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# A spectral sequence for Iwasawa adjoints

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(Communicated by Christopher Deninger)

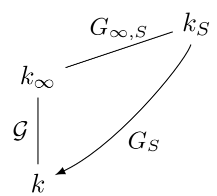
*To my friend Peter Schneider for his 60th birthday*

**Abstract.** We establish a purely algebraic tool for studying the Iwasawa adjoints of some natural Iwasawa modules for $p$-adic Lie group extensions of number fields.

## 1. Introduction

This paper is dedicated to Peter Schneider on the occasion of his sixtieth birthday. It does not correspond to my talk at the conference in his honor, but rather is a slightly edited and corrected version of a long time unpublished but several times quoted preprint from 1994. I feel that it fits better to Peter's domain of work, and that it may be time to make this paper better accessible. The aim of this paper was and is to give a purely algebraic tool for treating so-called (generalized) Iwasawa adjoints of some naturally occurring Iwasawa modules for $p$-adic Lie group extensions, by relating them to certain continuous Galois cohomology groups via a spectral sequence.

Let $k$ be a number field, fix a prime $p$, and let $k_\infty$ be some Galois extension of $k$ such that $\mathcal{G} = \mathrm{Gal}(k_\infty/k)$ is a $p$-adic Lie-group (e.g., $\mathcal{G} \cong \mathbb{Z}_p^r$ for some $r \geqslant 1$). Let $S$ be a finite set of primes containing all primes above $p$ and $\infty$, and all primes ramified in $k_\infty/k$, and let $k_S$ be the maximal $S$-ramified extension of $k$; by assumption, $k_\infty \subseteq k_S$. Let $G_S = \mathrm{Gal}(k_S/k)$ and $G_{\infty,S} = \mathrm{Gal}(k_S/k_\infty)$.




Let $A$ be a discrete $G_S$-module which is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^r$ for some $r \geqslant 1$ as an abelian group (e.g., $A = \mathbb{Q}_p/\mathbb{Z}_p$ with trivial action, or $A = E[p^\infty]$, the group of $p$-power torsion points of an elliptic curve $E/k$ with good reduction outside $S$). We are *not* assuming that $G_{\infty,S}$ acts trivially.

Let $\Lambda = \mathbb{Z}_p[[\mathcal{G}]]$ be the completed group ring. For a finitely generated $\Lambda$-module $M$ we put

$$E^i(M) = \mathrm{Ext}^i_\Lambda(M, \Lambda).$$

Hence $E^0(M) = \mathrm{Hom}_\Lambda(M, \Lambda) =: M^+$ is just the $\Lambda$-dual of $M$. This has a natural structure of a $\Lambda$-module, by letting $\sigma \in \mathcal{G}$ act via

$$(\sigma f)(m) = \sigma f(\sigma^{-1} m)$$

for $f \in M^+$, $m \in M$. It is known that $\Lambda$ is a noetherian ring (here we use that $\mathcal{G}$ is a $p$-adic Lie group), by results of Lazard [6]. Hence $M^+$ is a finitely generated $\Lambda$-module again (choose a projection $\Lambda^r \twoheadrightarrow M$; then we have an injection $M^+ \hookrightarrow (\Lambda^r)^+ = \Lambda^r$). By standard homological algebra, the $E^i(M)$ are finitely generated $\Lambda$-modules for all $i \geqslant 0$ which we call the (generalized) Iwasawa adjoints of $M$. They can also be seen as some kind of homotopy invariants of $M$, see [5], and also [7, V §4 and §5].

**Examples.** (a) If $\mathcal{G} = \mathbb{Z}_p$, then $\Lambda = \mathbb{Z}_p[[\mathcal{G}]] \cong \mathbb{Z}_p[[X]]$ is the classical Iwasawa algebra, and, for a $\Lambda$-torsion module $M$, it is known (see [9, I.2.2], or [1, 1.2 and rémarque] or [7, Prop. (5.5.6)]) that $E^1(M)$ is isomorphic to the classical Iwasawa adjoint, which was defined by Iwasawa ([2, 1.3]) as

$$\mathrm{ad}(M) = \varprojlim_n (M/\alpha_n M)^\vee$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is any sequence of elements in $\Lambda$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $(\alpha_n)$ is prime to the support of $M$ for every $n \geqslant 1$, and where

$$N^\vee = \mathrm{Hom}(N, \mathbb{Q}_p/\mathbb{Z}_p)$$

is the Pontrjagin dual of a discrete or compact $\mathbb{Z}_p$-module $N$. For any finitely generated $\Lambda$-module $M$, $E^1(M)$ is quasi-isomorphic to the $\Lambda$-module $\mathrm{Tor}_\Lambda(M)^\sim$, where $\mathrm{Tor}_\Lambda(M)$ is the $\Lambda$-torsion submodule of $M$, and $M^\sim$ is the "Iwasawa twist" of a $\Lambda$-module $M$: the action of $\gamma \in \mathcal{G}$ is changed to the action of $\gamma^{-1}$.

