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Ping-pong in Hadamard manifolds

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Abstract. In this paper, we prove a quantitative version of the Tits alternative for negatively pinched manifolds X. Precisely, we prove that a nonelementary discrete isometry subgroup of $\mathrm{Isom}(X)$ generated by two non-elliptic isometries g, f contains a free subgroup of rank 2 generated by isometries f^N, h of uniformly bounded word length. Furthermore, we show that this free subgroup is convex-cocompact when f is hyperbolic.

1. Introduction

Let X be an n-dimensional negatively curved Hadamard manifold, with sectional curvature ranging between $-\kappa^2$ and -1, for some $\kappa \geq 1$. The main result of this note is the following quantitative version of the Tits alternative for X, which answers a question asked by Filippo Cerocchi during the Oberwolfach Workshop "Differentialgeometrie im Grossen", 2017, see also [10].

Theorem 1.1. *There exists a function $\mathcal{L} = \mathcal{L}(n, \kappa)$ such that the following holds: Let f, g be non-elliptic isometries of X generating a nonelementary discrete subgroup Γ of $\mathrm{Isom}(X)$. Then there exists an element $h \in \Gamma$ whose word length (with respect to the generators f, g) is $\leq \mathcal{L}$ and a number $N \leq \mathcal{L}$ such that the subgroup of Γ generated by f^N, h is free of rank two.*

One can regard this theorem as a quantitative version of the Tits alternative for discrete subgroups of $\mathrm{Isom}(X)$. For other forms of the quantitative Tits alternative, we refer to [2, 5, 6, 8].

After replacing g with the element $g' := gfg^{-1}$, and noticing that the subgroup generated by f, g' is still discrete and nonelementary, we reduce the problem to the case when the isometries f and g are conjugate in $\mathrm{Isom}(X)$, which we will assume from now on.

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The proof of Theorem 1.1 breaks into two cases which are handled by different arguments:

Case 1. f (and, hence, g) has translation length bounded below by some positive number λ. We discuss this case in Section 4.

Case 2. f has translation length bounded above by some positive number λ. We discuss this case in Section 5.

Remark 1.2. (i) For the constant λ, we take $\varepsilon(n,\kappa)/10$, where $\varepsilon(n,\kappa)$ is a positive lower bound for the Margulis constant of X.
(ii) We need to use a power of f only in Case 1, while in Case 2 we can take $N = 1$.

We also note that if f is hyperbolic, the free group $\langle f^N, h\rangle$ constructed in our proof is convex-cocompact. See Proposition 3.22 and Corollary 4.9. One can also show that this subgroup is geometrically finite if f is parabolic but we will not prove it.

2. Definitions and notation

In a metric space (Y, d), we will be using the notation $B(a, R)$ to denote the open R-ball centered at $a \in Y$, and the notation $\bar{N}_R(A)$ to denote the closed R-neighborhood of a subset $A \subset X$. By

$$d(A, B) := \inf\{d(a, b) : a \in A, b \in B\},$$

we denote the minimal distance between subsets $A, B \subset Y$.

If (Y, d) is a geodesic δ-hyperbolic metric space or a CAT(0) space, then $\partial_\infty Y$ will denote the visual boundary equipped with the visual topology, and we write $\bar{Y} := Y \cup \partial_\infty Y$. If Y is proper, then $\bar{Y}$ is a compactification of Y. Given a pair of points x, y in (Y, d), we will use the notation xy to denote a geodesic segment in Y connecting x to y. For general δ-hyperbolic spaces, this segment is *not* unique, but, since any two such segments are within distance δ from each other, this abuse of notation is harmless. We let $|xy| = d(x, y)$ denote the length of xy. Given points $A, B, C \in Y$, we let $\triangle ABC$ denote a geodesic triangle in Y with vertices A, B, C. Similarly, if $y \in Y, \xi \in \partial_\infty Y$, then $y\xi$ will denote a geodesic ray emanating from y which is asymptotic to ξ.

A subset A of Y is called λ-*quasiconvex* if every geodesic segment xy with the end-points in A is contained in $\bar{N}_\lambda(A)$.

A subset A in a metric space Y is called *starlike* with respect to a point $a \in A$ if for every $y \in A$, every geodesic segment ya is contained in A. More generally, if Y is δ-hyperbolic or a CAT(0) space, then $A \subset Y$ is called *starlike* with respect to a point $\xi \in \partial_\infty Y$ if for every $y \in A$, every geodesic ray $y\xi$ is contained in A.

Throughout the paper, X will denote an n-dimensional Hadamard manifold with sectional curvature ranging between $-\kappa^2$ and -1, unless otherwise stated. Let d denote the Riemannian distance function on X. We use $\partial_\infty X$ to denote the visual boundary of X, and $\bar{X} := X \cup \partial_\infty X$ the visual compactification of X. Let $\mathrm{Isom}(X)$ denote the isometry group of X. We use $\varepsilon(n,\kappa)$ to denote a fixed positive lower bound on the Margulis constant for X; this number is known to depend only on n and κ, see, e.g., [1].

Given a pair of points p, q in X, we let $H(p,q)$ denote the closed half-space in X given by

$$H(p,q) = \{x \in X : d(x,p) \leq d(x,q)\}.$$

Then $\mathrm{Bis}(p,q) = \mathrm{Bis}(q,p) := H(p,q) \cap H(q,p)$ is the equidistant set of p, q.

We use the notation $\mathrm{Hull}(A)$ for the closed convex hull of a subset $A \subset X$ which is the intersection of all closed convex subsets of X containing A.

For each isometry g of X, we define its *translation length* $\tau(g)$ as

$$\tau(g) = \inf_{x \in X} d(x, g(x)). \tag{1}$$

The isometries of X are classified in terms of their translation lengths, see Section 3.7.

A discrete subgroup $\Gamma < \mathrm{Isom}(X)$ is called *elementary* if either it fixes a point in $\bar{X}$ or preserves a geodesic in X.

3. Preliminary material

3.1. **Some CAT(−1) computations.** Let X be a CAT(-1) space. Recall that the hyperbolicity constant of X is $\leq \delta = \cosh^{-1}(\sqrt{2})$.

Lemma 3.2. *Let $\triangle A_1A_2C$ be a triangle in X such that $\angle A_1CA_2 \geq \pi/2$. Then*

$$|A_1A_2| \geq |A_1C| + |A_2C| - 2\delta.$$

Proof. Let $D \in A_1A_2$ be the point closest to C. Then at least one of the angles $\angle A_iCD$, $i = 1, 2$, is $\geq \pi/4$. The CAT(-1) property and the dual cosine law for the hyperbolic plane imply that

$$\cosh(|CD|)\sin\Big(\frac{\pi}{4}\Big) \leq 1,$$

i.e.,

$$|CD| \leq \cosh^{-1}(\sqrt{2}) = \delta.$$

The rest follows from the triangle inequalities. □

Corollary 3.3. *Suppose that $x, x_+, \hat{x}_+, x'_+$ are points in X which lie on a common geodesic and appear on this geodesic in the given order. Assume that*

$$d(\hat{x}_+, x'_+) \geq d(x, x_+) + 2\cosh^{-1}(\sqrt{2}).$$

Then $H(x_+, \hat{x}_+) \subset H(x, x'_+)$.

