Mathematik

The Behavior of Nil-Groups under Localization

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vorgelegt von
Joachim Grunewald
aus Bielefeld
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Dekan:	Prof. Dr. Klaus Hinrichs
Erster Gutachter:	Prof. Dr. Wolfgang Lück
Zweiter Gutachter:	PD Dr. Holger Reich
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Abstract

This thesis centers around the study of the behavior of Nil-groups under localization and its effect on the Farrell-Jones conjecture. We prove that under mild assumptions we can always write the Nil-groups of the localized ring as a certain colimit over the Nil-groups of the ring, generalizing a result of Vorst. For applications it is fruitful to improve this result. For this purpose, we define Frobenius and Verschiebung operations on certain Nil-groups. These operations provide the tool to prove that Nil-groups are modules over the ring of Witt-vectors and are either trivial or not finitely generated as abelian groups. We use the improved localization results to show that Nil-groups of polycyclic-by-finite groups are torsion. As an important corollary, we obtain the result that the Nil-groups appearing in the decomposition of the K-groups of virtually cyclic groups are torsion groups. This implies that the Farrell-Jones conjecture predicts that rationally the building blocks of the K-groups of groups are the K-groups of finite groups.

Contents

Introduction

In general, Nil-groups measure the difference between the K-groups of a ground ring and the K-groups of a certain ring built up out of this ring. We start by briefly recalling the importance of the various kinds of Nil-groups.

The most basic kind of Nil-group is given by Bass's definition of the abelian groups $\mathrm{Nil}_i(R)$ [Bas68, Section 12.6]. The so called *Fundamental Theorem of algebraic K-theory* [BHS64] gives a description of the K-groups of the Laurent polynomial ring of a ring R in terms of the K-groups of R and the groups $\mathrm{Nil}_i(R)$ for all $i \in \mathbb{Z}$:

$$K_i(R[t,t^{-1}]) = K_i(R) \oplus K_{i-1}(R) \oplus \mathrm{Nil}_{i-1}(R) \oplus \mathrm{Nil}_{i-1}(R).$$

If α is a ring automorphism of R Farrell introduced the abelian groups $\mathrm{Nil}_i(R;\alpha)$ [Far72]. He generalized, together with Hsiang, the Fundamental Theorem of algebraic K-theory to K_1 of the twisted Laurent polynomial ring [FH70]. They proved that the sequence

$$K_1(R) \xrightarrow{1-\alpha_*} K_1(R) \longrightarrow K_1(R_\alpha[t,t^{-1}])/\,\mathrm{Nil}_0(R,\alpha) \oplus \mathrm{Nil}_0(R,\alpha^{-1}) \longrightarrow$$

$$\longrightarrow K_0(R) \xrightarrow{1-\alpha_*} K_0(R)$$

is exact. This decomposition was extended by Grayson to higher algebraic K-theory [Gra88].

For applications in topology, the K-groups of group rings are of special importance. If $\Gamma = G \rtimes_\alpha \mathbb{Z}$ is the semidirect product of a group G, an automorphism α of G and the infinite cyclic group, the group ring of Γ can be seen as a twisted Laurent polynomial ring. In this context, the statement gives that the sequence

$$\longrightarrow K_i(RG) \xrightarrow{1-\alpha_*} K_i(RG) \longrightarrow K_i(R\Gamma)/\,\mathrm{Nil}_{i-1}(RG,\alpha) \oplus \mathrm{Nil}_{i-1}(RG,\alpha^{-1}) \longrightarrow$$

$$\longrightarrow K_{i-1}(RG) \xrightarrow{1-\alpha_*} K_{i-1}(RG) \longrightarrow$$

is exact.

An inclusion $\alpha\colon C \to A$ of rings is called *pure* if $A = \alpha(C) \oplus A'$ as C-bimodules. It is called *pure* and *free* if in addition A' is free as a left C-module. Let $\alpha\colon C \to A$ and $\beta\colon C \to B$ both be pure and free. The *generalized free product* of A and B is the push-out of α and β in the category of rings. Waldhausen introduced, for R-bimodules X and Y, the abelian groups $\mathrm{Nil}_i(R;X,Y)$ [Wal78a, Wal78b]. Nil-groups of this kind relate the K-groups of the generalized free product to the K-groups of the ground rings.

Theorem (Waldhausen [Wal78a, Wal78b]). *Let $\alpha\colon C \hookrightarrow A$ and $\beta\colon C \hookrightarrow B$ both be pure and free. Write $A = \alpha(C) \oplus A'$ and $B = \beta(C) \oplus B'$. Let R be the generalized free product of A and B. The groups $\mathrm{Nil}_i(C; A', B')$ are direct summands of $K_{i+1}(R)$ and there is a long exact sequence of groups*

$$\cdots \longrightarrow K_{i+1}(C) \longrightarrow K_{i+1}(A) \oplus K_{i+1}(B) \longrightarrow K_{i+1}(R)/\,\mathrm{Nil}_i\left(C; A', B'\right) \longrightarrow K_i(C) \longrightarrow \cdots,$$

where $i \geq 0$.

For $i < 0$ the result is extended by Bartels and Lück [BL04].

A special case of a generalized free product comes from the amalgamated product of groups. Let G_1 and G_2 be groups with a common subgroup H. The group ring of $G_1 *_H G_2$ is the generalized free product of the group rings of H, G_1 and G_2. In the context of group rings, the previous theorem reads as follows:

Corollary. *Let $\Gamma = G_1 *_H G_2$ be the amalgamated product of the groups H, G_1 and G_2. The groups $\mathrm{Nil}_i\left(\mathbb{Z}H; \mathbb{Z}[G_1 - H], \mathbb{Z}[G_2 - H]\right)$ are direct summands of $K_{i+1}(\Gamma)$ and there is a long exact sequence of groups*

$$\cdots \longrightarrow K_{i+1}(\mathbb{Z}H) \longrightarrow K_{i+1}(\mathbb{Z}G_1) \oplus K_{i+1}(\mathbb{Z}G_2) \longrightarrow$$

$$\longrightarrow K_{i+1}(\Gamma)/\,\mathrm{Nil}_i\left(\mathbb{Z}H; \mathbb{Z}[G_1 - H], \mathbb{Z}[G_2 - H]\right) \longrightarrow K_i(\mathbb{Z}H) \longrightarrow \cdots,$$

where $i \in \mathbb{Z}$.

If the Nil-groups are ignored, the sequence given above is the analog of the Mayer-Vietoris sequence for K-groups. It is a general belief that Nil-groups are the obstruction for K-theory to be a homology theory.

Let $\alpha, \beta\colon C \to A$ be pure and free. The *generalized Laurent extension* with respect to α, β is the universal ring $R = A_{\alpha,\beta}\{t^{\pm 1}\}$ which contains A and an invertible element t which satisfies

$$\alpha(c)t = t\beta(c) \;\text{ for all}\, c \in C.$$

The existence of such a ring is explained in [Wal78a]. For R-bimodules X, Y, Z and W Waldhausen introduced the abelian groups $\mathrm{Nil}_i(R; X, Y, Z, W)$ [Wal78a, Wal78b], which are the most general kind of Nil-groups. Nil-groups of this kind relate the K-groups of a generalized Laurent extension to the K-groups of the ground rings.

Theorem (Waldhausen [Wal78a, Wal78b]). *Let R be the generalized Laurent extension of pure and free maps $\alpha, \beta\colon C \hookrightarrow A$. Write $A = \alpha(C) \oplus A'$ and $A = \beta(C) \oplus A''$. Denote by ${}_\beta A_\alpha$ the C-bimodule A with C-action from the left via β and from the right via α. Let ${}_\alpha A'_\alpha$, ${}_\beta A''_\beta$ and ${}_\alpha A_\beta$ be defined similarly. The groups $\mathrm{Nil}_i\left(C; {}_\alpha A'_\alpha, {}_\beta A''_\beta, {}_\beta A_\alpha, {}_\alpha A_\beta\right)$ are direct summands of $K_{i+1}(R)$ and there is a long exact sequence of groups*

$$\cdots \longrightarrow K_{i+1}(C) \xrightarrow{\alpha - \beta} K_{i+1}(A) \longrightarrow$$

$$\longrightarrow K_{i+1}(R)/\operatorname{Nil}_i\big(C;_\alpha A'_\alpha,_\beta A''_\beta,_\beta A_\alpha,_\alpha A_\beta\big) \longrightarrow K_i(C) \longrightarrow \cdots,$$

where $i \geq 0$.

Again, this is extended to $i < 0$ by Bartels and Lück [BL04].

A special case of a generalized Laurent extension comes from the HNN-extension of groups. Let H be a group which is embedded into another group G in two different ways. The group ring of the HNN-extension of H and G is the generalized Laurent extension of RH and RG. In the setting of group rings, the previous theorem reads as follows:

Corollary. *Let* Γ *be the HNN-extension of the embeddings* $\alpha, \beta\colon H \hookrightarrow G$. *The groups* $\operatorname{Nil}_i\big(\mathbb{Z}H; \mathbb{Z}[G-\alpha(H)], \mathbb{Z}[G-\beta(H)],_\beta \mathbb{Z}G_\alpha,_\alpha \mathbb{Z}G_\beta\big)$ *are direct summands of* $K_{i+1}(\mathbb{Z}\Gamma)$ *and there is a long exact sequence of groups*

$$\cdots \longrightarrow K_{i+1}(\mathbb{Z}H) \xrightarrow{\alpha-\beta} K_{i+1}(\mathbb{Z}G) \longrightarrow$$

$$\longrightarrow K_{i+1}(\mathbb{Z}\Gamma)/\operatorname{Nil}_i\big(\mathbb{Z}H; \mathbb{Z}[G-\alpha(H)], \mathbb{Z}[G-\beta(H)],_\beta \mathbb{Z}G_\alpha,_\alpha \mathbb{Z}G_\beta\big) \longrightarrow \cdots,$$

where $i \in \mathbb{Z}$.

To avoid confusion, Nil-groups of the form $\operatorname{Nil}_i(R)$ are called *Bass Nil-groups*, Nil-groups of the form $\operatorname{Nil}_i(R;\alpha)$ are called *Farrell Nil-groups*, Nil-groups of the form $\operatorname{Nil}_i(R;X,Y)$ are called *Waldhausen Nil-groups of generalized free products* and Nil-groups of the form $\operatorname{Nil}_i(R;X,Y,Z,W)$ are called *Waldhausen Nil-groups of generalized Laurent extensions*. In general, Nil-groups are subgroups of the K-groups of NIL-categories. Bass Nil-groups are subgroups of the K-groups of the NIL-category $\operatorname{NIL}(R)$, for Farrell Nil-groups the NIL-category is denoted by $\operatorname{NIL}(R;\alpha)$, for Waldhausen Nil-groups of generalized free products the NIL-category is denoted by $\operatorname{NIL}(R;X,Y)$ and for Waldhausen Nil-groups of generalized Laurent extensions the NIL-category is denoted by $\operatorname{NIL}(R;X,Y,Z,W)$. For a definition of the different kind of Nil-groups and NIL-categories see Chapter 1.

The Behavior of Nil-Groups under Localization

Nil-groups seem to be hard to compute. It is for example known that higher Bass Nil-groups are either trivial or not finitely generated as abelian groups [Far77, Wei81]. However, in the first chapter of this thesis, it is proven that all these kinds of Nil-groups behave nicely under localization.

Definition. Let R be a ring.

1. Let $T \subseteq R$ be a multiplicatively closed subset of central non zero divisors. The ring $T^{-1}R$ is denoted by R_T.

2. Let s be a central non zero divisor and let S be the multiplicatively closed set generated by s. We use the short hand notation R_s for R_S.

3. Let X be an R-bimodule. We define X_T to be the R_T-bimodule $R_T \otimes_R X \otimes_R R_T$.
4. If s is a central non zero divisor, we use the short hand notation X_s for $R_s \otimes_R X \otimes_R R_s$.

The main theorem of the first chapter is the following theorem:

Theorem. *Let R be a ring and let X, Y, Z and W be left flat R-bimodules. Let s be an element of the center of R which is not a zero divisor and satisfies $s \cdot x = x \cdot s$ for all elements $x \in X$ and similar conditions for Y, Z and W. We obtain an isomorphism*

$$\mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}[t]} \mathrm{Nil}_i(R; X, Y, Z, W) \cong \mathrm{Nil}_i(R_s; X_s, Y_s, Z_s, W_s),$$

for $i \in \mathbb{Z}$, and t acts on $\mathrm{Nil}_i(R; X, Y, Z, W)$ via the map induced by the functor

$$\begin{aligned} F_s\colon\ \mathrm{NIL}(R; X, Y, Z, W) &\to \mathrm{NIL}(R; X, Y, Z, W) \\ (P, Q, p, q) &\mapsto (P, Q, p \cdot s, q \cdot s). \end{aligned}$$

The condition that the bimodules X, Y, Z and W are left flat does not seem to be overly restrictive since in all the cases considered by Waldhausen X, Y, Z and W are left free by the purity and freeness condition. The condition $s \cdot x = x \cdot s$ translates in Waldhausen's setting of a generalized Laurent extension to the assumption that s is mapped, under the maps α and β, to central elements.

In Remark 1.2.17, it is explained that Waldhausen Nil-groups of generalized Laurent extensions are a generalization of the other kind of Nil-groups. Thus we get the following corollaries:

Corollary. *Let R be a ring and let X and Y be left flat R-bimodules. Let s be an element of the center of R which is not a zero divisor and satisfies $s \cdot x = x \cdot s$ for all elements $x \in X$ and a similar condition for Y. We obtain an isomorphism*

$$\mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}[t]} \mathrm{Nil}_i(R; X, Y) \cong \mathrm{Nil}_i(R_s; X_s, Y_s),$$

for $i \in \mathbb{Z}$, and t acts on $\mathrm{Nil}_i(R; X, Y)$ via the map induced by the functor

$$\begin{aligned} F_s\colon\ \mathrm{NIL}(R; X, Y) &\to \mathrm{NIL}(R; X, Y) \\ (P, Q, p, q) &\mapsto (P, Q, p \cdot s, q \cdot s). \end{aligned}$$

Again, the assumption that X and Y are left flat modules is not overly restrictive since in the case of a generalized free product X and Y are left free by the purity and freeness condition. The condition $s \cdot x = x \cdot s$ translates in the setting of a generalized free product to the condition that s is mapped to central elements.

Corollary. *Let R be a ring and let $\alpha\colon R \to R$ be an automorphism. Let s be an element of the center of R which is not a zero divisor and is fixed under α. We obtain an isomorphism*

$$\mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}[t]} \mathrm{Nil}_i(R; \alpha) \cong \mathrm{Nil}_i(R_s; \alpha),$$

for $i \in \mathbb{Z}$, and t acts on $\mathrm{Nil}_i(R,\alpha)$ via the map induced by the functor

$$\begin{aligned} F_s\colon\ \mathrm{NIL}(R;\alpha) &\to \mathrm{NIL}(R;\alpha)\\ (P,\nu) &\mapsto (P,\nu\cdot s). \end{aligned}$$

Corollary. *Let R be a ring. Let s be an element of the center of R which is not a zero divisor. We obtain an isomorphism*

$$\mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}[t]} \mathrm{Nil}_i(R) \cong \mathrm{Nil}_i(R_s),$$

for all $i \in \mathbb{Z}$, and t acts on $\mathrm{Nil}_i(R)$ via the map induced by the the functor

$$\begin{aligned} F_s\colon\ \mathrm{NIL}(R) &\to \mathrm{NIL}(R)\\ (P,\nu) &\mapsto (P,\nu\cdot s). \end{aligned}$$

For $\mathrm{Nil}_i(R)$ the result was already known [Vor79].

Nil-Groups as Modules over the Ring of Witt vectors

For applications, it is fruitful to improve these results to an isomorphism between $R_s \otimes_R \mathrm{Nil}(R)$ and $\mathrm{Nil}(R_s)$. To get this isomorphism, we develop in the third chapter a Witt vector module structure on certain Nil-groups. In this chapter, the ring is always assumed to be a either a group ring or more generally an algebra over a commutative ring R. Let us briefly recall the definition of the ring of Witt vectors. For an introduction to the ring of Witt vectors see [Blo77]. The *ring of (big) Witt vectors* is the ring $1 + tR[\![t]\!]$ of power series with constant term 1. The underlying additive group of the ring of Witt vectors is the multiplicative group of $1 + tR[\![t]\!]$. The multiplication is the unique continuous functorial operation $*$ for which

$$(1-at) * (1-bt) = (1-abt)$$

holds for all a, $b \in R$. In the sequel, the ring of Witt vectors is denoted by $W(R)$. We define ideals $I_N := (1 + t^N R[\![t]\!])$ for all $N \in \mathbb{N}$. The resulting topology on the ring of Witt vectors is called the *t-adic topology.*

There is a different approach to the ring of Witt vectors which emphasizes the relation to K-theory. Let $\mathrm{END}(R)$ be the exact *category of endomorphisms of finitely generated projective right R-modules.* The objects of this category are pairs (B,φ) where B is a finitely generated projective R-module and φ is an endomorphism of B. Maps from (B,φ) to (B',φ') are module homomorphisms $f\colon B \to B'$ such that $\varphi' \circ f = f \circ \varphi$. A sequence is called *exact* if the underlying sequence of module homomorphisms is exact. Since R is assumed to be commutative, the tensor product induces a ring structure on $K_0\big(\mathrm{END}(R)\big)$. Denote by $\mathrm{End}_0(R)$ the quotient of $K_0\big(\mathrm{END}(R)\big)$ by the ideal generated by elements of the form $[(B,0)]$. Since the characteristic polynomial

$$\chi\big((B,\varphi)\big) := \det(\mathrm{id}_B - t\cdot\varphi)$$

defines a map from $\mathrm{END}(R)$ into $W(R)$ which is additive with respect to short exact sequences and sends elements of the form $[(B,0)]$ to zero, it induces a map from $\mathrm{End}_0(R)$ into $W(R)$. A theorem of Almkvist [Alm73, Alm74] states that this map is an injective ring homomorphism whose image is dense in the ring of Witt vectors with respect to the t-adic topology. Grayson gives an overview over this approach to the ring of Witt vectors [Gra78].

In Section 3.4, we prove that certain kind of Nil-groups are modules over the ring of Witt vectors. The most important Nil-groups of this kind are the Nil-groups considered in the following proposition.

Proposition. *Let R be a commutative ring, let G be a group and let $\alpha, \beta \colon G \to G$ be inner group automorphism. The groups $\mathrm{Nil}_i(RG;\alpha)$ and $\mathrm{Nil}_i(RG;RG_\alpha,RG_\beta)$ are modules over the ring of Witt vectors of R.*

To get a Witt vector-module structure on Nil-groups, we first define an $\mathrm{End}_0(R)$-module structure on the Nil-groups $\mathrm{Nil}_i(\Lambda;X,Y,Z,W)$ where Λ is an algebra over a commutative ring R. To obtain this module structure, we define in Section 3.1 an exact pairing between the categories $\mathrm{END}(R)$ and $\mathrm{NIL}(\Lambda;X,Y,Z,W)$. For Bass Nil-groups this pairing is induced by the tensor product

$$\begin{aligned} \otimes\colon\ \mathrm{END}(R) \times \mathrm{NIL}(\Lambda) &\to \mathrm{NIL}(\Lambda) \\ (B,\varphi)\times(P,\nu) &\mapsto (B \otimes_R P, \varphi\otimes\nu). \end{aligned}$$

The machinery developed by Waldhausen [Wal78a, Wal78b] gives us a pairing on K-groups:

$$K_0\big(\mathrm{END}(R)\big) \times K_i\big(\mathrm{NIL}(\Lambda;X,Y,Z,W)\big) \to K_i\big(\mathrm{NIL}(\Lambda;X,Y,Z,W)\big).$$

It is easily seen that this structure restricts to an $\mathrm{End}_0(R)$-module structure on $\mathrm{Nil}_i(\Lambda;X,Y,Z,W)$. In the sequel, this module multiplication is denoted by $*$.

For the proof that the $\mathrm{End}_0(R)$-module structure can be extended to a $W(R)$-module structure we have to restrict to a certain class of Nil-groups containing Nil-groups of the form $\mathrm{Nil}_i(RG;\alpha)$ or $\mathrm{Nil}_i(RG;RG_\alpha,RG_\beta)$ where G is a group and α and β are inner group automorphisms. In Section 3.2, we define Verschiebung and Frobenius operations for Nil-groups of this kind. For Bass Nil-groups and End_0 Verschiebung and Frobenius operations are well understood [Blo78, CdS95, Sti82, Wei81].

The strength of the Verschiebung and Frobenius operations is that they satisfy certain relations. If for a natural number n the Verschiebung operation is denoted by V_n and the Frobenius operation by F_n, we prove in Section 3.3 that

$$\mathrm{F}_n\mathrm{V}_n(x) = x \cdot n$$

for every x in $\mathrm{Nil}_i(RG;\alpha)$ and a similar identity for $\mathrm{Nil}_i(RG;RG_\alpha,RG_\beta)$. This implies the following corollary which is a generalization of a result by Farrell [Far77].

Corollary. *Let R be a ring, let G be a group, let X and Y be arbitrary RG-bimodules and let α and β be inner group automorphisms of G. Then* $\mathrm{Nil}_i(RG;\alpha)$ *and* $\mathrm{Nil}_i(RG; RG_\alpha \otimes X, RG_\beta \oplus Y)$ *are either trivial or not finitely generated as an abelian group for $i \in \mathbb{Z}$.*

If y is an element of $\mathrm{End}_0(R)$ and x an element of $\mathrm{Nil}_i(RG;\alpha)$ or $\mathrm{Nil}_i(RG; RG_\alpha, RG_\beta)$, we obtain that

$$V_n(y * F_n x) = (V_n y) * x.$$

This relation is the main ingredient of the proof that the $\mathrm{End}_0(R)$-module structure can be extended to a $W(R)$-module structure on $\mathrm{Nil}_i(RG;\alpha)$ and $\mathrm{Nil}_i(RG; RG_\alpha, RG_\beta)$.

Torsion Results

In the fourth chapter, we apply the at first glance totally unrelated results of the previous chapters to prove that in important cases the Nil-groups are torsion groups. To do so, we first improve the localization result into the following theorem:

Theorem. *Let R be $\mathbb{Z}_T$ for some multiplicatively closed set $T \subseteq \mathbb{Z} - \{0\}$, $\hat{\mathbb{Z}}_p$ or a commutative $\mathbb{Q}$-algebra. Let G be a group and let α and β be inner automorphism of G. Then for every multiplicatively closed set $S \subset R$ of non zero divisors there are isomorphisms of R_S-modules*

$$R_S \otimes_R \mathrm{Nil}_i(RG;\alpha) \cong \mathrm{Nil}_i(R_S G;\alpha)$$

and

$$R_S \otimes_R \mathrm{Nil}_i(RG; RG_\alpha, RG_\beta) \cong \mathrm{Nil}_i(R_S G; R_S G_\alpha, R_S G_\beta),$$

for all $i \in \mathbb{Z}$.

The main application of this result are torsion results. A group G is called *poly-(infinite)cyclic* if it has a finite chain of normal subgroups

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_m = G$$

such that every G_i/G_{i-1} is (infinite) cyclic. A group is called *polycyclic-by-finite* if it has a normal subgroup of finite index which is polycyclic.

Every polycyclic-by-finite group has a poly-infinite cyclic subgroup of finite index [Pas85, page 422]. In Section 4.3, we prove that if G is a polycyclic-by-finite group with poly-infinite cyclic subgroup of finite index n then $\mathrm{Nil}_i(\mathbb{Z}[1/n]G;\alpha)$ and $\mathrm{Nil}_i(\mathbb{Z}[1/n]G; \mathbb{Z}[1/n]G_\alpha, \mathbb{Z}[1/n]G_\beta)$ vanish. Thus we get as an immediate corollary:

Corollary. *Let G be a polycyclic-by-finite group. Then there exists a poly-infinite cyclic subgroup of finite index n. If α and β are inner group automorphism, then the groups* $\mathrm{Nil}_i(\mathbb{Z}G;\alpha)$ *and* $\mathrm{Nil}_i(\mathbb{Z}G; \mathbb{Z}G_\alpha, \mathbb{Z}G_\beta)$ *are n-torsion groups for $i \in \mathbb{Z}$.*

In this corollary, the assumption that α is an inner automorphism is too restrictive. To get around this problem, we develop in Section 4.2 induction and transfer maps for Nil-groups. These maps are generalizations of the induction and transfer maps in algebraic K-theory [Mil71, Chapter 14]. They enable us to prove that the Nil-groups are torsion groups even if α and β are not inner automorphism.

Theorem. *Let G be a polycyclic-by-finite group. Then there exists a poly-infinite cyclic subgroup of finite index n. Let α and β be group automorphisms of finite order m and m'.*

1. *The group $\mathrm{Nil}_i(\mathbb{Z}G;\alpha)$ is an $(n\cdot m)$-torsion group for $i\in\mathbb{Z}$.*
2. *The group $\mathrm{Nil}_i(\mathbb{Z}G;\mathbb{Z}G_\alpha,\mathbb{Z}G_\beta)$ is an $(n\cdot m\cdot m')$-torsion group for $i\in\mathbb{Z}$.*

As an important application of this theorem, we get that the Nil-groups appearing in the calculation of the K-groups of virtually cyclic groups are torsion groups. A group is called *virtually (infinite) cyclic* if it has a (infinite) cyclic subgroup of finite index. Virtually cyclic groups are of special importance because of the Farrell-Jones conjecture [FJ93]. The conjecture implies that the building blocks of algebraic K-theory of group rings are the K-groups of virtually cyclic groups.

There are three kinds of virtually cyclic groups:

I. the semidirect product $G\rtimes\mathbb{Z}$ of a finite group G and the infinite cyclic group;

II. the amalgamated product $G_1 *_H G_2$ of two finite groups G_1 and G_2 over a subgroup H such that $[G_1:H]=[G_2:H]=2$;

III. finite groups.

For a calculation of the K-groups of the first two kinds of virtually cyclic groups, the Nil-groups of finite groups are needed. In the case of a virtually cyclic group of the first type Farrell Nil-groups of finite groups appear. If we consider a virtually cyclic group of the second type the Nil-groups $\mathrm{Nil}_i(\mathbb{Z}H,\mathbb{Z}[G_1-H],\mathbb{Z}[G_2-H])$ relate the K-groups of $G_1 *_H G_2$ to the K-groups of H, G_1 and G_2. The group H is an index two subgroup. Thus we can find automorphisms α and β of H such that the $\mathbb{Z}H$-modules $\mathbb{Z}[G_1-H]$ and $\mathbb{Z}[G_2-H]$ are isomorphic to $\mathbb{Z}H_\alpha$ and $\mathbb{Z}H_\beta$. Since every finite group is polycyclic-by-finite and every group automorphism is of finite order, we get the following result about this kind of Nil-groups:

Corollary. *Let G be a finite group of order n and let α and β be group automorphisms of finite order m and m'. The group $\mathrm{Nil}_i(\mathbb{Z}G;\mathbb{Z}G_\alpha,\mathbb{Z}G_\beta)$ is an $(n\cdot m\cdot m')$-torsion group for $i\in\mathbb{Z}$.*

The result that Nil-groups of finite groups are torsion groups was already known in some cases. For Bass Nil-groups Weibel proved that if G is a finite group of order n, then $\mathrm{Nil}_i(\mathbb{Z}G)$ is n-torsion for $i\geq 0$ [Wei81]. The torsion result is known not to be sharp. Harmon proved that $\mathrm{Nil}_0(\mathbb{Z}G)$ and $\mathrm{Nil}_{-1}(\mathbb{Z}G)$ are the trivial group if n is square-free [Har87]. If $i\leq -2$, Bass Nil-groups of $\mathbb{Z}G$ are known to vanish.

Connolly and Prassidis [CP02] took an approach which goes back to Carter [Car80] and Farrell and Jones [FJ95] to prove that $\mathrm{Nil}_{-1}(\mathbb{Z}G;\alpha)$ is torsion. Kuku and Tang generalized this concept to prove that $\mathrm{Nil}_i(\mathbb{Z}G,\alpha)$ are n-torsion groups for $i \geq -1$ [KT03]. In their paper it is also proven that $\mathrm{Nil}_i(\mathbb{Z}G,\alpha)$ is the trivial group for $i \leq -2$.

In the paper of Kuku and Tang it is proven that $\mathrm{Nil}_0(\mathbb{Z}H, \mathbb{Z}[G_1 - H], \mathbb{Z}[G_2 - H])$ is torsion. Note that in this paper the Nil-groups appearing in the decomposition of Waldhausen are denoted by $\widetilde{\mathrm{Nil}}_i^W(R; R^\alpha, R^\beta)$ and called Waldhausen Nil-groups.

It was not known that the groups $\mathrm{Nil}_i(\mathbb{Z}H, \mathbb{Z}[G_1 - H], \mathbb{Z}[G_2 - H])$ are torsion groups for i bigger than zero, as was incorrectly stated in [LR04].

Another important application is that for an arbitrary group G the groups $\mathrm{Nil}_i(\mathbb{Q}G;\alpha)$ and $\mathrm{Nil}_i(\mathbb{Q}G;\mathbb{Q}G_\alpha,\mathbb{Q}G_\beta)$ are almost torsion free.

Theorem. *Let G be a arbitrary group and let α and β be group automorphisms of finite order m and m'.*

1. *The group $\mathbb{Z}[1/m] \otimes_{\mathbb{Z}} \mathrm{Nil}_i(\mathbb{Q}G;\alpha)$ is an $\mathbb{Q}$-algebra and therefore torsion free for $i \in \mathbb{Z}$. If G is a polycyclic-by-finite group, then $\mathrm{Nil}_i(\mathbb{Q}G;\alpha)$ is the trivial group.*

2. *The group $\mathbb{Z}[1/(m \cdot m')] \otimes_{\mathbb{Z}} \mathrm{Nil}_i(\mathbb{Q}G;\mathbb{Q}G_\alpha,\mathbb{Q}G_\beta)$ is a $\mathbb{Q}$-algebra and therefore torsion free for $i \in \mathbb{Z}$. If G is a polycyclic-by-finite group, then $\mathrm{Nil}_i(\mathbb{Q}G;\mathbb{Q}G_\alpha,\mathbb{Q}G_\beta)$ is the trivial group.*

The Farrell-Jones Conjecture

Before we start discussing the effect of the torsion results on the Farrell-Jones Conjecture, let us briefly recall the relevant notions. Let $\mathcal{F}$ be a family of subgroups of a group G. A G-CW-complex, all whose isotropy groups belong to $\mathcal{F}$ and whose H-fixed point sets are contractible for all $H \in \mathcal{F}$, is called a *classifying space for the family* $\mathcal{F}$ and will be denoted by $E_{\mathcal{F}}(G)$. Such a space always exists and is unique up to G-homotopy because it is characterized by the property that for any G-CW-complex X, all whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map from X to $E_{\mathcal{F}}(G)$. These spaces were introduced by tom Dieck [tD72, tD87]. Lück wrote a survey concerning recent results [Lüc03].

