On homogeneous composed Clifford foliations

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Abstract. We complete the classification, initiated by the second named author, of homogeneous singular Riemannian foliations of spheres that are lifts of foliations produced from Clifford systems.

A singular Riemannian foliation of a Riemannian manifold M is, roughly speaking, a partition $\mathcal{F}$ of M into connected complete submanifolds, not necessarily of the same dimension, that locally stay at a constant distance one from another. Singular Riemannian foliations of round spheres $(\mathbb{S}^n, \mathcal{F})$ are of special importance since, other than producing submanifolds with interesting geometrical properties, they provide local models around a point of general singular Riemannian foliations.

The special case of singular Riemannian foliations in spheres whose leaves of maximal dimension have codimension 1 is better known as the case of *isoparametric foliations*, and its study dates back to É. Cartan, who showed the existence of a number of nontrivial examples. However, his examples were all *homogeneous*, i.e., given as orbits of isometric group actions on $\mathbb{S}^n$. The first inhomogeneous examples were found much later by Ozeki and Takeuchi [16]. A while later, Ferus, Karcher and Münzner [6] developed an algebraic framework based on Clifford algebras (or, equivalently, Clifford systems, see Section 1.3) to construct a large family of examples of isoparametric foliations (so-called of *FKM type*), including many inhomogeneous examples, and completely classified the homogeneous ones.

Whereas the theory and classification of isoparametric foliations of spheres are by now rather well understood, the situation of singular Riemannian foliations in higher codimensions is still largely *terra incognita*. In [18], inspired by the ideas in [6], two new classes of foliations were introduced. Namely, the class

of *Clifford foliations*, and the class of *composed foliations* which properly contains the first one. A Clifford foliation $(\mathbb{S}^n, \mathcal{F}_C)$ is constructed from a Clifford system C, and a composed foliation $(\mathbb{S}^n, \mathcal{F}_0 \circ \mathcal{F}_C)$ is constructed from C and a singular Riemannian foliation $(\mathbb{S}^m, \mathcal{F}_0)$ of a lower-dimensional sphere. The natural question of determining which ones are homogeneous was also solved in [18], with the exception of composed foliations based on Clifford systems of type $C_{8,1}$ and $C_{9,1}$ (see Section 1.3). The goal of the present work is to deal with these two remaining, more involved cases.

Theorem 0.1. *Let $(\mathbb{S}^n, \mathcal{F}_0 \circ \mathcal{F}_C)$ be a homogeneous composed foliation, with either $C = C_{8,1}$ or $C = C_{9,1}$.*

- *If $C = C_{8,1}$, then $n = 15$, and there are exactly six examples of homogeneous foliations $(\mathbb{S}^{15}, \mathcal{F}_0 \circ \mathcal{F}_{C_{8,1}})$, listed in Tables 1 and 2.*
- *If $C = C_{9,1}$, then $n = 31$, and the only homogeneous foliation $(\mathbb{S}^{31}, \mathcal{F}_0 \circ \mathcal{F}_{C_{9,1}})$ is the isoparametric one induced by the action of $\mathrm{Spin}(10)$ on $\mathbb{S}^{31}$ via the spin representation. In this case, $m = 9$ and the corresponding foliation $(\mathbb{S}^9, \mathcal{F}_0)$ consists of one leaf.*

There is a general idea that it should be possible to recover many geometric properties of the singular Riemannian foliations from the geometry of the underlying leaf (or quotient) space, compare, e.g., [13, 14, 21, 8, 7, 9, 1]. In this regard, it was shown in [18] that Clifford foliations are characterized as those singular Riemannian foliations of spheres whose leaf space is isometric to either a sphere or a hemi-sphere of constant curvature 4. More generally, it was believed that any foliation whose leaf space has constant curvature 4 should be a composed foliation. Our result shows that this belief is now dismissed. Comparing our Main Theorem with [20, Tab. II] and [8, Tab. 1], we observe that there are exactly two homogeneous foliations on $\mathbb{S}^{31}$ whose quotient space has constant sectional curvature 4 and which are not composed, namely, those given by the orbits of $\mathrm{Spin}(9)$ and $\mathrm{Spin}(9) \cdot \mathrm{SO}(2)$ actions on $\mathbb{S}^{31}$ with quotient a quarter of a round sphere $\frac{1}{2}\mathbb{S}^3_{++}$ and an eighth of a round sphere $\frac{1}{2}\mathbb{S}^3_{+++}$, respectively. Together with results in [18, 8] this implies:

Corollary 0.2. *The foliations given by the $\mathrm{Spin}(9)$ and $\mathrm{Spin}(9) \cdot \mathrm{SO}(2)$ actions on $\mathbb{S}^{31}$ are the only homogeneous foliations of spheres whose leaf space has constant curvature 4 and which are not composed.*

The case of composed foliations $(\mathbb{S}^{15}, \mathcal{F}_0 \circ \mathcal{F}_{C_{8,1}})$ is also very interesting, as they coincide with those foliations that contain the fibers of the octonionic Hopf fibration $\mathbb{S}^{15} \to \mathbb{S}^8$. Based on the fact that the Cayley projective plane $\mathbb{O}P^2$ is the mapping cone of $\mathbb{S}^{15} \to \mathbb{S}^8$, it was shown in [18] that there corresponds to any singular Riemannian foliation $(\mathbb{S}^8, \mathcal{F}_0)$ a singular Riemannian foliation $(\mathbb{O}P^2, \tilde{\mathcal{F}}_0)$ which is homogeneous if and only if $\mathcal{F}_0 \circ \mathcal{F}_{C_{8,1}}$ is homogeneous. It thus follows from our Main Theorem that there is a large amount of inhomogeneous foliations of $\mathbb{O}P^2$:

Corollary 0.3. *The foliation $(\mathbb{O}P^2, \tilde{\mathcal{F}}_0)$ is inhomogeneous for any foliation $(\mathbb{S}^8, \mathcal{F}_0)$ except for those six (homogeneous) examples listed in Tables 1 and 2.*

About this paper: After a section on preliminaries, we first consider the case of foliations with closed leaves and treat the cases $C_{9,1}$ and $C_{8,1}$ in separate sections, as they have very different features. The short, last section is devoted to foliations with nonclosed leaves.

1. Preliminaries

In this section, we quickly review some definitions and results from [18].

1.1. Singular Riemannian foliations.

Definition 1.2. Let M be a Riemannian manifold, and $\mathcal{F}$ a partition of M into complete, connected, injectively immersed submanifolds, called *leaves*. The pair $(M, \mathcal{F})$ is called:

- a *singular foliation* if there is a family of smooth vector fields $\{X_i\}$ that spans the tangent space of the leaves at each point.
- a *transnormal system* if any geodesic starting perpendicular to a leaf stays perpendicular to all the leaves it meets. Such geodesics are called *horizontal geodesics*.
- a *singular Riemannian foliation* if it is both a singular foliation and a transnormal system.

Given a singular foliation $(M, \mathcal{F})$, the *space of leaves*, denoted by $M/\mathcal{F}$, is the set of leaves of $\mathcal{F}$ endowed with the topology induced by the *canonical projection* $\pi : M \to M/\mathcal{F}$ that sends a point $p \in M$ to the leaf $L_p \in \mathcal{F}$ containing it. If in addition the leaves of $\mathcal{F}$ are closed, then $M/\mathcal{F}$ inherits the structure of a Hausdorff metric space, by declaring the distance $d(\pi(p), \pi(q))$ to be equal to the distance $d(L_p, L_q)$ in M between the corresponding leaves. Moreover, $M/\mathcal{F}$ is stratified by smooth Riemannian manifolds, and the projection π is global submetry, and a Riemannian submersion along each stratum.

