Gunnar Dietz

The Braid Group Representation on Intersection Matrices and Monodromy of Singularities

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The Braid Group Representation on Intersection Matrices and Monodromy of Singularities

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vorgelegt von
Gunnar Dietz
aus Hamburg
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Dekan:	Prof. Dr. Klaus Hinrichs
Erster Gutachter:	Prof. Dr. Helmut A. Hamm
Zweiter Gutachter:	Prof. Dr. Wolfgang Ebeling
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To Angie and my parents

Abstract

Abstract. It is a well-known fact that the monodromy of the Milnor fibration of an isolated singularity is quasiunipotent. This holds no longer true if a non-local monodromy around several singularities is considered. Here the case of families of (finitely many) Morse singularities will be studied. For the case that such a family arises from a morsification of an isolated singularity it will be proven that all monodromies corresponding to simple loops around a subfamily of the corresponding critical values are already quasiunipotent if and only if this is always the case for simple loops around only two critical values. We conjecture that this is (for purely combinatorial reasons) also true for the general case and prove a weaker analogon of this conjecture.

Zusammenfassung. Es ist bekannt, dass die Monodromie der Milnor-Faserung einer isolierten Singularität quasiunipotent ist. Dies ist nicht länger der Fall, wenn man eine nicht-lokale Monodromie um mehrere Singularitäten betrachtet. Wir studieren hier den Fall von Familien von (endlich vielen) Morse-Singularitäten. Für den Fall, dass eine solche Familie eine Morsifikation einer isolierten Singularität ist, zeigen wir, dass sämtliche Monodromien, die zu einfachen Schleifen um eine Teilfamilie der zugehörigen kritischen Punkte gehören, schon dann quasiunipotent sind, wenn dies stets für Schleifen um nur zwei kritische Punkte gilt. Wir stellen die Vermutung auf, dass dies auch (aus rein kombinatorischen Gründen) im allgemeinen Fall gilt und beweisen eine abgeschwächte Form dieser Vermutung.

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Introduction

石の上にも三年

Sitting on a stone for three years

One of the important theorems of singularity theory says that the monodromy of the Milnor fibration of an isolated singularity is quasiunipotent, i.e. all eigenvalues of the monodromy are roots of unity. This has been proven in several ways and has many consequences (see Section 1.7).

This fact holds no longer true if one considers non-local monodromies for paths around several singularities. The simplest case for this is a function $f : X \to S$ (with X an $(n+1)$-dimensional complex manifold and $S \subset \mathbb{C}$ open) which has only nondegenerate critical points with distinct critical values (i.e. f is a Morse function) and which satisfies some regularity conditions regarding the "boundary" of X (e.g. in the case that $X = B \cap f^{-1}(S)$ for an open ball $B \subset \mathbb{C}^{n+1}$ we assume that f is transversal to $\partial X := \partial B \cap f^{-1}(S)$ and $f : \overline{B} \cap f^{-1}(S) \to S$ is proper; or in the case that $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a polynomial we assume that f has no singularities at infinity). The main example for this situation is a morsification of an isolated singularity.

One immediately gets examples for a non-quasiunipotent monodromy if one admits non-simple loops; for instance take a morsification of the cusp singularity:

$$f(x, y) = x^3 + y^2 - 3x.$$

If one moves around the critical values $s_1 = 2$, $s_2 = -2$ in form of an eight, one gets

$$m = m_2^{-1} m_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

which is not quasiunipotent.[*] We will therefore admit only simple loops for our investigations here.

In the above situation we get a basis of the homology of the generic fibre consisting of vanishing cycles. For this we have to fix a system of paths from the fixed generic point to the critical values. Then the monodromy around one critical value is given by the Picard-Lefschetz formulas

$$m_\delta(\alpha) = \alpha - (-1)^{\frac{1}{2}n(n-1)}(\alpha, \delta)\delta,$$
$$(\delta, \delta) = (-1)^{\frac{1}{2}n(n-1)} + (-1)^{\frac{1}{2}n(n+1)}.$$

[*]This example is due to a personal conversation to N. A'Campo.

Here δ is the vanishing cycle corresponding to the critical point and $(\cdot,\cdot)$ denotes the intersection product.

The intersection products of the elements of (distinguished) bases of vanishing cycles are encoded in Coxeter-Dynkin diagrams. To be more precise: Each distinguished system of paths defines a Coxeter-Dynkin diagram, hence there is not only one but several Coxeter-Dynkin diagrams which encode the geometric situation. We have an operation of the braid group on distinguished systems of paths which yields an operation on distinguished bases of vanishing cycles and Coxeter-Dynkin diagrams. Given two distinguished bases or two Coxeter-Dynkin diagrams encoding the same geometric situation, one always gets the second one from the first one by an operation of an element of the braid group.

The monodromy around one critical value is always quasiunipotent as it follows from the theory, but in our case this follows immediately, since $m_\delta^2 = \mathbb{1}$ (hence it is even finite in our case). The next step is the case of a monodromy around two critical values. From the above formulas it follows that such a monodromy is quasiunipotent if and only if $|(\delta_1, \delta_2)| \leq 2$ for the vanishing cycles δ_1, δ_2 corresponding to the corresponding two critical points.

Now for each simple loop around two values we can find a distinguished system of paths such that this loop is the product of two simple loops corresponding to two paths of the distinguished system. Conversely, two paths of a distinguished system define a simple loop around two critical values.

By this remark and the above we see that the following two statements are equivalent:

(i) All monodromies of simple loops around two critical values are quasiunipotent.
(ii) All Coxeter-Dynkin diagrams contain only lines with a weight of absolute value ≤ 2.

The next and final step is to ask for a similar criterion for arbitrary simple loops. It is surprising that the answer seems to be the same as for simple loops around only two values. At least in the case of a morsification of an isolated singularity we show that the statement

(iii) All monodromies of arbitrary simple loops are quasiunipotent.

is also equivalent to the statement (ii) above.

The proof for this uses a deformation argument, namely that each isolated singularity with a modularity greater than 1 deforms into an exceptional hyperbolic singularity.

This leads to the following question: Is the above equivalence true in the general situation as well? Or to put it in another way: Does the above equivalence really depend on this deformation argument, or is it just a combinatorial property of the underlying data, namely intersection matrices, monodromy matrices, Coxeter-Dynkin diagrams and the operation of the braid group?

We conjecture that the equivalence is true in the general situation as well. We give abstract definitions of all terms used in the above geometric situation and discuss their algebraic and combinatorial properties. This allows us to formulate the conjecture in purely algebraic terms.

Even though the conjecture remains unproven, we are able to prove some weaker analogue of it. Whereas the conjecture is formulated in the framework of distinguished bases of vanishing cycles and the operation of the braid group on them, the weaker analogue is formulated in terms of weakly distinguished bases of weakly vanishing cycles and the operation of a larger group (namely the Gabrielov group) on them. Translated back into the geometric situation it says that all monodromies of arbitrary simple loops are quasiunipotent if and only if for every pair α, β of weakly vanishing cycles one has $|(\alpha, \beta)| \leq 2$.

We also show that statement (iii) above is equivalent to the condition that the intersection matrix is semidefinite. This is true in the general context, even for a fixed distinguished system of paths resp. a fixed distinguished basis of vanishing cycles.

In the same way the above assertions remain true if one replaces "quasiunipotent" by "finite", "≤ 2" by "≤ 1", and "semidefinite" by "definite". This will be discussed along with the above.

In Chapter 1 we introduce and discuss the geometric framework — however, most calculations will be postponed to Chapter 3. We recall the definitions of the Milnor fibration and its monodromy, unfoldings and morsifications, distinguished bases of vanishing cycles and the operation of the braid group on them, and Picard-Lefschetz theory and intersection matrices. Two (different) proofs of the quasiunipotence of the monodromy of the Milnor fibration of an isolated singularity will be indicated.

In Chapter 2 we then state and prove our theorems in the context of a morsification of an isolated singularity.

In Chapter 3 we abstract some ideas of the first chapter. We give algebraic definitions of distinguished bases of vanishing cycles and the operation of the braid group on them, and of intersection matrices and monodromy. In this context we reformulate our above theorem, while some part of it remains conjectured.

For the weaker analogues of the conjectures we also have to introduce the framework of weakly distinguished bases of weakly vanishing cycles and the operation of the Gabrielov group on them. Furthermore we introduce the notion of admissible families of cycles. In this context we then state and prove these theorems.

In the appendix we recall the definition of the braid group and its properties; the same will be done for the Gabrielov group. Also some calculations involving matrices and some figures were shifted to the appendix.

Chapter 1

Isolated Singularities

芸術は長く、人生は短い

Art is long, life is short

Introduction

Singularities are a source of many topologically interesting spaces. While all manifolds (of same dimension) look like the same locally, the local topology of a complex space in the neighborhood of a singularity has many interesting properties.

The simplest type of (analytic) singularities are hypersurface singularities, i.e. singularities that arise in the zero set of only one holomorphic function on a manifold.

In the first section we discuss the theory of the Milnor fibration. Given a holomorphic function which defines an isolated singularity, this theory shows that — after selection sufficiently small neighborhoods — such a function defines a fibration outside the singular fibre. (We will discuss this theorem in some more general setting.) Such a fibration leads to the notion of monodromy which describes what happens when going around the singular fibre. Understanding this monodromy means to understand some part of the local topological behavior of the singularity.

Of course a function on a manifold may have more than one singularity. We discuss how the local descriptions fit together in such a case focussing on the case that the function behaves well in the "outer region" of the manifold.

In the next section we then discuss Picard-Lefschetz theory which describes the local monodromy for the most simple singularity, namely nondegenerate critical points of the function. This theory becomes useful when considering a so-called morsification of an isolated singularity: By disturbing the function the original singularity splits into some "less complicated" singularities. A morsification is the special case when the singularity splits completely into nondegenerate singularities. This allows to introduce a special type of bases of the homology of the Milnor fibre of the singularity, namely distinguished bases of vanishing cycles. Then from Picard-Lefschetz theory it follows that it is sufficient to understand how these cycles intersect to know the monodromy. This intersection data can be visualized in some diagram, a so-called Coxeter-Dynkin diagram.

However, the choice of a distinguished basis of vanishing cycles is not well-defined. In particular there are several Coxeter-Dynkin diagrams for a given singularity. But it can be shown that all different choices are connected by the operation of some group, namely the braid group. Given one Coxeter-Dynkin diagram for an isolated singularity, the operation of the braid group allows to find all possible Coxeter-Dynkin diagrams for that singularity.

In the next section the famous monodromy theorem will be discussed. It says that the monodromy operation on the homology of the Milnor fibre of an isolated singularity is always quasiunipotent. This theorem is the cornerstone for the investigations in this thesis.

We close this chapter by indicating the beginning of the story of the classification of isolated hypersurface singularities. Also Coxeter-Dynkin diagrams are given for all these singularities (namely the simple and unimodal singularities).

1.1 Milnor Fibration

Let (X, x) be a reduced complex space germ and $f : (X, x) \to (\mathbb{C}^k, 0)$ a holomorphic function. We assume that (after selecting a sufficiently small representative $f : X \to \mathbb{C}^k$) the space X is non-singular along $f^{-1}(0) \setminus \{x\}$ and that f is a submersion in these points. Note that these conditions are satisfied in the case that $(X, x) = (\mathbb{C}^{n+1}, 0)$, $k = 1$, and $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ defines an isolated singularity.

Choose a real-analytic function $r : X \to \mathbb{R}_{\geq 0}$ with $r^{-1}(0) = \{x\}$. Such a function always exists (for $(X, x) \subset (\mathbb{C}^N, 0)$ one can take $r(z) = \|z\|^2$) and has the meaning of some kind of distance function.

The Milnor fibration is characterized in the following theorem:

Theorem 1.1.1. *There exists an $\varepsilon > 0$ and a (small) contractible representative S of $(\mathbb{C}^k, 0)$ such that for*

$$\mathcal{X} := f^{-1}(S)_{r<\varepsilon}, \quad \overline{\mathcal{X}} := f^{-1}(S)_{r\leq\varepsilon}, \quad \partial\mathcal{X} := f^{-1}(S)_{r=\varepsilon},$$

and

$$C_f := \{p \in \mathcal{X} \mid \mathcal{X} \text{ is singular at } p \text{ or } f \text{ is not submersive at } p\},$$
$$D_f := f(C_f)$$

the following statements are true:

(i) *$f : \overline{\mathcal{X}} \to S$ is proper and $f : \partial\mathcal{X} \to S$ is a differentiable fibration.*

(ii) *C_f is analytic in $\mathcal{X}$ and closed in $\overline{\mathcal{X}}$. Furthermore $f|_{C_f}$ is finite.*

(iii) *$\mathcal{X}_{sing}$ has dimension $\leq k$ and $C_f \setminus \mathcal{X}_{sing}$ has pure dimension $k - 1$.*

(iv) *D_f is analytic with the same dimension as C_f. In particular, if $\mathfrak{X}_{sing}$ is nowhere dense in C_f, then D_f is a hypersurface.*

(v) *$f : (\overline{\mathfrak{X}}_{S\setminus D_f}, \partial\mathfrak{X}_{S\setminus D_f}) \to S \setminus D_f$ is a differentiable fibration pair. Each fibre is a complex manifold with boundary.*

By a complex manifold with boundary we mean a real analytic manifold with boundary whose interior carries a complex structure. A fibration is always understood to be a local trivial fibration.

As immediate consequences we get that also $f : \mathfrak{X}_{S\setminus D_f} \to S \setminus D_f$ is a differentiable fibration. Furthermore, since S is contractible, $f : \partial\mathfrak{X} \to S$ is a trivial fibration. The fibrations $f : \mathfrak{X}_{S\setminus D_f} \to S \setminus D_f$ and $f : \overline{\mathfrak{X}}_{S\setminus D_f} \to S \setminus D_f$ are called the the *Milnor fibration* of f resp. the *proper Milnor fibration* of f. Sometimes the whole maps $f : \mathfrak{X} \to S$ resp. $f : \overline{\mathfrak{X}} \to S$ are called the *Milnor fibration* resp. *proper Milnor fibration* of f as well. The fibres $\mathfrak{X}_s$ resp. $\overline{\mathfrak{X}}_s$ for $s \in S \setminus D_f$ are called the *Milnor fibres* resp. *proper Milnor fibres* of f. The Milnor fibres are also called the *generic fibres* of the Milnor fibration of f.

The main ingredients for the above theorem are the curve selection lemma and the Ehresmann theorem.

Theorem 1.1.2 (Curve Selection Lemma)**.** *Let U be an open neighborhood of $p \in \mathbb{R}^n$ and $f_1, \dots, f_k, g_1, \dots, g_l$ be real analytic functions on U such that p is contained in the closure of*

$$Z := \{x \in U \mid f_1(x) = \cdots = f_k(x) = 0,\ g_1(x) > 0, \dots, g_l(x) > 0\}.$$

Then there exists a real analytic curve $\gamma : [0, \varepsilon[\to U$ with $\gamma(0) = p$ and $\gamma(t) \in Z$ for $t > 0$.

Theorem 1.1.3 (Ehresmann theorem)**.** *Let X and S be real differentiable manifolds where X may have a boundary and S is connected, and let $f : X \to S$ be a proper submersion such that also $f|_{\partial X}$ is submersive. Then $f : (X, \partial X) \to S$ is a differentiable fibration pair.*

Proofs for both theorems are hard to find in the literature. For the curve selection lemma in most cases a reference to Milnor [42] is given who first formulated and proved the curve selection lemma for the algebraic case. The proof of the analytic analogue is nearly the same, however, one needs an analytic (and local) version of the fact that the difference of two algebraic subsets has finitely many connected components. See e.g. [22] for a proof. There are also some other proofs for the curve selection lemma using other methods, e.g. in the context of the theory of subanalytic sets.

The Ehresmann theorem which is due to Ehresmann [99] is much easier to prove. The proof uses the fact that a submersion allows to lift vector fields (however, of course not uniquely), and that these lifted vector fields are globally integrable if the original vector fields are, since f is proper. (Note that some care is needed since we lift more than one vector field.)

The roadmap of the proof of the Milnor fibration theorem is then as follows: We want to use the Ehresmann theorem, so we first restrict the function to some closed neighborhood of the form $X_{r\leq\varepsilon}$ to make it a proper map. However, we need that f is still submersive on the boundary $X_{r=\varepsilon}$ which fails in almost every case. But by the curve selection lemma we can show that for small ε the intersection of $f^{-1}(0)$ and $X_{r=\varepsilon}$ is transversal. Therefore we can choose a small (contractible) open neighborhood S of $0 \in \mathbb{C}^k$ such that f is submersive on $f^{-1}(S)_{r=\varepsilon}$. Now we can use the Ehresmann theorem to get the fibration.

It can be shown that the Milnor fibration is unique up to diffeomorphism, and that C_f and D_f can be endowed with structures of complex spaces which are invariant under base changes.

The most important case for us is the case of one function (i.e. $k = 1$) on $(X, x) = (\mathbb{C}^{n+1}, 0)$ which defines an isolated singularity. In this case $C_f = \{0\}$ and $D_f = \{0\}$, and we can choose for S a small open disc

$$S = B_\eta = \{s \in \mathbb{C} \mid |s| < \eta\}.$$

This gives us the usual formulation of the Milnor fibration theorem that there exist $0 < \eta \ll \varepsilon \ll 1$ such that

$$f : B_\varepsilon \cap f^{-1}(B^*_\eta) \to B^*_\eta$$

is a fibration (with $B^*_\eta = B_\eta \setminus \{0\}$).

The standard picture for this is shown in Figure 1.1.

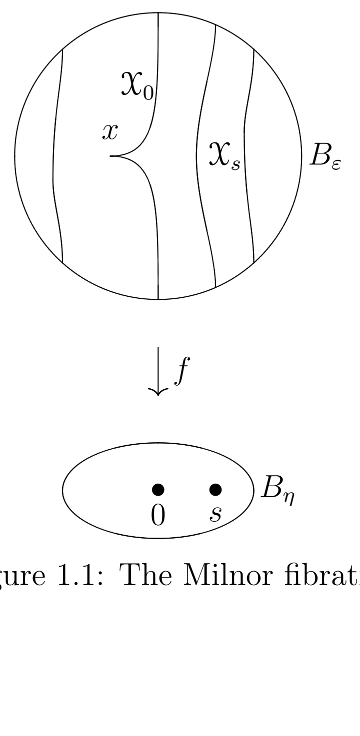


Figure 1.1: The Milnor fibration

1.2 Monodromy

In the case of the Milnor fibration $f : (\overline{\mathcal{X}}_{S\setminus D_f}, \partial\mathcal{X}_{S\setminus D_f}) \to S \setminus D_f$ we get by the local trivializations of the fibration representations

$$\pi_1(S \setminus D_f, s) \to \operatorname{Isot}(\overline{\mathcal{X}}_s) \qquad \text{and}$$
$$\pi_1(S \setminus D_f, s) \to \operatorname{Isot}(\mathcal{X}_s)$$

(after selecting a base point $s \in S \setminus D_f$), where $\operatorname{Isot}(Y)$ denotes the set of isotopy classes of diffeomorphisms from Y to itself (for a manifold Y).

Since we also have that $\partial\mathcal{X}$ is trivially fibred over the whole set S via f, we can choose the above representation such that the automorphisms of $\overline{\mathcal{X}}_s$ are trivial on $\partial\mathcal{X}$. Therefore we even get a representation

$$\pi_1(S \setminus D_f, s) \to \operatorname{Isot}^+(\overline{\mathcal{X}}_s; \partial\mathcal{X}_s)$$

where $\operatorname{Isot}^+(Y; Z)$ denotes the set of relative isotopy classes of diffeomorphisms from Y to itself which are the identity on Z (for a manifold Y and a subset $Z \subset Y$).

The above representations are called the *geometric monodromy* of the Milnor fibration of f. For each closed path $[\gamma] \in \pi_1(S \setminus D_f)$ we denote the corresponding (isotopy class of the) diffeomorphism by $h_\gamma \in \operatorname{Isot}^+(\overline{\mathcal{X}}_s; \partial\mathcal{X}_s)$.

The geometric monodromy induces representations on homology and cohomology

$$\pi_1(S \setminus D_f, s) \to \operatorname{Aut}\big(H_i(\mathcal{X}_s; \mathbb{Z})\big) \quad \text{and}$$
$$\pi_1(S \setminus D_f, s) \to \operatorname{Aut}\big(H_i(\overline{\mathcal{X}}_s, \partial\mathcal{X}; \mathbb{Z})\big),$$

resp.

$$\pi_1(S \setminus D_f, s) \to \operatorname{Aut}\big(H^i(\mathcal{X}_s; \mathbb{Z})\big) \quad \text{and}$$
$$\pi_1(S \setminus D_f, s) \to \operatorname{Aut}\big(H^i(\overline{\mathcal{X}}_s, \partial\mathcal{X}; \mathbb{Z})\big).$$

(Note that $H_i(\mathcal{X}_s; \mathbb{Z}) = H_i(\overline{\mathcal{X}}_s; \mathbb{Z})$ and $H^i(\mathcal{X}_s; \mathbb{Z}) = H^i(\overline{\mathcal{X}}_s; \mathbb{Z})$ which follows from the existence of a collar for manifolds with boundary.)

Fix a $[\gamma] \in \pi_1(S \setminus D_f)$. Since the diffeomorphism h_γ is the identity on $\partial\mathcal{X}$, we get variation morphisms

$$\operatorname{var}(h_\gamma)_* : H_i(\overline{\mathcal{X}}_s, \partial\mathcal{X}_s; \mathbb{Z}) \to H_i(\overline{\mathcal{X}}_s; \mathbb{Z})$$

resp.

$$\operatorname{var}(h_\gamma)^* : H^i(\overline{\mathcal{X}}_s; \mathbb{Z}) \to H^i(\overline{\mathcal{X}}_s, \partial\mathcal{X}_s; \mathbb{Z})$$

which fit into the following diagrams:

$$\begin{array}{ccc} H_i(\overline{\mathcal{X}}_s; \mathbb{Z}) & \longrightarrow & H_i(\overline{\mathcal{X}}_s, \partial\mathcal{X}_s; \mathbb{Z}) \\ {\scriptstyle h_{\gamma *}-1\!\!1}\downarrow & \swarrow{\scriptstyle \operatorname{var}(h_\gamma)_*} & \downarrow{\scriptstyle h_{\gamma *}-1\!\!1} \\ H_i(\overline{\mathcal{X}}_s; \mathbb{Z}) & \longrightarrow & H_i(\overline{\mathcal{X}}_s, \partial\mathcal{X}_s; \mathbb{Z}) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} H^i(\overline{\mathcal{X}}_s, \partial\mathcal{X}_s; \mathbb{Z}) & \longrightarrow & H^i(\overline{\mathcal{X}}_s; \mathbb{Z}) \\ {\scriptstyle h_\gamma^*-1\!\!1}\downarrow & \swarrow{\scriptstyle \operatorname{var}(h_\gamma)^*} & \downarrow{\scriptstyle h_\gamma^*-1\!\!1} \\ H^i(\overline{\mathcal{X}}_s, \partial\mathcal{X}_s; \mathbb{Z}) & \longrightarrow & H^i(\overline{\mathcal{X}}_s; \mathbb{Z}) \end{array}$$

where the horizontal arrows are part of the long exact sequences for relative homology resp. cohomology.

Note that $H^i(\overline{\mathfrak{X}}_s, \partial\mathfrak{X}_s; \mathbb{Z}) = H^i_c(\mathfrak{X}_s; \mathbb{Z})$, since $\overline{\mathfrak{X}}_s$ is compact.

Lemma 1.2.1.

(i) *The monodromy respects the intersection product, i.e.*

$$\big(h_{\gamma_*}a, h_{\gamma_*}b\big) = (a, b)$$

for a and b in $H_n(\overline{\mathfrak{X}}_s; \mathbb{Z})$ or $H_n(\overline{\mathfrak{X}}_s, \partial\mathfrak{X}_s; \mathbb{Z})$.

(ii) *For $a, b \in H_n(\overline{\mathfrak{X}}_s, \partial\mathfrak{X}_s; \mathbb{Z})$ one has*

$$\big(\mathrm{var}(h_\gamma)_* a, \mathrm{var}(h_\gamma)_* b\big) + \big(\mathrm{var}(h_\gamma)_* a, b\big) + \big(a, \mathrm{var}(h_\gamma)_* b\big) = 0.$$

Proof. (i) is geometrically obvious.

(ii): By the definition of the variation map it follows that

$$\big(\mathrm{var}(h_\gamma)_* a, b\big) = (h_{\gamma_*}a, b) - (a, b)$$

for all $a \in H_n(\overline{\mathfrak{X}}_s, \partial\mathfrak{X}_s; \mathbb{Z})$ and $b \in H_n(\overline{\mathfrak{X}}_s; \mathbb{Z})$. The equation of the lemma follows by a small calculation from this and (i). □

1.3 Families of Singularities

In this section we consider a complex manifold X which embeds as an open submanifold into a larger real analytic manifold Y which may have a boundary. Denote the complement by $Z = Y \setminus X$. Furthermore, let $S \subset \mathbb{C}$ be open and contractible, and let $f : Y \to S$ be a proper and surjective real analytic map which is holomorphic on X with only isolated critical points.

We demand that the restriction of f to Z is non-singular in some sense we have to specify. For this we consider two cases:

Case 1: Z is a real analytic manifold. In this case we assume that $f|_Z$ is a real submersion.

Case 2: Y is a complex manifold and Z is a complex analytic subset. Then there exists a Whitney stratification of Y such that X is an open stratum. We assume that $f|_Z$ is a stratified submersion.

Set

$$C_f := \{x \in X \mid x \text{ is a crit. point of } f\} \quad \text{and} \quad D_f := f(C_f).$$

Then D_f consists of isolated points in S. We assume that D_f is finite.

As in the case of the Milnor fibration, we get by the assumptions that

$$f : (Y_{S\setminus D_f}, Z_{S\setminus D_f}) \to S \setminus D_f$$

is a fibration pair and that
$$f : Z \to S$$
is trivially fibred over the whole set S. (For Case 2 one has to use Thom's isotopy lemma which generalizes the Ehresmann theorem.) In particular, also
$$f : X_{S \setminus D_f} \to S \setminus D_f$$
is a fibration.

The main example for the first case is a Milnor fibration $f : \overline{\mathfrak{X}} \to S$, or a morsification of it (see Section 1.5) where $Y = \overline{\mathfrak{X}}$, $X = \mathfrak{X}$, and $Z = \partial\mathfrak{X}$.

For the second case assume that $p : \mathbb{C}^{n+1} \to \mathbb{C}$ is a polynomial, say
$$p(z) = \sum_{j=1}^{r} a_j z^{k_j}$$
where we write $z = (z_1, \dots, z_{n+1})$, $k_j = (k_1, \dots, k_{n+1})$ and $z^{k_j} = z_1^{k_{1,j}} z_2^{k_{2,j}} \cdots z_{n+1}^{k_{n+1,j}}$. Also write $|k_j| = k_{1,j} + k_{2,j} + \cdots + k_{n+1,j}$, and let $d = \max_j |k_j|$ be the degree of p. Now consider the homogenization P of p:
$$P : \mathbb{C}^{n+1} \times \mathbb{C} \to \mathbb{C}$$
$$P(z, z_0) = \sum_{j=1}^{r} a_j z^{k_j} z_0^{d-|k_j|}.$$
Then the polynomial P_0 defined by
$$P_0(z) := P(z, 0)$$
is the principal part of the original polynomial p, i.e. the sum of all monomials of highest degree d.

We would like to define a map $\tilde{P} : \mathbb{P}^{n+1} \to \mathbb{P}^1$ by setting $\tilde{P}([z_1 : \cdots : z_{n+1} : z_0]) = [P(z, z_0) : z_0^d]$ which is of course impossible, since the right side is not defined if $z_0 = 0$ and z is a zero of P_0. But by blowing up $\mathbb{P}^{n+1}$ some times in these points we get a complex manifold $\tilde{Y}$ (which projects onto $\mathbb{P}^{n+1}$) such that P induces a map
$$\tilde{P} : \tilde{Y} \to \mathbb{P}^1.$$
Set $S = \mathbb{C} \subset \mathbb{P}^1$, $Y = \tilde{Y}_S$, $X = \mathbb{C}^{n+1} \subset Y$, and $f = \tilde{P}|_Y$. Then $f|_X$ is the original polynomial p.

Now our assumption that $f|_Z$ is non-singular (for some stratification) gives an exact meaning to the statement "p has no singularities at infinity". We will not discuss the further details. (Note that one also could choose other compactifications of p. Furthermore the meaning of "$f|_Z$ has no singularities" is not well-defined since Z itself is not smooth in general — to consider singularities in the stratified sense (for some stratification of Z) is only one possibility. See Massey [40] for a nice discussion on this.)

1.3.1 One singular fibre

Assume that $0 \in D_f$ and let $\Delta \subset S$ be an open disc with $\Delta \cap D_f = \{0\}$. Then the inclusion $Y_0 \hookrightarrow Y_\Delta$ then induces isomorphisms on homology

$$\begin{aligned} H_i(Y_0;\mathbb{Z}) &\xrightarrow{\sim} H_i(Y_\Delta;\mathbb{Z}), & H_i(X_0;\mathbb{Z}) &\xrightarrow{\sim} H_i(X_\Delta;\mathbb{Z}),\\ H_i(Z_0;\mathbb{Z}) &\xrightarrow{\sim} H_i(Z_\Delta;\mathbb{Z}), & H_i(Y_0,Z_0;\mathbb{Z}) &\xrightarrow{\sim} H_i(Y_\Delta,Z_\Delta;\mathbb{Z}). \end{aligned}$$

This can be shown by a retraction arguments, or by looking at the constructible sheaf $Rf_*\mathbb{Z}$ and duality (see e.g. [22]).

This allows to define a map

$$\lambda : H_i(X_s;\mathbb{Z}) \to H_i(X_0;\mathbb{Z}),$$

called *specialization map* for an $s \in \Delta \setminus \{0\}$ such that the following diagram commutes:

$$\begin{array}{ccc} H_i(X_s;\mathbb{Z}) & \xrightarrow{\lambda} & H_i(X_0;\mathbb{Z}) \\ & \searrow \quad \swarrow{\scriptstyle\cong} & \\ & H_i(X_\Delta;\mathbb{Z}) & \end{array}$$

where the downwards arrays are induced by the inclusions. In the same way we get a specialization map

$$\lambda : H_i(Y_s;\mathbb{Z}) \to H_i(Y_0;\mathbb{Z})$$

These maps compare the singular and the generic fibres.

The fibre X_0 contains finitely many singularities $x_1,\ldots,x_r \in C_f$. For each such x_ν select a Milnor fibration

$$x \in \overline{\mathcal{X}}^\nu \xrightarrow{f} \Delta \ni 0$$

such that the $\overline{\mathcal{X}}^\nu$'s do not meet each other and do not meet Z, and where Δ is small enough to be a base space for each of these Milnor fibrations. Then the restriction

$$f : \Big(Y_\Delta \setminus \bigcup_\nu \mathcal{X}^\nu, \bigcup_\nu \partial\mathcal{X}^\nu, Z_\Delta\Big) \to \Delta$$

is again by the Ehresmann theorem a fibration triple. In particular, we get isomorphisms

$$H_i\big(X_s, \bigcup_\nu \overline{\mathcal{X}}^\nu_s;\mathbb{Z}\big) \cong H_i\big(X_0, \bigcup_\nu \overline{\mathcal{X}}^\nu_0;\mathbb{Z}\big).$$

Therefore we get a homology ladder

$$\begin{array}{ccccccc} H_{i+1}\big(X_s,\bigcup \overline{\mathcal{X}}^\nu_s;\mathbb{Z}\big) & \longrightarrow & \bigoplus H_i(\overline{\mathcal{X}}^\nu_s;\mathbb{Z}) & \longrightarrow & H_i(X_s;\mathbb{Z}) & \longrightarrow & H_i\big(X_s,\bigcup \overline{\mathcal{X}}^\nu_s;\mathbb{Z}\big) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\sum\lambda_\nu} & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\cong} \\ H_{i+1}\big(X_0,\bigcup \overline{\mathcal{X}}^\nu_0;\mathbb{Z}\big) & \longrightarrow & \bigoplus H_i(\overline{\mathcal{X}}^\nu_0;\mathbb{Z}) & \longrightarrow & H_i(X_0;\mathbb{Z}) & \longrightarrow & H_i\big(X_0,\bigcup \overline{\mathcal{X}}^\nu_0;\mathbb{Z}\big) \end{array}$$

where λ_ν is the specialization map for the Milnor fibration at x_ν. This homology ladder says that λ carries exactly the information of all λ_ν's.

1.3.2 Monodromy around one singular fibre

We have seen that the restriction of f to Y_Δ is fibred outside the Milnor fibrations at the critical points $x_1, \dots, x_r$. Therefore there exists a geometric monodromy $h \in \mathrm{Isot}^+(Y_s, Z_s)$ (that corresponds to the generator of $\pi_1(\Delta \setminus \{0\}, s)$) which is trivial outside the Milnor fibrations of the x_ν's.

This means that the variation of the monodromy

$$\mathrm{var}(h)_* : H_i(Y_s, Z_s; \mathbb{Z}) \to H_i(Y_s; \mathbb{Z})$$

is given by the variations of the monodromy of the Milnor fibrations at the x_ν's as the composition of the following maps:

$$\begin{array}{ccccc} H_i(Y_s, Z_s; \mathbb{Z}) & \longrightarrow & H_i(Y_s, Y_s \setminus \bigcup \mathcal{X}_s^\nu; \mathbb{Z}) & & \\ & & \uparrow \cong & & \\ & & \bigoplus H_i(\overline{\mathcal{X}}_s^\nu, \partial \mathcal{X}_s^\nu; \mathbb{Z}) & \xrightarrow{\sum \mathrm{var}(h_\nu)_*} \bigoplus H_i(\overline{\mathcal{X}}_s; \mathbb{Z}) & \longrightarrow H_i(Y_s; \mathbb{Z}) \end{array}$$

where

$$h_\nu = h|_{(\overline{\mathcal{X}}_s^\nu, \partial \mathcal{X}_s^\nu)} \in \mathrm{Isot}^+(\overline{\mathcal{X}}_s^\nu, \partial \mathcal{X}_s^\nu)$$

is the monodromy of the Milnor fibration at x_ν. The vertical map is an excision isomorphism.

1.3.3 Global monodromy

By the Riemann mapping theorem we may assume w.l.o.g. that S is an open disc in $\mathbb{C}$, or that $S = \mathbb{C}$. Since we assumed that D_f is finite, S contains a closed disc $\overline{\Delta} \subset S$ with $D_f \subset \Delta$ (with $\Delta = \overline{\Delta} \setminus \partial\Delta$). We change the notations and denote $Y_{\overline{\Delta}}$, $Z_{\overline{\Delta}}$, $X_{\overline{\Delta}}$ and $\overline{D}$ again by Y, Z, X resp. S. Select a base point $s \in \partial\Delta$ and write $D_f = \{z_1, \dots, z_m\}$.

Now $S \setminus D_f$ is a disc with m points removed, hence $\pi_1(S \setminus D_f, s)$ is the free group $\mathcal{F}_m$ with m generators.

In the appendix, Section A.1.9 we introduced the notion of a *geometric basis* for $\pi_1(S \setminus D_f, s)$. For this we have to select a *distinguished system of paths* $(\gamma_1, \dots, \gamma_m)$ where γ_μ starts at s and ends at z_μ. The word "distinguished" means that these paths do not have self-intersections, do not intersect each other (besides the starting point s), and that the starting vectors are ordered counter-clockwise. Then the (classes of the) corresponding loops $\omega_1, \dots, \omega_\mu$ (as defined in the appendix) form a basis of $\pi_1(S \setminus D_f, s)$, and their product is homotopic to the boundary of Δ:

$$[\omega_m] \cdots [\omega_2] \cdot [\omega_1] = [\partial\Delta].$$

(See Section A.1.9.)

Now take for each $z_\mu \in D_f$ a small open disc $\Delta_\mu \subset \Delta$ in the sense of the previous section (i.e. in such a way that Δ_μ forms the basis for all Milnor fibrations at all critical points in X_{z_μ}) such that all these discs are distinct. Furthermore select for each z_μ a base point $s_\mu \in \Delta_\mu \setminus \{z_\mu\}$ that lies in the image of γ_μ. Then the paths γ_μ define by parallel transport isomorphisms

$$\tau_\mu : H_i(X_s;\mathbb{Z}) \xrightarrow{\sim} H_i(X_{s_\mu};\mathbb{Z}).$$

(Of course these isomorphisms depend on the choice of the distinguished system of paths.)

For each z_μ now define

$$V_i^\mu := \ker\bigl(H_i(X_s;\mathbb{Z}) \xrightarrow{\lambda_\mu \circ \tau_\mu} H_i(X_{z_\mu};\mathbb{Z})\bigr)$$

where $\lambda_\mu : H_i(X_{s_\mu};\mathbb{Z}) \to H_i(X_{z_\mu};\mathbb{Z})$ is the specialization map defined in the previous section. V_i^μ is called the *vanishing homology* of the fibre X_{z_μ}.

The following has been first proven by Broughton [20].

Proposition 1.3.1. *The following canonical map is an isomorphism:*

$$H_{i+1}(X, X_s;\mathbb{Z}) \xrightarrow{\sim} \bigoplus_\mu V_i^\mu.$$

In particular, if X is contractible, then we have an isomorphism on reduced homology

$$\tilde{H}_i(X_s;\mathbb{Z}) \cong \bigoplus_\mu V_i^\mu.$$

Proof. This follows from the fact that Δ can be retracted to

$$\bigcup_\mu \bigl(\operatorname{im}\gamma_\mu \cup \Delta_\mu\bigr)$$

and a Mayer-Vietoris argument. (Note that $H_{i+1}(X_{\Delta_\mu}, X_{s_\mu};\mathbb{Z}) \xrightarrow{\sim} V_i^\mu$.)

See e.g. [44]. □

For each z_μ we have a monodromy of

$$f : X_{\Delta_\mu} \to \Delta_\mu$$

which we denote by $\tilde{h}_\mu \in \operatorname{Isot}(X_{s_\mu})$, and which has been discussed in the previous section. This map can be composed with the isomorphism τ_μ:

$$h_\mu := \tau_\mu^{-1} \circ \tilde{h}_\mu \circ \tau_\mu \in \operatorname{Isot}(X_s)$$

which is just the monodromy of the loop ω_μ corresponding to γ_μ.

We then have

Proposition 1.3.2.

$$\operatorname{im}\bigl(h_{\mu_*} - \mathbb{1}\bigr) \subset V_i^\mu.$$

Proof. This follows by the same methods as in the proof of the previous proposition. See e.g. [44]. □

Now let γ be a simple loop in S with base point s (as defined in the appendix, Section A.4). Then by Proposition A.4.2 there exist a distinguished basis of paths $(\gamma_1, \dots, \gamma_m)$ and $1 \le \mu_1 < \mu_2 < \cdots < \mu_r \le m$ such that

$$[\gamma] = [\omega_{\mu_r}] \cdots [\omega_{\mu_2}] \cdot [\omega_{\mu_1}]$$

where ω_μ is the loop corresponding to γ_μ. This means that for the corresponding monodromy we have

$$h_\gamma = h_{\mu_r} \cdots h_{\mu_2} h_{\mu_1}.$$

In particular, for the monodromy around all critical values we have

$$h_{\partial\Delta} = h_m \cdots h_2 h_1.$$

We often call the monodromy around all critical values "*the* monodromy" (without referring to the path) of the family of singularities. In the case of a polynomial $p : \mathbb{C}^{n+1} \to \mathbb{C}$ this monodromy is normally called the monodromy *at infinity* (do not confuse this with the monodromy of a singularity at infinity for a polynomial which may have atypical values which are not critical values).

1.4 Picard-Lefschetz Theory

1.4.1 Nondegenerate singularities

Let X be an $(n+1)$-dimensional complex manifold and $f : X \to \mathbb{C}$ holomorphic. A critical point $x \in X$ of f is called nondegenerate if its Hessian $\left(\frac{\partial^2 f}{\partial z_i \partial z_j}\right)_{ij}$ (for some coordinate system $(z_1, \dots, z_{n+1})$) is nondegenerate. (This is in fact independent of the coordinate system.)

The following proposition shows that all nondegenerate critical points look like the same:

Proposition 1.4.1 (Complex Morse Lemma)**.** *Let X be an $(n+1)$-dimensional complex manifold and $f : X \to \mathbb{C}$ holomorphic with a nondegenerate critical point in $x \in X$. Set $s = f(x)$. The there exists a (local) coordinate system $(z_1, \dots, z_{n+1})$ with $z_i(x) = 0$ $(i = 1, \dots, n+1)$ such that*

$$f(z_1, \dots, z_{n+1}) = z_1^2 + \cdots + z_{n+1}^2 + s.$$

Proof. See e.g. [30]. □

1.4.2 The local case

Assume that $n > 0$ and let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be the function

$$f(z_1, \dots, z_{n+1}) = z_1^2 + \cdots + z_{n+1}^2.$$

By the complex Morse lemma all nondegenerate critical point are of this form.

The Milnor fibration can be easily described: Select $\varepsilon > 0$ arbitrary (in fact ε may be arbitrary large for quasihomogenous functions) and $\eta > 0$ small enough (in this case $\eta < \varepsilon^2$). Then

$$\begin{aligned}\overline{\mathcal{X}} &= \{z \in \mathbb{C}^{n+1} \mid \|z\| \leq \varepsilon \text{ and } |f(z)| \leq \eta\},\\ S &= \overline{B}_\eta = \{s \in \mathbb{C} \mid |s| \leq \eta\},\\ C_f &= \{0\} \quad \text{and} \quad D_f = \{0\}.\end{aligned}$$

Take $s = \eta$ as a base point. The fibre then can be written as

$$\overline{\mathcal{X}}_\eta = \{x + iy \in \mathbb{C}^{n+1} \mid \|x\|^2 + \|y\|^2 \leq \varepsilon^2,\ \|x\|^2 - \|y\|^2 = \eta, \text{ and } \langle x, y\rangle = 0\}.$$

By a (real) reparametrization it can be shown that $\overline{\mathcal{X}}_\eta$ is isomorphic to

$$E = \{u + iv \in \mathbb{C}^{n+1} \mid \|u\| = 1,\ \|v\| \leq 1,\ \text{and } \langle u, v\rangle = 0\}$$

by a map $h_0 : E \to \overline{\mathcal{X}}_s$. E is the "unit ball bundle" over a sphere. Hence E can be retracted onto its zero section

$$\{u + i \cdot 0 \in \mathbb{C}^{n+1} \mid \|u\| = 1\} = S^n$$

by the retracting map $r : E \to S^n$, $u + iv \mapsto u$. For a base point $u_0 \in S^n$ in the zero section denote its fibre by

$$D^n = \{u_0 + iv \in \mathbb{C}^{n+1} \mid \|v\| \leq 1,\ \langle u_0, v\rangle = 0\}$$

which is isomorphic to a closed ball. Denote by $j : D^n \to E$ the inclusion. Then r and j induce isomorphisms on homology resp. relative homology, hence we have

$$H_i(E;\mathbb{Z}) \xrightarrow[r_*]{\sim} H_i(S^n;\mathbb{Z}) = \begin{cases}\mathbb{Z} & \text{for } i = 0, n\\ 0 & \text{otherwise,}\end{cases}$$

and

$$H_i(E, \partial E;\mathbb{Z}) \xleftarrow[j_*]{\sim} H_i(D^n, \partial D^n;\mathbb{Z}) = \begin{cases}\mathbb{Z} & \text{for } i = 0, n\\ 0 & \text{otherwise.}\end{cases}$$

Define

$$g_\vartheta(u+iv) = \big(u\cos(\pi\vartheta\|v\|) + \tfrac{v}{\|v\|}\sin(\pi\vartheta\|v\|)\big) + i\big(-u\|v\|\sin(\pi\vartheta\|v\|) + v\cos(\pi\vartheta\|v\|)\big).$$

This defines a one-parameter family of homeomorphisms $g_\vartheta : E \to E$. Now define homeomorphisms

$$h_\vartheta = e^{\pi i\vartheta} h_0 \circ g_\vartheta : E \to \overline{\mathcal{X}}_{\eta e^{2\pi i\vartheta}}.$$

One calculates that these maps are in fact homeomorphisms onto the fibres $\overline{\mathcal{X}}_{\eta e^{2\pi i\vartheta}}$, and that h_0 and h_1 are equal on ∂E. Therefore the map $h := h_1 \circ h_0^{-1} : \overline{\mathcal{X}}_0 \to \overline{\mathcal{X}}_1$ represents the geometric monodromy. Translated back to E this map is

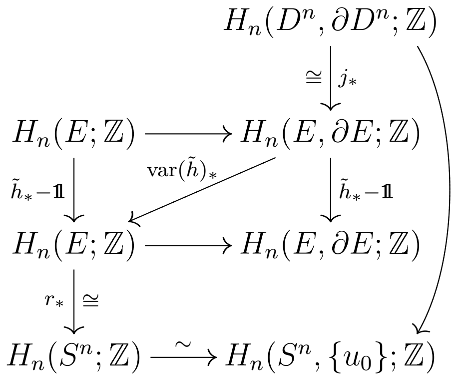
$$h_0^{-1} \circ h \circ h_0 = -g_1 =: \tilde{h}.$$

To calculate its operation on homology it suffices to look at the map

$$\psi := r \circ \tilde{h} \circ j : D^n \to S^n \qquad \psi(u_0 + iv) = -u_0 \cos(\pi\|v\|) - \tfrac{v}{\|v\|} \sin(\pi\|v\|).$$

One easily sees that $\deg(-\psi) = 1$, i.e. $\deg \psi = (-1)^{n+1}$.

Now consider the following diagram which shows that ψ_* calculates the variation of the monodromy:



Now consider the generators of the (relative) homology

$$\tilde{\delta} = i_*([S^n]) \in H_n(E;\mathbb{Z}) \quad \text{and} \quad d = j_*([D^n, \partial D^n]) \in H_n(E, \partial E; \mathbb{Z})$$

where $i : S^n \to E$ is the inclusion and $[\,\cdot\,]$ denotes the fundamental class. To calculate their intersection products $(d, \tilde{\delta})$ and $(\tilde{\delta}, \tilde{\delta})$ one has to take care with orientations (on the one hand E carries an orientation coming from the complex structure of $\overline{\mathfrak{X}}_\eta$, on the other hand it gets an orientation as the unit ball bundle over S^n). One gets

$$(d, \tilde{\delta}) = (-1)^{\frac{1}{2}n(n+1)}$$

and

$$(\tilde{\delta}, \tilde{\delta}) = (-1)^{\frac{1}{2}n(n-1)}\chi(S^n) = (-1)^{\frac{1}{2}n(n-1)}(1 + (-1)^n) = (-1)^{\frac{1}{2}n(n-1)} + (-1)^{\frac{1}{2}n(n+1)}.$$

Now we have

$$\mathrm{var}(\tilde{h})_*(d) = \deg(\psi) \cdot \tilde{\delta} = (-1)^{n+1}\tilde{\delta}$$

which can be rewritten as

$$\mathrm{var}(\tilde{h})_*(c) = (-1)^{n+1}\frac{(c, \tilde{\delta})}{(d, \tilde{\delta})}\tilde{\delta} = (-1)^{n+1} \cdot (-1)^{\frac{1}{2}n(n+1)}(c, \tilde{\delta})\tilde{\delta} = -(-1)^{\frac{1}{2}n(n-1)}(c, \tilde{\delta})\tilde{\delta}$$

for $c \in H_n(E, \partial E; \mathbb{Z})$, since d generates $H_n(E, \partial E; \mathbb{Z})$.

If we transfer this back to $\overline{\mathfrak{X}}_\eta$, we get the following theorem:

Theorem 1.4.2 (Picard-Lefschetz Formulas)**.** *Let δ be a generator of $H_n(\overline{\mathfrak{X}}_s; \mathbb{Z})$. Then*

$$(\delta, \delta) = (-1)^{\frac{1}{2}n(n-1)} + (-1)^{\frac{1}{2}n(n+1)}, \tag{1.1}$$

and the variation of the monodromy is given by

$$\mathrm{var}(h)_*(c) = -(-1)^{\frac{1}{2}n(n-1)}(c, \delta)\delta \tag{1.2}$$

for $c \in H_n(\overline{\mathfrak{X}}_s, \partial\mathfrak{X}_s)$ □

Corollary 1.4.3. *Using the notation of the theorem, the monodromy is given by*

$$h_* a = a - (-1)^{\frac{1}{2}n(n-1)}(a, \delta)\delta \tag{1.3}$$

for $a \in H_n(\overline{\mathfrak{X}}_s)$. *In particular,*

$$h_* \delta = (-1)^{n+1}\delta. \tag{1.4}$$

□

The cycle $\delta \in H_n(\overline{\mathfrak{X}}_s; \mathbb{Z})$ is called a *vanishing cycle*. This has the following reason: It can be easily seen that the special fibre

$$\begin{aligned} \overline{\mathfrak{X}}_0 &= \{z \in \mathbb{C}^{n+1} \mid z_1^2 + \cdots + z_{n+1}^2 = 0,\ \|z\| \le \varepsilon\} \\ &= \{x + iy \in \mathbb{C}^{n+1} \mid \|x\|^2 + \|y\|^2 \le \varepsilon^2,\ \|x\|^2 = \|y\|^2 \text{ and } \langle x, y\rangle = 0\} \end{aligned}$$

is contractible, i.e.

$$H_i(\overline{\mathfrak{X}}_0; \mathbb{Z}) = 0 \quad \text{for } i \neq 0.$$

This means that δ vanishes when considering the specialization map $\lambda : H_i(\overline{\mathfrak{X}}_s; \mathbb{Z}) \to H_i(\overline{\mathfrak{X}}_0; \mathbb{Z})$. Or to be more geometrical: Under the isomorphism $\overline{\mathfrak{X}}_\eta \xrightarrow{\sim} E$, we have that the zero section S^n corresponds to

$$\{x + iy \in \overline{\mathfrak{X}}_\eta \mid y = 0\} = \{x + i \cdot 0 \in \mathbb{C}^{n+1} \mid \|x\|^2 = \eta\}.$$

If η goes to zero, this sphere contracts to a point, i.e. its homology class vanishes.

1.4.3 The global case, families of nondegenerate singularities

We assume the same situation as in Section 1.3, and use the same notations, i.e. X is a complex manifold which embeds into a real analytic manifold Y which may have a boundary, and $f : Y \to S$ is a proper and surjective analytic map which is holomorphic on X with only isolated critical points such that the restriction of f to the complement $Z = Y \setminus X$ is non-singular (in the sense given there).

We make the additional assumption that all critical points of f are nondegenerate. Again we assume that D_f is finite and write $D_f = \{z_1, \ldots, z_m\}$.

We want to describe the monodromy h_μ around one critical value z_μ. For this it suffices to describe the maps $\tilde{h}_\mu$ (notations as in Section 1.3).

For each critical point x_ν with critical value z_μ we have a Milnor fibration

$$f : \overline{\mathfrak{X}}^\nu \to \Delta_\mu$$

(again with same notations as in Section 1.3). In the previous section we have seen that the (reduced) homology of its generic fibre is generated by the vanishing cycle $\delta_\nu = i_{\nu *}([S^n]) \in H_n(\overline{\mathfrak{X}}^\nu_{s_\mu}; \mathbb{Z})$.

