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On convex hulls of orbits of Coxeter groups and Weyl groups

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Abstract. The notion of a linear Coxeter system introduced by Vinberg generalizes the geometric representation of a finite Coxeter system. Our main theorem asserts that if v is an element of the Tits cone of a linear Coxeter system and $\mathcal{W}$ is the corresponding Coxeter group, then $\mathcal{W}v \subseteq v - C_v$, where C_v is the convex cone generated by the coroots $\check{\alpha}$, for which $\alpha(v) > 0$. This implies that the convex hull of $\mathcal{W}v$ is completely determined by the image of v under the reflections in $\mathcal{W}$. We also obtain an analogous result for convex hulls of $\mathcal{W}$-orbits in the dual space, although this action need not correspond to a linear Coxeter system. Motivated by the applications in representation theory, we further extend these results to Weyl group orbits of locally finite and locally affine root systems. In the locally affine case, we also derive some applications on minimizing linear functionals on Weyl group orbits.

Introduction

The present paper is motivated by the unitary representation theory of locally finite, resp., locally affine Lie algebras and their analytic counterparts [6, 7, 8]. In the algebraic context, these Lie algebras $\mathfrak{g}$ contain a maximal abelian subalgebra $\mathfrak{t}$ for which the complexification $\mathfrak{g}_\mathbb{C}$ has a root decomposition $\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\mathbb{C}^\alpha$ and there is a distinguished class of unitary representations (ρ, V) of $\mathfrak{g}$ on a pre-Hilbert space V for which the operators $\rho(x)$, $x \in \mathfrak{g}$, are skew-symmetric, on which $\rho(\mathfrak{t})$ is diagonalizable, and the corresponding weight set $\mathcal{P}_V \subseteq i\mathfrak{t}^*$ has the form

$$\mathcal{P}_V = \mathrm{conv}(\mathcal{W}\lambda) \cap (\lambda + \mathcal{Q}),$$

where $\mathcal{W} \subseteq \mathrm{GL}(\mathfrak{t})$ is the *Weyl group* of the pair $(\mathfrak{g}, \mathfrak{t})$, and $\mathcal{Q} \subseteq i\mathfrak{t}^*$ is the *root group.* Then

$$\mathrm{Ext}(\mathrm{conv}(\mathcal{P}_V)) = \mathcal{W}\lambda$$

is the set of *extremal weights.* For finite-dimensional compact Lie algebras $\mathfrak{g}$ and the unitary (=compact) forms of Kac–Moody algebras, this follows from

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the well-known description of the weight set of unitary highest weight modules [3]. For the generalization to the locally finite, resp., locally affine case we refer to [6], resp., [7] for details. Motivated by these representation theoretic issues, the present paper addresses a better understanding of the convex hulls of Weyl group orbits in $\mathfrak{t}$ and $\mathfrak{t}^*$.

In the locally finite, resp., locally affine case, the Weyl group is a direct limit of finite, resp., affine Coxeter groups, and it is the geometry of linear actions of Coxeter groups that provides the key to the crucial information on convex hulls of orbits. Therefore this paper is divided into two parts. The first part deals with *linear Coxeter systems* [10]. These are realizations of Coxeter groups by groups generated by a finite set $(r_s)_{s\in S}$ of reflections of a finite-dimensional vector space V satisfying the following conditions. We write $r_s(v) = v - \alpha_s(v)\check{\alpha}_s$ with elements $\alpha_s \in V^*$ and $\check{\alpha}_s \in V$ and $\mathrm{cone}(M)$ for the convex cone generated by the subset M of a real vector space. Then the conditions for a linear Coxeter system are:

(LCS1) The polyhedral cone $K := \{v \in V \mid \text{ for all } \alpha_s(v) \geq 0\}$ has interior points.

(LCS2) $\alpha_s \notin \mathrm{cone}(\{\alpha_t \mid t \neq s\})$ holds for all $s \in S$.

(LCS3) $wK^0 \cap K^0 = \varnothing$ holds for all $w \in \mathcal{W} \setminus \{\mathbf{1}\}$, where K^0 denotes the interior of K.

Note that the conditions (LCS1/3) refer to the canonical topology on a finite-dimensional real vector space. This is why the set S is assumed to be finite for a linear Coxeter system (cp. the discussion in Example 3.6).

For any linear Coxeter system, the set $T := \mathcal{W}K \subseteq V$ is a convex cone, called the *Tits cone*. A *reflection in* $\mathcal{W}$ is an element conjugate to one of the r_s, $s \in S$. Any reflection can be written as $r_\alpha(v) = v - \alpha(v)\check{\alpha}$ with $\alpha \in V^*$ and $\check{\alpha} \in V$, where α belongs to the set $\Delta := \mathcal{W}\{\alpha_s \mid s \in S\}$ of *roots* and $\check{\alpha} \in \check{\Delta} := \mathcal{W}\{\check{\alpha}_s \mid s \in S\}$ is a *coroot*. In these terms, our first main result (Theorem 2.7) asserts that

(1) $$\mathcal{W}v \subseteq v - C_v \text{ for } C_v := \mathrm{cone}\{\check{\alpha} \mid \alpha(v) > 0\} \text{ and } v \in T = \mathcal{W}K.$$

As an inspection of two-dimensional examples shows, the restriction to elements of the Tits cone is crucial and that the boundary ∂T may contain elements v for which $\mathcal{W}v$ is not contained in $v - C_v$.[1]

For the applications to orbits of weights, we also need a corresponding result for elements in the dual space. Here a difficulty arises from the fact that the concept of a linear Coxeter system is not preserved by exchanging the role of V and V^*, so that Theorem 2.7 cannot be applied directly. However, this can be overcome by reduction to the subspace $U \subseteq V^*$ generated by the dual cone $C_S^\star$ of $C_S := \mathrm{cone}\{\check{\alpha}_s \mid s \in S\}$. Then $\mathcal{W}C_S^\star$ is the Tits cone for a linear Coxeter system on U (Theorem 2.12).

[1] The proof of Theorem 2.7 relies on various results from the unpublished diploma thesis of Georg Hofmann [1] which contains already a proof for the special case where $v \in K^0$, i.e., where the stabilizer of v is trivial.

Structure of this paper: In Section 1 we recall some basics on linear Coxeter systems and generalize some results which are well-known for the geometric representation of a finite Coxeter system to linear Coxeter systems. Here the main results are Theorem 1.10 relating positivity conditions to the length function and Proposition 1.12 asserting that the stabilizer of an element in the fundamental chamber K is generated by reflections. Our main results, (1) and its dual version, are proved in Section 2. In Section 3 we turn to the applications to Weyl group orbits of linear functionals for locally finite and locally affine root systems (Theorems 2.7 and 2.12). The two final subsections are motivated by the representation theoretic problem to determine maximal and minimal eigenvalues of elements of the Cartan subalgebra in extremal weight representations. In this context we provide in the locally affine case a complete classification of the set $\mathcal{P}_d^+$ of those linear functionals λ for which $\lambda(d) = \min(\widehat{\mathcal{W}}\lambda)(d)$ holds for a certain distinguished element d and show that any orbit of the affine Weyl group $\widehat{\mathcal{W}}$ intersects $\mathcal{P}_d^+$ in an orbit of the corresponding locally finite Weyl group $\mathcal{W}$. Writing $\widehat{\mathcal{W}}$ as a semidirect product $\mathcal{N} \rtimes \mathcal{W}$, where $\mathcal{N}$ acts by unipotent isometries of a Lorentzian form, the minimization of the d-value on a $\widehat{\mathcal{W}}$-orbit turns into a problem of minimizing a quadratic form on an infinite-dimensional analog of a lattice in a euclidean space. These issues are briefly discussed in Subsection 3.17.

The results of the present paper constitute a crucial ingredient in the classification of semibounded unitary representations of double extensions of loop groups with values in Hilbert–Lie groups carried out in [8].

Notation: For a subset E of the real vector space V we write $\mathrm{conv}(E)$ for its convex hull and $\mathrm{cone}(E)$ for the convex cone generated by E. For a convex cone $C \subseteq V$ we write $H(C) := C \cap -C$ for the largest subspace contained in C. The cone C is said to be *pointed* if $H(C) = \{0\}$. We also write

$$E^\star := \{\alpha \in V^* \mid \alpha(v) \geq 0 \text{ for all } v \in E\}$$

for its dual cone in V^*. For $E \subseteq V^*$, we define its dual cone by

$$E^\star := \{v \in V \mid \alpha(v) \geq 0 \text{ for all } \alpha \in E\}.$$

Contents

1. Linear Coxeter systems

In this section we recall Vinberg's concept of a linear Coxeter system, generalizing the geometric representation of a Coxeter group.

Definition 1.1. (a) Let V be a real vector space. A *reflection data on V* consists of a family $(\alpha_s)_{s\in S}$ of linear functionals on V and a family $(\check{\alpha}_s)_{s\in S}$ of elements of V satisfying

$$\alpha_s(\check{\alpha}_s) = 2 \text{ for } s \in S.$$

Then $r_s(v) := v - \alpha_s(v)\check{\alpha}_s$ is a reflection on V. We write $\mathcal{W} := \langle r_s \mid s \in S\rangle \subseteq \mathrm{GL}(V)$ for the subgroup generated by these reflections and

$$\mathrm{co}(v) := \mathrm{conv}(\mathcal{W}v) \tag{2}$$

for the convex hull of a $\mathcal{W}$-orbit. We say that a reflection data is *of finite type* if S is finite and $\dim V < \infty$.

(b) We consider the following polyhedral cones in V resp. V^*:

$$C_S := \mathrm{cone}\{\check{\alpha}_s \mid s \in S\} \subseteq V, \qquad \check{C}_S := \mathrm{cone}\{\alpha_s \mid s \in S\} \subseteq V^*,$$

and the *fundamental chamber*

$$K := \{v \in V \mid \alpha_s(v) \geq 0 \text{ for all } s \in S\} = (\check{C}_S)^\star.$$

(c) A reflection data of finite type is called a *linear Coxeter system* (cp. [10]) if

(LCS1) K has interior points, i.e., the cone $\check{C}_S \subseteq V^*$ is pointed.
(LCS2) $\alpha_s \notin \mathrm{cone}(\{\alpha_t \mid t \neq s\})$ holds for all $s \in S$.
(LCS3) $wK^0 \cap K^0 = \varnothing$ holds for all $w \in \mathcal{W} \setminus \{\mathbf{1}\}$.

Then $T := \mathcal{W}K$ is called the associated *Tits cone*.

Definition 1.2. Let V be a real vector space endowed with a symmetric bilinear form β. A *symmetric reflection data on V* consists of a family $(\check{\alpha}_s)_{s\in S}$ of nonisotropic elements of V for which the linear functionals

$$\alpha_s(v) := \frac{2\beta(v, \check{\alpha}_s)}{\beta(\check{\alpha}_s, \check{\alpha}_s)}$$

define a reflection data on V. Then the reflections

$$r_s(v) = v - \alpha_s(v)\check{\alpha}_s = v - 2\frac{\beta(v, \check{\alpha}_s)}{\beta(\check{\alpha}_s, \check{\alpha}_s)}\check{\alpha}_s$$

preserve β, so that $\mathcal{W} \subseteq \mathrm{O}(V, \beta)$.

Remark 1.3. (a) Condition (LCS2) means that, for each $s \in S$, the dual cone of $\mathrm{cone}\{\alpha_t \mid t \neq s\}$ is strictly larger than K, i.e., there exists an element $v \in K \cap \ker \alpha_s$ with $\alpha_t(v) > 0$ for $t \neq s$. Then $K \cap \ker \alpha_s$ is a codimension 1 face of the polyhedral cone K.

(b) Typical examples of linear Coxeter systems arise from the geometric representation of a Coxeter system $(\mathcal{W}, S)$ (cp. [2, §5.3], [10]).

The following criterion for the recognition of linear Coxeter systems will be convenient in many situations because conditions (C1) and (C2) are preserved by the passage to the dual reflection data obtained by exchanging V and V^*.