1.3. Clifford systems and Clifford foliations.

A Clifford system, denoted by C, is an $(m + 1)$-tuple $C = (P_0, \ldots, P_m)$ of symmetric transformations of a Euclidean vector space V such that

$$P_i^2 = \mathrm{Id} \quad \text{for all } i$$

and

$$P_i P_j = -P_j P_i \quad \text{for all } i \neq j.$$

Two Clifford systems $(P_0, \ldots, P_m)$, $(Q_0, \ldots, Q_m)$ are called *geometrically equivalent* if there exists an element $A \in \mathrm{O}(V)$ such that $(AP_0A^{-1}, \ldots, AP_mA^{-1})$ and $(Q_0, \ldots, Q_m)$ span the same subspace in $\mathrm{Sym}^2(V)$. Geometric equivalence classes of Clifford systems are completely classified, and the following statements hold.

- A Clifford system $\{P_0, \ldots, P_m\}$ on V exists if and only if $\dim V = 2k\delta(m)$, where k is a positive integer and $\delta(m)$ is given by:

m	1	2	3	4	5	6	7	8	$8+n$
$\delta(m)$	1	2	4	4	8	8	8	8	$16\delta(n)$

Given integers m, k, we denote by $C_{m,k}$ any Clifford system consisting of $m+1$ symmetric matrices on a vector space of dimension $2k\delta(m)$.

- If $m \not\equiv 0 \bmod 4$, there exists a unique geometric equivalence class of Clifford system of type $C_{m,k}$, for any fixed k.
- If $m \equiv 0 \bmod 4$, there exist exactly $\lfloor \frac{k}{2} \rfloor + 1$ equivalence classes of Clifford systems of type $C_{m,k}$. They are distinguished by the invariant $|\mathrm{tr}(P_0 P_1 \cdots P_m)|$.

Given a Clifford system $C = (P_0, \ldots, P_m)$ on $\mathbb{R}^{2l}$, $l = k\delta(m)$, we can define a map

$$\pi_C : \mathbb{S}^{2l-1} \subset \mathbb{R}^{2l} \to \mathbb{R}^{m+1},$$

$$x \mapsto (\langle P_0 x, x \rangle, \ldots, \langle P_m x, x \rangle).$$

The *Clifford foliation* $(\mathbb{S}^{2l-1}, \mathcal{F}_C)$ associated to C is given by the preimages of the map π_C. This foliation is a singular Riemannian foliation, it only depends on the geometric equivalence class of C, and its quotient is isometric to either a round sphere $\frac{1}{2}\mathbb{S}^m$ if $l = m$, or a round hemisphere $\frac{1}{2}\mathbb{S}^{m+1}_+$ if $l \geq m+1$.

1.4. Composed foliations.

Fix a Clifford system $C = C_{m,k} = (P_0, \ldots, P_m)$ with associated Clifford foliation $(\mathbb{S}^n, \mathcal{F}_C)$, and fix a singular Riemannian foliation $(\mathbb{S}^m, \mathcal{F}_0)$. Alternatively, we can view $\mathcal{F}_0$ as:

- a foliation of the boundary of the leaf space of $\mathcal{F}_C$, namely $\partial(\mathbb{S}^n/\mathcal{F}_C) = \partial(\frac{1}{2}\mathbb{S}^{m+1}_+)$, in case $l \geq m+1$,
- a foliation of $\frac{1}{2}\mathbb{S}^m$ in case $l = m$.

Such a foliation can be extended by homotheties to a foliation $(\frac{1}{2}\mathbb{S}^{m+1}_+, \mathcal{F}_0^h)$. The *composed foliation* $(\mathbb{S}^n, \mathcal{F}_0 \circ \mathcal{F}_C)$ is then defined by taking the π_C-preimages of the leaves of $\mathcal{F}_0^h$.

Given any Clifford system $C = C_{m,k}$ and any singular Riemannian foliation $(\mathbb{S}^m, \mathcal{F}_0)$, the composed foliation $(\mathbb{S}^n, \mathcal{F}_0 \circ \mathcal{F}_C)$ is a singular Riemannian foliation.

1.5. Homogeneous composed foliations.

Recall that a singular Riemannian foliation $(M, \mathcal{F})$ is called *homogeneous* if its leaves are orbits of an isometric Lie group action $G \to \mathrm{Isom}(M)$. In [18] appears a complete classification of homogeneous Clifford foliation and a partial classification of composed foliations:

Theorem 1.6 ([18]). *Let $C = C_{m,k} = (P_0, \ldots, P_m)$ be a Clifford system on $\mathbb{R}^{2l}$ and let $(\mathbb{S}^m, \mathcal{F}_0)$ be a singular Riemannian foliation. Then:*

(1) *The Clifford foliation $(\mathbb{S}^{2l-1}, \mathcal{F}_C)$ is homogeneous if and only if $m = 1, 2$ or $m = 4$ and $P_0 P_1 \cdots P_4 = \pm \mathrm{Id}$, in which cases it is respectively spanned by the orbits of the diagonal action of $\mathrm{SO}(k)$ on $\mathbb{R}^k \times \mathbb{R}^k$ ($m = 1$), $\mathrm{SU}(k)$ on $\mathbb{C}^k \times \mathbb{C}^k$ ($m = 2$) or $\mathrm{Sp}(k)$ on $\mathbb{H}^k \times \mathbb{H}^k$ ($m = 4$).*

(2) *If $C \neq C_{8,1}, C_{9,1}$, then $(\mathbb{S}^{2l-1}, \mathcal{F}_0 \circ \mathcal{F}_C)$ is homogeneous if and only if both $\mathcal{F}_0$ and $\mathcal{F}_C$ are homogeneous. If $C = C_{9,1}$ and $(\mathbb{S}^{2l-1}, \mathcal{F}_0 \circ \mathcal{F}_C)$ is homogeneous, then $\mathcal{F}_0$ is homogeneous.*

By the classification of Clifford systems, both $C_{8,1}$ and $C_{9,1}$ consist of a unique geometric equivalence class of Clifford systems. Moreover, for $C = C_{8,1}$ the corresponding Clifford foliation $(\mathbb{S}^{15}, \mathcal{F}_C)$ is given by the fibers of the octonionic Hopf fibration $\mathbb{S}^{15} \to \frac{1}{2}\mathbb{S}^8$, while for $C = C_{9,1}$ the Clifford foliation $(\mathbb{S}^{31}, \mathcal{F}_C)$ is given by the fibers of $\pi_C : \mathbb{S}^{31} \to \frac{1}{2}\mathbb{S}^{10}_+$.

2. The case $C = C_{9,1}$

In this section we will show that there are no new examples of homogeneous composed foliations originating from the Clifford system $C = C_{9,1}$. More precisely, we will see that a composed foliation $(\mathbb{S}^{31}, \mathcal{F}_0 \circ \mathcal{F}_C)$ is homogeneous if and only if $\mathcal{F}_0^h$ is the codimension-one foliation of $\mathbb{S}^{10}_+$ consisting of concentric 9-spheres; recall that in that case, the composed foliation is the isoparametric foliation $\tilde{\mathcal{F}}_C$ of FKM type given by the orbits of the spin representation $\mathrm{Spin}(10) \to \mathrm{SO}(32)$ (see [6]). Recall also that the maximal connected Lie subgroup of $\mathrm{SO}(32)$ whose orbits coincide with the leaves of $\tilde{\mathcal{F}}_C$ is $\mathrm{Spin}(10) \cdot \mathrm{U}(1) = \mathrm{Spin}(10) \times_{\mathbb{Z}_4} \mathrm{U}(1)$ (see [3, 5]).