By Section 1.3.2 the variation $\text{var}(\tilde{h}_\mu)_*$ is the sum of the variations $\text{var}(h_\nu)_*$ (besides some canonical maps) where the maps h_ν are the monodromies of the

Milnor fibrations of the critical points x_ν with critical value z_μ. Since all critical values are nondegenerate, these maps are determined by the local Picard-Lefschetz formulas. Since the intersection product on X is compatible with the restrictions to the Milnor fibrations $\mathfrak{X}^\nu_{s_\mu}$, we get a formula for $\tilde{h}_\mu$. But since the intersection product is also compatible with parallel transport, this gives the formula for h_μ. Together we get:

Theorem 1.4.4 (Picard-Lefschetz Formulas)**.** *For each ν let $\delta_\nu \in H_n(X_s;\mathbb{Z})$ be the vanishing cycle corresponding to the critical point x_ν.*

Then for ν, ν' such that x_ν and $x_{\nu'}$ have the same critical value one has

$$(\delta_\nu, \delta_{\nu'}) = \begin{cases} (-1)^{\frac{1}{2}n(n-1)} + (-1)^{\frac{1}{2}n(n+1)} & \text{for } \nu = \nu' \\ 0 & \text{for } \nu \neq \nu', \end{cases} \tag{1.5}$$

and the variation of the monodromy around the critical value z_μ is given by

$$\operatorname{var}(h_\mu)_*(c) = -(-1)^{\frac{1}{2}n(n-1)} \sum_{\substack{\nu \\ f(x_\nu)=z_\mu}} (c, \delta_\nu)\delta_\nu \tag{1.6}$$

for $c \in H_n(X_s, Z_s)$. □

Corollary 1.4.5. *Using the notation of the theorem, the monodromy around the critical value z_μ is given by*

$$h_{\mu_*} a = a - (-1)^{\frac{1}{2}n(n-1)} \sum_{\substack{\nu \\ f(x_\nu)=z_\mu}} (a, \delta_\nu)\delta_\nu \tag{1.7}$$

for $a \in H_n(X_s)$. □

Note that we also have, as it follows from the above:

Proposition 1.4.6. *For each μ those δ_ν which have critical value z_μ generate freely V_i^μ.*

In particular, if X is contractible, then the reduced homology of X_s is concentrated in degree n, and $H_n(X_s;\mathbb{Z})$ is freely generated by all δ_ν. □

In the case that X is contractible it follows from the Picard-Lefschetz formulas that one knows the monodromy (for all $[\gamma] \in \pi_1(S \setminus D_f, s)$) if one knows all intersection products of the δ_μ's. We will show in Chapter 3 that also the converse is true (see Corollary 3.1.5).

1.5 Unfoldings and Morsifications

1.5.1 Unfoldings

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function-germ. The theory of unfoldings describes how f behaves under small disturbances.

Definition 1.5.1. An *unfolding* of a holomorphic function-germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a holomorphic function-germ

$$F : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0)$$

such that $F(z, 0) = f(z)$.

If $F : U \times V \to \mathbb{C}$ is a representative of an unfolding F, for each $t \in V$ we write

$$F_t := F(\cdot, t) : V \to \mathbb{C}.$$

Definition 1.5.2. Let F and F' be two unfoldings of f.

F' is *equivalent* to F if

$$F'(x, t) = F(g(x, t), t)$$

for a holomorphic function-germ $g : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \to (\mathbb{C}^{n+1}, 0)$.

F' is *induced* from F if

$$F'(x, t) = F(x, h(t))$$

for a holomorphic function-germ $h : (\mathbb{C}^{k'}, 0) \to (\mathbb{C}^k, 0)$.

Definition 1.5.3. An unfolding F of f is called *versal* if every unfolding F' of f is equivalent to an unfolding induced from F.

F is called *miniversal* if it is versal with minimal dimension k of the parameter space.

F is called *infinitesimal versal* if the images of the functions

$$\left.\frac{\partial F}{\partial t_i}\right|_{t=0} : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$$

under the canonical projection

$$\mathcal{O}_{n+1} \to \mathcal{O}_{n+1} / \langle \tfrac{\partial f}{\partial z_1}, \dots, \tfrac{\partial f}{\partial z_{n+1}} \rangle =: \mathcal{J}_f$$

generate the ideal $\mathcal{J}_f$.

The ideal $\mathcal{J}_f$ is called the *Jacobian ideal* of f. Its dimension

$$\mu(f) := \dim_{\mathbb{C}} \mathcal{J}_f$$

is called the *Milnor number* of f. We have:

Proposition 1.5.4. *If f defines an isolated singularity, then μ is finite.*

Proof. This follows from the fact that for an isolated singularity the map grad f is finite. □

Infinitesimal versality can be checked easily. This shows the importance of the following theorem:

Theorem 1.5.5. *Any infinitesimal versal unfolding is versal.*

Proof. See [41] and [5]. □

Corollary 1.5.6. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function-germ which defines an isolated singularity. If the images of $\varphi_0, \dots, \varphi_{\mu-1} \in \mathcal{O}_{n+1}$ generate the Jacobian ideal $\mathfrak{J}_f$, then*

$$F(x,t) = f(x) + \sum_{j=0}^{\mu-1} t_i \varphi_i(x)$$

is a miniversal unfolding of f. □

1.5.2 Truncated unfoldings

Instead of looking at arbitrary unfoldings $F : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ of $f : (\mathbb{C}^{n+1}, 0) \to \mathbb{C}$, one can look at unfoldings such that $F(0,t) = 0$ for all t. This means that one disturbs f within the maximal ideal $\mathfrak{m} \subset \mathcal{O}_{n+1}$. Such an unfolding is called a *truncated unfolding.*

The definitions of *equivalent, induced, versal*, and *miniversal* remain the same.

Proposition 1.5.7. *A truncated unfolding F' of f is versal if and only if the unfolding*

$$F(z,t) = F'(z,t') - t_0 \qquad t = (t_0, t')$$

is versal.

Proof. See [12]. □

Corollary 1.5.8. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function-germ which defines an isolated singularity. If the images of $\varphi_1, \dots, \varphi_{\mu-1} \in \mathcal{O}_{n+1}$ and $\varphi_0 \equiv -1$ generate the Jacobian ideal $\mathfrak{J}_f$, then*

$$F'(x,t') = f(x) + \sum_{j=1}^{\mu-1} t'_i \varphi_i(x)$$

is a miniversal truncated unfolding of f. □

1.5.3 The discriminant and the bifurcation set

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ which defines an isolated singularity, and let $F : (\mathbb{C}^{n+1} \times \mathbb{C}^{\mu}, 0) \to (\mathbb{C}, 0)$ and $F' : (\mathbb{C}^{n+1} \times \mathbb{C}^{\mu-1}, 0) \to (\mathbb{C}, 0)$ be its versal unfolding resp. versal truncated unfolding in the sense of Corollaries 1.5.6 and 1.5.8 (with the same functions φ_i and $\varphi_0 \equiv -1$). Write $t = (t_0, t')$.

Let $F : U \times V \to \mathbb{C}$ be a representative of the unfolding F with $V = V_0 \times V'$ and set

$$Y = \{(z,t) \in U \times V \mid F(z,t) = 0\}.$$

Then Y is non-singular. By the Milnor fibration theorem there exist $0 < \eta, \rho \ll \varepsilon \ll 1$ such that for

$$\overline{\mathcal{Y}} = \{(z,t) \in U \times V \mid F(z,t) = 0,\ \|z\| \le \varepsilon,\ |t_0| < \eta,\ \|t'\| < \rho\},$$
$$\Lambda = \{t \in V \mid |t_0| < \eta,\ \|t'\| < \rho\},$$

the restriction $p : \overline{\mathcal{Y}} \to \Lambda$ of the projection $U \times V \to V$ is a Milnor fibration.

Now the set of critical points of p is exactly

$$C := C_p = \{(z,t) \in \overline{\mathcal{Y}} \mid z \text{ is a critical point of } F_t\}.$$

The discriminant $D := D_p = p(C)$ is called the *discriminant of the versal unfolding* of f. By the Milnor fibration theorem D is a hypersurface in the parameter space Λ of the versal unfolding.

Definition 1.5.9. Let X be a complex manifold and $f : X \to \mathbb{C}$ a holomorphic function. g is called a *Morse function* if all critical points of g are nondegenerate and have distinct critical values.

Now set

$$\overline{\mathcal{Y}}' = \{(z,t') \in U \times V' \mid \|z\| \le \varepsilon,\ |F(z,t')| < \eta,\ \|t'\| < \rho\},$$
$$\Lambda' = \{t' \in V' \mid \|t'\| < \rho\}$$
$$S = \{t_0 \in V_0 \mid |t_0| < \eta\}$$

(with the same η, ρ and ε as above).

$F' : \overline{\mathcal{Y}}' \to S$ is a representative of F' such that $f = F'_0 : \overline{\mathcal{Y}}'_{t'=0} \to S$ is a Milnor fibration and for each $t' \in \Lambda'$ the map $F'_{t'} : \overline{\mathcal{Y}}'_{t'} \to S$ is still non-singular at the boundary $\partial \mathcal{Y}'_{t'}$ and transversal to it.

Define

$$\Xi := \{t' \in \Lambda' \mid F'_{t'} \text{ is not a Morse function}\} \subset \Lambda'.$$

This set is called the *bifurcation set* of (the versal truncated unfolding of) f.

Theorem 1.5.10. *Ξ is a hypersurface in the parameter space Λ'.*

Moreover, the restriction $\pi : D \to \Lambda'$ of the projection $\pi : \Lambda \to \Lambda'$ to the discriminant D is a μ-sheeted ramified covering whose set of ramification points is exactly Ξ. (Here μ is the Milnor number of f).

Proof. See [32]. □

1.5.4 Morsifications

Consider the versal truncated unfolding

$$F' : \overline{\mathcal{Y}}' \to S$$

of a holomorphic map-germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ which defines an isolated singularity.

Since $\Xi \subset \Lambda'$ is a hypersurface, for each generic complex line

$$\ell : \Delta \to \Lambda'$$

(where $\Delta \subset \mathbb{C}$ is the unit disc) 0 lies isolated in the intersection $\operatorname{im} \ell \cap \Xi$. This means that for the induced unfolding

$$\tilde{F} : \overline{\mathcal{Y}}' \times_{\Lambda'} \Delta \to S \qquad \tilde{F}(z, \tau) = F'(z, \ell(\tau))$$

each $\tilde{F}_\tau$ is a Morse function for all $\tau \neq 0$ small enough.

$\tilde{F}$, or sometimes $\tilde{F}_\tau$ for a $\tau \neq 0$ small enough, is called a *morsification* of f.

From Theorem 1.5.10 it follows:

Proposition 1.5.11. *The Milnor number μ of f is exactly the number of critical points in a morsification $\tilde{F}_\tau$ of f.*

Note that a morsification $\tilde{F}_\tau$ of f satisfies our conditions for a family of singularities in the sense of Section 1.3 (as mentioned there).

1.5.5 Braid monodromy

If $F' : \overline{\mathcal{Y}}' \to S$ is the versal truncated unfolding of f, then for each $t' \in \Lambda' \setminus \Xi$ the map $F'_{t'}$ is a morsification of f.

By Theorem 1.5.10 the restriction

$$\pi : D_{\Lambda' \setminus \Xi} \to \Lambda' \setminus \Xi$$

of the projection $\pi : \Lambda \to \Lambda'$ from the parameter space of the versal unfolding to that of the versal truncated unfolding (where D is the discriminant) is a μ-fold unbranched covering. That means that for $t' \in \Lambda' \setminus \Xi$ the set $D_{t'}$ contains μ points in $\Lambda_{t'} = B_\eta$. Therefore, if γ is a closed path in $\Lambda' \setminus \Xi$, then the family $D_{\gamma(\tau)} \subset B_\eta$ defines an element in $\pi_1(X^\mu)$ where X^μ is the space of μ-configurations as defined in Section A.1.1. Since $\pi_1(X^\mu) = \mathrm{Br}_\mu$, the braid group with μ strands (see Section A.1.1), we get the so-called *braid monodromy*

$$h_{\text{braid}} : \pi_1(\Lambda' \setminus \Xi, t') \to \mathrm{Br}_\mu$$

(after selecting a base point t').

1.6 Distinguished Bases and Coxeter-Dynkin Diagrams

1.6.1 Distinguished bases and the intersection matrix

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function-germ which defines an isolated singularity, and let

$$\tilde{f} := \tilde{F}_\tau : \overline{\mathcal{X}}' \to S \qquad \text{with } \overline{\mathcal{X}}' := \overline{\mathcal{Y}}'_\tau$$

be a morsification of f (where we use the notation from Section 1.5.3). By shrinking η a bit, we may replace S by the closed disc $\overline{\Delta} = \overline{B}_\eta$ which again we denote by S. Select a base point $s \in \partial\Delta$.

Now as seen above, $\tilde{f}$ has μ critical points with distinct critical values. Write $D_{\tilde{f}} = \{z_1, \dots, z_\mu\} \subset \Delta = \overline{\Delta} \setminus \partial\Delta$. Now, following the methods in Section 1.3.3 we may choose a distinguished system of paths $(\gamma_1, \dots, \gamma_\mu)$. The corresponding loops define a basis $[\omega_1], \dots, [\omega_\mu]$ of $\pi_1(S \setminus D_{\tilde{f}}, s)$. By Section 1.4.3 we have that for each $k = 1, \dots, \mu$ the vanishing homology is generated by the vanishing cycle corresponding to the critical value z_k:

$$V_n^k = \langle \delta_k \rangle.$$

Since $\overline{\mathfrak{X}}'$ is contractible, by Proposition 1.3.1 we get

$$H_n(\overline{\mathfrak{X}}'_s; \mathbb{Z}) = \bigoplus_{k=1}^{\mu} V_n^k = \langle \delta_1, \dots, \delta_\mu \rangle \cong \mathbb{Z}^\mu.$$

The basis

$$(\delta_1, \dots, \delta_\mu)$$

is called a *distinguished basis* (of vanishing cycles) of the homology of the Milnor fibre of f.

The distinguished basis depends on two choices: The first is the choice of the morsification, the second is the choice of the distinguished system of paths.

Two different choices for the distinguished system of paths can be compared by an operation of the braid group, as seen in the appendix, Section A.1.9: The braid group acts simply transitive on the set of distinguished systems of paths, hence there is an element of the braid group Br_μ which carries the first system over to the second.

Two different choices of the morsification are connected via braid monodromy: Given $t'_1, t'_2 \in \Lambda' \setminus \Xi$, one can select a path from t'_1 to t'_2 (since $\Lambda' \setminus \Xi$ is connected). This defines an "open" braid from the critical points of $\tilde{F}_{t'_1}$ to those of $\tilde{F}_{t'_2}$. This corresponds — as for ordinary braids outlined in the appendix, Section A.1.3 — to a relative isotopy class of a diffeomorphism from $S \setminus D_{F_{t'_1}}$ to $S \setminus D_{F_{t'_2}}$ which respects the boundary. This diffeomorphism maps a given distinguished system of paths for the parameter t'_1 to a distinguished system of paths for the parameter t'_2. One easily checks that the complete geometry is transferred isomorphically in this process.

The monodromy of a simple loop ω_k corresponding to a path γ_k of a distinguished system of paths is given by the Picard-Lefschetz formulas. If $(\delta_1, \dots, \delta_\mu)$ is the distinguished basis of vanishing cycles defined by the distinguished system of paths $(\gamma_1, \dots, \gamma_\mu)$, then

$$m_k \delta_j = \delta_j - (-1)^{\frac{1}{2}n(n-1)} (\delta_j, \delta_k) \delta_k \qquad m_k := h_{k*} \tag{1.8}$$

The monodromy around all critical values is given by

$$m := m_\mu \cdots m_2 m_1. \tag{1.9}$$

However, by the construction of the morsification, this monodromy corresponds to the monodromy of the Milnor fibration

$$f = \tilde{F}_0 : \overline{\mathfrak{X}} \to S$$

of the original function f.

Equations (1.8) and (1.9) show that one can calculate m if one knows all intersection products (δ_i, δ_j) of the vanishing cycles. This motivates the introduction of the intersection matrix

$$S_{\mathrm{IS}} := \big((\delta_i, \delta_j)\big)_{\substack{1\le i\le\mu \\ 1\le j\le\mu}}. \tag{1.10}$$

By the properties of the intersection product, S_{IS} is symmetric if n is even, and antisymmetric if n is odd. Moreover, the diagonal entries are

$$(S_{\mathrm{IS}})_{ii} = (-1)^{\frac{1}{2}n(n-1)} + (-1)^{\frac{1}{2}n(n+1)}$$

as follows from the self-intersection products of the vanishing cycles given in the Picard-Lefschetz theorems.

1.6.2 The Seifert matrix

A matrix presentation of the variation of the monodromy can be obtained as follows: The intersection product

$$(\cdot,\cdot) : H_n(\overline{\mathfrak{X}}_s, \partial\mathfrak{X}_s; \mathbb{Z}) \times H_n(\overline{\mathfrak{X}}_s; \mathbb{Z}) \to \mathbb{Z}$$

is a perfect pairing, hence there exists a basis $(\delta_1^*, \dots, \delta_\mu^*)$ of $H_n(\overline{\mathfrak{X}}_s, \partial\mathfrak{X}_s; \mathbb{Z})$ dual to the distinguished basis $(\delta_1, \dots, \delta_\mu)$. From the Picard-Lefschetz formulas it follows that

$$\mathrm{var}(h_i)_*\delta_j^* = -(-1)^{\frac{1}{2}n(n-1)} \cdot \begin{cases} \delta_i & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases} \tag{1.11}$$

Let $\iota_* : H_n(\overline{\mathfrak{X}}_s; \mathbb{Z}) \to H_n(\overline{\mathfrak{X}}_s, \partial\mathfrak{X}_s; \mathbb{Z})$ be the canonical map of the long exact sequence for relative homology. Note that

$$\iota_*(\delta_i) = \sum_{j=1}^{\mu} (\delta_i, \delta_j)\delta_j^* \tag{1.12}$$

as it follows from $(\iota_*(\delta_i), \delta_j) = (\delta_i, \delta_j)$.

For two closed paths γ_1, γ_2 one has $h_{\gamma_2\circ\gamma_1} = h_{\gamma_2}h_{\gamma_1}$. From this and the definition of the variation it follows that

$$\mathrm{var}(h_{\gamma_2\circ\gamma_1})_* = \mathrm{var}(h_{\gamma_2})_* \circ \iota_* \circ \mathrm{var}(h_{\gamma_1})_* + \mathrm{var}(h_{\gamma_1})_* + \mathrm{var}(h_{\gamma_2})_*.$$

From this it follows inductively for the case of a distinguished system of paths that

$$\mathrm{var}(h_{\partial\Delta}) = \sum_{r=1}^{\mu} \sum_{1\le i_1<\cdots<i_r\le\mu} \mathrm{var}(h_{i_r})_* \circ \iota_* \circ \cdots \circ \iota_* \circ \mathrm{var}(h_{i_2})_* \circ \iota_* \circ \mathrm{var}(h_{i_1}). \tag{1.13}$$

From equations (1.11) and (1.13) it follows that the matrix for $\mathrm{var}(h_{\partial\Delta})$ with respect to the bases (δ_i^*) and (δ_j) is a lower triangle matrix whose diagonal entries are all $-(-1)^{\frac{1}{2}n(n-1)}$. Now define V as the transpose of the inverse of this matrix multiplied by $-(-1)^{\frac{1}{2}n(n-1)}$: Set

$$v(a,b) := -(-1)^{\frac{1}{2}n(n-1)}\big(\mathrm{var}(h_{\partial\Delta})_*^{-1}a, b\big) \qquad \text{for } a,b \in H_n(\overline{\mathfrak{X}}_s;\mathbb{Z})$$

and set

$$V_{ij} = v(\delta_i, \delta_j).$$

Then V is an upper triangle matrix with 1's on the diagonal.

In the following calculation we follow [30]. Let $a, b \in H_n(\overline{\mathfrak{X}}_s;\mathbb{Z})$. Set $a' = \mathrm{Var}^{-1} a$ and $b' = \mathrm{Var}^{-1} b$ where we write $\mathrm{Var} := \mathrm{var}(h_{\partial\Delta})_*$. We then have

$$\begin{aligned}(a,b) &= (\mathrm{Var}\, a', \mathrm{Var}\, b') \\ &= -(\mathrm{Var}\, a', b') - (a', \mathrm{Var}\, b') && \text{(by Lemma 1.2.1)} \\ &= -(a, \mathrm{Var}^{-1} b) - (\mathrm{Var}^{-1} b, a) \\ &= -(-1)^{\frac{1}{2}n(n-1)}\big(-(-1)^n v(b,a) - v(a,b)\big) \\ &= (-1)^{\frac{1}{2}n(n-1)}\big(v(a,b) + (-1)^n v(b,a)\big).\end{aligned}$$

From this it follows that

$$S_{\mathrm{IS}} = (-1)^{\frac{1}{2}n(n-1)}(V + (-1)^n V^t). \tag{1.14}$$

V is called the *Seifert matrix* of f (with respect to the distinguished basis $(\delta_1, \ldots, \delta_\mu)$).

Now by the above definitions the matrices of $\mathrm{Var} : H_n(\overline{\mathfrak{X}}_s, \partial\mathfrak{X}_s;\mathbb{Z}) \to H_n(\overline{\mathfrak{X}}_s;\mathbb{Z})$ and $\iota_* : H_n(\overline{\mathfrak{X}}_s;\mathbb{Z}) \to H_n(\overline{\mathfrak{X}}_s, \partial\mathfrak{X}_s;\mathbb{Z})$ with respect to the bases (δ_i) of $H_n(\overline{\mathfrak{X}}_s;\mathbb{Z})$ and (δ_j^*) of $H_n(\overline{\mathfrak{X}}_s, \partial\mathfrak{X}_s;\mathbb{Z})$ are

$$\begin{aligned}\iota_* &\cong S_{\mathrm{IS}}^t = (-1)^n S_{\mathrm{IS}} \qquad \text{and} \\ \mathrm{Var} &\cong -(-1)^{\frac{1}{2}n(n-1)}(V^t)^{-1}.\end{aligned}$$

(The first equation follows from equation (1.12).) Since we have for $h := h_{\partial\Delta *}$ that

$$h = \mathrm{Var} \circ \iota_* + \mathbb{1},$$

the matrix m for the monodromy h is

$$\begin{aligned}m &= -(-1)^{\frac{1}{2}n(n-1)}(V^t)^{-1} \cdot (-1)^n \cdot (-1)^{\frac{1}{2}n(n-1)}(V + (-1)^n V^t) + \mathbb{1} \\ &= (-1)^n (V^t)^{-1} V.\end{aligned} \tag{1.15}$$

1.6.3 Stabilization

The following theorem is due to Gabrielov.

Theorem 1.6.1. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function-germ which defines an isolated singularity, and let $\tilde{f}$ be a morsification of f. Select a distinguished system of paths $(\gamma_1, \ldots, \gamma_\mu)$ for $\tilde{f}$ and let $(\delta_1, \ldots, \delta_\mu)$ be the corresponding distinguished basis of vanishing cycles.*

Then also $g : (\mathbb{C}^{n+1} \times \mathbb{C}^m, 0) \to (\mathbb{C}, 0)$, $g(z, w) = f(z) + w_1^2 + \cdots + w_m^2$ defines an isolated singularity and $\tilde{g}(z, w) = \tilde{f}(z) + w_1^2 + \cdots + w_m^2$ is a morsification of g with the same critical values then $\tilde{f}$. Let $(\tilde{\delta}_1, \ldots, \tilde{\delta}_\mu)$ be the distinguished bases of vanishing cycles corresponding to the same distinguished system of paths $(\gamma_1, \ldots, \gamma_\mu)$.

Then both intersection products are connected by the following relation:

$$(\tilde{\delta}_i, \tilde{\delta}_j) = \big(\mathrm{sign}(j - i)\big)^m (-1)^{(n+1)m + \frac{1}{2}m(m+1)} (\delta_i, \delta_j).$$

Proof. See [19]. □

The function g in the theorem is called a *stabilization* of f.

Remark 1.6.2. The signs in this theorem depend on the convention how the paths in a distinguished system of paths are numbered. Our convention is to order the starting vectors counter-clockwise, but the opposite convention is also found often in the literature. The formula of the theorem differs by a factor $(-1)^m$ in this case.

From the theorem it follows particularly that the monodromy of f and g is the same (up to a sign). Since all signs in the formulas of our interest only depend on the congruence class of n modulo 4, we can restrict our analysis to some specific congruence class. Most common is to take $n \equiv 2 \bmod 4$.

1.6.4 Coxeter-Dynkin diagrams

The intersection data encoded in the intersection matrix S_{IS} can be encoded in a diagram, called a *Coxeter-Dynkin* diagram. For each vanishing cycle δ_i $(i = 1, \ldots, \mu)$ draw a vertex and label it by its number i. Now for $i < j$ draw $|(\delta_i, \delta_j)|$ lines between the two vertices i and j. These lines are drawn dashed in the case that $(-1)^{\frac{1}{2}n(n-1)}(\alpha_i, \alpha_j) < 0$ (we think of a number of dashed lines as the corresponding negative number of lines). This sign-convention is chosen in such way that it is stable under stabilizations (as defined in the previous section), and that for the case $n \equiv 2 \bmod 4$ the number of lines is positive if and only if the corresponding intersection product is positive. If two vertices are connected by l lines (with sign) we also say that these vertices are connected by a line of weight l.

See also the Chapter 3 for further discussions on this topic.

The best known Coxeter-Dynkin diagrams are those of the simple singularities, namely the singularities of type A_μ, D_μ, and E_μ as shown if Table 1.1. Note that each possible numbering of the vertices is a valid diagram of these singularities, so we may omit the labels.

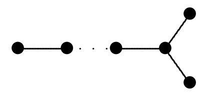
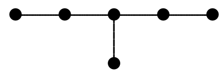
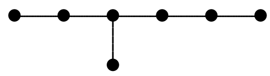
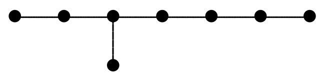
Type	f	Coxeter-Dynkin diagram
A_k	$x^{k+1} + y^2 \quad (k \geq 1)$	
D_k	$x^2y + y^{k-1} \quad (k \geq 4)$	
E_6	$x^3 + y^4$	
E_7	$x^3 + xy^3$	
E_8	$x^3 + y^5$	

Table 1.1: Simple singularities and their Coxeter-Dynkin diagrams A_k, D_k and E_k

1.6.5 The operation of the braid group

As seen in Section 1.6.1, the intersection matrix, the Seifert matrix, the monodromy matrix (but not the monodromy itself) and the Coxeter-Dynkin diagram is dependent on the choice of the distinguished basis of cycles which depends on the choice of the distinguished system of paths (and the morsification, but we have seen that this choice can be reduced to a choice of a distinguished system of paths also).

As mentioned, the braid group Br_μ on μ strands operates on the set of distinguished systems of paths, see the appendix, Section A.1.9. This operation induces an operation on distinguished bases of vanishing cycles and therefore also to an operation on all data defined by these.

Let $(\gamma_1, \ldots, \gamma_\mu)$ be a distinguished system of paths (for some morsification $\tilde{f}$ of f). The corresponding loops denote by $(\omega_1, \ldots, \omega_\mu)$. As seen in the appendix, a generator σ_i maps the system (γ_j) to a new system (γ'_j) where

$$\gamma'_j \text{ is homotopic to } \begin{cases} \gamma_j & \text{for } j \neq i, i+1 \\ \gamma_{i+1} & \text{for } j = i \\ \gamma_i \circ \omega_{i+1}^{-1} & \text{for } j = i+1. \end{cases}$$

Now one gets the corresponding new distinguished basis of vanishing cycles $(\delta'_1, \ldots, \delta'_\mu)$ by transporting back the vanishing cycles of the Milnor fibres near the critical points to the base point via the new distinguished system of paths. These paths are the old ones, besides γ'_{i+1} (and besides the numbering). Transporting a cycle back via

$$\gamma'_{i+1} \simeq \gamma_i \circ \omega_{i+1}^{-1}$$

means that one first transports back via γ_i to get the old cycle δ_i which one then has to transport back via ω_{i+1}^{-1}. But this means that one has to apply $\left(h_{\omega_{i+1}^{-1}}{}_*\right)^{-1}$

to this cycle. If we write

$$m_{\delta_k} := h_{\omega_k *}, \tag{1.16}$$

then the new cycle is

$$\delta'_{i+1} = m_{\delta_{i+1}} \delta_i.$$

We therefore get:

$$\sigma_i \cdot (\delta_1, \ldots, \delta_i, \delta_{i+1}, \ldots, \delta_\mu) = (\delta_1, \ldots, \delta_{i+1}, m_{\delta_{i+1}} \delta_i, \ldots, \delta_\mu). \tag{1.17}$$

Besides this, for each vanishing cycle we have two different possible orientations. Changing this orientation maps the corresponding vanishing cycle δ_i to $-\delta_i$. This can be expressed by an action of $(\mathbb{Z}/2\mathbb{Z})^\mu$ where a generator ξ_i of this group acts as follows:

$$\xi_i \cdot (\delta_1, \ldots, \delta_i, \ldots, \delta_\mu) = (\delta_1, \ldots, -\delta_i, \ldots, \delta_\mu). \tag{1.18}$$

Combining these two actions together, we get an action of the semidirect product $\mathrm{Br}_\mu \ltimes (\mathbb{Z}/2\mathbb{Z})^\mu$ on the set of distinguished bases of vanishing cycles. See also Section 3.1.3 in Chapter 3.

This action induces an action on intersection matrices for f and Coxeter-Dynkin diagrams for f. We postpone the description of these action to Chapter 3.

In the same way one can introduce weakly distinguished bases using the notion of weakly distinguished systems of paths (see the appendix, Section A.1.9). The Gabrielov group acts on them (see Section A.2). A detailed discussion on this we also postpone to Chapter 3.

1.7 Quasiunipotence of the Monodromy

Consider a commutative ring R. We simply call an element $a \in R$ *finite* if it has finite order, i.e. there exists a $k \in \mathbb{N}$ such that $a^k = 1$. Recall that a is called *unipotent* if $a - 1$ is nilpotent, i.e. there exists a $m \in \mathbb{N}$ such that $(a-1)^m = 0$.

Quasiunipotence is a simultaneous generalization of finiteness and unipotence:

Definition 1.7.1. $a \in R$ is called *quasiunipotent* if there exist $k, m \in \mathbb{N}$ such that

$$(a^k - 1)^m = 0.$$

Sometimes a is also called *quasifinite* instead of quasiunipotent.

The most important case for us is that $R = \mathrm{End}_{\Bbbk}(M)$ where M is a free module of finite rank over a commutative ring $\Bbbk$. Note that in the case of $\Bbbk = \mathbb{C}$ we have that $\varphi \in \mathrm{End}_{\Bbbk}(M)$ is quasiunipotent if and only if all eigenvalues of φ are roots of unity. Therefore, if $\Bbbk = \mathbb{Z}$, then φ is quasiunipotent if and only if the characteristic polynomial of φ is the product of cyclotomic polynomials.

Now the following important theorem is true:

Theorem 1.7.2 (Monodromy Theorem). *The monodromy of the Milnor fibration of an isolated singularity is quasiunipotent.*

This theorem has first been proven by BRIESKORN and DELIGNE, see [16] and [21].

The proof uses the fact that the canonical connection ∇ on the cohomology bundle

$$\mathcal{H} = R^n f_* \mathbb{C}_{\mathcal{X}^*} \otimes \mathcal{O}_{S^*}$$

(for $f : \overline{\mathcal{X}} \to S$ the Milnor fibration of a holomorphic map germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ which defines an isolated singularity, $S^* = S \setminus \{0\}$, and $\mathcal{X}^* = \mathcal{X}_{S^*}$) can be extended over $0 \in S$ to a meromorphic connection on $\mathcal{H}^n_{\mathrm{DR}}(\mathcal{X}/S)$ (which is defined by relative differential forms), the so-called *local Gauß-Manin connection.* The monodromy of this connection is exactly the monodromy of the (Milnor fibration of the) singularity.

It can be shown that this connection is regular, i.e. the corresponding system of differential equations has only regular singularities. From this result one can get an explicit description of the solutions of the connection. In particular it follows that all eigenvalues of the monodromy are of the form

$$\lambda_k = e^{2\pi i \alpha_k}$$

for some algebraic numbers α_k (this is the so-called algebraicity theorem for the Gauß-Manin connection). But the λ_k are also algebraic as they are the zeros of an integer polynomial, namely the characteristic polynomial of the monodromy in integer (co-)homology.

Now the seventh Hilbert problem (which has a positive answer) says: If $\lambda = e^{2\pi i \alpha}$ for two algebraic numbers α, λ, then $\alpha \in \mathbb{Q}$, i.e. λ is a root of unity.

This shows the monodromy theorem. (See also [35] for details.) □

There is another (shorter) proof of the monodromy theorem using resolution of singularities.

If $f : \overline{\mathcal{X}} \to S$ is a Milnor representative of a function $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ defining an isolated singularity, then there exists a resolution

$$\pi : \overline{\mathcal{Y}} \to \overline{\mathcal{X}}, \qquad g := f \circ \pi : \overline{\mathcal{Y}} \to S$$

such that π is proper, $\pi : \overline{\mathcal{Y}}_{S^*} \to \overline{\mathcal{X}}_{S^*}$ is biholomorphic, and $\overline{\mathcal{Y}}_0$ is a divisor with normal crossings. This means that at each point $y \in \overline{\mathcal{Y}}_0$ the germ $g : (\overline{\mathcal{Y}}, y) \to (S, g(y))$ is equivalent to the function

$$z_1^{k_1} z_2^{k_2} \cdots z_r^{m_r}.$$

These functions all have finite monodromy.

Denote by $E = \pi^{-1}(0) \subset \mathcal{Y}_0$ the exceptional divisor. Then the monodromies of g for all points $y \in E$ and the monodromy of f at 0 are connected via a spectral sequence. (This is best seen in derived categories when using the nearby cycle functor and vanishing cycle functor — we will not go into details here). Since by the following lemma quasiunipotence is stable under exact sequences (and hence under spectral sequences), the theorem follows.

Lemma 1.7.3. *Let V, V' and V'' be vector spaces over a field $\Bbbk$, such there is a diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \xrightarrow{\iota} & V & \xrightarrow{\pi} & V'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi''} & & \\ 0 & \longrightarrow & V' & \xrightarrow{\iota} & V & \xrightarrow{\pi} & V'' & \longrightarrow & 0 \end{array}$$

where the rows are exact sequences. Then φ is quasiunipotent if and only if φ' and φ'' are quasiunipotent.

Proof. The exact sequence splits, hence we can assume that $V = V' \oplus V''$. Then using matrix notation this means that φ is of the form

$$\varphi = \begin{pmatrix} \varphi' & * \\ 0 & \varphi'' \end{pmatrix}.$$

Now the lemma easily follows. □

The monodromy theorem is valid in a much larger context (e.g. for non-isolated singularities or in a p-adic algebraic context) with partially complete different proofs.

1.8 Classification of Isolated Singularities

In order to classify singularities we first need to know how to compare them.

We call a holomorphic function-germ $f : (\mathbb{C}^{n+1}, x) \to (\mathbb{C}, 0)$ simply a *singularity*, or to be more precise a *hypersurface singularity*. In the same way we simply call f an *isolated singularity* if f defines an isolated singularity.

Definition 1.8.1. Let $f : (\mathbb{C}^{n+1}, x) \to (\mathbb{C}, s)$ and $g : (\mathbb{C}^{m+1}, x') \to (\mathbb{C}, s')$ two singularities.

We call f and g *equivalent* (or to be more precise *right-equivalent*) if $n = m$ and there exists a biholomorphic map germ $\varphi : (\mathbb{C}^{n+1}, y) \to (\mathbb{C}^{n+1}, x)$ such that $g - s' = (f - s) \circ \varphi$.

f and g are called *stably equivalent* if there exist stabilizations $\tilde{f}$ and $\tilde{g}$ of $f - s$ resp. $g - s'$ that are equivalent.

When unfolding singularity it decomposes in some smaller singularities that are “less complicated” than the original one. This will be made precise in the following definitions:

Definition 1.8.2. Let $f, g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ two singularities. g is called adjacent to f if for the versal truncated unfolding $F' : \overline{\mathcal{Y}}' \to S$ of f there exist arbitrary small $t' \in \Lambda'$ and and $x \in \overline{\mathcal{Y}}'_{t'}$ such that the germ

$$F'_{t'} : (\overline{\mathcal{Y}}'_{t'}, x) \to (S, F'_{t'}(x))$$

is equivalent to g.

One also says in this case that f *deforms into* g.

Definition 1.8.3. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an isolated singularity. The modality (or the modulus) of f is the smallest number m such that for a representative $F' : \overline{\mathcal{Y}}' \to S$ of the versal truncated unfolding all singularities

$$F'_{t'} : (\overline{\mathcal{Y}}'_{t'}, x) \to (S, f(x))$$

for a $t' \in \Lambda'$ and a $x \in \overline{\mathcal{Y}}'_{t'}$ belong to finitely many families of equivalence classes which depend on at most m parameters.

A singularity with modulus 0 is called *simple*, one with modulus 1 is called *unimodal* or *unimodular*, and one with modulus 2 is called *bimodal* or *bimodular*.

Therefore, the "least complicated" singularities are the simple singularities. These have been classified by Arnol'd [5] and are the famous ADE-singularities. The complete list is given above in Table 1.1.

Also the classification of unimodal and bimodal singularities is due to Arnol'd, see [6], [7] and [8].

There are three kinds of unimodal singularities: three families of parabolic singularities, a three-parameter infinite series of families of hyperbolic singularities and 14 families of exceptional hyperbolic singularities. The corresponding Coxeter-Dynkin diagrams were obtained by Ebeling, see [27] and [28]. These are shown in Tables 1.2, 1.3 and 1.4 together with Figure 1.2.

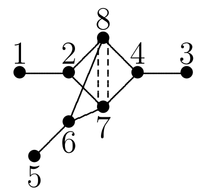
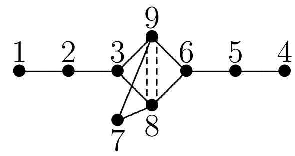
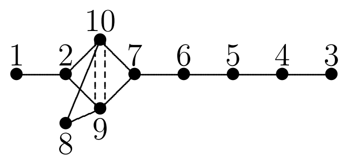
Type	f		Coxeter-Dynkin diagram
P_8	$x^3 + y^3 + z^3 + axyz$	$a^3 + 27 \neq 0$	
X_9	$x^4 + y^4 + ax^2y^2$	$a^2 - 4 \neq 0$	
J_{10}	$x^3 + y^6 + ax^2y^2$	$4a^3 + 27 \neq 0$	

Table 1.2: Parabolic singularities

Type	f
T_{pqr}	$x^p + y^q + z^r + axyz \quad a \neq 0,\ \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,\ p \leq q \leq r$

Table 1.3: Hyperbolic singularities

At last, there are eight infinite series and 14 families of bimodal singularities, but we forbear from listing them here, since we do not need them.

Type	f	C.-D. diag.	Type	f	C.-D. diag.
E_{12}	$x^3 + y^7 + axy^5$	$\tilde{T}_{2,3,7}$	W_{12}	$x^4 + y^5 + ax^2y^3$	$\tilde{T}_{2,5,5}$
E_{13}	$x^3 + xy^5 + ay^8$	$\tilde{T}_{2,3,8}$	W_{13}	$x^4 + xy^4 + ay^6$	$\tilde{T}_{2,5,6}$
E_{14}	$x^3 + y^8 + axy^6$	$\tilde{T}_{2,3,9}$	Q_{10}	$x^3 + y^4 + yz^2 + axy^3$	$\tilde{T}_{3,3,4}$
Z_{11}	$x^3y + y^5 + axy^4$	$\tilde{T}_{2,4,5}$	Q_{11}	$x^3 + y^2z + xz^3 + az^5$	$\tilde{T}_{3,3,5}$
Z_{12}	$x^3y + xy^4 + ax^2y^3$	$\tilde{T}_{2,4,6}$	Q_{12}	$x^3 + y^5 + yz^2 + axy^4$	$\tilde{T}_{3,3,6}$
Z_{13}	$x^3y + y^6 + axy^5$	$\tilde{T}_{2,4,7}$	S_{11}	$x^4 + y^2z + xz^2 + ax^3z$	$\tilde{T}_{3,4,4}$
U_{12}	$x^3 + y^3 + z^4 + axyz^2$	$\tilde{T}_{4,4,4}$	S_{12}	$x^2y + y^2z + xz^3 + az^5$	$\tilde{T}_{3,4,5}$

Table 1.4: Exceptional hyperbolic singularities

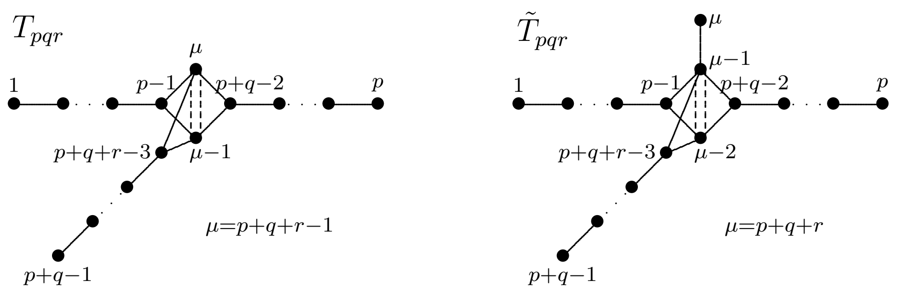


Figure 1.2: Coxeter-Dynkin diagrams T_{pqr} and $\tilde{T}_{pqr}$

Chapter 2

The Main Theorem in the Singularity Case

事実は小説より奇なり

Fact is stranger than fiction

Introduction

In this chapter we formulate and prove the main theorem. The situation we consider in the theorem is that we are given a morsification $\tilde{f} : \overline{\mathcal{X}}' \to S$ of an isolated singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ (with n even). This is the most simple example for a family of singularities as defined in the previous chapter.

As it follows from the monodromy theorem, the monodromy corresponding to a simple loop around all critical points of $\tilde{f}$ (which is exactly the monodromy of the Milnor fibration of f) is quasiunipotent. This holds no longer true for monodromies corresponding to simple loops around only a part of these singularities.

However, there are situations in which all these monodromies are still quasiunipotent. The theorem says that to check this condition it already suffices to check the quasiunipotence of monodromies corresponding to simple loops around only two critical points. Moreover, the latter condition is equivalent to some condition on the intersection products — or equivalently on the weights of lines in Coxeter-Dynkin diagrams of f. Furthermore, it follows from the theorem that the above condition is true exactly for the simple and parabolic singularities.

One also can ask for finiteness of the monodromy instead of quasiunipotence. The same theorem remains true with similar conditions in this case. It follows that all monodromies corresponding to arbitrary simple loops are finite exactly for the simple singularities.

2.1 The Theorem

In the theorem we use the notations of the previous chapter. For simple loops we use the definition of Section A.4 in the appendix.

Theorem 2.1.1. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an isolated singularity with n even, and let $\tilde{f} = \tilde{F}_\tau : \overline{\mathfrak{X}}' \to S = \overline{\Delta}$ be a morsification of f and $D_{\tilde{f}} = \{z_1, \ldots, z_\mu\}$ its discriminant. Select a base point $s \in \partial\Delta$.*

Then the following statements are equivalent:

(i) *The intersection matrix S_{IS} (with respect to an arbitrary distinguished basis of vanishing cycles) is semidefinite.*

(ii) *For each simple loop $\gamma : [0, 1] \to S \setminus D_{\tilde{f}}$ with base point s the corresponding monodromy h_{γ_*} on $H_n(\overline{\mathfrak{X}}'_s; \mathbb{Z})$ is quasiunipotent.*

(iii) *Statement (ii) is true for all γ which go around exactly two critical values.*

(iv) *Each Coxeter-Dynkin diagram of f contains only lines with a weight of absolute value ≤ 2.*

The same equivalence is true with "definite" instead of "semidefinite", "finite" instead of "quasiunipotent", and "≤ 1" instead of "≤ 2".

For the proof we need the following theorem and proposition.

Theorem 2.1.2. *Each isolated singularity with a modality greater than one deforms into an exceptional hyperbolic singularity.*

Proof. This statement is found in Brieskorn [18]. It is obtained in the course of the classification of simple, unimodular and bimodular singularities and their adjacency, see e.g. also [5], [6], [8], [17]. □

Proposition 2.1.3. *Let f and g be isolated singularities such that f deforms into g. Then for each Coxeter-Dynkin diagram of g there exists a Coxeter-Dynkin diagram of f that contains the first one as a subdiagram.*

Proof. If g is nondegenerate, this is trivial. So, let g be degenerate.

Let $F' : \overline{\mathfrak{Y}}' \to S$ be the versal truncated unfolding of f. By assumption there exist $t' \in \Xi$ and $x \in \overline{\mathfrak{Y}}'_{t'}$ such that g is equivalent to the germ

$$g' := F'_{t'} : (\overline{\mathfrak{Y}}'_{t'}, x) \to (S, f(x)).$$

Now F' is also an unfolding of g'. Select a good representative

$$F' : \overline{\mathfrak{X}} \to \tilde{S}$$

of the germ of F' at x in the sense of Section 1.5.3 with $x \in \overline{\mathfrak{X}} \subset \overline{\mathfrak{Y}}'$ and $f(x) \in \tilde{S} \subset S$, and with base space $\tilde{\Lambda}$ such that $t' \in \tilde{\Lambda} \subset \Lambda'$.

Since Ξ is thin in S, there exists a $\tilde{t}' \in \tilde{\Lambda} \setminus \Xi$. Then $F'_{\tilde{t}'}$ is a morsification and induces a morsification

$$\tilde{g}' = F'_{\tilde{t}'} : \overline{\mathfrak{X}}_{\tilde{t}'} \to \tilde{S}$$

of g'.

Now, given a Coxeter-Dynkin diagram of g, this is also one of g' and corresponds to the choice of a base point $\tilde{s} \in \partial\tilde{S}$ and a distinguished system of paths in $\tilde{S}$ which start at $\tilde{s}$ and end at one of those critical values of $F'_{\tilde{t}'}$ which lie in $\tilde{S}$.

Now choose a base point $s \in \partial S$ and an injective path from s to $\tilde{s}$ (which does not meet $\tilde{S}$ and the critical values). If one joins this path with the paths of the distinguished system of paths of g' chosen above, this system can be completed to a distinguished system of paths for f. The corresponding Coxeter-Dynkin diagram then contains that of g by construction. □

Brieskorn conjectured in [18] that also the converse is true, i.e. that adjacency of singularities can be detected by Coxeter-Dynkin diagrams. While this is true for simple singularities (see [39]), it is not true in general as known today. Note that for a family of singularities of a specific type it may even depend on the values of the moduli if the singularity deforms into a singularity of another given type (see e.g. [34] for the case of parabolic singularities).

2.1.1 Proof of the theorem

Since some parts are true in the general case, we will postpone their proof to the next chapter. Note that we assume $n \equiv 2 \bmod 4$ in the next chapter, but the case $n \equiv 0 \bmod 4$ only differs by a sign.

First note that (as already mentioned in Section 1.3.3) by Proposition A.4.2 for each simple loop γ as in the theorem there exists a distinguished system of paths $(\gamma_1, \dots, \gamma_\mu)$ such that

$$[\gamma] = [\omega_{i_r}] \cdots [\omega_{i_2}] \cdot [\omega_{i_1}]$$

for some $1 \leq i_1 < i_2 < \cdots < i_r \leq \mu$ where ω_i is the loop corresponding to γ_i. But this means the following: The monodromy

$$h_{\gamma*} = m_{\delta_{i_r}} \cdots m_{\delta_{i_2}} m_{\delta_{i_1}}$$

(with the notion of m_{δ_i} of equation (1.16)) is exactly the monodromy of the subdiagram of the Coxeter-Dynkin diagram (corresponding to the distinguished system of paths) consisting of the vertices $i_1, i_2, \dots, i_r$, see Section 3.1.5 in the next chapter.

Vice versa, each such "submonodromy" of a subdiagram of a Coxeter-Dynkin diagram of f corresponds to a choice of a distinguished system of paths and a simple loop around the corresponding singularities defined by the distinguished system of paths.

It follows from the Picard-Lefschetz formulas that the monodromy of a simple loop around exactly two singularities is quasiunipotent (resp. finite) if and only if the intersection product $(\delta_{i_1}, \delta_{i_2})$ of the corresponding vanishing cycles has absolute value ≤ 2 (resp. ≤ 1). This easy calculation can be found in Section 3.2.2 in the next chapter. Since this intersection product is exactly the weight of the line between the vertices i_1 and i_2 in the Coxeter-Dynkin diagram corresponding to the distinguished basis (which again corresponds to the distinguished system of paths), we get the equivalence of (iii) and (iv).

With the same reformulation "simple loops" ↔ "subdiagrams of Coxeter-Dynkin diagrams" the equivalence of (i) and (ii) is proven in the next chapter, Sections 3.2.5 and 3.2.6 (as part of Theorems 3.2.1 and 3.2.2).

Since (ii) ⇒ (iii) is trivial, it remains to prove (iv) ⇒ (i).

(iv) ⇒ (i), case "definite" and "≤ 1": It is an easy calculation that for all simple singularities the intersection matrix S_{IS} is always definite (namely positive definite for $n \equiv 0 \bmod 4$ and negative definite for $n \equiv 2 \bmod 4$).

So assume that S_{IS} is not definite. We have to show that there is a Coxeter-Dynkin diagram of f that contains a line with a weight of absolute value ≥ 2.

Since S_{IS} is not definite, the singularity f cannot be simple. If it is unimodal, then it has a Coxeter-Dynkin diagram as given in Section 1.8. Each of those diagrams contains a line of weight -2. If the singularity has a modality greater than one, then by Theorem 2.1.2 it deforms into an exceptional hyperbolic singularity. Hence by Proposition 2.1.3 it has a Coxeter-Dynkin diagram which contains a Coxeter-Dynkin diagram of an exceptional hyperbolic singularity which again contains a line of weight -2.

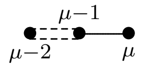
(iv) ⇒ (i), case "semidefinite" and "≤ 2": The proof is quite the same as in the previous case, however it needs some more work.

First note that for all three parabolic (families of) singularities the intersection matrix is semidefinite — in fact it is parabolic, i.e. the inertia (n_+, n_-, n_0) (where n_+, n_- and n_0 are the numbers of eigenvalues that are positive, negative, resp. zero) is $(\mu - 1, 0, 1)$ for $n \equiv 0 \bmod 4$ resp. $(0, \mu - 1, 1)$ for $n \equiv 2 \bmod 4$.

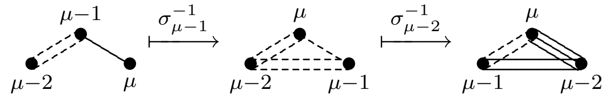
Now assume that S_{IS} is not semidefinite. Then the singularity f cannot be simple or of parabolic type. We have to show that there exists a Coxeter-Dynkin diagram of f which contains a line with a weight of absolute value greater or equal than 3.

As in the previous case, if f has a modality greater than one, then it deforms into an exceptional hyperbolic singularity. Hence we only have to look at hyperbolic and exceptional hyperbolic singularities.

The easier case is that of exceptional hyperbolic singularities. The Coxeter-Dynkin diagrams of these given in Section 1.8 all contain the following subdiagram:



By an operation of the braid group this subdiagram can be modified as follows:



Of course this operation does affect the other lines not contained in this subdiagram, but this does not matter, since we now have found a diagram which contains a line of weight 3.

The case of hyperbolic singularities is much more difficult. We have:

Lemma 2.1.4. *Let* $p, q, r \in \mathbb{N}$ *with*

$$\tfrac{1}{p} + \tfrac{1}{q} + \tfrac{1}{r} < 1$$

and $p \leq q \leq r$. *Then one of the following cases is true:*

- $p \geq 2$, $q \geq 3$ *and* $r \geq 7$, *or*
- $p \geq 2$, $q \geq 4$ *and* $r \geq 5$, *or*
- $p \geq 3$, $q \geq 3$ *and* $r \geq 4$.

Proof. This is easy to prove. □

From this lemma it follows that each diagram T_{pqr} of a hyperbolic singularity contains one of the diagrams $T_{2,3,7}$, $T_{2,4,5}$, or $T_{3,3,4}$ as a subdiagram. So we can restrict our attention to these three diagrams.

