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On the Statistical Mechanics of Dense Instanton Gases[1]

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Abstract. Instanton gases of two-dimensional $\mathbb{C}P^{n-1}$ and four-dimensional $SU(n)$ Yang-Mills theories are considered. The presumable denseness of instanton gases in these models and the corresponding statistics of instantons lead to a theormodynamic limit in which the coupling constant dependence of non-perturbative quantities is modified by a factor proportional to $\frac{1}{n}$ compared to the case of a dilute gas. As a consequence the large n limit and the infinite volume limit do not appear to commute. We present a naive droplet model for dense instanton gases which exhibits these features. Possible consequences for the large order behaviour of perturbation series are discussed.

1. Introduction

In this article we consider the instanton gases of two-dimensional $\mathbb{C}P^{n-1}$ and four-dimensional $SU(n)$ Yang-Mills theories. Instantons are solutions of the classical equations of motions in Euclidean space-time [1, 2]. As stationary points of the Euclidean action they give contributions to the saddle point approximation of Euclidean path integrals [3, 4]. Both models possess multi-instanton and multi-anti-instanton solutions, labelled by a topological charge $K \in \mathbb{Z}$, with action $S_K = K \cdot S_1$. A K-instanton solution depends on several parameters. The contribution to the saddle point approximation involves the determinant of the fluctuation operator around the saddle point and integrations over the parameters.

To evaluate these contributions is an extremely difficult task [5–7]. Therefore one tries to estimate the multi-instanton contributions in some approximation scheme. In particular it has been proposed to consider a dilute gas of instantons and anti-instantons [8]. The basic idea of the dilute gas approximation is the following. The one-instanton solution looks like a localized object and can be parametrized by the location vector a_μ of its center and a scale size λ. The multi-instanton solutions are then approximated by a superposition of single instantons with coordinates $a_\mu^{(j)}$ and scale sizes λ_j. Also quasi saddle points consisting of both instantons and anti-instantons are included. If this ensemble is treated as a dilute gas the contribution to the partition function Z is just the exponentiated one-instanton contribution. The pressure of this grand-canonical ensemble is then given by

$$p = \lim_{V \to \infty} \frac{1}{V} \ln Z = D(g) \exp\left(-\frac{S_1}{g^2}\right) \tag{1.1}$$

where $D(g)$ is the integral of the fluctuation determinant over the zero modes (without translations) for the one-instanton solution and g is a renormalized coupling constant. Now it turns out that the integration over the instanton scale size λ is infrared divergent like

$$\int_0^\infty d\lambda\, \lambda^b, \quad b \geqq -1. \tag{1.2}$$

But this means that the dilute gas does not have a thermodynamic limit. According to (1.2) large instantons contribute much more to the partition function than small ones, which suggests that the instanton gas is rather dense [5, 6, 9]. For the $\mathbb{C}P^1$ case this can be shown to be the case [5, 6].

It is well known in statistical mechanics that a dense gas behaves quantitatively differently from a dilute one. It would therefore be interesting to study the consequences of the presumable denseness of the instanton gas. In order to do this it is necessary to take the correct thermodynamic limit. This means that the infinite volume limit has to be taken only after summing the instanton gas in a finite volume. In Sect. 2 we shall discuss the consequences of this

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limit on the basis of renormalization group considerations. The main result will be a correction factor κ^{-1} proportional to $\frac{1}{n}$ which multiplies the instanton action S_1 in non-perturbative contributions to the pressure such that

$$\exp\left(-\frac{S_1}{\kappa g^2}\right) = \exp\left(-\frac{1}{2\beta_0 g^2}\right) \tag{1.3}$$

where β_0 is the first coefficient in the Callan–Symanzik β-function.

In Sect. 3 we present a naive droplet model for a dense instanton gas in order to give an explicit example for the mechanism that leads to the behaviour (1.3) in the correct thermodynamic limit. Section 4 contains critical remarks and a discussion of possible consequences for the large n limit and the large-order behaviour of perturbation series. In particular the large n limit and the infinite volume limit appear not to commute.