In this section we will only consider closed Lie subgroups of $\mathrm{SO}(32)$, which correspond to *proper* isometric actions on $\mathbb{S}^{31}$, and postpone the case of non-closed Lie subgroups to Section 4. So suppose the leaves of $\mathcal{F}_0 \circ \mathcal{F}_C$ are orbits of a closed connected Lie subgroup G of $\mathrm{SO}(32)$. Since $\mathcal{F}_0 \circ \mathcal{F}_C$ is contained in $\tilde{\mathcal{F}}_C$, i.e., the leaves of $\mathcal{F}_0 \circ \mathcal{F}_C$ are contained in those of $\tilde{\mathcal{F}}_C$, G preserves each leaf of $\tilde{\mathcal{F}}_C$. By the above maximality property, $G \subset \mathrm{Spin}(10) \cdot \mathrm{U}(1)$.

Lemma 2.1. *The foliation $(\mathbb{S}^{31}, \mathcal{F})$ induced by a given $G \subset \mathrm{Spin}(10) \cdot \mathrm{U}(1)$ is of the form $\mathcal{F}_0 \circ \mathcal{F}_C$ if and only if $\mathcal{F}_C$ is contained in $\mathcal{F}$.*

Proof. The only if part is clear. Suppose now that the orbits of G contain the leaves of $\mathcal{F}_C$. Any element in $\mathrm{Spin}(10) \cdot \mathrm{U}(1)$ preserves the submanifold $M_+ \subset \mathbb{S}^{31}$ defined as the preimage of the north pole of $\mathbb{S}^{31}/\mathcal{F}_C = \frac{1}{2}\mathbb{S}^{10}_+$, and therefore so does G. Since G acts by isometries, the projection of any G-orbit to the quotient $\frac{1}{2}\mathbb{S}^{10}_+$ is either entirely contained in the interior of $\frac{1}{2}\mathbb{S}^{10}_+$ or entirely contained in the boundary. It follows that for every leaf L of $\mathcal{F}$, the restriction $(L, \mathcal{F}_C|_L)$ is a regular foliation, and its quotient $L/\mathcal{F}_C \subset \frac{1}{2}\mathbb{S}^{10}_+$ is a submanifold. The partition $\{L/\mathcal{F}_C\}_{L \in \mathcal{F}}$ is easily seen to form a singular Riemannian foliation $\mathcal{F}_0^h$ on $\frac{1}{2}\mathbb{S}^{10}_+$ with the north pole as a 0-dimensional leaf and, by the Homothetic Transformation Lemma (see, e.g., [17, Lem. 1.1]), this foliation is determined by its restriction $\mathcal{F}_0$ on the boundary $\frac{1}{2}\mathbb{S}^9$. By the definition of composed foliation, $\mathcal{F}$ is of the form $\mathcal{F}_0 \circ \mathcal{F}_C$. $\square$

It follows from Lemma 2.1 that we need only consider *maximal* connected closed subgroups of $\mathrm{Spin}(10) \cdot \mathrm{U}(1)$.

The orbital geometry of the spin representation $\mathrm{Spin}(10) \to \mathrm{SO}(32)$ (or its extension to $\mathrm{Spin}(10) \cdot \mathrm{U}(1)$) is well understood. The orbit space $\mathbb{S}^{31}/\mathrm{Spin}(10)$ is isometric to an interval of length $\pi/4$, where the endpoints parametrize singular orbits M_+, M_- of dimensions 21 and 24 (cf. [11, p. 436]; see also [2, pp. 8–9] for a more elementary discussion). The orbit M_+ is particularly interesting, as

it is also a leaf of $\mathcal{F}_0 \circ \mathcal{F}_C$ for any homogeneous foliation $\mathcal{F}_0$ of $\frac{1}{2}\mathbb{S}_+^{10}$, namely, the π_C-fiber over the origin of $\frac{1}{2}\mathbb{S}_+^{10}$. As a homogeneous space,

$$M_+ \cong \mathrm{Spin}(10)/\mathrm{SU}(5) \cong \mathrm{Spin}(10) \cdot \mathrm{U}(1)/\mathrm{U}(5)$$

(this also follows from the fact that M_+ is the orbit of a highest weight vector of the spin representation). As G is transitive on M_+, we must have $\dim G \geq 21$.

The maximal connected closed subgroups of $\mathrm{Spin}(10) \cdot \mathrm{U}(1)$ are, up to conjugacy,

$$\mathrm{Spin}(10), \ \mathrm{U}(5) \cdot \mathrm{U}(1), \ \mathrm{Spin}(10-k) \cdot \mathrm{Spin}(k) \cdot \mathrm{U}(1)$$

for $k = 1, \ldots, 5$, and

$$\rho(H) \cdot \mathrm{U}(1),$$

where H is simple and ρ is irreducible of real type and degree 10 (cf. [4]; see also [12, Prop. 8]). We have already remarked that $\mathrm{Spin}(10)$ is an orbit equivalent subgroup of $\mathrm{Spin}(10) \cdot \mathrm{U}(1)$; we shall not need to discuss its subgroups, because they are subgroups of the other maximal subgroups of $\mathrm{Spin}(10) \cdot \mathrm{U}(1)$. In the sequel, we will first analyze which of the other maximal subgroups of $\mathrm{Spin}(10) \cdot \mathrm{U}(1)$ can act transitively on M_+.

The group $\mathrm{U}(5) \cdot \mathrm{U}(1)$ cannot act transitively on M_+ since its semisimple part $\mathrm{SU}(5)$ is coincides with an isotropy subgroup of $\mathrm{Spin}(10)$ on M_+.

The simply-connected compact connected simple Lie groups H of rank at most 5 and dimension between 20 and 44 are $\mathrm{Spin}(7)$, $\mathrm{Spin}(8)$, $\mathrm{Spin}(9)$, $\mathrm{Sp}(3)$, $\mathrm{Sp}(4)$, $\mathrm{SU}(5)$ and $\mathrm{SU}(6)$; none admits irreducible representations of real type and degree 10.

In order to determine if the groups $\mathrm{Spin}(10-k) \cdot \mathrm{Spin}(k) \cdot \mathrm{U}(1)$ can act transitively on M_+, one can compute the intersection of the Lie algebra $\mathfrak{so}(10-k) \oplus \mathfrak{so}(k)$ with the $\mathfrak{so}(10)$-isotropy subalgebra $\mathfrak{su}(5)$. It does not matter that the subalgebras are defined only up to conjugacy (corresponding to the fact that one can choose a different basepoint in M_+). We view $\mathfrak{su}(5)$ inside $\mathfrak{so}(10)$ as consisting of matrices of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

where A and B are real 5×5 matrices, A is skew-symmetric, B is symmetric of trace zero. A standard choice of embedding of $\mathfrak{so}(10-k) \oplus \mathfrak{so}(k)$ into $\mathfrak{so}(10)$ is given by matrices of the form

$$\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix},$$

where C and D are skew-symmetric $(10-k) \times (10-k)$, resp. $k \times k$, matrix blocks. Then their intersection is isomorphic to $\mathfrak{su}(5-k) \oplus \mathfrak{so}(k)$. Therefore the dimension of the $\mathrm{Spin}(10-k) \cdot \mathrm{Spin}(k)$-orbit through the basepoint is $21 - \frac{k(k-1)}{2}$ for $k \leq 4$, and 10 for $k = 5$. We deduce that $\mathrm{Spin}(9)$ and $\mathrm{Spin}(9) \cdot \mathrm{U}(1)$ act transitively on M_+; besides those, only $\mathrm{Spin}(8) \cdot \mathrm{SO}(2) \cdot \mathrm{U}(1)$ has a chance of acting transitively on that manifold. In order to discard the latter group, we choose a different embedding of $\mathfrak{so}(8) \oplus \mathfrak{so}(2)$ into $\mathfrak{so}(10)$, namely, that in

which the (i,j)-entry is zero if $i \in \{1,2,3,4,6,7,8,9\}$ and $j \in \{5,10\}$ or $j \in \{1,2,3,4,6,7,8,9\}$ and $i \in \{5,10\}$. Now $(\mathfrak{so}(8) \oplus \mathfrak{so}(2)) \cap \mathfrak{su}(5) \cong \mathfrak{s}(\mathfrak{u}(4) \oplus \mathfrak{u}(1))$ and the corresponding $\mathrm{Spin}(8) \cdot \mathrm{SO}(2)$-orbit has dimension $29 - 16 = 13$, showing that $\mathrm{Spin}(8) \cdot \mathrm{SO}(2) \cdot \mathrm{U}(1)$ is not transitive on M_+.

