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› Foliations in moduli spaces of bounded global G-shtukas

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Abstract

We establish a foliation structure for Newton strata in moduli spaces of bounded global G-shtukas. We describe these strata as a product of "truncated" affine Deligne-Lusztig varieties with so-called Igusa varieties up to a finite surjective morphism. This is a function field analogue to Oort's and Mantovan's foliation for moduli spaces of abelian varieties and generalizes the foliation structure for deformation spaces of local shtukas to parahoric group schemes and to global G-shtukas.

Zusammenfassung

Wir führen eine Blätterungsstruktur für Newton-Strata in Modulräumen von beschränkten globalen G-Shtukas ein. Wir beschreiben diese Strata als ein Produkt von "abgeschnittenen" affinen Deligne-Lusztig-Varietäten mit so genannten Igusa-Varietäten bis auf einen endlichen surjektiven Morphismus. Dies ist ein Funktionenkörper-Analogon zu Oort's und Mantovan's Blätterung für Modulräume abelscher Varietäten und verallgemeinert die Blätterungsstruktur für Deformationsräume von lokalen Shtukas auf den Fall parahorischer Gruppenschemata und auf globale G-Shtukas.

Introduction

As a function field analogue to abelian varieties (with additional structures) Arasteh Rad and Hartl introduced global G-shtukas in [AH14a] and [AH14b]. We recall their definition. Let $\mathbb{F}_q$ be a finite field with q elements, C a smooth projective geometrically irreducible curve over $\mathbb{F}_q$ and let $Q = \mathbb{F}_q(C)$ be the function field of C. For a closed point $c \in C$ we denote by $\mathbb{F}_c$ the residue field at c, $A_c \cong \mathbb{F}_c[\![z_c]\!]$ the completion of the local ring $\mathcal{O}_{C,c}$ and by Q_c the fraction field of A_c. We let $\mathcal{N}\text{ilp}\, A_c$ be the category of A_c-schemes on which the image ζ_c of z_c is locally nilpotent. Let G be a parahoric Bruhat-Tits group scheme over C as defined by Bruhat and Tits [BT72, Définition 5.2.6] and denote by $G_c := G \times_C \operatorname{Spec} A_c$ the base change of G to $\operatorname{Spec} A_c$, which is a parahoric group scheme over A_c. A *global G-shtuka* over an $\mathbb{F}_q$-scheme S is a tuple $\underline{\mathcal{G}} = (\mathcal{G}, s_1, \dots, s_n, \tau)$ consisting of a G-torsor $\mathcal{G}$ over $C \times_{\mathbb{F}_q} S =: C_S$, a tuple of characteristic sections $(s_1, \dots, s_n) \in C^n(S)$, and a Frobenius connection τ, which is an isomorphism $\tau : \sigma^*\mathcal{G}\,|_{C_S \smallsetminus \cup_i \Gamma_{s_i}} \xrightarrow{\sim} \mathcal{G}\,|_{C_S \smallsetminus \cup_i \Gamma_{s_i}}$ outside the graph of the s_i's with $\sigma^* := (\mathrm{id}_C \times \mathrm{Frob}_q)^*$. An isomorphism between $\underline{\mathcal{G}}$ and another global G-shtuka $\underline{\mathcal{G}}' = (\mathcal{G}', s_1', \dots, s_n', \tau')$ is an isomorphism $f : \mathcal{G} \tilde{\to} \mathcal{G}'$ of G-torsors with $\tau' \circ \sigma^* f = f \circ \tau$, and $s_i = s_i'$ for all $i = 1, \dots, n$. The category of global G-shtukas with isomorphisms as morphisms is fibered over the category of $\mathbb{F}_q$-schemes and is an ind-algebraic stack, that is an inductive limit of Deligne-Mumford stacks (together with closed immersions), which are separated and locally of finite type over C (see [AH14b, Theorem 3.14]). If global G-shtukas are provided with H-level structures (see Definition 2.52) and with boundedness conditions (see Definition 2.29 and Definition 2.32) the moduli stack of bounded global G-shtukas is a separated Deligne-Mumford stack and locally of finite type over C^n (see [AH14b, Theorem 3.14]). These stacks are the function field analogues of Shimura varieties over number fields. Therefore, one can expect that the Langlands correspondence for function fields can be realized in the cohomology of those moduli spaces. Moduli stacks of global G-shtukas, constructed by Arasteh Rad and Hartl in [AH14b] are generalizations of moduli spaces of *Drinfeld shtukas*, which were investigated and studied by Drinfeld [Dri74] and Lafforgue [Laf02] for $G = \mathrm{GL}_2$ (resp. $G = \mathrm{GL}_r$). Moduli spaces of global G-shtukas also generalize the moduli stacks FBun studied by Varshavsky [Var04], which treat the case of a constant split reductive group G, and also the stacks $\mathcal{E}\ell\ell_{C,\mathcal{D},I}$ of Laumon, Rapoport and Stuhler [LRS93] who used

them to prove the Local Langlands Correspondence for GL_r in the function field case.
Analogously to p-divisible groups (also called Barsotti-Tate groups), which are an important tool in the theory of Shimura varieties and abelian varieties, global G-shtukas give rise to so-called *local* G_c-shtukas. For an explicit definition of local G_c-shtukas see Definition 2.8. As a function field analogue to the functor that assigns a p-divisible group to an abelian variety, Arasteh Rad and Hartl established the so-called *global-local functor* which associates a tupel of local G_c-shtukas to a global G-shtuka. In [HV11] and [HV12] Hartl and Viehmann studied deformation spaces of local G_c-shtukas for constant split connected reductive groups G_c. Morphisms between local shtukas are called quasi-isogenies. An important invariant of a local G_c-shtuka (over an algebraically closed field) is its isogeny class. This gives rise to a so-called *Newton stratification* of those deformation spaces by quasi-isogeny classes. By [HV12], each Newton stratum in the deformation space of local G_c-shtukas for constant split G_c possesses a foliation structure in the following sense: Onto each Newton stratum there is a finite morphism from the product of a *central leaf* with (the completion at a point of) the corresponding *affine Deligne-Lusztig variety* (which is the underlying topological space of a Rapoport-Zink space, that is the moduli space for local G_c-shtukas). By Arasteh Rad's analogue of the Serre-Tate theorem (see Section 2.3) the deformation spaces of global G-shtukas are isomorphic to products of deformation spaces of the local G_{c_i}-shtukas, and hence also have a foliation structure.
The matter of this thesis is to obtain such a foliation structure for Newton strata moduli spaces of bounded global G-shtukas with H-level structure for an arbitrary parahoric Bruhat-Tits group G. Therefore, we define and study Newton strata of these moduli spaces. We describe such a stratum up to a finite surjective morphism as a product of the reduced fibres of the corresponding (truncated) Rapoport-Zink spaces with some algebraic stacks, which we call Igusa varieties. This foliation is a function field analogue to Oort's foliations for p-divisible groups and abelian varieties ([Oor04]) and to Mantovan's product structure for Newton polygon strata in Shimura varieties of (PEL) type and generalizes Hartl's and Viehmann's foliation structure for deformation spaces of local shtukas to parahoric group schemes and to global G-shtukas. We proceed as follows:

In **Chapter 1** we present definitions and notations that are used throughout this paper and recall some general facts about reductive groups.

Chapter 2 treats global G-shtukas and their relations to local G_c-shtukas. The results there are already shown in [AH14a] and [AH14b]. We recall the definitions of global and local G-shtukas in Section 2.1. We describe the global-local functor which associates a tupel of local G_c-shtukas to a global G-shtuka $\underline{\hat{\Gamma}} := \prod_{i=1,\dots,n} \hat{\Gamma}_{c_i}$

for a closed point $\underline{c} = (c_1, \ldots, c_n) \in C^n$ with $c_i \neq c_j$ for $i \neq j$ (see Definition 2.18) in Section 2.2. Section 2.3 presents the Serre-Tate theorem. In Section 2.4 we introduce Rapoport-Zink spaces for (bounded) local G_c-shtukas and define the *truncated Rapoport-Zink spaces* as follows: Let $\mathbb{L}_0$ be a fixed local G_c-shtuka over a field $\tilde{k}$, and $\underline{\mathcal{L}}$ a local G_c-shtuka over a scheme S. Denote by $\mathcal{RZ}_{\mathbb{L}_0}^{\hat{Z}}$ the (bounded) Rapoport-Zink space associated to $\mathbb{L}_0$. We fix a representation $\varrho : G_c \longrightarrow \mathrm{SL}_{r,\operatorname{Spec}\mathbb{F}_c[\![z]\!]}$ and denote by $2\check{\varrho}$ the sum of all positive coroots of SL_r with respect to the Borel-subgroup of upper triangular matrices in SL_r. For a non-negative integer d_i the truncated Rapoport-Zink space is the closed ind-subscheme $\mathcal{RZ}_{\hat{Z}}^{d_i}$ of $\mathcal{RZ}_{\mathbb{L}_0}^{\hat{Z}}$ defined by the functor

$$\begin{aligned} \mathcal{RZ}_{\hat{Z}}^{d_i} : (\mathcal{N}\mathrm{ilp}_{\tilde{k}[\![\xi]\!]})^\circ &\longrightarrow (Sets) \\ S &\mapsto \Big\{\text{Isomorphism classes of } (\underline{\mathcal{L}}, \delta) := (\mathcal{L}_+, \hat{\tau}, \delta) \in \mathcal{RZ}_{\mathbb{L}_0}^{\hat{Z}}(S), \\ &\qquad \text{s.t. } \varrho_*(\delta) \text{ is bounded by } 2d_i\check{\varrho}\Big\} \end{aligned}$$

where $\delta : \underline{\mathcal{L}}_{\bar{S}} \to \mathbb{L}_{0,\bar{S}}$ is a quasi-isogeny; see Definition 2.41. Arasteh Rad and Hartl used the Rapoport-Zink spaces of local G_c-shtukas to uniformize the moduli spaces of global G-shtukas. We will recall this uniformization morphism (established by Arasteh Rad and Hartl in [AH14b]) in Section 2.5.

In **Chapter 3** we introduce our generalization of Newton strata inside the special fibre of the moduli space of global G-shtukas. We denote by $\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$ the base change of the special fibre $\nabla\mathcal{H}^1$ over $\underline{c}$ of the moduli space of bounded global G-shtukas (with H-level structure) to $\mathbb{F}_q^{\mathrm{alg}}$. The stratification is given via the quasi-isogeny classes of local G_c-shtukas. We show that Newton strata exist as locally closed substacks in the special fibre of the moduli space of global G-shtukas. We give a short description of Newton strata here: Denote by $\mathcal{B}(G_{c_i})$ the set of quasi-isogeny classes of local G_{c_i}-shtukas $\underline{\hat{\mathcal{G}}}_i$. Choose a tuple of quasi-isogeny classes $\underline{b} = (b_1, \ldots, b_n) \in \prod_i \mathcal{B}(G_{c_i})$. Then the reduced scheme

$$\mathcal{N}_{\leq \underline{b}} := \Big\{ s \in S \;:\; [\underline{\hat{\mathcal{G}}}_{i,s}] \leq b_i \text{ for all } i \Big\}$$

is a closed subscheme of S, and

$$\mathcal{N}_{\underline{b}} := \Big\{ s \in s \;:\; [\underline{\hat{\mathcal{G}}}_{i,s}] = b_i \text{ for all } i \Big\}$$

is a locally closed subscheme; see Corollary 3.6.

In analogy to Mantovan's foliation for moduli spaces of abelian varieties we want to give a foliation structure on the Newton Strata but not only in deformation spaces but also for the whole Newton strata in the moduli space of (bounded) global G-shtukas (with H-level structure) and for relatively general groups. This foliation structure is decribed as a product of a covering of central leaves by Igusa varieties with truncated Rapoport-Zink spaces. Therefore, in **Chapter 4** we start by considering leaves.

In the classical case Oort introduced the notion of a central leaf. In our case a *central leaf* inside a Newton stratum is defined to be the locus where the associated local G_c-shtukas (to a global G-shtuka) are isomorphic to a chosen fixed local G_c-shtuka. For our purposes we choose this isomorphism class to be given by a so-called *fundamental element*; see Definition 4.3. Let K be a field extension of $\mathbb{F}_{c_i}$ and let $\underline{\hat{\mathbb{G}}}_i$ be a local G_{c_i}-shtuka over K. For a K-scheme S and a local G_{c_i}-shtuka $\underline{\hat{\mathcal{G}}}_i$ over S the *central leaf* is defined as the subset

$$\mathcal{C}_{\underline{\hat{\mathbb{G}}}_i,S} := \left\{ s \in S : \underline{\hat{\mathbb{G}}}_{i,L} \cong \underline{\hat{\mathcal{G}}}_{i,L} \text{ over an algebraically closed field extension } L/\kappa(s) \right\}.$$

If $\underline{\mathcal{G}}$ is a global G-shtuka over a K-scheme S, the *central leaves* are defined as the subsets

$$\mathcal{C}_{\underline{\hat{\mathbb{G}}}_i,S} := \left\{ s \in S : \underline{\hat{\mathbb{G}}}_{i,L} \cong \hat{\Gamma}_{c_i}(\underline{\mathcal{G}})_L \text{ over an algebraically closed field extension } L/\kappa(s) \right\}$$

and

$$\mathcal{C}_{(\underline{\hat{\mathbb{G}}}_i)_i,S} := \bigcap_i \mathcal{C}_{\underline{\hat{\mathbb{G}}}_i,S}\,;$$

see Definition 4.19.

The results in [Oor04] and [Man04] (resp. in [HV12]) concerning the existence of a foliation structure with the desired properties are based on the statement that a family of p-divisible groups (resp. local G_c-shtukas) with constant Newton polygon is quasi-isogenous to a completely slope divisble family ([HV12], [OZ02]). To get a similar result in our case we choose a *fundamental element* x which is defined over a field $\mathbb{F}_x$ (for more details see Section 4.1). In Section 4.2 we introduce the notion of a *complete slope division* of a local G_c-shtuka $\underline{\mathcal{G}}$ over S (see Definition 4.16) and clarify the existence of complete slope divisions (see Theorem 4.17). In the following, we describe this briefly: Denote by I an *Iwahori group* as defined in [HR08, Definition 1]. Let $P = MN$ be a parabolic of G_Q with Levifactor M and unipotent radical N. Let $\bar{P} = M\bar{N}$ be its opposite. Furthermore, we use the notation $I_{\bar{P}} := I_M \cdot I_{\bar{N}}$ for the intersection with I. We do not assume that I, M or N is fixed by the $\mathbb{F}_c$-Frobenius $\hat{\sigma}$. Roughly speaking, a *complete slope division* $(\overline{\mathcal{P}}, \eta)$ *with respect to* x of a local G_c-shtuka $\underline{\mathcal{G}} = (\mathcal{G}, \hat{\tau}_{\mathcal{G}})$ over an $\mathbb{F}_q^{\mathrm{alg}}$-scheme is described by the existence of a "trivializtion" of the Frobenius $\hat{\tau}_{\mathcal{G}}$ to an element in $x \cdot L^+ I_{\bar{P}}(S')$ (here, $L^+ I$ denotes the groop of positive loops, see Definition 1.5). If $\underline{\mathcal{G}}$ is a local G_c-shtuka over an $\mathbb{F}_q^{\mathrm{alg}}$-scheme S with constant isogeny class $[x]$, then there is scheme Y_x which is finite over S and represents isomorphism classes of complete slope divisions of $\underline{\mathcal{G}}$. We show that for a tuple $\underline{x} = (x_1, \ldots, x_n)$ of fundamental elements the central leaf $\mathcal{C}_{x_i,S} := \mathcal{C}_{(L^+G_{c_i}, x_i),S}$ is closed in the Newton stratum $\mathcal{N}_{x_i} \subset S$ and that it exists as a

locally closed substack of the special fibre $\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$, and hence is a Deligne-Mumford stack locally of finite type.
We prove that a local G_c-shtuka over an $\mathbb{F}_q^{\mathrm{alg}}$- scheme S with constant isogeny class $[x]$ is completely slope divisible over a finite surjective morphism $Y := Y_x \to \mathcal{C}_{x,S}$. If $G_c = I$ is an Iwahori group, then we can take $Y = \mathcal{C}_{x,S}$; see Corollary 4.22. The schemes Y_{x_i} from above define Deligne-Mumford stacks $\mathcal{Y}_{x_i}$ for all $i = 1, \dots, n$ which are finite over $\mathcal{N}_{x_i} \subset \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$. We set $\mathcal{Y}_{\underline{x}} := \mathcal{Y}_{x_1} \times_{\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} \dots \times_{\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} \mathcal{Y}_{x_n}$. We will see in Theorem 4.29 that $\mathcal{Y}_{\underline{x}} \to \operatorname{Spec}\mathbb{F}_q^{\mathrm{alg}}$ is a smooth stack, and we will compute its dimension. Furthermore, all central leaves $\mathcal{C}_{(\hat{\mathbb{G}}_i)_i}$ in the Newton stratum $\mathcal{N}_{\underline{x}} \subset \nabla\mathcal{H}^1$ are closed substacks and smooth; see Proposition 6.20. This property is the anologue of [Oor04, Theorem 3.13] resp. [Man04, Proposition 2.7].

We introduce in **Chapter 5** the *Igusa varieties* Ig_{e_i} *of level* e_i (for an integer $e_i > 0$) as coverings of the central leaves ; see Definition 5.4. Our definition is inspired by Mantovan's definition of Igusa varieties in [Man04] and by the one of Harris and Taylor in [HT01]. Mantovan describes ([Man04, §3]) Igusa varieties in the setting of abelian varieties as a sheaf of isomorphisms between truncated Barsotti-Tate groups. We modify this definition by using a trivialization of the Levi part of a complete slope division. More precisely, we choose a presentation $f : S \to \mathcal{Y}_{x_i}$ such that the universal complete slope division $(\overline{\mathcal{P}}_i, \eta_i)$ of $\hat{\Gamma}_{c_i}(f^*\underline{\mathcal{G}}^{\mathrm{univ}}) = (\mathcal{G}_i, \hat{\tau}_{\mathcal{G}_i})$ over $\mathcal{Y}_{x_i}$ possesses a trivialization such that, loosely speaking, $\hat{\tau}_{\mathcal{G}_i}$ can be written as an element $x_i m_i \bar{n}_i \in x_i \cdot L^+ I_{\bar{P}_i}(S)$. For an integer $e_i \geq 0$ we define the *Igusa variety of level* e_i as the functor Ig_{e_i} on the category of S-schemes T given by

$$\mathrm{Ig}_{e_i}(T) := \left\{ j_{e_i} \in (L^+\hat{\sigma}_i^{-1} I_{M_i} \big/ L^+\hat{\sigma}_i^{-1} I_{e_i,M_i})(T) \;:\; \hat{\sigma}_i^*(j_{e_i}^{-1}) x_i^{-1} j_{e_i} x_i m_i \in L^+ I_{e_i,M_i}(T) \right\}.$$

For a tuple of non-negative integers $\underline{e} = (e_i)_i$ we define the *Igusa variety* $\mathrm{Ig}_{\underline{e}}$ *of level* $\underline{e}$ as the functor on the category of S-schemes T

$$\mathrm{Ig}_{\underline{e}}(T) := \left\{ (j_{e_i})_i \;:\; j_{e_i} \in \mathrm{Ig}_{e_i}(T) \right\};$$

see Definition 5.4. We show that there exist algebraic stacks

$$\mathfrak{Ig}_{e_i} \to \mathcal{Y}_{x_i} \to \mathcal{C}_{x_i} \subset \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$$

with $\mathfrak{Ig}_{e_i} \times_{\mathcal{Y}_{x_i}} S = \mathrm{Ig}_{e_i}$ and $\mathfrak{Ig}_{\underline{e}} \times_{\mathcal{Y}_{\underline{x}}} S = \mathrm{Ig}_{\underline{e}}$; see Proposition 5.6. The morphisms $\mathfrak{Ig}_{e_i} \to \mathcal{Y}_{x_i}$ and $\mathfrak{Ig}_{\underline{e}} \to \mathcal{Y}_{\underline{x}}$ are finite étale Galois coverings. In particular, the morphisms $\mathfrak{Ig}_{e_i} \to \mathcal{C}_{x_i} \subset \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$ and $\mathfrak{Ig}_{\underline{e}} \to \mathcal{C}_{\underline{x}} \subset \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$ are finite, but in general not étale; see Proposition 5.7.

The main result of this thesis, which is proven in **Chapter 6**, is the following:

Main Theorem 0.1. *Let* $\underline{d} = (d_1, \ldots, d_n)$ *and denote by* $\underline{e} = (e_i)_i$ *the level of the Igusa variety* $\mathfrak{Ig}_{\underline{e}}$*, such that* $e_i + 2d_i(1 - r_i) \geq 1$ *for all* $i = 1, \ldots, n$*. Let* l *be a multiple of all* $\deg c_i := [\mathbb{F}_{c_i} : \mathbb{F}_q]$ *for the number* $l_i(e_i)$ *from Definition 4.3 (vi). There is a morphism of stacks*

$$\pi^{\underline{d},\underline{e},l} : \mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}}(\underline{x}) \to \sigma^{l*}\mathcal{N}_{\underline{x}} \times_{\mathbb{F}_{\underline{c}}} \mathbb{F}_q^{\mathrm{alg}} \subset \sigma^{l*}\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}},$$

which is finite. If we consider the compositum of $\pi^{\underline{d},\underline{e},l}$ *with the projection* $\sigma^{l*}\mathcal{N}_{\underline{x}} \to \mathcal{N}_{\underline{x}}$ *then for every point of* $\mathcal{N}_{\underline{x}}$ *with values in an algebraically closed field there is a tuple* $(\underline{d}, \underline{e}, l)$ *and a point in the preimage under* $\pi^{\underline{d},\underline{e},l}$*.*

Here, $X_{Z_i}(x_i)$ denotes the affine Deligne-Lusztig variety from Definition 2.38. Furthermore, we denote by $X_{Z_i}^{d_i}(x_i)$ the reduced subscheme of $\mathcal{RZ}_{\hat{Z}_i}^{d_i}$ and set

$$X_{\underline{Z}}^{\underline{d}} := (X_{Z_1}^{d_1}(x_1) \times_{\mathbb{F}_q^{\mathrm{alg}}} \ldots \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{Z_n}^{d_n}(x_n)).$$

The proof of Main Theorem 0.1 is devided into several steps and the construction is very technical as we need suitable "trivializations", and several quasi-isogenies. But once we have chosen those we construct the product morphism $\pi^{\underline{d},\underline{e},l}$ by using Arasteh Rad's uniformization morphism. Then we can show that $\pi^{\underline{d},\underline{e},l}$ is quasi-finite and satisfies the valuation criterion for properness and hence is finite and "surjective" in the sense of the statement of the Main Theorem. The tuple of trivializations α is chosen in Section 6.1, the tuple of quasi-isogenies $h(l)$ is constructed in Section 6.2. In Section 6.3 we show that the tuple $(\alpha, h(l))$ descends to the Igusa variety. These techniques are needed to construct the product morphism $\pi^{\underline{d},\underline{e},l}$ and prove its desired properties in Section 6.4.
Finally, we conclude this thesis with an application of the Main Theorem. Namely in Proposition 6.20 we deduce, that each leaf inside one Newton stratum has the same dimension and we compute this dimension.

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Contents

1 Preliminaries

1.1 Notations

Following [AH14a] and [AH14b], we denote by

$\mathbb{F}_q$	a finite field with q elements and characteristic p,
C	a smooth projective geometrically irreducible curve over $\mathbb{F}_q$,
$Q := \mathbb{F}_q(C)$	the function field of C,
c	a closed point of C, also called a *place* of C,
$\mathbb{F}_c$	the residue field at the place c on C,
A_c	the completion of the stalk $\mathcal{O}_{C,c}$ at c,
$Q_c := \mathrm{Frac}(A_c)$	its fraction field,
$\mathbb{D}_{c,R} := \mathrm{Spec}\, R[\![z]\!]$	the spectrum of the ring of formal power series in z with coefficients in an $\mathbb{F}_c$-algebra R,
$\hat{\mathbb{D}}_{c,R} := \mathrm{Spf}\, R[\![z]\!]$	the formal spectrum of $R[\![z]\!]$ with respect to the z-adic topology.

We let z be a uniformizer of A_c. Therfore, there are canonical isomorphisms $A_c = \mathbb{F}_c[\![z]\!]$ and $Q_c = \mathbb{F}_c(\!(z)\!)$. When $R = \mathbb{F}_c$ we drop the subscript R from the notation of $\mathbb{D}_{c,R}$ and $\hat{\mathbb{D}}_{c,R}$.

For a formal scheme $\hat{T}$ we denote by $\mathcal{N}\mathrm{ilp}_{\hat{T}}$ the category of schemes over $\hat{T}$ on which an ideal of definition of $\hat{T}$ is locally nilpotent. We equip $\mathcal{N}\mathrm{ilp}_{\hat{T}}$ with the $*$-topology where $* \in \{$étale, fppf, fpqc$\}$.

Let $n \in \mathbb{N}_{>0}$ be a positive integer, and denote by $\underline{c} := (c_i)_{i=1\ldots n}$ an n-tuple of closed points of C with $c_i \neq c_j$ for $i \neq j$, and by

$A_{\underline{c}}$	the completion of the local ring $\mathcal{O}_{C^n,\underline{c}}$ of C^n at the closed point $\underline{c} = (c_i)$,
$\mathbb{O}^{\underline{c}} := \prod_{c \notin \underline{c}} A_c$	the ring of integral adeles of C outside c,
$\mathbb{A}^{\underline{c}} := \mathbb{O}^{\underline{c}} \otimes_{\mathcal{O}_C} Q$	the ring of adeles of C outside c,
$\mathcal{N}\mathrm{ilp}_{A_{\underline{c}}} := \mathcal{N}\mathrm{ilp}_{\mathrm{Spf}\, A_{\underline{c}}}$	the category of schemes over C^n on which the ideal defining the closed point $\underline{c} \in C^n$ is locally nilpotent,
$\mathcal{N}\mathrm{ilp}_{\mathbb{F}_c[\![\zeta]\!]} := \mathcal{N}\mathrm{ilp}_{\hat{\mathbb{D}}_c}$	the category of $\mathbb{D}_c$-schemes S for which the image of z in $\mathcal{O}_S$ is locally nilpotent. We denote the image of z by ζ since we need to distinguish it from $z \in \mathcal{O}_{\mathbb{D}_c}$.

We further denote by

G	a flat affine group scheme of finite type over C, which is parahoric in the sense of Definition 1.1,
$G_c := G \times_C \operatorname{Spec} A_c$	the base change of G to $\operatorname{Spec} A_c$,
$G_{Q_c} := G \times_C \operatorname{Spec} Q_c$	the generic fibre of G_c over $\operatorname{Spec} Q_c$.

Independently of G, we also consider more generally

G_c	a flat affine group scheme of finite type over $\mathbb{D}_c$,
$G_{Q_c} := G_c \times_{\mathbb{D}_c} \operatorname{Spec} \mathbb{F}_c(\!(z)\!)$	the generic fibre of G_c over $\operatorname{Spec} \mathbb{F}_c(\!(z)\!)$.

For any $\mathbb{F}_q$-scheme S we denote by $\sigma_S : S \to S$ its $\mathbb{F}_q$-Frobenius endomorphism, which acts as the identity on the points of S and as the q-power map on the structure sheaf.
We denote likewise by $\hat{\sigma}_S : S \to S$ the $\mathbb{F}_c$-Frobenius endomorphism of an $\mathbb{F}_c$-scheme S with $\hat{\sigma}_S := \sigma_S^{\deg c}$ for $\deg c := [\mathbb{F}_c : \mathbb{F}_q]$.
We set

$$C_S := C \times_{\operatorname{Spec} \mathbb{F}_q} S,$$
$$\sigma := \mathrm{id}_C \times \sigma_S.$$

1.2 Further Definitions

Let $\mathbb{F}_q$ be a finite field with q elements, C a smooth projective geometrically irreducible curve over $\mathbb{F}_q$ and $Q := \mathbb{F}_q(C)$ the function field of C. For any closed point c denote by $\mathbb{F}_c$ the residue field at the place c on C.
Let $A_c \cong \mathbb{F}_c[\![z_c]\!]$ be the completion of the local ring $\mathcal{O}_{C,c}$ and $Q_c := \mathrm{Frac}(A_c)$ its fraction field.
By $\mathcal{N}\mathrm{ilp}_{A_c}$ we denote the category of A_c-schemes on which (the image) ζ_c (of z_c) is locally nilpotent.
Let G be a *parahoric Bruhat-Tits group scheme over* C as defined in the following:

Definition 1.1. *A* parahoric (Bruhat-Tits) group scheme over C *is a smooth affine group scheme* G *over* C *such that*

(i) all geometric fibres of G *are connected and the generic fibre of* G *is reductive over* $\mathbb{F}_q(C)$,

(ii) for any ramification point c *of* G *(that is, those points* c *of* C *for which the fibre above* c *is not reductive) the group scheme* $G_c := G \times_C \operatorname{Spec} A_c$ *is a parahoric group scheme over* A_c *as defined by Bruhat and Tits in [BT72, Definition 5.2.6], (or see also [HR08]).*

For more details on parahoric group schemes, see e.g. [HR08].

Remark 1.2. *One can define and show the extistence of the moduli space of global* G*-shtukas without assuming that* G *is parahoric, but we need in this thesis that the affine flag variety at each* c *is proper and this is the case if the group is parahoric.*

Proposition 1.3. *(i) The parahoric group scheme* G *has a faithful presentation*

$$\varrho : G \hookrightarrow \mathrm{GL}(V)$$

for a vector bundle V *on* C *such that the quotient* $\mathrm{GL}(V)/G$ *is quasi affine.*

(ii) If all fibres of G *over* C *are reductive then the quotient* $\mathrm{GL}(V)/G$ *as above is affine.*

Proof. (i) [Hei10, Example (1), page 504],

(ii) [AH14b, Proposition 2.2.(c)]. □

Definition 1.4. *For a sheaf of groups* H *(for the* $*$*-topology) on a scheme* X *we define a* (right) H-torsor *(or an* H-bundle*) on* X *to be a sheaf* $\mathcal{F}$ *for the* $* \in \{\text{étale}, \text{fppf}, \text{fpqc}\}$*-topology on* X *together with a (right) action of the sheaf* H *such that* $\mathcal{F}$ *is isomorphic to* H *on a* $*$*-covering of* X.
H *is viewed as an* H*-torsor by right multiplication.*

Definition 1.5. *(cf. [AH14a, Definition 2.1])*
The group of positive loops associated with G_c *is the infinite dimensional affine group scheme* L^+G_c *over* $\mathbb{F}_c$ *whose* R*-valued points for an* $\mathbb{F}_c$*-algebra* R *are*

$$L^+G_c(R) := G_c(R[\![z]\!]) := \operatorname{Hom}_{\mathbb{D}_c}(\mathbb{D}_{c,R}, G_c).$$

The group of loops associated with G_c *is the* fpqc*-sheaf of groups* LG_c *over* $\mathbb{F}_c$ *whose* R*-valued points for an* $\mathbb{F}_c$*-algebra* R *are*

$$LG_c(R) := G_c(R(\!(z)\!)),$$

with $R(\!(z)\!) := R[\![z]\!]\left[\frac{1}{z}\right]$.

Definition 1.6. *(i) The* affine flag variety $\mathcal{F}\ell_{G_c}$ *is the* fpqc-*sheaf associated with the presheaf*

$$R \mapsto LG_c(R)/L^+G_c(R) = G_c(R((z)))/G_c(R[\![z]\!])$$

on the category of $\mathbb{F}_c$-algebras.

(ii) When G_c is hyperspecial*, that is, when the special fibre of G_c is reductive, we call $\mathcal{F}\ell_{G_c}$* the affine Grassmannian*.*

In the case of L^+G_c-torsors (which is defined in Definition 1.5, see also [Fal03, Definition 1]) we have that the L^+G_c-torsors for the étale-, fppf- and fpqc-topology are equivalent:

Lemma 1.7. *(cf. [AH14a, Proposition 2.4])*

$$\check{H}^1(S_{\text{ét}}, L^+G_c) = \check{H}^1(S_{\text{fppf}}, L^+G_c) = \check{H}^1(S_{\text{fpqc}}, L^+G_c).$$

Definition 1.8. *Let S be a scheme and denote by $(\mathrm{Sch}/S)_{\text{fppf}}$ the big* fppf *site. An* algebraic stack over S *(also called* Artin-stack*) is a category*

$$\mathcal{X} \to (\mathrm{Sch}/S)_{\text{fppf}}$$

over $(\mathrm{Sch}/S)_{\text{fppf}}$ with the following properties:

(i) The category $\mathcal{X}$ is a stack in groupoids over $(\mathrm{Sch}/S)_{\text{fppf}}$.

(ii) The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.

(iii) There exists a scheme X in $(\mathrm{Sch}/S)_{\text{fppf}}$ and a 1-morphism $(\mathrm{Sch}/X)_{\text{fppf}} \to \mathcal{X}$ which is smooth and surjective.

Definition 1.9. *Let S be a scheme contained in $\mathrm{Sch}_{\text{fppf}}$ and let $\mathcal{X}$ be an algebraic stack over S. $\mathcal{X}$ is called a* Deligne-Mumford stack *if there exists a scheme X and a surjective étale morphism $(\mathrm{Sch}/X)_{\text{fppf}} \to \mathcal{X}$.*

Following [AH14b, Definition 2.3] we denote by $\mathcal{H}^1(C,G)$ the category fibered in groupoids over the category of $\mathbb{F}_q$-schemes with objects

$$\mathcal{H}^1(C,G)(S) := \{\text{G-torsors } \textit{over } C_S\} \tag{1.1}$$

and isomorphisms of G-torsors as its morphisms.
By [AH14b, Theorem 2.4] it is a *smooth Artin-stack locally of finite type over* $\mathbb{F}_q$:

Proposition 1.10. *(i) Let G be a parahoric group scheme over the curve C. Then the stack $\mathcal{H}^1(C,G)$ is a smooth Artin-stack locally of finite type over $\mathbb{F}_q$. It admits a covering by connected open substacks of finite type over $\mathbb{F}_q$.*

(ii) *Let X be a projective scheme over a field k. Let V be a vector bundle over X and let $\varrho : G \hookrightarrow \mathrm{GL}(V)$ be a closed subgroup with quasi-affine quotient $\mathrm{GL}(V)/G$. Then the natural morphism of k-stacks*

$$\varrho_* : \mathcal{H}^1(X, G) \longrightarrow \mathcal{H}^1(X, \mathrm{GL}(V))$$

is representable, quasi-affine and of finite presentation.

Proof. (i) [AH14b, Theorem 2.4],

(ii) [AH14b, Theorem 2.5]. □

By [Ser94, III, 2.3, Théorème 1′ and Remarque 1] the analogue of the Theorem of Steinberg holds. That is, by loc. cit. follows that any connected reductive algebraic group G_{Q_c} over $\mathbb{F}_c^{\mathrm{alg}}((z))$ is quasi-split, i.e. it contains a Borel subgroup defined over $\mathbb{F}_c^{\mathrm{alg}}((z))$, or eqivalently, the centralizer of a maximal $\mathbb{F}_c^{\mathrm{alg}}((z))$-split torus in G_{Q_c} is a torus. Hence, the following definition makes sense:

Definition 1.11. *(cf. [AH16a, Definition 1.1])*
Assume that G_c is parahoric. Then the generic fibre G_{Q_c} of G_c over $\operatorname{Spec}\mathbb{F}_c((z))$ *is connected reductive. Consider the base change G_L of G_{Q_c} to $L = \mathbb{F}_c^{\mathrm{alg}}((z))$. Let S be a maximal split torus in G_L and let T be its centralizer. Since $\mathbb{F}_c^{\mathrm{alg}}$ is algebraically closed, G_L is quasi-split, and so T is a maximal torus in G_L. Let $N = N(T)$ be the normalizer of T and let $\mathcal{T}^0$ be the identity component of the Néron model of T over $\mathcal{O}_L = \mathbb{F}_c^{\mathrm{alg}}[\![z]\!]$.*
The Iwahori-Weyl group *associated with S is the quotient group $\tilde{W} = N(L)/\mathcal{T}^0(\mathcal{O}_L)$. It is an extension of the finite Weyl group $W_0 = N(L)/T(L)$ by the coinvariants $X_*(T)_I$ under $I =$* $\mathrm{Gal}(L^{\mathrm{sep}}|L)$*:*

$$0 \to X_*(T)_I \to \tilde{W} \to W_0 \to 1.$$

By [HR08, Proposition 8], there is a bijection

$$L^+G_c(\mathbb{F}_c^{\mathrm{alg}})\backslash LG_{Q_c}(\mathbb{F}_c^{\mathrm{alg}})/L^+G_c(\mathbb{F}_c^{\mathrm{alg}}) \xrightarrow{\sim} \tilde{W}^{G_c}\backslash\tilde{W}/\tilde{W}^{G_c}$$

where $\tilde{W}^{G_c} := (N(L) \cap G_c(\mathcal{O}_L))/\mathcal{T}^0(\mathcal{O}_L)$. Here, $LG_{Q_c}(R) = LG_c(R) = G_c(R((z)))$ and $L^+G_c(R) = G_c(R[\![z]\!])$ are the loop group, resp. the group of positive loops of G_c; see [PR08, § 1.a], or [BD91, §4 .5], [NP01] and [Fal03] when G_c is constant.
Let $\omega \in \tilde{W}^{G_c}\backslash\tilde{W}/\tilde{W}^{G_c}$ and let $\mathbb{F}_\omega$ be the fixed field in $\mathbb{F}_c^{\mathrm{alg}}$ of

$$\left\{\gamma \in \mathrm{Gal}(\mathbb{F}_c^{\mathrm{alg}}|\mathbb{F}_c) \;:\; \gamma(\omega) = \omega\right\}.$$

There is a representative $g_\omega \in LG_c(\mathbb{F}_\omega)$ of ω; see [AH14a, Example 4.12]. The Schubert variety *$\mathcal{S}(\omega)$ associated with ω is the ind-scheme theoretic closure of the L^+G_c-orbit of g_ω in $\mathcal{F}\ell_{G_c} \hat{\times}_{\mathbb{F}_c}\mathbb{F}_\omega$. It is a reduced projective variety over $\mathbb{F}_\omega$. For further details see [PR08] and [Ric13b].*

1.3 Some facts about reductive groups

Definition 1.12. *A smooth connected algebraic group is* reductive *if its geometric unipotent radical is trivial.*

Definition 1.13. *A* parabolic subgroup P *of an algebraic group* G *is a subgroup such that* G/P *is projective.*

Proposition 1.14. *(cf. [Con11,* Chevalley*,Theorem 1.1.9])*
The parabolic subgroups of any connected linear algebraic group G *are connected.*

Definition 1.15. *A* Levi subgroup *of a parabolic* P *is a subgroup* M *such that* P *is the semidirect product* $P = N \rtimes M$*, where* N *is the unipotent radical (that is the maximal connected, unipotent, normal linear algebraic subgroup of the linear algebaric group* P*).*

By [Mil15, 22.148 and Definition 22.161] (resp. [Bor91, Corollary 14.19], [CGP10]), such Levi subgroups exist. They are reductive and smooth by [Con11, Definition 5.4.2].

Definition 1.16. *Two parabolic subgroups* P_1 *and* P_2 *are called* opposite *if* $P_1 \cap P_2$ *is a Levi subgroup of* P_1 *and* P_2*.*

2 Global G-shtukas and local G_c-shtukas

Throughout this chapter we use the following notations:

Definition 2.1. *For a closed point $c \in C$ we write*

$$\mathbb{D}_c := \operatorname{Spec} A_c \quad \text{and}$$
$$\hat{\mathbb{D}}_c := \operatorname{Spf} A_c.$$

Set $\deg c := [\mathbb{F}_c : \mathbb{F}_q]$ *and choose an inclusion $\mathbb{F}_c \subset A_c$ which induces the identity on $\mathbb{F}_c$ (the inclusion is uniquely defined by Hensels Lemma). We assume that there is a section $s : S \to C$ factoring through* $\operatorname{Spf} A_c$ *(thus, the image of a uniformizer of A_c in $\mathcal{O}_S$ is locally nilpotent). Set*

$$\mathfrak{a}_{c,l} := \left\langle a \otimes 1 - 1 \otimes s^*(a)^{q^l} \ : \ a \in \mathbb{F}_c \right\rangle.$$

There are isomorphisms

$$\hat{\mathbb{D}}_c \hat{\times}_{\mathbb{F}_q} S \cong \coprod_{l \in \mathbb{Z}/(\deg c)} V(\mathfrak{a}_{c,l}).$$

2.1 Global G-shtukas and their relation to local G_c-shtukas

Function field analogues of abelian varieties are global G-shtukas. They were investigated by Arasteh Rad and Hartl in [AH14a] and [AH14b]. They showed that there is a Deligne-Mumford stack $\nabla_n^{H,\hat{Z}}\mathcal{H}^1(C,G)$ which parametrizes global *bounded* G-shtukas with *H-level structure*. Moreover, this algebraic stack is separated and of finite type over C^n ([AH14b, Theorem 3.14]). The moduli space generalizes the space of F-sheaves $\mathrm{FSh}_{D,r}$ introduced by Drinfeld ([Dri87]) and Lafforgue ([Laf02]) for $G = \mathrm{GL}_2$ (rsp. $G = \mathrm{GL}_r$) and further studied and generalized by Varshavsky's moduli stacks FBun ([Var04]).

We recall some of the definitions and notations given in [AH14a] and [AH14b]. First we give the definition of a *global G-shtuka* (without level structure).

Definition 2.2. *A* global G-shtuka $\underline{\mathcal{G}}$ *over an* $\mathbb{F}_q$*-scheme* S *is a tuple*

$$\underline{\mathcal{G}} = (\mathcal{G}, s_1, \dots, s_n, \tau)$$

consisting of

(i) a G*-torsor* $\mathcal{G} \in \mathcal{H}^1(C,G)(S)$ *over* $C_S \coloneqq C \times_{\mathbb{F}_q} S$*,*

(ii) an n*-tuple of* characteristic sections $(s_1, \dots, s_n) \in C^n(S)$*, and*

(iii) a Frobenius connection τ *defined outside the graph* $\Gamma_{s_i} \subset C_S$ *of the morphisms* s_i*, that is an isomorphism*

$$\tau : \sigma^* \mathcal{G} \mid_{C_S \smallsetminus \cup_i \Gamma_{s_i}} \xrightarrow{\sim} \mathcal{G} \mid_{C_S \smallsetminus \cup_i \Gamma_{s_i}},$$

where $\sigma^* = (id_C \times \mathrm{Frob}_{q,S})^*$*.*

Definition 2.3. *We denote by* $\nabla_n \mathcal{H}^1(C,G)$ *the* moduli stack of global G-shtukas *(without level structure). It parametrizes global* G*-shtukas as defined in 2.2, where two global* G*-shtukas* $\underline{\mathcal{G}} = (\mathcal{G}, \tau)$ *and* $\underline{\mathcal{G}}' = (\mathcal{G}', \tau')$ *are said to be* isomorphic*, if there is an isomporphism*

$$f : \mathcal{G} \xrightarrow{\sim} \mathcal{G}'$$

of G*-torsors with*

$$f \circ \tau = \tau' \circ \sigma^* f.$$

There are relations to local G_c-shtukas, where G_c is a parahoric group scheme. Global G-shtukas induce local G_c-shtukas by “completing” at the closed point $c \in C$. We prefer to work with the associated loop groups torsors. It is very useful to study those local G_c-shtukas because the global G-shtukas are controlled by their local behaviour at their characteristic places. We explain this relation in more detail: For a finite field $\mathbb{F}_c$ containing $\mathbb{F}_q$ we denote by $\mathbb{F}_c[\![z]\!]$ the power series ring over $\mathbb{F}_c$ in the variable z. We consider a smooth affine group scheme over $\operatorname{Spec} \mathbb{F}_c[\![z]\!]$ which we denote by G_c with connected fibres (this holds in particular if G_c is parahoric), and we let $G_{Q_c} \coloneqq G_c \times_{\operatorname{Spec} \mathbb{F}_c[\![z]\!]} \operatorname{Spec} \mathbb{F}_c(\!(z)\!)$ the generic fibre of G_c over $\operatorname{Spec} \mathbb{F}_c(\!(z)\!)$.
We are especially interested in the local G_c-shtukas associated to a global G-shtuka. Therefore, in the situation where there is an isomorphism $\operatorname{Spec} \mathbb{F}_c[\![z]\!] \cong \operatorname{Spec} A_c$ for a place c of C and $G_c \coloneqq G \times_C \operatorname{Spec} A_c$, $G_{Q_c} \coloneqq G \times_C \operatorname{Spec} Q_c$.
We assume now that G_c is a parahoric group scheme over $\operatorname{Spec} \mathbb{F}_c[\![z]\!]$ in the sense of Bruhat and Tits ([BT72]) and G_{Q_c} is reductive over $\mathbb{F}_c(\!(z)\!)$. If one wants to consider the theory of local G-shtukas independent (or without the notion) of global G-shtukas this is possible by writing $\mathbb{D}_c \coloneqq \operatorname{Spec} k[\![z]\!]$ for a finite field k instead of $\operatorname{Spec} A_c$. We will define local G_c-shtukas in Definition 2.8.

2.1.1 Loop groups and local G_c-shtukas

Let LG_c resp. L^+G_c be the *group of loops* resp. of *positive loops* of G_c, that is the sheaves of groups defined on affine schemes $S = \operatorname{Spec} R$ by

$$L^+G_c(S) := G_c(R[\![z]\!]) \quad \textit{and}$$
$$LG_c(S) := G_c(R[\![z]\!]\left[\frac{1}{z}\right]) = G_c(R(\!(z)\!)).$$

In order to introduce local G_c-shtukas we investigate some definitions and constructions (already given e.g. in [AH14a] or [HV11]).
LG_c (resp. L^+G_c) is representable by an ind-scheme of ind-finite type (resp. affine group scheme) over $\mathbb{F}_c$; see also [PR08, § 1.a], or [BD91, § 4.5], [NP01], [Fal03].
The stack

$$\mathcal{H}^1(\operatorname{Spec}\mathbb{F}_c, L^+G_c) := [\operatorname{Spec}\mathbb{F}_c/L^+G_c],$$

resp.

$$\mathcal{H}^1(\operatorname{Spec}\mathbb{F}_c, LG_c) := [\operatorname{Spec}\mathbb{F}_c/LG_c],$$

classifying L^+G_c-torsors, resp. LG_c-torsors, is a stack fibered in groupoids over the category of $\mathbb{F}_c$-schemes S.

Induced by the natural inclusion $L^+G_c \subset LG_c$ there is a morphism

$$L : \mathcal{H}^1(\operatorname{Spec}\mathbb{F}_c, L^+G_c) \longrightarrow \mathcal{H}^1(\operatorname{Spec}\mathbb{F}_c, LG_c)$$
$$\mathcal{L}_+ \mapsto \mathcal{L}.$$

Definition 2.4. *(cf. [AH14a, Definition 2.2])*
Let $\hat{G}_c$ be the formal group scheme over $\hat{\mathbb{D}}_c := \operatorname{Spf}\mathbb{F}_c[\![z]\!]$ obtained by the formal completion of G_c along $V(z)$. A formal $\hat{G}_c$-torsor *over an $\mathbb{F}_c$-scheme S is a z-adic formal scheme $\hat{\mathcal{P}}$ over $\hat{\mathbb{D}}_{c,S} := \hat{\mathbb{D}}_c \hat{\times}_{\mathbb{F}_c} S$ together with an action $\hat{G}_c \hat{\times}_{\hat{\mathbb{D}}_c} \hat{\mathcal{P}} \to \hat{\mathcal{P}}$ of $\hat{G}_c$ on $\hat{\mathcal{P}}$ such that there is a covering $\hat{\mathbb{D}}_{c,S'} \to \hat{\mathbb{D}}_{c,S}$ where $S' \to S$ is an* fpqc*-covering and such that there is a $\hat{G}_c$-equivariant isomorphism*

$$\hat{\mathcal{P}} \hat{\times}_{\hat{\mathbb{D}}_{c,S}} \hat{\mathbb{D}}_{c,S'} \xrightarrow{\sim} \hat{G}_c \hat{\times}_{\hat{\mathbb{D}}_c} \hat{\mathbb{D}}_{c,S'}.$$

Here $\hat{G}_c$ acts on itself by right multiplication. We denote by $\mathcal{H}^1(\hat{\mathbb{D}}_c, \hat{G}_c)$ the category fibered in groupoids that assigns to an $\mathbb{F}_c$-scheme S the groupoid consisting of all formal $\hat{G}_c$-torsors over $\hat{\mathbb{D}}_{c,S}$.

Proposition 2.5. *(cf. [AH14a, Proposition 2.4] or [HV11, Proposition 2.2(a)] for a split reductive group G_c)*
There is an isomorphism

$$\mathcal{H}^1(\hat{\mathbb{D}}_c, \hat{G}_c) \xrightarrow{\sim} \mathcal{H}^1(\operatorname{Spec}\mathbb{F}_c, L^+G_c)$$

of groupoids. If moreover G_c is smooth over $\mathbb{D}_c$, then each L^+G_c-torsor for the fpqc*-topology on S is étale locally trivial on S.*

Definition 2.6. *Assume that there are two morphisms $f,g : X \to Y$ of schemes or stacks. We denote by* $\mathrm{equi}(f,g : X \rightrightarrows Y)$ *the pull back of the diagonal under the morphsim $(f,g) : X \to Y \times_{\mathbb{Z}} Y$, that is, we let*

$$\mathrm{equi}(f,g : X \rightrightarrows Y) := X \times_{(f,g),Y\times Y,\Delta} Y$$

where $\Delta := \Delta_{Y/\mathbb{Z}} : Y \to Y \times_{\mathbb{Z}} Y$ is the diagonal morphism.

Definition 2.7. *Let $\mathcal{X}$ be the fibre product*

$$\mathcal{H}^1(\mathrm{Spec}\,\mathbb{F}_c, L^+G_c) \times_{\mathcal{H}^1(\mathrm{Spec}\,\mathbb{F}_c, LG_c)} \mathcal{H}^1(\mathrm{Spec}\,\mathbb{F}_c, L^+G_c)$$

of groupoids.
Denote by pr_i the projection onto the i-th factor. We define the groupoid of local G_c-shtukas $\mathrm{Sht}_{G_c}^{\mathbb{D}_c}$ *to be*

$$\mathrm{Sht}_{G_c}^{\mathbb{D}_c} := equi(\hat{\sigma} \circ \mathrm{pr}_1, pr_2 : \mathcal{X} \rightrightarrows \mathcal{H}^1(\mathrm{Spec}\,\mathbb{F}_c, L^+G_c) \hat{\times}_{\mathrm{Spec}\,\mathbb{F}_c} \mathrm{Spf}\,\mathbb{F}_c[\![\zeta]\!]),$$

where $\hat{\sigma} := \hat{\sigma}_{\mathcal{H}^1(\mathrm{Spec}\,\mathbb{F}_c, L^+G_c)}$ is the absolute $\mathbb{F}_c$-Frobenius of $\mathcal{H}^1(\mathrm{Spec}\,\mathbb{F}_c, L^+G_c)$.
$\mathrm{Sht}_{G_c}^{\mathbb{D}_c}$ *is the category fibered in groupoids over $\mathcal{N}\mathrm{ilp}_{\mathbb{F}_c[\![\zeta]\!]}$ (the category of $\mathbb{F}_c[\![\zeta]\!]$-schemes on which ζ is locally nilpotent). We call the objects in $\mathrm{Sht}_{G_c}^{\mathbb{D}_c}(S)$ the* local G_c-shtukas over *S.*

More precisely we define a local G_c-shtuka as in [AH14a, Definition 2.6]:

Definition 2.8. A local G_c-shtuka *over a scheme $S \in \mathcal{N}\mathrm{ilp}_{A_c}$ is a pair $\underline{\mathcal{L}} = (\mathcal{L}_+, \hat{\tau})$ consisting of an L^+G_c-torsor $\mathcal{L}_+$ on S and an isomorphism of LG_c-torsors*

$$\hat{\tau} : \hat{\sigma}^*\mathcal{L} \xrightarrow{\sim} \mathcal{L},$$

where $\mathcal{L}$ denotes the LG_c-torsor associated with $\mathcal{L}_+$ and $\hat{\sigma}^*\mathcal{L}$ the pullback of $\mathcal{L}$ under the absolute $\mathbb{F}_c$-Frobenius endomorphism $\hat{\sigma} := \mathrm{Frob}_{(\#\mathbb{F}_c),S} : S \to S$.

