp(z\tilde{D}) = S^{-1}A(z)S$ is again a diagonal matrix with entries $\exp(z\alpha_j)$, satisfying $(\star)$. But this implies that

$$1 = \exp(2\pi i r \alpha_j) \qquad \forall r \in \mathbb{Z},\ j \in \{1, ..., n\}$$

from which it follows that $\alpha_j \in \mathbb{Z}$ for all $j \in \{1, ..., n\}$. So we showed that

$$S^{-1} f(\exp(z)) S = \begin{bmatrix} \exp(z)^{\alpha_1} & & & \\ & \exp(z)^{\alpha_2} & & \\ & & \ddots & \\ & & & \exp(z)^{\alpha_n} \end{bmatrix},$$

where again all nonspecified entries are 0. This completes the proof. $\square$

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[1] Saying 'my advisor organizes metal-concerts in his free time' when someone asks you about thesis-related factoids is also a far more effective way of combating math clichés than saying 'well, I think a lot about complexes'.

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