Finally, we need to show that the $\mathrm{Spin}(9) \cdot \mathrm{U}(1)$-orbits cannot coincide with the leaves of $\mathcal{F}_0 \circ \mathcal{F}_C$ for any $\mathcal{F}_0$ consisting of more than one leaf. Suppose the contrary. Since $\pi_C : \mathbb{S}^{31} \to \frac{1}{2}\mathbb{S}^{10}_+$ is equivariant with respect to the double covering $\mathrm{Spin}(10) \to \mathrm{SO}(10)$, we see that $\mathrm{SO}(9)$ preserves the leaves of $\mathcal{F}_0$. We already know that $\mathcal{F}_0$ is homogeneous (Theorem 1.6(2)), and $\mathrm{SO}(9)$ is a maximal connected subgroup of $\mathrm{SO}(10)$. Therefore $\mathcal{F}_0$ must be given by the orbits of $\mathrm{SO}(9)$. It follows that the leaf space of $\mathcal{F}_0 \circ \mathcal{F}_C$ is $\frac{1}{2}\mathbb{S}^{10}_+/\mathrm{SO}(9)$, which is isometric to $\frac{1}{2}\mathbb{S}^2_{++}$. On the other hand, the quotient space $\mathbb{S}^{31}/\mathrm{Spin}(9) \cdot \mathrm{U}(1)$ is one-eighth of a round sphere $\frac{1}{2}\mathbb{S}^2_{+++}$ (see [20, Tab. II, Type III$_4$]). We reach a contradiction and deduce that $(\mathbb{S}^{31}, \mathcal{F}_0 \circ \mathcal{F}_C)$ cannot be homogeneous under $\mathrm{Spin}(9) \cdot \mathrm{U}(1)$.

Remark 2.2. Let $(\mathbb{S}^9, \mathcal{F}_0)$ denote the homogeneous foliation given by the orbits of $\mathrm{SO}(9)$, and let $(\mathbb{S}^{31}, \mathcal{F}_0 \circ \mathcal{F}_C)$ be the corresponding composed foliation. By the result above, $\mathcal{F}_0 \circ \mathcal{F}_C$ is not homogeneous and, in particular, it is different from the homogeneous foliation induced by the orbits of $\mathrm{Spin}(9) \cdot \mathrm{U}(1)$. Nevertheless, both foliations have cohomogeneity 2, and both have quotients of constant curvature 4. Moreover, they both contain the homogeneous foliation induced by the orbits of $\mathrm{Spin}(9)$. Since $\mathbb{S}^{31}/\mathrm{Spin}(9) = \frac{1}{2}\mathbb{S}^3_{++}$, it follows that the orbits of $\mathrm{Spin}(9)$ have codimension 1 in the leaves of $\mathcal{F}_0 \circ \mathcal{F}_C$, which makes $\mathcal{F}_0 \circ \mathcal{F}_C$ very close to a homogeneous foliation.

3. THE CASE $C = C_{8,1}$

In this section, we determine the list of homogeneous composed foliations originating from the Clifford system $C = C_{8,1}$. Namely, we determine the orbit equivalence classes of the isometric group actions that yield such foliations. In this section we only consider closed subgroups of $\mathrm{SO}(16)$ and defer the analysis of nonclosed Lie subgroups to Section 4. The foliation $\mathcal{F}_C$ is given by the fibers of the inhomogeneous octonionic Hopf fibration $\mathbb{S}^{15} \to \frac{1}{2}\mathbb{S}^8$. Fix a singular Riemannian foliation $(\mathbb{S}^8, \mathcal{F}_0)$, and suppose that $\mathcal{F}_0 \circ \mathcal{F}_C$ is homogeneous, given by the orbits of a closed connected subgroup G of $\mathrm{SO}(16)$. Recall that if X denotes the leaf space

$$X = \mathbb{S}^8/\mathcal{F}_0,$$

then the orbit space $\mathbb{S}^{15}/G$ is isometric to $\frac{1}{2}X$. In particular, the sectional curvature of (the regular part of) $\mathbb{S}^{15}/G$ is everywhere ≥ 4 and hence G cannot act polarly, unless it acts with cohomogeneity 1.

3.1. **Criteria to recognize composed foliations.** Before we start the classification in detail, we want to present some results that will be helpful to identify foliations that can be written as $\mathcal{F}_0 \circ \mathcal{F}_C$, where $C = C_{8,1}$. We start with the straight-forward remark that a foliation $\mathcal{F}$ can be written in the form $\mathcal{F}_0 \circ \mathcal{F}_C$

if and only if every fiber of the Hopf fibration $\mathbb{S}^{15} \to \mathbb{S}^8$ is contained in a leaf of $\mathcal{F}$ (compare Lemma 2.1). In particular, if $\mathcal{F}$ is a homogeneous composed foliation induced by the action of a group $G \subset \mathrm{SO}(16)$, then any other group $\overline{G}$ with $G \subset \overline{G} \subset \mathrm{SO}(16)$ will also generate a homogeneous composed foliation.

As a special case of the above situation, which will be useful later on, suppose that $(\mathbb{S}^{15}, \mathcal{F}_0 \circ \mathcal{F}_C)$ is homogeneous given by the orbits of $G \subset \mathrm{SO}(16)$, and suppose that $(\mathbb{S}^8, \mathcal{F}_0)$ is homogeneous given by the orbits of $H \subset \mathrm{SO}(9)$. Then for any group $\overline{H} \subset \mathrm{SO}(9)$ containing H, there is a canonical enlargement $\overline{G} \subset \mathrm{SO}(16)$ of G whose orbits yield a composed foliation, as follows. Since the Hopf fibration $\mathbb{S}^{15} \to \mathbb{S}^8$ is equivariant with respect to the covering map $\mathrm{Spin}(9) \to \mathrm{SO}(9)$, we can lift $\overline{H}$ to a group $\tilde{H} \subset \mathrm{Spin}(9) \subset \mathrm{SO}(16)$. Now $\overline{G}$ is defined as the closure of the subgroup in $\mathrm{SO}(16)$ generated by G and $\tilde{H}$. By the discussion above, the orbits of $\overline{G}$ define a homogeneous composed foliation on $\mathbb{S}^{15}$.

Next we prove a criterion to distinguish some foliations that cannot be written in the form $\mathcal{F}_0 \circ \mathcal{F}_C$.

