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A remark on the isomorphism conjectures

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Abstract. We show that for various natural classes of groups and appropriately defined K- and L-theoretic functors, injectivity or bijectivity of the assembly map follows from the isomorphism conjecture being true for acyclic groups lying within that class.

1. Introduction

A group G is acyclic if the reduced homology $\tilde{H}_*(G;\mathbb{Z})$ is 0. It is well known that every (torsion-free) group embeds as a subgroup into a (torsion-free) acyclic group. It follows that Kaplansky's idempotent conjecture (cp. [16, p. 55]) holds for every torsion-free group if and only if it holds for every torsion-free acyclic group. Berrick, Chatterji and Mislin [4] prove that every (torsion-free) group G embeds as a subgroup into a (torsion-free) acyclic group $A(G)$ such that the conjugacy relations are preserved, i.e. $g_1 \sim_G g_2$ in G if and only if $g_1 \sim_{A(G)} g_2$ in $A(G)$ for any two elements $g_1, g_2 \in G$. This implies that the Bass conjecture (cp. [16, p. 66]) holds for any torsion-free group if and only if it holds for any torsion-free acyclic group. In the note, we consider the isomorphism conjectures, such as Baum–Connes conjecture and Farrell–Jones conjecture. For more information on these conjectures, see Mislin–Vallete [16] and Lück–Reich [15]. We prove that the fact that isomorphism conjectures hold for any torsion-free acyclic group implies that the assembly maps are injective for any torsion-free group. One interesting corollary is that the isomorphism conjectures hold for any torsion-free group if and only if the assembly maps are surjective for any torsion-free group.

Note that the isomorphism conjectures considered in this note are not the fibered versions with coefficients (cp. [3]), which are stable under passage to subgroups. Since every group embeds into an acyclic group, the corresponding results for fibered isomorphism conjectures are obviously true.

2. Statement of results

We will use the set-up of [9], with which we assume familiarity. For a discrete group G, a set $\mathcal{E}$ of subgroups of G is called a *family of subgroups* if it is closed under conjugation and taking subgroups. In other words, for any $H \in \mathcal{E}$, $K \le H$ and any $g \in G$, we have $gHg^{-1} \in \mathcal{E}$ and $K \in \mathcal{E}$. Typical examples of $\mathcal{E}$ are

$$\{1\} = \{\text{trivial subgroup}\}, \quad \mathcal{F}\text{in} = \{\text{finite subgroups}\},$$
$$\mathcal{VCY} = \{\text{virtually cyclic subgroups}\}, \quad \mathcal{ALL} = \{\text{all subgroups}\}.$$

For a family $\mathcal{E}$ of subgroups, the classifying space $E_{\mathcal{E}}(G)$ is uniquely characterized up to equivariant homotopy by the property that the fixed-point set $E_{\mathcal{E}}(G)^H$ is contractible for any $H \in \mathcal{E}$ and is empty for any $H \notin \mathcal{E}$. Let $H_*^G(-;\mathbb{K}^t)$ denote the equivariant homology theory associated to the topological K-theory $\mathrm{Or}(G)$-spectrum $\mathbb{K}^t$. Let $E_{\mathcal{F}\text{in}}(G)$ be the space classifying proper actions of G. The Baum–Connes conjecture (as reformulated in [9]) asserts that the assembly map

$$H_*^G(E_{\mathcal{F}\text{in}}(G);\mathbb{K}^t) \to K_*^t(C_r^*(G)) \tag{1}$$

is an isomorphism for all $*$, where the groups on the right are the topological K-groups of the reduced C^*-algebra of G. We will write BC for the Baum–Connes conjecture, MBC resp. EBC for the conjecture that the assembly map in (1) is a monomorphism resp. epimorphism, and R-BC (resp. R-MBC resp. R-EBC) for the conjecture that the Baum–Connes assembly map becomes an isomorphism (resp. monomorphism resp. epimorphism) after tensoring both sides of (1) with a subring $R \subseteq \mathbb{Q}$. Finally, R-BC(G) resp. R-MBC(G) resp. R-EBC(G) will denote the conjecture that R-BC resp. R-MBC resp. R-EBC holds for a particular group G. Let $\mathcal{G}$ be the class of all groups. Given a subclass $\mathcal{C} \subset \mathcal{ALL}$, we say that R-IC, R-EC, or R-MC holds over $\mathcal{C}$ if the conjecture is true for all groups in $\mathcal{C}$. The subclasses of interest here are:

(i) $\mathcal{TF} \subset \mathcal{G}$, consisting of all torsion-free discrete groups, and
(ii) $\mathcal{FF} \subset \mathcal{TF}$, the subcollection of groups G for which $BG \simeq X$ is a finite complex (called FF groups).