Definition 2.9. *A local G_c-shtuka $(\mathcal{L}_+, \hat{\tau})$ is called* étale *if $\hat{\tau}$ comes from an isomorphism of L^+G_c-torsors $\hat{\sigma}^*\mathcal{L}_+ \xrightarrow{\sim} \mathcal{L}_+$.*
We denote by $\text{Ét}\,\mathrm{Sht}(S)$ *the category of étale local G_c-shtukas over S.*

The second part of the above Proposition 2.5 and [AH14a, Lemma 2.8] imply the following:

Remark 2.10. *If G_c is smooth over $\mathbb{D}_c$ with connected special fibre, we can trivialize each étale local G_c-shtuka to $((L^+G_c)_k, 1 \cdot \hat{\sigma}^*)$ over a separably closed field k.*

2.1.2 Quasi-isogenies

Now we consider the morphisms in the category of global G-shtukas (resp. local G_c-shtukas), the *quasi-isogenies* and recall some important properties, especially the rigidity of quasi-isogenies.

Definition 2.11. *(i) A* quasi-isogeny

$$f : \underline{\mathcal{L}} \to \underline{\mathcal{L}}^{'}$$

between two local G_c-shtukas $\underline{\mathcal{L}} := (\mathcal{L}_+, \hat{\tau})$ and $\underline{\mathcal{L}}^{'} := (\mathcal{L}'_+, \hat{\tau}')$ over a scheme $S \in \mathcal{N}\mathrm{ilp}_{\mathbb{F}_c[\![\zeta]\!]}$ is an isomorphism

$$f : \mathcal{L} \xrightarrow{\sim} \mathcal{L}^{'}$$

of the associated LG_c-torsors such that $f \cdot \hat{\tau} = \hat{\tau}' \cdot \hat{\sigma}^* f$, that is that the diagram

$$\begin{array}{ccc} \hat{\sigma}^*\mathcal{L} & \xrightarrow{\hat{\tau}} & \mathcal{L} \\ {\scriptstyle \hat{\sigma}^* f}\downarrow & & \downarrow{\scriptstyle f} \\ \hat{\sigma}^*\mathcal{L}' & \xrightarrow[\hat{\tau}^{'}]{} & \mathcal{L}' \end{array}$$

is commutative.

(ii) We write $\mathrm{QIsog}_S(\underline{\mathcal{L}}, \underline{\mathcal{L}}^{'})$ for the set of quasi-isogenies between $\underline{\mathcal{L}}$, and $\underline{\mathcal{L}}^{'}$ and let $\mathrm{QIsog}_S(\underline{\mathcal{L}}) := \mathrm{QIsog}_S(\underline{\mathcal{L}}, \underline{\mathcal{L}})$ be the set of quasi-selfisogenies, which is in fact a group, called the quasi-isogeny group *of $\underline{\mathcal{L}}$.*

One important property of quasi-isogenies is their *rigidity*, a lifting property:

Proposition 2.12. *(*Rigidity of quasi-isogenies for local G_c-shtukas*) (cf. [AH14a, Proposition 2.11])*
Let S be a scheme in $\mathcal{N}\mathrm{ilp}_{\mathbb{F}_c[\![\zeta]\!]}$ and let $i : \overline{S} \to S$ be a closed immersion defined by a sheaf of ideals $\mathcal{I}$ which is locally nilpotent. For any two local G_c-shtukas $\underline{\mathcal{L}}$ and $\underline{\mathcal{L}}^{'}$ over S there is a bijection of sets:

$$\mathrm{QIsog}_S(\underline{\mathcal{L}}, \underline{\mathcal{L}}^{'}) \longrightarrow \mathrm{QIsog}_{\overline{S}}(i^*\underline{\mathcal{L}}, i^*\underline{\mathcal{L}}^{'}),$$
$$f \mapsto i^* f$$

(with $\mathcal{O}_S \xrightarrow{i^*} \mathcal{O}_{\overline{S}}$).

There is also a definition of quasi-isogenies for global G-shtukas:

Definition 2.13. *(i) A* quasi-isogeny

$$f : \underline{\mathcal{G}} \to \underline{\mathcal{G}}'$$

between two global G-shtukas $\underline{\mathcal{G}} := (\mathcal{G}, \tau)$ and $\underline{\mathcal{G}}' := (\mathcal{G}', \tau')$ over a scheme S with the same characteristics $s_i : S \to C$, for $i = 1, \dots, n$, is an isomorphism

$$f : \mathcal{G}|_{C_S \smallsetminus D_S} \xrightarrow{\sim} \mathcal{G}'|_{C_S \smallsetminus D_S},$$

such that $\tau' \circ \sigma^* f = f \circ \tau$, where D is an effective divisor on C.

(ii) Likewise as for local G_c-shtukas we denote by $\mathrm{QIsog}_S(\underline{\mathcal{G}}, \underline{\mathcal{G}}')$ the set of quasi-isogenies between $\underline{\mathcal{G}}$ and $\underline{\mathcal{G}}'$ and by $\mathrm{QIsog}_S(\underline{\mathcal{G}})$ the group of quasi-isogenies *of $\underline{\mathcal{G}}$ to itself.*

Proposition 2.14. *(*Rigidity of quasi-isogenies for global G-shtukas*) (cf. [AH14a, Proposition 5.9])*
Let $\underline{\mathcal{G}} := (\mathcal{G}, \tau)$ and $\underline{\mathcal{G}}' := (\mathcal{G}', \tau')$ be two global G-shtukas over a scheme S with $S \in \mathcal{N}\mathrm{ilp}_{A_{\underline{c}}}$ and let $i : \bar{S} \to S$ be a closed immersion defined by a sheaf of ideals $\mathcal{I}$ which is locally nilpotent.

(i) There is a bijection of sets

$$\mathrm{QIsog}_S(\underline{\mathcal{G}}, \underline{\mathcal{G}}') \longrightarrow \mathrm{QIsog}_{\bar{S}}(i^*\underline{\mathcal{G}}, i^*\underline{\mathcal{G}}'),$$
$$f \mapsto i^* f$$

(ii) f is an isomorphism at a place $c \notin \underline{c}$ if and only if $i^* f$ is an isomorphism at c.

For more details on local G_c-shtukas and their relation to global G-shtukas, see [AH14a, § 2] .

2.2 Global-local functor

As a function field analogue to the functor that assigns a p-divisible group to an abelian variety Arasteh Rad and Hartl established the so-called *global-local functor* which associates a tupel of local G_c-shtukas to a global G-shtuka. In the following, we want to recall this construction which we will use in the sequel for many times. In order to formulate the functor we need some further notations and preperations: For an arbitrary closed point $c \in C$ set $C' := C \backslash \{c\}$, $C'_S := C' \times_{\mathrm{Spec}\,\mathbb{F}_q} S$ and let $\mathcal{H}^1_e(C', G)$ be the category fibered in groupoids over $\mathbb{F}_q$-schemes (see [AH14a, § 5.1]) such that the objects are given by:

$$\mathcal{H}^1_e(C', G)(S) = \left\{ \text{G-torsors } \textit{over } C'_S \textit{ that can be extended to a } \text{G-torsor } \textit{over } C_S \right\}.$$

There is a restriction morphism

$$\begin{aligned} \mathrm{res} : \mathcal{H}^1(C,G) &\longrightarrow \mathcal{H}^1_e(C',G) \\ \mathcal{G} &\mapsto \mathrm{res}(\mathcal{G}) := \mathcal{G} \times_{C_S} C'_S. \end{aligned}$$

The group scheme $\tilde{G}_c$ defined by the Weil restrictions

$$\tilde{G}_c := \mathrm{Res}_{\mathbb{F}_c|\mathbb{F}_q} G_c$$

is a flat affine group scheme of finite type over $\operatorname{Spec} \mathbb{F}_q[\![z]\!]$. Consider the c-adic completion

$$\hat{\tilde{G}}_c := \tilde{G}_c \hat{\times}_{\operatorname{Spec} \mathbb{F}_q[\![z]\!]} \operatorname{Spf} \mathbb{F}_q[\![z]\!] = \mathrm{Res}_{\mathbb{F}_c|\mathbb{F}_q} \hat{G}_c,$$

so we have loop groups:

$$\begin{aligned} L^+\hat{\tilde{G}}_c(R) &= \tilde{G}_c(R[\![z]\!]) \quad \textit{and} \\ L\hat{\tilde{G}}_c(R) &= \tilde{G}_c(R(\!(z)\!)) \end{aligned}$$

for an $\mathbb{F}_q$-algebra R.

If $\mathcal{G} \in \mathcal{H}^1(C,G)(S)$, then, its completion $\hat{\mathcal{G}}_c := \mathcal{G} \hat{\times}_{C_S} (\operatorname{Spf} A_c \hat{\times}_{\mathbb{F}_q} S)$ is a formal $\hat{G}_c$-torsor over $\operatorname{Spf} A_c \hat{\times}_{\mathbb{F}_q} S$. The Weil restriction $\mathrm{Res}_{\mathbb{F}_c|\mathbb{F}_q} \hat{\mathcal{G}}_c$ is a formal $\hat{\tilde{G}}_c$-torsor over $\operatorname{Spf} \mathbb{F}_q[\![z]\!] \hat{\times}_{\mathbb{F}_q} S$.

Definition 2.15. *We define $L^+_c(\mathcal{G})$ to be the $L^+\tilde{G}_c$-torsor over S corresponding by Proposition 2.5 to* $\mathrm{Res}_{\mathbb{F}_c|\mathbb{F}_q} \hat{\mathcal{G}}_c$. *So there are morphisms*

$$\begin{aligned} L^+_c : \mathcal{H}^1(C,G) &\longrightarrow \mathcal{H}^1(\operatorname{Spec} \mathbb{F}_q, L^+\tilde{G}_c) \\ \mathcal{G} &\mapsto L^+_c(\mathcal{G}) \end{aligned}$$

and

$$\begin{aligned} L_c : \mathcal{H}^1_e(C',G)(S) &\longrightarrow \mathcal{H}^1(\operatorname{Spec} \mathbb{F}_q, L\tilde{G}_c)(S) \\ \mathrm{res}(\mathcal{G}) &\mapsto L_c(\mathrm{res}(\mathcal{G})) := L(L^+_c(\mathcal{G})). \end{aligned}$$

For more details see [AH14a, §5.1].

Remark 2.16. *We have a cartesian diagram (of groupoids) (see [AH14a, Lemma 5.1]):*

$$\begin{array}{ccc} \mathcal{H}^1(C,G) & \xrightarrow{\mathrm{res}} & \mathcal{H}^1_e(C',G) \\ \downarrow & & \downarrow \\ \mathcal{H}^1(\operatorname{Spec} \mathbb{F}_q, L^+\tilde{G}_c) & \xrightarrow{L} & \mathcal{H}^1(\operatorname{Spec} \mathbb{F}_q, L\tilde{G}_c) \end{array} \qquad (2.1)$$

By [AH14a, Lemma 5.1], we know that the loop group version of the Beauville-Laszlo gluing lemma *holds.*

Definition 2.17. *(cf. [AH14a, Definition 5.4])*
Let $\underline{c} := (c_i)_{i=1,\dots,n}$, $c_i \neq c_j$ *for* $i \neq j$, *be a fixed tuple of places on* C. *We write* $A_{\underline{c}}$ *for the completion of the local ring* $\mathcal{O}_{C^n,\underline{c}}$ *of* C^n *at the closed point* $\underline{c}$ *and* $\mathbb{F}_{\underline{c}}$ *for the residue field of* $\underline{c}$. *So we can consider* $\mathbb{F}_{\underline{c}}$ *as the compositum of the fields* $\mathbb{F}_{c_i}$ $(i = 1,\dots,n)$ *inside* $\mathbb{F}_q^{\mathrm{alg}}$ *and have* $A_{\underline{c}} \cong \mathbb{F}_{\underline{c}}[\![\zeta_1,\dots,\zeta_n]\!]$, *where* ζ_i *denotes the image (under the characteristic morphism) of a uniformizing parameter* z_i *of* C *at* c_i.

(i) We define the stack

$$\nabla_n\mathcal{H}^1(C,G)^{\underline{c}} := \nabla_n\mathcal{H}^1(C,G)\hat{\times}_{C^n}\mathrm{Spf}(A_{\underline{c}})$$

to be the formal completion of the stack $\nabla_n\mathcal{H}^1(C,G)$ *at* $\underline{c} \in C^n$.
By [AH14b, Theorem 3.14] it is an ind-algebraic stack, ind-separated and locally of ind-finite type over $\mathrm{Spf}\,A_{\underline{c}}$.

(ii) Let $(\mathcal{G}, s_1,\dots,s_n,\tau) \in \nabla_n\mathcal{H}^1(C,G)^{\underline{c}}(S)$ *(i.e.* $s_i : S \to C$ *factors through* $\mathrm{Spf}\,A_{c_i}$*). Set* $G_{c_i} := G \times_C \mathrm{Spec}\,A_{c_i}$ *and* $\hat{G}_{c_i} := G \times_C \mathrm{Spf}\,A_{c_i}$. *Consider the decomposition (induced by Definition 2.1)*

$$\mathcal{G}\hat{\times}_{C_S}(\mathrm{Spf}\,A_{c_i}\hat{\times}_{\mathbb{F}_q}S) \cong \coprod_{l\in\mathbb{Z}/(\deg c_i)} \mathcal{G}\hat{\times}_{C_S}V(\mathfrak{a}_{c_i,l}),$$

where $\mathcal{G}\hat{\times}_{C_S}V(\mathfrak{a}_{c_i,l}) \in \mathcal{H}^1(\hat{\mathbb{D}}_{c_i},\hat{G}_{c_i})$.

(iii) We set $\hat{\sigma} := \sigma^{\deg c_i}$ *and define the local* G_{c_i}*-shtuka as the pair*

$$\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}) := (\mathcal{L}_i^+, \tau^{\deg c_i}),$$

where $\mathcal{L}_i^+$ *is the* $L^+G_{c_i}$*-torsor associated to* $\mathcal{G}\hat{\times}_{C_S}V(\mathfrak{a}_{c_i,0})$ *by Proposition 2.5 with*

$$\tau^{\deg c_i} : \hat{\sigma}^*\mathcal{L}_i := (\sigma^{\deg c_i})^*\mathcal{L}_i \xrightarrow{\sim} \mathcal{L}_i,$$

the $\mathbb{F}_{c_i}$*-Frobenius on the loop group torsor associated to* $\mathcal{G}\hat{\times}_{C_S}V(\mathfrak{a}_{c_i,0})$.

Now we can give the construction of the *global-local functor*:

Definition 2.18. *Fix a closed point* $\underline{c} = (c_1,\dots,c_n) \in C^n$ *with* $c_i \neq c_j$ *for* $i \neq j$. *We only consider global* G*-shtukas* $\underline{\mathcal{G}}$ *whose characteristic sections factor through* $\mathrm{Spf}\,A_{c_i}$. *Then there is a* global-local-functor $\underline{\hat{\Gamma}}$ *which associates with* $\underline{\mathcal{G}}$ *an* n*-tuple of local* G_{c_i}*-shtukas* $\underline{\hat{\Gamma}}(\underline{\mathcal{G}}) = (\hat{\Gamma}_i(\underline{\mathcal{G}}))_i$ *defined by:*

$$\begin{aligned}\hat{\Gamma}_i := \hat{\Gamma}_{c_i} : \nabla_n\mathcal{H}^1(C,G)^{\underline{c}}(S) &\longrightarrow \mathrm{Sht}_{G_{c_i}}^{\mathrm{Spec}\,A_{c_i}}(S)\\ (\mathcal{G},\tau) &\mapsto \hat{\Gamma}_{c_i}(\underline{\mathcal{G}}),\\ \underline{\hat{\Gamma}} := \prod_{i=1,\dots,n}\hat{\Gamma}_{c_i} : \nabla_n\mathcal{H}^1(C,G)^{\underline{c}}(S) &\longrightarrow \prod_{i=1,\dots,n}\mathrm{Sht}_{G_{c_i}}^{\mathrm{Spec}\,A_{c_i}}(S)\\ (\mathcal{G},\tau) &\mapsto (\hat{\Gamma}_{c_i}(\underline{\mathcal{G}})_i.\end{aligned}$$

Remark 2.19. *Note that by [AH14a, Lemma 5.1] both* $\hat{\Gamma}_i$ *and* $\hat{\underline{\Gamma}}$ *transform quasi-isogenies into quasi-isogenies.*

Definition 2.20. *Recall the* $L^+\tilde{G}_c$*-torsor* $L^+_c(\mathcal{G})$ *from Definition 2.15. For a global* G*-shtuka* $\underline{\mathcal{G}} = (\mathcal{G}, \tau) \in \nabla_n\mathcal{H}^1(C,G)^{\underline{c}}(S)$ *over* S *and a closed point* $c \in C$ *we define the* local $\tilde{G}_c := \operatorname{Res}_{\mathbb{F}_c|\mathbb{F}_q} G_c$-shtuka associated with $\underline{\mathcal{G}}$ at c *in the following way:*

(i) Let $c = c_i \in \underline{c} = (c_1, \ldots, c_n)$*. In the sense of [AH14a, Remark 5.5] we call the local* $\tilde{G}_{c_i}$*-shtuka* $(L^+_{c_i}(\mathcal{G}), \tau\hat{\times}\mathrm{id})$ *over* S *the* local $\tilde{G}_{c_i}$-shtuka associated with $\underline{\mathcal{G}}$ at c_i*. Note that this is well defined as*

$$\operatorname{Res}_{\mathbb{F}_{c_i}|\mathbb{F}_q} \hat{\mathcal{G}}_{c_i} := \operatorname{Res}_{\mathbb{F}_{c_i}|\mathbb{F}_q}(\mathcal{G}\hat{\times}_{C_S}(\operatorname{Spf} A_{c_i}\hat{\times}_{\mathbb{F}_q}S))$$

corresponds to an $L^+\tilde{G}_{c_i}$*-torsor denoted by* $L^+_{c_i}(\mathcal{G})$ *and*

$$\tau\hat{\times}\mathrm{id} : \sigma^*(\mathcal{G}\hat{\times}_{C_S}V(\mathfrak{a}_{c_i,l-1}) = (\sigma^*\mathcal{G})\hat{\times}_{C_S}V(\mathfrak{a}_{c_i,l}) \xrightarrow{\sim} \mathcal{G}\hat{\times}_{C_S}V(\mathfrak{a}_{c_i,l})$$

is an isomorphism for $l \neq 0$*.*

(ii) Let $c \notin \underline{c}$*. As* τ *is an isomorphism at* c*, we call the associated* $\tilde{G}_c$*-shtuka* $L^+_c(\underline{\mathcal{G}})$ *the* étale local $\tilde{G}_c$-shtuka associated with $\underline{\mathcal{G}}$ at the place $c \notin \underline{c}$ *(for more details see Definition 2.47 or [AH14a, Remark 5.6]).*

The next proposition is very important for some of our constructions we do in the following chapters. It tells how one can construct a “new” global G-shtuka from quasi-isogenies between local G_c-shtukas, resp. that one can pull back global G-shtukas along quasi-isogenies between their associated local G_c-shtukas.

Proposition 2.21. *(cf. [AH14a, Proposition 5.7])*
Let $\underline{\mathcal{G}} \in \nabla_n\mathcal{H}^1(C,G)^{\underline{c}}(S)$ *be a global* G*-shtuka over* S *and let* $c \in C$ *be a closed point. Consider the local* $\tilde{G}_c$*-shtuka* $L^+_c(\underline{\mathcal{G}})$ *associated with* $\underline{\mathcal{G}}$ *at* c*. Let*

$$f : \underline{\tilde{\mathcal{L}}}' \longrightarrow L^+_c(\underline{\mathcal{G}})$$

be a quasi-isogeny of local $\tilde{G}_c$*-shtukas over* S*. If* $c \in \underline{c}$*, we assume that the Frobenius of* $\underline{\tilde{\mathcal{L}}}'$ *is an isomorphism outside* $V(\mathfrak{a}_{c,0})$*. If* $c \notin \underline{c}$*, we assume that* $\underline{\tilde{\mathcal{L}}}'$ *is étale. Then there exists a unique global* G*-shtuka* $\underline{\mathcal{G}}' \in \nabla_n\mathcal{H}^1(C,G)^{\underline{c}}(S)$ *and a unique quasi-isogeny*

$$g : \underline{\mathcal{G}}' \longrightarrow \underline{\mathcal{G}}$$

which is an isomorphism outside c *such that the local* $\tilde{G}_c$*-shtuka associated with* $\underline{\mathcal{G}}'$ *is* $\underline{\tilde{\mathcal{L}}}'$ *and the quasi-isogeny of local* $\tilde{G}_c$*-shtukas induced by* g *is* f*.*

Definition 2.22. *We denote the constructed global* G*-shtuka* $\underline{\mathcal{G}}'$ *from the above Proposition 2.21 by* $f^*\underline{\mathcal{G}}$*.*

2.3 Serre-Tate for global G-shtukas

By [AH14a, Theorem 5.10], we know that the *Serre-Tate theorem* holds, i.e. the infinitesimal deformation space of $\underline{\mathcal{G}}$ is isomorphic to the product of the infinitesimal deformation spaces of the $\hat{\Gamma}_i(\underline{\mathcal{G}})$.

Definition 2.23. *Let S be a scheme in $\mathcal{N}\mathrm{ilp}_{A_{\underline{c}}}$ and $j : \overline{S} \to S$ a closed subscheme defined by a locally nilpotent sheaf of ideals.*

(i) We denote by $\mathrm{Defo}_S(\overline{\underline{\mathcal{G}}})$ the category of lifts of a global G-shtuka $\overline{\underline{\mathcal{G}}} \in \nabla_n\mathcal{H}^1(C,G)^{\underline{c}}(\overline{S})$ to S, that is of pairs $(\underline{\mathcal{G}}, \alpha : j^*\underline{\mathcal{G}} \xrightarrow{\sim} \overline{\underline{\mathcal{G}}})$ with $\underline{\mathcal{G}} \in \nabla_n\mathcal{H}^1(C,G)^{\underline{c}}(S)$ and α an isomorphism of global G-shtukas over $\overline{S}$ as objects and isomorphisms of global G-shtukas that are compatible with the α's as morphisms.

(ii) We denote by $\mathrm{Defo}_S(\overline{\underline{\mathcal{L}}})$ the category of lifts of local G_c-shtuka $\overline{\underline{\mathcal{L}}} \in \mathrm{Sht}_{G_c}^{\mathbb{D}_c}(S)$ to S.

Proposition 2.24. *(*Serre-Tate*) (cf. [AH14a, Theorem 5.10])*
There is an equivalence of categories

$$\mathrm{Defo}_S(\overline{\underline{\mathcal{G}}}) \longrightarrow \prod_i \mathrm{Defo}_S(\overline{\underline{\mathcal{L}}}_i)$$

$$(\underline{\mathcal{G}}, \alpha) \mapsto (\underline{\hat{\Gamma}}(\underline{\mathcal{G}}), \underline{\hat{\Gamma}}(\alpha)),$$

where $\overline{\underline{\mathcal{G}}} \in \nabla_n\mathcal{H}^1(C,G)^{\underline{c}}(\overline{S})$ and $(\overline{\underline{\mathcal{L}}}_i)_i \coloneqq \underline{\hat{\Gamma}}(\overline{\underline{\mathcal{G}}})$.

2.4 Rapoport-Zink-spaces

In this section we consider moduli spaces of (bounded) local G_c-shtuks. They are affine Deligne-Lusztig varieties and function field analogues of Rapoport-Zink spaces of p-divisible groups.

2.4.1 Rapoport-Zink spaces for local G_c-shtukas

We recall the construction of *Rapoport-Zink spaces* for local G_c-shtukas where G_c is a smooth affine group scheme over $\mathbb{D}_c$. They were first constructed by Hartl and Viehmann in [HV11] for the case where G_c is a constant split reductive group, that is $G_c = G_0 \times \mathbb{D}_c$ for a split reductive group G_0 over $\mathbb{F}_c$, and then generalized in [AH14a] for parahoric groups. The Rapoport-Zink spaces can be used to (partly) uniformize the moduli stacks $\nabla_n\mathcal{H}^1(C,G)$ for global G-shtukas.
We will recall certain results and definitions with respect to those moduli spaces.
The right analogue of p-divisible groups are *bounded* local G_c-shtukas. Arasteh Rad

and Hartl showed in [AH14a] that their Rapoport-Zink spaces are formal schemes locally formally of finite type (over $\operatorname{Spf}\mathbb{F}_c[\![\zeta]\!]$). To summarize the results in a special case we can say: Fix a local G_c-shtuka $\underline{\mathbb{L}}_0 = ((L^+G_c)_k, \hat{\tau} = b\hat{\sigma}^*)$ over a field k that contains $\mathbb{F}_c$ with $b \in LG_c(k)$ and $\hat{\sigma} := \sigma^{\deg c}$. The *Rapoport-Zink space* classifies *bounded* local G_c-shtukas together with a quasi-isogeny to $\underline{\mathbb{L}}_0$. It is a formal scheme locally formally of finite type whose underlying reduced subscheme is an *affine Deligne-Lusztig variety.* We present this result more precisely:
Let again be G_c a smooth affine group scheme over $\mathbb{D}_c$. Following the notation of [AH14a, §4.1], we set

$$\hat{T} := \overline{T}\hat{\times}_{\operatorname{Spec}\mathbb{F}_c}\operatorname{Spf}\mathbb{F}_c[\![\zeta]\!], \tag{2.2}$$

where $\overline{T}$ is a scheme over $\mathbb{F}_c$.
Let $S \in \mathcal{N}\mathrm{ilp}_{\mathbb{F}_c[\![\zeta]\!]}$ and $\overline{S} := V_S(\zeta) \subseteq S$. We denote by $\mathcal{N}\mathrm{ilp}_{\hat{T}}$ the category of $\hat{T}$-schemes on which ζ is locally nilpotent.

Proposition 2.25. *(cf. [AH14a, Theorem 4.4])*
We fix a local G_c-shtuka $\underline{\mathcal{L}}_0$ over an $\mathbb{F}_c$-scheme $\overline{T}$.
Consider the functor

$$\begin{aligned} \mathcal{RZ}_{\underline{\mathcal{L}}_0} : (\mathcal{N}\mathrm{ilp}_{\hat{T}})^\circ &\longrightarrow \mathit{Sets} \\ S &\mapsto \left\{ \text{ Isomorphism classes of pairs } (\underline{\mathcal{L}}, \overline{\delta}) \right\} \end{aligned}$$

with $\underline{\mathcal{L}}$ a local G_c-shtuka over S and

$$\overline{\delta} : \underline{\mathcal{L}}_{\overline{S}} \to \underline{\mathcal{L}}_{0,\overline{S}}$$

a quasi-isogeny over $\overline{S} := V(\zeta)$. Here, two tuples $\underline{\mathcal{L}} = (\mathcal{L}, \overline{\delta})$ and $\underline{\mathcal{L}}' = (\mathcal{L}, \overline{\delta}')$ are said to be isomorphic *if $\overline{\delta}^{-1} \circ \overline{\delta}'$ lifts via rigidity to an isomorphism $\underline{\mathcal{L}}' \xrightarrow{\sim} \underline{\mathcal{L}}$. Then the functor is representable by an ind-scheme, ind-quasi-projective, ind-separated and of ind-finite type over $\hat{T}$.*

If $T = \operatorname{Spec} k$ for a field k and if one assume that the Hodge polygon of the G_c-shtukas satisfies a boundedness condition then $\mathcal{RZ}_{\underline{\mathcal{L}}_0}$ is represented by a formal scheme locally formally of finite type over $\operatorname{Spf} k[\![\zeta]\!]$, that is, it is locally noetherian and adic and its reduced subscheme is locally of finite type over k.

Proposition 2.26. *(cf. [AH14a, Theorem 4.4])*
Using the notations from Proposition 2.25 above and assuming that the fibres of G_c are connected over $\mathbb{D}_c$ we get that $\mathcal{RZ}_{\underline{\mathcal{L}}_0}$ is ind-projective if and only if G_c is parahoric.

Proposition 2.27. *(cf. [AH14a, Theorem 4.4])*
Using the notations in Proposition 2.25 and assuming that there is a trivialization

$$\underline{\mathcal{L}}_0 \xrightarrow{\sim} ((L^+G_c)_{\overline{T}}, b\hat{\sigma}^*)$$

over $\overline{T}$ *with* $b \in LG_c(\overline{T})$ *we get that* $\underline{\mathcal{M}}_{\underline{\mathcal{L}}_0}$ *is represented by the ind-scheme*

$$\mathcal{F}\ell_{G_c,\hat{T}} \coloneqq \mathcal{F}\ell_{G_c} \hat{\times}_{\mathbb{F}_c} \hat{T},$$

where $\mathcal{F}\ell_{G_c}$ *denotes the* affine flag variety *as defined in Definition 1.6.*

By rigidity of quasi-isogenies, the functor $\mathcal{RZ}_{\underline{\mathcal{L}}_0}$ is naturally isomorphic to the functor that assigns to a scheme S tuples $(\underline{\mathcal{L}}, \delta)$ (up to isomorphism) where $\underline{\mathcal{L}}$ is a local G_c-shtuka over S and $\delta : \underline{\mathcal{L}}_S \to \underline{\mathcal{L}}_{0,S}$ a quasi-isogeny over S. Therefore, there are no other automorphisms of $(\underline{\mathcal{L}}, \delta)$ than the identity, and $\mathcal{RZ}_{\underline{\mathcal{L}}_0}$ does not have to be viewed as a stack.

2.4.2 Rapoport-Zink spaces for *bounded* local G_c-shtukas

As moduli spaces for local resp. global G-shtukas are in general “only” ind-schemes we “add” a *boundedness-condition* to get moduli spaces locally of finite type.
In this section we introduce those “bounds” as defined in [AH14a, § 4], but first we need some further notations:

Fix an algebraic closure $\mathbb{F}_c((\zeta))^{\mathrm{alg}}$ of $\mathbb{F}_c((\zeta))$. Note that its ring of integers is not complete, so we consider a finite extension of discrete valuation rings R over $\mathbb{F}_c[\![\zeta]\!]$ (write $R|\mathbb{F}_c[\![\zeta]\!]$) with $R \subset \mathbb{F}_c((\zeta))^{\mathrm{alg}}$. Denote by κ_R its residue field, by $\mathcal{N}\mathrm{ilp}_R$ the category of R-schemes on which ζ is locally nilpotent and let

$$\hat{\mathcal{F}}\ell_{G_c,R} \coloneqq \mathcal{F}\ell_{G_c} \hat{\times}_{\mathbb{F}_c} \mathrm{Spf}\, R \tag{2.3}$$

and

$$\hat{\mathcal{F}}\ell_{G_c} \coloneqq \hat{\mathcal{F}}\ell_{G_c,\mathbb{F}_c[\![\zeta]\!]}. \tag{2.4}$$

Definition 2.28. *(i) For a finite extension of discrete valuation rings* $\mathbb{F}_c[\![\zeta]\!] \subset R \subset \mathbb{F}_c((\zeta))^{\mathrm{alg}}$ *let* $\hat{Z}_R \subset \hat{\mathcal{F}}\ell_{G_c,R}$ *be a closed ind-subscheme. Two such subschemes* $\hat{Z}_R \subset \hat{\mathcal{F}}\ell_{G_c,R}$ *and* $\hat{Z}'_{R'} \subset \hat{\mathcal{F}}\ell_{G_c,R'}$ *are said to be* equivalent *if there exists a finite extension of discrete valuation rings* $\mathbb{F}_c[\![\zeta]\!] \subset \tilde{R} \subset \mathbb{F}_c((\zeta))^{\mathrm{alg}}$ *containing* R *and* R' *such that*

$$\hat{Z}_R \hat{\times}_{\mathrm{Spf}\, R} \mathrm{Spf}\, \tilde{R} = \hat{Z}'_{R'} \hat{\times}_{\mathrm{Spf}\, R'} \mathrm{Spf}\, \tilde{R}$$

as closed ind-subschemes of $\hat{\mathcal{F}}\ell_{G_c,\tilde{R}}$.

(ii) *For an equivalence class $\hat{Z} = [\hat{Z}_R]$ of closed ind-subschemes $\hat{Z}_R \subset \hat{\mathcal{F}\ell}_{G_c,R}$ and for*

$$G_{\hat{Z}} := \left\{ \gamma \in Aut_{\mathbb{F}_c[\![\zeta]\!]}(\mathbb{F}_c((\zeta))^{\mathrm{alg}}) \ : \ \gamma(\hat{Z}) = \hat{Z} \right\}$$

we define the ring of definition $R_{\hat{Z}}$ of $\hat{Z}$ *to be the intersection of the fixed field of $G_{\hat{Z}}$ in $\mathbb{F}_c((\zeta))^{\mathrm{alg}}$ with all finite extensions $R \subset \mathbb{F}_c((\zeta))^{\mathrm{alg}}$ of $\mathbb{F}_c[\![\zeta]\!]$ over which a representative $\hat{Z}_R$ of $\hat{Z}$ exists.*

Now we are able to define bounds. For more details with respect to the above definitions see [AH14a, Remark 4.6 and Remark 4.7].

Definition 2.29. (i) *A* bound $\hat{Z}$ *is defined to be an equivalence class $[\hat{Z}_R]$ of closed subschemes $\hat{Z}_R \subset \hat{\mathcal{F}\ell}_{G_c,R}$ such that for all R each ind-subscheme $\hat{Z}_R$ is stable under the left L^+G_c-action on $\mathcal{F}\ell_{G_c}$ and the special fibres $Z_R := \hat{Z}_R \hat{\times}_{\mathrm{Spf} R} \mathrm{Spec}\, \kappa_R$ are quasi-compact subschemes of $\mathcal{F}\ell_{G_c} \hat{\times}_{\mathbb{F}_c} \mathrm{Spec}\, \kappa_R$.*

(ii) *The ring of definition $R_{\hat{Z}}$ of $\hat{Z}$ is called* the reflex ring *associated to the bound $\hat{Z}$.*

(iii) *Let $\hat{Z}$ be a bound with reflex ring $R_{\hat{Z}}$. Let $\mathcal{L}_+$ and $\mathcal{L}'_+$ be L^+G_c-torsors over a scheme S in $\mathcal{N}\mathrm{ilp}_{R_{\hat{Z}}}$ and let $\delta : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ be an isomorphism of the associated LG_c-torsors. Let $S' \to S$ be an étale covering over which trivializations*

$$\alpha : \mathcal{L}_+ \xrightarrow{\sim} (L^+G_c)_{S'} \quad \text{and} \quad \alpha' : \mathcal{L}'_+ \xrightarrow{\sim} (L^+G_c)_{S'}$$

exist. Then the automorphism $\alpha' \circ \delta \circ \alpha^{-1}$ of $(LG_c)_{S'}$ corresponds to a morphism

$$S' \to LG_c \hat{\times}_{\mathbb{F}_c} \mathrm{Spf}\, R_{\hat{Z}}.$$

We say that δ is bounded by $\hat{Z}$ *if for any such trivialization and for all finite extensions R of $\mathbb{F}_c[\![\zeta]\!]$ over which a representative $\hat{Z}_R$ of $\hat{Z}$ exists the induced morphism*

$$S' \hat{\times}_{R_{\hat{Z}}} \mathrm{Spf}\, R \to LG_c \hat{\times}_{\mathbb{F}_c} \mathrm{Spf}\, R \to \hat{\mathcal{F}\ell}_{G_c,R}$$

factors through $\hat{Z}_R$.
A local G_c-shtuka $(\mathcal{L}_+, \hat{\tau})$ over a scheme $S \in \mathcal{N}\mathrm{ilp}_{R_{\hat{Z}}}$ is said to be bounded by $\hat{Z}$ *if the isomorphism $\hat{\tau}$ is bounded by $\hat{Z}$.*

Remark 2.30. (i) *The Z_R are given by a base change from a unique closed subscheme $Z \subset \mathcal{F}\ell_{G_c} \hat{\times}_{\mathbb{F}_c} \kappa_{R_{\hat{Z}}}$ which is a projective scheme over $\kappa_{R_{\hat{Z}}}$ (see [AH14a, Definition 4.8]). It is called* the special fibre *of $\hat{Z}$.*

(ii) *The definition of the bound is independent of the chosen trivialization in Definition 2.29(iii). Moreover, the condition in Definition 2.29(iii) is satisfied for all trivializations and for all such finite extensions R of $\mathbb{F}_c[\![\zeta]\!]$ if and only if it is satisfied for one trivialization and for one such finite extension (see [AH14a, Remark 4.9]).*

Remark 2.31. *Observe that this definition of bounds generalizes the defintion of bounds given in [HV11] (see [AH14a, Example 4.13 and Example 4.12]).*

Definition 2.32. *Consider a global G-shtuka $\underline{\mathcal{G}}$ in $\nabla_n\mathcal{H}^1(C,G)^{\underline{c}}$ and an n-tuple of bounds $\hat{Z}_{\underline{c}} := (\hat{Z}_{c_i})_i$. We say that $\underline{\mathcal{G}}$ is* bounded *by $\hat{Z}_{\underline{c}}$ if for every $i \in \{1,\dots,n\}$ the associated local G_{c_i}-shtuka $\hat{\Gamma}_i(\underline{\mathcal{G}})$ is bounded by $\hat{Z}_{c_i}$.*
We denote the substack of $\nabla_n\mathcal{H}^1(C,G)^{\underline{c}}$ consisting of bounded global G-shtukas by $\nabla_n^{\hat{Z}_{\underline{c}}}\mathcal{H}^1(C,G)^{\underline{c}}$.

We recall some results with respect to the representability of the Rapoport-Zink spaces of bounded local G_c-shtukas.
For this purpose assume G_c to be a smooth affine group scheme over $\mathbb{D}_c$ with connected reductive generic fibre G_{Q_c}.

Definition 2.33. *Fix some $b \in LG_{Q_c}(k) = LG_c(k)$, $k \in \mathcal{N}\mathrm{ilp}_{\mathbb{F}_c[\![\zeta]\!]}$ a field. Kottwitz defines a* slope homomorphism associated to b *which we denote due to Kottwitz by*

$$\nu_b : D_{k((z))} \longrightarrow (G_{Q_c})_{k((z))},$$

called the Newton polygon of b *(see [Kot85, 4.2]), where D is the diagonalizable pro-algebraic group over $k((z))$ with character group $\mathbb{Q}$.*

Definition 2.34. *We call an element $b \in LG_c(k)$* decent *if it satisfies a* decency equation for a positive integer *s:*

$$(b\hat{\sigma})^s = s\nu_b(z)\hat{\sigma}^s \quad \text{in } LG_c(k) \rtimes \langle \hat{\sigma}^s \rangle$$

with $(b\hat{\sigma})^s := b \cdot \hat{\sigma}(b) \cdot \ldots \cdot (\hat{\sigma}^{s-1})(b) \cdot (\hat{\sigma}^s)$.

Remark 2.35. *Let $b \in LG_c(k)$ be decent with integer s, $\tilde{k} \subset k^{\mathrm{alg}}$ a finite field extension of $\mathbb{F}_c$ with degree s. Denote by ν_b the Newton polygon of b.*

(i) Since b has values in $\tilde{k}$ which is the fixed field of $\hat{\sigma}^s$ we have $b \in LG_c(\tilde{k})$.

(ii) By [Kot85, 4.3] and [Ser94, III, 2.3, Théorème 1′ and Remarque 1] we know that each $\hat{\sigma}$-conjugacy class, that is each quasi-isogeny class, in $LG_c(k)$ contains a decent element if k is algebraically closed.

(iii) Denote by $F_s := \tilde{k}((z))$ the fixed field of $\hat{\sigma}^s$ in $\mathbb{F}_c^{\mathrm{alg}}((z))$ and consider the connected linear algebraic group J_b over $\mathbb{F}_c((z))$, defined by

$$J_b(R) := \left\{ g \in G_c(R \otimes_{\mathbb{F}_c((z))} F_s) \ : \ g^{-1} b \hat{\sigma}(g) = b \right\} \tag{2.5}$$

for an $\mathbb{F}_c((z))$*-algebra* R.
Then ν_b *is defined over* F_s *and* $J_b \times_{\mathbb{F}_c((z))} F_s$ *is the centralizer of the* 1*-parameter group* $s\nu_b$ *of* G_c *and hence a Levi subgroup of* $(G_c)_{F_s}$*. In particular,*

$$J_b(\mathbb{F}_c((z))) \subset G_c(F_s) \subset LG_c(\tilde{k})$$

(see [RZ96, Corollary 1.14]).

In the following we recall the results of [AH14a, § 4.3] about the representability of the Rapoport-Zink space for bounded local G_c-stukas in some special case which we use in the sequel many times.
Let $\hat{Z}$ be a bound with reflex ring $R_{\hat{Z}} = \kappa[\![\xi]\!]$ and special fibre $Z \subset \mathcal{F}\ell_{G_c} \hat{\times}_{\mathbb{F}_c} \operatorname{Spec} \kappa$. Furthermore, let $\underline{\mathbb{L}}_0 = (L^+G_c, b\hat{\sigma}^*)$ be a trivialized local G_c-shtuka over $k \in \mathcal{N}\mathrm{ilp}_{\mathbb{F}_c[\![\zeta]\!]}$ and decent b (with integer s). We denote by $\tilde{k} \subset k^{\mathrm{alg}}$ the compositum of the residue field κ of $R_{\hat{Z}}$ and the finite field extension of $\mathbb{F}_c$ of degree s, so that we can assume $b \in LG_c(\tilde{k})$. Note that $\tilde{k}[\![\xi]\!]$ is the unramified extension of $R_{\hat{Z}}$ with residue field $\tilde{k}$. With these notations and assumptions we define:

Definition 2.36. *Consider the base change* $\mathcal{RZ}_{\underline{\mathbb{L}}_0} \hat{\times}_{\tilde{k}[\![\zeta]\!]} \operatorname{Spf} \tilde{k}[\![\xi]\!]$ *of* $\mathcal{RZ}_{\underline{\mathbb{L}}_0}$*. It is represented by the ind-scheme* $\hat{\mathcal{F}}\ell_{G_c, \tilde{k}[\![\xi]\!]} := \mathcal{F}\ell_{G_c} \hat{\times}_{\mathbb{F}_c} \operatorname{Spf} \tilde{k}[\![\xi]\!]$*. We define the subfunctor* $\mathcal{RZ}_{\underline{\mathbb{L}}_0}^{\hat{Z}}$ *of* $\underline{\mathcal{M}}_{\underline{\mathbb{L}}_0} \hat{\times}_{\tilde{k}[\![\zeta]\!]} \operatorname{Spf} \tilde{k}[\![\xi]\!]$ *by:*

$$\mathcal{RZ}_{\underline{\mathbb{L}}_0}^{\hat{Z}} : (\mathcal{N}\mathrm{ilp}_{\tilde{k}[\![\xi]\!]})^{\circ} \longrightarrow (Sets)$$
$$S \mapsto \left\{ (\underline{\mathcal{L}}, \bar{\delta}) \in \mathcal{RZ}_{\underline{\mathbb{L}}_0}(S) : \underline{\mathcal{L}} \text{ is bounded by } \hat{Z} \right\} \big/ \cong \ .$$

Proposition 2.37. *(cf. [AH14a, Theorem 4.18] resp. [HV11, Theorem 6.3] for split, constant* G_c*)*
Let G_c *be a smooth affine group scheme over* $\mathbb{D}_c$ *with connected reductive generic fibre then* $\mathcal{RZ}_{\underline{\mathbb{L}}_0}^{\hat{Z}}$ *as defined in Definition 2.36 above is ind-represented by a formal scheme, which is locally formally of finite type over* $\operatorname{Spf} \tilde{k}[\![\xi]\!]$ *and separated.*

Definition 2.38. (i) *We call the formal scheme that represents the functor* $\mathcal{RZ}_{\underline{\mathbb{L}}_0}^{\hat{Z}}$ *a* bounded Rapoport-Zink space for local G_c-shtukas.

(ii) *Its underlying reduced subscheme is the* affine Deligne-Lusztig variety *(ADLV)* $X_Z(b) \subset \mathcal{F}\ell_{G_c} \hat{\times}_{\mathbb{F}_c} \operatorname{Spec} \tilde{k}$ *which is defined by*

$$X_Z(b)(E) := \left\{ g \in \mathcal{F}\ell_{G_c}(E) : g^{-1} b \hat{\sigma}^*(g) \in Z(E) \right\}$$

for any field extension E *of* $\tilde{k}$*. Thus,* $X_Z(b)$ *is a scheme locally of finite type and separated over* $\tilde{k}$*.*

Remark 2.39. *By applying the uniformization theorem (see Proposition 2.53) we can relate the rational points of the Newton stratum of the moduli stack of global G-shtukas to the rational points of (certain) ADLV's.*

Remark 2.40. *Observe that the quasi-isogeny group* $\mathrm{QIsog}_{\tilde{k}}(\underline{\mathbb{L}}_0)$ *equals the group* $J_b(\mathbb{F}_c((z)))$ *and acts on* $\mathcal{RZ}^{\hat{Z}}_{\underline{\mathbb{L}}_0}$ *via*

$$(\underline{\mathcal{L}}, \bar{\delta}) \mapsto (\underline{\mathcal{L}}, g \circ \bar{\delta}) \tag{2.6}$$

for $g \in \mathrm{QIsog}_{\tilde{k}}(\underline{\mathbb{L}}_0)$.

Now we introduce *truncated Rapoport-Zink spaces*, a construction that we need in the following chapters.

In order to do this we fix a representation

$$\varrho : G_c \longrightarrow \mathrm{SL}_{r,\mathbb{D}_c} \tag{2.7}$$

such that the quotient $\mathrm{SL}_{r,\mathbb{D}_c} / G_c$ is quasi-affine (this can be done by 1.3). Consider the induced morphism

$$\varrho_* : \mathcal{H}^1(\operatorname{Spec}\mathbb{F}_c, L^+G_c)(S) \longrightarrow \mathcal{H}^1(\operatorname{Spec}\mathbb{F}_c, L^+\mathrm{SL}_{r,\mathbb{D}_c})(S). \tag{2.8}$$

Let $\mathcal{O}_S[\![z]\!]$ be the sheaf of $\mathcal{O}_S$-algebras on S for the fpqc-topology whose rings of sections on an S-scheme Y is the ring of power series $\mathcal{O}_S[\![z]\!](Y) := \Gamma(Y, \mathcal{O}_Y[\![z]\!])$. Since it is a countable direct product of $\mathcal{O}_S$ it is a sheaf. By $\mathcal{O}_S((z))$ we denote the fpqc-sheaf of $\mathcal{O}_S$-algebras on S associated with the presheaf $Y \mapsto \Gamma(Y, \mathcal{O}_Y)[\![z]\!]\left[\frac{1}{z}\right]$. Let $\mathcal{L}_+$ be any L^+G_c-torsor over S and $\mathcal{V}(\varrho_*\mathcal{L}_+)$ the sheaf of $\mathcal{O}_S[\![z]\!]$-modules associated to $\varrho_*\mathcal{L}_+$ in $\mathcal{H}^1(\operatorname{Spec}\mathbb{F}_c, L^+\mathrm{GL}_r)(S)$. In the special case $\mathcal{L}_+ = L^+G_c$ we get

$$\mathcal{V}(\varrho_*(L^+G_c)_S) = \mathcal{O}_S[\![z]\!]^{\oplus r}.$$

By $2\check{\varrho} = (r-1, \dots, 1-r)$ we denote the sum of all positive coroots of SL_r with respect to the Borel-subgroup of upper triangular matrices in SL_r.

Definition 2.41. *We define the closed ind-subscheme* $\mathcal{RZ}^d_{\hat{Z}}$ *of* $\mathcal{RZ}^{\hat{Z}}_{\underline{\mathbb{L}}_0}$ *by the functor:*

$$\begin{aligned} \mathcal{RZ}^d_{\hat{Z}} : (\mathcal{N}\mathrm{ilp}_{\tilde{k}[\![\xi]\!]})^\circ &\longrightarrow (Sets) \\ S &\mapsto \Big\{ \textit{Isomorphism classes of } (\underline{\mathcal{L}}, \delta) := (\mathcal{L}_+, \hat{\tau}, \delta) \in \mathcal{RZ}^{\hat{Z}}_{\underline{\mathbb{L}}_0}(S), \\ &\qquad \textit{s.t. } \varrho_*(\delta) \textit{ is bounded by } 2d\check{\varrho} \Big\} \end{aligned}$$

for $d \in \mathbb{N}_0$ *where* $\varrho_*(\delta)$ *is* bounded by $2d\check{\varrho}$ *if for all* $j = 1, \dots, r$*:*

$$\bigwedge^j_{\mathcal{O}_S[\![z]\!]} \varrho_*(\delta)(\mathcal{V}(\varrho_*\mathcal{L}_+)) \subset z^{d(j^2-jr)} \cdot \bigwedge^j_{\mathcal{O}_S[\![z]\!]} \mathcal{V}(\varrho_*(L^+G_c)_S). \tag{2.9}$$

Proposition 2.42. *(cf. [AH14a, Lemma 4.20])*
$\mathcal{RZ}^d_{\hat{Z}}$ *is a ξ-adic noetherian formal scheme over* $\tilde{k}[\![\xi]\!]$. *Its underlying topological space* $(\mathcal{RZ}^d_{\hat{Z}})_{red}$ *is a quasi-projective scheme over* $\operatorname{Spec}\tilde{k}$.
Moreover, if G_c is parahoric, then $(\mathcal{RZ}^d_{\hat{Z}})_{red}$ is projective.

Proposition 2.43. *(cf. [AH14a, Corollary 4.26])*
The irreducible components of $\mathcal{RZ}^{\hat{Z}}_{\mathbb{L}_0}$ (as a topological space) are quasi-projective schemes over $\tilde{k}$, and thus, quasi-compact. If G_c is parahoric, they are even projective.

2.5 The uniformization morphism for bounded global G-shtukas

Fix an n-tuple $\underline{c} := (c_i)_{i=1,\dots,n}$ of closed points of C with $c_i \neq c_j$ for $i \neq j$. Let $C' := C \smallsetminus \{c_1, \dots c_n\}$ and let S be a scheme in $\mathcal{N}\mathrm{ilp}_{A_{\underline{c}}}$. Recall the substack $\nabla_n^{\hat{Z}_{\underline{c}}}\mathcal{H}^1(C,G)^{\underline{c}}$ of bounded global G-shtukas (see Definition 2.32) of $\nabla_n\mathcal{H}^1(C,G)^{\underline{c}}$. For a finite subscheme D of C we set $D_S := D \times_{\mathbb{F}_q} S$ and denote by $\underline{\mathcal{G}}|_{D_S} := \underline{\mathcal{G}} \times_{C_S} D_S$ the pullback of $\underline{\mathcal{G}}$ to D_S.
Now we want to equip a global G-shtuka $\underline{\mathcal{G}}$ with a *(rational) H-level structure.* In order to define this we need some further notations and definitions which we introduce in the following.
We denote by $\mathrm{Rep}_{\mathbb{O}^{\underline{c}}}G$ the category of $\mathbb{O}^{\underline{c}}$-morphisms

$$\rho : G \times_C \operatorname{Spec}\mathbb{O}^{\underline{c}} \longrightarrow \mathrm{GL}_{\mathbb{O}^{\underline{c}}}(V),$$

where $\mathbb{O}^{\underline{c}} := \prod_{c \notin \underline{c}} A_{\underline{c}}$ is the ring of integral adeles of C outside $\underline{c}$ and V a finite free module over $\mathbb{O}^{\underline{c}}$. We assume S to be connected and let $\bar{s}$ be a geometric base point on S. By $\mathrm{Funct}^{\otimes}(\mathrm{Rep}_{\mathbb{O}^{\underline{c}}}G, \mathfrak{Mod}_{\mathbb{O}^{\underline{c}}[\pi_1(S,\bar{s})]})$ we denote the category of tensor functors from $\mathrm{Rep}_{\mathbb{O}^{\underline{c}}}G$ to the category $\mathfrak{Mod}_{\mathbb{O}^{\underline{c}}[\pi_1(S,\bar{s})]}$ of $\mathbb{O}^{\underline{c}}[\pi_1(S,\bar{s})]$-modules.
Remember the definition of the (étale) local $\tilde{G}_c$-shtuka associated with a global G-shtuka from Definition 2.20.