(b) If $\mathcal{G} = \mathbb{Z}_p^r$, $r \geqslant 1$, then the $E^i(M)$ are the standard groups considered in local duality. By duality for the ring $\mathbb{Z}_p[[\mathcal{G}]] = \mathbb{Z}_p[[x_1, \dots, x_r]]$, they can be computed in terms of local cohomology groups (with support) or by a suitable Koszul complex. More precisely, $E^i(M) \cong H^{r+1-i}_\mathfrak{m}(M)^\vee$, as recalled in [1, 1].

The main result of this note is the following observation.

**Theorem 1.** *There is a spectral sequence of finitely generated $\Lambda$-modules*

$$E_2^{p,q} = E^p(H^q(G_{\infty,S}, A)^\vee) \Rightarrow \varprojlim_{k',m} H^{p+q}(G_S(k'), A[p^m]) = \varprojlim_{k'} H^{p+q}(G_S(k'), T_p A).$$

*Here the inverse limits runs through the finite extensions $k'/k$ contained in $k_\infty$, and the natural numbers $m$, via the corectrictions and the natural maps*

$$H^n(G_S(k'), A[p^{m+1}]) \to H^n(G_S(k'), A[p^m]),$$

*respectively. The groups*

$$H^{p+q}(G_S(k'), T_pA) = \varprojlim_m H^{p+q}(G_S(k'), A[p^m])$$

*are the continuous cohomology groups of the Tate module* $T_pA = \varprojlim_m A[p^m]$.

## 2. Some consequences

Before we give the proof of a slightly more general result (cp. Theorem 11 below), we discuss what this spectral sequence gives in more down-to-earth terms. First of all, we always have the 5-low-terms exact sequence

$$0 \longrightarrow E^1(H^0(G_{\infty,S}, A)^\vee) \xrightarrow{\inf^1} \varprojlim_{k'} H^1(G_S(k'), T_pA) \longrightarrow (H^1(G_{\infty,S}, A)^\vee)^+$$
$$\longrightarrow E^2(H^0(G_{\infty,S}, A)^\vee) \xrightarrow{\inf^2} \varprojlim_{k'} H^2(G_S(k'), T_pA).$$

To say more, we make the following assumption.

**A.1** Assume that $p > 2$ or that $k_\infty$ is totally imaginary.

It is well-known that this implies

**A.2** $H^r(G_{\infty,S}, A) = 0 = \varprojlim_{k'} H^r(G_S(k'), T_pA)$ for all $r > 2$.

**Corollary 2.** *Assume in addition that $H^2(G_{\infty,S}, A) = 0$. (This is the so-called "weak Leopoldt conjecture" for $A$. It is stated classically for $A = \mathbb{Q}_p/\mathbb{Z}_p$ with trivial action, and there are precise conjectures when this is expected to hold for modules $A$ coming from algebraic geometry, cp.* [3].*) Then the cokernel of* $\inf^2$ *is*

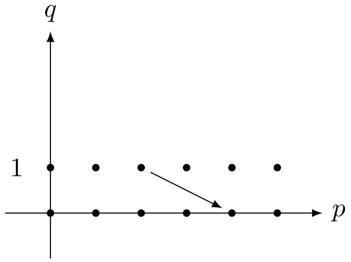
$$\ker(E^1(H^1(G_{\infty,S}, A)^\vee) \twoheadrightarrow E^3(H^0(G_{\infty,S}, A)^\vee)),$$

*and there are isomorphisms*

$$E^i(H^1(G_{\infty,S}, A)^\vee) \xrightarrow{\sim} E^{i+2}(H^0(G_{\infty,S}, A)^\vee)$$

*for $i \geqslant 2$.*

*Proof.* This comes from A.2 and the following picture of the spectral sequence




□

**Corollary 3.** *Assume that $H^0(G_{\infty,S}, A) = 0$. Then*

*(a)*

$$\varprojlim_{k'} H^1(G_S(k'), T_pA) \xrightarrow{\sim} (H^1(G_{\infty,S}, A)^\vee)^+.$$

*(b) There is an exact sequence*

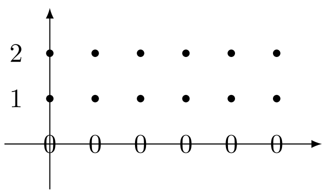
$$0 \longrightarrow E^1(H^1(G_{\infty,S}, A)^\vee) \longrightarrow \varprojlim_{k'} H^2(G_S(k'), T_pA) \longrightarrow (H^2(G_{\infty,S}, A)^\vee)^+ \longrightarrow E^2(H^1(G_{\infty,S}, A)^\vee) \longrightarrow 0.$$

*(c) There are isomorphisms*

$$E^i(H^2(G_{\infty,S}, A)^\vee) \xrightarrow{\sim} E^{i+2}(H^1(G_{\infty,S}, A)^\vee)$$

*for $i \geqslant 1$.*

*Proof.* In this case, the spectral sequence looks like

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**Corollary 4.** *Assume that $\mathcal{G}$ is a $p$-adic Lie group of dimension 1 (equivalently: an open subgroup is $\cong \mathbb{Z}_p$). Then $E^i(-) = 0$ for $i \geqslant 3$. Let*

$$B = \operatorname{im}(\inf{}^2 : E^2(H^0(G_{\infty,S}, A)^\vee) \to \varprojlim_{k'} H^2(G_S(k'), T_pA))$$

