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Münster, Gernot

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NUMERICAL INVESTIGATION OF THE FOUR-DIMENSIONAL NONLINEAR SIGMA-MODEL

Gernot MÜNSTER

II. Institut für Theoretische Physik der Universität Hamburg
Luruper Chaussee 149, 2000 Hamburg 50, FRG

Results from a numerical simulation of the four-dimensional O(4)-symmetric ϕ^4-model in the phase with unbroken symmetry at infinite bare quartic coupling are presented. Using a recently developed efficient cluster updating algorithm the renormalized mass and coupling are determined with high precision. Finite size effects are studied with special care in order to achieve a reliable infinite volume extrapolation. The finite volume behaviour of masses and couplings is well approximated by one-loop renormalized perturbation theory. The obtained infinite volume results agree with a recent analytical calculation based on a high order hopping parameter expansion.

1. INTRODUCTION

The work discussed below has been done in collaboration with Ch. Frick, K. Jansen, J. Jersák, I. Montvay and P. Seuferling and is presented in more detail in ref. 1.

The general framework of our investigation is a non-perturbative study of ϕ^4-theory in four dimensions. In the continuum the Lagrange density reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}(\phi^2)^2, \qquad (1.1)$$

where ϕ represents a real N-component scalar field $\phi^\alpha(x)$. The physical relevance of ϕ^4-theory with a four-component scalar field is based on the fact that it describes the Higgs-sector of the standard model of electro-weak interactions. In particular due to the smallness of the gauge coupling questions about the Higgs mechanism can be studied in the context of the pure ϕ^4-theory.[2] One of these questions is about the mass m_H of the Higgs particle. It is related to the value of the renormalized quartic coupling g_R and the field expectation value $v \approx 250$ GeV through

$$m_H = \sqrt{\frac{g_R}{3}}\, v.$$

Thus non-perturbative upper bounds on the coupling g_R in the phase with broken symmetry yield upper bounds on the Higgs-mass.

In our numerical work we concentrated on the limit of an infinitely strong bare quartic self-coupling g_0, where the field ϕ is constrained to unit length:

$$\phi \cdot \phi = 1.$$

The theory is then called the O(4)-symmetric non-linear sigma model. This limit is relevant for the Higgs-mass bounds as will be discussed below. On the other hand in this limit accurate Monte Carlo calculations are feasible. We studied the model in the phase with unbroken O(4)-symmetry which is better accessible to numerical methods. The results bear information about the phase with broken symmetry due to the scaling connection between both phases.[3]

2. ϕ^4-THEORY

In order to regularize the model it is defined on a hypercubical lattice $\mathbb{Z}^4$. The euclidean action is parametrized as

$$S = \sum_x \{-2\kappa \sum_{\mu=1}^{4} \phi(x)\cdot\phi(x+\hat{\mu}) + \phi(x)\cdot\phi(x) + \lambda[\phi(x)\cdot\phi(x) - 1]^2\}, \qquad (2.1)$$

where the lattice spacing a is set to 1 and $\hat{\mu}$ denotes the unit vector in the positive μ-direction. The parameters κ and λ are related to the bare mass m_0 and the bare coupling g_0 through

$$m_0^2 = \frac{1-2\lambda}{\kappa} - 8\,, \quad g_0 = \frac{6\lambda}{\kappa^2}. \qquad (2.2)$$

The non-linear sigma model is obtained in the limit $\lambda \to \infty$. For values of κ below a certain $\kappa_c(\lambda)$ all expectation values respect the O(4)-symmetry of the action, whereas for values of κ above κ_c the symmetry is broken spontaneously.

In the symmetric phase the spectrum has a gap m corresponding to the mass of an O(N)-vector multiplet of particles. The physical mass m is numerically close to the renormalized mass m_R, which together with the wave function renormalization Z_R is defined through the small momentum behaviour of the propagator:

$$\tilde{G}(p)^{-1}_{\alpha\beta} = 2\kappa Z_R^{-1}\delta_{\alpha\beta}\{m_R^2 + p^2 + \mathcal{O}(p^4)\}. \qquad (2.3)$$

The renormalized coupling g_R is defined in terms of the 4-point vertex function by

$$\Gamma^{(4)}(0,0,0,0)_{\alpha\beta\gamma\delta} = -g_R\, Z_R^{-2}\,\frac{1}{3}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}). \qquad (2.4)$$

The renormalized coupling and the bare coupling are numerically quite different. In particular g_R remains finite even in the limit where λ goes to infinity. Other physical quantities of interest include the six-point coupling h_R and masses M of different two-particle states.