Proposition 3.2. *Let $(\mathbb{S}^{15}, \overline{\mathcal{F}})$ denote the homogeneous, codimension-one foliation given by the orbits of $\mathrm{Sp}(2) \cdot \mathrm{Sp}(2)$, under the representation $\nu_2 \hat{\otimes}_{\mathbb{H}} \nu_2^*$. Then any foliation $(\mathbb{S}^{15}, \mathcal{F})$ which is contained in $\overline{\mathcal{F}}$ (i.e., every leaf of $\mathcal{F}$ is contained in a leaf of $\overline{\mathcal{F}}$) cannot be written in the form $\mathcal{F}_0 \circ \mathcal{F}_{C_{8,1}}$.*

Proof. If $\mathcal{F}$ could be written as $\mathcal{F}_0 \circ \mathcal{F}_{C_{8,1}}$, by the remarks above, so could $\overline{\mathcal{F}}$. Therefore it is enough to prove the proposition for $\overline{\mathcal{F}}$ and, to do so, it is enough to provide a leaf of $\overline{\mathcal{F}}$ that cannot be foliated by totally geodesic 7-spheres. We thus consider the singular orbit M_+ containing the point $\mathrm{Id} \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{H}^2, \mathbb{H}^2) \cong \mathbb{H}^2 \otimes_{\mathbb{H}} \mathbb{H}^{2*}$, which is diffeomorphic to $\mathrm{Sp}(2)$.

Suppose now that $M_+ = \mathrm{Sp}(2)$ is foliated by totally geodesic $\mathbb{S}^7$. Then the leaves are all simply connected, which implies that there is no leaf holonomy, and thus the quotient $M_+/\mathcal{F}$ is a manifold B and $M_+ \to B$ is a Riemannian submersion with totally geodesic fibers. Then it is also a fibration, and from the long exact sequence in homotopy, B is simply connected (and 3-dimensional). Therefore it must be $B = \mathbb{S}^3$, and we have a fibration

$$\mathbb{S}^7 \to \mathrm{Sp}(2) \to \mathbb{S}^3.$$

Again from the long exact sequence in homotopy, we have

$$\pi_6(\mathrm{Sp}(2)) \to \pi_6(\mathbb{S}^3) \to \pi_5(\mathbb{S}^7) = 0.$$

However, on the one hand $\pi_6(\mathrm{Sp}(2)) = 0$ (for example, compare [15]), and on the other $\pi_6(\mathbb{S}^3) \neq 0$, which gives a contradiction. □

As an application of Proposition 3.2, consider the Clifford foliation $\mathcal{F}_{\overline{C}}$ generated by $\overline{C} = (P_0, \dots, P_4)$ with $P_0 P_1 P_2 P_3 P_4 = \pm \mathrm{Id}$. This foliation is homogeneous and given by the orbits of the diagonal action of $\mathrm{Sp}(2)$ on $\mathbb{H}^2 \oplus \mathbb{H}^2$ (Theorem 1.6) and thus, by Proposition 3.2, it cannot be written as $\mathcal{F}_0 \circ \mathcal{F}_{C_{8,1}}$. In fact, this is the only Clifford foliation of $\mathbb{S}^{15}$ with this property.

Proposition 3.3. *For any Clifford system C' on $\mathbb{R}^{16}$ with $C' \neq \bar{C}$, the foliation $\mathcal{F}_{C'}$ can be written in the form $\mathcal{F}_{C'} = \mathcal{F}_0 \circ \mathcal{F}_{C_{8,1}}$, for some foliation $(\mathbb{S}^8, \mathcal{F}_0)$.*

Proof. Let $C_{8,1} = (P_0, \dots, P_8)$ and, for every $i = 1, \dots, 7$, let C_i denote the sub-Clifford system $(P_0, \dots, P_i)$. Since $\mathcal{F}_{C_i}$ is given by the preimages of the map

$$\pi_{C_i} : \mathbb{S}^{15} \to \mathbb{R}^{i+1}, \quad \pi_{C_i}(x) = (\langle P_0 x, x \rangle, \dots, \langle P_i x, x \rangle),$$

it is clear that π_{C_i} factors as $\pi_i \circ \pi_{C_{8,1}}$, where $\pi_{C_{8,1}} : \mathbb{S}^{15} \to \mathbb{S}^8$ is the Hopf fibration, and $\pi_i : \mathbb{S}^8 \subset \mathbb{R}^9 \to \mathbb{D}^{i+1} \subset \mathbb{R}^{i+1}$ is the projection onto the first $i+1$ components. In particular, $\mathcal{F}_{C_i}$ can be written as $\mathcal{F}_0 \circ \mathcal{F}_{C_{8,1}}$, where $(\mathbb{S}^8, \mathcal{F}_0)$ is given by the fibers of π_i. Notice that $\mathcal{F}_0$ in this case is homogeneous and given by the orbits of $\mathrm{SO}(8-i)$, embedded in $\mathrm{SO}(9)$ as the lower diagonal block.

Moreover, any Clifford system $C' = C_{m,k}$ on $\mathbb{R}^{16}$ must satisfy the equation $k\delta(m) = 8$, and the only possibilities are

$$(m, k) = (8, 1), (7, 1), (6, 1), (5, 1), (4, 2), (3, 2), (2, 4), (1, 8).$$

For any $m \not\equiv 0 \mod 4$ there is only one geometric equivalence class of Clifford systems, and therefore $C_{m,k}$ can be identified with the sub-Clifford system $C_m \subset C_{8,1}$. For $m \equiv 0 \mod 4$ there are exactly $\lfloor \frac{k}{2} \rfloor + 1$ geometrically distinct Clifford systems of type $C_{m,k}$. Therefore, there is a unique $C_{8,1}$, and two distinct classes of type $C_{4,2}$. One of them is $C_4 \subset C_{8,1}$, which is composed by the discussion above, and the other is $\bar{C}$. Since this exhausts all possible Clifford systems on $\mathbb{R}^{16}$, it follows that all of them are composed, with the exception of $\bar{C}$. $\square$

Gathering all the information together, we obtain the following corollary.

Corollary 3.4. *A composed foliation $(\mathbb{S}^{15}, \mathcal{F}_0 \circ \mathcal{F}_{C_{m,k}})$ can also be written as $\mathcal{F}'_0 \circ \mathcal{F}_{C_{8,1}}$ for some $(\mathbb{S}^8, \mathcal{F}'_0)$, if and only if $C_{m,k} \neq \bar{C}$.*

Proof. If $C_{m,k} \neq \bar{C}$, then, by Proposition 3.3, $\mathcal{F}_{C_{m,k}}$ can be written as $\mathcal{F}'_0 \circ \mathcal{F}_{C_{8,1}}$ and, by the initial remark, the same holds for $\mathcal{F}_0 \circ \mathcal{F}_{C_{m,k}}$ since it contains $\mathcal{F}_{C_{m,k}}$. On the other hand, any composed foliation $\mathcal{F}_0 \circ \mathcal{F}_{\bar{C}}$ is contained in the foliation $\mathcal{F}_1 \circ \mathcal{F}_{\bar{C}}$, where $(\mathbb{S}^8, \mathcal{F}_1)$ is the trivial foliation with one leaf. Since $\mathcal{F}_1 \circ \mathcal{F}_{\bar{C}}$ coincides with the foliation $\overline{\mathcal{F}}$ of Proposition 3.2, it follows that $\mathcal{F}_0 \circ \mathcal{F}_C$ cannot be written as $\mathcal{F}'_0 \circ \mathcal{F}_{C_{8,1}}$ for any $(\mathbb{S}^8, \mathcal{F}'_0)$. $\square$

We can now proceed with the classification of composed foliations of $\mathbb{S}^{15}$ homogeneous under a closed Lie group G. The diameter of $X = \mathbb{S}^8 / \mathcal{F}_0$ is either equal to π, or it is at most $\pi/2$. We will consider these two cases separately.