Proposition 1.4. *Let* $(V, (\alpha_s)_{s\in S}, (\check{\alpha}_s)_{s\in S})$ *be a reflection data of finite type satisfying* (LCS1). *Then it defines a linear Coxeter system if and only if the following conditions are satisfied for* $s \neq t \in S$*:*

(C1) $\alpha_s(\check{\alpha}_t)$ *and* $\alpha_t(\check{\alpha}_s)$ *are either both negative or both zero.*
(C2) $\alpha_s(\check{\alpha}_t)\alpha_t(\check{\alpha}_s) \geq 4$ *or* $\alpha_s(\check{\alpha}_t)\alpha_t(\check{\alpha}_s) = 4\cos^2\frac{\pi}{k}$ *for some natural number* $k \geq 2$*.*

In this case $(\mathcal{W}, (r_s)_{s\in S})$ *is a Coxeter system.*

Proof. According to [10, p. 1085], for a reflection data of finite type satisfying (LCS1/2), condition (LCS3) is equivalent to both conditions (C1/2).

Therefore it remains to show that (C1) implies (LCS2). So we assume that $\alpha_s = \sum_{t\neq s}\lambda_t\alpha_t$ with each $\lambda_t \geq 0$. Then (C1) leads to the contradiction

$$2 = \alpha_s(\check{\alpha}_s) = \sum_{t\neq s}\lambda_t\alpha_t(\check{\alpha}_s) \leq 0.$$

Therefore (C1) implies (LCS2). That $(\mathcal{W}, (r_s)_{s\in S})$ is a Coxeter system follows from [10, Thm. 2(6)]. □

Remark 1.5. As a consequence of the preceding proposition, we obtain for every subset $S_0 \subseteq S$ and every linear Coxeter system $(V, (\alpha_s)_{s\in S}, (\check{\alpha}_s)_{s\in S})$ a linear Coxeter system

$$(V, (\alpha_s)_{s\in S_0}, (\check{\alpha}_s)_{s\in S_0}).$$

Remark 1.6. (a) If $(V, (\alpha_s)_{s\in S}, (\check{\alpha}_s)_{s\in S})$ is a reflection data, then we also consider the elements $\check{\alpha}_s$ as linear functionals on V^*. We thus obtain a reflection data $(V^*, (\check{\alpha}_s)_{s\in S}, (\alpha_s)_{s\in S})$. Suppose that $(V, (\alpha_s)_{s\in S}, (\check{\alpha}_s)_{s\in S})$ is a linear Coxeter system. Then (C1/2) also hold for the dual reflection data $(V^*, (\check{\alpha}_s)_{s\in S}, (\alpha_s)_{s\in S})$. Hence it is a linear Coxeter system if and only if (LCS1) holds, i.e., if the convex cone C_S is pointed, i.e., $H(C_S) = \{0\}$.

(b) If C_S is not pointed, then we still obtain a linear Coxeter system by replacing V^* by the smaller subspace

$$U := \operatorname{span}(C_S^\star) = H(C_S)^\perp \subseteq V^*.$$

Let $q : V \to V/U^\perp \cong U^*$ denote the canonical projection. We put

$$\widetilde{S} := \{s \in S \mid \check{\alpha}_s \notin U^\perp = H(C_S)\}.$$

Proposition 1.4 implies that $(U, (q(\check{\alpha}_s))_{s\in\widetilde{S}}, (\alpha_s)_{s\in\widetilde{S}})$ is a linear Coxeter system because the convex cone $\check{C}_{\widetilde{S}} \subseteq \check{C}_S$ is pointed. We write

$$\mathcal{W}_U := \langle r_s \mid s \in \widetilde{S}\rangle \subseteq \mathrm{GL}(U)$$

for the corresponding reflection group.

To relate this group to $\mathcal{W}$, we first claim that U is $\mathcal{W}$-invariant. For $s \in S\setminus\widetilde{S}$ the relation $\check{\alpha}_s \in U^\perp$ implies that the reflection r_s acts trivially on the subspace $U \subseteq V^*$. Next we observe that, if an element $\sum_{s\in S}\lambda_s\check{\alpha}_s \in C_S$ with $\lambda_s \geq 0$ is contained in $H(C_S)$, then $\lambda_s > 0$ implies that $\check{\alpha}_s \in H(C_S)$, i.e., $s \in S \setminus \widetilde{S}$. We conclude that

$$H(C_S) = \mathrm{cone}\{\check{\alpha}_s \mid s \in S \setminus \widetilde{S}\} = C_{S\setminus\widetilde{S}}.$$

For $s \in \widetilde{S}$ we now derive from (C1) that $\alpha_s \in -(C_{S\setminus\widetilde{S}})^\star = -H(C_S)^\star = U$. Therefore U is also invariant under r_s. Hence U is $\mathcal{W}$-invariant. As the reflections r_s, $s \in S \setminus \widetilde{S}$, act trivially on U, we see that the restriction map

$$R : \mathcal{W} \to \mathcal{W}_U,\ w \mapsto w|_U$$

is a surjective homomorphism.

Lemma 1.7. *Let $a \geq 2$ and*

$$r_1 := \begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix}, \qquad r_2 := \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}).$$

If e_1 and e_2 denote the standard basis of $\mathbb{R}^2$, then

$$(r_1r_2)^n e_1, r_2(r_1r_2)^n e_1 \in \mathrm{cone}\{e_1, e_2\} \textit{ for } n \in \mathbb{N}_0.$$

The matrix r_1r_2 is of infinite order.

Proof. We define a linear functional on $\mathbb{R}^2$ by $\beta(x) := x_1 - \frac{1}{2}ax_2$. First we show by induction on n that

$$(r_1r_2)^n e_1 \in \mathrm{cone}\{e_1, e_2\} \text{ and } \beta((r_1r_2)^n e_1) \geq 0 \text{ hold for any } n \in \mathbb{N}_0.$$

For $n = 0$ this is trivial. So let $v := (r_1r_2)^n e_1$ and assume that the assertion holds for some $n \in \mathbb{N}_0$. With

$$r_1r_2 = \begin{pmatrix} a^2 - 1 & -a \\ a & -1 \end{pmatrix}$$

we then obtain

$$\begin{aligned} \beta(r_1r_2v) &= (a^2-1)v_1 - av_2 - \frac{1}{2}a(av_1 - v_2) = (\frac{a^2}{2} - 1)v_1 - \frac{a}{2}v_2 \\ &= \underbrace{(\frac{a^2}{2} - 2)}_{\geq 0} v_1 + \beta(v) \geq 0. \end{aligned}$$

We also have

$$(a^2-1)v_1 - av_2 = (a^2-3)v_1 + 2\Big(v_1 - \frac{a}{2}v_2\Big) \geq v_1 + 2\beta(v) \geq 0$$

and

$$av_1 - v_2 = \frac{1}{a}\big((a^2-2)v_1 + 2(v_1 - \frac{a}{2}v_2)\big) \geq \frac{1}{a}(2v_1 + 2\beta(v)) \geq 0. \tag{3}$$

This proves that $r_1r_2v \in \mathrm{cone}\{e_1, e_2\}$ and completes our induction.

For $v' := r_2v = r_2(r_1r_2)^n e_1$, we finally obtain with (3)

$$v_1' = v_1 \geq 0 \text{ and } v_2' = av_1 - v_2 \geq 0,$$

hence that $v' \in \mathrm{cone}\{e_1, e_2\}$.

From $\det(r_1r_2) = 1$ and $\mathrm{tr}(r_1r_2) = a^2 - 2 \geq 2$ it follows that, for $a > 2$, r_1r_2 is diagonalizable with real eigenvalues $0 < \lambda_1 < 1 < \lambda_2 := \lambda^{-1}$, and for $a = 2$ it is unipotent but different from the identity matrix. Hence r_1r_2 is of infinite order for every $a \geq 2$. $\square$

Lemma 1.8. *Let $(V, (\alpha_s, \alpha_t), (\check{\alpha}_s, \check{\alpha}_t))$ be a reflection data satisfying* (C1/2) *and $\mathcal{W} \subseteq \mathrm{GL}(V)$ be the subgroup generated by r_s and r_t. Then $(\mathcal{W}, \{r_s, r_t\})$ is a Coxeter system. Its length function ℓ satisfies for $g \in \mathcal{W}$:*

$$\ell(gr_s) \geq \ell(g) \;\Rightarrow\; g\check{\alpha}_s \in C_S = \mathrm{cone}\{\check{\alpha}_s, \check{\alpha}_t\}.$$

Proof. Let $g \in \mathcal{W}$ with $\ell(gr_s) \geq \ell(g)$. Then every reduced expression for g is an alternating product of r_s and r_t ending in r_t. After normalizing the pair $(\alpha_s, \check{\alpha}_s)$ suitably, we may assume that $\alpha_s(\check{\alpha}_t) = \alpha_t(\check{\alpha}_s)$. Let

$$a := -\alpha_s(\check{\alpha}_t) \geq 0 \text{ and } U := \mathrm{span}\{\check{\alpha}_s, \check{\alpha}_t\}.$$

Then

$$\det \begin{pmatrix} \alpha_s(\check{\alpha}_s) & \alpha_s(\check{\alpha}_t) \\ \alpha_t(\check{\alpha}_s) & \alpha_t(\check{\alpha}_t) \end{pmatrix} = 4 - a^2$$

shows that, if $a^2 \neq 4$, then $\check{\alpha}_s$ and $\check{\alpha}_t$ are linearly independent and the same holds for the restrictions of α_s and α_t to U.

If $\check{\alpha}_s$ and $\check{\alpha}_t$ are linearly independent, then the corresponding matrices of the restrictions of r_s and r_t to U are given by

$$r_1 := \begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix} \text{ and } r_2 := \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}.$$

Moreover, r_1 and r_2 are orthogonal with respect to the symmetric bilinear form $\langle \cdot, \cdot \rangle$ defined with respect to the basis $(\check{\alpha}_s, \check{\alpha}_t)$ by the matrix

$$\begin{pmatrix} 1 & -\frac{a}{2} \\ -\frac{a}{2} & 1 \end{pmatrix}$$

because $\alpha_s(v) = 2\langle \check{\alpha}_s, v \rangle$ and $\alpha_t(v) = 2\langle \check{\alpha}_t, v \rangle$ hold for each $v \in U$.

Case 1: $a^2 < 4$. In this case the bilinear form on U is positive definite and the assertion follows from the argument in [2, p. 112].

Case 2: $a^2 > 4$. In this case the bilinear form on U is indefinite and the assertion follows immediately from Lemma 1.7.

Case 3: $a^2 = 4$. Here we have to distinguish two cases. If $\check{\alpha}_s$ and $\check{\alpha}_t$ are linearly dependent, then $C_S = U = \mathbb{R}\check{\alpha}_s$ follows from $\alpha_t(\check{\alpha}_s) = -a < 0$ and $\alpha_t(\check{\alpha}_t) = 2 > 0$. In this case the assertion follows from the invariance of U under $\mathcal{W}$. If $\check{\alpha}_s$ and $\check{\alpha}_t$ are linearly independent, then the assertion follows from Lemma 1.7. $\square$

The following proposition generalizes the corresponding well known result for the geometric representation of a Coxeter group to arbitrary linear Coxeter systems (cp. [2, Thm. 5.4]).

Proposition 1.9. *Let $(V, (\alpha)_{s\in S}, (\check{\alpha}_s)_{s\in S})$ be a reflection data of finite type satisfying* (C1/2)*. Let $\mathcal{W} \subseteq \mathrm{GL}(V)$ be the subgroup generated by the reflections $(r_s)_{s\in S}$ and assume that $(\mathcal{W}, (r_s)_{s\in S})$ is a Coxeter system. Let $\ell : \mathcal{W} \to \mathbb{N}_0$ denote its length function and $g \in \mathcal{W}$, $s \in S$.*

(i) *If $\ell(gr_s) > \ell(g)$, then $g\check{\alpha}_s \in C_S$.*
(ii) *If $\ell(gr_s) < \ell(g)$, then $g\check{\alpha}_s \in -C_S$.*

Proof. First we note that (ii) is a consequence of (i), applied to the element $g' = gr_s$ and using $g\check{\alpha}_s = -g'\check{\alpha}_s$.