2. Renormalization Group and Thermodynamic Limit of an Instanton Gas

We write the dependence of the action on the bare coupling g_0 in the form

$$\text{action: } \int d^\nu x\, \mathscr{L} = \frac{1}{g_0^2} S \tag{2.1}$$

where S does not contain $g_0^\star$. Let us first consider the one-instanton solution in the infinite volume space-time. It depends on several parameters amongst which are a position vector a_μ and a scale size λ. Let α_i denote the remaining dimensionless collective coordinates which give rise to zero modes in the fluctuation operator Δ around the classical solution. In the saddle point approximation the contribution of the instanton to the partition function normalized to the vacuum contribution is given by (see eg. [3])

$$W_1 = (\sqrt{2\pi}\, g_0)^{-m} \int d a_\mu \frac{d\lambda}{\lambda} d\alpha_i J(\alpha_i) \left(\frac{\det' \Delta}{\det \Delta_0}\right)^{-1/2} \cdot e^{-(S_1/g_0^2)} \tag{2.2}$$

Here m is the number of collective coordinates, J is a collective coordinate Jacobian and the determinants include only non-zero modes of the fluctuation operators Δ and Δ_0 (vacuum fluctuations)**. For details see e.g. [10]. The determinants are ultraviolet divergent as in ordinary perturbation theory and require renormalization. With an ultraviolet cutoff M one finds in one loop order [3–5]

$$J(\alpha_i)\left(\frac{\det' \Delta}{\det \Delta_0}\right)_M^{-1/2} = \exp(\kappa \ln M) f(\alpha_i, \lambda) \tag{2.3}$$

with some constant κ and a finite function f. This leads to an integrand

$$\exp\left\{-S_1\left(\frac{1}{g_0^2} - \frac{\kappa}{S_1}\ln M\right)\right\} f(\alpha_i) \tag{2.4}$$

On the other hand one knows from the renormalization group that the coupling constant gets renormalized according to

$$\frac{1}{g^2(\mu)} = \frac{1}{g_0^2} - 2\beta_0 \ln\frac{M}{\mu} \tag{2.5}$$

in one loop order, where a renormalization mass scale μ has been introduced. That renormalization around instantons is the same as for the perturbative saddle point has been shown in [11]. Therefore in order for a renormalized determinant to exist we must have

$$\kappa = 2\beta_0 S_1 \tag{2.6}$$

which actually is found by explicit calculation. From that we get a factor

$$\exp\left\{-S_1\left(\frac{1}{g^2(\mu)} - \frac{\kappa}{S_1}\ln\mu\right)\right\} = \left\{\mu \exp\left(-\frac{1}{2\beta_0 g^2(\mu)}\right)\right\}^\kappa = \Lambda^\kappa \tag{2.7}$$

in the integrand. Λ is the renormalization group invariant mass scale, which is given by (2.7) in one loop order. Different renormalization schemes of course lead to different scales Λ, but this ambiguity is not important for the present discussion. When above is inserted in (2.2) the λ-dependence of the integrand is fixed on dimensional grounds (the α_i are dimensionless)

$$\begin{aligned} W_1 &= (\sqrt{2\pi} g)^{-m} \int d a_\mu d\lambda \lambda^{-\nu-1} (\lambda\Lambda)^\kappa \int d\alpha_i f(\alpha_i) \\ &= C \Lambda^\kappa \int d a_\mu d\lambda \lambda^{\kappa-\nu-1} g^{-m} \end{aligned} \tag{2.8}$$

(ν space-time dimensions)

In one loop order the power of g is not renormalized but it is expected that higher orders lead to a proper renormalization. So it might be replaced by a $g(\lambda^{-1})$ inside the integral, but we are interested in the dominant λ-dependence, which is given by the power $\kappa-\nu-1$. In $SU(n)$ Yang–Mills theory we have

$$S_1 = 8\pi^2, \quad \beta_0 = \frac{11}{3}\frac{n}{16\pi^2}, \quad \kappa = \frac{11}{3} n \tag{2.9}$$

and in the $\mathbb{C}P^{n-1}$ model

$$S_1 = \pi, \quad \beta_0 = \frac{n}{2\pi}, \quad \kappa = n \tag{2.10}$$

In both cases

$$\kappa \geqq \nu \tag{2.11}$$

leads to the well known infrared divergence.