Now $T_{2,3,7}$, $T_{2,4,5}$, and $T_{3,3,4}$ contain again the diagram $E_{2,3,7}$, $E_{2,4,5}$, resp. $E_{3,3,4}$ where the diagram E_{pqr} is shown in Figure 2.1. The size of this diagram is $\tilde{\mu} = \mu - 1$.

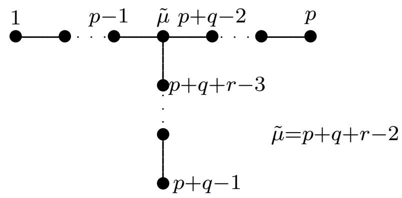
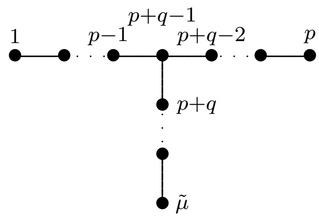


Figure 2.1: The diagram E_{pqr}

By applying the element $\sigma^2_{\tilde{\mu}-r-1} \cdots \sigma^2_{\tilde{\mu}-1}$ to E_{pqr} we get the same diagram with the numbering of the vertical part conversed:



Now look at the subdiagram of this diagram which one gets by deleting the vertex $\tilde{\mu}$: It is a diagram of type $\tilde{E}_8$, $\tilde{E}_7$, resp. $\tilde{E}_6$, see Figure 2.2.

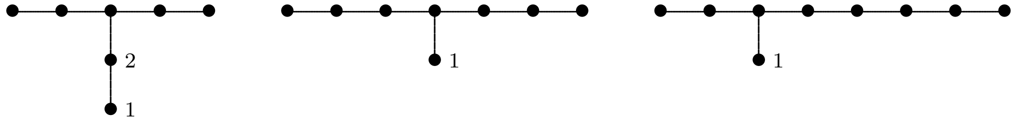
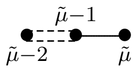


Figure 2.2: The diagrams $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$

The idea is now the following: Set $\tilde{\mu}' = \tilde{\mu} - 1 = \mu - 2$. By an operation of a braid word not containing $\sigma_{\tilde{\mu}'-1}$ or its inverse on the diagrams $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$ one can get a diagram such that the line between the vertices $\tilde{\mu}' - 1$ and $\tilde{\mu}'$ has weight -2.

If one considers the same operation on the relabelled diagrams $E_{3,3,4}$, $E_{2,4,5}$, resp. $E_{2,3,7}$, one then gets a diagram which contains the following subdiagram:



As in the previous case, one then can operate on this subdiagram to get a line of weight 3.

Hence it remains to show the following lemma:

Lemma 2.1.5. *If D is a Dynkin diagram of type $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$, then by an operation of a braid word not containing $\sigma_{\tilde{\mu}'-1}$ or its inverse (for $\tilde{\mu}' = 7$, 8, resp. 9) one can produce a diagram which has a line of weight 2 between the vertices $\tilde{\mu}' - 1$ and $\tilde{\mu}'$.*

The idea to prove this lemma is the following: The Dynkin diagrams $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ occur as so-called "affine Dynkin diagrams" in the classification of root systems. The vertex $\tilde{\mu}'$ corresponds to the negative of the highest root in the root system corresponding to the Dynkin diagrams E_6, E_7, resp. E_8. This highest root is given as a linear combination of the roots in the Dynkin diagrams E_6, E_7, resp. E_8 as denoted in Figure 2.3.

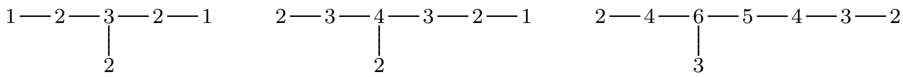
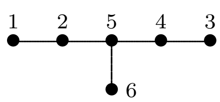


Figure 2.3: Finding the highest root for E_6, E_7, and E_8

Then the idea is to get the corresponding linear combination of the vanishing cycles corresponding to the vertices of the diagram by an operation of the braid group. If one applies the same operation to the diagram $\tilde{E}_6$, $\tilde{E}_7$, resp. $\tilde{E}_8$, then this new cycle has intersection product 2 with the cycle corresponding to the vertex $\tilde{\mu}'$ (which is not affected by the operation).

That this is possible can be easily checked — however, there are several operations of the basis elements $\sigma_k \in \mathrm{Br}_\mu$ needed to achieve the result. To facilitate such calculations, the author has written a C++ computer program (running under Windows™) that grants a user interface to apply the operations of the basis elements $\sigma_k \in \mathrm{Br}_\mu$ to arbitrary Coxeter-Dynkin diagrams — with help of this program one can easily check this assertion.

Let us illustrate the case of E_6: We start with the Coxeter-Dynkin diagram



with corresponding distinguished basis of vanishing cycles $(\delta_1, \ldots, \delta_6)$.

Now apply the element

$$\sigma_5^{-1}\sigma_4\sigma_3\sigma_2\sigma_1^{-1}\sigma_5\sigma_4^{-1}\sigma_5\sigma_3\sigma_2\sigma_1\sigma_3\sigma_4\sigma_3.$$

On the distinguished basis of vanishing cycles this element operates as follows:

$$\begin{aligned}(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6) &\xmapsto{\sigma_3} (\delta_1, \delta_2, \delta_4, \delta_3 + \delta_4, \delta_5, \delta_6)\\ &\xmapsto{\sigma_4} (\delta_1, \delta_2, \delta_4, \delta_5, \delta_3 + \delta_4 + \delta_5, \delta_6)\\ &\xmapsto{\sigma_3} (\delta_1, \delta_2, \delta_5, \delta_4 + \delta_5, \delta_3 + \delta_4 + \delta_5, \delta_6)\\ &\vdots\\ &\xmapsto{\sigma_5^{-1}} (\dots, \delta_1 + 2\delta_2 + \delta_3 + 2\delta_4 + 3\delta_5 + 2\delta_6).\end{aligned}$$

In the last step we get a new distinguished basis $(\delta_1', \dots, \delta_6')$ where δ_6' is indeed the cycle

$$1\delta_1 + 2\delta_2 + 3\delta_5 + 2\delta_4 + 1\delta_3 + 2\delta_6.$$

The corresponding procedure for E_7 and E_8 is even more complicated, but is still quite similar.

Hence the theorem is proven. □

Chapter 3

The Algebraic Formulation

塵も積もれば山なる

Piled-up specks of dust become a mountain

Introduction

The goal of this chapter is to figure out to what extent the main theorem given in the previous chapter is true in the general setting for families of nondegenerate singularities.

Since the proof of the main theorem in the previous section was dependent of the deformation argument given in Theorem 2.1.2 which we cannot use in the general context, we need other methods to examine the general situation. Furthermore, it is an interesting question if the statement of the theorem really depends on some geometric properties of the given situation, or if it is in fact already true for only combinatorial or algebraic reasons.

We therefore give abstract definitions of all ingredients for a reformulation of the theorem and discuss their properties. The fact that the braid group acts simply transitive on distinguished systems of paths (resp. that the Gabrielov group acts simply transitive on weakly distinguished systems of paths) will be used to give abstract definitions for (weakly) vanishing cycles and the intersection product which correspond to the old definitions. We can then use the Picard-Lefschetz formulas as well as equations 1.14 and 1.15 of Chapter 1 to define monodromy maps. This is done in the first section.

In the second section we reformulate the theorem in the new context. However, some part of it remains unproven and will be conjectured. (In fact the theorem is divided into two parts which leads to the formulation of two conjectures.) The case of small μ is discussed (where we also prove the conjectures for this case). After this we prove both theorems.

We then show that the first conjecture is a consequence of the second conjecture, i.e. both conjectures are in fact only one conjecture.

After that we discuss some weaker analogues of both conjectures which we are able to prove. This establishes a substitute for the main theorem of the previous chapter for the general situation (however in a weaker version).

We close the chapter with a discussion on approaches for proving the conjectures.

3.1 Vanishing Cycles

For the following fix an $n \in \mathbb{Z}$ (which corresponds to the dimension of the Milnor fibre in the singularity case — only its class in $\mathbb{Z}/4\mathbb{Z}$ is important) and define the following signs:

$$\varepsilon = (-1)^n \quad \text{and} \quad \eta = (-1)^{\frac{1}{2}n(n-1)}.$$

In this chapter we always consider a lattice, i.e. a free $\mathbb{Z}$-module M of rank μ which is equipped with a bilinear form

$$(\cdot,\cdot) : M \times M \to \mathbb{Z}.$$

We require that $(\cdot,\cdot)$ is symmetric if $\varepsilon = 1$ and antisymmetric if $\varepsilon = -1$ (i.e. $(\alpha,\beta) = \varepsilon(\beta,\alpha)$), in other words that the lattice is symmetric resp. antisymmetric. In the following we refer to $\varepsilon = 1$ as the "symmetric case" and to $\varepsilon = -1$ as the "antisymmetric case".

We assume that M has a basis $\underline{\delta} := (\delta_1, \ldots, \delta_\mu)$, such that

$$(\delta_i, \delta_i) = \eta(1+\varepsilon) = (-1)^{\frac{1}{2}n(n-1)} + (-1)^{\frac{1}{2}n(n+1)} \quad \text{for} \quad i = 1, \ldots, \mu. \tag{3.1}$$

(We have seen in the Picard-Lefschetz theorems that the vanishing cycles in the singularity case satisfy this equation. Note that $1+\varepsilon$ is the Euler characteristic of S^n. In the antisymmetric case this assumption is tautological.)

Fix a such a basis $\underline{\delta}$.

By abuse of language we call a vector $\alpha \in M$ a *cycle*, and we think of $(\cdot,\cdot)$ as an *intersection product*. Thus for $\alpha, \beta \in M$ we call (α,β) the *intersection product* of α and β and (α,α) the *self-intersection (product)* of α.

Such data $\big(M, (\cdot,\cdot), \underline{\delta}\big)$ (to be more precise $\big(M, (\cdot,\cdot), \underline{\delta}, \varepsilon, \eta\big)$) we call an *intersection datum*. The rank of M we denote by

$$\mu\big(M, (\cdot,\cdot), \underline{\delta}\big) := \operatorname{rank} M$$

and call it the *Milnor number* (or sometimes simply the *size*) of the intersection datum. A *morphism* of intersection data $\big(M, (\cdot,\cdot), \underline{\delta}\big)$ and $\big(M', (\cdot,\cdot), \underline{\delta}'\big)$ is a morphism of $\mathbb{Z}$-modules $\varphi : M \to M'$ with $(\varphi(\alpha), \varphi(\beta)) = (\alpha,\beta)$ and $\varphi(\delta_i) = \delta'_i$. Each morphism is automatically an isomorphism. This defines for each $\mu \geq 1$ a category $\mathcal{I}nt\mathcal{D}at_\mu(\varepsilon,\eta)$ of intersection data with Milnor number μ which is skeletal (i.e. each morphism is an isomorphism). The (disjoint) union of all $\mathcal{I}nt\mathcal{D}at_\mu(\varepsilon,\eta)$ we denote by $\mathcal{I}nt\mathcal{D}at(\varepsilon,\eta)$. If ε and η are clear from the context we simply write $\mathcal{I}nt\mathcal{D}at$ instead of $\mathcal{I}nt\mathcal{D}at(\varepsilon,\eta)$ (resp. $\mathcal{I}nt\mathcal{D}at_\mu$ instead of $\mathcal{I}nt\mathcal{D}at_\mu(\varepsilon,\eta)$).

3.1.1 The intersection matrix, the Seifert matrix, and the monodromy

For each basis $\underline{\alpha} := (\alpha_1, \ldots, \alpha_\mu)$ of M consisting of cycles with $(\alpha_i, \alpha_i) = \eta(1+\varepsilon)$ $(i = 1, \ldots, \mu)$ (especially for $(\delta_1, \ldots, \delta_\mu)$) we can define the matrix

$$S_{\underline{\alpha}} = \big((\alpha_i, \alpha_j)\big)_{\substack{1\le i\le\mu \\ 1\le j\le\mu}}$$

which we call the *intersection matrix* of M with respect to the basis $\underline{\alpha}$. To $S_{\underline{\alpha}}$ we assign in the standard way a Coxeter-Dynkin diagram $D_{\underline{\alpha}}$: For each cycle α_i $(i = 1, \ldots, \mu)$ we draw a vertex which we label with its number i, and for $i < j$ we draw $|(\alpha_i, \alpha_j)|$ lines between the two vertices α_i, α_j which we draw dashed in case $\eta(\alpha_i, \alpha_j) < 0$.

In the case that $\underline{\alpha} = \underline{\delta}$ we simply write $S := S_{\underline{\delta}}$ and $D := D_{\underline{\delta}}$ and call this *the* intersection matrix resp. *the* Coxeter-Dynkin diagram of M.

Given S, we can write the intersection product as

$$(\alpha, \beta) = \langle \vec{\alpha}, S\vec{\beta} \rangle,$$

with $\langle \cdot, \cdot \rangle : \mathbb{Z}^\mu \times \mathbb{Z}^\mu \to \mathbb{Z}$ the standard scalar product and $\vec{\alpha}, \vec{\beta} \in \mathbb{Z}^\mu$ the vectors of α, β with respect to the basis $\underline{\delta}$.

For each $\mu \in \mathbb{N}$ define the set

$$\mathcal{I}nt\mathcal{M}at_\mu(\varepsilon, \eta) = \{S \in \mathrm{Mat}(\mu \times \mu, \mathbb{Z}) \mid S_{ii} = \eta(1+\varepsilon) \text{ and } S \text{ is } \varepsilon\text{-symmetric}\}$$

of all possible intersection matrices of size μ where ε-symmetric means symmetric for $\varepsilon = 1$ and antisymmetric for $\varepsilon = -1$. Also define the set

$$\mathcal{D}ynkin_\mu = \{D \mid D \text{ is a Coxeter-Dynkin diagram of size } \mu\}$$

of all Coxeter-Dynkin diagrams (i.e. a graph of μ vertices, labelled by $1, \ldots, \mu$, and two vertices are connected by an integer number of lines where a negative number l of lines is drawn as $-l$ dotted lines) of size μ. Set

$$\mathcal{I}nt\mathcal{M}at(\varepsilon, \eta) := \bigcup_{\mu\ge1} \mathcal{I}nt\mathcal{M}at_\mu(\varepsilon, \eta) \quad \text{and} \quad \mathcal{D}ynkin := \bigcup_{\mu\ge1} \mathcal{D}ynkin_\mu .$$

Again we simply write $\mathcal{I}nt\mathcal{M}at$ resp. $\mathcal{I}nt\mathcal{M}at_\mu$ if ε and η are clear by the context.

Then we have mappings

$$\mathcal{I}nt\mathcal{D}at_\mu(\varepsilon, \eta) \longrightarrow \mathcal{I}nt\mathcal{M}at_\mu(\varepsilon, \eta) \longrightarrow \mathcal{D}ynkin_\mu \tag{3.2}$$

which are equivalences of categories (the last map is in fact an isomorphism) if we consider $\mathcal{I}nt\mathcal{M}at_\mu(\varepsilon, \eta)$ and $\mathcal{D}ynkin_\mu$ as categories with only trivial arrows. We also get equivalences of categories

$$\mathcal{I}nt\mathcal{D}at(\varepsilon, \eta) \longrightarrow \mathcal{I}nt\mathcal{M}at(\varepsilon, \eta) \longrightarrow \mathcal{D}ynkin \tag{3.3}$$

(again the last map is an isomorphism).

Because of this an intersection datum $\big(M, (\cdot,\cdot), \underline{\delta}\big)$ (for which we simply write M), its intersection matrix S and its Coxeter-Dynkin diagram all represent the same data, so we may switch between them without further explanation.

Since S is a symmetric resp. antisymmetric matrix with all diagonal entries equal to $\eta(1+\varepsilon)$, we can write

$$S = \eta(V + \varepsilon V^t) \tag{3.4}$$

with an upper triangular matrix V which has 1's on the diagonal. We call V the *Seifert matrix* of M. Now we can define the *monodromy matrix* m of M by

$$m = -\varepsilon (V^t)^{-1} V \tag{3.5}$$

This matrix defines an automorphism on M which we denote by the same symbol.

Let $\alpha \in M$ be a cycle with $(\alpha,\alpha) = \eta(1+\varepsilon)$. We can define a mapping

$$\begin{aligned} m_\alpha : M &\to M \\ \beta &\mapsto \beta - \eta(\beta,\alpha)\alpha. \end{aligned} \tag{3.6}$$

We call m_α (again by abuse of language) the *monodromy around* α. In case $\alpha = \delta_i$ $(i = 1, \dots, \mu)$ we simply write

$$m_i := m_{\delta_i}$$

Lemma 3.1.1. *One has $m_\alpha \alpha = -\varepsilon\alpha$. m_α is isomorphic with inverse*

$$m_\alpha^{-1}\beta = \beta - \eta\varepsilon(\beta,\alpha)\alpha = \beta - \eta(\alpha,\beta)\alpha,$$

and preserves the intersection product. In particular, in the symmetric case one has $m_\alpha^2 = \mathbb{1}$.

Proof. We have

$$m_\alpha\alpha = \alpha - \eta(\alpha,\alpha)\alpha = \alpha - (1+\varepsilon)\alpha = -\varepsilon\alpha$$

and

$$\begin{aligned} (m_\alpha\beta, m_\alpha\gamma) &= \big(\beta - \eta(\beta,\alpha)\alpha, \gamma - \eta(\gamma,\alpha)\alpha\big) \\ &= (\beta,\gamma) + \big(-\eta - \eta\varepsilon + (\alpha,\alpha)\big)(\alpha,\beta)(\alpha,\gamma) = (\beta,\gamma), \end{aligned}$$

Setting

$$\tilde{m}_\alpha\beta = \beta - \eta\varepsilon(\beta,\alpha)\alpha$$

we get

$$m_\alpha\tilde{m}_\alpha\beta = m_\alpha\big(\beta - \eta\varepsilon(\beta,\alpha)\alpha\big) = \beta - \eta(\beta,\alpha)\alpha - \eta\varepsilon(\beta,\alpha)(-\varepsilon\alpha) = \beta$$

and similarly for $\tilde{m}_\alpha m_\alpha$. □

Lemma 3.1.2. *One has $m_{-\alpha} = m_\alpha$ and $m_{m_\alpha\beta} = m_\alpha m_\beta m_\alpha^{-1}$.*

Proof. We have

$$m_{-\alpha}\beta = \beta - \eta(\beta, -\alpha)(-\alpha) = \beta - \eta(\beta, \alpha)\alpha = m_\alpha\beta.$$

Furthermore

$$\begin{aligned} m_{m_\alpha\beta}\gamma &= \gamma - \eta\big(\gamma, \beta - \eta(\beta,\alpha)\alpha\big)\big(\beta - \eta(\beta,\alpha)\alpha\big) \\ &= \gamma - \eta(\gamma,\beta)\beta + (\gamma,\beta)(\beta,\alpha)\alpha + (\beta,\alpha)(\gamma,\alpha)\beta - \eta(\beta,\alpha)^2(\gamma,\alpha)\alpha, \end{aligned}$$

and

$$\begin{aligned} &m_\alpha m_\beta m_\alpha^{-1}\gamma \\ &\quad= m_\alpha m_\beta\big(\gamma - \eta\varepsilon(\gamma,\alpha)\alpha\big) \\ &\quad= m_\alpha\big(\gamma - \eta(\gamma,\beta)\beta - \eta\varepsilon(\gamma,\alpha)(\alpha - \eta(\alpha,\beta)\beta)\big) \\ &\quad= m_\alpha\big(\gamma - \eta\varepsilon(\gamma,\alpha)\alpha + \big(-\eta(\gamma,\beta) + \varepsilon(\gamma,\alpha)(\alpha,\beta)\big)\beta\big) \\ &\quad= \gamma - \eta\varepsilon(\gamma,\alpha)\alpha - \eta(\gamma,\alpha)(-\alpha) + (-\eta(\gamma,\beta) + \varepsilon(\gamma,\alpha)(\alpha,\beta))(\beta - \eta(\beta,\alpha)\alpha) \\ &\quad= \gamma - \eta(\gamma,\beta)\beta + \varepsilon(\gamma,\alpha)(\alpha,\beta)\beta + (\gamma,\beta)(\beta,\alpha)\alpha - \eta(\gamma,\alpha)(\alpha,\beta)^2\alpha, \end{aligned}$$

hence we get $m_{m_\alpha\beta} = m_\alpha m_\beta m_\alpha^{-1}$ as desired. □

By this lemma we get the following: Let

$$A_{\mathrm{sis}} := \{\alpha \in M \mid (\alpha,\alpha) = \eta(1+\varepsilon)\}$$

be the set of cycles with the correct self intersection product. Then this set becomes an automorphic set (as defined in the appendix, Section A.3) when equipped with the product

$$\alpha \triangleright \beta := m_\alpha\beta,$$

since m_α is isomorphic and

$$\alpha \triangleright (\beta \triangleright \gamma) = m_\alpha m_\beta\gamma = m_\alpha m_\beta m_\alpha^{-1} m_\alpha\gamma = m_{m_\alpha\beta} m_\alpha\gamma = (\alpha \triangleright \beta) \triangleright (\alpha \triangleright \gamma)$$

by the lemma.

Lemma 3.1.3.

(i)$_\mathrm{s}$ *Suppose $\varepsilon = 1$ (symmetric case). Then $m_\alpha = m_\beta$ if and only if $\alpha = \pm\beta$.*

(i)$_\mathrm{a}$ *Suppose $\varepsilon = -1$ (antisymmetric case). If $m_\alpha = m_\beta$, then $(\alpha,\beta) = 0$.*

(ii) *$m_\alpha m_\beta = m_\beta m_\alpha$ if and only if $\alpha = \pm\beta$ or $(\alpha,\beta) = 0$.*

(iii) *$m_\alpha m_\beta m_\alpha = m_\beta m_\alpha m_\beta$ if and only if $m_\alpha = m_\beta$ or $(\alpha,\beta) = \pm 1$.*

Proof. (i): Suppose $m_\alpha = m_\beta$. Then

$$-\varepsilon\alpha = m_\alpha\alpha = m_\beta\alpha = \alpha - \eta(\alpha,\beta)\beta \quad \Rightarrow \quad (1+\varepsilon)\alpha - \eta(\alpha,\beta)\beta = 0.$$

In the symmetric case it follows

$$\alpha = \tfrac{1}{2}\eta(\alpha,\beta)\beta,$$

hence $\alpha = \pm\beta$ since $(\alpha,\alpha) = (\beta,\beta) = 2\eta$.

In the antisymmetric case it follows

$$(\alpha,\beta)\beta = 0,$$

therefore $(\alpha,\beta) = 0$.

(ii): By the previous lemma we have

$$m_\alpha m_\beta = m_\beta m_\alpha \quad \Leftrightarrow \quad m_{m_\alpha\beta} = m_\alpha m_\beta m_\alpha^{-1} = m_\beta.$$

In the symmetric case we get by (i) that $m_\alpha\beta = \kappa\beta$ for some $\kappa = \pm 1$. Therefore

$$\beta - \eta(\beta,\alpha)\alpha = \kappa\beta.$$

Now either $\kappa = 1$, then $(\beta,\alpha)\alpha = 0$, hence $(\alpha,\beta) = 0$, or $\kappa = -1$, then we get (as above) $\alpha = \pm\beta$.

In the antisymmetric case we get by (i) that $(m_\alpha\beta,\beta) = 0$. Therefore

$$0 = (\beta - \eta(\beta,\alpha)\alpha, \beta) = \eta(\alpha,\beta)^2,$$

hence $(\alpha,\beta) = 0$.

(iii): By the previous lemma we have

$$m_\alpha m_\beta m_\alpha = m_\beta m_\alpha m_\beta \Leftrightarrow m_\beta^{-1} m_\alpha m_\beta = m_\alpha m_\beta m_\alpha^{-1} \Leftrightarrow m_{m_\beta^{-1}\alpha} = m_{m_\alpha\beta}.$$

In the symmetric case we get by (i) that $m_\beta\alpha = m_\beta^{-1}\alpha = \kappa m_\alpha\beta$ for some $\kappa = \pm 1$, i.e.

$$\alpha - \eta(\alpha,\beta)\beta = \kappa(\beta - \eta(\alpha,\beta)\alpha),$$

hence

$$\big(1 + \kappa\eta(\alpha,\beta)\big)\alpha = \big(\kappa + \eta(\alpha,\beta)\big)\beta.$$

Now either α and β are linearly dependent, then $m_\alpha = m_\beta$, or $1 + \kappa\eta(\alpha,\beta) = 0$, i.e. $(\alpha,\beta) = \pm 1$.

In the antisymmetric case we get by (i) that $(m_\beta^{-1}\alpha, m_\alpha\beta) = 0$, hence

$$0 = \big(\alpha - \eta(\beta,\alpha)\beta, \beta - \eta(\beta,\alpha)\alpha\big) = (\alpha,\beta) + (\beta,\alpha)^3 = x - x^3,$$

for $x = (\alpha,\beta)$, therefore $(\alpha,\beta) = 0$ or $(\alpha,\beta) = \pm 1$. Now if $(\alpha,\beta) = 0$, then by (ii) we get

$$m_\alpha m_\beta m_\alpha = m_\beta m_\alpha m_\beta = m_\alpha m_\beta m_\beta,$$

hence $m_\alpha = m_\beta$.

Conversely, assume that $(\alpha,\beta) =: \kappa = \pm 1$. Then

$$m_\beta^{-1}\alpha = \alpha - \eta\varepsilon\kappa\beta = -\eta\varepsilon\kappa(\beta - \eta\varepsilon\kappa\alpha) = -\eta\varepsilon\kappa m_\alpha\beta,$$

therefore $m_\beta^{-1}\alpha = \pm m_\alpha\beta$, and we get $m_\alpha m_\beta m_\alpha = m_\beta m_\alpha m_\beta$. □

Proposition 3.1.4. *$m = m_\mu \cdots m_2 m_1$. Furthermore $m_1, \ldots, m_\mu$ are determined by m.*

Proof. First observe that as a matrix for the basis $(\delta_1, \ldots, \delta_\mu)$

$$m_i = \mathbb{1}_\mu - \eta\varepsilon L_i \tag{3.7}$$

where L_i is the matrix whose i-th row is exactly the i-th row of S and with all other entries 0. This follows from

$$m_i(\delta_j) = \delta_j - \eta(\delta_j, \delta_i)\delta_i = \delta_j - \eta S_{ji}\delta_i = \delta_j - \eta\varepsilon S_{ij}\delta_i.$$

The entry $(m_i)_{ii}$ is $-\varepsilon$. Therefore we can apply Lemma B.1.5 of the appendix with $a = -\varepsilon$ and $X = -(V - \mathbb{1})^t$, $Y = -\varepsilon(V - \mathbb{1})^*$ to get

$$m_\mu \cdots m_2 m_1 = a(-X + \mathbb{1})^{-1}(a^{-1}Y + \mathbb{1}) = -\varepsilon(V^t)^{-1}V = m.$$

We also get from that lemma that the above product determines its factors m_i. □

The automorphism m_i determines all intersection products (α, δ_i), thus all m_i together determine the intersection product $(\cdot, \cdot)$. So by the previous proposition one gets:

Corollary 3.1.5. *The monodromy m determines the intersection matrix S.* □

Definition 3.1.6. The *monodromy group* Γ is defined as the subgroup of $\mathrm{Aut}(M)$ generated by $m_1, \ldots, m_\mu$.

3.1.2 Stabilization

For each pair of signs (ε, η) we have equivalences of categories

$$\mathcal{I}nt\mathcal{D}at(\varepsilon, \eta) \longrightarrow \mathcal{I}nt\mathcal{M}at(\varepsilon, \eta) \xrightarrow{\sim} \mathcal{D}ynkin$$

where the last map is an isomorphism. Therefore, given two pairs of signs (ε, η) and (ε', η') we can put them together to get an equivalence

$$\mathcal{I}nt\mathcal{D}at(\varepsilon, \eta) \longrightarrow \mathcal{I}nt\mathcal{D}at(\varepsilon', \eta') \tag{3.8}$$

resp. an isomorphism

$$\mathcal{I}nt\mathcal{M}at(\varepsilon, \eta) \xrightarrow{\sim} \mathcal{I}nt\mathcal{M}at(\varepsilon', \eta'). \tag{3.9}$$

Let us describe this equivalence resp. this isomorphism here.

Let $S \in \mathcal{I}nt\mathcal{M}at(\varepsilon, \eta)$ be an intersection matrix. We have $S = \eta(V + \varepsilon V^t)$. The corresponding Coxeter-Dynkin diagram is as follows: If $i < j$, two vertices

*Note that this is in fact the correct choice for X and Y since $X + Y + a\mathbb{1} = -V^t - \varepsilon V + (1 + \varepsilon)\mathbb{1} - \varepsilon\mathbb{1} = \mathbb{1} - \eta\varepsilon S$, hence the rows of the matrix $X + Y + a\mathbb{1}$ are the non-trivial rows of the m_i, as needed in the lemma.

i and j are connected by $|S_{ij}|$ lines, dashed in the case that $\eta S_{ij} < 0$. In terms of V this means that these vertices are connected by $|V_{ij}|$ lines, dashed in the case that $V_{ij} > 0$. This means that the Seifert matrix V is invariant under the isomorphism (3.9). Hence the isomorphism (3.9) is

$$\mathcal{I}nt\mathcal{M}at(\varepsilon, \eta) \ni S = \eta(V + \varepsilon V^t) \mapsto S' = \eta'(V + \varepsilon' V^t) \in \mathcal{I}nt\mathcal{M}at(\varepsilon', \eta').$$

Now let $\big(M, (\cdot,\cdot), \underline{\delta}\big) \in \mathcal{I}nt\mathcal{D}at(\varepsilon, \eta)$ be an intersection datum corresponding to S. This maps to an intersection datum $\big(M', (\cdot,\cdot)', \underline{\delta}'\big)$ corresponding to S' (here one can take $M' = M$ and $\underline{\delta}' = \underline{\delta}$). Then the above shows that

$$(\delta_i', \delta_j')' = \eta\eta'(\delta_i, \delta_j) \quad \text{for } i < j.$$

3.1.3 The operation of the (extended) braid group

We now define an action of the semidirect product $\mathrm{Br}_\mu \ltimes (\mathbb{Z}/2\mathbb{Z})^\mu$ of the braid group (see the appendix, Section A.1) with the group of "sign-changes" on bases $(\alpha_1, \ldots, \alpha_\mu)$ of M consisting of cycles with $(\alpha_i, \alpha_i) = \eta(1 + \varepsilon)$ $(i = 1, \ldots, \mu)$. This group can be written by generators and relations as follows: As generators we take

$$\sigma_1, \ldots, \sigma_{\mu-1} \quad \text{and} \quad \xi_1, \ldots, \xi_\mu.$$

These satisfy the following relations:

$$\begin{aligned}
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && \\
\sigma_i\sigma_j &= \sigma_j\sigma_i && \text{for } |i-j| \geq 2 \\
\xi_i\xi_j &= \xi_j\xi_i && \text{for } i \neq j \\
\xi_i^2 &= 1 && \\
\sigma_i\xi_i &= \xi_{i+1}\sigma_i \qquad \sigma_{i-1}\xi_i = \xi_{i-1}\sigma_{i-1} && \\
\sigma_j\xi_i &= \xi_i\sigma_j && \text{for } |i-j| \geq 2
\end{aligned}$$

The operation of this group is defined as follows:

$$\begin{aligned}
\sigma_i(\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots \alpha_\mu) &= (\alpha_1, \ldots, \alpha_{i+1}, m_{\alpha_{i+1}}\alpha_i, \ldots \alpha_\mu) \\
\sigma_i^{-1}(\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots \alpha_\mu) &= (\alpha_1, \ldots, m_{\alpha_i}^{-1}\alpha_{i+1}, \alpha_i, \ldots \alpha_\mu) \\
\xi_i(\alpha_1, \ldots, \alpha_i, \ldots \alpha_\mu) &= (\alpha_1, \ldots, -\alpha_i, \ldots \alpha_\mu)
\end{aligned}$$

We call $\mathrm{Br}_\mu \ltimes (\mathbb{Z}/2\mathbb{Z})^\mu$ the *extended braid group*. However, by abuse of language, we often call this group simply "the braid group" if it is clear by context that this group is meant. Indeed:

Lemma 3.1.7. *This defines an action on bases of M consisting of cycles with self-intersection $\eta(1 + \varepsilon)$.*

Proof. To see that the above operations respect the relations of the group is a straightforward calculation. For example, we have

$$\begin{aligned}\sigma_i\sigma_i^{-1}(\alpha_1,\dots,\alpha_i,\alpha_{i+1},\dots\ \alpha_\mu) &= \sigma_i(\alpha_1,\dots,m_{\alpha_i}^{-1}\alpha_{i+1},\alpha_i,\dots,\alpha_\mu)\\ &= (\alpha_1,\dots,\alpha_i,m_{\alpha_i}m_{\alpha_i}^{-1}\alpha_{i+1},\dots,\alpha_\mu)\\ &= (\alpha_1,\dots,\alpha_i,\alpha_{i+1},\dots\ \alpha_\mu)\end{aligned}$$

and

$$\begin{aligned}&\sigma_i\sigma_{i+1}\sigma_i(\alpha_1,\dots,\alpha_i,\alpha_{i+1},\alpha_{i+2},\dots\ \alpha_\mu)\\ &\quad= \sigma_i\sigma_{i+1}(\alpha_1,\dots,\alpha_{i+1},m_{\alpha_{i+1}}\alpha_i,\alpha_{i+2},\dots,\alpha_\mu)\\ &\quad= \sigma_i(\alpha_1,\dots,\alpha_{i+1},\alpha_{i+2},m_{\alpha_{i+2}}m_{\alpha_{i+1}}\alpha_i,\dots,\alpha_\mu)\\ &\quad= (\alpha_1,\dots,\alpha_{i+2},m_{\alpha_{i+2}}\alpha_{i+1},m_{\alpha_{i+2}}m_{\alpha_{i+1}}\alpha_i,\dots,\alpha_\mu)\\ &\quad= (\alpha_1,\dots,\alpha_{i+2},m_{\alpha_{i+2}}\alpha_{i+1},m_{m_{\alpha_{i+2}}\alpha_{i+1}}m_{\alpha_{i+2}}\alpha_i,\dots,\alpha_\mu)\\ &\quad= \sigma_{i+1}(\alpha_1,\dots,\alpha_{i+2},m_{\alpha_{i+2}}\alpha_i,m_{\alpha_{i+2}}\alpha_{i+1},\dots,\alpha_\mu)\\ &\quad= \sigma_{i+1}\sigma_i(\alpha_1,\dots,\alpha_i,\alpha_{i+2},m_{\alpha_{i+2}}\alpha_{i+1},\dots,\alpha_\mu)\\ &\quad= \sigma_{i+1}\sigma_i\sigma_{i+1}(\alpha_1,\dots,\alpha_i,\alpha_{i+1},\alpha_{i+2},\dots\ \alpha_\mu)\end{aligned}$$

by Lemma 3.1.2

Furthermore, if $\underline{\alpha} := (\alpha_1,\dots,\alpha_\mu)$ is a basis of cycles with self-intersection $\eta(1+\varepsilon)$, also $\sigma_i\cdot\underline{\alpha}$ and $\xi_i\cdot\underline{\alpha}$ are bases for all $i=1,\dots,\mu-1$ resp. $i=1,\dots,\mu$, and by Lemma 3.1.1 they also contain cycles with self-intersection $\eta(1+\varepsilon)$. □

Definition 3.1.8. An element of the orbit of $(\delta_1,\dots,\delta_\mu)$ under the operation of $\mathrm{Br}_\mu \ltimes (\mathbb{Z}/2\mathbb{Z})^\mu$ is called a *distinguished basis.*

A cycle α is called a *vanishing cycle* if there is a distinguished basis $(\alpha_1,\dots,\alpha_\mu)$ with $\alpha=\alpha_1$.

Hence all vanishing cycles have self-intersection $\eta(1-\varepsilon)$ by definition.

The action of the braid group on bases of M consisting of cycles with self-intersection-product $\eta(1+\varepsilon)$ restricts by definition to an action on distinguished bases. From this we get an action on intersection data: For $g\in \mathrm{Br}_\mu\ltimes(\mathbb{Z}/2\mathbb{Z})^\mu$ and an intersection datum $\big(M,(\cdot,\cdot),\underline{\delta}\big)$ with Milnor number μ define

$$g\big(M,(\cdot,\cdot),\underline{\delta}\big) := \big(M,(\cdot,\cdot),g\cdot\underline{\delta}\big)$$

(with unchanged intersection product $(\cdot,\cdot)$). This extends to an functorial action of $\mathrm{Br}_\mu\ltimes(\mathbb{Z}/2\mathbb{Z})^\mu$ on $\mathcal{I}nt\mathcal{D}at_\mu$. From this we also get actions of $\mathrm{Br}_\mu\ltimes(\mathbb{Z}/2\mathbb{Z})^\mu$ on $\mathcal{I}nt\mathcal{M}at_\mu$ and $\mathcal{D}ynkin_\mu$ by the equivalences of categories (3.2). For example, the element $\sigma_{\mu-1}\in \mathrm{Br}_\mu\ltimes(\mathbb{Z}/2\mathbb{Z})^\mu$ acts on $\mathcal{I}nt\mathcal{M}at_\mu$ as follows: An intersection matrix $S\in\mathcal{I}nt\mathcal{M}at_\mu$ write as follows:

$$S=\begin{pmatrix} S_2 & y & z\\ \varepsilon y^t & \eta(1+\varepsilon) & u\\ \varepsilon z^t & \varepsilon u & \eta(1+\varepsilon)\end{pmatrix}$$

with $S_2 \in \mathcal{I}nt\mathcal{M}at_{\mu-2}$, $y, z \in \mathbb{Z}^{\mu-2}$ and $u \in \mathbb{Z}$. If this matrix comes from an intersection datum $\big(M, (\cdot,\cdot), \underline{\delta}\big)$, then

$$\underline{\alpha} := \sigma_{\mu-1}\underline{\delta} = \big(\delta_1, \dots, \delta_{\mu-2}, \delta_\mu, \delta_{\mu-1} - \eta(\delta_{\mu-1}, \delta_\mu)\delta_\mu\big)$$

Therefore (for $i \leq \mu - 2$) we get

$$\begin{aligned} y_i' &:= (\alpha_i, \alpha_{\mu-1}) = (\delta_i, \delta_\mu) && = z_i, \\ z_i' &:= (\alpha_i, \alpha_\mu) \quad = (\delta_i, \delta_{\mu-1} - \eta u \delta_\mu) = (\delta_i, \delta_{\mu-1}) - \eta u (\delta_i, \delta_\mu) && = y_i - \eta u z_i, \\ u' &:= (\alpha_{\mu-1}, \alpha_\mu) = (\delta_\mu, \delta_{\mu-1} - \eta u \delta_\mu) = (\delta_\mu, \delta_{\mu-1}) - \eta u (\delta_\mu, \delta_\mu) && = -u, \end{aligned}$$

hence we get

$$\sigma_{\mu-1} S = \begin{pmatrix} S_2 & z & y - \eta u z \\ \varepsilon z^t & \eta(1+\varepsilon) & -u \\ \varepsilon y^t - \eta\varepsilon u z^t & -\varepsilon u & \eta(1+\varepsilon) \end{pmatrix}.$$

If we describe this in terms of Seifert matrices, we get

$$V = \begin{pmatrix} V_2 & -\tilde{y} & -\tilde{z} \\ 0 & 1 & -\tilde{u} \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \sigma_{\mu-1} V = \begin{pmatrix} V_2 & -\tilde{z} & -(\tilde{y} + \tilde{u}\tilde{z}) \\ 0 & 1 & -(-\tilde{u}) \\ 0 & 0 & 1 \end{pmatrix}$$

where $\tilde{y} = -\eta y$, $\tilde{z} = -\eta z$ and $\tilde{u} = -\eta u$. We see that the operation on the Seifert matrices is independent of ε, η and therefore respects the equivalences of categories (3.2) (see Section 3.1.2).

We also get the operation on the Coxeter-Dynkin diagrams: The number (with sign) of lines between the i-th and the j-th vertex is exactly $-V_{ij}$ for $i < j$. By the above we see that $\sigma_{\mu-1}$ operates as follows: Let l be the number of lines between the $(\mu-1)$-th and the μ-th vertex. To get the new diagram, first erase the $(\mu-1)$-th vertex and all lines connected to this vertex. Then relabel the old μ-th vertex with the number $\mu - 1$. Draw a new μ-th vertex and draw $-l$ lines between the new $(\mu-1)$-th and the new μ-th vertex. A vertex k with $k < \mu - 1$ is connected to the new μ-th vertex by the following number of lines: The number of lines between the k-th and the old $(\mu-1)$-th vertex plus l times the number of lines between the k-th and the old μ-th vertex.

Two intersection data, two intersection matrices resp. two Coxeter-Dynkin diagrams are called *equivalent* if they lie in the same orbit under the operation of the extended braid group.

Lemma 3.1.9. *Two equivalent intersection matrices are similar as matrices in* $\mathrm{Mat}(\mu \times \mu, \mathbb{Q})$.

Proof. We have to prove that the base change matrices for two distinguished bases are orthogonal. It suffices to prove this for base changes $\underline{\alpha} \rightarrowtail \sigma_i \cdot \underline{\alpha}$ $(i = 1, \dots, \mu-1)$ and $\underline{\alpha} \rightarrowtail \xi_i \cdot \underline{\alpha}$ $(i = 1, \dots, \mu)$ for which the assertion is trivial. □

Lemma 3.1.10. *Let* $(\alpha_1, \ldots, \alpha_\mu)$ *be a distinguished basis and* $\{i_1, i_2, \ldots, i_r\} \subset \{1, \ldots, \mu\}$ *with* $i_1 < \cdots < i_r$ $(r = 1, \ldots, \mu)$.

Then there exists a distinguished basis $(\alpha'_1, \ldots, \alpha'_\mu)$ *with* $\alpha'_j = \alpha_{i_j}$ *for all* $j = 1, \ldots, r$.

In particular, all cycles of a distinguished basis are vanishing cycles.

Proof. For $i \leq j$ consider the element

$$g_{i,j} := \begin{cases} \sigma_i \sigma_{i+1} \cdots \sigma_{j-1} & i < j \\ \mathbb{1} & i = j. \end{cases}$$

One easily calculates that for $j > i$ one has

$$\begin{aligned} & g_{i,j}(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i, \ldots, \alpha_{j-1}, \alpha_j, \alpha_{j+1}, \ldots, \alpha_\mu) \\ & \qquad = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_j, m_{\alpha_j}\alpha_i, \ldots, m_{\alpha_j}\alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_\mu). \end{aligned}$$

Hence the basis $g \cdot (\alpha_1, \ldots, \alpha_\mu)$ has the desired property for $g = g_{r,i_r} \cdots g_{1,i_1}$.

The last remark follows from the case $r = 1$. □

As a generalization of Proposition 3.1.4 we have:

Proposition 3.1.11. *For each distinguished basis* $(\alpha_1, \ldots, \alpha_\mu)$ *one has*

$$m = m_{\alpha_\mu} \cdots m_{\alpha_2} m_{\alpha_1}.$$

Proof. Set $\underline{\alpha} = (\alpha_1, \ldots, \alpha_\mu)$. By Proposition 3.1.4 the assertion is true for $\underline{\alpha} = \underline{\delta}$. Since the set of distinguished bases is by definition the orbit of $\underline{\delta}$ under the operation of the braid group, we just have to show the following:

$$m_{\alpha'_\mu} \cdots m_{\alpha'_2} m_{\alpha'_1} = m_{\alpha_\mu} \cdots m_{\alpha_2} m_{\alpha_1}$$

if $\underline{\alpha}' = \sigma_i \cdot \underline{\alpha}$ $(i = 1, \ldots, \mu - 1)$ or $\underline{\alpha}' = \xi_i \cdot \underline{\alpha}$ $(i = 1, \ldots, \mu)$ (with $\underline{\alpha}' = (\alpha'_1, \ldots, \alpha'_\mu)$).

If $\underline{\alpha}' = \sigma_i \cdot \underline{\alpha}$, we have

$$(\alpha'_1, \ldots, \alpha'_i, \alpha'_{i+1}, \ldots, \alpha'_\mu) = (\alpha_1, \ldots, \alpha_{i+1}, m_{\alpha_{i+1}}\alpha_i, \ldots \alpha_\mu),$$

hence

$$\begin{aligned} m_{\alpha'_\mu} \cdots m_{\alpha'_{i+1}} m_{\alpha'_i} \cdots m_{\alpha'_1} &= m_{\alpha_\mu} \cdots m_{m_{\alpha_{i+1}}\alpha_i} m_{\alpha_{i+1}} \cdots m_{\alpha_1} \\ &= m_{\alpha_\mu} \cdots m_{\alpha_{i+1}} m_{\alpha_i} m_{\alpha_{i+1}}^{-1} m_{\alpha_{i+1}} \cdots m_{\alpha_1} \\ &= m_{\alpha_\mu} \cdots m_{\alpha_{i+1}} m_{\alpha_i} \cdots m_{\alpha_1} \end{aligned}$$

by Lemma 3.1.2.

The case $\underline{\alpha}' = \xi_i \cdot \underline{\alpha}$ follows trivially by Lemma 3.1.1. □

Proposition 3.1.12. *For each distinguished basis* $(\alpha_1, \ldots, \alpha_\mu)$, *the subgroup of* $\mathrm{Aut}(M)$ *generated by* $m_{\alpha_1}, \ldots, m_{\alpha_\mu}$ *is exactly the monodromy group* Γ.

Proof. This proof is nearly the same as in the previous proposition. We have to prove that the statement of this proposition is invariant under the operation of the braid group.

Suppose $\underline{\alpha} := (\alpha_1, \ldots, \alpha_\mu)$ generates the monodromy group Γ. If $\underline{\alpha}' = \sigma_i \cdot \underline{\alpha}$, then

$$\begin{aligned}(m_{\alpha'_1}, \ldots, m_{\alpha'_\mu}) &= (m_{\alpha_1}, \ldots, m_{\alpha_{i+1}}, m_{m_{\alpha_{i+1}}\alpha_i}, \ldots\ m_{\alpha_\mu})\\ &= (m_{\alpha_1}, \ldots, m_{\alpha_{i+1}}, m_{\alpha_{i+1}} m_{\alpha_i} m^{-1}_{\alpha_{i+1}}, \ldots\ m_{\alpha_\mu})\end{aligned}$$

by Lemma 3.1.2. This obviously also generates Γ.

The case $\underline{\alpha}' = \xi_i \cdot \underline{\alpha}$ follows again trivially by Lemma 3.1.1. □

Lemma 3.1.13. *If $(\alpha_1, \ldots, \alpha_\mu)$ is a distinguished basis, then $(m\alpha_1, \ldots, m\alpha_\mu)$ is also one. In particular, m maps vanishing cycles to vanishing cycles.*

Proof. In Br_μ consider the following elements:

$$\begin{aligned}\overline{d} &= \sigma_{\mu-1} \cdots \sigma_1,\\ \Delta &= \sigma_1(\sigma_2\sigma_1) \cdots (\sigma_{\mu-1} \cdots \sigma_1).\end{aligned}$$

In the appendix, Section A.1 it is shown that

$$\Delta^2 = \overline{d}^\mu.$$

The operation of $\overline{d}$ on distinguished bases is as follows:

$$\begin{aligned}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{\mu-1}, \alpha_\mu) &\overset{\sigma_1}{\longmapsto} (\alpha_2, m_{\alpha_2}\alpha_1, \alpha_3, \ldots, \alpha_{\mu-1}, \alpha_\mu)\\ &\overset{\sigma_2}{\longmapsto} (\alpha_2, \alpha_3, m_{\alpha_3}m_{\alpha_2}\alpha_1, \ldots, \alpha_{\mu-1}, \alpha_\mu)\\ &\ \vdots\\ &\overset{\sigma_{\mu-1}}{\longmapsto} (\alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_\mu, m_{\alpha_\mu} \cdots m_{\alpha_3}m_{\alpha_2}\alpha_1)\\ &= (\alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_\mu, m_{\alpha_\mu} \cdots m_{\alpha_3}m_{\alpha_2}m_{\alpha_1}(-\varepsilon\alpha_1))\\ &= (\alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_\mu, -\varepsilon m\alpha_1).\end{aligned}$$

The last equality follows from Proposition 3.1.11. It follows that

$$(\alpha_1, \alpha_2, \ldots, \alpha_\mu) \overset{\overline{d}^\mu = \Delta^2}{\longmapsto} (-\varepsilon m\alpha_1, -\varepsilon m\alpha_2, \ldots, -\varepsilon m\alpha_\mu).$$

In the symmetric case apply afterwards the element $\prod_{i=1}^{\mu} \xi_i$ to the right side to get the correct signs. □

3.1.4 Gabrielov transformations

After we have defined an operation of the braid group on bases $(\alpha_1, \ldots, \alpha_\mu)$ of M consisting of cycles α_i with self-intersection product $(\alpha_i, \alpha_i) = \eta(1 + \varepsilon)$ in the last section, we now define an operation of a larger group on such bases.

The group in question is the semidirect product of the Gabrielov group Gabr_μ (i.e. the semidirect product $\mathcal{S}_\mu \ltimes \mathrm{PGabr}_\mu$ of the symmetric group with the pure Gabrielov group) defined in the appendix, Section A.2 and (as in the braid group case) the group $(\mathbb{Z}/2\mathbb{Z})^\mu$ of sign-changes.

$\mathrm{Gabr}_\mu \ltimes (\mathbb{Z}/2\mathbb{Z})^\mu$ can be described by generators and relations as follows: As generators take

$$\begin{aligned} &\rho_j^i \quad \text{for} \quad 1 \le i, j \le n,\ i \ne j, \\ &\tau_i \quad \text{for} \quad 1 \le i \le \mu - 1, \quad \text{and} \\ &\xi_i \quad \text{for} \quad 1 \le i \le \mu. \end{aligned}$$

These satisfy the following relations:

$$\begin{aligned}
\rho_j^i \rho_l^k &= \rho_l^k \rho_j^i && && \text{for } |\{i,j,k,l\}| = 4, \\
\rho_j^i \rho_k^i &= \rho_k^i \rho_j^i && && \text{for } |\{i,j,k\}| = 3, \\
\rho_j^i (\rho_i^k \rho_j^k) &= (\rho_i^k \rho_j^k) \rho_j^i && && \text{for } |\{i,j,k\}| = 3 \\
\tau_i \tau_{i+1} \tau_i &= \tau_{i+1} \tau_i \tau_{i+1} && && \\
\tau_i \tau_j &= \tau_j \tau_i && && \text{for } |i-j| \ge 2 \\
\tau_i^2 &= 1 && && \\
\tau_i \rho_{i+1}^i &= \rho_i^{i+1} \tau_i & \tau_i \rho_i^{i+1} &= \rho_{i+1}^i \tau_i && \\
\tau_i \rho_j^i &= \rho_j^{i+1} \tau_i & \tau_{j-1} \rho_j^i &= \rho_{j-1}^i \tau_{j-1} && \text{for } j \ne i+1 \\
\tau_{i-1} \rho_j^i &= \rho_j^{i-1} \tau_{i-1} & \tau_j \rho_j^i &= \rho_{j+1}^i \tau_j && \text{for } j \ne i-1 \\
\tau_k \rho_j^i &= \rho_j^i \tau_k && && \text{for } k \ne i-1, i, j-1, j \\
\xi_i \xi_j &= \xi_j \xi_i && && \text{for } i \ne j \\
\xi_i^2 &= 1 && && \\
\rho_j^i \xi_k &= \xi_k \rho_j^i && && \\
\tau_i \xi_i &= \xi_{i+1} \tau_i & \tau_{i-1} \xi_i &= \xi_{i-1} \tau_{i-1} && \\
\tau_j \xi_i &= \xi_i \tau_j && && \text{for } |i-j| \ge 2
\end{aligned}$$

Again we call $\mathrm{Gabr}_\mu \ltimes (\mathbb{Z}/2\mathbb{Z})^\mu$ the *extended Gabrielov group* and also we often call this group simply "the Gabrielov group" (again by abuse of language) if no confusion can arise.