Suppose we are given a family of subgroups $\mathcal{F}$ and a subfamily $\mathcal{F}' \subseteq \mathcal{F}$. By the universal property of $E_{\mathcal{F}}(G)$ we obtain a map $E_{\mathcal{F}'}(G) \to E_{\mathcal{F}}(G)$ which is unique up to G-homotopy. Thus for every G-homology theory $\mathcal{H}_*^G$ we obtain a *relative assembly map*

$$A_{\mathcal{F}' \to \mathcal{F}} \colon \mathcal{H}_*^G(E_{\mathcal{F}'}(G)) \to \mathcal{H}_*^G(E_{\mathcal{F}}(G)).$$

By the universal property we obtain for every chain of families

$$\mathcal{F}'' \subseteq \mathcal{F}' \subseteq \mathcal{F}$$

the following commutative diagram:

$$\mathcal{H}_i^G(E_{\mathcal{F}''}G) \xrightarrow{A_{\mathcal{F}''\to\mathcal{F}'}} \mathcal{H}_i^G(E_{\mathcal{F}'}G) \xrightarrow{A_{\mathcal{F}'\to\mathcal{F}}} \mathcal{H}_i^G(E_{\mathcal{F}}G), \quad A_{\mathcal{F}''\to\mathcal{F}} \colon \mathcal{H}_i^G(E_{\mathcal{F}''}G) \to \mathcal{H}_i^G(E_{\mathcal{F}}G).$$

We denote the family of finite subgroups by $\mathcal{FIN}$, the family of virtually cyclic subgroups by $\mathcal{VCY}$ and the family of all subgroups by $\mathcal{ALL}$.

In [DL98] it is explained how the K-theory spectrum of a ring R, in the sequel denoted by $\mathbf{K}_R$, gives an equivariant homology theory, in the sequel denoted by $H_i^?(-;\mathbf{K}_R)$.

The Farrell-Jones conjecture [FJ93] predicts that the *assembly map*

$$A_{\mathcal{VCY}\to\mathcal{ALL}} \colon H_i^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R) \to H_i^G(E_{\mathcal{ALL}}(G);\mathbf{K}_R) \cong H_i^G(\mathrm{pt};\mathbf{K}_R) \cong K_i(RG)$$

is an isomorphism. Assuming the Farrell-Jones conjecture is true, the computation of $K_i(RG)$ reduces to the computation of $H_i^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R)$. This is much easier since here we can use standard methods form algebraic topology such as spectral sequences, Mayer-Vietoris sequences and Chern characters. The Farrell-Jones conjecture is known to be true for a wide class of groups. For a survey on the Farrell-Jones conjecture see for example [LR04].

It is known that the relative assembly map

$$A_{\mathcal{FIN}\to\mathcal{VCY}} \colon H_i^G(E_{\mathcal{FIN}};\mathbf{K}_R) \to H_i^G(E_{\mathcal{VCY}};\mathbf{K}_R)$$

is split injective [Bar03]. In the last section, we prove that rationally this relative assembly map is an isomorphism. The main ingredient of the proof is that Nil-groups of finite groups are torsion. Since the $\mathcal{FIN} \to \mathcal{ALL}$-assembly map factors over the relative assembly map we obtain the following statement.

Theorem. *Let G be a group for which the Farrell-Jones conjecture is known to be true. Then the rationalized $\mathcal{FIN}$-assembly map*

$$A_{\mathcal{FIN}} \colon H_i^G(E_{\mathcal{FIN}};\boldsymbol{K}_R) \otimes \mathbb{Q} \to K_n(\mathbb{Z}G) \otimes \mathbb{Q}$$

is an isomorphism.

This result is of special importance because the smaller the family $\mathcal{F}$ of subgroups is, the easier it is to compute $H_i^G(E_{\mathcal{F}}(G);\mathbf{K}_R)$.

This thesis is built up in the following manner. In the first chapter we recall the basic definitions of End- and Nil-groups and END- and NIL-categories. The concept of End- and Nil-groups END and NIL-categories is also extended.

The second chapter is devoted to the proof that $\mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}[t]} \mathrm{Nil}_i(R;X,Y,Z,W)$ and $\mathrm{Nil}_i(R_s;X_s,Y_s,Z_s,W_s)$ are isomorphic. The proof is an application of the long exact localization sequence of NIL-categories, which is developed in Section 2.3. This sequence is similar to the localization sequence of algebraic K-theory [Gra76]. To get the sequence, we need to apply the Resolution Theorem and for this application it is necessary to lift nilpotent endomorphisms (Section 2.1). The proof also requires knowledge of the shape of a colimit of a functor from a small filtering category into the category of exact categories. These colimits are studied in Section 2.2.

In Chapter 3, we first define an exact pairing which gives rise to an End_0-module structure on Nil-groups (Section 3.1). In the second section, we restrict to a class of Nil-groups containing Nil-groups of the form $\mathrm{Nil}_i(RG;\alpha)$ and $\mathrm{Nil}_i(RG;RG_\alpha,RG_\beta)$ where α and β are inner automorphism. We define Frobenius and Verschiebung operations on these kind of Nil-groups and prove that certain relations are fulfilled (Section 3.2). The relations have two important applications. First of all, they give the non finiteness results. Secondly, we use them to prove in Section 3.4 that the $\mathrm{End}_0(R)$-module structure can be extended to a Witt vector-module structure.

In the first section of Chapter 4, we combine the results of Chapter 2 and 3 to sharpen the localization results for Nil-groups. The main application of these improved results are the torsion results of Section 4.4. To get the torsion results, we generalize the transfer and induction maps of algebraic K-theory to Nil-groups (Section 4.2). Section 4.3 is concerned with the question whether we can find for a given group G an $n \in \mathbb{N}$ such that the Nil-groups of $\mathbb{Z}[1/n]G$ vanish. The main result is that for a polycyclic-by-finite group we can always find such an n. In Section 4.5 we proof that rationally the relative assembly from the family of finite subgroup to the family of virtually cyclic subgroups is an isomorphism. The main ingredient of the proof is the torsion results of Section 4.4.

Notations and Conventions

In the following, all rings are assumed to be associative and to have a unit. Ring homomorphisms are always unital. If not stated otherwise, modules are always assumed to be right modules.

A ring is usually denoted by R; only if we have an algebra over a commutative ring in mind, the ring is denoted by Λ.

A bimodule is called left free if it is free considered as a left module. Left flat is defined similarly.

The letters $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{I}$, $\mathcal{M}$ and $\mathcal{P}$ are reserved for categories.

Acknowledgements

1 End- and Nil-Groups

Various kinds of End- and Nil-groups have been defined [Bas68, Far72, Gra77, Wal78a, Wal78b, Wal73] as subgroups of the K-groups of END- and NIL-categories. In the following section, we give the general definition of an End- and Nil-group and of an END and NIL-category.

1.1 End-Groups and END-categories

We begin by generalizing End-groups and END-categories.

Definition 1.1.1 (END-category). Let $\mathcal{A}$ be an abelian category and let $\mathrm{A}\colon \mathcal{A} \to \mathcal{A}$ be an exact functor. Let $\mathcal{C} \subseteq \mathcal{A}$ be a full subcategory which is *closed under extension*, i.e. if

$$0 \longrightarrow M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow 0$$

is exact and both M_0 and M_2 belong to $\mathcal{C}$ then M_1 belongs to $\mathcal{C}$. Let $\mathrm{END}(\mathcal{C};\mathrm{A})$ be the following category. Objects are pairs (M, m) consisting of an object M of $\mathcal{C}$ and a morphism

$$m\colon M \to \mathrm{A}(M)$$

in $\mathcal{A}$. A morphism $f\colon (M, m) \to (M', m')$ in $\mathrm{END}(\mathcal{C};\mathrm{A})$ consist of a morphism $f\colon M \to M'$ in $\mathcal{C}$ satisfying $m' \circ f = \mathrm{A}(f) \circ m$, i.e. the following diagram commutes.

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ \mathrm{A}(M) & \xrightarrow{\mathrm{A}(f)} & \mathrm{A}(M'). \end{array}$$

Let $\mathrm{S}\colon \mathrm{END}(\mathcal{C};\mathrm{A}) \to \mathcal{C}$ be the forgetful functor mapping (M, m) to M and a morphism $f\colon (M, m) \to (M', m')$ to $f\colon M \to M'$. A sequence

$$(M_0, m_0) \xrightarrow{f} (M_1, m_1) \xrightarrow{g} (M_2, m_2)$$

is called *exact* at (M_1, m_1) if the sequence

$$M_0 \xrightarrow{f} M_1 \xrightarrow{g} M_2$$

in $\mathcal{C}$ obtained by applying S is exact at M_1. Short exact sequences, surjective and injective morphism in $\mathrm{END}(\mathcal{C};\mathrm{A})$ are defined in the obvious way.

Proposition 1.1.2. *Let the notation be as in the preceding definition. We additionally assume that $\mathcal{C}$ has a small skeleton. The category* $\mathrm{END}(\mathcal{A};A)$ *is an abelian category and the category* $\mathrm{END}(\mathcal{C};A)$ *is an exact category.*

Proof. One easily cheeks the required identities. □

Let $\mathrm{S}\colon \mathrm{END}(\mathcal{C};\mathrm{A}) \to \mathcal{C}$ be the forgetful functor sending an object (M,m) to M and let $\mathrm{T}\colon \mathcal{C} \to \mathrm{END}(\mathcal{C};\mathrm{A})$ be the functor sending M to $(M,0)$. The functors S and T are exact and we have $\mathrm{S}\circ\mathrm{T} = \mathrm{id}$. Thus we get maps

$$K_i(S)\colon K_i\big(\mathrm{END}(\mathcal{C};\mathrm{A})\big) \to K_i(\mathcal{C})$$
$$K_i(T)\colon K_i(\mathcal{C}) \to K_i\big(\mathrm{END}(\mathcal{C};\mathrm{A})\big)$$

such that $K_i(\mathrm{S})\circ K_i(\mathrm{T}) = \mathrm{id}$. We define

$$\mathrm{End}_i(\mathcal{C};\mathrm{A}) := \mathrm{Ker}\,\big(K_i(\mathrm{S})\colon K_i\big(\mathrm{END}(\mathcal{C};\mathrm{A})\big) \to K_i(\mathcal{C})\big)$$

for $i \geq 0$. These groups are called the *End-groups of* $\mathcal{C}$ *and* A.

We continue by defining maps between End-groups. Suppose that we have for $i = 0,1$ abelian categories $\mathcal{A}_i$ together with full subcategories $\mathcal{C}_i \subseteq \mathcal{A}_i$ which are closed under extension and exact functors $\mathrm{A}_i\colon \mathcal{A}_i \to \mathcal{A}_i$. Suppose that $u\colon \mathcal{A}_0 \to \mathcal{A}_1$ is a functor and $U\colon u\circ\mathrm{A}_0 \to \mathrm{A}_1\circ u$ is a natural transformation of exact functors such that u sends an object of $\mathcal{C}_0$ to an object of $\mathcal{C}_1$. Then we obtain an exact functor

$$\mathrm{END}(u,U)\colon\ \mathrm{END}(\mathcal{C}_0,\mathrm{A}_0) \to \mathrm{END}(\mathcal{C}_1,\mathrm{A}_1)$$

which sends an object $m\colon M \to \mathrm{A}(M)$ to the object given by the composition

$$u(M) \xrightarrow{u(m)} u(\mathrm{A}_0(M)) \xrightarrow{U(M)} \mathrm{A}_1(u(M)).$$

A morphism $f\colon (M,m) \to (M',m')$ is sent to the morphism $u(M,m) \to u(M',m')$ whose underlying morphism in $\mathcal{C}$ is $u(f)\colon u(M) \to u(M')$. This is a well defined functor since the following diagram commutes.

$$\begin{array}{ccccc} u(M) & \xrightarrow{u(m)} & u(\mathrm{A}_0(M)) & \xrightarrow{U(M)} & \mathrm{A}_1(u(M)) \\ \downarrow{\scriptstyle u(f)} & & \downarrow{\scriptstyle u(\mathrm{A}(f))} & & \downarrow{\scriptstyle \mathrm{A}(u(f))} \\ u(M') & \xrightarrow{u(m')} & u(\mathrm{A}_0(M')) & \xrightarrow{U(M')} & \mathrm{A}_1(u(M')). \end{array}$$

Since also the diagram

$$\begin{array}{ccc} \mathrm{END}(\mathcal{C}_0,\mathrm{A}_0) & \xrightarrow{\mathrm{END}(u,U)} & \mathrm{END}(\mathcal{C}_1,\mathrm{A}_1) \\ {\scriptstyle \mathrm{S}}\downarrow & & \downarrow{\scriptstyle \mathrm{S}} \\ \mathcal{C}_0 & \xrightarrow{\quad u \quad} & \mathcal{C}_1 \end{array}$$

commutes we obtain for the given pair (u,U) a homomorphism

$$\mathrm{End}_i(u,U)\colon\ \mathrm{End}_i(\mathcal{C}_0,\mathrm{A}_0) \to \mathrm{End}_i(\mathcal{C}_1,\mathrm{A}_1).$$

1.1.1 End(R)-Groups

In the following we show how the definition of $\mathrm{End}_i(R)$ fits into the given setting.

Definition 1.1.3 ($\mathrm{END}(\mathcal{C})$). Let $\mathcal{C}$ be an exact category and let id be the identity functor. We define $\mathrm{END}(\mathcal{C})$ to be $\mathrm{END}(\mathcal{C};\mathrm{id})$.

Remark 1.1.4. The objects of $\mathrm{END}(\mathcal{C})$ are pairs (M,ν) where M is an object in $\mathcal{C}$ and ν is an endomorphism of M. Maps from (M,ν) to (M',ν') are morphisms $f\colon M \to M'$ such that $f \circ \nu = \nu' \circ f$.

Definition 1.1.5 ($\mathcal{M}^{all}(R)$,$\mathcal{P}(R)$). Let R be a ring.

1. We define $\mathcal{M}^{all}(R)$ to be the category of all right modules.

2. We define $\mathcal{P}(R)$ to be the category of all finitely generated projective right R-modules.

As usual, the category $\mathrm{END}\big(\mathcal{P}(R)\big)$ is denoted by $\mathrm{END}(R)$.

Definition 1.1.6 (Higher End-groups). Let R be a ring. For $i \geq 0$, we define

$$\mathrm{End}_i(R) := \mathrm{End}_i(\mathcal{P}(R);\mathrm{id}).$$

The *cone ring* $\Lambda\mathbb{Z}$ of $\mathbb{Z}$ is the ring of matrices over $\mathbb{Z}$ such that every column and every row contains only finitely many non-zero entries. The *suspension ring* $\Sigma\mathbb{Z}$ is the quotient of $\Lambda\mathbb{Z}$ by the ideal of finite matrices. For a natural number $n \geq 2$ we define inductively $\Sigma^n\mathbb{Z} = \Sigma\Sigma^{n-1}\mathbb{Z}$. For an arbitrary ring R we define the n-fold suspension ring $\Sigma^n R$ to be $\Sigma^n\mathbb{Z} \otimes R$. For an R bimodule X we define $\Sigma^n X$ to be the $\Sigma^n R$-bimodule $\Sigma^n R \otimes X$ where the right $\Sigma^n R$-module structure is given by $x \cdot z \otimes r = zxr$ for $x \in X$, $z \in \Sigma^n\mathbb{Z}$ and $r \in R$.

Definition 1.1.7 (Lower End-groups). Let R be a ring. For $i < 0$, we define

$$\mathrm{End}_i(R) := \mathrm{End}_0(\mathcal{P}(\Sigma^{-i}R);\mathrm{id}).$$

1.1.2 Waldhausen End-Groups

Definition 1.1.8 ($\mathrm{END}(\mathcal{C};X,Y)$). Let R be a ring and let X and Y be left flat R-bimodules. Let $\mathcal{C}$ be a subcategory of $\mathcal{M}^{all}(R)$ which is closed under extension. Define

$$\mathrm{F}_{X,Y}\colon \mathcal{C} \times \mathcal{C} \to \mathcal{M}^{all}(R) \times \mathcal{M}^{all}(R)$$

by sending (M,N) to $(N \otimes X, M \otimes Y)$. We define $\mathrm{END}(\mathcal{C};X,Y)$ to be the category $\mathrm{END}(\mathcal{C} \times \mathcal{C};\mathrm{F}_{X,Y})$.

The category $\mathrm{END}(\mathcal{C};X,Y)$ is called *Waldhausen* END*-category.*

Remark 1.1.9. Objects of $\mathrm{END}(\mathcal{C}; X, Y)$ are quadruples (M, N, m, n) with M and N objects in $\mathcal{C}$ and m and n are R-module morphisms

$$\begin{aligned} m\colon M &\longrightarrow N \otimes X \\ n\colon N &\longrightarrow M \otimes Y. \end{aligned}$$

A morphism from (M, N, m, n) to (M', N', m', n') consists of module homomorphisms $f\colon M \to M'$ and $g\colon N \to N'$ such that

$$\begin{array}{ccc} M & \xrightarrow{m} & N \otimes X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g\otimes \mathrm{id}_X} \\ M' & \xrightarrow{m'} & N' \otimes X \end{array}$$

and a similar diagram involving the morphisms n and n' commutes.

As usual, the category $\mathrm{END}\big(\mathcal{P}(R); X, Y\big)$ is denoted by $\mathrm{END}(R; X, Y)$.

Definition 1.1.10 (Waldhausen End-groups of generalized free products). Let R be a ring and let X and Y be left flat R-bimodules. We define *Waldhausen End-groups*, for $i \geq 0$, by

$$\mathrm{End}_i(R; X, Y) := \mathrm{End}_i(R; \mathrm{F}_{X,Y})$$

and for $i < 0$ by

$$\mathrm{End}_i(R; X, Y) := \mathrm{End}_0(\Sigma^{-i}R; \mathrm{F}_{\Sigma^{-i}X, \Sigma^{-i}RY}.)$$

1.2 Nil-Groups and NIL-categories

In the sequel the full subcategory of $\mathrm{END}(\mathcal{C}; \mathrm{A})$ consisting of "nilpotent" endomorphisms will become important.

Definition 1.2.1 (NIL-category). Let the notation be as in Definition 1.1.1, let A^0 be the identity functor $\mathcal{C} \to \mathcal{C}$ and for a natural number $\ell \geq 1$ let A^ℓ be the ℓ-fold composite $\mathrm{A} \circ \mathrm{A} \circ \cdots \circ \mathrm{A}$. We define $\mathrm{Fr}_\ell\colon \mathrm{END}(\mathcal{C}; \mathrm{A}) \to \mathrm{END}(\mathcal{C}; \mathrm{A}^\ell)$ to be the exact functor which sends an object $m\colon M \to \mathrm{A}(M)$ to the object given by the composite

$$\mathrm{A}^0(M) = M \xrightarrow{\mathrm{A}^0(m)=m} \mathrm{A}(M) \xrightarrow{\mathrm{A}(m)} \mathrm{A}^2(M) \xrightarrow{\mathrm{A}^2(m)} \cdots \xrightarrow{\mathrm{A}^{\ell-1}(m)} \mathrm{A}^\ell(M)$$

and a morphism $f\colon (M, m) \to (M', m')$ in $\mathrm{END}(\mathcal{C}; \mathrm{A})$ to the morphism $\mathrm{Fr}_\ell(M) \to \mathrm{Fr}_\ell(M')$ given by the underling morphism $f\colon M \to M'$ in $\mathcal{C}$. An object (M, m) in $\mathrm{END}(\mathcal{C}; \mathrm{A})$ is called *nilpotent* if for some natural number $\ell \geq 1$ the map $\mathrm{Fr}_\ell(M, m)$ is given by the zero morphism from M to $\mathrm{A}^\ell(M)$. We call the smallest number with this property the *nilpotency degree* of (M, m). Let $\mathrm{NIL}(\mathcal{C}; \mathrm{A})$ be the full subcategory of $\mathrm{END}(\mathcal{C}; \mathrm{A})$ given by nilpotent objects.

Proposition 1.2.2. *Let the notation be as in the preceding definition. We additionally assume that $\mathcal{C}$ has a small skeleton. The category $\mathrm{NIL}(\mathcal{A};A)$ is an abelian category and the category $\mathrm{NIL}(\mathcal{C};A)$ is an exact category.*

Proof. One easily checks the required identities. □

Let $\mathrm{S}\colon \mathrm{NIL}(\mathcal{C};\mathrm{A}) \to \mathcal{C}$ be the forgetful functor sending an object (M,m) to M and let $\mathrm{T}\colon \mathcal{C} \to \mathrm{NIL}(\mathcal{C};\mathrm{A})$ be the functor sending M to $(M,0)$. The functors S and T are exact and we have $\mathrm{S}\circ\mathrm{T} = \mathrm{id}$. Thus we get maps

$$K_i(\mathrm{S})\colon K_i\big(\mathrm{NIL}(\mathcal{C};\mathrm{A})\big) \to K_i(\mathcal{C})$$
$$K_i(\mathrm{T})\colon K_i(\mathcal{C}) \to K_i\big(\mathrm{NIL}(\mathcal{C};\mathrm{A})\big)$$

such that $K_i(\mathrm{S})\circ K_i(\mathrm{T}) = \mathrm{id}$. We define

$$\mathrm{Nil}_i(\mathcal{C};\mathrm{A}) := \mathrm{Ker}\,\Big(K_i(\mathrm{S})\colon K_i\big(\mathrm{NIL}(\mathcal{C};\mathrm{A})\big) \to K_i(\mathcal{C})\Big)$$

for $i \geq 0$. These groups are called the *Nil-groups of $\mathcal{C}$ and A.*

Note that by similar reasons as above we obtain for a pair (u,U) a morphism

$$\mathrm{Nil}_i(u,U)\colon\ \mathrm{Nil}_i(\mathcal{C}_0,\mathrm{A}_0) \to \mathrm{Nil}_i(\mathcal{C}_1,\mathrm{A}_1).$$

1.2.1 Bass Nil-Groups

The $\mathrm{Nil}_i(R)$-groups appearing in the Fundamental Theorem of algebraic K-theory are subgroups of the K-groups of the category of nilpotent endomorphisms of finitely generated projective R-modules. We use the setup of the preceding section to extent this definition to nilpotent endomorphisms of an arbitrary exact category (compare [Bas68, Chapter 12.6]).

Definition 1.2.3 ($\mathrm{NIL}(\mathcal{C})$). Let $\mathcal{C}$ be an exact category and id the identity functor. We define $\mathrm{NIL}(\mathcal{C})$ to be $\mathrm{NIL}(\mathcal{C};\mathrm{id})$. The category $\mathrm{NIL}(\mathcal{C})$ is called *Bass* NIL*-category.*

Remark 1.2.4. The objects of $\mathrm{NIL}(\mathcal{C})$ are pairs (M,ν) where M is an object in $\mathcal{C}$ and ν is an nilpotent endomorphism of M. Maps from (M,ν) to (M',ν') are morphisms $f\colon M \to M'$ such that $f\circ\nu = \nu'\circ f$.

As usual, the category $\mathrm{NIL}\,\big(\mathcal{P}(R)\big)$ is denoted by $\mathrm{NIL}(R)$.

Definition 1.2.5 (Bass Nil-groups). Let R be a ring. We define *Bass Nil-groups*, for $i \geq 0$, by

$$\mathrm{Nil}_i(R) := \mathrm{Nil}_i(R;\mathrm{id})$$

and for $i < 0$ by

$$\mathrm{Nil}_i(R) := \mathrm{Nil}_0(\Sigma^{-i}R;\mathrm{id}).$$

Remark 1.2.6. Notice that the given definition for lower Nil-groups coincides with the definition of NK given by Bass [Bas68]. This can be seen since for an arbitrary ring R and a natural number $i < 0$ we have $K_i(R) = K_0(\Sigma^{-i}R)$ and $\Sigma R[t] = (\Sigma R)[t]$.

1.2.2 Farrell Nil-Groups

Farrell and Hsiang showed that for the calculation of the K-groups of a twisted Laurent polynomial ring similar Nil-groups can be defined [FH70]. These Nil-groups are subgroups of the K-groups of the category of semilinear nilpotent endomorphisms of finitely generated projective modules. In the following, this concept is extended to arbitrary modules.

Definition 1.2.7 ($\mathrm{NIL}(\mathcal{C}; X)$). Let R be a ring and let X be a left flat R-bimodule, let $\mathcal{C}$ be a subcategory of $\mathcal{M}^{all}(R)$ which is closed under extension and let F_X be the exact functor from $\mathcal{P}(R)$ to $\mathcal{M}^{all}(R)$ which is induced by tensoring with X. We define $\mathrm{NIL}(\mathcal{C}; X)$ to be $\mathrm{NIL}(\mathcal{C}; \mathrm{F}_X)$. The category $\mathrm{NIL}(\mathcal{C}; X)$ is called *Farrell* NIL-*category*.

Remark 1.2.8. Objects of $\mathrm{NIL}(\mathcal{C}; X)$ are pairs (M, ν) with M a module in $\mathcal{C}$ and ν an R-module homomorphism

$$\nu \colon M \to M \otimes X$$

such that ℓ-fold composition

$$M \xrightarrow{\quad \nu \quad} M \otimes X \xrightarrow{\quad \nu\otimes \mathrm{id} \quad} M \otimes X \otimes X \xrightarrow{\quad \nu\otimes \mathrm{id}\otimes\mathrm{id} \quad} \cdots$$

is zero after finitely map steeps.

Morphisms from (M, ν) to (M', ν') are module homomorphism $f \colon M \to M'$ such that $\nu' \circ f = (f \otimes \mathrm{id}) \circ \nu$.

As usual, the category $\mathrm{NIL}\big(\mathcal{P}(R); X\big)$ is denoted by $\mathrm{NIL}(R; X)$. The bimodules appearing in the decomposition of Farrell and Hsiang are of the form $X = R$ where the left R-module structure is given by multiplication and the right R-module structure comes from an automorphism α of the ring R. These kind of bimodules are denoted by R_α and $\mathrm{NIL}(R; R_\alpha)$ is denoted by $\mathrm{NIL}(R; \alpha)$. Now we define Farrell Nil-groups $\mathrm{Nil}_i(R; \alpha)$ appearing in the decomposition of the K-groups of a twisted Laurent polynomial ring.

Definition 1.2.9 (Farrell Nil-groups). Let α be an automorphism of a ring R. We define *Farrell Nil-groups*, for $i \geq 0$, by

$$\mathrm{Nil}_i(R; \alpha) := \mathrm{Nil}_i(R; \mathrm{F}_{R_\alpha})$$

and for $i < 0$ by

$$\mathrm{Nil}_i(R; \alpha) := \mathrm{Nil}_0(\Sigma^{-i} R; \mathrm{F}_{\Sigma^{-i} R_\alpha}).$$

Remark 1.2.10. Notice that the given definition for lower Farrell Nil-groups coincides with the usual definition of lower Farrell Nil-groups. Again, this can be seen since for an arbitrary ring R and an arbitrary endomorphism α of R we have $\Sigma R_\alpha[t] = (\Sigma R)_{\mathrm{id}\otimes\alpha}[t]$.

1.2.3 Waldhausen Nil-Groups of Generalized Free Products

Waldhausen defines Nil-groups which relate the K-groups of a generalized free product of two rings to the K-groups of the ground rings [Wal78a, Wal78b]. These kind of Nil-groups are subgroups of the K-groups of categories of certain morphisms of finitely generated projective modules. In the following, this concept is extended to arbitrary modules.

Definition 1.2.11 ($\mathrm{NIL}(\mathcal{C}; X, Y)$)**.** Let R be a ring and let X and Y be left flat R-bimodules. Let $\mathcal{C}$ be a subcategory of $\mathcal{M}^{all}(R)$. Define

$$\mathrm{F}_{X,Y} \colon \mathcal{C} \times \mathcal{C} \to \mathcal{M}^{all}(R) \times \mathcal{M}^{all}(R)$$

by sending (M, N) to $(N \otimes X, M \otimes Y)$. We define $\mathrm{NIL}(\mathcal{C}; X, Y)$ to be the category $\mathrm{NIL}(\mathcal{C} \times \mathcal{C}; \mathrm{F}_{X,Y})$.

The category $\mathrm{NIL}(\mathcal{C}; X, Y)$ is called *Waldhausen* NIL*-category of generalized free products*.

Remark 1.2.12. Objects of $\mathrm{NIL}(\mathcal{C}; X, Y)$ are quadruples (M, N, m, n) with M and N objects in $\mathcal{C}$ and m and n are R-module morphisms

$$m \colon M \longrightarrow N \otimes X$$
$$n \colon N \longrightarrow M \otimes Y$$

such that the sequences

$$M \xrightarrow{\quad m \quad} N \otimes X \xrightarrow{\quad n\otimes\mathrm{id} \quad} M \otimes Y \otimes X \xrightarrow{m\otimes\mathrm{id}\otimes\mathrm{id}} \cdots$$

and

$$N \xrightarrow{\quad n \quad} M \otimes Y \xrightarrow{\quad m\otimes\mathrm{id} \quad} N \otimes X \otimes Y \xrightarrow{n\otimes\mathrm{id}\otimes\mathrm{id}} \cdots$$

are zero after finitely many steeps.

A morphism from (M, N, m, n) to (M', N', m', n') consists of module homomorphisms $f \colon M \to M'$ and $g \colon N \to N'$ such that

$$\begin{array}{ccc} M & \xrightarrow{m} & N \otimes X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g\otimes\mathrm{id}_X} \\ M' & \xrightarrow{m'} & N' \otimes X \end{array}$$

and a similar diagram involving the morphisms n and n' commutes.

As usual, the category $\mathrm{NIL}\big(\mathcal{P}(R); X, Y\big)$ is denoted by $\mathrm{NIL}(R; X, Y)$. In the following, we define the Nil-groups $\mathrm{Nil}_i(R; X, Y)$ appearing in the decomposition of Waldhausen.

Definition 1.2.13 (Waldhausen Nil-groups of generalized free products). Let R be a ring and let X and Y be left flat R-bimodules. We define *Waldhausen Nil-groups of generalized free products*, for $i \geq 0$, by

$$\mathrm{Nil}_i(R; X, Y) := \mathrm{Nil}_i(R; \mathrm{F}_{X,Y})$$

and for $i < 0$ by

$$\mathrm{Nil}_i(R; X, Y) := \mathrm{Nil}_0(\Sigma^{-i} R; \mathrm{F}_{\Sigma^{-i} X, \Sigma^{-i} Y}).$$

1.2.4 Waldhausen Nil-Groups of Generalized Laurent Extensions

Waldhausen defines Nil-groups which relate the K-groups of a generalized Laurent extension of two rings to the K-groups of the ground rings [Wal78a, Wal78b]. Nil-groups of this kind are subgroups of the K-groups of categories of certain morphisms of finitely generated projective modules. In the following, this concept is extended to arbitrary modules.