Definition 2.44. (i) *A* local shtuka *over $S \in \mathcal{N}\mathrm{ilp}_{\mathbb{F}_q[\![\zeta]\!]}$ of rank r is a pair (M, φ) wehre M is a sheaf of $\mathcal{O}_S[\![z]\!]$-modules on S which Zariski-locally is free of rank r, together with an isomorphism of $\mathcal{O}_S((z))$-modules $\varphi : \sigma^*M \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S((z)) \tilde{\to} M \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S((z))$.*

(ii) *A* quasi-isogeny *between local shtukas $(M, \varphi) \to (M', \varphi')$ is an isomorphism of $\mathcal{O}_S((z))$-modules $f : M \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S((z)) \tilde{\to} M' \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S((z))$ with $\varphi'\sigma^*(f) = f\varphi$.*

Remark 2.45. *By [HV12, Lemma 4.2], there is an equivalence of categories between the category of local GL_r-shtukas over S an the category of (rank r) local shtukas over S.*

Definition 2.46. *Assume that S is connected. Let $\bar{s}$ be a geometric point of S and let $\pi_1(S,\bar{s})$ denote the algebraic fundamental group of S at $\bar{s}$. We define the* Tate functor *from the category of local* GL_r*-shtukas* $\mathrm{Sht}_{\mathrm{GL}_r}(S)$ *over S to the category of* $\mathbb{F}_q[\![z]\!]\,[\pi_1(S,\bar{s})]$*-modules* $\mathfrak{Mod}_{\mathbb{F}_q[\![z]\!][\pi_1(S,\bar{s})]}$ *as follows*

$$\begin{aligned} T_- : \mathrm{Sht}_{\mathrm{GL}_r}(S) &\to \mathfrak{Mod}_{\mathbb{F}_q[\![z]\!][\pi_1(S,\bar{s})]} \\ \underline{M} := (M,\tau) &\mapsto T_{\underline{M}} := (M \otimes_{\mathcal{O}_{S[\![z]\!]}} \kappa(\bar{s})[\![z]\!])^\tau \end{aligned}$$

where the superscript τ represents the τ-invariants.

Definition 2.47. *Let c be a place with $c \neq c_i$ for all i. For every representation $\rho : G_c \to \mathrm{GL}_{r,A_c}$ in $\mathrm{Rep}_{A_c} G_c$ we consider the representation*

$$\tilde{\rho} \in \mathrm{Rep}_{\mathbb{F}_q[\![z]\!]} \tilde{G}_c$$

wich is the composition of

$$\mathrm{Res}_{\mathbb{F}_c|\mathbb{F}_q}(\rho) : \tilde{G}_c \to \mathrm{Res}_{\mathbb{F}_c|\mathbb{F}_q} \mathrm{GL}_{r,A_c}$$

followed by the natural inclusion $\mathrm{Res}_{\mathbb{F}_c|\mathbb{F}_q} \mathrm{GL}_{r,A_c} \subset \mathrm{GL}_{r\cdot[\mathbb{F}_c:\mathbb{F}_q],\mathbb{F}_q[\![z]\!]}$.
Consider a global G-shtuka $\underline{\mathcal{G}}$. We define

$$\begin{aligned} \check{\mathcal{T}}_{L_c^+(\underline{\mathcal{G}})}(\rho) &:= \check{\mathcal{T}}_{L_c^+(\underline{\mathcal{G}})}(\tilde{\rho}) \\ &\cong \varprojlim_n \rho_* \big(\underline{\mathcal{G}} \times_C \operatorname{Spec} A_c/(c^n)\big)^\tau, \end{aligned}$$

by [AH14a, Remark 5.6].

Definition 2.48. *Following [AH14b, (6.1) and (6.2)], we define the* (dual) Tate functors

$$\begin{aligned} \check{\mathcal{T}}_- : \nabla_n \mathcal{H}^1(C,G)^{\underline{c}}(S) &\longrightarrow \mathrm{Funct}^{\otimes}(\mathrm{Rep}_{\mathbb{O}^{\underline{c}}} G, \mathfrak{Mod}_{\mathbb{O}^{\underline{c}}[\pi_1(S,\bar{s})]}) \\ \underline{\mathcal{G}} &\mapsto (\check{\mathcal{T}}_{\underline{\mathcal{G}}} : \rho \mapsto \varprojlim_{D \subset C'} (\rho_* \underline{\mathcal{G}}|_{D_{\bar{s}}})^\tau), \\ \check{\mathcal{V}}_- : \nabla_n \mathcal{H}^1(C,G)^{\underline{c}}(S) &\longrightarrow \mathrm{Funct}^{\otimes}(\mathrm{Rep}_{\mathbb{O}^{\underline{c}}} G, \mathfrak{Mod}_{\mathbb{A}^{\underline{c}}[\pi_1(S,\bar{s})]}) \\ \underline{\mathcal{G}} &\mapsto (\check{\mathcal{V}}_{\underline{\mathcal{G}}} : \rho \mapsto \varprojlim_{D \subset C'} (\rho_* \underline{\mathcal{G}}|_{D_{\bar{s}}})^\tau \otimes_{\mathbb{O}^{\underline{c}}} \mathbb{A}^{\underline{c}}). \end{aligned}$$

Remark 2.49. *(cf. [AH14a, Remark 5.5 and Remark 5.6])*

(i) For $\rho = (\rho_c)$ and $L_c^+(\underline{\mathcal{G}})$ the étale local $\tilde{G}_c$-shtuka associated with $\underline{\mathcal{G}} = (\mathcal{G},\tau)$ at $c \in C$ as defined in Definition 2.47 above in the sense of [AH14a, Remark 5.6] we have

$$\varprojlim_{D \subset C'} (\rho_* \underline{\mathcal{G}}|_{D_{\bar{s}}})^\tau) \cong \prod_{c \in C'} \check{\mathcal{T}}_{L_c^+(\underline{\mathcal{G}})}(\rho_c).$$

(ii) $\check{\mathcal{V}}_-$ *transforms quasi-isogenies into quasi-isogenies.*

Definition 2.50. *(i) We denote by*

$$\omega^\circ_{\mathbb{O}^{\underline{c}}} : \mathrm{Rep}_{\mathbb{O}^{\underline{c}}} G \longrightarrow \mathfrak{Mod}_{\mathbb{O}^{\underline{c}}}$$

resp. by

$$\omega^\circ := \omega^\circ_{\mathbb{O}^{\underline{c}}} \otimes_{\mathbb{O}^{\underline{c}}} \mathbb{A}^{\underline{c}} : \mathrm{Rep}_{\mathbb{O}^{\underline{c}}} G \longrightarrow \mathfrak{Mod}_{\mathbb{A}^{\underline{c}}}$$

the forgetful functors with $\mathbb{A}^{\underline{c}} := \mathbb{O}^{\underline{c}} \otimes_{\mathcal{O}_C} Q$.

(ii) We denote by $\mathrm{Isom}^{\otimes}(\omega^\circ_{\mathbb{O}^{\underline{c}}}, \check{\mathcal{T}}_{\underline{\mathcal{G}}})$ *resp. by* $\mathrm{Isom}^{\otimes}(\omega^\circ, \check{\mathcal{V}}_{\underline{\mathcal{G}}})$ *the set of isomorphisms of tensor functors.*

Remark 2.51. *By the generalized Tannakian formalism ([Wed04, Corollary 5.20] or [AH14b, Definition 6.1]), we get*

$$G(\mathbb{O}^{\underline{c}}) \cong \mathrm{Aut}^{\otimes}(\omega^\circ_{\mathbb{O}^{\underline{c}}}),$$

because $\mathbb{O}^{\underline{c}}$ *is a Prüfer ring.*

Definition 2.52. *(i) Let* $H \subset G(\mathbb{A}^{\underline{c}}) \cong \mathrm{Aut}^{\otimes}(\omega^\circ)$ *be a compact open subgroup. We define a* (rational) H-level structure *on a global G-shtuka* $\underline{\mathcal{G}}$ *over* $S \in \mathcal{N}\mathrm{ilp}_{A_{\underline{c}}}$ *as a* $\pi_1(S, \bar{s})$*-invariant H-orbit* $\bar{\gamma} = \gamma H$ *in* $\mathrm{Isom}^{\otimes}(\omega^\circ, \check{\mathcal{V}}_{\underline{\mathcal{G}}})$.

(ii) By $\nabla_n^H \mathcal{H}^1(C,G)^{\underline{c}}$ *we denote the category fibered in groupoids where the objects* $(\underline{\mathcal{G}}, \bar{\gamma})$ *of the category* $\nabla_n^H \mathcal{H}^1(C,G)^{\underline{c}}(S)$ *are given by* $\underline{\mathcal{G}} \in \nabla_n \mathcal{H}^1 C, G)^{\underline{c}}(S)$ *and* $\bar{\gamma}$ *a H-level structure on* $\underline{\mathcal{G}}$*. The morphisms are quasi-isogenies of global G-shtukas that are compatible with the H-level structures and that are isomorphisms at* $\underline{c}$ *(see [AH14b, Definition 6.3(b)]).*

(iii) By $\nabla_n^{H,\hat{Z}_{\underline{c}}} \mathcal{H}^1(C,G)^{\underline{c}}$ *we denote the closed ind-substack of* $\nabla_n^H \mathcal{H}^1(C,G)^{\underline{c}}$ *parametrizing global G-shtukas that are bounded by* $\hat{Z}_{\underline{c}}$ *and equipped with a H-level structure. It is an ind-algebraic stack over* $\mathrm{Spf}\, A_{\underline{c}}$ *which is ind-separated and locally of ind-finite type (see [AH14b, Corollary 6.7]).*

Now we can introduce the uniformization morphism:

Proposition 2.53. *(cf. [AH14b, Theorem 7.4])*
For a fixed global G-shtuka $\underline{\mathcal{G}}_0 \in \nabla_n^{H,\hat{Z}_{\underline{c}}} \mathcal{H}^1(C,G)^{\underline{c}}(k)$ *over an algebraically closed field k with characteristic* $\underline{c}$ *denote by* $I_{\underline{\mathcal{G}}_0}(Q) = \mathrm{QIsog}_k(\underline{\mathcal{G}}_0)$ *its quasi-isogeny group. Let*

$$\omega^\circ : \mathrm{Rep}_{\mathbb{A}^{\underline{c}}} G \longrightarrow \mathfrak{Mod}_{\mathbb{A}^{\underline{c}}}$$

be the forgetful functor. Then there is a uniformization morphism

$$\theta = \theta(\underline{\mathcal{G}}_0) : I_{\underline{\mathcal{G}}_0}(Q)\backslash(\prod_i \mathcal{RZ}^{\hat{Z}_i}_{\hat{\Gamma}_i(\underline{\mathcal{G}}_0)} \times \mathrm{Isom}^{\otimes}(\omega^\circ, \check{\mathcal{V}}_{\underline{\mathcal{G}}_0})/H) \to \nabla_n^{H,\hat{Z}_{\underline{c}}}\mathcal{H}^1(C,G)^{\underline{c}}\hat{\times}_{\mathbb{F}_{\underline{c}}}\mathrm{Spec}(k)$$

of ind-algebraic stacks over $\mathrm{Spf}\, k[\![\underline{\zeta}]\!]$ *which is ind-proper and formally étale. It is an isomorphism onto the formal completion of* $\nabla_n^{H,\hat{Z}_{\underline{c}}}\mathcal{H}^1(C,G)^{\underline{c}}$ *along its image. Furthermore, the morphism* θ *is* $G(\mathbb{A}^{\underline{c}})$*-equivariant for the action through Hecke correspondences on the source and the target.*

Remark 2.54. *(i) Note that the group* $I_{\underline{\mathcal{G}}_0}(Q)$ *acts on* $\prod_i \mathcal{RZ}^{\hat{Z}_i}_{\underline{\mathbb{L}}_i}$ *via the morphism*

$$\begin{aligned} I_{\underline{\mathcal{G}}_0}(Q) &\longrightarrow \prod_i J_{\underline{\mathbb{L}}_i}(Q_{c_i}) \\ \alpha &\mapsto (\hat{\Gamma}_{c_i}(\alpha))_i (=: (\alpha_i)_i) \end{aligned}$$

where $J_{\underline{\mathbb{L}}_i}(Q_{c_i}) = \mathrm{QIsog}_k(\underline{\mathbb{L}}_i)$ *and* $(\underline{\mathbb{L}}_i)_{i=1,\dots,n} := \underline{\hat{\Gamma}}(\underline{\mathcal{G}}_0)$ *denotes the associated* n*-tuple of local* G_{c_i}*-shtukas* $\underline{\mathbb{L}}_i = ((L^+G_{c_i})_k, b_i\hat{\sigma}^*)$ *with* $b_i \in LG_{c_i}(k)$ *over* k.

(ii) The uniformization morphism is based on the "unbounded" uniformization morphism which is defined by:

$$\begin{aligned} \Psi_{\underline{\mathcal{G}}_0} : \prod_i \mathcal{RZ}_{\underline{\mathcal{L}}_i}(S) &\longrightarrow \nabla_n\mathcal{H}^1(C,G)^{\underline{c}}(S) \\ ((\underline{\mathcal{L}}_i'), \delta_i) &\mapsto \delta_n^* \circ \ldots \circ \delta_1^*\underline{\mathcal{G}}_{0,S} \end{aligned}$$

where $(\underline{\mathcal{L}})_i := \underline{\hat{\Gamma}}(\underline{\mathcal{G}}_0)$ *for* $\underline{\mathcal{G}}_0 \in \nabla_n\mathcal{H}^1(C,G)^{\underline{c}}(T)$ *and* T *a scheme over* $\mathrm{Spec}\, A_{\underline{c}}/(\zeta_1,\dots,\zeta_n)$ *and* $S \in \mathcal{N}\mathrm{ilp}_{T\hat{\times}_{\mathbb{F}_{c_i}}\mathrm{Spf}\, A_{c_i}}$. *It is* $\hat{\Gamma}_{c_i}(\delta_n^* \circ \ldots \circ \delta_1^*\underline{\mathcal{G}}_{0,S}) = \underline{\mathcal{L}}_i'$.

(iii) Let $(\underline{\mathcal{G}}_i, \delta_i)$ *be a* k*-valued point of* $\prod_i \mathcal{RZ}^{\hat{Z}_i}_{\hat{\Gamma}_i(\underline{\mathcal{G}}_0)}$ *and denote by* $\underline{\mathcal{G}} := \delta_n^* \circ \ldots \circ \delta_1^*\underline{\mathcal{G}}_{0,k}$ *its image under* $\Psi_{\underline{\mathcal{G}}_0}$. *By [AH14b, Remark 5.4] there is a unique quasi-isogeny* $\delta : \underline{\mathcal{G}} \to \underline{\mathcal{G}}_{0,k}$ *which is an isomorphism outside the* c_i *and satisfies* $\underline{\hat{\Gamma}}(\delta) = (\delta_i)_i$. *This induces a functor* $\check{\mathcal{T}}_\delta : \check{\mathcal{T}}_{\underline{\mathcal{G}}} \to \check{\mathcal{T}}_{\underline{\mathcal{G}}_0}$ *(see Definition 2.48). Let* $\gamma_0 \in \mathrm{Isom}^{\otimes}(\omega^\circ, \check{\mathcal{V}}_{\underline{\mathcal{G}}_0})$ *be a representative of the* H*-level structure* $\gamma_0 H$ *on* $\underline{\mathcal{G}}_0$. *Then* θ *is induced by the morphism*

$$(\underline{\mathcal{L}}_i, \delta_i)_i \times hH \mapsto (\underline{\mathcal{G}}, Hh^{-1}\gamma_0\check{\mathcal{T}}_\delta).$$

Let $\eta \in I_{\underline{\mathcal{G}}_0}(Q)$ *and consider*

$$\begin{aligned} \epsilon : I_{\underline{\mathcal{G}}_0}(Q) &\to \mathrm{Isom}^{\otimes}(\omega^\circ, \check{\mathcal{V}}_{\underline{\mathcal{G}}_0}) \\ \eta &\mapsto \gamma_0 \circ \check{\mathcal{V}}_\eta \circ \gamma_0^{-1}. \end{aligned}$$

Then $I_{\underline{\mathcal{G}}_0}(Q)$ *acts on* $\prod_i \mathcal{RZ}^{\hat{Z}_i}_{\hat{\Gamma}_i(\underline{\mathcal{G}}_0)} \times \mathrm{Isom}^{\otimes}(\omega^\circ, \check{\mathcal{V}}_{\underline{\mathcal{G}}_0})/H$ *via*

$$(\underline{\mathcal{L}}_i, \delta_i)_i \times hH \mapsto (\underline{\mathcal{L}}_i, \hat{\Gamma}_{c_i}(\eta)\delta_i)_i \times \epsilon(\eta)hH.$$

(iv) For more details, see [AH14b, § 5 and § 7].

We have the following lemma:

Lemma 2.55. *(cf. [AH16b, Lemma 3.18])*
Let $\underline{\mathcal{G}}_0$ *and* $\underline{\mathcal{G}}_0^{'}$ *be two global* G*-shtukas in* $\nabla_n\mathcal{H}^1(C,G)^{\underline{c}}(\mathbb{F}_q^{\mathrm{alg}})$ *for an algebraic closure* $\mathbb{F}_q^{alg}$ *of* $\mathbb{F}_q$*. Then the following are equivalent:*

(i) $\mathrm{im}\theta(\underline{\mathcal{G}}_0) \cap \mathrm{im}\theta(\underline{\mathcal{G}}_0^{'}) \neq \varnothing$.

(ii) $\mathrm{im}\theta(\underline{\mathcal{G}}_0) = \mathrm{im}\theta(\underline{\mathcal{G}}_0^{'})$.

(iii) There is a quasi-isogeny $\underline{\mathcal{G}}_0 \to \underline{\mathcal{G}}_0^{'}$ *over* $\mathbb{F}_q^{\mathrm{alg}}$.

3 Newton strata

In [HV11, § 7], the Newton stratification on deformation spaces of local G_c-shtukas was given. In the following we give a definition of such a stratification of the moduli space of global G-shtukas by quasi-isogeny classes of their associated local G_c-shtukas.

The construction considered in [HV11] is related to σ-conjugacy classes due to Kottwitz. We first recall the Newton stratification described in [HV11, § 7]. In this situation let k be an algebraically closed field (over $\mathbb{F}_q$) and denote by $\underline{\hat{\mathcal{G}}} = (\hat{\mathcal{G}}, \hat{\tau})$ a local G_c-shtuka over k. Then (e.g. by [HV11, Remark 3.2]) $\underline{\hat{\mathcal{G}}}$ is isomorphic to a local G_c-shtuka $(L^+G_c, b\hat{\sigma}^*)$ for some element $b \in LG_c(k)$. There is a bijection between the set of quasi-isogeny classes (of local G_c-shtukas over k) and

$$\mathcal{B}(G_c) := \{[b] \,:\, b \in LG_c(k)\},$$

the set of σ-conjugacy classes of elements b in $LG_c(k)$ where $[b]$ denotes the class of b which contains all $g^{-1}b\hat{\sigma}^*(g)$ for $g \in LG_c(k)$. Note that $\mathcal{B}(G_c)$ is independent of the choice of k as long as k is algebraically closed by [RR96, Lemma 1.3].

Now let S be a scheme in $\mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$, $\underline{\hat{\mathcal{G}}} = (\hat{\mathcal{G}}, \hat{\tau})$ a local G_c-shtuka over S and $\bar{s} \in S(k)$ a geometric point.

Then in [HV11, § 7] the *quasi-isogeny class of* $\underline{\hat{\mathcal{G}}}$ *in* s is defined as the class $[\underline{\hat{\mathcal{G}}}](\bar{s}) := [\underline{\hat{\mathcal{G}}}_{\bar{s}}]$ in $\mathcal{B}(G_c)$ which corresponds to the specialization $\underline{\hat{\mathcal{G}}}_{\bar{s}}$. The *Newton polygon of* $\underline{\hat{\mathcal{G}}}$ *in* s is defined as $\nu_{G_c}([\underline{\hat{\mathcal{G}}}_{\bar{s}}]) =: \nu_{G_c}(\underline{\hat{\mathcal{G}}}_{\bar{s}})$. $[\underline{\hat{\mathcal{G}}}_{\bar{s}}]$ is independent of the geometric point $\bar{s}$ that is chosen above the point $s \in S$. So there is a map

$$\begin{aligned}[\underline{\hat{\mathcal{G}}}] : S &\longrightarrow \mathcal{B}(G_c)\\ s &\mapsto [\underline{\hat{\mathcal{G}}}](\bar{s}).\end{aligned}$$

For further details on $\mathcal{B}(G_c)$, see [RR96] or [Kot97]. By [Kot85] we have an embedding

$$\mathcal{B}(G_c) \hookrightarrow (X_*(T)_{\mathbb{Q}}/W_{G_c})^{\mathrm{Gal}(\mathbb{F}_c((z))^{\mathrm{sep}}|\mathbb{F}_c((z)))} \times \pi_1(G_c)_{\mathrm{Inertia}} \tag{3.1}$$

given by

$$[b] \mapsto (\nu_{G_c}(b), \kappa_{G_c}(b))$$

where $\mathrm{Inertia} \subset \mathrm{Gal}(\mathbb{F}_c((z))^{\mathrm{sep}} \mid \mathbb{F}_c((z)))$ is the inertia subgroup and W_{G_c} denotes the finite Weyl group of T and $X_*(T)_{\mathbb{Q}} := X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ for $X_*(T) = \underline{\mathrm{Hom}}(\mathbb{G}_m, T)$. Here,

these homomorphisms are defined over an algebraic closure $\mathbb{F}_c((z))^{\mathrm{alg}}$ of $\mathbb{F}_c((z))$ and T is a maximal torus in $G_c(\mathbb{F}_c((z))^{\mathrm{alg}})$. $\pi_1(G_c)$ denotes the fundamental group of G_c. We call $\nu_{G_c}(b)_{\mathrm{polygon}}$ the *Newton point* of b and $\kappa_{G_c}(b)$ the *Kottwitz point.*
Now, we will define the Newton stratum associated to a local G_c-shtuka. Two local G_c-shtukas are in the same Newton stratum if both their Newton polygon and their Kottwitz point coincide. Therefore, two of each local G_c-shtukas in the same Newton stratum are quasi-isogenous, that is they belong to the same class in $\mathcal{B}(G_c)$.

Definition 3.1. *(cf. [HV11, page 116])*
Let $\underline{\hat{\mathcal{G}}}$ be a local G_c-shtuka over $S \in \mathcal{N}\mathrm{ilp}_{\mathbb{F}_c[\![\zeta]\!]}$ and fix a $\hat{\sigma}$-conjugacy class $[b] \in \mathcal{B}(G_c)$. Denote by $\nu_{G_c}(\underline{\hat{\mathcal{G}}}_s)$ the Newton polygon (or Newton point) of $\underline{\hat{\mathcal{G}}}$ in s, and by $\kappa_{G_c}(\underline{\hat{\mathcal{G}}})$ its Kottwitz point. Then,

$$\begin{aligned}\mathcal{N}_b := \mathcal{N}_{[b]} &= \left\{s \in S \ :\ \nu_{G_c}(\underline{\hat{\mathcal{G}}}_s) = \nu_{G_c}(b) \ \textit{and}\ \kappa_{G_c}(\underline{\hat{\mathcal{G}}}_s) = \kappa_{G_c}(b)\right\} \\ &= \left\{s \in S \ :\ \left[\underline{\hat{\mathcal{G}}}\right](s) = [b]\right\}\end{aligned}$$

is the Newton stratum associated to $\underline{\hat{\mathcal{G}}}$ and $[b]$.

We give now an analogous description of Newton strata in the special fibre of the moduli space of bounded global G-shtukas.

Notation 3.2. *We denote by $\nabla\mathcal{H}^1$ the special fibre of the stack $\nabla_n^{H,\hat{Z}_{\underline{c}}}\mathcal{H}^1(C,G)^{\underline{c}}$. $\nabla\mathcal{H}^1$ is a Deligne-Mumford stack locally of finite type and separated over $\mathbb{F}_{\underline{c}}$ by [AH14b, Corollary 6.7 and Remark 7.2]. For a field extension K of $\mathbb{F}_{\underline{c}}$ we write $\nabla\mathcal{H}^1_K := \nabla\mathcal{H}^1 \times_{\mathbb{F}_{\underline{c}}} \operatorname{Spec} K$.*

We define the *Newton stratum* associated to a global G-shtuka $\underline{\mathcal{G}} = (\mathcal{G}, \tau) \in \nabla\mathcal{H}^1$ as a locally closed substack of $\nabla\mathcal{H}^1$ in the sense of [LMB00, Lemma 4.10]. Therefore, we need some preperations.

Proposition 3.3. *(cf. [HV11, Theorem 7.3], resp. [RR96, Theorem 3.6])*
Let $S \in \mathcal{N}\mathrm{ilp}_{\mathbb{F}_c[\![\zeta]\!]}$ be a scheme and $\underline{\hat{\mathcal{G}}} \in \mathrm{Sht}_{G_c}^{\operatorname{Spec} A_c}(S)$ a local G_c-shtuka over S. Using the notations from Definition 3.1, the reduced subschme $\mathcal{N}_{\preceq b}$ of S with

$$\mathcal{N}_{\preceq b} := \left\{s \in S \ :\ \left[\underline{\hat{\mathcal{G}}}\right](s) \preceq [b]\right\}$$

is a closed subscheme of S.
Furthermore,

$$\begin{aligned}\mathcal{N}_b := \mathcal{N}_{[b]} &= \left\{s \in S \ :\ \nu_{G_c}(\underline{\hat{\mathcal{G}}}_s) = \nu_{G_c}(b) \ \textit{and}\ \kappa_{G_c}(\underline{\hat{\mathcal{G}}}_s) = \kappa_{G_c}(b)\right\} \\ &= \left\{s \in S \ :\ \left[\underline{\hat{\mathcal{G}}}\right](s) = [b]\right\}\end{aligned}$$

is a locally closed subscheme of S and $\mathcal{N}_b$ is open in $\mathcal{N}_{\preceq b}$.

Proof. Hartl and Viehmann consider in loc. cit. the case where G_c is split connected reductive, but the proof of [RR96, Theorem 3.6] (in the equal characteristic case) holds in our analogous situation for a connected reductive group G_c. □

Remark 3.4. *(cf. [RR96, Theorem 3.6]*
Note that for each $b_0 \in \mathcal{B}(G_c)$*, the subset*

$$\left\{s \in S \;:\; \nu_{G_c}(\underline{\hat{\mathcal{G}}}_s) \preceq \nu_{G_c}(b_0)\right\}$$

is Zariski-closed on S *and locally on* S *the zero set is a finitely generated ideal.*

Notation 3.5. *From now on we denote by* $\left[\underline{\hat{\mathcal{G}}}\right]$ *the quasi-isogeny class of a local* G_c*-shtuka* $\underline{\hat{\mathcal{G}}}$*.*

By Proposition 3.3, follows directly:

Corollary 3.6. *Let* $\underline{c} = (c_1, \ldots, c_n) \in C^n$ *be a tuple of places and denote by* $\underline{\mathcal{G}} = (\mathcal{G}, s_1, \ldots, s_n, \tau) \in \nabla_n^{H,\hat{Z}_{\underline{c}}}\mathcal{H}^1(C,G)^{\underline{c}}(S)$ *a global* G*-shtuka. Denote by* $\mathcal{B}(G_{c_i})$ *the set of quasi-isogeny classes of local* G_{c_i}*-shtukas. Choose a tuple* $\underline{b} = (b_i, \ldots, b_n) \in \prod_i \mathcal{B}(G_{c_i})$*. Set* $\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}) =: \underline{\hat{\mathcal{G}}}_i$*. Then, the reduced scheme* $\mathcal{N}_{\preceq \underline{b}}$ *with*

$$\mathcal{N}_{\preceq \underline{b}} := \left\{s \in S \;:\; \left[\underline{\hat{\mathcal{G}}}_i\right](s) \preceq b_i \textit{ for all } i\right\}$$

is a closed subscheme of S*, and*

$$\mathcal{N}_{\underline{b}} := \left\{s \in s \;:\; \left[\underline{\hat{\mathcal{G}}}_i\right](s) = b_i \textit{ for all } i\right\}$$

is a locally closed subscheme.

Consider now the complement of $\mathcal{N}_{\preceq b}$ (resp. of $\mathcal{N}_{\preceq \underline{b}}$) in S and denote it by $(\mathcal{N}_{\preceq b})^o$ (resp. by $\left(\mathcal{N}_{\preceq \underline{b}}\right)^o$). Then, $(\mathcal{N}_{\preceq b})^o$ (resp. $\left(\mathcal{N}_{\preceq \underline{b}}\right)^o$) is an open subscheme of S.

Definition 3.7. *(i) Let* $i \in \{1, \ldots, n\}$ *and fix a quasi-isogeny class of local* G_{c_i}*-shtukas given by* b_i*. Consider the stack consisting of tuples*

$$\left\{(S, \underline{\mathcal{G}}) \;:\; S \textit{ a scheme}, \underline{\mathcal{G}} \in \nabla\mathcal{H}^1(S) \textit{ s.t. } (\mathcal{N}_{\preceq b_i})^o = S\right\}.$$

This stack is an open stack of $\nabla\mathcal{H}^1$ *and equals the stack*

$$\left\{(S, \underline{\mathcal{G}}) \;:\; S \textit{ a scheme}, \underline{\mathcal{G}} \in \nabla\mathcal{H}^1(S) \textit{ s.t. } (3.2) \textit{ holds}\right\}$$

with (3.2) *is the following condition:*
For all $s : \operatorname{Spec} k \to S$ *for a field* k *the following condition holds:*

$$\left[(s^*\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}))\right] \npreceq b_i. \tag{3.2}$$

Denote its reduced complement by $\mathcal{N}_{\preceq b_i}$*. Then, by [LMB00, Lemma 4.10],* $\mathcal{N}_{\preceq b_i}$ *is a closed substack of* $\nabla\mathcal{H}^1$*. We call the reduced locally closed substack* $\mathcal{N}_{b_i}$ *the* Newton stratum associated to c_i and b_i.

(ii) *Choose a tuple of quasi-isogeny classes* $\underline{b} := (b_1, \dots, b_n) \in \prod_i \mathcal{B}(G_{c_i})$. *Consider the stack consisting of tuples*

$$\left\{(S, \underline{\mathcal{G}}) \ : \ S \text{ a scheme}, \underline{\mathcal{G}} \in \nabla\mathcal{H}^1 \text{ s.t. } \left(\mathcal{N}_{\leq \underline{b}}\right)^o = S\right\}.$$

This stack is open in $\nabla\mathcal{H}^1$ *and equals the stack*

$$\left\{(S, \underline{\mathcal{G}}) \ : \ S \text{ a scheme}, \underline{\mathcal{G}} \in \nabla\mathcal{H}^1 \text{ s.t. } (3.3)) \text{ holds}\right\}$$

with (3.3) *is the following condition:*
For all $s : \operatorname{Spec} k \to S$ *for a field* k *there exists an* $i \in \{1, \dots, n\}$ *such that the following condition holds:*

$$\left[\left(\hat{\Gamma}_{c_i}(s^*\underline{\mathcal{G}})\right)\right] \npreceq b_i. \tag{3.3}$$

Denote its reduced complement by $\mathcal{N}_{\leq \underline{b}}$. *Then, by [LMB00, Lemma 4.10],* $\mathcal{N}_{\leq \underline{b}}$ *is a (reduced) closed substack of* $\nabla\mathcal{H}^1$. *We call the reduced locally closed substack* $\mathcal{N}_{\underline{b}}$ *the* Newton stratum associated to $\underline{\mathcal{G}}$ and $\underline{b}$.

Remark 3.8. *More properties of the Newton strata are shown in [HV11, § 7] (in the case where* G_c *is a split connected reductive group).*

4 Central Leaves

In this chapter we present our generalization of notion of the *central leaves* introduced by Oort in [Oor04] and further studied by Mantovan in [Man04] for p-divisible groups and by Hartl and Viehmann in [HV12] in the function field case for local G_c-shtukas if G_c is a split connected reductive group over $\mathbb{F}_c$.
We define the central leaf as the subset of a Newton stratum corresponding to a global G-shtuka such that its associated local G_{c_i}-shtukas have constant isomorphism class. It will turn out that the central leaf is a closed substack of the Newton stratum and in particular, locally closed in the stack $\nabla\mathcal{H}^1$ (see Notation 3.2). Furthermore, it is smooth and locally of finite tpe.
Hartl and Viehmann studied the central leaf by using that the isogeny class of a local G_c-shtuka is given by *P-fundamental alcoves* in the case where G_c is a constant split connected reductive group over $\mathbb{F}_c$. For more gerneral G_c it is not clear wether such an element exists. Therefore, we introduce a similar "fundamental alcove" by the axioms given in Definition 4.3. In order to define this we need some further notations which we introduce first. In the second part of the chapter we introduce *completely slope divsible local G_c-shtukas*. Then we consider the central leaf associated to a chosen *fundamental element*.

4.1 Choice of a fundamental element

Consider a complete discrete valuation ring $\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]$ with field of fractions $\mathbb{F}_c^{\mathrm{alg}}(\!(z)\!)$. Let G_c be a parahoric group scheme over $\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]$ with generic fibre $G_{\mathbb{F}_c^{\mathrm{alg}}(\!(z)\!)}$. Then $G_{\mathbb{F}_c^{\mathrm{alg}}(\!(z)\!)}$ is reductive.

Definition 4.1. *Let $L^+G_c =: L_0^+G_c$ be the infinite dimensional affine group scheme over $\mathbb{F}_c$ with $L_0^+G_c(\operatorname{Spec} R) = G_c(R[\![z]\!])$ for any $\mathbb{F}_c$-algebra R (resp. $L_0^+G_c(S) := L^+G_c(S)$ for a scheme S).*
For $m \geq 0$ we define the following subgroup scheme of $L_0^+G_c$:

$$L_m^+G_c(S) = \{g \in L_0^+G_c(S) : \ g \equiv 1 \mod z^m\}\,.$$

We recall the definition of an *Iwahori subgroup* given in [HR08, Definition 1]:

Definition 4.2. *We denote by* $\mathcal{B}(G_c|\mathbb{F}_c^{\mathrm{alg}}((z)))$ *the Bruhat-Tits building of* G_c *over* $\mathbb{F}_c^{\mathrm{alg}}((z))$*. Bruhat and Tits associate to a facet* F *in* $\mathcal{B}(G_c, \mathbb{F}_c^{\mathrm{alg}}((z)))$ *a smooth group scheme* $\mathfrak{G}_F$ *over* $\operatorname{Spec} \mathbb{F}_c^{\mathrm{alg}}[\![z]\!]$ *with generic fibre* G_c*. Denote by* $\mathfrak{G}_F^\circ$ *the open subgroup of it with the same generic fibre and with connected special fibre and define the parahoric subgroup associated to* F *as* $P_F^\circ = \mathfrak{G}_F^\circ(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!])$*. It is equal to the* parahoric subgroup associated to a facet F *is*

$$P_F^\circ = Fix(F) \cap \ker \kappa_{G_c}$$

with κ_{G_c} *the functorial surjective homomorphism defined by Kottwitz in [Kot97] (see formula* (3.1)*). If* F *is an alcove, that is a facet of maximal dimension,* P_F° *is called an* Iwahori subgroup of $G_c(\mathbb{F}_c^{\mathrm{alg}}((z)))$*. This definition is due to Haines and Rapoport in [HR08, Definition 1] and coincides with the one given by Bruhat and Tits (see [HR08, Proposition 3]).*

Let $P = MN$ be a parabolic of $G_{\mathbb{F}_c^{\mathrm{alg}}((z))}$ with Levifactor M and unipotent radical N. Let $\bar{P} = M\bar{N}$ its opposite. From now on, we consider M, N, and $\bar{N}$ as smooth group schemes over $\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]$ (not only over $\mathbb{F}_c^{\mathrm{alg}}((z))$) with

$$M(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]) = G_c(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]) \cap M(\mathbb{F}_c^{\mathrm{alg}}((z))) \tag{4.1}$$
$$N(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]) = G_c(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]) \cap N(\mathbb{F}_c^{\mathrm{alg}}((z))) \tag{4.2}$$
$$\bar{N}(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]) = G_c(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]) \cap \bar{N}(\mathbb{F}_c^{\mathrm{alg}}((z))) \tag{4.3}$$

Definition 4.3. *Consider tuples* (I, S, P, M, x) *such that*

(i) S *is a maximal* $\mathbb{F}_c^{\mathrm{alg}}((z))$*-split torus of* $G_{\mathbb{F}_c^{\mathrm{alg}}((z))}$*,*

(ii) $I \subseteq G_{\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]}$ *an Iwahori subgroup such that the simplices corresponding to the parahoric group* G_c *and* I *lie on an apartment which is given by* $S \subseteq M$*,*

(iii) for a Levisubgroup M *of a parabolic* P *in* $G_{\mathbb{F}_c^{\mathrm{alg}}((z))}$ *with*

(iv) $P = MN$ *(with Levifactor* M*,* N *the unipotent radical) and opposite* $\bar{P} = M\bar{N}$ *and*

(v) $x \in LG_c(\mathbb{F}_c^{\mathrm{alg}})$ *a decent element, such that* $x \in LG_c(\mathbb{F}_x)$ *for a finite extension* $\mathbb{F}_x$ *of* $\mathbb{F}_c$ *(which exists by Remark 2.35(i)) with*

$$\hat{\sigma}^*(x(L^+I(R) \cap L^+M(R))x^{-1}) = L^+I(R) \cap L^+M(R) =: L^+I_M(R),$$
$$\hat{\sigma}^*(x(L^+I(R) \cap L^+N(R))x^{-1}) \subset L^+I(R) \cap L^+N(R) =: L^+I_N(R),$$
$$\hat{\sigma}^*(x(L^+I(R) \cap L^+\bar{N}(R))x^{-1}) \supset L^+I(R) \cap L^+\bar{N}(R) =: L^+I_{\bar{N}}(R),$$

for every $\mathbb{F}_c$-algebra R,

$$\begin{aligned} \hat{\sigma}^*(xL^+I_M x^{-1}) &= L^+I_M \\ \hat{\sigma}^*(xL^+I_N x^{-1}) &\subset L^+I_N \\ \hat{\sigma}^*(xL^+I_{\bar{N}} x^{-1}) &\supset L^+I_{\bar{N}}. \end{aligned}$$

(vi) *Under the assumption that the Iwahori I and G_c lie on an apartment given by $S \subseteq M$ with M a Levisubgroup of $G_{\mathbb{F}_c^{\mathrm{alg}}((z))}$ follows by [MP96], that there exist smooth $\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]$-group schemes I_M, I_N and $I_{\bar{N}}$ with*

$$\begin{aligned} I_M(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]) &= I(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]) \cap M(\mathbb{F}_c^{\mathrm{alg}}((z))) \\ I_N(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]) &= I(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]) \cap N(\mathbb{F}_c^{\mathrm{alg}}((z))) \\ I_{\bar{N}}(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]) &= I(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]) \cap \bar{N}(\mathbb{F}_c^{\mathrm{alg}}((z))). \end{aligned}$$

$I(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]) \cap M$ is an Iwahori subgroup of $M(\mathbb{F}_c^{\mathrm{alg}}((z)))$ and I_M is the Iwahori subgroup scheme corresponding to this Iwahori subgroup of $M(\mathbb{F}_c^{\mathrm{alg}}((z)))$. There is also a construction of the corresponding group schemes I_N and $I_{\bar{N}}$ by [BT84, 4.3].
We further assume (see [HV12, Remark 4.2.e)]) that for any $d \in \mathbb{N}$ there is an $l(d) \in \mathbb{N}$ that for every $l \geq l(d)$ we have

$$\begin{aligned} \hat{\sigma}^{l*}(x)^{-1} \cdot (\ldots \cdot (\hat{\sigma}^*(x)^{-1} L^+ I_{\bar{N}} \hat{\sigma}^*(x)) \cdot \ldots) \cdot \hat{\sigma}^{l*}(x) &\subseteq L_d^+ \hat{\sigma}^{l*} I_{\bar{N}}, \\ x \cdot (\ldots \cdot (\hat{\sigma}^{l*}(x) L^+ \hat{\sigma}^{l*} I_N \hat{\sigma}^{l*}(x)^{-1}) \cdot \ldots) \cdot (x)^{-1} &\subseteq L_d^+ I_N. \end{aligned}$$

(vii) *We assume that $L^+I\hat{\sigma}(xL^+I) \subset [x]$ is closed in the quasi-isogeny class $[x]$ of x in LG_c.*

(viii) *We require that for every $m \in \mathbb{N}$ there exists a smooth $\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]$-group scheme $I_{m,M}$ which are filtration subgroups as defined in [MP96, § 3.2], and such that $L^+I_{m,M} \subset L^+I_M$ is a closed normal subgroup with $L^+I_{m,M} \subset L_m^+M$ and $L^+I_{m+1,M} \subset L^+I_{m,M}$, such that the fppf-sheaf $L^+I_M / L^+I_{m,M}$ on $\mathbb{F}_c^{\mathrm{alg}}$-schemes is representable by a scheme of finite type over $\mathbb{F}_c^{\mathrm{alg}}$ and*

$$x \cdot L^+ I_{m,M} \cdot x^{-1} \;=\; L^+ \hat{\sigma}^{-1} I_{m,M}\,,$$

(ix) *and that for every algebraically closed field k the étale map*

$$\begin{aligned} (L^+\hat{\sigma}^{-1} I_{m,M} / L^+\hat{\sigma}^{-1} I_{m+1,M})(k) &\longrightarrow (L^+\sigma^{-1} I_{m,M} / L^+\sigma^{-1} I_{m+1,M})(k) \\ h &\mapsto hx\,\hat{\sigma}^*(h^{-1})x^{-1} \end{aligned}$$

is surjective.

Notation 4.4. *We denote by $\mathbb{F}_x$ the finite extension field of $\mathbb{F}_c$ over which x is defined.*

Remark 4.5. *(i) We do not require that $\hat{\sigma}^*P = P$ nor that $\hat{\sigma}^*M = M$.*

(ii) We do not require that $\hat{\sigma}^*I = I$.

(iii) Note that in Definition 4.3(vii) we could also require that $(L^+\hat{\sigma}^{-1}I)xL^+I \subset [x]$ is closed because $[x] = [\hat{\sigma}^*(x)]$ (because $x^{-1}x\hat{\sigma}(x) = \hat{\sigma}(x)$).

Proposition 4.6. *Let $F \subset L^+G_c$ be a closed subgroup scheme containing $L^+_dG_c$ for some $d \geq 0$. Then the fppf-quotient sheaf L^+G_c/F is representable by a smooth scheme over $\mathbb{F}_c$ and the morphism $L^+G_c/L^+_dG_c \to L^+G_c/F$ is faithfully flat and of finite presentation. If $F \subset L^+G_c$ is normal, then L^+G_c/F is an affine group scheme.*

Proof. This is explained in [CGP10, p. 470 after Definition A.1.11] in view of the fact that $F/L^+_dG_c \subset L^+G_c/L^+_dG_c \cong \operatorname{Res}_{A_c/\mathfrak{m}_c^d|\mathbb{F}_c}(G_{c,A_c/\mathfrak{m}_c^d})$ is a closed subgroup scheme, for more details see Lemma 5.2. □

Let L be the completion of the maximal unramified field extension of Q_c and recall from Definition 2.33 the Newton cocharacter $\nu_x : D_L \to G_{c,L}$ where D_L is the diagonalizable pro-algebraic group over L with character group $\mathbb{Q}$. Over an algebraic closure L^{alg} of L we can choose a maximal torus T and a Borel subgroup B of G_c. With respect to this Borel subgroup we let $\nu_{x,\mathrm{dom}} : D_{L^{\mathrm{alg}}} \to T_{L^{\mathrm{alg}}}$ be the unique dominant homomorphism, which is conjugate to ν_x in G_c and we let 2ϱ be the sum of all positive roots of T. Then

$$\langle \nu_{x,\mathrm{dom}}\,,\, 2\varrho \rangle \in \mathbb{N}_0 \tag{4.4}$$

is well defined. The axioms in Definition 4.3 imply that x is "$\hat{\sigma}$-straight" in the following sense; compare [He14, § 2.4].

Proposition 4.7. *For all $l \geq 1$ abbreviate $x^{(l)} := \hat{\sigma}^{1-l*}(x) \cdot \ldots \cdot x$. Then $x^{(1)} = x$ and the quotient*

$$H_l \;:=\; L^+I_{\bar{N}} \,/\, \big((x^{(l)})^{-1} \cdot L^+\hat{\sigma}^{-l*}I_{\bar{N}} \cdot x^{(l)}\big)$$

is representable by a smooth scheme over $\mathbb{F}_c^{\mathrm{alg}}$ of relative dimension $l \cdot \langle \nu_{x,\mathrm{dom}}\,,\, 2\varrho \rangle$.

Proof. The smoothness of $H_l \to \operatorname{Spec} \mathbb{F}_c^{\mathrm{alg}}$ follows from Proposition 4.6. Let d_l be its relative dimension. Fix an integer d such that $L^+_d\bar{N} \subset \big((x^{(l)})^{-1} \cdot L^+\hat{\sigma}^{-l*}I_{\bar{N}} \cdot x^{(l)}\big)$. We consider the isomorphism of fppf-sheaves

$$\begin{array}{rcl} L^+I_{\bar{N}}/L^+_d\bar{N} \times_{H_l} H_{l+1} & \xrightarrow{\sim} & L^+I_{\bar{N}}/L^+_d\bar{N} \times_{\mathbb{F}_c^{\mathrm{alg}}} \hat{\sigma}^{-l*}H_1 \\ (h\,,\,\bar{h}_{l+1}) & \mapsto & (h\,,\, x^{(l)} \cdot h^{-1}\bar{h}_{l+1} \cdot (x^{(l)})^{-1}) \\ (h\,,\, h \cdot (x^{(l)})^{-1} \cdot \bar{h}_1 \cdot x^{(l)}) & \leftarrow & (h\,,\bar{h}_1)\,. \end{array}$$

Since $\hat\sigma^{-l*}H_1 \to \operatorname{Spec}\mathbb{F}_c^{\mathrm{alg}}$ is obtained from $H_1 \to \operatorname{Spec}\mathbb{F}_c^{\mathrm{alg}}$ by base change, it is smooth of relative dimension d_1. Since $L^+I_{\bar N}/L_d^+\bar N \to H_l$ is faithfully flat by Proposition 4.6, [Gro67, IV$_4$, Corollaire 17.7.3] implies that $H_{l+1} \to H_l$ is smooth of relative dimension d_1. This implies that $d_l = l \cdot d_1$.

We compute it by using that x is decent for some integer s, that is $\hat\sigma^{(s-1)*}(x^{(s)}) = (s\,\nu_x)(z)$ where $z \in A_c$ is a uniformizing parameter. Let $g \in G_c(L^{\mathrm{alg}})$ be such that $g \cdot (s\,\nu_x)(z) \cdot g^{-1} = (s\,\nu_{x,\mathrm{dom}})(z) \in T(L^{\mathrm{alg}})$. Then for every root α of $G_c(L^{\mathrm{alg}})$, the conjugation with $(s\,\nu_{x,\mathrm{dom}})(z)$ equals multiplication with $z^{\langle s\,\nu_{x,\mathrm{dom}}\,,\,\alpha\rangle}$ on the root subgroup U_α of α. Now Definition 4.3 (v) and (vi) imply that

$$\langle s\,\nu_{x,\mathrm{dom}}\,,\,\alpha\rangle \left\{\begin{matrix} >0 \\ =0 \\ <0 \end{matrix}\right\} \text{ for all roots } \alpha \text{ of } \left\{\begin{matrix} g\cdot \bar N\cdot g^{-1}:, \\ g\cdot M\cdot g^{-1}\,, \\ g\cdot N\cdot g^{-1}\,. \end{matrix}\right.$$

Therefore all roots of $g\cdot\bar N\cdot g^{-1}$ are positive and the remaining positive roots belong to $g\cdot M\cdot g^{-1}$. This implies that $s\cdot d_1 = \dim H_s = \sum_{\alpha\text{ positive}}\langle s\,\nu_{x,\mathrm{dom}}\,,\,\alpha\rangle = s\cdot\langle \nu_{x,\mathrm{dom}}\,,\,2\varrho\rangle$ and the proposition is proved. □

Lemma 4.8. *With the notations from Definition 4.3 there is an Iwahori factorization*

$$\begin{aligned} I(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]) &= I_N(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!])I_M(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!])I_{\bar N}(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]) \\ &= I_{\bar N}(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!])I_M(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!])I_N(\mathbb{F}_c^{\mathrm{alg}}[\![z]\!]) \end{aligned}$$

and

$$\begin{aligned} L^+I(R) &= L^+I_N(R)L^+I_M(R)L^+I_{\bar N}(R) \\ &= L^+I_{\bar N}(R)L^+I_M(R)^+I_N(R) \end{aligned}$$

for every $\mathbb{F}_c$-algebra R.

Proof. This follows from [MP96, Theorem 4.2] under the assumption that I and G_c lie on an apartment given by a maximal $\mathbb{F}_c^{\mathrm{alg}}((z))$-split torus contained in a Levi M. □

Lemma 4.9. *Using the notations and under the assumptions in Definition 4.3, for any $r>0$ there is an Iwahori decomposition for every filtration subgroup I_r of I, that is*

$$\begin{aligned} L^+I_r(R) &= L^+I_{r,N}(R)\cdot L^+I_{r,M}(R)\cdot L^+I_{r,\bar N}(R), \\ L^+I_r(R) &= L^+I_{r,\bar N}(R)\cdot L^+I_{r,M}(R)\cdot L^+I_{r,N}(R). \end{aligned}$$

Proof. This follows directly from [HV12, Lemma 4.6]. Note that in loc. cit. the case that G_c is split, constant, connected, reductive is considered. But the proof holds also in our more general case by our assumptions on the filtration subgroup I_r in Definition 4.3. □

Remark 4.10. *In the case of a split constant reductive group G_c we could set*

$$I_r(S) := \{g \in L_r^+(S) : (g \mod z^{r+1}) \in B(\Gamma(S, \mathcal{O}_S)[\![z]\!]/(z^{r+1}))\}$$

and $I := I_0$ for the Iwahori subgroup. Here $B \supset T$ denote a Borelsubgroup and a maximal split torus of G_c.