*Then $B$ is finite, and there is an exact sequence*

$$0 \to E^1(H^1(G_{\infty,S}, A)^\vee) \to \varprojlim_{k'} H^2(G_S(k'), T_pA)/B \to (H^2(G_{\infty,S}, A)^\vee)^+ \to E^2(H^1(G_{\infty,S}, A)^\vee) \to 0,$$

*and*

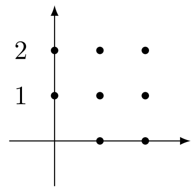
$$E^1(H^2(G_{\infty,S}, A)^\vee) = 0 = E^2(H^2(G_{\infty,S}, A)^\vee),$$

*i.e., $H^2(G_{\infty,S}, A)^\vee$ is a projective $\Lambda$-module.*

*Proof.* Quite generally, for a $p$-adic Lie group $\mathcal{G}$ of dimension $n$ one has $vcd_p(\mathcal{G}) = n$ for the virtual cohomological $p$-dimension of $\mathcal{G}$, and hence $E^i(-) = 0$ for $i > n+1$, cp. [5, Cor. 2.4]. The finiteness of $E^2(M)$, for a $\Lambda$-module $M$ which is finitely generated over $\mathbb{Z}_p$ (like our module $H^0(G_{\infty,S}, A)^\vee$) follows from Lemma 5 below. In fact, the exact sequence $0 \to M_{\mathrm{tor}} \to M \to \tilde{M} \to 0$, in which $M_{\mathrm{tor}}$ is the torsion submodule of $M$, induces a long exact sequence

$$\ldots \to E^i(\tilde{M}) \to E^i(M) \to E^i(M_{\mathrm{tor}}) \to \ldots$$

in which we have $E^i(M_{\mathrm{tor}}) = 0$ for $i \neq n+1$ and finiteness of $E^{n+1}(M_{\mathrm{tor}})$ by Lemma 5 (b), and $E^i(\tilde{M}) = 0$ for $i \neq n$ for the torsion-free module $\tilde{M}$ by Lemma 5 (a). In our case we have $n = 1$ and therefore the finiteness of $E^2(M)$. The remaining claims follow from the following shape of the spectral sequence:

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**Lemma 5.** *Assume that $\mathcal{G}$ is a $p$-adic Lie group of dimension $n$ (this holds, e.g., if $\mathcal{G}$ contains an open subgroup $\cong \mathbb{Z}_p^n$), and let $M$ be a $\Lambda$-module which is finitely generated as a $\mathbb{Z}_p$-module. Then the following holds.*

*(a)* $E^i(M) = 0$ *for* $i \neq n,\ n+1$.

*(b) If $M$ is torsion-free, then*

$$E^i(M) = \begin{cases} 0, & \text{for } i \neq n, \\ \mathrm{Hom}(D, M^\vee), & \text{for } i = n, \end{cases}$$

*where $D$ is the dualizing module for $\mathcal{G}$ (which is a divisible cofinitely generated $\mathbb{Z}_p$-module, e.g., $D = \mathbb{Q}_p/\mathbb{Z}_p$ if $\mathcal{G} = \mathbb{Z}_p^n$). In particular, $E^n(M)$ is a torsion-free finitely generated $\mathbb{Z}_p$-module.*

*(c) If $M$ is finite, then*

$$E^i(M) = \begin{cases} 0, & \text{for } i \neq n+1, \\ \mathrm{Hom}(M^\vee, D)^\vee, & \text{for } i = n+1. \end{cases}$$

*In particular, $E^{n+1}(M)$ is a finite $\mathbb{Z}_p$-module.*

*Proof.* See [5, Cor. 2.6]. For (b) note the isomorphism

$$\mathrm{Hom}(D \otimes_{\mathbb{Z}_p} M, \mathbb{Q}_p/\mathbb{Z}_p) \cong \mathrm{Hom}(D, \mathrm{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)) = \mathrm{Hom}(D, M^\vee). \quad \square$$

**Corollary 6.** *Let $\mathcal{G}$ be a $p$-adic Lie group of dimension 2 (e.g., $\mathcal{G}$ contains an open subgroup $\cong \mathbb{Z}_p^2$). If $G_{\infty,S}$ acts trivially on $A$, then there are exact* *sequences*

$$0 \to \varprojlim_{k'} H^1(G_S(k'), T_pA) \to (H^1(G_{\infty,S}, A)^\vee)^+ \\ \to T_pA \xrightarrow{\inf^2} \varprojlim_{k'} H^2(G_S(k'), T_pA)$$

*and*

$$0 \to E^1(H^1(G_{\infty,S}, A)^\vee) \to \varprojlim_{k'} H^2(G_S(k'), T_pA)/\operatorname{im}\inf^2 \\ \to (H^1(G_{\infty,S}, A)^\vee)^+ \to E^2(H^1(G_{\infty,S}, A)^\vee) \to 0,$$

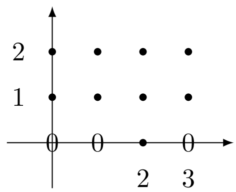
*an isomorphism*

$$E^1(H^2(G_{\infty,S}, A)^\vee) \xrightarrow{\sim} E^3(H^1(G_{\infty,S}, A)^\vee),$$