Definition 1.2.14 ($\mathrm{NIL}(\mathcal{C}; X, Y, Z, W)$). Let R be a ring and let X, Y, Z and W be left flat R-bimodules. Let $\mathcal{C}$ be a subcategory of the category $\mathcal{M}^{all}(R)$ which is closed under extension. Define

$$\mathrm{F}_{X,Y,Z,W} \colon \mathcal{C} \times \mathcal{C} \to \mathcal{M}^{all}(R) \times \mathcal{M}^{all}(R)$$

by sending (M, N) to $(N \otimes X \oplus M \otimes Z, M \otimes Y \oplus N \otimes W)$. We define $\mathrm{NIL}(\mathcal{C}; X, Y, Z, W)$ to be the category $\mathrm{NIL}(\mathcal{C} \times \mathcal{C}; \mathrm{F}_{X,Y,Z,W})$. The category $\mathrm{NIL}(\mathcal{C}; X, Y, Z, W)$ is called *Waldhausen* NIL*-category of generalized Laurent extensions*.

Remark 1.2.15. Objects of $\mathrm{NIL}(\mathcal{C}; X, Y, Z, W)$ are quadruples (M, N, m, n), with M and N objects in $\mathcal{C}$ and m and n are R-module homomorphisms

$$m \colon M \longrightarrow N \otimes X \oplus M \otimes Z$$
$$n \colon N \longrightarrow M \otimes Y \oplus N \otimes W$$

such that the sequences

$$M \xrightarrow{\ m\ } \begin{matrix} N \otimes X \\ \oplus \\ M \otimes Z \end{matrix} \xrightarrow{\begin{matrix} n \otimes \mathrm{id} \\ \oplus \\ m \otimes \mathrm{id} \end{matrix}} \begin{matrix} \begin{pmatrix} M \otimes Y \\ \oplus \\ N \otimes W \end{pmatrix} \otimes X \\ \bigoplus \\ \begin{pmatrix} N \otimes X \\ \oplus \\ M \otimes Z \end{pmatrix} \otimes Z \end{matrix} \longrightarrow \cdots$$

and

$$N \xrightarrow{\ n\ } \begin{matrix} M \otimes Y \\ \oplus \\ N \otimes W \end{matrix} \xrightarrow{\begin{matrix} m \otimes \mathrm{id} \\ \oplus \\ n \otimes \mathrm{id} \end{matrix}} \begin{matrix} \begin{pmatrix} N \otimes X \\ \oplus \\ M \otimes Z \end{pmatrix} \otimes Y \\ \bigoplus \\ \begin{pmatrix} M \otimes Y \\ \oplus \\ N \otimes W \end{pmatrix} \otimes W \end{matrix} \longrightarrow \cdots$$

are zero after finitely many steeps.

A morphism from (M, N, m, n) to (M', N', m', n') consists of module homomorphisms $f\colon M \to M'$ and $g\colon N \to N'$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ m\ } & N \otimes X \oplus M \otimes Z \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g \otimes \mathrm{id} \oplus f \otimes \mathrm{id}} \\ M' & \xrightarrow{\ m'\ } & N' \otimes X \oplus M' \otimes Z \end{array}$$

and a similar diagram for n and n' commutes.

As usual, $\mathrm{NIL}\big(\mathcal{P}(R); X, Y, Z, W\big)$ is denoted by $\mathrm{NIL}(R; X, Y, Z, W)$. In the following, we define the Nil-groups appearing in the decomposition of Waldhausen.

Definition 1.2.16 (Waldhausen Nil-groups of generalized Laurent extensions). Let R be a ring and let X, Y, Z and W be left flat R-bimodules. We define *Waldhausen Nil-groups of generalized Laurent extensions*, for $i \geq 0$, by

$$\mathrm{Nil}_i(R; X, Y, Z, W) := \mathrm{Nil}_i(\mathcal{P}(R); \mathrm{F}_{X,Y,Z,W})$$

and for $i < 0$ by

$$\mathrm{Nil}_i(R; X, Y, Z, W) := \mathrm{Nil}_0(\mathcal{P}(\Sigma^{-i}R), \mathrm{F}_{\Sigma^{-i}X, \Sigma^{-i}Y, \Sigma^{-i}Z, \Sigma^{-i}W}).$$

Remark 1.2.17. Note that the $\mathrm{NIL}(R; X, Y, Z, W)$-category is a generalization of the other kinds of NIL-categories. To see this, let X and Y be the trivial module and let Z and W be R seen as an R-bimodule. The category $\mathrm{NIL}(R; 0, 0, R, R)$ is equivalent to $\mathrm{NIL}(R) \times \mathrm{NIL}(R)$. Let α be an R-automorphism. The category $\mathrm{NIL}(R; 0, 0, R_\alpha, R_{\alpha^{-1}})$ is equivalent to $\mathrm{NIL}(R; \alpha) \times \mathrm{NIL}(R; \alpha^{-1})$. To obtain the Waldhausen NIL-category of generalized free products let X and Y be R-bimodules. The category $\mathrm{NIL}(R; X, Y, 0, 0)$ is equivalent to $\mathrm{NIL}(R; X, Y)$.

On the other hand we can ask when $\mathrm{NIL}(R; X, Y, Z, W)$ is completely determined by $\mathrm{NIL}(R; X)$, $\mathrm{NIL}(R; Y)$ and $\mathrm{NIL}(R; X, Y)$. This question is answered by Waldhausen [Wal78a, page 157]. Let $\alpha, \beta\colon C \hookrightarrow A$ be pure and free maps of rings. The category $\mathrm{NIL}(C; {}_\alpha A'_\alpha, {}_\beta A_\beta, {}_\beta A_\alpha, {}_\alpha A_\beta)$ has as retracts the categories $\mathrm{NIL}(C; {}_\alpha A'_\alpha, {}_\beta A''_\beta)$, $\mathrm{NIL}(C; {}_\beta A_\alpha)$ and $\mathrm{NIL}(C; {}_\alpha A_\beta)$. If α and β are both isomorphisms, $\mathrm{NIL}(C; {}_\alpha A'_\alpha, {}_\beta A_\beta, {}_\beta A_\alpha, {}_\alpha A_\beta)$ reduces to that product.

2 The Behavior of Nil-Groups under Localization

In this chapter, we prove that Nil-groups behave nicely under localization. The main result is Theorem 2.4.1 saying that

$$\mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}[t]} \mathrm{Nil}_i(R;X,Y,Z,W) \cong \mathrm{Nil}_i(R_s;X_s,Y_s,Z_s,W_s)$$

for all $i \in \mathbb{Z}$. The $\mathbb{Z}[t]$-module structure on $\mathrm{Nil}_i(R;X,Y,Z,W)$ is given by the map which is induced by the functor F_s. This result is an application of the long exact localization sequence of NIL-categories which is developed in Section 2.3. In Section 2.1 we provide basic homological facts about NIL-categories which are needed for the proof of the exactness of the localization sequence. The proof also requires knowledge about the shape of a colimit in the category of exact categories. Colimits of this kind are studied in Section 2.2.

2.1 Homological Facts about NIL-categories

To prove the exactness of the localization sequence of Nil-categories, basic homological facts about NIL-categories are needed. In this section, we provide these facts.

Lemma 2.1.1. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{C}$ be a subcategory which is closed under extension. Let $A\colon \mathcal{C} \to \mathcal{A}$ be an exact functor. The subcategory $\mathrm{NIL}(\mathcal{C};A)$ of $\mathrm{END}(\mathcal{C};A)$ is closed under extension.*

Proof. Let

$$0 \longrightarrow (P,p) \longrightarrow (M,m) \longrightarrow (P',p') \longrightarrow 0$$

be a short exact sequence in $\mathrm{END}(\mathcal{C};\mathrm{A})$ with (P,q) and (P',p') in $\mathrm{NIL}(\mathcal{C};\mathrm{A})$. Let (P,p) be of nilpotency degree L and let (P',p') be of nilpotency degree L'. We are going to show that (M,m) is of nilpotency degree at most $L+L'$. For $n \in \mathbb{N}$ we define $m^n\colon M \to \mathrm{A}^n$ to be the composite of

$$m^n\colon M \xrightarrow{m} \mathrm{A}(M) \xrightarrow{\mathrm{A}(m)} \cdots \xrightarrow{\mathrm{A}^{n-1}(m)} \mathrm{A}^n(M).$$

The morphism p^n and p'^n are defined similarly. Since A is exact, we get a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \longrightarrow & M & \longrightarrow & P' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p^{L'}} & & \downarrow{\scriptstyle m^{L'}} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathrm{A}^{L'}(P) & \longrightarrow & \mathrm{A}^{L'}(M) & \longrightarrow & \mathrm{A}^{L'}(P') & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle (m^{L'})^L} & & \downarrow{\scriptstyle (p'^{L'})^L} & & \\
0 & \longrightarrow & \mathrm{A}^{L'+L}(P) & \longrightarrow & \mathrm{A}^{L'+L}(M) & \longrightarrow & \mathrm{A}^{L'+L}(P') & \longrightarrow & 0.
\end{array}$$

The obvious diagram chase gives that $m^{L'}\colon M \to \mathrm{A}^{L'}(M)$ factors over $\mathrm{A}^{L'}(P)$ and therefore the lemma. □

Lemma 2.1.2. *Let the notation be as in the preceding lemma. Let*

$$0 \longrightarrow (M', m') \xrightarrow{f'} (M, m) \xrightarrow{f''} (M'', m'') \longrightarrow 0$$

be a short exact sequence in $\mathrm{NIL}(\mathcal{C}; A)$. *Let* (P', p') *and* (P'', p'') *be projective objects in* $\mathrm{NIL}(\mathcal{C}; A)$ *admitting surjections*

$$\pi'_P\colon (P', p') \twoheadrightarrow (M', m')$$
$$\pi''_P\colon (P'', p'') \twoheadrightarrow (M'', m'').$$

Since $P' \oplus P''$ *is projective, we can construct a surjection*

$$\pi_P\colon P' \oplus P'' \twoheadrightarrow M$$

out of π'_P *and* π''_P.

There is an object $(P' \oplus P'', p)$ *in* $\mathrm{NIL}(\mathcal{C}; A)$ *such that the diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (P', p') & \longrightarrow & (P' \oplus P'', p) & \longrightarrow & (P'', p'') & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi'_P} & & \downarrow{\scriptstyle \pi_P} & & \downarrow{\scriptstyle \pi''_P} & & \\
0 & \longrightarrow & (M', m') & \xrightarrow{f'} & (M, m) & \xrightarrow{f''} & (M'', m'') & \longrightarrow & 0
\end{array}$$

commutes.

Proof. Consider the map

$$\sigma := m \circ \pi_P - \mathrm{A}(\pi_P) \circ \begin{pmatrix} p' & \\ & p'' \end{pmatrix}$$

from $P' \oplus P''$ to $\mathrm{A}(M)$. A diagram chase gives us that the image of σ is in the image of $\mathrm{A}(P' \oplus 0)$ under $\mathrm{A}(\pi_P)$. Since P'' is projective, we obtain a lift σ_P such that the diagram

$$\begin{array}{ccc} P'' & \xrightarrow{\sigma_P} & \mathrm{A}(P') \\ & \searrow^{\sigma} & \downarrow{\scriptstyle \mathrm{A}(\pi_P)} \\ & & \mathrm{A}(M) \end{array}$$

commutes. We define

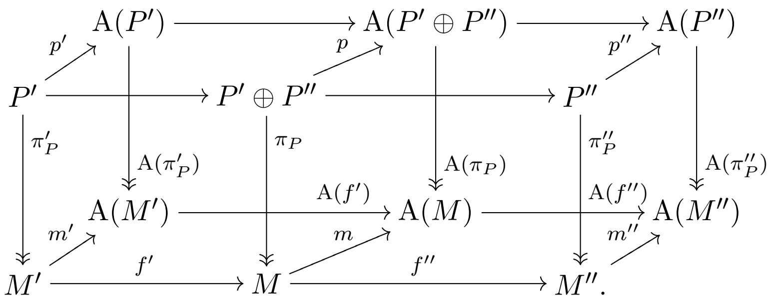
$$p := \begin{pmatrix} p' & \sigma_P \\ & p'' \end{pmatrix}.$$

Another diagram chase gives us that P' is in the kernel of σ. Thus the diagram

$$\begin{array}{ccc} P' \oplus P'' & \xrightarrow{p} & \mathrm{A}(P' \oplus P'') \\ \downarrow{\scriptstyle \pi_P} & & \downarrow{\scriptstyle \mathrm{A}(\pi_P)} \\ M & \xrightarrow{m} & \mathrm{A}(M) \end{array}$$

commutes.

The following diagram commutes by construction of p and the reasoning given above.



The object $(P' \oplus P'', p)$ is, by Proposition 2.1.1, in $\mathrm{NIL}(\mathcal{C}; \mathrm{A})$. □

Lemma 2.1.3. *Assume the following conditions for an abelian category $\mathcal{A}$ with subcategory $\mathcal{C}$ which is closed under extension and an exact functor A:*

1. *For any object M in $\mathcal{C}$ there exists an object P in $\mathcal{P}$ which is projective and admits an epimorphism $c\colon P \to M$;*

2. *Any object in $\mathcal{A}$ can be written as a colimit of a directed system $\{M_i | i \in I\}$ such that each structure map is injective and each object belongs to $\mathcal{C}$;*

3. *The functor A commutes with colimits over directed systems and injective structure maps;*

4. *For any epimorphism $f\colon M \to N$ in $\mathcal{A}$ for which M belongs to $\mathcal{C}$ also N belongs to $\mathcal{C}$;*

5. *Suppose that the object N in $\mathcal{A}$ is the colimit of a directed system $\{M_i | i \in I\}$ such that each structure map is injective and each object belongs to $\mathcal{C}$. Let $k\colon N' \to N$ be an injective morphism with $N' \in \mathcal{C}$. Then there exists an index $i \in I$ such that the image of k is contained in the image of $N_i \to N$.*

Then we can find for any object (M, m) in $\mathrm{NIL}(\mathcal{C}; A)$ an object (P, p) in $\mathrm{NIL}(\mathcal{C}; A)$ together with an epimorphism from (P, p) onto (M, m) such that P is projective. The nilpotency degree of (P, p) is smaller or equal to one plus the nilpotency degree of (M, m).

Proof. The result is proven by induction on the nilpotency degree L. For $L = 1$ we have $m = 0$. Let P_M be a projective object surjecting onto M. The trivial map from P_M to $\mathrm{A}(P_M)$ is a lift of m.

For the induction step from L to $L + 1$ let (M, m) be an object of $\mathrm{NIL}(\mathcal{C}; \mathrm{A})$ of nilpotency degree $L + 1$. For $\ell \geq 1$ define $m^\ell\colon M \to \mathrm{A}^\ell(M)$ to be the composite

$$m^\ell\colon M \xrightarrow{m} \mathrm{A}(M) \xrightarrow{\mathrm{A}(m)} \cdots \xrightarrow{\mathrm{A}^{\ell-1}(m)} \mathrm{A}^\ell(M).$$

Let N be the kernel of m^ℓ. Consider the exact sequence

$$0 \longrightarrow N \xrightarrow{i} M \xrightarrow{m^\ell} \mathrm{A}^\ell(M) \longrightarrow 0$$

where i is the inclusion. Since F is exact, the following sequence is also exact.

$$0 \longrightarrow \mathrm{A}(N) \xrightarrow{\mathrm{A}(i)} \mathrm{A}(M) \xrightarrow{\mathrm{A}(m^\ell)} \mathrm{A}^{\ell+1}(M) \longrightarrow 0.$$

The image of m is contained in the kernel of m^ℓ, since $m^{\ell+1} = 0$. The given exact sequence implies that the image of m is contained in the image of $\mathrm{A}(i)\colon \mathrm{A}(N) \to \mathrm{A}(M)$. By assumption we can write

$$N = \varinjlim_{i \in I} N_i$$

for a directed system with injective structure maps such that each N_i belongs to $\mathcal{C}$. By assumption the canonical map

$$\varinjlim_{i \in I} \mathrm{A}(N_i) = \mathrm{A}(N)$$

is bijective. The image of $m\colon M \to \mathrm{A}(M)$ belongs, by assumption, to $\mathcal{C}$ since M belongs to $\mathcal{C}$. Thus we can find an index $i_0 \in I$ such that the image of m belongs to the image of $\mathrm{A}(N_{i_0}) \to \mathrm{A}(N)$. Put $M_{\mathrm{Im}} = N_{i_0}$.

The object M_{Im} has the property that the image of m is contained in $\mathrm{A}(M_{\mathrm{Im}})$. This has two implications. First of all we can restrict m to a morphisms of M_{Im}. The object $(M_{\mathrm{Im}}, m|_{M_{\mathrm{Im}}})$ is of nilpotency degree L. By construction, M_{Im} is an object in

$\mathrm{NIL}(\mathcal{C};\mathrm{A})$. Thus, by our induction hypothesis, we get a projective object $P_{M_{\mathrm{Im}}}$ and a nilpotent lift $\tilde{m}_{\mathrm{Im}}$ such that the diagram

$$\begin{array}{ccc} P_{M_{\mathrm{Im}}} & \xrightarrow{\tilde{m}_{\mathrm{Im}}} & \mathrm{A}(P_{M_{\mathrm{Im}}}) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ M_{\mathrm{Im}} & \xrightarrow{m|_{M_{\mathrm{Im}}}} & \mathrm{A}(M_{\mathrm{Im}}) \\ \hookdownarrow & & \hookdownarrow \\ M & \xrightarrow{m} & \mathrm{A}(M) \end{array}$$

commutes.

Let P_M be a projective object surjecting onto M. The second implication is that since P_M is projective there is a map $\tilde{m}_M$ making the diagram

$$\begin{array}{ccc} P_M & \xrightarrow{\tilde{m}_M} & \mathrm{A}(P_{M_{\mathrm{Im}}}) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ M & \xrightarrow{m} & \mathrm{A}(M_{\mathrm{Im}}) \end{array}$$

commutative. Plugging these maps together in the matrix

$$\tilde{m} := \begin{pmatrix} 0 & 0 \\ \tilde{m}_M & \tilde{m}_{\mathrm{Im}} \end{pmatrix}$$

we get a commutative diagram

$$\begin{array}{ccc} P_M \oplus P_{\mathrm{Im}_M} & \xrightarrow{\tilde{m}} & \mathrm{A}(P_M \oplus P_{\mathrm{Im}_M}) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ M & \xrightarrow{\quad m \quad} & \mathrm{A}(M). \end{array}$$

This lift of m is nilpotent since $\tilde{m}_{\mathrm{Im}}$ is. Thus the object $\big(P_M \oplus P_{\mathrm{Im}_M}, \tilde{m}\big)$ is a projective object of nilpotency degree $L+2$ surjecting onto (M, m). □

Definition 2.1.4 ($\mathcal{M}(R)$). Let R be a ring. The category $\mathcal{M}(R)$ is the category of all finitely generated right R-modules.

Corollary 2.1.5. *Let R be a ring, let X, Y, Z and W be left flat R-bimodules and let F be one of the functors id, F_X, $F_{X,Y}$ and $F_{X,Y,Z,W}$. Let (M,m) be an object in* $\mathrm{NIL}\big(\mathcal{M}(R);F\big)$. *There exists an object (P,p) in the subcategory* $\mathrm{NIL}\big(\mathcal{P}(R);F\big)$ *admitting a surjection onto (M,m).*

Proof. One easily checks that F has the required properties. □

2.2 Colimits in the Category of Exact Categories

It is well known that the category of exact categories is closed under colimits over small filtering categories [Qui73, page 104]. In the following, we review basic definitions and properties of these kind of colimits.

Definition 2.2.1 (Filtering category). A nonempty category $\mathcal{I}$ is called *filtering* if the following conditions hold.

1. For every $i, j \in \operatorname{Obj} \mathcal{I}$, there are arrows

$$\begin{array}{ccc} i & & \\ & \searrow & \\ & & k \\ & \nearrow & \\ j & & \end{array}$$

   to some $k \in \operatorname{Obj} \mathcal{I}$.

2. For every two parallel arrows $u, v \colon i \rightrightarrows j$, there is an arrow $w \colon j \to k$ such that $w \circ u = w \circ v$.

The dual notion is called *cofiltering*.

Example 2.2.2. Let ω be the category where objects are non-negative integers and there is exactly one morphism from i to j if and only if i is smaller or equal to j. The category ω is small and filtering.

Let us briefly recall the definition of a colimit [ML98, page 67].

Definition 2.2.3. Let $\mathcal{I}$ and $\mathcal{E}$ be categories.

1. Denote the functor category from $\mathcal{I}$ to $\mathcal{E}$ by $\mathcal{E}^{\mathcal{I}}$. The *diagonal functor*

$$\Delta \colon \mathcal{E} \to \mathcal{E}^{\mathcal{I}}$$

   sends each object x to the constant functor Δx, whose value on each $i \in \mathcal{I}$ is the object x.

2. Let F be a functor from $\mathcal{I}$ into $\mathcal{E}$. The *colimit* of F is an object in $\mathcal{E}$, written $\varinjlim_{i \in \mathcal{I}} \mathrm{F}(i)$, together with a universal natural transformation S from F to the functor $\Delta \varinjlim_{i \in \mathcal{I}} \mathrm{F}(i)$.

3. The natural transformation S induces maps from $\mathrm{F}(i)$ into $\varinjlim_{i \in \mathcal{I}} \mathrm{F}(i)$. These maps are denoted by S_i and called *structure maps*.

In an arbitrary category a colimit need not to exist.

Proposition 2.2.4. *Let $\mathcal{I}$ be a small filtering category and let F be a functor from $\mathcal{I}$ into the category of exact categories. The colimit over F exists in the category of exact categories.*

Quillen constructs such a colimit [Qui73, page 104]. This is done by defining a new category $\mathcal{C}$ by

$$\operatorname{Obj}\mathcal{C} = \varinjlim_{i\in\mathcal{I}} \operatorname{Obj}\mathrm{F}(i)$$
$$\operatorname{Ar}\mathcal{C} = \varinjlim_{i\in\mathcal{I}} \operatorname{Ar}\mathrm{F}(i).$$

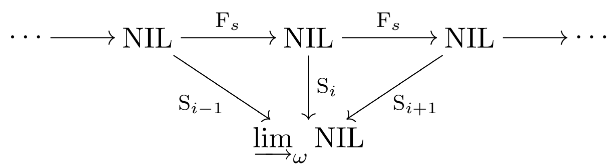
The colimit in the definition given above is the set-theoretical colimit. A sequence in $\mathcal{C}$ is exact if we can find an exact sequence in some $\mathrm{F}(i)$ mapping onto it under the structure maps.

Example 2.2.5. In Section 2.3, colimits of functors from ω to the category of exact categories $\mathcal{E}$ of the following type will become important. Let F_s be an exact endofunctor of an exact category NIL. Define a functor F from ω to $\mathcal{E}$ by sending every object of ω to the category NIL and the morphisms from i to j to the $(i-j)$-fold suspension of F_s.

A colimit in this case consist of an exact category $\varinjlim_\omega \mathrm{NIL}$ and structure functors

$$\mathrm{S}_i \colon \mathrm{NIL} \to \varinjlim_\omega \mathrm{NIL}$$

which are universal among functors making the diagram



commutative.

Note that every object and morphism in $\varinjlim_\omega \mathrm{NIL}$ needs to be in the image of some S_i. In the following, we use the short hand notation x_i for $\mathrm{S}_i(x)$.

Let $i \le j$. Morphisms in $\varinjlim_\omega \mathrm{NIL}$ from x_i to y_j are for example elements of $\varinjlim_{n\in\omega} \operatorname{mor}_{\mathrm{NIL}}\big(\mathrm{F}_s^{n+i-j}(x), \mathrm{F}_s^n(y)\big)$. A sequence is exact if it becomes exact after applying F_s sufficiently often. In particular, if F_s has the property that for every morphism f the image $\mathrm{F}_s(f)$ is an isomorphism iff f is a isomorphism, a morphism in $\varinjlim_\omega \mathrm{NIL}$ is an isomorphism iff the inverse image under the structure functors contains just isomorphisms.

2.3 The Long Exact Localization Sequence of NIL-Categories

The behavior of algebraic K-theory under localization is well understood. If $s \in R$ is a central non zero divisor, we define $H_s(R)$ to be the exact category of finitely generated R-modules with the property that modules have projective dimension smaller or equal to 1 and are s-torsion. Quillen proved [Gra76] that there is a long exact sequence

$$\cdots \longrightarrow K_{i+1}(R_s) \longrightarrow K_i\big(H_s(R)\big) \longrightarrow K_i(R) \longrightarrow K_i(R_s) \longrightarrow \cdots .$$

This sequence is called *localization sequence of algebraic K-theory.* In the following, we develop a similar sequence for NIL-categories. More precisely, Theorem 2.3.7 states that the sequence

$$\cdots \longrightarrow K_{i+1}\big(\operatorname{NIL}(R_s; X_s, Y_s, Z_s, W_s)\big) \longrightarrow K_i\big(H_s(R)\big) \oplus K_i\big(H_s(R)\big) \xrightarrow{\mathrm{I}}$$
$$\xrightarrow{\mathrm{I}} \varinjlim K_i\big(\operatorname{NIL}(R; X, Y, Z, W)\big) \xrightarrow{\mathrm{L}} K_i\big(\operatorname{NIL}(R_s; X_s, Y_s, Z_s, W_s)\big) \longrightarrow \cdots ,$$

is exact. The colimit is of the form considered in Example 2.2.5. To obtain this statement, we first prove that a similar sequence (Lemma 2.3.10) is exact. In the second part of this section the different K-groups appearing in the exact sequence of Lemma 2.3.10 are identified with the K-groups in the given sequence.

The proof that the localization sequence of NIL-categories is exact follows a proof of the exactness of the localization sequence of algebraic K-theory [Gra87, Sta89]. Let us briefly recall how the exactness of this sequence is proven.

Lemma 2.3.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be exact categories. The categories $\mathcal{C}$ and $\mathcal{D}$ can be seen as Waldhausen categories, by defining a map to be a weak equivalence if it is an isomorphism and to be a cofibration if it is an admissible monomorphism. Denoted by iso$\mathcal{C}$ and iso$\mathcal{D}$ the categories of weak equivalences and by co$\mathcal{C}$ and co$\mathcal{D}$ the categories of cofibrations. If not stated otherwise $\mathcal{C}$ and $\mathcal{D}$ are always considered with this Waldhausen category structure.*

Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor. We give $\mathcal{C}$ a new Waldhausen structure by defining a map to be a weak equivalence if it maps to a weak equivalence under F. Denote this new category of weak equivalences by $w\mathcal{C}$. Let $\operatorname{Ker}(F)$ be the kernel of F, i.e., the full Waldhausen subcategory of all objects C in $\mathcal{C}$ such that there exists a weak equivalence $0 \xrightarrow{\simeq} F(C)$ in $\mathcal{D}$.

If the functor F induces an equivalence

$$coS_r\mathcal{C} \cap wS_r\mathcal{C} \to isoS_r\mathcal{D}$$

after taking the nerve and realization for every $r \in \mathbb{N}$, then

$$\operatorname{Ker}(F) \longrightarrow \mathcal{C} \xrightarrow{F} \mathcal{D}$$

induces a long exact sequence on K-groups.

Proof. We denote the inclusion functor

$$\mathrm{I}\colon \operatorname{Ker(F)} \to \mathcal{C}$$

by I. Proposition 1.5.5. in [Wal85] gives that

$$\big|s_\bullet \operatorname{Ker}(\mathrm{F})\big| \xrightarrow{\mathrm{I}} \big|s_\bullet \mathcal{C}\big| \longrightarrow \big|s_\bullet S_\bullet\big(\operatorname{Ker}(\mathrm{F}) \xrightarrow{\mathrm{I}} \mathcal{C}\big)\big|$$

is a fibration up to homotopy. The main step is to prove that F identifies

$$\big|s_\bullet S_\bullet\big(\operatorname{Ker}(\mathrm{F}) \xrightarrow{\mathrm{I}} \mathcal{C}\big)\big|$$

with

$$\big|N_\bullet isoS_\bullet \mathcal{D}\big|,$$

where $N_\bullet$ is the $\bullet$-th part of the nerve of a category. First observe that by reversal of priorities we can replace the bisimplicial set

$$(m,n) \longmapsto s_m S_n\big(\operatorname{Ker}(\mathrm{F}) \xrightarrow{\mathrm{I}} \mathcal{C}\big)$$

by the equivalent bisimplicial set

$$(m,n) \longmapsto s_n\big(S_m \operatorname{Ker}(\mathrm{F}) \xrightarrow{\mathrm{I}} S_m\mathcal{C}\big).$$

Since Ker(F) is a Waldhausen subcategory of $\mathcal{C}$, the category $S_m \operatorname{Ker}(\mathrm{F})$ is a Waldhausen subcategory of $S_m\mathcal{C}$. Thus following Waldhausen [Wal85, page 344], we can replace

$$s_n\big(S_m \operatorname{Ker}(\mathrm{F}) \xrightarrow{\mathrm{I}} S_m\mathcal{C}\big)$$

by

$$F_n\big(S_m \operatorname{Ker}(\mathrm{F}), S_m\mathcal{C}\big).$$

The functor F induces a map of bisimplicial sets from

$$(m,n) \longmapsto F_n\big(S_m \operatorname{Ker}(\mathrm{F}), S_m\mathcal{C}\big)$$

to

$$(m,n) \longmapsto N_n\big(isoS_m\mathcal{D}\big).$$

The next step is to identify $F_n\big(S_m \operatorname{Ker}(\mathrm{F}), S_m\mathcal{C}\big)$ also with the nerve of a category. Consider the category

$$coS_m\mathcal{C} \cap wS_m\mathcal{C}.$$

The bisimplicial set

$$(m,n) \longmapsto F_n\big(S_m \operatorname{Ker}(\mathrm{F}), S_m\mathcal{C}\big)$$

is equivalent to

$$(m,n) \longmapsto N_n(coS_m\mathcal{C} \cap wS_m\mathcal{C}).$$

Thus by the realization lemma, to prove the statement it suffices to show that for each $r \geq 0$ the induced functor

$$\mathrm{F}_r\colon\, coS_r\mathcal{C} \cap wS_r\mathcal{C} \longrightarrow isoS_r\mathcal{D}$$

realizes to a homotopy equivalence. But this is one of our assumptions. □

Recall the categories which give rise to the localization sequence of algebraic K-theory.