From now on we write $x := (I, S, P, M, x)$ for the tuple as defined in Definition 4.3. If G_c is split connected reductive every isogeny class contains a *P-fundamental element* as defined in [HV12]. We recall the definition of the "usual" P-fundamental alcove given in [HV12, Definition 4.1] or [Vie14, Definition 6.1] which was first defined and studied in [GHKR10]:

Definition 4.11. *Let G_c be a split connected reuctive group over $\mathbb{F}_c$. Let P be a semistandard parabolic subgroup of G_c containing a maximal split torus T of G_c. Denote by N its unipotent radical and M a Levi sbgroup. Let $\bar{N}$ be the unipotent radical of the opposite to P. Let $I = I_N I_M I_{\bar{N}}$ be the Iwahori decompostion. An element $x \in \tilde{W}$ of the extended affine Weyl group of G_c is called P-*fundamental *if*

$$\hat{\sigma}(x I_M x^{-1}) = I_M,$$
$$\hat{\sigma}(x I_N x^{-1}) \subset I_N,$$
$$\hat{\sigma}(x^{-1} I_{\bar{N}} x) \subset I_{\bar{N}}.$$

Remark 4.12. *(i) Note that for P-fundamental alcoves defined as in [HV12] or [Vie14] the properties in 4.3 hold if G_c is constant split (see [HV12, Remark 4.2 and Lemma 4.5]).*

(ii) If x is a P-fundamental alcove in the sense of [HV12] by [Vie14, Remark 6.2 and Lemma 6.3] one can enlarge M, P and $\bar{P}$, such that M is the centrailzer of the M-dominant Newton point of x if x is decent.

(iii) In the case of split groups G_c each σ-conjugacy class in $LG_c(k)$ contains a P-fundamental alcove for an algebraically closed field k by [Vie14, Theorem 6.5] (even for unramified groups) or [GHKR10, Corollary 13.2.4]. Further properties of P-fundamental alcoves are given for example in [HV12, Remark 4.2], [GHKR10, § 13] and [Vie14, § 6].

Lemma 4.13. *Let k be an algebraically closed field.*

(i) For every element $g \in (L^+\hat{\sigma}^{-1}I)(k) \cdot x \cdot (L^+I)(k)$ there is an element $h \in (L^+\hat{\sigma}^{-1}I)(k)$ with $h^{-1}g\,\hat{\sigma}^*(h) = x$.

(ii) If $g \in x \cdot (L^+I_{d,M})(k) \cdot (L^+I_{\bar{N}})(k)$, then we can find $h \in (L^+\hat{\sigma}^{-1}I_{d,M})(k) \cdot (L^+\hat{\sigma}^{-1}I_{\bar{N}})(k)$.

Proof. (i) follows from (ii) by the argument of [HV12, Proposition 4.8]. Note that, although loc. cit. assumes that the group G_c is constant split, its argument works in our more general situation with the following modifications (in the notation of loc. cit.):

— $g \in (L^+\hat{\sigma}^{-1*}I_N)xL^+I_N$,

— $h, h_l \in L^+\hat{\sigma}^{-1*}I_N$,

— $r_l \in \hat{\sigma}^*(x) \cdot \ldots \cdot \hat{\sigma}^{l*}(x) \cdot (L^+\hat{\sigma}^{l*}I_N) \cdot \hat{\sigma}^{l*}(x^{-1}) \cdot \ldots \cdot \hat{\sigma}^*(x^{-1}) \subset L^+I_N$,

— $h_l^{-1} \cdot h_{l+1} \in x \cdot \ldots \cdot \hat{\sigma}^{l*}(x) \cdot (L^+\hat{\sigma}^{l*}I_N) \cdot \hat{\sigma}^{l*}(x^{-1}) \cdot \ldots \cdot x^{-1} \subset L^+\hat{\sigma}^{-1*}I_N$.

(ii) If $g = xm\bar{n}$ with $m \in (L^+I_{d,M})(k)$ and $\bar{n} \in (L^+I_{\bar{N}})(k)$, then Definition 4.3(ix) yields an element $h_d \in (L^+\hat{\sigma}^{-1}I_{d,M})(k)$ with $h_d x\,\hat{\sigma}^*(h_d^{-1})x^{-1} = xmx^{-1} \cdot \tilde{m}_{d+1}$ with $\tilde{m}_{d+1} \in (L^+\hat{\sigma}^{-1}I_{d+1,M})(k)$, that is $h_d^{-1}xm\,\hat{\sigma}^*(h_d) = xm_{d+1}$ with

$$m_{d+1} := \hat{\sigma}^*(h_d^{-1})x^{-1}(\tilde{m}_{d+1})^{-1}x\,\hat{\sigma}^*(h_d) \in (L^+I_{d+1,M})(k)$$

by Definition 4.3(viii). We may now iterate this process to obtain $h := h_d \cdot h_{d+1} \cdot \ldots \in (L^+\hat{\sigma}^{-1}I_{d,M})(k)$ with $h^{-1}xm\,\hat{\sigma}^*h = x$. Note that the product converges, because $L^+\hat{\sigma}^{-1}I_{d,M} \subset L_d^+\hat{\sigma}^{-1}M$.

This yields

$$h^{-1}g\,\hat{\sigma}^*(h) = h^{-1}xm\bar{n}\,\hat{\sigma}^*(h) = x \cdot \hat{\sigma}^*(h^{-1})\bar{n}\,\hat{\sigma}^*(h).$$

We set $\bar{n}_0 := \hat{\sigma}^*(h^{-1})\bar{n}\,\hat{\sigma}^*(h) \in L^+I_{\bar{N}}(k)$. Now $\tilde{h}_0 := \hat{\sigma}^{-1}(\bar{n}_0) \in (L^+\hat{\sigma}^{-1}I_{\bar{N}})(k)$ satisfies

$$\tilde{h}_0 x\bar{n}_0\,\hat{\sigma}^*(\tilde{h}_0^{-1}) = x \cdot x^{-1}\tilde{h}_0 x\ .$$

When we set $\bar{n}_1 := x^{-1}\tilde{h}_0 x = x^{-1}\hat{\sigma}^{-1}(\bar{n}_0)x$ then $\tilde{h}_1 := \hat{\sigma}^{-1}(\bar{n}_1) \cdot \tilde{h}_0$ satisfies

$$\tilde{h}_1 x\bar{n}_0\,\hat{\sigma}^*(\tilde{h}_1^{-1}) = x \cdot x^{-1}\hat{\sigma}^{-1}(\bar{n}_1)x\,.$$

Iterating this, we set $\bar{n}_l := x^{-1}\hat{\sigma}^{-1}(\bar{n}_{l-1})x$ and $\tilde{h}_l = \hat{\sigma}^{-1}(\bar{n}_l \cdot \ldots \cdot \bar{n}_0)$ in order to obtain $\tilde{h}_l x\bar{n}_0\,\hat{\sigma}^*(\tilde{h}_l^{-1}) = x\bar{n}_{l+1}$ for every $l \geq 0$. By Definition 4.3(vi) the sequence $\bar{n}_l \in L^+I_{\bar{N}}(k)$ converges to 1 and the sequence $\tilde{h}_l \in (L^+\hat{\sigma}^{-1}I_{\bar{N}})(k)$ converges to an element $\tilde{h} \in (L^+\hat{\sigma}^{-1}I_{\bar{N}})(k)$ with $\tilde{h}x\bar{n}_0\,\hat{\sigma}^*(\tilde{h}^{-1}) = x$, that is $\tilde{h}h^{-1}g\,\hat{\sigma}^*(h\tilde{h}^{-1}) = x$ as desired. □

Remark 4.14. *The proof of Lemma 4.13 also shows that one can find h in $(L^+\hat{\sigma}_i^{-1}I_{\bar{P}})(k)$ if $g \in x \cdot (L^+I_{\bar{P}})(k)$.*

Remark 4.15. *If G_c is constant split reductive Lemma 4.13 holds by [HV12, Remark 4.2.(b)] or by [GHKR10, Proposition 6.3.1] for a P-fundamental x.*

4.2 Completely slope divisible local G_c-shtukas

In this section we define *completely slope divisible local G_c-shtukas.* We follow the definition given in [HV12, Definition 4.7]. We fix an element x with the data given in Definition 4.3. Furthermore, we use the following notation:

$$I_{\bar{P}} := I_M \cdot I_{\bar{N}}.$$

We consider the natural morphism of group schemes $\iota : \hat{\sigma}^{-1} I_{\bar{P}} \to G_c$ and the induced functor ι_* from $L^+\hat{\sigma}^{-1} I_{\bar{P}}$-torsors to L^+G_c-torsors.

Definition 4.16. *A local G_c-shtuka $\underline{\mathcal{G}} = (\mathcal{G}, \hat{\tau}_{\mathcal{G}})$ over an $\mathbb{F}_c^{\text{alg}}$-scheme S is called* completely slope divisible with respect to x *if there exists*

- *an $L^+\hat{\sigma}^{-1} I_{\bar{P}}$-torsor $\overline{\mathcal{P}}$ over S and*
- *an isomorphism of L^+G_c-torsors $\eta : \iota_* \overline{\mathcal{P}} \xrightarrow{\sim} \mathcal{G}$ over S,*

such that there exists an étale covering $S' \to S$ and a trivialization $\alpha : \overline{\mathcal{P}}_{S'} \xrightarrow{\sim} (L^+\hat{\sigma}^{-1} I_{\bar{P}})_{S'}$ satisfying

$$\iota_* \alpha \, \eta^{-1} \hat{\tau}_{\mathcal{G}} \, \hat{\sigma}^*(\eta \, \iota_* \alpha^{-1}) \in x \cdot L^+ I_{\bar{P}}(S').$$

The pair $(\overline{\mathcal{P}}, \eta)$ is called a complete slope division *of $\underline{\mathcal{G}}$ over S.*

The next results clarify the existence of complete slope divisions.

Theorem 4.17. *Let $\underline{\mathcal{G}}$ be a local G_c-shtuka over an $\mathbb{F}_c^{\text{alg}}$-scheme S with constant isogeny class $[x]$. Then the functor Y_x on the category of S-schemes T*

$$\begin{aligned} Y_x(T) \;=\; \Big\{ & \textit{Isomorphism classes of pairs } (\overline{\mathcal{P}}, \eta) \textit{ where } \overline{\mathcal{P}} \textit{ is an } L^+\hat{\sigma}^{-1} I_{\bar{P}}\textit{-torsor} \\ & \textit{over } T \textit{ and } \eta : \iota_* \overline{\mathcal{P}} \xrightarrow{\sim} \mathcal{G}_T \textit{ is an isomorphism of } L^+G_c\textit{-torsors} \\ & \textit{over } T, \textit{ such that there exists an étale covering } T' \to T \\ & \textit{and a trivialization } \alpha : \overline{\mathcal{P}}_{T'} \xrightarrow{\sim} (L^+\hat{\sigma}^{-1} I_{\bar{P}})_{T'} \\ & \textit{satisfying } \iota_* \alpha \, \eta^{-1} \hat{\tau}_{\mathcal{G}} \, \hat{\sigma}^*(\eta \, \iota_* \alpha^{-1}) \in x \cdot L^+ I_{\bar{P}}(T') \Big\} \end{aligned} \tag{4.5}$$

is representable by a scheme which is finite over S. Here $(\overline{\mathcal{P}}, \eta)$ and $(\overline{\mathcal{P}}', \eta')$ are isomorphic, *if there is an isomorphism $\delta : \overline{\mathcal{P}} \xrightarrow{\sim} \overline{\mathcal{P}}'$ of $L^+\hat{\sigma}^{-1} I_{\bar{P}}$-torsors with $\eta = \eta' \circ \iota_* \delta$.*

Remark 4.18. *Note that Y_x is well defined, because the condition is independent of the choice of T' and α. Indeed, if $\widetilde{T}' \to T$ and $\tilde{\alpha} : \overline{\mathcal{P}}_{\widetilde{T}'} \xrightarrow{\sim} (L^+\hat{\sigma}^{-1} I_{\bar{P}})_{\widetilde{T}'}$ is another étale covering, (resp. another trivialization,) then the element $H := \alpha \circ \tilde{\alpha}^{-1} \in L^+\hat{\sigma}^{-1} I_{\bar{P}}(T' \times_T \widetilde{T}')$ satisfies $\iota_* \tilde{\alpha} \, \eta^{-1} \hat{\tau}_{\mathcal{G}} \, \hat{\sigma}^*(\eta \, \iota_* \tilde{\alpha}^{-1}) = H^{-1} \iota_* \alpha \, \eta^{-1} \hat{\tau}_{\mathcal{G}} \, \hat{\sigma}^*(\eta \, \iota_* \alpha^{-1}) \hat{\sigma}^* H \in H^{-1} x \cdot L^+ I_{\bar{P}} \cdot \hat{\sigma}^* H = x(x^{-1} H^{-1} x) \cdot L^+ I_{\bar{P}} \subset x \cdot L^+ I_{\bar{P}}$, by Condition (v) of Definition 4.3.*

Proof. of Theorem 4.17

1. Choose an étale covering $S' \to S$ over which a trivialization

$$\beta : ((L^+G_c)_{S'}, g\hat{\sigma}^*) \xrightarrow{\sim} \underline{\mathcal{G}}_{S'}$$

with $g = \beta^{-1}\hat{\tau}_{\mathcal{G}}\,\hat{\sigma}^*\beta \in LG_c(S')$ exists. Consider the functor Y'_x on the category of S'-schemes T

$$Y'_x(T) \;=\; \left\{ a : T \to L^+G_c/L^+\hat{\sigma}^{-1}I_{\bar{P}} \;\text{ such that } a^{-1}g_{_T}\,\hat{\sigma}^*a \in x \cdot L^+I_{\bar{P}} \right\}. \qquad (4.6)$$

The condition $a^{-1}g_{_T}\,\hat{\sigma}^*a \in x \cdot L^+I_{\bar{P}}$ means that over an étale covering $T' \to T$ the morphism a is represented by an element $a \in L^+G_c(T')$ which satisfies $a^{-1}g_{_{T'}}\,\hat{\sigma}^*a \in x \cdot L^+I_{\bar{P}}(T')$. Then the condition is independent of the covering T' and the choice of the element a, because Condition (v) of Definition 4.3 implies $L^+\hat{\sigma}^{-1}I_{\bar{P}} \cdot x \cdot L^+I_{\bar{P}} = x \cdot L^+I_{\bar{P}}$.

We claim that $(Y_x \times_S S')(T)$ is isomorphic to $Y'_x(T)$. Namely, we may choose an étale covering $T' \to T$ over which a trivialization α as in (4.5), respectively an element $a \in L^+G_c(T')$ exists. Then the isomorphism is defined by sending $(\overline{\mathcal{P}}, \eta)_{_{T'}}$ to $a := \beta^{-1}\,\eta\,\iota_*\alpha^{-1} \in L^+G_c(T')$ and conversely sending $a \in L^+G_c(T')$ to $((L^+\hat{\sigma}^{-1}I_{\bar{P}})_{T'}, \beta a)$. Both assignments descend to T, are well defined and inverse to each other.

2. To prove that Y'_x is representable by a scheme which is projective over S' we use yet another description of Y'_x. We consider the functor on the category of S'-schemes T:

$$\widetilde{Y}'_x(T) = \left\{ f : T \to L^+G_c/L^+\hat{\sigma}^{-1}I \;\text{ such that } f^{-1}g_{_T}\,\hat{\sigma}^*f \in L^+\hat{\sigma}^{-1}I \cdot x \cdot L^+I \right\}. \qquad (4.7)$$

The condition $f^{-1}g_{_T}\,\hat{\sigma}^*f \in L^+\hat{\sigma}^{-1}I \cdot x \cdot L^+I$ means that over an étale covering $T' \to T$ the morphism f is represented by an element $f \in L^+G_c(T')$ which satisfies $f^{-1}g_{_{T'}}\,\hat{\sigma}^*f \in (L^+\hat{\sigma}^{-1}I)(T') \cdot x \cdot L^+I(T')$. Then the condition is independent of the covering T' and the choice of the element f.

The element $f^{-1}g_{_{T'}}\,\hat{\sigma}^*f$ lies in the Newton stratum $\mathcal{N}_x(T')$ by our assumption on the constancy of the isogeny class of $\underline{\mathcal{G}}$ (notice Remark 4.5 (iii)). Therefore, Condition (vii) from Definition 4.3 implies that $\widetilde{Y}'_x \subset (L^+G_c/L^+\hat{\sigma}^{-1}I) \times_{\mathbb{F}_c^{\mathrm{alg}}} S'$ is representable by a closed subscheme. Since I and so also $\hat{\sigma}^{-1}I$ are parahoric, $L\hat{\sigma}^{-1}I/L^+\hat{\sigma}^{-1}I$ is ind-projective by [Ric13a, Corollary 1.3]. Therefore $L^+G_c/L^+\hat{\sigma}^{-1}I$ is a projective scheme over $\mathbb{F}_c^{\mathrm{alg}}$ by [HV11, Lemma 5.4], because $L^+G_c \subset LG_c = L\hat{\sigma}^{-1}I$ is a quasi-compact ind-closed subscheme. It follows that $\widetilde{Y}'_x \to S'$ is projective.

We claim that Y'_x and $\widetilde{Y}'_x$ are isomorphic over S'. First of all there is a canonical morphism $Y'_x \to \widetilde{Y}'_x$ comming from the inclusion $L^+\hat{\sigma}^{-1}I_{\bar{P}} \subset L^+\hat{\sigma}^{-1}I$. To prove that this morphism is an isomorphism, we may choose an étale covering $\widetilde{X}' \to \widetilde{Y}'_x$ over which the universal morphism $f : \widetilde{Y}'_x \to L^+G_c/L^+\hat{\sigma}^{-1}I$ is represented by an element $f \in L^+G_c(\widetilde{X}')$. By faithfully flat descent [Gro67, IV$_2$, Proposition 2.7.1] it suffices to show that $\mathrm{pr}_2 : X' := Y'_x \times_{\widetilde{Y}'_x} \widetilde{X}' \to \widetilde{X}'$ is an isomorphism. We now apply [HV12, Proposition 4.8], which assumes that G_c is constant split, but whose argument goes through in our general case if one makes the following modifications (in the notation of [HV12, Proposition 4.8]): $g \in (L^+\hat{\sigma}^{-1}I_N) \cdot x \cdot L^+I_N$, and $h \in L^+\hat{\sigma}^{-1}I$, as well as $h_l \in L^+\hat{\sigma}^{-1}I_N$ and $r_l \in \hat{\sigma}^*x \cdot \ldots \cdot \hat{\sigma}^{l*}x \cdot (L^+\hat{\sigma}^{l*}I_N) \cdot \hat{\sigma}^{l*}x^{-1} \cdot \ldots \cdot \hat{\sigma}^*x^{-1} \subset L^+I_N$ and $h_{l-1}^{-1} \cdot h_l \in x \cdot \ldots \cdot \hat{\sigma}^{(l-1)*}x \cdot (L^+\hat{\sigma}^{(l-1)*}I_N) \cdot \hat{\sigma}^{(l-1)*}x^{-1} \cdot \ldots \cdot x^{-1} \subset L^+\hat{\sigma}^{-1}I_N$. By this argument we may multiply $f \in L^+G_c(\widetilde{X}')$ on the right with an element of $L^+\hat{\sigma}^{-1}I(\widetilde{X}')$, called h in [HV12, Proposition 4.8], to obtain a new element $f \in L^+G_c(\widetilde{X}')$ satisfying $f^{-1}g_{\widetilde{X}'}\hat{\sigma}^*f =: x \cdot \bar{p}_f \in x \cdot L^+I_{\bar{P}}(\widetilde{X}')$. Then f defines a morphism $\widetilde{X}' \to Y'_x$, and hence a section $s : \widetilde{X}' \to X'$ of pr_2, that is $\mathrm{pr}_2 \circ s = \mathrm{id}_{\widetilde{X}'}$.

To show that $s \circ \mathrm{pr}_2 = \mathrm{id}_{X'}$ we consider a T-valued point $(a, \tilde{x}') \in X'(T)$ with $a \in Y'_x(T)$ and $\tilde{x}' \in \widetilde{X}'(T)$ for a scheme T. Since $\mathrm{pr}_2 \circ (s \circ \mathrm{pr}_2) \circ (a, \tilde{x}') = \mathrm{pr}_2 \circ (a, \tilde{x}')$, it suffices to show that $f_T = \mathrm{pr}_1 \circ (s \circ \mathrm{pr}_2) \circ (a, \tilde{x}') \overset{!}{=} \mathrm{pr}_1 \circ (a, \tilde{x}') = a$ in $Y'_x(T)$. Since f_T and a map to the same element in $\widetilde{Y}'_x(T)$, we can write $a^{-1}f_T = \bar{p}n$ with $\bar{p} \in L^+\hat{\sigma}^{-1}I_{\bar{P}}(T)$ and $n \in L^+\hat{\sigma}^{-1}I_N(T)$ by Lemma 4.8. We define $\bar{p}_a \in L^+I_{\bar{P}}(T)$ by $a^{-1}g_T\hat{\sigma}^*a = x \cdot \bar{p}_a \in x \cdot L^+I_{\bar{P}}(T)$ and consider the equation

$$x^{-1}n^{-1}x(x^{-1}\bar{p}^{-1}x\bar{p}_a\,\hat{\sigma}^*\bar{p})\hat{\sigma}^*n \;=\; x^{-1}f_T^{-1}a\,x\bar{p}_a\,\hat{\sigma}^*a^{-1}\hat{\sigma}^*f_T \;=\; x^{-1}f_T^{-1}g_T\hat{\sigma}^*f_T \;=\; p_f$$

which yields

$$(x^{-1}\bar{p}^{-1}x\bar{p}_a\,\hat{\sigma}^*\bar{p}) \cdot \hat{\sigma}^*n \;=\; x^{-1}nx \cdot p_f \quad \text{in } L^+I(T)\,. \tag{4.8}$$

Observe that $x^{-1}\bar{p}^{-1}x \cdot \bar{p}_a \cdot \hat{\sigma}^*\bar{p} \in L^+I_{\bar{P}}(T)$ by Condition (v) of Definition 4.3. Applying Lemma 4.8 to (4.8) we conclude that $x^{-1}nx \in L^+I_N(T)$, and hence $n \in x \cdot L^+I_N(T) \cdot x^{-1} \subset (L^+\hat{\sigma}^{-1}I_N)(T)$ and $\hat{\sigma}^*n \in \hat{\sigma}^*(x \cdot L^+I_N(T) \cdot x^{-1}) \subset L^+I_N(T)$. Now we successively apply Lemma 4.9 for the group

$$I \cap \hat{\sigma}^*(x \cdot \ldots \cdot \hat{\sigma}^{l*}x \cdot \hat{\sigma}^{l*}I \cdot \hat{\sigma}^{l*}x^{-1} \cdot \ldots \cdot x^{-1}) \;=\; \hat{\sigma}^*(x \cdot \ldots \cdot \hat{\sigma}^{l*}x \cdot \hat{\sigma}^{l*}I_N \cdot \hat{\sigma}^{l*}x^{-1} \cdot \ldots \cdot x^{-1}) \cdot I_{\bar{P}}$$

to (4.8). For $l = 0$ we derive that $x^{-1}nx \in \hat{\sigma}^*(x \cdot L^+I_N \cdot x^{-1})(T)$, and therefore we obtain $n \in x \cdot \ldots \cdot \hat{\sigma}^{(l+1)*}x \cdot (L^+\hat{\sigma}^{(l+1)*}I_N)(T) \cdot \hat{\sigma}^{(l+1)*}x^{-1} \cdot \ldots \cdot x^{-1} \subset (L^+\hat{\sigma}^{-1}I_N)(T)$ for $l = 0$ and successively for all $l \geq 0$. By Condition (vi) of Definition 4.3, this implies $n = 1$, and so $a^{-1}f_T = \bar{p} \in L^+\hat{\sigma}^{-1}I_{\bar{P}}(T)$. This shows that $a = f_T$ in $Y'_x(T)$ and so $X' \to \widetilde{X}'$ and $Y'_x \to \widetilde{Y}'_x$ are isomorphisms.

3. We next prove that the morphism $\widetilde{Y}'_x \to S'$ is quasi-finite, and hence finite, and in particular affine. Let k be an algebraically closed field and let $f \in \widetilde{Y}'_x(k) \subset L^+G_c(k)/(L^+\hat{\sigma}^{-1}I)(k)$. We denote by $s \in S'(k)$ its image in S'. Then $f^{-1}g_s\hat{\sigma}^*f \in (L^+\hat{\sigma}^{-1}I)(k) \cdot x \cdot L^+I(k)$. By Lemma 4.13 there exists an element $h \in (L^+\hat{\sigma}^{-1}I)(k)$ such that $(fh)^{-1}g_s\hat{\sigma}^*(fh) = x$. Note that $fh \in \widetilde{Y}'_x(k)$ defines the same point as f. Now if $\tilde{f} \in \widetilde{Y}'_x(k)$ is any other point mapping to the same $s \in S'$, we represent $\tilde{f}$ by

an element $\tilde{f} \in L^+G_c(k)$ with $\tilde{f}^{-1} g_s \hat{\sigma}^* \tilde{f} = x$. This implies that $(f^{-1}\tilde{f})^{-1} x \hat{\sigma}^*(f^{-1}\tilde{f}) = \tilde{f}^{-1} g_s \hat{\sigma}^* \tilde{f} = x$, and so $i := f^{-1}\tilde{f} \in \mathrm{Aut}_k\big((L^+G_c)_k, x\hat{\sigma}^*\big)$ and $\tilde{f} = fi$. Since x is decent, this automorphism group is contained in $L^+G_c(\mathbb{F}')$ for a finite field extension $\mathbb{F}'$ of $\mathbb{F}_c$ by Remark 2.35. Since $L^+G_c/L^+\hat{\sigma}^{-1}I$ is a projective scheme, $(L^+G_c/L^+\hat{\sigma}^{-1}I)(\mathbb{F}')$ is finite, and so we have proved that there are only finitely many points $\tilde{f} \in \widetilde{Y}'_x(k)$ mapping to s.

4. Since faithfully flat descent for affine schemes is effective by [BLR90, § 6.1, Theorem 6] we conclude that Y_x is representable by an S-scheme which is finite by [Gro67, IV_2, Proposition 2.7.1]. □

4.3 Definition and properties of central leaves

Denote by $\nabla\mathcal{H}^1$ the special fibre of the stack $\nabla_n^{H,\hat{Z}_{\underline{c}}}\mathcal{H}^1(C,G)^{\underline{c}}$ (see Notation 3.2). It is a separated Deligne-Mumford stack locally of finite type over $\mathbb{F}_{\underline{c}}$.
We will introduce the so-called *central leaf* as a reduced closed substack of $\nabla\mathcal{H}^1$. In order to do this we first give the definition of the central leaf in the sense of [HV12] as a scheme. Our notion of central leaves in the moduli space of global G-shtukas is closely related to the definition of central leaves for local G_{c_i}-shtukas intruduced in [HV12, Definition 6.1].
Denote by $\mathbb{F}_x := \mathbb{F}_{x_i}$ a field such that $x := x_i$ as defined in Definition 4.3 exists over $\mathbb{F}_x$.

Definition 4.19. *Let K be a field extension of $\mathbb{F}_{c_i}$ and let $\underline{\hat{\mathbb{G}}}_i$ be a local G_{c_i}-shtuka over K.*

(i) For a K-scheme S and a local G_{c_i}-shtuka $\underline{\hat{\mathcal{G}}}_i$ over S the central leaf *is defined as the subset*

$$\mathcal{C}_{\underline{\hat{\mathbb{G}}}_i,S} := \big\{ s \in S \; : \underline{\hat{\mathbb{G}}}_{i,L} \cong \underline{\hat{\mathcal{G}}}_{i,L} \text{ over an algebraically closed field extension } L/\kappa(s) \big\}.$$

(ii) If $\underline{\mathcal{G}}$ is a global G-shtuka over a K-scheme S, the central leaves *are defined as the subsets*

$$\mathcal{C}_{\underline{\hat{\mathbb{G}}}_i,S} := \big\{ s \in S \; : \underline{\hat{\mathbb{G}}}_{i,L} \cong \hat{\Gamma}_{c_i}(\underline{\mathcal{G}})_L \text{ over an algebraically closed field extension } L/\kappa(s) \big\}$$

and

$$\mathcal{C}_{(\underline{\hat{\mathbb{G}}}_i)_i,S} := \bigcap_i \mathcal{C}_{\underline{\hat{\mathbb{G}}}_i,S}\,.$$

(iii) If $\underline{\hat{\mathbb{G}}}_i = \big((L^+G_{c_i})_K, x_i\hat{\sigma}_i^*\big)$ *and* $\underline{x} = (x_i)_i$*, we also write*

$$\mathcal{C}_{x_i,S} := \mathcal{C}_{\underline{\hat{\mathbb{G}}}_i,S}$$

and

$$\mathcal{C}_{\underline{x},S} := \bigcap_i \mathcal{C}_{x_i,S}\,.$$

The definition is independent of the level stucture of a global G-shtuka because the property for a point of belonging to a central leaf depends only on the local behavior of a global G-shtuka at its characteristic places.

For each i we fix a tuple $(I_i, S_i, P_i, M_i, x_i)$ as defined in Definition 4.3. Denote by $\mathbb{F}_{\underline{x}} \in \mathcal{N}\mathrm{ilp}_{A_{\underline{c}}}$ a field such that each x_i as defined in Definition 4.3 exists over $\mathbb{F}_{\underline{x}}$ and is decent.

In the following, we will show that $\mathcal{C}_x$ is closed in $\mathcal{N}_x$. This is the analogue to [Oor04, Theorem 2.2]. Then we get that the central leaf $\mathcal{C}_{x_i}$ (and $\mathcal{C}_{\underline{x}}$) exists as a locally closed substack of $\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$, which is in particular a Deligne-Mumford stack locally of finite type.

By Theorem 4.17 follows:

Corollary 4.20. *Let* $\underline{\mathcal{G}}$ *be a local* G_c*-shtuka over an* $\mathbb{F}_q^{\mathrm{alg}}$*-scheme* S *with constant isogeny class* $[x]$*. Then the central leaf* $\mathcal{C}_{x,S}$ *is the (scheme theoretic) image of* $Y_x \to S$*. In particular, it is closed in* S*.*

Proof. The image of the morphism $Y_x \to S$ from Theorem 4.17 is closed. By construction, this image equals $\mathcal{C}_{x,S}$. Indeed, if $s \in \mathcal{C}_{x,S}$, then there is an isomorphism $\eta : ((L^+G_c)_{\kappa(s)^{\mathrm{alg}}}, x\hat{\sigma}^*) \xrightarrow{\sim} \underline{\mathcal{G}}_{\kappa(s)^{\mathrm{alg}}}$ and then $((L^+\hat{\sigma}^{-1}I_{\bar{P}})_{\kappa(s)^{\mathrm{alg}}}, \eta)$ with $T' = \operatorname{Spec}\kappa(s)^{\mathrm{alg}}$ and $\alpha = \mathrm{id}$ is a $\kappa(s)^{\mathrm{alg}}$-valued point of Y_x mapping to s.

Conversely, if $s = (\overline{\mathcal{P}}, \eta) \in Y_x(k)$ is a point with values in an algebraically closed field k, we obtain an isomorphism α over k with $g := \iota_*\alpha\eta^{-1}\hat{\tau}_{\mathcal{G}}\,\hat{\sigma}^*(\eta\,\iota_*\alpha^{-1}) \in x \cdot L^+I_{\bar{P}}(k)$. By Lemma 4.13 there exists an element $h \in (L^+\hat{\sigma}^{-1}I)(k)$ such that $h^{-1}g\,\hat{\sigma}^*h = x$. Then $\eta\,\iota_*\alpha^{-1}h : ((L^+G_c)_k, x) \xrightarrow{\sim} \underline{\mathcal{G}}_s$, and hence the image of s in S belongs to $\mathcal{C}_{x,S}$. □

Remark 4.21. *As the scheme theoretic image of* Y_x *in* S*, the central leaf* $\mathcal{C}_{x,S}$ *carries a natural scheme structure. If* $S \to \mathcal{N}_{\underline{x}} \subset \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$ *is an étale presentation of the Newton stratum, we will see in Theorem 4.29 below that* $Y_{\underline{x}}$ *is smooth over* $\mathbb{F}_q^{\mathrm{alg}}$ *and in particular reduced. This shows that the natural scheme structure on* $\mathcal{C}_{\underline{x},S}$ *is reduced.*

Corollary 4.22. *Let* $\underline{\mathcal{G}}$ *be a local* G_c*-shtuka over an* $\mathbb{F}_q^{\mathrm{alg}}$*- scheme* S *with constant isogeny class* $[x]$*. Then there is a finite surjective morphism* $Y \to \mathcal{C}_{x,S}$ *such that* $\underline{\mathcal{G}}_Y$ *is completely slope divisible in the sense of Definition 4.16. If* $G_c = I$ *is an Iwahori group, then we can take* $Y = \mathcal{C}_{x,S}$*.*

Remark 4.23. *If G_c is not an Iwahori group, it can happen that $Y \to \mathcal{C}_{x,S}$ does not have a section, and also that $Y \to \mathcal{C}_{x,S}$ is not étale. See Examples 4.24 and 4.25 below, in which S is the spectrum of a field.*

Proof. of Corollary 4.22
The morphism $Y := Y_x \to \mathcal{C}_{x,S}$ from Theorem 4.17 is finite and surjective by the proof of Corollary 4.20 and $\underline{\mathcal{G}}_Y$ is completely slope divisible by definition. If $G_c = I$, then also $L^+G_c = L^+\hat{\sigma}^{-1}G_c = L^+\hat{\sigma}^{-1}I$, because G_c is defined over $\mathbb{F}_c$. Then $\widetilde{Y}' \to S'$ is a closed immersion in (4.7) and therefore $Y \to \mathcal{C}_{x,S}$ is an isomorphism by Corollary 4.20. □

Example 4.24. *We show that it can happen, that a local G_c-shtuka over a field K becomes completely slope divisible over a field extension of K, but not over K itself. Let $K = \mathbb{F}_c(\lambda)$ where λ is transcendental over $\mathbb{F}_c$. Let $G_c = \mathrm{GL}_{2,\mathbb{F}_c[\![z]\!]}$ and let I be the subgroup of G_c whose elements are upper triangular modulo z. Let $S = \operatorname{Spec} K$ and $\underline{\mathcal{G}} = \big((L^+G_c)_K, g\hat{\sigma}^*\big)$ for $g = \left(\begin{smallmatrix} \lambda & \lambda \\ 1 & \lambda \end{smallmatrix}\right)$. Then $x = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $M = G_c$. Indeed, by Lang's theorem [Lan56, Corollary on p. 557] there is a matrix $f \in \mathrm{GL}_2(K^{\mathrm{alg}})$ with $g\,\hat{\sigma}^* f = f$, that is with $f^{-1}g\,\hat{\sigma}^* f = x$. This matrix f defines a K^{alg}-valued point of $\widetilde{Y}'_x$ from (4.7). And so $\mathcal{C}_{x,S} = S$ by the proof of Theorem 4.17.*

On the other hand, we claim that there is no K-valued point f of Y_x. Note that here we may take $S' = S$ and $Y_x = \widetilde{Y}'_x$ in the proof of Theorem 4.17. Writing $f = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we want $f^{-1}g\,\hat{\sigma}^* f \in L^+I$, that is $a^{\hat{q}+1} - \lambda a^{\hat{q}}c + \lambda a c^{\hat{q}} - \lambda c^{\hat{q}+1} = 0$. We must have $c \neq 0$, because if $c = 0$, then $a = 0$ and f is not invertible. By Eisenstein's irreducibility criterion, a/c does not lie in K, but generates a field extension of K of degree $\hat{q}+1$. So the point $[a : c] \in \mathbb{P}^1_{\mathbb{F}_c} = L^+G_c/L^+I$ is not K-rational. This proves the claim.

Example 4.25. *We show that $Y_x \to \mathcal{C}_{x,S}$ does not need to be smooth. Let $G_c = \mathrm{GL}_{2,\mathbb{F}_c[\![z]\!]}$ and let I be the subgroup of G_c whose elements are upper triangular modulo z. Let $x = \left(\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right) \in LG_c(\mathbb{F}_c)$. Then $I_{\bar{P}} = \big\{\left(\begin{smallmatrix} \alpha & 0 \\ \gamma & \delta \end{smallmatrix}\right) : z|\gamma\big\}$. In the fibre above $g = x \in \mathcal{C}_{x,S}(\mathbb{F}_c)$ the morphism $Y_x \to \mathcal{C}_{x,S}$ is neither (formally) smooth nor (formally) unramified at the point corresponding to $1 \in Y'_x(\mathbb{F}_c)$. Indeed, let $B = \mathbb{F}_c[\varepsilon]/(\varepsilon^{\hat{q}+1})$ and $\overline{B} := B/(\varepsilon^{\hat{q}})$. Then the point $\bar{a} = 1 \in Y_x(\overline{B})$ has two different lifts $a = \left(\begin{smallmatrix} 1 & 0 \\ \gamma & 1 \end{smallmatrix}\right)$ in $Y_x(B)$ with $\gamma = 0$ or $\gamma = \varepsilon^{\hat{q}}$, because both satisfy $\hat{\sigma}^* a = 1$ and $x^{-1}a^{-1}g\,\hat{\sigma}^* a = \left(\begin{smallmatrix} 1 & 0 \\ \gamma z & 1 \end{smallmatrix}\right) \in L^+I_{\bar{P}}(B)$. So $Y_x \to \mathcal{C}_{x,S}$ is not unramified at 1.*

Also, the point $\bar{a} = \left(\begin{smallmatrix} 1 & 0 \\ \varepsilon & 1 \end{smallmatrix}\right)$ lies in $Y'_x(\overline{B})$, because $\hat{\sigma}^*\bar{a} = 1$ and $x^{-1}\bar{a}^{-1}g\,\hat{\sigma}^*\bar{a} = \left(\begin{smallmatrix} 1 & 0 \\ \varepsilon z & 1 \end{smallmatrix}\right) \in L^+I_{\bar{P}}(\overline{B})$. It has no lift $a \in Y'_x(B)$, because any such lift would be of the form $a = \left(\begin{smallmatrix} 1 & 0 \\ \varepsilon & 1 \end{smallmatrix}\right)\cdot(1+\varepsilon^{\hat{q}}\tilde{a}) \in Y'_x(B)$ with $\tilde{a} \in \mathbb{F}_c[\![z]\!]^{2\times 2}$, but then $\hat{\sigma}^* a = \left(\begin{smallmatrix} 1 & 0 \\ \varepsilon^{\hat{q}} & 1 \end{smallmatrix}\right)$ and $x^{-1}a^{-1}g\,\hat{\sigma}^* a = x^{-1}(1-\varepsilon^{\hat{q}}\tilde{a})\cdot\left(\begin{smallmatrix} z & 0 \\ \varepsilon^{\hat{q}}-\varepsilon z & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ \varepsilon^{\hat{q}}-\varepsilon z & 1 \end{smallmatrix}\right) - \varepsilon^{\hat{q}}\left(\begin{smallmatrix} z^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\cdot\tilde{a}\cdot\left(\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right) \notin x\cdot L^+I_{\bar{P}}(B)$. So $Y_x \to \mathcal{C}_{x,S}$ is not smooth at 1.

Now we are going to give the notion of the *central leaf* in the Newton stratum of $\nabla\mathcal{H}^1$. Let x_i be as defined in Definition 4.3. The schemes Y_{x_i} from Theorem 4.17 above define Deligne-Mumford stacks $\mathcal{Y}_{x_i}$ for all $i = 1, \ldots, n$:

Proposition 4.26. *Theorem 4.17 gives rise to separated, algebraic Deligne-Mumford stacks of finite type*

$$\mathcal{Y}_{x_i} \longrightarrow \mathcal{C}_{x_i} \subset (\mathcal{N}_{x_i} \times_{\mathbb{F}_{c_i}} \mathbb{F}_q^{\mathrm{alg}}) \subset \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}} \tag{4.9}$$

and

$$\mathcal{Y}_{\underline{x}} \coloneqq (\mathcal{Y}_{x_1} \times_{\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} \ldots \times_{\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} \mathcal{Y}_{x_n}) \longrightarrow \mathcal{C}_{\underline{x}} \subset (\mathcal{N}_{\underline{x}} \times_{\mathbb{F}_{\underline{c}}} \mathbb{F}_q^{\mathrm{alg}}) \subset \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}} \tag{4.10}$$

which are defined with respect to $\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}^{\mathrm{univ}})$. The stacks $\mathcal{C}_{x_i} \subset \mathcal{N}_{x_i}$ and $\mathcal{C}_{\underline{x}} \subset \mathcal{N}_{\underline{x}}$ are closed substacks and are the central leaves *in the sense that for every $\mathbb{F}_q^{\mathrm{alg}}$-scheme S and every morphism $S \to \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$ the fibre product $S \times_{\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} \mathcal{C}_{\underline{x}}$ equals the central leaf $\mathcal{C}_{\underline{x},S}$ in S. The morphisms $\mathcal{Y}_{x_i} \to \mathcal{C}_{x_i}$ and $\mathcal{Y}_{\underline{x}} \to \mathcal{C}_{\underline{x}}$ are finite.*

Proof. Let $S_i \to \mathcal{N}_{x_i} \times_{\mathbb{F}_{\underline{c}}} \mathbb{F}_q^{\mathrm{alg}}$, be an étale presentation and let $\underline{\mathcal{G}}^{\mathrm{univ}}$ be the universal global G-shtuka over S_i. We observe that $\mathcal{N}_{\underline{x}} = \big(\mathcal{N}_{x_1} \times_{\nabla\mathcal{H}^1} \ldots \times_{\nabla\mathcal{H}^1} \mathcal{N}_{x_n}\big)_{\mathrm{red}}$. Therefore, the scheme $S \coloneqq \big(S_1 \times_{\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} \ldots \times_{\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} S_n\big)_{\mathrm{red}}$ is an étale presentation of $\mathcal{N}_{\underline{x}} \times_{\mathbb{F}_{\underline{c}}} \mathbb{F}_q^{\mathrm{alg}}$. Let $Y_{x_i} \to S_i$ be the scheme from Theorem 4.17 with respect to the local G_{c_i}-shtuka $\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}^{\mathrm{univ}})$ over S_i and let $Y_{\underline{x}} \coloneqq (Y_{x_1} \times_{S_1} S) \times_S \ldots \times_S (Y_{x_n} \times_{S_n} S)$. Let $\mathcal{C}_{x_i,S_i} \subset S_i$, respectively $\mathcal{C}_{\underline{x},S} \subset S$, be the scheme theoretic images of $Y_{x_i} \to S_i$, respectively of $Y_{\underline{x}} \to S$. Then Y_{x_i} and $Y_{\underline{x}}$, and hence also $\mathcal{C}_{x_i,S_i}$ and $\mathcal{C}_{\underline{x},S}$, are defined as moduli problems. Therefore they descend to stacks $\mathcal{Y}_i \to \mathcal{C}_{x_i} \subset \mathcal{N}_{x_i}$ and $\mathcal{Y}_{\underline{x}} \to \mathcal{C}_{\underline{x}} \subset \mathcal{N}_{\underline{x}}$. The properties of these stacks follow from Theorem 4.17 and Corollary 4.20. □

We will see in Theorem 4.29 below that $\mathcal{Y}_{\underline{x}} \to \operatorname{Spec}\mathbb{F}_q^{\mathrm{alg}}$ is a smooth stack, and we will compute its dimension. As a motivation for this result, we want to give an interpretation of $\mathcal{Y}_{x_i}$ and $\mathcal{Y}_{\underline{x}}$ as the central leaves in a different moduli space $\nabla_n^H\mathcal{H}^1(C, I)$ in Corollary 4.28 below. For this purpose we begin with the following theorem.

Theorem 4.27. *Let $\rho : G \to G'$ be a morphism of parahoric group schemes over C, which is an isomorphism over $C \smallsetminus \underline{c}$. Let $H \subset G(\mathbb{A}^{\underline{c}}) = G'(\mathbb{A}^{\underline{c}})$ be a compact open subgroup. Then the induced morphism*

$$\begin{array}{rccc} \rho_* : & \nabla_n^H\mathcal{H}^1(C, G)^{\underline{c}} & \longrightarrow & \nabla_n^H\mathcal{H}^1(C, G')^{\underline{c}} \\ & (\mathcal{G}, s_1, \ldots, s_n, \tau_{\mathcal{G}}, H\gamma) & \longmapsto & (\rho_*\mathcal{G}, s_1, \ldots, s_n, \rho_*\tau_{\mathcal{G}}, H\gamma) \end{array} \tag{4.11}$$

is relatively representable by a projective morphism. More precisely, it is relatively representable by the morphism

$$\big((L^+G'_{c_1}/L^+G_{c_1}) \times_{\mathbb{F}_{c_1}} \mathbb{F}_{\underline{c}}\big) \times_{\mathbb{F}_{\underline{c}}} \ldots \times_{\mathbb{F}_{\underline{c}}} \big((L^+G'_{c_n}/L^+G_{c_n}) \times_{\mathbb{F}_{c_n}} \mathbb{F}_{\underline{c}}\big) \longrightarrow \operatorname{Spec}\mathbb{F}_{\underline{c}}\,.$$

Proof. Let $S \to \nabla_n^H\mathcal{H}^1(C,G')^{\underline{c}}$ be a morphism corresponding to a global G'-shtuka with H-level structure $(\mathcal{G}', \tau_{\mathcal{G}'}, H\gamma')$ over S. Let $T = \operatorname{Spec} R$ be an affine scheme. Then the category of T-valued points of the fibre product

$$S' := \nabla_n^H\mathcal{H}^1(C,G)^{\underline{c}} \times_{\nabla_n^H\mathcal{H}^1(C,G')^{\underline{c}}} S$$

is the category of tuples $\big((\mathcal{G},\tau_{\mathcal{G}},H\gamma),f,\alpha\big)$ where $(\mathcal{G},\tau_{\mathcal{G}},H\gamma) \in \nabla_n^H\mathcal{H}^1(C,G)^{\underline{c}}(T)$, where $f : T \to S$ is a morphism, and $\alpha : \rho_*(\mathcal{G},\tau_{\mathcal{G}},H\gamma) \xrightarrow{\sim} f^*(\mathcal{G}',\tau_{\mathcal{G}'},H\gamma')$ is an isomorphism in $\nabla_n^H\mathcal{H}^1(C,G')^{\underline{c}}(T)$, that is, a quasi-isogeny $\alpha : \rho_*(\mathcal{G},\tau_{\mathcal{G}}) \to f^*(\mathcal{G}',\tau_{\mathcal{G}'})$ which is an isomorphism at $\underline{c}$ and compatible with the H-level structures.

The G-structure on $\rho_*\mathcal{G}$ over T is equivalent to a C_T-morphism

$$C_T \to \rho_*\mathcal{G}/(G\times_C C_T)\,;$$

see for example [AH14b, Proof of Proposition 3.9]. Since $G = G'$ over $C \smallsetminus \underline{c}$, the restriction of $\rho_*\mathcal{G}/(G\times_C C_T)$ to $(C\smallsetminus\underline{c})_T$ is canonically isomorphic to $(C\smallsetminus\underline{c})_T$. Note that $G(C_T) \subset G'(C_T)$ via ρ as one sees by restricting to $C\smallsetminus\underline{c}$. Both contain $G'_d(C_T) := \{g \in G'(C_T) : g \equiv 1 \mod D\}$ as a normal subgroup for some divisor $D = d\cdot[\underline{c}]$ with $d \in \mathbb{Z}$, and $(G'/G'_d)(C_T) = G'(D_T)$. Therefore $(G'/G)(C_T) = (G'/G)(D_T)$ and $\big(\rho_*\mathcal{G}/(G\times_C C_T)\big)(C_T) = \big(\rho_*\mathcal{G}/(G\times_C C_T)\big)(D_T)$. Since α is an isomorphism over a neighborhood of $\underline{c}$ we conclude that the G-structure on $\rho_*\mathcal{G}$ over T corresponds to a D_S-morphism $a : D_T \to (\mathcal{G}'\times_{C_S} D_S)/(G\times_C D_S)$. By [AH14a, Remark 5.2 and Lemma 5.3], the space $D_T = \coprod_{i=1}^n \operatorname{Spec}(\mathbb{F}_{c_i}[\![z_{c_i}]\!]/(z_{c_i}^d)\otimes_{\mathbb{F}_q} R)$ decomposes into the disjoint union of $\operatorname{Spec}(\mathbb{F}_{c_i}[\![z_{c_i}]\!]/(z_{c_i}^d)\otimes_{\mathbb{F}_q} R)/\mathfrak{a}_j$ for $\mathfrak{a}_j = (\lambda\otimes 1 - 1\otimes(s_i^*\lambda)^{q^j} : \lambda \in \mathbb{F}_{c_i})$ and $j \in \mathbb{Z}/[\mathbb{F}_{c_i} : \mathbb{F}_q]\mathbb{Z}$. The $\mathfrak{a}_j$ are cyclically permuted by σ and the graph Γ_{s_i} is contained in the component for $j = 0$. The compatibility $\alpha\circ\rho_*\tau_{\mathcal{G}} = f^*\tau_{\mathcal{G}'}\circ\sigma^*\alpha$ outside Γ_{s_i} thus implies that the value of a on the component for $j = 0$ uniquely determines a on all the components for $j \neq 0$. On the former component it corresponds to an $\mathbb{F}_{c_i}$-morphism $a : T \to \hat{\Gamma}_{c_i}(\mathcal{G}')/L^+G_{c_i}$. After trivializing $\hat{\Gamma}_{c_i}(\mathcal{G}')$ on an étale covering of S, we see that the fibre product S' has the form claimed in the theorem. Since $L^+G_{c_i}$ is parahoric, $LG_{c_i}/L^+G_{c_i}$ is ind-projective by [Ric13a, Corollary 1.3]. Since $L^+G'_{c_i} \subset LG_{c_i}$ is a quasi-compact ind-closed subscheme, $L^+G'_{c_i}/L^+G_{c_i}$ is a projective scheme over $\mathbb{F}_{c_i}$ by [HV11, Lemma 5.4]. This proves the theorem. □

Corollary 4.28. *If $\hat{\sigma}_i^* I_i = I_i$ for one i, let I be the parahoric group scheme over C which coincides with G outside c_i and with I_i at c_i. Then $\mathcal{Y}_{x_i}$ equals the central leaf $\mathcal{C}_{I,x_i}$ in $\nabla_n^H\mathcal{H}^1(C,I)^{\underline{c}}\times_{\mathbb{F}_{c_i}}\mathbb{F}_q^{\mathrm{alg}}$.*

If $\hat{\sigma}_i^* I_i = I_i$ for all i, let I be the parahoric group scheme over C which coincides with G outsice $\underline{c}$ and with I_i at every c_i. Then $\mathcal{Y}_{\underline{x}}$ equals the central leaf $\mathcal{C}_{I,\underline{x}}$ in $\nabla_n^H\mathcal{H}^1(C,I)^{\underline{c}}\times_{\mathbb{F}_{\underline{c}}}\mathbb{F}_q^{\mathrm{alg}}$.

Proof. The group scheme I comes with a morphism $\rho : I \to G$. The projective morphism ρ_* from (4.11) induces morphisms of Newton strata $(\mathcal{N}_I)_{x_i} \to (\mathcal{N}_G)_{x_i}$ and $(\mathcal{N}_I)_{\underline{x}} \to (\mathcal{N}_G)_{\underline{x}}$ in the two moduli stacks. We choose an étale presentation $S \to (\mathcal{N}_G)_{x_i}$, respectively $S \to (\mathcal{N}_G)_{\underline{x}}$ over which $\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}^{\mathrm{univ}})$ can be trivialized. Then S is reduced and $(\mathcal{N}_I)_{x_i} \times_{(\mathcal{N}_G)_{x_i}} S = (L^+G_{c_i}/L^+I_{c_i}) \times_{\mathbb{F}_{c_i}} S$, because $L^+G_{c_i}/L^+I_{c_i}$ is smooth over $\mathbb{F}_{c_i}$; see Proposition 4.6. The description of $\mathcal{Y}_{G,x_i}$ in terms of $\widetilde{Y}'_{G,x_i}$ in (4.7) shows that $\widetilde{Y}'_{I,x_i} = \widetilde{Y}'_{G,x_i}$, and hence $\mathcal{Y}_{G,x_i} = \mathcal{Y}_{I,x_i} = \mathcal{C}_{I,x_i}$. □

We will now prove the smoothness of $\mathcal{Y}_{\underline{x}}$ and compute its dimension without using Theorem 4.27 and Corollary 4.28. Our proof is an adaption of [HV11, Theorem 5.6], which is inspired by Corollary 4.28. For every i, let L_i be the completion of the maximal unramified field extension of Q_{c_i} and recall from Definition 2.33 the Newton cocharacter $\nu_{x_i} : D_{L_i} \to G_{c_i,L_i}$ where D_{L_i} is the diagonalizable pro-algebraic group over L_i with character group $\mathbb{Q}$. Over an algebraic closure L_i^{alg} of L_i we can choose a maximal torus T and a Borel subgroup B of G_{c_i}. With respect to this Borel subgroup we let $\nu_{x_i,\mathrm{dom}} : D_{L_i^{\mathrm{alg}}} \to T_{L_i^{\mathrm{alg}}}$ be the unique dominant homomorphism, which is conjugate to ν_{x_i} in G_{c_i} and we let $2\varrho_i$ be the sum of all positive roots of T. Then

$$\langle \nu_{x_i,\mathrm{dom}}\,,\,2\varrho_i\rangle \in \mathbb{N}_0 \tag{4.12}$$

is well defined.