The elements operate as follows: First the elements of the pure Gabrielov group operate as the so-called *Gabrielov transformations* as follows:

$$\begin{aligned} \rho_j^i(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_j, \ldots, \alpha_\mu) &= (\alpha_1, \ldots, \alpha_i, \ldots, m_{\alpha_i}\alpha_j, \ldots, \alpha_\mu) \qquad \text{or} \\ \rho_j^i(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_i, \ldots, \alpha_\mu) &= (\alpha_1, \ldots, m_{\alpha_i}\alpha_j, \ldots, \alpha_i, \ldots, \alpha_\mu) \end{aligned}$$

depending on whether $i < j$ or $i > j$. The permutations operate by permuting the elements of the basis, i.e.

$$\tau_i(\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_\mu) = (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_\mu),$$

and the sign-changes are defined as before:

$$\xi_i(\alpha_1, \dots, \alpha_i, \dots, \alpha_\mu) = (\alpha_1, \dots, -\alpha_i, \dots, \alpha_\mu).$$

Remark 3.1.14. We have seen in Section A.2 that the braid group is a subgroup of the Gabrielov group. Thus the above operations include the operations of the braid group. One has

$$\sigma_i = \tau_i \rho_i^{i+1} = \rho_{i+1}^i \tau_i.$$

It is easily checked that the corresponding operators satisfy this equation.

Note that Gabr_μ is also generated by $\mathrm{Br}_\mu \cup \mathcal{S}_\mu$ where $\mathrm{Br}_\mu \subset \mathrm{Gabr}_\mu$ as in the above remark.

In analogue to Definition 3.1.8 we can now make the following definition:

Definition 3.1.15. An element of the orbit of $(\delta_1, \dots, \delta_\mu)$ under the operation of $\mathrm{Gabr}_\mu \ltimes (\mathbb{Z}/2\mathbb{Z})^\mu$ is called a *weakly distinguished basis.*

A cycle α is called a *weakly vanishing cycle* if there is a weakly distinguished basis $(\alpha_1, \dots, \alpha_\mu)$ with $\alpha = \alpha_1$.

By Remark 3.1.14 a distinguished basis is also a weakly distinguished basis, and a vanishing cycle is a weakly vanishing cycle.

Again as in the case of the (extended) braid group the operation restricts to an operation on weakly distinguished bases and again we get an operation on intersection data, intersection matrices and Coxeter-Dynkin diagrams.

Two intersection data, two intersection matrices resp. two Coxeter-Dynkin diagrams are called *weakly equivalent* if they lie in the same orbit under the extended Gabrielov group. Since the operation of the braid group is part of the operation of the Gabrielov group, weak equivalence is in fact a weaker property than equivalence.

As in the previous section we have:

Lemma 3.1.16. *Two weakly equivalent intersection matrices are similar as matrices in* $\mathrm{Mat}(\mu \times \mu, \mathbb{Q})$.

Proof. The same as in Lemma 3.1.9. □

As a generalization of Proposition 3.1.12 we get

Proposition 3.1.17. *For each weakly distinguished basis* $(\alpha_1, \dots, \alpha_\mu)$ *the subgroup of* $\mathrm{Aut}(M)$ *generated by* $m_{\alpha_1}, \dots, m_{\alpha_\mu}$ *is exactly the monodromy group* Γ.

Proof. This follows immediately from Proposition 3.1.12 and the fact that Gabr_μ is generated by $\mathrm{Br}_\mu \cup \mathcal{S}_\mu$. □

By the discussion in Section A.2 we get:

Proposition 3.1.18. $\underline{\alpha} = (\alpha_1, \dots, \alpha_\mu)$ *is a weakly distinguished basis if and only if for each* k

$$\alpha_k = \iota_k \cdot m_{i_{k,r_k}}^{\kappa_{k,r_k}} \cdots m_{i_{k,2}}^{\kappa_{k,2}} m_{i_{k,1}}^{\kappa_{k,1}} \delta_{j_k}$$

for some $1 \leq i_{k,1}, \dots, i_{k,r_k}, j_k \leq \mu$ *and* $\kappa_{k,1}, \dots, \kappa_{k,r_k}, \iota_k = \pm 1$ *such that the following holds:*

(i) $\{j_1, \dots, j_\mu\} = \{1, \dots, \mu\}$.

(ii) *Let* $\mathcal{F}_\mu$ *be the free group generated by* $x_1, \dots, x_\mu$*. Set*

$$y_k = x_{i_{k,r_k}}^{\kappa_{k,r_k}} \cdots x_{i_{k,2}}^{\kappa_{k,2}} x_{i_{k,1}}^{\kappa_{k,1}} x_{j_k} x_{i_{k,1}}^{-\kappa_{k,1}} x_{i_{k,2}}^{-\kappa_{k,2}} \cdots x_{i_{k,r_k}}^{-\kappa_{k,r_k}}.$$

Then $y_1, \dots, y_\mu$ *also generate* $\mathcal{F}_\mu$.

Proof. This follows from the description of the Gabrielov group given in the appendix, Section A.2 in equation (A.14): We have that $\mathrm{Gabr}_\mu \subset \mathrm{Aut}(\mathcal{F}_\mu)$, and a $\varphi \in \mathrm{Aut}(\mathcal{F}_\mu)$ is in Gabr_μ if and only if $y_k := \varphi(x_k)$ is conjugated to x_{j_k} for some $j_k = 1, \dots, \mu$. Moreover φ is a product of some generators ρ_j^i and a permutation $\pi \in \mathcal{S}_\mu$.

If one compares the operation of these generators on $\mathcal{F}_n$ and the set of weakly distinguished bases, the proposition follows from this description. □

Corollary 3.1.19. $\alpha \in M$ *is a weakly vanishing cycle if and only if*

$$\alpha = \iota \cdot m_{i_r}^{\kappa_r} \cdots m_{i_2}^{\kappa_2} m_{i_1}^{\kappa_1} \delta_j \tag{3.10}$$

for some $1 \le i_1, \dots, i_r, j \le \mu$ *and* $\kappa_1, \dots, \kappa_r, \iota = \pm 1$.

Proof. It is clear that a vanishing cycle must be of this form by the proposition.

Conversely, let α be of that form, i.e.

$$\alpha = \iota \cdot m_{i_r}^{\kappa_r} \cdots m_{i_2}^{\kappa_2} m_{i_1}^{\kappa_1} \delta_j$$

Then let π be a permutation of $\{1, \dots, \mu\}$ such that $\pi(1) = j$ and set

$$\alpha_k = \iota \cdot m_{i_r}^{\kappa_r} \cdots m_{i_2}^{\kappa_2} m_{i_1}^{\kappa_1} \delta_{\pi(k)}.$$

Then clearly $\underline{\alpha} := (\alpha_1, \dots, \alpha_\mu)$ satisfies the conditions of the proposition, thus $\underline{\alpha}$ is a weakly distinguished basis. Since $\alpha = \alpha_1$, α is a weakly vanishing cycle. □

In the symmetric case this corollary has an easier formulation:

Corollary 3.1.20. *In the symmetric case* ($\varepsilon = 1$) α *is a weakly vanishing cycle if and only if it is of the form*

$$\alpha = m_{i_r} \cdots m_{i_2} m_{i_1} \delta_j$$

for some $1 \le i_1, \dots, i_r, j \le \mu$.

Proof. If $\varepsilon = 1$, we have $m_i^{-1} = m_i$, hence we do not need the κ's in the previous corollary. Furthermore, we have $m_j \delta_j = -\delta_j$ in the symmetric case, thus in case of $\iota = -1$ in the previous corollary we can write

$$\alpha = -m_{i_r} \cdots m_{i_2} m_{i_1} \delta_j = m_{i_r} \cdots m_{i_2} m_{i_1} m_j \delta_j,$$

which also is of the desired form. □

Proposition 3.1.21. *The set of weakly vanishing cycles is finite if and only if* Γ *has finite order.*

Proof. Suppose that the set of weakly vanishing cycles is finite. We have to show that each element $g \in \Gamma$ has finite order. g is of the form

$$g = m_{i_r} \cdots m_{i_2} m_{i_1}.$$

By Proposition 3.1.18 we have that

$$(g^k \cdot \delta_1, g^k \cdot \delta_2, \ldots, g^k \cdot \delta_\mu)$$

is a weakly distinguished basis for all $k \in \mathbb{Z}$. Since there are only finitely many weakly vanishing cycles, there are also only finitely many weakly distinguished bases. Hence there must exist a $k \neq 0$ with

$$g^k \cdot \delta_i = \delta_i \quad \forall i,$$

i.e. $g^k = \mathbb{1}$.

Conversely, suppose that Γ has finite order. By Corollary 3.1.19 each weakly vanishing cycle α is of the form

$$\alpha = \iota \cdot g \cdot \delta_j$$

for some $\iota = \pm 1$, $g \in \Gamma$ and $j = 1, \ldots, \mu$. Since Γ is finite, there are only finitely many possibilities for this, thus the set of weakly vanishing cycles must be finite. □

Proposition 3.1.22. *Suppose* D *is a connected*[†] *Coxeter-Dynkin diagram that only contains lines with a weight of absolute value* ≤ 1. *Then the monodromy group operates transitively on the set of weakly vanishing cycles.*

Proof. Consider two weakly vanishing cycles α and β with $(\alpha, \beta) = \theta = \pm 1$. Then

$$m_\beta \alpha = \alpha - \eta(\alpha, \beta)\beta = \alpha - \eta\theta\beta, \quad m_\alpha \beta = \beta - \eta(\beta, \alpha)\alpha = \beta - \eta\varepsilon\theta\alpha,$$

hence we have

$$\begin{aligned} m_\alpha m_\beta \alpha &= m_\alpha(\alpha - \eta\theta\beta) = -\varepsilon\alpha - \eta\theta(\beta - \eta\varepsilon\theta\alpha) \ = -\eta\theta\beta, \\ m_\beta m_\alpha \beta &= m_\beta(\beta - \eta\varepsilon\theta\alpha) = -\varepsilon\beta - \eta\varepsilon\theta(\alpha - \eta\theta\beta) = -\eta\varepsilon\theta\alpha \end{aligned}$$

and therefore

$$\begin{aligned} m_\beta m_\alpha m_\beta m_\alpha m_\beta \alpha &= m_\beta m_\alpha m_\beta(-\eta\theta\beta) = m_\beta m_\alpha(\varepsilon\eta\theta\beta) = -\alpha, \\ m_\alpha m_\beta m_\alpha m_\beta m_\alpha \beta &= m_\alpha m_\beta m_\alpha(-\eta\varepsilon\theta\alpha) = m_\alpha m_\beta(\eta\theta\alpha) = -\beta. \end{aligned}$$

[†] This means that D is connected as a graph, see Section 3.1.6.

Combining this we get the following: If $(\delta_j, \delta_k) = \pm 1$, then $-\delta_j$ and δ_k can be expressed as

$$-\delta_j = g \cdot \delta_j, \quad \delta_k = g' \cdot \delta_j$$

where $g, g' \in \Gamma$ are the following elements in the monodromy group:

$$g = (m_k m_j)^2 m_k, \quad g' = \begin{cases} m_j m_k & \text{if } (\delta_j, \delta_k) = -\eta \\ (m_j m_k)^2 m_j^2 m_k & \text{if } (\delta_j, \delta_k) = \eta. \end{cases}$$

Since D is connected, we get for each δ_j an expression of the form

$$\delta_j = g_j' \cdot \delta_1,$$

with an element $g_j' \in \Gamma$, as we can connect the vertices δ_1 and δ_j through other vertices with non-vanishing lines which must have weight ± 1 by assumption. Furthermore we have a $g \in \Gamma$ with

$$-\delta_1 = g \cdot \delta_1$$

by the same argument.

Now let α be an arbitrary weakly vanishing cycle. By Corollary 3.1.19 we can express α as

$$\alpha = \iota \cdot m_{i_r}^{\kappa_r} \cdots m_{i_2}^{\kappa_2} m_{i_1}^{\kappa_1} \delta_j,$$

i.e. $\alpha = \iota \cdot g'' \cdot \delta_j$ for an element $g \in \Gamma$. By the above we can write

$$\alpha = \begin{cases} g'' g_j' \cdot \delta_1 & \text{if } \iota = 1 \\ g'' g_j' g \cdot \delta_1 & \text{if } \iota = -1 \end{cases}$$

which shows that Γ acts transitively on the set of weakly vanishing cycles. □

3.1.5 Subdiagrams

For a subset $J \subset \{1, \dots, \mu\}$ one can draw a corresponding subdiagram $\tilde{D}$ of the Coxeter-Dynkin diagram as follows: Write $J = \{j_1, \dots, j_{\tilde{\mu}}\}$ with $j_1 < \cdots < j_{\tilde{\mu}}$. Delete from D all vertices δ_i with $i \notin J$ and all lines that start or end at a deleted vertex. The remaining vertices $\delta_{j_1}, \dots, \delta_{j_{\tilde{\mu}}}$ are newly labelled by 1 to $\tilde{\mu}$ (i.e. δ_{j_i} is labelled with i).

To the subdiagram $\tilde{D}$ there corresponds a (principal) submatrix $\tilde{S}$ of S. Just delete all i-th rows and i-th columns for $i \notin J$. In the same way we get the corresponding Seifert matrix $\tilde{V}$ from V. Also the submodule $\tilde{M}$ of M spanned by $\underline{\tilde{\delta}} := (\delta_{j_1}, \dots, \delta_{j_{\tilde{\mu}}})$ (with induced intersection form and fixed basis $\underline{\tilde{\delta}}$) corresponds to $\tilde{D}$.

To J we can assign two "submonodromies": On the one hand the matrix

$$m^{(J)} := m_{j_{\tilde{\mu}}} \cdots m_{j_2} m_{j_1} \in GL(\mu, \mathbb{Z}),$$

on the other hand the monodromy of $\tilde{D}$:

$$\tilde{m} = -(\tilde{V}^t)^{-1} \tilde{V} \in GL(\tilde{\mu}, \mathbb{Z}).$$

The following lemma states that these two matrices are essentially the same.

Lemma 3.1.23. *$\tilde{m}$ is the principal submatrix of $m^{(J)}$ which one gets by deleting the i-th rows and i-th columns for $i \notin J$. Furthermore the i-th row of $m^{(J)}$ is trivial for $i \notin J$, i.e. $m_{ii}^{(J)} = 1$ and $m_{ij}^{(J)} = 0$ for $j \neq i$.*

In particular, one has $\det m^{(J)} = \det \tilde{m}$, $\chi_{m^{(J)}}(t) = (1-t)^{\mu-\tilde{\mu}}\chi_{\tilde{m}}(t)$ and $m^{(J)}$ is finite (resp. quasiunipotent) if and only if $\tilde{m}$ is finite (resp. quasiunipotent).

Proof. See Lemma B.1.6. □

3.1.6 Connectedness of Coxeter-Dynkin diagrams

We say that a Coxeter-Dynkin diagram D is connected if it is connected as a graph (if one forgets all the weights of the lines). That means the following: On the set $\{1,\ldots,\mu\}$ consider the following equivalence relation: $i \sim j$ if and only if there exists $1 \leq k_1,\ldots,k_r \leq \mu$ such that

$$(\delta_i,\delta_{k_1}) \neq 0, \quad (\delta_{k_l},\delta_{k_{l+1}}) \neq 0 \ (l=1,\ldots,r), \quad (\delta_{k_r},\delta_j) \neq 0$$

Then D is connected if and only if there is only one equivalence class (namely $\{1,\ldots,\nu\}$ itself). In general, we call an equivalence class *the index set of a connected component* of D.

Lemma 3.1.24. *Let D and D' be two weakly equivalent Coxeter-Dynkin diagrams. Then D is connected if and only if D' is connected.*

Proof. Assume that D is not connected, so using the above equivalence relation there exist at least two index sets of connected components of D. We have to prove that for all generators g of $\mathrm{Gabr}_\mu \ltimes (\mathbb{Z}/2\mathbb{Z})^\mu$ the diagram $g \cdot D$ is also not connected.

For $g \in \mathcal{S}_\mu$ and for $g \in (\mathbb{Z}/2\mathbb{Z})^\mu$ this is clear, hence it remains to prove that for all $1 \leq i,j \leq \mu$, $i \neq j$ the diagram $\rho_j^i \cdot D$ is also not connected. Set $\underline{\delta'} = \rho_j^i \cdot \underline{\delta}$, thus

$$\delta'_k = \delta_k \text{ for } k \neq j, \quad \text{and} \quad \delta'_j = \delta_j - \eta(\delta_j,\delta_i)\delta_i.$$

Now assume $k \not\sim l$.

Case 1: $k,l \neq j$. Then

$$(\delta'_k,\delta'_l) = (\delta_k,\delta_l) = 0.$$

Case 2: $k = j$ (and therefore $l \neq j$): Then we have $j \not\sim l$, hence

$$(\delta'_k,\delta'_l) = (\delta_j - \eta(\delta_j,\delta_i)\delta_i,\delta_l) = (\delta_j,\delta_l) - \eta(\delta_j,\delta_i)(\delta_i,\delta_l) = -\eta(\delta_j,\delta_i)(\delta_i,\delta_l).$$

Now suppose $(\delta_j,\delta_i) \neq 0$ and $(\delta_i,\delta_l) \neq 0$. That means $j \sim i$ and $i \sim l$, but then also $j \sim l$, a contradiction. Hence we get also in this case that

$$(\delta'_k,\delta'_l) = 0.$$

Case 3: $l = j$. Same arguments as in Case 2. □

Remark 3.1.25. The proof shows in fact a little bit more: Consider the canonical mapping

$$\Phi : \mathrm{Gabr}_\mu \ltimes (\mathbb{Z}/2\mathbb{Z})^\mu \to \mathcal{S}_\mu$$

Then, if $g \in \mathrm{Gabr}_\mu \ltimes (\mathbb{Z}/2\mathbb{Z})^\mu$ and $J \subset \{1, \dots, \mu\}$ is the index set of a connected component of D, then $\Phi(g)(J)$ is the index set of a connected component of $g \cdot D$.

By this remark and Lemma 3.1.10 we immediately get:

Lemma 3.1.26. *Let D be a Coxeter-Dynkin diagram which has r connected components of size $\mu_1, \dots, \mu_r$. Then there exists an equivalent Coxeter-Dynkin diagram D' whose index sets of the connected components are*

$$\{1, \dots, \mu_1\},\ \{\mu_1 + 1, \dots, \mu_1 + \mu_2\},\ \dots,\ \{\mu_1 + \dots + \mu_{r-1} + 1, \dots, \mu\}.$$

In particular, the corresponding intersection matrix S', the Seifert matrix V' and the monodromy m' are block matrices with blocks $S'_1, \dots, S'_r$ resp. $V'_1, \dots, V'_r$ resp. $m'_1, \dots, m'_r$ of size $\mu_1, \dots, \mu_r$. These satisfy

$$S'_i = \eta(V'_i + \varepsilon V_i'^t), \quad m'_i = -\varepsilon(V_i'^t)^{-1} V'_i$$

□

3.2 Criteria for Definiteness

From now on we specialize to the symmetric case for which we choose $n \equiv 2 \bmod 4$, i.e.

$$\varepsilon = 1, \qquad \eta = -1.$$

All results are also true (with some changes of signs) for the case $n \equiv 0 \bmod 4$.

3.2.1 The results

The goal of this section is to show the following theorems:

Theorem 3.2.1. *The following statements are equivalent:*

(i) *S is (negative) semidefinite.*

(ii) *All subdiagrams of the Coxeter-Dynkin diagram D have quasiunipotent monodromy.*

(iii) *All subdiagrams of all Coxeter-Dynkin diagrams which are equivalent to D have quasiunipotent monodromy.*

(iv) *There exists a Coxeter-Dynkin diagram D' which is equivalent to D such that all subdiagrams of D' have quasiunipotent monodromy.*

If these conditions are true, then also the following statement holds:

(∗) *All Coxeter-Dynkin diagrams D' which are equivalent to D contain only lines with a weight of absolute value ≤ 2.*

Theorem 3.2.2. *The following statements are equivalent:*

(i) *S is (negative) definite.*

(ii) *All subdiagrams of the Coxeter-Dynkin diagram D have finite monodromy.*

(iii) *All subdiagrams of all Coxeter-Dynkin diagrams which are equivalent to D have finite monodromy.*

(iv) *There exists a Coxeter-Dynkin diagram D' which is equivalent to D such that all subdiagrams of D' have finite monodromy.*

(v) *The monodromy group of D has finite order.*

(vi) *The set of vanishing cycles is finite.*

(vii) *The set of weakly vanishing cycles is finite.*

If these conditions are true, then also the following statement holds:

(∗) *All Coxeter-Dynkin diagrams D' which are equivalent to D contain only lines with a weight of absolute value ≤ 1.*

We will also discuss the following conjectures:

Conjecture 1. *In Theorem 3.2.1 the statement* (∗) *is equivalent to the statements* (i) *to* (iv).

Conjecture 2. *In Theorem 3.2.2 the statement* (∗) *is equivalent to the statements* (i) *to* (vii).

Furthermore, we will prove the following weaker versions of the above conjectures:

Theorem 3.2.3. *Statements* (i) *to* (iv) *of Theorem 3.2.1 are equivalent to:*

(∗∗) *For each pair α, β of weakly vanishing cycles one always has $|(\alpha, \beta)| \leq 2$.*

Theorem 3.2.4. *Statements* (i) *to* (vii) *of Theorem 3.2.2 are equivalent to:*

(∗∗) *For each pair α, β of weakly vanishing cycles with $\alpha \neq \pm\beta$ one always has $|(\alpha, \beta)| \leq 1$.*

3.2.2 The case $\mu = 2$

In the case $\mu = 2$ the only possible intersection matrices are of the following form:

$$S = \begin{pmatrix} -2 & u \\ u & -2 \end{pmatrix},$$

with $u \in \mathbb{Z}$.

In this case we have the following lemma:

Lemma 3.2.5. *Assume that $\mu = 2$ and let $S = \left(\begin{smallmatrix} -2 & u \\ u & -2 \end{smallmatrix}\right)$. Then S' is equivalent to S if and only if S' is weakly equivalent to S if and only if $S' = S$ or $S' = \left(\begin{smallmatrix} -2 & -u \\ -u & -2 \end{smallmatrix}\right)$.*

Furthermore, in this case the following statements are equivalent:

(i) *S is (negative) semidefinite,*

(ii) *m is quasiunipotent,*

(iii) $|u| \leq 2$.

Also the following statements are equivalent:

(i') *S is (negative) definite,*

(ii') *m is finite,*

(iii') $|u| \leq 1$.

Proof. One has

$$\det(-S) = 4 - u^2,$$

thus by Proposition B.1.1 we get (i) ⇔ (iii) resp. (i)' ⇔ (iii)'.

We have

$$m = \begin{pmatrix} -1 & u \\ -u & u^2 - 1 \end{pmatrix}.$$

For $|u| \leq 2$ one easily sees that m is equivalent to the following Jordan matrices:

$$m_{u=0} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad m_{u=\pm 1} \cong \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}, \quad m_{u=\pm 2} \cong \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

where $\zeta_3 = e^{\frac{2\pi i}{3}}$ is a third primitive root of unity. In particular, m is finite for $|u| \leq 1$ and not finite for $|u| = 2$, but all m are quasiunipotent for $|u| \leq 2$.

The characteristic polynomial of m is

$$\chi(t) = t^2 + (2 - u^2)t + 1,$$

hence the eigenvalues of m are

$$\lambda_{1,2} = \tfrac{1}{2}u^2 - 1 \pm \tfrac{1}{2}u\sqrt{u^2 - 4}.$$

For $|u| \geq 3$ they are real with $\lambda_1 > 1$ and $\lambda_2 = \lambda_1^{-1}$, so m is not quasiunipotent (and even not finite) for $|u| \geq 3$. □

From this lemma we can derive the parts (i) $\Rightarrow$ $(*)$ of Theorem 3.2.1 and Theorem 3.2.2:

Proposition 3.2.6.

(i) *If S is semidefinite, all Coxeter-Dynkin diagrams D' which are equivalent to D contain only lines with a weight of absolute value ≤ 2.*

(ii) *If S is definite, all Coxeter-Dynkin diagrams D' which are equivalent to D contain only lines with a weight of absolute value ≤ 1.*

Proof. If S is (semi-)definite, also all intersection matrices S' which are equivalent to S are so, and also all submatrices $\tilde{S}'$ of S'. Hence the corollary follows from the previous lemma if one considers all (2×2)-submatrices $\tilde{S}'$. $\square$

3.2.3 The case $\mu = 3$

In this section we want to determine all intersection matrices for $\mu = 3$ which are semidefinite or definite. We also want to determine the orbits of the braid group for these matrices. By Proposition 3.2.6 the intersection matrices in question do only contain lines with a weight of absolute value ≤ 2, hence there are only a finite number (namely 125) of matrices we have to look at.

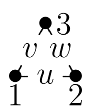
Each intersection matrix S is of the form

$$S = \begin{pmatrix} -2 & u & v \\ u & -2 & w \\ v & w & -2 \end{pmatrix}.$$

Thus we can identify $\mathcal{I}nt\mathcal{M}at_3$ with $\mathbb{Z}^3$ by mapping the above matrix to (u, v, w). With this identification the operation of the braid group is as follows:

$$\begin{aligned} \sigma_1(u,v,w) &= (-u, w, v + uw), & \sigma_1^{-1}(u,v,w) &= (-u, uv + w, v), \\ \sigma_2(u,v,w) &= (v, u + wv, -w), & \sigma_2^{-1}(u,v,w) &= (v + uw, u, -w), \\ \xi_1(u,v,w) &= (-u, -v, w), & \xi_2(u,v,w) &= (-u, v, -w), \\ \xi_3(u,v,w) &= (u, -v, -w). \end{aligned}$$

The Coxeter-Dynkin diagram corresponding to (u, v, w) is as follows:



In the following pictures the ordering of the vertices will be understood as in the diagram above.

First observe that

$$\xi_1 \sigma_1 \sigma_2(u, v, w) = (v, w, u),$$

thus the orbits under the operation of $\mathrm{Br}_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3$ are invariant under cyclic permutations of u, v and w. Furthermore, if one intersection number is zero (we can assume w.l.o.g. that $w = 0$ by the above), we have

$$\sigma_2(u, v, 0) = (v, u, 0),$$

hence in this case the orbit is invariant under arbitrary permutations. By these remarks we get:

Lemma 3.2.7. *Assume* $-2 \leq u, v, w \leq 2$. *Then in the orbit of* (u, v, w) *there is exactly one element* (u', v', w') *such that* $(|u'|, |v'|, |w'|)$ *is a permutation of* $(|u|, |v|, |w|)$, $uvwu'v'w' \geq 0$ *and*

$$u' \geq v' \geq |w'|.$$

Proof. If one of u, v or w is zero, then we can permute them to get them sorted by absolute values. One the other hand, if u, v and w are all nonzero, then two of them must have the same absolute value, hence we can sort them by absolute values by cyclic permutations.

After that, apply ξ_1, ξ_2 or ξ_3 if needed, to get the first two values positive.

The uniqueness follows easily. □

This lemma shows that to understand all the 125 cases of (u, v, w) with $-2 \leq u, v, w \leq 2$, it suffices to look at the 14 cases with $u \geq v \geq |w|$. Table 3.1 and Table 3.2 show all 125 Coxeter-Dynkin diagrams with lines of absolute value ≤ 2, sorted into their corresponding orbits under the operation of the braid group, together with the determinant of the (negative of the) intersection matrix and the Jordan matrix of the monodromy m.

From this one gets the following lemma which continues Lemma 3.2.5:

Lemma 3.2.8. *Assume* $\mu = 3$ *and let* $S \in \mathfrak{IntMat}_3$ *be an intersection matrix with Coxeter-Dynkin diagram* D. *In this case the following statements are equivalent:*

(i) *S is (negative) semidefinite,*

(ii) *The monodromies of all subdiagrams of D are quasiunipotent,*

(iii) *All Coxeter-Dynkin diagrams D' equivalent to D contain only lines with a weight of absolute value ≤ 2.*

Also the following statements are equivalent:

(i') *S is (negative) definite,*

(ii') *The monodromies of all subdiagrams of D are finite,*

(iii') *All Coxeter-Dynkin diagrams D' equivalent to D contain only lines with a weight of absolute value ≤ 1.*

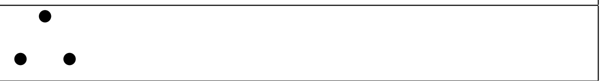
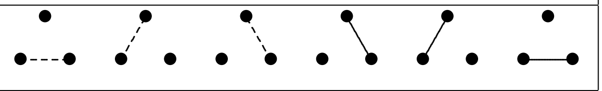
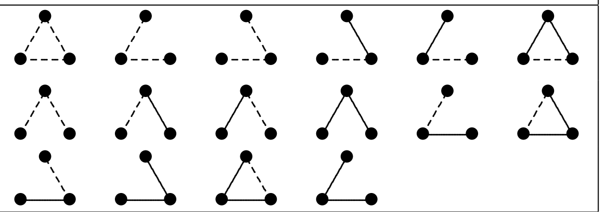
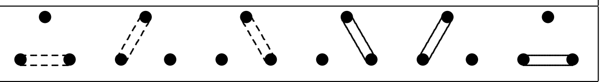
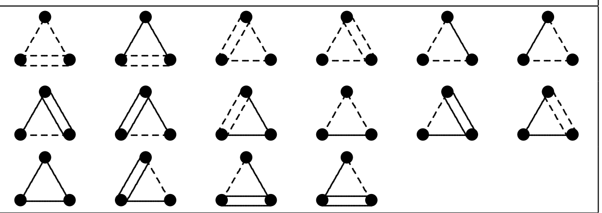
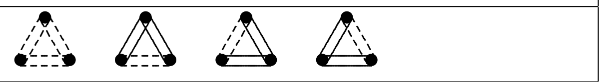
det$(-S)$	Jordan(m)	tr m	Coxeter-Dynkin diagrams
8	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	-3	
6	$\begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3^{-1} & 0 \\ 0 & 0 & -1 \end{pmatrix}$	-2	
4	$\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -1 \end{pmatrix}$	-1	
0	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	1	
0	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	1	
0	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	1	

Table 3.1: Orbits under the operation of $\mathrm{Br}_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3$, Part 1

Proof. Let S be represented by (u, v, w) as above.

Suppose $-2 \leq u, v, w \leq 2$. In this case look at Table 3.1 and 3.2. The first three entries of Table 3.1 have definite intersection matrices and finite monodromies. The last three entries have semidefinite intersection matrices and quasiunipotent submonodromies. Furthermore, each orbit in exactly the last three cases contains a diagram with a line of weight 2. The entries in Table 3.2 have indefinite intersection matrices and the monodromy is not quasiunipotent. All orbits contain diagrams which contain a line with a weight ≥ 3. By all this, together with Lemma 3.2.5 (for the size-2-subdiagrams), the lemma follows.

On the other hand, if u, v or w have absolute value ≥ 3, then the monodromy of the corresponding size-2-subdiagram is not quasiunipotent by Lemma 3.2.5 and the corresponding 2×2 principal submatrix of S is not semidefinite. So again the lemma follows. □

Proposition 3.2.9. *Assume $\mu \geq 3$ and let $S \in \mathcal{I}nt\mathcal{M}at_\mu$ be an intersection matrix with Coxeter-Dynkin diagram D. Suppose D is connected. Then the following statements are equivalent:*

($*$) *All Coxeter-Dynkin diagrams D' equivalent to D contain only lines with a weight of absolute value ≤ 2.*

($*'$) *The weights of all lines of all Coxeter-Dynkin diagrams D' equivalent to D are bounded.*

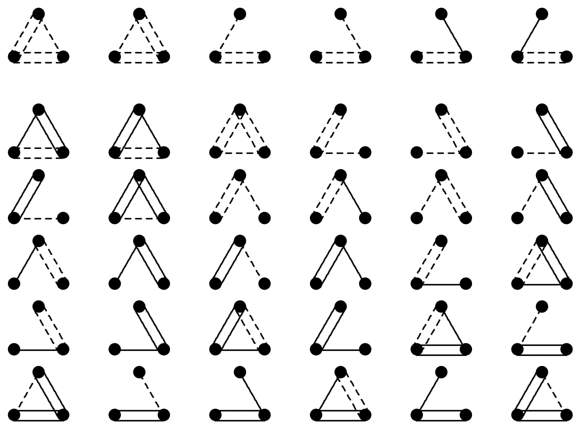
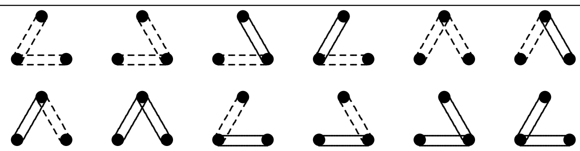
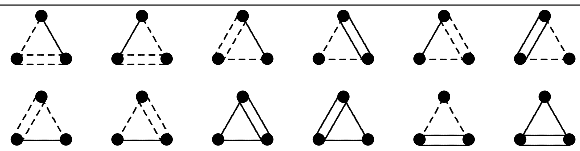
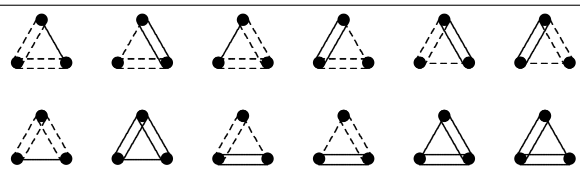
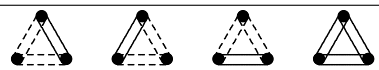
$\det(-S)$	Jordan(m) $\operatorname{tr} m$	Coxeter-Dynkin diagrams
-2	$\begin{pmatrix} \frac{3}{2}-\frac{1}{2}\sqrt{5} & 0 & 0 \\ 0 & \frac{3}{2}+\frac{1}{2}\sqrt{5} & 0 \\ 0 & 0 & -1 \end{pmatrix}$ 2	 + diagrams with lines of higher weight
-8	$\begin{pmatrix} 3-2\sqrt{2} & 0 & 0 \\ 0 & 3+2\sqrt{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$ 5	 + diagrams with lines of higher weight
-8	$\begin{pmatrix} 3-2\sqrt{2} & 0 & 0 \\ 0 & 3+2\sqrt{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$ 5	 + diagrams with lines of higher weight
-18	$\begin{pmatrix} \frac{11}{2}-\frac{3}{2}\sqrt{13} & 0 & 0 \\ 0 & \frac{11}{2}+\frac{3}{2}\sqrt{13} & 0 \\ 0 & 0 & -1 \end{pmatrix}$ 10	 + diagrams with lines of higher weight
-32	$\begin{pmatrix} 9-4\sqrt{5} & 0 & 0 \\ 0 & 9+4\sqrt{5} & 0 \\ 0 & 0 & -1 \end{pmatrix}$ 17	 + diagrams with lines of higher weight

Table 3.2: Orbits under the operation of $\mathrm{Br}_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3$, Part 2

Proof. $(*) \Rightarrow (*')$ is trivial.

For the converse, assume first that $\mu = 3$. Suppose $(*)$ does not hold. If S is represented by (u, v, w) as above, we can assume w.l.o.g. that $|u| \geq 3$ (first replace S by a equivalent S' with "high lines", then permute cyclicly if needed). We show that there exists (u', v', w') equivalent to (u, v, w) with $|u'| = |u|$ and $\max(|v'|, |w'|) > \max(|v|, |w|)$. Inductively we then can produce lines with an arbitrary high weight.

Since D is connected (and therefore all equivalent diagrams, too, by Lemma 3.1.24) we have $v \neq 0$ or $w \neq 0$.

Case 1: $|v| \geq |w|$. In this case set

$$(u', v', w') := \sigma_1^{-1}(u, v, w) = (-u, uv + w, -w).$$

Then

$$|v'| = |uv + w| \geq 3|v| - |w| \geq 2|v|.$$

Case 2: $|v| \leq |w|$. Now set

$$(u', v', w') := \sigma_1(u, v, w) = (-u, w, v + uw).$$

Then

$$|w'| = |v + uw| \geq -|v| + 3|w| \geq 2|w|.$$

Now assume $\mu > 3$. Assume again that $(*)$ does not hold. Then we can assume w.l.o.g. that there exists $1 \leq i, j \leq \mu$ with $S_{ij} \geq 3$ (in the same way as above). Since D is connected, there must exist a $1 \leq k \leq \mu$, $k \neq i, j$ such that $S_{ik} \neq 0$ or $S_{jk} \neq 0$. By Lemma 3.1.10 we can assume w.l.o.g. that $\{i, j, k\} = \{1, 2, 3\}$.

Now we can use the above for the principal submatrix containing the first three rows and columns of S to get arbitrary high entries in intersection matrices equivalent to S. □

Proposition 3.2.10. *Assume $\mu \geq 2$. For each $1 \leq i, j \leq \mu$, $i \neq j$ there exists an isomorphism*

$$\{S \in \mathcal{I}nt\mathcal{M}at_{\mu-1} \mid S \text{ semidefinite}\} \xrightarrow{\sim} \{S \in \mathcal{I}nt\mathcal{M}at_{\mu} \mid S \text{ semidefinite},\ S_{ij} = 2\}$$

Proof. We can assume w.l.o.g. that $i = \mu - 1$, $j = \mu$; the other cases are proven quite similarly.

Let S be a matrix of the right hand side, i.e. we can write

$$S = \begin{pmatrix} S_2 & y & z \\ y^t & -2 & 2 \\ z^t & 2 & -2 \end{pmatrix}$$

($S_2 \in \mathcal{I}nt\mathcal{M}at_{\mu-2}$, $y, z \in \mathbb{Z}^{\mu-2}$), and S is semidefinite.

Assume $k = 1, \ldots, \mu - 2$ and look at the principal submatrix

$$\begin{pmatrix} -2 & y_k & z_k \\ y_k & -2 & 2 \\ z_k & 2 & -2 \end{pmatrix}$$

of size 3 which must be semidefinite, too. Therefore the corresponding Coxeter-Dynkin diagram of this matrix must be contained in Table 3.1. From this table we get that there must be $z_k = -y_k$.

Together we get $z = -y$.

On the other hand, let S be an intersection matrix of the form

$$S = \begin{pmatrix} S_2 & y & -y \\ y^t & -2 & 2 \\ -y^t & 2 & -2 \end{pmatrix},$$

such that

$$S_1 := \begin{pmatrix} S_2 & y \\ y^t & -2 \end{pmatrix}$$

is semidefinite. Then S is semidefinite too: We have $\det(-S) = 0$, since the last two rows of S are linearly dependent. Moreover it follows from the assumptions that all proper principal submatrices are semidefinite, therefore S is semidefinite by Proposition B.1.1.

It follows that the mapping

$$S_1 = \begin{pmatrix} S_2 & y \\ y^t & -2 \end{pmatrix} \in \mathcal{I}nt\mathcal{M}at_{\mu-1} \mapsto \begin{pmatrix} S_2 & y & -y \\ y^t & -2 & 2 \\ -y^t & 2 & -2 \end{pmatrix} \in \mathcal{I}nt\mathcal{M}at_{\mu}$$

restricts to the desired isomorphism. □

3.2.4 The case $\mu \leq 6$

The case $\mu = 3$ could have been done by handwork. For $\mu \geq 4$ however a computer becomes handy since there are many cases to calculate.

However, we can first improve the situation before that. As we have seen in the case of $\mu = 3$, some kind of cyclic permutation does not leave the orbit under the operation of the braid group. This is a general fact:

Lemma 3.2.11. *(η, ε arbitrary.) Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_\mu)$ be a distinguished basis. Then $\underline{\alpha}' := (m_{\alpha_\mu}\alpha_\mu, m_{\alpha_\mu}\alpha_1, \ldots, m_{\alpha_\mu}\alpha_{\mu-1})$ is also one.*

In particular, since m_{α_μ} preserves the intersection product, one gets the intersection matrix $S_{\underline{\alpha}'}$ by cyclically permutating the rows and columns of $S_{\underline{\alpha}}$.

Proof. The proof is very similar to the proof of Lemma 3.1.13.

Consider the element $d = \sigma_1 \cdots \sigma_{\mu-1} \in \mathrm{Br}_\mu$. Let us see how this element operates on distinguished bases:

$$\begin{aligned}(\alpha_1, \alpha_2, \ldots, \alpha_{\mu-2}, \alpha_{\mu-1}, \alpha_\mu) &\xmapsto{\sigma_{\mu-1}} (\alpha_1, \alpha_2, \ldots, \alpha_{\mu-2}, \alpha_\mu, m_{\alpha_\mu}\alpha_{\mu-1}) \\ &\xmapsto{\sigma_{\mu-2}} (\alpha_1, \alpha_2, \ldots, \alpha_\mu, m_{\alpha_\mu}\alpha_{\mu-2}, m_{\alpha_\mu}\alpha_{\mu-1}) \\ &\vdots \\ &\xmapsto{\sigma_1} \quad (\alpha_\mu, m_{\alpha_\mu}\alpha_1, \ldots, m_{\alpha_\mu}\alpha_{\mu-2}, m_{\alpha_\mu}\alpha_{\mu-1}).\end{aligned}$$

In the antisymmetric case $m_{\alpha_\mu}\alpha_\mu = \alpha_\mu$, so we are ready. In the symmetric case $m_{\alpha_\mu}\alpha_\mu = -\alpha_\mu$, hence we get $\underline{\alpha}'$ by applying ξ_1 afterwards. □

The case $\mu = 4$

An intersection matrix of size 4 has the form

$$S = \begin{pmatrix} -2 & u & v & x \\ u & -2 & w & y \\ v & w & -2 & z \\ x & y & z & -2 \end{pmatrix}.$$

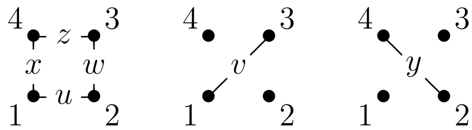
We identify it with the vector $(u, v, w, x, y, z) \in \mathbb{Z}^6$. With this identification the operation of the braid group is as follows:

$$\begin{aligned}
\sigma_1(u, v, w, x, y, z) &= (-u, w, v + uw, y, x + uy, z), \\
\sigma_1^{-1}(u, v, w, x, y, z) &= (-u, uv + w, v, ux + y, x, z), \\
\sigma_2(u, v, w, x, y, z) &= (v, u + vw, -w, x, z, y + wz), \\
\sigma_2^{-1}(u, v, w, x, y, z) &= (v + uw, u, -w, x, wy + z, y), \\
\sigma_3(u, v, w, x, y, z) &= (u, x, y, v + xz, w + yz, -z), \\
\sigma_3^{-1}(u, v, w, x, y, z) &= (u, x + vz, y + wz, v, w, -z), \\
\xi_1(u, v, w, x, y, z) &= (-u, -v, w, -x, y, z), \\
\xi_2(u, v, w, x, y, z) &= (-u, v, -w, x, -y, z), \\
\xi_3(u, v, w, x, y, z) &= (u, -v, -w, x, y, -z), \\
\xi_4(u, v, w, x, y, z) &= (u, v, w, -x, -y, -z).
\end{aligned}$$

Moreover the "cyclic permutation" of Lemma 3.2.11 is

$$\xi_1\sigma_1\sigma_2\sigma_3(u, v, w, x, y, z) = (x, y, u, z, v, w).$$

The Coxeter-Dynkin diagram corresponding to (u, v, w, x, y, z) consists of the following lines:



Again in the following pictures the ordering of the vertices will be understood as above.

In comparison to Lemma 3.2.7 for the case $\mu = 3$ we get here:

Lemma 3.2.12. *In the orbit of (u, v, w, x, y, z) there is an element (u', v', w', x', y', z') which is the result of applying some number of "cyclic permutations" $\xi_1\sigma_1\sigma_2\sigma_3$ and "sign-changes" ξ_1, ξ_2, ξ_3 and ξ_4 such that one of the following conditions is satisfied:*

$$\begin{aligned}
&u = 0,\ w = 0,\ z = 0,\ x = 0,\ v \geq y \geq 0 && \textit{or} \\
&u > 0,\ w = 0,\ z = 0,\ x = 0,\ v \geq 0,\ y \geq 0 && \textit{or} \\
&u > 0,\ w > 0,\ z = 0,\ x = 0,\ y \geq 0 && \textit{or} \\
&u > 0,\ w = 0,\ z > 0,\ x = 0,\ u \geq z && \textit{or} \\
&u > 0,\ w > 0,\ z > 0.
\end{aligned}$$

Proof. First consider the "outer lines" u, w, z, x of the Coxeter-Dynkin diagram: The number of nonzero outer lines is 0, 1, 2, 3 or 4. By rotating the diagram (that is exactly what $\xi_1\sigma_1\sigma_2\sigma_3$ does) one can obtain that the position of the nonzero lines is `----`, `x---`, `xx--`, `x-x-`, `xxx-` or `xxxx`. In the first case we can do an additional rotation to interchange v and y, in the fourth case we can interchange u and z by a double rotation. After that one can apply the sign-changes to achieve positivity of certain lines. □

However, even with this kind of "normal form" there are still 1159 possibilities in question for $-2 \leq u, v, w, x, y, z \leq 2$. Moreover, the above normal form is not unique. This improves when we restrict to the case $-1 \leq u, v, w, x, y, z \leq 1$. There are 49 normal forms in the sense of the lemma for this case. Then delete for those diagrams which do not have a unique normal form all but one diagram. (For example, the two diagrams

are normal forms in the sense of the lemma, but the second one gets by applying $\xi_1\xi_2$ to the first.) As a result we get the 36 diagrams in Figure 3.1.

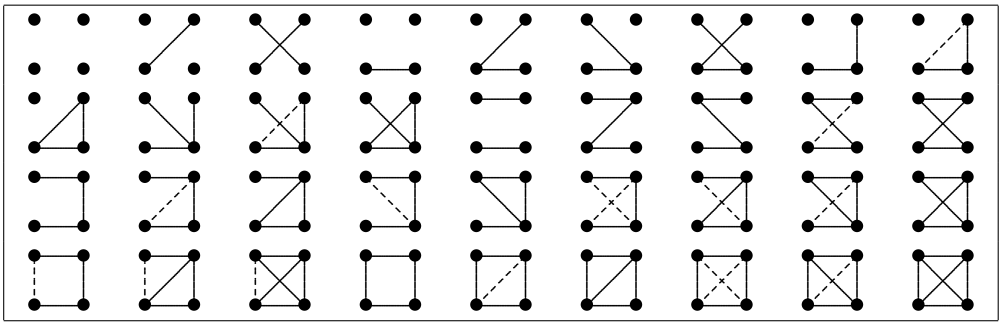
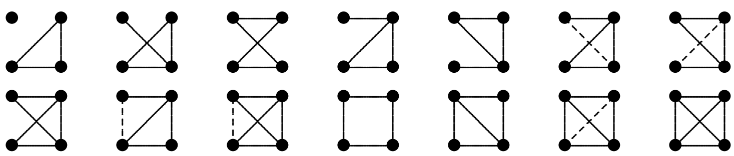


Figure 3.1: Normal forms for the case $-1 \leq u, v, w, x, y, z \leq 1$

We will later show that Conjecture 1 is a consequence of Conjecture 2. So in order to prove both conjectures for the case $\mu = 4$ we only have to prove Conjecture 2. For this we have to do the following: For each of the 36 Coxeter-Dynkin diagrams of Figure 3.1 such that S is not definite we have to prove that there is an equivalent diagram with a line with a weight of absolute value ≥ 2. There are exactly 14 diagrams of these such that S is not definite:



It is easy to show that for each of these diagrams one can produce an equivalent diagram with a line of weight 2 by the operation of the braid group.

Of course Conjecture 1 also can be checked directly for the case $\mu = 4$, but here are many more Coxeter-Dynkin diagrams to check.

Table 3.3 shows the classification (computed with aid of a computer) of all Coxeter-Dynkin diagrams of size 4 with semidefinite intersection matrix by the operation of the braid group. For each orbit only typical diagrams are shown — all other diagrams of this orbit can be produced by cyclic permutation and sign-changes applied to the given ones. The list of the eigenvalues of m also shows the Jordan blocks: All eigenvalues of one Jordan block are put in parentheses. For example, $(1,1),(1,1)$ stands for the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that the two orbits with $\det(-S) = 4$ (one orbit consists of the Coxeter-Dynkin diagram of a singularity of type D_4) show that the orbits are *not* stable under permutations of the vertices.

The case $\mu = 5$

An intersection matrix $S \in \mathcal{I}nt\mathcal{M}at_5$ can be identified with the vector

$$(S_{12}, S_{13}, S_{23}, S_{14}, S_{24}, S_{34}, S_{15}, S_{25}, S_{35}, S_{45}) \in \mathbb{Z}^{10}.$$

Similarly to Lemma 3.2.7 and Lemma 3.2.12 we can find in each orbit under the operation braid group some kind of normal form by rotating the Coxeter-Dynkin diagram (which we can do by Lemma 3.2.11) and by changing the sign of some vertices: Consider the "outer" lines $S_{12}, S_{23}, S_{34}, S_{45}, S_{15}$ of the Coxeter-Dynkin diagram. By rotating the diagram we can achieve that the nonzero lines are at the following positions: `-----`, `x----`, `xx---`, `x-x--`, `xxx--`, `xx-x-`, `xxxx-`, `xxxxx`. Then we can make in all but the last cases these lines positive by changing the signs of some vertices if needed. In the last case we can make the first four lines positive. After that we can also ask for the positivity of some "inner" lines, but we will not do this here since too many cases would arise (however, this leaves us with a larger number of normal forms).

The number of normal forms is still quite large, so at least from now on we should use a computer for the calculations.

To prove the conjectures for $\mu = 5$ we again only have to prove the second conjecture. Hence we have to check all diagrams with lines with a weight of absolute value ≤ 1 whose intersection matrix is not definite. We can restrict to normal forms in the above sense. There exist 2817 normal forms with lines with a weight of absolute value ≤ 1 of which 1432 are not definite. We have to show that for all these diagrams there exists an equivalent diagram with a line of weight 2. However, from Lemma 3.1.10 the following remark follows:

Remark 3.2.13. Suppose D is a Coxeter-Dynkin diagram. Suppose D contains a subdiagram $\tilde{D}$, and $\tilde{D}'$ is a diagram equivalent to $\tilde{D}$. Then there exists a diagram D' which is equivalent to D and contains $\tilde{D}'$ an a subdiagram.

In particular, a Coxeter-Dynkin diagram has an equivalent diagram with a line of weight 2 if it contains a subdiagram for which there is an equivalent diagram with a line of weight 2.