Definition 2.3.2. Let R be a ring and let s be a central non zero divisor of R.

1. Let $\mathcal{P}^{\mathrm{Im}}(R_s)$ be the full exact subcategory of $\mathcal{P}(R_s)$ consisting of those objects isomorphic to $P \otimes_R R_s$ for some $P \in \mathcal{P}(R)$.
2. Let $\mathcal{P}^{d1}(R)$ be the exact category of finitely generated R-modules of projective dimension smaller or equal to 1 with the additional assumption that $P \otimes_R R_s \in \mathcal{P}^{\mathrm{Im}}(R_s)$ for all objects P in $\mathcal{P}^{d1}(R)$.

Remark 2.3.3. These categories are closed under extension in either the category of all R-modules or the category of all R_s-modules.

Let L be the functor from $\mathcal{P}^{d1}(R)$ into $\mathcal{P}^{\mathrm{Im}}(R_s)$ whose value at P is $P \otimes_R R_s$. Note that the kernel of L is the category $H_s(R)$. One way of proving the exactness of the localization sequence of algebraic K-theory is to apply Lemma 2.3.1 to

$$H_s(R) \longrightarrow \mathcal{P}^{d1}(R) \xrightarrow{\ \mathrm{L}\ } \mathcal{P}^{\mathrm{Im}}(R_s).$$

This is possible since we can apply Quillen's Theorem A to the functor

$$\mathrm{L}_r\colon\, coS_r\mathcal{P}^{d1}(R) \cap wS_r\mathcal{P}^{d1}(R) \to isoS_r\mathcal{P}^{\mathrm{Im}}(R_s),$$

by proving that the comma category

$$\mathrm{L}_r/M$$

is nonempty and cofiltering for every $M \in isoS_r\mathcal{P}^{\mathrm{Im}}$ [Sta89]. Note that if the functor L would not be surjective on isomorphism classes of objects, then the comma category is empty for a ceratin $M \in iso\mathcal{P}^{\mathrm{Im}}$ and therefore it is not possible to apply Quillen's Theorem A.

In the second step, $K_i\big(\mathcal{P}^{d1}(R)\big)$ is identified with $K_i\big(\mathcal{P}(R)\big)$ via the Resolution Theorem and $K_i\big(\mathcal{P}^{\mathrm{Im}}(R)\big)$ with $K_i\big(\mathcal{P}(R_s)\big)$ since $\mathcal{P}^{\mathrm{Im}}(R)$ is cofinal in $\mathcal{P}(R_s)$.

To transfer the ideas to the setting of the NIL-categories, we use the generalized notion of $\mathrm{NIL}(R; X, Y, Z, W)$ introduced in Chapter 1. Proposition 1.2.2 gives us that for left flat R-bimodules X, Y, Z and W the categories $\mathrm{NIL}\,\big(H_s(R); X, Y, Z, W\big)$, $\mathrm{NIL}\,\big(\mathcal{P}^{d1}(R); X, Y, Z, W\big)$ and $\mathrm{NIL}\,\big(\mathcal{P}^{\mathrm{Im}}(R); X, Y, Z, W\big)$ are exact categories.

As a next step, we need to define localization and inclusion functors on NIL-categories. The generalization of the inclusion functor is obvious. In the following, all different inclusion functors are denoted by I. For the definition of a localization functor a little bit more work is needed. If $s \in R$ is an central nonzero divisor, denote the exact functor from $\mathcal{M}^{all}(R)$ to $\mathcal{M}^{all}(R_s)$ given by localization at s by u_s. We denote the obvious natural transformation between the functors $u_s \circ \mathrm{F}_{X,Y,Z,W}$ and $\mathrm{F}_{X_s,Y_s,Z_s,W_S} \circ u_s$ by U_s.

Definition 2.3.4. Let R be a ring, let $s \in R$ be a central non zero divisor and let X, Y, Z and W be left flat R-bimodules.

1. Define
$$\mathrm{L}\colon \operatorname{NIL}(R; X, Y, Z, W) \to \operatorname{NIL}(R_s; X_s, Y_s, Z_s, W_s)$$
to be the functor induced by the tuple (u_s, U_s).
2. The functor L restricts to a functor from $\operatorname{NIL}\big(\mathcal{P}^{d1}(R); X, Y, Z, W\big)$ to $\operatorname{NIL}\big(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\big)$ which is also denoted by L.

We can not transfer the ideas of the proof of the long exact localization sequence of algebraic K-theory one to one to the NIL-categories. In general, we can not hope that

$$\operatorname{NIL}\big(H_s(R); X, Y, Z, W\big) \xrightarrow{\mathrm{I}} \operatorname{NIL}\big(\mathcal{P}^{d1}(R); X, Y, Z, W\big) \xrightarrow{\mathrm{L}} \operatorname{NIL}\big(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\big)$$

induces a long exact sequence in K-theory. The problem is that in the category $\operatorname{NIL}\big(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\big)$ there are no requirements on the nilpotent morphisms. This implies that the localization functor is not necessarily surjective on isomorphism classes of objects. Therefore the comma category

$$\mathrm{L}_1/M$$

might be empty for some $M \in \mathit{iso}S_1 \operatorname{NIL}\big(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\big)$. Hence we can not apply Quillen's Theorem A to the functor

$$\mathrm{L}_1\colon \mathit{co}S_1 \operatorname{NIL}\big(\mathcal{P}^{d1}(R); X, Y, Z, W\big) \cap \mathit{w}S_1 \operatorname{NIL}\big(\mathcal{P}^{d1}(R); X, Y, Z, W\big)$$
$$\to \mathit{iso}S_1 \operatorname{NIL}\big(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\big).$$

To get around this problem, we have to pass to the colimit.

Definition 2.3.5 (F_s). Let $\mathcal{C}$ be an exact subcategory of $\mathcal{M}^{all}(R)$ and let s be a central element in R. We define F_s to be the exact endofunctor of $\operatorname{NIL}(\mathcal{C}; X, Y, Z, W)$ defined by

$$\mathrm{F}_s\colon \operatorname{NIL}(\mathcal{C}; X, Y, Z, W) \to \operatorname{NIL}(\mathcal{C}; X, Y, Z, W)$$
$$(P, Q, p, q) \mapsto (P, Q, p \cdot s, q \cdot s).$$

Thus F_s is defined on all the different NIL-categories considered above. In the following, all colimits are colimits taken over the functor from the small filtering category ω into the category of exact categories which sends an object of ω to one of the different NIL-categories and the morphism from i to j to the $(j-i)$-fold composition of the functor F_s (compare Example 2.2.5).

Definition 2.3.6. Let R be ring, let $s \in R$ be a central non zero divisor and let X, Y, Z and W be R-bimodules.

1. Since the inclusion functor

$$\mathrm{I}\colon\ \mathrm{NIL}\left(H_s(R); X, Y, Z, W\right) \to \mathrm{NIL}\left(\mathcal{P}^{d1}(R); X, Y, Z, W\right)$$

   commutes with F_s, we get a functor

$$\mathrm{I}\colon\ \varinjlim \mathrm{NIL}\left(H_s(R); X, Y, Z, W\right) \to \varinjlim \mathrm{NIL}\left(\mathcal{P}^{d1}(R); X, Y, Z, W\right).$$

   By abuse of language both of these functors are denoted by I.

2. In the same way as above *localization functors*

$$\mathrm{L}\colon\ \varinjlim \mathrm{NIL}(R; X, Y, Z, W) \to \varinjlim \mathrm{NIL}(R_s; X_s, Y_s, Z_s, W_s)$$

   and

$$\mathrm{L}\colon\ \varinjlim \mathrm{NIL}\left(\mathcal{P}^{d1}(R); X, Y, Z, W\right) \to \varinjlim \mathrm{NIL}\left(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\right)$$

   are defined.

3. Since F_s is "invertible" on $\mathrm{NIL}(R_s; X_s, Y_s, Z_s, W_s)$ we get a map

$$\mathrm{L}\colon\ \varinjlim K_i\big(\mathrm{NIL}(R; X, Y, Z, W)\big) \to K_i\big(\mathrm{NIL}(R_s; X_s, Y_s, Z_s, W_s)\big).$$

Theorem 2.3.7. *Let R be a ring and let s be a central non zero divisor of R. Let X, Y, Z and W be left free R bimodules with $s \cdot x = x \cdot s$ for all elements $x \in X$ and similar assumptions for Y, Z and W.*

Localization at s induces a long exact sequence

$$\cdots \longrightarrow K_i\big(H_s(R)\big) \oplus K_i\big(H_s(R)\big) \xrightarrow{I} \varinjlim K_i\big(\mathrm{NIL}(R; X, Y, Z, W)\big) \xrightarrow{L}$$

$$\xrightarrow{L} K_i\big(\mathrm{NIL}(R_s; X_s, Y_s, Z_s, W_s)\big) \longrightarrow K_{i-1}\big(H_s(R)\big) \oplus K_{i-1}\big(H_s(R)\big) \longrightarrow \cdots,$$

where $K_0\big(\mathrm{NIL}(R; X, Y, Z, W)\big) \to K_0\big(\mathrm{NIL}(R_s; X_s, Y_s, Z_s, W_s)\big)$ is not necessarily surjective.

For the proof of this theorem, a whole bunch of lemmas is needed. Without further mentioning we will always use the notation of the preceding theorem.

Lemma 2.3.8. *The functor*

$$L\colon\ \varinjlim \mathrm{NIL}\left(\mathcal{P}^{d1}(R); X, Y, Z, W\right) \to \varinjlim \mathrm{NIL}\left(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\right)$$

is surjective on isomorphism classes of objects.

Proof. Let $(M, N, p, q)_k$ be an object of $\varinjlim \mathrm{NIL}\left(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\right)$. By construction, (M, N, p, q) is isomorphic to some element $(P \otimes R_s, Q \otimes R_s, p, q)$ where P and Q are finitely generated projective R-modules.

The modules P and Q are finitely generated and the relation $s \cdot x = x \cdot s$ holds for every $x \in X$ and similar conditions hold for the bimodules Y, Z and W. Thus

$$p(a \otimes \mathrm{id}) \cdot s^{\ell} = \sum_j q_j(a) \otimes_R \mathrm{id} \otimes_{R_s} \mathrm{id} \otimes_R x_j(a) \otimes_R \mathrm{id} \oplus p_j(a) \otimes_R \mathrm{id} \otimes_{R_s} \mathrm{id} \otimes_R z_j(a) \otimes_R \mathrm{id}$$

$$q(b \otimes \mathrm{id}) \cdot s^{\ell} = \sum_j p_j(b) \otimes_R \mathrm{id} \otimes_{R_s} \mathrm{id} \otimes_R y_j(b) \otimes_R \mathrm{id} \oplus q_j(b) \otimes_R \mathrm{id} \otimes_{R_s} \mathrm{id} \otimes_R w_j(b) \otimes_R \mathrm{id}$$

for every $a \in P$ and $b \in Q$ and some $\ell \in \mathbb{N}$, $x_j(a) \in X$, $y_j(b) \in Y$, $z_j(a) \in Z$ and $w_j(b) \in W$. The maps

$$\begin{aligned} P &\to Q \otimes X \oplus P \otimes Z \\ a &\mapsto \sum q_j(a) \otimes x_j(a) \oplus p_j(a) \otimes z_j(a) \\ Q &\to P \otimes Y \oplus Q \otimes W \\ b &\mapsto \sum p_j(b) \otimes y_j(b) \oplus q_j(b) \otimes w_j(b) \end{aligned}$$

are not necessarily R-module morphisms. But, by construction, every relation is satisfied up to s-torsion. Since the modules P and Q are finitely generated projective and therefore finitely presented the maps defined above become an R-module homomorphism after multiplication with a certain multiple of s. Thus $\mathrm{F}_s^{\ell'}\big((M,N,p,q)\big)$ is isomorphic to an element in the image of the localization functor from $\mathrm{NIL}\big(\mathcal{P}^{d1}(R); X,Y,Z,W\big)$ to $\mathrm{NIL}\big(\mathcal{P}^{\mathrm{Im}}(R_s); X_s,Y_s,Z_s,W_s\big)$.

In the preceding section, we have seen that $\mathrm{F}_s^{\ell'}\big((M,N,p,q)\big)_{k+\ell'}$ and $(M,N,p,q)_k$ represent the same object in $\varinjlim \mathrm{NIL}\big(\mathcal{P}^{\mathrm{Im}}(R_s); X_s,Y_s,Z_s,W_s\big)$. Thus $(M,N,p,q)_k$ is isomorphic to an element in the image of L. □

Lemma 2.3.9. *Let P and Q be finitely generated projective R-modules. Let (f,g) be a monomorphism in $\mathrm{NIL}\big(\mathcal{P}^{d1}(R); X,Y,Z,W\big)$ from (P,Q,p,q) to an arbitrary object. If the cokernel of f and g is s-torsion, then (f,g) is admissible.*

Proof. To prove this statement, first observe that the quadruple $(\mathrm{Coker}(f), \mathrm{Coker}(g), p|_{\mathrm{Coker}(f)}, q|_{\mathrm{Coker}(g)})$ is well-defined. The long exact sequence of Ext-groups associated to the maps f and g gives that $\mathrm{Coker}(f)$ and $\mathrm{Coker}(g)$ are of projective dimension smaller or equal to 1. Since $\mathrm{Coker}(f) \otimes_R R_s = \mathrm{Coker}(g) \otimes_R R_s = 0$, we get that $(\mathrm{Coker}(f), \mathrm{Coker}(g), p|_{\mathrm{Coker}(f)}, q|_{\mathrm{Coker}(g)})$ is an object in $\mathrm{NIL}\big(\mathcal{P}^{d1}(R); X,Y,Z,W\big)$. □

Lemma 2.3.10. *Let R be a ring and let s be a central non zero divisor. Let X, Y, Z and W be left free R bimodules with $s \cdot x = x \cdot s$ for all elements x in X and similar assumptions for Y, Z and W. Let I and L be the functors defined above. The sequence*

$$\varinjlim \mathrm{NIL}\big(H_s(R); X,Y,Z,W\big) \xrightarrow{I} \varinjlim \mathrm{NIL}\big(\mathcal{P}^{d1}(R); X,Y,Z,W\big) \xrightarrow{L}$$

$$\xrightarrow{L} \varinjlim \mathrm{NIL}\big(\mathcal{P}^{\mathrm{Im}}(R_s); X_s,Y_s,Z_s,W_s\big)$$

of functors induces a long exact sequence on K-theory.

Proof. For the proof, $\mathrm{NIL}\left(H_s(R); X, Y, Z, W\right)$, $\mathrm{NIL}\left(\mathcal{P}^{d1}(R); X, Y, Z, W\right)$ and $\mathrm{NIL}\left(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\right)$ are denoted by NIL^{H_s}, NIL^{d1} and $\mathrm{NIL}^{\mathrm{Im}}$.

The kernel of the functor $\mathrm{L}\colon \varinjlim \mathrm{NIL}^{d1} \to \varinjlim \mathrm{NIL}^{\mathrm{Im}}$ is $\varinjlim \mathrm{NIL}^{H_s}$. Thus to apply Lemma 2.3.1, it suffices to show that for each $M \in isoS_r \varinjlim \mathrm{NIL}^{\mathrm{Im}}$ the comma category L_r/M is contractible. This will follow since L_r/M is nonempty and cofiltering [BK72, page 318].

For $r = 0$, there is nothing to prove.

For $r = 1$, the objects of $\mathrm{L}_1/(M, N, m, n)_j$ can be seen in two different ways. One way is by simply taking an object to be a map f in $iso \varinjlim \mathrm{NIL}^{\mathrm{Im}}$ between $(M, N, m, n)_j$ and an object $\mathrm{L}\big((V, W, v, w)_i\big)$ with $(V, W, v, w)_i$ in $\varinjlim \mathrm{NIL}^{d1}$. Since F_s has the property that a morphism is an isomorphism iff the image under F_s is an isomorphism, these are the maps such that the inverse image of f under the structure functors are tuples (f_1, f_2) which are an isomorphism in $\mathrm{NIL}^{\mathrm{Im}}$. Since the diagram

$$\begin{array}{ccccc}
W \otimes X \oplus V \otimes Z & \hookrightarrow & W \otimes R_s \otimes X_s \oplus V \otimes R_s \otimes Z_s & \xrightarrow{f_2 \otimes \mathrm{id} \oplus f_1 \otimes \mathrm{id}} & N \otimes X_s \oplus M \otimes Z_s \\
\uparrow{\scriptstyle v} & & \uparrow{\scriptstyle v_s} & & \uparrow{\scriptstyle m} \\
V & \hookrightarrow & V \otimes R_s & \xrightarrow{f_1} & M
\end{array}$$

commutes we get that (f_1, f_2) is induced by a tuple of R-module homomorphism between (V, W, v, w) and (M, N, m, n) which become a isomorphisms under localization at s. Thus a second way to see an object in $\mathrm{L}_1/(M, N, m, n)_j$ is to see it as a morphism f from $\mathrm{L}\big((V, W, v, w)_i\big)$ to $(M, N, m, n)_j$ such that the inverse image of f under the structure functors are maps which are R-module homomorphisms which become isomorphisms under localization at s.

Arrows are morphisms in $\varinjlim \mathrm{NIL}^{d1}$ such that the inverse images of the structure functors are monomorphism such that triangles like

$$\begin{array}{ccc}
(V, W, v, w) & \xrightarrow{\quad g \quad} & (V', W', v', w') \\
& {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle f'} & \\
& (M, N, m, n), &
\end{array}$$

commutes and the the cokernel of g is in NIL^{H_s}.

The category $\mathrm{L}_1/(M, N, m, n)_j$ is non empty since L is surjective on isomorphism classes (Lemma 2.3.8).

Given two objects

$$\mathrm{L}\big((V, W, v, w)_i\big) \xrightarrow{f} (M, N, m, n)_j \xleftarrow{f'} \mathrm{L}\big((V', W', v', w')_{i'}\big)$$

to prove that $\mathrm{L}_1/(M, N, m, n)_j$ is cofiltering we need to find an object which maps on both. Since the category ω is filtering we can find an object k in ω such that the inverse image of f and f' under the k-th structure functor are induced by tuples

(f_1, f_2) and (f'_1, f'_2) of R-modules homomorphism which become isomorphisms under localization. Additional we can assume that $(M, N, m, n)_j$ is isomorphic to an object $\mathrm{L}\big((P_M, P_N, p_m, p_n)_k\big)$ with (P_M, P_N, p_m, p_n) in $\mathrm{NIL}(R; X, Y, Z, W)$. To avoid confusion, we assume that $j = i = i' = k$. For the general case, we have to multiply the nilpotent morphisms by a certain multiple of s.

With this assumption, we get R-module morphisms

$$V \xrightarrow{f_1} M \xleftarrow{f'_1} V'$$

and

$$W \xrightarrow{f_2} N \xleftarrow{f'_2} W'.$$

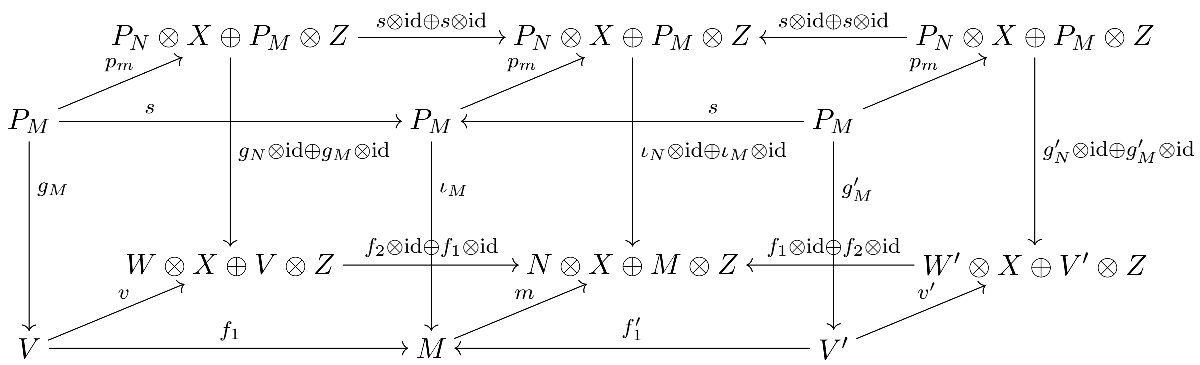
There exists an $s \in \{s^i\}_i$ such that we can find g_M, g'_M, g_N and g'_N making the diagrams

$$\begin{array}{ccccc} P_M & \xrightarrow{s} & P_M & \xleftarrow{s} & P_M \\ \downarrow{\scriptstyle g_M} & & \downarrow{\scriptstyle \iota_M} & & \downarrow{\scriptstyle g'_M} \\ V & \xrightarrow{f_1} & M & \xleftarrow{f'_1} & V' \end{array}$$

and

$$\begin{array}{ccccc} P_N & \xrightarrow{s} & P_N & \xleftarrow{s} & P_N \\ \downarrow{\scriptstyle g_N} & & \downarrow{\scriptstyle \iota_N} & & \downarrow{\scriptstyle g'_N} \\ W & \xrightarrow{f_2} & N & \xleftarrow{f'_2} & W' \end{array}$$

commute. The point is now that the diagram



and a similar diagram for N commutes after a slight modification. The lower part of the diagram commutes by construction of (f_1, f_2) and (f'_1, f'_2). The front and back parts commutes by construction of the g's. The middle part commutes by the definition of (P_M, P_N, p_m, p_n). The upper part of the diagram commutes since p_m is an R-linear map. The right and the left hand side will commute after a modification. In the following, the right hand side is treated, similar arguments work for the left hand side. Since $f_1 \otimes \mathrm{id} \oplus f_2 \otimes \mathrm{id}$ becomes an isomorphism after localization at s, the elements in the kernel need to be s-torsion. Thus the difference between $v' \circ g'_M$

and $(g'_N \otimes \mathrm{id} \oplus g'_M \otimes \mathrm{id}) \circ p_m$ need to be s-torsion. Since P_M is finitely generated, we can choose $s \in \{s^i\}_i$ such that $v' \circ g'_M \cdot s$ and $(g'_N \cdot s \otimes \mathrm{id} \oplus g'_M \cdot s \otimes \mathrm{id}) \circ p_m$ agree. Now replace g'_M, g'_N, ι_N and ι_M by $g'_M \cdot s$, $g'_N \cdot s$, $\iota_N \cdot s$ and $\iota_M \cdot s$. It is easily seen that the new diagram commutes.

Let g and g' be the maps in NIL^{d1} induced by g_M, g_N and g'_M, g'_N. By construction, these maps become isomorphisms after localization at s. Since P_M and P_N are projective, g and g' are monomorphisms with a cokernel which is s-torsion. Since in NIL^{d1} such a monomorphism is by Lemma 2.3.9 admissible, we get exact sequences

$$0 \longrightarrow (P_M, P_N, p_m, p_n) \xrightarrow{g} (V, W, v, w) \longrightarrow (H, K, h, k) \longrightarrow 0$$

$$0 \longrightarrow (P_M, P_N, p_m, p_n) \xrightarrow{g'} (V', W', v', w') \longrightarrow (H', K', h', k') \longrightarrow 0,$$

where (H, K, h, k) and (H', K', h', k') are in NIL^{H_s}. The image of this construction under the j-th structure functor constructs an object

$$(P_M, P_N, p_m, p_n)_j \xrightarrow{s} (P_M, P_N, p_m, p_n)_j \xrightarrow{\iota} (M, N, m, n)_j$$

in $\mathrm{L}_1/(M, N, m, n)_j$ which maps to the given objects

$$\mathrm{L}\big((V, W, v, w)_j\big) \xrightarrow{f} (M, N, m, n)_j$$

and

$$\mathrm{L}\big((V', W', v', w')_j\big) \xrightarrow{f'} (M, N, m, n)_j\ .$$

Suppose that we have two maps

$$h_1\colon \Big(\mathrm{L}\big((V, W, v, w)_i\big) \xrightarrow{f_k} (M, N, m, n)_j\Big) \longrightarrow \Big(\mathrm{L}\big((V', W', v', w')_{i'}\big) \xrightarrow{f'_{k'}} (M, N, m, n)_j\Big)$$

and

$$h_2\colon \Big(\mathrm{L}\big((V, W, v, w)_i\big) \xrightarrow{f_k} (M, N, m, n)_j\Big) \longrightarrow \Big(\mathrm{L}\big((V', W', v', w')_{i'}\big) \xrightarrow{f'_{k'}} (M, N, m, n)_j\Big).$$

By the same arguments as above, we can assume that $j = i = i' = k = k'$ and that the maps are induced by R-module morphisms in degree j. We get maps

$$h_1\colon \big((V, W, v, w) \xrightarrow{f} (M, N, m, n)\big) \longrightarrow \big((V', W', v', w') \xrightarrow{f'} (M, N, m, n)\big)$$

and

$$h_2\colon \big((V, W, v, w) \xrightarrow{f} (M, N, m, n)\big) \longrightarrow \big((V', W', v', w') \xrightarrow{f'} (M, N, m, n)\big).$$

To show that the category is cofiltering, we need to find a map g in $\mathrm{L}_1/(M, N, m, n)_j$ such that $h_1 \circ g = h_2 \circ g$. As above, we find g

$$g\colon \Big((P_M, P_N, p_m, p_n) \xrightarrow{s} (P_M, P_N, p_m, p_n) \xrightarrow{\iota} (M, N, m, n)\Big) \to \Big((V, W, v, w) \xrightarrow{f} (M, N, m, n)\Big).$$

Since all maps become isomorphisms after localization at s, elements of $\operatorname{Ker}(h_1 \circ g - h_2 \circ g)$ are s-torsion. Since P_M and P_N are projective and therefore s-torsion free, we get that $h_1 \circ g = h_2 \circ g$ as desired. Define an element in $\mathrm{L}_1/(M, N, m, n)_j$ which is the image of (P_M, P_N, p_m, p_n) under the j-th structure functor and a map g_j from $(P_M, P_N, p_m, p_n)_j \to (M, N, m, n)_j$ to $(V, W, v, w)_j \to (M, N, m, n)_j$. This map has the property that $h_1 \circ g_j = h_2 \circ g_j$.

For $r \geq 2$, the arguments can be extended in the following manner. An object of L_r/M amounts to a map f of a diagram

$$\begin{array}{ccccccc}
0 & \rightarrowtail & \mathrm{L}\big((V,W,v,w)^{1,0}_{i_{1,0}}\big) & \rightarrowtail & \cdots & \rightarrowtail & \mathrm{L}\big((V,W,v,w)^{r,0}_{i_{r,0}}\big) \\
 & & \downarrow\!\!\!\downarrow & & & & \downarrow\!\!\!\downarrow \\
 & & 0 & \rightarrowtail & \cdots & \rightarrowtail & \mathrm{L}\big((V,W,v,w)^{r,1}_{i_{r,1}}\big) \\
 & & & & & & \downarrow\!\!\!\downarrow \\
 & & & & & & \cdots \\
 & & & & & & \downarrow\!\!\!\downarrow \\
 & & & & 0 & \rightarrowtail & \mathrm{L}\big((V,W,v,w)^{r,r-1}_{i_{r,r-1}}\big) \\
 & & & & & & \downarrow\!\!\!\downarrow \\
 & & & & & & 0
\end{array}$$

to the diagram

$$\begin{array}{cccccccc}
M = & 0 & \rightarrowtail & (M,N,m,n)^{1,0}_{j_{1,0}} & \rightarrowtail & \cdots & \rightarrowtail & (M,N,m,n)^{r,0}_{j_{r,0}} \\
 & & & \downarrow\!\!\!\downarrow & & & & \downarrow\!\!\!\downarrow \\
 & & & 0 & \rightarrowtail & \cdots & \rightarrowtail & (M,N,m,n)^{r,1}_{j_{r,1}} \\
 & & & & & & & \downarrow\!\!\!\downarrow \\
 & & & & & & & \cdots \\
 & & & & & & & \downarrow\!\!\!\downarrow \\
 & & & & & 0 & \rightarrowtail & (M,N,m,n)^{r,r-1}_{j_{r,r-1}} \\
 & & & & & & & \downarrow\!\!\!\downarrow \\
 & & & & & & & 0,
\end{array}$$

where the inverse image of the maps in the different components under the structure functors are induced by R-module morphisms which become isomorphisms under localization at s. By the same arguments as above, we can assume that all the i's and j's are equal to some j and that there are $(P_M^{i,i-1}, P_N^{i,i-1}, p_m^{i,i-1}, p_n^{i,i-1}) \in \mathrm{NIL}(R;X,Y,Z,W)$ such that

$$\mathrm{L}\big((P_M^{i,i-1}, P_N^{i,i-1}, p_m^{i,i-1}, p_n^{i,i-1})\big) \cong (M,N,m,n)^{i,i-1}.$$

To show that L_r/M is non empty, an object with $(P_M^{i,i-1}, P_N^{i,i-1}, p_m^{i,i-1}, p_n^{i,i-1})$ in position $(i,i-1)$ is constructed inductively. To define components in the positions $(i+1,i-1)$, consider the exact sequences

$$(M,N,m,n)^{i,i-1} \rightarrowtail (M,N,m,n)^{i+1,i-1} \twoheadrightarrow (M,N,m,n)^{i+1,i}\ .$$

By Lemma 2.1.2 we can find an element $\big((P_M^{i,i-1} \oplus P_M^{i+1,i}) \otimes R_s, (P_N^{i,i-1} \oplus P_N^{i+1,i}) \otimes R_s, p_M, p_N\big)$ of $\mathrm{NIL}(R_s; X_s, Y_s, Z_s, W_s)$ such that

$$\begin{array}{ccc}
\mathrm{L}\big((P_M^{i,i-1}, P_N^{i,i-1}, p_m^{i,i-1}, p_n^{i,i-1})\big) & \xrightarrow{\cong} & (M,N,m,n)^{i,i-1} \\
\downarrow & & \downarrow \\
\big((P_M^{i,i-1} \oplus P_M^{i+1,i}) \otimes R_s, (P_N^{i,i-1} \oplus P_N^{i+1,i}) \otimes R_s, p_M, p_N\big) & \longrightarrow & (M,N,m,n)^{i+1,i-1} \\
\downarrow & & \downarrow \\
\mathrm{L}\big((P_M^{i+1,i}, P_N^{i+1,i}, p_m^{i+1,i}, p_n^{i+1,i})\big) & \xrightarrow{\cong} & (M,N,m,n)^{i+1,i}
\end{array}$$

commutes. By the same arguments as Lemma 2.3.8 is proven we can prove that the localization functor from $\varinjlim \mathrm{NIL}(R;X,Y,Z,W)$ to $\varinjlim \mathrm{NIL}(R_s;X_s,Y_s,Z_s,W_s)$ is surjective on isomorphism classes. This implies that we can find an object $(P_M^{(i+1,i-1)}, P_N^{(i+1,i-1)}, p_m^{(i+1,i-1)}, p_n^{(i+1,i-1)})$ in $\mathrm{NIL}(R;X,Y,Z,W)$ such that

$$\begin{aligned}
&\big((P_M^{i,i-1} \oplus P_M^{i+1,i}) \otimes R_s, (P_N^{i,i-1} \oplus P_N^{i+1,i}) \otimes R_s, p_M, p_N)\big)_j \\
&\qquad \cong \mathrm{L}\big((P_M^{(i+1,i-1)}, P_N^{(i+1,i-1)}, p_m^{(i+1,i-1)}, p_n^{(i+1,i-1)})\big)_k
\end{aligned}$$

W.l.o.g. we can assume that $k = j$ (otherwise multiply the nilpotent morphism by a certain multiple of s). The map from $(P_M^{(i+1,i-1)}, P_N^{(i+1,i-1)}, p_m^{(i+1,i-1)}, p_n^{(i+1,i-1)})$ to $(M,N,m,n)^{i+1,i-1}$ becomes an isomorphism after localization at s, because the other morphisms in the short exact sequence have this property. If we apply this construction r-times, we get an object of $S_r\,\mathrm{NIL}^{d1}$ and a map to the diagram M which becomes an isomorphism after localization at s. Thus L_r/M is non empty.