Now the central question is about the dependence of m_R, g_R and the other quantities on κ and λ. Of particular interest is the scaling region near κ_c where the mass m_R (in lattice units) vanishes as $\kappa \to \kappa_c$. Most important for these considerations is the fact that ϕ^4-theory is almost certainly trivial, as work by various authors in recent years indicates, including ref. 3. This means that in the continuum limit, where the cutoff $\Lambda/m_R = 1/am_R$ goes to infinity, the renormalized coupling g_R vanishes and free field theory is obtained. Therefore ϕ^4-theory can only be used as a low-energy effective field theory with some finite cutoff Λ. The coupling g_R will then also be finite but will decrease with increasing cutoff. Since this concept is only meaningful if the cutoff is larger than, say, twice the renormalized mass, upper bounds on g_R result. To obtain numerical values for these upper bounds is the aim of recent nonperturbative investigations. Lüscher and Weisz[3] addressed this problem by means of a combination of high-temperature expansions in the symmetric phase and renormalization group methods, which allowed them to get control over m_R and g_R in the whole symmetric phase. These functions could then be continued to the scaling region ($\Lambda/m_R > 2$) in the phase with broken symmetry.

3. OUR CALCULATIONS

3.1. AIMS

The aims of our investigation are

1. A high precision calculation of m, g_R, Z_R and two-particle masses M in the symmetric phase.
   The results are to be used to check the scaling behaviour of physical quantities and to make a comparison with the results of Lüscher and Weisz. Since g_R is maximal for $\lambda \to \infty$ but still finite as follows from ref. 3, this is the relevant limit for obtaining upper bounds on g_R. Therefore the Monte Carlo calculations are done for the non-linear sigma model.

2. A study of finite size effects.
   The calculations are done for a system in a finite volume $L^3 \cdot T$. This implies finite size effects for all quantities under consideration. The L-dependence of m, g_R etc. can be measured accurately in the Monte Carlo calculation. On the other hand, if the coupling g_R is small enough, these finite size effects can be calculated in renormalized lattice perturbation theory, which we did in the one-loop approximation. This then allows an extrapolation of the Monte Carlo results to $L = \infty$.
   Two-particle masses are of particular interest. The leading finite volume corrections for two-particle masses are due to scattering effects and can be expressed in terms of the S-wave scattering length a_0 as has been shown by Lüscher.[4] For a two-particle state with zero relative momentum and mass M the shift due to the finite volume is given by

$$M(L) - 2m(L) = -\frac{4\pi a_0}{m_* L^3}\left(1 + c_1\frac{a_0}{L} + c_2\left(\frac{a_0}{L}\right)^2\right) + \mathcal{O}(L^{-6}), \qquad (3.1)$$

with the kinetic mass

$$m_* = \sinh m + \mathcal{O}(g_R^2) \qquad (3.2)$$

and certain known constants c_1 and c_2. A precise determination of the volume dependence of two-particle masses would thus allow to obtain information about the scattering length.

3.2. MEASUREMENTS

The numerical simulations have been performed partly with the conventional Metropolis algorithms and partly with Wolff's cluster algorithm for continuous spin models.[5] The use of the cluster algorithm was essential for achieving high precision and Karl Jansen will talk about this aspect in more detail at this symposium.

The calculations have been done for two values of κ and various lattice sizes, namely

$$\kappa = 0.290 \qquad (m \approx 0.45)$$

$$L = 4, 6, 8, 10, 12, \qquad T = 12$$

and

$$\kappa = 0.297 \qquad (m \approx 0.3)$$

$$L = 8, 10, 12, 14, 16, \qquad T = 16.$$

4. RESULTS

With the cluster algorithm accurate numerical results could be obtained. For the largest lattices the numbers are

$$\kappa = 0.290 \qquad L = 12 \qquad (4.1)$$

$$m = 0.4465(6), \quad g_R = 26.6(2.1), \quad Z_R = 0.988(2)$$

$$M_1 = 0.91(1), \quad M_9 = 0.898(8)$$

and

$$\kappa = 0.297 \qquad L = 16 \qquad (4.2)$$

$$m = 0.3039(4), \quad g_R = 21.9(1.7), \quad Z_R = 0.981(2)$$

$$M_1 = 0.61(1), \quad M_9 = 0.61(1),$$

where M_1 and M_9 are the two-particle masses for the scalar and tensor representation of O(4) respectively.