Proof. We observe that the CAT(-1) condition implies that for each z equidistant from $x_+, \hat{x}_+$, we have

$$\angle z x_+ \hat{x}_+ \leq \pi/2, \quad \angle z \hat{x}_+ x_+ \leq \pi/2.$$

Hence,

$$\angle x x_+ z \geq \pi/2, \quad \angle x'_+ \hat{x}_+ z \geq \pi/2.$$

Then the lemma and the triangle inequality implies that

$$d(z,x) \le d(z,x'_+),$$

and thus

$$\mathrm{Bis}(x_+,\hat{x}_+) \subset H(x,x'_+).$$

Since every geodesic connecting $w \in H(x_+,\hat{x}_+)$ to x'_+ passes through some point $z \in \mathrm{Bis}(x_+,\hat{x}_+)$, it follows that

$$d(x,w) \le d(w,x'_+). \qquad \square$$

3.4. **Quasiconvex and starlike subsets.**

Lemma 3.5. *Starlike subsets in a δ-hyperbolic space Y are δ-quasiconvex.*

Proof. We prove this for subsets $A \subset Y$ starlike with respect to $a \in A$; the proof in the case of starlike subsets with respect to $\xi \in \partial_\infty Y$ is similar and is left to the reader. Take $z_1, z_2 \in A$. Then, by the δ-hyperbolicity,

$$z_1z_2 \subset \bar{N}_\delta(az_1 \cup az_2) \subset \bar{N}_\delta(A). \qquad \square$$

Suppose now that X is a Hadamard manifold of negatively pinched curvature as above. Then, according to [4, Proposition 2.5.4], there exists $\mathfrak{q} = \mathfrak{q}(\kappa,\lambda)$ such that for every λ-quasiconvex subset $A \subset X$, we have

$$\mathrm{Hull}(A) \subset \bar{N}_{\mathfrak{q}}(A).$$

In particular, the following proposition holds.

Proposition 3.6. *For every starlike subset A in a Hadamard manifold X of negatively pinched curvature, the closed convex hull* $\mathrm{Hull}(A)$ *is contained in the* $\mathfrak{q} = \mathfrak{q}(\kappa,\delta)$*-neighborhood of A.*

In what follows, we will suppress the dependence of $\mathfrak{q}$ on κ and δ, since these are fixed for our space X.

3.7. **Classification of isometries.** Let X be a negatively curved Hadamard manifold. The isometries of X are classified into three types according to their translation lengths τ, see [1, 2].

(i) An isometry g of X is *hyperbolic* if $\tau(g) > 0$. Equivalently, the infimum in (1) is attained and is positive. In this case, the infimum is attained on a g-invariant geodesic, called the *axis* of g, and denoted by A_g.
(ii) An isometry g of X is *elliptic* of $\tau(g) = 0$ and the infimum in (1) is attained; the set where the infimum is attained is a totally geodesic submanifold of X fixed pointwise by g.
(iii) An isometry g of X is *parabolic* if the infimum in (1) is not attained. In this case, the infimum is necessarily equal to zero.

Thus, only parabolic and elliptic isometries have zero translation lengths. For any $g \in \mathrm{Isom}(X)$ and $m \in \mathbb{Z}$, we have

$$\tau(g^m) = |m|\tau(g). \tag{2}$$

The following theorem provides an alternative characterization of types of isometries of X, see [7].

Theorem 3.8. *Suppose that g is an isometry of X. Then*

(i) *g is hyperbolic if and only if for some (equivalently, every) $x \in X$, the orbit map $N \to g^N x$ is a quasiisometric embedding $\mathbb{Z} \to X$;*
(ii) *g is elliptic if and only if for some (equivalently, every) $x \in X$, the orbit map $N \to g^N x, N \in \mathbb{Z}$ has bounded image;*
(iii) *g is parabolic if and only if for some (equivalently, every) $x \in X$, the orbit map $N \to g^N x, N \in \mathbb{Z}$ is proper and*

$$\lim_{N\to\infty} \frac{d(x, g^N(x))}{N} = 0.$$

If f, g are hyperbolic isometries of X generating a discrete subgroup of $\mathrm{Isom}(X)$, then either the ideal boundaries of the axes A_f, A_g are disjoint or $A_f = A_g$ (see [3], the argument for negatively curved Hadamard manifolds is similar).

3.9. **Margulis cusps and tubes.** Take $g \in \mathrm{Isom}(X)$. For each $\varepsilon \geq \tau(g)$, we define the following nonempty closed convex subset of X:

$$T_\varepsilon(g) = \{x \in X \mid d(x, g(x)) \leq \varepsilon\}.$$

Of primary importance are subsets $T_\varepsilon(g)$ for $\varepsilon < \varepsilon(n, \kappa)$. For any two isometries g, h of X, we have

$$T_\varepsilon(hgh^{-1}) = h(T_\varepsilon(g)). \tag{3}$$

In particular, if g, h commute, then h preserves $T_\varepsilon(g)$.

For parabolic isometries g of X define the *Margulis cusp*

$$\mathcal{T}_\varepsilon(g) := \bigcup_{i\in\mathbb{Z}-\{0\}} T_\varepsilon(g^i).$$

(The same definition works for elliptic isometries of X, except the above region is not called a *cusp.*) This subset is $\langle g\rangle$-invariant.

Suppose that g is a hyperbolic isometry of X. Define m_g to be the (unique) positive integer such that

$$\tau(g^{m_g}) \leq \varepsilon/10, \quad \tau(g^{m_g+1}) > \varepsilon/10, \tag{4}$$

and set

$$\mathcal{T}_\varepsilon(g) := \bigcup_{1\leq i\leq m_g} T_\varepsilon(g^i) \subset X.$$

If $\tau(g) > \varepsilon/10$, then $\mathcal{T}_\varepsilon(g) = \varnothing$.

Since the subgroup $\langle g\rangle$ is abelian, in view of (3), we obtain the following lemma.

Lemma 3.10. *The subgroup $\langle g\rangle$ preserves $\mathcal{T}_\varepsilon(g)$ and, hence, also preserves* $\mathrm{Hull}(\mathcal{T}_\varepsilon(g))$.

By the convexity of the distance function, for any isometry $g \in \mathrm{Isom}(X)$, $T_\varepsilon(g)$ is convex. In particular, $\mathcal{T}_\varepsilon(g)$ is a starlike region with respect to any fixed point $p \in \bar{X}$ of g for general g, and with respect to any point on the axis of g if g is hyperbolic. As a corollary to Lemma 3.5, one obtains the following corollary.

Corollary 3.11. *For every isometry $g \in \mathrm{Isom}(X)$, the set $\mathcal{T}_\varepsilon(g)$ is δ-quasi-convex.*

Proposition 3.6 then implies the following.