We prove (i) by induction on the length of g. The case $\ell(g) = 0$, i.e., $g = \mathbf{1}$ is trivial. If $\ell(g) > 0$, then there exists a $t \in S$ with $\ell(gr_t) = \ell(g) - 1$. Then $t \neq s$ follows from $\ell(gr_s) > \ell(g)$. Let $\widetilde{S} := \{s, t\}$ and consider the subgroup $\widetilde{\mathcal{W}} := \langle r_s, r_t\rangle \subseteq \mathcal{W}$ with the length function $\widetilde{\ell}$. Let

$$A := \{f \in \mathcal{W} \mid f \in g\widetilde{\mathcal{W}}, \ell(g) = \ell(f) + \widetilde{\ell}(f^{-1}g)\}.$$

Obviously $g \in A$, so that A is not empty. Pick $f \in A$ with minimal length $\ell(f)$ and put $f' := f^{-1}g \in \widetilde{\mathcal{W}}$. Then $g = ff'$ with $\ell(g) = \ell(f) + \widetilde{\ell}(f')$.

We also note that $gr_t \in A$ follows from $\ell(g) = \ell(gr_t) + 1 = \ell(gr_t) + \widetilde{\ell}(r_t)$. Hence the choice of f implies that $\ell(f) \leq \ell(gr_t) = \ell(g) - 1$. We now want to apply the induction hypothesis to the pair (f, r_s). To this end, we have to compare the length or f and fr_s. If $\ell(fr_s) < \ell(f)$, then $\ell(fr_s) = \ell(f) - 1$ and we have

$$\begin{aligned}\ell(g) &\leq \ell(fr_s) + \ell(r_s f^{-1}g) \leq \ell(fr_s) + \widetilde{\ell}(r_s f^{-1}g)\\ &\leq \ell(f) - 1 + \widetilde{\ell}(f^{-1}g) + 1 = \ell(f) + \widetilde{\ell}(f^{-1}g) = \ell(g).\end{aligned}$$

We conclude that $\ell(g) = \ell(fr_s) + \widetilde{\ell}(r_s f^{-1}g)$, so that $fr_s \in A$, contradicting the minimality of $\ell(f)$. This implies that $\ell(fr_s) > \ell(f)$ because $\ell(fr_s) \neq \ell(f)$ (see [2, p. 108]), so that the induction hypothesis leads to $f\check{\alpha}_s \in C_S$. By the same argument we obtain $\ell(fr_t) > \ell(f)$ and therefore $f\check{\alpha}_t \in C_S$.

From

$$\ell(f) + \widetilde{\ell}(f') = \ell(g) < \ell(gr_s) = \ell(ff'r_s) \leq \ell(f) + \ell(f'r_s) \leq \ell(f) + \widetilde{\ell}(f'r_s)$$

we further derive that $\widetilde{\ell}(f') \leq \widetilde{\ell}(f'r_s)$. Now Lemma 1.8 implies that

$$f'\check{\alpha}_s \in C_{\widetilde{S}} = \mathrm{cone}\{\check{\alpha}_s, \check{\alpha}_t\}.$$

and thus $g\check{\alpha}_s = ff'\check{\alpha}_s \in fC_{\widetilde{S}} \subseteq C_S$. □

Theorem 1.10. *Let $(V, (\alpha_s)_{s\in S}, (\check{\alpha}_s)_{s\in S})$ be a linear Coxeter system and $\ell : \mathcal{W} \to \mathbb{N}_0$ be the length function with respect to the generating set $\{r_s \mid s \in S\}$. Then the following assertions hold for $g \in \mathcal{W}$:*

(i) *If $\ell(gr_s) > \ell(g)$, then $g\check{\alpha}_s \in C_S$ and $g\alpha_s \in \check{C}_S$.*
(ii) *If $\ell(gr_s) < \ell(g)$, then $g\check{\alpha}_s \in -C_S$ and $g\alpha_s \in -\check{C}_S$.*

Proof. In view of Proposition 1.4, the linear Coxeter system $(V, (\alpha_s)_{s\in S}, (\check{\alpha}_s)_{s\in S})$ and its dual $(V^*, (\check{\alpha}_s)_{s\in S}, (\alpha_s)_{s\in S})$ satisfy the assumptions of Proposition 1.9 because the fact that $(\mathcal{W}, (r_s)_{s\in S})$ is a Coxeter system implies that the adjoints $r_s^* \in \mathrm{GL}(V^*)$ define a Coxeter system in the subgroup $\mathcal{W} \cong \langle r_s^* \mid s \in S\rangle \subseteq \mathrm{GL}(V^*)$. This implies the assertion. $\square$

Remark 1.11. For $s \in S$ and $g \in \mathcal{W}$ the condition $g\alpha_s \in \check{C}_S = K^\star$ is equivalent to the linear functional $g\alpha_s$ taking nonnegative values on K. Therefore Theorem 1.10 implies in particular that each linear functional

$$\alpha \in \mathcal{W}\{\alpha_s \mid s \in S\}$$

either is positive or negative on K^0.

Proposition 1.12. *If $(V, (\alpha_s)_{s\in S}, (\check{\alpha}_s)_{s\in S})$ is a linear Coxeter system, $v \in K$ and*

$$I := \{s \in S \mid \alpha_s(v) = 0\},$$

then the subgroup $\mathcal{W}_I \subseteq \mathcal{W}$ generated by the reflections $\{r_s \mid s \in I\}$ coincides with the stabilizer $\mathcal{W}_v = \{w \in \mathcal{W} \mid wv = v\}$.

Proof. Clearly $\mathcal{W}_I \subseteq \mathcal{W}_v$ because the generators of $\mathcal{W}_I$ fix v. If $\mathcal{W}_I$ is strictly smaller than $\mathcal{W}_v$, there exists an element $g \in \mathcal{W}_v \setminus \mathcal{W}_I$ of minimal positive length. Then $\ell(gr_s) > \ell(g)$ for every $s \in I$ (recall that $\ell(gr_s) \neq \ell(g)$ by [2, p. 108]) implies that $g\alpha_s \in \check{C}_S$ (Theorem 1.10). If $s \notin I$, then $\alpha_s(v) > 0$ implies that $(g\alpha_s)(v) = \alpha_s(g^{-1}v) = \alpha_s(v) > 0$. Therefore $g\alpha_s$ takes positive values on K^0 which entails $g\alpha_s \in \check{C}_S$ (Remark 1.11). We thus arrive at $g\check{C}_S \subseteq \check{C}_S$. As the element $g^{-1} \in \mathcal{W}_v \setminus \mathcal{W}_I$ has the same length, we also obtain $g^{-1}\check{C}_S \subseteq \check{C}_S$, and thus $g\check{C}_S = \check{C}_S$. Now $K = (\check{C}_S)^\star$ leads to $gK = K$, and by (LCS3) further to $g = \mathbf{1}$, contradicting $g \notin \mathcal{W}_I$. $\square$

2. A convexity theorem for linear Coxeter systems

Before we come to our main theorem in this section (Theorem 2.7), we have to define the roots and coroots of a linear Coxeter system. This is crucial to obtain a formulation of the theorem which does not depend on the generating system. This will be essential for the infinite-dimensional generalization where roots and coroots still make sense but $\mathcal{W}$ need not be a Coxeter group (cp. Example 3.6).

2.1. **Roots and coroots.**

Definition 2.2. Let $(V, (\alpha_s)_{s\in S}, (\check{\alpha}_s)_{s\in S})$ be a linear Coxeter system.

(a) We define the set of *roots* by

$$\Delta := \mathcal{W}\{\alpha_s \mid s \in S\} \subseteq V^* \text{ and put } \Delta^\pm := \Delta \cap \pm K^\star = \Delta \cap \pm \check{C}_S.$$

Roots in Δ^+ are called *positive* and roots in Δ^- are called *negative*. Remark 1.11 shows that

$$\Delta = \Delta^+ \dot\cup \Delta^-.$$

We likewise define the corresponding sets of *coroots*

$$\check{\Delta} := \mathcal{W}\{\check{\alpha}_s \mid s \in S\} \subseteq V.$$

(b) The subset

$$\mathcal{R} := \{wr_s w^{-1} \mid w \in W, s \in S\} \subseteq \mathcal{W}$$

is called the *set of reflections* in $\mathcal{W}$. If $r = wr_s w^{-1}$ is a reflection, $\alpha := w\alpha_s$ and $\check{\alpha} := w\check{\alpha}_s$, then

$$r(v) = v - \alpha(v)\check{\alpha} \text{ for } v \in V \tag{4}$$

and $\mathrm{Fix}(r) = \ker \alpha = w \ker \alpha_s$.

Remark 2.3. We claim that a reflection $r \in \mathcal{W}$ is uniquely determined by its hyperplane of fixed points. So let $r \in \mathcal{R}$. Then the hyperplane $\mathrm{Fix}(r)$ intersects the interior T^0 of the Tits cone $T = \mathcal{W}K$ (cp. Remark 1.3(a)). Since each $\mathcal{W}$-orbit in T^0 meets K exactly once (see [10, Thm. 2]), there exists a unique face $F \subseteq K$ of codimension one and some $w \in \mathcal{W}$ with $\mathrm{Fix}(w^{-1}rw) \cap K = F$. Then there exists a uniquely determined $s \in S$ with $F = \ker \alpha_s \cap K$ (Remark 1.3). Now $w^{-1}rw$ and r_s are two reflections in the same hyperplane $\ker \alpha_s$, hence fixing F pointwise. Therefore $w^{-1}rw(K^0) \cap r_s(K^0) \neq \varnothing$ leads to $r_s = w^{-1}rw$, so that $r = wr_s w^{-1}$.

Next we note that, if $\alpha = w\alpha_s = w'\alpha_t$ for some $w, w' \in \mathcal{W}$ and $s, t \in S$, then $wr_s w^{-1}$ and $w'r_t(w')^{-1}$ are both reflections with the same sets of fixed points, so that the preceding argument implies that they are equal: $wr_s w^{-1} = w'r_t(w')^{-1}$. This in turn shows that $w\alpha_s \otimes w\check{\alpha}_s = w'\alpha_t \otimes w'\check{\alpha}_t = w\alpha_s \otimes w'\check{\alpha}_t$, and hence that $w\check{\alpha}_s = w'\check{\alpha}_t$.

Definition 2.4. In view of the preceding remark, we can associate to each root $\alpha \in \Delta$ a well-defined coroot $\check{\alpha} \in \check{\Delta}$ such that the map $\Gamma : \Delta \to \check{\Delta}, \alpha \mapsto \check{\alpha}$ is $\mathcal{W}$-equivariant and the reflections in $\mathcal{W}$ have the form (4).

Remark 2.5. Since the roots α_s, $s \in S$, are positive by definition, Theorem 1.10 implies that

$$\mathrm{cone}(\Delta^+) = \mathrm{cone}\{\alpha_s \mid s \in S\} = \check{C}_S,$$

and hence that

$$K = (\check{C}_S)^\star = \{v \in V \mid \alpha(v) \geq 0 \text{ for all } \alpha \in \Delta^+\}.$$

2.6. **Convex hulls of orbits in the Tits cone.** The following theorem strengthens the corresponding assertion for elements $v \in K^0$ in [1, p. 20] substantially because it provides also sharp information if the stabilizer $\mathcal{W}_v$ is nontrivial.