4.1. FINITE SIZE EFFECTS

A comparison of the measured values of m and g_R for different lattice sizes L showed that finite size effects are well reproduced by one-loop renormalized lattice perturbation theory. This means that the renormalized coupling is small enough for perturbation theory to be applicable. As a consequence the extrapolation to $L = \infty$ with the help of the perturbative formulae is reliable.

The finite-size effects on the two-particle masses M_1 ans M_9 were fitted as a function of L with the formula (3.1) in order to obtain the scattering lengths a_0^s and a_0^t respectively. As an example the numbers for $\kappa = 0.29$ are

$$a_0^s = -0.74(10), \quad a_0^t = -0.32(7). \qquad (4.3)$$

On the other hand two-loop perturbation theory using the measured value of g_R yields

$$a_0^s = -0.89(7), \quad a_0^t = -0.35(7).$$

Although the errors are not very small, the agreement is remarkable, showing that the lattices considered in this paper are large enough in order to contain also the lowest two-particle states in a relatively undistorted form. In other words, information about low energy scattering can be obtained from these lattice studies, similarly to the case of a single-component ϕ^4-theory.[6]

4.2. COMPARISON WITH LÜSCHER AND WEISZ

The extrapolated infinite volume results for m_R and g_R agree well with the analytical results of Lüscher and

Weisz.[3] In fig. 1 this is shown for the case of the renormalized coupling. The figure also includes the results of a previous numerical simulation by Kuti et al.[7] which also agrees within errors with ref. 3 and with us.

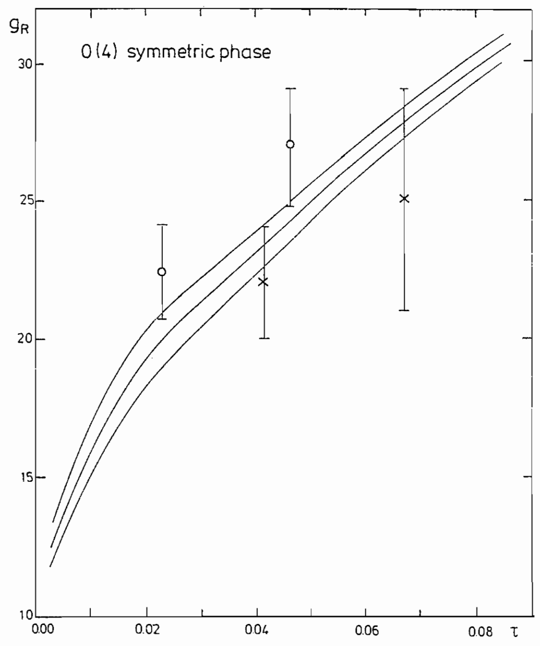

Figure 1

Comparison of our results (open circles) to the analytical work in ref. 3 (strip given by the three lines) and to the numerical results of ref. 7. The renormalized coupling g_R is shown as a function of $\tau = 1 - \kappa/\kappa_c$ with $\kappa_c \equiv 0.30411$.

The estimated relative errors for our values for m_R are up to a factor of 5–10 smaller than those of ref. 3, but the errors of the renormalized coupling are somewhat worse here, as is shown by the figure. In fig. 1 the slope of the solid curve below our data points reflects the scaling prediction from the renormalization group. As can be seen from the figure our data are very well consistent with scaling behaviour.

5. CONCLUSION

- ϕ^4-theory can be fully understood by means of present day methods.
- Finite size effects are under control.
- Presently available results confirm the existing theoretical picture based on triviality and scaling. The qualitative behaviour of the four-component model in the symmetric phase is very similar to the one-component case.[6]
- Our calculations agree with the work of Lüscher and Weisz[3], leading to the bound $m_H < 650$ GeV on the Higgs mass.

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