To show that L_r/M is cofiltering, assume there are two objects

$$P \xrightarrow{f} M \xleftarrow{f'} P'.$$

Again, we can assume that P, P' and M are induced by diagrams in $S_r\,\mathrm{NIL}^{d1}/S_r\,\mathrm{NIL}^{\mathrm{Im}}$ in degree j. The components of the diagram which induces P

are denoted by $(V,W,v,w)^{i,k}$ and the components of the diagram which induces P' are denoted by $(V',W',v',w')^{i,k}$. As above, we can construct for every part

$$(V,W,v,w)^{i,i-1} \xrightarrow{f_{i,i-1}} (M,N,m,n)^{i,i-1} \xleftarrow{f'_{i,i-1}} (V',W',v',w')^{i,i-1}$$

of the diagram an object $(P_M,P_N,p_m,p_n)^{i,i-1} \in \mathrm{NIL}(R;X,Y,Z,W)$ such that

$$\begin{array}{ccccc} & & (P_M,P_N,p_m,p_n)^{i,i-1} & & \\ & \swarrow & & \searrow & \\ (V,W,v,w)^{i,i-1} & \xrightarrow{f_{i,i-1}} & (M,N,m,n)^{i,i-1} & \xleftarrow{f'_{i,i-1}} & (V',W',v',w')^{i,i-1} \end{array}$$

commutes. In the same matter as above, an element in $S_r\,\mathrm{NIL}^{d1}$ is constructed which maps on both P, P'. The image under the j-th structure functor gives an element in L_r/M mapping on both.

Given two maps

$$h_1\colon \left(P \xrightarrow{f} M\right) \longrightarrow \left(P' \xrightarrow{f'} M\right)$$

$$h_2\colon \left(P \xrightarrow{f} M\right) \longrightarrow \left(P' \xrightarrow{f'} M\right),$$

as above, we find a diagram P_M in $S_r\,\mathrm{NIL}(R;X,Y,Z,W)$ and a map g

$$g\colon \left(\, P_M \xrightarrow{s} P_M \xrightarrow{\iota} M \,\right) \longrightarrow \left(\, P \xrightarrow{f} M \,\right)$$

such that $h_1 \circ g = h_2 \circ g$, using the argument above pointwise.

This proves that L_r/M is non empty and cofiltering and therefore the lemma. □

In the last part of this section we derive Theorem 2.3.7 out of Lemma 2.3.10. To do so we prove the following three identities

$$\begin{aligned} K_i\Big(\varinjlim \mathrm{NIL}\left(H_s(R);X,Y,Z,W\right)\Big) &\cong K_i\big(H_s(R)\times H_s(R)\big) \\ K_i\Big(\mathrm{NIL}\left(\mathcal{P}^{d1}(R);X,Y,Z,W\right)\Big) &\cong K_i\big(\mathrm{NIL}(R;X,Y,Z,W)\big) \\ K_i\Big(\varinjlim \mathrm{NIL}\left(\mathcal{P}^{\mathrm{Im}}(R_s);X_s,Y_s,Z_s,W_s\right)\Big) &\cong K_i\big(\mathrm{NIL}(R_s;X_s,Y_s,Z_s,W_s)\big). \end{aligned}$$

We start with $K_i\Big(\mathrm{NIL}\left(H_s(R);X,Y,Z,W\right)\Big)$.

Lemma 2.3.11. *We have*

$$K_i\Big(\varinjlim \mathrm{NIL}\left(H_s(R);X,Y,Z,W\right)\Big) \cong K_i\big(H_s(R)\times H_s(R)\big)$$

for $i \geq 0$.

Proof. First, we prove that we have an equivalence of categories

$$\varinjlim \mathrm{NIL}\left(H_s(R); X, Y, Z, W\right) \cong \varinjlim \left(H_s(R) \times H_s(R)\right).$$

Let

$$\mathrm{I}\colon \varinjlim \left(H_s(R) \times H_s(R)\right) \to \varinjlim \mathrm{NIL}\left(H_s(R); X, Y, Z, W\right)$$

be the inclusion functor and let

$$\mathrm{P}\colon \varinjlim \mathrm{NIL}\left(H_s(R); X, Y, Z, W\right) \to \varinjlim \left(H_s(R) \times H_s(R)\right)$$

be the forgetful functor whose value at $(H, K, h, k)_i$ is $(H, K)_i$.

Clearly, $\mathrm{P} \circ \mathrm{I} = \mathrm{id}$. Thus it remains to prove that $\mathrm{I} \circ \mathrm{P} \cong \mathrm{id}$. Let $(H, K, h, k)_j$ be an object of $\varinjlim \mathrm{NIL}\left(H_s(R); X, Y, Z, W\right)$. We have

$$\mathrm{I} \circ \mathrm{P}\big((H, K, h, k)_j\big) = (H, K, 0, 0)_j.$$

Since H and K are finitely generated and s-torsion for ℓ big enough, we have

$$\mathrm{F}_s^{\ell}\big((H, K, h, k)\big) \cong (H, K, 0, 0).$$

This implies that

$$(H, K, h, k)_j \cong (H, K, 0, 0)_{j+\ell} = (H, K, 0, 0)_j$$

and therefore $\mathrm{I} \circ \mathrm{P} \cong \mathrm{id}$ on objects.

Morphisms stay the same under the functor $\mathrm{I} \circ \mathrm{P}$. This implies that the categories $\varinjlim \mathrm{NIL}\left(H_s(R); X, Y, Z, W\right)$ and $\varinjlim \left(H_s(R) \times H_s(R)\right)$ are equivalent.

Since K-theory commutes with colimits over small filtering categories, we get

$$K_i\Big(\varinjlim \mathrm{NIL}\left(H_s(R); X, Y, Z, W\right)\Big) \cong \varinjlim K_i\big(H_s(R) \times H_s(R)\big).$$

To prove the lemma, note that the functor F_s induces the identity on $H_s(R) \times H_s(R)$ and therefore

$$\varinjlim K_i\big(H_s(R) \times H_s(R)\big) = K_i\big(H_s(R) \times H_s(R)\big).$$ □

Lemma 2.3.12. *We have*

$$K_i\Big(\varinjlim \mathrm{NIL}\left(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\right)\Big) \cong K_i\big(\,\mathrm{NIL}(R_s; X_s, Y_s, Z_s, W_s)\big)$$

for $i \geq 1$ and there is an injective map on K_0.

Proof. Since K-theory commutes with colimits over small filtering categories, we have

$$K_i\Big(\varinjlim \mathrm{NIL}\left(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\right)\Big) \cong \varinjlim K_i\Big(\,\mathrm{NIL}\left(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\right)\Big).$$

Since F_s is "invertible" on $\mathrm{NIL}\left(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\right)$, we have

$$\varinjlim K_i\Big(\mathrm{NIL}\left(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\right)\Big) \cong K_i\Big(\mathrm{NIL}\left(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\right)\Big).$$

Since $\mathrm{NIL}\left(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\right)$ contains all quadruples, where P and Q are free modules, $\mathrm{NIL}\left(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\right)$ is cofinal in $\mathrm{NIL}(R_s; X_s, Y_s, Z_s, W_s)$. Therefore

$$K_i\Big(\mathrm{NIL}\left(\mathcal{P}^{\mathrm{Im}}(R_s); X_s, Y_s, Z_s, W_s\right)\Big) \cong K_i\big(\mathrm{NIL}(R_s; X_s, Y_s, Z_s, W_s)\big)$$

for $i \geq 1$ and there is an injective map on K_0. □

Last but not least, we have to take care of $K_i\Big(\mathrm{NIL}\left(\mathcal{P}^{d1}(R); X, Y, Z, W\right)\Big)$. To apply the Resolution Theorem, some work is needed:

Lemma 2.3.13. *Let (M, N, m, n) be an object in $\mathrm{NIL}\left(\mathcal{P}^{d1}(R); X, Y, Z, W\right)$ and let (P, Q, p, q) be an object in the subcategory $\mathrm{NIL}(R; X, Y, Z, W)$. Let*

$$(f, g)\colon (P, Q, p, q) \to (M, N, m, n)$$

be a surjective morphism in $\mathrm{NIL}\left(\mathcal{P}^{d1}(R); X, Y, Z, W\right)$. The quadruple $(\mathrm{Ker}(f_1), \mathrm{Ker}(f_2), p|_{\mathrm{Ker}(f_1)}, q|_{\mathrm{Ker}(f_2)})$ is a well defined object in $\mathrm{NIL}(R; X, Y, Z, W)$.

Proof. The exact categories $\mathrm{NIL}\left(\mathcal{P}^{d1}(R); X, Y, Z, W\right)$ and $\mathrm{NIL}(R; X, Y, Z, W)$ are subcategories of the abelian category $\mathrm{NIL}\left(\mathcal{M}^{all}(R); X, Y, Z, W\right)$. Thus $(\mathrm{Ker}(f_1), \mathrm{Ker}(f_2), p|_{\mathrm{Ker}(f_1)}, q|_{\mathrm{Ker}(f_2)})$ is a well defined object in $(\mathrm{Ker}(f_1), \mathrm{Ker}(f_2), p|_{\mathrm{Ker}(f_1)}, q|_{\mathrm{Ker}(f_2)})$. Since

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow P \xrightarrow{f} M \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Ker}(g) \longrightarrow Q \xrightarrow{g} N \longrightarrow 0$$

are exact, Schanuel's Lemma gives that $\mathrm{Ker}(f)$ and $\mathrm{Ker}(g)$ are finitely generated projective R-modules. Thus $(\mathrm{Ker}(f), \mathrm{Ker}(g), p|_{\mathrm{Ker}(f)}, q|_{\mathrm{Ker}(g)})$ is an object in the subcategory $\mathrm{NIL}(R; X, Y, Z, W)$ □

Now everything is set up to apply the Resolution Theorem.

Lemma 2.3.14. *We have*

$$K_i\Big(\mathrm{NIL}\left(\mathcal{P}^{d1}(R); X, Y, Z, W\right)\Big) \cong K_i\big(\mathrm{NIL}(R; X, Y, Z, W)\big)$$

for $i \geq 0$.

Proof. The preceding lemma and the lift constructed in Section 2.1 (Corollary 2.1.5) give that any object in $\mathrm{NIL}\big(\mathcal{P}^{d1}(R);X,Y,Z,W\big)$ has a length one $\mathrm{NIL}(R;X,Y,Z,W)$-resolution. To apply the Resolution Theorem, it is necessary to check that $\mathrm{NIL}(R;X,Y,Z,W)$ is closed under kernels in $\mathrm{NIL}\big(\mathcal{P}^{d1}(R);X,Y,Z,W\big)$, i.e., if

$$0 \longrightarrow (M,N,m,n) \longrightarrow (P',Q',p',q') \longrightarrow (P,Q,p,q) \longrightarrow 0$$

is an exact sequence in $\mathrm{NIL}\big(\mathcal{P}^{d1}(R);X,Y,Z,W\big)$ with (P',Q',p',q') and (P,Q,p,q) in $\mathrm{NIL}(R;X,Y,Z,W)$, then (M,N,m,n) is also in $\mathrm{NIL}(R;X,Y,Z,W)$. This follows since the category $\mathcal{P}(R)$ has this property in $\mathcal{P}^{d1}(R)$. □

Now everything is set up to proof the main theorem of this section.

Proof of Theorem 2.3.7. Combining the Lemmas 2.3.11, 2.3.12 and 2.3.14, we get that Theorem 2.3.7 is implied by Lemma 2.3.10. □

Remark 2.3.15. The reason for the long exact sequence stated in Theorem 2.3.7 not to be surjective in degree zero is that the groups $\varinjlim K_i\Big(\mathrm{NIL}\big(\mathcal{P}^{\mathrm{Im}}(R_s);X_s,Y_s,Z_s,W_s\big)\Big)$ and $K_i\big(\mathrm{NIL}(R_s;X_s,Y_s,Z_s,W_s)\big)$ are not necessarily the same in degree zero.

2.4 The Behavior of Nil-Groups under Localization

In this section, the main theorem of this chapter is proven.

Theorem 2.4.1. *Let R be a ring and let X, Y, Z and W be left flat R-bimodules. If s is an element of the center of R which is not a zero divisor and satisfies $s\cdot x = x\cdot s$ for all $x\in X$ and similar conditions for Y, Z and W, we obtain an isomorphism*

$$\mathbb{Z}[t,t^{-1}]\otimes_{\mathbb{Z}[t]}\mathrm{Nil}_i(R;X,Y,Z,W)\cong \mathrm{Nil}_i(R_s;X_s,Y_s,Z_s,W_s),$$

for all $i\in\mathbb{Z}$, and t acts on $\mathrm{Nil}_i(R;X,Y,Z,W)$ via the map induced by the functor

$$\begin{aligned} F_s\colon\ \mathrm{NIL}(R;X,Y,Z,W) &\to \mathrm{NIL}(R;X,Y,Z,W)\\ (P,Q,p,q) &\mapsto (P,Q,p\cdot s,q\cdot s). \end{aligned}$$

From now on, $\mathbb{Z}[t,t^{-1}]\otimes_{\mathbb{Z}[t]}\mathrm{Nil}_i(R;X,Y,Z,W)$ is denoted by $\mathrm{Nil}_i(R;X,Y,Z,W)_s$. For the proof of Theorem 2.4.1, we first need a general lemma which implies that $\mathrm{Nil}_i(R;X,Y,Z,W)_s$ can be seen as a certain colimit.

If F is a functor from the category ω into the category of abelian groups the colimit over F exists and a model for $\varinjlim_{\omega} F(i)$ is for example given by

$$\bigoplus_{i=0}^{\infty}\mathrm{F}(i)/\langle g+f_{i\,j}(g)\rangle,\ f_{i\,j}\colon \mathrm{F}(i)\to\mathrm{F}(j),$$

[Lan02, page 160].

Lemma 2.4.2. *Let G be an abelian group and let φ be an endomorphism of G. Define a functor F from ω into the category of abelian groups by sending every object of ω to the abelian group G and a morphism from i to j to the $(j-i)$-fold composition of φ with itself. Define a $\mathbb{Z}[t]$-module structure on G where t operates via φ.*

The map

$$\phi\colon \varinjlim_{\omega} F(i) \to \mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}[t]} G$$

induced by

$$\bigoplus_{i=0}^{\infty} g_i \mapsto t^{-i} \otimes g_i$$

is a group isomorphism.

Proof. The proof that ϕ is a group homomorphism is left to the reader.

The morphism ϕ is surjective since the image contains the elements $t^{-i} \otimes g$ which form a generating set of the group $\mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}[t]} G$.

Define a map

$$\xi\colon \mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}[t]} G \to \varinjlim F(i)$$

by

$$\big(\sum \lambda_i t^{-i}\big) \otimes g \mapsto \bigoplus_{i=0}^{\infty} \lambda_i g.$$

Again, the proof that ξ is a group homomorphism is left to the reader.

It remains to prove that $\xi \circ \phi = \mathrm{id}$. This is easily seen on generators:

$$\lambda_i t^{-i} \otimes g \overset{\phi}{\longmapsto} \lambda_i g \overset{\xi}{\longmapsto} t^{-i} \otimes \lambda_i g = \lambda_i t^{-i} \otimes g\ . \qquad \square$$

As a corollary of the preceding lemma we obtain that $\mathrm{Nil}_i(R;X,Y,Z,W)_s$ is isomorphic to $\varinjlim_{\omega} \mathrm{Nil}_i(R;X,Y,Z,W)$.

In the following, all colimits are of the type considered above. After this general lemma we start with the proof of Theorem 2.4.1.

Proof of Theorem 2.4.1. The following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Nil}_i(R;X,Y,Z,W) & \hookrightarrow & K_i\big(\mathrm{NIL}(R;X,Y,Z,W)\big) & \xrightarrow{\iota} & K_i(R) \oplus K_i(R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{F}_s} & & \downarrow{\scriptstyle \mathrm{F}_s} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathrm{Nil}_i(R;X,Y,Z,W) & \hookrightarrow & K_i\big(\mathrm{NIL}(R;X,Y,Z,W)\big) & \xrightarrow{\iota} & K_i(R) \oplus K_i(R) & \longrightarrow & 0 \end{array}$$

and $\mathrm{Nil}_i(R;X,Y,Z,W)_s$ is the colimit of the left hand side by Lemma 2.4.2. Since taking colimits over a filtered system preserves exact sequences, we obtain

$$\mathrm{Nil}_i(R;X,Y,Z,W)_s = \mathrm{Ker}\left(\varinjlim K_i\big(\mathrm{NIL}(R;X,Y,Z,W)\big) \to K_i(R) \oplus K_i(R)\right).$$

In Theorem 2.3.7 it is proven that the following commutative diagram has an exact row in the middle. The lowest row is the localization sequence of algebraic K-theory and therefore exact

$$\begin{array}{ccccccc}
\longrightarrow & 0 & \longrightarrow & \mathrm{Nil}_i(R;X,Y,Z,W)_s & \longrightarrow & \mathrm{Nil}_i(R_s;X_s,Y_s,Z_s,W_s) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & K_i\big(H_s(R)\big) \oplus K_i\big((H_s(R)\big) & \longrightarrow & \varinjlim K_i\big(\,\mathrm{NIL}(R;X,Y,Z,W)\big) & \longrightarrow & K_i\big((\mathrm{NIL}(R_s;X_s,Y_s,Z_s,W_s)\big) & \longrightarrow \\
 & \downarrow\cong & & \downarrow & & \downarrow & \\
\longrightarrow & K_i\big(H_s(R)\big) \oplus K_i\big(H_s(R)\big) & \longrightarrow & K_i(R) \oplus K_i(R) & \longrightarrow & K_i(R_s) \oplus K_i(R_s) & \longrightarrow \cdot
\end{array}$$

The snake lemma implies now that $\mathrm{Nil}_i(R;X,Y,Z,W)_s$ and $\mathrm{Nil}_i(R_s;X_s,Y_s,Z_s,W_s)$ are isomorphic, which is the statement of Theorem 2.4.1 for $i \geq 1$.

To obtain the isomorphism for $i = 0$ note that

$$\mathrm{Ker}\,\Big(K_0\big(\,\mathrm{NIL}(\mathcal{P}^{\mathrm{Im}}(R_s);X_s,Y_s,Z_s,W_s)\big) \to K_0\big(\mathcal{P}^{\mathrm{Im}}(R_s)\times\mathcal{P}^{\mathrm{Im}}(R_s)\big)\Big)$$

and $\mathrm{Nil}_0(R_s;X_s,Y_s,Z_s,W_s)$ are isomorphic since elements of the form $[(P,Q,0,0)]$ are trivial in $\mathrm{Nil}_0(R_s;X_s,Y_s,Z_s,W_s)$.

To obtain the statement for $i < 0$ note that the suspension construction commutes with localization, i.e. $(\Sigma R)_s = \Sigma R_s$. □

Remark 2.4.3. We have not used the fact that the morphism are nilpotent in the preceding two section. Thus Theorem 2.4.1 stays valid if we replace Nil by End.

3 Farrell and Waldhausen Nil-Groups as Modules over the Ring of Witt vectors

In this chapter, we develop a Witt vector-module structure on a certain class of Nil-groups including the important cases $\mathrm{Nil}_i(RG;\alpha)$ and $\mathrm{Nil}_i(RG;RG_\alpha,RG_\beta)$ where G is a group, R is a commutative ring and α and β are inner group automorphism. In the first section, we prove that if Λ is an algebra over a commutative ring R, then $\mathrm{Nil}_i(\Lambda;X,Y,Z,W)$ carries a natural $\mathrm{End}_0(R)$-module structure. As pointed out in the introduction, $\mathrm{End}_0(R)$ is a dense subring of the ring of Witt vectors. To extend the $\mathrm{End}_0(R)$-module structure to a Witt vector-module structure, we define Frobenius and Verschiebung operations on Nil-groups in Section 3.2. For this definition we have to restrict to Nil-groups of the form $\mathrm{Nil}_i(\mathcal{C};\mathrm{A})$ where we have natural transformations between the identity and A. Examples are $\mathrm{Nil}_i(RG;\alpha)$ or $\mathrm{Nil}_i(RG;RG_\alpha,RG_\beta)$ where α and β are inner group automorphism. The strength of the Frobenius and Verschiebung operations is that they satisfy certain relations (Section 3.3). One implication of these relations is that Nil-groups of the given kind are either trivial or not finitely generated as an abelian group. Another relation is the main ingredient of the proof that Nil-groups of the given kind are modules over the ring of Witt vectors (Section 3.4).

3.1 Nil-Groups as Modules over End_0

In this section we define a End_0-module structure on certain Nil-groups. As an application we obtain an $\mathrm{End}_0(R)$-module structure on $\mathrm{Nil}_i(\Lambda;X,Y,Z,W)$ if Λ is an algebra over a commutative ring R. On $\mathrm{Nil}_i(\Lambda)$ a similar module structure is defined by Weibel [Wei81]. In this case, the module structure is induced by the tensor product. For general Nil-groups we have to be slightly more careful.

Definition 3.1.1. Let $\mathcal{A}$ be an abelian category and let $\mathrm{A}\colon \mathcal{A}\to\mathcal{A}$ be an exact functor. Let $\mathcal{C}\subseteq\mathcal{A}$ be a full subcategory which is closed under extension. Let $\mathcal{B}$ be an exact category with an exact pairing

$$\mathcal{B}\times\mathcal{A}\to\mathcal{A}$$

which restricts to $\mathcal{C}$. Let U be a natural transformation between the functors

$$\begin{aligned} &\mathcal{B} \times \mathcal{C} \to \mathcal{A} \\ \mathrm{F}_1 \colon &(b,c) \mapsto b \times \mathrm{A}(c) \\ \mathrm{F}_2 \colon &(b,c) \mapsto \mathrm{A}(b \times c). \end{aligned}$$

We define a an exact pairing

$$\begin{aligned} \mathrm{END}(\mathcal{B}) \times \mathrm{NIL}(\mathcal{C};\mathrm{A}) &\to \mathrm{NIL}(\mathcal{C};\mathrm{A}) \\ (B,b) \times (M,m) &\mapsto (B \times M, U(B,M) \circ (b \times m)). \end{aligned}$$

Remark 3.1.2. To see that the given pairing is well-defined on morphisms, note that the diagram

$$\begin{array}{ccccc} B \times M & \xrightarrow{b\times m} & B \times \mathrm{A}(M) & \xrightarrow{U(B,M)} & \mathrm{A}(B \times M) \\ \big\downarrow{\scriptstyle h\times f} & & \big\downarrow{\scriptstyle h\times \mathrm{A(f)}} & & \big\downarrow{\scriptstyle \mathrm{A}(h\times f)} \\ B' \times M' & \xrightarrow{b'\times m'} & B' \times \mathrm{A}(M') & \xrightarrow{U(B',M')} & \mathrm{A}(B' \times M') \end{array}$$

commutes.

Proposition 3.1.3. *Let the notation be as in the preceding definition. If we additionally assume that we have a pairing*

$$\mathcal{B} \times \mathcal{B} \to \mathcal{B}$$

satisfying

$$b \times (b' \times c) \cong (b \times b') \times c$$

for b*,* $b' \in \mathcal{B}$ *and* $c \in \mathcal{C}$*. Then* $\mathrm{End}_0(\mathcal{B})$ *carries a ring structure and* $\mathrm{Nil}_i(\mathcal{C};A)$ *is an* $\mathrm{End}_0(\mathcal{B})$*-module.*

Proof. The machinery developed by Waldhausen [Wal78a, Wal78b] implies that we get pairings

$$K_0\big(\mathrm{END}(\mathcal{B})\big) \times K_i\big(\mathrm{NIL}(\mathcal{C};\mathrm{A})\big) \to K_i\big(\mathrm{NIL}(\mathcal{C};\mathrm{A})\big).$$

By our assumption we have $(B,b) \times (B',b') \times (M,m) \cong \big((B,b) \times (B',b')\big) \times (M,m)$. This implies that we get a $K_0\big(\mathrm{END}(\mathcal{B})\big)$-module structure on $K_i\big(\mathrm{NIL}(\mathcal{C};\mathrm{A})\big)$. Pairing with objects of the form $(B,0)$ reflects $\mathrm{END}(\mathcal{B})$ into $\mathcal{B}$ and $\mathrm{NIL}(\mathcal{C};\mathrm{A})$ into $\mathcal{C}$. Thus the $K_0(\mathrm{END}(\mathcal{B}))$-module structure restricts to an $\mathrm{End}_0(\mathcal{B})$-module structure on $\mathrm{Nil}_i(\mathcal{C};\mathrm{A})$. □

Corollary 3.1.4. *Let* Λ *be an algebra over a commutative ring* R*. Let* X*,* Y*,* Z *and* W *be arbitrary* Λ*-bimodules. The groups* $\mathrm{Nil}_i(\Lambda)$*,* $\mathrm{Nil}_i(\Lambda;X)$*,* $\mathrm{Nil}_i(\Lambda;X,Y)$ *and* $\mathrm{Nil}_i(\Lambda;X,Y,Z,W)$ *are modules over the ring* $\mathrm{End}_0(R)$ *for all* $i \in \mathbb{Z}$*.*

Proof. We proof the result for $\mathrm{Nil}_i(\Lambda; X)$. Similar arguments work for the other kind of Nil-groups. Let Ψ be the canonical isomorphism between $B \otimes_R (P \otimes_\Lambda X)$ and $(B \otimes_R P) \otimes_\Lambda X$. The map Ψ induces the required natural transformation to obtain a pairing

$$\mathrm{END}(\mathcal{P}(R)) \times \mathrm{NIL}(\mathcal{P}(\Lambda); F_X) \to \mathrm{NIL}(\mathcal{P}(\Lambda); F_X).$$

Since R is assumed to be commutative, $\mathcal{P}(R)$ carries a product structure

$$\begin{aligned} \mathcal{P}(R) \times \mathcal{P}(R) &\to \mathcal{P}(R) \\ P \times P' &\mapsto P \otimes_R P'. \end{aligned}$$

The tensor product also induces a pairing between $\mathcal{P}(R)$ and $\mathcal{P}(\Lambda)$ which satisfies the assumptions of the preceding proposition. Thus we obtain the corollary for higher Nil-groups. To prove the same statement for lower Nil-groups note that if Λ is an algebra over a ring R then $\Sigma\Lambda$ is also an algebra over R. □

In the following, this module multiplication is denoted by $*$.

3.2 Operations on Farrell and Waldhausen Nil-Groups

We define Verschiebung and Frobenius operations on $\mathrm{Nil}_i(\mathcal{C}; \mathrm{A})$ if there is a natural transformations between A and the identity or the identity and A. As a special case we obtain Verschiebung and Frobenius operations on $\mathrm{Nil}_i(RG; \alpha)$ and $\mathrm{Nil}_i(RG; RG_\alpha, RG_\beta)$ where R is a not necessarily commutative ring, G is a group and α and β are inner group automorphism. On Bass Nil-groups and $\mathrm{End}_0(R)$ similar operations are well studied [Blo78, CdS95, Sti82, Wei81].

3.2.1 Frobenius

Frobenius Operations on Nil-Groups

On Bass Nil-groups, for a natural number n the n-th Frobenius is defined to be the map induced by the functor whose value at (P, ν) is (P, ν^n). The main problem with the definition of the Frobenius operation on more general Nil-groups is that in general $\mathrm{A}(\nu) \circ \nu$ is an object in $\mathrm{NIL}(\mathcal{C}; \mathrm{A}^2)$ and not in $\mathrm{NIL}(\mathcal{C}; \mathrm{A})$. To get around this problem, we use the assumption that we have a natural transformation between A^ℓ and the identity.

Definition 3.2.1. Let $\mathcal{A}$ be an abelian category and let $\mathrm{A}\colon \mathcal{A} \to \mathcal{A}$ be an exact functor. Let $\mathcal{C} \subseteq \mathcal{A}$ be a full subcategory which is closed under extension. For a natural number $n \geq 1$, we define a functor

$$\mathrm{Fr}_n\colon \mathrm{NIL}(\mathcal{C}; \mathrm{A}) \to \mathrm{NIL}(\mathcal{C}; \mathrm{A}^n)$$

by sending an object $m \to \mathrm{A}(M)$ to the object given by the composite

$$M \xrightarrow{m} \mathrm{A}(M) \xrightarrow{\mathrm{A}(m)} \mathrm{A}^2(M) \xrightarrow{\mathrm{A}^2(m)} \cdots \xrightarrow{\mathrm{A}^{n-1}(m)} \mathrm{A}^n(M).$$

A morphism f from (M,m) to (M',m') is sent to the morphism from $\mathrm{Fr}_n(M,m)$ to $\mathrm{Fr}_n(M',m')$ whose underlying morphism in $\mathcal{C}$ is f again.

For applications it is useful to have operators which are endomorphisms. For this purpose one needs a natural transformation $U\colon \mathrm{A}^{\ell} \to \mathrm{id}_{\mathcal{A}}$.