* For the $\mathbb{C}P^{n-1}$ models $f = ng^2$ is the coupling constant used in [2, 7]

** For complex variables in the path integral det Δ actually stands for the square of the fluctuation determinant

Let us now consider the case of a finite space-time volume V. For appropriate boundary conditions, e.g. periodic for gauge invariant quantities, it is supposed that instanton solutions exist which tend to the usual ones in the limit $V \to \infty$. In the $\mathbb{C}P^{n-1}$ case for example the self-duality equations are identical to Cauchy–Riemann equations in certain coordinates [2]. In a finite volume with periodic boundary conditions these are solved (at least for even topological charge) by Jacobian elliptic functions.

In general we should expect that the former instantons turn into quasi saddle points whose action depend on $\lambda V^{-1/\nu}$. The dilatational zero mode is then changed into a quasi zero mode with a very small eigenvalue, which goes to zero for $\lambda V^{-1/\nu} \to 0$ and has to be treated as a collective coordinate. We get from (2.8)

$$W_1 = \Lambda^\kappa h(V) \tag{2.12}$$

and on dimensional grounds this must be

$$W_1 = C_1 (\Lambda^\nu V)^{\kappa/\nu} \tag{2.13}$$

Now consider the K-instanton contribution W_K. From the action we get a factor $\exp\left(-\frac{K S_1}{g_0^2}\right)$. The renormalization of the fluctuation determinants converts this into

$$\exp\left\{-K S_1\left(\frac{1}{g^2(\mu)} - 2\beta_0 \ln\mu\right)\right\} = \Lambda^{K \cdot \kappa} \tag{2.14}$$

Therefore we shall get as in (2.13) for dimensional reasons

$$W_K = C_K (\Lambda^\nu V)^{K(\kappa/\nu)} \tag{2.15}$$

In order to perform the thermodynamic limit we first have to sum all contributions to W for finite V. Then we have to consider

$$p = \lim_{V\to\infty} \frac{1}{V} \ln W \tag{2.16}$$

Let us assume for the moment that p exists. Then (2.15) implies

$$p = \gamma\Lambda^\nu = \gamma\mu^\nu \exp\left(-\frac{\nu}{2\beta_0 g^2(\mu)}\right) \tag{2.17}$$

where γ is some constant. Comparing this with the coupling constant dependence of the one-instanton contribution (2.8)

$$W_1 \sim \exp\left(-\frac{S_1}{g^2(\mu)}\right) \tag{2.18}$$

one recognizes in (2.17) the appearance of a correction factor $\frac{\nu}{\kappa}$ proportional to $\frac{1}{n}$, which multiplies the instanton action S_1. Consequences of this will be discussed in Sect. 4.

3. Simple Droplet Model for a Dense Instanton Gas

The question if the thermodynamic limit of the instanton gas exists depends on the behaviour of the coefficients c_K in (2.15) for large K. The existence of the pressure p in the infinite volume limit requires

$$C_K = (K!)^{-\kappa/\nu} q^{K(\kappa/\nu)} K^a \left(1 + \mathcal{O}\left(\frac{1}{K}\right)\right) \tag{3.1}$$

$$q = \frac{\nu}{\kappa}\gamma$$

for large K. One might wonder how such odd powers of $K!$ could arise in the calculation of the K-instanton contribution. In this section we shall see that a simple model for a dense gas of instantons produces (3.1) and therefore supports the possibility that the thermodynamic limit of an exact instanton gas exists.

In general the contribution of a K-instanton solution to the partition function normalized to the vacuum contribution is (cp. 2.2) [5–7, 10]

$$W_K = (\sqrt{2\pi} g_0)^{-mK} e^{-K(S_1/g_0^2)} \int d\xi_i J(\xi_i) \left(\frac{\det' \Delta_K}{\det \Delta_0}\right)^{-1/2} \tag{3.2}$$

where ξ_i are collective coordinates with Jacobian $J(\xi_i)$ and Δ_K the K-instanton fluctuation determinant. The renormalization of the determinants leads as in the case of W_1 to the replacement

$$\frac{1}{g_0^2} \to \frac{1}{g^2(\mu)} - 2\beta_0 \ln\mu \tag{3.3}$$

in one loop order. Thus

$$W_K = \Lambda^{K \cdot \kappa} \int d\xi_i e^{-U(\xi_i)} (\sqrt{2\pi} g)^{-mK} \tag{3.4}$$

and $U(\xi_i)$ is come complicated potential.