*and one has*

$$E^2(H^2(G_{\infty,S}, A)^\vee) \;=\; 0 \;=\; E^3(H^2(G_{\infty,S}, A)^\vee).$$

*Proof.* The spectral sequence looks like

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**Corollary 7.** *Let $\mathcal{G}$ be a $p$-adic Lie group of dimension 2 (so $E^i(-) = 0$ for $i \geqslant 4$). If $H^0(G_{\infty,S}, A)$ is finite, then*

$$\varprojlim_{k'} H^1(G_S(k'), T_pA) \cong (H^1(G_{\infty,S}, A)^\vee)^+.$$

*If*

$$d_2^{1,1} : E^1(H^1(G_{\infty,S}, A)^\vee) \;\longrightarrow\; E^3(H^0(G_{\infty,S}, A)^\vee)$$

*is the differential of the spectral sequence in the theorem, then one has an exact sequence*

$$0 \;\longrightarrow\; \ker d_2^{1,1} \longrightarrow \varprojlim_{k'} H^2(G_S(k'), T_pA) \\ \longrightarrow \ker(d_2^{0,2}\colon (H^2(G_{\infty,S}, A)^\vee)^+ \twoheadrightarrow E^2(H^1(G_{\infty,S}, A)^\vee)) \longrightarrow \operatorname{coker} d_2^{1,1} \longrightarrow 0,$$

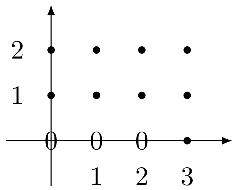
*an isomorphism*

$$E^1(H^2(G_{\infty,S}, A)^\vee) \xrightarrow{\sim} E^3(H^1(G_{\infty,S}, A)^\vee),$$

*and the vanishing*

$$E^2(H^2(G_{\infty,S}, A)^\vee) \;=\; 0 \;=\; E^3(H^2(G_{\infty,S}, A)^\vee).$$

*Proof.* The spectral sequence looks like



□

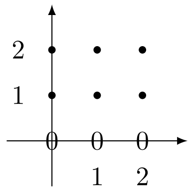
**Remark.** In the situation of Corollary 5, one has an exact sequence up to *finite modules*:

$$0 \to E^1(H^1(G_{\infty,S},A)^\vee) \to \varprojlim_{k'} H^2(G_S(k'),T_pA) \\ \to (H^2(G_{\infty,S},A)^\vee)^+ \to E^2(H^1(G_{\infty,S},A)^\vee) \to 0.$$

**Corollary 8.** *Let $\mathcal{G}$ be a $p$-adic Lie group of dimension $> 2$. Then*

$$(H^1(G_{\infty,S},A)^\vee)^+ \;\cong\; \varprojlim_{k'} H^1(G_S(k'),T_pA)$$

*Proof.* The first three columns of the spectral sequence look like



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## 3. Proof of the main theorem

We will now prove Theorem 1, by proving a somewhat more general result. For any profinite group $G$, let $\Lambda(G) = \mathbb{Z}_p[[G]]$ be the completed group ring over $\mathbb{Z}_p$, and let $M_G = M_{G,p}$ be the category of discrete (left) $\Lambda(G)$-modules. These are the discrete $G$-modules $A$ which are $p$-primary torsion abelian groups. For such a module $A$, its Pontrjagin dual $A^\vee = Hom(A,\mathbb{Q}_p/\mathbb{Z}_p)$ is a compact $\Lambda(G)$-module. In fact, Pontrjagin duality gives an anti-equivalence between $M_G$ and the category $C_G = C_{G,p}$ of compact (right) $\Lambda(G)$-modules.

Let $M_G^{\mathbb{N}}$ be the category of inverse systems

$$(A_n):\ \ldots \to A_3 \to A_2 \to A_1$$

in $M_G$ as in [4]. Denote by $H^i_{\mathrm{cont}}(G,(A_n))$ the continuous cohomology of such a system and recall that one has an exact sequence for each $i$

$$0 \;\to\; R^1\varprojlim_n H^{i-1}(G,A_n) \;\to\; H^i_{\mathrm{cont}}(G,(A_n)) \;\to \varprojlim_n H^i(G,A_n) \;\to\; 0,$$

in which the first derivative $R^1\varprojlim_n$ of the inverse limit, also noted as $\varprojlim_n^1$, vanishes if the groups $H^{i-1}(G,A_n)$ are finite for all $n$ (cp. *loc.cit.*).

**Definition 9.** For a closed subgroup $H \leq G$ and a discrete $G$-module $A$ in $M_G$ define the relative cohomology $H^m(G,H;A)$ as the value at $A$ of the $m$-th derived functor of the left exact functor (with $Ab$ being the category of abelian groups)

$$\begin{array}{rcl} H^0(G,H;-) : M_G & \to & Ab \\ A & \mapsto & \varprojlim_U H^0(U,A), \end{array}$$

where $U$ runs through all open subgroups $U \subset G$ containing $H$, and the transition maps are the corestriction maps. For an inverse system $(A_n)$ of modules in $M_G$ define the continuous relative cohomology $H^m_{\mathrm{cont}}(G,H;(A_n))$ as the value at $(A_n)$ of the $m$-th right derivative of the functor

$$\begin{array}{rcl} H^0_{\mathrm{cont}}(G,H;-) : M_G^{\mathbb{N}} & \to & Ab \\ (A_n) & \mapsto & \varprojlim_n \varprojlim_U H^0(U,A_n), \end{array}$$

where the limit over $U$ is as before, and the limit over $n$ is induced by the transition maps $A_{n+1} \to A_n$.