Definition 3.2.2 (Frobenius). Let the notation be as in the preceding definition. Let U be a natural transformation $U\colon \mathrm{A}^{\ell} \to \mathrm{id}_{\mathcal{A}}$ for some $\ell \in \mathbb{N}$. For a natural number n we define

$$\mathrm{F}_{\ell n+1}\colon \mathrm{NIL}(\mathcal{C};\mathrm{A}) \to \mathrm{NIL}(\mathcal{C};\mathrm{A})$$

to be the functor which is the composition of $\mathrm{Fr}_{n\ell+1}\colon \mathrm{NIL}(\mathcal{C};\mathrm{A}) \to \mathrm{NIL}(\mathcal{C};\mathrm{A}^{\ell n+1})$ and the functor $\mathrm{NIL}(\mathrm{id},U^{\ell n})\colon \mathrm{NIL}(\mathcal{C};\mathrm{A}^{\ell+1}) \to \mathrm{NIL}(\mathcal{C};\mathrm{A})$. The maps induced by $\mathrm{F}_{\ell n+1}$ on $\mathrm{Nil}_i(\mathcal{C};\mathrm{A})$, for $i \geq 0$, are also denoted by $\mathrm{F}_{\ell n+1}$ and called *Frobenius operations.*

Example 3.2.3 (Frobenius on Farrell Nil-groups). Let R be a ring, let G be a group, let X be an arbitrary RG bimodule and let α be an inner group automorphism of G. Since

$$\alpha(x) = gxg^{-1}$$

for some group elements g, we can define the RG-module homomorphism

$$.g\colon P \otimes_{RG} (RG_{\alpha} \oplus X) \to P$$

by

$$p \otimes (r \oplus x) \mapsto prg$$

for $p \in P$, $r \in RG$ and $x \in X$. The map $.g$ induces a natural transformation between the functor $\mathrm{F}_{RG_{\alpha}\oplus X}$ and the identity. Thus we obtain for every $n \in \mathbb{N}$ and $i \in \mathbb{N}$ a Frobenius operation on $\mathrm{Nil}_i(RG; RG_{\alpha} \oplus X)$. Since $\Sigma RG = (\Sigma R)G$ we can define Frobenius operations on lower Nil-groups in the obvious way.

Example 3.2.4 (Frobenius on Waldhausen Nil-groups). Let R be a ring, let G be a group, let X and Y be arbitrary RG-bimodules and let α and β be inner group automorphism of G induced by group elements g and g'. The maps $.(gg')$ and $.(g'g)$ induce a natural transformation between $\mathrm{F}^2_{RG_{\alpha}\oplus X, RG_{\beta}\oplus Y}$ and the identity. Thus we obtain for every odd $n \in \mathbb{N}$ and $i \in \mathbb{N}$ a Frobenius operation on $\mathrm{Nil}_i(RG; RG_{\alpha} \oplus X, RG_{\beta} \oplus Y)$. The Frobenius operations on lower Nil-groups are defined in the obvious way.

By the definition of F_n, it is obvious that the functors satisfy the following two relations.

Proposition 3.2.5. *With the notation of the preceding definition, we obtain:*

1. *Let (M,m) be an object of $\mathrm{NIL}(\mathcal{C};A)$. If (M,m) is of nilpotency degree L, then*

$$F_n\big((M,m)\big) = (M,0)$$

for $n \geq L$.

2. For n_1, $n_2 \in \mathbb{N}$, we have

$$F_{n_1} F_{n_2} = F_{n_1 \cdot n_2}.$$

3.2.2 Verschiebung

The main problem with the definition of Verschiebung operations is that in general we do not have a map from M to $\mathrm{A}(M)$ which plays the role of the identity. We use the assumption that we have a natural transformation between the identity and A to get such a map.

Verschiebung on Farrell Nil-Groups

Definition 3.2.6 (Verschiebung). Let $\mathcal{A}$ be an abelian category and let $\mathrm{A}\colon \mathcal{A} \to \mathcal{A}$ be an exact functor. Let $\mathcal{C} \subseteq \mathcal{A}$ be a full subcategory which is closed under extension. Suppose that we have a natural transformation $U\colon \mathrm{id}_{\mathcal{A}} \to \mathrm{A}$ of exact functors $\mathcal{A} \to \mathcal{A}$. Then we define, for $n \in \mathbb{N}$,

$$V_n\colon \mathrm{NIL}(\mathcal{C};\mathrm{A}) \to \mathrm{NIL}(\mathcal{C};\mathrm{A})$$

by sending an object $m\colon M \to \mathrm{A}(M)$ to the object

$$\left(M^n, \begin{pmatrix} 0 & & & m \\ U(M) & \ddots & & \\ & \ddots & 0 & \\ & & U(M) & 0 \end{pmatrix}\right).$$

A morphism f from (M,m) to (M',m') is mapped to the morphism $f^{\oplus n}$. The maps induced by this functors on $\mathrm{Nil}_i(\mathcal{C};\mathrm{A})$, for $i \geq 0$, are also denoted by V_n and called *Verschiebung operations.*

Remark 3.2.7. To see that the functors V_n are well-defined, note that for every morphism f from M to M' the diagram

$$\begin{array}{ccc} M & \xrightarrow{U(M)} & \mathrm{A}(M) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{A}(f)} \\ M' & \xrightarrow{U(M')} & \mathrm{A}(M') \end{array}$$

commutes. In particular

$$\begin{array}{ccc} M & \xrightarrow{U(M)} & \mathrm{A}(M) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \mathrm{A}(m)} \\ \mathrm{A}(M) & \xrightarrow{U(\mathrm{A}(M))} & \mathrm{A}^2(M) \end{array}$$

commutes. This together with the fact that m is nilpotent implies that the morphism

$$\begin{pmatrix} 0 & & & m \\ U(M) & \ddots & & \\ & \ddots & 0 & \\ & & U(M) & 0 \end{pmatrix}$$

is nilpotent.

Example 3.2.8 (Verschiebung on Farrell Nil-groups). Let R be a ring, let G be a group, let X be an arbitrary RG bimodule and let α be an inner group automorphism of G. Since

$$\alpha(x) = gxg^{-1}$$

for some group element $g \in G$, we can define the RG-module homomorphism

$$.g^{-1}\colon P \to P \otimes_{RG} (RG_\alpha \oplus X)$$

by

$$p \mapsto p \otimes (g^{-1} \oplus 0)$$

for $p \in P$. The map $.g^{-1}$ induces the required natural transformation between the identity and $\mathrm{F}_{RG_\alpha \oplus X}$. Thus we obtain Verschiebung operations on $\mathrm{Nil}_i(RG; RG_\alpha \oplus X)$ for $i \in \mathbb{N}$. The Verschiebung operations on lower Nil-groups are defined in the obvious way. Note that $.g^{-1}$ is the right inverse of $.g$. If X is the trivial module the natural transformation $.g^{-1}$ is also a left inverse of $.g$.

Consider $\mathbb{N}$ with the multiplication. This gives $\mathbb{N}$ the structure of a semigroup. We define $\mathbb{Z}\mathbb{N}$ to be the "semigroup ring" of $\mathbb{N}$ with coefficients in $\mathbb{Z}$. The next identity implies that we get a $\mathbb{Z}\mathbb{N}$-module structure on $\mathrm{Nil}_i(\mathcal{C}; \mathrm{A})$, where $n \in \mathbb{N}$ operates on $\mathrm{Nil}_i(\mathcal{C}; \mathrm{A})$ via V_n.

Proposition 3.2.9. *Let n_1 and n_2 be natural numbers. With the assumptions of the preceding definition, we have*

$$V_{n_1} V_{n_2} = V_{n_1 \cdot n_2}$$

as operations on $\mathrm{Nil}_i(\mathcal{C}; A)$*.*

Proof. This relation follows since there is an isomorphism between the two functors. (Isomorphism is meant in the sense that there is a natural transformation between the functors $\mathrm{V}_{n_1}\mathrm{V}_{n_2}$ and $\mathrm{V}_{n_1 \cdot n_2}$ such that the map $\mathrm{V}_{n_1}\mathrm{V}_{n_2}(x) \to \mathrm{V}_{n_1 \cdot n_2}(x)$ is an isomorphism for every object x in $\mathrm{NIL}(\mathcal{C}; \mathrm{A})$). An application of Proposition 1.3.1. in [Wal85] yields the identity on $K_i\big(\mathrm{NIL}(\mathcal{C}; \mathrm{A})\big)$ and therefore on $\mathrm{Nil}_i(\mathcal{C}; \mathrm{A})$. □

Remark 3.2.10. The given definition of the Verschiebung operation is a generalization of the Verschiebung operation on Bass Nil-groups. In the case of Bass Nil-groups, it is known that if G is a finite group, $\mathrm{Nil}_{-1}(\mathbb{Z}G)$ is finitely generated as a $\mathbb{Z}\mathbb{N}$-module [CdS95]. We do not know whether $\mathrm{Nil}_i(\mathcal{C}; \mathrm{A})$ is finitely generated as a $\mathbb{Z}\mathbb{N}$-module.

Verschiebung on Waldhausen Nil-Groups of generalized free products

To define a Verschiebung operation on Waldhausen Nil-groups of generalized free products we proceed in a similar manner as for Farrell Nil-groups.

Definition 3.2.11 (Verschiebung on Waldhausen Nil-groups). Let $\mathcal{A}$ be an abelian category and let $\mathrm{A}_1, \mathrm{A}_2 \colon \mathcal{A} \to \mathcal{A}$ be exact functors. Let A_W be the endofunctor of $\mathcal{A} \times \mathcal{A}$ whose value at (a, a') is $(\mathrm{F}_1(a'), \mathrm{F}_2(a))$. Let $\mathcal{C} \subseteq \mathcal{A}$ be a full subcategory which is closed under extension. Suppose that we have natural transformations

$$U_1 \colon \mathrm{id}_{\mathcal{A}} \to \mathrm{A}_1$$
$$U_2 \colon \mathrm{id}_{\mathcal{A}} \to \mathrm{A}_2$$

of exact functors $\mathcal{A} \to \mathcal{A}$. For $\ell \in \mathbb{N}$ we define an exact functor

$$V_{2\ell+1} \colon \mathrm{NIL}(\mathcal{C} \times \mathcal{C}; \mathrm{A}_W) \to \mathrm{NIL}(\mathcal{C} \times \mathcal{C}; \mathrm{A}_W)$$

$$(M, N, m, n)$$

$$\mapsto \left((M \oplus N)^{\ell} \oplus M, (N \oplus M)^{\ell} \oplus N, \begin{pmatrix} 0 & & & m \\ U_1(M) & \ddots & & \\ & \ddots & 0 & \\ & & U_1(N) & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & n \\ U_2(N) & \ddots & & \\ & \ddots & 0 & \\ & & U_2(M) & 0 \end{pmatrix}\right).$$

If (f, g) is a morphism from (M, N, m, n) to (M', N', n', m') of $\mathrm{NIL}(\mathcal{C} \times \mathcal{C}; \mathrm{A}_W)$, we define $\mathrm{V}_{2\ell+1}\big((f, g)\big)$ to be the morphism $\big((f \oplus g)^{\oplus \ell} \oplus f, (g \oplus f)^{\oplus \ell} \oplus g\big)$.

The maps induced by these functors on $\mathrm{Nil}_i(\mathcal{C} \times \mathcal{C}; \mathrm{A}_W)$, for $i \geq 0$, are also denoted by $\mathrm{V}_{2\ell+1}$ and called *Verschiebung operations*.

The functors $\mathrm{V}_{2\ell+1}$ are well-defined by the same reasoning as above.

Example 3.2.12 (Verschiebung on Waldhausen Nil-groups). Let R be a ring, let G be a group, let X and Y be arbitrary RG-bimodules and let α and β be inner group automorphism of G induced by group elements g and g'. The maps $.g^{-1}$ and $.g'^{-1}$ induce a natural transformations between the identity and $\mathrm{F}_{RG_\alpha \oplus X}$ and $\mathrm{F}_{RG_\beta \oplus Y}$. Thus we obtain for every odd $n \in \mathbb{N}$ and $i \in \mathbb{N}$ a Verschiebung operation on $\mathrm{Nil}_i(RG; RG_\alpha \otimes X, RG_\beta \oplus Y)$. Verschiebung operations on lower Nil-groups are defined in the obvious way. Note that the natural transformations induced by $.g^{-1}$ and g'^{-1} are a right inverse of the natural transformation induced by $.g$ and $.g'$. If X and Y are the trivial module the natural transformations induced by $.g^{-1}$ and g'^{-1} are also a left inverse.

Proposition 3.2.13. *Let the notation be as in the preceding definition. Let n_1 and n_2 be odd natural numbers. With the assumptions of the preceding definition, we have*

$$V_{n_1} V_{n_2} = V_{n_1 \cdot n_2}$$

as operations on $\mathrm{Nil}_i(\mathcal{C} \times \mathcal{C}; A_W)$.

Proof. The identity follows in the same manner as above. □

3.3 Relations

In the first part of this section, we prove that the Frobenius and Verschiebung operations satisfy the relations

$$\mathrm{F}_n\mathrm{V}_n(x) = x \cdot n$$

on $\mathrm{Nil}_i(\mathcal{C};\mathrm{A})$ and a similar relation on $\mathrm{Nil}_i(\mathcal{C}\times\mathcal{C};\mathrm{A}_W)$. We also apply this relation to prove that the given kind of Nil-groups are either trivial or not finitely generated as abelian groups.

In the second part the relation

$$\mathrm{V}_n(y * \mathrm{F}_n x) = (\mathrm{V}_n y) * x.$$

is proven.

3.3.1 Non Finiteness Results

We start with the proof of the first identity.

Definition 3.3.1 (σ). Let the notation be as in Definition 3.2.11. We additionally assume that U_1 and U_2 have a left inverse U_1^{-1} and U_2^{-1}. We define an exact endofunctor

$$\begin{aligned} \mathrm{S}\colon \mathrm{NIL}(\mathcal{C}\times\mathcal{C};\mathrm{A}_W) &\to \mathrm{NIL}(\mathcal{C}\times\mathcal{C};\mathrm{A}_W) \\ (M,N,m,n) &\mapsto (N,M,U_1(N)\circ U_2^{-1}(N)\circ n, U_2(M)\circ U_1^{-1}(M)\circ m) \end{aligned}$$

The induced map on $\mathrm{Nil}_i(\mathcal{C}\times\mathcal{C};\mathrm{A}_W)$ is denoted by σ.

Note that σ is an group automorphism of order two.

Proposition 3.3.2. *Let $\mathcal{A}$ be an abelian category.*

1. *Let $A\colon \mathcal{A}\to\mathcal{A}$ be an exact functor. Let $\mathcal{C}\subseteq\mathcal{A}$ be a full subcategory which is closed under extension. Suppose that we have a natural transformation $U\colon id_{\mathcal{A}}\to A$ of exact functors $\mathcal{A}\to\mathcal{A}$ together with an left inverse natural transformation U^{-1}. For $n\in\mathbb{N}$, we have*

$$F_n V_n(x) = x\cdot n$$

for all x in $\mathrm{Nil}_i(\mathcal{C};A)$ and $i\in\mathbb{N}$.

2. *Let $A_1, A_2\colon \mathcal{A}\to\mathcal{A}$ be exact functors. Let A_W be the endofunctor of $\mathcal{A}\times\mathcal{A}$ whose value at (a,a') is $(A_1(a'),A_2(a))$. Let $\mathcal{C}\subseteq\mathcal{A}$ be a full subcategory which is closed under extension. Suppose that we have natural transformations*

$$\begin{aligned} U_1&\colon id_{\mathcal{A}}\to A_1 \\ U_2&\colon id_{\mathcal{A}}\to A_2 \end{aligned}$$

of exact functors $\mathcal{A} \to \mathcal{A}$ together with left inverse natural transformations U_1^{-1} and U_2^{-1}. For odd $n = 2\ell + 1$, we have

$$F_n V_n(x) = x \cdot (\ell + 1) + \sigma(x) \cdot \ell$$

for all x in $\mathrm{Nil}_i(\mathcal{C} \times \mathcal{C}; A_W)$ and $i \in \mathbb{N}$.

Proof. We prove the identity for $\mathrm{Nil}_i(\mathcal{C} \times \mathcal{C}; \mathrm{A}_W)$, similar arguments work for $\mathrm{Nil}_i(\mathcal{C}; \mathrm{A})$.

Let (M, N, m, n) be an object in $\mathrm{NIL}(\mathcal{C} \times \mathcal{C}; \mathrm{A}_W)$. Then

$$\begin{aligned}
&\mathrm{F}_{2\ell+1}\mathrm{V}_{2\ell+1}\big((M, N, m, n)\big) = \\
&= \mathrm{F}_{2\ell+1}\Big((M \oplus N)^\ell \oplus M, (N \oplus M)^\ell \oplus N, \\
&\begin{pmatrix} 0 & & & m \\ U_1(M) & \ddots & & \\ & \ddots & 0 & \\ & & U_1(N) & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & n \\ U_2(N) & \ddots & & \\ & \ddots & 0 & \\ & & U_2(M) & 0 \end{pmatrix}\Big) \\
&\cong \Big((M \oplus N)^\ell \oplus M, (N \oplus M)^\ell \oplus N, \\
&\begin{pmatrix} m & & \\ & U_1(N) \circ U_2^{-1}(N) \circ n & \\ & & \ddots & \\ & & & m \end{pmatrix}, \begin{pmatrix} n & & \\ & U_2(M) \circ U_1^{-1}(M) \circ m & \\ & & \ddots & \\ & & & n \end{pmatrix}\Big) \\
&\cong (\ell + 1)(M, N, m, n) \oplus \ell\, \mathrm{S}\big((M, N, m, n)\big).
\end{aligned}$$

The last two $\cong$-signs are meant in the sense that the corresponding functors are isomorphic.

Thus we obtain the identity on $K_i\big(\mathrm{NIL}(\mathcal{C} \times \mathcal{C}; \mathrm{A}_W)\big)$ and therefore on $\mathrm{Nil}_i(\mathcal{C} \times \mathcal{C}; \mathrm{A}_W)$. □

Corollary 3.3.3. *Let R be a ring, let G be a group, let X and Y be arbitrary RG-bimodules and let α and β be inner group automorphism of G. For $n \in \mathbb{N}$ and $i \in \mathbb{Z}$, we have*

$$F_n V_n(x) = x \cdot n$$

for all x in $\mathrm{Nil}_i(RG; RG_\alpha \oplus X)$. For odd $n = 2\ell + 1$, we have

$$F_n V_n(x) = x \cdot (\ell + 1) + \sigma(x) \cdot \ell$$

for all x in $\mathrm{Nil}_i(RG; RG_\alpha \oplus X, RG_\beta \oplus Y)$.

Proof. By the preceding proposition we obtain the identity on $\mathrm{Nil}(RG; RG_\alpha \oplus X)$ and $\mathrm{Nil}_i(RG; RG_\alpha \oplus X, RG_\beta \oplus Y)$ for $i \geq 0$. Since $\Sigma RG = (\Sigma R)G$ the statement is also true for $i < 0$. □

Lemma 3.3.4. *Let G be an abelian group with finite torsion subgroup T. Let σ be a group automorphism of order two. If $n \in \mathbb{N}$ is -1 modulo $|T|$ then*

$$\Phi_n(x) := x \cdot (n+1) + \sigma(x) \cdot n$$

is a monomorphism.

Proof. First we prove that $\Phi_n(x) = 0$ implies $x \in T$. We define $u_x := \sigma(x) + x$ and $v_x(x) := -\sigma(x) + x$. We have

$$\begin{aligned} u_x + v_x &= 2x \\ \sigma(u_x) &= u_x \\ \sigma(v_x) &= -v_x. \end{aligned}$$

Since $\Phi_n(x) = 0$ implies that $\Phi_n(2x) = 0$, we have

$$\begin{aligned} 0 &= \Phi_n(u_x + v_x) \\ &= (n+1)(u_x + v_x) + n(u_x - v_x) \\ &= (2n+1)u_x + v_x. \end{aligned}$$

Applying σ to this equality yields

$$(2n+1)u_x - v_x = 0.$$

Hence $u_x \in T$. Therefore, $v_x \in T$ and also $u_x + v_x \in T$. In particular $2x \in T$ and $x \in T$.

Assume now $n = k|T| - 1$. Let $x \in G$, such that $\Phi_n(x) = 0$. By the arguments given above we have $x \in T$. We have

$$\begin{aligned} 0 = \Phi_n(x) &= (n+1)x + n\sigma(x) \\ &= k|T|x + (k|T| - 1)\sigma(x) \\ &= 0 + -\sigma(x). \end{aligned}$$

Thus $x = 0$ which implies the lemma. □

Corollary 3.3.5. *Let the notation be as in the preceding proposition. The groups* $\mathrm{Nil}_i(\mathcal{C}; A)$ *and* $\mathrm{Nil}_i(\mathcal{C} \times \mathcal{C}; A_W)$ *are either trivial or not finitely generated as abelian groups for* $i \in \mathbb{N}$.

Proof. Since F_n and V_n satisfy the relation stated in Proposition 3.3.2 and $\mathrm{F}_n(x) = 0$ for n bigger than a certain number M, we can apply a trick which is due to Farrell [Far77]. We prove the result for $\mathrm{Nil}_i(\mathcal{C} \times \mathcal{C}; \mathrm{A}_W)$, similar arguments work for $\mathrm{Nil}_i(\mathcal{C}; \mathrm{A})$.

We denoted the category which is used in Quillen's Q-construction by $Q\,\mathrm{NIL}(\mathcal{C} \times \mathcal{C}; \mathrm{A}_W)$. Let $Q_m\,\mathrm{NIL}(\mathcal{C} \times \mathcal{C}; \mathrm{A}_W)$ be the full subcategories of $Q\,\mathrm{NIL}(\mathcal{C} \times \mathcal{C}; \mathrm{A}_W)$

consisting of objects of nilpotency degree smaller or equal to m. For a category $\mathcal{C}$ we denote the classifying space by $B\mathcal{C}$. We have

$$\begin{aligned} K_i\big(\operatorname{NIL}(\mathcal{C}\times\mathcal{C};\mathrm{A}_W)\big) &= \pi_i\Omega BQ\operatorname{NIL}(\mathcal{C}\times\mathcal{C};\mathrm{A}_W) \\ &= \varinjlim_m \pi_i\Omega BQ_m\operatorname{NIL}(\mathcal{C}\times\mathcal{C};\mathrm{A}_W) \end{aligned}$$

and

$$K_i\big(\mathcal{C}\times\mathcal{C}\big) = \pi_i\Omega BQ_0\operatorname{NIL}(\mathcal{C}\times\mathcal{C};\mathrm{A}_W).$$

Assume now that $\operatorname{Nil}_i(\mathcal{C}\times\mathcal{C};\mathrm{A}_W)$ is a finitely generated abelian group. Thus we can find an L such that the generators of $K_i\big(\operatorname{NIL}(\mathcal{C}\times\mathcal{C};\mathrm{A}_W)\big)$ are contained in $\pi_i\Omega BQ_L\operatorname{NIL}(\mathcal{C}\times\mathcal{C};\mathrm{A}_W)$. Thus there is an L such that F_n, whenever defined, is the trivial map for $n\geq L$ (Proposition 3.2.5). Let T be the torsion subgroup of $\operatorname{Nil}_i(\mathcal{C}\times\mathcal{C};\mathrm{A}_W)$. Since $\operatorname{Nil}_i(\mathcal{C}\times\mathcal{C};\mathrm{A}_W)$ is finitely generated we have that $|T|$ is finite. Choose $\ell\in\mathbb{N}$ such that $2\ell|T|-1\geq L$. By the preceding lemma $\mathrm{V}_{2\ell|T|-1}\mathrm{F}_{2\ell|T|-1}$ is a monomorphism. On the other hand, since $2\ell|T|-1\geq L$, the group $\operatorname{Nil}_i(\mathcal{C}\times\mathcal{C};\mathrm{A}_W)$ is in the kernel of $\mathrm{F}_{2\ell|T|-1}$. Thus $\operatorname{Nil}_i(\mathcal{C}\times\mathcal{C};\mathrm{A}_W)$ is the trivial group. □

Corollary 3.3.6. *Let R be a ring, let G be a group, let X and Y be arbitrary RG-bimodules and let α and β be inner group automorphisms of G. The groups $\operatorname{Nil}_i(RG;RG_\alpha\oplus X)$ and $\operatorname{Nil}_i(RG;RG_\alpha\oplus X,RG_\beta\oplus Y)$ are either trivial or not finitely generated as abelian groups for $i\in\mathbb{Z}$.*

Proof. The preceding corollary proves the statement for higher Nil-groups. The same statement for lower Nil-groups follows in the obvious way. □

3.3.2 The Relation $\mathbf{V}_n(y{*}\mathbf{F}_n x) = (\mathbf{V}_n y) * x$ on Farrell Nil-Groups

In this section we restrict to the case that we have an algebra Λ over a ring R, a left flat Λ-bimodule X and a natural transformation between the identity and F_X together with an right inverse natural transformation. For example if R is a ring, G is a group and α is an inner group automorphism induced by $g\in G$, then $.g^{-1}$ induces a natural transformation between the identity and F_{RG_α} with a right inverse natural transformation induced by $.g$.

The proof of the relation

$$\mathrm{V}_n(y*\mathrm{F}_n x) = (\mathrm{V}_n y)*x$$

requires a little bit more machinery. The basic idea, which is due to Stienstra [Sti82], is to define some exact category $\operatorname{END}(\mathbb{Z}[x,y];S_n)$ and an exact pairing

$$\begin{aligned} \operatorname{END}(\mathbb{Z}[x,y];S_n)\times\operatorname{END}(R)\times\operatorname{NIL}(\Lambda;X) &\longrightarrow \operatorname{NIL}(\Lambda;X) \\ c\times x\times y &\longrightarrow (c,x,y). \end{aligned}$$

Then we prove that there are objects C_1 and C_2 of $\operatorname{END}(\mathbb{Z}[x,y];S_n)$ such that $\mathrm{V}_n(x\times \mathrm{F}_n y) = (C_1,x,y)$ and $(\mathrm{V}_n x)\times y = (C_2,x,y)$. Using the product map developed by Waldhausen [Wal78a, Wal78b], we can now prove the relation by proving that $[C_1] = [C_2]$ as elements in $\tilde{K}_0\big(\mathrm{END}(\mathbb{Z}[x,y];S_n)\big)$.

To start this program, we define the category $\mathrm{END}(R;T)$.

Definition 3.3.7 ($\mathrm{END}(R;T)$)**.** Let R be a ring and let $T \subset R[t]$ be a multiplicatively closed subset containing t.

1. We define $\mathrm{END}(R;T)$ to be the the full subcategory of $\mathrm{END}(R)$ consisting of tuples (C,γ) such that there exist a polynomial $\sum t^i\lambda_i \in T$ with
$$\sum \gamma^i \lambda_i = 0.$$
A sequence in $\mathrm{END}(R;T)$ is called *exact* if the underlying sequence of R-modules is exact.

2. We define $\mathrm{End}_0(R;T)$ to be the kernel of the map on K_0 which is induced by the forgetful functor from $\mathrm{END}(R;T)$ onto $\mathcal{P}(R)$ whose value at (C,γ) is C.

To see that the exact sequences in $\mathrm{END}(R;T)$ define the structure of an exact category on $\mathrm{END}(R;T)$ consider an exact sequence

$$0 \longrightarrow (C_1,\gamma_1) \longrightarrow (C,\gamma) \longrightarrow (C_2,\gamma_2) \longrightarrow 0$$

in $\mathrm{END}(R)$, with (C_1,γ_1) and (C_2,γ_2) in $\mathrm{END}(R;T)$. The morphisms γ_1 and γ_2 satisfy polynomial equations $p_1(t) = \sum t^i\lambda_i$ and $p_2(t) = \sum t^j\mu_j$ in T. The arguments given in the proof of Proposition 2.1.1 imply that, that $p_1(\gamma)\circ p_2(\gamma)$ vanishes. Thus for an arbitrary $x \in C$

$$\begin{aligned} 0 &= p_1(\gamma)\circ p_2(\gamma)(x) \\ &= \sum \gamma^i\big(\sum \gamma^j(x)\mu_j\big)\lambda_i \\ &= \sum \gamma^{i+j}(x)\mu_j\lambda_i \\ &= p_1\cdot p_2(\gamma)(x). \end{aligned}$$

This implies that γ is annihilated by a polynomial in T and therefore that $\mathrm{END}(R;T)$ is closed under extension in $\mathrm{END}(R)$. We obtain that $\mathrm{END}(R;T)$ is an exact category.

For commutative rings R the K-groups of the exact categories $\mathrm{END}(R;T)$ are well-studied [Gra77, Sti82].

In the following $\mathrm{END}(\mathbb{Z}[x,y];S_n)$, where $S_n \subseteq \mathbb{Z}[x,y][t]$ is the multiplicatively closed set generated by t and $t^n - x^n\cdot y$ will become important. Objects of this category are for example

$$\left(\mathbb{Z}[x,y]^n, \begin{pmatrix} 0 & & & x^n y \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{pmatrix}\right)$$

and

$$\left(\mathbb{Z}[x,y]^n, \begin{pmatrix} 0 & & & xy \\ x & \ddots & & \\ & \ddots & 0 & \\ & & x & 0 \end{pmatrix}\right)$$

which are annihilated by $t^n - x^n y$.

Let Λ be an algebra over a ring R, let X be an Λ bimodule and let U be a natural transformation between the identity and F_X with a right inverse natural transformation U^{-1}. We define an exact pairing

$$\mathrm{END}(\mathbb{Z}[x,y];S_n) \times \mathrm{END}(R) \times \mathrm{NIL}(\Lambda;X) \to \mathrm{NIL}(\Lambda;X)$$

$$(C,\gamma)\times(B,\varphi)\times(P,\nu) \mapsto \big(C \otimes_{\mathbb{Z}[x,y]} B \otimes_R P, \gamma \otimes \mathrm{id} \otimes U(P)\big),$$

where x acts on $B \otimes_R P$ via $\mathrm{id} \otimes U^{-1}(P)^{-1} \circ \nu$ and y acts on $B \otimes_R P$ via $\varphi \otimes \mathrm{id}$. To show that the pairing is a well-defined exact pairing we need to prove two things. First of all, we need to check that this defines a pairing, i.e., that $C \otimes_{\mathbb{Z}[x,y]} B \otimes_R P$ is a finitely generated projective RG-module and that $(\gamma \otimes \mathrm{id} \otimes U(P))$ is a nilpotent morphism. Secondly, we need to verify the exactness.

The morphism $(\gamma \otimes \mathrm{id} \otimes U(P))$ is nilpotent since γ satisfies a relation in S_n. The module $C \otimes_{\mathbb{Z}[x,y]} B \otimes_R P$ is a finitely generated projective Λ-module since C is a finitely generated projective $\mathbb{Z}[x,y]$-module and $B \otimes_R P$ is a finitely generated projective Λ-module. The second part follows since C is a projective $\mathbb{Z}[x,y]$-module and therefore flat.