Because the potential $U(\xi_i)$ is in general intractable we now introduce some simplifications that lead us to a droplet model of instantons. First of all we introduce locations $a_\mu^{(j)}$ and scale sizes λ_j, $j = 1, \ldots, K$ as in the dilute gas approximation and write

$$\begin{aligned} W_K = \Lambda^{K\kappa} \frac{1}{K!} \int \prod_j \frac{d a^{(j)} d\lambda_j}{\lambda_j^{\nu+1}} \int \prod_{i,j} d\alpha_i^{(j)} \prod_j \lambda_j^\kappa \\ e^{-\tilde{U}(a^{(j)}, \lambda_j, \alpha_i^{(j)})} (\sqrt{2\pi} g(\lambda_j^{-1}))^{-m} \end{aligned} \tag{3.5}$$

where $\alpha_i^{(j)}$ are the remaining collective coordinates. For separated instantons we know that the scale dependence is proportional to $\lambda_j^{\kappa-\nu-1} d\lambda_j$ (see (2.8)) which grows with λ_j. In this case the potential $\tilde{U}$ is negligible. The largest contribution to W_K will come from configurations where the volume is densely filled with instantons and $\tilde{U}$ becomes important. This leads to the next simplification, in which the complicated potential $\tilde{U}$ in (3.5) is just replaced by a droplet potential. This potential excludes all those configurations in which some instantons overlap. In a box of volume V it is defined by

$$e^{-\tilde{U}} = \Theta(a^{(j)}, \lambda_j)$$

$$\equiv \begin{cases} 1, \text{ if i) } \|a^{(j)} - a^{(i)}\| > \left(\frac{\tau}{v_1}\right)^{(1/\nu)} (\lambda_j + \lambda_i) \quad \forall i,j \\ \quad \text{ii) } |a_\mu^{(j)}| < \frac{1}{2} V^{(1/\nu)} \quad \forall j, \mu \\ 0, \text{ otherwise} \end{cases} \tag{3.6}$$

where

$$v_1 = \frac{\pi^{\nu/2}}{\Gamma\left(\frac{\nu}{2}+1\right)} \tag{3.7}$$

is the volume of a unit ball in ν dimensions and the parameter τ determines the effective volume $\tau\lambda_j^\nu$ of a single instanton. If we denote the collective coordinate integral of a single instanton by C, cp. (2.8), we get

$$W_K = C^K \Lambda^{\kappa K} \int \prod_j d a^{(j)} d\lambda_j \prod_j \lambda_j^{\kappa-\nu-1} g(\lambda_j^{-1})^{-m} \cdot \Theta(a^{(j)},\lambda_j)\frac{1}{K!} \tag{3.8}$$

$$= C^K \Lambda^{\kappa K}\frac{1}{K!}\int_0^V dv \prod_j da^{(j)} d\lambda_j \prod_j \lambda_j^{\kappa-\nu-1} g(\lambda_j^{-1})^{-m} \cdot\delta\left(v - \tau\sum_j \lambda_j^\nu\right)\Theta \tag{3.9}$$

The total effective volume

$$v = \tau\sum_{j=1}^K \lambda_j^\nu \tag{3.10}$$

of course obeys

$$v \leqq V \tag{3.11}$$

For fixed v the dependence of the integrand on the λ_j is dominated by $\prod_j \lambda_j^{\kappa-\nu-1}$ because $g(\lambda_j^{-1})$ varies only logarithmically with λ_j. If K is large the function $\prod_j \lambda_j^{\kappa-\nu-1}$ at fixed v has a sharp maximum at

$$\lambda_j = \left(\frac{v}{K\tau}\right)^{1/\nu} \equiv \lambda_0, \quad j=1,\dots,K, \quad \text{for } \kappa-\nu-1>0. \tag{3.12}$$