Corollary 3.12. *For every isometry $g \in \mathrm{Isom}(X)$,*

$$\mathrm{Hull}(\mathcal{T}_\varepsilon(g)) \subset \bar{N}_{\mathfrak{q}}(\mathcal{T}_\varepsilon(g)),$$

where $\mathfrak{q}$ is as in Proposition 3.6.

For a more detailed discussion of the regions $\mathcal{T}_\varepsilon(g)$, see [4, 14].

3.13. **Displacement estimates.** In this subsection, we let X be a CAT(-1) geodesic metric space. For each pair of points $A, B \in \mathbb{H}^2$ and each circle $S \subset \mathbb{H}^2$ passing through these points, we let $\widehat{AB}^S$ denote the (hyperbolic) length of the shorter arc into which A, B divide the circle S.

Lemma 3.14. *If $d(A,B) \le D$, then, for every circle S as above, the length ℓ of $\widehat{AB}^S$ satisfies the inequality:*

$$d(A,B) \le \ell \le \frac{2\pi \tanh(D/4)}{1 - \tanh^2(D/4)}.$$

Proof. The first inequality is clear, so we verify the second. We want to maximize the length of $\widehat{AB}^S$ among all circles S passing through A, B. We claim that the maximum is achieved on the circle S_o whose center o is the midpoint of AB. This follows from the fact that given any other circle S, we have the radial projection from $\widehat{AB}^{S_o}$ to $\widehat{AB}^S$ (with the center of the projection at o). Since this radial projection is distance-decreasing (by convexity), the claim follows. The rest of the proof amounts to a computation of the length of the hyperbolic half-circle with the given diameter. □

Lemma 3.15. *There exists a function $c(D)$ so that the following holds: Consider an isosceles triangle ABC in X with $d(A,C) = d(B,C)$, $d(A,B) \le D$, and an isosceles subtriangle $A'B'C$ with $A' \in AC$, $B' \in BC$, $d(A,A') = d(B,B') = \tau$. Then*

$$d(A', B') \le c(D) e^{-\tau}.$$

Proof. In view of the CAT(-1) assumption, it suffices to consider the case when $X = \mathbb{H}^2$. We will work with the unit disk model of the hyperbolic plane, where C is the center of the disk as in Figure 1. Let α denote the angle $\angle(ACB)$. Set $T := d(C,A) = d(C,B)$. For points $A_t \in CA$, $B_t \in CB$ such that $d(C, A_t) = d(C, B_t) = t$, we let l_t denote the hyperbolic length of the (shorter) circular arc $\widehat{A_tB_t} = \widehat{A_tB_t}^{S_t}$ of the angular measure α, centered at C

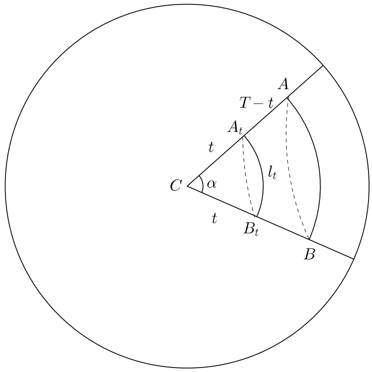

FIGURE 1

and connecting A_t to B_t. (Here S_t is the circle centered at C and of hyperbolic radius t.) Let R_t denote the Euclidean distance between C and A_t (same for B_t). Then

$$l_t = \frac{2\alpha R_t}{1-R_t^2}, \quad R_t = \tanh(t/2).$$

Thus, for $\tau = T - t$,

$$\begin{aligned}\frac{l_t}{l_T} &= \frac{R_t}{R_T}\frac{1-R_T^2}{1-R_t^2} \leq \frac{1-R_T^2}{1-R_t^2}\\ &\leq 2\frac{1-R_T}{1-R_t} = 2\frac{1-\tanh(T/2)}{1-\tanh(t/2)}\\ &= 2\frac{e^t+1}{e^T+1} = 2\frac{e^{-T}+e^{-\tau}}{e^{-T}+1} \leq 4e^{-\tau}.\end{aligned}$$

In other words,

$$d(A_t, B_t) \leq l_t \leq 4e^{-\tau} l_T.$$

Combining this inequality with Lemma 3.14, we obtain

$$l_t \leq 4e^{-\tau}\frac{2\pi\tanh(d(A,B)/4)}{1-\tanh^2(d(A,B)/4)} \leq 4e^{-\tau}\frac{2\pi\tanh(D/4)}{1-\tanh^2(D/4)}.$$

Lastly, setting $A' = A_t$, $B' = B_t$, $A = A_T$, $B = B_T$, we get

$$d(A', B') \leq 4\frac{2\pi\tanh(D/4)}{1-\tanh^2(D/4)}e^{-\tau} = c(D)e^{-\tau}. \qquad \square$$

Corollary 3.16. *There exists a function $\mathfrak{r}(\varepsilon)$ such that for any hyperbolic isometry $h \in \mathrm{Isom}(X)$ with translation length $\tau(h) = l \leq \varepsilon/10$, if $A \in X$ satisfies $d(A, h(A)) = \varepsilon$, then there exists $B \in X$ such that $d(B, h(B)) = \varepsilon/3$, $d(A, B) \leq \mathfrak{r} = \mathfrak{r}(\varepsilon)$ and B lies on the shortest geodesic segment connecting A to the axis A_h of h.*

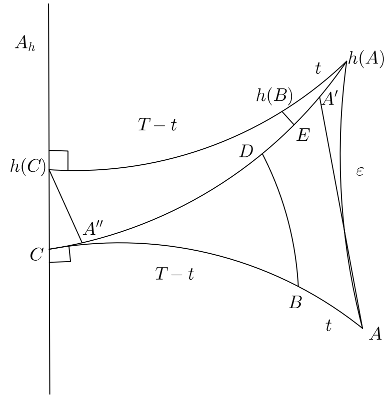

FIGURE 2

Proof. Let $C \in A_h$ be the closest point to A in A_h. By the convexity of the distance function, there exists a point $B \in AC$ such that $d(B, h(B)) = \varepsilon/3$. Suppose that $d(A, B) = d(h(A), h(B)) = t$ and $d(A, C) = d(h(A), h(C))) = T$, as shown in Figure 2. Then $d(C, h(A)) \leq T + l \leq T + \varepsilon/10$. There exist points D, E in the segment $h(A)C$ such that $d(C, D) = d(C, B) = T - t$, $d(h(A), E) = t$ and $d(A', C) = d(A, C) = T$.