Theorem 3.3.8. *Let R be a ring, let Λ be an algebra over R, let X be an Λ-bimodule and let U be a natural transformation between the identity and F_X with an right inverse natural transformation U^{-1}. Let y be an element of $\mathrm{End}_0(R)$ and let x be an element of $\mathrm{Nil}_i(\Lambda;X)$ for $i \in \mathbb{Z}$. For $n \in \mathbb{N}$, we have*

$$V_n(y * F_n x) = (V_n y) * x$$

where $*$ is the module multiplication defined in Section 3.1.

Proof. We prove the statement for higher Nil-groups. The arguments extend to lower Nil-groups in the obvious way. The pairing between the categories $\mathrm{END}(R) \times \mathrm{NIL}(\Lambda;X)$ and $\mathrm{END}(\mathbb{Z}[x,y];S_n)$ defined above gives that for every object (C,γ) of $\mathrm{END}(\mathbb{Z}[x,y];S_n)$ we get an exact endofunctor of $\mathrm{END}(R)\times\mathrm{NIL}(\Lambda;X)$. Since objects of the form $(C,0)$ reflect $\mathrm{END}(R) \times \mathrm{NIL}(\Lambda;X)$ into the subcategory where objects are of the form $(P,0)$ we get that every element in $\mathrm{End}_0(\mathbb{Z}[x,y];S_n)$ gives rise to an operation on $\mathrm{End}_0(R) \times \mathrm{Nil}_i(\Lambda;X)$.

Let (B,φ) be an object of $\mathrm{END}(R)$, let (P,ν) be an object of $\mathrm{NIL}(\Lambda;X)$. We define ν^n to be $\nu \circ (U^{-1}(P) \circ \nu)^{n-1}$ We have

$$\mathrm{V}_n\big((B,\varphi) \times \mathrm{F}_n(P,\nu)\big) = \left((B \otimes P)^n, \begin{pmatrix} 0 & & & \varphi \otimes \nu^n \\ U(B\otimes P) & \ddots & & \\ & \ddots & 0 & \\ & & U(B \otimes P) & 0 \end{pmatrix}\right)$$
$$\cong \left(\mathbb{Z}[x,y]^n, \begin{pmatrix} 0 & & & x^n y \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{pmatrix}\right) \times (B,\varphi) \times (P,\nu)$$

and

$$\big(\mathrm{V}_n(B,\varphi)\big) \times (P,\nu) = \left(B^n \otimes P, \begin{pmatrix} 0 & & & \varphi \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{pmatrix} \otimes \nu\right)$$
$$\cong \left(\mathbb{Z}[x,y]^n, \begin{pmatrix} 0 & & & xy \\ x & \ddots & & \\ & \ddots & 0 & \\ & & x & 0 \end{pmatrix}\right) \times (B,\varphi) \times (P,\nu).$$

Thus it remains to show that

$$\left[\left(\mathbb{Z}[x,y]^n, \begin{pmatrix} 0 & & & x^n y \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{pmatrix}\right)\right]$$

and

$$\left[\left(\mathbb{Z}[x,y]^n, \begin{pmatrix} 0 & & & xy \\ x & \ddots & & \\ & \ddots & 0 & \\ & & x & 0 \end{pmatrix}\right)\right]$$

represent the same element in $\mathrm{End}_0(\mathbb{Z}[x,y]; S_n)$.

For $\mathrm{End}_0(\mathbb{Z}[x,y]; S_n)$ it is known [Sti82, page 87] that the characteristic polynomial induces an injective map into the ring of Witt vectors. Thus, since the two elements have the same characteristic polynomial, we obtain the required identity. □

3.3.3 The Relation $\mathbf{V}_n(y * \mathbf{F}_n x) = (\mathbf{V}_n y) * x$ on Waldhausen Nil-Groups

Similarly we proceed for Waldhausen Nil-groups.

Definition 3.3.9 ($\mathrm{END}(R; \mathbf{id}, \mathbf{id}; T)$). Let R be a ring. Let $T \subset R[t]$ be a multiplicatively closed set containing t. We define $\mathrm{END}(R; \mathrm{id}, \mathrm{id}; T)$ to be the full subcategory of $\mathrm{END}(R; R, R)$ consisting of quadruples (P, Q, p, q) such that there exist polynomials $p_1(t) = \sum t^i \lambda_i$ and $p_2(t) = \sum t^i \mu_i$ in T with $\sum (p \circ q)^i \lambda_i = 0$ and $\sum (q \circ p)^i \mu_i = 0$.

We obtain, in the same manner as above that $\mathrm{END}(R; \mathrm{id}, \mathrm{id}; T)$ is an exact category.

Remark, the category $\mathrm{NIL}(R; R, R)$ is the category $\mathrm{END}(R; \mathrm{id}, \mathrm{id}; S)$, where $S \subseteq R[t]$ is the multiplicatively closed set generated by t.

Let δ be the ring automorphism of $\mathbb{Z}[y, v, w]$ induced by mapping v to w and w to v. Let S'_n be the multiplicatively closed subset of the polynomial ring of the twisted polynomial ring $\mathbb{Z}[y, v, w]_\delta[x]$ generated by t, $t^n - v^{2n} \cdot y^2 \cdot x^{2n}$ and $t^n - w^{2n} \cdot y^2 \cdot x^{2n}$. In the following the category $\mathrm{END}(\mathbb{Z}[y, v, w]_\delta[x]; \mathrm{id}, \mathrm{id}; S'_n)$ will become important. Objects are for example

$$\left(\mathbb{Z}[y,v,w]_\delta[x]^n, \mathbb{Z}[y,v,w]_\delta[x]^m, \begin{pmatrix} 0 & & & & yx^n v \\ v & 0 & & & \\ & w & \ddots & & \\ & & \ddots & 0 & \\ & & & w & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & yx^n w \\ w & 0 & & & \\ & v & \ddots & & \\ & & \ddots & 0 & \\ & & & v & 0 \end{pmatrix}\right)$$

and

$$\left(\mathbb{Z}[y,v,w]_\delta[x]^n, \mathbb{Z}[y,v,w]_\delta[x]^n, \begin{pmatrix} 0 & & & & yxv \\ xv & 0 & & & \\ & xv & \ddots & & \\ & & \ddots & 0 & \\ & & & xv & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & yxw \\ xw & 0 & & & \\ & xw & \ddots & & \\ & & \ddots & 0 & \\ & & & xw & 0 \end{pmatrix}\right).$$

which are annihilated by $(t^n - v^{2n} \cdot y^2 \cdot x^{2n}) \cdot (t^n - w^{2n} \cdot y^2 \cdot x^{2n})$.

Let Λ be an algebra over a ring R, let X and Y be left flat Λ-bimodules and let U_X and U_Y be natural transformations between the identity and F_X and F_Y with right inverse natural transformation U_X^{-1} and U_Y^{-1}. We define an exact pairing

$$\mathrm{END}(\mathbb{Z}[y,v,w]_\delta[x]; \mathrm{id}, \mathrm{id}; S'_n) \times \mathrm{END}(R) \times \mathrm{NIL}(\Lambda; X, Y) \to \mathrm{NIL}(\Lambda; X, Y)$$

where

$$(C, D, c, d) \times (B, \varphi) \times (P, Q, p, q)$$

is mapped to

$$\Big(C \otimes_{\mathbb{Z}[y,v,w]_\delta[x]} B \otimes_R (P \oplus Q), D \otimes_{\mathbb{Z}[y,v,w]_\delta[x]} B \otimes_R (P \oplus Q),$$

$$c \otimes \mathrm{id} \otimes U_X(P \oplus Q), d \otimes \mathrm{id} \otimes U_Y(P \oplus Q)\Big),$$

x acts on $B \otimes_R (P \oplus Q)$ via $\mathrm{id} \otimes \begin{pmatrix} 0 & U_Y^{-1}(Q) \circ q \\ U_X^{-1}(P) \circ p & 0 \end{pmatrix}$, y acts via $\varphi \otimes \mathrm{id}$, v acts via $\mathrm{id} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and w acts via $\mathrm{id} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

We obtain in the same manner as above that this is a well defined exact paring.

Before we can start with the proof that the Frobenius and Verschiebung operations satisfy the relation stated above we need to define the group $\mathrm{End}_0^{D1}(\Lambda; \mathrm{id}, \mathrm{id}; S'_n)$. This is necessary since we do not have something like an injective characteristic polynomial for $\mathrm{End}_0(\Lambda; \mathrm{id}, \mathrm{id}; S'_n)$.

Definition 3.3.10 ($\mathrm{END}^{D1}(\Lambda;\mathbf{id},\mathbf{id};T)$). Let Λ be a ring and let $T\subseteq\Lambda[t]$ be a multiplicatively closed set containing t. Let $\mathrm{END}^{D1}(\Lambda;\mathrm{id},\mathrm{id};T)$ be the full subcategory of $\mathrm{END}(\mathcal{M}^{all}(\Lambda);\Lambda,\Lambda)$ consisting of those objects which have a $\mathrm{END}(\Lambda;\mathrm{id},\mathrm{id};T)$-resolution of length one.

Proposition 3.3.11. *Let the notation be as above. The category* $\mathrm{END}^{D1}(\Lambda;id,id;T)$ *is closed under extension in the abelian category* $\mathrm{END}(\mathcal{M}^{all}(\Lambda);\Lambda,\Lambda)$ *and therefore an exact category.*

Proof. The proposition is proven in the same manner as Lemma 2.1.2 is proven. □

Definition 3.3.12 ($\mathrm{End}_0(\Lambda;\mathbf{id},\mathbf{id};T)$, $\mathrm{End}_0^{D1}(\Lambda;\mathbf{id},\mathbf{id};T)$). Let the notation be as above.

1. We define $\mathrm{End}_0(\Lambda;\mathrm{id},\mathrm{id};T)$ to be the kernel of the map on K_0 which is induced by the forgetful functor from $\mathrm{END}(\Lambda;\mathrm{id},\mathrm{id};T)$ onto $\mathcal{P}(\Lambda)\times\mathcal{P}(\Lambda)$ whose value at (C,D,c,d) is (C,D).
2. Let $\mathcal{P}^{D1}(\Lambda)$ be the full subcategory of $\mathcal{M}^{all}(\Lambda)$ consisting of those object which have a $\mathcal{P}(\Lambda)$-resolution of length one.
3. We define $\mathrm{End}_0^{D1}(\Lambda;\mathrm{id},\mathrm{id};T)$ to be the kernel of the map on K_0 which is induced by the forgetful functor from $\mathrm{END}^{D1}(\Lambda;\mathrm{id},\mathrm{id};T)$ onto $\mathcal{P}^{D1}(\Lambda)\times\mathcal{P}^{D1}(\Lambda)$ whose value at (M,N,m,n) is (M,N).

Lemma 3.3.13. *Let the notation be as above. The inclusion map*

$$\mathrm{End}_0(\Lambda;id,id;T)\longrightarrow\mathrm{End}_0^{D1}(\Lambda;id,id;T)$$

is an isomorphisms.

Proof. The category $\mathcal{P}(\Lambda)$ is a subcategory of $\mathcal{P}^{D1}(\Lambda)$. Every object in $\mathcal{P}^{D1}(\Lambda)$ has, by construction, a finite $\mathcal{P}(\Lambda)$-resolution. To apply the Resolution Theorem it remains to show that $\mathcal{P}(\Lambda)$ is closed under kernels. Let

$$0\longrightarrow M\longrightarrow P\longrightarrow P'\longrightarrow 0$$

be a short exact sequence in $\mathcal{P}^{D1}(\Lambda)$ with P and P' in $\mathcal{P}(\Lambda)$. The sequence splits since P' is a projective module. Since P is projective we get that M is projective. Thus $\mathcal{P}(\Lambda)$ is closed under kernels in $\mathcal{P}^{D1}(R)$. An application of the Resolution Theorem gives now that the inclusion map from $K_0\big(\mathcal{P}(\Lambda)\big)$ into $K_0\big(\mathcal{P}^{D1}(\Lambda)\big)$ is an isomorphism.

Since the exact structure on $\mathrm{END}(\Lambda;\mathrm{id},\mathrm{id};T)$ and $\mathrm{END}^{D1}(\Lambda;\mathrm{id},\mathrm{id};T)$ comes from the underlying exact structure of $\mathcal{P}(\Lambda)$ and $\mathcal{P}^{D1}(\Lambda)$, we obtain that the inclusion map from $K_0\big(\mathrm{END}(\Lambda;\mathrm{id},\mathrm{id};T)\big)$ into $K_0\big(\mathrm{END}^{D1}(\Lambda;\mathrm{id},\mathrm{id};T)\big)$ is an isomorphism by the same reasoning as above.

Combining these two results we get the lemma. □

Theorem 3.3.14. *Let Λ be an algebra over a ring R, let X and Y be left flat Λ-bimodules and let U_X and U_Y be natural transformations between the identity and F_X and F_Y with right inverse natural transformations U_X^{-1} and U_Y^{-1}. Let y be an element of $\mathrm{End}_0(R)$ and let x be an element of $\mathrm{Nil}_i(\Lambda; X, Y)$ for $i \in \mathbb{Z}$. For odd $n \in \mathbb{N}$, we have*

$$V_n(y * F_n x) = (V_n y) * x$$

where $*$ is the module multiplication defined in Section 3.1.

Proof. We prove the statement for higher Nil-groups. The arguments extend to lower Nil-groups in the obvious way. The pairing between the categories $\mathrm{END}(R) \times \mathrm{NIL}(\Lambda; X, Y)$ and $\mathrm{END}(\mathbb{Z}[y, v, w]_\delta[x]; \mathrm{id}, \mathrm{id}; S'_n)$ defined above gives that for every object (C, D, c, d) of $\mathrm{END}(\mathbb{Z}[y, v, w]_\delta[x]; \mathrm{id}, \mathrm{id}; S'_n)$ we get an exact endofunctor of $\mathrm{END}(R) \times \mathrm{NIL}(\Lambda; X, Y)$. Since objects of the form $(C, D, 0, 0)$ reflect $\mathrm{END}(R) \times \mathrm{NIL}(\Lambda; X, Y)$ into the subcategory where elements are of the form $(P, Q, 0, 0)$ we get that every element in $\mathrm{End}_0(\mathbb{Z}[y, v, w]_\delta[x]; \mathrm{id}, \mathrm{id}; S'_n)$ gives rise to an operation on $\mathrm{End}_0(R) \times \mathrm{Nil}_i(\Lambda; X, Y)$.

Let (B, φ) be an object of $\mathrm{END}(R)$, let (P, Q, p, q) be an object of $\mathrm{NIL}(\Lambda; X, Y)$, let $n = 2\ell + 1$ for $\ell \in \mathbb{N}$. We define p^n to be $p \circ (U_Y^{-1}(P) \circ q \circ U_X^{-1}(P) \circ p)^{2\ell}$. The morphism q^n is defined similarly. We have

$$\mathrm{V}_n\big((B, \varphi) \times \mathrm{F}_n(P, Q, p, q)\big) \oplus (B \otimes Q, B \otimes P, 0, 0)^{\ell+1} \oplus (B \otimes P, B \otimes Q, 0, 0)^{\ell}$$

$$= \mathrm{V}_n\big((B \otimes P), (B \otimes Q), \varphi \otimes p^n, \varphi \otimes q^n \cdot g^{-\ell})\big) \oplus (B \otimes Q, B \otimes P, 0, 0)^{\ell+1} \oplus (B \otimes P, B \otimes Q, 0, 0)^{\ell}$$

$$= \Big((B \otimes P \oplus B \otimes Q)^{\ell} \oplus (B \otimes P), (B \otimes Q \oplus B \otimes P)^{\ell} \oplus (B \otimes Q),$$
$$\begin{pmatrix} 0 & & & \varphi \otimes p^n \\ U_X(B \otimes P) & \ddots & & \\ & \ddots & 0 & \\ & & U_X(B \otimes Q) & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & \varphi \otimes q^n \\ U_Y(B \otimes Q) & \ddots & & \\ & \ddots & 0 & \\ & & U_Y(B \otimes P) & 0 \end{pmatrix}\Big)$$
$$\oplus (B \otimes Q, B \otimes P, 0, 0)^{\ell+1} \oplus (B \otimes P, B \otimes Q, 0, 0)^{\ell}$$

$$\cong \Big(\mathbb{Z}[y, v, w]_\delta[x]^n, \mathbb{Z}[y, v, w]_\delta[x]^n, \begin{pmatrix} 0 & & & & yx^n v \\ v & 0 & & & \\ & w & \ddots & & \\ & & \ddots & 0 & \\ & & & w & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & yx^n w \\ w & 0 & & & \\ & v & \ddots & & \\ & & \ddots & 0 & \\ & & & v & 0 \end{pmatrix}\Big)$$
$$\times (B, \varphi) \times (P, Q, p, q)$$

and

$$\big(\mathrm{V}_n(B,\varphi)\big) \times (P,Q,p,q) \oplus (B\otimes Q, B\otimes P, 0,0)^n$$

$$= \left(B^n \otimes P, B^n \otimes Q, \begin{pmatrix} 0 & & & \varphi \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{pmatrix} \otimes p, \begin{pmatrix} 0 & & & \varphi \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{pmatrix} \otimes q\right) \oplus (B\otimes Q, B\otimes P, 0,0)^n$$

$$\cong \left(\mathbb{Z}[y,v,w]_\delta[x]^n, \mathbb{Z}[y,v,w]_\delta[x]^n, \begin{pmatrix} 0 & & & & yxv \\ xv & 0 & & & \\ & xv & \ddots & & \\ & & \ddots & 0 & \\ & & & xv & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & yxw \\ xw & 0 & & & \\ & xw & \ddots & & \\ & & \ddots & 0 & \\ & & & xw & 0 \end{pmatrix}\right)$$

$$\times (B,\varphi) \times (P,Q,p,q).$$

Thus it remains to show that

$$C_1 := \left(\mathbb{Z}[y,v,w]_\delta[x]^n, \mathbb{Z}[y,v,w]_\delta[x]^n, \begin{pmatrix} 0 & & & & yx^nv \\ v & 0 & & & \\ & w & \ddots & & \\ & & \ddots & 0 & \\ & & & w & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & yx^nw \\ w & 0 & & & \\ & v & \ddots & & \\ & & \ddots & 0 & \\ & & & v & 0 \end{pmatrix}\right)$$

and

$$C_2 := \left(\mathbb{Z}[y,v,w]_\delta[x]^n, \mathbb{Z}[y,v,w]_\delta[x]^n, \begin{pmatrix} 0 & & & & yxv \\ xv & 0 & & & \\ & xv & \ddots & & \\ & & \ddots & 0 & \\ & & & xv & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & yxw \\ xw & 0 & & & \\ & xw & \ddots & & \\ & & \ddots & 0 & \\ & & & xw & 0 \end{pmatrix}\right)$$

represent the same elements in $\mathrm{End}_0(\mathbb{Z}[y,v,w]_\delta[x]; \mathrm{id}, \mathrm{id}; S'_n)$.

Since $\mathrm{End}_0^{D1}(\mathbb{Z}[y,v,w]_\delta[x]; \mathrm{id}, \mathrm{id}; S'_n)$ and $\mathrm{End}_0(\mathbb{Z}[y,v,w]_\delta[x]; \mathrm{id}, \mathrm{id}; S'_n)$ are naturally isomorphic it is enough to prove that the objects represent the same element in $\mathrm{End}_0^{D1}(\mathbb{Z}[y,v,w]_\delta[x]; \mathrm{id}, \mathrm{id}; S'_n)$.

Consider the map $\iota\colon C_1 \to C_2$ in $\mathrm{End}_0^{D1}(\mathbb{Z}[y,v,w]_\delta[x]; \mathrm{id}, \mathrm{id}; S'_n)$ which is induce by the injective maps

$$\iota_1 := \begin{pmatrix} 1 & & & \\ & x & & \\ & & \ddots & \\ & & & x^{n-1} \end{pmatrix},$$

$$\iota_2 := \begin{pmatrix} 1 & & & \\ & x & & \\ & & \ddots & \\ & & & x^{n-1} \end{pmatrix}.$$

The object

$$C := \Big(\bigoplus_{i=0}^{n-1} \mathbb{Z}[y,v,w]_\delta[x]/x^i, \bigoplus_{i=0}^{n-1} \mathbb{Z}[y,v,w]_\delta[x]/x^i,$$

$$\left(\begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & xv & \ddots & & \\ & & \ddots & 0 & \\ & & & xv & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & xw & \ddots & & \\ & & \ddots & 0 & \\ & & & xw & 0 \end{pmatrix} \right)$$

is the cokernel of ι in $\mathrm{END}^{D1}(\mathbb{Z}[y,v,w]_\delta[x];\mathrm{id},\mathrm{id};S'_n)$. Thus we get a short exact sequence

$$0 \longrightarrow C_1 \xrightarrow{\iota} C_2 \longrightarrow C \longrightarrow 0$$

in $\mathrm{END}^{D1}(\mathbb{Z}[y,v,w]_\delta[x];\mathrm{id},\mathrm{id};S'_n)$.

Notice that objects of the form $\left[\left(\bigoplus_{i=0}^n M_i, \bigoplus_{i=0}^n M_i, m_1, m_2\right)\right]$ with m_1 and m_2 lower triangular matrices vanish in $\mathrm{End}_0^{D1}(\mathbb{Z}[y,v,w]_\delta[x];\mathrm{id},\mathrm{id};S'_n)$. Thus

$$\big[(C)\big] = 0$$

in $\mathrm{End}_0^{D1}(\mathbb{Z}[y,v,w]_\delta[x];\mathrm{id},\mathrm{id};S'_n)$, which implies that $[C_1]$ and $[C_2]$ represent the same element in $\mathrm{End}_0^{D1}(\mathbb{Z}[y,v,w]_\delta[x];\mathrm{id},\mathrm{id};S'_n)$ and therefore the required identity. □

3.4 Farrell and Waldhausen Nil-Groups as Modules over the Ring of Witt vectors

In this section, we prove that if we have natural transformations U_X and U_Y between the identity and F_X and F_Y which have a right inverse then $\mathrm{Nil}_i(\Lambda;X)$ and $\mathrm{Nil}_i(\Lambda;X,Y)$ are modules over the ring of Witt vectors of R.

In Section 3.1, we have seen that Nil-groups are modules over $\mathrm{End}_0(R)$. As mentioned in the introduction, $\mathrm{End}_0(R)$ is a dense subring of the ring of Witt vectors with the t-adic topology. The ideals which give rise to the topology are $I_N := \{1+t^N R[\![t]\!]\}$. To extend the $\mathrm{End}_0(R)$-module structure to a Witt vector-module structure we need to prove that for each $x \in \mathrm{Nil}$ there is an N such that x is annihilated by I_N. For Bass Nil-groups this is proven by Stienstra [Sti82].

Theorem 3.4.1. *Let Λ be an algebra over a commutative ring R, let X and Y be left flat Λ-bimodules. If we have natural transformations U_X and U_Y between the identity and F_X and F_Y which have right inverse natural transformations then for every x in $\mathrm{Nil}_i(\Lambda;X)$ or $\mathrm{Nil}_i(\Lambda;X,Y)$ there is an N such that x is annihilated by the ideal I_N. Thus $\mathrm{Nil}_i(\Lambda;X)$ and $\mathrm{Nil}_i(\Lambda;X,Y)$ are modules over the ring of Witt vectors.*

Proof. This result follows from Theorem 3.3.8/3.3.14 in the same manner as Corollary 3.3.5 is obtained from Proposition 3.2.5. □

4 Applications

In the second chapter, the behavior of Nil-groups under localization was studied. The main result is Theorem 2.4.1 saying that

$$\mathbb{Z}[t,t^{-1}] \otimes_{\mathbb{Z}[t]} \mathrm{Nil}_i(R;X,Y,Z,W) \cong \mathrm{Nil}_i(R_s;X_s,Y_s,Z_s,W_s).$$

for all $i \in \mathbb{Z}$. In the third chapter, we developed a Witt vector-module structure on certain Nil-groups. In Section 4.1, we combine these at first glance totally unrelated results to improve Theorem 2.4.1 to the statement that if R is $\mathbb{Z}_T$ for some multiplicatively closed set $T \subseteq \mathbb{Z} - \{0\}$, $\hat{\mathbb{Z}}_p$ or a commutative $\mathbb{Q}$-algebra, if S is a multiplicatively closed subset of non zero divisors of R, if Λ is an algebra over R and X and Y are Λ-bimodules with natural transformations from the identity to F_X and F_Y which have a left inverse, we have

$$R_S \otimes_R \mathrm{Nil}_i(\Lambda;X) \cong \mathrm{Nil}_i(\Lambda_S;X_S)$$

and

$$R_S \otimes_R \mathrm{Nil}_i(\Lambda;X,Y) \cong \mathrm{Nil}_i(\Lambda_S;X_S,Y_S).$$

for all $i \in \mathbb{Z}$. The given assumptions are, for example, satisfied if G is a group and Λ is RG, X is RG_α and Y is RG_β for inner automorphism α and β.

The main applications are torsion results. Whenever we know that the Nil-groups of the ring $\mathbb{Z}[1/n]G$ vanish, we get that the Nil-groups of $\mathbb{Z}G$ are n-torsion. In Section 4.3, it is proven that for every polycyclic-by-finite group we can find such an n.

The assumption that α and β are inner group automorphisms is quite restrictive. To weaken the assumptions on α, we define in Section 4.2 induction and transfer maps on Nil-groups which are similar to the induction and transfer maps in algebraic K-theory [Mil71, Chapter 14].

In the Section 4.4, we use this machinery to prove that Nil-groups of polycyclic-by-finite groups are torsion groups if α and β are group automorphism of finite order. As an important corollary we get that the Nil-groups of finite groups are torsion groups.

As an application of the torsion results we prove in Section 4.5 that the relative assembly map from the family of finite subgroups to the family of virtually cyclic subgroups is rationally an isomorphism.

4.1 Improved Localization Results

In this section, we use the Witt vector-module structure developed in the preceding chapter to improve Theorem 2.4.1.

Lemma 4.1.1. *Let Λ be an algebra over a commutative ring R and let X and Y be Λ-bimodules such that there are natural transformations from the identity to the functor F_X and F_Y which have right inverse natural transformations. Then for every multiplicatively closed set $S \subset R$ of non zero divisors satisfying $s \cdot x = x \cdot s$ for all $s \in S$ and $x \in X$ and a similar condition for Y we have*

$$\{(1 - st) \mid s \in S\}^{-1} W(R) \otimes_{W(R)} \mathrm{Nil}_i(\Lambda; X) \cong \mathrm{Nil}_i(\Lambda_G; X_s)$$

and

$$\{(1 - st) \mid s \in S\}^{-1} W(R) \otimes_{W(R)} \mathrm{Nil}_i(\Lambda; X, Y) \cong \mathrm{Nil}_i(\Lambda_S; X_S, Y_S),$$

for all $i \in \mathbb{Z}$.

Proof. In the following, $\mathrm{Nil}_i(\Lambda; X)$ is treated, the same arguments work for $\mathrm{Nil}_i(\Lambda; X, Y)$.

Lemma 4.1.1 is proven for the special case that S is generated by a single element s. Iterated use of the argument shows that

$$\{(1 - st) \mid s \in \langle s_1, \dots, s_m\rangle\}^{-1} W(R) \otimes_{W(R)} \mathrm{Nil}_i(\Lambda; X) \cong \mathrm{Nil}_i(\Lambda_{\langle s_1, \dots, s_m\rangle}; X_{\langle s_1, \dots, s_m\rangle})$$

for finitely generated $S = \langle s_1, \dots, s_m\rangle$. To get the general case we need to pass to the colimit. Checking on generators we obtain that

$$\begin{array}{ccc}
\{(1 - st) \mid s \in \langle s_1\rangle\}^{-1} W(R) \otimes_{W(R)} \mathrm{Nil}_i(\Lambda; X) & \xrightarrow{\cong} & \mathrm{Nil}_i(\Lambda_{\langle s_1\rangle}; X_{\langle s_1\rangle}) \\
\big\downarrow & & \big\downarrow \mathrm{L} \\
\{(1 - st) \mid s \in \langle s_1, s_2\rangle\}^{-1} W(R) \otimes_{W(R)} \mathrm{Nil}_i(\Lambda; \alpha) & \xrightarrow{\cong} & \mathrm{Nil}_i(\Lambda_{\langle s_1, s_2\rangle}; X_{\langle s_1, s_2\rangle}) \\
\big\downarrow & & \big\downarrow \mathrm{L} \\
\vdots & & \vdots
\end{array}$$

commutes. Thus we get an isomorphism between

$$\varinjlim \Big(\{(1 - st) \mid s \in \langle s_1, \dots, s_m\rangle\}^{-1} W(R) \otimes_{W(R)} \mathrm{Nil}_i(\Lambda; X)\Big)$$

and

$$\varinjlim \mathrm{Nil}_i(\Lambda_{\langle s_1, \dots, s_m\rangle}; X_{\langle s_1, \dots, s_m\rangle}),$$

which are $\{(1 - st) \mid s \in S\}^{-1} W(R) \otimes_{W(R)} \mathrm{Nil}_i(\Lambda; \alpha)$ and $\mathrm{Nil}_i(\Lambda_S; X_S)$.

As pointed out in the introduction, $\mathrm{End}_0(R)$ is a dense subring of $W(R)$. The elements $(1 - st)$ belong to the objects $[(R, s)]$ in $\mathrm{End}_0(R)$. Since $(R, s) \times ? \cong \mathrm{F}_s$, Theorem 2.4.1 implies that

$$\{(1 - st) \mid s \in \langle s\rangle\}^{-1} W(R) \otimes_{W(R)} \mathrm{Nil}_i(\Lambda; X) \cong \mathrm{Nil}_i(\Lambda_s; X_s). \qquad \square$$

Theorem 4.1.2. *Let R be $\mathbb{Z}_T$ for some multiplicatively closed set $T \subseteq \mathbb{Z} - \{0\}$, $\hat{\mathbb{Z}}_p$ or a commutative $\mathbb{Q}$-algebra and let Λ be an R-algebra. Let X and Y be Λ-bimodules such that there are natural transformations from the identity to the functor F_X and F_Y which have a right inverse. Then for every multiplicatively closed set $S \subset R$ of non zero divisors satisfying $s \cdot x = x \cdot s$ for all $s \in S$ and $x \in X$ and a similar condition for Y there are isomorphisms of R_S-modules*

$$R_S \otimes_R \mathrm{Nil}_i(\Lambda; X) \cong \mathrm{Nil}_i(\Lambda_S; X_S)$$

and

$$R_S \otimes_R \mathrm{Nil}_i(\Lambda; X, Y) \cong \mathrm{Nil}_i(\Lambda_S; X_S, Y_S),$$

for all $i \in \mathbb{Z}$.