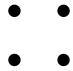
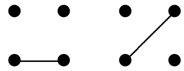
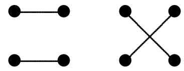
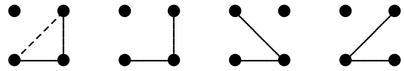
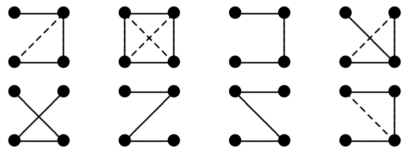
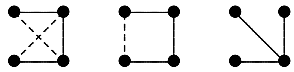
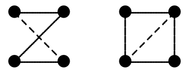
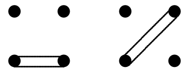
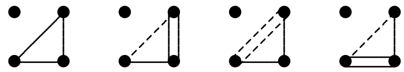
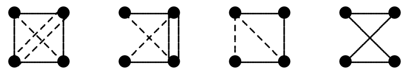
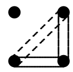
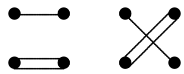
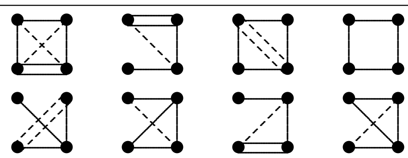
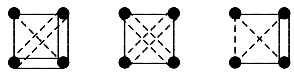
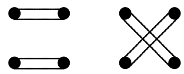
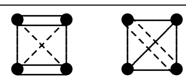
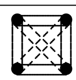
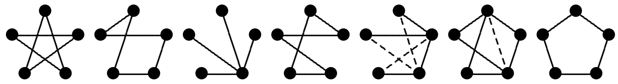
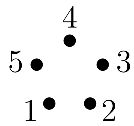
$\det(-S)$	eig.val.(m)	$\operatorname{tr} m$	Coxeter-Dynkin diagrams
16	$-1, -1, -1, -1$	-4	
12	$\zeta_3, \zeta_3^{-1}, -1, -1$	-3	
9	$\zeta_3, \zeta_3^{-1}, \zeta_3, \zeta_3^{-1}$	-2	
8	$i, -i, -1, -1$	-2	
5	$\zeta_5, \zeta_5^2, \zeta_5^{-2}, \zeta_5^{-1}$	-1	
4	$\zeta_6, \zeta_6^{-1}, -1, -1$	-1	
4	$i, -i, i, -i$	0	
0	$(1,1), -1, -1$	0	
0	$(1,1), -1, -1$	0	
0	$(1,1), -1, -1$	0	
0	$1, 1, -1, -1$	0	
0	$(1,1), \zeta_3, \zeta_3^{-1}$	1	
0	$(1,1), \zeta_3, \zeta_3^{-1}$	1	
0	$1, 1, \zeta_3, \zeta_3^{-1}$	1	
0	$(1,1), (1,1)$	4	
0	$(1,1), (1,1)$	4	
0	$(1,1), 1, 1$	4	

Table 3.3: Orbits under the operation of $\mathrm{Br}_4 \ltimes (\mathbb{Z}/2\mathbb{Z})^4$

By the previous case $\mu = 4$ we only need to check diagrams where the intersection matrix S is not definite, but where all principal submatrices of size 4 of S are definite. Of all normal forms there remain 30 cases which satisfy this condition. After deleting all unnecessary diagrams which can be produced by applying cyclic permutations and sign-changes to already given ones, the following 7 cases remain:



Here the ordering of the vertices is understood as follows:



For these diagrams it is easy to show that there exists an equivalent diagram with a line of weight 2, hence the conjectures are proven for $\mu = 5$.

To classify all Coxeter-Dynkin diagrams with semidefinite intersection form we can restrict to connected diagrams. The classification is shown in the appendix, Section C. As in the case $\mu = 4$ only some special elements of the orbits are shown, all others can be produced by cyclic permutations and sign-changes.

The case $\boldsymbol{\mu = 6}$

From now on the number of diagrams is quite large even for a computer. Even the normal form problem (in the same sense as in the previous cases) becomes rather complex: For the nonzero outer lines of the Coxeter-Dynkin diagrams we get the following possibilities after identifying by cyclic permutations: `------`, `x-----`, `xx----`, `x-x---`, `x--x--`, `xxx---`, `xx-x--`, `x-xx--`, `x-x-x-`, `xxxx--`, `xxx-x-`, `xx-xx-`, `xxxxx-`, `xxxxxx`.

As in the previous case for $\mu = 5$, in order to prove the conjectures for $\mu = 6$ we have to check all Coxeter-Dynkin diagrams where the intersection matrix S is not definite, but where each principal submatrix of size 5 of S is definite. After the identifications by cyclic permutations and sign-changes there remain 110 diagrams to check which are to be found in the appendix, Section C. For all these diagrams an equivalent diagram with a line of weight 2 exists which proves the conjectures for $\mu = 6$.

3.2.5 The proof of Theorem 3.2.1

Lemma 3.2.14. *Suppose* $\det(-S) < 0$. *Then the monodromy* m *has an eigenvalue* λ *with* $\lambda \in \mathbb{R}$ *and* $\lambda > 1$.

In particular, if $\det(-S) < 0$, then m is not quasiunipotent.

Proof. Let $\lambda_1, \ldots, \lambda_\mu$ be the eigenvalues of m. From equations (3.4) and (3.5) (see Section 3.1.1) we get

$$S = V^t(m - \mathbb{1}).$$

Since $\det V = 1$, we have

$$\det(-S) = \det(\mathbb{1} - m) = \prod_{i=1}^{\mu}(1 - \lambda_i).$$

Now S is a real (in fact integer) matrix, hence the eigenvalues of m are either real, or they occur in complex conjugated pairs. Since always $(1-\lambda)(1-\overline{\lambda}) \geq 0$, it follows from $\det(-S) < 0$ that a real eigenvalue λ with $1 - \lambda < 0$ must exist. □

Remark 3.2.15. In the case of an isolated singularity it follows that we always have

$$\det(-S) \geq 0,$$

since in this case the monodromy is always quasiunipotent, see Section 1.7.

From this lemma we derive the following proposition which is part of Theorem 3.2.1:

Proposition 3.2.16. *If S has a positive eigenvalue (i.e. if S is not semidefinite), then the Coxeter-Dynkin diagram contains a subdiagram with non-quasiunipotent monodromy.*

Proof. Since $-S$ is not positive definite, it follows from Proposition B.1.1 that there must exist a principal submatrix $-\tilde{S}$ from $-S$ with $\det(-\tilde{S}) < 0$. Hence the corresponding monodromy $\tilde{m}$ is not quasiunipotent as it follows from the previous lemma. □

This proposition proves the part (ii) ⇒ (i) of Theorem 3.2.1. To prove the opposite, we need some preparation.

Lemma 3.2.17. *Let $P \in \mathbb{Z}[X]_{\text{monic}}$ be an integer and monic polynomial. If for every complex zero x of $P_{\mathbb{C}}$ one has that $|x| \leq 1$, then every zero of $P_{\mathbb{C}}$ is in fact a root of unity (i.e. P is the product of cyclotomic polynomials).*

Proof. Set $d := \deg P$ and let $x_1, \ldots, x_d$ be the zeros of $P_{\mathbb{C}}$. Then, for each $k \in \mathbb{Z}_{\geq 1}$, the set $\{x_1^k, \ldots, x_d^k\}$ is again the set of (complex) zeros of an integer and monic polynomial $(P_k)_{\mathbb{C}}$, $P_k \in \mathbb{Z}[X]_{\text{monic}}$ of the same degree d: Write

$$\begin{aligned} P(x) &= x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 &&= \prod_{i=1}^{d}(x - x_i), \\ P_k(x) &= x^d + a_{k,d-1}x^{d-1} + \cdots + a_{k,1}x + a_{k,0} &&= \prod_{i=1}^{d}(x - x_i^k). \end{aligned}$$

The coefficients of these polynomials are the values of the elementary symmetric polynomials applied to the zeros, i.e.

$$a_l = (-1)^l \sum_{1 \leq j_1 < \cdots < j_l \leq d} x_{j_1} \cdots x_{j_d},$$

and therefore

$$a_{k,l} = (-1)^l \sum_{1\le j_1<\cdots<j_l\le d} x_{j_1}^k \cdots x_{j_d}^k = (-1)^l \Big(\sum_{1\le j_1<\cdots<j_l\le d} x_{j_1}\cdots x_{j_d}\Big)^k + [\dots]$$
$$= a_l^k + [\dots]$$

where it is not difficult to see that the term $[\dots]$ is an integer polynomial expression in $a_{l-1}, \dots, a_0$.

Now, since by assumption $|x_i^k| \le 1$ for all $i = 1, \dots, d$, the coefficients $a_{i,k}$ of P_k are bounded (in fact, $|a_{i,k}| \le d$ for all i, k). That means that the set $\{P_k \mid k \in \mathbb{N}\}$ is finite. Hence there is an infinite set $\{k_1, k_2, \dots\} \subset \mathbb{N}$ (with $k_p \neq k_q$ for $p \neq q$) such that $P_{k_p} = P_{k_q}$ for all p, q, i.e. $\{x_1^{k_p}, \dots, x_d^{k_p}\} = \{x_1^{k_q}, \dots, x_d^{k_q}\}$ for all p, q. Moreover, it follows from this that there must exist $p \neq q$ with $(x_1^{k_p}, \dots, x_d^{k_p}) = (x_1^{k_q}, \dots, x_d^{k_q})$, i.e. $x_i^{k_p-k_q} = 1$ for all $i = 1, \dots, d$. □

From this it follows:

Lemma 3.2.18. *An integer quadratic matrix* $A \in \mathrm{Mat}(\mu \times \mu, \mathbb{Z})$ *is quasiunipotent if and only if all (complex) eigenvalues* λ *of* $A_\mathbb{C}$ *satisfy* $|\lambda| \le 1$.

Proof. The characteristic polynomial χ of A is integer and monic, hence the lemma follows from the previous lemma. □

Lemma 3.2.19. *Let* $A \in GL(\mu, \mathbb{C})$ *and let* M *be defined as* $M = -(A^t)^{-1}A$. *Then* λ *is an eigenvalue of* M *if and only if* $\det(A + \lambda A^t) = 0$.

Proof. $\det(A + \lambda A^t) = \det(-A^t)\det(M - \lambda \mathbb{1})$. □

Now, with Lemma 3.2.18, the part (i) ⇒ (ii) of Theorem 3.2.1 is a special case of the following proposition:

Proposition 3.2.20. *Let* $A \in GL(\mu, \mathbb{R})$ *and let* M *be defined as* $M = -(A^t)^{-1}A$. *If* $A + A^t$ *is semidefinite, then for every eigenvalue* λ *of* M *one has* $|\lambda| = 1$.

Proof. Case 1: $A + A^t$ is definite. This is equivalent to:

$$v^t A v \neq 0 \quad \forall v \in \mathbb{R}^\mu \setminus \{0\}. \tag{3.11}$$

If λ is an eigenvalue of M, by Lemma 3.2.19 there exists a vector $v \in \mathbb{C}^\mu \setminus \{0\}$ such that

$$(A + \lambda A^t)v = 0. \tag{3.12}$$

Since it does not matter if we multiply v by a complex number $\neq 0$, we can assume w.l.o.g. that

$$\mathrm{re}\, v \neq 0. \tag{3.13}$$

From (3.12) we get

$$0 = v^t(A + \lambda A^t)v = (1 + \lambda)v^t A v.$$

Now either $\lambda = -1$ which would finish the proof, or we have

$$v^t A v = 0. \tag{3.14}$$

In the same way we have

$$0 = \overline{v}^t (A + \lambda A^t) v = \overline{v}^t A v + \lambda \overline{\overline{v}^t A v},$$

hence

$$0 = \sigma + \lambda \overline{\sigma} \quad \text{with} \quad \sigma = \overline{v}^t A v. \tag{3.15}$$

By (3.11) and (3.13) we now get

$$0 \neq (\operatorname{re} v)^t A (\operatorname{re} v) = \tfrac{1}{4}(v^t A v + \overline{v}^t A v + v^t A \overline{v} + \overline{v}^t A \overline{v}) = \tfrac{1}{4}(0 + \sigma + \overline{\sigma} + 0) = \tfrac{1}{2} \operatorname{re} \sigma.$$

In particular, we have that $\sigma \neq 0$, hence by (3.15) it follows that

$$|\lambda| = \left| \frac{-\sigma}{\overline{\sigma}} \right| = 1.$$

<u>Case 2:</u> $A + A^t$ is semidefinite.

W.l.o.g. let $A + A^t$ be positive semidefinite (if not take $-A$ instead of A).

Define for $t \in \mathbb{C}$:

$$A_t := A + t \cdot \mathbb{1}_\mu,$$
$$\varphi(\lambda, t) := \det(A_t + \lambda A_t^t).$$

If $t \in]0, \varepsilon[$ (ε short enough such that A has no eigenvalue in $]-\varepsilon, 0]$) we have that A_t is invertible and $A_t + A_t^t = A + A^t + 2t \cdot \mathbb{1}_\mu$ is positive definite.

From Case 1 it follows now that

$$\varphi(\lambda, t) = 0 \Rightarrow |\lambda| = 1 \quad (t \in]0, \varepsilon[).$$

Since φ is analytic in λ and t, and $\varphi(\cdot, 0)$ is not constantly zero, it follows from the continuity of the zeros (e.g. Weierstraß' Preparation Theorem) that also

$$\varphi(\lambda, 0) = 0 \Rightarrow |\lambda| = 1$$

which proves the proposition. □

Collecting the above, we get Theorem 3.2.1:

Proof of Theorem 3.2.1. (i) ⇒ (ii) has been proven by the above proposition.

(ii) ⇒ (i) has been proven by Proposition 3.2.16.

But since statement (i) does not depend on the (distinguished) basis, the equivalence of (i) and (ii) also shows the equivalence of (i) and (iii) resp. of (i) and (iv).

The part (i) ⇒ (∗) has been proven by Proposition 3.2.6. □

3.2.6 The proof of Theorem 3.2.2

Lemma 3.2.21. *Let* $A \in GL(\mu, \mathbb{R})$ *be an invertible matrix such that for* $B := A + A^t$ *one has* $\dim\ker B = 1$*. Then there exist* $v, w \in \mathbb{R}^\mu \setminus \{0\}$ *such that*

$$Bv = 0, \quad Bw = Av. \tag{3.16}$$

Proof. Since B is a real symmetric matrix one can find a $C \in GL(\mu, \mathbb{R})$ such that

$$B' := CBC^t = \operatorname{diag}(b_1, \dots, b_{\mu-1}, 0), \qquad b_1, \dots, b_{\mu-1} \neq 0.$$

(In fact C could be chosen in $O(\mu, \mathbb{R})$, then $b_1, \dots, b_{\mu-1}, 0$ would be the eigenvalues of B; or by Sylvester's Law of Inertia C could be chosen such that all b_i ($i = 1, \dots, \mu-1$) have values ± 1.)

Set $A' := CAC^t$ and $v' := (0, \dots, 0, 1)$. Then $B' = A' + A'^t$ and therefore $A'_{nn} = 0$. In particular, $(A'v')_n = 0$, i.e.

$$A'v' = (u_1, \dots, u_{\mu-1}, 0)$$

for some $u_1, \dots, u_{\mu-1} \in \mathbb{R}$. Set

$$w' := \Big(\frac{u_1}{b_1}, \dots, \frac{u_{n-1}}{b_{n-1}}, 0\Big).$$

Then $A'v' = B'w'$. Hence $v := C^{-1}v'(C^t)^{-1}$ and $w := C^{-1}w'(C^t)^{-1}$ satisfy (3.16). □

From this lemma we derive the following proposition:

Proposition 3.2.22. *If* $\dim\ker S = 1$*, then the monodromy* m *has a Jordan block of size* ≥ 2 *for the eigenvalue* 1*. In particular,* m *is not finite in this case.*

Proof. In the previous lemma take $A = -V$ (i.e. $B = S$). This gives us $v, w \neq 0$ such that

$$Sv = 0, \quad Sw = -Vv. \tag{3.17}$$

From equations (3.4) and (3.5) we get

$$(V^t)^{-1}S = m - \mathbb{1},$$

hence we get together with (3.17) that

$$(m - \mathbb{1})v = 0, \quad \text{and} \quad (m - \mathbb{1})w = -(V^t)^{-1}Vv = mv = v.$$

□

Proposition 3.2.23. *If* S *is definite, then the set of vanishing cycles and the set of weakly vanishing cycles are finite.*

Proof. Each (weakly) vanishing cycle δ has self-intersection $(\delta, \delta) = -2$.

Since S is negative definite, the set

$$\{v \in M_{\mathbb{R}} \mid (v, v) \geq -2\}$$

is compact (because in this case $-(\cdot, \cdot)$ is a scalar product on $M_{\mathbb{R}}$). Therefore

$$\{v \in M \mid (v, v) \geq -2\} = M \cap \{v \in M_{\mathbb{R}} \mid (v, v) \geq -2\}$$

is finite as a discrete and compact set. □

In a similar way as in Proposition 3.1.21 we get

Lemma 3.2.24. *If the set of vanishing cycles is finite, then m is finite.*

Proof. By Lemma 3.1.13 for a distinguished basis $(\delta_1, \dots, \delta_\mu)$ also

$$\big(m^k(\delta_1), \dots, m^k(\delta_\mu)\big)$$

are distinguished bases for all $k \in \mathbb{Z}$. But since there are only finitely many vanishing cycles, there are also only finitely many distinguished bases. Therefore there must exist a $k \neq 0$ with

$$m^k(\delta_i) = \delta_i \quad \forall i,$$

i.e. $m^k = \mathbb{1}$. □

We now can prove Theorem 3.2.2:

Proof of Theorem 3.2.2. (ii) ⇒ (i): Suppose that S is not definite. We have to show that there exists a subdiagram $\tilde{D}$ of D whose monodromy is not finite.

Case 1: S is even not semidefinite. By Theorem 3.2.1 we then get a subdiagram $\tilde{D}$ of D whose monodromy is not quasiunipotent, therefore also not finite.

Case 2: S is semidefinite, but not definite. Then S must have non-trivial kernel. Therefore S has a principal submatrix $\tilde{S}$ with $\dim\ker \tilde{S} = 1$. By Proposition 3.2.22 the monodromy of $\tilde{S}$ then is not finite.

(i) ⇒ (vii): This is Proposition 3.2.23.

(vii) ⇒ (vi) is trivial, since each vanishing cycle is also a weakly vanishing cycle.

(vi) ⇒ (ii): Consider a subdiagram $\tilde{D}$ of D. By Lemma 3.1.10 we can assume that $\tilde{D}$ consists of $\delta_1, \dots, \delta_{\tilde{\mu}}$ for some $\tilde{\mu} \leq \mu$. Then one gets the set of distinguished bases for $\tilde{D}$ by looking at the operation of the braid group $\mathrm{Br}_{\tilde{\mu}} \ltimes (\mathbb{Z}/2\mathbb{Z})^{\tilde{\mu}}$ which acts as a subgroup of $\mathrm{Br}_\mu \ltimes (\mathbb{Z}/2\mathbb{Z})^\mu$. By this one sees that the set of vanishing cycles of $\tilde{D}$ can be regarded as a subset of the vanishing cycles of D.

Now, if the set of vanishing cycles of D is finite, so is the set of vanishing cycles for each subdiagram of D by the above. By Lemma 3.2.24 we therefore get that the monodromy of each subdiagram must be finite.

(i) ⇔ (iii), (iv): We have shown the equivalence between (i) and (ii). But as in the proof of Theorem 3.2.1 this also shows the equivalence of (i) and (iii) resp. of (i) and (iv), since statement (i) does not depend on the (distinguished) basis.

(vii) ⇔ (v) is Proposition 3.1.21.

Again the part (i) ⇒ (∗) has been proven by Proposition 3.2.6. □

3.2.7 Conjecture 1 is a consequence of Conjecture 2

In this section we will prove that Conjecture 1 follows from Conjecture 2, thus these two conjectures are in fact only one conjecture.

The main tool to prove this is the "higher quasiinverse" QI_2 introduced in the appendix, Section B.1.3, and the following lemma:

Lemma 3.2.25. *Let $A \in \mathrm{Mat}(\mu \times \mu, \mathbb{R})$ be a symmetric matrix ($\mu \geq 2$) such that each proper principal submatrix of A is positive semidefinite. Write*

$$A = \begin{pmatrix} A_1 & x \\ x^t & a \end{pmatrix}$$

with $A_1 \in \mathrm{Mat}((\mu-1) \times (\mu-1), \mathbb{R})$, $x \in \mathbb{R}^{\mu-1}$ and $a \in \mathbb{R}$.

Then the following statements are equivalent:

(i) *A is (positive) semidefinite.*

(ii) $\det A \geq 0$.

(iii) *A satisfies one of the following conditions:*

  (a) *A_1 is invertible and $x^t A_1^{-1} x \leq a$.*

  (b) $\dim\ker A_1 = 1$ *and* $x \in \operatorname{im} A_1$.

  (c) $\dim\ker A_1 \geq 2$.

(iv) *There exists $z \in \mathbb{R}^{\mu-1}$ with $A_1 z = x$ such that $\alpha := x^t z$ satisfies $\alpha \leq a$ (in this case α is independent of the choice of z).*

Furthermore, if A satisfies the above conditions and $\det A_1 = 0$ *(i.e. if* (iii) (b) *or* (iii) (c) *is satisfied), then also* $\det A = 0$.

Proof. (i) $\Leftrightarrow$ (ii) is obvious (see Proposition B.1.1).

(ii) $\Leftrightarrow$ (iii): We use Lemma B.1.9 from which we get

$$\det A = a \det A_1 - x^t \widehat{A_1} x \tag{3.18}$$

where $\widehat{A_1}$ is the quasiinverse of A_1. We distinguish according to whether A_1 is invertible.

A_1 is invertible: In this case we have $\widehat{A_1} = \det A_1 \cdot A_1^{-1}$, hence

$$\det A = \det A_1 \cdot \big(a - x^t A_1^{-1} x\big).$$

Since A_1 is positive semidefinite (and invertible, hence in fact positive definite), we have $\det A_1 > 0$, and therefore

$$\det A \geq 0 \Leftrightarrow x^t A_1^{-1} x \leq a.$$

A_1 is not invertible: Since A_1 is positive semidefinite, also $\widehat{A_1}$ is positive semidefinite by Lemma B.1.8. But $\det A_1 = 0$, hence equation (3.18) becomes

$$\det A = -x^t \widehat{A_1} x,$$

and we get $\det A \leq 0$. Hence

$$\det A \geq 0 \Leftrightarrow \det A = 0 \Leftrightarrow x^t \widehat{A_1} x = 0 \Leftrightarrow \widehat{A_1} x = 0.$$

Now either $\dim \ker A_1 = 1$, then by Lemma B.1.8, (ker) (iii) we have

$$\widehat{A_1} x = 0 \Leftrightarrow x \in \operatorname{im} A_1,$$

or $\dim \ker A_1 \geq 2$, then by the same Lemma, (ker) (ii) we get $\widehat{A_1} = 0$, so in this case $\widehat{A_1} x = 0$ is always true.

We also see that in case (iii) (b) and (iii) (c) we get $\det A = 0$.

(iv) $\Rightarrow$ (iii) is trivial.

(i) $\Rightarrow$ (iv): If A_1 is invertible, then set $z := A_1^{-1} x$. By (iii) (a) (we already showed (i) $\Rightarrow$ (iii)) we have $\alpha \leq a$ (and in this case z is unique).

So, let A_1 be not invertible. Since A_1 is real symmetric, there exists an orthogonal matrix C such that

$$C^{-1} A_1 C = \operatorname{diag}(0, \ldots, 0, \lambda_1, \ldots, \lambda_s) =: D$$

(where $s = \operatorname{rank} A_1 = \mu - 1 - \dim \ker A_1$). Set $C^{-1} x = C^t x =: w$. Then we have

$$\begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} A_1 & x \\ x^t & a \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} C^{-1} A_1 C & C^{-1} x \\ x^t C & a \end{pmatrix} = \begin{pmatrix} D & w \\ w^t & a \end{pmatrix}.$$

If we can show (iv) for this matrix now, then (iv) also follows for A: If $z \in \mathbb{R}^{\mu-1}$ satisfies $Dz = w$ then set $y = Cz$. Then we have $A_1 y = CDC^{-1}Cz = CDz = Cw = x$ and $\lambda = x^t y = (Cw)^t (Cz) = w^t z$.

Now look at the following $(s+2) \times (s+2)$ principal submatrices of $\left(\begin{smallmatrix} D & w \\ w^t & a \end{smallmatrix}\right)$ which are positive semidefinite (since A is positive semidefinite):

$$\begin{pmatrix} 0 & & & & w_k \\ & \lambda_1 & & & w_{\mu-s} \\ & & \ddots & & \vdots \\ & & & \lambda_s & w_{\mu-1} \\ w_k & w_{\mu-s} & \cdots & w_{\mu-1} & a \end{pmatrix}$$

where $1 \leq k \leq \mu - s - 1$ and $\lambda_i \neq 0$ $(i = 1, \ldots, s)$. Since we already showed (i) $\Rightarrow$ (iii), these matrices must satisfy (iii) (b), therefore $(w_k, w_{\mu-s}, \ldots, w_{\mu-1})^t \in \operatorname{im} \operatorname{diag}(0, \lambda_1, \ldots, \lambda_s)$, i.e. $w_k = 0$ for all $1 \leq k \leq \mu - s - 1$. Together it follows $w \in \operatorname{im} D$. Hence there exists a $z \in \mathbb{R}^{\mu-1}$ with $Dz = w$ (and one can see that $w^t z$ is in fact independent of the choice of z).

Now look at the following $(s+1)\times(s+1)$ principal submatrix of $\left(\begin{smallmatrix} D & w \\ w^t & a \end{smallmatrix}\right)$:

$$\begin{pmatrix} \lambda_1 & & & w_{\mu-s} \\ & \ddots & & \vdots \\ & & \lambda_s & w_{\mu-1} \\ w_{\mu-s} & \cdots & w_{\mu-1} & a \end{pmatrix}.$$

This matrix now must satisfy (iii) (a), i.e. we get the desired inequality

$$w^t z = w^t\big(\operatorname{diag}(0,\ldots,0,\lambda_1^{-1},\ldots,\lambda_s^{-1})\big)w \le a.$$

□

Inductively from the last remark of this lemma one gets:

Lemma 3.2.26. *Let* $A \in \operatorname{Mat}(\mu\times\mu,\mathbb{R})$ *be a symmetric matrix* $(\mu \ge 2)$ *such that each proper principal submatrix of* A *is positive semidefinite. If* A *contains a principal submatrix* $\tilde{A}$ *with* $\det\tilde{A} = 0$*, then* $\det A = 0$. □

An immediate consequence of this lemma is the following:

Proposition 3.2.27. *If* S *is semidefinite and the Coxeter-Dynkin diagram* D *contains a line of weight* ± 2*, then* $\det S = 0$.

Proof. Consider the 2×2 principal submatrix of S which contains the line of weight ± 2. It is $\left(\begin{smallmatrix} -2 & \pm 2 \\ \pm 2 & -2 \end{smallmatrix}\right)$ which has determinant 0. By the previous lemma we get $\det S = 0$. □

However, we need to improve this result, since we do not want to assume yet that S is semidefinite. As said above, the main preparations for this are done in the appendix, Section B.1.3.

Proposition 3.2.28. *Suppose that each proper principal submatrix of* S *is semidefinite. If the Coxeter-Dynkin diagram* D *contains a line of weight* ± 2 *in this case, then* $\det S = 0$*. In particular, in this case* S *itself is semidefinite.*

Proof. The case $\mu = 2$ is trivial.

For the case $\mu = 3$ this follows from Section 3.2.3 — all cases in question are listed in Table 3.1.

Now assume $\mu \ge 4$. We can assume w.l.o.g. that the line of weight ± 2 connects the first and second vertex by Lemma 3.1.10.

Write

$$S = \begin{pmatrix} S_2 & y & z \\ y & -2 & u \\ z & u & -2 \end{pmatrix}$$

with $S_2 \in \operatorname{Mat}((\mu-2)\times(\mu-2),\mathbb{Z})$, $y, z \in \mathbb{Z}^{\mu-2}$ and $u \in \mathbb{Z}$. By Lemma B.1.13 we have

$$\begin{aligned} \det(-S) &= (4-u^2)\det(-S_2) + \mathrm{QI}_2(z,y,-S_2,y,z) \\ &\quad - 2y^t\widehat{(-S_2)}y - 2z^t\widehat{(-S_2)}z - 2uy^t\widehat{(-S_2)}z. \end{aligned} \tag{3.19}$$

Now look at the submatrices

$$S_1 = \begin{pmatrix} S_2 & y \\ y^t & -2 \end{pmatrix} \quad \text{and} \quad S_1' = \begin{pmatrix} S_2 & z \\ z^t & -2 \end{pmatrix}$$

which are semidefinite by assumption. By the previous proposition we get $\det S_1 = \det S_1' = 0$.

Now, if $\dim\ker S_1 \geq 2$ or $\dim\ker S_1' \geq 2$, we get from Lemma 3.2.25 (iii) (c) that $\det S = 0$.

The remaining case is $\dim\ker S_1 = \dim\ker S_1' = 1$. By applying Lemma 3.2.25 to S_1 and S_1' we get $y \in \operatorname{im} S_2$ resp. $z \in \operatorname{im} S_2$. By Lemma B.1.8 (ker) it follows $y, z \in \ker \widehat{S_2}$. Also, we get by Lemma B.1.17 (ker) that $\mathrm{QI}_2(y, x, S_2, y, z) = 0$. Hence all terms of the right hand side of equation (3.19) vanish, therefore $\det S = 0$. □

We now state for each of both conjectures an equivalent formulation:

Conjecture 1a. *Assume $\mu \geq 3$. Suppose S has the property that each proper principal submatrix $\tilde{S}'$ of each intersection matrix S' which is equivalent to S is semidefinite. Then S itself is semidefinite.*

Conjecture 2a. *Assume $\mu \geq 3$. Suppose S has the property that each proper principal submatrix $\tilde{S}'$ of each intersection matrix S' which is equivalent to S is definite. Then S itself is definite.*

Proof of the equivalence of Conjecture 1 and Conjecture 1a. Conjecture 1 ⇒ Conjecture 1a: Suppose that Conjecture 1 holds and let S be an intersection matrix such that each proper principal submatrix $\tilde{S}'$ of each intersection matrix S' which is equivalent to S is semidefinite. By Proposition 3.2.6 we then get that each corresponding proper subdiagram $\tilde{D}'$ contains only lines with a weight of absolute value ≤ 2. But then of course each diagram D' contains only such lines and by Conjecture 1 we get that S is semidefinite.

Conjecture 1a ⇒ Conjecture 1: Suppose conversely that Conjecture 1a holds. We then prove Conjecture 1 by induction on μ. The case $\mu = 2$ is true by trivial reasons (see Lemma 3.2.5). Now let S be an intersection matrix such each Coxeter-Dynkin diagram D' which is equivalent to D contains only lines with a weight of absolute value ≤ 2. By induction hypothesis we can then apply Conjecture 1 to each proper subdiagram $\tilde{D}'$ of a D', hence we get that each $\tilde{S}'$ is semidefinite. By Conjecture 1a we get that S is semidefinite. □

Proof of the equivalence of Conjecture 2 and Conjecture 2a. This is the same as the previous proof with "1" instead of "2" and "definite" instead of "semidefinite". □

Now we are prepared to prove that Conjecture 1 is a consequence of Conjecture 2.

Proof of Conjecture 2 $\Rightarrow$ Conjecture 1. Suppose that Conjecture 2 holds. We then show that Conjecture 1a holds. So, assume $\mu \geq 3$ and let S be an intersection matrix such that each proper principal submatrix $\tilde{S}'$ of each intersection matrix S' which is equivalent to S is semidefinite. We have to show that S is semidefinite.

First note that in this case all D' contain only lines with a weight of absolute value ≤ 2 (see the proof of the equivalence of Conjecture 1 and Conjecture 1a).

Case 1: One of the D' contains a line of weight ± 2. Then by Proposition 3.2.28 we get that $\det S = 0$, particularly that S is semidefinite.

Case 2: All D' only contain lines with a weight of absolute value ≤ 1. Then S is even definite by Conjecture 2. □

3.2.8 The weaker versions of the conjectures

In this section we will formulate and prove two theorems (namely Theorem 3.2.43 and Theorem 3.2.44) which are weaker versions of Conjecture 1 and Conjecture 2.

Definition 3.2.29. A subset $A \subset M$ of cycles is called an *admissible family of cycles* if the following properties hold:

(i) For all $\alpha \in A$ one has $(\alpha, \alpha) = -2$.

(ii) A is a generating set for $M \otimes \mathbb{Q}$.

(iii) If $\alpha \in A$, then also $-\alpha \in A$.

(iv) If $\alpha, \beta \in A$ with $(\alpha, \beta) = 1$, then also $\alpha + \beta \in A$.

Lemma 3.2.30. *The set*

$$A_{sis} := \{\alpha \in M \mid (\alpha, \alpha) = -2\}$$

is an admissible family of cycles.

Proof. Property (i), (ii) and (iii) of the definition are clear.

Property (iv):

$$(\alpha, \alpha) = -2,\ (\beta, \beta) - 2,\ (\alpha, \beta) = 1 \quad \Rightarrow \quad (\alpha + \beta, \alpha + \beta) = -2 - 2 + 2 \cdot 1 = -2.$$

□

Lemma 3.2.31. *The set A_{wvc} of weakly vanishing cycles is an admissible family of cycles.*

Proof. Property (i), (ii) and (iii) of the definition are clear.

Property (iv): Let α_1 and α_2 be weakly vanishing cycles with $(\alpha_1, \alpha_2) = 1$. By Corollary 3.1.20 we can write for $k = 1, 2$

$$\alpha_k = m_{i_{k,r_k}} \cdots m_{i_{k,2}} m_{i_{k,1}} \delta_{j_k}$$

Since $(\alpha_1, \alpha_2) = 1$, we have

$$\alpha_1 + \alpha_2 = m_{\alpha_1}\alpha_2 = m_{i_{1,r_1}} \cdots m_{i_{1,2}} m_{i_{1,1}} m_{j_1} m_{i_{1,1}} m_{i_{1,2}} \cdots m_{i_{1,r_1}} m_{i_{2,r_2}} \cdots m_{i_{2,2}} m_{i_{2,1}} \delta_{j_2}$$

by Lemma 3.1.2. Hence again by Corollary 3.1.20 we get that $\alpha_1 + \alpha_2$ is also a weakly vanishing cycle. □

Definition 3.2.32. Let A be an admissible family of cycles. A is called *simple* if

$$|(\alpha, \beta)| \leq 1 \quad \forall \alpha \neq \pm\beta \quad (\alpha, \beta \in A).$$

The goal of this section is to prove the following theorem.

Theorem 3.2.33. *Let A be an admissible family of cycles and suppose A is simple. Then A is finite; more precisely,*

$$|A| \leq 52\mu^2 + 17\mu.$$

The proof will be done in some steps.

Let A be an admissible family of cycles and $\alpha_1, \ldots, \alpha_k \in A$. Then this defines an intersection matrix

$$S_{\underline{\alpha}} = \big((\alpha_i, \alpha_j)\big)_{\substack{1 \leq i \leq k, \\ 1 \leq j \leq k}}$$

(with $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$) and therefore an intersection datum $\big(M_{\underline{\alpha}}, (\cdot, \cdot), \underline{\alpha}\big)$ and a Coxeter-Dynkin diagram $D_{\underline{\alpha}}$. If $\alpha_1, \ldots, \alpha_k$ are linearly independent, then $M_{\underline{\alpha}}$ is a submodule of M (with induced intersection product), namely

$$M_{\underline{\alpha}} = \langle \alpha_1, \ldots, \alpha_k \rangle_{\mathbb{Z}}.$$

If furthermore $k = \mu$, then $M_{\underline{\alpha}} \otimes \mathbb{Q} = M \otimes \mathbb{Q}$. However generally $S_{\underline{\alpha}}$ is *not* equivalent (or weakly equivalent) to S as an intersection matrix even in the case $M_{\underline{\alpha}} = M$. But $S_{\underline{\alpha}}$ and S are congruent matrices as matrices in $\mathrm{Mat}(\mu \times \mu, \mathbb{Q})$ if $k = \mu$ and $\alpha_1, \ldots, \alpha_\mu$ are linearly independent. In particular, S is (semi-)definite if and only if $S_{\underline{\alpha}}$ is.

Lemma 3.2.34. *Let A be an admissible family of cycles and $\alpha_1, \ldots, \alpha_k \in A$. Suppose $S_{\underline{\alpha}}$ is invertible. Then $\alpha_1, \ldots, \alpha_k$ are linearly independent. In particular, $k \leq \mu$.*

In particular, this is the case if one of the following conditions is satisfied:

(i) $(\alpha_i, \alpha_j) = 0$ *for all* $i \neq j$ *($1 \leq i, j \leq k$).*

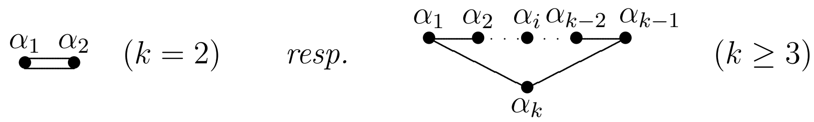
(ii) $(\alpha_i, \alpha_j) = -1$ *for all* $i \neq j$ *($1 \leq i, j \leq k$).*

Proof. Let $\beta = n_1\alpha_1 + \cdots + n_k\alpha_k$ and suppose $\beta = 0$. Then for all $i = 1, \ldots, k$ one gets

$$0 = (\beta, \alpha_i) = n_1(\alpha_1, \alpha_i) + \cdots + n_k(\alpha_k, \alpha_i) = \big(S_{\underline{\alpha}} \cdot \underline{n}\big)_i$$

with $\underline{n} := (n_1, \ldots, n_k)^t$, therefore $S_{\underline{\alpha}} \cdot \underline{n} = 0$, and we get $\underline{n} = 0$ since $S_{\underline{\alpha}}$ is invertible. □

Lemma 3.2.35. *Let A be an admissible family of cycles which is simple, and let $\alpha_1, \dots, \alpha_k \in A$. Suppose that the Coxeter-Dynkin diagram $D_{\underline{\alpha}}$ is of the following form:*

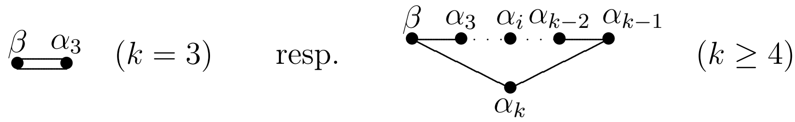


Then $\alpha_1 + \cdots + \alpha_k = 0$.

Proof. We will prove this by induction over k.

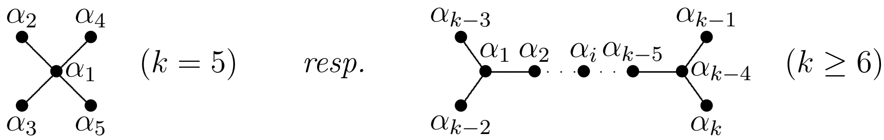
The case $k = 2$ is clear, since $\alpha_1 = \pm\alpha_2$ since A is simple, and the sign must be a minus since $(\alpha_1, \alpha_2) = 2$.

So, assume $k \geq 3$. Set $\beta = \alpha_1 + \alpha_2$. Since $(\alpha_1, \alpha_2) = 1$ we have that $\beta \in A$. Now the diagram for $\beta, \alpha_3, \dots, \alpha_k$ is again



Hence by the induction hypothesis $\beta + \alpha_3 + \cdots + \alpha_k = 0$. $\square$

Lemma 3.2.36. *Let A be an admissible family of cycles which is simple, and let $\alpha_1, \dots, \alpha_k \in A$. Suppose that the Coxeter-Dynkin diagram $D_{\underline{\alpha}}$ is of the following form:*

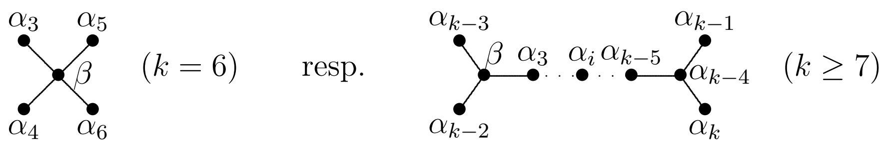


Then $2(\alpha_1 + \cdots + \alpha_{k-4}) + (\alpha_{k-3} + \cdots + \alpha_k) = 0$.

Proof. We prove again inductively over k.

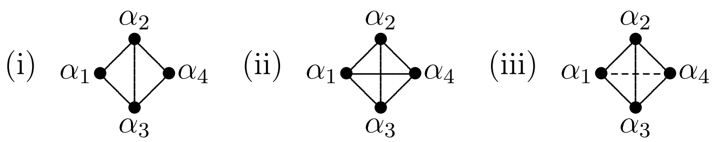
$k = 5$: Set $\beta_1 = \alpha_1$ and inductively $\beta_i = \beta_{i-1} + \alpha_i$ $(i = 2, \dots, 5)$. Then one gets $(\beta_{i-1}, \alpha_i) = 1$ $(i = 2, \dots, 5)$ and therefore $\beta_i \in A$ for all $i = 1, \dots, 5$. Now $(\beta_5, \alpha_1) = -2 + 1 + 1 + 1 + 1 = 2$, hence by the previous lemma $\beta_5 + \alpha_1 = 0$.

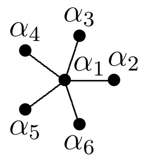
$k \geq 6$: Set $\beta = \alpha_1 + \alpha_2$. Then the diagram for $\beta, \alpha_3, \dots, \alpha_k$ is again



hence by the induction hypothesis $2(\beta + \alpha_3 + \cdots + \alpha_{k-4}) + (\alpha_{k-3} + \cdots + \alpha_k) = 0$. $\square$

Lemma 3.2.37. *Let A be an admissible family of cycles which is simple. Then A cannot contain cycles $\alpha_1, \dots, \alpha_k$ such that the Coxeter-Dynkin diagram $D_{\underline{\alpha}}$ is of one of the following forms:*



(iv) 

Proof. (i), (ii) and (iii): By looking at the subdiagrams for $\alpha_1, \alpha_2, \alpha_3$ and for $\alpha_2, \alpha_3, \alpha_4$ one gets from Lemma 3.2.35 that $\alpha_1 + \alpha_2 + \alpha_3 = 0 = \alpha_2 + \alpha_3 + \alpha_4$, therefore that $\alpha_1 = \alpha_4$. But then (α_1, α_4) could not be 0, 1 or -1.

(iv): By looking at the subdiagrams for $\alpha_1, \alpha_2, \dots, \alpha_5$ and for $\alpha_1, \alpha_3, \dots, \alpha_6$ one gets from Lemma 3.2.36 that $2\alpha_1 + \alpha_2 + \dots + \alpha_5 = 0 = 2\alpha_1 + \alpha_3 + \dots + \alpha_6$, therefore that $\alpha_2 = \alpha_6$. But then (α_2, α_6) could not be 0. □

Lemma 3.2.38. *Let A be an admissible family of cycles which is simple. For $\alpha \in A$ set*

$$A_\alpha^1 := \{\beta \in A \mid (\alpha, \beta) = 1\}.$$

Then

$$|A_\alpha^1| \leq 26\mu + 8.$$

Proof. Let m be the maximal cardinality of sets $B \subset A_\alpha^1$ such that for $\beta_1, \beta_2 \in B$ with $\beta_1 \neq \beta_2$ one has $(\beta_1, \beta_2) = 0$ and let B be such a set (with $|B| = m$). By Lemma 3.2.37 we know that $m \leq 4$, since $m \geq 5$ would imply that B contains diagram (iv) of Lemma 3.2.37 as a subdiagram.

Now for each $\gamma \in A_\alpha^1 \setminus B$ there must exist a $\beta \in B$ with $(\gamma, \beta) = \pm 1$: If $(\gamma, \beta) = 0$ for all $\beta \in B$, then $B \cup \{\gamma\}$ would be a set contradicting the maximality of m.

Furthermore for each $\beta \in B$ there exists maximally one $\gamma \in A_\alpha^1 \setminus B$ with $(\gamma, \beta) = 1$, since in that case one has $\alpha + \beta + \gamma = 0$ by Lemma 3.2.35. Set

$$C := \{\gamma \in A_\alpha^1 \setminus B \mid \exists \beta \in B : (\gamma, \beta) = 1\}.$$

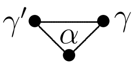
Then by the above one has $|C| \leq m$.

Now, if $\gamma \in A_\alpha^1 \setminus (B \cup C)$, then for each $\beta \in B$ we have $(\gamma, \beta) \in \{0, -1\}$, and there must exist at least one $\beta \in B$ with $(\gamma, \beta) = -1$. In the following we will count how many $\gamma \in A_\alpha^1 \setminus (B \cup C)$ are possible.

Case 1: $(\gamma, \beta) = -1$ for exactly one $\beta \in B$: Let $\gamma \in A_\alpha^1 \setminus (B \cup C)$ and $\beta \in B$ such that $(\gamma, \beta) = -1$ and $(\gamma, \beta') = 0$ for each $\beta' \in B$ with $\beta' \neq \beta$. Suppose $\gamma' \in A_\alpha^1 \setminus (B \cup C)$ has the same property (i.e. $(\gamma', \beta) = -1$, $(\gamma', \beta') = 0$ for $B \ni \beta' \neq \beta$).

Case 1a: $(\gamma, \gamma') = 0$: This is impossible, since in this case the set $(B \setminus \{\beta\}) \cup \{\gamma, \gamma'\}$ would contradict the maximality of m.

Case 1b: $(\gamma, \gamma') = 1$: In this case we have the following diagram:



hence we get $\alpha + \gamma + \gamma' = 0$ by Lemma 3.2.35. Therefore for each γ there exists at most one γ' belonging to this case.

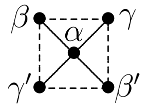
It remains Case 1c: $(\gamma, \gamma') = -1$.

By this we are able to see the following: For fixed β there exist at most 2μ γ's belonging to Case 1: If we have one such γ there exists at most one γ' not belonging to Case 1b by the above, hence all but at most one γ' that belong to Case 1 must belong to Case 1b, i.e. $(\gamma, \gamma') = -1$. So, if there would be more than 2μ such γ's, there would be more than μ γ's which are all connected by -1-lines. But this is impossible by Lemma 3.2.34 (ii).

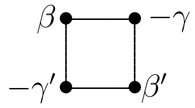
Since the number of β's in B is m we see that there are at most $2m\mu$ γ's belonging to Case 1.

Case 2: $(\gamma, \beta) = -1$ for at least two $\beta \in B$: We proceed similarly as above. Let $\gamma \in A^1_\alpha \setminus (B \cup C)$ and $\beta, \beta' \in B$ such that $(\gamma, \beta) = (\gamma, \beta') = -1$. Suppose again that $\gamma' \in A^1_\alpha \setminus (B \cup C)$ has the same property (i.e. $(\gamma', \beta) = (\gamma', \beta') = -1$).

Case 2a: $(\gamma, \gamma') = 0$. In this case we have the following diagram:



therefore



hence we get $\beta + \beta' - \gamma - \gamma' = 0$ by Lemma 3.2.35.

Case 2b: $(\gamma, \gamma') = 1$. As in Case 1b we get that $\alpha + \gamma + \gamma' = 0$.

It remains again Case 2c: $(\gamma, \gamma') = -1$.

Now by the same arguments as above we see the following: For fixed $\beta, \beta' \in B$ there are at most 3μ γ's belonging to Case 2 (maximally μ γ's that are all connected through -1-lines, and for each of these at most one for Case 2a and one for Case 2b). Thus together there are at most $3\binom{m}{2}\mu$ γ's belonging to Case 2.

By combining this we get

$$\begin{aligned} |A^1_\alpha| &\leq |B| + |C| + |\{\gamma \mid \text{Case 1}\}| + |\{\gamma \mid \text{Case 2}\}| \\ &\leq m + m + 2m\mu + 3\tbinom{m}{2}\mu \\ &\leq 4 + 4 + 2 \cdot 4\mu + 3 \cdot 6\mu = 26\mu + 8. \end{aligned}$$

□

Remark 3.2.39. If we set

$$A^{-1}_\alpha := \{\beta \in A \mid (\alpha, \beta) = -1\},$$

we get the same bound

$$|A^{-1}_\alpha| \leq 26\mu + 8,$$

since $A^{-1}_\alpha = A^1_{-\alpha}$.

Proof of Theorem 3.2.33. Let m_0 be the maximal cardinality of sets $A_0 \subset A$ such that for $\alpha_1, \alpha_2 \in A_0$ with $\alpha_1 \neq \alpha_2$ one has $(\alpha_1, \alpha_2) = 0$ and let A_0 be such a set (with $|A_0| = m_0$). By Lemma 3.2.34 we know that $m_0 \leq \mu$.

Now for each $\beta \in A \setminus A_0$ there must exist an $\alpha \in A_0$ with $\beta \in A^1_\alpha$ or $\beta \in A^{-1}_\alpha$ (otherwise this would contradict the maximality of m_0).

We thus get

$$|A| \le |A_0| + \sum_{\alpha \in A_0} \left(|A_\alpha^1| + |A_\alpha^{-1}|\right)$$
$$\le \mu + \mu\big((26\mu + 8) + (26\mu + 8)\big) = 52\mu^2 + 17\mu.$$

□

Remark 3.2.40. The bound for the size of A given in Theorem 3.2.33 seems to be far from being optimal. For example, one has for $\mu = 2$ that $|A_{\text{sis}}| \le 6$, while the theorem only shows that $|A_{\text{sis}}| \le 242$. (All admissible families of cycles are subsets of A_{sis}.)

Proposition 3.2.41. *Let A be an admissible family of cycles with $\delta_i \in A$ ($i = 1, \dots, \mu$). Suppose that for each $\alpha \in A$ and each $i = 1, \dots, \mu$ one has*

$$|(\alpha, \delta_i)| \le 1 \quad \textit{or} \quad \alpha = \pm\delta_i.$$

Then

$$A \subset A_{wvc}.$$

Proof. Define a function

$$g : M \ni n_1\delta_1 + \cdots + n_\mu\delta_\mu \mapsto |n_1| + \cdots + |n_\mu| \in \mathbb{Z}.$$

Then for $\alpha \in M$ one has

$$\alpha = \pm\delta_j \text{ for some } j = 1, \dots, \mu \quad \Leftrightarrow \quad g(\alpha) = 1.$$

We will now prove the following: If $\alpha \in A$, then either $g(\alpha) = 1$, or we can find an $i = 1, \dots, \mu$ such that

$$g(\alpha') = g(\alpha) - 1 \quad \text{for} \quad \alpha' = m_i\alpha \in A.$$

(Note that $\alpha' = m_i\alpha \Leftrightarrow \alpha = m_i\alpha'$.) This proves the proposition, since then inductively each $\alpha \in A$ can be written in the form

$$\alpha = \pm m_{i_r} \cdots m_{i_2} m_{i_1} \delta_j$$

which means exactly that α is a weakly vanishing cycle by Corollary 3.1.20.

So, let $\alpha \in A$ with $g(\alpha) > 1$, $\alpha = n_1\delta_1 + \cdots + n_\mu\delta_\mu$. Then

$$-2 = (\alpha, \alpha) = n_1(\alpha, \delta_1) + \cdots + n_\mu(\alpha, \delta_\mu),$$

hence there must exist an $i = 1, \dots, \mu$ with

$$n_i(\alpha, \delta_i) < 0.$$

In particular, we have $|(\alpha, \delta_i)| = 1$. Then

$$\alpha' = m_i\alpha = \alpha + (\alpha, \delta_i)\delta_i = n_1\delta_1 + \cdots + (n_i + (\alpha, \delta_i))\delta_i + \cdots + n_\mu\delta_\mu.$$

Since
$$|n_i + (\alpha, \delta_i)| = |n_i| - 1,$$
this proves the desired.

Note that $\alpha' \in A$: Either $(\alpha, \delta_i) = 1$, then $\alpha' = \alpha + \delta_i \in A$ by the definition of A, or $(\alpha, \delta_i) = -1$ which is equivalent to $(\alpha, -\delta_i) = 1$, thus again $\alpha' \in A$. □

Proposition 3.2.42. *Let A be an admissible family of cycles with $\delta_i \in A$ $(i = 1, \ldots, \mu)$. If A is simple, then*
$$A = A_{wvc}.$$

Proof. By the previous proposition we only have to prove that $A_{\mathrm{wvc}} \subset A$.

If A is simple and $\alpha, \beta \in A$, then $m_\alpha \beta \in A$: Either $(\alpha, \beta) = 0$, then $m_\alpha \beta = \beta \in A$, or $(\alpha, \beta) = 1$, then $m_\alpha \beta = \alpha + \beta \in A$ by the definition of A, or $(\alpha, \beta) = -1$, but this is equivalent to $(-\alpha, \beta) = 1$, and $-\alpha$ is also in A by definition.

Now, if $\alpha \in A_{\mathrm{wvc}}$ is a weakly vanishing cycle, then by Corollary 3.1.20
$$\alpha = m_{i_r} \cdots m_{i_2} m_{i_1} \delta_j$$
for some $1 \le i_1, \ldots, i_r, j \le \mu$. Hence $\alpha \in A$ by the above. □

We are now ready to state the two theorems and to prove the second one. The first one needs some further preparations after that.