When

$$\lambda_j = \lambda_0 + \delta_j, \quad \sum_j \delta_j = 0 \tag{3.13}$$

is a small deviation from the maximum for v fixed, we get

$$\prod_j \lambda_j^{\kappa-\nu-1} \approx \left(\frac{v}{K\tau}\right)^{(K/\nu)(\kappa-\nu-1)} \cdot \exp\left\{-\frac{1}{2}(\kappa-\nu-1)\lambda_0^{-2}\sum_j\delta_j^2\right\} \tag{3.14}$$

and we approximate the λ_j-integrals by the corresponding Gaussian integral

$$\lambda_0^{K(\kappa-\nu-1)}\int\prod_j d\delta_j\, \delta\left(\tau\nu\lambda_0^{\nu-1}\sum_j\delta_j\right) \cdot \exp\left\{-\frac{1}{2}(\kappa-\nu-1)\lambda_0^{-2}\sum_j\delta_j^2\right\}$$

$$= (\tau\nu)^{-1}\lambda_0^{K(\kappa-\nu-1)-\nu+1}\left(\frac{2\pi\lambda_0^2}{\kappa-\nu-1}\right)^{(K-1)/2} \tag{3.15}$$

$$= (\tau\nu)^{-1}\left(\frac{2\pi}{\kappa-\nu-1}\right)^{(K-1)/2}\left(\frac{v}{K\tau}\right)^{K(\kappa/\nu)-K-1}$$

We are left with

$$W_K = C^K\Lambda^{\kappa K}(\tau\nu)^{-1}\left(\frac{2\pi}{\kappa-\nu-1}\right)^{(K-1)/2} \times\frac{1}{K!}\int_0^V dv\prod_j da^{(j)}\left(\frac{v}{K\tau}\right)^{(K\kappa/\nu)-K-1} g(\lambda_0^{-1})^{-mK}\,\Theta(a^{(j)}) \tag{3.16}$$

where

$$\Theta(a^{(j)}) \equiv \begin{cases}1, \text{ if i) } \|a^{(j)}-a^{(i)}\| > 2\left(\frac{v}{Kv_1}\right)^{(1/\nu)}, \quad \forall i,j\\ \quad\text{ii) } |a_\mu^{(j)}| < \frac12 V^{(1/\nu)}, \quad\forall j,\mu \\ 0, \text{ otherwise}\end{cases} \tag{3.17}$$

The integral has been reduced to an ensemble of equally large balls in V. Since $g(\lambda_0^{-1})$ varies slowly compared to the other function in the integrand we shall replace it by $g(\bar\lambda^{-1})$ where $\bar\lambda$ is the mean value of λ_0 to be determined later. If we keep v fixed the $a^{(j)}$ integrations produce the effective free volume of hard spheres in a volume V. A good approximation is

$$\int\prod_j da^{(j)}\,\Theta(a^{(j)}) \approx (V-v)^K \tag{3.18}$$

as used by Ornstein in his treatment of the van der Waals gas [12] (This is exact in 1 dimension.)

The remaining integral is

$$\frac{1}{K!}\int_0^V dv(V-v)^K\left(\frac{v}{K\tau}\right)^{(K\kappa/\nu)-K-1}$$

$$= V^{(K\kappa/\nu)}(K\tau)^{-(K\kappa/\nu)+K+1}\frac{\Gamma\left(\frac{K\kappa}{\nu}-K\right)}{\Gamma\left(\frac{K\kappa}{\nu}+1\right)} \tag{3.19}$$

Using Stirlings formula we obtain for large K

$$(3.19) = \text{const.}\,(A\,V)^{K(\kappa/\nu)}(K!)^{-\kappa/\nu}K^{\kappa/2\nu}$$

$$A = \left(\frac{1-\frac{\nu}{\kappa}}{e\tau}\right)^{1-(\nu/\kappa)}\left(\frac{\nu}{\kappa}\right)^{\nu/\kappa} \tag{3.20}$$

The resulting expression for W_K is

$$W_K = \text{const.}\,(q\,V\Lambda^\nu)^{K\kappa/\nu}(K!)^{-(\kappa/\nu)}K^{\kappa/2\nu} \tag{3.21}$$

where

$$q = A\left[C\left(\frac{2\pi}{\kappa-\nu-1}\right)^{1/2} g(\bar\lambda^{-1})^{-m}\right]^{\nu/\kappa} \tag{3.22}$$

In (3.21) we recognize the desired power of $K!$ which is appropriate for the existence of the thermodynamic limit.