Then $d(A, A') \leq \varepsilon + l \leq 11\varepsilon/10$. By Lemma 3.15, $c(11\varepsilon/10)$ (defined in that lemma) satisfies

$$d(B, D) \leq c(d(A, A'))e^{-t} \leq c(11\varepsilon/10)e^{-t}.$$

Similarly, by taking the point $A'' \in h(A)C$ satisfying $d(A'', h(A)) = T$ and $d(h(C), A'') \leq 2l$, considering the isosceles triangle $\triangle h(C)A''h(A)$ and its sub-triangle $\triangle h(B)Eh(A)$, we obtain

$$d(h(B), E) \leq c(2l)e^{t-T}.$$

Since $l \leq \varepsilon/10$ and $d(B, h(B)) = \varepsilon/3$, the convexity of the distance function implies that $T - t > t$. Thus,

$$\begin{aligned}\varepsilon/3 = d(B, h(B)) &\leq d(B, D) + d(D, E) + d(E, h(B)) \\ &\leq c(11\varepsilon/10)e^{-t} + l + c(2l)e^{t-T} \\ &\leq c(11\varepsilon/10)e^{-t} + \frac{\varepsilon}{10} + c(\varepsilon/5)e^{-t},\end{aligned}$$

which simplifies to

$$\frac{7}{30}\varepsilon \leq \big(c(11\varepsilon/10) + c(\varepsilon/5)\big)e^{-t},$$

and consequently

$$d(A, B) = t \leq \mathfrak{r}(\varepsilon) := \log\Big([c(11\varepsilon/10) + c(\varepsilon/5)]\frac{30}{7}\varepsilon^{-1}\Big). \qquad \square$$

3.17. **Local-to-global principle for quasigeodesics in X.** For a piecewise-geodesic path consisting of alternating 'long' arcs and 'short' segments such that adjacent geodesic segments meet at angles $\geq \pi/2$, we construct a quasi-geodesic in X by making the long segments sufficiently long, given a lower bound on the lengths of the short arcs. More precisely, according to [14, Proposition 7.3], we have the following result.

Proposition 3.18. *There are functions $\lambda = \lambda(\varepsilon) \geq 1, \alpha = \alpha(\varepsilon) \geq 0$ and $L = L(\varepsilon) > \varepsilon > 0$ such that the following holds. Suppose that $\gamma = \cdots \gamma_{-1} * \gamma_0 * \gamma_1 * \cdots * \gamma_n \cdots \subseteq X$ is a piecewise geodesic path such that:*

(i) *Each geodesic arc γ_j has length either at least ε or at least L.*
(ii) *If γ_j has length $< L$, then the adjacent geodesic arcs γ_{j-1} and γ_{j+1} have lengths at least L.*
(iii) *All adjacent geodesic segments meet at angles $\geq \pi/2$.*

Then γ is a (λ, α)-quasigeodesic in X.

3.19. **Ping-pong.**

Proposition 3.20. *Suppose that $g, h \in \operatorname{Isom}(X)$ are parabolic/hyperbolic elements with equal translation lengths $\leq \varepsilon/10$, and*

$$d\big(\operatorname{Hull}(\mathcal{T}_\varepsilon(g)), \operatorname{Hull}(\mathcal{T}_\varepsilon(h))\big) \geq L,$$

where $L = L(\varepsilon/10)$ is as in Proposition 3.18. Then $\Phi := \langle g, h\rangle < \operatorname{Isom}(X)$ is a free subgroup of rank 2.

Proof. To simplify the notation, for a non-elliptic element $f \in \operatorname{Isom}(X)$, we denote $\operatorname{Hull}(\mathcal{T}_\varepsilon(f))$ by $\widehat{\mathcal{T}}_\varepsilon(f)$.

Using Lemma 3.10, (3), and the definition of $\mathcal{T}_\varepsilon$, we obtain

$$d(\widehat{\mathcal{T}}_\varepsilon(g), g^k\widehat{\mathcal{T}}_\varepsilon(h)) = d(\widehat{\mathcal{T}}_\varepsilon(g), \widehat{\mathcal{T}}_\varepsilon(h)) \geq L, k \in \mathbb{Z}.$$

Our goal is to show that every nonempty word $w(g, h)$ represents a nontrivial element of $\operatorname{Isom}(X)$. It suffices to consider cyclically reduced words w which are not powers of g, h.

We will consider a cyclically reduced word

$$w = w(g, h) = g^{m_k} h^{m_{k-1}} g^{m_{k-2}} h^{m_{k-3}} \cdots g^{m_2} h^{m_1},$$

words with the last letter g are treated by relabeling. Since w is cyclically reduced and is not a power of g, h, the number k is ≥ 2 and all of the m_i's in this equation are nonzero.

For each $N \geq 1$, we define the *r-suffix* of w^N as the following sub-word of w^N:

$$w_r = \begin{cases} g^{m_r} h^{m_{r-1}} g^{m_{r-2}} h^{m_{r-3}} \cdots g^{m_2} h^{m_1}, & r \text{ even},\\ h^{m_r} g^{m_{r-1}} h^{m_{r-2}} \cdots g^{m_2} h^{m_1}, & r \text{ odd},\end{cases}$$

where, of course, $m_i \equiv m_j$ modulo N. Since w is reduced, each w_r is reduced as well.

We will prove that the map

$$\mathbb{Z} \to X, \quad N \mapsto w^N x,$$

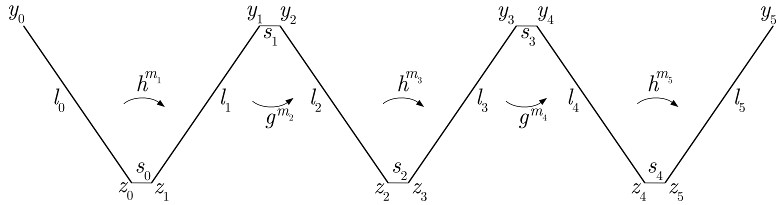

FIGURE 3

is a quasiisometric embedding. This will imply that $w(g,h)$ is nontrivial. In fact, this will also show that $w(g,h)$ is hyperbolic, see Theorem 3.8.

Let $l = yz$ be the unique shortest geodesic segment connecting points in $\widehat{\mathcal{T}}_\varepsilon(g)$ and $\widehat{\mathcal{T}}_\varepsilon(h)$, where $y \in \widehat{\mathcal{T}}_\varepsilon(g)$ and $z \in \widehat{\mathcal{T}}_\varepsilon(h)$. For $r \geq 0$, we denote $w_r l$, $w_r y$ and $w_r z$ by l_r, y_r and z_r, respectively. In particular, $y_0 = y$, $z_0 = z$ and $l_0 = l$.