Theorem 2.1. *Let R be a subring of $\mathbb{Q}$. Let $\mathcal{C} = \mathcal{G}$, $\mathcal{TF}$ or $\mathcal{FF}$, the class of all groups, torsion-free groups or groups with finite classifying spaces. If* R-BC(G) *holds true for all acyclic groups in $\mathcal{C}$, then* R-MBC *is true for all groups in $\mathcal{C}$.*

The assembly map considered above is a special case of a much more general construction. For suitably defined functors F on the class $\mathcal{G}$ of discrete groups, one has an assembly map

$$HF_*(G) \to F_*(G) \tag{2}$$

and the *isomorphism conjecture* (IC) [9] asserts that this map is an isomorphism, where $HF_*(-)$ denotes the appropriate homology group associated to F. For any G, there is a unique G-map from $E_{\mathcal{F}\text{in}}(G)$ to a point. If $\mathcal{H}_*$

is any equivariant homology theory (cp. [15]), then the assembly conjecture for the triple $\mathcal{H}_*$, $\mathcal{F}\mathrm{in}$ and G asserts that the induced map from $\mathcal{H}_*(E_{\mathcal{F}\mathrm{in}}(G))$ to $\mathcal{H}_*(\mathrm{pt})$ is an isomorphism, where pt denotes a point with trivial G-action. Following the definitions given above, the *epimorphism conjecture* (EC) resp. *monomorphism conjecture* (MC) for the theory being considered states that the assembly map in (2) is a monomorphism resp. epimorphism. Again, given a subring $R \subset \mathbb{Q}$, the conjecture R-IC resp. R-EC resp. R-MC is the conjecture that the assembly map is an isomorphism resp. epimorphism resp. monomorphism after tensoring with R, with the appendage "(G)" indicating the conjecture for a particular group G.

Theorem 2.2. *Let $F_*(G) = L_*^{\langle -\infty\rangle}(\mathbb{Z}[G])$, with*

$$HF_*(G) := H_*^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbb{L}^{\langle -\infty\rangle}(\mathbb{Z}))$$

the equivariant homology group associated to the algebraic L-theory $\mathrm{Or}(G)$-*spectrum $\mathbb{L}^{\langle -\infty\rangle}(\mathbb{Z})$. Let $\mathcal{C} = \mathcal{G}$, $\mathcal{TF}$ or $\mathcal{FF}$. Fix $R \subset \mathbb{Q}$. If $\frac{1}{2} \in R$ and* R-IC(G) *is true for the functor F for all acyclic groups in $\mathcal{C}$, then* R-IC *holds for F over $\mathcal{C}$. If $\mathcal{C} \subseteq \mathcal{T}F$, the implication holds without restriction on R. In particular, the Novikov conjecture holds for all groups in $\mathcal{C}$ if the assembly map for F is a rational isomorphism for all acyclic $G \in \mathcal{C}$.*

Let $KH(S)$ denote the homotopy K-theory spectrum of the discrete ring S, as defined by Weibel in [22].

Theorem 2.3. *Let $F_*(G) = KH_*(\mathbb{Z}[G])$, with*

$$HF_*(G) := H_*^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbb{KH}(\mathbb{Z})).$$

Let $\mathcal{C} = \mathcal{G}$, $\mathcal{T}F$ or $\mathcal{FF}$. Let R be a subring of $\mathbb{Q}$. If R-IC *holds for F for all acyclic groups in $\mathcal{C}$, then* R-IC *holds for F over $\mathcal{C}$.*

For ordinary algebraic K-theory, a slightly weaker result can be shown.