Proof. This proof follows Weibel's proof of the same identity for Bass Nil-groups [Wei81, page 489]. Again just the groups $\mathrm{Nil}_i(\Lambda; X)$ are treated, the same arguments work for $\mathrm{Nil}_i(\Lambda; X, Y)$.

Let $W_N(R) := W(R)/I_N$ be the *truncated ring of Witt vectors.*

The group $\mathrm{Nil}_i(\Lambda; X)$ is a colimit over the family of finitely generated $W(R)$-submodules M. Since M is assumed to be finitely generated, Theorem 3.3.8 and Proposition 3.2.5 imply that the $W(R)$-module structure restricts to a $W_N(R)$-module structure for a certain N. For a ring R which is $\mathbb{Z}_T$ for some multiplicatively closed set $T \subseteq \mathbb{Z} - \{0\}$, $\hat{\mathbb{Z}}_p$ or a commutative $\mathbb{Q}$-algebra Weibel [Wei81, Proposition 6.2] proves the identity

$$\{(1 - st) \,|\, s \in S\}^{-1} W_N(R) \cong W_N(R_S).$$

Since R is a λ-ring, M caries an R-module structure. We have

$$\begin{aligned} R_S \otimes_R M &\cong W(R_S) \otimes_{W(R)} M \\ &\cong W_N(R_S) \otimes_{W(R)} M \\ &\cong \{(1 - st) \,|\, s \in S\}^{-1} W_N(R) \otimes_{W(R)} M \\ &\cong \{(1 - st) \,|\, s \in S\}^{-1} W(R) \otimes_{W(R)} M. \end{aligned}$$

Taking the colimit of both sides gives

$$R_S \otimes_R \mathrm{Nil}_i(\Lambda; X) \cong \{(1 - st) \,|\, s \in S\}^{-1} W(R) \otimes_{W(R)} \mathrm{Nil}_i(\Lambda; X).$$

Thus Lemma 4.1.1 implies

$$R_S \otimes_R \mathrm{Nil}_i(\Lambda; X) \cong \mathrm{Nil}_i(\Lambda_S; X_S).$$

□

4.2 Transfer and Induction on Nil-Groups

In algebraic K-theory, it is well-known that if we have a ring Γ and a subring Λ such that Γ is finitely generated projective as a module over Λ, we can define maps

$\iota^i\colon K_i(\Gamma) \to K_i(\Lambda)$. These so called *transfer maps* are induced by the functor whose value at a projective Γ-module is the same module, seen as a Λ-module. The inclusion map $\iota\colon \Lambda \hookrightarrow \Gamma$ induces maps $\iota_i\colon K_i(\Lambda) \to K_i(\Gamma)$, the so called *induction maps.* We generalize this concept to Nil-groups.

Definition 4.2.1 (Induction and Transfer). Let Γ be a ring with a subring Λ. Let $\iota\colon \Lambda \hookrightarrow \Gamma$ be the inclusion map and let α and β be a ring automorphisms of Γ which restricts to Λ.

1. Define a functor u from $\mathcal{M}^{all}(\Lambda)$ to $\mathcal{M}^{all}(\Gamma)$ which sends a Λ-module M to the Γ-module $M \otimes_\Lambda \Gamma$. We denote the natural transformation between $u \circ \mathrm{F}_{\Lambda_\alpha}$ and $\mathrm{F}_{\Gamma_\alpha} \circ u$ which is induced by the map
$$\begin{aligned} \Lambda_\alpha \otimes_\Lambda \Gamma &\to \Gamma_\alpha \\ \lambda \otimes \gamma &\mapsto \lambda \cdot \alpha(\gamma), \end{aligned}$$
by U. We define ι_i to be $\mathrm{Nil}_i(u, U)\colon \mathrm{Nil}_i(\Lambda;\alpha) \to \mathrm{Nil}_i(\Gamma;\alpha)$. We call these maps *induction maps.*

2. We denote the natural transformation between $u \circ \mathrm{F}_{\Lambda_\alpha,\Lambda_\beta}$ and $\mathrm{F}_{\Gamma_\alpha,\Gamma_\beta} \circ u$ which is induced by the map
$$\begin{aligned} \Lambda_\alpha \otimes_\Lambda \Gamma &\to \Gamma_\alpha \\ \lambda \otimes \gamma &\mapsto \lambda \cdot \alpha(\gamma), \\ \Lambda \otimes_\Lambda \Gamma &\to \Gamma \\ \lambda \otimes \gamma &\mapsto \lambda \cdot \beta(\gamma), \end{aligned}$$
by U_W. We define ι_i to be $\mathrm{Nil}_i(u, U_W)\colon \mathrm{Nil}_i(\Lambda;\Lambda_\alpha,\Lambda_\beta) \to \mathrm{Nil}_i(\Gamma;\Gamma_\alpha,\Gamma_\beta)$. We call these maps *induction maps.*

If we have additionally that Γ_ι is a finitely generated projective right Λ-module, we can define transfer maps.

1. We define a functor T from $\mathrm{NIL}(\Gamma;\alpha)$ to $\mathrm{NIL}(\Lambda;\alpha)$ whose value at an object (P,ν) is (P_ι,ν). On morphisms, T is the identity. The map which is induced on the i-th Nil-groups is denoted by ι^i and called *transfer map.*

2. We define a functor T from $\mathrm{NIL}(\Gamma;\Gamma_\alpha,\Gamma_\beta)$ to $\mathrm{NIL}(\Lambda;\Lambda_\alpha,\Lambda_\beta)$ whose value at an object (P,Q,p,q) is (P_ι,Q_ι,p,q). On morphisms, T is the identity. The map which is induced on the i-th Nil-groups is denoted by ι^i and called *transfer map.*

Example 4.2.2. The situation which will become important is that we have a group G and a group automorphism α of finite order m. In this case we can form the semidirect product of G and the cyclic group of order m, which is denoted by C_m. Conjugation with the element $(\mathrm{id}, 1)$ extends α to an inner group automorphism of

$G \rtimes_\alpha C_m$, which is denoted by $\tilde{\alpha}$. The group ring $RG \rtimes_\alpha C_m$, seen as a bimodule over RG, is isomorphic to $\bigoplus_{i=0}^{n-1} RG_{\alpha^i}$. Thus we get induction maps

$$\iota_i \colon \operatorname{Nil}_i(RG;\alpha) \to \operatorname{Nil}_i(RG \rtimes C_m;\tilde{\alpha})$$

and transfer maps

$$\iota^i \colon \operatorname{Nil}_i(RG \rtimes C_m;\tilde{\alpha}) \to \operatorname{Nil}_i(RG;\alpha).$$

If β is a group automorphism of finite order m'. By the same reasoning as above we can define induction maps

$$\iota_i \colon \operatorname{Nil}_i(RG;RG_\alpha,RG_\beta) \to \operatorname{Nil}_i(RG \rtimes C_{m\cdot m'};RG \rtimes C_{m\cdot m'\,\tilde{\alpha}},RG \rtimes C_{m\cdot m'\,\tilde{\beta}})$$

and transfer maps

$$\iota^i \colon \operatorname{Nil}_i(RG \rtimes C_{m\cdot m'};RG \rtimes C_{m\cdot m'\,\tilde{\alpha}},RG \rtimes C_{m\cdot m'\,\tilde{\beta}}) \to \operatorname{Nil}_i(RG;RG_\alpha,RG).$$

Proposition 4.2.3. *Let the notation be as in the preceding example. We have*

$$\iota^i \circ \iota_i(x) = x \cdot m,$$

for x in $\operatorname{Nil}_i(RG;\alpha)$ and $i \in \mathbb{Z}$. We have

$$\iota^i \circ \iota_i(x) = x \cdot m \cdot m',$$

for x in $\operatorname{Nil}_i(RG;RG_\alpha,RG_\beta)$ and $i \in \mathbb{Z}$.

Proof. We prove the identity for $\operatorname{Nil}_i(RG;\alpha)$ the same arguments hold for $\operatorname{Nil}_i(RG;RG_\alpha,RG)$. The identity is proven for higher Nil-groups. Since $\Sigma RG = (\Sigma R)G$ we obtain the identity for lower Nil-groups in the obvious way.

Let (P,ν) be an object of $\operatorname{NIL}(RG;\alpha)$. We have

$$\mathrm{T} \circ \operatorname{NIL}(U,u)\big((P,\nu)\big) = \Big(P \otimes \bigoplus_{i=0}^{m-1} RG_{\alpha^i}, U(P) \circ (\nu \otimes \mathrm{id})\Big).$$

Right multiplication with the element $(\mathrm{id}, m-i)$ induces an RG-bimodule isomorphism between RG_{α^i} and RG which commutes with α. The induced RG-bimodule isomorphism between $\bigoplus_{i=0}^{m-1} RG_{\alpha^i}$ and $\bigoplus_{i=0}^{m-1} RG$ is denoted by $\tilde{g}$. We have

$$\tilde{g} \circ \tilde{\alpha} \circ \tilde{g}^{-1}(x) = \tilde{\alpha}(x)$$

for all $x \in RG \rtimes C_m$. The map $\tilde{g}$ induces an isomorphism between

$$\Big(P \otimes \bigoplus_{i=0}^{m-1} RG_{\alpha^i}, U(P) \circ (\nu \otimes \mathrm{id})\Big)$$

and

$$\Big(P \otimes \bigoplus_{i=0}^{m-1} RG, (\mathrm{id} \otimes \tilde{g}) \circ U(P) \circ (\nu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tilde{g}^{-1})\Big).$$

Using the identity $\tilde{g} \circ \tilde{\alpha} \circ \tilde{g}^{-1} = \tilde{\alpha}$ we obtain that the canonical isomorphism between $P \otimes \bigoplus_{i=0}^{n-1} RG$ and P^n provides an isomorphism to

$$\left(P^n, \begin{pmatrix} \nu & & & \\ & \nu & & \\ & & \ddots & \\ & & & \nu \end{pmatrix}\right).$$

Thus we obtain the identity on $K_i\big(\operatorname{NIL}(RG;\alpha)\big)$ and therefore on $\operatorname{Nil}_i(RG;\alpha)$. □

4.3 Regular Group Rings

Let G be an arbitrary group. In view of Theorem 4.1.2, the question whether we can find $n \in \mathbb{Z}$ such that $\operatorname{Nil}_i(\mathbb{Z}[1/n]G; X_{[1/n]})$ or $\operatorname{Nil}_i(\mathbb{Z}[1/n]G; X_{[1/n]}, Y_{[1/n]})$ vanishes arises. The main result of this section is that for polycyclic-by-finite groups we can always find such an n.

Definition 4.3.1 (Regular). A ring is called *regular* if it is right noetherian and every finitely generated right module has a finite resolution of finitely generated projective right modules.

Examples of regular rings are fields, $\mathbb{Z}$ and $\mathbb{Z}_T$ if T is a multiplicatively closed subset of $\mathbb{Z}$.

It is a result of Waldhausen that for $i \geq 0$ and for a regular ring R the group $\operatorname{Nil}_i(R; X, Y, Z, W)$ vanishes [Wal78a, Wal78b]. The result is extend to lower Nil-groups in [BL04]. Thus we can rephrase the question, stated at the beginning, to the question whether we can find an n such that $\mathbb{Z}[1/n]G$ is regular. To find a class of groups with this property we need the following definition:

Definition 4.3.2. Suppose $\mathcal{P}$ and $\mathcal{S}$ are properties of groups. A group is called *poly-$\mathcal{P}$* if there exists a finite chain of normal subgroups:

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_m = G$$

such that every G_i/G_{i-1} has property $\mathcal{P}$.

A group is called *$\mathcal{P}$-by-$\mathcal{S}$* if G has a normal subgroup N such that N has property $\mathcal{P}$ and G/N has property $\mathcal{S}$.

Lemma 4.3.3. *Let R be a ring, let G be a group and let H be a subgroup of finite index n with the property that RH is regular. If n is invertible in R, then RG is also regular.*

Proof. The ring RG is right-noetherian since RH is right noetherian and RG is a finitely generated right module over RH. Let M be a finitely generated RG-module. The module M seen as an RH-module is denoted by $\operatorname{res} M$. Since RH is regular, res M has a finite projective RH-resolution. Applying $-\otimes_{RH} RG$ yields an RG-resolution of res $M \otimes_{RH} RG$. Let S be a set of representatives of right cosets. Since n is invertible in R, we can define the following RG-module map:

$$\begin{aligned} M &\to \operatorname{res} M \otimes_{RH} RG \\ m &\mapsto 1/n \sum_{g \in S} mg^{-1} \otimes g. \end{aligned}$$

This is an RG-module map since for $\lambda \in G$ we have

$$\begin{aligned} 1/n \sum_{g\in S} m\lambda g^{-1} \otimes g &= 1/n \sum_{g \in S} m(g\lambda^{-1})^{-1} \otimes g(\lambda\lambda^{-1}) \\ &= 1/n \sum_{g \in S} m(g\lambda^{-1})^{-1} \otimes (g\lambda^{-1})\lambda \\ &= 1/n \sum_{g\in S} mg^{-1} \otimes g\lambda. \end{aligned}$$

Since the composition of this map with the canonical map from res $M \otimes_{RH} RG$ to M is the identity, we get that M, as an RG-module, is a direct summand of res $M \otimes_{RH} RG$. Hence M has a finite projective resolution. □

Proposition 4.3.4. *Let G be a polycyclic-by-finite group. Then there exists a poly-infinite cyclic subgroup of finite index n. If n is invertible in a regular ring R, then the group ring RG is also regular.*

Proof. The family of polycyclic-by-finite groups coincides with the family of poly infinite cyclic-by-finite groups [Pas85, page 422]. Thus we can always find a poly infinite cyclic subgroup of finite index n. If we can show that for every poly infinite cyclic group H and regular ring R the group ring RH is regular, Lemma 4.3.3 implies the statement. This is done by induction on the length of the chain of normal subgroups with infinite cyclic quotient. In the case of a chain of length one we have $RH_0 = R$ which is regular. For the induction step from m to $m+1$ let

$$1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_{m+1} = H$$

be a chain of subgroups such that H_{i+1}/H_i is infinite cyclic. By our induction hypothesis RH_m is regular. Since H_{m+1}/H_m is cyclic, we get an exact sequence

$$1 \longrightarrow H_m \longrightarrow H_{m+1} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

This sequence splits since $\mathbb{Z}$ is free. Thus we get

$$H_{m+1} \cong H_m \rtimes H_{m+1}/H_m$$

and therefore

$$RH_{m+1} \cong RH_{m\,\alpha}[t, t^{-1}]$$

for some automorphism α. It is a result of Farrell and Hsiang that a twisted Laurent polynomial ring of a regular ring is regular [FH70]. Therefore RH is regular, which proves the statement. □

4.4 Torsion Results

In this section, we prove that the Nil-groups of polycyclic-by-finite groups are torsion groups. We obtain as an important corollary that the Nil-groups appearing in the calculation of the K-theory of virtually cyclic groups are torsion groups.

Theorem 4.4.1. *Let G be a polycyclic-by-finite group. Then there exists a poly-infinite cyclic subgroup of finite index n. Let α and β be group automorphisms of finite order m and m'.*

1. *The group $\mathrm{Nil}_i(\mathbb{Z}G;\alpha)$ is an $(n \cdot m)$-torsion group for $i \in \mathbb{Z}$.*
2. *The group $\mathrm{Nil}_i(\mathbb{Z}G;\mathbb{Z}G_\alpha,\mathbb{Z}G_\beta)$ is an $(n \cdot m \cdot m')$-torsion group for $i \in \mathbb{Z}$.*

Proof. In the following, $\mathrm{Nil}_i(\mathbb{Z}G;\alpha)$ is treated, the same arguments work for $\mathrm{Nil}_i(\mathbb{Z}G;\mathbb{Z}G_\alpha,\mathbb{Z}G_\beta)$. Since α is of finite order m, we can form the semidirect product of G and the cyclic group with m elements C_m. Conjugation with the element $(\mathrm{id},1)$ extends α to an inner automorphism of $G \rtimes C_m$, which is denoted by $\tilde{\alpha}$. Since $\tilde{\alpha}$ is an inner automorphism, we can apply Theorem 4.1.2. We have

$$\mathbb{Z}[1/n,1/m] \otimes_{\mathbb{Z}} \mathrm{Nil}_i\left(\mathbb{Z}G \rtimes C_m;\tilde{\alpha}\right) \cong \mathrm{Nil}_i\left(\mathbb{Z}[1/n,1/m]G \rtimes C_m;\tilde{\alpha}\right).$$

The right hand side is the trivial group by Proposition 4.3.4. Thus $\mathrm{Nil}_i(\mathbb{Z}G \rtimes C_m;\tilde{\alpha})$ is $(n \cdot m)$-torsion. An application of Proposition 4.2.3 yields that $\mathrm{Nil}_i(\mathbb{Z}G;\alpha)$ is $(n \cdot m)$-torsion. □

Corollary 4.4.2. *Let G be a finite group of order n. Let α and β be group automorphisms of finite order m and m'.*

1. *The group $\mathrm{Nil}_i(\mathbb{Z}G;\alpha)$ is an $(n \cdot m)$-torsion group for $i \in \mathbb{Z}$.*
2. *The group $\mathrm{Nil}_i(\mathbb{Z}G;\mathbb{Z}G_\alpha,\mathbb{Z}G_\beta)$ is an $(n \cdot m \cdot m')$-torsion group for $i \in \mathbb{Z}$.*

Theorem 4.4.3. *Let G be an arbitrary group and let α and β be group automorphisms of finite order m and m'.*

1. *The group $\mathbb{Z}[1/m] \otimes_{\mathbb{Z}} \mathrm{Nil}_i(\mathbb{Q}G;\alpha)$ is a $\mathbb{Q}$-algebra and therefore torsion free for $i \in \mathbb{Z}$. If G is a polycyclic-by-finite group then $\mathrm{Nil}_i(\mathbb{Q}G;\alpha)$ is the trivial group.*
2. *The group $\mathbb{Z}[1/(m \cdot m')] \otimes_{\mathbb{Z}} \mathrm{Nil}_i(\mathbb{Q}G;\mathbb{Q}G_\alpha,\mathbb{Q}G_\beta)$ is a $\mathbb{Q}$-algebra and therefore torsion free for $i \in \mathbb{Z}$. If G is a polycyclic-by-finite group then $\mathrm{Nil}_i(\mathbb{Q}G;\mathbb{Q}G_\alpha,\mathbb{Q}G_\beta)$ is the trivial group.*

Proof. The proof of this result follows the same pattern as Theorem 4.4.2 is proven. □

4.5 The Relative Assembly Map

In the final section, we prove that the relative assembly map from the family of finite groups to the family of virtually cyclic groups is rationally an isomorphism. The main ingredient of the proof is the torsion result of the preceding section. We begin by briefly recalling the relevant notions. For a survey on the various isomorphism conjectures see [LR04].

Definition 4.5.1 (Families of Subgroups). Let G be a group. A *family $\mathcal{F}$ of subgroups of* G is a set of subgroups of G closed under conjugation and finite intersection.

We denote the family of finite subgroups by $\mathcal{FIN}$, the family of virtually cyclic subgroup by $\mathcal{VCY}$ and the family of all subgroups by $\mathcal{ALL}$.

Definition 4.5.2 (Classifying space for a family of subgroups). Let $\mathcal{F}$ be a family of subgroups of a group G. A model $E_{\mathcal{F}}G$ for the *classifying G-CW-complex for the family* $\mathcal{F}$ is a G-CW-complex $E_{\mathcal{F}}G$ which has the following properties:

1. All isotropy groups of $E_{\mathcal{F}}G$ belong to $\mathcal{F}$.
2. For any G-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to E_{\mathcal{F}}G$.

The classifying G-CW-complex of a family always exists and is unique up to G-homotopy.

Example 4.5.3. 1. Let G be a finite group. The point with the trivial G-action is a model for $E_{\mathcal{FIN}}G$.

2. Let V be a virtually cyclic group of the first type, i.e. V is the semidirect product of a finite group H and $\mathbb{Z}$. Since V admits a surjection onto $\mathbb{Z}$ with finite kernel, a model for $E_{\mathcal{FIN}}\mathbb{Z}$ gives a model for $E_{\mathcal{FIN}}V$. The $\mathbb{Z}$-push out of
$$\begin{array}{ccc} S^0 \times \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \\ D^1 \times \mathbb{Z} & & \end{array}$$
is a model for $E_{\mathcal{FIN}}\mathbb{Z}$ and gives therefore a model for $E_{\mathcal{FIN}}V$. This model can be pictured by the real line
$$\text{——} \circ \text{——} \circ \text{——} \circ \text{——} \circ \text{——} \circ \text{——}$$
where $\mathbb{Z}$ acts via translation.

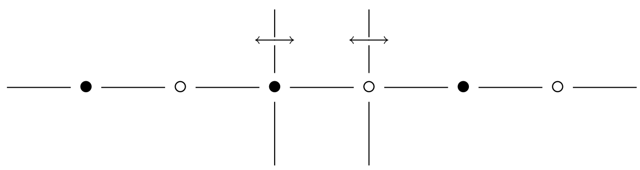
3. Let V be a virtually cyclic group of the second type, i.e. V is the amalgamated product of two finite groups G_1 and G_2 over a subgroup H of index

2. Since V admits a surjection onto $\mathbb{Z}_2 * \mathbb{Z}_2$ with finite kernel [FJ95], a model for $E_{\mathcal{FIN}}\mathbb{Z}_2 * \mathbb{Z}_2$ gives a model for $E_{\mathcal{FIN}}V$. The group $\mathbb{Z}_2 * \mathbb{Z}_2$ admits two canonical surjections onto $\mathbb{Z}_2$, let k_1 and k_2 be the kernels of these surjections. In the following $\mathbb{Z}_2 * \mathbb{Z}_2/k_j$ is identified with $\{+/-1\}$ and S^0 with the set of two points $\{pt_1, pt_2\}$. We define $q_1^i \colon \mathbb{Z}_2 * \mathbb{Z}_2/k_1 \times S^0 \to \mathbb{Z}$ to be $z2i$ for $z \in \mathbb{Z}_2 * \mathbb{Z}_2/k_1$ and $pt_1 \in S^0$ and $z(2i+2)$ for $z \in \mathbb{Z}_2 * \mathbb{Z}_2/k_1$ and $pt_2 \in S^0$ and we define $q_2^i \colon \mathbb{Z}_2 * \mathbb{Z}_2/k_2 \times S^0 \to \mathbb{Z}$ to be $z2i+1$ for $z \in \mathbb{Z}_2 * \mathbb{Z}_2/k_2$ and $pt_1 \in S^0$ and $z(2i+2)+1$ for $z \in \mathbb{Z}_2 * \mathbb{Z}_2/k_2$ and $pt_2 \in S^0$.

The $\mathbb{Z}_2 * \mathbb{Z}_2$-push out of

$$\begin{array}{ccc} \coprod_{i\in\mathbb{Z}} \left(\mathbb{Z}_2 * \mathbb{Z}_2/k_1 \times S^0 \coprod \mathbb{Z}_2 * \mathbb{Z}_2/k_2 \times S^0\right) & \xrightarrow{\coprod_{i\in\mathbb{Z}}(q_1^i \coprod q_2^i)} & \mathbb{Z} \\ \downarrow & & \\ \coprod_{i\in\mathbb{Z}} \left(\mathbb{Z}_2 * \mathbb{Z}_2/k_1 \times D^1 \coprod \mathbb{Z}_2 * \mathbb{Z}_2/k_2 \times D^1\right) & & \end{array}$$

is a model for $E_{\mathcal{FIN}}\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ and gives therefore a model for $E_{\mathcal{FIN}}V$. This model can be pictured by the real line



where the two copies of $\mathbb{Z}_2$ act via reflection on the different vertical axes.

A G-homology theory is the obvious generalization of a homology theory to the equivariant setting. In particular if G is the trivial group than a G-homology theory is just a homology theory. For a definition of a G-homology theory see [LR04]. Suppose we are given a family of subgroups $\mathcal{F}$ and a subfamily $\mathcal{F}' \subseteq \mathcal{F}$. By the universal property of $E_{\mathcal{F}}(G)$ we obtain a map $E_{\mathcal{F}'}(G) \to E_{\mathcal{F}}(G)$ which is unique up to G-homology. Thus for every G-homology theory $\mathcal{H}_*^G$ we obtain a *relative assembly map*

$$A_{\mathcal{F}'\to\mathcal{F}} \colon \mathcal{H}_*^G(E_{\mathcal{F}'}(G)) \to \mathcal{H}_*^G(E_{\mathcal{F}}(G)).$$

An *equivariant homology theory* consist of G-homology theories for every group G with an so called *induction structure* which connects the different homology theories. Again, for a definition we refer to [LR04]. In [DL98] it is explained how the K-theory spectrum of a ring R, in the sequel denoted by $\mathbf{K}_R$, gives a equivariant homology theory, in the sequel denoted by $H_i^?(-;\mathbf{K}_R)$.

The Farrell-Jones conjecture [FJ93] predicts that the $\mathcal{VCY}$*-assembly map*

$$A_{\mathcal{VCY}} \colon H_i^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R) \to H_i^G(E_{\mathcal{ALL}}(G);\mathbf{K}_R) \cong H_i^G(\mathrm{pt};\mathbf{K}_R) \cong K_i(RG)$$

is an isomorphism. It is known that the relative assembly map

$$A_{\mathcal{FIN}\to\mathcal{VCY}}\colon H_i^G(E_{\mathcal{FIN}};\mathbf{K}_R)\to H_i^G(E_{\mathcal{VCY}};\mathbf{K}_R)$$

is split injective [Bar03]. In this section, we prove that rationally the relative assembly map is an isomorphism.

Theorem 4.5.4. *Let G be a group and let i be a natural number. If $i \geq 0$ then the rationalized relative assembly map*

$$A_{\mathcal{FIN}\to\mathcal{VCY}}\colon H_i^G(E_{\mathcal{FIN}};\boldsymbol{K}_{\mathbb{Z}})\otimes\mathbb{Q}\to H_i^G(E_{\mathcal{VCY}};\boldsymbol{K}_{\mathbb{Z}})\otimes\mathbb{Q}$$

is an isomorphism. For $i < 0$, the relative assembly map is an isomorphism even integrally.

Proof. The proof follows closely a proof of the statement that the relative assembly map is an isomorphism for a regular ring R in which the orders of all finite subgroups of G are invertible [LR04, Proposition 2.14]. Because of the Transitivity Principle [LR04, Theorem 2.9] we need to prove that the $\mathcal{FIN}$-assembly map is an isomorphism for virtually cyclic groups V. As mentioned in the introduction we can assume that either $V\cong H\rtimes\mathbb{Z}$ or $V\cong G_1 *_H G_2$ with finite groups H, G_1 and G_2. In both cases we obtain long exact sequences involving the algebraic K-theory of the constituents, the algebraic K-theory of V and additional Nil-groups. If V is a virtually cyclic group of the first type or a virtually cyclic group of the second type the Nil-groups vanish rationally by Corollary 4.4.2. Thus we get long exact sequences

$$\cdots\longrightarrow K_i(RH)\otimes\mathbb{Q}\longrightarrow K_i(RH)\otimes\mathbb{Q}\longrightarrow$$

$$\longrightarrow K_i(RV)\otimes\mathbb{Q}\longrightarrow K_{i-1}(RH)\otimes\mathbb{Q}\longrightarrow K_{i-1}(RH)\otimes\mathbb{Q}\longrightarrow\cdots$$

and

$$\cdots\longrightarrow K_i(RH)\otimes\mathbb{Q}\longrightarrow\big(K_i(RG_1)\oplus K_i(RG_2)\big)\otimes\mathbb{Q}\longrightarrow K_i(RV)\otimes\mathbb{Q}\longrightarrow$$

$$\longrightarrow K_{i-1}(RH)\otimes\mathbb{Q}\longrightarrow\big(K_{i-1}(RG_1)\oplus K_{i-1}(RG_2)\big)\otimes\mathbb{Q}\longrightarrow\cdots.$$

One obtains analogous exact sequences for the sources of the various assembly maps from the fact that the sources are equivariant homology theories and the models for $E_{\mathcal{FIN}}V$ given in Example 4.5.3. These sequences are compatible with the assembly maps. The assembly maps for finite groups H, G_1 and G_2 are bijective. Now a Five-Lemma argument shows that also the one for V is bijective.

We obtain the stronger statement for $i < 0$ because the lower Nil-groups are known to vanish [FJ95]. □

Theorem 4.5.5. *Let G be a group for which the Farrell-Jones conjecture is known to be true. If $i \geq 0$ then the rationalized $\mathcal{FIN}$-assembly map*

$$A_{\mathcal{FIN}} \colon H_i^G(E_{\mathcal{FIN}}; \boldsymbol{K}_R) \otimes \mathbb{Q} \to K_i(\mathbb{Z}G) \otimes \mathbb{Q}$$

is an isomorphism. For $i < 0$, the $\mathcal{FIN}$-assembly map is an isomorphism even integrally.

Proof. The statement is implied by the preceding theorem since the $\mathcal{FIN}$-assembly map factors over the relative assembly map. □

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Curriculum Vitae

Joachim Grunewald

Geburtsdatum und -ort:	20.07.1978, Bielefeld
Familienstand:	ledig
Eltern:	Prof. Dr. Fritz Grunewald und Prof. Dr. Barbara Grunewald, geb. Gernhuber

Schulbildung	08/1984 – 08/1988	Clemens-August-Grundschule, Bonn
	08/1988 – 06/1997	Clara-Schumann-Gymnasium, Bonn
Hochschulreife	20.06.1997	Clara-Schumann-Gymnasium, Bonn
Zivildienst	09/1997 – 10/1998	Kinderheim Maria im Walde, Bonn
Studium	10/1998 – 08/2000	Mathematik und Physik, Rheinische Friedrich Wilhelms-Universität Bonn
	08/2000 – 08/2001	Physik, Washington State University, Pullman(USA)
	08/2001 – 09/2002	Mathematik, Yale University, New Haven (USA)
Prüfungen	08/2000	Vordiplom in Mathematik, Rheinische Friedrich Wilhelms-Universität Bonn
	08/2000	Vordiplom in Physik, Rheinische Friedrich Wilhelms-Universität Bonn
	09/2002	Master of Science in Mathematik, Yale University, New Haven (USA)
Tätigkeiten	02/2000 – 03/2000	studentische Hilfskraft, Rheinische Friedrich Wilhelms-Universität Bonn
	10/2003 – 03/2004, 10/2004 – 03/2005	wissenschaftliche Hilfskraft, Westfälische Wilhelms-Universität Münster
Beginn der Dissertation	10/2002	Westfälische Wilhelms-Universität Münster, Mathematisches Institut, Betreuer: Prof. Dr. W. Lück