Theorem 4.29. *The algebraic stack $\mathcal{Y}_{\underline{x}}$ obtained in Proposition 4.26 is smooth over* $\operatorname{Spec}\mathbb{F}_q^{\mathrm{alg}}$ *of relative dimension* $\sum_{i=1}^n \langle \nu_{x_i,\mathrm{dom}}\,,\,2\varrho_i\rangle$.

Note that the dimension formula is independent of the parahoric model of G_Q.

Proof of Theorem 4.29. Our proof proceeds by computing the complete local ring $\widehat{\mathcal{O}}_y := \widehat{\mathcal{O}}_{\mathcal{Y}_{\underline{x}} \times_{\mathbb{F}_q^{\mathrm{alg}}} K, y}$ of $\mathcal{Y}_{\underline{x}}$ at a point $y \in \mathcal{Y}_{\underline{x}}(K)$ for an algebraically closed field K. Note that if $S \to \mathcal{Y}_{\underline{x}}$ is an étale presentation and $s \in S(K)$ maps to y, then this local ring is canonically isomorphic to the completion $\widehat{\mathcal{O}}_{S \times_{\mathbb{F}_q^{\mathrm{alg}}} K, s}$ of the local ring at s. By Propositions 4.30 and 4.7, all the complete local rings $\widehat{\mathcal{O}}_y$ are regular of dimension $\sum_{i=1}^n \langle \nu_{x_i,\mathrm{dom}}\,,\,2\varrho_i\rangle$. Since $\mathbb{F}_q^{\mathrm{alg}}$ is perfect, regularity is the same as smoothness, and the theorem follows. □

It remains to give the proof of the following proposition.

Proposition 4.30. *The complete local ring $\widehat{\mathcal{O}}_y$ (see the proof of Theorem 4.29) is isomorphic to the complete tensor product $\mathcal{D} := \mathcal{D}_1 \widehat{\otimes}_K \ldots \widehat{\otimes}_K \mathcal{D}_n$ of the completions $\mathcal{D}_i$ of the local ring of* $(L^+I_{\bar{N}_i}/x_i^{-1} \cdot L^+\hat{\sigma}_i^{-1} I_{\bar{N}_i} \cdot x_i) \times_{\mathbb{F}_q^{\mathrm{alg}}} K$ *at the element* 1.

Proof. 1. We first define a K-homomorphism $v : \widehat{\mathcal{O}}_y \to \mathcal{D}$. Let $\mathfrak{m} \subset \mathcal{D}$ be the maximal ideal. The point y maps to a point in $\nabla\mathcal{H}^1(K)$, which corresponds to a global G-shtuka with H-level structure $(\underline{\mathcal{G}}_0, H\gamma_0)$ over K. The point $y \in \mathcal{Y}_{\underline{x}}(K)$ corresponds to complete slope divisions $(\overline{\mathcal{P}}_{i,0}, \eta_{i,0})$ of $\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}_0)$ over K. Since K is algebraically closed, there are trivializations $\beta_{i,0} : \big((L^+G_{c_i})_K, g_{i,0}\hat{\sigma}_i^*\big) \xrightarrow{\sim} \hat{\Gamma}_{c_i}(\underline{\mathcal{G}}_0)$, and then y defines an element $\bar{a}_{i,0} \in Y'_{x_i}(K)$ for all i. Let $a_{i,0} \in L^+G_{c_i}(K)$ be a representative of $\bar{a}_{i,0}$. By Lemma 4.13 we may replace this representative without changing $\bar{a}_{i,0}$ such that $a_{i,0}^{-1} \cdot g_{i,0} \cdot \hat{\sigma}_i^*(a_{i,0}) = x_i$. We choose a lift $a'_i \in L^+G_{c_i}(\mathcal{D})$ of $a_{i,0}$. Since $\mathcal{D}$ is a K-algebra, we may take for example $a'_i = a_{i,0}$. Since $\mathcal{D}$ has no non-trivial étale coverings, we may choose representatives $\bar{n}'_i \in L^+I_{\bar{N}_i}(\mathcal{D})$ with $\bar{n}'_i \equiv 1 \mod \mathfrak{m}$ of the structure morphisms $\operatorname{Spec}\mathcal{D}_i \to L^+I_{\bar{N}_i}/x_i^{-1} \cdot L^+\hat{\sigma}_i^{-1}I_{\bar{N}_i} \cdot x_i$. We consider the local G_{c_i}-shtukas $\hat{\underline{\mathcal{G}}}'_i := \big((L^+G_{c_i})_{\mathcal{D}}, g'_i\hat{\sigma}_i^*\big)$ over $\mathcal{D}$ where $g'_i := a'_i \cdot x_i \cdot \bar{n}_i'^{-1} \cdot \hat{\sigma}_i^*(a'_i)^{-1} \in LG_{c_i}(\mathcal{D})$. Since $a'_i \equiv a_{i,0} \mod \mathfrak{m}$, and hence $g'_i \equiv g_{i,0} \mod \mathfrak{m}$, the local shtuka $\hat{\underline{\mathcal{G}}}'_i$ is a deformation of $\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}_0)$. By the Serre-Tate theorem 2.24 these deformations induce a global G-shtuka $\underline{\mathcal{G}}'$ over $\mathcal{D}$, which is a deformation of $\underline{\mathcal{G}}_0$ with isomorphism $\beta'_i : \hat{\underline{\mathcal{G}}}'_i \xrightarrow{\sim} \hat{\Gamma}_{c_i}(\underline{\mathcal{G}}')$. With the H-level structure $\gamma' := \gamma_0$, the pair $(\underline{\mathcal{G}}', H\gamma')$ lies in $\nabla\mathcal{H}^1(\mathcal{D})$. Moreover, a'_i induces a point in $Y'_{x_i}(\mathcal{D})$, because $a_i'^{-1}g'_i\,\hat{\sigma}_i^*(a'_i) = x_i\,\bar{n}_i'^{-1}$. This yields the point $(\overline{\mathcal{P}}'_i, \eta'_i) := \big((L^+\hat{\sigma}^{-1}I_{\bar{P}_i})_{\mathcal{D}}, \beta'_i a'_i\big) \in \mathcal{Y}_{x_i}(\mathcal{D})$. Alltogether we obtain a morphism $\operatorname{Spec}\mathcal{D} \to \mathcal{Y}_{\underline{x}} \times_{\mathbb{F}_q^{\mathrm{alg}}} K$ such that the maximal ideal $\mathfrak{m}$ maps to our initial point y. Since $\mathcal{D}$ is a complete local ring, this morphism factors through a K-homomorphism $v : \widehat{\mathcal{O}}_y \to \mathcal{D}$. In particular, the maximal ideal $\mathfrak{m}_y \subset \widehat{\mathcal{O}}_y$ equals $v^{-1}(\mathfrak{m})$.

2. To prove that $v : \widehat{\mathcal{O}}_y \to \mathcal{D}$ is an isomorphism, we consider local Artinian K-algebras A with residue field K and points y_A in $\mathcal{Y}_{\underline{x}}(A)$ which are deformations of $y \in \mathcal{Y}_{\underline{x}}(K)$. The image of y_A in $\nabla\mathcal{H}^1(A)$ corresponds to a global G-shtuka with H-level structure $(\underline{\mathcal{G}}, H\gamma)$ over A which lifts $(\underline{\mathcal{G}}_0, H\gamma_0)$. The point $y_A \in \mathcal{Y}_{\underline{x}}(A)$ corresponds to complete slope divisions $(\overline{\mathcal{P}}_i, \eta_i)$ of $\hat{\Gamma}_{c_i}(\underline{\mathcal{G}})$ over A lifting $(\overline{\mathcal{P}}_{i,0}, \eta_{i,0})$ for every i. We make the following

Claim. For every local Artinian K-algebra A with residue field K and for every deformation $y_A = (\underline{\mathcal{G}}, H\gamma, \overline{\mathcal{P}}_i, \eta_i) \in \mathcal{Y}_{\underline{x}}(A)$ of $y \in \mathcal{Y}_{\underline{x}}(K)$, there is a uniquely determined K-homomorphism $u : \mathcal{D} \to A$ and an equality $(\underline{\mathcal{G}}, H\gamma, \overline{\mathcal{P}}_i, \eta_i) = u^*(\underline{\mathcal{G}}', H\gamma', \overline{\mathcal{P}}'_i, \eta'_i)$ in $\mathcal{Y}_{\underline{x}}(A)$.

3. The claim implies that $v : \widehat{\mathcal{O}}_y \to \mathcal{D}$ is an isomorphism as follows. It suffices to prove for all l that $v_l := v \mod \mathfrak{m}^{\hat{q}^l} : \widehat{\mathcal{O}}_y/\mathfrak{m}_y^{\hat{q}^l} \to \mathcal{D}/\mathfrak{m}^{\hat{q}^l}$ is an isomorphism. We first take $A = \widehat{\mathcal{O}}_y/\mathfrak{m}_y^{\hat{q}^l}$ and the pull-back $(\underline{\mathcal{G}}, H\gamma, \overline{\mathcal{P}}_i, \eta_i)$ under $\operatorname{Spec} A \to \mathcal{Y}_{\underline{x}}$ of the universal tupel over $\mathcal{Y}_{\underline{x}}$. Then the claim provides a K-homomorphism $u : \mathcal{D} \to \widehat{\mathcal{O}}_y/\mathfrak{m}_y^{\hat{q}^l}$ which automatically satisfies $u^{-1}(\mathfrak{m}_y) = \mathfrak{m}$. It induces the K-homomorphism $u_l := u \mod \mathfrak{m}^{\hat{q}^l} : \mathcal{D}/\mathfrak{m}^{\hat{q}^l} \to \widehat{\mathcal{O}}_y/\mathfrak{m}_y^{\hat{q}^l}$. It suffices to show that u_l is inverse to v_l. By definition of the morphism v, we have $v_l^*(\underline{\mathcal{G}}, H\gamma, \overline{\mathcal{P}}_i, \eta_i) = (\underline{\mathcal{G}}', H\gamma', \overline{\mathcal{P}}'_i, \eta'_i)$ in $\mathcal{Y}_{\underline{x}}(\mathcal{D}/\mathfrak{m}^{\hat{q}^l})$, and so

$u_l^* v_l^*(\underline{\mathcal{G}}, H\gamma, \overline{\mathcal{P}}_i, \eta_i) = (\underline{\mathcal{G}}, H\gamma, \overline{\mathcal{P}}_i, \eta_i)$ in $\mathcal{Y}_{\underline{x}}(A)$. By the universal property of $\mathcal{Y}_{\underline{x}}$, the compositions of $\operatorname{Spec} A \to \mathcal{Y}_{\underline{x}}$ with either $\operatorname{Spec}(\mathrm{id}_A)$ or $\operatorname{Spec}(v_l u_l)$ coincide. Since $\widehat{\mathcal{O}}_y \to A$ is surjective, we conclude $v_l u_l = \mathrm{id}_A$.

To prove that $u_l v_l = \mathrm{id}_{\mathcal{D}/\mathfrak{m}^{\hat{q}^l}}$ we take $A' = \mathcal{D}/\mathfrak{m}^{\hat{q}^l}$ and the tupel $(\underline{\mathcal{G}}', H\gamma', \overline{\mathcal{P}}'_i, \eta'_i)$ in $\mathcal{Y}_{\underline{x}}(\mathcal{D}/\mathfrak{m}^{\hat{q}^l})$. Then both K-homomorphisms $u_l v_l$ and $\mathrm{id}_{\mathcal{D}/\mathfrak{m}^{\hat{q}^l}}$ satisfy the assertions of the claim. By the uniqueness, we obtain $u_l v_l = \mathrm{id}_{\mathcal{D}/\mathfrak{m}^{\hat{q}^l}}$. This shows that v is an isomorphism.

4. It remains to prove the claim. We let $\mathfrak{m}_A \subset A$ be the maximal ideal and set $A_l := A/\mathfrak{m}_A^{\hat{q}^l}$. By induction on l we construct the uniquely determined K-homomorphism $u_l : \mathcal{D} \to A_l$. For $l = 0$ we can take $u_l = \mathrm{id}_K : \mathcal{D} \to \mathcal{D}/\mathfrak{m} = K \to K = A_1$, because $(\underline{\mathcal{G}}, H\gamma, \overline{\mathcal{P}}_i, \eta_i)_K = (\underline{\mathcal{G}}_0, H\gamma_0, \overline{\mathcal{P}}_{i,0}, \eta_{i,0}) = (\underline{\mathcal{G}}', H\gamma', \overline{\mathcal{P}}'_i, \eta'_i)_K$.

Let us assume as induction hypothesis that we have constructed a uniquely determined K-homomorphism $u_l : \mathcal{D} \to A_l$ with $(\underline{\mathcal{G}}, H\gamma, \overline{\mathcal{P}}_i, \eta_i)_{A_l} = u_l^*(\underline{\mathcal{G}}', H\gamma', \overline{\mathcal{P}}'_i, \eta'_i)$ in $\mathcal{Y}_{\underline{x}}(A_l)$. The latter means that there is a quasi-isogeny $\varphi_l : u_l^*\underline{\mathcal{G}}' \to \underline{\mathcal{G}}_{A_l}$ of global G-shtukas, which is compatible with $H\gamma'$ and $H\gamma$ and induces isomorphisms at all c_i of local G_{c_i}-shtukas $\hat{\Gamma}_{c_i}(\varphi_l) : u_l^*\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}') \to \hat{\Gamma}_{c_i}(\underline{\mathcal{G}})_{A_l}$, and there is an isomorphism $\delta_{i,l} : (L^+\hat{\sigma}^{-1}I_{\bar{P}_i})_{A_l} = u_l^*\overline{\mathcal{P}}'_i \xrightarrow{\sim} \overline{\mathcal{P}}_{i,A_l}$ of $L^+\hat{\sigma}^{-1}I_{\bar{P}_i}$-torsors over A_l satisfying $\eta_i \mod \mathfrak{m}_A^{\hat{q}^l} \circ \iota_*\delta_{i,l} = \hat{\Gamma}_{c_i}(\varphi_l) \circ u_l^*\eta'_i$,

$$\begin{array}{ccc} u_l^*\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}') & \xrightarrow{\hat{\Gamma}_{c_i}(\varphi_l)} & \hat{\Gamma}_{c_i}(\underline{\mathcal{G}})_{A_l} \\ {\scriptstyle u_l^*\eta'_i}\uparrow & & \uparrow{\scriptstyle \eta_i} \\ \iota_* u_l^*\overline{\mathcal{P}}'_i & \xrightarrow{\iota_*\delta_{i,l}} & \iota_*\overline{\mathcal{P}}_{i,A_l}\,. \end{array} \tag{4.13}$$

We choose a lift $\delta_{i,l+1} : (L^+\hat{\sigma}^{-1}I_{\bar{P}_i})_{A_{l+1}} \xrightarrow{\sim} \overline{\mathcal{P}}_{i,A_{l+1}}$ over A_{l+1} of the trivialization $\delta_{i,l}$ and consider the isomorphism of local G_{c_i}-shtukas

$$\psi_{i,l+1} := \eta_i \mod \mathfrak{m}_A^{\hat{q}^l} \circ \iota_*\delta_{i,l+1} : \big((L^+G_{c_i})_{A_{l+1}}, x_i\,\bar{p}_{i,l+1}\hat{\sigma}_i^*\big) \xrightarrow{\sim} \hat{\Gamma}_{c_i}(\underline{\mathcal{G}})_{A_{l+1}}$$

with $\bar{p}_{i,l+1} \in L^+I_{\bar{P}}(A_{l+1})$. Here we use that A_{l+1} has no non-trivial étale coverings and so the isomorphism α from (4.5) for the complete slope division $\overline{\mathcal{P}}_{i,A_{l+1}}$ can be chosen over A_{l+1}. Then $\iota_*\alpha\,\eta_i^{-1}\hat{\Gamma}_{c_i}(\tau_{\mathcal{G}})\,\hat{\sigma}_i^*(\eta_i\,\iota_*\alpha^{-1}) \in x_i \cdot L^+I_{\bar{P}_i}(A_{l+1})$, and $\psi_{i,l+1}^{-1}\,\hat{\Gamma}_{c_i}(\tau_{\mathcal{G}})\,\hat{\sigma}_i^*\psi_{i,l+1} \in x_i \cdot L^+I_{\bar{P}_i}(A_{l+1})$, because $\alpha \circ \delta_{i,l+1} \in L^+\hat{\sigma}_i^{-1}I_{\bar{P}_i}(A_{l+1})$. We write $\bar{p}_{i,l+1} = m_{i,l+1}\,\bar{n}_{i,l+1}^{-1}$ with $m_{i,l+1} \in L^+I_{M_i}(A_{l+1})$ and $\bar{n}_{i,l+1} \in L^+I_{\bar{N}_i}(A_{l+1})$. Since $\psi_{i,l+1} \equiv \hat{\Gamma}_{c_i}(\varphi_l) \circ u_l^*\eta'_i \mod \mathfrak{m}_A^{\hat{q}^l}$, they satisfy $m_{i,l+1} \equiv 1 \mod \mathfrak{m}_A^{\hat{q}^l}$ and $\bar{n}_{i,l+1} \equiv u_l(\bar{n}'_i) \mod \mathfrak{m}_A^{\hat{q}^l}$. Observe that $\hat{\sigma}_i^*(m_{i,l+1}) = 1$ implies $(x_i\,m_{i,l+1}\,x_i^{-1})^{-1} \cdot x_i\,\bar{p}_{i,l+1} \cdot \hat{\sigma}_i^*(x_i\,m_{i,l+1}\,x_i^{-1}) = x_i\,\bar{n}_{i,l+1}^{-1}$. Using that $x_i\,m_{i,l+1}\,x_i^{-1} \in L^+\hat{\sigma}^{-1}I_{M_i}(A_{l+1})$ by Condition (v) from Definition 4.3, we replace $\delta_{i,l+1}$ by the isomorphism $\delta_{i,l+1} \circ x_i\,m_{i,l+1}\,x_i^{-1}$, and hence $\psi_{i,l+1}$ by the isomorphism

$$\psi_{i,l+1} \circ x_i\,m_{i,l+1}\,x_i^{-1} : \big((L^+G_{c_i})_{A_{l+1}}, x_i\,\bar{n}_{i,l+1}^{-1}\hat{\sigma}_i^*\big) \xrightarrow{\sim} \hat{\Gamma}_{c_i}(\underline{\mathcal{G}})_{A_{l+1}}$$

without changing $\delta_{i,l}$ and $\psi_{i,l}$. Consider the following diagram

$$\begin{array}{ccccc}
\operatorname{Spec} A_{l+1} & \xrightarrow{\bar{n}_{i,l+1}} & L^+ I_{\bar{N}_i} & \longrightarrow & L^+ I_{\bar{N}_i} / x_i^{-1} \cdot L^+ \hat{\sigma}_i^{-1} I_{\bar{N}_i} \cdot x_i \\
\uparrow & \searrow^{\operatorname{Spec} u_{i,l+1}} & & & \uparrow \\
\operatorname{Spec} A_l & \xrightarrow[\operatorname{Spec} u_l]{} & \operatorname{Spec} \mathcal{D} & \longrightarrow & \operatorname{Spec} \mathcal{D}_i
\end{array} \tag{4.14}$$

in which the top row is defined by the element $\bar{n}_{i,l+1} \in L^+ I_{\bar{N}_i}(A_{l+1})$. The commutativity of the outer rectangle follows from $\bar{n}_{i,l+1} \equiv u_l(\bar{n}'_i) \mod \mathfrak{m}_A^{\hat{q}^l}$. Since $\bar{n}'_i \equiv 1 \mod \mathfrak{m}$, the morphism in the top row maps the closed point $\mathfrak{m}_A$ to 1, and hence this morphism factors through $\operatorname{Spec} \mathcal{D}_i$, because A_{l+1} is a complete local ring. This defines morphisms $u_{i,l+1} : \mathcal{D}_i \to A_{l+1}$ for all i. Together they yield a morphism $u_{l+1} : \mathcal{D} \to A_{l+1}$ with $u_{l+1}(\bar{n}'_i) = \bar{n}_{i,l+1} \cdot x_i^{-1} \cdot h_i \cdot x_i$ for some $h_i \in L^+ \hat{\sigma}_i^{-1} I_{\bar{N}_i}(A_{l+1})$ with $h_i \equiv 1 \mod \mathfrak{m}_A^{\hat{q}^l}$. Observe that $\hat{\sigma}_i^*(h_i) = 1$ implies $h_i \cdot x_i\, u_{l+1}(\bar{n}'_i)^{-1} \cdot \hat{\sigma}_i^*(h_i)^{-1} = x_i\, \bar{n}_{i,l+1}^{-1}$. We now replace $\delta_{i,l+1}$ by $\delta_{i,l+1} \circ h_i$, and hence $\psi_{i,l+1}$ by $\psi_{i,l+1} \circ h_i : \big((L^+ G_{c_i})_{A_{l+1}}, x_i\, u_{l+1}(\bar{n}'_i)^{-1} \hat{\sigma}_i^* \big) \xrightarrow{\sim} \hat{\Gamma}_{c_i}(\underline{\mathcal{G}})_{A_{l+1}}$ without changing $\delta_{i,l}$ and $\psi_{i,l}$.

The quasi-isogeny $\varphi_l : u_l^* \underline{\mathcal{G}}' \to \underline{\mathcal{G}}_{A_l}$ lifts by rigidity (Proposition 2.14) uniquely to a quasi-isogeny $\varphi_{l+1} : u_{l+1}^* \underline{\mathcal{G}}' \to \underline{\mathcal{G}}_{A_{l+1}}$ of global G-shtukas over A_{l+1}, which is compatible with the H-level structures. Since $\hat{\Gamma}_{c_i}(\varphi_{l+1})$ and $\eta_i \circ \iota_* \delta_{i,l+1} \circ u_{l+1}^*(\eta'_i)^{-1}$ are quasi-isogenies $u_{l+1}^* \hat{\Gamma}_{c_i}(\underline{\mathcal{G}}') \to \hat{\Gamma}_{c_i}(\underline{\mathcal{G}})_{A_{l+1}}$ which both lift $\hat{\Gamma}_{c_i}(\varphi_l)$, they coincide by rigidity (Proposition 2.12). This shows that $\hat{\Gamma}_{c_i}(\varphi_{l+1})$ is an isomorphism of local G_{c_i}-shtukas and the lift to A_{l+1} of diagram (4.13) commutes for all i. So $(\underline{\mathcal{G}}, H\gamma, \overline{\mathcal{P}}_i, \eta_i)_{A_{l+1}} = u_{l+1}^*(\underline{\mathcal{G}}', H\gamma', \overline{\mathcal{P}}'_i, \eta'_i)$ in $\mathcal{Y}_{\underline{x}}(A_{l+1})$ and this proves the existence of u_{l+1} in our claim.

5. To prove the uniqueness of u_{l+1} assume there is another K-homomorphism $\tilde{u}_{l+1} : \mathcal{D} \to A_{l+1}$ which satisfies the assertions of the claim. Then there is a quasi-isogeny $\tilde{\varphi}_{l+1} : \tilde{u}_{l+1}^* \underline{\mathcal{G}}' \to \underline{\mathcal{G}}_{A_{l+1}}$ of global G-shtukas and an isomorphism $\tilde{\delta}_{i,l+1} : \tilde{u}_{l+1}^* \overline{\mathcal{P}}'_i \xrightarrow{\sim} \overline{\mathcal{P}}_{i,A_{l+1}}$ of $L^+ \hat{\sigma}^{-1} I_{\bar{P}_i}$-torsors over A_{l+1} which make the following diagram commutative

$$\begin{array}{ccccc}
u_{l+1}^* \hat{\Gamma}_{c_i}(\underline{\mathcal{G}}') & \xrightarrow{\hat{\Gamma}_{c_i}(\varphi_{l+1})} & \hat{\Gamma}_{c_i}(\underline{\mathcal{G}})_{A_{l+1}} & \xleftarrow{\hat{\Gamma}_{c_i}(\tilde{\varphi}_{l+1})} & \tilde{u}_{l+1}^* \hat{\Gamma}_{c_i}(\underline{\mathcal{G}}') \\
{\scriptstyle u_{l+1}^* \eta'_i} \uparrow & & \uparrow {\scriptstyle \eta_i} & & \uparrow {\scriptstyle \tilde{u}_{l+1}^* \eta'_i} \\
\iota_* u_{l+1}^* \overline{\mathcal{P}}'_i & \xrightarrow{\iota_* \delta_{i,l+1}} & \iota_* \overline{\mathcal{P}}_{i,A_{l+1}} & \xleftarrow{\iota_* \tilde{\delta}_{i,l+1}} & \iota_* \tilde{u}_{l+1}^* \overline{\mathcal{P}}'_i\,.
\end{array}$$

The uniqueness assumption of our induction hypothesis implies $u_{l+1} \equiv u_l \equiv \tilde{u}_{l+1} \mod \mathfrak{m}_A^{\hat{q}^l}$, and hence $\varphi_{l+1} \equiv \varphi_l \equiv \tilde{\varphi}_{l+1} \mod \mathfrak{m}_A^{\hat{q}^l}$ and $\delta_{i,l+1} \equiv \delta_{i,l} \equiv \tilde{\delta}_{i,l+1} \mod \mathfrak{m}_A^{\hat{q}^l}$. Thus $h := \tilde{\delta}_{i,l+1}^{-1} \circ \delta_{i,l+1} \in L^+ \hat{\sigma}^{-1} I_{\bar{P}_i}(A_{l+1})$ satisfies $h \equiv 1 \mod \mathfrak{m}_A^{\hat{q}^l}$ and

$$h \cdot x_i\, u_{l+1}(\bar{n}'_i)^{-1} = x_i\, \tilde{u}_{l+1}(\bar{n}'_i)^{-1} \cdot \hat{\sigma}_i^*(h) = x_i\, \tilde{u}_{l+1}(\bar{n}'_i)^{-1}.$$

So in fact

$$h = x_i\, \tilde{u}_{l+1}(\bar{n}'_i)^{-1} u_{l+1}(\bar{n}'_i)\, x_i^{-1} \in L\hat{\sigma}^{-1} \bar{N}_i(A_{l+1}) \cap L^+ \hat{\sigma}^{-1} I_{\bar{P}_i}(A_{l+1}) = L^+ \hat{\sigma}^{-1} I_{\bar{N}_i}(A_{l+1})$$

by Definition 4.3(v). This shows that $u_{l+1}(\bar{n}'_i) = \tilde{u}_{l+1}(\bar{n}'_i) \cdot x_i^{-1} h\, x_i$, and therefore the compositions of both $\operatorname{Spec} u_{i,l+1}$ and $\operatorname{Spec} \tilde{u}_{i,l+1}$ with $\operatorname{Spec} \mathcal{D}_i \to L^+ I_{\bar{N}_i} \big/ x_i^{-1} \cdot L^+ \hat{\sigma}_i^{-1} I_{\bar{N}_i} \cdot x_i$ in diagram (4.14) are equal, because $u_{i,l+1}$ and $\tilde{u}_{i,l+1}$ are uniquely determined by their value on $\bar{n}'_i$. By the definition of $\mathcal{D}_i$ this shows that $u_{i,l+1} = \tilde{u}_{i,l+1} : \mathcal{D}_i \to A_{l+1}$ and $u_{l+1} = \tilde{u}_{l+1} : \mathcal{D} \to A_{l+1}$ as desired. □

Furthermore, the central leaves are smooth. This property is the anologue of [Oor04, Theorem 3.13] resp. [Man04, Proposition 2.7]:

Proposition 4.31. *Denote by $\mathbb{F}_{\underline{x}}$ a field in $\mathcal{N}\mathrm{ilp}\, A_{\underline{c}}$ such that $\underline{x}$ is defined over $\mathbb{F}_{\underline{x}}$. The central leaf is smooth and locally of finite type over $\mathbb{F}_{\underline{x}}$.*

Proof. We follow the idea of the proof of Oort (see [Oor04, Theorem 3.13]) resp. Mantovan (see [Man04, Proposition 2.7]).
We only have to show that $\mathcal{C}_{\underline{x}}$ is regular because over perfect fields this implies the smoothness. Since $\mathcal{C}_{\underline{x}}$ is a locally closed substack of $\nabla\mathcal{H}^1_{\mathbb{F}_{\underline{x}}}$, which is a stack locally of finite type over $\mathbb{F}_{\underline{x}}$, we get $\mathcal{C}_{\underline{x}}$ is locally of finite type over $\mathbb{F}_{\underline{x}}$.
Consider an étale presentation $X \to \mathcal{C}_{\underline{x}}$, where X is a scheme (which is reduced by the construction of the central leaf). Furthermore, we choose a presentation $Y \to \nabla\mathcal{H}^1$, where Y is a scheme. Now we have to show that X is a smooth scheme. We show that X is regular at each closed point.
First, we proof that X is regular in each point if it is regular in one point:
Fix an algebraic closure $\mathbb{F}_{\underline{x}}^{\mathrm{alg}}$ of $\mathbb{F}_{\underline{x}}$ and let

$$y = (\mathcal{G}, \tau) \in X(\mathbb{F}_{\underline{x}}^{\mathrm{alg}})$$

and

$$y' = (\mathcal{G}', \tau') \in X(\mathbb{F}_{\underline{x}}^{\mathrm{alg}})$$

be closed points.
Consider the completion $\hat{\mathcal{O}}_{X,y}$ of the local ring of X at y, resp. $\hat{\mathcal{O}}_{X,y'}$ at y' and likewise $\hat{\mathcal{O}}_{Y,y}$ and $\hat{\mathcal{O}}_{Y,y'}$.
Remember the deformation functor Defo investigated in [AH14a, § 5.3] (see Definition 2.23) and use Serre-Tate (Theorem 2.24) to get:

$$\mathrm{Defo}(\mathcal{G}, \tau) \cong \prod_i \mathrm{Defo}(\hat{\Gamma}_{c_i}(\mathcal{G}, \tau))$$

and

$$\mathrm{Defo}(\mathcal{G}', \tau') \cong \prod_i \mathrm{Defo}(\hat{\Gamma}_{c_i}(\mathcal{G}', \tau')).$$

The definition of the central leaf implies:

$$\mathrm{Defo}(\mathcal{G}, \tau) \cong \prod_i \mathrm{Defo}(L^+ G_{c_i}, x_i \hat{\sigma}^*) \cong \mathrm{Defo}(\mathcal{G}', \tau').$$

Since

$$\mathrm{Defo}(\mathcal{G},\tau) \cong \hat{\mathcal{O}}_{Y,y} \quad \text{and} \quad \mathrm{Defo}(\mathcal{G}',\tau') \cong \hat{\mathcal{O}}_{Y,y'},$$

we get

$$\hat{\mathcal{O}}_{Y,y} \cong \hat{\mathcal{O}}_{Y,y'}.$$

Via the latter isomorphism, the definition that defines $\hat{\mathcal{O}}_{X,y}$ as a quotient is the same as the one that defines $\hat{\mathcal{O}}_{X,y'}$. In fact, this condition is defined on the universal deformation of the central leaf. Therefore, we get an isomorphism

$$\hat{\mathcal{O}}_{X,y} \cong \hat{\mathcal{O}}_{X,y'}.$$

By [Gro67, IV_2, Corollaire 6.12.5] the regular locus is open. Since the $\mathbb{F}_{\underline{x}}^{\mathrm{alg}}$-valued points are dense, it is sufficient to consider them. It remains to show that there is at least one regular point:
This follows from the fact that $\mathcal{C}_{\underline{x}}$ is reduced locally of finite type over $\operatorname{Spec}\mathbb{F}_{\underline{x}}$ as it is locally closed in $\nabla\mathcal{H}^1_{\mathbb{F}_{\underline{x}}}$ and $\nabla\mathcal{H}^1_{\mathbb{F}_{\underline{x}}}$ is locally of finite type over $\mathbb{F}_{\underline{x}}$. So $\mathcal{C}_{\underline{x}}$ is generically regular and contains a smooth point (and even an open dense subset of smooth points). □

5 Igusa varieties

We want to decompose Newton strata in the special fibre of moduli spaces of global G-shtukas into a product of so-called *Igusa varieties* and *truncated* Rapoport-Zink spaces.

Following an idea of Mantovan in [Man04] and of Harris and Taylor in [HT01], we introduce the Igusa varieties as coverings of the central leaves. Therefore, our decomposition is an analogue of the covering of the moduli space of abelian varieties considered by Oort in [Oor04] and which was generalized by Mantovan as a product of the central leaf and the "isogeny leaf". In the case of a constant, split, reductive group G_c this was done by Hartl and Viehmann ([HV12]) for the deformation space of local G_c-shtukas.

Mantovan considers in [Man04] abelian varieties and their associated p-divisible groups. Their p^m-torsion define the so-called *truncated Barsotti-Tate groups*, which are finite flat group schemes. Mantovan described in [Man04, §3] (see also [Man05, §4]) Igusa varieties as schemes representing isomorphisms between (two of) those BT-groups. For global G-shtukas, the idea analogous to Harris and Taylor [HT01] is to parametrize partial trivializations of the local G_{c_i}-shtukas at the characteristic places.

We define in Definition 5.4 the Igusa variety by using a trivialization of the Levi part of a complete division of the local G_{c_i}-shtuka $\underline{\mathcal{G}}_i := (\mathcal{G}_i, \hat{\tau}_{\mathcal{G}_i}) := \hat{\Gamma}_{c_i}(f^*\underline{\mathcal{G}}^{\text{univ}})$ associated to the universal global G-shtuka $\underline{\mathcal{G}}^{\text{univ}}$. Here f denotes a presentation $f : S \to \mathcal{Y}_{x_i}$ with $\mathcal{Y}_{x_i}$ as in Proposition 4.26.

We fix an element $x = x_i$ where the index i means that we fix x_i associated to G_{c_i} for a characteristic place c_i with the conditions given in Definition 4.3. We write $A = \mathbb{F}_{c_i}^{\text{alg}}[\![z]\!]$. We remember that the Levi M_i is defined over $\mathbb{F}_c^{\text{alg}}(\!(z)\!)$. By our notation, M_i is a smooth group scheme over $\mathbb{F}_{c_i}[\![z]\!]$. Under this convention we have:

$$L_m^+ M_i(R) = L_m^+ G_{c_i}(R) \cap LM_i(R)$$
$$L_m^+ N_i(R) = L_m^+ G_{c_i}(R) \cap LN_i(R)$$
$$L_m^+ \bar{N}_i(R) = L_m^+ G_{c_i}(R) \cap L\bar{N}_i(R)$$
$$L_m^+ \bar{P}_i(R) = L_m^+ G_{c_i}(R) \cap L\bar{P}_i(R)$$

with $\bar{P}_i = M_i \bar{N}_i$ for every $\mathbb{F}_{c_i}^{\text{alg}}$-algebra R.

Lemma 5.1. *(i) For all $g_i \in L^+ I_{M_i}$ we have $g_i^{-1} \cdot L_m^+ I_{\bar{N}_i} \cdot g_i = L_m^+ I_{\bar{N}_i}$.*

(ii) For all $g_i \in L^+M_i$ *we have* $g_i^{-1} \cdot L_m^+\bar{N}_i \cdot g_i = L_m^+\bar{N}_i$.

(iii) For all $g_i \in L^+M_i$ *we have* $g_i^{-1} \cdot L_m^+G_{c_i} \cdot g_i = L_m^+G_{c_i}$.

Proof. Calculate in $L^+G_{c_i}/L_m^+G_{c_i}(R) = G_{c_i}(R[\![z]\!]/(z^m))$ for an $\mathbb{F}_{c_i}^{\text{alg}}$-algebra R: Let $\bar{n} \in L_m^+\bar{N}_i$. Then $g^{-1}\bar{n}g = g^{-1}g = 1$ there. □

We define

$$(L^+M_i)(T) := M_i(\Gamma(T, \mathcal{O}_T)[\![z]\!])$$

and

$$(L_m^+M_i \backslash L^+M_i)(T) := M_i(\Gamma(T, \mathcal{O}_T)[\![z]\!]/(z^m)).$$

The second definition makes sense because we have the following isomorphism:

Lemma 5.2.

$$\phi : \{g \in L^+M_i(T) : g \equiv 1 \mod z^m\} \backslash M_i(\Gamma(T, \mathcal{O}_T)[\![z]\!]) \to M_i(\Gamma(T, \mathcal{O}_T)[\![z]\!]/(z^m)$$

is an isomorphism.

Proof. We show the above equality

$$\{g \in L^+M_i(T) : g \equiv 1 \mod z^m\} \backslash M_i(\Gamma(T, \mathcal{O}_T)[\![z]\!]) = M_i(\Gamma(T, \mathcal{O}_T)[\![z]\!]/(z^m)$$

and the equality to the corresponding quotient of sheaves.
In fact, ϕ is injective because we can consider $M_i \subseteq \mathrm{GL}_{r,\mathbb{F}[\![z]\!]}$ closed for some integer $r := r_i$, that is as $V(F_1, \ldots, F_s)$. We can write the elements in $\mathrm{GL}_{r,\mathbb{F}[\![z]\!]}$ as $\sum_{j=0}^{\infty} b_i j z^j$ with $b_j \in \Gamma(T, \mathcal{O}_T)^{r \times r}$, where b_0 is invertible. So $F_\nu(\sum b_j z^j) = 0$. So

$$\phi \text{ injective} \Leftrightarrow \phi^{-1}(1) = 1 \Leftrightarrow g \equiv 1 \mod z^m.$$

ϕ is also surjective:
Since $M_i \to \operatorname{Spec} \mathbb{F}_q[\![z]\!]$ is smooth, there is a lift.
In fact we have

$$M_i(\Gamma(T, \mathcal{O}_T)[\![z]\!]/(z^m)) = \mathrm{Hom}_{\mathbb{F}_q[\![z]\!]}(\operatorname{Spec} \Gamma(T, \mathcal{O}_T)[\![z]\!]/(z^m), M_i).$$

Note that we have the general fact

$$\mathrm{Mor}(Y, X) = \mathrm{Mor}(A, \Gamma(Y, \mathcal{O}_Y)) = \mathrm{Mor}(\operatorname{Spec} \Gamma(Y, \mathcal{O}_Y), X)$$

for $X = \operatorname{Spec} A$ affine.
Since M_i is smooth over $\mathbb{F}_q[\![z]\!]$ we get that for all $g \in M_i(\Gamma(T, \mathcal{O}_T)[\![z]\!]/(z^m))$ there is a $g' \in M_i(\Gamma(T, \mathcal{O}_T)[\![z]\!]/(z^{m+1}))$. □

Proposition 5.3. *Let $\mathcal{M}_i$ be an L^+M_i-torsor on S.*
The functor

$$m\backslash \mathrm{Isom} : (\textit{Schemes on } S) \to (\textit{Sets})$$

with

$$m\backslash \mathrm{Isom}(T) := \{ j : \mathcal{M}_{i,T} \tilde{\to} (L^+M_i)_T \textit{ is an isomorphism} \mod L_m^+M_i(T)\}$$

for an S-scheme T is represented by an affine scheme of finite type over S.

Proof. We choose an étale covering $S' \to S$ together with an isomorphism

$$\alpha : \mathcal{M}_{i,S'} \tilde{\to} (L^+M_i)_{S'} .$$

For $T \to S'$ let $\alpha_T : \mathcal{M}_{i,T} \to (L^+M_i)_T$ be the induced isomorphism. We consider the following isomorphism

$$\beta : m\backslash \mathrm{Isom} \times_S S' \xrightarrow{\sim} (L_m^+M_i\backslash L^+M_i)_{S'}$$

given by

$$T \to m\backslash \mathrm{Isom} \times_S S' : (j : \mathcal{M}_{i,T} \xrightarrow{\sim} (L^+M_i)_T) \mapsto j \circ \alpha_T^{-1}$$

and define the inverse map by

$$\tilde{j} \mapsto \tilde{j} \circ \alpha_T.$$

This is well-defined as it is independent of the choice of the representative $\tilde{j}$: Let $\tilde{j}'$ be in the same equivalence class of $\tilde{j}$, that is $\tilde{j}' = g \cdot \tilde{j}$ for an element $g \in L_m^+M_i$. Then

$$g \cdot \tilde{j} \mapsto g \cdot \tilde{j} \cdot \alpha_T \equiv \tilde{j} \cdot \alpha_T$$

Now we show that the scheme on the right hand side is a scheme of finite type and then we can deduce that also the one on the left hand side is a scheme of finite type over S'.
By the above Lemma 5.2,

$$\phi : \{g \in L^+M_i(T) : g \equiv 1 \mod z^m\} \backslash M_i(\Gamma(T, \mathcal{O}_T)[\![z]\!]) \to M_i(\Gamma(T, \mathcal{O}_T)[\![z]\!]/(z^m)$$

is an isomorphism. Now consider the Weil-restriction $\mathrm{Res}_{\mathbb{F}[\![z]\!]/(z^m)|\mathbb{F}} \mathrm{GL}_{r,\mathbb{F}[\![z]\!]/(z^m)}$. Since it is equal to $\mathrm{GL}_r \times M_r^{m-1}$ it is affine and of finite type over $\mathbb{F}_q$. Because of the definition of $m\backslash \mathrm{Isom}(T)$ we have

$$m\backslash \mathrm{Isom} \cong L_m^+M_i\backslash L^+M_i \subseteq \mathrm{Res}_{\mathbb{F}[\![z]\!]/(z^m)|\mathbb{F}} \mathrm{GL}_{r,\mathbb{F}[\![z]\!]/(z^m)}$$

is closed as it is described by the above vanishing-ideal. Since it is a closed subset $m\backslash \mathrm{Isom}$ is affine of finite type over $\mathbb{F}$.
Now we want to show that $m\backslash \mathrm{Isom}$ is affine of finite type over S. We do this with

a descent argument:
We have to show that there is a scheme X over S with $X' := m\backslash \mathrm{Isom} \times_{\mathbb{F}} S' = X \times_S S'$. So consider the fibre product $S'' := S' \times_S S'$ and the associated projections $\mathrm{pr}_i : S' \times_S S' \to S$ resp. $\mathrm{pr}_{ij} : S''' := S' \times_S S' \times_S S' \to S' \times_S S'$ the projection on the (i,j)-th factor and let $d : S' \to S' \times_S S'$ be the diagonal.
Set $h := \mathrm{pr}_1^* \alpha \circ \mathrm{pr}_2^* \alpha^{-1} : (L^+ M_i)_{S''} \tilde{\to} (L^+ M_i)_{S''}$, so h defines an element in $(L^+ M_i)(S'')$. Then $\mathrm{pr}_{13}^* h = \mathrm{pr}_1^* \alpha \circ \mathrm{pr}_3^* \alpha^{-1} = \mathrm{pr}_1^* \alpha \circ \mathrm{pr}_2^* \alpha^{-1} \circ \mathrm{pr}_2^* \alpha \circ \mathrm{pr}_3^* \alpha^{-1} = \mathrm{pr}_{12}^* h \circ \mathrm{pr}_{23}^* h$ and $d^*(h) = \mathrm{id}$. Therefore, we have the desired cocycle condition.
Over S'' we consider the following diagram:

$$\begin{array}{ccc} \mathrm{pr}_1^* X' & \xrightarrow{\;\;f\;\;} & \mathrm{pr}_2^* X' \\ {\scriptstyle g_1} \uparrow \downarrow {\scriptstyle \tilde{g}_1} & & {\scriptstyle g_2} \uparrow \downarrow {\scriptstyle \tilde{g}_2} \\ \mathrm{pr}_1^*(m\backslash \mathrm{Isom}) & & \mathrm{pr}_2^*(m\backslash \mathrm{Isom}) \end{array}$$

where $\tilde{g}_1$ is given by $\tilde{j} \mapsto \tilde{j} \circ \mathrm{pr}_1^* \alpha$ resepectively g_1 by $j \mapsto j \circ \mathrm{pr}_1^* \alpha^{-1}$ and analogously $\tilde{g}_2$ resp. g_2. So f is given by $\tilde{j} \mapsto \tilde{j} \circ \mathrm{pr}_1^* \alpha \circ \mathrm{pr}_2^* \alpha^{-1} = \tilde{j} \circ h \mod z^m$ where h is given by the above equation. As before the cocycle condition holds over S''' and we have $X' = X \times_S S'$ for a scheme X over S. It is affine of finite type (because this property descends by [Gro67, IV §2, 2.7.1]. $\square$

In the following we fix for every i the data (I_i, P_i, M_i, x_i) for the group G_{c_i} as in Definition 4.3. We consider the natural morphism of group schemes $\iota : \hat{\sigma}_i^{-1} I_{\bar{P}_i} \to G_{c_i}$ and the induced functor ι_* from $L^+ \hat{\sigma}_i^{-1} I_{\bar{P}_i}$-torsors to $L^+ G_{c_i}$-torsors. We recall the stacks

$$\mathcal{Y}_{x_i} \longrightarrow \mathcal{C}_{x_i} \subset \nabla \mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}} \qquad \text{and}$$

$$\mathcal{Y}_{\underline{x}} := (\mathcal{Y}_{x_1} \times_{\nabla \mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} \dots \times_{\nabla \mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} \mathcal{Y}_{x_n}) \to \mathcal{C}_{\underline{x}} \subset \nabla \mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$$

from Proposition 4.26.

Definition 5.4. *(i) We choose a presentation $f : S \to \mathcal{Y}_{x_i}$ such that the universal complete slope division $(\overline{\mathcal{P}}_i, \eta_i)$ of $\hat{\Gamma}_{c_i}(f^* \underline{\mathcal{G}}^{\mathrm{univ}}) =: \underline{\mathcal{G}}_i = (\mathcal{G}_i, \hat{\tau}_{\mathcal{G}_i})$ over $\mathcal{Y}_{x_i}$ possesses a trivialization $\alpha_i : \overline{\mathcal{P}}_{i,S} \xrightarrow{\sim} (L^+ \hat{\sigma}_i^{-1} I_{\bar{P}_i})_S$ satisfying*

$$\iota_* \alpha_i \, \eta_i^{-1} \hat{\tau}_{\mathcal{G}_i} \, \hat{\sigma}_i^* (\eta_i \, \iota_* \alpha_i^{-1}) = x_i m_i \bar{n}_i \in x_i \cdot L^+ I_{\bar{P}_i}(S) \,. \tag{5.1}$$

For an integer $e_i \geq 0$ we define the Igusa variety of level e_i *as the functor* Ig_{e_i} *on the category of S-schemes T by*

$$\mathrm{Ig}_{e_i}(T) := \\ \left\{ j_{e_i} \in (L^+ \hat{\sigma}_i^{-1} I_{M_i} / L^+ \hat{\sigma}_i^{-1} I_{e_i, M_i})(T) \; : \; \hat{\sigma}_i^* (j_{e_i}^{-1}) x_i^{-1} j_{e_i} x_i m_i \in L^+ I_{e_i, M_i}(T) \right\}.$$

Here we choose a representative $j_{e_i} \in (L^+\hat{\sigma}_i^{-1}I_{M_i})(T')$ *over an fppf-covering* $T' \to T$*. Note that then* $\hat{\sigma}_i^*(j_{e_i}), x_i^{-1}j_{e_i}x_i, m_i \in L^+I_{M_i}(T')$ *by Condition (vi) from Definition 4.3. Also the condition is well defined, because if* j_{e_i} *is multiplied by* $h \in L^+\hat{\sigma}_i^{-1}I_{e_i,M_i}(T')$ *on the right, then* $\hat{\sigma}_i^*(j_{e_i}^{-1})x_i^{-1}j_{e_i}x_im_i$ *is multiplied on the left with* $\hat{\sigma}_i^*(h^{-1})$ *and on the right with* $m_i^{-1}x_i^{-1}hx_im_i$*, which lies in* $L^+I_{e_i,M_i}(T')$ *by Condition (viii) from Definition 4.3.*

(ii) Likewise, we choose a presentation $f : S \to \mathcal{Y}$*, such that Condition* (5.1) *holds for all i. For a tuple of non-negative integers* $\underline{e} = (e_i)_i$ *we define the* Igusa variety $\mathrm{Ig}_{\underline{e}}$ of level $\underline{e}$ *as the functor on the category of S-schemes T*

$$\mathrm{Ig}_{\underline{e}}(T) := \left\{ (j_{e_i})_i \in \prod_i \mathrm{Ig}_{e_i}(T) \right\}.$$

Proposition 5.5. Ig_{e_i} *is representable by a closed subscheme of* $(L^+\hat{\sigma}_i^{-1}I_{M_i}/L^+\hat{\sigma}_i^{-1}I_{e_i,M_i}) \times_{\mathbb{F}_q^{\mathrm{alg}}} S$*, and is hence of finite presentation over S.*

Proof. This follows from Condition (viii) from Definition 4.3. □

Proposition 5.6. *There exist algebraic stacks* $\mathfrak{Ig}_{e_i} \to \mathcal{Y}_{x_i} \to \mathcal{C}_{x_i} \subset \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$ *and* $\mathfrak{Ig}_{\underline{e}} \to \mathcal{Y}_{\underline{x}} \to \mathcal{C}_{\underline{x}} \subset \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$ *with* $\mathfrak{Ig}_{e_i} \times_{\mathcal{Y}_{x_i}} S = \mathrm{Ig}_{e_i}$ *and* $\mathfrak{Ig}_{\underline{e}} \times_{\mathcal{Y}} S = \mathrm{Ig}_{\underline{e}}$ *for the presentations* $S \to \mathcal{Y}_{x_i}$*, respectively* $S \to \mathcal{Y}$ *from Definition 5.4. They satisfy*

$$\mathfrak{Ig}_{\underline{e}} = (\mathfrak{Ig}_{e_1} \times_{\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} \dots \times_{\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} \mathfrak{Ig}_{e_n}).$$

Proof. Let $S'' := S \times_{\mathcal{Y}_{x_i}} S$ and consider the two projections $pr_j : S'' \to S$. Let $pr_j^* \mathrm{Ig}_{e_i} = \mathrm{Ig}_{e_i} \times_{S,pr_j} S''$. Then the image h of $\tilde{h} := pr_2^*\alpha_i \circ pr_1^*\alpha_i^{-1} \in L^+\hat{\sigma}_i^{-1}I_{\bar{P}_i}(S'')$ in $L^+\hat{\sigma}_i^{-1}I_{M_i}(S'')$ satisfies the cocycle condition $pr_{23}(h) \circ p_{12}(h) = pr_{13}(h)$ where p_{lk} denotes the projection on the (l,k)-th factor of $S'' := S \times_{\mathcal{Y}_{x_i}} S \times_{\mathcal{Y}_{x_i}} S$ for $l,k \in \{1,2,3\}$. Moreover, it satisfies

$$\begin{aligned} x_i \cdot pr_1^*(m_i\bar{n}_i) &= \iota_* pr_1^*\alpha_i\, \eta_i^{-1}\hat{\tau}_{\mathcal{G}_i}\, \hat{\sigma}_i^*(\eta_i\, \iota_* pr_1^*\alpha_i^{-1}) \\ &= \tilde{h}^{-1}\iota_* pr_2^*\alpha_i\, \eta_i^{-1}\hat{\tau}_{\mathcal{G}_i}\, \hat{\sigma}_i^*(\eta_i\, \iota_* pr_2^*\alpha_i^{-1})\hat{\sigma}_i^*\tilde{h} \\ &= \tilde{h}^{-1}x_i \cdot pr_2^*(m_i\bar{n}_i)\hat{\sigma}_i^*\tilde{h} \end{aligned}$$

We obtain an isomorphism $pr_1^* \mathrm{Ig}_{e_i} \to pr_2^* \mathrm{Ig}_{e_i}, j_{e_i} \mapsto j_{e_i}h$ which satisfies the cocycle condition as h satisfies it, and so this gives a descend datum on Ig_{e_i}. The groupoid $(pr_1^* \mathrm{Ig}_{e_i}, \mathrm{Ig}_{e_i})$ is an algebraic stack $\mathfrak{Ig}_{e_i}$ over $\mathcal{Y}_{x_i}$ as desired.