**Lemma 10.** *If $G/H$ has a countable basis of neighborhoods of identity, i.e., if there is a countable family $U_\nu$ of open subgroup, $H \leq U_\nu \leq G$, with $\bigcap_\nu U_\nu = H$, and if, in addition, $H^i(U,A_n)$ is finite for all these $U$ and all $n$, then*

$$H^i_{\mathrm{cont}}(G,H;(A_n)) = \varprojlim_n \varprojlim_U H^i(U,A_n).$$

*Proof.* More generally, without assuming the finiteness of the groups $H^i(U,A)$, we claim that we have a Grothendieck spectral sequence for the composition of the functors $(A_n)_n \rightsquigarrow (H^0(U,A_n))_{U,n}$ with the functor $\lim_{\leftarrow n,U}$

$$E_2^{p,q} = R^p \lim_{\leftarrow n,U} H^q(U,A_n) \Rightarrow H^{p+q}_{\mathrm{cont}}(G,H;(A_n)).$$

For this we have to show that the first functor sends injective objects to acyclics for the second functor. But if $(I_n)$ is an injective system, then all $I_n$ are injective and all morphisms $I_{n+1} \to I_n$ are split surjections, see [4, (1.1)], so $H^0(U,I_{n+1}) \to H^0(U,I_n)$ is surjective for any open subgroup $U \subset G$.

On the other hand, if $I$ is an injective $G$-module, then for any pair of open subgroups $U' \subset U$ the corestriction $\mathrm{cor} : H^0(U',I) \to H^0(U,I)$ is surjective. In fact, we may assume that $I = \mathrm{Ind}_1^G(B)$ is an induced module for a divisible abelian group $B$. (Any such module is injective, and any discrete $G$-module can be embedded into such a module, see [10, p.28 and p.29], so that any injective is a direct factor of such a module). Moreover, since the formation of corestrictions is transitive, we may consider an open subgroup $U'' \subset U'$ which is normal in $G$. Then $\mathrm{Ind}_1^G(B)^{U''} \cong \mathrm{Ind}_1^{\overline{G}}(B)$ for $\overline{G} = G/U''$, and it is known that this is a cohomologically trivial $\overline{G}$-module. Therefore, letting $\overline{U} = U/U''$, we have

$$\mathrm{Ind}_1^G(B)^U = \mathrm{Ind}_1^{\overline{G}}(B)^{\overline{U}} = \mathrm{tr}_{\overline{U}} \mathrm{Ind}_1^{\overline{G}}(B) = \mathrm{cor}_{U''/U} \mathrm{Ind}_1^G(B)^{U''}$$

as claimed.

By assumption, the inverse limit over the open subgroups $U$ containing $H$ can be replaced by a cofinal set of subgroups $U_m$ with $m \in \mathbb{N}$ and $U_{m+1} \subset U_m$, and then the limit over these $U_m$ and over $n$ can be replaced by the 'diagonal' limit over the pairs $(U_n, n)$ for $n \in \mathbb{N}$. For such an inverse limit it is well-known that $R^p \lim_{\leftarrow, n} = 0$ for $p > 1$, and that $R^1 \lim_{\leftarrow, n} H^0(U_n, I_n) = 0$, since the transition maps $H^0(U_{n+1}, I_{n+1}) \to H^0(U_n, I_n)$ are surjective, as shown above (so the system trivially satisfies the Mittag-Leffler condition).

This shows the existence of the above spectral sequence. If, in addition, all $H^q(U, A_n)$ are finite, then, reasoning as above,

$$R^1 \lim_{\leftarrow n, U} H^q(U, A_n) = 0$$

by the Mittag-Leffler property, and we get the claimed isomorphisms. □

Now we come to the spectral sequence in Theorem 1. Any module $A$ in $M_G$ gives rise to two inverse systems, viz., the system $(A[p^n])$, where the transition maps $A[p^{n+1}] \to A[p^n]$ are induced by multiplication with $p$ in $A$, and the system $(A/p^n)$, where the transition maps are induced by the identity of $A$. For reasons explained later, denote by $H^m_{\mathrm{cont}}(G, H; R\underline{T}_p A)$ the value at $A$ of the $m$-th derived functor of the left exact functor

$$F : A \rightsquigarrow \varprojlim_n \varprojlim_U H^0(U, A[p^n])$$

where $U$ runs through all open subgroups $U \subset G$ containing $H$, and the transition maps are the corestriction maps and those coming from $A[p^{n+1}] \to A[p^n]$, respectively. If $H$ is a normal subgroup, then we may restrict to normal open subgroups $U \leq G$ containing $H$ in the above inverse limit, and the limit is a (left) $\Lambda(G/H)$-module in a natural way.