Theorem 3.2.43. *Let A be an admissible family of cycles. The following conditions are equivalent:*

(i) *S is semidefinite.*

(∗∗) *For each $\alpha, \beta \in A$ one has $|(\alpha, \beta)| \le 2$.*

Theorem 3.2.44. *Let A be an admissible family of cycles. The following conditions are equivalent:*

(i) *S is definite.*

(∗∗) *A is simple.*

Proof. (i) ⇒ (∗∗): If S is definite, then A_{sis} is simple, and therefore also $A \subset A_{\mathrm{sis}}$: If $\alpha, \beta \in A_{\mathrm{sis}}$ with $\alpha \ne \pm\beta$, then
$$0 > (\alpha \pm \beta, \alpha \pm \beta) = -2 - 2 \pm 2(\alpha, \beta) \quad \text{therefore} \quad \pm(\alpha, \beta) < 2,$$
hence we get $|(\alpha, \beta)| \le 1$.

(∗∗) ⇒ (i): First suppose that $\delta_i \in A$ $(i = 1, \ldots, \mu)$. By Proposition 3.2.42 we get that $A = A_{\mathrm{wvc}}$. By Theorem 3.2.33 A is finite. Hence the set of weakly vanishing cycles is finite which is part (vii) of Theorem 3.2.2, thus by this theorem we get that (i) holds.

Now let A be arbitrary. Then there exist $\alpha_1, \ldots, \alpha_\mu \in A$ such that $\underline{\alpha} = (\alpha_1, \ldots, \alpha_\mu)$ is a basis of $M \otimes \mathbb{Q}$. The corresponding matrix $S_{\underline{\alpha}}$ is congruent to S, so if we prove that $S_{\underline{\alpha}}$ is definite, then also S is definite. But $S_{\underline{\alpha}}$ is definite by the above since A is also an admissible family of cycles for $\big(M_{\underline{\alpha}}, (\cdot,\cdot), \underline{\alpha}\big)$. □

Corollary 3.2.45. *Suppose A is an admissible family of cycles. If A is simple, then $A = A_{sis}$.*

Proof. Let $\alpha_1, \ldots, \alpha_\mu \in A$ such that $\underline{\alpha} = (\alpha_1, \ldots, \alpha_\mu)$ is a basis of $M \otimes \mathbb{Q}$. Then A is also an admissible family of cycles for $\big(M_{\underline{\alpha}}, (\cdot,\cdot), \underline{\alpha}\big)$.

By Theorem 3.2.44 we get that S is definite, hence A_{sis} is also simple by the theorem again. By Proposition 3.2.42 we then get $A = A_{\mathrm{wvc}(\underline{\alpha})}$ and $A_{\mathrm{sis}} = A_{\mathrm{wvc}(\underline{\alpha})}$ (where $A_{\mathrm{wvc}(\underline{\alpha})}$ denotes the set of weakly vanishing cycles for $M_{\underline{\alpha}}$) and therefore $A = A_{\mathrm{sis}}$. □

Corollary 3.2.46. *If S is definite and $\alpha_1, \ldots, \alpha_\mu$ are cycles with $(\alpha_i, \alpha_i) = -2$ $(i = 1, \ldots, \mu)$ that are linearly independent, then they are already a $\mathbb{Z}$-basis for M.*

Proof. $\alpha_1, \ldots, \alpha_\mu$ generate an admissible family of cycles: Set $A_0 := \{\alpha_1, \ldots, \alpha_\mu\}$ and

$$A_{k+1} := A_k \cup (-A_k) \cup \{\alpha + \beta \mid \alpha, \beta \in A_k,\ (\alpha, \beta) = 1\}.$$

Then $A := \bigcup_{k=0}^{\infty} A_k$ is an admissible family of cycles.

By the previous corollary $A = A_{\mathrm{sis}}$, particularly $\delta_i \in A$ for all $i = 1, \ldots, \mu$. But all elements of A are $\mathbb{Z}$-linear combinations of $\alpha_1, \ldots, \alpha_\mu$. □

As said above, for the proof of Theorem 3.2.43 we need some further preparation. Then Theorem 3.2.43 is a consequence of Theorem 3.2.44 by the same methods as in the case of the conjectures.

Lemma 3.2.47. *Let A be an admissible family of cycles and $\alpha_1, \alpha_2 \in A$ with $\alpha_2 \neq \pm\alpha_1$. Then there exist $\alpha_3, \ldots, \alpha_\mu \in A$ such that $(\alpha_1, \alpha_2, \ldots, \alpha_\mu)$ is a basis of $M \otimes \mathbb{Q}$.*

Proof. For $\mu \leq 2$ there is nothing to prove. So, assume $\mu \geq 3$. Set $M_{\mathbb{Q}} := M \otimes \mathbb{Q}$ and let B be the image A under the canonical map

$$M \to M_{\mathbb{Q}} / \langle \alpha_1, \alpha_2 \rangle_{\mathbb{Q}}.$$

Then B generates $M_{\mathbb{Q}} / \langle \alpha_1, \alpha_2 \rangle_{\mathbb{Q}}$, thus B contains a basis $\beta_3, \ldots, \beta_\mu$. If $\alpha_3, \ldots, \alpha_\mu$ are preimages of $\beta_3, \ldots, \beta_\mu$, then $(\alpha_1, \alpha_2, \ldots, \alpha_\mu)$ generates $M_{\mathbb{Q}}$. □

The following lemma is trivial to prove:

Lemma 3.2.48. *Let $\alpha_1, \ldots, \alpha_k \in A$ be linearly independent. Then $A \cap M_{\underline{\alpha}}$ is an admissible family of cycles for $M_{\underline{\alpha}}$.* □

Proof of Theorem 3.2.43. (i) $\Rightarrow$ $(**)$: If S is semidefinite, then for $\alpha, \beta \in A_{\mathrm{sis}}$ one has

$$0 \geq (\alpha \pm \beta, \alpha \pm \beta) = -2 - 2 \pm 2(\alpha, \beta) \quad \text{therefore} \quad \pm(\alpha, \beta) \leq 2,$$

hence we get $|(\alpha, \beta)| \leq 2$. Thus $A \subset A_{\mathrm{sis}}$ has the desired property.

$(**)$ $\Rightarrow$ (i): This is similar to the proof that Conjecture 2 implies Conjecture 1. We will prove by induction on μ.

The case $\mu = 2$ follows from Lemma 3.2.5: Take $\alpha_1, \alpha_2 \in A$ that generate $M \otimes \mathbb{Q}$. Then $|(\alpha_1, \alpha_2)| \leq 2$, hence we get by the lemma that $S_{\underline{\alpha}}$ is semidefinite (with $\underline{\alpha} = (\alpha_1, \alpha_2)$). Since $S_{\underline{\alpha}}$ is congruent to S also S is semidefinite.

Now assume $\mu \geq 3$.

Case 1: There exist $\alpha_1, \alpha_2 \in A$ with $(\alpha, \beta) = \pm 2$ but $\alpha \neq \pm\beta$. By Lemma 3.2.47 they are part of a basis $(\alpha_1, \ldots, \alpha_\mu)$ of $M \otimes \mathbb{Q}$. Consider the intersection matrix $S_{\underline{\alpha}}$. We can apply the induction hypothesis to all proper principal submatrices of $S_{\underline{\alpha}}$: If $J \subset \{1, \ldots, \mu\}$ is a proper subset we get from Lemma 3.2.48 for the corresponding principal submatrix of $S_{\underline{\alpha}}$ an admissible family of cycles which, as a subset of A, also satisfies the property $(**)$. Hence this submatrix of $S_{\underline{\alpha}}$ must be semidefinite by the induction hypothesis.

So, $S_{\underline{\alpha}}$ is an intersection matrix such that each proper principal submatrix is semidefinite and whose Coxeter-Dynkin diagram contains a line of weight ± 2. Therefore by Proposition 3.2.28 we get that $S_{\underline{\alpha}}$ is semidefinite. Since $S_{\underline{\alpha}}$ is congruent to S, also S is semidefinite.

Case 2: There does not exist $\alpha_1, \alpha_2 \in A$ with $(\alpha, \beta) = \pm 2$ but $\alpha \neq \pm\beta$. In this case A is simple, hence S is even definite by Theorem 3.2.44. □

Note that both theorems are true particularly for the case that $A = A_{\text{wvc}}$. This proves Theorem 3.2.3 and Theorem 3.2.4.

If we could prove that two weakly vanishing cycles $\alpha_1, \alpha_2 \in A_{\text{wvc}}$ are always part of a weakly distinguished basis $\alpha_1, \alpha_2, \ldots, \alpha_\mu$, then both theorems would state that the conjectures would be true when "equivalent" is replaced by "weakly equivalent" in $(*)$. If we even could prove that the set of vanishing cycles is an admissible family of cycles and that two vanishing cycles are always part of a distinguished basis, then the conjectures would follow.

3.3 Approaches for Proving the Conjectures

3.3.1 Some quotients of the braid group

Consider the following quotient of the braid group:

$$\begin{aligned} G(\mu, m) := \langle \sigma_1, \ldots, \sigma_{\mu-1} \mid\ & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} && \text{for } i = 1, \ldots, \mu - 2, \\ & \sigma_i \sigma_j = \sigma_j \sigma_i && \text{for } |i - j| \geq 2, \\ & \sigma_i^m = \mathbb{1} && \text{for } i = 1, \ldots, \mu - 1 \ \rangle \end{aligned}$$

It is a remarkable fact that some of these groups are finite.

For $\mu = 2$ we have $G(2, m) = \mathbb{Z}/m\mathbb{Z}$, and for $m = 2$ we have that $G(\mu, 2)$ is simply the permutation group $\mathcal{S}_\mu$, so in both of these cases $G(\mu, m)$ is finite.

In Table 3.4 some further cases for which $G(\mu, m)$ is finite are shown. These calculations were done with Magma [111].

However, the group $G(4, 4)$ is infinite. More generally we have

Lemma 3.3.1. *$G(\mu, 4m)$ is infinite for all $\mu \geq 4$ and $m \geq 1$.*

μ	m	$\lvert G(\mu,m)\rvert$	$\operatorname{ord}\delta$	$\operatorname{ord}\Delta$	$\operatorname{ord}(\prod_{i=1}^{\mu-1}\sigma_i^2)$	$\operatorname{ord}(\prod_{i=1}^{\mu-1}\sigma_i^3)$
2	m	m	m	m	$\frac{m}{\gcd(m,2)}$	$\frac{m}{\gcd(m,3)}$
μ	2	$m!$	m	2	1	2
3	3	24	6	4	6	1
3	4	96	12	8	4	12
3	5	600	30	20	10	10
4	3	648	12	6	12	1
5	3	155520	30	12	30	1

Table 3.4: Order of $G(\mu,m)$ for small μ and m

Proof. First consider the case $\mu = 4$.

The Burau representation (see Section A.1.8) of Br_4 is as follows:

$$\sigma_1 \mapsto \begin{pmatrix} 1-t & t & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_3 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-t & t \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We get a representation of $G(4,4m)$ from this by setting $t = i$.

One easily calculates:

$$\begin{aligned} d = \sigma_1\sigma_2\sigma_3 &\mapsto \begin{pmatrix} 1-i & 1+i & -1+i & -i \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ \overline{d} = \sigma_3\sigma_2\sigma_1 &\mapsto \begin{pmatrix} 1-i & i & 0 & 0 \\ 1-i & 0 & i & 0 \\ 1-i & 0 & 0 & i \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ \Delta = \sigma_1(\sigma_2\sigma_1)(\sigma_3\sigma_2\sigma_1) &\mapsto \begin{pmatrix} 1-i & 1+i & -1+i & -i \\ 1-i & 1+i & -1 & 0 \\ 1-i & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

We get

$$d^4 = \overline{d}^4 = \Delta^2 \mapsto \begin{pmatrix} 2-i & 1+i & -1+i & -1-i \\ 1-i & 2+i & -1+i & -1-i \\ 1-i & 1+i & i & -1-i \\ 1-i & 1+i & -1+i & -i \end{pmatrix} = \begin{pmatrix} 1-i & 1+i & -1+i & -1-i \\ 1-i & 1+i & -1+i & -1-i \\ 1-i & 1+i & -1+i & -1-i \\ 1-i & 1+i & -1+i & -1-i \end{pmatrix} + \mathbb{1}_4,$$

and furthermore

$$\Delta^{2k} \mapsto \begin{pmatrix} k-ki & k+ki & -k+ki & -k-ki \\ k-ki & k+ki & -k+ki & -k-ki \\ k-ki & k+ki & -k+ki & -k-ki \\ k-ki & k+ki & -k+ki & -k-ki \end{pmatrix} + \mathbb{1}_4.$$

This shows that Δ maps to an element of infinite order, hence Δ must be of infinite order itself. Therefore $G(4,4m)$ cannot be finite.

If $\mu > 4$, then we have an embedding $G(4,4m) \hookrightarrow G(\mu,4m)$ which shows that $G(\mu,4m)$ is also infinite. □

The groups of our interest are the groups $G(\mu,12)$, because of the following lemma:

Lemma 3.3.2. *Suppose D is a Coxeter-Dynkin diagram such that all Coxeter-Dynkin diagrams D' which are equivalent to D contain only lines with a weight of absolute value ≤ 1.*

Then σ_i^{12} acts trivially on the sets of distinguished bases for all $i = 1,\ldots,\mu-1$.

Proof. Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_\mu)$ be a distinguished basis, $i = 1, \dots, \mu - 1$, and set $u = (\delta_i, \delta_{i+1})$.

Depending on u the operation on $\underline{\alpha}$ is as follows where we set $\beta := \alpha_i$, $\gamma := \alpha_{i+1}$:

	$u=-1$		$u=0$		$u=1$	
k	$(\sigma_i^k \cdot \underline{\alpha})_i$	$(\sigma_i^k \cdot \underline{\alpha})_{i+1}$	$(\sigma_i^k \cdot \underline{\alpha})_i$	$(\sigma_i^k \cdot \underline{\alpha})_{i+1}$	$(\sigma_i^k \cdot \underline{\alpha})_i$	$(\sigma_i^k \cdot \underline{\alpha})_{i+1}$
0	β	γ	β	γ	β	γ
1	γ	$\beta-\gamma$	γ	β	γ	$\beta+\gamma$
2	$\beta-\gamma$	β	β	γ	$\beta+\gamma$	$-\beta$
3	β	$-\gamma$	γ	β	$-\beta$	γ
4	$-\gamma$	$\beta-\gamma$	β	γ	γ	$-\beta-\gamma$
5	$\beta-\gamma$	$-\beta$	γ	β	$-\beta-\gamma$	$-\beta$
6	$-\beta$	$-\gamma$	β	γ	$-\beta$	$-\gamma$
7	$-\gamma$	$-\beta+\gamma$	γ	β	$-\gamma$	$-\beta-\gamma$
8	$-\beta+\gamma$	$-\beta$	β	γ	$-\beta-\gamma$	β
9	$-\beta$	γ	γ	β	β	$-\gamma$
10	γ	$-\beta+\gamma$	β	γ	$-\gamma$	$\beta+\gamma$
11	$-\beta+\gamma$	β	γ	β	$\beta+\gamma$	β
12	β	γ	β	γ	β	γ

□

This lemma shows that under the conditions of the lemma the operation of the braid group on distinguished bases factors over an operation of the group $G(\mu, 12)$. So, if this group would be finite (however, it is not, at least for $\mu \geq 4$, as shown above), Conjecture 2 would be proven, since then the set of vanishing cycles would be finite too, and by Theorem 3.2.2 S would be definite.

However, this does not work since $G(\mu, 12)$ is not finite for $\mu \geq 4$. But we have more relations than only $\sigma_i^{12} = \mathbb{1}$ in the above case. As a continuation of Lemma 3.3.2 we have:

Lemma 3.3.3. *Suppose D is a Coxeter-Dynkin diagram such that all Coxeter-Dynkin diagrams D' which are equivalent to D contain only lines with a weight of absolute value ≤ 1.*

Then the operation of the extended braid group $\mathrm{Br}_\mu \ltimes (\mathbb{Z}/\mathbb{Z})^\mu$ factors over a quotient

$$\langle \sigma_1, \dots, \sigma_{n-1}, \xi_1, \dots, \xi_\mu \mid \textit{Relations } R \rangle$$

of this group where R contains the relations of $\mathrm{Br}_\mu \ltimes (\mathbb{Z}/\mathbb{Z})^\mu$ given in Section 3.1.3 and furthermore the following relations where we set $a := \sigma_i$, $b := \sigma_{i+1}$ $(i =$

$1, \dots, \mu - 2$):

$$
\begin{array}{rclcrcl}
a^{12} &=& \mathbb{1} && b^{12} &=& \mathbb{1} \\
a^6 b^6 &=& b^6 a^6 && (a^6 b^3)^2 &=& (b^3 a^6)^2 \\
(a^3 b^3)^3 &=& (b^3 a^3)^3 && (a^3 b^3)^6 &=& \mathbb{1} \\
(a^4 b^4)^3 &=& (b^4 a^4)^3 && (a^4 b^8)^4 &=& (b^8 a^4)^4 \\
(a^2 b^2)^6 &=& (b^2 a^2)^6 && a^4 (a^2 b^2)^3 &=& (b^2 a^2)^3 a^4 \\
a^6 b^2 a^4 b^2 &=& b^2 a^4 b^2 a^6 && b^6 a^2 b^4 a^2 &=& a^2 b^4 a^2 b^6 \\
a^6 b^2 a^6 b^4 &=& b^2 a^6 b^4 a^6 && b^6 a^2 b^6 a^4 &=& a^2 b^6 a^4 b^6
\end{array}
$$

All other relations between a^2 and b^2 resp. between a^3 and b^3 can be derived from these.

In addition to these relations, R contains the following relations:

$$(a^k b^l)^p = \mathbb{1} \qquad \text{and} \qquad (a^k b^l)^q = (b^l a^k)^q$$

for k, l, p, q given in the following tables:

p		l										
		1	2	3	4	5	6	7	8	9	10	11
k	1	36	24	18	12	24	12	36	24	18	12	24
	2	24	12	12	24	12	6	24	12	12	24	12
	3	18	12	6	12	18	4	18	12	6	12	18
	4	12	24	12	12	24	6	12	24	12	12	24
	5	24	12	18	24	36	12	24	12	18	24	36
	6	12	6	4	6	12	2	12	6	4	6	12
	7	36	24	18	12	24	12	36	24	18	12	24
	8	24	12	12	24	12	6	24	12	12	24	12
	9	18	12	6	12	18	4	18	12	6	12	18
	10	12	24	12	12	24	6	12	24	12	12	24
	11	24	12	18	24	36	12	24	12	18	24	36

q		l										
		1	2	3	4	5	6	7	8	9	10	11
k	1	3	2	3	12	12	6	6	4	6	6	12
	2	2	6	6	4	6	3	4	6	6	4	6
	3	3	6	3	6	6	2	3	6	3	6	6
	4	12	4	6	3	4	3	12	4	6	6	4
	5	12	6	6	4	3	6	12	12	3	4	6
	6	6	3	2	3	6	1	6	3	2	3	6
	7	6	4	3	12	12	6	3	4	6	6	12
	8	4	6	6	4	12	3	4	3	6	4	12
	9	6	6	3	6	3	2	6	6	3	6	3
	10	6	4	6	6	4	3	6	4	6	6	2
	11	12	6	6	4	6	6	12	12	3	2	3

Proof. All relations that are given can be proven by direct calculation. To do this one has to operate with the corresponding elements on distinguished basis for all 23 Coxeter-Dynkin diagrams for the case $\mu = 3$ which have a definite intersection matrix.

The hard part is the remark that all relations between a^2 and b^2 resp. a^3 and b^3 can be derived from the given ones. This was proven with help of MAGMA [111] and of a computer program written in C++: The program first determines the order of the subgroup generated by a^2 and b^2 resp. a^3 and b^3 by operating with these subgroups simultaneously on the distinguished bases for all 23 diagrams mentioned above. The result is that these subgroups have order 147456 resp. 384. If one now represents the subgroups by generators $x = a^2$ and $y = b^2$ resp. $x = a^3$ and $y = b^3$ and relations given in the lemma, one can calculate the order of these groups with MAGMA, and gets the same orders which proves the remark. □

What is still missing in the lemma is a complete set of relations for a and b, as well as all relations that involve more than two σ's. The goal would be to improve this lemma in such a way that one has enough relations to show that these groups are finite. That would prove the conjectures in the same way as mentioned above. However, these groups are quite large (even if one finds more relations — as mentioned in the proof, the subgroup generated by a^2 and b^2 is already rather large) which makes it hard to handle them.

It would suffice to show that the elements

$$\Delta = \sigma_1(\sigma_2\sigma_1)\cdots(\sigma_{\mu-1}\cdots\sigma_1)$$

have finite order in these groups (for all μ) for our purpose, as the following lemma shows:

Lemma 3.3.4. *Let S be an intersection matrix such that the element*

$$\Delta = \sigma_1(\sigma_2\sigma_1)\cdots(\sigma_{\mu-1}\cdots\sigma_1)$$

operates of finite order on distinguished bases.

Then m is finite.

Proof. As the proof of Lemma 3.1.13 shows, the element Δ^4 maps a distinguished basis $(\alpha_1,\ldots,\alpha_\mu)$ to $(m^2\alpha_1,\ldots,m^2\alpha_\mu)$.

Therefore, if Δ acts of finite order, then for some $k \neq 0$ we have $(m^{2k}\alpha_1,\ldots,m^{2k}\alpha_m u) = (\alpha_1,\ldots,\alpha_\mu)$, i.e. $m^{2k} = \mathbb{1}$. □

So, if Δ would be of finite order for all μ in the groups in question, then m would be finite for all diagrams in question (i.e. satisfying the condition the all equivalent diagrams have only lines with a weight of absolute value ≤ 1). But that would be also true for all subdiagrams of all equivalent diagrams, hence Theorem 3.2.2 again would show that S would be definite.

Let us finish this section with some lemmata for the case $\mu = 3$. In the following we set

$$a = \sigma_1, \quad b = \sigma_2$$

to simplify the notation.

Lemma 3.3.5. *In* Br_3 *we have*

$$(ab)^{3m+1} = a^{2m+1}b(a^2b^2)^m.$$

Therefore in $G(3,2m)$ we have

$$\Delta^{2m} = (aba)^{2m} = (ab)^{3m} = (a^2b^2)^m.$$

Proof. We prove by induction on m. The case $m = 0$ is trivial.

For $\Delta = aba = bab$ we know that $a\Delta = \Delta b$ and $b\Delta = \Delta a$. Hence we can "move" occurring Δ's appearing in a word to another position by changing those a's and b's between the old and new position to b resp. a.

By using this and the induction hypothesis for $m - 1$ we get

$$\begin{aligned}(ab)^{3m+1} &= \Delta^2 (ab)^{3(m-1)+1} = \Delta^2 a^{2(m-1)+1} b(a^2b^2)^{m-1} \\ &= \Delta b^{2m-1}(bab)b(a^2b^2)^{m-1} = a^{2m-1}a(aba)ab\, b(a^2b^2)^{m-1} \\ &= a^{2m+1}b(a^2b^2)^m.\end{aligned}$$

□

This lemma shows that in each $G(3, 2m)$ (and each quotient of this group) one has that Δ is of finite order if and only if a^2b^2 has finite order.

Lemma 3.3.6. *In* Br_3 *we have*

$$(ab)^{6m+1}b = (a^3b^3)^m ab\, b(a^3b^3)^m.$$

In particular, if G is a quotient of Br_3 *in which* $(a^3b^3)^m = \mathbb{1}$*, then* $(ab)^{6m} = \mathbb{1}$ *in G.*

Proof. This is proven similarly as in the previous lemma. We again prove by induction on m. The case $m = 0$ is trivial again.

By using the induction hypothesis for $m - 1$ we get

$$\begin{aligned}(ab)^{6m+1}b &= \Delta^4(ab)^{6(m-1)+1}b = \Delta^4(a^3b^3)^{m-1}ab\, b(a^3b^3)^{m-1} \\ &= \Delta^3(b^3a^3)^{m-1}b(bab)bb(a^3b^3)^{m-1} \\ &= \Delta^2(a^3b^3)^{m-1}aa(aba)abbb(a^3b^3)^{m-1} \\ &= \Delta(b^3a^3)^{m-1}bbba(aba)aabbb(a^3b^3)^{m-1} \\ &= (a^3b^3)^{m-1}aaabb(bab)baaabbb(a^3b^3)^{m-1} \\ &= (a^3b^3)^m ab\, b(a^3b^3)^m.\end{aligned}$$

□

3.3.2 Minimal corners

Let $S \in \mathcal{I}nt\mathcal{M}at_\mu$ be an intersection matrix and write S in the following two forms:

$$S = \begin{pmatrix} S_1 & x \\ x^t & -2 \end{pmatrix} = \begin{pmatrix} S_2 & y & z \\ y^t & -2 & u \\ z^t & u & -2 \end{pmatrix},$$

with $S_1 \in \mathcal{I}nt\mathcal{M}at_{\mu-1}$, $S_2 \in \mathcal{I}nt\mathcal{M}at_{\mu-2}$, $x \in \mathbb{Z}^{\mu-1}$, $y, z \in \mathbb{Z}^{\mu-2}$ and $u \in \mathbb{Z}$.

By Lemma B.1.18 we have

$$\det(-S) = \det(-S_1)(2 + x^t S_1^{-1} x)$$

if S_1 is invertible. Hence we get that

$$0 \leq -x^t S_1^{-1} x < 2, \tag{3.20}$$

if S is definite (which also follows from Lemma 3.2.25).

Furthermore, by Lemma B.1.19, we have

$$\begin{aligned}\det(-S) = \det(-S_2) \cdot \big(4 - u^2 + (y^t S_2^{-1} y)(z^t S_2^{-1} z) - (y^t S_2^{-1} z)^2 \\ + 2y^t S_2^{-1} y + 2z^t S_2^{-1} z + 2uy^t S_2^{-1} z\big),\end{aligned} \tag{3.21}$$

if S_2 is invertible.

Now regarding the conjectures consider the case that S has the property that each proper principal submatrix $\tilde{S}'$ of each intersection matrix S' which is equivalent to S is definite. We want to prove that $\det(-S) > 0$ in this case which would prove the conjectures.

By equation (3.21) we therefore have to show that

$$4 - u^2 + (y^t S_2^{-1} y)(z^t S_2^{-1} z) - (y^t S_2^{-1} z)^2 + 2y^t S_2^{-1} y + 2z^t S_2^{-1} z + 2uy^t S_2^{-1} z > 0. \tag{3.22}$$

Let us analyze the different terms in this inequality. First, u must be 0 or ± 1, hence $4 - u^2$ is 4 or 3. The term $(y^t S_2^{-1} y)(z^t S_2^{-1} z) - (y^t S_2^{-1} z)^2$ is positive by the Cauchy-Schwarz inequality (this term is equal to $\frac{\mathrm{QI}_2(z,y,-S_2,y,z)}{\det(-S_2)}$, hence this follows also from Lemma B.1.17).

Now some problems arise: The terms $y^t S_2^{-1} y$ and $z^t S_2^{-1} z$ are negative, since S_2 is negative definite. However, if we apply equation (3.20) to S_1, we get

$$0 \leq -y^t S_2^{-1} y < 2.$$

Also we can apply this equation to the principal submatrix

$$S_1' = \begin{pmatrix} S_2 & z \\ z^t & -2 \end{pmatrix}$$

(which is the upper-left corner of $\sigma_{\mu-1} S$), to get also

$$0 \leq -z^t S_2^{-1} z < 2.$$

It remains the term $uy^t S_2^{-1} z$ of which we do not know anything at first glance.

The determinant $\det(-S)$ does not change if we switch to an equivalent matrix. Hence we can assume that S is chosen in such a way that $\det(-S_1)$ is minimal if S runs through all equivalent intersection matrices.

By applying $\sigma_{\mu-1}$ to S we get

$$S' = \begin{pmatrix} S_2 & z & y + uz \\ z^t & -2 & -u \\ y^t + uz^t & -u & -2 \end{pmatrix}$$

and by applying $\sigma_{\mu-1}^2$ to S we get

$$S'' = \begin{pmatrix} S_2 & y+uz & -uy+(1-u)^2z \\ y^t+uz^t & -2 & -u \\ -uy^t+(1-u)^2z^t & -u & -2 \end{pmatrix}.$$

The upper corners of these matrices are

$$S_1' = \begin{pmatrix} S_2 & z \\ z^t & -2 \end{pmatrix}$$

and

$$S_1'' = \begin{pmatrix} S_2 & y+uz \\ y^t+uz^t & -2 \end{pmatrix}.$$

By assumption we have $\det(-S_1) \le \det(-S_1')$ and $\det(-S_1) \le \det(-S_1'')$. We have

$$\begin{aligned} \det(-S_1) \le \det(-S_1') \quad &\Leftrightarrow \quad \det(-S_2)(2+y^tS_2^{-1}y) \le \det(-S_2)(2+z^tS_2^{-1}z) \\ &\Leftrightarrow \quad -z^tS_2^{-1}z \le -y^tS_2^{-1}y \end{aligned}$$

and similarly

$$\begin{aligned} \det(-S_1) \le \det(-S_1'') \quad &\Leftrightarrow \quad -(y+uz)^tS_2^{-1}(y+uz) \le -y^tS_2^{-1}y \\ &\Leftrightarrow \quad 2uyS_2^{-1}z \ge -u^2zS_2^{-1}z, \end{aligned}$$

so by our choice of S we now get some information on the last term of equation (3.22) (in particular, it is positive).

For the following we set

$$p = -y^tS_2^{-1}y \qquad q = -z^tS_2^{-1}z \qquad t = y^tS_2^{-1}z.$$

The left hand side of equation (3.22) we denote by

$$\Theta = \frac{\det(-S)}{\det(-S_2)}.$$

Thus we have

$$\Theta = 4 - u^2 + pq - t^2 - 2p - 2q + 2ut,$$

and we know

$$0 \le q \le p < 2 \qquad 2ut \ge u^2q \qquad pq - t^2 \ge 0.$$

We consider the cases $u = 0$ and $u = 1$ (the case $u = -1$ can be derived from this by applying ξ_μ).

Case 1: $u = 0$. In this case

$$\Theta = 4 + pq - t^2 - 2p - 2q \ge 4 - 4p.$$

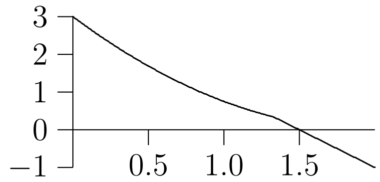
Case 2: $u = 1$. In this case

$$\Theta = 3 + pq - t^2 - 2p - 2q + 2t = 3 + pq - t^2 - 2p - q + (2t - q) \ge 3 - 3p.$$

But there is a better inequality: One can show that under the above conditions on p, q, t one has

$$\Theta \geq \begin{cases} \frac{3}{4}(p-2)^2 & 0 \leq p \leq \frac{4}{3} \\ 3-2p & \frac{4}{3} \leq p < 2. \end{cases}$$

The graph of the right hand side is as follows:



So the good case is $u = 1$, since the inequality for Θ is much better in this case. If we could show that $p \leq \frac{3}{2}$, it would follow that $\Theta \geq 0$ (however, we want to show $\Theta > 0$).

On the other hand, $p \leq \frac{3}{2}$ is the best we can hope of, as the example of a diagram of type E_8 shows: All subdiagrams of size 7 of all diagrams equivalent to a diagram of type E_8 are equivalent to a diagram of type A_7, D_7 or E_7 (this is not easy to show if one tries this purely combinatorial, but this follows from singularity theory, see [39]). The diagram with the smallest determinant is an E_7-diagram. We have

$$\det(-S) = 1 \quad \text{for } S = S_{E_8} \qquad \text{and} \qquad \det(-S_1) = 2 \quad \text{for } S_1 = S_{E_7}.$$

Hence in this case

$$-x^t S_1^{-1} x = 2 - \tfrac{\det(-S)}{\det(-S_1)} = \tfrac{3}{2}.$$

The idea to make p as small as possible is to force also $\det(-S_2)$ to be minimal (after making $\det(-S_1)$ minimal).

However, the first problem when doing this, is the following: Without the requirement that $\det(-S_2)$ is minimal, we could assume w.l.o.g. that $u = 1$: Assume $u = 0$, i.e.

$$S = \begin{pmatrix} S_2 & y & z \\ y^t & -2 & 0 \\ z^t & 0 & -2 \end{pmatrix}.$$

Now, if $z \neq 0$, this vector contains an entry ± 1, say $z_i = \pm 1$. By applying $\sigma_{\mu-2}^{-1} \cdots \sigma_i^{-1}$ to S we can "shift down" this entry to the position of u without destroying the minimality of $\det(-S_1)$. On the other hand, if $z = 0$, it would follow $0 \leq q \leq p = 0$, and $0 \cdot q - t^2 \geq 0$, thus $p = q = t = 0$ and therefore $\Theta = 4$.

But if we also want that $\det(-S_2)$ is minimal we cannot apply $\sigma_{\mu-2}$ in general without destroying this minimality.

One could try to prove inductively that we can assume $p \leq \frac{3}{2}$. In other words the idea would be to prove that S always can be chosen such that $-x^t S_1 x \leq \frac{3}{2}$ if S is definite. This then can by used inductively for S_1 to get $p \leq \frac{3}{2}$.

The roadmap for this would be as follows: We have

$$-x^t S_1^{-1} x = 2 - \frac{\det(-S)}{\det(-S_1)} = 2 - \frac{4 - u^2 + pq - t^2 - 2p - 2q + 2ut}{2-p}.$$

hence

$$\begin{aligned} -x^t S_1^{-1} x \le \omega \quad &\Leftrightarrow \quad 4 - u^2 + pq - t^2 - 2p - 2q + 2ut \ge (2-\omega)(2-p) \\ &\Leftrightarrow \quad 2\omega - u^2 + pq - t^2 - \omega p - 2q + 2ut \ge 0 \end{aligned}$$

So we want to prove (for $\omega = \frac{3}{2}$) that

$$\tilde{\Theta} := 2\omega - u^2 + pq - t^2 - \omega p - 2q + 2ut \ge 0$$

under the hypothesis that $p \le \omega$. We again know that

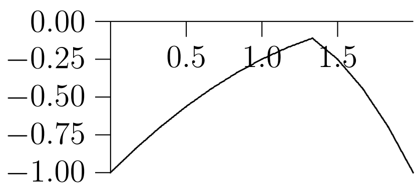
$$0 \le q \le p < \omega \qquad 2ut \ge u^2 q \qquad pq - t^2 \ge 0.$$

But even by ignoring the problem that u could be 0 this does not work. This also does not work for other values for ω as $\frac{3}{2}$.

Assume that $u = 1$. In this case it only follows that

$$\tilde{\Theta} \ge \begin{cases} -\frac{1}{2}(\omega - 2)^2 & 0 \le \omega \le \frac{4}{3} \\ -(\omega - 1)^2 & \frac{4}{3} \le \omega < 2. \end{cases}$$

The graph of the right hand side is as follows:



So it fails to get positive at $\omega = \frac{3}{2}$, but it nearly does.

But even if this does not work, one may conjecture:

Conjecture 3. *Let S be a definite intersection matrix. Then there exists an equivalent intersection matrix S' of the form*

$$S' = \begin{pmatrix} S_1 & x \\ x^t & -2 \end{pmatrix}$$

with $\det(-S_1) \le 2\det(-S)$.

(Note that $\det(-S_1) \le 2\det(-S)$ is equivalent to $-x^t S_1^{-1} x \le \frac{3}{2}$.)

Besides the problem that one has to show that one can assume $u = 1$, we need more information for p, q and t. This can perhaps be done by forcing all corners to have minimal determinant, not only S_1 and S_2, and by then using not only determinant formulas involving the ordinary quasiinverse and the "second quasiinverse" QI_2 but higher analogues.

3.3.3 Completion of distinguished bases

As mentioned at the end of Section 3.2.8, another way to prove the conjectures would be to prove that the family of distinguished bases is an admissible family of cycles, and that two vanishing cycles which are linearly independent are part of a distinguished bases. From these two facts the conjectures would follow from their weaker versions.

So we would have to prove the following:

Let α, β be two vanishing cycles with $(\alpha, \beta) = 1$. Then $\alpha + \beta$ is a vanishing cycle.

and

Let α, β be two vanishing cycles with $\alpha \neq \pm\beta$. Then there exists a distinguished basis $(\alpha_1, \ldots, \alpha_\mu)$ with $\alpha_1 = \alpha$, $\alpha_2 = \beta$.

However, it is not clear if these two statements are true, at least in general context. It may be conjectured that both statements are true in the context of Conjecture 2, i.e. in the case that all Coxeter-Dynkin diagrams equivalent to the given one contain only lines with a weight of absolute value ≤ 1.

It is also interesting to investigate the second statement for weakly vanishing cycles, even if it does not suffice to prove the conjecture (however, the analog versions of the conjectures with “equivalent” replaced by “weakly equivalent” in $(*)$ would be proven by this):

Let α, β be two weakly vanishing cycles with $\alpha \neq \pm\beta$. Then there exists a weakly distinguished basis $(\alpha_1, \ldots, \alpha_\mu)$ with $\alpha_1 = \alpha$, $\alpha_2 = \beta$.

With help of Proposition 3.1.18 and Corollary 3.1.20 this can be reformulated as follows:

Given two weakly vanishing cycles α_1, α_2 with $\alpha_1 \neq \pm\alpha_2$, there exists some $1 \leq i_{k,1}, \ldots, i_{k,r_k}, j_k \leq \mu$ and some $\kappa_{k,1}, \ldots, \kappa_{k,r_k}$ for $k = 1, \ldots, \mu$, and some $\iota_1, \iota_2 = \pm 1$ such that

$$\alpha_k = \iota_k \cdot m_{i_{k,r_k}} \cdots m_{i_{k,2}} m_{i_{k,1}} \delta_{j_k} \quad \textit{for } k = 1, 2$$

and the following holds: Let $\mathcal{F}_\mu$ be the free group generated by $x_1, \ldots, x_\mu$. Set

$$y_k = x_{i_{k,r_k}}^{\kappa_{k,r_k}} \cdots x_{i_{k,2}}^{\kappa_{k,2}} x_{i_{k,1}}^{\kappa_{k,1}} x_{j_k} x_{i_{k,1}}^{-\kappa_{k,1}} x_{i_{k,2}}^{-\kappa_{k,2}} \cdots x_{i_{k,r_k}}^{-\kappa_{k,r_k}}.$$

Then $y_1, \ldots, y_\mu$ also generate $\mathcal{F}_\mu$.

We only need this statement to be true in the case that the admissible family of weakly vanishing cycles is simple.

Appendix A

The Braid Group and the Gabrielov Group

脳ある鷹は爪を隠す

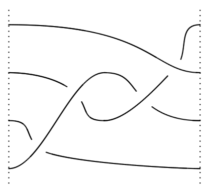
The hawk with talents hides its talons

A.1 The Braid Group

The famous braid group was first brought to attention by E. ARTIN [52]. Nowadays it has several applications in numerous fields of mathematics.

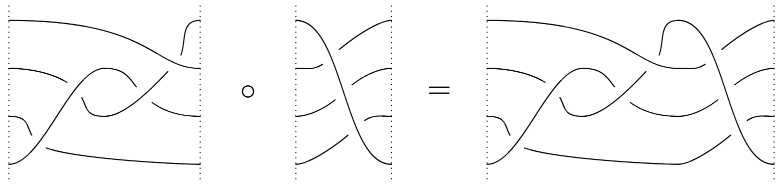
A.1.1 Definitions of the braid group

The most direct definition of the braid group Br_n of n strands is to define it as the group of braid diagrams modulo homotopy. A braid diagram is a diagram as follows:



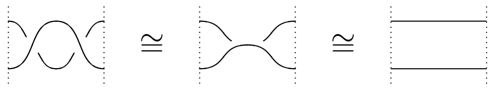
Take a vertical bar with n distinguished points. Each braid diagram has this bar at the left and at the right side. Now a strand is a path moving strictly from left to right starting at one of the distinguished points of the left bar and ending at one of the distinguished points of the right bar. The diagram consists of n strands such that at each distinguished point of the left bar exactly one strand starts and at each distinguished point of the right bar one strand ends. If two strands cross, they do not meet each other, but one strand crosses in front of the other strand, giving the diagram a virtual third dimension. Moreover we demand that at each horizontal position at most two strands cross. Strands are not allowed to meet nontransversally.

These braid diagrams form a monoid by concatenating (and removing the bar between them). For example:

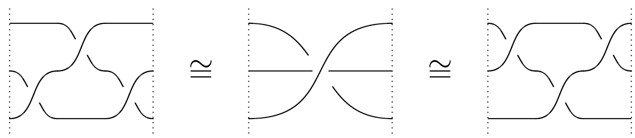


A homotopy of these diagrams is a continuous family of diagrams (with parameter $t \in [0,1]$) fixing the vertical bars which starts at the first diagram and ends at the second diagram. However, this homotopy is allowed to go through diagrams where at some position not only two strands may cross, but also three (with correct ordering in the virtual third dimension), or where two times two strands cross simultaneously at some horizontal position, or where two strands meet nontransversal (but are in fact separated by the virtual third dimension).

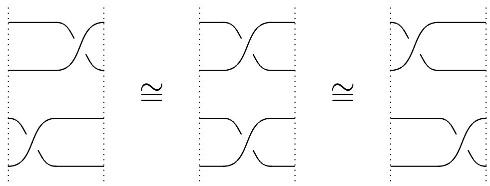
The two main examples for this are:



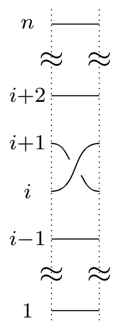
and

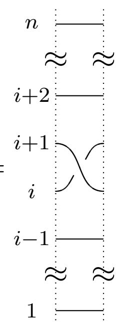


and



Now these examples and the discussion above show the following: Define the following elements of Br_n for $i = 1, \ldots, n-1$:

$\sigma_i :=$ 

$\sigma_i^{-1} :=$ 

Then the elements $\sigma_1, \ldots, \sigma_{n-1}$ generate Br_n, and σ_i^{-1} is in fact the inverse of σ_i as the first of the above examples shows. Furthermore we have

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad \text{for } i = 1, \ldots, n-2,$$

as the second of the above examples shows, and

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{for } |i-j| \geq 2 \quad (i,j = 1,\dots,n-1),$$

as the third of the above examples shows. Furthermore the description of the homotopies given above shows that these relations generate all other relations.

As a result we get the following algebraic definition of the braid group:

$$\begin{aligned}\mathrm{Br}_n := \langle \sigma_1,\dots,\sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i = 1,\dots,n-2, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \qquad\quad \text{for } |i-j| \geq 2 \qquad \rangle\end{aligned} \tag{A.1}$$

On the other hand we can get another definition of the braid group by "making the virtual third dimension real": Consider an open disc $D \subset \mathbb{C}$ that contains the points $1,\dots,n$. Now consider the set

$$\bar{X} := \big\{\{x_1,\dots,x_n\} \;\big|\; x_i \in D,\ x_i \neq x_j \text{ for } i \neq j\big\} \tag{A.2}$$

of sets of n pairwise distinct points in D. This set can be identified with

$$X := \hat{X}/\,\mathcal{S}_n \tag{A.3}$$

where

$$\hat{X} := \big\{(x_1,\dots,x_n) \in D^n \;\big|\; x_i \neq x_j \text{ for } i \neq j\big\} \tag{A.4}$$

is the set of tuples of pairwise distinct points in D, and where the permutation group $\mathcal{S}_n$ operates on $\hat{X}$ by permuting the x_i's. Therefore all these sets carry natural topologies. X is called the *space of n-configurations* in D.

Now a braid can be defined as a closed (continuous) path in X, starting and ending at $\{1,\dots,n\}$, modulo homotopy (the usual homotopy of paths). It is not hard to see that this gives an identification of braids defined in this way and braid diagrams modulo homotopy of braid diagrams defined above (see e.g. [52]). One has to check that each braid can be modified homotopically such that its projection yields a braid diagram and that each homotopy of braids can by represented by a homotopy of braid diagrams.

This now gives a third definition of the braid group as follows:

$$\mathrm{Br}_n = \pi_1(X)$$

where X is the space defined above (equation (A.3)). In fact this definition is independent of the base point $\{1,\dots,n\} \in X$. Furthermore it is independent of the choice of the disc D. In fact one also can use $\mathbb{C}$ instead of a disc D.

A.1.2 The pure braid group

The braid group carries a canonical morphism to the symmetric group

$$\begin{aligned}\theta : \mathrm{Br}_n &\to \mathcal{S}_n \\ \sigma_i &\mapsto \tau_i\end{aligned}$$

where τ_i is the transposition interchanging i and $i+1$. This morphism simply forgets how the strands cross and only remembers where the strands start and end.

Algebraically this morphism follows immediately from the well know presentation

$$\begin{aligned}\mathcal{S}_n := \langle \tau_1, \ldots, \tau_{n-1} \mid\ & \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1} && \text{for } i = 1, \ldots, n-2, \\ & \tau_i\tau_j = \tau_j\tau_i && \text{for } |i-j| \geq 2, \\ & \tau_i^2 = \mathbb{1} && \text{for } i = 1, \ldots, n-1 \ \rangle\end{aligned}$$

Obviously θ is surjective. The kernel of this map is denoted by $\mathcal{P}_n$ and called the *pure braid group*. We get an exact sequence

$$1 \longrightarrow \mathcal{P}_n \longrightarrow \mathrm{Br}_n \longrightarrow \mathcal{S}_n \longrightarrow 1.$$

If D is again an open disc in $\mathbb{C}$ (or $\mathbb{C}$ itself) one easily sees by the above description of Br_n that

$$\mathcal{P}_n = \pi_1(\hat{X})$$

where $\hat{X}$ is the space defined above (equation (A.4)).

A.1.3 The braid group as a mapping class group

Consider the group Γ_g of relative isotopy classes of diffeomorphism of a surface S of genus g with one boundary component which are the identity on the boundary. In case of $g = 0$ we can take a closed disc $\overline{D} \subset \mathbb{C}$ for S. It is well known that Γ_0 is trivial.

Now fix n points $x_1, \ldots, x_n$ in $S \setminus \partial S$ and consider the subgroup $\Gamma_{g,n}$ of (isotopy classes of) diffeomorphisms $\varphi \in \Gamma_g$ with $\varphi\big(\{x_1, \ldots, x_n\}\big) = \{x_1, \ldots, x_n\}$. (For $g = 0$ we can again assume $x_i = i$ for D large enough.) Then one has that

$$\mathrm{Br}_n = \Gamma_{0,n}.$$

Similarly, take the subgroup $\Gamma_{g,\hat{n}}$ of (isotopy classes of) diffeomorphisms $\varphi \in \Gamma_g$ with $\varphi(x_i) = x_i$ for all i. Then one also has that

$$\mathcal{P}_n = \Gamma_{0,\hat{n}}.$$

For a proof see e.g. [60]. The geometric idea behind this is the following: Take a $\varphi \in \Gamma_{0,n}$ which is represented by a diffeomorphism of $\overline{D}$ which fixes $\{x_1, \ldots, x_n\}$ (which again is denoted by φ). Now consider its image in Γ_0 which is trivial. Therefore φ must be isotopic to the identity in $\mathrm{Diff}(\overline{D}, \partial D)$, i.e. there is an isotopy

$$H : \overline{D} \times [0,1] \to \overline{D}$$

with $H_0 = \mathbb{1}$ and $H_1 = \varphi$. Set

$$\gamma_i(t) := H(x_i, t).$$

Then γ_i is a path from x_i to $\varphi(x_i)$ (which must be equal to some x_j), and these paths together form a braid.

This describes the above map $\Gamma_{0,n} \to \mathrm{Br}_n$ which has to be checked to be an isomorphism.

A.1.4 The Hurwitz action

Consider a group G. Then we can define the Hurwitz action of the braid group Br_n on the set G^n of n-tuples of elements of G as follows: Fix a $\kappa = \pm 1$ and set

$$\sigma_i \cdot (g_1, \ldots, g_i, g_{i+1}, \ldots, g_n) := (g_1, \ldots, g_{i+1}, g_{i+1}^{\kappa} g_i g_{i+1}^{-\kappa}, \ldots, g_n) \qquad \text{(A.5)}$$

If G is abelian, this action factors over the action of the symmetric group.

Note that equation (A.5) not only defines a (left-)action of Br_n on G^n, but also a right-action (by setting $(g_k) \cdot \sigma_i := \sigma_i \cdot (g_k)$).

We consider the case $\kappa = 1$, the other case is symmetrical. The following is true for the left- and right-action of Br_n (we only formulate it for the left-action).

Lemma A.1.1. *Let $(g_1, \ldots, g_n) \in G$ and $b \in \mathrm{Br}_n$ and set $(g_1', \ldots, g_n') := b \cdot (g_1, \ldots, g_n)$. Then*

(i) *There is a permutation $\pi \in \mathcal{S}_n$ such that each g_i' is conjugated to $g_{\pi(i)}$.*

(ii) $g_n \cdots g_2 g_1 = g_n' \cdots g_2' g_1'$.

Proof. Follows directly from the definition. □

(For the case $\kappa = -1$ one has to reverse the order of products $g_n \cdots g_2 g_1$ in (ii).)

The interesting fact is that this lemma has a converse which one could use to define Br_n:

Proposition A.1.2. *Consider the Hurwitz action of Br_n on $\mathcal{F}_n^n$ for the free group $\mathcal{F}_n$ with n generators. Then this action is faithful. Furthermore $(x_1', \ldots, x_n')$ lies in the orbit of $(x_1, \ldots, x_n)$ if and only if*

(i) *There is a permutation $\pi \in \mathcal{S}_n$ such that each x_i' is conjugated to $x_{\pi(i)}$.*

(ii) $x_n \cdots x_2 x_1 = x_n' \cdots x_2' x_1'$.

Furthermore this action restricts to a faithful action on the set of bases of $\mathcal{F}_n$.

For a proof see Artin [52], Theorem 16.

By this proposition we can define the braid group as a subgroup of $\mathrm{Aut}_{\mathcal{S}et}(\mathcal{F}_n^n)$ (where $\mathrm{Aut}_{\mathcal{S}et}$ denotes the set of automorphisms of sets, i.e. the set of bijective maps): Br_n is the set of those $\varphi \in \mathrm{Aut}_{\mathcal{S}et}(\mathcal{F}_n^n)$ such that for all $g = (g_1, \ldots, g_n)$ the elements g and $g' = \varphi(g)$ satisfy (i) and (ii) of the above proposition.

Now fix a basis $(x_1, \ldots, x_n)$ of $\mathcal{F}_n$. Then we have the following lemma:

Lemma A.1.3. *The Hurwitz left-action of Br_n on the set of bases of $\mathcal{F}_n$ defines a right-action of Br_n on $\mathcal{F}_n$ as follows: For $b \in \mathrm{Br}_n$ set $(x_1', \ldots, x_n') := b \cdot (x_1, \ldots, x_n)$. Then set*

$$x_i \cdot b := x_i' \qquad i = 1, \ldots, n.$$

Similarly the Hurwitz right-action of Br_n defines a left-action on $\mathcal{F}_n$.

Proof. Let b be a braid word in $\sigma_1, \ldots, \sigma_{n-1}$ and their inverses. We prove by induction on the word length of b. The case of length 1 is trivial.

So, assume that $b = \sigma b'$ where $\sigma \in \{\sigma_1, \ldots, \sigma_{n-1}, \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}\}$ and the word length of b' is smaller than that of b. W.l.o.g. $\sigma = \sigma_1$, the other cases are similar.

Set $(\tilde{x}_1, \ldots, \tilde{x}_n) := b' \cdot (x_1, \ldots, x_n)$. Then

$$b \cdot (x_1, x_2, \ldots, x_n) = \sigma_1 \cdot (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) = (\tilde{x}_2, \tilde{x}_2 \tilde{x}_1 \tilde{x}_2^{-1}, \ldots, \tilde{x}_n).$$

Similarly

$$x_i \cdot b = (x_i \cdot \sigma_1) \cdot b' = \begin{cases} x_2 \cdot b' = \tilde{x}_2 & i = 1 \\ (x_2 x_1 x_2^{-1}) \cdot b' = \tilde{x}_2 \tilde{x}_1 \tilde{x}_2^{-1} & i = 2 \\ x_i \cdot b' = \tilde{x}_i & i \geq 3 \end{cases}.$$

□

The (left-)action on $\mathcal{F}_n$ of this lemma is also denoted by the *Hurwitz action.* It follows now from the above that we have an embedding

$$\mathrm{Br}_n \hookrightarrow \mathrm{Aut}(\mathcal{F}_n)$$

given by the Hurwitz action.