Theorem 2.7. *Let $(V, (\alpha_s)_{s\in S}, (\check{\alpha}_s)_{s\in S})$ be a linear Coxeter system and $T = \mathcal{W}K \subseteq V$ be its Tits cone. For $v \in T$ we have*

$$\mathcal{W}v \subseteq v - C_v, \ \textit{where } C_v := \mathrm{cone}\{\check{\alpha} \in \check{\Delta} \mid \alpha(v) > 0\}.$$

Proof. As $T = \mathcal{W}K$ and $C_{gv} = gC_v$, we may w.l.o.g. assume that $v \in K$. We put $I := \{s \in S \mid \alpha_s(v) = 0\}$ and recall from Proposition 1.12 that the corresponding parabolic subgroup $\mathcal{W}_I \subseteq \mathcal{W}$ coincides with the stabilizer $\mathcal{W}_v$ of v. Let

$$\mathcal{W}^I := \{g \in \mathcal{W} \mid \ell(gr_s) > \ell(g) \text{ for all } s \in I\},$$

so that $\mathcal{W} = \mathcal{W}^I\mathcal{W}_I$ by [2, p. 123].

We now show $gv - v \in -C_v$ by induction on the length $\ell(g)$ of g. The assertion is trivial for $g = \mathbf{1}$, i.e., $\ell(g) = 0$. Suppose that $\ell(g) > 0$. Then $g^{-1} = h^{-1}g_I$ with $h^{-1} \in \mathcal{W}^I$ and $g_I \in \mathcal{W}_I$ satisfying

$$\ell(g) = \ell(g^{-1}) = \ell(h^{-1}) + \ell(g_I) = \ell(h) + \ell(g_I)$$

(see [2, p. 123]). If $g_I \neq \mathbf{1}$, then $\ell(h) < \ell(g)$ and our induction hypothesis leads to

$$gv - v = g_I^{-1}hv - v = g_I^{-1}(hv - v) \in -g_I^{-1}C_v = -C_v$$

because the stabilizer $\mathcal{W}_I = \mathcal{W}_v$ of v preserves the cone C_v. We may therefore assume that $g_I = \mathbf{1}$, i.e., $g^{-1} \in \mathcal{W}^I$. By Theorem 1.10, $g^{-1}\check{C}_I \subseteq \check{C}_S$. In particular, g^{-1} maps the set

$$\Delta_I^+ := \Delta \cap \operatorname{cone}(\{\alpha_s \mid s \in I\}) = \{\alpha \in \Delta^+ \mid \alpha(v) = 0\}$$

into Δ^+.

Pick $s \in S$ with $\ell(gr_s) < \ell(g)$. If $\alpha_s(v) = 0$, then $gv - v = gr_sv - v \in -C_v$ by the induction hypothesis. We may therefore assume that $\alpha_s(v) > 0$. Then our induction hypothesis implies

$$\begin{aligned} gv - v &= (gr_s)r_sv - v = (gr_s)(v - \alpha_s(v)\check{\alpha}_s) - v \\ &= (gr_s)v - v - \alpha_s(v)(gr_s)\check{\alpha}_s \in -C_v + \alpha_s(v)g\check{\alpha}_s. \end{aligned}$$

In view of $\alpha_s(v) > 0$, it remains to see that $g\check{\alpha}_s = (g\alpha_s)\check{} \in -C_v$ (cp. Definition 2.2(c)), so that it suffices to verify that $(g\alpha_s)(v) < 0$. As $g(-\alpha_s) \in \Delta^+ \subseteq \check{C}_S$ by Theorem 1.10, we have $(g\alpha_s)(v) \leq 0$. If $(g\alpha_s)(v) = 0$, then

$$-\alpha_s \in g^{-1}\{\beta \in \Delta^+ \mid \beta(v) = 0\} = g^{-1}\Delta_I^+ \subseteq \Delta^+$$

by construction of g; but this leads to the contradiction $-\alpha_s \in \Delta^+$. This proves that $(g\alpha_s)(v) < 0$, which completes the proof. □

Corollary 2.8. *For $v \in T$, the following assertions hold:*

(i) $\operatorname{cone}(\mathcal{W}v - v) = -C_v$.
(ii) *v is an extreme point of* $\operatorname{co}(v)$ *if and only if the cone C_v is pointed.*
(iii) *For $\lambda \in V^*$,*

$$\lambda(v) = \min \lambda(\mathcal{W}v) \ \Leftrightarrow\ \lambda \in -C_v^\star.$$

Proof. (i) For any $v \in V$ with $\alpha(v) > 0$ the relation $r_\alpha(v) = v - \alpha(v)\check{\alpha}$ implies that

$$-\check{\alpha} \in \mathbb{R}_+(\mathcal{W}v - v),$$

so that Theorem 2.7 implies for $v \in T$ that $\operatorname{cone}(\mathcal{W}v - v) = -C_v$.

(ii) and (iii) follow immediately from (i). □

Corollary 2.9. *The following conditions are equivalent:*

(i) *The cone* C_S *is pointed, i.e.,* $(V^*, (\check{\alpha}_s)_{s\in S}, (\alpha_s)_{s\in S})$ *is a linear Coxeter system.*
(ii) *There exists a* $v \in K^0$ *which is an extreme point of* $\mathrm{co}(v)$*.*
(iii) *Each* $v \in K^0$ *is an extreme point of* $\mathrm{co}(v)$*.*

Proof. This is immediate from Corollary 2.8 and the fact that $C_S = C_v$ for $v \in K^0$. $\square$

Problem 2.10. From Theorem 2.7 we obtain for $v \in T$ the relation

$$\mathrm{co}(v) \subseteq \bigcap_{w\in\mathcal{W}} w(v - C_v).$$

When do we have equality?

The following example shows that, in general, not every element $v \in V$ satisfies $\mathcal{W}v \subseteq v - C_v$.

Example 2.11. We take a closer look at linear Coxeter systems with a 2-element set $S = \{s, t\}$ and $\dim V = 2$.

First we show that α_s and α_t are linearly independent. If this is not the case, then $\alpha_s(\check{\alpha}_t) \leq 0$ implies that $\alpha_s = \lambda\alpha_t$ for some $\lambda < 0$, but this leads to the contradiction $K = \varnothing$. Therefore α_s and α_t are linearly independent.

(a) Suppose that $\alpha_s(\check{\alpha}_t) = \alpha_t(\check{\alpha}_s) = -2$. Then α_s and α_t vanish on $\check{\alpha}_s + \check{\alpha}_t$, and since V^* is spanned by α_s and α_t, it follows that $\check{\alpha}_t = -\check{\alpha}_s$. In particular, the cone $C_S = \mathbb{R}\check{\alpha}_s$ is not pointed.

This implies that the action of $\mathcal{W}$ on V leaves all affine subspaces of the form $v + C_S$ invariant. If $\alpha_s^*, \alpha_t^* \in V$ is the dual basis of α_s, α_t, then

$$K = \mathbb{R}_+\alpha_s^* + \mathbb{R}_+\alpha_t^* \text{ and } \check{\alpha}_s = -\check{\alpha}_t = 2(\alpha_s^* - \alpha_t^*).$$

The linear map $r_t r_s$ fixes the line C_S pointwise and induces on V/C_S the identity, hence is unipotent. Moreover,

$$\begin{aligned} r_t r_s(\alpha_s^*) &= r_t(\alpha_s^* - \check{\alpha}_s) = r_t(-\alpha_s^* + 2\alpha_t^*) = -\alpha_s^* + 2\alpha_t^* - 2\check{\alpha}_t \\ &= -5\alpha_s^* + 6\alpha_t^* = \alpha_s^* - 3\check{\alpha}_s, \end{aligned}$$

so that the convexity of the Tits cone implies that

$$T^0 = C_S + \mathbb{R}_+^\times \alpha_s^*$$

is an open half plane and $T = T^0 \cup \{0\}$. We further have

$$\mathrm{co}(v) = v + C_S \text{ for } v \in T^0$$

and

$$\check{\Delta} = \{\check{\alpha}_s, \check{\alpha}_t\} = \{\pm\check{\alpha}_s\},$$

whereas $\Delta = \mathcal{W}\{\alpha_s, \alpha_t\}$ is infinite. More precisely, we have

$$r_t r_s \alpha_s = r_t(-\alpha_s) = -\alpha_s + \alpha_s(\check{\alpha}_t)\alpha_t = -\alpha_s - 2\alpha_t = \alpha_s - 2(\alpha_s + \alpha_t),$$

and since $\mathcal{W} = \{r_s(r_t r_s)^n, (r_t r_s)^n \mid n \in \mathbb{Z}\}$, it follows that

$$\Delta = \{\pm\alpha_s, \pm\alpha_t\} + 2\mathbb{Z}(\alpha_s + \alpha_t).$$

The two coroots $\check{\alpha}_s$ and $\check{\alpha}_t$ lie in the boundary of the Tits cone and $\mathcal{W}\check{\alpha}_s = \{\pm\check{\alpha}_s\}$. Moreover,

$$\{\alpha \in \Delta \mid \alpha(\check{\alpha}_s) > 0\} = \{\alpha_s, -\alpha_t\} + 2\mathbb{Z}(\alpha_s + \alpha_n),$$

so that

$$C_{\check{\alpha}_s} = \operatorname{cone}\{\check{\alpha} \mid \alpha(\check{\alpha}_s) > 0\} = \mathbb{R}_+\check{\alpha}_s$$

has the property that

$$\mathcal{W}\check{\alpha}_s \subseteq \check{\alpha}_s - C_{\check{\alpha}_s}.$$

That this remains true on the boundary ∂T is due to the fact that the $\mathcal{W}$-action on this line can also be obtained from the one-dimensional reflection datum $(\partial T, \alpha_s, \check{\alpha}_s)$.

(b) Suppose that $\alpha_s(\check{\alpha}_t) = \alpha_t(\check{\alpha}_s) = -3$. Then

$$\det\begin{pmatrix} \alpha_s(\check{\alpha}_s) & \alpha_s(\check{\alpha}_t) \\ \alpha_t(\check{\alpha}_s) & \alpha_t(\check{\alpha}_t) \end{pmatrix} = 4 - 9 < 0$$

implies that $\check{\alpha}_s$ and $\check{\alpha}_t$ are linearly independent. In this case the symmetric bilinear form $(\cdot,\cdot)$ on V represented by the matrix

$$A = \begin{pmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{pmatrix}$$

with respect to the basis $\check{\alpha}_s, \check{\alpha}_t$ is $\mathcal{W}$-invariant. If $\alpha_s^*, \alpha_t^* \in V$ is the dual basis of α_s, α_t, then

$$K = \mathbb{R}_+\alpha_s^* + \mathbb{R}_+\alpha_t^* \text{ and } \alpha_s^* = -\frac{2}{5}\check{\alpha}_s - \frac{3}{5}\check{\alpha}_t,\ \alpha_t^* = -\frac{3}{5}\check{\alpha}_s - \frac{2}{5}\check{\alpha}_t.$$

Now $(\alpha_s^*, \alpha_s^*) = (\alpha_t^*, \alpha_t^*) < 0$ implies that

$$T = \mathcal{W}K \subseteq \{v \in V \mid (v,v) \leq 0, (v, \alpha_s^*) \leq 0\}.$$

From the case $a = 3$ in Lemma 1.7 we derive that the matrix of $r_s r_t$ with respect to the basis $(\check{\alpha}_s, \check{\alpha}_t)$ is

$$B := \begin{pmatrix} a^2 - 1 & -a \\ a & -1 \end{pmatrix} = \begin{pmatrix} 8 & -3 \\ 3 & -1 \end{pmatrix}.$$

From $\det B = 1$ and $\operatorname{tr} B = 7$ it follows that it has two eigenvalues λ and λ^{-1} with $6 < \lambda < 7$. We conclude that, for every $v_0 \in K$ and any norm $\|\cdot\|$ on V, the limit of $\frac{1}{\|(r_s r_t)^n v_0\|}(r_s r_t)^n v_0$ for $n \to \infty$ is a unit eigenvector v_1 for λ, and that, for $n \to -\infty$, we obtain a unit eigenvector v_2 for λ^{-1}. From the $\mathcal{W}$-invariance of T, we derive that $v_1, v_2 \in \overline{T}$. As $r_s r_t$ is orthogonal w.r.t. $(\cdot,\cdot)$, we have $(v_1, v_1) = 0 = (v_2, v_2)$. This implies that

$$\overline{T} = \mathbb{R}_+ v_1 + \mathbb{R}_+ v_2 = \{v \in V \mid (v,v) \leq 0, (v, \alpha_s^*) \leq 0\}.$$

We claim that the element $v_1 \in \partial T$ satisfies $\mathcal{W}v_1 \not\subseteq v_1 - C_{v_1}$. Pick $\alpha \in \Delta$. Then α is $\mathcal{W}$ conjugate to α_s or α_t, so that $\ker\alpha$ intersects the interior of T, which implies that $(\check{\alpha}, \check{\alpha}) > 0$ and hence further $(\check{\alpha}, v_j) \neq 0$ for $j = 1, 2$.