**Theorem 11.** *Let $H$ be a closed subgroup of a profinite group $G$ such that $G/H$ has a countable basis of neighborhoods of identity (see Lemma 10), and let $A$ be a discrete $\Lambda(G)$-module.*

*(a) There are short exact sequences*

$$0 \to H^i_{\mathrm{cont}}(G, H; (A[p^n])) \to H^i_{\mathrm{cont}}(G, H; R\underline{T}_p A) \to H^{i-1}_{\mathrm{cont}}(G, H; (A/p^n)) \to 0.$$

*If $H$ is a normal subgroup, then these are exact sequences of $\Lambda(G/H)$-modules.*

*(b) Let $H'$ be a normal subgroup of $G$, with $H' \subset H$. There is a spectral sequence*

$$E_2^{p,q} = H^p_{\mathrm{cont}}(G/H', H/H'; R\underline{T}_p H^q(H', A)) \Rightarrow H^{p+q}_{\mathrm{cont}}(G, H; R\underline{T}_p A).$$

*If $H$ is a normal subgroup, too, this is a spectral sequence of $\Lambda(G/H)$-modules.*

(c) *If $H$ is a normal subgroup of $G$, then for every discrete $\Lambda(G)$-module $A$ one has canonical isomorphisms of $\Lambda(G/H)$-modules*

$$H^m(G,H;R\underline{T}_pA) \ \cong\ \mathrm{Ext}^m_{\Lambda(G)}(A^\vee,\Lambda(G/H))$$

*for all $m \geq 0$, where $\Lambda(G/H)$ is regarded as a $\Lambda(G)$-module via the ring homomorphism $\Lambda(G) \to \Lambda(G/H)$. More precisely, the $\delta$-functor*

$$M_G \ \to\ Mod_{\Lambda(G/H)}\ ,\ A \ \rightsquigarrow\ (H^m(G,H;R\underline{T}_pA) \mid m \geq 0)$$

*is canonically isomorphic to the $\delta$-functor*

$$M_G \ \to\ Mod_{\Lambda(G/H)}\ ,\ A \ \rightsquigarrow\ (\mathrm{Ext}^m_{\Lambda(G)}(A^\vee,\Lambda(G/H)) \mid m \geq 0).$$

*Here and in the following, the* Ext*-groups* $\mathrm{Ext}_{\Lambda(G)}(-,-)$ *are taken in the category $C_G$ of compact $\Lambda(G)$-modules. We note that these* Ext*-groups are $\Lambda(G)$-modules, but not necessarily compact.*

(d) *In particular, let $H$ be a normal subgroup of $G$, and let $\mathcal{G} = G/H$. If $A$ is a discrete $\Lambda(G)$-module, then one has a spectral sequence of $\Lambda(\mathcal{G})$-modules*

$$E_2^{p,q} = \mathrm{Ext}^p_{\Lambda(\mathcal{G})}(H^q(H,A)^\vee,\Lambda(\mathcal{G})) \Rightarrow\ H^{p+q}_{\mathrm{cont}}(G,H;R\underline{T}_pA) = \mathrm{Ext}^{p+q}_{\Lambda(G)}(A^\vee,\Lambda(\mathcal{G})).$$

Before we give the proof of Theorem 11, we note that it implies Theorem 1. In fact, we apply Theorem 11 to $G = G_S$ and $H = G_{\infty,S}$. If $A$ is a $G_S$-module of cofinite type as in Theorem 1, then $A/p^n = 0$ and $A[p^n]$ is finite, for all $n$. Moreover, $H^i(U,B)$ is known to be finite for all open subgroups $U \leq G_S$ and all finite $U$-modules $B$. By (a) and Lemma 10 we deduce

$$H^m_{\mathrm{cont}}(G_S,G_{\infty,S};RT_pA) = \varprojlim_{n,U} H^m(U,A[p^n]) = \varprojlim_{n,k'} H^m(G_S(k'),A[p^n]),$$

where $k'$ runs through all finite subextensions of $k_\infty/k$. Moreover, one has canonical isomorphisms

$$\varprojlim_{n} H^m(U,A[p^n]) \ \cong\ H^m(U,T_pA)$$

where the latter group is continuous cochain group cohomology, cp. [4]. By applying Theorem 11 (d) we thus get the desired spectral sequence. Finally, $H^m(H,A)^\vee$ is a finitely generated $\Lambda(\mathcal{G})$-module for all $m \geq 0$, so that the initial terms of the spectral sequence are finitely generated $\Lambda(\mathcal{G})$-modules as well, and so are the limit terms. In fact, let $N$ be the kernel of the homomorphism $G_S \to Aut(A)$ given by the action of $G_S$ on $A$, and let $H' = H \cap N$. Then $G/H'$ is a $p$-adic analytic Lie group, since $G/H$ and $G/N$ are. It is well-known that $H^m(H',\mathbb{Q}_p/\mathbb{Z}_p)$ is a cofinitely generated discrete $\Lambda(G/H')$-module for all $m \geq 0$; hence the same is true for $H^m(H',A) \cong H^m(H',\mathbb{Q}_p/\mathbb{Z}_p) \otimes T_pA$. The claim then follows from the Hochschild–Serre spectral sequence $H^p(H/H',H^q(H',A)) \Rightarrow H^{p+q}(H,A)$.