A.1.5 Some special elements of the braid group

Consider the following elements in Br_n:

$$d := \sigma_1 \cdots \sigma_{n-1}, \qquad \overline{d} := \sigma_{n-1} \cdots \sigma_1,$$
$$\Delta := \sigma_1(\sigma_2\sigma_1) \cdots (\sigma_{n-1} \cdots \sigma_1).$$

Furthermore consider the following automorphism of Br_n:

$$\overline{\cdot} : \mathrm{Br}_n \to \mathrm{Br}_n$$
$$\sigma_i \mapsto \sigma_{n-i}$$

(With this notation $\overline{d}$ is in fact the image of d under this automorphism.)

Lemma A.1.4. *The above elements satisfy the following relations:*

$$d\sigma_i = \sigma_{i+1} d \qquad i = 1, \ldots, n-2, \tag{A.6}$$
$$\overline{d}\sigma_i = \sigma_{i-1} \overline{d} \qquad i = 2, \ldots, n-1, \tag{A.7}$$
$$\Delta = \overline{\Delta} = (\sigma_1 \cdots \sigma_{n-1}) \cdots (\sigma_1\sigma_2)\sigma_1, \tag{A.8}$$
$$\Delta\sigma_i = \sigma_{n-i}\Delta \qquad i = 1, \ldots, n-1, \tag{A.9}$$
$$\Delta^2 = d^n = \overline{d}^n. \tag{A.10}$$

Proof. Equation (A.6):

$$\begin{aligned} d\sigma_i &= (\sigma_1 \cdots \sigma_{i-1})\sigma_i\sigma_{i+1}(\sigma_{i+2}\cdots\sigma_{n-1})\sigma_i \\ &= (\sigma_1 \cdots \sigma_{i-1})\sigma_i\sigma_{i+1}\sigma_i(\sigma_{i+2}\cdots\sigma_{n-1}) \\ &= (\sigma_1 \cdots \sigma_{i-1})\sigma_{i+1}\sigma_i\sigma_{i+1}(\sigma_{i+2}\cdots\sigma_{n-1}) \\ &= \sigma_{i+1}(\sigma_1 \cdots \sigma_{i-1})\sigma_i\sigma_{i+1}(\sigma_{i+2}\cdots\sigma_{n-1}) = \sigma_{i+1}d. \end{aligned}$$

Equation (A.7) is essentially the same.

Equation (A.8): We will prove this inductively. The case $n = 2$ is trivial. Set

$$\Delta_{n-1} := \sigma_1(\sigma_2\sigma_1)\cdots(\sigma_{n-2}\cdots\sigma_1).$$

This is the element Δ in the smaller braid group Br_{n-1} which embeds in Br_n. The induction hypothesis for Br_{n-1} now translates to

$$\Delta_{n-1} = \sigma_{n-2}(\sigma_{n-3}\sigma_{n-2})\cdots(\sigma_1\cdots\sigma_{n-2}) = (\sigma_1\cdots\sigma_{n-2})\cdots(\sigma_1\sigma_2)\sigma_1.$$

Hence we have

$$\begin{aligned} \overline{\Delta_{n-1}} &= \sigma_{n-1}(\sigma_{n-2}\sigma_{n-1})\cdots(\sigma_2\cdots\sigma_{n-1}) \\ &= \sigma_2(\sigma_3\sigma_2)\cdots(\sigma_{n-1}\cdots\sigma_2) = (\sigma_{n-1}\cdots\sigma_2)\cdots(\sigma_{n-1}\sigma_{n-2})\sigma_{n-1}. \end{aligned}$$

Comparing this with equations (A.6) and (A.7) we get

$$d\Delta_{n-1} = \overline{\Delta_{n-1}}d \qquad \text{and} \qquad \overline{d}\,\overline{\Delta_{n-1}} = \Delta_{n-1}\overline{d}.$$

Since we have $\Delta = \Delta_{n-1}\overline{d}$, we get

$$\overline{\Delta} = \overline{\Delta_{n-1}}d = d\Delta_{n-1}$$

which proves the second equality of equation (A.8).

Now

$$\begin{aligned} d\Delta_{n-1} &= \sigma_1\cdots\sigma_{n-2}\sigma_{n-1}\cdot\sigma_1(\sigma_2\sigma_1)\cdots(\sigma_{n-3}\cdots\sigma_1)(\sigma_{n-2}\cdots\sigma_1) \\ &= \sigma_1\cdots\sigma_{n-2}\cdot\sigma_1(\sigma_2\sigma_1)\cdots(\sigma_{n-3}\cdots\sigma_1)\sigma_{n-1}(\sigma_{n-2}\cdots\sigma_1) \\ &= d_{n-1}\Delta_{n-2}\overline{d} \end{aligned}$$

where d_i denotes the d of Br_i and Δ_i denotes the Δ of Br_i. Inductively we get

$$d\Delta_{n-1} = d_{n-1}\Delta_{n-2}\overline{d} = d_{n-2}\Delta_{n-3}\overline{d_{n-1}}\,\overline{d} = \cdots = \overline{d_1}\cdots\overline{d_{n-1}}\,\overline{d} = \Delta$$

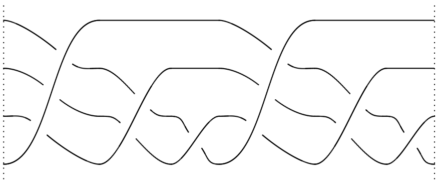
which proves equation (A.8).

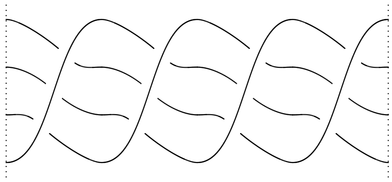
Equation (A.9): We will again prove this inductively. The case $n = 2$ is trivial. By the induction hypothesis we know $\Delta_{n-1}\sigma_i = \sigma_{n-1-i}\Delta_{n-1}$ for $i = 1, \ldots, n-2$. Therefore

$$\Delta\sigma_i = \Delta_{n-1}\overline{d}\sigma_i = \Delta_{n-1}\sigma_{i-1}\overline{d} = \sigma_{n-1-(i-1)}\Delta_{n-1}\overline{d} = \sigma_{n-i}\Delta$$

for $i = 2, \dots, n-1$. One the other hand, for $i = 1$ we can use a symmetric argument by equation (A.8).

Equation (A.10): The algebraic verification is somewhat tedious, so we prove this geometrically by braid diagrams. The following diagrams show that both Δ^2 and d^n represent the braid which one gets by twisting all strands twice:

$\Delta^2 =$ 

$d^n =$ 

Hence the first equation follows. The second equation follows from the first and equation (A.8). □

From this lemma follows that

$$x\Delta = \Delta\overline{x}$$

for all $x \in \mathrm{Br}_n$ and therefore that $\Delta^2 = d^n = \overline{d}^n$ lies in the center of Br_n. In fact one can show that this element generates the center (see below, Proposition A.1.18).

A.1.6 The braid group is a Garside group

An interesting question is how to decide if two given words in $\sigma_1, \dots, \sigma_{n-1}$ and their inverses define the same braid (the so-called "word problem"). As a first step one can ask the same question for the positive braid monoid Br_n^+ which is defined as the monoid of all positive braid words in $\sigma_1, \dots, \sigma_n$, i.e.:

$$\begin{aligned}\mathrm{Br}_n^+ := \langle \sigma_1, \dots, \sigma_{n-1} \mid\ & \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1} \text{ for } i = 1, \dots, n-2, \\ & \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{ for } |i-j| \geq 2 \qquad \rangle_{\text{monoid}}.\end{aligned}$$

From the works of Garside [74] it follows that two words in $\sigma_1, \dots, \sigma_{n-1}$ represent the same element in Br_n^+ if and only if they represent the same element in Br_n (see also [62]). To be more precise, he proved the following cancellation property:

Proposition A.1.5 (Garside [74], Theorem H)**.** *Let x, y be two elements in Br_n^+, and suppose that*

$$\sigma_i x = \sigma_j y$$

Then the following holds:

(i) *If $i = j$ then $x = y$.*

(ii) *If* $|i-j|=1$ *then* $x=\sigma_j\sigma_i w$ *and* $y=\sigma_i\sigma_j w$ *for some* $w\in \mathrm{Br}_n^+$.

(iii) *If* $|i-j|\geq 2$ *then* $x=\sigma_j w$ *and* $y=\sigma_i w$ *for some* $w\in \mathrm{Br}_n^+$.

□

GARSIDE also proved the following

Proposition A.1.6 (GARSIDE [74], Theorem 5)**.** *Let* $x\in \mathrm{Br}_n$. *Then there exists a* $k\in\mathbb{Z}_{\geq 0}$ *and a positive braid* $y\in\mathrm{Br}_n^+$ *such that*

$$x=\Delta^{-k}y.$$

□

By this proposition one can reduce the word problem of Br_n to the one of Br_n^+. In fact, GARSIDE proved that y can be expressed in some kind of normal form in the above theorem, such that the factorization $x=\Delta^{-k}y$ becomes unique.

We will not further discuss the word problem here but we will discuss some additional properties of the braid group, namely that it satisfies the axioms of a Garside group.

From Proposition A.1.5 (i) it follows that the monoid Br_n^+ satisfies the left-cancellation property:

$$wx=wy \quad\Rightarrow\quad x=y$$

Moreover, since we have an antiautomorphism from Br_n^+ to itself by reversing the order of a positive braid word, we also get that this monoid satisfies the right-cancellation property.

This cancellation property is the first step to define a Garside monoid. A Garside monoid has some additional properties, we will define below.

Definition A.1.7. A monoid M is called *atomic* if it is finitely generated and satisfies the following property: For any element $x\in M$ and any generating set A of M the length of words consisting of letters $a\in A$ representing the element g is bounded.

The following lemma is easy to prove (see e.g. [71]):

Lemma A.1.8. *A monoid* M *is atomic if and only if there exists a function* $\nu: M\to\mathbb{Z}_{\geq 0}$ *such that* $\nu(x)>0$ *for all* $x\in M\setminus\{1\}$ *and*

$$\nu(xy)=\nu(x)+\nu(y)$$

for all $x,y\in M$.

The positive braid monoid is atomic: All relations for Br_n^+ are homogenous, i.e. we have a well-defined length function

$$\ell:\mathrm{Br}_n^+\to\mathbb{Z}_{\geq 0}$$

which maps each braid word in $\sigma_1, \dots, \sigma_{n-1}$ to its length.

If M is atomic, left and right divisibility are partial orderings (if there is an equality $x = yz$ then y is called a left divisor of x, z a right divisor of x, and x is called a right multiple of y and a left multiple of z). Thus we can ask for infima and suprema for these orderings, i.e. for left-l.c.m.s and left-g.d.c.s resp. right-l.c.m.s and right-g.d.c.s. If they exist, for two elements $x, y \in M$ we denote by $a \vee_L b$, $a \wedge_L b$, $a \vee_R b$ and $a \wedge_R b$ the left-l.c.m., the left-g.d.c., the right-l.c.m. resp. the right-g.d.c. of x and y.

Definition A.1.9. A *Gaussian monoid* M is a monoid which satisfies the left- and right-cancellation property, which is atomic, and which contains $a \vee_L b$ and $a \vee_R b$ for all $a, b \in M$.

A Gaussian monoid M also contains $a \wedge_L b$ and $a \wedge_R b$ for all $a, b \in M$: $a \wedge_L b$ is the right-l.c.m. of all common left divisors of a and b, and similarly for $a \wedge_R b$.

One can show that a Gaussian monoid embeds in its group of fractions (this follows from the fact that it satisfies Ore's conditions, see [107]). However, for the positive braid monoid this already follows from GARSIDE's results.

The positive braid monoid is indeed a Gaussian monoid, as it follows from the above and from GARSIDE [74], Lemma 7, which states that the positive braid monoid contains all finite l.c.m.s.

Definition A.1.10. If M is an atomic monoid, an *atom* $x \in M$ is an element such that an equation $yz = x$ for $y, z \in M$ implies $y = 1$ or $z = 1$.

In the case of Br_n^+ the atoms are exactly the elements $\sigma_1, \dots, \sigma_{n-1}$.

Lemma A.1.11. *The set of atoms generates M.*

Proof. Let $x \in M$. We will prove by induction over $\nu(x)$ that x is a finite product of atoms. Either x is an atom itself, or there are $y, z \in M \setminus \{1\}$ with $x = yz$. Then $\nu(y), \nu(z) < \nu(x)$, hence y and z are finite products of atoms. Therefore x is a finite product of atoms too. □

In particular, the set of atoms is finite.

Definition A.1.12. In a Gaussian monoid, the left-l.c.m. of all atoms is denoted by Δ_L and the right-l.c.m. of all atoms is denoted by Δ_R.

A left divisor or Δ_R is called *left simple* and a right divisor of Δ_L is called *right simple.*

Definition A.1.13. A *Garside monoid* is a Gaussian monoid where the set of left simple elements is equal to the set of right simple elements.

A *Garside group* is the group of fractions of a Garside monoid.

It is now easy to see that the positive braid monoid is in fact a Garside monoid in the sense of this definition, hence the braid group is a Garside group.

For a Garside monoid one has that $\Delta_L = \Delta_R$. This element is simply denoted by Δ and is called the *fundamental element* of M or the *Garside element* of M.

In the case of the braid group, the element Δ introduced above is in fact the fundamental element.

A Garside group has many good properties. For example, we have

Proposition A.1.14. *A Garside group is torsion free.*

Proof. See e.g. Dehornoy [70]. □

Also we have

Proposition A.1.15. *For a Garside group the word problem is always solvable. To be more precise, for each Garside group there exists a language of normal forms.*

Proof. For a discussion of this see e.g. Dehornoy [71]. □

Moreover, the fundamental element Δ of a Garside group plays a central role:

Proposition A.1.16 (Dehornoy [71], Proposition 2.6)**.** *Let M be a Garside monoid and let A be the set of its atoms. Then there exists a permutation $\pi : A \to A$ such that*

$$\Delta x = \pi(x)\Delta$$

for all x in A. In particular, if k is the order of π then Δ^n lies in the center of the Garside group defined by M. □

This generalizes equation (A.9) of Lemma A.1.4 to general Garside groups.

A.1.7 Other presentations of the braid group

The presentation in equation (A.1) above involving the generators σ_i is due to Artin and is therefore called the *Artin presentation* of Br_n. But there are other presentations besides this.

In [62] Birman, Koo, and Lee introduced another presentation, called the *BKL presentation* of the braid group (see [62]). They introduced elements

$$\sigma_{ij} := (\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\dots\sigma_{i+1}^{-1})\sigma_i(\sigma_{i+1}\dots\sigma_{j-2}\sigma_{j-1})$$

for each $1 \le i < j \le n$. This element crosses the i^{th} strand over the j^{th} strand in front of the other strands.

With this notation $\sigma_i = \sigma_{i,i+1}$, hence these elements obviously also generate the braid group. Birman, Koo, and Lee showed that these elements together with the following relations form a presentation of Br_n:

$$\begin{aligned} &\sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} && \text{for } (i-k)(i-l)(j-k)(j-l) > 0 \quad (\text{and } i<j,\ k<l) \\ &\sigma_{ij}\sigma_{jk} = \sigma_{jk}\sigma_{ik} = \sigma_{ik}\sigma_{ij} && \text{for } i<j<k \end{aligned}$$

One can also look at the monoid BKL_n^+ of positive words in these elements. It is also a Garside monoid and the braid group is the group of its fractions. This shows that a Garside group can be the group of fractions of more than one Garside monoid.

In analogue to Proposition A.1.6 Birman, Koo, and Lee showed:

Proposition A.1.17. *Let $x \in \mathrm{Br}_n$. Then there exists a $k \in \mathbb{Z}_{\geq 0}$ and a $y \in \mathrm{BKL}_n^+$ such that*

$$x = d^{-k}y.$$

Here d is the same element as defined above, i.e.

$$d = \sigma_{12}\sigma_{23}\ldots\sigma_{n-1,n}$$

in this presentation. With this new presentation they also presented a new solution to the word problem for Br_n.

Another representation of the braid groups is already due to ARTIN [51]. By equation (A.6) of Lemma A.1.4 it follows that all braid groups are in fact generated by only two elements, namely σ_1 and d, since by this equation one has

$$\sigma_i = d^{i-1}\sigma_1 d^{-(i-1)}.$$

ARTIN showed that these elements together with the following relations form a presentation of the braid group:

$$\begin{aligned} \sigma_1 d^i \sigma_1 d^{-i} &= d^i \sigma_1 d^{-1} \sigma_1 \quad \text{for } 2 \leq i \leq \tfrac{n}{2} \\ d^n &= (\sigma\sigma_1)^{n-1} \end{aligned}$$

Using this presentation, CHOW showed first in [67] the following:

Proposition A.1.18 (CHOW [67])**.** *For $n \geq 3$ the center of the braid group Br_n is generated by Δ^2.* □

There are several other proofs of this fact, e.g. using GARSIDE's normal forms, or e.g. a geometrical proof (using hyperplane arrangements) in [68].

Since Br_n is torsion free (Proposition A.1.14) this shows that the center of Br_n is infinitely cyclic (of course also for $\mathrm{Br}_2 = \mathbb{Z}$).

A.1.8 Representations of the braid group

Since we have the canonical morphism $\mathrm{Br}_n \to \mathcal{S}_n$, each representation of the symmetric group yields a representation of the braid group. The irreducible representations of the symmetric group are in 1-to-1 correspondence to so-called Young diagrams which furthermore are in 1-to-1 correspondence to partitions of n.

For example, the partition $n = 1 + 1 + \cdots + 1$ corresponds to the parity representation, mapping each $\pi \in \mathcal{S}_n$ to its parity ± 1 (even or odd). The partition $n = n$ corresponds to the trivial representation.

Consider the standard representation of the symmetric group, given by permuting the entries of a vector in $\mathbb{C}^n$, i.e.

$$\tau_i \mapsto \mathbb{1}_{i-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{1}_{n-i-1}.$$

This representation is reducible, since it fixes the diagonal in $\mathbb{C}^n$. The induced representation on the $(n-1)$-dimensional subspace of all $(z_1, \ldots, z_n) \in \mathbb{C}^n$ with

$z_1 + \cdots + z_n = 0$ is irreducible. This representation corresponds to the partition $n = (n-1) + 1$.

However, there are of course other representations. The most important ones are the so-called Burau representation and the Lawrence-Krammer representation.

The Burau representation

The Burau representation was first introduced 1935 by BURAU [65]. It can be introduced ad-hoc as a matrix representation over the ring $\mathbb{Z}[t, t^{-1}]$ as follows:

$$\sigma_i \mapsto \mathbb{1}_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus \mathbb{1}_{n-i-1}.$$

By setting by $t = 1$ one gets the standard representation of the symmetric group, hence the Burau representation can be thought of as some kind of deformation of this representation.

It is straightforward to check that these matrices satisfy the braid relations which shows that the above mapping indeed defines a representation of Br_n.

While it was known early that the Burau representation is faithful for $n \leq 3$, it was long time unknown if it is faithful for all n. However, in 1991 MOODY [84] showed that it is not faithful for large n (to be more precise for $n \geq 9$). His result was later improved by LONG and PATON [82] to $n \geq 6$ and by BIGELOW [54] to $n \geq 5$. This question remains still open for $n = 4$.

As for the standard representation of the symmetric group, the Burau representation is reducible and splits as above into a 1-dimensional representation and a $(n-1)$-dimensional representation. The latter is denoted by the *reduced Burau representation*. If can be expressed as a matrix representation as follows:

$$\sigma_i \mapsto \mathbb{1}_{i-2} \oplus \begin{pmatrix} 1 & -t & 0 \\ 0 & -t & 0 \\ 0 & -1 & 1 \end{pmatrix} \oplus \mathbb{1}_{n-i-2}$$

where one has to cut off the first resp. last row and column of the 3×3-matrix for $i = 1$ resp. $i = n-1$; i.e.

$$\sigma_1 \mapsto \begin{pmatrix} -t & 0 \\ -1 & 1 \end{pmatrix} \oplus \mathbb{1}_{n-3} \qquad \text{and} \qquad \sigma_{n-1} \mapsto \mathbb{1}_{n-3} \oplus \begin{pmatrix} 1 & -t \\ 0 & -t \end{pmatrix}.$$

The reduced Burau representation is often called simply the Burau representation throughout the literature.

There is a nice geometrical interpretation of the (reduced) Burau representation. Take again a closed disc $\overline{D} \subset \mathbb{C}$ and fix n points $x_1, \ldots, x_n$ in $D = \overline{D} \setminus \partial D$. Also take a base point $y \in \partial D$. Set $D_n = \overline{D} \setminus \{x_1, \ldots, x_n\}$.

Then $\pi_1(D_n, y)$ is the free group with n generators where these n generators are represented by a path from y nearly to x_i, then circling counterclockwise one time around x_i and then back to y. This path is denoted by ω_i.

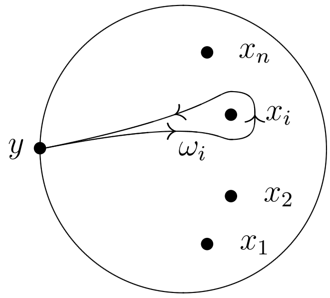


Figure A.1: Generators for $\pi_1(D_n, y)$

For a $\pi_1(D_n, y) \ni [\gamma] = [\omega_{i_1}]^{k_1} \cdots [\omega_{i_r}]^{k_r}$ we define the *total winding number* to be $w([\gamma]) := k_1 + \cdots + k_r$. This defines a surjective map

$$w : \pi_1(D_n, y) \to \mathbb{Z}.$$

Its kernel now defines a regular covering $\tilde{D}_n$ of D_n whose group of covering transformations is $\mathrm{Aut}_{D_n}(\tilde{D}_n) = \mathbb{Z}$. Therefore the homology $H_1(\tilde{D}_n)$ has a $\mathbb{Z}[t, t^{-1}]$-structure where t denotes the operation of a generator of the group of covering transformations. It can be shown that $H_1(\tilde{D}_n)$ is a free $\mathbb{Z}[t, t^{-1}]$-module of rank $n - 1$.

Since, as discussed above, Br_n can be thought of as the mapping class group of the n-punctured disc, any element $b \in \mathrm{Br}_n$ can be represented by a diffeomorphism $\varphi : D_n \to D_n$ which respects the boundary ∂D. Now (after fixing a base point $\tilde{y} \in \tilde{D}_n$ mapping to y) φ lifts uniquely to a diffeomorphism $\tilde{\varphi} : \tilde{D}_n \to \tilde{D}_n$ which yields an $\mathbb{Z}[t, t^{-1}]$-module automorphism $\tilde{\varphi}_* : H^1(\tilde{D}_n) \to H^1(\tilde{D}_n)$. Since $H^1(\tilde{D}_n) \cong (\mathbb{Z}[t, t^{-1}])^{n-1}$ we get in this way a map

$$\mathrm{Br}_n \to GL_{n-1}(\mathbb{Z}[t, t^{-1}])$$

which can be shown to be equivalent to the reduced Burau representation.

The Lawrence-Krammer representation

The Lawrence-Krammer representation can be presented — as the Burau representation — either algebraically as a matrix representation or by similar methods as in the case of the Burau representation topologically. It was first introduced (among other representations of Br_n) 1990 by Lawrence [80] by topological methods. Krammer [78] then gave in 2000 a purely algebraic definition of this representation. He also showed that it is faithful for $n = 4$. Bigelow [56] shortly later showed that it is faithful for all n, by topological methods which was proven again by Krammer [79] by algebraic methods. Thus by these results the braid group is a linear group for all n.

Set $R = \mathbb{Z}[q, q^{-1}, t, t^{-1}]$ and $m = \frac{n(n-1)}{2}$. The Lawrence-Krammer representation is a map

$$\mathrm{Br}_n \to GL_m(R).$$

The elements of a basis of R^m we denote by x_{ij} for $1 \le i < j \le n$. On these elements the braid group operates as follows:

$$
\begin{aligned}
\sigma_k x_{k,k+1} &= tq^2 x_{k,k+1} && \\
\sigma_k x_{ik} &= (1-q)x_{ik} + qx_{i,k+1} && \text{for } i < k \\
\sigma_k x_{i,k+1} &= x_{ik} + tq^{k-i+1}(q-1)x_{k,k+1} && \text{for } i < k \\
\sigma_k x_{kj} &= tq(q-1)x_{k,k+1} + qx_{k+1,j} && \text{for } k+1 < j \\
\sigma_k x_{k+1,j} &= x_{kj} + (1-q)x_{k+1,j} && \text{for } k+1 < j \\
\sigma_k x_{ij} &= x_{ij} && \text{for } i < j < k \text{ or } k+1 < i < j \\
\sigma_k x_{ij} &= x_{ij} + tq^{k-i}(q-1)^2 x_{k,k+1} && \text{for } i < k < k+1 < j
\end{aligned}
$$

For example, for $n = 3$ we have

$$
\sigma_1 \mapsto \begin{pmatrix} tq^2 & tq(q-1) & 0 \\ 0 & 0 & 1 \\ 0 & q & 1-q \end{pmatrix}, \qquad \sigma_2 \mapsto \begin{pmatrix} 1-q & 1 & 0 \\ q & 0 & 0 \\ 0 & tq^2(q-1) & tq^2 \end{pmatrix}
$$

and for $n = 4$ we have

$$
\sigma_1 \mapsto \begin{pmatrix} tq^2 & tq(q-1) & tq(q-1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & q & 0 & 1-q & 0 & 0 \\ 0 & 0 & q & 0 & 1-q & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \sigma_2 \mapsto \begin{pmatrix} 1-q & 1 & 0 & 0 & 0 & 0 \\ q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & tq^2(q-1) & tq(q-1)^2 & tq^2 & tq(q-1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & q & 1-q \end{pmatrix},
$$

$$
\sigma_3 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-q & 1 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-q & 1 & 0 \\ 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & tq^3(q-1) & 0 & tq^2(q-1) & tq^2 \end{pmatrix}.
$$

A.1.9 The action of the braid group on distinguished systems of paths

Recall that the braid group can be defined as the mapping class group of the n-punctured disc. So, let again $\overline{D} \subset \mathbb{C}$ be a closed disc and fix n points $x_1, \dots, x_n$ in $D = \overline{D} \setminus \partial D$ and a base point $y \in \partial D$. Set again $D_n = \overline{D} \setminus \{x_1, \dots, x_n\}$.

Now let $\gamma : [0,1] \to \overline{D}$ be a piecewise differentiable path with $\gamma(0) = y$ and $\gamma(1) = x_i$ for an $i = 1, \dots, n$ such that γ is injective with $\gamma(]0,1[) \subset D \cap D_n$.

Such a path γ defines an element $\omega(\gamma) \in \pi_1(D_n, y)$: Follow the path γ up to nearly its endpoint x_i, then circle around x_i anti-clockwise, then follow γ back to y, see Figure A.2.

Now consider a family of paths $(\gamma_j)_{j=1,\dots,n}$ which all have the properties as above such that each γ_j ends at a different x_i (i.e. there is a permutation π such that γ_j ends at $x_{\pi(j)}$). Set $\omega_j := \omega(\gamma_j)$.

Definition A.1.19. A family of paths (γ_j) as above is called a *weakly distinguished system of paths* if $[\omega_1], \dots, [\omega_n]$ generate $\pi_1(D_n, y)$.

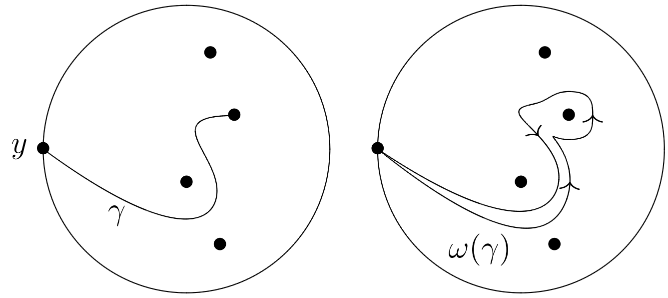


Figure A.2: The loop $\omega(\gamma) \in \pi_1(D_n, y)$ associated to γ

It is clear that a sufficient condition for being a weakly distinguished system of paths is that the γ_j do not intersect each other (apart from the starting point y; i.e. from $\gamma_j(t) = \gamma_k(s)$ it follows that $j = k$ or $t = s = 0$).

Definition A.1.20. A family of paths (γ_j) as above is called a *distinguished system of paths* if the γ_j do not intersect each other (as above) and furthermore the following is true: The starting vectors $\frac{d}{dt}\gamma_j(0)$ are ordered anti-clockwise, i.e.

$$\arg \tfrac{d}{dt}\gamma_k(0) > \arg \tfrac{d}{dt}\gamma_j(0) \quad \text{for} \quad k > j.$$

(Take here a branch of arg that is defined continuously on a half-space $H \subset \mathbb{C}$ that contains the disc when y is moved to the origin.)

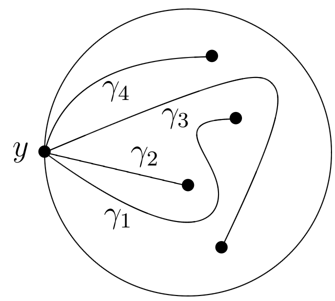


Figure A.3: A distinguished system of paths

Define an *isotopy* of two (weakly) distinguished systems of paths (γ_j) and (γ'_j) to be a differentiable map

$$H : \{1, \dots, n\} \times [0,1] \times [0,1] \to \overline{D}$$

such that

$$H(j,t,0) = \gamma_j(t), \quad H(j,t,1) = \gamma'_j(t),$$

and $\big(H(j,\cdot,s)\big)_{j=1,\dots,n}$ forms a (weakly) distinguished system of paths for all $s \in [0,1]$. Two isotopic (weakly) distinguished systems of paths define the same basis $([\omega_1], \dots, [\omega_n])$ of $\pi_1(D_n, y)$.

An element $b \in \mathrm{Br}_n$ now can be represented by a diffeomorphism $\varphi : \overline{D} \to \overline{D}$ which respects the boundary ∂D and the set $\{x_1, \dots, x_n\}$. If now $(\gamma_j)_{j=1,\dots,n}$ is a (weakly) distinguished system of paths, then $(\varphi \circ \gamma_j)_{j=1,\dots,n}$ is again a (weakly) distinguished system of paths which is well-defined up to isotopy. This defines an action of the braid group Br_n on the set of (weakly) distinguished systems of paths up to isotopy.

If (γ_j) is a distinguished system of paths, the operation of σ_i on (γ_j) can be described as follows: For the paths of the new distinguished system of paths $(\gamma'_j) = \sigma_i \cdot (\gamma_j)$ one has that

$$\gamma'_j \text{ is homotopic to } \begin{cases} \gamma_j & \text{for } j \neq i, i+1 \\ \gamma_{i+1} & \text{for } j = i \\ \gamma_i \circ \omega_{i+1}^{-1} & \text{for } j = i+1. \end{cases}$$

(where a concatenated path $\beta \circ \alpha$ runs first through α), see Figure A.4.

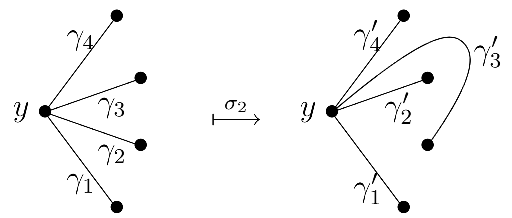


Figure A.4: The operation of the braid group on isotopy classes of distinguished systems of paths

We will use the following notation introduced by Garber, Kaplan and Teicher [73]:

Definition A.1.21. For a distinguished system of paths (γ_j) the corresponding basis $([\omega_j])$ of $\pi_1(D_n, y)$ is called a *geometric basis* or short *g-base.*

With this notation, the action of the braid group on isotopy classes of (weakly) distinguished bases induces an action on g-bases. The above description shows

$$[\omega'_j] = \begin{cases} [\omega_j] & \text{for } j \neq i, i+1 \\ [\omega_{i+1}] & \text{for } j = i \\ [\omega_{i+1}] \cdot [\omega_i] \cdot [\omega_{i+1}]^{-1} & \text{for } j = i+1, \end{cases}$$

thus this action is just the (restriction of the) Hurwitz action on $\mathcal{F}_n^n$.

Proposition A.1.22. *The action of the braid group on g-bases is simply transitive.*

Proof. If $([\omega_j])$ and $([\omega'_j])$ are two g-bases of $\pi_1(D_n, y)$, then

$$[\omega_n] \cdots [\omega_1] = [\partial D] = [\omega'_n] \cdots [\omega'_1].$$

Since $\pi_1(D_n, y) \cong \mathcal{F}_n$, by Proposition A.1.2 there exists a braid $b \in \mathrm{Br}_n$ such that $b \cdot ([\omega_j]) = ([\omega'_j])$, hence transitivity follows. That the action is faithful follows from the fact that the Hurwitz action on bases of the free group is faithful. □

A.2 The Gabrielov Group

Recall that Proposition A.1.2 and Lemma A.1.3 give a possibility to redefine the braid group as the subgroup of $\mathrm{Aut}(\mathcal{F}_n)$ consisting of Hurwitz operations (when fixing a basis $(x_1, \ldots, x_n)$ of $\mathcal{F}_n$:

$$\begin{aligned}\mathrm{Br}_n = \{\varphi \in \mathrm{Aut}(\mathcal{F}_n) \mid \exists \pi \in \mathcal{S}_n \text{ s.t. } \varphi(x_i) \text{ is conjugated to } x_{\pi(i)}\ \forall i \quad \text{and} \\ \varphi(x_n) \cdots \varphi(x_2)\varphi(x_1) = x_n \cdots x_2 x_1 \qquad \}.\end{aligned} \tag{A.11}$$

Since the pure braid group is the kernel under the canonical map $\mathrm{Br}_n \to \mathcal{S}_n$, we get from equation (A.11) a similar (re-)definition of $\mathcal{P}_n$:

$$\begin{aligned}\mathcal{P}_n = \{\varphi \in \mathrm{Aut}(\mathcal{F}_n) \mid \varphi(x_i) \text{ is conjugated to } x_i\ \forall i \quad \text{and} \\ \varphi(x_n) \cdots \varphi(x_2)\varphi(x_1) = x_n \cdots x_2 x_1\}.\end{aligned} \tag{A.12}$$

By the same ideas we can define the pure Gabrielov group and the Gabrielov group as subgroups of $\mathrm{Aut}(\mathcal{F}_n)$ consisting of *basis-conjugating* automorphisms: Define the pure Gabrielov group as

$$\mathrm{PGabr}_n = \{\varphi \in \mathrm{Aut}(\mathcal{F}_n) \mid \varphi(x_i) \text{ is conjugated to } x_i\ \forall i\}. \tag{A.13}$$

and the Gabrielov group as

$$\mathrm{Gabr}_n = \{\varphi \in \mathrm{Aut}(\mathcal{F}_n) \mid \exists \pi \in \mathcal{S}_n \text{ s.t. } \varphi(x_i) \text{ is conjugated to } x_{\pi(i)}\ \forall i\}. \tag{A.14}$$

It is not very common in the literature to call these groups the pure Gabrielov group resp. the Gabrielov group, but they will play here the role of acting on weakly distinguished systems of paths (and weakly distinguished bases in the main chapters) by so-called Gabrielov transformations, hence this name suggests itself.

The pure Gabrielov group was investigated by Humphries [76] and McCool [83]. Humphries found a set of generators for this group and McCool found a little bit later a presentation of this group (using the same generators).

Let G be an arbitrary group and consider the following map $\tilde{\rho}^i_j \in \mathrm{Aut}_{\mathcal{S}et}(G^n)$ for $1 \leq i, j \leq n$, $i \neq j$:

$$\begin{aligned}\tilde{\rho}^i_j(g_1, \ldots, g_i, \ldots, g_j, \ldots, g_n) &:= (g_1, \ldots, g_i, \ldots, g_i g_j g_i^{-1}, \ldots, g_n) \qquad \text{or} \\ \tilde{\rho}^i_j(g_1, \ldots, g_j, \ldots, g_i, \ldots, g_n) &:= (g_1, \ldots, g_i g_j g_i^{-1}, \ldots, g_i, \ldots, g_n)\end{aligned} \tag{A.15}$$

depending on whether $i < j$ or $i > j$.

Now take $G = \mathcal{F}_n$ and fix a basis $(x_1, \ldots, x_n)$ of $\mathcal{F}_n$. Then replacing $g_1, \ldots, g_n$ by $x_1, \ldots, x_n$ in equations (A.15) defines automorphisms

$$\rho^i_j \in \mathrm{Aut}(\mathcal{F}_n) \qquad \rho^i_j \cdot x_k = \begin{cases} x_k & \text{for } k \neq j \\ x_i x_j x_i^{-1} & \text{for } k = j \end{cases}$$

which are obviously elements of PGabr_n. Humphries showed that these elements generate the pure Gabrielov group. McCool then showed that the pure Gabrielov group has the following presentation:

$$\begin{aligned}\mathrm{PGabr}_n := \langle \rho^i_j\ (1\le i,j\le n,\ i\ne j) \mid \rho^i_j\rho^k_l &= \rho^k_l\rho^i_j && \text{for } |\{i,j,k,l\}|=4,\\ \rho^i_j\rho^i_k &= \rho^i_k\rho^i_j && \text{for } |\{i,j,k\}|=3,\\ \rho^i_j(\rho^k_i\rho^k_j) &= (\rho^k_i\rho^k_j)\rho^i_j && \text{for } |\{i,j,k\}|=3 \quad \rangle\end{aligned}$$

Equations (A.15) then define an action of the pure Gabrielov group on G^n by $\rho^i_j\cdot(g_1,\dots,g_n) := \tilde\rho^i_j(g_1,\dots,g_n)$. Note that the same equations also define a right action of PGabr_n on $\mathcal{F}_n^n$ (by setting $(g_k)\cdot\rho^i_j := \rho^i_j\cdot(g_k)$) and that an analogue of Lemma A.1.3 is true in this setting.

The symmetric group $\mathcal{S}_n$ also acts on G^n simply by permuting the elements of an n-tuple. Analogously we get an embedding

$$\mathcal{S}_n \hookrightarrow \mathrm{Aut}_{\mathcal{S}et}(\mathcal{F}_n^n)$$

after fixing again a basis $(x_1,\dots,x_n)$ of the free group $\mathcal{F}_n$ by

$$\tau_i x_k := \begin{cases} x_k & \text{for } k\ne i,i+1\\ x_{i+1} & \text{for } k=i\\ x_i & \text{for } k=i+1\end{cases}$$

where τ_i is the transposition which interchanges i and $i+1$ ($i=1,\dots,n-1$).

It is now easy to see that the Gabrielov group Gabr_n is the semidirect product

$$\mathrm{Gabr}_n := \mathcal{S}_n \ltimes \mathrm{PGabr}_n\,.$$

Here $\mathcal{S}_n$ acts on PGabr_n by operating on the indices of the generators, i.e.

$$\pi\star\rho^i_j = \rho^{\pi(i)}_{\pi(j)}$$

for $\pi\in\mathcal{S}_n$. By this we see that Gabr_n can be written by generators and relations as follows: As generators take

$$\begin{aligned} &\rho^i_j \quad \text{for} \quad 1\le i,j\le n,\ i\ne j \quad \text{and}\\ &\tau_i \quad \text{for} \quad 1\le i\le n-1.\end{aligned}$$

These satisfy the following relations:

$$\begin{aligned}
\rho^i_j\rho^k_l &= \rho^k_l\rho^i_j && && \text{for } |\{i,j,k,l\}|=4,\\
\rho^i_j\rho^i_k &= \rho^i_k\rho^i_j && && \text{for } |\{i,j,k\}|=3,\\
\rho^i_j(\rho^k_i\rho^k_j) &= (\rho^k_i\rho^k_j)\rho^i_j && && \text{for } |\{i,j,k\}|=3\\
\tau_i\tau_{i+1}\tau_i &= \tau_{i+1}\tau_i\tau_{i+1} && && \\
\tau_i\tau_j &= \tau_j\tau_i && && \text{for } |i-j|\ge 2\\
\tau_i^2 &= 1 && && \\
\tau_i\rho^i_{i+1} &= \rho^{i+1}_i\tau_i & \tau_i\rho^{i+1}_i &= \rho^i_{i+1}\tau_i & \\
\tau_i\rho^i_j &= \rho^{i+1}_j\tau_i & \tau_{j-1}\rho^i_j &= \rho^i_{j-1}\tau_{j-1} & \text{for } j\ne i+1\\
\tau_{i-1}\rho^i_j &= \rho^{i-1}_j\tau_{i-1} & \tau_j\rho^i_j &= \rho^i_{j+1}\tau_j & \text{for } j\ne i-1\\
\tau_k\rho^i_j &= \rho^i_j\tau_k && && \text{for } k\ne i-1,i,j-1,j.
\end{aligned}$$

The braid group is then a subgroup of the Gabrielov group:

$$\sigma_i = \tau_i \rho_i^{i+1} = \rho_{i+1}^i \tau_i.$$

Similar to the action of the braid group on the set of distinguished systems of paths there is an action of the Gabrielov group on the set of weakly distinguished systems of paths in the obvious way. If (γ_k) is a distinguished system of paths, then $(\gamma'_k) = \rho_j^i \cdot (\gamma_k)$ is given by

$$\gamma'_k \text{ is homotopic to } \begin{cases} \gamma_k & \text{for } k \neq j \\ \gamma_j \circ \omega_i^{-1} & \text{for } k = j. \end{cases}$$

From the definition of the Gabrielov group is follows immediately that this action is simply transitive.

There are some other groups that can be defined as subgroups of $\mathrm{Aut}(\mathcal{F}_n)$ in a similar way. For a discussion see BRIESKORN [63].

A.3 Automorphic Sets

Consider the category of sets with a product: An object of this category is a pair $(A, \triangleright)$ consisting of a set A and a map

$$\triangleright : A \times A \to A.$$

A morphism $\varphi : (A, \triangleright) \to (B, \triangleright)$ is a map $\varphi : A \to B$ with $\varphi(a \triangleright b) = \varphi(a) \triangleright \varphi(b)$. φ is an isomorphism in this category if and only if its underlying map of sets is bijective.

Definition A.3.1. An automorphic set is a set with product $(A, \triangleright)$ such that for all $a \in A$ the mapping $a \triangleright \cdot : A \to A$ is an automorphism of sets with product.

This definition is equivalent to the following:

(i) Given $a, b \in A$ there exists exactly one $c \in A$ with $a \triangleright c = b$.

(ii) For $a, b, c \in A$ one has

$$a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c). \tag{A.16}$$

As an important example each group G defines an automorphic set $(G, \triangleright)$ as follows:

$$g \triangleright h := ghg^{-1} \qquad g, h \in G.$$

For each automorphic set $(A, \triangleright)$ and each $n \in \mathbb{N}$ we have an operation of the braid group and the Gabrielov group on A^n as follows:

$$\sigma_i \cdot (a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) := (a_1, \ldots, a_{i+1}, a_{i+1} \triangleright a_i, \ldots, g_n)$$

and

$$\rho_j^i \cdot (a_1, \dots, a_i, \dots, a_j, \dots, a_n) := (a_1, \dots, a_i, \dots, a_i \triangleright a_j, \dots, a_n) \qquad \text{or}$$
$$\rho_j^i \cdot (a_1, \dots, a_j, \dots, a_i, \dots, a_n) := (a_1, \dots, a_i \triangleright a_j, \dots, a_i, \dots, a_n)$$

depending on whether $i < j$ or $i > j$.

If one takes $(A, \triangleright) = (G, \triangleright)$ for a group G as above, this action is precisely the Hurwitz action.

For a nice discussion of automorphic sets and applications see e.g. Brieskorn [63].

A.4 Simple Loops in D_n

Definition A.4.1. Let X be a real manifold. A *simple loop* in X with base point $y \in X$ is a piecewise differentiable path $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = y$ which yields an homeomorphism of S^1 onto its image.

Let again $\overline{D}$ be a closed disc in $\mathbb{C}$ and fix n points $x_1, \dots, x_n$ in $D = \overline{D} \setminus \partial D$ and a base point $y \in \partial D$. Set again $D_n = \overline{D} \setminus \{x_1, \dots, x_n\}$.

Proposition A.4.2. *If γ is a simple loop in D_n, then there exists a geometric basis $([\omega_1], \dots, [\omega_n])$ of $\pi_1(D_n, y)$ such that*

$$[\gamma] = [\omega_{j_r}] \cdots [\omega_{j_2}] \cdot [\omega_{j_1}]$$

for some $1 \leq j_1 < j_2 < \cdots < j_r \leq n$.

Proof. By the Jordan Curve Theorem and the Riemann Mapping Theorem γ encloses a subset $\overline{E} \subset \overline{D}$ which is homeomorphic to the unit disc in $\mathbb{C}$, and with $\partial E = \operatorname{im} \gamma$ (this is in fact the Schönflies Theorem). Set $E = \overline{E} \setminus \partial E$.

Then E contains some of the x's, say $x_{j_1}, \dots, x_{j_r}$ for some $1 \leq j_1 < j_2 < \cdots < j_r \leq n$. Set $E_r = \overline{E} \setminus \{x_{j_1}, \dots, x_{j_r}\}$.

Now select a distinguished system of paths for E_r (with base point y) that respects the numbering of the x's. This can be completed to a distinguished system of paths for D_n (eventually we have to deform γ homotopically if it starts or ends in tangential direction at $y \in \overline{D}$ to make place for the additional paths). This yields a geometric base $([\omega_1], \dots, [\omega_n])$ for D_n such that $([\omega_{j_1}], \dots, [\omega_{j_r}]$ is one for E_r.

Since for a geometric basis one always has that the product over the elements of the basis is homotopic to the boundary of the disc, this is also true for E_r, hence we get

$$[\gamma] = [\partial E_r] = [\omega_{j_r}] \cdots [\omega_{j_2}] \cdot [\omega_{j_1}].$$

□

Appendix B

Some Auxiliary Lemmas

聞(き)くは一時(いっとき)の恥(はじ)、聞(き)かぬは一生(いっしょう)の恥(はじ)

To ask may bring momentary shame, but not to ask brings everlasting shame

B.1 Some Matrix Lemmas

B.1.1 Definite and semidefinite matrices

The criteria for definiteness and semidefiniteness given in the following proposition are well-known:

Proposition B.1.1. *A symmetric matrix* $A \in \mathrm{Mat}(n \times n, \mathbb{R})$ *is*

(i) *positive definite if and only if*

$$\det A_k > 0 \quad \forall k = 1, \ldots, n$$

for

$$A_k := \big(A_{ij}\big)_{\substack{1 \le i \le k \\ 1 \le j \le k}} \in \mathrm{Mat}(k \times k, \mathbb{C}).$$

(ii) *positive semidefinite if and only if*

$$\det \tilde{A} \ge 0$$

for all quadratic principal submatrices $\tilde{A}$ *of* A *(i.e.* $\tilde{A} = \big(A_{ij}\big)_{\substack{i \in J \\ j \in J}}$ *for some* $J \subset \{1, \ldots, n\}$*).*

□

B.1.2 Triangular matrices

In this section we denote by $\Bbbk$ a commutative ring with unit and with $\Bbbk^*$ the multiplicative group of the units of $\Bbbk$.

Lemma B.1.2. *Let $M \in GL(n, \Bbbk)$ such that $M = AB$ with A and B as follows: A is an invertible lower triangular matrix and B an upper triangular matrix which has 1's on the diagonal.*

Then A and B are uniquely determined by M.

Proof. We prove inductively. The case $n = 1$ is trivial. Write

$$A = \begin{pmatrix} A_1 & 0 \\ x^t & a \end{pmatrix} \qquad B = \begin{pmatrix} B_1 & y \\ 0 & 1 \end{pmatrix}$$

(where $A_1, B_1 \in GL(n-1, \Bbbk)$ have the same properties as A and B, and $x, y \in \Bbbk^{n-1}$, $a \in \Bbbk^*$). Then

$$M = \begin{pmatrix} A_1B_1 & A_1y \\ x^tB_1 & x^ty + a \end{pmatrix}.$$

By the induction hypothesis A_1 and B_1 are determined by A_1B_1; therefore we get x and y from x^tB_1 resp. A_1y, and finitely we get a from $x^ty + a$. □

Lemma B.1.3. *Let $A \in GL(n \times n, \Bbbk)$ be an invertible lower triangular matrix. Then its inverse $B := A^{-1}$ is determined by the following property: Define*

$$\begin{aligned} B_k &:= \bigl(B_{ij}\bigr)_{\substack{1\le i\le k\\1\le j\le k}} \in \mathrm{Mat}(k \times k, \Bbbk) \quad &&(k = 1, \dots, n) \\ a_k &:= (A_{k+1,1}, \dots, A_{k+1,k}) \in \Bbbk^k &&(k = 1, \dots, n-1) \\ c_k &:= A_{kk} &&(k = 1, \dots, n) \end{aligned}$$

Then $B_1 = \bigl(c_1^{-1}\bigr)$ and for $k = 1, \dots, n-1$

$$B_{k+1} = \begin{pmatrix} B_k & 0 \\ -c_{k+1}^{-1}a_kB_k & c_{k+1}^{-1} \end{pmatrix}.$$

Proof. Define also

$$A_k := \bigl(A_{ij}\bigr)_{\substack{1\le i\le k\\1\le j\le k}} \in \mathrm{Mat}(k \times k, \Bbbk) \quad (k = 1, \dots, n).$$

We will show inductively that $A_kB_k = \mathbb{1}_k$ for $k = 1, \dots, n$. $A_1B_1 = \mathbb{1}_1$ is clear. So assume that we have shown that $A_kB_k = \mathbb{1}_k$ for $k = 1, \dots, n-1$. We then have

$$\begin{aligned} A_{k+1}B_{k+1} &= \begin{pmatrix} A_k & 0 \\ a_k & c_{k+1} \end{pmatrix} \begin{pmatrix} B_k & 0 \\ -c_{k+1}^{-1}a_kB_k & c_{k+1}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} A_kB_k & 0 \\ a_kB_k - c_{k+1}c_{k+1}^{-1}a_kB_k & c_{k+1}c_{k+1}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

□

Lemma B.1.4. *Let $M, N \in GL(n, \Bbbk)$ and let $1 \le k \le n$ be an integer. Suppose that M is a lower triangular matrix with $M_{ij} = 0$ for $j \ne i \ge k$, and $M_{ii} = 1$ for $i \ge k$, and that $N_{ij} = 0$ for $j \ne i \ne k$ and $N_{ii} = 1$ for $i \ne k$. Set $a := N_{kk}$ (then $a \in \Bbbk^*$).*

Then $P := NM = M'B$ with M' a lower triangular matrix with $M'_{ij} = 0$ for $j \ne i > k$, $M'_{kk} = a$, $M'_{ii} = M_{ii}$ for $i < k$ and $M'_{ii} = 1$ for $i > k$, and B an upper triangular matrix with 1's on the diagonal such that $B_{ij} = 0$ for $j \ne i > k$. Moreover, N, M, M' and B are uniquely determined by P.