If $\alpha(v_1) > 0$, then $r_\alpha v_1 \in \overline{T}$ leads to

$$0 \geq (v_1, r_\alpha v_1) = (v_1, v_1) - \alpha(v_1)(v_1, \check{\alpha}) = \underbrace{-\alpha(v_1)}_{<0}(v_1, \check{\alpha}),$$

so that $(v_1, \check{\alpha}) > 0$. This implies that $\{v \in C_{v_1} \mid (v, v_1) = 0\} = \{0\}$, and thus $\pm v_1 \notin C_{v_1}$. Now $r_s r_t v_1 = \lambda v_1$ leads to $\mathcal{W} v_1 \not\subseteq v_1 - C_{v_1}$.

The element $\check{\alpha}_s$ satisfies $(\check{\alpha}_s, \check{\alpha}_s) > 0$, so that it is not contained in $\pm\overline{T}$. For this element we have

$$r_s r_t(\check{\alpha}_s) = r_s(\check{\alpha}_s + 3\check{\alpha}_t) = -\check{\alpha}_s + 3(\check{\alpha}_t + 3\check{\alpha}_s) = 8\check{\alpha}_s + 3\check{\alpha}_t$$

and

$$r_t r_s(\check{\alpha}_s) = r_t(-\check{\alpha}_s) = -\check{\alpha}_s - 3\check{\alpha}_t,$$

so that

$$\frac{7}{2}\check{\alpha}_s = \frac{1}{2}(r_t r_s(\check{\alpha}_s) + r_s r_t(\check{\alpha}_s)) \in \mathrm{co}(\check{\alpha}_s)$$

shows that $\check{\alpha}_s$ is an interior point of $\mathrm{co}(\check{\alpha}_s)$.

On the other hand, the cone

$$C_{\check{\alpha}_s} = \mathrm{cone}\{\check{\alpha} \mid \alpha(\check{\alpha}_s) > 0\} = \mathrm{cone}\{\check{\alpha} \mid (\check{\alpha}, \check{\alpha}_s) > 0\}$$

is proper, so that $\mathcal{W}\check{\alpha}_s \not\subseteq \check{\alpha}_s - C_{\check{\alpha}_s}$.

In general the dual of a linear Coxeter system is not a linear Coxeter system. However, we have seen in Remark 1.6(b) that we always obtain a linear Coxeter system on the subspace $U = \mathrm{span}(C_S^\star) \subseteq V^*$. For orbits in the corresponding Tits cone, we have the following variant of Theorem 2.7.

Theorem 2.12. (Convexity Theorem for V^*) *Let $(V, (\alpha_s)_{s\in S}, (\check{\alpha}_s)_{s\in S})$ be a linear Coxeter system and $\check{T} = \mathcal{W} C_S^\star \subseteq V^*$. For any $\lambda \in \check{T}$ we then have*

$$\mathcal{W}\lambda \subseteq \lambda - C_\lambda \ \textit{for}\ C_\lambda := \mathrm{cone}\{\alpha \in \Delta \mid \lambda(\check{\alpha}) > 0\}.$$

Proof. We may w.l.o.g. assume that $\lambda \in C_S^\star$. Consider the subspace

$$U := \mathrm{span}(C_S^\star) = H(C_S)^\perp \subseteq V^*$$

and recall from Remark 1.6(b) that $(U, (q(\check{\alpha}_s))_{s\in\widetilde{S}}, (\alpha_s)_{s\in\widetilde{S}})$ is a linear Coxeter system with fundamental chamber $C_S^\star$ for which we have the surjective restriction map

$$R : \mathcal{W} \to \mathcal{W}_U,\ w \mapsto w|_U.$$

It follows in particular that, for $\lambda \in C_S^\star \subseteq U$, we have $\mathcal{W}\lambda = \mathcal{W}_U\lambda$. Applying Theorem 2.7, we obtain

$$\mathcal{W}\lambda = \mathcal{W}_U\lambda \subseteq \lambda - C_\lambda^U,\ \text{where}\ \ C_\lambda^U = \mathrm{cone}\{\alpha \in \check{\Delta}_U \mid \lambda(\check{\alpha}) > 0\}. \tag{5}$$

On the other hand $\mathcal{W}|_U = \mathcal{W}_U$ implies

$$\check{\Delta}_U = \mathcal{W}_U\{\alpha_s \mid s \in \widetilde{S}\} = \mathcal{W}\{\alpha_s \mid s \in \widetilde{S}\} \subseteq U.$$

For $s \in S\backslash\widetilde{S}$ we have $\check{\alpha}_s \in U^\perp$, hence also $\mathcal{W}\check{\alpha}_s \in U^\perp$ because U is $\mathcal{W}$-invariant and thus $\lambda(\mathcal{W}\check{\alpha}_s) = \{0\}$. This shows that

$$C_\lambda = \mathrm{cone}\{\alpha \in \check{\Delta} \mid \lambda(\check{\alpha}) > 0\} = C_\lambda^U,$$

and by (5), the proof is complete. ☐

The following proposition extends Proposition 1.12 to stabilizers of elements in $C_{\widetilde{S}}^\star$.

Proposition 2.13. *If $\lambda \in C_{\widetilde{S}}^\star$, then $\mathcal{W}_\lambda = \langle r_s \mid \lambda(\check{\alpha}_s) = 0\rangle$.*

Proof. We recall the subspace $U := (\check{C}_S)^\star - (\check{C}_S)^\star \subseteq V^*$ and the related objects also used in the preceding proof. We then have a surjective homomorphism $R : \mathcal{W} \to \mathcal{W}_U, w \mapsto w|_U$, and $\lambda \in U$ implies that $\ker R \subseteq \mathcal{W}_\lambda$.

For $s \in S \setminus \widetilde{S}$ we have $\check{\alpha}_s \in H(C_S) \subseteq \ker\lambda$, which leads to $r_s \in \ker R \subseteq \mathcal{W}_\lambda$. Therefore

$$S_\lambda := \{s \in S \mid \lambda(\check{\alpha}_s) = 0\} \supseteq S \setminus \widetilde{S}.$$

Let $\widetilde{S}_\lambda := \widetilde{S} \cap S_\lambda$. Since $C_{\widetilde{S}}^\star$ is the fundamental chamber of the linear Coxeter system in U, Proposition 1.12 yields

$$\mathcal{W}_{U,\lambda} = \langle r_s \mid s \in \widetilde{S}, \lambda(\check{\alpha}_s) = 0\rangle.$$

This implies that $\mathcal{W}_\lambda \subseteq \langle r_s \mid s \in \widetilde{S}_\lambda\rangle \cdot \ker R$.

Next we observe that, for $s \in \widetilde{S}$ and $t \in S \setminus \widetilde{S}$ we have $\alpha_s \in U$ and $\check{\alpha}_t \in H(C_S) = U^\perp$, so that $\alpha_s(\check{\alpha}_t) = 0$. From (C1) we now also obtain $\alpha_t(\check{\alpha}_s) = 0$ and this implies that $r_s r_t = r_t r_s$:

$$\begin{aligned} r_s r_t(v) &= r_s(v - \alpha_t(v)\check{\alpha}_t) = v - \alpha_t(v)\check{\alpha}_t - \alpha_s(v)\check{\alpha}_s + \alpha_t(v)\alpha_s(\check{\alpha}_t)\check{\alpha}_s \\ &= v - \alpha_t(v)\check{\alpha}_t - \alpha_s(v)\check{\alpha}_s + \alpha_s(v)\alpha_t(\check{\alpha}_s)\check{\alpha}_t = r_t r_s(v). \end{aligned}$$

Therefore

$$\mathcal{W} = \langle r_s \mid s \in S\rangle = \langle r_s \mid s \in \widetilde{S}\rangle\langle r_t \mid t \in S \setminus \widetilde{S}\rangle = \mathcal{W}_{\widetilde{S}}\mathcal{W}_{S\setminus\widetilde{S}}$$

is a product of two commuting subgroups. Since the subgroup $\mathcal{W}_{\widetilde{S}}$ of $\mathcal{W}$ is a Coxeter group with Coxeter system $\{r_s \mid s \in \widetilde{S}\}$, the restriction homomorphism $R : \mathcal{W} \to \mathcal{W}_U$ maps $\mathcal{W}_{\widetilde{S}}$ bijectively onto $\mathcal{W}_U$. On the other hand, $\mathcal{W}_{S\setminus\widetilde{S}} \subseteq \ker R$, so that $\ker R \cap \mathcal{W}_{\widetilde{S}} = \{\mathbf{1}\}$ leads to $\ker R = \mathcal{W}_{S\setminus\widetilde{S}}$. We finally arrive at

$$\mathcal{W}_\lambda = \mathcal{W}_{\widetilde{S}_\lambda} \ker R = \mathcal{W}_{\widetilde{S}_\lambda}\mathcal{W}_{S\setminus\widetilde{S}} = \mathcal{W}_{S_\lambda}. \qquad \square$$

The following proposition describes the subset of a $\mathcal{W}$-orbit in T on which a linear functional $\lambda \in C_{\widetilde{S}}^\star$ takes its maximal values as the orbit of the stabilizer of $\mathcal{W}_\lambda$ and we also provide a dual version.

Proposition 2.14. *Let $\lambda \in C_{\widetilde{S}}^\star$ and $v \in K = (\check{C}_S)^\star$, so that*

$$\lambda(v) = \max\lambda(\mathcal{W}v).$$

If $g \in \mathcal{W}$ satisfies $\lambda(gv) = \max\lambda(\mathcal{W}v)$, then $gv \in \mathcal{W}_\lambda v$ and $g^{-1}\lambda \in \mathcal{W}_v\lambda$.

Proof. (a) First we show that $gv \in \mathcal{W}_\lambda v$. In Proposition 2.13 we have seen that $\mathcal{W}_\lambda = \mathcal{W}_{S_\lambda}$ is a parabolic subgroup of $\mathcal{W}$. Let

$$\mathcal{W}^\lambda := \{w \in \mathcal{W} \mid \ell(r_s w) > \ell(w) \text{ for all } s \in S_\lambda\},$$

so that $\mathcal{W} = \mathcal{W}_\lambda\mathcal{W}^\lambda$ by [2, p. 123].

If there exists a $g \in \mathcal{W}$ with $gv \notin \mathcal{W}_\lambda v$ and $\lambda(gv) = \max \lambda(\mathcal{W}v)$, we choose such an element g with minimal length. Then $g \in \mathcal{W}^\lambda$ with $\ell(g) > 0$. We pick an $s \in S$ with $\ell(g^{-1}r_s) = \ell(r_s g) < \ell(g)$ and observe that this implies that $s \notin S_\lambda$, i.e., $\lambda(\check{\alpha}_s) \neq 0$ and therefore $\lambda(\check{\alpha}_s) > 0$ because $\lambda \in C_S^\star$. Then $g^{-1}\alpha_s \in -\check{C}_S$ by Theorem 1.10, which leads to $0 \geq (g^{-1}\alpha_s)(v) = \alpha_s(gv)$. We thus arrive at

$$\lambda(r_s gv) = \lambda(gv) - \underbrace{\alpha_s(gv)}_{\leq 0}\underbrace{\lambda(\check{\alpha}_s)}_{>0} \geq \lambda(gv) \geq \lambda(r_s gv),$$

where the last inequality follows from the maximality of $\lambda(gv)$. We conclude that $\alpha_s(gv) = 0$, so that $r_s gv = gv \notin \mathcal{W}_\lambda v$. This contradicts the minimality of the length of g.