The proof for $\mathfrak{Ig}_{\underline{e}}$ proceeds analogously and the last equation for $\mathfrak{Ig}_{\underline{e}}$ follows directly from $\mathcal{Y} = (\mathcal{Y}_{x_1} \times_{\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} \dots \times_{\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} \mathcal{Y}_{x_n}) \to \mathcal{C}_{\underline{x}} \subset \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$. □

In the next chapter we will prove in Lemma 6.12 that $\mathrm{Aut}\big((L^+I_i)_{\mathbb{F}_{x_i}^{\mathrm{alg}}}, x_i\hat{\sigma}_i^*\big) \subset L^+\hat{\sigma}_i^{-1}I_{M_i}(\mathbb{F}_{x_i}^{\mathrm{alg}})$, but we already use this result here.

Proposition 5.7. *Let* Γ_i *be the image of* $\operatorname{Aut}\big((L^+I_i)_{\mathbb{F}_{x_i}^{\mathrm{alg}}}, x_i\hat{\sigma}_i^*\big)$ *in* $\big(L^+\hat{\sigma}_i^{-1}I_{M_i}\big/L^+\hat{\sigma}_i^{-1}I_{e_i,M_i}\big)(\mathbb{F}_{x_i}^{\mathrm{alg}})$*. Then the morphisms* $\mathfrak{Ig}_{e_i} \to \mathcal{Y}_{x_i}$ *and* $\mathfrak{Ig}_{\underline{e}} \to \mathcal{Y}$ *are finite étale Galois coverings with Galois groups* Γ_i*, respectively* $\Gamma := \prod_i \Gamma_i$*. In particular, the morphisms* $\mathfrak{Ig}_{e_i} \to \mathcal{C}_{x_i} \subset \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$ *and* $\mathfrak{Ig}_{\underline{e}} \to \mathcal{C}_{\underline{x}} \subset \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$ *are finite, but in general not étale.*

Proof. We use the presentations $S \to \mathcal{Y}_{x_i}$, respectively $S \to \mathcal{Y}$ from Definition 5.4. Then we have to show that $\mathrm{Ig}_{e_i} \to S$, respectively $\mathrm{Ig}_{\underline{e}} \to S$ are finite étale Galois coverings. The surjectivity follows from Lemma 4.13, which shows that for every point $s \in S(k)$ over an algebraically closed field k there is an element $h \in (L^+\hat{\sigma}_i^{-1}I_{\bar{P}_i})(k)$ with $h^{-1}x_i m_i(s)\bar{n}_i(s)\hat{\sigma}^*h = x_i$ (see Remark 4.14). Then the image j_{e_i} of h^{-1} in $\big(L^+\hat{\sigma}_i^{-1}I_{M_i}\big/L^+\hat{\sigma}_i^{-1}I_{e_i,M_i}\big)(k)$ lies in Ig_{e_i} and maps to s.

To compute the fiber above s, we take the representative of j_{e_i} from the previous paragraph with $\hat{\sigma}_i^*(j_{e_i}^{-1})x_i^{-1}j_{e_i}x_i m_i(s) = 1$. For every $g_i \in \operatorname{Aut}\big((L^+I_i)_{\mathbb{F}_{x_i}^{alg}}, x_i\hat{\sigma}_i^*\big)$, the element $\tilde{j}_{e_i} := g_i j_{e_i} \in (L^+\hat{\sigma}_i^{-1}I_{M_i})(k)$ satisfies $\hat{\sigma}_i^*(\tilde{j}_{e_i}^{-1})x_i^{-1}\tilde{j}_{e_i}x_i m_i(s) = 1$, and hence gives a point in $\mathrm{Ig}_{e_i}(k)$ above s. For two such g_i the points in $\mathrm{Ig}_{e_i}(k)$ coincide if and only if the images of the two g_i in Γ_i coincide. Conversely, let $\tilde{j}_{e_i} \in \mathrm{Ig}_{e_i}(k)$ be any point mapping to s and choose a representative $\tilde{j}_{e_i} \in (L^+\hat{\sigma}_i^{-1}I_{M_i})(k)$. Then $\tilde{m}_i' := \hat{\sigma}_i^*(\tilde{j}_{e_i}^{-1})x_i^{-1}\tilde{j}_{e_i}x_i m_i(s) \in L^+I_{e_i,M_i}(k)$. By Condition (viii) of Definition 4.3 we have $\hat{\sigma}_i^*(\tilde{j}_{e_i})\tilde{m}_i'\hat{\sigma}_i^*(\tilde{j}_{e_i}^{-1}) \in L^+I_{e_i,M_i}(k)$, and by Lemma 4.13 there is an element $\tilde{h} \in (L^+\hat{\sigma}_i^{-1}I_{e_i,M_i})(k)$ with $\tilde{h}^{-1}\big(x_i\,\hat{\sigma}_i^*(\tilde{j}_{e_i})\tilde{m}_i'\,\hat{\sigma}_i^*(\tilde{j}_{e_i}^{-1})\big)\hat{\sigma}_i^*(\tilde{h}) = x_i$, whence

$$\hat{\sigma}_i^*(\tilde{j}_{e_i}^{-1}\tilde{h})x_i^{-1}\tilde{h}^{-1}\tilde{j}_{e_i}x_i m_i(s) \;=\; \hat{\sigma}_i^*(\tilde{j}_{e_i}^{-1})\,\hat{\sigma}_i^*(\tilde{j}_{e_i})(\tilde{m}_i')^{-1}\,\hat{\sigma}_i^*(\tilde{j}_{e_i}^{-1})x_i^{-1}\tilde{j}_{e_i}x_i m_i(s) \;=\; 1\,.$$

Replacing the representative $\tilde{j}_{e_i}$ by $\tilde{h}^{-1}\tilde{j}_{e_i}$ yields $\hat{\sigma}_i^*(\tilde{j}_{e_i}^{-1})x_i^{-1}\tilde{j}_{e_i}x_i m_i(s) = 1$ without changing the point $\tilde{j}_{e_i} \in \mathrm{Ig}_{e_i}(k)$. We now take the representative of j_{e_i} from the previous paragraph with $\hat{\sigma}_i^*(j_{e_i}^{-1})x_i^{-1}j_{e_i}x_i m_i(s) = 1$ and set $g_i := \tilde{j}_{e_i}j_{e_i}^{-1}$. Then $g_i x_i\,\hat{\sigma}_i^*(g_i^{-1}) = x_i$ and $g_i \in \operatorname{Aut}\big((L^+I_i)_{\mathbb{F}_{x_i}^{alg}}, x_i\hat{\sigma}_i^*\big)$. This shows that Γ_i acts simply transitively on the fiber of Ig_{e_i} above s.
To show that the maps are étale, we use the infinitesimal lifting criterion [BLR90, §2.2, Proposition 6]. Let B be a ring and $\overline{B} = B/I$ for an ideal $I \subset B$ with $I^2 = (0)$. In particular $\hat{\sigma}_i^*I = (0)$ and the $\hat{q}_i := \#\mathbb{F}_{c_i}$-Frobenius $\hat{\sigma}_i^*$ on B factors

$$\begin{array}{ccccc} B & \longrightarrow & \overline{B} & \xrightarrow{f^*} & B \\ b & \mapsto & b \bmod I & \mapsto & b^{\hat{q}_i}\,. \end{array}$$

Consider a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}\overline{B} & \xrightarrow{\bar{j}_{e_i}} & \mathrm{Ig}_{e_i} \\ \downarrow & \overset{?}{\nearrow} & \downarrow \\ \operatorname{Spec}B & \longrightarrow & S \end{array}$$

in which we claim the existence of a uniquely determined dashed arrow. By the uniqueness we may replace B by an étale covering. After replacing $\overline{B}$ by an étale covering and B by the unique induced étale covering [Gro71, Théorème I.8.3], $\bar{j}_{e_i}$ is represented by an element $\bar{j}_{e_i} \in L^+\hat{\sigma}_i^{-1}I_{M_i}(\overline{B})$ with $\hat{\sigma}_i^*(\bar{j}_{e_i}^{-1})x_i^{-1}\bar{j}_{e_i}x_im_i =: \overline{m}_i' \in L^+I_{e_i,M_i}(\overline{B})$. Since I_{e_i,M_i} is smooth over A_{c_i} by Condition (viii) from Definition 4.3, there is a lift $m_i' \in L^+I_{e_i,M_i}(B)$ of $\overline{m}_i'$. Then $j_{e_i} := x_if^*(\bar{j}_{e_i})m_i'm_i^{-1}x_i^{-1}$ lies in $L^+\hat{\sigma}^{-1}I_{M_i}(B)$ by Condition (viii) from Definition 4.3 and lifts $\bar{j}_{e_i}$. It satisfies $j_{e_i} \in \mathrm{Ig}_{e_i}(B)$ and makes the above diagram commutative, because $\hat{\sigma}_i^*(j_{e_i}) = f^*(\bar{j}_{e_i})$ and so $\hat{\sigma}_i^*(j_{e_i}^{-1})x_i^{-1}j_{e_i}x_im_i = m_i' \in L^+I_{e_i,M_i}(B)$. Moreover, this lift $j_{e_i} \in \mathrm{Ig}_{e_i}(B)$ of $\bar{j}_{e_i} \in \mathrm{Ig}_{e_i}(\overline{B})$ is uniquely determined, because any other lift $\tilde{j}_{e_i} \in \mathrm{Ig}_{e_i}(B)$ with $\hat{\sigma}_i^*(\tilde{j}_{e_i}^{-1})x_i^{-1}\tilde{j}_{e_i}x_im_i = \widetilde{m}_i' \in L^+I_{e_i,M_i}(B)$ satisfies $\hat{\sigma}_i^*(j_{e_i}) = f^*(\bar{j}_{e_i}) = \hat{\sigma}_i^*(\tilde{j}_{e_i})$, and hence

$$j_{e_i}^{-1}\tilde{j}_{e_i} = x_im_i(m_i')^{-1}\widetilde{m}_i'm_i^{-1}x_i^{-1} \in L^+\hat{\sigma}_i^{-1}I_{e_i,M_i}(B)$$

by Condition (viii) from Definition 4.3. So $j_{e_i} = \tilde{j}_{e_i}$ in $\mathrm{Ig}_{e_i}(B)$. This shows that $\mathfrak{Ig}_{e_i} \to \mathcal{Y}_{x_i}$ is étale.

Using the argument given by Harris and Taylor in [HT01, Proposition II.1.7., p.69], resp. [Man04, Proposition 3.3] the above is sufficient to show the claim of the Proposition. □

Proposition 5.8. *For every pair of integers $e_i' > e_i$ the projection*

$$\left(L^+\hat{\sigma}_i^{-1}I_{M_i}\big/L^+\hat{\sigma}_i^{-1}I_{e_i',M_i}\right) \to \left(L^+\hat{\sigma}_i^{-1}I_{M_i}\big/L^+\hat{\sigma}_i^{-1}I_{e_i,M_i}\right)$$

induces finite, étale, surjective morphisms $\mathfrak{Ig}_{e_i'} \to \mathfrak{Ig}_{e_i}$ and $\mathfrak{Ig}_{\underline{e}'} \to \mathfrak{Ig}_{\underline{e}}$.

Proof. That these morphisms are finite étale follows from Proposition 5.7. To prove surjectivity, let $j_{e_i} \in \mathfrak{Ig}_{e_i}(k)$ be a point with values in an algebraically closed field k. By Condition (ix) from Definition 4.3 there exists an element $h \in (L^+\hat{\sigma}_i^{-1}I_{e_i,M_i})(k)$ with $h^{-1}x_i\,\hat{\sigma}_i^*h\,x_i^{-1} \equiv j_{e_i}x_im_i\,\hat{\sigma}_i^*(j_{e_i}^{-1})x_i^{-1} \mod L^+I_{e_i+1,M_i}(k)$. Then $j_{e_i+1} := hj_{e_i}$ lies in $\mathfrak{Ig}_{e_i+1}(k)$ and is mapped to j_{e_i}, because

$$\begin{aligned}\hat{\sigma}_i^*(hj_{e_i})^{-1}x_i^{-1}hj_{e_i}x_im_i &= \hat{\sigma}_i^*(j_{e_i})^{-1}\big(\hat{\sigma}_i^*(h)^{-1}x_i^{-1}hj_{e_i}x_im_i\hat{\sigma}_i^*h^{-1}\big)\hat{\sigma}_i^*(j_{e_i})\\ &\in L^+I_{e_i+1,M_i}(k)\,.\end{aligned}$$

. □

Remark 5.9. *We use a definition of Igusa varieties different to the one given by Mantovan in [Man04], because in loc. cit. it is not clear wether the analogue to Proposition 5.7 holds there as it is not clear (and not proven in loc. cit.) that the morphisms $\mathrm{Ig}_{e_i+1} \to \mathrm{Ig}_{e_i}$ between the Igusa varieties are surjective. Furthermore, we do not want to use the intersection of the scheme theorestic images of some morphisms similar to [Man04, 3.1.2] to define the Igusa variety as this implies some problems to show the finiteness in Proposition 5.7.*

Remark 5.10. *Note that in our definition of the Igusa variety we do* not *require that over the complete local ring $\hat{\mathcal{O}}_{\mathcal{C},s}$ of the central leaf the local G_{c_i}-shtuka $\underline{\hat{\mathcal{G}}}_i$ is isomorphic to the fixed $\underline{\mathbb{G}}_i = (L^+G_{c_i}, x_i\hat{\sigma}_i^*)$. If we would have such an isomorphism then we would get that the central leaf is zero-dimensional. This would be a contradiction to [HV12, Corollary 6.8] and the proof of loc.cit.*
In fact, fix $\underline{c} : \operatorname{Spec}\mathbb{F}_{\underline{c}} \to C^n$ and denote by S a presentation of

$$\nabla\mathcal{H}^1 := \nabla_n^{H,\hat{Z}_{\underline{c}}}\mathcal{H}^1(C,G) \times_{C^n} \operatorname{Spec}\mathbb{F}_{\underline{c}}$$

the special fibre of the moduli space of global G-shtuks. Write

$$\mathcal{C}_{\underline{x},S} = \left\{ s \in S \,:\, \underline{\hat{\mathcal{G}}}_{i,\kappa(s)^{\mathrm{alg}}} \cong \underline{\mathbb{G}}_{i,\kappa(s)^{\mathrm{alg}}} \, for\, all\, i \right\}$$

for the presentation of the central leaf associated to $\underline{\mathbb{G}}_i$ which is locally closed in $\nabla\mathcal{H}^1$.
Let $s \in S$ be a point. Consider the complete local ring $\hat{\mathcal{O}}_{\nabla\mathcal{H}^1,s}$. By Serre-Tate it is isomorphic to the product of the universal deformation rings:

$$\hat{\mathcal{O}}_{\nabla\mathcal{H}^1,s} \cong \prod \operatorname{Defo}(\hat{\mathcal{G}}_{i,s}).$$

Consider the projection onto the i-th factor. Without loss of generality assume $i = 1$:

$$\prod \operatorname{Defo}(\hat{\mathcal{G}}_{i,s}) \twoheadrightarrow \operatorname{Defo}(\hat{\mathcal{G}}_{1,s})$$

and consider

$$\hat{\mathcal{O}}_{\nabla\mathcal{H}^1,s} \twoheadrightarrow \hat{\mathcal{O}}_{\mathcal{C},s} = \hat{\mathcal{O}}_{\nabla\mathcal{H}^1,s}/\mathfrak{a} \twoheadrightarrow R$$

with $R := \hat{\mathcal{O}}_{\nabla\mathcal{H}^1,s}/\mathfrak{a} + (0) \times (1) \times \ldots \times (1)$. Now, if there was an isomorphism

$$\hat{\mathcal{G}}_{1,R} \cong \mathbb{G}_{1,R}$$

this would correspond to the equation:

$$(\operatorname{Defo}(\hat{\mathcal{G}}_{1,s}) \twoheadrightarrow R) = (\operatorname{Defo}(\hat{\mathcal{G}}_{1,s}) \twoheadrightarrow \kappa(s) \subseteq R).$$

This would imply R is equal to the residue field $\kappa(s)$ which is a contradiction.

6 Product structure

Now we want to construct a morphism from the product of our Igusa variety (that covers the central leaf) and "truncated" Rapoport-Zink-spaces to the (Frobenius-pullback of the) moduli space of global G-shtukas.
We show that the morphism from Main Theorem 0.1 is finite as it is quasi-finite and satisfies the valuative criterion for properness.
The construction of the morphism is based on the one given by Mantovan [Man04]. We give a concrete construction of the morphism which is based on the uniformization morphism established by Arasteh Rad and Hartl in [AH14a] and [AH14b] and especially using [AH14a, Proposition 5.7] to get a "new" global G-shtuka (in the same Newton-Stratum).
Throughout this chapter we let $\underline{c} = (c_1, \dots, c_n)$ be a tupel of characteristic places. We denote by $\mathbb{F}_{c_i}$ the residue field at c_i and by $\mathbb{F}_{\underline{c}}$ the compositum of the fields $\mathbb{F}_{c_i}$ inside $\mathbb{F}_q^{\text{alg}}$. We denote by $\nabla\mathcal{H}^1$ the special fibre of $\nabla_n^{H,\hat{Z}_{\underline{c}}}\mathcal{H}^1(C,G)$ above $\underline{c}$.
We fix a tuple of quasi-isogeny classes $(b_i)_i$ and assume that to each of those $b_i \in \mathcal{B}(G_{c_i})$ there exists an element x_i with the properties as defined in 4.3. Remember that we do not assume that $\hat{\sigma}_i^* x_i = x_i$. We write again $\underline{x} := (x_1, \dots, x_n)$ and consider the associated central leaf $\mathcal{C}_{\underline{x}} \subset \mathcal{N}_{\underline{x}}$ in the Newton Stratum of $[\underline{x}]$. Set $\underline{\mathbb{G}}_i = (L^+G_{c_i}, x_i\hat{\sigma}_i^*)$, defined over a field $\mathbb{F}_{x_i}$ in $\mathcal{N}\text{ilp}_{\mathbb{F}_{c_i}[\![\zeta]\!]}$. Let $\underline{\mathcal{G}}^{\text{univ}}$ be the universal global G-shtuka over $\mathcal{C}_{\underline{x}}$ and denote by $\underline{\mathcal{G}}_i^{\text{univ}} := (\mathcal{G}_i^{\text{univ}}, \hat{\tau}_i)_{i=1,\dots,n}$ the tupel of the associated local G_{c_i}-shtukas (via the "global-local functor"). Let $\mathfrak{Ig}_{e_i}$ be the Igusa variety of level e_i associated to the characteristic place c_i and denote by

$$\mathfrak{Ig}_{\underline{e}} = (\mathfrak{Ig}_{e_1} \times_{\nabla\mathcal{H}^1_{\mathbb{F}_q^{\text{alg}}}} \dots \times_{\nabla\mathcal{H}^1_{\mathbb{F}_q^{\text{alg}}}} \mathfrak{Ig}_{e_n})$$

the Igusa variety defined in 5.6. Consider the functor $\mathcal{RZ}_{\underline{\hat{\mathcal{G}}}}^{\hat{Z}_i}$. It is pro-representable by a formal scheme over $\operatorname{Spf} \mathbb{F}_{x_i}((\xi_i))$ (where $\kappa_i((\xi))$ is the reflex ring $R_{\hat{Z}_i}$) which is locally formally of finite type and its underlying reduced subscheme equals $X_{Z_i}(x_i)$. We consider $\mathcal{RZ}_{\underline{\hat{\mathcal{G}}}}^{\hat{Z}_i}$ and the ADLV $X_{Z_i}(x_i)$ as (formal) schemes over $\mathbb{F}_q^{\text{alg}}$. For each i we choose a faithful representation $\varrho : G_{c_i} \to \mathrm{GL}_{r_i}$. Remember the *truncated Rapoport-Zink space* $\mathcal{RZ}_{\hat{Z}_i}^{d_i}$ for a fixed $d_i \geq 0$ from Defintion 2.41 consisting of isomorphism classes of tuples $(\underline{\mathcal{G}}_i, g_i) = (\mathcal{G}_i, \hat{\tau}_i, g_i)$, such that $\hat{\tau}_i = g_i^{-1} x_i \hat{\sigma}_i^* g_i$ is bounded by $\hat{Z}_{c_i}$, and such that $\varrho_*(g_i)$ is bounded by $2d_i\check{\varrho}$, especially $\varrho_*(g_i), \varrho_*(g_i^{-1}) \in z^{d_i(1-r_i)}\mathcal{O}_S[\![z]\!]^{r_i \times r_i}$. We choose the level $\underline{e}$ of the Igusa variety such that $\underline{e} = (e_i)_i$ with

$e_i > 2d_i(r_i - 1)$. This will play a role in Proposition 6.9.
To make it more readable we split the preperations of the construction of the desired product morphism into different sections:
First, we choose a trivialization α_i of each local G_{c_i}-shtuka associated to the universal global G-shtuka over the Igusa-variety in the first part of this chapter. In the second section we construct a quasi-isogeny h_i and show in the third section that h_i is independent of the choice of α_i and descends to the Igusa-variety. In the last part of the chapter we construct the "product morphism" from Main Theorem 0.1 and prove its properties.

6.1 Definition of the trivialization α_i

Definition 6.1. *We choose a presentation $f : S \to \mathcal{Y}_i$ such that the universal complete slope division $(\overline{\mathcal{P}}_i, \eta_i)$ of $\hat{\Gamma}_{c_i}(f^*\underline{\mathcal{G}}^{\text{univ}}) =: \underline{\mathcal{G}}_i$ over $\mathcal{Y}_i$ possesses a trivialization $\alpha_i : \overline{\mathcal{P}}_{i,S} \xrightarrow{\sim} (L^+\hat{\sigma}_i^{-1} I_{\bar{P}_i})_S$ satisfying*

$$\iota_*\alpha_i\, \eta_i^{-1}\hat{\tau}_{\mathcal{G}_i}\, \hat{\sigma}_i^*(\eta_i\, \iota_*\alpha_i^{-1}) = x_i m_i \bar{n}_i \in x_i \cdot L^+ I_{\bar{P}_i}(S)$$

as in Definition 5.4. We have $\mathfrak{Ig}_{e_i} \times_{\mathcal{Y}_i} S = \mathrm{Ig}_{e_i}$. Over Ig_{e_i} there exists the universal j_{e_i}. Let $S' \to \mathrm{Ig}_{e_i}$ be an étale covering over which a representative $j_{e_i} \in (L^+\hat{\sigma}_i^{-1} I_{M_i})(S')$ of the universal j_{e_i} on Ig_{e_i} exists. Since $S' \to S$ is an étale covering by Proposition 5.7, we may replace the original presentation $S \to \mathcal{Y}_{x_i}$ by $S' \to \mathcal{Y}_{x_i}$. It is then also a presentation of $\mathfrak{Ig}_{e_i}$. Now, we write $j_{e_i} \circ \alpha_i$ instead of α_i, where we consider j_{e_i} as an element in $L^+\hat{\sigma}_i^{-1} I_M(S) \subset L^+\hat{\sigma}_i^{-1} I_{\bar{P}}(S)$. Then, the new tuple $(m_i^{\text{new}}, \bar{n}_i^{\text{new}})$ is given by:

$$\begin{aligned} &j_{e_i}\iota_*\alpha_i\eta_i^{-1}\hat{\tau}_{\mathcal{G}_i}\hat{\sigma}_i^*(\eta_i\, \iota_*\alpha_i^{-1} j_{e_i}^{-1}) \\ &= x_i x_i^{-1} j_{e_i} x_i m_i \hat{\sigma}_i^*(j_{e_i}^{-1})\hat{\sigma}_i^*(j_{e_i})\bar{n}_i\hat{\sigma}_i^*(j_{e_i}^{-1}), \end{aligned}$$

so by

$$m_i^{\text{new}} = x_i^{-1} j_{e_i} x_i m_i \hat{\sigma}_i^*(j_{e_i}^{-1}) \in L^+ I_{e_i, M_i}(S) \tag{6.1}$$

and

$$\bar{n}_i^{\text{new}} = \hat{\sigma}_i^*(j_{e_i})\bar{n}_i\hat{\sigma}_i^*(j_{e_i}^{-1}), \tag{6.2}$$

which is an element in $L^+ I_{\bar{N}}(S)$ by Lemma 5.1.

Notation 6.2. *We will from now on shorten the notation: instead of $\iota_*\alpha_i\eta_i^{-1}$ we simply write α_i.*

6.2 Construction of the quasi-isogeny $h_i(l_i)$

Now we construct $h_i(l_i)$ such that we also get a (almost) trivialization of the $\bar{N}_i$-part and show that it (together with the trivialization α_i) descends to the Igusa-variety and that we have some independences.

Lemma 6.3. *In the situation introduced in Definition 6.1 let $e_i \in \mathbb{N}$ and let $l_i(e_i)$ be the number from Condition (vi) from Definition 4.3. Then for every $l_i \geq l_i(e_i)$ there exists a quasi-isogeny $h_i(l_i) \in L^+\hat{\sigma}_i^{(l_i-1)*} I_{\bar{N}_i}(S)$ of local G_{c_i}-shtukas*

$$h_i(l_i) : ((L^+G_{c_i})_S, \hat{\sigma}_i^{l_i*}(x_i m_i)\bar{n}_i'(l_i)) \to ((L^+G_{c_i})_S, \hat{\sigma}_i^{l_i*}(x_i m_i \bar{n}_i)),$$

with $\bar{n}_i'(l_i) \in L^+_{e_i}\hat{\sigma}_i^{l_i*} I_{\bar{N}_i}(S)$, that is

$$\hat{\sigma}_i^{l_i*}(x_i m_i \bar{n}_i)\hat{\sigma}^* h_i(l_i) = h_i(l_i)\hat{\sigma}_i^{l_i*}(x_i m_i)\bar{n}_i'(l_i).$$

Proof. We give an explicit construction of $h_i(l_i)$. For simplicity we write everything without the Index i in this proof.
Start with $h(0) := 1$,

$$\bar{n}'(0) := \bar{n} \in L^+ I_{\bar{N}}(S).$$

Then:

$$\hat{\sigma}^*(xm\bar{n})\hat{\sigma}^* h(1) = h(1)\hat{\sigma}^*(xm)\bar{n}'(1),$$

so set $h(1) := \bar{n}^{-1}$,

$$\begin{aligned}\bar{n}'(1) &= \hat{\sigma}^*(xm)^{-1}\bar{n}\hat{\sigma}^*(xm\bar{n})\hat{\sigma}^*\bar{n}^{-1} \\ &= \hat{\sigma}^*(xm)^{-1}\bar{n}\hat{\sigma}^*(xm) \\ &= \hat{\sigma}^*(m)^{-1}\hat{\sigma}^*(x)^{-1}\bar{n}\hat{\sigma}^*(x)\hat{\sigma}^*(m),\end{aligned}$$

with $\bar{n}'(1) \in L^+_e\hat{\sigma}^* I_{\bar{N}}(S)$. Using Condition (vi) from Definition 4.3 Lemma 5.1 this pushes $\bar{n}'$ into the right direction.
Inductively we set:

$$\begin{aligned}h(l) &:= \hat{\sigma}^{l*}(xm) \prod_{i=l-1\to 0} (\hat{\sigma}^{(i+1)*}(xm)^{-1}\hat{\sigma}^{i*}(\bar{n})^{-1}) \cdot \prod_{i=1\to l-1} (\hat{\sigma}^{i*}(xm)) \\ &= \hat{\sigma}^{(l-1)*}(\bar{n})^{-1} \cdot \hat{\sigma}^{(l-2)*}(\bar{n}'(1)^{-1}) \cdot \ldots \cdot \hat{\sigma}^{0*}\bar{n}'(l-1)^{-1} \\ &\in L^+\hat{\sigma}^{(l-1)*} I_{\bar{N}},\end{aligned}$$

where $\bar{n} = \bar{n}'(0)$. The second equation follows by the definition of $\bar{n}(l)$ beyond.

$$\begin{aligned}\bar{n}'(l) &:= \hat{\sigma}^{l*}(xm)^{-1} \prod_{i=l-1\to 1} (\hat{\sigma}^{i*}(xm)^{-1}) \prod_{i=0\to l-1} (\hat{\sigma}^{i*}(\bar{n})\hat{\sigma}^{(i+1)*}(xm))\cdot \\ &\quad \cdot \hat{\sigma}^{l*}(xm)^{-1} \cdot \hat{\sigma}^{l*}(xm\bar{n})\hat{\sigma}^{(l+1)*}(xm)\cdot \\ &\quad \cdot \prod_{i=l\to 1} (\hat{\sigma}^{(i+1)*}(xm)^{-1}\hat{\sigma}^{i*}(\bar{n})^{-1}) \prod_{i=2\to l} \hat{\sigma}^{i*}(xm) \\ &= (\hat{\sigma}^{l*}(xm)^{-1} \cdot \ldots \cdot \hat{\sigma}^{*}(xm)^{-1})\bar{n}(\hat{\sigma}^{*}(xm) \cdot \ldots \cdot \hat{\sigma}^{l*}(xm)) \\ &= (\hat{\sigma}^{l*}(m)^{-1} \cdot \ldots \cdot (\hat{\sigma}^{l*}(x)^{-1} \cdot \ldots \cdot \hat{\sigma}^{2*}(x)^{-1}\hat{\sigma}^{*}(m)^{-1}\hat{\sigma}^{2*}(x) \cdot \ldots \hat{\sigma}^{l*}(x)))\cdot \\ &\quad \cdot (\hat{\sigma}^{l*}(x)^{-1} \cdot \ldots \cdot \hat{\sigma}^{*}(x)^{-1} \cdot \bar{n} \cdot \hat{\sigma}^{*}(x) \cdot \ldots \cdot \hat{\sigma}^{l*}(x))\cdot \\ &\quad \cdot ((\hat{\sigma}^{l*}(x)^{-1} \cdot \ldots \cdot \hat{\sigma}^{2*}(x)^{-1}\hat{\sigma}^{*}(m)\hat{\sigma}^{2*}(x) \cdot \ldots \hat{\sigma}^{l*}(x)) \cdot \ldots \cdot \hat{\sigma}^{l*}(m)) \\ &\in L_e^+ \hat{\sigma}^{l*} I_{\bar{N}}(S).\end{aligned}$$

"$\in$" holds for $l \geq l(e)$ from Condition (vi) from Definition 4.3 and using Lemma 5.1. (Note in the case where G_c is split connected reductive and x chosen to be P-fundamental this would follow by [HV12, Remark 4.2,e)].

It follows

$$h(l) = \hat{\sigma}^{*} h(l-1) \cdot \bar{n}'(l-1)^{-1}.$$

We have:

$$\begin{aligned}h(l) &= \hat{\sigma}^{(l-1)*}(\bar{n}^{-1})\cdot \\ &\quad \cdot (\hat{\sigma}^{(l-1)*}(xm)^{-1}\hat{\sigma}^{(l-2)*}(\bar{n})^{-1}\hat{\sigma}^{(l-1)*}(xm))\cdot \\ &\quad \cdot \ldots \cdot (\hat{\sigma}^{(l-1)*}(xm)^{-1}) \cdot \ldots \cdot (\hat{\sigma}^{*}(xm)^{-1})\bar{n}^{-1}(\hat{\sigma}^{*}(xm)) \cdot \ldots \cdot \hat{\sigma}^{(l-1)*}(xm).)\end{aligned}$$

and

Lemma 6.4. *In the situation introduced in Definition 6.1 let $d_i \in \mathbb{N}$ with $d_i \leq e_i$ and let $l_i \geq l_i(d_i)$ and $\bar{n}'(l_i)$ be the data from Lemma 6.3. Then over an algebraically closed field K there exists a*

$$\theta_i : ((L^+G_{c_i})_K, \hat{\sigma}_i^{(l_i)*} x_i) = \hat{\sigma}_i^{(l_i)*} \underline{\mathbb{G}}_i \xrightarrow{\sim} ((L^+G_{c_i})_K, \hat{\sigma}_i^{(l_i)*}(x_i) w_i)$$

with $w_i := \hat{\sigma}_i^{(l_i)*}(m_i)\bar{n}_i'(l_i)$ such that $\theta_i \in L_{d_i}^+ \hat{\sigma}_i^{(l_i-1)*} I_{\bar{P}_i}(K)$ for $l_i \geq l_i(d_i)$ from Lemma 6.3.

Proof. For simplicity we write everything without the index i in this proof.
Consider:

$$w_0 := \hat{\sigma}^{(1)*}(x) \cdot \ldots \cdot \hat{\sigma}^{(l)*}(x)\hat{\sigma}^{(l)*}(m) \cdot \bar{n}'(l) \cdot \hat{\sigma}^{(l)*}(x)^{-1} \cdot \ldots \cdot \hat{\sigma}^{(1)*}(x)^{-1},$$

then

$$\begin{aligned}w_0 &:= \hat{\sigma}^{(1)*}(x) \cdot \ldots \cdot \hat{\sigma}^{(l)*}(x)\hat{\sigma}^{(l)*}(m) \cdot \hat{\sigma}^{(l)*}(m)^{-1}\cdot \\ &\quad \cdot \hat{\sigma}^{(l)*}(x)^{-1} \cdot \ldots \cdot \hat{\sigma}^{*}(m)^{-1}\hat{\sigma}^{*}(x)^{-1} \cdot \bar{n}\cdot \\ &\quad \cdot \hat{\sigma}^{*}(x)\hat{\sigma}^{*}(m) \cdot \ldots \cdot \hat{\sigma}^{(l)*}(x)\hat{\sigma}^{(l)*}(m)\cdot \\ &\quad \cdot \hat{\sigma}^{(l)*}(x)^{-1} \cdot \ldots \cdot \hat{\sigma}^{(1)*}(x)^{-1}.\end{aligned}$$

$$\begin{aligned} w_0 \; = \; & \hat{\sigma}^{(1)*}(x)\cdot\ldots\cdot\hat{\sigma}^{(l-1)*}(x)\cdot\hat{\sigma}^{(l-1)*}(m^{-1})\cdot\hat{\sigma}^{(l-1)*}(x^{-1})\cdot\ldots\cdot\hat{\sigma}^{(1)*}(x^{-1})\cdot \\ & \hat{\sigma}^{(1)*}(x)\cdot\ldots\cdot\hat{\sigma}^{(l-2)*}(x)\cdot\hat{\sigma}^{(l-2)*}(m^{-1})\cdot\hat{\sigma}^{(l-2)*}(x^{-1})\cdot\ldots\cdot\hat{\sigma}^{(1)*}(x^{-1})\cdot \\ & \cdot\ldots\cdot\hat{\sigma}^{(1*)}(xm^{-1}x^{-1})\cdot\bar{n}'\cdot\hat{\sigma}^{(1*)}(xmx^{-1})\cdot\ldots\cdot \\ & \hat{\sigma}^{(1)*}(x)\cdot\ldots\cdot\hat{\sigma}^{(l-2)*}(x)\cdot\hat{\sigma}^{(l-2)*}(m)\cdot\hat{\sigma}^{(l-2)*}(x^{-1})\cdot\ldots\cdot\hat{\sigma}^{(1)*}(x^{-1})\cdot \\ & \hat{\sigma}^{(1)*}(x)\cdot\ldots\cdot\hat{\sigma}^{(l-1)*}(x)\cdot\hat{\sigma}^{(l-1)*}(m)\cdot\hat{\sigma}^{(l-1)*}(x^{-1})\cdot\ldots\cdot\hat{\sigma}^{(1)*}(x^{-1})\cdot \\ & \hat{\sigma}^{(1)*}(x)\cdot\ldots\cdot\hat{\sigma}^{(l)*}(x)\cdot\hat{\sigma}^{(l)*}(m)\cdot\hat{\sigma}^{(l)*}(x^{-1})\cdot\ldots\cdot\hat{\sigma}^{(1)*}(x^{-1}) \\ & \in L^+I_{\bar{N}}(K)\cdot L^+I_{d,M}(K). \end{aligned}$$

So we can write

$$\begin{aligned} w &= \hat{\sigma}^{(l)*}(m)\bar{n}'(l) \\ &= \hat{\sigma}^{(l)*}(x)^{-1}\cdot\ldots\cdot\hat{\sigma}^{(2)*}(x)^{-1}\hat{\sigma}^{(1)*}(x)^{-1}\cdot w_0\cdot\hat{\sigma}^{(1)*}(x)\hat{\sigma}^{(2)*}(x)\cdot\ldots\cdot\hat{\sigma}^{(l)*}(x). \end{aligned}$$

By Lemma 4.13 there is a $\theta_0 \in L^+\hat{\sigma}^{-1}I_{\bar{N}}(K)\cdot L^+\hat{\sigma}^{-1}I_{d,M}(K)$, such that $\theta_0 x = x w_0\hat{\sigma}^*\theta_0$.
Now we can set

$$\theta := \hat{\sigma}^{(l-1)*}(x)^{-1}\cdot\ldots\cdot\hat{\sigma}^{0*}(x)^{-1}\cdot\theta_0\cdot\hat{\sigma}^{0*}(x)\cdot\ldots\cdot\hat{\sigma}^{(l-1)*}(x),$$

with $\theta \in L_d^+\hat{\sigma}^{(l-1)*}I_{\bar{N}}(K)\cdot L^+\hat{\sigma}^{(l-1)*}I_{d,M}(K)$ because of Conditions (vi) and (viii) from Definition 4.3. then

$$\begin{aligned} \theta\cdot\hat{\sigma}^{(l)*}(x) &= \hat{\sigma}^{(l-1)*}(x)^{-1}\cdot\ldots\cdot\hat{\sigma}^{(0)*}(x)^{-1}\cdot\theta_0\cdot\hat{\sigma}^{(0)*}(x)\cdot\ldots\cdot\hat{\sigma}^{(l-1)*}(x)\cdot\hat{\sigma}^{(l)*}(x) \\ &= \hat{\sigma}^{(l-1)*}(x)^{-1}\cdot\ldots\cdot\hat{\sigma}^{(0)*}(x)^{-1}\cdot\hat{\sigma}^{(0)*}(x)w_0\hat{\sigma}^*\theta_0\cdot\hat{\sigma}^{(1)*}(x)\cdot\ldots\cdot\hat{\sigma}^{(l)*}(x) \\ &= \hat{\sigma}^{(l)*}(x)\cdot\hat{\sigma}^{(l)*}(m)\cdot\bar{n}'(l)\cdot\hat{\sigma}^*\theta \\ &= \hat{\sigma}^{(l)*}(x)w\hat{\sigma}^*\theta \end{aligned}$$

with $\theta \in L_d^+\hat{\sigma}^{(l-1)*}I_{\bar{P}}(K)$. □

6.3 Independence of $h_i(l_i)$ of the choice of α_i and descend to $\mathfrak{Ig}_{e_i}$

Lemma 6.5. *The quasi-isogeny $h_i(l_i)$ in Lemma 6.3 is independent of the choice of the trivialization α_i for $l_i \geq l(e_i)$.*

Remark 6.6. *We show also that $h_i(l_i)$ together with the trivialization α_i descend to the Igusa variety Ig_{e_i}. Since there α_i is determined by the isomorphism j_{e_i} we get that on the Igusa variety $h_i(l_i)$ is uniquely defined.*

Proof. of Lemma 6.5.
For our fixed presentation $S \to \mathfrak{J}\mathfrak{g}_{e_i}$ we let $S'' := S \times_{\mathfrak{J}\mathfrak{g}_{e_i}} S$ and denote by pr_1 and pr_2 the projection onto the first and second factor. For simplicity we write everything without the index i in this proof. Let $\mathrm{pr}_1^*\alpha\hat{\tau}\hat{\sigma}^*(\mathrm{pr}_1^*\alpha)^{-1} = xm'\bar{n}'$ and $\mathrm{pr}_2^*\alpha\hat{\tau}\hat{\sigma}^*(\mathrm{pr}_2^*\alpha)^{-1} = xm''\bar{n}''$ with $\bar{n}', \bar{n}'' \in L^+I_{\bar{N}}$. Since $l \geq l(e)$ there are quasi-isogenies $h'(l)$, resp. $h(l)$, of local G_c-shtukas and an element $\bar{n}''(l) \in L_e^+\hat{\sigma}^{l*}I_{\bar{N}}(S'')$, resp. $\bar{n}'(l) \in L_e^+\hat{\sigma}^{l*}I_{\bar{N}}(S'')$, with

$$h'(l)^{-1}\hat{\sigma}^{(l)*}(\mathrm{pr}_2^*\alpha\hat{\tau}\hat{\sigma}^*(\mathrm{pr}_2^*\alpha)^{-1})\hat{\sigma}^*h'(l) = \hat{\sigma}^{(l)*}(x)\hat{\sigma}^{(l)*}(m'')\bar{n}''(l),$$

resp.

$$h(l)^{-1}\hat{\sigma}^{(l)*}(\mathrm{pr}_1^*\alpha\hat{\tau}\hat{\sigma}^*(\mathrm{pr}_1^*\alpha)^{-1})\hat{\sigma}^*h(l) = \hat{\sigma}^{(l)*}(x)\hat{\sigma}^{(l)*}(m')\bar{n}'(l).$$

Set

$$\begin{aligned} H'' &:= h'(l)^{-1}\hat{\sigma}^{(l)*}(\mathrm{pr}_2^*\alpha)\hat{\sigma}^{(l)*}(\mathrm{pr}_1^*\alpha)^{-1}h(l) \\ &= h'(l)^{-1}\hat{\sigma}^{(l)*}(\mathrm{pr}_2^*\alpha\mathrm{pr}_1^*\alpha^{-1})h(l). \end{aligned}$$

We have $\mathrm{pr}_2^*\alpha \cdot \mathrm{pr}_1^*\alpha^{-1} \in L^+\hat{\sigma}^{-1}I_{\bar{P}}$ and the image of $\tilde{h} := \mathrm{pr}_2^*\alpha \cdot \mathrm{pr}_1^*\alpha^{-1}$ in $L^+\hat{\sigma}^{-1}I_M/L^+\hat{\sigma}^{-1}I_{e,M}$ is 1. Hence, $H'' \in L_e^+\hat{\sigma}^{l-1*}I_M \cdot L\hat{\sigma}^{(l-1)*}I_{\bar{N}}$.
We have:

$$\begin{aligned} H''\hat{\sigma}^{(l)*}(x)\hat{\sigma}^{(l)*}(m')\bar{n}'(l) &= h'(l)^{-1}\hat{\sigma}^{(l)*}\mathrm{pr}_2^*\alpha(\hat{\sigma}^{(l)*}\mathrm{pr}_1^*\alpha^{-1}h(l)\hat{\sigma}^{(l)*}(x)\hat{\sigma}^{(l)*}(m')\bar{n}'(l)) \\ &= h'(l)^{-1}\hat{\sigma}^{(l)*}\mathrm{pr}_2^*\alpha\sigma^{l*}\hat{\tau}\hat{\sigma}^{(l+1)*}(\mathrm{pr}_1^*\alpha)^{-1}\hat{\sigma}^*h(l) \\ &= (\hat{\sigma}^{(l)*}(x)\hat{\sigma}^{(l)*}(m'')\bar{n}''(l)\hat{\sigma}^*h'(l)^{-1}\hat{\sigma}^{(l+1)*}\mathrm{pr}_2^*\alpha)\cdot \\ &\quad\cdot \hat{\sigma}^{(l+1)*}\mathrm{pr}_1^*\alpha^{-1}\hat{\sigma}^*h(l) \\ &= \hat{\sigma}^{(l)*}(x)\hat{\sigma}^{(l)*}(m'')\bar{n}''(l)\hat{\sigma}^*H''. \end{aligned}$$

So

$$\hat{\sigma}^{(l)*}(x)^{-1}H''\hat{\sigma}^{(l)*}(x)\underbrace{\hat{\sigma}^{(l)*}(m')}_{\in L_e^+\hat{\sigma}^{l*}I_M}\underbrace{\bar{n}'(l)}_{\in L_e^+\hat{\sigma}^{l*}I_{\bar{N}}} = \underbrace{\hat{\sigma}^{(l)*}(m'')}_{\in L_e^+\hat{\sigma}^{l*}I_M}\underbrace{\bar{n}''(l)}_{\in L_e^+\hat{\sigma}^{l*}I_{\bar{N}}}\hat{\sigma}^*H''.$$

Hence

$$\hat{\sigma}^*H'' = \bar{n}''(l)^{-1}\hat{\sigma}^{(l)*}(m'')^{-1}\hat{\sigma}^{(l)*}(x)^{-1}H''\hat{\sigma}^{(l)*}(x)\hat{\sigma}^{(l)*}(m')\cdot\bar{n}'(l).$$

This implies

$$\hat{\sigma}^{2*}H'' = \hat{\sigma}^*\bar{n}''(l)^{-1}\hat{\sigma}^{(l+1)*}(m'')^{-1}\hat{\sigma}^{(l+1)*}(x)^{-1}\hat{\sigma}^*H''\hat{\sigma}^{(l+1)*}(x)\hat{\sigma}^{(l+1)*}(m')\hat{\sigma}^*\bar{n}'(l).$$

That is:

$$\begin{aligned} &\hat{\sigma}^{2*}H'' = \\ &\quad = \hat{\sigma}^*\bar{n}''(l)^{-1}\hat{\sigma}^{(l+1)*}(m'')^{-1}\hat{\sigma}^{(l+1)*}(x)^{-1}\cdot \\ &\quad\cdot \bar{n}''(l)^{-1}\hat{\sigma}^{(l)*}(m'')^{-1}\hat{\sigma}^{(l)*}(x)^{-1}H''\hat{\sigma}^{(l)*}(x)\hat{\sigma}^{(l)*}(m')\bar{n}'(l)\cdot \\ &\quad\cdot \hat{\sigma}^{(l+1)*}(x)\hat{\sigma}^{(l+1)*}(m')\hat{\sigma}^*\bar{n}'(l). \end{aligned}$$

Thus we get:

$$
\begin{aligned}
&\hat{\sigma}^{2*}H'' = \\
&\quad= \hat{\sigma}^{*}\bar{n}''(l)^{-1}\hat{\sigma}^{(l+1)*}(m'')^{-1}\hat{\sigma}^{(l+1)*}(x)^{-1}\bar{n}''(l)^{-1}\hat{\sigma}^{(l+1)*}(x)\cdot \\
&\quad\cdot \hat{\sigma}^{(l+1)*}(x)^{-1}\hat{\sigma}^{(l)*}(m'')^{-1}\hat{\sigma}^{(l)*}(x)^{-1}H''\hat{\sigma}^{(l)*}(x)\hat{\sigma}^{(l)*}(m')\hat{\sigma}^{(l+1)*}(x)\cdot \\
&\quad\cdot \hat{\sigma}^{(l+1)*}(x)^{-1}\bar{n}'(l)\hat{\sigma}^{(l+1)}(x)\hat{\sigma}^{(l+1)*}(m')\hat{\sigma}^{*}\bar{n}'(l).
\end{aligned}
$$

We can iterate this to (for $k \geq 2$):

$$
\begin{aligned}
&\hat{\sigma}^{k*}H'' = \\
&\quad= \underbrace{\hat{\sigma}^{(k-1)*}\bar{n}''(l)^{-1}}_{\in L_e^+}\underbrace{\hat{\sigma}^{(l+k-1)*}(m'')^{-1}}_{\in L_e^+}\cdot \\
&\quad\cdot \underbrace{\hat{\sigma}^{(l+k-1)*}(x)^{-1}\cdot\hat{\sigma}^{(k-2)*}\bar{n}''(l)^{-1}\cdot\hat{\sigma}^{(l+k-1)*}(x)}_{\in L_e^+}\cdot \\
&\quad\cdot \hat{\sigma}^{(l+k-1)*}(x^{-1})\underbrace{\hat{\sigma}^{(l+k-2)*}(m'')^{-1}}_{\in L_e^+}\hat{\sigma}^{(l+k-1)*}(x)\underbrace{\cdot\ldots\cdot}_{\in L_e^+} \\
&\quad\cdot \hat{\sigma}^{(l+k-1)*}(x)^{-1}\hat{\sigma}^{(l+k-2)*}(x)^{-1}\cdot\ldots\cdot\hat{\sigma}^{(l)*}(x)^{-1}H''\hat{\sigma}^{(l)*}(x)\cdot\ldots\cdot \\
&\quad\hat{\sigma}^{(l+k-1)*}(x)\cdot\underbrace{\ldots\cdot}_{\in L_e^+}\cdot\hat{\sigma}^{(l+k-1)*}(x)^{-1}\cdot\underbrace{\hat{\sigma}^{(l+k-2)*}(m')}_{\in L_e^+}\cdot\hat{\sigma}^{(l+k-1)*}(x)\cdot \\
&\quad\cdot \underbrace{\hat{\sigma}^{(l+k-1)*}(x)^{-1}\hat{\sigma}^{(k-2)*}\bar{n}'(l)\hat{\sigma}^{(l+k-1)*}(x)}_{\in L_e^+}\underbrace{\hat{\sigma}^{(l+k-1)*}(m')}_{\in L_e^+}\underbrace{\hat{\sigma}^{(k-1)*}\bar{n}'(l)}_{\in L_e^+}.
\end{aligned}
$$

Write $H'' = H''_M H''_{\bar{N}}$ and consider the term

$$\hat{\sigma}^{(l+k-1)*}(x)^{-1}\hat{\sigma}^{(l+k-2)*}(x)^{-1}\cdot\ldots\cdot\hat{\sigma}^{(l)*}(x)^{-1}H''\hat{\sigma}^{(l)*}(x)\cdot\ldots\cdot\hat{\sigma}^{(l+k-1)*}(x).$$

Then:

$$
\begin{aligned}
&\prod_{i=k-1,\ldots,0}\hat{\sigma}^{(l+i)*}(x)^{-1}\cdot H''\cdot\prod_{i=0,\ldots,k-1}\hat{\sigma}^{(l+i)*}(x) = \\
&= \underbrace{\prod_{i=k-1,\ldots,0}\hat{\sigma}^{(l+i)*}(x)^{-1}\cdot H''_M\cdot\prod_{i=0,\ldots,k-1}\hat{\sigma}^{(l+i)*}(x)}_{\in L^+\hat{\sigma}^{l-1*}I_M}\cdot \\
&\cdot\underbrace{\prod_{i=k-1,\ldots,0}\hat{\sigma}^{(l+i)*}(x)^{-1}\cdot H''_{\bar{N}}\cdot\prod_{i=0,\ldots,k-1}\hat{\sigma}^{(l+i)*}(x)}_{\in L_e^+\hat{\sigma}^{l-1*}I_{\bar{N}}\ for\ k>>0}
\end{aligned}
$$

because of the assumptions on x in Definition 4.3. Then we get $\hat{\sigma}^{k*}H'' \in L^+\hat{\sigma}^{(l-1)*}I_M\cdot L_e^+\hat{\sigma}^{(l-1)*}I_{\bar{N}}$ and so $H'' = H''_M\cdot H''_{\bar{N}} \in L^+\hat{\sigma}^{(l-1)*}I_M\cdot L_e^+\hat{\sigma}^{l-1)*}I_{\bar{N}}$. By Lemma 5.1 we get $H'' \in L^+\hat{\sigma}^{(l-1)*}I_{e,M}\cdot L_e^+\hat{\sigma}^{(l-1)*}I_{\bar{N}}$ and also $H''_M = \hat{\sigma}^{(l)*}(\mathrm{pr}_2^*\alpha\cdot\mathrm{pr}_1^*\alpha^{-1}) \in L^+\hat{\sigma}^{(l-1)*}I_{e,M}$.□

Now we show that $h_i(l_i)$ descends (together with α_i) to the Igusa variety. The proof is very similar to the one of the previous lemma. In order to do this we choose $l_i \geq l_i(e_i)$, such that $\bar{n}'(l_i) \in L_{e_i}^+\hat{\sigma}^{l_i*}I_{\bar{N}_i}$ with the data from Lemma 6.3.