Theorem 2.4. *For a discrete ring S, set $FS_*(G) = K_*(S[G])$, with*

$$HFS_*(G) := H_*^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbb{K}(S)).$$

Let $\mathcal{C} = \mathcal{G}$ or $\mathcal{TF}$ and R an arbitrary subring of $\mathbb{Q}$.

(i) *If* $\mathbb{Q}$-IC *holds for $F\mathbb{Z}$ for all acyclic groups in $\mathcal{C}$, then* $\mathbb{Q}$-MC *holds for $F\mathbb{Z}$ over $\mathcal{C}$.*
(ii) *Let S be a regular ring containing the rationals $\mathbb{Q}$. If* R-IC *holds for FS for all acyclic groups in $\mathcal{C}$, then* R-MC *holds for FS over $\mathcal{C}$.*
(iii) *Let S be a regular ring. If* R-IC *holds for FS for all acyclic groups in $\mathcal{FF}$, then* R-MC *holds for FS over $\mathcal{FF}$.*

3. Proofs of the theorems

The proof in all cases is based on the method of [11, §6.5]. For any discrete group G, a classical construction allows us to embed G in an acyclic group $A(G)$ (its acyclic envelope), with the inclusion $i_G : G \hookrightarrow A(G)$ being functorial in G. Now the variation of the Kan–Thurston construction detailed in

[5, Thm. 2.4] produces a group $T(G)$ together with a surjective homomorphism $p_G : T(G) \to G$ inducing an homology equivalence. The association $G \mapsto T(G)$ is functorial in G; moreover $T(G)$ lies in the *Waldhausen–Cappell class* $\mathfrak{C}$ consisting of those groups which can be constructed from free groups by (i) amalgamated free products, (ii) HNN extensions, and (iii) taking direct unions. Additionally, as shown in [5, Thm. 2.4], starting with a group $G' \in \mathfrak{C}$, the acyclic envelope $A(G')$ can be formed so as to remain inside of $\mathfrak{C}$. In the case $\mathcal{C} = \mathcal{G}$ or $\mathcal{TF}$, $A(T(G))$ will denote Block's construction of this envelope. Let $A_1 = G \times A(T(G))$ and $A_2 = A(T(G))$. There are inclusions

$$\begin{aligned} T(G) &\hookrightarrow A_1, \quad g \mapsto (p_G(g), i_{T(G)}(g)),\\ T(G) &\hookrightarrow A_2, \quad g \mapsto i_{T(G)}(g). \end{aligned}$$

Let $A_3 = A_1 *_{T(G)} A_2$. By an application of Mayer–Vietoris sequence, the group A_3 is acyclic.

In what follows, we will, for all of the functors considered above, write $HF_*(G)$ for $H_*^G(E_{\mathcal{F}\mathrm{in}}G; \mathbb{F})$, where $\mathbb{F}$ denotes the $\mathrm{Or}(G)$-spectrum associated to F. There is a homomorphism of sequences where the horizontal arrows are given by assembly:

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ HF_{n+1}(A_3) & \xrightarrow{\phi^3_{n+1}} & F_{n+1}(A_3) \\ \downarrow \partial & & \downarrow \partial \\ HF_n(T(G)) & \xrightarrow{\phi^T_n} & F_n(T(G)) \\ \downarrow & & \downarrow \\ HF_n(A_1) \oplus HF_n(A_2) & \xrightarrow{\phi^1_n \oplus \phi^2_n} & F_n(A_1) \oplus F_n(A_2) \\ \downarrow & & \downarrow \\ HF_n(A_3) & \xrightarrow{\phi^3_n} & F_n(A_3) \\ \downarrow \partial & & \downarrow \partial \\ HF_{n-1}(T(G)) & \xrightarrow{\phi^T_{n-1}} & F_{n-1}(T(G)) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array}$$