3.5. Case I: diam $X = \pi$. Suppose first that the diameter of X is π. Then there is a copy of $\mathbb{S}^0 \subset \mathbb{S}^8$ consisting of 0-dimensional leaves, and $(\mathbb{S}^8, \mathcal{F}_0)$ decomposes as a spherical join

$$(\mathbb{S}^8, \mathcal{F}_0) = \mathbb{S}^0 \star (\mathbb{S}^7, \mathcal{F}_1),$$

for some foliation $\mathcal{F}_1$. In particular, X is isometric to a spherical join $\mathbb{S}^0 \star Y$, where $Y = \mathbb{S}^7/\mathcal{F}_1$. In this case, $\frac{1}{2}X$ has diameter $\pi/2$ (thus G acts reducibly) and it contains two points x_+, x_- at distance $\pi/2$. Moreover, any unit speed geodesic in $\frac{1}{2}X$ starting from x_- meets x_+ at the same time $t = \pi/2$. Therefore the preimages $S_\pm$ of $x_\pm$ are orthogonal round spheres of curvature 1, i.e., they are the unit spheres of subspaces $V_\pm$ of $\mathbb{R}^{16}$ such that $\mathbb{R}^{16} = V_+ \oplus V_-$. Since we are assuming that the G-orbits contain the fibers of the Hopf fibration, it must be $\dim S_\pm \geq 7$. Therefore equality must hold, and $\dim V_+ = \dim V_- = 8$. Moreover, G acts transitively on $S_\pm$. Given $p \in S_+$, the isotropy G_p acts on the unit sphere in the normal space $\nu_p S_+$, which is isometric to S_- via the map $v \mapsto \exp_p \frac{\pi}{2}v$. Moreover, the foliation (S_-, G_p) coincides with the infinitesimal foliation of $\mathcal{F}_0$ at $\pi_C(p) \in \mathbb{S}^8$, which in turn coincides with $(\mathbb{S}^7, \mathcal{F}_1)$. In particular, $\mathcal{F}_0$ is homogeneous and given by the action of G_p on $\mathbb{R}^9 = \mathbb{R} \oplus V_-$ given by $\epsilon \oplus \lambda|_{G_p}$, where $\epsilon : G \to \mathbb{R}$ is the trivial representation, and $\lambda : G \to \mathrm{SO}(8)$ denotes the representation of G on V_- (or V_+).

Remark 3.6. Since the infinitesimal foliation of $\mathcal{F}_0 \circ \mathcal{F}_C$ at any point of S_- coincides with the infinitesimal foliation at a point in S_+ (because they both coincide with $(\mathbb{S}^7, \mathcal{F}_1)$), the slice representations at S_+ and S_- must be orbit equivalent.

If the G action on $S_\pm$ is not effective, then the kernels $K_\pm$ of $G \to \mathrm{SO}(V_\pm)$ are normal subgroups of G with $K_+ \cap K_- = \{e\}$. Since G is compact, it admits a normal subgroup L such that $G = K_+ \cdot L \cdot K_-$, where $K_+ \cdot L$ acts effectively on S_- and $L \cdot K_-$ acts effectively on S_+. Let $\mathfrak{k}_+, \mathfrak{k}_-, \mathfrak{l}$ denote the Lie algebras of K_+, K_-, L respectively. From the list of all groups acting transitively on the 7-sphere, we get the following possibilities:
Case 1: $\mathfrak{k}_+ = \mathfrak{k}_- = 0$. Then $G = L$ up to a finite cover, and the possible such representations are:

Type	G	$G \to \mathrm{SO}(16)$
I.1	SO(8)	$\rho_8 \oplus \rho_8$
I.2	SU(4)	$\mu_4 \oplus \mu_4$
I.3	U(4)	$\mu_4 \oplus \mu_4$
II.1	Spin(8)	$\Delta_8^+ \oplus \Delta_8^-$
II.2	Spin(7)	$\Delta_7 \oplus \Delta_7$
II.3	SU(4) $\cdot$ U(1)	$\mu_4 \,\hat{\otimes}\, (\mu_1^r \oplus \mu_1^s)$ $(r \neq s)$
III.1	Sp(2)	$\nu_2 \oplus \nu_2$
III.2	Sp(2) $\cdot$ Sp(1)	$(\nu_2 \hat{\otimes} \nu_1)^{\oplus 2}$
III.3	Sp(2) $\cdot$ U(1)	$\nu_2 \hat{\otimes} (\mu_1^r \oplus \mu_1^s)$

The actions of type I induce the Clifford foliations $\mathcal{F}_{C_{1,8}}$ and $\mathcal{F}_{C_{2,4}}$ respectively (actions I.2 and I.3 are orbit equivalent) and, by Proposition 3.3, they indeed can be written as $\mathcal{F}_0 \circ \mathcal{F}_{C_{8,1}}$. Therefore, the same is true for the foliations coming from actions of type II, since each of them contains a foliation

G for $\mathcal{F}_0 \circ \mathcal{F}_C$	$G \to \mathrm{SO}(16)$	H for $\mathcal{F}_0$	$H \to \mathrm{SO}(9)$	X
Spin(8)	$\Delta_8^+ \oplus \Delta_8^-$	Spin(7)	$\epsilon \oplus \Delta_7$	$[0, \pi]$
SO(8)	$\rho_8 \oplus \rho_8$	SO(7)	$\epsilon^2 \oplus \rho_7$	$\mathbb{S}_+^2$
Spin(7)	$\Delta_7 \oplus \Delta_7$	G_2	$\epsilon^2 \oplus \phi_7$	
SU(4)	$\mu_4 \oplus \mu_4$	SU(3)	$\epsilon^3 \oplus \mu_3$	$\mathbb{S}_+^3$
U(4)	$\mu_4 \oplus \mu_4$	U(3)	$\epsilon^3 \oplus \mu_3$	
SU(4) · U(1)	$\mu_4 \hat{\otimes} (\mu_1^r \oplus \mu_1^s)$ $(r \neq s)$	U(3) · U(1)	$\epsilon \oplus \mu_1^{r-s} \oplus \mu_3 \otimes \mu_1^{-s}$	$\mathbb{S}_{++}^2$

TABLE 1. diam $X = \pi$, and $\mathfrak{k}_+ = \mathfrak{k}_- = 0$.

of type I. On the other hand, the foliations of type III are contained in the orbits of the representation of Sp(2) · Sp(2) given by $\nu_2 \hat{\otimes}_{\mathbb{H}} \nu_2^*$, and therefore are not of the form $\mathcal{F}_0 \circ \mathcal{F}_{C_{8,1}}$ by Proposition 3.2. Therefore, the homogeneous composed foliations in this case are in number of four and given by the orbits of the groups listed in Table 1, where we have grouped together orbit equivalent actions since these give rise to the same foliation. As we have seen, the foliation $(\mathbb{S}^8, \mathcal{F}_0)$ is also homogeneous, given by the orbits of the isotropy group H of G at a certain point.

Remark 3.7. Any pair of equivalent or inequivalent 8-dimensional irreducible representations of Spin(8) could occur in the table, but some are not listed since they differ from the two listed by an outer automorphism of Spin(8). In particular, those representations are not only orbit equivalent to a representation in the list, but their image in SO(16) is the same as the image of a representation in the list.

Case 2: $\mathfrak{l} = 0$. Then $G = K_+ \cdot K_-$, and each $K_\pm$ acts transitively on $\mathbb{S}^7$. All these cases are orbit equivalent among themselves, and also to the first entry in Table 1, so we get no new examples.

Case 3: $\mathfrak{l} \neq 0$ and $\mathfrak{k}_+ \neq 0$. Since $L \cdot K_+$ is a nontrivial product, and it acts effectively and transitively on S_-, it must be

$$L \cdot K_+ \in \{\mathrm{Sp}(2) \cdot \mathrm{Sp}(1), \mathrm{SU}(4) \cdot \mathrm{U}(1), \mathrm{Sp}(2) \cdot \mathrm{U}(1)\}.$$

If $L = \mathrm{Sp}(2)$, then the foliation is contained in the foliation of Proposition 3.2, so it is not composed.