(b) Now we show that $g^{-1}v \in \mathcal{W}_v \lambda$. In Proposition 1.12 we have seen that $\mathcal{W}_v$ is a parabolic subgroup of $\mathcal{W}$ generated by the reflections r_{α_s} with

$$s \in S_v := \{s \in S \mid \alpha_s(v) = 0\}.$$

Let

$$\mathcal{W}^v := \{w \in \mathcal{W} \mid \ell(wr_s) > \ell(w) \text{ for all } s \in S_v\},$$

so that $\mathcal{W} = \mathcal{W}^v \mathcal{W}_v$ by [2, p. 123].

If there exists a $g \in \mathcal{W}$ with $g^{-1}\lambda \notin \mathcal{W}_v \lambda$ and $\lambda(gv) = \max \lambda(\mathcal{W}v)$, we choose such an element $g \in \mathcal{W}$ of minimal length. Then $g \in \mathcal{W}^v$ with $\ell(g) > 0$. We pick an $s \in S$ with $\ell(gr_s) < \ell(g)$ and observe that this implies that $s \notin S_v$, i.e., $\alpha_s(v) \neq 0$ and therefore $\alpha_s(v) > 0$ because $v \in K$. We further obtain $g\check{\alpha}_s \in -C_S$ from Proposition 1.9, which leads to $0 \geq \lambda(g\check{\alpha}_s) = (g^{-1}\lambda)(\check{\alpha}_s)$. We thus arrive at

$$\lambda(gr_s v) = \lambda(gv) - \underbrace{\alpha_s(v)}_{>0}\underbrace{\lambda(g\check{\alpha}_s)}_{\leq 0} \geq \lambda(gv) \geq \lambda(gr_s v),$$

where the last inequality follows from the maximality of $\lambda(gv)$. We conclude that $\lambda(g\check{\alpha}_s) = 0$, so that $r_s(g^{-1}\lambda) = g^{-1}\lambda \notin \mathcal{W}_v\lambda$. This contradicts the minimality of the length of g. $\square$

Problem 2.15. For $v \in K$ and $\alpha \in \Delta$, the relation $\alpha(v) > 0$ implies $\alpha \in \Delta^+ \subseteq \check{C}_S$, so that $C_v \subseteq C_S$ and $C_S^\star \subseteq C_v^\star$. Is the conclusion of Proposition 2.14 still valid under the weaker assumption $\lambda \in C_v^\star$?

3. Orbits of locally finite and locally affine Weyl groups

Now we turn to the applications of Theorems 2.7 and 2.12 to Weyl group orbits of linear functionals for locally finite and locally affine root systems.

3.1. **Locally finite root systems.** First we describe the irreducible locally finite root systems of infinite rank (cp. [4, §8], [9]). Let J be a set and $V := \mathbb{R}^{(J)}$ denote the free vector space over J, endowed with the canonical scalar product, given by

$$(x, y) := \sum_{j \in J} x_j y_j.$$

We write $(e_j)_{j\in J}$ for the canonical orthonormal basis and we realize the root systems in the dual space $V^* \cong \mathbb{R}^J$ which contains the linearly independent system $\varepsilon_j := e_j^*$, defined by $\varepsilon_j^*(e_k) = \delta_{jk}$. On $\operatorname{span}\{\varepsilon_j \mid j \in J\}$ we also have a positive definite scalar product defined by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$ for which the canonical inclusion $V \hookrightarrow V^*$ is isometric. The infinite irreducible locally finite root systems are given by

$$\begin{aligned} A_J &:= \{\varepsilon_j - \varepsilon_k \mid j,k \in J, j \neq k\}, \\ B_J &:= \{\pm\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k \mid j,k \in J, j \neq k\}, \\ C_J &:= \{\pm 2\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k \mid j,k \in J, j \neq k\}, \\ D_J &:= \{\pm\varepsilon_j \pm \varepsilon_k \mid j,k \in J, j \neq k\}, \\ BC_J &:= \{\pm\varepsilon_j, \pm 2\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k \mid j,k \in J, j \neq k\}. \end{aligned}$$

Let $\Delta \subseteq V^* \cong \mathbb{R}^J$ be a locally finite root system of type X_J with $X \in \{A, B, C, D, BC\}$. For $\alpha \in \operatorname{span}\Delta$, we write $\alpha^\sharp \in V$ for the unique element determined by

$$\alpha(v) = (v, \alpha^\sharp) \text{ for } v \in V.$$

For $\alpha \in \Delta$ we define its *coroot* by

$$\check{\alpha} = \frac{2}{(\alpha, \alpha)}\alpha^\sharp.$$

This leads to a reflection data $(V, \Delta, \check{\Delta})$ and a corresponding group $\mathcal{W}$, called in this context the *Weyl group*.

Theorem 3.2. *For $\lambda \in V^*$ we have*

$$\mathcal{W}\lambda \subseteq \lambda - C_\lambda \textit{ for } C_\lambda := \operatorname{cone}\{\alpha \in \Delta \mid \lambda(\check{\alpha}) > 0\}.$$

Proof. Let $w \in \mathcal{W}$ and observe that w is a finite product of reflections $r_{\alpha_1}, \ldots, r_{\alpha_n}$. We pick a finite dimensional subset $F \subseteq J$ such that $\alpha_j \in \mathbb{R}^F$ for $j = 1, \ldots, n$. Accordingly, we have an orthogonal direct sum

$$V = V_0 \oplus V_1 \text{ with } V_0 = \mathbb{R}^F \text{ and } V_1 := V_0^\perp = \mathbb{R}^{J \setminus F}$$

which is invariant under w.

Next we observe that $\Delta_0 := \Delta \cap \mathbb{R}^F$ is a finite root system of type X_F, which implies that the finite reflection system $(V_0, \Delta_0, \check{\Delta}_0)$ comes from a finite Coxeter system with finite Coxeter group $\mathcal{W}_0$ containing w. In this case the Tits cones in V_0 and V_0^* coincide with the whole space, so that

$$\mathcal{W}_0\lambda_0 \subseteq \lambda_0 - C^0_{\lambda_0}$$

holds for $\lambda_0 := \lambda|_{V_0}$ and

$$C^0_{\lambda_0} := \operatorname{cone}\{\alpha \in \Delta_0 \mid \lambda_0(\check{\alpha}) = \lambda(\check{\alpha}) > 0\} \subseteq C_\lambda.$$

Writing $\lambda = \lambda_0 \oplus \lambda_1$ according to the decomposition $V = V_0 \oplus V_1$, we now obtain

$$w\lambda = w\lambda_0 \oplus \lambda_1 \in (\lambda_0 - C^0_{\lambda_0}) \oplus \lambda_1 = \lambda - C^0_{\lambda_0} \subseteq \lambda - C_\lambda. \qquad \square$$

Corollary 3.3. *For $d \in V$ and $\lambda \in V^*$, the following are equivalent*

(i) $\lambda(d) = \min\langle \mathcal{W}\lambda, d\rangle$.
(ii) $d \in -C_\lambda^\star$.
(iii) $\lambda(\check{\alpha}) > 0 \Rightarrow \alpha(d) \leq 0$ *holds for all* $\alpha \in \Delta$.

Remark 3.4. Since the canonical inclusion $V = \mathbb{R}^{(J)} \hookrightarrow V^* \cong \mathbb{R}^J$ is $\mathcal{W}$-equivariant, Theorem 3.2 implies the corresponding result for $\mathcal{W}$-orbits in V itself:

$$\mathcal{W}v \subseteq v - C_v \text{ for } C_v = \operatorname{cone}\{\check{\alpha} \mid \alpha(v) > 0\}.$$

We conclude this subsection with a brief discussion of the fact that the Weyl groups of uncountable irreducible locally finite root systems are not Coxeter groups.

Proposition 3.5. *Let $(\mathcal{W}, (r_s)_{s\in S})$ be an irreducible Coxeter system for which the group $\mathcal{W}$ is locally finite, i.e., each finite subset generates a finite subgroup. Then S is at most countable.*

Proof. Let Γ be the corresponding Coxeter graph. Its vertices are the elements of S and two such elements s, t are connected by an edge if $\operatorname{ord}(r_s r_t) > 2$. The irreducibility of $(\mathcal{W}, S)$ is equivalent to the connectedness of this graph. The classification of the finite Coxeter systems $(\mathcal{W}, S)$ for which $\mathcal{W}$ is finite (see [2, Thm. 2.7]) now implies that the order of any vertex in the graph Γ is at most 3. Therefore Γ must be at most countable. □

Example 3.6. For the root system A_J, the Weyl group $\mathcal{W}$ is the subgroup $S_{(J)}$ of the group S_J of all permutations of J which is generated by all transpositions $\tau_{i,j}$ exchanging i and j. Let $A_{(J)}$ denote the subgroup of index 2 consisting of all even permutations in $S_{(J)}$.

If $|J| \geq 5$, then $A_{(J)}$ is a direct limit of simple groups, hence simple. This implies that, for any subset $S \subseteq \mathcal{W}$ for which $(\mathcal{W}, S)$ is a Coxeter system, this Coxeter system is irreducible. Now Proposition 3.5 implies that $\mathcal{W} = S_{(J)}$ is not a Coxeter group if J is uncountable.

If J is countable, we may assume $J = \mathbb{N}$, and then it is easy to see that the transpositions $S := \{\tau_{i,i+1} \mid i \in \mathbb{N}\}$ lead to the Coxeter system $(\mathcal{W}, S)$.

Let $V = \mathbb{R}^{(J)}$ be the real vector space with basis $(e_j)_{j\in J}$, where $e_j = (\delta_{ji})_{i\in J}$. Then $\alpha_{ij}(x) = x_i - x_j$ and $\check{\alpha}_{ij} = e_i - e_j$ form the reflection data of the root system A_J.

For $J = \mathbb{N}$ and $S := \{\tau_{i,i+1} \mid i \in \mathbb{N}\}$ we obtain a smaller reflection data. The corresponding cone

$$K := \{v \in V \mid \alpha_s(v) \geq 0 \text{ for all } s \in S\} = \{x \in \mathbb{R}^{(\mathbb{N})} \mid x_{n+1} \leq x_n \text{ for all } n \in \mathbb{N}\}$$

has no interior points in any vector topology on V. This is the main reason why the concept of a linear Coxeter system is problematic for infinite sets S.

3.7. **Locally affine root systems.** Let $V = \mathbb{R}^{(J)}$ be as above and $\Delta \subseteq V^* \cong \mathbb{R}^J$ be a locally finite root system of type X_J. We put

$$\widehat{V} := \mathbb{R} \times V \times \mathbb{R} \text{ and } \Delta^{(1)} := \{0\} \times \Delta \times \mathbb{Z} \subseteq \mathbb{R} \times V^* \times \mathbb{R} \cong \widehat{V}^*.$$

We also define a Lorentzian form on $\widehat{V}$ by

$$((z,x,t),(z,x',t')) := (x,x') - zt' - z't.$$

Suppressing the first component, we have Yoshii's classification [11, Cor. 13].