*Proof of Theorem 11.* (a): We can write $F$ as the composition of the two left exact functors

$$\begin{array}{rcl} \underline{T}_p : M_G & \to & M_G^{\mathbb{N}} \\ A & \rightsquigarrow & (A[p^n]) \end{array}$$

and

$$\begin{array}{rcl} H^0_{\text{cont}}(G,H;-) : M_G^{\mathbb{N}} & \to & Ab \\ (A_n) & \rightsquigarrow & \varprojlim_n \varprojlim_U H^0(U,A_n), \end{array}$$

where the limit over $U$ runs through all open (normal) subgroups of $G$ containing $H$, with the corestrictions as transition maps. With the arguments in the proof of Lemma 10, we can deduce that $\underline{T}_p$ maps injectives to $H^0_{\text{cont}}(G,H;-)$-acyclics. In fact, we may assume injective $G$-modules given as induced modules $I = \text{Ind}_1^G(B)$ with a divisible abelian group $B$. Then each module $I[p^n] = \text{Ind}_1^G(B[n])$ is induced, hence acyclic for the functor $H^0(U,-)$, and since $I$ is divisible, we have exact sequences

$$0 \to I[p] \longrightarrow I[p^{r+1}] \longrightarrow I[p^r] \to 0.$$

Therefore the transition maps $H^0(U, I[p^{r+1}]) \to H^0(U, I[p^r])$ are surjective, and as in the proof of Lemma 10 we conclude that the corestrictions cor : $H^0(U', I[p^n]) \to H^0(U, I[p^n])$ are surjective for open subgroups $U' \subset U$ of $G$. Therefore the system $I[p^n]$ is acyclic for $\varprojlim_{U,n}$, noting that

$$\varprojlim_{U,n} H^0(U, I[p^n]) = \varprojlim_{n} H^0(U_n, I[p^n])$$

for a cofinal family $(U_n)$ of subgroups between $H$ and $G$.

Therefore we get a spectral sequence

$$E_2^{p,q} = H^p_{\text{cont}}(G,H; R^q\underline{T}_p A) \Rightarrow H^{p+q}_{\text{cont}}(G,H; R\underline{T}_p A).$$

From the snake lemma one immediately gets

$$R^q\underline{T}_p A = \begin{cases} (A/p^m A), & q = 1, \\ 0, & q > 1. \end{cases}$$

(Note that the described functor $A \rightsquigarrow (A/p^m)$ is effacable, since $A$ embeds into an injective, hence divisible $G$-module.) Hence we get a short exact sequences

$$0 \to H^n_{\text{cont}}(G,H;\underline{T}_p A) \to H^n_{\text{cont}}(G,H;R\underline{T}_p A) \to H^{n-1}_{\text{cont}}(G,H;R^1\underline{T}_p A) \to 0.$$

This shows (a) and also explains the notation $H^n_{cont}(G,H;R\underline{T}_p A)$ for $R^n F(A)$. In fact, $R^n F(A)$ is the hypercohomology with respect to $H^0_{\text{cont}}(G,H;-)$ of a complex $R\underline{T}_p A$ in $M_G^{\mathbb{N}}$ computing the $R^i\underline{T}_p A$.

(b): If $H$ is a normal subgroup, we can regard the functor $F$ as a functor from $M_G$ to the category $Mod_{\Lambda(G/H)}$ of $\Lambda(G/H)$-modules. On the other hand, we can also write $F$ as the composition of the left exact functors

$$H^0(H,-): \ M_G \to M_{G/H} \ , \ A \rightsquigarrow A^H$$

and

$$\begin{array}{rcl}\tilde{F} : M_{G/H} & \to & Mod_{\Lambda(G/H)},\\ B & \rightsquigarrow & \varprojlim_{n} \varprojlim_{U/H} H^0(U/H, B[p^n]) = H^0(G/H, \{1\}; RT_pB).\end{array}$$

(Note that $U/H$ runs through all open (normal) subgroups of $G/H$.) This immediately gives the spectral sequence in (b).

(c): We claim that the functor $F$ is isomorphic to the functor

$$\begin{array}{rcl}M_G & \to & Mod_{\Lambda(G/H)}\\ B & \rightsquigarrow & \mathrm{Hom}_{\Lambda(G)}(B^\vee, \Lambda(G/H)).\end{array}$$

In fact, writing $\mathrm{Hom}_{\Lambda(G)}(-,-)$ for the homomorphism groups of compact $\Lambda(G)$-modules, we have (cp. [5, p. 179])

$$\begin{aligned}\mathrm{Hom}_{\Lambda(G)}(B^\vee, \Lambda(G/H)) &= \varprojlim_{U} \mathrm{Hom}_{\Lambda(G)}(B^\vee, \mathbb{Z}_p[G/U])\\ &= \varprojlim_{n} \varprojlim_{U} \mathrm{Hom}_{\mathrm{cont}}(H^0(U,B)^\vee, \mathbb{Z}/p^n\mathbb{Z})\\ &= \varprojlim_{n} \varprojlim_{U} \mathrm{Hom}_{\mathrm{cont}}(H^0(U,B[p^n])^\vee, \mathbb{Z}/p^n\mathbb{Z})\\ &= \varprojlim_{n} \varprojlim_{U} H^0(U, B[p^n]),\end{aligned}$$

where $U$ runs through all open subgroups of $G$ containing $H$, and hence

$$\mathrm{Hom}_{\Lambda(G)}(B^\vee, \Lambda(G/H)) = H^0(G, H; RT_pB).$$

Since taking Pontrjagin duals is an exact functor $M_G \to C_G$ taking injectives to projectives, the derived functors of the functor $B \rightsquigarrow \mathrm{Hom}_{\Lambda(G)}(B^\vee, \Lambda(G/H))$ are the functors $B \rightsquigarrow \mathrm{Ext}^i_{\Lambda(G)}(B^\vee, \Lambda(G/H))$, and we get (c). Finally, by applying (b) for $H' = H$ and (c) for $H = \{1\}$ we get (d). □