If $L = \mathrm{SU}(4)$, then we have that $K_+ = \mathrm{U}(1)$, and K_- can be either U(1) or trivial. Then G is given by U(1) · SU(4) · U(1), resp., U(1) · SU(4), and it acts via $(\mu_1 \hat{\otimes} \mu_4) \oplus (\mu_4 \hat{\otimes} \mu_1)$, resp., $(\mu_1 \hat{\otimes} \mu_4) \oplus \mu_4$. Those actions are orbit equivalent to the representation of SU(4) · U(1) in Table 1 given by $\mu_4 \hat{\otimes} (\mu_1^r \oplus \mu_1^s)$ with $r \neq s$ (including the case $(r, s) = (1, 0)$).

Finally, if $L = \mathrm{Sp}(1)$ or U(1), then we have $K_+, K_- \in \{\mathrm{Sp}(2), \mathrm{SU}(4)\}$ and the action has cohomogeneity 1, and they are all orbit equivalent to the first entry in Table 1.

Hence we get no new examples in this case.

3.8. **Case II: diam $X \leq \pi/2$.** In this case the diameter of $\mathbb{S}^{15}/G = \frac{1}{2}X$ is at most $\pi/4$ and thus G acts irreducibly. We distinguish between possible cases, according to the dimension of X.

Suppose first that $\dim X = 1$, i.e., $\mathcal{F}_0 \circ \mathcal{F}_C$ is an isoparametric family in $\mathbb{S}^{15}$. It follows from the classification of cohomogeneity-one actions in spheres that the only possible actions on $\mathbb{S}^{15}$ with quotient of diameter at most $\pi/4$ are given by $\nu_2 \hat{\otimes} \nu_2^*$ for $G_1 = \mathrm{Sp}(2) \cdot \mathrm{Sp}(2)$, $\mu_2 \hat{\otimes} \mu_4$ for $G_2 = \mathrm{S}(\mathrm{U}(2) \cdot \mathrm{U}(4))$, and $\rho_2 \hat{\otimes} \rho_8$ for $G_3 = \mathrm{SO}(2) \cdot \mathrm{SO}(8)$ (or $\mathrm{SU}(2) \cdot \mathrm{SU}(4)$ and $\mathrm{SO}(2) \cdot \mathrm{Spin}(7)$, which are orbit equivalent subgroups of G_2, G_3, resp.). By Proposition 3.2, the action of G_1 is ruled out, but the other two actions give rise to composed foliations; in fact, those actions yield foliations containing foliations given in Table 1. Since G_2 and G_3 are contained in $\mathrm{Spin}(9)$, in each case they project to a subgroup H of $\mathrm{SO}(9)$ which generates a codimension-one isoparametric foliation $\mathcal{F}_0$ in $\mathbb{S}^8$. We summarize the discussion above in Table 2.

G for $\mathcal{F}_0 \circ \mathcal{F}_C$	$G \to \mathrm{SO}(16)$	H for $\mathcal{F}_0$	$H \to \mathrm{SO}(9)$	X
$\mathrm{SU}(2) \cdot \mathrm{SU}(4)$	$\mu_2 \otimes_{\mathbb{C}} \mu_4$	$\mathrm{SO}(3) \cdot \mathrm{SO}(6)$	$\rho_3 \hat{\oplus} \rho_6$	$[0, \pi/2]$
$\mathrm{SO}(2) \cdot \mathrm{SO}(8)$	$\rho_2 \hat{\otimes}_{\mathbb{R}} \rho_8$	$\mathrm{SO}(2) \cdot \mathrm{SO}(7)$	$\rho_2 \hat{\oplus} \rho_7$	$[0, \pi/2]$

Table 2. $\mathrm{diam}\, X = \pi/2$.

If $2 \leq \dim X \leq 4$, then G acts irreducibly on $\mathbb{S}^{15}$ with cohomogeneity ≤ 4, and the action is not polar. From the classification of low cohomogeneity representations in [10, 20, 7], it follows that G must act on $\mathbb{S}^{15}$ with cohomogeneity 2, and there are exactly two possible actions, $\mu_2 \hat{\otimes}_{\mathbb{C}} \nu_2$ for $G_1 = \mathrm{U}(2) \cdot \mathrm{Sp}(2)$, and $S^3(\mu_1) \hat{\otimes}_{\mathbb{H}} \nu_2^*$ for $G_2 = \mathrm{SU}(2) \cdot \mathrm{Sp}(2)$ (see [20, Tab. II]). Again these actions are ruled out by Proposition 3.2. In fact, it is clear that G_2 is contained in $\mathrm{Sp}(2) \cdot \mathrm{Sp}(2)$. As for G_1 being contained in that group, note that the $\mathrm{Sp}(2)$-representation $\mathbb{C}^4$ restricts to $\mathbb{C}^2 \oplus \mathbb{C}^{2*}$ along the embedding $\mathrm{U}(2) \subset \mathrm{Sp}(2)$, so the result follows from the following representation theoretic lemma.

Lemma 3.9. *If V and W are representations of complex, resp., quaternionic type, then $(V \oplus V^*) \otimes_{\mathbb{H}} W^*$ is equivalent as a real representation to the realification of $V \otimes_{\mathbb{C}} W$.*

Proof. The representations have equivalent complexifications. Indeed the complexification of the first representation is $(V \oplus V^*) \otimes_{\mathbb{C}} W$ whereas that of the second is $(V \otimes_{\mathbb{C}} W) \oplus (V \otimes_{\mathbb{C}} W)^*$, where $W \cong W^*$ over $\mathbb{C}$. $\square$

If $\dim X \geq 5$, then the foliation $(\mathbb{S}^8, \mathcal{F}_0)$ has leaves of dimension ≤ 3 and, by [17], it is homogeneous and generated by a closed connected subgroup H of $\mathrm{SO}(9)$. We claim that there are no composed homogeneous foliations in this case.

First of all, H cannot be an abelian group, for otherwise it would be contained in a maximal torus of $\mathrm{SO}(9)$, but such tori act on $\mathbb{S}^8$ fixing at least two

antipodal points. In particular, the diameter of X would be π which contradicts our assumption. This already implies that the regular leaves of $\mathcal{F}_0$ cannot have dimension 1 or 2 (i.e., $\dim X \neq 6, 7$) (that H is abelian in case they have dimension 2 is shown in [17]).

We are thus left with the case in which $(\mathbb{S}^8, \mathcal{F}_0)$ is homogeneous under a closed connected subgroup H of $\mathrm{SO}(9)$ and $\dim X = 5$. The principal orbits are 3-dimensional submanifolds with effective, transitive actions of H. Therefore a principal isotropy group H_{princ} does not contain a normal subgroup of H, H_{princ} is a subgroup of $\mathrm{O}(3)$, $\dim H \leq 6$ and equality holds if and only if H is locally isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)$. We deduce that H is one of $\mathrm{SU}(2)$, $\mathrm{SU}(2) \times T^1$, $\mathrm{SU}(2) \times \mathrm{SU}(2)$, up to cover (the first condition on H_{princ} above precludes the case $H = \mathrm{SU}(2) \times T^2$).

The only almost faithful 9-dimensional representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ without fixed directions is $\rho_3 \hat{\otimes} \rho_3$, which has 6-dimensional principal orbits.