Proposition 3.8. *The irreducible reduced locally affine root systems of infinite rank are the following, where J is an infinite set: $A_J^{(1)}, B_J^{(1)}, C_J^{(1)}, D_J^{(1)}$, or*

$$\begin{aligned} B_J^{(2)} &:= \big(B_J \times 2\mathbb{Z}\big) \cup \big(\{\pm\varepsilon_j \mid j \in J\} \times (2\mathbb{Z}+1)\big), \\ C_J^{(2)} &:= (C_J \times 2\mathbb{Z}) \cup \big(D_J \times (2\mathbb{Z}+1)\big) \\ (BC_J)^{(2)} &:= (B_J \times 2\mathbb{Z}) \cup \big(BC_J \times (2\mathbb{Z}+1)\big). \end{aligned}$$

Let $\widehat{\Delta} \subseteq \widehat{V}^*$ be one of these locally affine root systems. We write

$$\Delta_n := \{\alpha \in \Delta \mid (0,\alpha,n) \in \widehat{\Delta}\},$$

so that

$$\widehat{\Delta} = (\{0\} \times \Delta_0 \times 2\mathbb{Z}) \dot\cup (\{0\} \times \Delta_1 \times (2\mathbb{Z}+1)).$$

A quick inspection shows that all reflections corresponding to roots in Δ_1 are also obtained from Δ_0. Therefore we obtain an injection

$$\iota_{\mathcal{W}} : \mathcal{W} \cong \langle r_{(0,\alpha,0)} : \alpha \in \Delta_0 \rangle_{\mathrm{grp}} \hookrightarrow \widehat{\mathcal{W}}.$$

Since $\widehat{\Delta}$ consists of nonisotropic vectors for the Lorentzian form, we can also define for $\underline{\alpha} = (0,\alpha,n) \in \widehat{\Delta}$ the *coroot* by

$$\check{\underline{\alpha}} = \frac{2}{(\underline{\alpha},\underline{\alpha})}\underline{\alpha}^\sharp = \frac{2}{(\alpha,\alpha)}(-n,\alpha^\sharp,0) = \Big(\frac{-2n}{(\alpha,\alpha)}, \check{\alpha}, 0\Big) \tag{6}$$

and obtain a reflection data $(\widehat{V}, \widehat{\Delta}, \check{\widehat{\Delta}})$ and a corresponding (affine) Weyl group $\widehat{\mathcal{W}}$. In the following we write

$$c := (1,0,0) \text{ and } d := (0,0,1)$$

for these two distinguished elements of $\widehat{V}$.

Theorem 3.9. *For $\lambda \in \widehat{V}^*$ with $\lambda(c) \neq 0$ the following assertions hold:*

(i) $\widehat{\mathcal{W}}\lambda \subseteq \lambda - C_\lambda$ *for* $C_\lambda := \mathrm{cone}\{\alpha \in \widehat{\Delta} \mid \lambda(\check{\alpha}) > 0\}$.

(ii) *If $\lambda(d) = \min(\widehat{\mathcal{W}}\lambda)(d)$, then any $\mu \in \widehat{\mathcal{W}}\lambda$ with $\mu(d) = \lambda(d)$ is contained in the orbit of $\mathcal{W} \cong \widehat{\mathcal{W}}_d$.*

Proof. (i) Since we can argue as in the locally finite case, it suffices to show that, if J is finite and $\widehat{\Delta}$ of type $X_J^{(t)}$ for $t = 1,2$, then $\lambda \in \widehat{V}^*$ is contained in the Tits cone, so that Theorem 2.12 applies. To this end, we have to recall the itemize of affine root systems with respect to a simple system of roots.

Since $\lambda(c) \neq 0$, the $\widehat{\mathcal{W}}$-orbit contains an element which is either dominant or antidominant (see [7, Prop. 4.9] and also [3, Prop. 6.6] and [5, Thm. 16]), i.e., there exists a simple systems of roots $\Pi \subseteq \widehat{\Delta}$ such that λ is dominant. This means that λ is contained in the corresponding fundamental chamber, hence in particular in the Tits cone. Now the assertion follows from Theorem 2.7.

(ii) With the same argument as before, it suffices it suffices to prove the assertion for the case where J is finite and λ is antidominant with respect to a given simple system Π of roots. So assume that $\widehat{\Delta}$ is of type $X_J^{(t)}$ for some finite set J. We write $\Pi = \{\alpha_0, \dots, \alpha_r\}$ for a set of simple roots, where $\alpha_1, \dots, \alpha_r$ are simple roots of the corresponding finite root system Δ and $\alpha_0 = (-\theta, 1)$, where θ is the "highest weight" in Δ_1 with respect to the positive system defined by $\{\alpha_1, \dots, \alpha_r\}$ (cp. [3]).

Now $\alpha_j(d) = 0$ for $j = 1, \dots, r$ and $\alpha_0(d) = 1$ imply that $d \in K = (\check{C}_S)^\star$. Further, the antidominance of λ with respect to Π means that $\lambda \in -C_S^\star$. If $\mu = w^{-1}\lambda$ satisfies $\mu(d) = \lambda(wd) = \lambda(d) = \min(\widehat{\mathcal{W}}\lambda)(d)$, we obtain $\mu \in \widehat{\mathcal{W}}_d\lambda$ from Proposition 2.14.

Finally, we note that the stabilizer group $\widehat{\mathcal{W}}_d$ is a parabolic subgroup of $\widehat{\mathcal{W}}$ generated by the fundamental reflections r_{α_j} fixing d, which is the case for $j > 0$. Therefore $\widehat{\mathcal{W}}_d \cong \mathcal{W}$ is the Weyl group of the corresponding finite root system X_J. $\square$

Corollary 3.10. *For $d = (0, 0, 1)$ and $\lambda \in \widehat{V}^*$, the following are equivalent*

(i) $\lambda(d) = \min\langle\widehat{\mathcal{W}}\lambda, d\rangle$.
(ii) $d \in -C_\lambda^\star$.
(iii) $\lambda(\underline{\check{\alpha}}) > 0 \Rightarrow \underline{\alpha}(d) \leq 0$ *holds for all* $\underline{\alpha} \in \widehat{\Delta}$.
(iv) $n > 0 \Rightarrow \frac{(\alpha,\alpha)}{2n}\lambda(\check{\alpha}) \leq \lambda(c)$ *holds for all* $\underline{\alpha} = (0, \alpha, n) \in \widehat{\Delta}$).

Proof. The equivalence of (i) and (ii) follows from Theorem 3.9(i), and (iii) is a reformulation of (ii). The equivalence of (iii) and (iv) follows by negating the implication, inserting the formula for the coroot and using $\underline{\alpha}(d) = n$. $\square$

Definition 3.11. Linear functionals $\lambda \in \widehat{V}^*$ satisfying the equivalent conditions in Corollary 3.10 are called *d-minimal.*

3.12. **Characterization of *d*-minimal weights.** For $\lambda \in \widehat{V}^*$, let $\lambda_c := \lambda(c)$.

Lemma 3.13. *If $(\widehat{\mathcal{W}}\lambda)(d)$ is bounded from below, then $\lambda_c \geq 0$. If, in addition, $\lambda_c = 0$, then λ is fixed by $\widehat{\mathcal{W}}$.*

Proof. If $(0, \alpha, n) \in \widehat{\Delta}$, then also $(0, \alpha, n + 2k) \in \widehat{\Delta}$ for every $k \in \mathbb{N}$, so that Corollary 3.10(iv) implies $\lambda_c \geq 0$.

If $\lambda_c = 0$, then $(0.\alpha, n) \in \widehat{\Delta}$ for some $n \neq 0$ implies the additional condition $\lambda(\check{\alpha}) = 0$. Hence $\lambda_c = 0$ leads to $\lambda(\underline{\check{\alpha}}) = 0$ for every $\underline{\alpha} \in \widehat{\Delta}$, so that λ is fixed by $\widehat{\mathcal{W}}$. $\square$

Proposition 3.14. *Suppose that $\widehat{\Delta}$ is one of the 7 irreducible locally affine root systems. For $\lambda \in V^*$ with $\lambda_c > 0$, the following are equivalent:*

(i) λ *is d-minimal.*

(ii) $(0,\alpha,n)\in\widehat{\Delta}\Rightarrow|\lambda(\check{\alpha})|\frac{(\alpha,\alpha)}{2n}\le\lambda_c$ *holds for all* $\alpha\in\Delta$, $n=1,2$.

Proof. That (ii) follows from (i) is a consequence of Corollary 3.10(iv) and the observation that $(0,\alpha,n)\in\widehat{\Delta}$ implies $(0,-\alpha,n)\in\widehat{\Delta}$.

If, conversely, (ii) holds, then the 2-periodic structure of the root system implies the condition in Corollary 3.10(iv). □

Theorem 3.15. *For the seven irreducible locally affine root systems* $X_J^{(r)}=\widehat{\Delta}$ *of infinite rank, a linear functional* $\lambda=(\lambda_c,\overline{\lambda},\lambda_d)\in\widehat{V}^*$ *with* $\lambda_c>0$ *is d-minimal if and only if the following conditions are satisfied by the corresponding function* $\overline{\lambda}:J\to\mathbb{R}, j\mapsto\lambda_j$*:*

$(A_J^{(1)})$ $\max\overline{\lambda}-\min\overline{\lambda}\le\lambda_c$.

$(B_J^{(1)})$ $|\lambda_j|+|\lambda_k|\le\lambda_c$ *for* $j\neq k$.

$(C_J^{(1)})$ $|\lambda_j|\le\lambda_c/2$ *for* $j\in J$.

$(D_J^{(1)})$ $|\lambda_j|+|\lambda_k|\le\lambda_c$ *for* $j\neq k$.

$(B_J^{(2)})$ $|\lambda_j|\le\lambda_c$ *for* $j\in J$.

$(C_J^{(2)})$ $|\lambda_j|+|\lambda_k|\le\lambda_c$ *for* $j\neq k$.

$(BC_J^{(2)})$ $|\lambda_j|\le\lambda_c/2$ *for* $j\in J$.

Proof. In the untwisted case $r=1$ we have $\widehat{\Delta}=\{0\}\times\Delta\times\mathbb{Z}=X_J^{(1)}$, so that Proposition 3.14 asserts that λ is d-minimal if and only if $|\lambda(\check{\alpha})|\frac{(\alpha,\alpha)}{2}\le\lambda_c$ for $\alpha\in\Delta$.

$A_J^{(1)}$: For the root system $\Delta=A_J$, all roots α satisfy $(\alpha,\alpha)=2$, so that the d-minimality condition on λ is

$$\lambda_j-\lambda_k\le\lambda_c \text{ for } j\neq k\in J.$$

This can also be written as $\max\overline{\lambda}-\min\overline{\lambda}\le\lambda_c$.

$B_J^{(1)}$: For $\Delta=B_J$, the roots ε_j satisfy $(\varepsilon_j,\varepsilon_j)=1$ and $\check{\varepsilon}_j=2e_j$. This leads to the d-minimality conditions

$$|\lambda_j|\le\lambda_c \text{ and } |\lambda_j\pm\lambda_k|\le\lambda_c$$

which is equivalent to $|\lambda_j|+|\lambda_k|\le\lambda_c$ for $j\neq k$.

$C_J^{(1)}$: For the root system C_J, the roots $2\varepsilon_j$ satisfy $(2\varepsilon_j)\check{}=e_j$ and $(2\varepsilon_j,2\varepsilon_j)=4$. The d-minimality thus implies $|\lambda_j|\le\lambda_c/2$. This also implies that $|\lambda_j\pm\lambda_k|\le\lambda_c$ for $j\neq k\in J$, so that it characterizes the d-minimal weights.

$D_J^{(1)}$: For the root system D_J, we find the conditions $|\lambda_j\pm\lambda_k|\le\lambda_c$ which are equivalent to $|\lambda_j|+|\lambda_k|\le\lambda_c$ for $j\neq k\in J$.

$B_J^{(2)}$: In this case $\Delta_0=B_J$ and $\Delta_1=\{\pm\varepsilon_j\mid j\in J\}$ with $\|\varepsilon_j\|=1$. In view of $\check{\varepsilon}_j=2e_j$, we obtain from the roots in Δ_1 the condition $|\lambda_j|=\frac{1}{2}|2\lambda_j|\le\lambda_c$. For the roots $\varepsilon_j\pm\varepsilon_k\in\Delta_0$ we obtain the additional condition $|\lambda_j\pm\lambda_k|\le 2\lambda_c$ which is redundant.

$C_J^{(2)}$: In this case $\Delta_1 = D_J$ and $\Delta_0 = C_J$ with $\|2\varepsilon_j\|^2 = 4$ lead to the conditions
$$|\lambda_j \pm \lambda_k| \leq \lambda_c \text{ and } |\lambda_j| \leq \lambda_c,$$
which is equivalent to $|\lambda_j| + |\lambda_k| \leq \lambda_c$ for $j \neq k \in J$.