Since l is the shortest segment between $\widehat{\mathcal{T}}_\varepsilon(g), \widehat{\mathcal{T}}_\varepsilon(h)$ and these are convex subsets of X, for every $y' \in \widehat{\mathcal{T}}_\varepsilon(g)$ (resp. $z' \in \widehat{\mathcal{T}}_\varepsilon(h)$),

$$\angle y'yz \geq \pi/2 \quad \text{(resp. } \angle yzz' \geq \pi/2). \tag{5}$$

Since g and h have equal translation lengths, h is parabolic (resp. hyperbolic) if and only if g is parabolic (resp. hyperbolic). When both of them are hyperbolic, since y and z are not in the interior of $\mathcal{T}_\varepsilon(g)$ and $\mathcal{T}_\varepsilon(h)$, respectively, $d(y, g^i y), d(z, h^j z) \geq \varepsilon$ for all $1 \leq i \leq m_g$, $1 \leq j \leq m_h$. Also, when $i > m_g$, $j > m_h$, it follows from (2) and (4) that

$$\min\bigl(d(y, g^i y), d(z, h^j z)\bigr) \geq \frac{\varepsilon}{10}.$$

Moreover, when both g and h are parabolic, $d(y, g^i y), d(z, h^j z) \geq \varepsilon$ for all $1 \leq i$, $1 \leq j$. Therefore, in the general case,

$$\min\bigl(d(y, g^i y), d(z, h^j z)\bigr) \geq \frac{\varepsilon}{10} \quad \text{for all } i \geq 1, \text{ and all } j \geq 1. \tag{6}$$

Let s_r be the segment

$$s_r = \begin{cases} y_r y_{r+1}, & \text{when } r \text{ is odd,} \\ z_r z_{r+1}, & \text{when } r \text{ is even.} \end{cases}$$

See the arrangement of the points and segments in Figure 3.

Let $\tilde{l}_N$ be the concatenation of the segments l_r's and s_r's as shown in Figure 3, $0 \leq r \leq kN$. According to (6), the length of each segment s_r is at least $\varepsilon/10$, while by assumption, the length of each l_r is $\geq L = L(\varepsilon/10)$. Moreover, according to (5), the angle between any two consecutive segments in $\tilde{l}_N$ is at least $\pi/2$. Using Proposition 3.18, we conclude that $\tilde{l}_N$ is a (λ, α)-quasigeodesic.

Consequently,

$$d(w^N x, x) \geq \frac{1}{\lambda}\left(\sum_{i=0}^{kN-1} |s_i| + NkL\right) - \alpha \geq \frac{kL}{\lambda} N - \alpha. \tag{7}$$

From this inequality it follows that the map $\mathbb{Z} \to X$, $N \mapsto w^N x$ is a quasiisometric embedding. □

Remark 3.21. In fact, this proof also shows that every nontrivial element of the subgroup $\Phi < \mathrm{Isom}(X)$ is either conjugate to one of the generators or is hyperbolic.

For the next proposition and the subsequent remark, one needs the notions of *convex-cocompact* and *geometrically finite* subgroups of $\mathrm{Isom}(X)$. We refer to [4] for several equivalent definitions, see also [13, Section 1]. For now, it suffices to say that a subgroup Γ in $\mathrm{Isom}(X)$ is *convex-cocompact* if it is finitely generated and for some (equivalently, every) $x \in X$, the orbit map $\Gamma \to \Gamma x \subset X$ is a quasiisometric embedding, where Γ is equipped with a word metric.

Proposition 3.22. *Let $g, h \in \mathrm{Isom}(X)$ be hyperbolic isometries satisfying the hypothesis of Proposition 3.20. Then the subgroup $\Phi = \langle g, h\rangle < \mathrm{Isom}(X)$ is convex-cocompact.*

Proof. We equip the free group $\mathbb{F}_2$ on two generators (denoted g, h) with the word metric corresponding to this free generating set. Since g, h are hyperbolic, by (2), the lengths of the segments s_r's in the proof of Proposition 3.20 are $\geq \tau|m_{r+1}|$, where

$$\tau = \tau(g) = \tau(h).$$

Then, for $N = 1$, $r = k$, and a reduced but not necessarily cyclically reduced word w, the inequality (7) becomes

$$d(wy, y) \geq \frac{1}{\lambda}\left(\sum_{i=0}^{k-1} |s_i|\right) - \alpha \geq \frac{\tau}{\lambda}|w| - \alpha,$$

where $|w| \geq |m_1| + |m_2| + \cdots + |m_k|$ is the (word) length of w. Therefore, the orbit map $\mathbb{F}_2 \to \Phi y \subset X$ is a quasiisometric embedding. □

Remark 3.23. One can also show that if g, h are parabolic, then the subgroup Φ is geometrically finite. We will not prove it in this paper, since a proof requires further geometric background material on geometrically finite groups.

4. Case 1: Displacement bounded below

In this section we consider discrete nonelementary subgroups of $\mathrm{Isom}(X)$ generated by two hyperbolic elements g, h whose translation lengths are equal to $\tau \geq \lambda$. Our goal is to show that in this case the subgroup $\langle g^N, h^N\rangle$ is free of rank 2 provided that N is greater than some constant depending only on the Margulis constant of X and on λ. The strategy is to bound from above

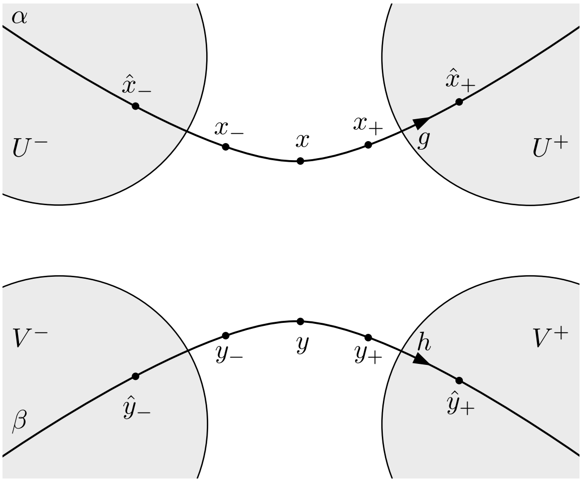

FIGURE 4

how 'long' the axes A_g, A_h of g and h can stay 'close to each other' in terms of the constant λ. Once we get such an estimate, we find a uniform upper bound on N such that the Dirichlet domains for $\langle g^N\rangle, \langle h^N\rangle$ (based at some points on A_g, A_h) have disjoint complements. This implies that g^N, h^N generate a free subgroup of rank two by a classical ping-pong argument.

Let α, β be complete geodesics in the Hadamard manifold X. These geodesics eventually will be the axes of g and h, hence we assume that these geodesics do not share ideal end-points. Let x_-x_+ denote the (nearest point) projection of β to α and let y_-y_+ denote the projection of x_-x_+ to β. Let x and y denote the mid-points of x_-x_+ and y_-y_+ respectively. Then

$$L_\beta := d(y_-, y_+) \le L_\alpha := d(x_-, x_+).$$

Fix some $T \ge 0$, and let $\hat{x}_-\hat{x}_+$ and $\hat{y}_-\hat{y}_+$ denote the subsegments of α and β containing x_-x_+ and y_-y_+, respectively, such that

$$d(x_\pm, \hat{x}_\pm) = T, \quad d(y_\pm, \hat{y}_\pm) = T. \tag{8}$$

We let $U_\pm$ and $V_\pm$ denote the 'half-spaces' in X equal to $H(\hat{x}_\pm, x_\pm)$ and $H(\hat{y}_\pm, y_\pm)$, respectively. See Figure 4.

The following is proven in [2, Appendix].