To simplify the discussion let us consider only a pure instanton gas, i.e. we take

$$W = \sum_{K=0}^\infty W_K \tag{3.23}$$

The problem of including anti-instantons and mixed instantion-anti-instantons configurations is non-trivial and we comment on this in Sect. 4.

From (3.21) we find the pressure

$$p = \lim_{V\to\infty}\frac{1}{V}\ln W = \frac{\kappa}{\nu}q\Lambda^\nu = \gamma\Lambda^\nu \tag{3.24}$$

as in (2.17).

The instanton density is

$$\rho = \lim_{V\to\infty} W^{-1}\sum_{K=0}^\infty\frac{K}{V}W_K = \frac{\nu}{\kappa}\gamma\frac{\partial}{\partial\gamma}p$$

$$= \frac{\nu}{\kappa}p = q\Lambda^\nu \tag{3.25}$$

and the mean instanton scale $\bar\lambda$ can be calculated to be

$$\bar\lambda = \left\{\tau^{-1}\rho^{-1}\left(1-\frac{\nu}{\kappa}\right)\right\}^{1/\nu}$$

$$= \left\{\tau^{-1}q^{-1}\left(1-\frac{\nu}{\kappa}\right)\right\}^{1/\nu}\Lambda^{-1} \tag{3.26}$$

For $SU(3)$ Yang–Mills theory in $\nu=4$ dimensions

$$\kappa = 11,\quad m=12,\quad C=3.6\times10^8\ [10]$$

$$A = 0.27\,\tau^{-7/11} = 0.1\left(\frac{\tau}{v_1}\right)^{-7/11} \tag{3.27}$$

$$q = 130\,g(\bar\lambda^{-1})^{-48/11}\left(\frac{\tau}{v_1}\right)^{-7/11}$$

and with $\tau\approx v_1$ we obtain

$$\bar\lambda = 0.1\,\Lambda^{-1}, \quad \frac{g^2(\bar\lambda)}{4\pi} = 0.25 \tag{3.28}$$

4. Critical Remarks and Discussion of Possible Implications

Let us start with a critical discussion of the dense gas approximation in Sect. 3. The dominant contribution to the pressure comes from those configurations where the volume is densely filled with instantons of a size $\bar\lambda \approx 0,1\,\Lambda^{-1}$. On this length scale one would expect that other field configurations distinct from multi-instanton solutions also start to play an important role in the path integral. Furthermore the description of multi-instantons by a superposition of several single instantons should break down when they start to come close to each other. Therefore the dense gas picture of the previous section is not believed to describe properly the physics of non-perturbative contributions to the path integral.

Instead we consider it more to be an illustration of the general features which are implied by the denseness of instantons. In particular the appearance of the correction factor $\frac{\nu}{\kappa}$ is supposed to be a general consequence of the statistics of a dense medium independent of the detailed dynamics.

If one asks for example for the differential density $\rho(\lambda)$ of instantons of size λ at small values of λ the result turns out to be

$$\rho(\lambda) \sim \frac{1}{\lambda}\lambda^{(\nu/\kappa)(\kappa-\nu)} \tag{4.1}$$

whereas in the dilute gas approximation

$$\rho_{DG}(\lambda) \sim \frac{1}{\lambda}\lambda^{\kappa-\nu} \tag{4.2}$$

Even in the case of an operator which is sensitive only to small instantons, the instanton contribution to its expectation value will thus be modified.