$BC_J^{(2)}$: Here $\Delta_1 = BC_J$ and $\Delta_0 = B_J$ with $\|2\varepsilon_j\|^2 = 4$ lead to the conditions $|\lambda_j| \leq \lambda_c/2$ for the roots $\alpha = \pm 2\varepsilon_j$, and the roots $\alpha = \pm\varepsilon_j$ provide no additional restriction. For the roots $\alpha = \varepsilon_j \pm \varepsilon_k$ we obtain $|\lambda_j \pm \lambda_j| \leq \lambda_c$, which also is redundant. □

Remark 3.16. (a) The preceding theorem implies that d-minimal weights $\lambda \in \widehat{V}^*$ define bounded functions $\overline{\lambda} : J \to \mathbb{R}$ and, moreover, that the boundedness of $\overline{\lambda}$ is equivalent to the existence of a $\lambda_c > 0$ such that $\lambda = (\lambda_c, \overline{\lambda}, \lambda_d) \in \widehat{V}^*$ is d-minimal.

(b) If $\lambda \in V^*$ satisfies $\lambda(\check{\alpha}) \in \mathbb{Z}$ for each $\alpha \in \widehat{\Delta}$, then the subset $\lambda + \widehat{\mathcal{Q}} \subseteq V^*$, where $\widehat{\mathcal{Q}} = \langle \widehat{\Delta} \rangle_{\text{grp}}$ is the *root group*, is invariant under the Weyl group $\widehat{\mathcal{W}}$. Therefore $(\widehat{\mathcal{W}}\lambda)(d) \subseteq \lambda(d) + \mathbb{Z}$. If $(\widehat{\mathcal{W}}\lambda)(d)$ is bounded from below, we thus obtain the existence of a d-minimal element in $\widehat{\mathcal{W}}\lambda$.

For general functionals which are not integral weights, the situation is more complicated, as Example 3.19 below shows.

3.17. **The affine Weyl group.** Recall the inclusion
$$\iota_{\mathcal{W}} : \mathcal{W} \cong \langle r_{(0,\alpha,0)} : \alpha \in \Delta_0 \rangle_{\text{grp}} \hookrightarrow \widehat{\mathcal{W}}$$
of the locally finite Weyl group $\mathcal{W}$ into $\widehat{\mathcal{W}}$ and note that it provides a section of the quotient homomorphism $q : \widehat{\mathcal{W}} \to \mathcal{W}$ corresponding to the passage from $\widehat{V}$ to V. For $n \in \mathbb{Z}$ and $(0, \alpha, n) \in \widehat{\Delta}$, the elements $r_{(0,\alpha,0)} r_{(0,\alpha,n)}$ generate the normal subgroup $\mathcal{N} := \ker q$.

To make the structure of $\mathcal{N}$ more explicit, we consider for $x \in V$ the endomorphism $\tau_x = \tau(x)$ of $\widehat{V}$, defined by
$$\tau_x(z, y, t) := \Big(z + \langle y, x \rangle + \frac{t\|x\|^2}{2}, y + tx, t\Big).$$
The maps τ_x are isometries with respect to the Lorentzian form and an easy calculation shows that
$$\tau_{x_1}\tau_{x_2} = \tau_{x_1+x_2} \text{ for } x_1, x_2 \in V$$
and that $r_{(0,\alpha,0)} r_{(0,\alpha,n)} = \tau_{n\check{\alpha}}$ for $(0, \alpha, n) \in \widehat{\Delta}$. This leads to
$$\mathcal{N} = \tau(\mathcal{T}) \text{ for } \mathcal{T} := \langle n\check{\alpha} : \alpha \in \Delta_n^\times, n \in \mathbb{N} \rangle_{\text{grp}}.$$

Proposition 3.18. *For the untwisted root systems of type $X_J^{(1)}$, the group $\mathcal{T}$ coincides with the group $\check{\mathcal{R}} := \langle \check{\Delta} \rangle_{grp}$ of coroots. For the three twisted cases, it is given in V in terms of the canonical basis elements $(e_j)_{j \in J}$ by:*

(i) $\mathcal{T} = 2\mathbb{Z}^{(J)}$ *for* $B_J^{(2)}$.

(ii) $\mathcal{T} = \big\{ \sum_{j \in J} n_j e_j \mid \sum_j n_j \in 2\mathbb{Z} \big\}$ *for* $C_J^{(2)}$.

(iii) $\mathcal{T} = \sum_J \mathbb{Z}e_j \cong \mathbb{Z}^{(J)}$ *for* $BC_J^{(2)}$.

Proof. (i) For $B_J^{(2)}$ we derive from $\Delta_1 = \{\pm\varepsilon_j \mid j \in J\}$, $\Delta_0 = B_J$ and $\check{\varepsilon}_j = 2e_j$, $(\varepsilon_j \pm \varepsilon_k)\check{} = e_j \pm e_k$ that $\mathcal{T}$ is the subgroup of V generated by the elements

$$2(e_j \pm e_k),\ j \neq k \text{ and } 2e_j,\ j \in J.$$

(ii) For $C_J^{(2)}$ we have $\Delta_1 = D_J$ and $\Delta_0 = C_J$, which leads to the generators

$$\pm e_j \pm e_k,\ j \neq k \text{ and } 2e_j,\ j \in J.$$

(iii) For $BC_J^{(2)}$ we obtain from $\Delta_1 = BC_J$ and $\Delta_0 = B_J$ the generators $\pm e_j, \pm e_j \pm e_k$ for $j \neq k$. □

Note that

$$\widehat{\mathcal{W}}d = \mathcal{N}d = \Big\{\Big(\frac{\|x\|^2}{2}, x, 1\Big) \mid x \in \mathcal{T}\Big\}$$

leads to

$$(\widehat{\mathcal{W}}\lambda)(d) = \Big\{\lambda_c\frac{\|x\|^2}{2} + \overline{\lambda}(x) + \lambda_d \mid x \in \mathcal{T}\Big\}. \tag{7}$$

This formula shows immediately that if $\lambda_c > 0$ and

$$\|\overline{\lambda}\|_2^2 := \sum_{j\in J} |\lambda_j|^2 < \infty,$$

which is in particular the case if $\operatorname{supp}(\overline{\lambda})$ is finite, then $(\widehat{\mathcal{W}}\lambda)(d)$ is bounded from below. Since $\mathcal{T}$ is not a vector space, we cannot expect that the condition that $(\widehat{\mathcal{W}}\lambda)(d)$ is bounded from below implies that $\|\overline{\lambda}\|_2 < \infty$, and Theorem 3.15 does indeed show that this is not the case. It only implies that $\overline{\lambda}$ is bounded. The following example illustrates the situation further.

Example 3.19. We provide an example of an element $\lambda \in \widehat{V}^*$ for which $(\widehat{\mathcal{W}}\lambda)(d)$ is bounded from below, but contains no minimum.

We consider the root system $A_{\mathbb{N}}$ ($J = \mathbb{N}$) and $\lambda = (1, \overline{\lambda}, 0) \in \widehat{V}^*$ defined by

$$\overline{\lambda} : \mathbb{N} \to \mathbb{R},\ \lambda_{2k} = 0 \text{ and } \lambda_{2k-1} = 1 + \frac{1}{k^2} \text{ for } k \in \mathbb{N}.$$

On $\mathcal{T} = \{x \in \mathbb{Z}^{(J)} \mid \sum_n x_n = 0\}$ we then consider the function $f : \mathcal{T} \to \mathbb{R}$, given by

$$f(x) := \tfrac{1}{2}\|x\|^2\lambda_c + \overline{\lambda}(x) = \tfrac{1}{2}\|x\|^2 + \sum_{n=1}^{\infty} x_n\lambda_n := \tfrac{1}{2}\|x\|^2 + \sum_{k=1}^{\infty} x_{2k-1}\Big(1 + \frac{1}{k^2}\Big).$$

We claim that f is bounded from below but that it does not have a minimal value.

If $x_{2k-1} \leq -3$ for some k, then we consider the element $\widetilde{x} := x + e_{2k-1} - e_{2\widetilde{k}}$, where $\widetilde{k}$ is such that $x_{2\widetilde{k}} = 0$. Then

$$f(x) - f(\widetilde{x}) = \tfrac{1}{2}(x_{2k-1}^2 - (x_{2k-1}+1)^2 - 1) - \Big(1 + \frac{1}{k^2}\Big) = -x_{2k-1} - 2 - \frac{1}{k^2} \geq 0.$$

To show that $f(\mathcal{T})$ is bounded from below, it therefore suffices to consider $f(x)$ for elements $x \in \mathcal{T}$ satisfying $x_{2k-1} \geq -2$ for every $k \in \mathbb{N}$. This leads to

$$f(x) = \tfrac{1}{2}\|x\|^2 + \sum_{k=1}^{\infty} x_{2k-1}\Big(1 + \frac{1}{k^2}\Big) \geq \tfrac{1}{2}\|x\|^2 + \sum_{k=1}^{\infty} x_{2k-1} - 2\sum_{k=1}^{\infty} \frac{1}{k^2},$$

and since $\frac{1}{2}\|x\|^2 + \sum_{k=1}^{\infty} x_{2k-1} \geq 0$ for every $x \in \mathcal{T}$ by Theorem 3.15($A_J^{(1)}$), we see that f is bounded from below.

If $f(x_0) = \min f(\mathcal{T})$, then

$$f(x_0 + x) = \tfrac{1}{2}\|x + x_0\|^2 + \langle \overline{\lambda}, x + x_0\rangle = \tfrac{1}{2}\|x\|^2 + \langle \overline{\lambda} + x_0, x\rangle + f(x_0)$$

implies that the function $\overline{\mu} := \overline{\lambda} + x_0$ defines a d-minimal functional $\mu = (1, \overline{\mu}, 0)$, so that

$$\sup \overline{\mu} - \inf \overline{\mu} \leq \mu_c = 1.$$

Since x_0 has finite support, $\sup \overline{\mu} > 1$, so that $\inf \underline{\mu} \leq 0$ leads to a contradiction. Therefore $\widehat{\mathcal{W}}\lambda$ contains no d-minimal element.

References

[1] G. Hofmann, Invariante konvexe Mengen für lineare Coxetergruppen, Diplomarbeit, TU Darmstadt, 1999.

[2] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, 29, Cambridge Univ. Press, Cambridge, 1990. MR1066460 (92h:20002)

[3] V. G. Kac, *Infinite-dimensional Lie algebras*, third edition, Cambridge University Press, Cambridge, 1990. MR1104219 (92k:17038)

[4] O. Loos and E. Neher, Locally finite root systems, Mem. Amer. Math. Soc. **171** (2004), no. 811, x+214 pp. MR2073220 (2005c:17016)

[5] R. V. Moody and A. Pianzola, *Lie algebras with triangular decompositions*, Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley, New York, 1995. MR1323858 (96d:17025)

[6] K.-H. Neeb, Holomorphic highest weight representations of infinite-dimensional complex classical groups, J. Reine Angew. Math. **497** (1998), 171–222. MR1617431 (99h:22027)

[7] K.-H. Neeb, Unitary highest weight modules of locally affine Lie algebras, in *Quantum affine algebras, extended affine Lie algebras, and their applications*, 227–262, Contemp. Math., 506, Amer. Math. Soc., Providence, RI, 2010. MR2642569 (2011e:17042)

[8] K. H. Neeb, Semibounded unitary representations of double extensions of Hilbert-loop groups, Ann. Inst. Fourier (Grenoble) **64** (2014), no. 5, 1823–1892. MR3330925

[9] K.-H. Neeb and N. Stumme, The classification of locally finite split simple Lie algebras, J. Reine Angew. Math. **533** (2001), 25–53. MR1823863 (2002b:17006)

[10] E. B. Vinberg, Discrete linear groups that are generated by reflections. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 1072–1112. MR0302779 (46 #1922)
English translation: E. B. Vinberg, Discrete linear groups generated by reflections, Math. USSR Izv. **5** (1971), 1083–1119

[11] Y. Yoshii, Locally extended affine root systems, in *Quantum affine algebras, extended affine Lie algebras, and their applications*, 285–302, Contemp. Math., 506, Amer. Math. Soc., Providence, RI, 2010. MR2642571 (2011f:17016)

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