Lemma 6.7. *Let $f : S \to \mathcal{Y}_i$ be a presentation such that the universal complete slope division $(\overline{\mathcal{P}}_i, \eta_i)$ of $\hat{\Gamma}_{c_i}(f^*\underline{\mathcal{G}}^{\text{univ}}) =: \underline{\mathcal{G}}_i$ over $\mathcal{Y}_i$ possesses a triviliaziation α_i as defined in Definition 6.1. The quasi-isogeny $h_i(l_i)$ from Lemma 6.3 with $l_i = l_i(e_i)$ descends to $\mathfrak{Ig}_{e_i}$.*

Proof. Fix a trivialization α_i (note that we already know the independence of the choice of such a trivialization) and for simplicity fix one characteristic c_i, that is fix an index i and write everything without that index. By Lemma 6.3 there is a quasi-isogeny $h(l)$:

$$h(l) : ((L^+G_c)_S, \hat{\sigma}^{(l)*}(xm)\bar{n}'(l)) \to ((L^+G_c)_S, \hat{\sigma}^{(l)*}(xm\bar{n})),$$

that is

$$h(l)^{-1}\hat{\sigma}^{(l)*}(\alpha\hat{\tau}\hat{\sigma}^*\alpha^{-1})\hat{\sigma}^*h(l) = \hat{\sigma}^{(l)*}(xm)\bar{n}'(l)$$

with $l \geq l(e)$ such that $\bar{n}'(l) \in L_e^+\hat{\sigma}^{l*}I_{\bar{N}}(S)$. Now we want to show the descend of $h(l)^{-1}\hat{\sigma}^{(l)*}\alpha$ to the Igusa variety $\mathfrak{Ig}_e$. Set

$$H := \mathrm{pr}_2^*(h(l)^{-1}\hat{\sigma}^{(l)*}\alpha)\mathrm{pr}_1^*(\hat{\sigma}^{(l)*}(\alpha)^{-1}h(l)),$$

where $S'' := S \times_{\mathfrak{Ig}_e} S$ and pr_i denotes the projection onto the i-th factor. Then

$$H \cdot \mathrm{pr}_1^*(\hat{\sigma}^{(l)*}(xm))\mathrm{pr}_1^*(\bar{n}'(l)) = \mathrm{pr}_2^*(\hat{\sigma}^{l*}(xm))\mathrm{pr}_2^*(\bar{n}'(l)) \cdot \hat{\sigma}^*H$$

and so

$$\mathrm{pr}_2^*(\hat{\sigma}^{(l)*}(x))^{-1} \cdot H \cdot \mathrm{pr}_1^*(\hat{\sigma}^{(l)*}(x)) \cdot \underbrace{\mathrm{pr}_1^*(\hat{\sigma}^{l*}(m))}_{\in L_e^+} \cdot \underbrace{\mathrm{pr}_1^*(\bar{n}'(l))}_{\in L_e^+}$$

$$= \underbrace{\mathrm{pr}_2^*(\sigma^{l*}(\hat{m}'))}_{\in L_e^+} \cdot \underbrace{\mathrm{pr}_2^*(\bar{n}'(l))}_{\in L_e^+} \cdot \hat{\sigma}^*H.$$

A similar calculation as in the Lemma 6.5 before (with H'' instead of H) shows that $H \in L^+\hat{\sigma}^{(l-1*)}I_{e,M} \cdot L_e^+\hat{\sigma}^{(l-1)*}I_{\bar{N}}$. □

6.4 The product morphism π

In this section we show that there is a finite product morphism $\pi := \pi^{\underline{d},\underline{e},l}$ as in Main Theorem 0.1. Recall the formal scheme $\mathcal{RZ}_{\hat{Z}_i}$ and the ADLV $X_{Z_i}(x_i)$ from Definition 2.38 which we consider as (formal) schemes over $\mathbb{F}_q^{\text{alg}}$.

Notation 6.8. *We denote by $X_{Z_i}^{d_i}(x_i)$ the reduced subscheme of $\mathcal{RZ}_{\hat{Z}_i}^{d_i}$ and set*

$$X_{\underline{Z}}^{\underline{d}} := (X_{Z_1}^{d_1}(x_1) \times_{\mathbb{F}_q^{\text{alg}}} \ldots \times_{\mathbb{F}_q^{\text{alg}}} X_{Z_n}^{d_n}(x_n)).$$

Note that we have fixed e_i, d_i such that $e_i > 2d_i(r_i - 1)$.

Proposition 6.9. *Let $S \to \mathfrak{Ig}_{e_i}$ (resp. $S \to \mathfrak{Ig}_{\underline{e}}$) and let α_i be the presentation and isomorphism from Definition 6.1. Let $h_i(l_i)$ be as in Lemma 6.3 with $l_i = l_i(e_i)$. Let $S' \to S \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{Z_i}^{d_i}(x_i)$ be an étale covering over which a trivialization*

$$\beta_i : \hat{\sigma}_i^{l_i *}\hat{\mathcal{G}}_i \tilde{\to} (L^+G_{c_i})_{S'}$$

for the universal local G_{c_i}-shtuka $(\hat{\mathcal{G}}_i, g_i)$ over $X_{Z_i}^{d_i}(x_i)$ exists. Then the isomorphism of LG_{c_i}-torsors

$$\hat{\sigma}_i^{l_i *}(\alpha_i)^{-1} h_i(l_i) \hat{\sigma}_i^{l_i *}(g_i)\beta_i^{-1} : LG_{c_i} \to \hat{\sigma}_i^{l_i *} L\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}^{\mathrm{univ}})$$

descends to a quasi-isogeny of local G_{c_i}-shtukas over $\mathfrak{Ig}_{e_i} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{Z_i}^{d_i}(x_i)$ (resp. $\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{Z_i}^{d_i}(x_i)$)

$$\delta_i : \underline{\tilde{\mathcal{G}}}_i \longrightarrow \hat{\sigma}_i^{l_i *}\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}^{\mathrm{univ}})$$

for a suitable local G_{c_i}-shtuka $\underline{\tilde{\mathcal{G}}}_i$ which is bounded by $\hat{Z}_{c_i}$.

Proof. Let $S'' := S \times_{\mathfrak{Ig}_{e_i}} S$ and denote by pr_i the projction onto the i-th factor. Set $l := l_i = l_i(e_i)$ in this proof. Note that $\hat{\sigma}_i^{(l)*}(\alpha_i)^{-1} h_i(l)\hat{\sigma}_i^{(l)*} g_i$ is independent of the choice of α_i up to an automorphism of $L^+G_{c_i}$.
Take $\tilde{\mathcal{G}}_i := (L^+G_{c_i})_S$. We have $\beta_i : \hat{\sigma}_i^{(l)*}\underline{\hat{\mathcal{G}}} \to (L^+G_{c_i}, \beta_i \hat{\sigma}_i^{l*}(g_i^{-1} x_i \hat{\sigma}_i^* g_i)\hat{\sigma}_i^* \beta_i^{-1})$. Set

$$\Gamma := \mathrm{pr}_2^*(\beta_i \hat{\sigma}_i^{(l)*}(g_i)^{-1} h_i(l)^{-1} \hat{\sigma}_i^{(l)*}(\alpha_i)) \cdot \mathrm{pr}_1^*((\beta_i \hat{\sigma}_i^{(l)*}(g_i)^{-1} h_i(l)^{-1}\hat{\sigma}_i^{(l)*}(\alpha_i))^{-1})$$

and then it sufficies to show that $\Gamma \in L^+G_{c_i}(S'')$, where pr_i and S'' are defined as above. Consider

$$\begin{aligned}\Gamma &= \mathrm{pr}_2^*(\beta_i \hat{\sigma}_i^{(l)*}(g_i)^{-1} h_i(l)^{-1}\hat{\sigma}_i^{(l)*}\alpha_i) \cdot \mathrm{pr}_1^*(\hat{\sigma}_i^{(l)*}\alpha_i^{-1} h_i(l)\hat{\sigma}_i^{(l)*}(g_i)\beta_i^{-1}) \\ &= \mathrm{pr}_2^*(\beta_i \hat{\sigma}_i^{(l)*}(g_i)^{-1}) \underbrace{\mathrm{pr}_2^*(h_i(l)^{-1}\hat{\sigma}_i^{(l)*}\alpha_i) \cdot \mathrm{pr}_1^*(\hat{\sigma}_i^{(l)*}\alpha_i^{-1} h_i(l))}_{=H \in L^+\hat{\sigma}_i^{(l-1)*} I_{e_i,M_i} \cdot L^+_{e_i}\hat{\sigma}_i^{(l-1)*} I_{\bar{N}_i}} \cdot \mathrm{pr}_1^*(\hat{\sigma}_i^{(l)*}(g_i)\beta_i^{-1})\end{aligned}$$

$\in L^+G_{c_i}$.

The element H lies in $L^+\hat{\sigma}_i^{(l-1)*} I_{e_i,M_i} \cdot L^+_{e_i}\hat{\sigma}_i^{(l-1)*} I_{\bar{N}_i}$ by the proof of Lemma 6.7 and $\Gamma \in L^+G_{c_i}$ holds by our choice $e_i > 2d_i(r_i - 1)$. In fact, by the definition of $\mathcal{RZ}_{\hat{Z}_i}^{d_i}$ (see Definition 2.41) $\varrho_*(\mathrm{pr}_2^*(\beta_i \hat{\sigma}_i^{(l)*}(g_i)^{-1})\mathrm{pr}_1^*(\hat{\sigma}_i^{(l)*}(g_i)\beta_i^{-1}))$ lies in $z^{2d_i(1-r_i)}\mathcal{O}_S[\![z]\!]^{r_i \times r_i}$. And since $\varrho_*(H)$ lies in $1 + z^{e_i}\mathcal{O}_S[\![z]\!]^{r_i \times r_i}$ we get that $\varrho_*(\Gamma) \in 1 + z\mathcal{O}_S[\![z]\!]^{r_i \times r_i}$. Hence, $\Gamma \in L^+G_{c_i}$.
We use Γ to define an $L^+G_{c_i}$-torsor $\tilde{\mathcal{G}}_i$ over $\mathfrak{Ig}_{e_i} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{Z_i}^{d_i}(x_i)$ together with a trivialization $\gamma_i : (\tilde{\mathcal{G}}_i)_S \tilde{\to} (L^+G_{c_i})_S$, such that $\mathrm{pr}_2^*\gamma_i \cdot \mathrm{pr}_1^*\gamma_i^{-1} = \Gamma$. This corresponds to a

change of the trivialization β_i to γ_i, between $\hat{\sigma}_i^{(l)*}\hat{\mathcal{G}}_i$ and $\tilde{\mathcal{G}}_i$. We make $\tilde{\mathcal{G}}_i$ into a local G_{c_i}-shtuka by equipping it with the following Frobenius:

$$\gamma_i : (\underline{\tilde{\mathcal{G}}}_i)_S \tilde{\to} ((L^+G_{c_i})_S, \beta_i \hat{\sigma}_i^{(l)*}(g_i)^{-1} h_i(l)^{-1} \hat{\sigma}_i^{(l)*}(\alpha_i \hat{\tau}_i \hat{\sigma}_i^* \alpha_i^{-1}) \hat{\sigma}_i^*(h_i(l) \hat{\sigma}_i^{(l)*}(g_i) \beta_i^{-1})).$$

Consider the quasi-isogeny over S

$$\gamma_i^{-1} \beta_i \hat{\sigma}_i^{(l)*} g_i^{-1} h_i(l)^{-1} \hat{\sigma}_i^{(l)*} \alpha_i : \hat{\sigma}_i^{(l)*} \hat{\Gamma}_{c_i}(\underline{\mathcal{G}}^{\text{univ}}) \to \underline{\tilde{\mathcal{G}}}_i.$$

Set

$$\delta_i := \hat{\sigma}_i^{(l)*}(\alpha_i)^{-1} h_i(l) \hat{\sigma}_i^{(l)*} g_i \beta_i^{-1} \gamma_i,$$

so

$$\delta_i : L\tilde{\mathcal{G}}_i \tilde{\to} L\hat{\sigma}_i^{(l)*} \hat{\Gamma}_{c_i}(\mathcal{G}^{\text{univ}}),$$

then

$$\mathrm{pr}_2^* \gamma_i \cdot \mathrm{pr}_2^*(\delta_i^{-1}) \cdot \mathrm{pr}_1^*(\delta_i) \cdot \mathrm{pr}_1^* \gamma_i^{-1} = \Gamma.$$

It follows that $\mathrm{pr}_1^*(\delta_i) = \mathrm{pr}_2^*(\delta_i)$. Moreover, δ_i is compatible with the Frobenii. Therefore, δ_i descends to a quasi-isogeny $\delta_i : \underline{\tilde{\mathcal{G}}}_i \to \hat{\sigma}_i^{(l)*} \hat{\Gamma}_{c_i}(\underline{\mathcal{G}}^{\text{univ}})$ as desired.
It remains to show that $\underline{\tilde{\mathcal{G}}}_i$ is bounded by $\hat{Z}_{c_i}$. We compute

$$\begin{aligned}
&\hat{\sigma}_i^{(l)*} g_i^{-1} h_i(l)^{-1} \hat{\sigma}_i^{(l)*}(\alpha_i \hat{\tau}_i \hat{\sigma}_i^*(\alpha_i)^{-1}) \hat{\sigma}_i^* h_i(l) \hat{\sigma}_i^*(\hat{\sigma}_i^{(l)*} g_i) \\
&= \hat{\sigma}_i^{(l)*} g_i^{-1} h_i(l)^{-1} \hat{\sigma}_i^{(l)*}(x_i m_i \bar{n}_i) \hat{\sigma}_i^* h_i(l) \hat{\sigma}_i^*(\hat{\sigma}_i^{(l)*} g_i) \\
&= \hat{\sigma}_i^{(l)*} g_i^{-1} \hat{\sigma}_i^{(l)*} x_i \hat{\sigma}_i^{(l)*}(m_i) \bar{n}_i'(l) \hat{\sigma}_i^* h_i(l)^{-1} \hat{\sigma}_i^* h_i(l) \hat{\sigma}_i^*(\hat{\sigma}_i^{(l)*} g_i) \\
&= \hat{\sigma}_i^{(l)*}(g_i^{-1} x_i) \underbrace{\hat{\sigma}_i^{(l)*}(m_i)}_{\in L^+_{e_i}} \underbrace{\bar{n}_i'(l)}_{\in L^+_{e_i}} \hat{\sigma}_i^*(\hat{\sigma}_i^{(l)*} g_i) \\
&= \underbrace{\hat{\sigma}_i^{(l)*}(g_i^{-1} x_i \hat{\sigma}_i^* g_i)}_{\in \hat{Z}_{c_i}} \underbrace{(\hat{\sigma}_i^*(\hat{\sigma}_i^{(l)*} g_i^{-1}) \hat{\sigma}_i^{(l)*}(m_i) \bar{n}_i'(l) \hat{\sigma}_i^*(\hat{\sigma}_i^{(l)*} g_i))}_{\in L^+G_{c_i} \quad (\textit{because } e_i + 2d_i(1-r_i) \geq 1,\ l \geq l(e_i))} \\
&\in \hat{Z}_{c_i}.
\end{aligned}$$

Remember that the inequality $e_i + 2d_i(1 - r_i) \geq 1$ holds by our choice of e_i at the beginning of this chapter. □

Corollary 6.10. *If $y \in \left(\mathfrak{Ig}_{e_i} \times_{\mathbb{F}_q^{\text{alg}}} X_{Z_i}^{d_i}(x_i)\right)(L)$ is a point with values in an algebraically closed field L, whose second component is $(\underline{\hat{\mathcal{G}}}_{i,y}, g_{i,y}) \in X_{Z_i}^{d_i}(x_i)(L)$ then the fibre $\underline{\tilde{\mathcal{G}}}_{i,y}$ at y of the local shtuka $\underline{\tilde{\mathcal{G}}}_i$ from Proposition 6.9 is isomorphic to $\hat{\sigma}_i^{l_i*} \underline{\hat{\mathcal{G}}}_{i,y}$.*

Proof. By Lemma 6.4 there exists an isomorphism $\theta_i \in L^+_{e_i} \hat{\sigma}_i^{(l_i-1)*} I_{\bar{P}_i}(L)$ in the following chain of isomorphisms

$$((L^+G_{c_i})_L, \hat{\sigma}_i^{l_i*} x_i) \xrightarrow[\sim]{\theta_i} ((L^+G_{c_i})_L, \hat{\sigma}_i^{l_i*}(x_i) w_i) \xrightarrow[\sim]{\hat{\sigma}_i^{l_i*}(\alpha_{i,y}^{-1}) h_i(l_i)_y} \hat{\sigma}_i^{l_i*} \hat{\Gamma}_{c_i}(\underline{\mathcal{G}}^{\text{univ}})_y,$$

that is $\hat{\sigma}_i^{l_i*}(\alpha_{i,y}^{-1})h_i(l_i)_y\,\theta_i\cdot\hat{\sigma}_i^{l_i*}(x_i) = \hat{\sigma}_i^{l_i*}(\hat{\tau}_{i,y})\circ\hat{\sigma}_i^*(\hat{\sigma}_i^{l_i*}(\alpha_{i,y}^{-1})h_i(l_i)_y\,\theta_i)$, where $\hat{\sigma}_i^{l_i*}(\hat{\tau}_{i,y})$ is the Frobenius of $\hat{\sigma}_i^{l_i*}\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}^{\text{univ}})_y$. As in the computation at the end of the proof of Proposition 6.9, the Frobenius of $\underline{\tilde{\mathcal{G}}}_{i,y}$ is computed as

$$\begin{aligned}
&\hat{\sigma}_i^{l_i*}(g_{i,y}^{-1})h_i(l_i)_y^{-1}\hat{\sigma}_i^{l_i*}(\alpha_{i,y}\hat{\tau}_{i,y}\hat{\sigma}_i^*(\alpha_{i,y}^{-1}))\hat{\sigma}_i^*h_i(l_i)_y\hat{\sigma}_i^*(\hat{\sigma}_i^{l_i*}g_{i,y})\\
&\qquad\qquad = \hat{\sigma}_i^{l_i*}(g_{i,y}^{-1})\cdot\theta_i\cdot\hat{\sigma}_i^{l_i*}(x_i)\cdot\hat{\sigma}^*(\theta_i^{-1})\cdot\hat{\sigma}_i^*(\hat{\sigma}_i^{l_i*}g_{i,y})\\
&\quad = \big(\hat{\sigma}_i^{l_i*}(g_{i,y}^{-1})\theta_i\hat{\sigma}_i^{l_i*}(g_{i,y})\big)\cdot\hat{\sigma}_i^{l_i*}(g_{i,y}^{-1}x_i\hat{\sigma}_i^*g_{i,y})\cdot\hat{\sigma}_i^*\big(\hat{\sigma}_i^{l_i*}(g_{i,y}^{-1})\theta_i^{-1}\hat{\sigma}_i^{l_i*}(g_{i,y})\big).
\end{aligned}$$

Since $\theta_i \in L_{e_i}^+\hat{\sigma}_i^{(l_i-1)*}I_{\bar{P}_i}(L)$ and $g_{i,y}$ is bounded by $2d_i\check{\varrho}$ the inequality $e_i+2d_i(1-r_i)\geq 1$ implies that $\hat{\sigma}_i^{l_i*}(g_{i,y}^{-1})\theta_i\hat{\sigma}_i^{l_i*}(g_{i,y})\in L^+G_{c_i}(L)$. Since $\hat{\sigma}_i^{l_i*}(g_{i,y}^{-1}x_i\hat{\sigma}_i^*g_{i,y})$ is the Frobenius of $\hat{\sigma}_i^{l_i*}\underline{\hat{\mathcal{G}}}_{i,y}$, the element $\hat{\sigma}_i^{l_i*}(g_{i,y}^{-1})\theta_i\hat{\sigma}_i^{l_i*}(g_{i,y})\in L^+G_{c_i}(L)$ provides an isomorphism of local G_{c_i}-shtukas from $\hat{\sigma}_i^{l_i*}\underline{\hat{\mathcal{G}}}_{i,y}$ to $\underline{\tilde{\mathcal{G}}}_i$. □

Now we will prove Main Theorem 0.1, the main result of this thesis.
We let $\underline{e} = (e_i)_i$ be such that $e_i + 2d_i(1-r_i)\geq 1$ and denote by l a multiple of all $\deg c_i$ such that $l\geq l_i(e_i)\cdot\deg c_i$ for the number $l_i(e_i)$ from Definition 4.3 (vi).

Remark 6.11. *The tuple $(\underline{d},\underline{e},l)$ is chosen large enough with respect to the partial order on $\mathbb{Z}^n$, that is*

$$(d_i,e_i,l)\leq(d_i',e_i',l') :\Leftrightarrow d_i\leq d_i',\ l\leq l',\ e_i\leq e_i'\ \forall i.$$

In the remaining part of this chapter we will show that there is a finite, surjective morphism of stacks as in Main Theorem 0.1

$$\begin{aligned}
\pi := \pi^{\underline{d},\underline{e},l} : \mathfrak{I}\mathfrak{g}_{\underline{e}}\times_{\mathbb{F}_q^{\text{alg}}} X_{\underline{Z}}^{\underline{d}}(\underline{x}) &\to \sigma^{l*}\mathcal{N}_{\underline{x}}\times_{\mathbb{F}_{\underline{x}}}\mathbb{F}_q^{\text{alg}}\subset\sigma^{l*}\nabla\mathcal{H}^1_{\mathbb{F}_q\text{alg}}\\
((\underline{\mathcal{G}}^{univ},(j_{e_i})_i),(g_i)_i) &\mapsto \delta^*\sigma^{l*}\underline{\mathcal{G}}^{univ}
\end{aligned}$$

with $\delta^*\sigma^{l*}\underline{\mathcal{G}}^{\text{univ}} := \delta_1^*\circ\ldots\circ\delta_n^*\sigma^{l*}\underline{\mathcal{G}}^{\text{univ}}$ where

$$\delta_i : \underline{\tilde{\mathcal{G}}}_i \xrightarrow{\text{qis}} \hat{\sigma}^{l_i*}\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}^{\text{univ}}) = \sigma^{l*}\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}^{\text{univ}})$$

was obtained in Proposition 6.9 for $l_i = l/[\mathbb{F}_{c_i}:\mathbb{F}_q]$ and $\delta^*\sigma^{l*}\underline{\mathcal{G}}^{\text{univ}}$ is based on the construction from Proposition 2.21. Here $X_{\underline{Z}}^{\underline{d}}$ denotes again the product

$$\begin{aligned}
X_{\underline{Z}}^{\underline{d}} &:= X_{\underline{Z}}^{\underline{d}}(\underline{x})\\
&:= (X_{Z_1}^{d_1}(x_1)\times_{\mathbb{F}_q^{\text{alg}}}\ldots\times_{\mathbb{F}_q^{\text{alg}}}X_{Z_n}^{d_n}(x_n)).
\end{aligned}$$

The proof of Main Theorem 0.1 is given in several steps and lemmas to make it more readable and comprehensible.

The next Lemma says that the quasi-isogeny group J_{x_i} is contained in the Levi. This result is useful to consider the fibres of the morphism π.

Lemma 6.12. *(i)* $J_{x_i}(\mathbb{F}_{c_i}((z))) \subseteq L\hat{\sigma}_i^{-1}M_i(\mathbb{F}_{x_i})$.

(ii) $\mathrm{Aut}\big((L^+I_i)_{\mathbb{F}_{x_i}^{alg}}, x_i\hat{\sigma}_i^*\big) \subset L^+\hat{\sigma}_i^{-1}I_{M_i}(\mathbb{F}_{x_i}^{alg})$

Proof. (i) $J_{x_i}(\mathbb{F}_{c_i}((z)))$ is defined as

$$J_{x_i}(\mathbb{F}_{c_i}((z))) := \big\{a \in LG_{c_i}(\mathbb{F}_q^{\mathrm{alg}}) \;:\; ax_i = x_i\hat{\sigma}_i^*a\big\}$$

(and acts on $\mathcal{RZ}_{\hat{Z}_i}^{d_i}$, $\hat{\tau}_i = g^{-1}x_i\hat{\sigma}_i^*g$ via $a : g \mapsto ag$).
Write $a = nm\bar{n}$ (decomposition in $LG_{c_i}(\mathbb{F}_q^{\mathrm{alg}})$ with $n \in L\hat{\sigma}_i^{-1}N_i$, $m \in L\hat{\sigma}_i^{-1}M_i$, $\bar{n} \in L\hat{\sigma}_i^{-1}\bar{N}_i$). Then $nm\bar{n} = x_i\hat{\sigma}_i^*(a)x_i^{-1}$ for an arbitrary $a \in J_{x_i}(\mathbb{F}_{c_i}((z)))$, so

$$\hat{\sigma}_i^*a = x_i^{-1}nx_ix_i^{-1}mx_ix_i^{-1}\bar{n}x_i.$$

This implies

a)

$$\begin{aligned}
\hat{\sigma}_i^{2*}(a) &= \hat{\sigma}_i^{2*}(nm\bar{n})\\
&= \hat{\sigma}_i^*(x_i)^{-1}\hat{\sigma}_i^*(a)\hat{\sigma}_i^*(x_i)\\
&= \hat{\sigma}_i^*(x_i)^{-1}x_i^{-1}ax_i\hat{\sigma}_i^*(x_i)\\
&= \hat{\sigma}_i^*(x_i)^{-1}x_i^{-1}nm\bar{n}x_i\hat{\sigma}_i^*(x_i)\\
&= \hat{\sigma}_i^*(x_i)^{-1}x_i^{-1}nx_i\hat{\sigma}_i^*(x_i)\hat{\sigma}_i^*(x_i)^{-1}x_i^{-1}mx_i\hat{\sigma}_i^*(x_i)\hat{\sigma}_i^*(x_i)^{-1}x_i^{-1}\bar{n}x_i\hat{\sigma}_i^*(x_i).
\end{aligned}$$

So for $l \geq 2$ we get:

$$\begin{aligned}
\hat{\sigma}_i^{l*}(a) = (&\hat{\sigma}_i^{(l-1)*}(x_i)^{-1}\cdot\ldots\cdot x_i^{-1}nx_i\cdot\ldots\cdot\hat{\sigma}_i^{(l-1)*}(x_i))\cdot\\
&\cdot(\hat{\sigma}_i^{(l-1)*}(x_i)^{-1}\cdot\ldots\cdot x_i^{-1}mx_i\cdot\ldots\cdot\hat{\sigma}_i^{(l-1)*}(x_i))\cdot\\
&\cdot(\hat{\sigma}_i^{(l-1)*}(x_i)^{-1}\cdot\ldots\cdot x_i^{-1}\bar{n}x_i\cdot\ldots\cdot\hat{\sigma}_i^{(l-1)*}(x_i)).
\end{aligned}$$

By assumption (see Definition 4.3) on the properties of x_i we have

$$\hat{\sigma}_i^{l*}(x_i)^{-1}\cdot\ldots\cdot\hat{\sigma}_i^*(x_i^{-1})\bar{N}_i\hat{\sigma}_i^*(x_i)\cdot\ldots\cdot\hat{\sigma}_i^{l*}(x_i) \subset \hat{\sigma}_i^{l*}\bar{N}_i,$$

for all l.
By Condition (vi) from Definition 4.3 we get

$$\hat{\sigma}_i^{(l-1)*}(x_i)^{-1}\cdot\ldots\cdot x_i^{-1}\bar{n}x_i\cdot\ldots\cdot\hat{\sigma}_i^{(l-1)*}(x_i) \to 1 \; for\; l \to \infty.$$

Thus $\bar{n} = 1$.

b) And on the other side

$$\begin{aligned}
a = nm\bar{n} = (&x_i\cdot\ldots\cdot\hat{\sigma}_i^{(l-1)*}(x_i)\hat{\sigma}_i^{l*}(n)\hat{\sigma}_i^{(l-1)*}(x_i)^{-1}\cdot\ldots\cdot x_i^{-1})\cdot\\
&\cdot(x_i\cdot\ldots\cdot\hat{\sigma}_i^{(l-1)*}(x_i)\hat{\sigma}_i^{l*}(m)\hat{\sigma}_i^{(l-1)*}(x_i)^{-1}\cdot\ldots\cdot x_i^{-1})\cdot\\
&\cdot(x_i\cdot\ldots\cdot\hat{\sigma}_i^{(l-1)*}(x_i)\hat{\sigma}_i^{l*}(\bar{n})\hat{\sigma}_i^{(l-1)*}(x_i)^{-1}\cdot\ldots\cdot x_i^{-1}).
\end{aligned}$$

As we have

$$\hat{\sigma}_i^*(x_i)\cdot\ldots\cdot\hat{\sigma}_i^{l*}(x_i)\hat{\sigma}_i^{l*}(N_i)\hat{\sigma}_i^{l*}(x_i)^{-1}\cdot\ldots\cdot\hat{\sigma}_i^* x_i^{-1} \subset N_i,$$

for all l, we get by Condition (vi) from Definition 4.3

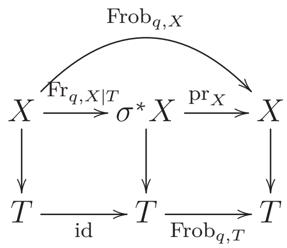
$$x_i\cdot\ldots\cdot\hat{\sigma}_i^{(l-1)*}(x_i)\hat{\sigma}_i^{l*}(n)\hat{\sigma}_i^{(l-1)*}(x_i)^{-1}\cdot\ldots\cdot x_i^{-1} \to 1,\ for\ l \to \infty.$$

Thus $n = 1$.

From (a) and (b) follows that $n = 1$, $\bar{n} = 1$, $a = m \in L\hat{\sigma}_i^{-1}M_i(\mathbb{F}_{x_i})$ as claimed.

(ii) This follows by the first part of the proof. □

Definition 6.13. *For an $\mathbb{F}_p$-scheme X over an $\mathbb{F}_p$-scheme T we denote by* $\mathrm{pr} := \mathrm{pr}_X$ *the projection as defined in the following diagram:*



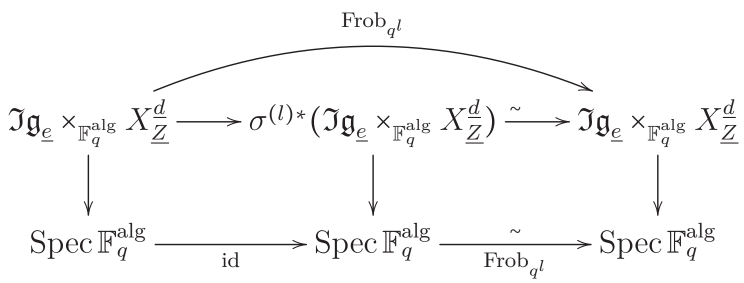
where the left diagram is commutative and the right one is cartesian and where Fr *(resp.* Frob*) denotes the relative (resp. absolute) Frobenius.*

Now we give the construction of the "product morphism":

Proposition 6.14. *We let $\underline{e} = (e_i)_i$ be such that $e_i + 2d_i(1 - r_i) \geq 1$ and denote by l a multiple of all $\deg c_i$ such that $l \geq l_i(e_i)\cdot \deg c_i$ for the number $l_i(e_i)$ from Definition 4.3 (vi) and let $\underline{d} := (d_i)_i$ for $i = 1, \ldots, n$. There exists an $\mathbb{F}_q^{\mathrm{alg}}$-morphism*

$$\pi^{\underline{d},\underline{e},l} : \mathfrak{I}\mathfrak{g}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}}(\underline{x}) \to \sigma^{l*}\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}.$$

Proof. Let $l_i := l/\deg c_i = l/[\mathbb{F}_{c_i} : \mathbb{F}_q]$ and $\sigma^{l*} = \hat{\sigma}_i^{l_i *}$. Note that this l_i depends on the charactersitic c_i. We use the abbreviation $X_{\underline{Z}}^{\underline{d}} := X_{\underline{Z}}^{\underline{d}}(\underline{x})$. Consider



where the composed map in the first line is Frob_{q^l} and the left square is commutative, the right one is cartesian.

We start with a point $f \in (\mathfrak{I}\mathfrak{g}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}})(R)$, that is a morphism

$$f : \operatorname{Spec} R \to \mathfrak{I}\mathfrak{g}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}},$$

which is given by a tuple $(\underline{\mathcal{G}}^{\mathrm{univ}}, \bar{\gamma}, (j_{e_i})_i, (g_i)_i)$, where $\underline{\mathcal{G}}^{\mathrm{univ}}$ is the universal global G-shtuka over the central leaf $\mathcal{C}_{\underline{x}}$, $\bar{\gamma}$ the associated H-level-structure and $(j_{e_i})_i \in \mathfrak{I}\mathfrak{g}_{\underline{e}}$ and $(g_i)_i \in X_{\underline{Z}}^{\underline{d}}$.

Now consider the ring R in a new way as an $\mathbb{F}_q^{\mathrm{alg}}$-algebra, namely as

$$\begin{aligned}\tilde{R} &:= (R, \mathrm{Frob}_{q^l} \circ (\mathbb{F}_q^{\mathrm{alg}} \to R)) \\ &= (R, (\mathbb{F}_q^{\mathrm{alg}} \to R) \circ \mathrm{Frob}_{q^l}) \\ (&= R).\end{aligned}$$

Remember that our aim is to construct a point in $\sigma^{l*} \nabla \mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}(R)$.

After passing to $\tilde{R}$ our tupel becomes accesible for the uniformization morphism, that is we construct a new tupel pulled back under the Frobenius and after pulling back we have the existence of $h_i(l_i)$, which (almost) trivializes the $\bar{N}$-part.

Therefore, consider the following diagram:

$$\begin{array}{ccccc} \operatorname{Spec} \tilde{R} & & \xrightarrow{\mathrm{Frob}_{q^l}} & & \operatorname{Spec} R \\ \Big\downarrow{\scriptstyle \bar{\gamma} = H\gamma \in H \backslash \mathrm{Isom}^{\otimes}(\check{\mathcal{V}}_{\underline{\mathcal{G}}^{\mathrm{univ}}}, \omega^{\circ})} & & \searrow{\scriptstyle f \circ \mathrm{Frob}_{q^l}} & & \Big\downarrow{\scriptstyle f} \\ \mathfrak{I}\mathfrak{g}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}} & \longrightarrow & \sigma^{l*}(\mathfrak{I}\mathfrak{g}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}}) & \xrightarrow{\sim} & \mathfrak{I}\mathfrak{g}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}} \end{array}$$

where the morphism f corresponds to the tuple $(\underline{\mathcal{G}}^{\mathrm{univ}}, \bar{\gamma}, (j_{e_i})_i, (g_i)_i)$ and $f \circ \mathrm{Frob}_{q^l}$ corresponds to $\sigma^{l*}(\underline{\mathcal{G}}^{\mathrm{univ}}, \bar{\gamma}, (j_{e_i})_i, (g_i)_i)$.

The reason why we passed to $\tilde{R}$ is the following advantage:

Over $\tilde{R}$ we can consider σ^{l*} and then exists $h_i(l_i)$ and we can define δ_i as

$$\delta_i := \hat{\sigma}_i^{(l_i)*} \alpha_i^{-1} \circ h_i(l_i) \circ \hat{\sigma}_i^{(l_i)*} g_i \circ \beta_i^{-1} \circ \gamma_i : \underline{\tilde{\mathcal{G}}}_i \longrightarrow \sigma^{l*} \hat{\Gamma}_{c_i}(\underline{\mathcal{G}}^{\mathrm{univ}})$$

which is the quasi-isogeny from Proposition 6.9.

We have a point, that is a global G-shtuka $\tilde{f} \in \nabla \mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}(\tilde{R})$, hence a morphism $\tilde{f}$: $\operatorname{Spec} \tilde{R} \to \nabla \mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$ given by the tuple $(\underline{\mathcal{G}}, H\sigma^{l*}\gamma \circ \check{\mathcal{V}}_{\delta})$ with $\underline{\mathcal{G}} := \delta_1^* \circ \ldots \circ \delta_n^*(\underline{\mathcal{G}}^{\mathrm{univ}})$, where

$$\delta : \underline{\mathcal{G}} \overset{\mathrm{qis}}{\to} \sigma^{l*} \underline{\mathcal{G}}^{\mathrm{univ}}$$

is the quasi-isogeny of global G-shtukas with $\hat{\Gamma}_{c_i}(\delta) = \delta_i$ by Proposition 2.21.

So $\delta_1^* \cdot \ldots \cdot \delta_n^*(\sigma^{l*}\underline{\mathcal{G}}^{\mathrm{univ}})$ is the "new" global G-shtuka and the level structure is given by

$$H\backslash \mathrm{Isom}^{\otimes}(\omega^{\circ}, \check{\mathcal{V}}_{\underline{\mathcal{G}}}) \longrightarrow H\backslash \mathrm{Isom}^{\otimes}(\omega^{\circ}, \check{\mathcal{V}}_{\sigma^{l*}\underline{\mathcal{G}}^{\mathrm{univ}}})$$
$$\sigma^{l*}\bar{\gamma} \circ \check{\mathcal{V}}_{\delta} \mapsto \sigma^{l*}\bar{\gamma}$$

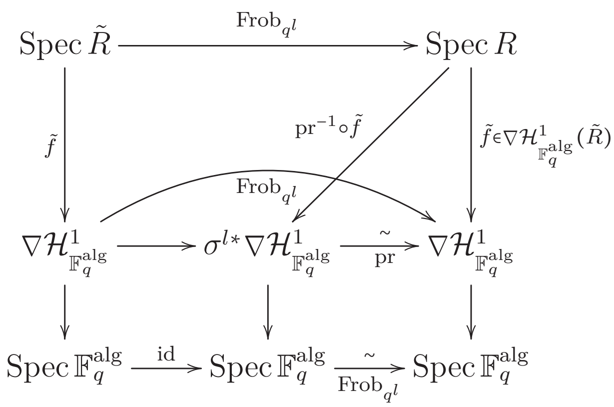
induced by the tensor-isomorphism $\check{\mathcal{V}}_{\delta} : \check{\mathcal{V}}_{\underline{\mathcal{G}}} \to \check{\mathcal{V}}_{\sigma^{l*}\underline{\mathcal{G}}^{\mathrm{univ}}}$. By the definition of the Tate-module (by τ-invariants) this is well-defined.
Thus, we have:

$$(\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}})(R) \to (\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}})(\tilde{R}) \to \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}(\tilde{R})$$
$$f \mapsto f \circ \mathrm{Frob}_{q^l} \mapsto \tilde{f}$$

but what we need is an R-valued point. We get that in the following way:
Briefly speaking:

$$\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}(\tilde{R}) \longrightarrow \sigma^{l*}\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}(R)$$
$$\tilde{f} \mapsto \mathrm{pr}^{-1} \circ \tilde{f},$$

that means:
Consider the following diagram:



That is we can define:

$$\pi^{\underline{d},\underline{e},l} : (\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}})(R) \longrightarrow (\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}})(\tilde{R}) \longrightarrow \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}(\tilde{R}) \longrightarrow \sigma^{l*}\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}(R)$$

$$f \mapsto f \circ \mathrm{Frob}_{q^l} \mapsto \tilde{f} \mapsto \mathrm{pr}^{-1} \circ \tilde{f}$$

So, there exists an $\mathbb{F}_q^{\mathrm{alg}}$-morphism

$$\pi^{\underline{d},\underline{e},l} : (\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}}) \to \sigma^{l*}\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$$

. $\square$

Now we show the properties of the morphism π.

Proposition 6.15. *With the previous constructions the following holds:*

(i) For $l' \geq l$ und l large enough suitable to $\underline{d}, \underline{e}$ (that is such that $\pi^{\underline{d},\underline{e},l}$ exists) the following holds on $\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}}(\underline{x})$:

$$\mathrm{Fr}_{q^{l'-l}, \sigma^{l*}\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} \circ \pi^{\underline{d},\underline{e},l} = \pi^{\underline{d},\underline{e},l'},$$

where $\pi^{l'} := \pi^{\underline{d},\underline{e},l'}$ is a morphism

$$\pi^{l'} : \mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}}(\underline{x}) \longrightarrow \sigma^{l'*}\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}.$$

(ii) For $\underline{d}' \geq \underline{d}$ let $\mathrm{incl}^{d,d'} : X_{\underline{Z}}^{\underline{d}}(\underline{x}) \hookrightarrow X_{\underline{Z}}^{\underline{d}'}(\underline{x})$ be the inclusion (and the identity on $\mathfrak{Ig}_{\underline{e}}$) and for $\underline{e}', l'$ suitable to $\underline{d}'$ (that is, such that $\pi^{\underline{d}',\underline{e}',l'}$ exists):

$$\pi^{\underline{d}',\underline{e}',l'} \circ \mathrm{incl}^{d,d'} = \pi^{\underline{d},\underline{e}',l'} =: \pi^{\underline{d}}$$

as a morphism

$$\pi^{\underline{d}} : \mathfrak{Ig}_{\underline{e}'} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}}(\underline{x}) \longrightarrow \sigma^{l'*}\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}.$$

(iii) For $\underline{e}' \geq \underline{e}$, l large enough (that is, such that $\pi^{\underline{d},\underline{e},l}$ exists) and for $q_{e',e} : \mathfrak{Ig}_{\underline{e}'} \twoheadrightarrow \mathfrak{Ig}_{\underline{e}}$ the projection (and the identity on $X_{\underline{Z}}^{\underline{d}}(\underline{x})$):

$$\pi^{\underline{d},\underline{e},l} \circ q_{e',e} = \pi^{\underline{d},\underline{e}',l} =: \pi^{\underline{e}'},$$

as a morphism

$$\pi^{\underline{e}'} : \mathfrak{Ig}_{\underline{e}'} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}}(\underline{x}) \longrightarrow \sigma^{l*}\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}.$$

Proof. (i) The idea is to show $\sigma^{(l'-l)*}\underline{\mathcal{G}} \cong \underline{\mathcal{G}}'$ in $\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$, where $\underline{\mathcal{G}}$ is the "associated" global G-shtuka to $(\underline{d}, \underline{e}, l)$ that is the image of $\pi^{\underline{d},\underline{e},l}$, and $\underline{\mathcal{G}}'$ the image under $\pi^{\underline{d},\underline{e},l'}$.

In terms of the notations given in the construction of $\pi^{\underline{d},\underline{e},l}$ it is sufficient to show:

$$\tilde{f} \circ \mathrm{Frob}_{q^{l'-l},R} = \tilde{f}',$$

where $\tilde{f}$ is defined with $h_i(l_i)$ and this is sufficient to show because $\tilde{f} \circ \mathrm{Frob}_{q^{l'-l},R}$ corresponds to $\sigma^{(l'-l)*}\underline{\mathcal{G}}$ and $\tilde{f}'$ corresponds to $\underline{\mathcal{G}}'$ and is defined with $h_i(l_i')$.

In order to show this consider

$$\delta : \underline{\mathcal{G}} \longrightarrow \sigma^{l*}\underline{\mathcal{G}}^{\mathrm{univ}},$$

with

$$\hat{\Gamma}_{c_i}(\delta) = \delta_i = \hat{\sigma}_i^{(l_i)*}\alpha_i^{-1} \circ h_i(l_i) \circ \hat{\sigma}_i^{(l_i)*} g_i \circ \beta_i^{-1} \circ \gamma_i$$

and

$$\delta' : \underline{\mathcal{G}}' \longrightarrow \sigma^{(l')*}\underline{\mathcal{G}}^{\mathrm{univ}},$$

with

$$\hat{\Gamma}_{c_i}(\delta') = \delta_i' = \hat{\sigma}_i^{(l_i')*}\alpha_i^{-1} \circ h_i(l_i') \circ \hat{\sigma}_i^{(l_i')*} g_i \circ (\beta_i')^{-1} \circ \gamma_i'.$$

By Lemma 6.3 we have

$$h_i(l_i') = \hat{\sigma}_i^{(l_i'-l_i)*} h_i(l_i) \cdot A$$

with

$$A = \hat{\sigma}_i^{(l_i'-l_i-1)*}\bar{n}'(l_i) \cdot \ldots \hat{\sigma}_i^{0*}\bar{n}'(l_i'-1) \in L_{e_i}^{+}\hat{\sigma}_i^{l_i'-1} I_{\bar{N}_i}\,.$$

So

$$\begin{aligned} h_i(l_i') \circ \hat{\sigma}_i^{(l_i')*} g_i &= \hat{\sigma}_i^{(l_i'-l_i)*} h_i(l_i) \cdot A \circ \hat{\sigma}_i^{(l_i')*} g_i \\ &= \hat{\sigma}_i^{(l_i'-l_i)*} h_i(l_i) \circ \hat{\sigma}_i^{(l_i')*} g_i \circ A' \end{aligned}$$

for an isomorphism

$$\begin{aligned} A' &:= (\hat{\sigma}_i^{(l_i')*}(g_i)^{-1}\hat{\sigma}_i^{(l_i'-l_i-1)*}\bar{n}'(l_i)\hat{\sigma}_i^{(l_i')*}(g_i)) \cdot \ldots \cdot (\hat{\sigma}_i^{(l_i')*}(g_i)^{-1}\hat{\sigma}_i^{0*}\bar{n}'(l_i'-1)\hat{\sigma}_i^{(l_i')*}(g_i)) \\ &\in L^{+}\hat{\sigma}_i^{(l_i'-1)*} I_{\bar{N}_i} \end{aligned}$$

because of $e_i > 2d_i(r_i - 1)$ and using Lemma 5.1. So we have:

$$\begin{array}{ccc} \sigma^{(l'-l)*}\underline{\mathcal{G}} & \longrightarrow & \sigma^{(l')*}\underline{\mathcal{G}}^{\mathrm{univ}} \\ \big\downarrow & & \big\downarrow \mathrm{id} \\ \underline{\mathcal{G}}' & \xrightarrow{\ \delta'\ } & \sigma^{(l')*}\underline{\mathcal{G}}^{\mathrm{univ}} \end{array}$$

where $\sigma^{l'*}\underline{\mathcal{G}}^{\mathrm{univ}}$ in the lower line is given by $\hat{\Gamma}_{c_i}(\delta')$ and the one in the upper line is given by:

$$\hat{\sigma}_i^{(l_i'-l_i)*}\hat{\Gamma}_{c_i}(\delta) = \hat{\sigma}_i^{(l_i')*}(\alpha_i)^{-1} \circ \hat{\sigma}_i^{(l_i'-l_i)*}(h_i(l_i)) \circ \hat{\sigma}_i^{(l_i'-l_i)*}(g_i) \circ \hat{\sigma}_i^{(l_i'-l_i)*}(\beta_i^{-1}\gamma_i).$$

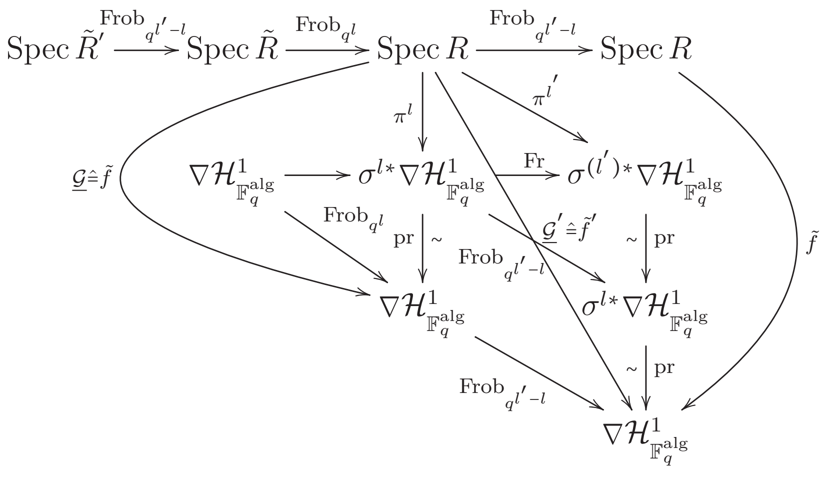
Therefore, $\hat{\sigma}_i^{(l_i')*}\underline{\mathcal{G}} \longrightarrow \underline{\mathcal{G}}'$ is given by:

$$(\gamma_i')^{-1} \circ \beta_i' \circ A' \circ \hat{\sigma}_i^{(l_i'-l_i)*}(\beta_i^{-1}\gamma_i).$$

This is an isomorphism at c_i and also outside of c_i.
Therefore,

$$\sigma^{(l'-l)*}\underline{\mathcal{G}} \cong \underline{\mathcal{G}}' \text{ in } \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}.$$

We illustrate this in the following diagram:



So:

$$
\begin{aligned}
\tilde{f}' &= \mathrm{pr} \circ \mathrm{pr} \circ \pi^{l'} \\
&= \mathrm{pr} \circ \mathrm{pr} \circ \mathrm{Fr}_{q^{(l'-l)},\sigma^{l*}\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} \circ \pi^l \\
&= \mathrm{pr} \circ \mathrm{Frob}_{q^{(l'-l)},\sigma^{l*}\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} \circ \pi^l \\
&= \mathrm{Frob}_{q^{(l'-l)},\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}} \circ \mathrm{pr} \circ \pi^l \qquad = \tilde{f} \circ \mathrm{Frob}_{q^{(l'-l)},R}.
\end{aligned}
$$

That is $\tilde{f}' = \tilde{f} \circ \mathrm{Frob}_{\hat{q}^{l'-l},R}$ and this corresponds to the desired isomorphism $\underline{\mathcal{G}}' \cong \sigma^{(l'-l)*}\underline{\mathcal{G}}$.

(ii) Since $\underline{d}' \geq \underline{d}$ we can choose $g_i' := g_i$ and thus $\delta_i = \delta_i'$ and so $\delta^*(\sigma^{l'*}\underline{\mathcal{G}}^{\mathrm{univ}}) = (\delta')^*(\sigma^{l'*}\underline{\mathcal{G}}^{\mathrm{univ}})$.

(iii) Since our construction is independent of the choice of α_i we can choose α_i as a trivialization also suitable to the level e_i' and this menas that again as in (ii) the global G-shtukas are the same. □

In the following, we prove that π is proper as a morphism to the Newton stratum $\sigma^{l*}\mathcal{N}_{\underline{x}}$. Note that $\mathcal{N}_{\underline{x}}$ is not closed in $\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$.

Proposition 6.16. *$\pi^{\underline{d},\underline{e},l}$ is proper as a morphism to the Newton stratum $\sigma^{l*}\mathcal{N}_{\underline{x}} \times_{\mathbb{F}_{\underline{c}}} \mathbb{F}_q^{\mathrm{alg}} \subset \sigma^{l*}\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$.*

Proof. By [LMB00, Théoréme 7.10] we can prove the claim with the valuative criterion for properness.
Let R be a complete discrete valuation ring with algebraically closed residue field. Denote its fraction field by K.