A doubtful point already mentioned in the previous section concerns the treatment of anti-instantons. We only considered a pure instanton gas. The only known real finite-action saddle points are pure multi-instantons and pure multi-anti-instantons. As is well known this leads to problems with the cluster property if one likes to take into account both. Usually these are avoided by including approximate mixed configurations [8]. Another solution has been proposed recently [13], namely to include complex saddle points which are a substitute for mixed configurations. As a different solution to the problem spontaneous breakdown of the symmetry between instantons and anti-instantons has been proposed [14]. Because these questions are still open we restrict the discussion here to a pure instanton gas and expect that qualitative results are not changed through some inclusion of anti-instantons.

Next we would like to discuss implications of the statistics of dense instantons.

4.1. *Implications for Large n*

The large n limit of $\mathbb{C}P^{n-1}$ models and $SU(n)$ Yang–Mills theory is believed to give considerable insight into the non-perturbative physics of these models also for moderate n. For reviews see [15]. The large n limit is defined by

$$n\to\infty, \quad g^2 n = x \quad \text{fixed} \tag{4.3}$$

Let us consider the K-instanton contribution in this limit. In a fixed renormalization scheme it depends on $g=g(\mu)$ in the following way, cp. (2.14)

$$W_K \sim \exp\left(-\frac{K S_1}{g^2}\right) = \exp\left(-\frac{K S_1}{x} n\right) \tag{4.4}$$

where S_1 is independent of n. From this we conclude that in the large n limit the instanton contributions W_K vanish exponentially with n. This is essentially also the basis of Witten's argument against instantons in the $\frac{1}{n}$ expansion [16].

On the other hand we have seen that the statistics of dense instantons produces a correction factor $\frac{v}{\kappa}$ proportional to $\frac{1}{n}$ in the thermodynamic limit. The coulpling constant dependence of the pressure is, cp. (2.17)

$$p \sim \exp\left(-\frac{v}{\kappa}\frac{S_1}{g^2}\right) = \exp\left(-\frac{\text{const.}}{x}\right) \tag{4.5}$$

and does not vanish in the large n limit. *Thus the infinite volume limit and the large n limit do not commute.* If the thermodynamic limit is taken first instantons will survive the large n limit.

In the case of $\mathbb{C}P^{n-1}$ models the fact that large n and infinite volume limit are not interchangeable can be shown explicitly. The topological susceptibility χ_t [22] in a finite volume goes to zero like e^{-n} for large n [23] and is thus not present in the $\frac{1}{n}$ expansion. On the other hand if the infinite volume limit is taken first χ_t is proportional to $\frac{1}{n}$ in the $\frac{1}{n}$ expansion [2].

4.2. *Implications for Large Order Estimates in Perturbation Theory*

For the large order behaviour of coefficients in perturbation series singularities in the Borel plane play an important role [17–19]. Instantons are supposed to produce singularities on the positive real axis at multiples of S_1 on the level of bare theories. For asymptotically free theories it is not clear if this is altered through renormalization [17]. Arguments have been given that the singularity at $2S_1$ dominates the large order behaviour im perturbation theory. Under this assumption the coefficients of the Callan–Symanzik β-function for $SU(n)$ Yang–Mills theory behave asymptotically like [20, 17]

$$\beta_{K-1} \underset{K\to\infty}{\sim} K!\,K^a (2S_1)^{-K} = K!\,K^a (16\pi^2)^{-K} \tag{4.6}$$

where a is some constant.

On the other hand singularities in the Borel plane are also related to the coupling constant dependence of non-perturbative contributions to path integrals [18]. This suggests that the factor $\frac{v}{\kappa}$ which appears in the coupling constant dependence of p is a signal for a shift of the instanton singularities from multiples of S_1 to multiples of $\kappa^{-1}S_1 = (2\beta_0)^{-1}$ in the Borel plane. If this is true (4.6) would be changed into

$$\beta_{K-1} \underset{K\to\infty}{\sim} K!\,K^a b^K \left(\frac{n}{16\pi^2}\right)^K \tag{4.7}$$

with b independent of n. Actual perturbative calculations yield a factor $\left(\frac{n}{16\pi^2}\right)^K$ from K-loop diagrams [21] and strongly support the correctness of (4.7).

In this case the first coefficient β_0 determines the behaviour of β_K for K large. This resembles a similar reciprocity law in semiclassical expansions for the quartic oscillator [18].

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