Proof. Write

$$M = \begin{pmatrix} \tilde{M} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathbb{1}_{n-k} \end{pmatrix}, \qquad N = \begin{pmatrix} \mathbb{1}_{k-1} & 0 & 0 \\ x & a & y \\ 0 & 0 & \mathbb{1}_{n-k} \end{pmatrix}$$

with $\tilde{M} \in GL(k-1, \Bbbk)$ a lower triangular matrix and $x \in \Bbbk^{k-1}$, $y \in \Bbbk^{n-k}$. Then

$$P = \begin{pmatrix} \tilde{M} & 0 & 0 \\ x\tilde{M} & a & y \\ 0 & 0 & \mathbb{1}_{n-k} \end{pmatrix} = \begin{pmatrix} \tilde{M} & 0 & 0 \\ x\tilde{M} & a & 0 \\ 0 & 0 & \mathbb{1}_{n-k} \end{pmatrix} \begin{pmatrix} \mathbb{1}_{k-1} & 0 & 0 \\ 0 & 1 & a^{-1}y \\ 0 & 0 & \mathbb{1}_{n-k} \end{pmatrix} =: M'B.$$

By Lemma B.1.2, M' and B are uniquely determined by P, and from M' and B one gets uniquely M and N back, as the above calculation shows. □

If we use this lemma (and the calculation in the proof) inductively, we get the following:

Lemma B.1.5. *Let $a \in \Bbbk^*$ and for each $1 \le k \le n$ let N_k be a matrix of the following form:*

$$N_k = \begin{pmatrix} \mathbb{1}_{k-1} & 0 & 0 \\ x_k & a & y_k \\ 0 & 0 & \mathbb{1}_{n-k} \end{pmatrix}$$

with $x_k \in \Bbbk^{k-1}$, $y_k \in \Bbbk^{n-k}$. Define

$$X_k = \begin{pmatrix} 0_{k-1} & 0 & 0 \\ x_k & 0 & 0 \\ 0 & 0 & 0_{n-k} \end{pmatrix}, \quad X = \sum_{k=1}^{n} X_k, \quad Y_k = \begin{pmatrix} 0_{k-1} & 0 & 0 \\ 0 & 0 & y_k \\ 0 & 0 & 0_{n-k} \end{pmatrix}, \quad Y = \sum_{k=1}^{n} Y_k.$$

Then the product $P := N_n \cdots N_2 N_1$ can be calculated as follows:

$$N_n \cdots N_2 N_1 = a\big(-X + \mathbb{1}\big)^{-1}\big(a^{-1}Y + \mathbb{1}\big).$$

Furthermore, the product P determines its factors $N_1, \dots, N_n$.

Proof. By the previous lemma (and the calculation in the proof of it), we have inductively that $N_k \cdots N_2 N_1 = A_k B_k$ where

$$A_k = \begin{pmatrix} \tilde{A}_k & 0 \\ 0 & \mathbb{1}_{n-k} \end{pmatrix} \quad \text{with} \quad \tilde{A}_1 = \big(a\big), \quad \tilde{A}_k = \begin{pmatrix} \tilde{A}_{k-1} & 0 \\ x_k A_{k-1} & a \end{pmatrix}$$

and

$$B_k = \tilde{B}_k \cdots \tilde{B}_2 \tilde{B}_1 \quad \text{with} \quad \tilde{B}_k = \begin{pmatrix} \mathbb{1}_{k-1} & 0 & 0 \\ 0 & 1 & a^{-1}y_k \\ 0 & 0 & \mathbb{1}_{n-k} \end{pmatrix}.$$

Hence one gets $B_n = a^{-1}Y + \mathbb{1}$ and by Lemma B.1.3 one sees that A_n is the inverse of $-a^{-1}X + a^{-1}\mathbb{1}$.

The last statement follows again from Lemma B.1.2 (or inductively from the last statement of the previous lemma). □

By looking a little more closer one gets the following more general lemma:

Lemma B.1.6. *Let $J \subset \{1, \dots, n\}$ be a set of indices, write $J = \{j_1, \dots, j_r\}$ with $j_1 < \cdots < j_r$, and N_k, X, Y and P be defined as in the previous lemma. Furthermore, let $\tilde{X}$, $\tilde{Y}$, and $\tilde{P}$ be the principal submatrices of X, Y resp. P which one gets by deleting the i-th row and i-th column for $i \notin J$. Set $P^{(J)} := N_{j_r} \cdots N_{j_2} N_{j_1}$. Then*

$$\tilde{P} = a\big(-\tilde{X} + \mathbb{1}\big)^{-1}\big(a^{-1}\tilde{Y} + \mathbb{1}\big),$$

and $\tilde{P}$ is the principal submatrix of $P^{(J)}$ which one gets by deleting the i-th rows and i-th columns for $i \notin J$. Furthermore the i-th row of $P^{(J)}$ is trivial for $i \notin J$, i.e. $P^{(J)}_{ii} = 1$ and $m^{(J)}_{ij} = 0$ for $j \neq i$.

Proof. This lemma follows from the previous lemma by deleting all i-th rows and i-th columns for $i \notin J$ and the fact that the matrices N_k are trivial except for the k-th row (i.e. they become the identity matrix when deleting the k-th row and the k-th column). □

B.1.3 Higher quasiinverses

In this section we will discuss some expansions for the determinant which can be derived from the Laplace expansion law for the determinant. For the sake of completeness we state Laplace's result here. (Again in this section $\mathbb{k}$ denotes an arbitrary commutative ring with unit.)

Proposition B.1.7 (Laplace's expansion law for the determinant)**.** *Let $A = \big(a_{ij}\big)_{\substack{1 \le i \le n \\ 1 \le j \le n}} \in \operatorname{Mat}(n \times n, \mathbb{k})$. Fix a row $i = 1, \dots, n$ resp. a column $j = 1, \dots, n$. Then we have*

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{(i;j)} \quad \textit{resp.} \quad \det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{(i;j)}.$$

Here and in the following, for a matrix $A \in \operatorname{Mat}(n \times m, \mathbb{k})$ we denote by $A_{(i_1,\dots,i_r;j_1,\dots,j_s)}$ for $1 \le i_1, \dots, i_r \le n$ pairwise distinct and $1 \le j_1, \dots, j_s \le m$ pairwise distinct the $(n-r) \times (m-s)$-matrix which one gets by deleting the rows $i_1, \dots, i_r$ and the columns $j_1, \dots, j_s$ from A. (We allow the case $r = 0$ resp. $s = 0$; in that case we would write $A_{(;j_1,\dots,j_s)}$ resp. $A_{(i_1,\dots,i_r;)}$.)

The following definitions still make sense if we allow determinants of "empty" matrices: If A is an $n \times n$-matrix, then define $\det A_{(1,\dots,n;1,\dots,n)} = 1$. Furthermore, in the following formulas there are terms $a \det A_{(i_1,\dots,i_r;j_1,\dots,j_s)}$ where not all $i_1, \dots, i_r$ or all $j_1, \dots, j_s$ are pairwise distinct; however, such undefined determinants only occur with a zero coefficient a, thus these undefined determinants can be ignored.

Notation. For a quadratic matrix $A \in \operatorname{Mat}(n \times n, \mathbb{k})$ and $k \in \mathbb{Z}_{\ge 0}$ we write

$$\operatorname{corank} A > k$$

if and only if $k < n$ and all principal $(n-k)$-minors vanish.

Moreover we write $\operatorname{corank} A \geq k$ if and only if either $k = 0$ or $\operatorname{corank} A > k-1$; and we write $\operatorname{corank} A = k$ if and only if $\operatorname{corank} A \geq k$ and $\operatorname{corank} A \not> k$.

Note that if $\mathbb{k}$ is a field we have

$$\operatorname{corank} A = n - \operatorname{rank} A.$$

As an easy application of Laplace's expansion law one has the existence of the so-called quasiinverse of a quadratic matrix: For $A \in Mat(n \times n, \mathbb{k})$ one defines a matrix $\widehat{A} \in \operatorname{Mat}(n \times n, \mathbb{k})$ as follows:

$$(\widehat{A})_{ij} = (-1)^{i+j} \det A_{(j;i)}.$$

Laplace's expansion law is now equivalent to

$$A\widehat{A} = \widehat{A}A = \det A \cdot \mathbb{1}. \tag{B.1}$$

The quasiinverse has the following properties (since we will later state a similar lemma we include some trivial properties here):

Lemma B.1.8. *For $A \in \operatorname{Mat}(n \times n, \mathbb{k})$, $\lambda \in \mathbb{k}$ and $C \in GL(n, \mathbb{k})$ we have:*

(lin) $\widehat{(\lambda A)} = \lambda^{n-1}\widehat{A}$.

(tr) $(\widehat{A})^t = \widehat{(A^t)}$.

(bc) $\widehat{(C^{-1}AC)} = C^{-1}\widehat{A}C$.

(inv) $\widehat{A}A = \det A \cdot \mathbb{1}$.

(ker) (i) *If* $\det A = 0$*, then* $\operatorname{im} A \subset \ker \widehat{A}$.

(ii) *If* $\operatorname{corank} A \geq 2$*, we have* $\widehat{A} = 0$.

(iii) *If* $\mathbb{k}$ *is a field and* $\dim\ker A = 1$*, then we have* $\ker \widehat{A} = \operatorname{im} A$.

(sd) *($\mathbb{k} = \mathbb{R}$.) If A is symmetric and positive semidefinite, so is $\widehat{A}$.*

(def) *($\mathbb{k} = \mathbb{R}$.) If A is symmetric and positive definite, so is $\widehat{A}$.*

Before we prove this we formulate a first formula for the determinant which involves the quasiinverse:

Lemma B.1.9. *Let $A \in \operatorname{Mat}(n \times n, \mathbb{k})$ be of the following form:*

$$A = \begin{pmatrix} A_1 & y \\ x^t & a \end{pmatrix}$$

with $A_1 \in \operatorname{Mat}((n-1) \times (n-1), \mathbb{k})$, $x, y \in \mathbb{k}^{n-1}$, $a \in \mathbb{k}$. Then

$$\det A = a \det A_1 - x^t \widehat{A_1} y. \tag{B.2}$$

Proof. By using Laplace's expansion law for the last row of A we get

$$\det A = \sum_{i=1}^{n} (-1)^{n+i} a_{ni} \det A_{(n;i)} = a \det A_1 + \sum_{i=1}^{n-1} (-1)^{n+i} x_i \det\Big(\begin{pmatrix} A_1 & y \end{pmatrix}_{(;i)} \Big).$$

Now use Laplace's expansion law for the last column of the remaining determinants. We get

$$\det A = a \det A_1 + \sum_{i=1}^{n-1} (-1)^{n+i} x_i \sum_{j=1}^{n-1} (-1)^{n-1+j} y_j \det(A_1)_{(j;i)} = a \det A_1 - x^t \widehat{A_1} y.$$

□

Remark B.1.10. As a special case of the previous lemma we get (for $A \in \mathrm{Mat}(n \times n, \Bbbk)$, $x, y \in \Bbbk^n$) the formula

$$x^t \widehat{A} y = -\det \begin{pmatrix} A & y \\ x^t & 0 \end{pmatrix} \tag{B.3}$$

Proof of Lemma B.1.8. (lin) and (tr) are trivial and (inv) is (as said above) a direct consequence of Laplace's expansion law.

To prove (bc) we use the previous lemma (resp. the remark after it): By equation (B.3), for all $x, y \in \Bbbk^n$ we have that

$$\begin{aligned}
(x^t C^{-1}) \widehat{A} (Cy) &= -\det \begin{pmatrix} A & Cy \\ x^t C^{-1} & 0 \end{pmatrix} \\
&= -\det \left(\begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} A & Cy \\ x^t C^{-1} & 0 \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix} \right) \\
&= -\det \begin{pmatrix} C^{-1}AC & y \\ x^t & 0 \end{pmatrix} \\
&= x^t (\widehat{C^{-1}AC}) y
\end{aligned}$$

for all x, y.

(ker) (i): Let $v \in \operatorname{im} A$. Then there exists a $w \in \Bbbk^n$ with $Aw = v$. By (inv) we then have that $\widehat{A} v = \widehat{A} A w = \det A \cdot w = 0$.

(ker) (ii): If $\operatorname{corank} A \geq 2$, then all $(n-1) \times (n-1)$-minors of A vanish, hence $\widehat{A} = 0$.

(ker) (iii): If $\Bbbk$ is a field and $\dim \ker A = 1$, we can by (bc) assume w.l.o.g. that A is of the form

$$A = \begin{pmatrix} a_{11} & \dots & a_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \dots & a_{n,n-1} & 0 \end{pmatrix}.$$

In this case we get that

$$\widehat{A} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$$

where the b's are all $(n-1)$-minors of the matrix $A_{(;n)}$. Since the rank of $A_{(;n)}$ is maximal, not all b's can be zero. Therefore (ker) (iii) follows.

(def): If A is definite, then it is invertible, hence $\widehat{A} = \det A \cdot A^{-1}$. But if A is definite, so is A^{-1}, and $\det A > 0$.

(sd): Note that the mapping $A \mapsto \widehat{A}$ is continuous. So (sd) can be deduced from (def) by considering a semidefinite matrix as a limit of definite matrices. Alternatively, since A is real and symmetric, there exists a $C \in GL(n, \mathbb{R})$ such that $C^{-1}AC = \operatorname{diag}(a_1, a_2, \ldots, a_n)$. By (bc) we can therefore assume w.l.o.g. that A is a diagonal matrix. The quasiinverse of $\operatorname{diag}(a_1, \ldots, a_n)$ is

$$\operatorname{diag}(a_2 \cdots a_n, a_1 a_3 \cdots a_n, \ldots, a_1 \cdots a_{n-1}).$$

Since a diagonal matrix is positive semidefinite resp. definite if and only if all diagonal entries are nonnegative resp. positive, we get that (sd) and (def) hold. □

We now go one step further. The entries of the quasiinverse were defined as the determinants of those matrices which one gets by deleting one row and column from A (and attaching the correct sign). We now define a similar construction which involves deleting two rows and columns.

Definition B.1.11. For $A \in Mat(n \times n, \Bbbk)$ and $x, y, z, w \in \Bbbk^n$ define a matrix $\mathrm{QI}_2(x, A, y) \in Mat(n \times n, \Bbbk)$ and $\mathrm{QI}_2(z, x, A, y, w) \in \Bbbk$ as follows:

$$\big(\mathrm{QI}_2(x, A, y)\big)_{ij} := \sum_{k,l=1}^{n} \varepsilon(l, j)\varepsilon(k, i) x_k y_l \det A_{(l,j;k,i)},$$

with

$$\varepsilon(i, j) := \begin{cases} (-1)^{i+j} & i < j \\ 0 & i = j \\ (-1)^{i+j+1} & i > j \end{cases}$$

and

$$\mathrm{QI}_2(z, x, A, y, w) := x^t \, \mathrm{QI}_2(z, A, y) z = \sum_{i,j,k,l=1}^{n} \varepsilon(l, j)\varepsilon(k, i) x_k z_i y_l w_j \det A_{(l,j;k,i)}.$$

Remark B.1.12. QI stands for "quasiinverse" — we also could introduce the notations $\mathrm{QI}_1(A) := \widehat{A}$ and $\mathrm{QI}_1(x, A, y) := x^t \widehat{A} y$, but we will not use these notations here. Also we could introduce analogously for each $m \in \mathbb{N}$ a matrix $\mathrm{QI}_m(x_{m-1}, \ldots, x_1, A, y_1, \ldots, y_{m-1}) \in \mathrm{Mat}(n \times n, \Bbbk)$ by

$$\begin{aligned}\big(\mathrm{QI}_m(x_{m-1}, \ldots, x_1, A, y_1, \ldots, y_{m-1})\big)_{ij} &:= \\ \sum_{k_1, \ldots, k_{m-1}, l_1, \ldots, l_{m-1} = 1}^{n} &\varepsilon(l_1, \ldots, l_{m-1}, j)\varepsilon(k_1, \ldots, k_{m-1}, i) \times \\ &\times (x_1)_{k_1} \cdots (x_{m-1})_{k_{m-1}} \cdot (y_1)_{l_1} \cdots (y_{m-1})_{l_{m-1}} \cdot \det A_{(l_1, \ldots, l_{m-1}, j; k_1, \ldots, k_{m-1}, i)},\end{aligned}$$

and a value $\mathrm{QI}_m(x_m, \ldots, x_1, A, y_1, \ldots, y_m) \in \Bbbk$ by

$$\mathrm{QI}_m(x_m, \ldots, x_1, A, y_1, \ldots, y_m) := x_m^t \, \mathrm{QI}_m(x_{m-1}, \ldots, x_1, A, y_1, \ldots, y_{m-1}) y_m.$$

However, we will only use the case $k = 1, 2$ here.

For this construction we have similar lemmata as for the quasiinverse.

Lemma B.1.13. *Let $A \in \mathrm{Mat}(n \times n, \Bbbk)$ be of the following form:*

$$A = \begin{pmatrix} A_2 & y & w \\ x^t & a & b \\ z^t & c & d \end{pmatrix}$$

with $A_2 \in \mathrm{Mat}((n-2) \times (n-2), \Bbbk)$, $x, y, z, w \in \Bbbk^{n-2}$, $a, b, c, d \in \Bbbk$. Then

$$\det A = (ad - bc) \det A_2 + \mathrm{QI}_2(z, x, A_2, y, w) - a z^t \widehat{A_2} w - d x^t \widehat{A_2} y + b z^t \widehat{A_2} y + c x^t \widehat{A_2} w, \tag{B.4}$$

Proof. The proof is the same as (but more tedious than) in Lemma B.1.9.

Use Laplace's expansion law for the last row of A to get

$$\begin{aligned} \det A &= \sum_{i=1}^{n} (-1)^{n+i} a_{ni} \det A_{(n;i)} \\ &= d \det \begin{pmatrix} A_2 & y \\ x^t & a \end{pmatrix} - c \det \begin{pmatrix} A_2 & w \\ x^t & b \end{pmatrix} + \sum_{i=1}^{n-2} (-1)^{n+i} z_i \det \left(\begin{pmatrix} A_2 & y & w \\ x^t & a & b \end{pmatrix}_{(;i)} \right). \end{aligned}$$

For the first two summands we use Lemma B.1.9. For the last summand we use again Laplace's expansion law for the last row. We have to be a little bit careful there, since the matrix of this summand has yet a deleted row (and therefore is a $(n-1) \times (n-1)$-matrix). We get:

$$\begin{aligned} \det A = {} & d\big(a \det A_2 - x^t \widehat{A_2} y\big) - c\big(b \det A_2 - x^t \widehat{A_2} w\big) \\ & + \sum_{i=1}^{n-2} (-1)^{n+i} z_i \Bigg(b \det\Big(\begin{pmatrix} A_2 & y \end{pmatrix}_{(;i)}\Big) - a \det\Big(\begin{pmatrix} A_2 & w \end{pmatrix}_{(;i)}\Big) \\ & + \Bigg(\sum_{k=1}^{i-1} - \sum_{k=i+1}^{n-2} \Bigg) (-1)^{n-1+k} x_k z_i \det\Big(\begin{pmatrix} A_2 & y & w \end{pmatrix}_{(;k,i)}\Big) \Bigg) \\ = {} & (ad - bc) \det A_2 - d x^t \widehat{A_2} y + c x^t \widehat{A_2} w \\ & + \sum_{i=1}^{n-2} (-1)^{n+i} z_i \Bigg(b \det\Big(\begin{pmatrix} A_2 & y \end{pmatrix}_{(;i)}\Big) - a \det\Big(\begin{pmatrix} A_2 & w \end{pmatrix}_{(;i)}\Big) \Bigg) \\ & + \sum_{i=1}^{n-2} \sum_{k=1}^{n-2} -\varepsilon(k, i) x_k z_i \det\Big(\begin{pmatrix} A_2 & y & w \end{pmatrix}_{(;k,i)}\Big). \end{aligned}$$

Now use Laplace's expansion law for the last column of the remaining determinants. As in the proof of Lemma B.1.9 we get for the middle line after the last equation sign terms involving the quasiinverse. Hence we get:

$$\begin{aligned}\det A &= (ad-bc)\det A_2 - dx^t\widehat{A_2}y + cx^t\widehat{A_2}w \\ &\quad + bz^t\widehat{A_2}y - az^t\widehat{A_2}w \\ &\quad + \sum_{i=1}^{n-2}\sum_{k=1}^{n-2} -\varepsilon(k,i)x_k z_i \sum_{j=1}^{n-2}(-1)^{n-j}w_j \det\Big(\begin{pmatrix}A_2 & y\end{pmatrix}_{(j;k,i)}\Big).\end{aligned}$$

Now use a last time Laplace's expansion law (again with some care for the correct sign) to get

$$\begin{aligned}\det A &= (ad-bc)\det A_2 - az^t\widehat{A_2}w - dx^t\widehat{A_2}y + bz^t\widehat{A_2}y + cx^t\widehat{A_2}w \\ &\quad + \sum_{i=1}^{n-2}\sum_{k=1}^{n-2} -\varepsilon(k,i)x_k z_i \sum_{j=1}^{n-2}\sum_{l=1}^{n-2} -\varepsilon(l,j)y_l w_j \det(A_2)_{(l,j;k,i)}.\end{aligned}$$

The last line is exactly $\mathrm{QI}_2(z,x,A,y,w)$. □

Remark B.1.14. Again as a special case of the previous lemma we get (for $A \in \mathrm{Mat}(n \times n, \Bbbk)$, $x, y, z, w \in \Bbbk^n$) the formula

$$\mathrm{QI}_2(z,x,A,y,w) = \det\begin{pmatrix} A & y & w \\ x^t & 0 & 0 \\ z^t & 0 & 0\end{pmatrix}. \tag{B.5}$$

Lemma B.1.15. *Let* $A \in \mathrm{Mat}(n \times n, \Bbbk)$ *be of the following form:*

$$A = \begin{pmatrix} A_1 & y \\ x^t & a\end{pmatrix}$$

with $A_1 \in \mathrm{Mat}((n-1)\times(n-1), \Bbbk)$, $x, y \in \Bbbk^{n-1}$, $a \in \Bbbk$. *Then*

$$\widehat{A} = \begin{pmatrix} a\widehat{A_1} - \mathrm{QI}_2(x, A_1, y) & -\widehat{A_1}y \\ -x^t\widehat{A_1} & \det A_1\end{pmatrix}. \tag{B.6}$$

Proof. Set $B = \begin{pmatrix} a\widehat{A_1} - \mathrm{QI}_2(x, A_1, y) & -\widehat{A_1}y \\ -x^t\widehat{A_1} & \det A_1\end{pmatrix}$. We have to show that $z^t\widehat{A}w = z^tBw$ for all $z, w \in \Bbbk^n$. Write $z = \begin{pmatrix} z_1 \\ c\end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ b\end{pmatrix}$.

Then by equation (B.3) we have

$$z^t\widehat{A}w = -\det\begin{pmatrix} A & w \\ z^t & 0\end{pmatrix} = -\det\begin{pmatrix} A_1 & y & w_1 \\ x^t & a & b \\ z_1^t & c & 0\end{pmatrix} \tag{B.7}$$

On the other hand

$$\begin{aligned} z^t B w &= \begin{pmatrix} z_1 \\ c \end{pmatrix}^t \begin{pmatrix} a\widehat{A_1} - \mathrm{QI}_2(x, A_1, y) & -\widehat{A_1} y \\ -x^t \widehat{A_1} & \det A_1 \end{pmatrix} \begin{pmatrix} w_1 \\ b \end{pmatrix} \\ &= a z_1^t \widehat{A_1} w_1 - \mathrm{QI}_2(z_1, x, A_1, y, w_1) - b z_1^t \widehat{A_1} y - c x^t \widehat{A_1} w_1 + bc \det A_1 \end{aligned} \tag{B.8}$$

But by the previous lemma the right hand sides of equations (B.7) and (B.8) are equal. □

Lemma B.1.16. *For $A \in \mathrm{Mat}(n \times n, \Bbbk)$ and $x, y \in \Bbbk^n$ one has*

$$A\,\mathrm{QI}_2(x, A, y) = (x^t \widehat{A} y) \cdot \mathbb{1} - y x^t \widehat{A},$$
$$\mathrm{QI}_2(x, A, y) A = (x^t \widehat{A} y) \cdot \mathbb{1} - \widehat{A} y x^t,$$

and

$$\mathrm{QI}_2(x, A, y) y = 0, \qquad x^t\,\mathrm{QI}_2(x, A, y) = 0$$

Proof. Set $B = \begin{pmatrix} A & y \\ x^t & 0 \end{pmatrix}$. We have $\det B \cdot \mathbb{1} = \widehat{B} B = B \widehat{B}$, hence by Lemma B.1.9 and the previous lemma we get

$$\begin{aligned} (-x^t \widehat{A} y) \cdot \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} -\mathrm{QI}_2(x, A, y) & -\widehat{A} y \\ -x^t \widehat{A} & \det A \end{pmatrix} \begin{pmatrix} A & y \\ x^t & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\mathrm{QI}_2(x, A, y) A - \widehat{A} y x^t & -\mathrm{QI}_2(x, A, y) y \\ -x^t \widehat{A} A + \det A \cdot x^t & -x^t \widehat{A} y \end{pmatrix} \end{aligned}$$

as well as

$$\begin{aligned} (-x^t \widehat{A} y) \cdot \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} A & y \\ x^t & 0 \end{pmatrix} \begin{pmatrix} -\mathrm{QI}_2(x, A, y) & -\widehat{A} y \\ -x^t \widehat{A} & \det A \end{pmatrix} \\ &= \begin{pmatrix} -A\,\mathrm{QI}_2(x, A, y) - y x^t \widehat{A} & -A \widehat{A} y + \det A \cdot y \\ -x^t\,\mathrm{QI}_2(x, A, y) & -x^t \widehat{A} y \end{pmatrix} \end{aligned}$$

□

Lemma B.1.17. *For $A \in \mathrm{Mat}(n \times n, \Bbbk)$, $x, y, z, w \in \Bbbk^n$, $\lambda \in \Bbbk$ and $C \in GL(n, \Bbbk)$ we have:*

(lin) $\mathrm{QI}_2(x, \lambda A, y) = \lambda^{n-2}\,\mathrm{QI}_2(x, A, y)$ *resp.*
$\mathrm{QI}_2(z, x, \lambda A, y, w) = \lambda^{n-2}\,\mathrm{QI}_2(z, x, A, y, w)$ *and*
$\mathrm{QI}_2(\cdot, A, \cdot)$ *resp.* $\mathrm{QI}_2(\cdot, \cdot, A, \cdot, \cdot)$ *are linear in all two resp. four arguments.*

(alt) $\mathrm{QI}_2(x, x, A, y, w) = 0 = \mathrm{QI}_2(z, x, A, y, y)$.

(tr) $\big(\mathrm{QI}_2(x, A, y)\big)^t = \mathrm{QI}_2(y, A^t, x)$ *resp.*
$\mathrm{QI}_2(z, x, A, y, w) = \mathrm{QI}_2(w, y, A^t, x, z)$.

(bc) $\mathrm{QI}_2(C^t x, C^{-1}AC, C^{-1}y) = C^{-1}\,\mathrm{QI}_2(x, A, y)C$, *resp.*
$\mathrm{QI}_2(C^t z, C^t x, C^{-1}AC, C^{-1}y, C^{-1}w) = \mathrm{QI}_2(z, x, B, y, w)$.

(inv) $\mathrm{QI}_2(x, A, Ay)A = \det A \cdot \big((x^t y) \cdot \mathbb{1} - yx^t\big)$, *resp.*
$\mathrm{QI}_2(z, x, A, Ay, Aw) = \det A \cdot \big((x^t y)(z^t w) - (z^t y)(x^t w)\big)$.

(ker) (i) *If* $\det A = 0$ *and* $y, w \in \operatorname{im} A$, *then* $\mathrm{QI}_2(\cdot, \cdot, A, y, w) = 0$.
(ii) *If* $\operatorname{corank} A \geq 3$, *we have* $\mathrm{QI}_2(\cdot, A, \cdot) = 0$.

(sd) *($\Bbbk = \mathbb{R}$.) If A is symmetric and positive semidefinite, then*

$$\mathrm{QI}_2(z, x, A, x, z) \geq 0$$

(def) *($\Bbbk = \mathbb{R}$.) If A is symmetric and positive definite, then*

$$\mathrm{QI}_2(z, x, A, x, z) > 0 \quad \textit{if} \quad x, z \textit{ are linearly independent.}$$

Proof. (lin), (alt), and (tr) are again immediate by the definition ((tr) is also proven in the previous lemma).

The proof of (bc) is similar as in Lemma B.1.8. By Lemma B.1.15 we have

$$\begin{pmatrix} \mathrm{QI}_2(x, A, y) & \widehat{A}y \\ x^t \widehat{A} & \det A \end{pmatrix} = -\widehat{\begin{pmatrix} A & y \\ x^t & 0 \end{pmatrix}}.$$

Together with (bc) for the quasiinverse (see Lemma B.1.8) we therefore have

$$\begin{aligned}
\begin{pmatrix} \mathrm{QI}_2(C^t x, C^{-1}AC, C^{-1}y) & C^{-1}\widehat{A}y \\ x^t \widehat{A}C & \det A \end{pmatrix} &= -\widehat{\begin{pmatrix} C^{-1}AC & C^{-1}y \\ x^t C & 0 \end{pmatrix}} \\
&= -\begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}^{-1} \widehat{\begin{pmatrix} A & y \\ x^t & 0 \end{pmatrix}} \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{QI}_2(x, A, y) & \widehat{A}y \\ x^t \widehat{A} & \det A \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} C^{-1}\,\mathrm{QI}_2(x, A, y)C & C^{-1}\widehat{A}y \\ x^t \widehat{A}C & \det A \end{pmatrix}.
\end{aligned}$$

The corresponding formula for $\mathrm{QI}_2(\cdot, \cdot, \cdot, \cdot, \cdot)$ is immediate.

(inv) is an immediate consequence of Lemma B.1.16 and $\widehat{A}A = \det A \cdot \mathbb{1}$.

Furthermore (ker) (i) is then an immediate consequence of (inv) (in the same way as in the proof of Lemma B.1.8).

For (ker) (ii) note that if $\dim \ker A \leq 3$, then all $(n-2) \times (n-2)$-minors of A are zero, therefore the claim follows directly from the definition.

(def): From Lemma B.1.16 we get

$$\det A \cdot \mathrm{QI}_2(x, A, y) = \mathrm{QI}_2(x, A, y)A\widehat{A} = (x^t \widehat{A}y) \cdot \widehat{A} - \widehat{A}yx^t\widehat{A},$$

therefore

$$\det A \cdot \mathrm{QI}_2(z,x,A,y,w) = (x^t\widehat{A}y)(z^t\widehat{A}w) - (z^t\widehat{A}y)(x^t\widehat{A}w).$$

If A is symmetric, we get

$$\det A \cdot \mathrm{QI}_2(z,x,A,x,z) = (x^t\widehat{A}x)(z^t\widehat{A}z) - (x^t\widehat{A}z)^2.$$

If A is definite, so is $\widehat{A}$ (see Lemma B.1.8), hence

$$(x,y) \mapsto x^t\widehat{A}y$$

is a scalar product, therefore we have the Cauchy-Schwarz inequality

$$(x^t\widehat{A}y)^2 \leq (x^t\widehat{A}x)(y^t\widehat{A}y),$$

and equality holds if and only if x and y are linearly dependent. Since $\det A > 0$, this proves the claim.

(sd): Again, as in the proof of Lemma B.1.8 note that

$$A \mapsto \mathrm{QI}_2(\cdot,\cdot,A,\cdot,\cdot) \in \big(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n\big)^\vee$$

(here $(\cdot)^\vee$ denotes the dual vector space) is continuous, hence (sd) can be deduced from (def) again (as in the proof of Lemma B.1.8) by considering a semidefinite matrix as a limit of definite matrices or alternatively by assuming (as well as in the proof of Lemma B.1.8 by (bc)) w.l.o.g. that A is a diagonal matrix and calculating $\mathrm{QI}_2(z,x,A,x,z)$ directly (which is of course more tedious than in the proof of Lemma B.1.8). □

If we collect, we get for the case of invertible upper-left corners the following lemmata:

Lemma B.1.18. *Let $A \in \mathrm{Mat}(n \times n, \Bbbk)$ be of the following form:*

$$A = \begin{pmatrix} A_1 & y \\ x^t & a \end{pmatrix}$$

with $A_1 \in \mathrm{Mat}((n-1)\times(n-1), \Bbbk)$, $x, y \in \Bbbk^{n-1}$, $a \in \Bbbk$. If A_1 is invertible, then

$$\det A = \det A_1 \cdot \big(a - x^t A_1^{-1} y\big). \tag{B.9}$$

Proof. With Lemma B.1.8 (inv) this is a special case of Lemma B.1.9. □

Lemma B.1.19. *Let $A \in \mathrm{Mat}(n \times n, \Bbbk)$ be of the following form:*

$$A = \begin{pmatrix} A_2 & y & w \\ x^t & a & b \\ z^t & c & d \end{pmatrix}$$

with $A_2 \in \mathrm{Mat}((n-2)\times(n-2), \Bbbk)$, $x, y, z, w \in \Bbbk^{n-2}$, $a, b, c, d \in \Bbbk$. If A_2 is invertible, then

$$\begin{aligned} \det A = \det A_2 \cdot \big((ad-bc) &+ (x^tA_2^{-1}y)(z^tA_2^{-1}w) - (z^tA_2^{-1}y)(x^tA_2^{-1}w) \\ &- az^tA_2^{-1}w - dx^tA_2^{-1}y + bz^tA_2^{-1}y + cx^tA_2^{-1}w\big), \end{aligned} \tag{B.10}$$

Proof. As in the proof of Lemma B.1.17 (ker) we get from Lemma B.1.16

$$\det A_2 \cdot \mathrm{QI}_2(z, x, A_2, y, w) = (x^t \widehat{A_2} y)(z^t \widehat{A_2} w) - (z^t \widehat{A_2} y)(x^t \widehat{A_2} w).$$

Therefore the claim follows from Lemma B.1.13 and Lemma B.1.8 (inv). □

Appendix C

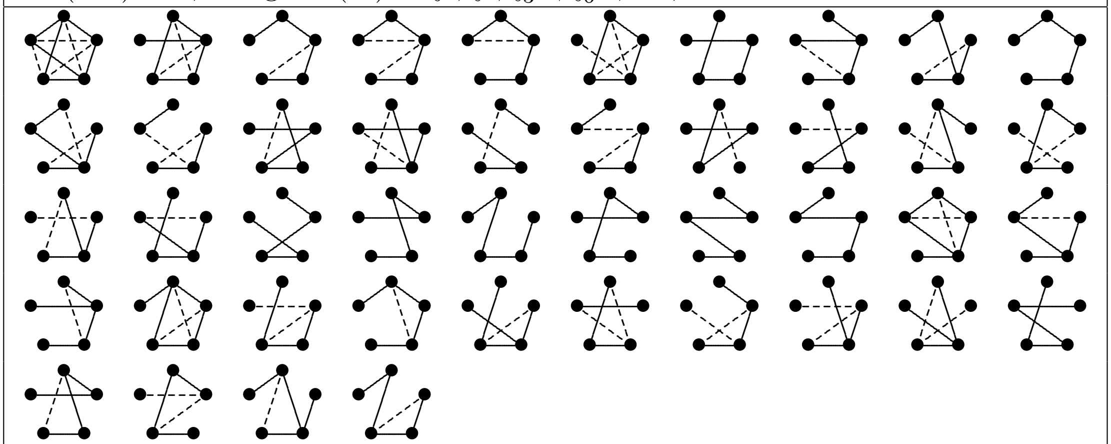
Some Figures

花より団子

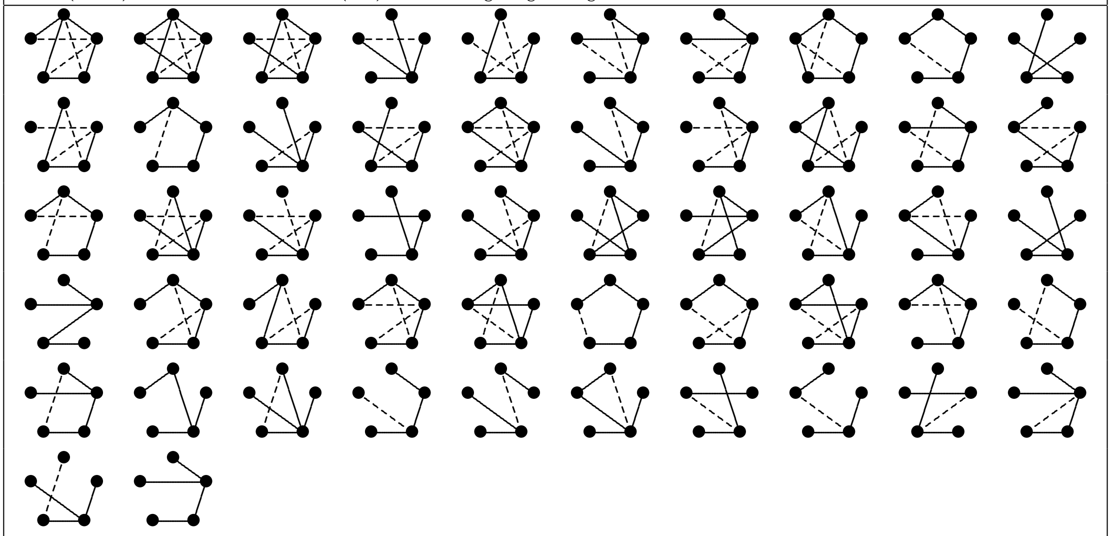
Dumplings rather than flowers

Orbits of connected Coxeter-Dynkin diagrams under the operation of $\mathbf{Br_5} \ltimes (\mathbb{Z}/2\mathbb{Z})^{\mathbf{5}}$

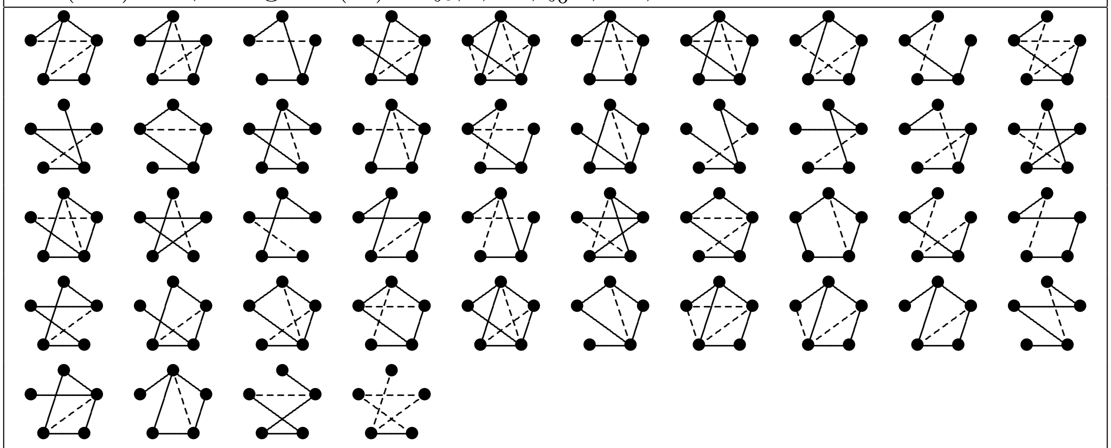
$\det(-S) = 6$, eig.val.$(m) = \zeta_6, \zeta_3, \zeta_3^{-1}, \zeta_6^{-1}, -1$, $\operatorname{tr} m = -1$



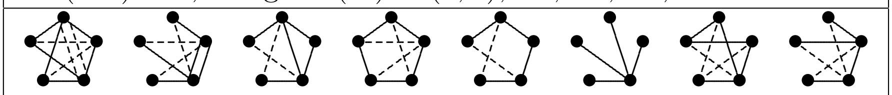
$\det(-S) = 4, \quad \text{eig.val.}(m) = \zeta_8, \zeta_8^3, \zeta_8^{-3}, \zeta_8^{-1}, -1, \quad \operatorname{tr} m = -1$



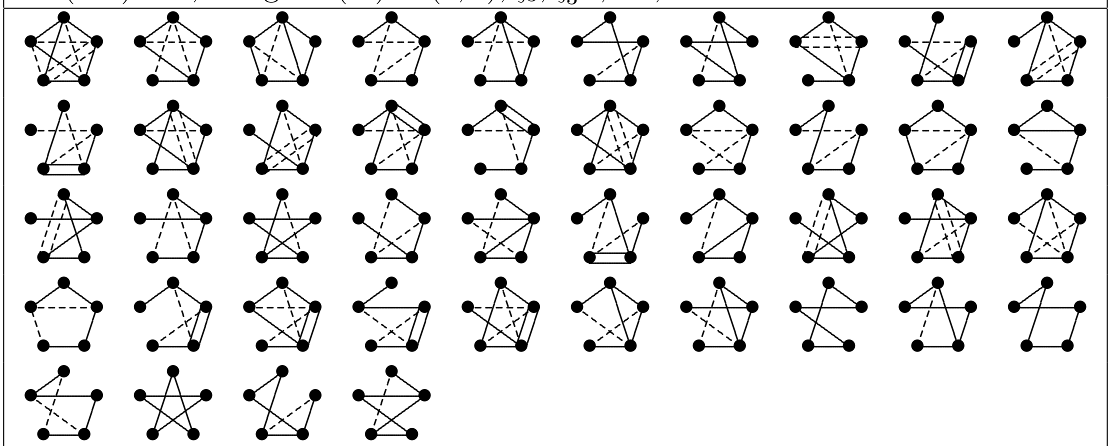
$\det(-S) = 4, \quad \text{eig.val.}(m) = \zeta_6, i, -i, \zeta_6^{-1}, -1, \quad \operatorname{tr} m = 0$



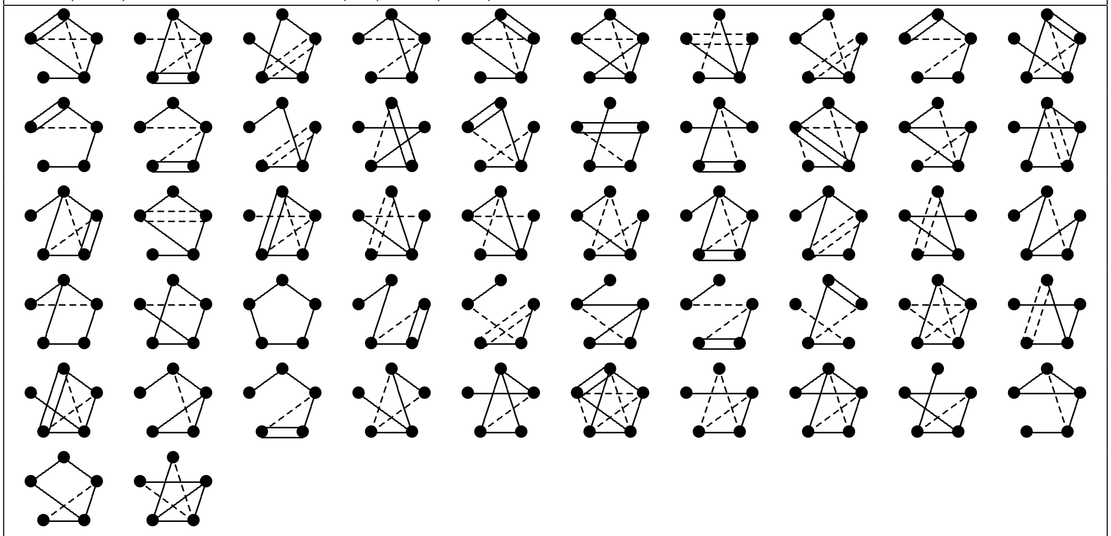
$\det(-S) = 0, \quad \text{eig.val.}(m) = (1,1), -1, -1, -1, \quad \operatorname{tr} m = -1$



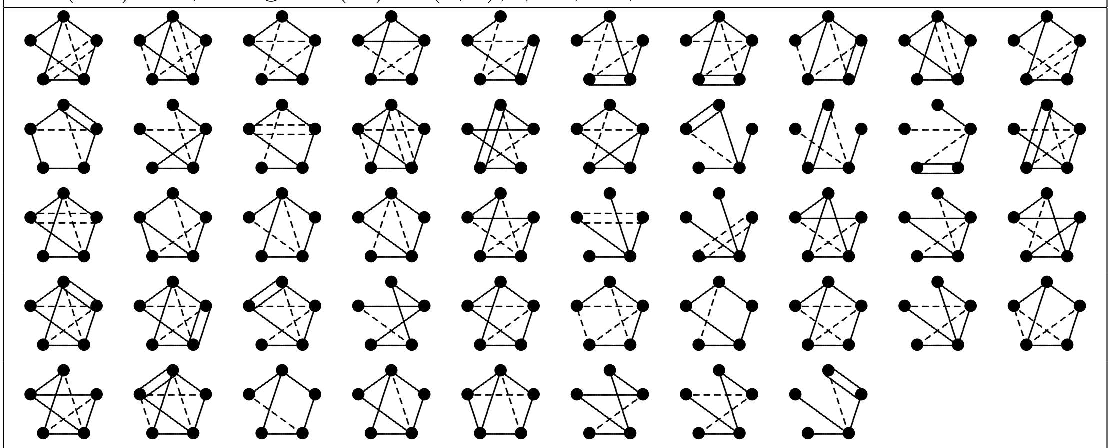
$\det(-S) = 0, \quad \text{eig.val.}(m) = (1,1), \zeta_3, \zeta_3^{-1}, -1, \quad \operatorname{tr} m = 0$



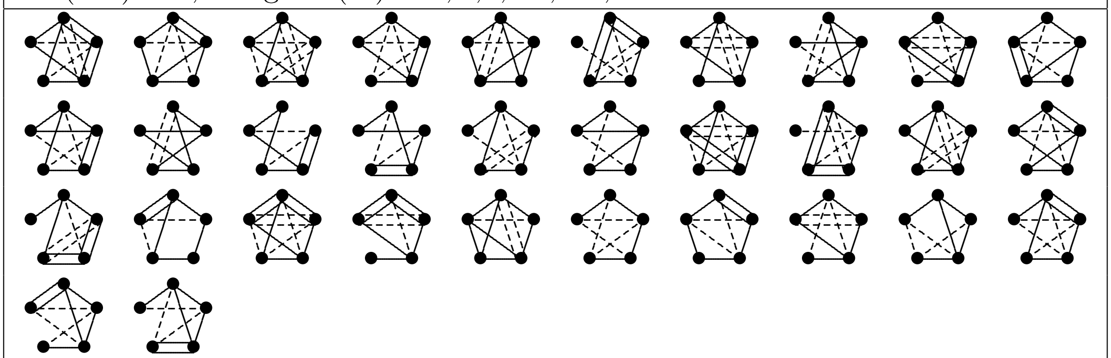
$\det(-S) = 0, \quad \text{eig.val.}(m) = (1,1), i, -i, -1, \quad \operatorname{tr} m = 1$



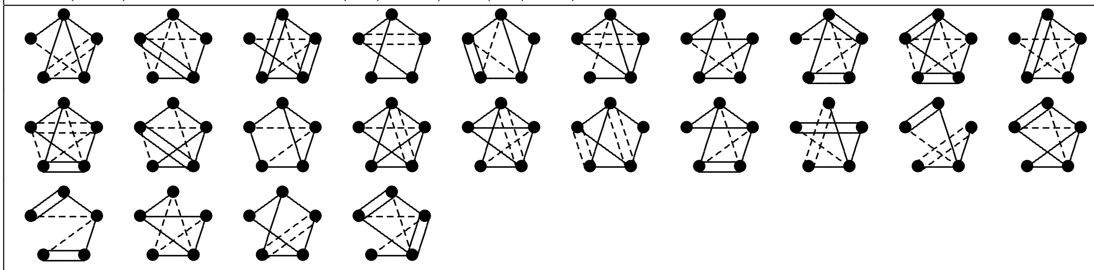
$\det(-S) = 0, \quad \text{eig.val.}(m) = (1,1), i, -i, -1, \quad \operatorname{tr} m = 1$



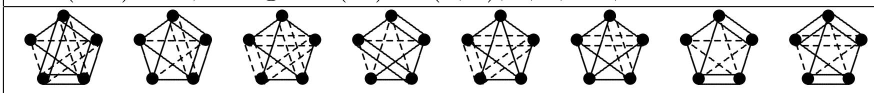
$\det(-S) = 0, \quad \text{eig.val.}(m) = 1, 1, i, -i, -1, \quad \operatorname{tr} m = 1$



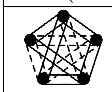
$\det(-S) = 0, \quad \text{eig.val.}(m) = (1,1),(1,1),-1, \quad \operatorname{tr} m = 3$



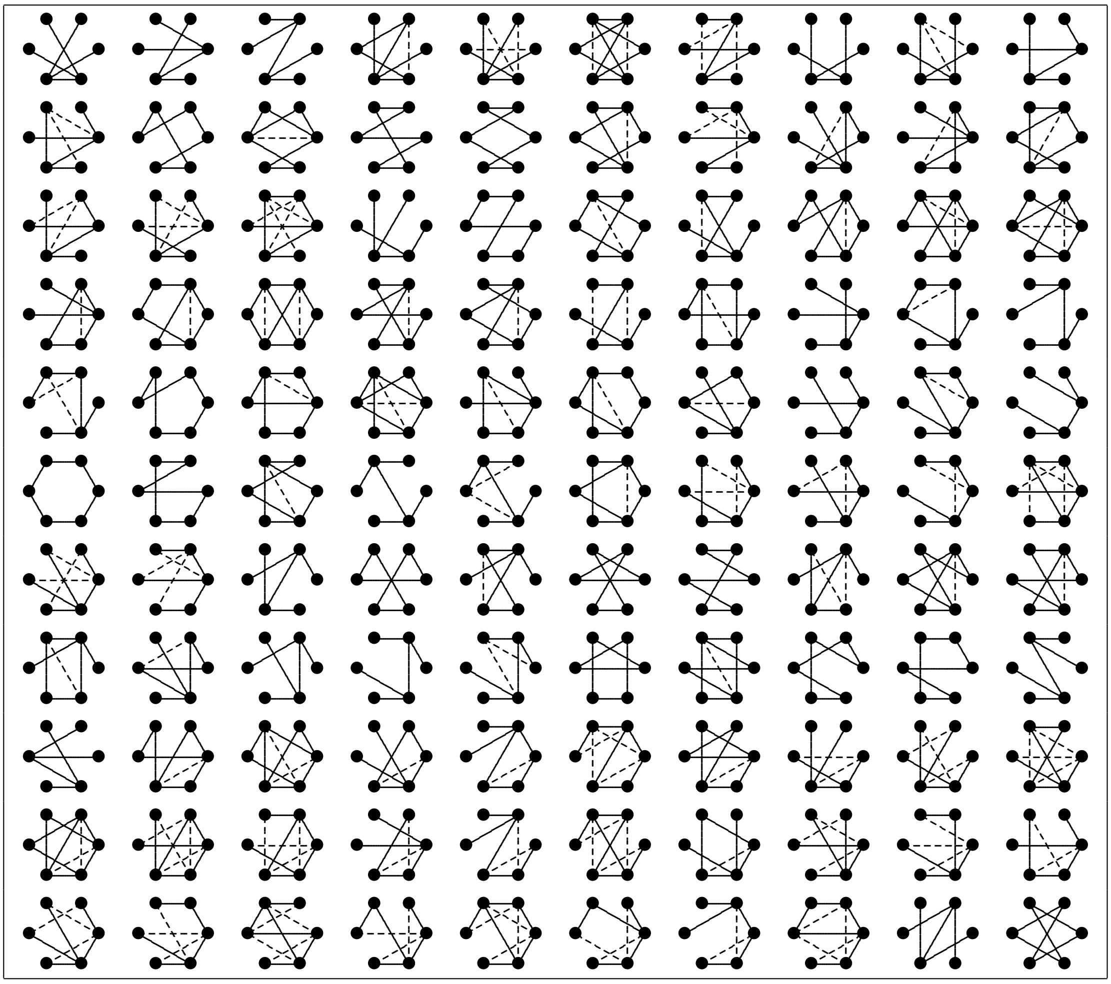
$\det(-S) = 0, \quad \text{eig.val.}(m) = (1,1),1,1,-1, \quad \operatorname{tr} m = 3$



$\det(-S) = 0, \quad \text{eig.val.}(m) = 1,1,1,1,-1, \quad \operatorname{tr} m = 3$



Coxeter-Dynkin diagrams for the proof of the conjectures for $\mu = 6$



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