Assume $H = \mathrm{SU}(2) \times T^1$ and V is a 9-dimensional representation with cohomogeneity 6 and no fixed directions. The identity component of H_{princ} on V is a circle with nontrivial projection into $\mathrm{SU}(2)$. It follows that the only admissible irreducible components of V are $(\mathrm{SU}(2), \mathbb{R}^3)$, $(\mathrm{U}(2), \mathbb{C}^2)$, $(T^1, \mathbb{C})$. Since 9 is odd, the first representation must occur exactly once. We get two possibilities: $\mathbb{R}^3 \oplus \mathbb{C}^2 \oplus \mathbb{C}$ and $\mathbb{R}^3 \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. The first one has trivial principal isotropy groups, so it is excluded. The second one can be extended to an action of $\overline{H} = \mathrm{SU}(2) \times T^3$ acting on $\mathbb{S}^8$ with cohomogeneity 3. If $\mathcal{F}_0 \circ \mathcal{F}_C$ were homogeneous, induced by some group G, then the extension $\overline{H}$ of H would induce an extension $\overline{G}$ of G that would act on $\mathbb{S}^{15}$ with cohomogeneity 3. This action would be non-polar and irreducible, however there is no such group (see [7, Tab. 1]).

The only 9-dimensional representations of $H = \mathrm{SU}(2)$ without fixed directions are λ_9, $\mu_2 \oplus \lambda_5$ and $\rho_3 \oplus \rho_3 \oplus \rho_3$.

The representation $\rho_3 \oplus \rho_3 \oplus \rho_3$ can be extended to an action of $\overline{H} = \mathrm{SO}(3)^3$ via the outer sum $\rho_3 \hat{\oplus} \rho_3 \hat{\oplus} \rho_3$, acting on $\mathbb{S}^8$ with cohomogeneity 2. If $\mathcal{F}_0 \circ \mathcal{F}_C$ were homogeneous, induced by some group G, then the extension $\overline{H}$ of H would induce an extension $\overline{G}$ of G, that would act on $\mathbb{S}^{15}$, with quotient isometric to $\frac{1}{2}\mathbb{S}^2_{+++}$ and three most singular orbits of dimension 9 (they would be preimages of most singular $\overline{H}$-orbits, of dimension 2). However, from the classification of non-polar irreducible isometric actions of cohomogeneity 2 on $\mathbb{S}^{15}$ there is no such group [20], and therefore $\mathcal{F}_0 \circ \mathcal{F}_C$ cannot be homogeneous in this case.

The representation $\mu_2 \oplus \lambda_5$ can be extended to an action of the group $\overline{H} = \mathrm{SU}(2) \times \mathrm{SU}(2)$ via the representation $\mu_2 \hat{\oplus} \lambda_5$, again acting on $\mathbb{S}^8$ with cohomogeneity 2. If $\mathcal{F}_0 \circ \mathcal{F}_C$ were homogeneous, induced by some group G, then the extension $\overline{H}$ of H would induce an extension $\overline{G}$ of G, with quotient isometric to $\frac{1}{2}\mathbb{S}^8/\overline{H} = \frac{1}{2}(\mathbb{S}^2_+/D_3)$, where D_3 denotes a dihedral group. The group $\overline{G}$ must then be $\mathrm{Sp}(1) \cdot \mathrm{Sp}(2)$ (compare [20]), which is 13-dimensional and thus acts on $\mathbb{S}^{15}$ with finite principal isotropy. In particular, G must act with finite principal isotropy as well, and since the cohomogeneity of H on $\mathbb{S}^8$ is 5, we have $\dim G = 10$. However, a quick check shows that there are no 10-dimensional

groups of rank at most 3 acting irreducibly (and non-polarly) on $\mathbb{R}^{16}$. In particular, in this case $\mathcal{F}_0 \circ \mathcal{F}_C$ cannot be homogeneous.

The representation λ_9 has isolated singular orbits, and therefore the quotient X has no boundary (compare [7, Sec. 11.2]). Now suppose that the composed foliation $\mathcal{F}_0 \circ \mathcal{F}_C$ is homogeneous, given by the action of G on $\mathbb{S}^{15}$. Since the quotient $\frac{1}{2}X$ has no boundary, there are no *nontrivial reductions* of $(G, \mathbb{R}^{16})$, i.e., there are no other representations $(G', \mathbb{R}^n)$ with $\dim G' < \dim G$ such that $\mathbb{S}^{n-1}/G'$ is isometric to $\mathbb{S}^{15}/G = \frac{1}{2}X$, compare [7, Prop. 5.2]. In particular, G must act with trivial principal isotropy, since otherwise we could produce a nontrivial reduction [7, p. 2]. Since the principal isotropy is trivial and $\dim X = 5$, again it must be $\dim G = 10$. The only 10-dimensional group acting irreducibly (and non-polarly) on $\mathbb{R}^{16}$ is $G = \mathrm{SU}(2)^3 \times \mathrm{U}(1)$ acting by $\hat{\otimes}^3(\mu_2) \hat{\otimes} \mu_1$; however a pure tensor $v_1 \otimes v_2 \otimes v_3$ has isotropy subgroup T^3, so this action has as an orbit of dimension 7. Since λ_9 has no fixed points in $\mathbb{S}^8$, this shows that the G-orbits cannot yield a foliation of the form $\mathcal{F}_0 \circ \mathcal{F}_C$.

4. Non-proper actions

We treat the cases of $C = C_{9,1}$ and $C = C_{8,1}$ simultaneously. Suppose that $\mathcal{F}_0 \circ \mathcal{F}_C$ is a homogeneous composed foliation of $\mathbb{S}^{31}$, resp. $\mathbb{S}^{15}$ given by the orbits of a nonclosed connected Lie subgroup G of $\mathrm{SO}(32)$, resp. $\mathrm{SO}(16)$. Then the closure of G is a closed connected subgroup whose orbits also comprise a homogeneous composed foliation, so it is already described in Sections 2 or 3. However, most of the groups therein listed admit no dense nonclosed connected Lie subgroups in view of the following:

Lemma 4.1. *A compact connected Lie group U with at most a 1-dimensional center admits no dense nonclosed connected Lie subgroups.*

Proof. Suppose, to the contrary, that G is a dense connected proper Lie subgroup of U. If G is a normal subgroup of U, then either G is contained in the semisimple part of U or it contains the center of U. Owing to [19], normal subgroups of semisimple Lie groups are closed. It follows that G cannot be normal in U. Let N be the normalizer of G in U. This is a proper subgroup of U, thus cannot be closed by denseness of G. On the other hand, N must be closed in U because it coincides with the normalizer in U of the Lie algebra of G (here we use connectedness of G), a contradiction. $\square$

The closed groups U yielding homogeneous composed foliations described in Sections 2 or 3 which do not satisfy the assumptions of Lemma 4.1 occur in case $C = C_{8,1}$ only and have 2-dimensional tori as centers, and they are of the following two types.
Case 1: $U = K_+ \cdot K_-$, where $K_\pm \in \{\mathrm{SU}(4) \cdot \mathrm{U}(1), \mathrm{Sp}(2) \cdot \mathrm{U}(1)\}$. In both cases there are dense connected Lie subgroups G which however yield orbit equivalent subactions.

Case 2: $U = K_+ \cdot L \cdot K_-$, where $K_\pm = \mathrm{U}(1)$, $L = \mathrm{SU}(4)$, and $K_+ \cdot L$ acts effectively on $\mathbb{C}^4 \oplus 0$ and $L \cdot K_-$ acts effectively on $0 \oplus \mathbb{C}^4$. The nonclosed dense connected Lie subgroups of U are of the form $G = \mathbb{R} \times \mathrm{SU}(4)$, where $\mathbb{R}$ is an irrational line in the center T^2 of U. Note that G and U share a common singular orbit through $p \in \mathbb{C}^4 \oplus 0$. Moreover the isotropy groups at p act with the same orbits in $0 \oplus \mathbb{C}^4$. It follows that G and U are orbit equivalent on $\mathbb{C}^4 \oplus \mathbb{C}^4$.

Finally, we get no homogenous composed foliations with nonclosed leaves.

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