Lemma 4.1. *If $T \ge 5$, then the sets $U_\pm$, $V_\pm$ are pairwise disjoint.*

Suppose now that g, h are hyperbolic isometries of X with the axes α, β, respectively, and equal translation length $\tau(g) = \tau(h) = \tau \ge \lambda > 0$. We let $\Gamma = \langle g, h\rangle < \mathrm{Isom}(X)$ denote the, necessarily nonelementary (but not necessarily discrete), subgroup of isometries of X generated by g and h.

As an application of the above lemma, as in [2, Appendix], we obtain the following lemma.

Lemma 4.2. *If $N\tau \geq L_\alpha + 5 + 2\delta$, then the half-spaces $H(g^{\pm N}x, x)$ and $H(h^{\pm N}y, y)$ are pairwise disjoint.*

Proof. The inequality

$$N\tau \geq L_\alpha + 5 + 2\delta \geq L_\beta + 5 + 2\delta.$$

implies that the quadruples

$$(x, x_+, \hat{x}_+, g^N(x)),\ (x, x_-, \hat{x}_-, g^{-N}(x)),\ (y, y_+, \hat{y}_+, h^N(y)),\ (y, y_-, \hat{y}_-, h^{-N}(y))$$

satisfy the assumptions of Corollary 3.3, where $\hat{x}_\pm$ and $\hat{y}_\pm$ are given by taking $T = 5$ in (8). Therefore, according to this corollary, we have

$$H(g^{\pm N}(x), x) \subset U^\pm, \quad H(h^{\pm N}(y), y) \subset V^\pm.$$

Now, the assertion of the lemma follows from Lemma 4.1. □

Corollary 4.3. *If*

$$N\tau \geq L_\alpha + 5 + 2\delta, \tag{9}$$

then the subgroup $\Gamma_N < \Gamma$ generated by g^N, h^N is free with the basis g^N, h^N.

Proof. We have

$$g^{\pm N}\big(H(h^{-N}(y), y) \cup H(h^{+N}(y), y)\big) \subset H(g^{\pm N}x, x)$$

and

$$h^{\pm N}\big(H(g^{-N}(x), x) \cup H(g^{+N}(x), x)\big) \subset H(h^{\pm N}y, y).$$

Thus, the conditions of the standard ping-pong lemma (see, e.g., [9, 11]) are satisfied and, hence, Γ_N is free with the basis g^N, h^N. □

Let $\eta = d(\alpha, \beta)$ denote the minimal distance between α, β and pick some $\eta_0 > 0$ (we will eventually take $\eta_0 = 0.01\varepsilon(n, \kappa)$). Let $\beta_0 = z_-^0 z_+^0 \subset \beta$ be the (possibly empty!) maximal closed subinterval such that the distance from the end-points of β_0 to α is $\leq \eta_0$. Thus, $\beta_0 \subset \bar{N}_{\eta_0}(\alpha)$.

Remark 4.4. Note that $\beta_0 = \varnothing$ if and only if $\eta_0 < \eta$.

Let $\alpha_0 = x_-^0 x_+^0$ denote the projection of β_0 to α, let $2L_0$ denote the length of α_0. Hence, the intervals α_0, β_0 are within Hausdorff distance η_0 from each other.

Furthermore, $\angle\beta(-\infty) z_-^0 x_-^0 \geq \pi/2$ and $\angle\beta(-\infty) z_+^0 x_+^0 \geq \pi/2$; see Figure 5. Hence, according to [14, Corollary 3.7], for

$$L_1 = \sinh^{-1}\Big(\frac{1}{\sinh(\eta_0)}\Big),$$

we have

$$d(x_-, x_-^0) \leq L_1, \quad d(x_+, x_+^0) \leq L_1.$$

Thus, the interval $x_- x_+$ breaks into the union of two subintervals of length $\leq L_1 = L_1(\eta_0)$ and the interval α_0 of length $2L_0$. In other words, $L_\alpha = 2(L_0 + L_1)$.

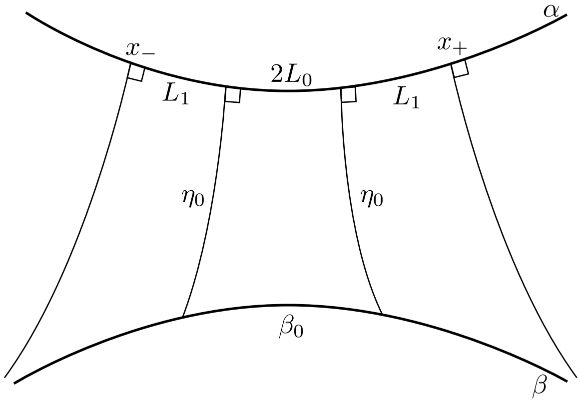

FIGURE 5

Most of our discussion below deals with the case when the interval β_0 is nonempty.

Our goal is to bound from above L_α in terms of λ, η_0 and the Margulis constant $\varepsilon(n,\kappa)$ of X, provided that $\eta_0 = 0.01\varepsilon(n,\kappa)$ and Γ is discrete.

Lemma 4.5. *Let $S \subset \Gamma$ be the subset consisting of elements of word-length ≤ 4 with respect to the generating set g, h. Let $P_-P_+ \subset \alpha_0$ be the middle subinterval of α_0 whose length is $\frac{2}{9}L_0$. Assume that $\tau \leq d(P_-, P_+)$. Then for each $\gamma \in S$, the interval $\gamma(P_-P_+)$ is contained in the $3\eta_0$-neighborhood of α_0.*

Proof. The proof is a straight-forward application of the triangle inequalities taking into account the fact that the Hausdorff distance between α_0 and β_0 is $\leq \eta_0$. ☐

Then, arguing as in the proof of [12, Theorem 10.24][1], we obtain that each of the commutators

$$[g^{\pm 1}, h^{\pm 1}], \quad [h^{\pm 1}, g^{\pm 1}]$$

moves each point of P_-P_+ by at most

$$28 \times 3\eta_0 \leq 100\eta_0.$$

Therefore, by applying the Margulis Lemma as in the proof of [12, Theorem 10.24], we obtain the following corollary.

Corollary 4.6. *If Γ is discrete and $\eta_0 = 0.01\varepsilon(n,\kappa)$, then*

$$\tau \geq \frac{2}{9}L_0 = \frac{1}{9}(L_\alpha - 2L_1).$$

Corollary 4.7. *If Γ is discrete and $\tau \geq \lambda$, then the subgroup $\langle g^N, h^N \rangle = \Gamma_N < \Gamma$ is free of rank 2 whenever one of the following holds:*

[1]In fact, the argument there is a variation on a proof due to Culler–Shalen–Morgan and Bestvina, Paulin

(i) *either* $L_\alpha \leq 3L_1$ *and*

$$N \geq \frac{5+2\delta+3L_1}{\lambda},$$

(ii) *or* $L_\alpha \geq 3L_1$ *and*

$$N \geq 27 + \frac{9(5+2\delta)}{L_1}.$$

Proof. In view of Corollary 4.3, it suffices to ensure that inequality (9) holds.