We have to show that for a K-valued point $y \in (\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}})(K)$ that maps to a point $\pi^{\underline{d},\underline{e},l}(y) \in \sigma^{l*}\mathcal{N}_{\underline{x}}(R)$ there is a unique lift of y to $\bar{y} \in (\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}})(R)$ that makes the following diagram commutative:

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{\ y\ } & (\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}}) \\ {\scriptstyle i}\big\downarrow & \overset{\bar{y}}{\nearrow} & \big\downarrow{\scriptstyle \pi^{\underline{d},\underline{e},l}} \\ \operatorname{Spec} R & \xrightarrow[\ s\ ]{} & \sigma^{l*}\mathcal{N}_{\underline{x}} \end{array}$$

Let $y = (j_{e_i}, g_i)_{i,K} \in (\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}})(K)$. The idea now is to use the fact that $\mathcal{RZ}_{\hat{Z}_i}^{d_i}$ is proper by Proposition 2.42. Now let $\underline{\mathcal{G}}$ be the image of y in $\sigma^{l*}\mathcal{N}_{\underline{x}}$, that is the global G-shtuka associated to $\pi^{\underline{d},\underline{e},l}(j_{e_i} g_i)$. By definition, that means the following:

$$\pi^{\underline{d},\underline{e},l}(j_{e_i}, g_i)_K = (\delta_1^* \cdot \ldots \cdot \delta_n^*)\sigma^{l*}(\underline{\mathcal{G}}_K^{\mathrm{univ}}) = \underline{\mathcal{G}}_K$$

and

$$\underline{\mathcal{G}}_K \underset{\mathrm{qis}}{\overset{\delta}{\longrightarrow}} \sigma^{l*}(\underline{\mathcal{G}}_K^{\mathrm{univ}}).$$

As before let

$$\tilde{R} := (R, (\mathbb{F}_q^{\mathrm{alg}} \to R) \circ \mathrm{Frob}_{q^l, \mathbb{F}_q^{\mathrm{alg}}})$$

and denote by $\tilde{K}$ its fraction field. Especially, one has:

$$\sigma^{l*}(\underline{\mathcal{G}}_K^{\mathrm{univ}}) = (\sigma^{l*}\underline{\mathcal{G}}^{\mathrm{univ}})_{\tilde{K}}.$$

So, by the assumption of the valuative criterion, there is a $\underline{\mathcal{G}} \in \sigma^{l*}(\mathcal{N}_{\underline{x}})(R)$, that is in the Newton stratum, given. This induces as before (that is, corresponds by the isomorphism pr to) $\underline{\mathcal{G}} \in \mathcal{N}_{\underline{x}}(\tilde{R})$. Then

$$z := \hat{\Gamma}_{c_i}(\sigma^{l*}\underline{\mathcal{G}}^{\mathrm{univ}}, \delta^{-1}) \in \mathcal{RZ}_{\hat{\Gamma}_{c_i}(\underline{\mathcal{G}})}^{d_i}(\tilde{K})$$

and by Proposition 2.42 and using that G_{c_i} is parahoric we know that $\mathcal{RZ}_{\hat{\Gamma}_{c_i}(\underline{\mathcal{G}})}^{d_i} \to \operatorname{Spec} \tilde{R}$ is proper, so there exists a unique lift $\bar{z}$ of z with $\bar{z} \in \mathcal{RZ}_{\hat{\Gamma}_{c_i}(\underline{\mathcal{G}})}^{d_i}(\tilde{R})$ and by Proposition 2.21 this implies that there exists a unique global G-shtuka

$$\tilde{\underline{\mathcal{G}}} \in \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}(\tilde{R}), \tilde{\delta}^{-1} : \tilde{\underline{\mathcal{G}}} \overset{\mathrm{qis}}{\to} \underline{\mathcal{G}}.$$

Over $\tilde{K}$ we have the following commutative diagram:

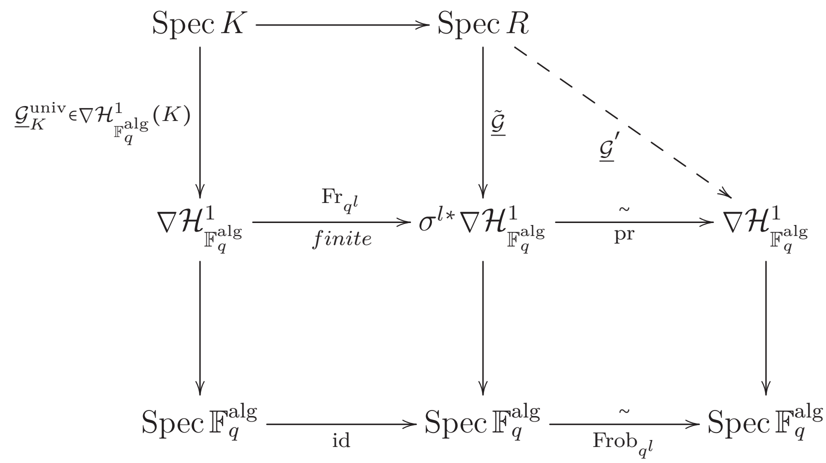
$$\begin{array}{ccc} \tilde{\delta}_{\tilde{K}}^{-1} : \tilde{\underline{\mathcal{G}}}_{\tilde{K}} & \xrightarrow{\ \mathrm{qis}\ } & \underline{\mathcal{G}}_{\tilde{K}} \\ {\scriptstyle \cong}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ \sigma^{l*}\underline{\mathcal{G}}_{\tilde{K}}^{\mathrm{univ}} & \longrightarrow & \underline{\mathcal{G}}_K \end{array}$$

The following holds: $\underline{\mathcal{G}} \in \mathcal{N}_{\underline{x}}(\tilde{R})$ and $\underline{\tilde{\mathcal{G}}} \in \nabla\mathcal{H}^1_{\mathbb{F}_q^{\text{alg}}}(\tilde{R})$ is quasi-isogenous to $\underline{\mathcal{G}}$, this implies

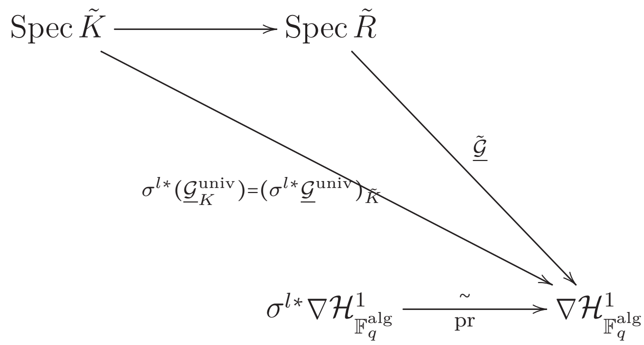
$$\underline{\tilde{\mathcal{G}}} \in \mathcal{N}_{\underline{x}}(\tilde{R}).$$

In the remaining part we construct our "new" point, that is the suitable tuple (j_{e_i}, g_i).
First, we construct (and this shows the existence of) $(\bar{j}_{e_i})_i \in \mathfrak{Ig}_{\underline{e}}(R)$ which is a lift of $(j_{e_i})_i \in \mathfrak{Ig}_{\underline{e}}(K)$ with $q_e(j_{e_i}) = \underline{\mathcal{G}}'$ which is defined in the next diagram where q_e is the projection $\mathfrak{Ig}_{\underline{e}} \to \mathcal{C}_{\underline{x}}$.
In order to do this consider the following diagrams:

(i)



The global G-shtuka $\underline{\tilde{\mathcal{G}}} \in \sigma^{l*}\nabla\mathcal{H}^1_{\mathbb{F}_q^{\text{alg}}}(R)$ is induced via pr^{-1}, more explicit by the following:



Since $\nabla\mathcal{H}^1_{\mathbb{F}_q^{\text{alg}}}$ is locally of finite type

$$\text{Fr}_{q^l} : \nabla\mathcal{H}^1_{\mathbb{F}_q^{\text{alg}}} \to \sigma^{l*}\nabla\mathcal{H}^1_{\mathbb{F}_q^{\text{alg}}}$$

is finite and so there exists a unique $\underline{\mathcal{G}}' \in \nabla\mathcal{H}^1_{\mathbb{F}_q^{\text{alg}}}(R)$ with

$$\underline{\tilde{\mathcal{G}}}_{\tilde{R}} = (\sigma^{l*}\underline{\mathcal{G}}')_{\tilde{R}},$$

so

$$\underline{\tilde{\mathcal{G}}}_K = (\hat{\sigma}^{(l)*}\underline{\mathcal{G}}')_{\tilde{K}} = \hat{\sigma}^{(l)*}(\underline{\mathcal{G}}'_K),$$

and with

$$\underline{\mathcal{G}}'_K = \underline{\mathcal{G}}^{\mathrm{univ}}_K.$$

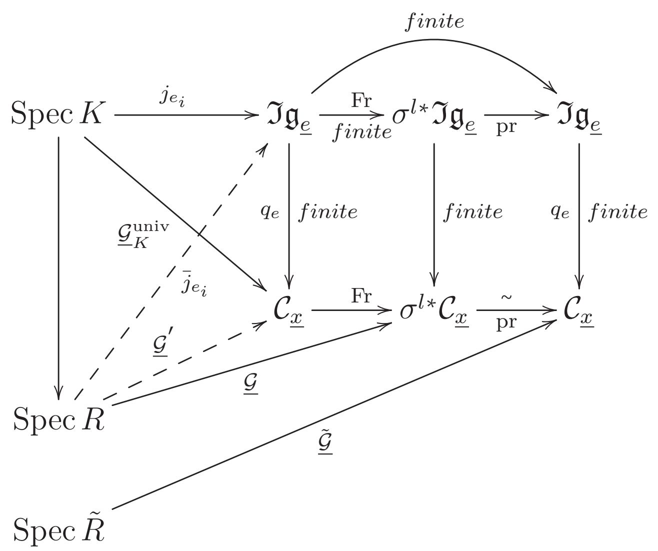
(ii) $\underline{\mathcal{G}}'_{\tilde{K}} = \underline{\mathcal{G}}^{\mathrm{univ}}_{\tilde{K}}$ and $\underline{\mathcal{G}}^{\mathrm{univ}}_{\tilde{K}} \in \mathcal{C}_{\underline{x}}(\tilde{K}) \subseteq \mathcal{N}_{\underline{x}}(\tilde{R})$. Since $\mathcal{C}_{\underline{x}}$ is closed in $\mathcal{N}_{\underline{x}}$, this implies

$$\underline{\mathcal{G}}' \in \mathcal{C}_{\underline{x}}(\tilde{R}).$$

Note that $\underline{\mathcal{G}}' \in \mathcal{N}_{\underline{x}}(R)$ because the Newton polygon at the algebraically closed residue field κ_R of R can be calculated as follows: The quasi-isogeny $\tilde{\delta}^{-1} : \underline{\mathcal{G}} \to \underline{\mathcal{G}}$ induces a quasi-isogeny $\eta : \hat{\Gamma}_{c_i}(\underline{\tilde{\mathcal{G}}}_{\kappa_R}) \to (L^+G_{c_i}, x_i)$ which in turn induces a quasi-isogeny

$$\hat{\sigma}_i^{-l_i*}\eta : \hat{\Gamma}_{c_i}(\underline{\mathcal{G}}'_{\kappa_R}) = \hat{\Gamma}_{c_i}(\hat{\sigma}_i^{-l_i*}\underline{\tilde{\mathcal{G}}}_{\kappa_R}) \to (L^+G_{c_i}, \hat{\sigma}_i^{-l_i*}x_i).$$

The latter is quasi-isogenuous to $(L^+G_{c_i}, x_i)$ via $\hat{\sigma}_i^{-1*}(x_i)\cdots\hat{\sigma}_i^{-l_i*}(x_i)$.
One could also use the above to get $\underline{\mathcal{G}}' =: \underline{\mathcal{G}}^{\mathrm{univ}}_R$ ($\underline{\mathcal{G}}' \in \mathcal{C}_{\underline{x}}(R)$) by using the facts that Frob and Fr are finite and that the Igusa variety is finite over the central leaf $\mathcal{C}_{\underline{x}}$:



Here $\operatorname{Spec} R \overset{\underline{\mathcal{G}}}{\to} \sigma^{l*}\mathcal{C}_{\underline{x}}$ is induced by the morphism $\operatorname{Spec}\tilde{R} \overset{\underline{\tilde{\mathcal{G}}}}{\to} \mathcal{C}_{\underline{x}}$ from part (i). It exists $(\bar{j}_{e_i})_i \in \mathfrak{Ig}_{\underline{e}}(R)$ because of the finiteness of

$$\mathfrak{Ig}_{\underline{e}} \to \sigma^{l*}\mathfrak{Ig}_{\underline{e}} \to \sigma^{l*}\mathcal{C}_{\underline{x}},$$

then $q_e(j_{e_i})$, $\underline{\mathcal{G}}' \in \nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}(R)$, $q_e(j_{e_i})_K = \underline{\mathcal{G}}^{\mathrm{univ}}_K \cong \underline{\mathcal{G}}'_K$ and $\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}$ is separated, so $q_e(j_{e_i}) = \underline{\mathcal{G}}'$ and $\bar{j}_{e_i} \in \mathfrak{Ig}_{e_i}(R)$ is a lift of $j_{e_i} \in \mathfrak{Ig}_{e_i}(K)$ with $q_e(j_{e_i}) = \underline{\mathcal{G}}'$ and we can set $\underline{\mathcal{G}}' =: \underline{\mathcal{G}}^{\mathrm{univ}}_R$.
Here $\mathfrak{Ig}_{\underline{e}} \overset{\mathrm{Fr}}{\to} \sigma^{l*}\mathfrak{Ig}_{\underline{e}}$ is finite because of the following general fact:

Assume S is a scheme of positive characteristic and denote by Frob the absolute Frobenius. Let $X \to S$ be étale, then the relative Frobenius Fr is an isomorphism and since the question to be finite is local on both X and S we may assume X is étale over $\mathbb{A}^r_S$ and both X and S affine. So we can write $S = \operatorname{Spec} A$, then $X = \operatorname{Spec} A[X_1, \dots, X_r]$ and the relative Frobenius is given by the A-linear map

$$\begin{aligned} A[Y_1, \dots, Y_r] &\longrightarrow A[X_1, \dots, X_r] \\ Y_i &\mapsto X_i^q, \end{aligned}$$

so an explicit basis is $\left\{X_i^j \;:\; i = 1, \dots, r;\, j = 1, \dots, q^r - 1\right\}$ and we get a finitely generated module.

Now we calculate the associated $(\bar{g}_i)_i \in X^{\underline{d}}_{\underline{Z}}(R)$. Since $X^{\underline{d}}_{\underline{Z}}$ is proper over $\mathbb{F}_q^{\mathrm{alg}}$, the point $(g_i)_i \in X^{\underline{d}}_{\underline{Z}}(K)$ lifts uniquely to a point $(\bar{g}_i)_i \in X^{\underline{d}}_{\underline{Z}}(R)$. It remains to show that $\pi^{\underline{d},\underline{e},l}(\bar{j}_{e_i}, \bar{g}_i) = \underline{\mathcal{G}}$ in $\mathcal{N}_{\underline{x}}(R)$.
Remember that R is a complete discrete valuation ring with algebraically closed residue field over which there exists a trivialization, that is there is an isomorphism $\gamma_i : \hat{\Gamma}_{c_i}(\underline{\mathcal{G}})_{\tilde{R}} \xrightarrow{\sim} (L^+G_{c_i})_{\tilde{R}}$. Note that we do not have to be careful about the trivialization β_i because we can assume that β_i is the identity. We have

$$(\underline{\mathcal{G}})_{\tilde{R}} \xrightarrow[\text{qis}]{(\tilde{\delta}^{-1})} \underline{\tilde{\mathcal{G}}}_{\tilde{R}} = \sigma^{l*}\underline{\mathcal{G}}'_R =: \sigma^{l*}\underline{\mathcal{G}}^{\mathrm{univ}}_R.$$

Set

$$\delta_i := \hat{\Gamma}_{c_i}((\tilde{\delta^{-1}})) : \hat{\Gamma}_{c_i}(\underline{\mathcal{G}}_{\tilde{R}}) \xrightarrow{\text{qis}} (\sigma^{l*}\underline{\mathcal{G}}')_{\tilde{R}}.$$

So, this is our “old” δ_i we started with and generically both coincide, so in particular g is generically the “old” one. The idea is to calculate g_i via:

$$\hat{\sigma}_i^{(l_i)*}(g_i) := h_i(l_i)^{-1} \circ \hat{\sigma}_i^{(l_i)*}(\alpha_i) \circ \delta_i \circ \gamma_i^{-1} : (L^+G_{c_i})_{\tilde{R}} \to (L^+G_{c_i})_{\tilde{R}},$$

where $\hat{\sigma}_i^{(l_i)*}\alpha_i$ is induced by $\hat{\sigma}_i^{(l_i)*}j_{e_i} \in \mathfrak{Ig}_{e_i}(\tilde{R})$ which exists, and $h_i(l_i)$ and $\hat{\sigma}_i^{(l_i)*}\alpha_i$ are uniquely determined on $\hat{\sigma}_i^{(l_i)*}\mathfrak{Ig}_{e_i}$.
Set

$$(\hat{\sigma}_i^{(l_i)*}\hat{\underline{\mathcal{G}}})_{\tilde{R}} := ((L^+G_{c_i})_{\tilde{R}}, \hat{\sigma}_i^{(l_i)*}(g_i^{-1}x_i\hat{\sigma}_i^* g_i)).$$

We have $\hat{\sigma}_i^{l_i*}g_i \in X^{d_i}_{Z_i}(x_i)$ because of the following: $X^{d_i}_{Z_i}(x_i)$ is defined over κ_i, so there is a lift of the K-valued point g_i to an R-valued point $\bar{g}_i$. Now, the δ_i constructed above is the same as the one constructed with this $\bar{g}_i$. This holds because

$$\begin{aligned} \hat{\sigma}_i^{l_i*}g_i &= h_i(l_i)^{-1}\hat{\sigma}_i^{l_i*}(\alpha_i)\delta_i\gamma_i^{-1} \\ &= \hat{\sigma}_i^{l_i*}\bar{g}_i \end{aligned}$$

at the generic K-valued point in the flag variety which is separated. This shows that $\pi^{\underline{d},\underline{e},l}(\bar{j}_{e_i}, \bar{g}_i) = \underline{\mathcal{G}}$ in $\mathcal{N}_{\underline{x}}(R)$ and proves the proposition. □

Proposition 6.17. *The morphism*

$$\pi := \pi^{\underline{d},\underline{e},l} : \mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X^{\underline{d}}_{\underline{Z}}(\underline{x}) \to \sigma^{l*}\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}.$$

is quasi-finite.

Proof. We calculate the fibres of π and show that it is a finite set.
Let $K = K^{\mathrm{alg}}$ be an algebraically closed field and fix a point

$$y = (j_{e_i,i}, g_i)_{i,K} \in (\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X^{\underline{d}}_{\underline{Z}}(\underline{x}))(K).$$

We calculate $\pi^{-1}(\pi(y))$, so let $y' = (j'_{e_i}, g'_i) \in \pi^{-1}(\pi(y))$ and $y'' = (j''_{e_i}, g''_i) \in \pi^{-1}(\pi(y))$ arbitrary points. Since $\pi(y') = \pi(y)$ there is a quasi-isogeny $\varphi' : \underline{\mathcal{G}}_y \to \underline{\mathcal{G}}_{y'}$ between the global G-shtukas $\underline{\mathcal{G}}_y$ and $\underline{\mathcal{G}}_{y'}$ corresponding to the points $\pi(y)$ and $\pi(y')$ in $\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}(K)$. This quasi-isogeny φ' is compatible with the H-level structure and an isomorphism at all c_i. We consider the following well-defind map:

$$\begin{aligned} \pi^{-1}(\pi(y)) &\longrightarrow \mathrm{SET} \\ y' &\mapsto (\overline{(b'_i)^{-1}}, \overline{(a'_i)^{-1}})_i, \end{aligned}$$

where $\overline{(b'_i)^{-1}}$ and $\overline{(a'_i)^{-1}}$ are defined beyond.
In order to show the finiteness of the fibres of π it is sufficient to show that the set "SET" is finite and the above map is injective.
Let $J_{\hat{\sigma}_i^{(l_i)*}x_i}$ be the quasi-isogeny group of $\hat{\sigma}_i^{(l_i)*}\underline{\mathbb{G}}_i = (L^+G_{c_i}, \hat{\sigma}_i^{l_i*}(x_i))$. We obtain $\overline{(b'_i)^{-1}}$ and $\overline{(a'_i)^{-1}}$ in the following way:

(i) Consider the quasi-isogeny

$$\hat{\sigma}_i^{(l_i)*}\underline{\mathbb{G}}_i \xrightarrow{\theta'_i b'_i} (L^+G_{c_i}, \hat{\sigma}_i^{(l_i)*}(x) \cdot w'_i)$$

which is defined by (see also Figure 6.1)

$$b'_i := \underbrace{(\theta'_i)^{-1}}_{\in L^+_{e_i}G_{c_i}} \hat{\sigma}^{(l_i)*} g'_i \beta^{-1}_{i,y'} \gamma_{i,y'} \hat{\Gamma}_{c_i}(\varphi') \gamma^{-1}_{i,y} \beta_{i,y} \hat{\sigma}_i^{(l_i)*} g_i^{-1} \theta_i,$$

where

$$\hat{\sigma}_i^{(l_i)*}(x_i) \cdot w'_i = h_i(l_i)^{-1}_{y'} \hat{\sigma}_i^{(l_i)*}(\alpha_{i,y'} \hat{\tau}_{y'} \hat{\sigma}_i^* \alpha^{-1}_{i,y'}) \hat{\sigma}_i^* h_i(l_i)_{y'}.$$

Here we fixed a $\theta_i \in L^+\hat{\sigma}_i^{(l_i-1)*} I_{e_i,M_i} \cdot L^+_{e_i}\hat{\sigma}_i^{(l_i-1)*} I_{\bar{N}_i}$ as in Lemma 6.4 and for each point y' with $\pi(y) = \pi(y')$ such a θ'_i (in the same group). We write

$$b'_i := (\theta'_i)^{-1} \cdot \theta'_i b'_i : \hat{\sigma}_i^{(l_i)*}\underline{\mathbb{G}}_i \xrightarrow{\mathrm{qis}} \hat{\sigma}^{(l_i)*}\underline{\mathbb{G}}_i$$

Here we define $\overline{b_i'}$ by

$$\begin{aligned} J_{\hat{\sigma}_i^{(l_i)*}x_i} &\longrightarrow L\hat{\sigma}_i^{(l_i-1)*}M_i/L^+\hat{\sigma}_i^{(l_i-1)*}I_{e_i,M_i} \\ (b_i')^{-1} &\mapsto \overline{(\theta_i' b_i')^{-1}} =: \overline{(b_i')^{-1}}. \end{aligned}$$

Since $g_i, g_i' \in X_{Z_i}^{d_i}$ the quasi-isogeny b_i' is bounded by $4d_i\check{\varrho}$. If one chooses another $\tilde{\theta}_i'$ at y' and the associated $\tilde{b}_i'$ then

$$(\tilde{\theta}_i')^{-1}\theta_i' \in L^+\hat{\sigma}_i^{(l_i-1)*}I_{e_i,M_i} \cdot L^+_{e_i}\hat{\sigma}_i^{(l_i-1)*}I_{\bar{N}_i}.$$

Since b_i' and $\tilde{b}_i' \in L\hat{\sigma}_i^{(l_i-1)*}M_i$ it is $(\tilde{\theta}_i')^{-1}\theta_i' \in L^+\hat{\sigma}_i^{(l_i-1)*}I_{e_i,M_i}$ by Lemma 6.12 (i). Hence, the image of b_i' in $L\hat{\sigma}_i^{(l_i-1)*}M_i/L^+\hat{\sigma}_i^{(l_i-1)*}I_{e_i,M_i}$ is well defined (as it is independent of the choice of θ_i'). Since it is in the image of $J_{\hat{\sigma}_i^{l_i*}x_i}$ and bounded by $4d_i\check{\varrho}$, there are only finitely many $\overline{b_i'}$.

(ii) For each i we fix isomorphisms (induced by the definition of the central leaf) $f_i : \underline{\mathbb{G}}_i \to \hat{\Gamma}_{c_i}(\underline{\mathcal{G}}^{\text{univ}})$ at y and $f_i' : \underline{\mathbb{G}}_i \to \hat{\Gamma}_{c_i}(\underline{\mathcal{G}}^{\text{univ}})$ at y' such that $\alpha_{i,y} \circ f_i$ and also $\alpha_{i,y'} \circ f_i' \in L^+\hat{\sigma}_i^{-1}I_{e_i,M_i} \cdot L^+\hat{\sigma}_i^{-1}I_{\bar{N}_i}$ which is possible by Lemma 4.13. Consider the quasi-isogeny

$$a_i' := \hat{\sigma}_i^{l_i*}(f_i')^{-1}\delta_i'\hat{\Gamma}_{c_i}(\varphi')\delta_i^{-1}\hat{\sigma}_i^{(l_i)*}(f_i)\,.$$

We write

$$a_i' := \hat{\sigma}_i^{(l_i)*}(\alpha_{i,y'}f_i')^{-1}\hat{\sigma}_i^{(l_i)*}(\alpha_{i,y'}f_i')a_i' : \hat{\sigma}_i^{(l_i)*}\underline{\mathbb{G}}_i \xrightarrow{\text{qis}} \hat{\sigma}_i^{(l_i)*}\underline{\mathbb{G}}_i$$

and define $\overline{a'}$ by:

$$\begin{aligned} J_{\hat{\sigma}_i^{(l_i)*}x_i} &\longrightarrow (L\hat{\sigma}_i^{(l_i-1)*}M_i)/(L^+\hat{\sigma}_i^{(l_i-1)*}I_{e_i,M_i}) \\ (a_i')^{-1} &\mapsto \overline{(a_i')^{-1}\hat{\sigma}_i^{(l_i)*}(\alpha_{i,y'}f_i')^{-1}} =: \overline{(a_i')^{-1}}. \end{aligned}$$

Since $g_i, g_i' \in X_{Z_i}^{d_i}(x_i)$ the quasi-isogeny a_i' is bounded by $4d_i\check{\varrho}$.
If one chooses another $\tilde{f}_i'$ at y' and the associated $\tilde{a}_i'$ then

$$(\tilde{f}_i')^{-1}f_i' \in L^+\hat{\sigma}_i^{(l_i-1)*}I_{e_i,M_i} \cdot L^+\hat{\sigma}_i^{(l_i-1)*}I_{\bar{N}_i}.$$

Since a_i' and also $\tilde{a}_i' \in L\hat{\sigma}_i^{(l_i-1)*}M_i$ we obtain $(\tilde{f}_i')^{-1}f_i' \in L^+\hat{\sigma}_i^{(l_i-1)*}I_{e_i,M_i}$ by Lemma 6.12 (i). Therefore, the image of a_i' in $(L\hat{\sigma}_i^{(l_i-1)*}M_i)/(L^+\hat{\sigma}_i^{(l_i-1)*}I_{e_i,M_i})$ is independent of the choice of f_i', thus well defined. As it is in the image of $J_{\hat{\sigma}_i^{(l_i)*}x_i}$ and bounded by $4d_i\check{\varrho}$ there are only finitely many $\overline{(a')^{-1}}$.

In order to finish the proof it remains to show the injectivity of the map

$$\begin{aligned} \pi^{-1}(\pi(y)) &\xrightarrow{\text{map}} \text{SET} \\ y' &\mapsto (\overline{(b_i')^{-1}}, \overline{(a_i')^{-1}}). \end{aligned}$$

So let $y', y'' \in \pi^{-1}(\pi(y))$ (i.e. $\pi(y) = \pi(y') = \pi(y'')$) with $(\overline{(b_i')^{-1}}, \overline{(a_i')^{-1}}) = \mathrm{map}(y') = \mathrm{map}(y'') = (\overline{(b_i'')^{-1}}, \overline{(a_i'')^{-1}})$. Therefore, we can write $b_i'' = B_i b_i'$, $a_i'' = A_i a_i'$ with $A_i, B_i \in L^+\hat{\sigma}_i^{(l_i-1)*} I_{e_i,M_i}$.
We have to show $(j'_{e_i}, g_i')_i = y'' = (j''_{e_i}, g_i'')_i$, thus to show $j'_{e_i} = j''_{e_i}$ and $g_i' = g_i''$.

(i) We have $\overline{(b_i')^{-1}} = \overline{(b_i'')^{-1}}$, so $b_i'' = B_i b_i'$ for a $B_i \in L^+\hat{\sigma}_i^{(l_i-1)*} I_{e_i,M_i}$. so we can calculate:

$$\begin{aligned}
&B_i(\theta_i')^{-1}\hat{\sigma}_i^{(l_i)*} g_i' \beta_{i,y'}^{-1}\gamma_{i,y'}\hat{\Gamma}_{c_i}(\varphi')\gamma_{i,y}^{-1}\beta_{i,y}\hat{\sigma}_i^{(l_i)*} g_i^{-1}\theta_i \\
&= B_i b_i' \\
&= b_i'' \\
&= (\theta_i'')^{-1}\hat{\sigma}_i^{(l_i)*} g_i'' \beta_{i,y''}^{-1}\gamma_{i,y''}\hat{\Gamma}_{c_i}(\varphi'')\gamma_{i,y}^{-1}\beta_{i,y}\hat{\sigma}_i^{(l_i)*} g_i^{-1}\theta_i.
\end{aligned}$$

So

$$\begin{aligned}
&\hat{\sigma}_i^{(l_i)*} g_i'' \beta_{i,y''}^{-1}\gamma_{i,y''}\hat{\Gamma}_{c_i}(\varphi'') \\
&= \hat{\sigma}_i^{(l_i)*}(g_i') \cdot (\hat{\sigma}_i^{(l_i)*}(g_i')^{-1}\theta_i'' B_i(\theta_i')^{-1}\hat{\sigma}_i^{(l_i)*}(g_i')) \cdot \beta_{i,y'}^{-1}\gamma_{i,y'}\hat{\Gamma}_{c_i}(\varphi')
\end{aligned}$$

with $\beta_{i,y''}^{-1}\gamma_{i,y''}\hat{\Gamma}_{c_i}(\varphi'') \in L^+G_{c_i}$, $\beta_{i,y'}^{-1}\gamma_{i,y'}\hat{\Gamma}_{c_i}(\varphi') \in L^+G_{c_i}$.
Since $B_i \in L^+\hat{\sigma}_i^{(l_i-1)*} I_{e_i,M_i}$ we have $(\hat{\sigma}_i^{(l_i)*}(g_i')^{-1}\theta_i'' B_i(\theta_i')^{-1}\hat{\sigma}_i^{(l_i)*}(g_i')) \in L^+G_{c_i}$ (by a similar calculation that we have done several times before, for example see the proof of Proposition 6.9) we get:

$$\hat{\sigma}^{(l_i)*} g_i'' = \hat{\sigma}_i^{(l_i)*} g_i' \text{ as elements in } X_{Z_i}^{d_i}.$$

Moreover, it follows that the quasi-isogeny

$$\delta''\varphi''(\varphi')^{-1}(\delta')^{-1} : \sigma^{l*}\underline{\mathcal{G}}_{y'}^{\mathrm{univ}} \to \sigma^{l*}\underline{\mathcal{G}}_{y''}^{\mathrm{univ}}$$

is an isomorphism at all c_i, because

$$\hat{\Gamma}_{c_i}(\delta''\varphi''(\varphi')^{-1}(\delta')^{-1}) = \hat{\sigma}_i^{l_i*}(\alpha_{i,y''}^{-1})h(l_i)_{y''}\theta_i'' B_i(\theta_i')^{-1}h(l_i)_{y'}^{-1}\hat{\sigma}_i^{l_i*}(\alpha_{i,y'})$$

is a compositum of isomorphisms. This shows that $\underline{\mathcal{G}}_{y'}^{\mathrm{univ}} = \underline{\mathcal{G}}_{y''}^{\mathrm{univ}}$ in $\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}(K)$, and so y'' and y' map to the same point in $\mathcal{C}_{\underline{x}}$.

(ii) We have $\overline{(a_i'')^{-1}} = \overline{(a_i')^{-1}}$, so there exists a $A_i \in L^+\hat{\sigma}_i^{(l_i-1)*} I_{e_i,M_i}$ such that $a_i'' = A_i \cdot a_i'$. So we can calculate

$$A_i \cdot \hat{\sigma}_i^{(l_i)*}(f_i')^{-1}\delta_i'\hat{\Gamma}_{c_i}(\varphi')\delta_i^{-1}\hat{\sigma}_i^{(l_i)*} f_i = \hat{\sigma}_i^{(l_i)*}(f_i'')^{-1}\delta_i''\hat{\Gamma}_{c_i}(\varphi'')\delta_i^{-1}\hat{\sigma}_i^{(l_i)*} f_i.$$

This implies

$$\hat{\sigma}_i^{(l_i)}(f_i'')^{-1} \cdot \delta_i''\hat{\Gamma}_{c_i}(\varphi'') = A_i \cdot \hat{\sigma}_i^{(l_i)}(f_i')^{-1} \cdot \delta_i'\hat{\Gamma}_{c_i}(\varphi').$$

Thus, we get

$$(\hat{\sigma}_i^{l_i*}(f_i'')A_i\hat{\sigma}_i^{l_i*}(f_i')^{-1})\delta_i'\hat{\Gamma}_{c_i}(\varphi') = \delta_i''\hat{\Gamma}_{c_i}(\varphi''),$$

and hence

$$\hat{\sigma}_i^{l_i*}(\alpha_{i,y''}^{-1})h(l_i)_{y''}(\theta_i''B_i(\theta_i')^{-1}) = (\hat{\sigma}_i^{l_i*}(f_i'')A_i\hat{\sigma}_i^{l_i*}(f_i')^{-1})\hat{\sigma}_i^{l_i*}(\alpha_{i,y'}^{-1})h(l_i)_{y'}.$$

The projection onto $L^+\hat{\sigma}_i^{(l_i-1)*}I_{M_i}$ yields

$$\hat{\sigma}_i^{l_i*}(\alpha_{i,y''}^{-1})(\theta_i''B_i(\theta_i')^{-1}) = (\hat{\sigma}_i^{l_i*}(f_i'')A_i\hat{\sigma}_i^{l_i*}(f_i')^{-1}\hat{\sigma}_i^{l_i*}(\alpha_{i,y'}^{-1}),$$

that is

$$\hat{\sigma}_i^{l_i*}(\alpha_{i,y'}) \equiv \hat{\sigma}_i^{l_i*}(\alpha_{i,y''}) \mod L^+\hat{\sigma}_i^{(l_i-1)*}I_{e_i,M_i}$$

and

$$\alpha_{i,y'} \equiv \alpha_{i,y''} \mod L^+\hat{\sigma}_i^{-1}I_{e_i,M_i}.$$

Since we already know that y' and y'' map to the same point in $\mathcal{C}_{\underline{x}}$, this implies that $y' = y''$ as points on $\mathfrak{I}\mathfrak{g}_{\underline{e}}(K)$ as desired. □

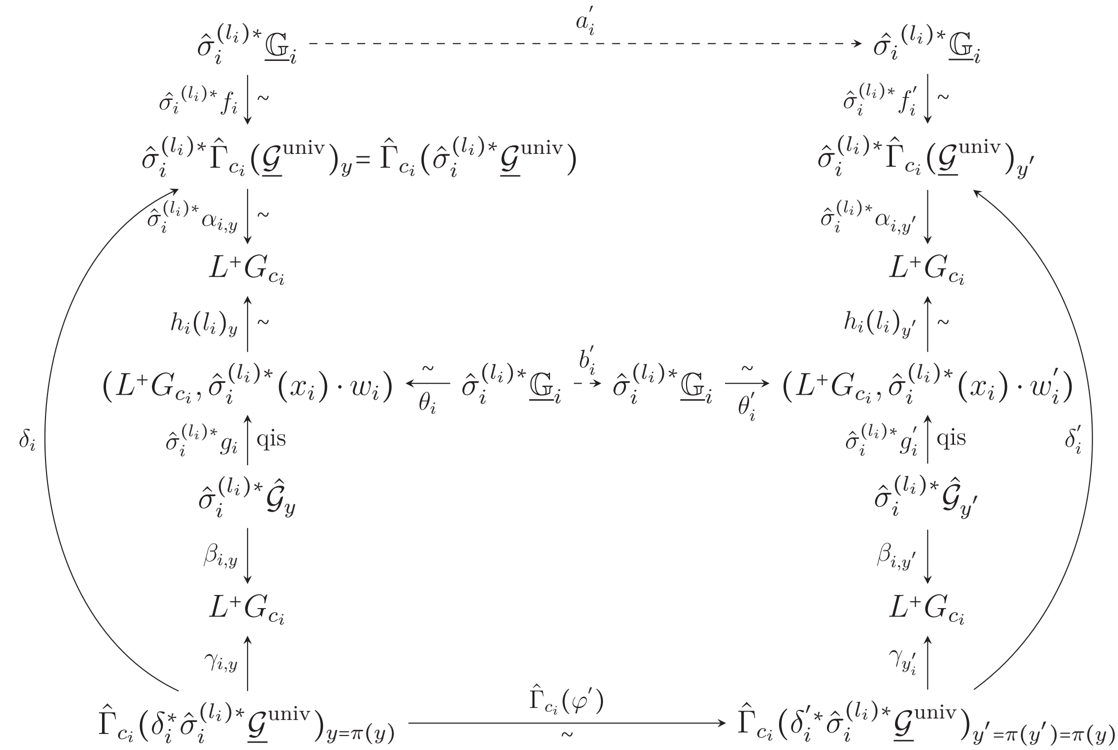


Figure 6.1: An illustrating diagram

with

$$\pi := \pi^{\underline{d},\underline{e},l} : \mathfrak{I}\mathfrak{g}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}} \to \sigma^{l*}\mathcal{N}_{\underline{x}} \times_{\mathbb{F}_{\underline{c}}} \mathbb{F}_q^{\mathrm{alg}} \subset \sigma^{l*}\nabla\mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}},$$

$y = (j_{e_i}, g_i)_i \in (\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}})(K)$, for $K = K^{\mathrm{alg}}$, and $y' = (j'_{e_i}, g'_i)_i \in \pi^{-1}(\pi(y))$, and

$$\varphi' : \delta^* \underline{\mathcal{G}}^{\mathrm{univ}} \longrightarrow \delta'^* \underline{\mathcal{G}}^{\mathrm{univ}}$$

where φ is a quasi-isogeny and an isomorphism at $\underline{c}$ that is comapatible with the H-level structure of $\underline{\mathcal{G}}$. Here $h_i(l_i)_y$ and $h_i(l_i)_{y'}$ are isomorphisms, not only quasi-isogenies and θ_i and θ'_i are defined as in Lemma 6.4.

Proposition 6.18. *Consider the composition of the morphism $\pi^{\underline{d},\underline{e},l}$ with the projection $\mathrm{pr} : \sigma^{l*}\mathcal{N}_{\underline{x}} \to \mathcal{N}_{\underline{x}}$. Then for every point $\underline{\mathcal{G}}_0$ of $\mathcal{N}_{\underline{x}}(K)$ with values in an algebraically closed field K there is a tuple $(\underline{d}, \underline{e}, l)$ and a point in the preimage of $\underline{\mathcal{G}}_0$ under $\pi^{\underline{d},\underline{e},l}$.*

Proof. We consider the morphism

$$\pi^{\underline{d},\underline{e},l} : \mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}}(\underline{x}) \to \sigma^{l*} \nabla \mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}.$$

on K-valued points. Let $\underline{\mathcal{G}}_0$ be a global G-shtuka in $\mathcal{N}_{\underline{x}}(K) \subseteq \nabla \mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}(K)$. Then by definition of the Newton stratum there is a quasi-isogeny

$$g_i : \Gamma_{c_i}(\underline{\mathcal{G}}_0) \to \underline{\mathbb{G}}_i \ .$$

We have to show that there is a tuple $(\underline{d}, \underline{e}, l)$ such that $(\pi^{\underline{d},\underline{e},l})^{-1}(\underline{\mathcal{G}}_0) \neq \varnothing$, that is that there exist $(j_{e_i})_i \in \mathfrak{Ig}_{\underline{e}}$ and $(g_i)_i \in X_{\underline{Z}}^{\underline{d}}(\underline{x})$ such that $\pi(j_{e_i}, g_i)_i = \underline{\mathcal{G}}_0$. Choose d_i large enough such that g_i is bounded by $2d_i\check{\rho}$, that is such that g_i defines an element in $X_{Z_i}^{d_i}(x_i)$. This choice is possible by the construction of Arasteh Rad and Hartl in [AH14a, § 4]. We choose integers $e_i > 2d_i(r_i - 1)$ and a multiple l of $\deg c_i$ for all i such that $l \geq l_i(e_i) \cdot \deg c_i$ for the numbers $l_i(e_i)$ from Definition 4.3 (vi). By Proposition 2.21 there is a global G-shtuka $\tilde{\underline{\mathcal{G}}} := (g_1^{-1})^* \circ \ldots \circ (g_n^{-1})^* \underline{\mathcal{G}}_0$ over K with $\hat{\Gamma}_{c_i}(\tilde{\underline{\mathcal{G}}}) = \underline{\mathbb{G}}_i$. In particular, $\tilde{\underline{\mathcal{G}}} \in \mathcal{C}_{\underline{x}}(K)$. Now we set $j_{e_i} = \mathrm{id}$ and $\alpha_i = \mathrm{id}$, as well as $h_i(l_i) = \mathrm{id}$. Then $y = (\tilde{\underline{\mathcal{G}}}, j_{e_i}, g_i) \in \big(\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}}(\underline{x})\big)(K)$. Since $\delta_i = \hat{\sigma}_i^{l_i*} g_i$, we obtain $\pi^{\underline{d},\underline{e},l}(y) = \sigma^{l*}\underline{\mathcal{G}}_0$ in $\sigma^{l*}\mathcal{N}_{\underline{x}}(K)$. So the point $y \circ \mathrm{Frob}^{-1}_{q^l, \mathrm{Spec}\, K} : \mathrm{Spec}\, K \to \mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}}(\underline{x})$ satisfies $\pi^{\underline{d},\underline{e},l}(y \circ \mathrm{Frob}^{-1}_{q^l, \mathrm{Spec}\, K}) = \underline{\mathcal{G}}_0$ in $\sigma^{l*}\mathcal{N}_{\underline{x}}(K)$ and $\mathrm{pr} \circ \pi^{\underline{d},\underline{e},l}(y \circ \mathrm{Frob}^{-1}_{q^l, \mathrm{Spec}\, K}) = \underline{\mathcal{G}}_0$ in $\mathcal{N}_{\underline{x}}(K)$. □

Proposition 6.19. *π is finite.*

Proof. The claim folows directly from the general fact that a morphism is finite if it is proper and quasi-finite. □

Thus, there is a finite morphism of stacks

$$\pi^{\underline{d},\underline{e},l} : \mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}}(\underline{x}) \to \sigma^{l*}\mathcal{N}_{\underline{x}} \times_{\mathbb{F}_{\underline{c}}} \mathbb{F}_q^{\mathrm{alg}} \subset \sigma^{l*} \nabla \mathcal{H}^1_{\mathbb{F}_q^{\mathrm{alg}}}.$$

which is "surjective" in the sense of the statement of Main Theorem 0.1 and this proves Main Theorem 0.1.
An application of the product structure is the possibility to calculate the dimension of the leaves inside a Newton stratum.

Proposition 6.20. *Let K be a perfect field extension of $\mathbb{F}_{\underline{x}}$. For each i let $\underline{\hat{\mathbb{G}}}_i$ be a local G_{c_i}-shtuka over K in the isogeny class of $(L^+G_{c_i}, x_i\hat{\sigma}_i^*)$ with x_i as in Definition 4.3.*

(i) Then for every global G-shtuka $\underline{\mathcal{G}}$ over a K-scheme S the central leaf $\mathcal{C}_{(\underline{\hat{\mathbb{G}}}_i)_i,S}$ is closed in the Newton stratum $\mathcal{N}_{\underline{x}} \subset S$, and hence locally closed in S. It carries a natural scheme structure.

(ii) Like in Proposition 4.26 this defines a closed reduced substack $\mathcal{C}_{(\underline{\hat{\mathbb{G}}}_i)_i}$ of $\mathcal{N}_{\underline{x},K} := \mathcal{N}_{\underline{x}} \times_{\mathbb{F}_{\underline{x}}} \operatorname{Spec} K \subset \nabla\mathcal{H}_K^1$. The stack $\mathcal{C}_{(\underline{\hat{\mathbb{G}}}_i)_i}$ is smooth over K of relative dimension $\dim \mathfrak{I}\mathfrak{g}_{\underline{e}} = \sum_{i=1}^n \langle \nu_{x_i,\mathrm{dom}}\,,\, 2\varrho_i \rangle$; see Theorem 4.29.

Proof. For each i, that is for each characteristic c_i, we choose a quasi-isogeny of local G_{c_i}-shtukas

$$g_i : \underline{\hat{\mathbb{G}}}_i \longrightarrow (L^+G_{c_i}, x_i\hat{\sigma}_i^*)$$

over K. These correspond to points g_i in $X_{Z_i}(x_i)$. We choose d_i large enough such that g_i is bounded by $2d_i\check{\varrho}$, that is such that $g_i \in X_{Z_i}^{d_i}(x_i)$. Now choose l_i and e_i large enough such that $\pi^{\underline{d},\underline{e},l}$ exists. Consider the morphism

$$\pi : \mathfrak{I}\mathfrak{g}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} \operatorname{Spec} K \longrightarrow \mathfrak{I}\mathfrak{g}_{\underline{x}} \times_{\mathbb{F}_q^{\mathrm{alg}}} \{(g_i)_i\} \xrightarrow{\pi^{\underline{d},\underline{e},l}} \sigma^{l*}\mathcal{N}_{\underline{x},K} = \sigma^{l*}\mathcal{N}_{\underline{x}} \times_{\mathbb{F}_{\underline{x}}} \operatorname{Spec} K\,,$$

which is finite, because $\pi^{\underline{d},\underline{e},l}$ is finite and $\{(g_i)_i\}$ is a closed K-valued point of $X_{\underline{Z}}^{\underline{d}}(\underline{x}) \times_{\mathbb{F}_q^{\mathrm{alg}}} \operatorname{Spec} K$, and hence $\mathfrak{I}\mathfrak{g}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} \operatorname{Spec} K \to \mathfrak{I}\mathfrak{g}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}}(\underline{x}) \times_{\mathbb{F}_q^{\mathrm{alg}}} \operatorname{Spec} K$ is a closed immersion. Since K is perfect, we may base change π under the inverse of the q^l-Frobenius on K to obtain the finite morphism

$$\sigma^{-l*}\pi : \sigma^{-l*}(\mathfrak{I}\mathfrak{g}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} \operatorname{Spec} K) \longrightarrow \mathcal{N}_{\underline{x},K}\,. \tag{6.3}$$

(i) Let $S' := \mathcal{N}_{\underline{x}} \subset S$ and consider the K-morphism $S' \to \mathcal{N}_{\underline{x},K} \subset \nabla\mathcal{H}_K^1$ corresponding to the global G-shtuka $\underline{\mathcal{G}}$. Let

$$\sigma^{-l*}\pi_{S'} : \sigma^{-l*}(\mathfrak{I}\mathfrak{g}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} \operatorname{Spec} K) \times_{\mathcal{N}_{\underline{x},K}} S' \longrightarrow S'$$

be obtained by base change from (6.3). We claim that $\mathcal{C}_{(\underline{\hat{\mathbb{G}}}_i)_i,S'}$ equals the image of $\sigma^{-l*}\pi_{S'}$, and hence is a closed subset of S' with a natural scheme structure as the scheme theoretic image.

To prove the claim, let first $s \in \mathcal{C}_{(\hat{\mathbb{G}}_i)_i, S'}$ be a point with values in an algebraically closed field L. By definition of the central leaf there are isomorphisms

$$f_i : \hat{\Gamma}_{c_i}(\underline{\mathcal{G}}_s) \xrightarrow{\sim} \underline{\hat{\mathbb{G}}}_i$$

for all i. Consider now the composition

$$\tilde{g}_i := g_i \circ f_i : \hat{\Gamma}_{c_i}(\underline{\mathcal{G}}_s) \longrightarrow (L^+ G_{c_i}, x_i \hat{\sigma}_i^*),$$

which also corresponds to the point g_i in the truncated Rapoport-Zink space $\mathcal{RZ}_{\hat{Z}_i}^{d_i}$ and hence in $X_{Z_i}^{d_i}(x_i)$. We consider the global G-shtuka $\underline{\mathcal{G}}' := (\tilde{g}_1^{-1})^* \circ \ldots \circ (\tilde{g}_n^{-1})^* \underline{\mathcal{G}}_s$ over L, which satisfies $\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}') = (L^+ G_{c_i}, x_i \hat{\sigma}_i^*)$. Thus, $\underline{\mathcal{G}}'$ together with $j_{e_i} = \mathrm{id}$ for all i defines an L-valued point of $\mathfrak{Ig}_{\underline{e}}$. By construction $y = (\underline{\mathcal{G}}', j_{e_i}, g_i)$ is an L-valued point of $\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}}(\underline{x})$ which factors through $\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} \operatorname{Spec} K$ and maps to $\sigma^{l*} \underline{\mathcal{G}}_s$ in $\mathcal{N}_{\underline{x}}$. So the point $y \circ \mathrm{Frob}_{q^l, \operatorname{Spec} L}^{-1} : \operatorname{Spec} L \to \mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} X_{\underline{Z}}^{\underline{d}}(\underline{x})$ is an L-valued point of $\sigma^{-l*}(\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} \operatorname{Spec} K)$ with $\sigma^{-l*}\pi(y \circ \mathrm{Frob}_{q^l, \operatorname{Spec} L}^{-1}) = \underline{\mathcal{G}}_0$ in $\mathcal{N}_{\underline{x}}$. This implies that $(y \circ \mathrm{Frob}_{q^l, \operatorname{Spec} L}^{-1},\ s)$ is an L-valued point of $\sigma^{-l*}(\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} \operatorname{Spec} K) \times_{\mathcal{N}_{\underline{x},K}} S'$ which maps to s. So s lies in the image of $\sigma^{-l*}\pi_{S'}$.

Conversely, let $(y, s) \in \sigma^{-l*}(\mathfrak{Ig}_{\underline{e}} \times_{\mathbb{F}_q^{\mathrm{alg}}} \operatorname{Spec} K) \times_{\mathcal{N}_{\underline{x},K}} S'$ be a point with values in an algebraically closed field L, that is $\sigma^{-l*}\pi(y) = \underline{\mathcal{G}}_s$ and $\pi^{\underline{d},\underline{e},l}(y, (g_i)_i) = \sigma^{l*}\underline{\mathcal{G}}_s$ in $\mathcal{N}_{\underline{x},K}(L)$. From the construction of $\pi^{\underline{d},\underline{e},l}$ and Corollary 6.10 it follows that $\sigma^{l*}\underline{\mathcal{G}}_s = \pi^{\underline{d},\underline{e},l}(y, (g_i)_i)$ satisfies $\hat{\Gamma}_{c_i}(\sigma^{l*}\underline{\mathcal{G}}_s) \cong \sigma^{l*}\underline{\hat{\mathbb{G}}}_i$. Therefore $\hat{\Gamma}_{c_i}(\underline{\mathcal{G}}_s) \cong \underline{\hat{\mathbb{G}}}_i$ and we conclude that $s \in \mathcal{C}_{(\hat{\mathbb{G}}_i)_i, S'}$. This proves the claim, shows that $\mathcal{C}_{(\hat{\mathbb{G}}_i)_i, S'}$ is a closed subset of S', and proves (i).

(ii) Since K is perfect, the reduced closed substack $\mathcal{C}_{(\hat{\mathbb{G}}_i)_i}$ of $\mathcal{N}_{\underline{x},K} \subset \nabla\mathcal{H}_K^1$ is geometrically reduced and the argument of Proposition 4.31 applies and shows that $\mathcal{C}_{(\hat{\mathbb{G}}_i)_i}$ is smooth over K. Since its connected components are irreducible and reduced, their dimension equals the dimension of $\mathfrak{Ig}_{\underline{e}}$, which is $\sum_{i=1}^n \langle \nu_{x_i, \mathrm{dom}}, 2\varrho_i \rangle$ by Theorem 4.29. □

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