Let us note that the proof of Theorem 11 gives the following $\mathbb{Z}/p^n$-analog (by "omitting the inverse limits over $n$"). For a profinite group $G$ let $\Lambda_n(G) = \Lambda(G)/p^n = \mathbb{Z}/p^n[[G]]$ be the completed group ring over $\mathbb{Z}/p^n$.

**Theorem 12.** *Let $H$ and $H'$ be normal subgroups of a profinite group $G$, with $H' \subset H$, and let $A$ be a discrete $\Lambda_n(G)$-module.*

*(a) There is a spectral sequence of $\Lambda_n(G/H)$-modules*

$$E_2^{p,q} = H^p(G/H', H/H'; H^q(H', A)) \;\Rightarrow\; H^{p+q}(G, H; A).$$

*(b) On the category of discrete $\Lambda_n(G)$-modules the $\delta$-functor $A \rightsquigarrow (H^m(G, H; A) \mid m \geq 0)$ with values in the category of $\Lambda_n(G/H)$-modules is canonically isomorphic to the $\delta$-functor $A \rightsquigarrow (\mathrm{Ext}^m_{\Lambda_n(G)}(A^\vee, \Lambda_n(G/H)) \mid m \geq 0)$, where the* Ext*-groups are taken in the category of compact $\Lambda_n(G)$-modules.*

(c) *In particular, if* $\mathcal{G} = G/H$*, and* $A$ *is a discrete* $\Lambda_n(G)$*-module, then one has a spectral sequence of* $\Lambda_n(\mathcal{G})$*-modules*

$$E_2^{p,q} = \mathrm{Ext}^p_{\Lambda_n(\mathcal{G})}(H^q(H,A)^\vee, \Lambda_n(\mathcal{G})) \Rightarrow H^{p+q}(G,H;A) = \mathrm{Ext}^{p+q}_{\Lambda_n(G)}(A^\vee, \Lambda(\mathcal{G})).$$

**Corollary 13.** *With the notations as for Theorem 1, let* $A$ *be a finite* $\Lambda_n(G_S)$*-module, and* $\Lambda_n = \Lambda(\mathcal{G})$*. Then there is a spectral sequence of finitely generated* $\Lambda_n$*-modules*

$$E_2^{p,q} = \mathrm{Ext}^p_{\Lambda_n}(H^q(G_{\infty,S}, A)^\vee, \Lambda_n) \Rightarrow \varprojlim_{k'} H^{p+q}(G_S(k'), A) = \mathrm{Ext}^{p+q}_{\Lambda_n(G_S)}(A^\vee, \Lambda_n),$$

*where* $k'$ *runs through the finite subextensions* $k'/k$ *of* $k_\infty/k$*.*

On the other hand, Theorem 1 also has the following counterpart for finite modules.

**Theorem 14.** *With notations as for Theorem 1, let* $A$ *be a finite* $p$*-primary* $G_S$*-module, of exponent* $p^n$*. Then there is a spectral sequence*

$$E_2^{p,q} = \mathrm{Ext}^p_{\Lambda}(H^q(G_{\infty,S}, A)^\vee, \Lambda) \Rightarrow \varprojlim_{k'} H^{p+q-1}(G_S(k'), A) = \mathrm{Ext}^{p+q-1}_{\Lambda_n(G_S)}(A^\vee, \Lambda_n),$$

*where, in the inverse limit,* $k'$ *runs through the finite extension* $k'$ *of* $k$ *inside* $k_\infty$ *and the transition maps are the corestrictions.*

*Proof.* As in the proof of Theorem 1, Theorem 11 (d) applies to $G = G_S$ and $H = G_{\infty,S}$. But now the inverse system $(A[p^n])$ is Mittag-Leffler-zero in the sense of [4]: if the exponent of $A$ is $p^d$, then the transition maps $A[p^{n+d}] \to A[p^n]$ are zero. This implies that $H^m_{\mathrm{cont}}(G_S, (A[p^n])) = 0$ for all $m \geq 0$, cp. [4]. On the other hand it is clear that the system $(A/p^n)$ is essentially constant ($A/p^n = A$ for $n \geq d$). From Theorem 11 (a) and Lemma 10 we immediately get

$$H^m_{\mathrm{cont}}(G_S, G_{\infty,S}; RT_pA) \cong H^{m-1}_{\mathrm{cont}}(G_S, G_{\infty,S}; (A/p^n)) \cong \varprojlim_{k'} H^{p+q-1}(G_S(k'), A),$$

and hence the claim, by applying Theorem 12 (b) in addition. □

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