(i) Suppose first that $L_\alpha \leq 3L_1$, hence $L_\beta \leq 3L_1$. Then, in view of the inequality $\tau \geq \lambda > 0$, inequality (9) will follow from

$$N \geq \frac{5+2\delta+3L_1}{\lambda}.$$

(ii) Suppose now that $L_\alpha \geq 3L_1$. The function

$$\frac{9(t+5+2\delta)}{t-2L_1}$$

attains its maximum on the interval $[3L_1, \infty)$ at $t = 3L_1$. Therefore,

$$\frac{9(L_\alpha+5+2\delta)}{L_\alpha-2L_1} \leq 27 + \frac{9(5+2\delta)}{L_1}.$$

Thus, the inequality

$$\tau \geq \frac{L_\alpha - 2L_1}{9}$$

implies that for any

$$N \geq 27 + \frac{9(5+2\delta)}{L_1},$$

we have $N\tau \geq L_\alpha + 5 + 2\delta$. □

Consider now the remaining case when for $\eta_0 := \frac{1}{100}\varepsilon(n,\kappa)$, the subinterval β_0 is empty, i.e., $\eta > \eta_0 = \frac{1}{100}\varepsilon(n,\kappa)$. Then, as above, the length L_α of the segment x_-x_+ is at most $2L_1$. Therefore, similarly to the case (i) of Corollary 4.7, in order for N to satisfy inequality (9), it suffices to get

$$N \geq \frac{5+2\delta+3L_1}{\lambda}.$$

Theorem 4.8. *Suppose that g, h are hyperbolic isometries of X generating a discrete nonelementary subgroup, whose translation lengths are equal to some $\tau \geq \lambda > 0$. Let L_1 be such that*

$$\sinh(L_1)\sinh\Big(\frac{1}{100}\varepsilon\Big) = 1,$$

where $\varepsilon = \varepsilon(n,\kappa)$. Then for every

$$N \geq \max\Big(\frac{5+2\delta+3L_1}{\lambda}, 27 + \frac{9(5+2\delta)}{L_1}\Big), \tag{10}$$

the group generated by g^N, h^N is free of rank 2.

We note that proving that (some powers of) g and h generate a free subsemigroup is easier, see [2] and [6, section 11].

Corollary 4.9. *Given g, h as in Theorem 4.8, and any N satisfying* (10)*, the free group $\Gamma_N = \langle g^N, h^N \rangle$ is convex-cocompact.*

Proof. Let $\mathcal{U}^{\pm} = H(g^{\pm N}x, x)$ and $\mathcal{V}^{\pm} = H(h^{\pm N}y, y)$. Observe that

$$g^{\pm N}(X \setminus \mathcal{U}^{\mp}) \subset \mathcal{U}^{\pm}$$

and

$$h^{\pm N}(X \setminus \mathcal{V}^{\mp}) \subset \mathcal{V}^{\pm}.$$

We let $\mathfrak{D}_{g^N}, \mathfrak{D}_{h^N}$ denote the closures in $\bar{X}$ of the domains

$$X \setminus (\mathcal{U}^{-} \cup \mathcal{U}^{+}), \quad X \setminus (\mathcal{V}^{-} \cup \mathcal{V}^{+}),$$

respectively, and set

$$\mathfrak{D} = \mathfrak{D}_{g^N} \cap \mathfrak{D}_{h^N}.$$

It is easy to see (cp. [15]) that this intersection is a fundamental domain for the action of Γ_N on the complement $\bar{X} \setminus \Lambda$ to its limit set Λ. Therefore, $(\bar{X} \setminus \Lambda)/\Gamma_N$ is compact. Hence, Γ_N is convex-cocompact (see [4]). ☐

Remark 4.10. It is also not hard to see directly that the orbit maps $\Gamma_N \to \Gamma_N x \subset X$ are quasiisometric embeddings by following the proofs in [14, Section 7] and counting the number of bisectors crossed by geodesics connecting points in Γx.

5. Case 2: Displacement bounded above

The strategy in this case is to find an element g' conjugate to g (by some uniformly bounded power of f) such that the Margulis regions of g, g' are sufficiently far apart, i.e., are at distance $\geq L$, where L is given by the local-to-global principle for piecewise-geodesic paths in X, see Proposition 3.20.

Proposition 5.1. *There exists a function*

$$\mathfrak{k}\colon [0, \infty) \times (0, \varepsilon] \to \mathbb{N},$$

for $0 < \varepsilon \leq \varepsilon(n, \kappa)$, with the following property: Let $g_1, \dots, g_k$ be nonelliptic isometries of the same type (hyperbolic or parabolic) with translation lengths $\leq \varepsilon/10$ and

$$k \geq \mathfrak{k}(L, \varepsilon).$$

Suppose that $\langle g_i, g_j \rangle$ are nonelementary discrete subgroups for all $i \neq j$. Then there exists a pair of indices $i, j \in \{1, \dots, k\}$, $i \neq j$, such that

$$d(\mathrm{Hull}(\mathcal{T}_{\varepsilon}(g_i)), \mathrm{Hull}(\mathcal{T}_{\varepsilon}(g_j))) > L.$$

Proof. If all the isometries g_i are parabolic, then the proposition is established in [14, Proposition 8.3]. Therefore, we only consider the case when all these isometries are hyperbolic. Our proof follows closely the proof of [14, Proposition 8.3].

Since for all $i \neq j$ the subgroup $\langle g_i, g_j \rangle$ is discrete and nonelementary, and $\varepsilon \leq \varepsilon(n, \kappa)$, we have

$$\mathcal{T}_\varepsilon(g_i) \cap \mathcal{T}_\varepsilon(g_j) = \varnothing.$$

Given $L > 0$, suppose that

$$d\big(\mathrm{Hull}(\mathcal{T}_\varepsilon(g_i)), \mathrm{Hull}(\mathcal{T}_\varepsilon(g_j))\big) \leq L \quad \text{for all } i, j \in \{1, \ldots, k\}.$$

Our goal is to get a uniform upper bound of k.

Consider the $L/2$-neighborhoods $\bar{N}_{L/2}(\mathrm{Hull}(\mathcal{T}_\varepsilon(g_i)))$. They are convex in X and have nonempty pairwise intersections. Thus, by [14, Proposition 8.2], there exists a point $x \in X$ such that

$$d(x, \mathcal{T}_\varepsilon(g_i)) \leq R_1 := n\delta + L/2 + \mathfrak{q}, \quad i = 1, \ldots, k,$$

where δ is the hyperbolicity constant of X and $\mathfrak{q}$ is as in Proposition 3.6. Then

$$\mathcal{T}_\varepsilon(g_i) \cap B(x, R_1) \neq \varnothing, \quad i = 1, \ldots, k.$$

For each $i = 1, \ldots, k$, we take a point $x_i \in \mathcal{T}_\varepsilon(g_i) \cap B(x, R_1)$ satisfying $d(x_i, g_i^{p_i}(x_i)) = \varepsilon$ for some $0 < p_i \leq m_{g_i}$. Since the translation lengths of the elements $g_i^{p_i}$ are $\leq \varepsilon/10$, by Corollary 3.16, there exist points $y_i \in X$ such that

$$d(y_i, g_i^{p_i}(y_i)) = \varepsilon/3, \quad d(x_i, y_i) \leq \mathfrak{r}(\varepsilon).$$

Consider the $\varepsilon/3$-balls $B(y_i, \varepsilon/3)$. Then $B(y_i, \varepsilon/3) \subset \mathcal{T}_\varepsilon(g_i)$, since

$$d(z, g_i^{p_i}(z)) \leq d(z, y_i) + d(y_i, g_i^{p_i}(y_i)) + d(g_i^{p_i}(y_i), g_i^{p_i}(z)) \leq \varepsilon$$

for any point $z \in B(y_i, \varepsilon/3)$. Thus, the balls $B(y_i, \varepsilon/3)$ are pairwise disjoint. Observe that $B(y_i, \varepsilon/3) \subset B(x, R_2)$, where $R_2 = R_1 + \mathfrak{r}(\varepsilon) + \varepsilon/3$.

Let $V(r, n)$ denote the volume of the r-ball in $\mathbb{H}^n$. Then for each $i = 1, \ldots, k$, $\mathrm{Vol}(B(y_i, \varepsilon/3))$ is at least $V(\varepsilon/3, n)$, see [4, Proposition 1.1.12]. Moreover, the volume of $B(x, R_2)$ is at most $V(\kappa R_2, n)/\kappa^n$, see [4, Proposition 1.2.4]. Let

$$\mathfrak{k}(L, \varepsilon) := \frac{V(\kappa R_2, n)/\kappa^n}{V(\varepsilon/3, n)} + 1.$$

Then $k < \mathfrak{k}(L, \varepsilon)$, because otherwise we would obtain

$$\mathrm{Vol}\bigg(\bigcup_{i=1}^{k} B(y_i, \varepsilon/3)\bigg) > \mathrm{Vol}(B(x, R_2)),$$

where the union of the balls on the left side of this inequality is contained in $B(x, R_2)$, which is a contradiction.

Therefore, whenever $k \geq \mathfrak{k}(L, \varepsilon)$, there exist a pair of indices i, j such that

$$d\big(\mathrm{Hull}(\mathcal{T}_\varepsilon(g_i)), \mathrm{Hull}(\mathcal{T}_\varepsilon(g_j))\big) > L. \qquad \square$$

Remark 5.2. Proposition 5.1 also holds for isometries of mixed types (i.e., some g_i's are parabolic and some are hyperbolic). The proof is similar to the one given above.

Theorem 5.3. *For every nonelementary discrete subgroup* $\Gamma = \langle g, h\rangle < \mathrm{Isom}(X)$*, with* g, h *nonelliptic isometries satisfying*

$$\tau(g) \le \varepsilon/10 \le \varepsilon(n,\kappa)/10,$$

there exists i*,* $1 \le i \le \mathfrak{k}(L(\varepsilon/10), \varepsilon)$*, such that* $\langle g, h^i g h^{-i}\rangle$ *is a free subgroup of rank 2, where* $\mathfrak{k}$ *is the function given by Proposition 5.1 and* $L(\varepsilon/10)$ *is the constant in Proposition 3.18.*

Proof. Consider the isometries $g_i := h^i g h^{-i}$, $i \ge 1$. We first claim that no pair g_i, g_j, $i \neq j$, generates an elementary subgroup of $\mathrm{Isom}(X)$. There are two cases to consider:

(i) Suppose that g is parabolic with the fixed point $p \in \partial_\infty X$. We claim that for all $i \neq j$, $h^i(p) \neq h^j(p)$. Otherwise, $h^{j-i}(p) = p$, and p would be a fixed point of h. But this would imply that Γ is elementary, contradicting our hypothesis.

(ii) The proof in the case when g is hyperbolic is similar. The axis of g_i equals $h^i(A_g)$. If the hyperbolic isometries g_i, g_j, $i \neq j$, generate a discrete elementary subgroup of Γ, then they have to share the axis, and we would obtain $h^i(A_g) = h^j(A_g)$. Then $h^{j-i}(A_g) = A_g$. Since h^{j-i} is nonelliptic, it cannot swap the fixed points of g, hence it fixes both of these points. Therefore, g, h have common axis, contradicting the hypothesis that Γ is nonelementary.

All the isometries g_i have equal translation lengths $\le \varepsilon/10$. Therefore, by Proposition 5.1, there exists a pair of natural numbers $i, j \le \mathfrak{k}(L(\varepsilon/10), \varepsilon)$ such that

$$d\big(\mathrm{Hull}(\mathcal{T}_\varepsilon(h^i g h^{-i})), \mathrm{Hull}(\mathcal{T}_\varepsilon(h^j g h^{-j}))\big) > L(\varepsilon/10),$$

where $\mathfrak{k}(L(\varepsilon/10), \varepsilon)$ is the function as in Proposition 5.1. It follows that

$$d\big(\mathrm{Hull}(\mathcal{T}_\varepsilon(h^{j-i} g h^{i-j})), \mathrm{Hull}(\mathcal{T}_\varepsilon(g))\big) > L(\varepsilon/10).$$

Setting $f := h^{j-i} g h^{i-j}$, and applying Proposition 3.20 to the isometries f, g, we conclude that the subgroup $\langle f, g\rangle < \Gamma$ is free of rank 2. The word length of f is at most $2|j-i| + 1 \le 2\mathfrak{k}(L(\varepsilon/10), \varepsilon) + 1$. ☐

6. Conclusion

Now we are in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We set $\lambda := \varepsilon/10$, where $\varepsilon = \varepsilon(n,\kappa)$ is the Margulis constant. Let g, h be non-elliptic isometries of X generating a discrete nonelementary subgroup of $\mathrm{Isom}(X)$ such that $\tau(g) = \tau(h) = \tau$.

If $\tau \ge \lambda$, then, by Theorem 4.8, the subgroup $\Gamma_N < \Gamma$ generated by g^N, h^N is free of rank 2, where

$$N := \left\lceil \max\left(\frac{5 + 2\delta + 3L_1}{\lambda}, 27 + \frac{9(5+2\delta)}{L_1}\right)\right\rceil.$$

Here $\delta = \cosh^{-1}(\sqrt{2})$, and

$$L_1 = \sinh^{-1}\left(\frac{1}{\sinh(\varepsilon/100)}\right).$$

If $\tau \leq \lambda$, then, by Theorem 5.3, there exists $i \in [1, \mathfrak{k}(L(\lambda), \varepsilon)]$ such that $\langle g, h^i g h^{-i} \rangle$ is free of rank 2, where $\mathfrak{k}(L(\lambda), \varepsilon)$ is a constant as in Theorem 5.3. The proof is complete. □

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