As noted in [16, p. 25], the space $E_{\mathcal{F}\mathrm{in}}(A_3)$ is equivalent (up to equivariant homotopy) to the homotopy push-out of the diagram

$$\begin{array}{ccc} A_3 \times_{T(G)} E_{\mathcal{F}\mathrm{in}}(T(G)) & \longrightarrow & A_3 \times_{A_1} E_{\mathcal{F}\mathrm{in}}(A_1) \\ \downarrow & & \\ A_3 \times_{A_2} E_{\mathcal{F}\mathrm{in}}(A_2) & & \end{array}$$

by which one may derive the exactness of the left sequence for coefficients in any $\mathrm{Or}(A_3)$-spectrum. The commutativity of the diagram, as well as the exactness of the right column, is the point that needs to be verified. We consider first the case $\mathcal{C} = \mathcal{G}$ or $\mathcal{TF}$ for the functor $F_*(G) = K_*^t(C_r^*(G))$; here exactness of the right column follows by the results of Pimsner [19], while the commutativity of the diagram has been shown by Oyono-Oyono [18]. As noted in [5], the result of [19] implies that ϕ_*^T is an isomorphism. By the same reasoning, ϕ_*^2 is an isomorphism, and ϕ_*^3 is an isomorphism by hypothesis. The five-lemma then implies that ϕ_*^1 must be an isomorphism as well.

For a $\mathbb{Z}[\mathrm{Or}(G)]$-module M and G-CW complex X, denote by $H_*^{\mathrm{Or}(G)}(X;M)$ the Bredon homology of X with coefficients M. Since the groups in $\mathfrak{C}$ are torsion-free, every finite subgroup of A_1 is contained in G and thus the family of finite subgroups of A_1 is the same as that of G. Taking $M = \pi_i(\mathbb{K}^{\mathrm{top}})$ viewed both as an $\mathbb{Z}[\mathrm{Or}(A_1)]$-module and as an $\mathbb{Z}[\mathrm{Or}(G)]$-module, one has isomorphisms

$$\begin{aligned} H_n^{\mathrm{Or}(A_1)}(E_{\mathcal{F}\mathrm{in}}(A_1);M) &\cong H_n^{\mathrm{Or}(A_1)}(E_{\mathcal{F}\mathrm{in}}(G) \times E(A(T(G)));M) \\ &\cong H_n^{\mathrm{Or}(G)}(E_{\mathcal{F}\mathrm{in}}(G) \times \mathrm{B}A(T(G));M) \\ &\cong H_n^{\mathrm{Or}(G)}(E_{\mathcal{F}\mathrm{in}}(G);M). \end{aligned}$$

By the equivariant Atiyah–Hirzebruch spectral sequence (cp. [9]), there is an isomorphism

$$H_n^{A_1}(E_{\mathcal{F}\mathrm{in}}(A_1);\mathbb{K}^{\mathrm{top}}) \cong H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbb{K}^{\mathrm{top}}), \quad n \in \mathbb{Z}.$$

Therefore, the inclusion map $G \to A_1$ induces an injection

$$\begin{aligned} &\ker(H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbb{K}^{\mathrm{top}}) \to K_n(C_r^*(G))) \\ &\qquad \subset \ker(H_n^{A_1}(E_{\mathcal{F}\mathrm{in}}(A_1);\mathbb{K}^{\mathrm{top}}) \to K_n(C_r^*(A_1))). \end{aligned}$$

This implies that the assembly map $H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbb{K}^{\mathrm{top}}) \to K_n(C_r^*(G))$ is injective, which completes the proof of Theorem 2.1 for $R = \mathbb{Z}$. Tensoring with any ring flat over $\mathbb{Z}$ yields the same result for all $R \subset \mathbb{Q}$.

For $\mathcal{C} = \mathcal{G}$ or $\mathcal{TF}$, the proofs of Theorems 2.2 and 2.3 follow exactly the same line of reasoning, after applying the following modifications:

- In the case $F_*(G) = L_*^{\langle -\infty \rangle}(\mathbb{Z}[G])$, the exactness of the right column follows by the results of [7], the one complication being the possible existence of $U\mathrm{Nil}$-terms. These terms vanish when tensoring with any R containing $\frac{1}{2}$, or in the case the groups in question are torsion-free. For this functor, the assembly map is an integral isomorphism for groups in the class $\mathfrak{C}$ by [7, 8].
- For $F_*(G) = KH_*(\mathbb{Z}[G])$, the corresponding results (exactness of right column and equivalence of assembly map for $\mathfrak{C}$-groups) are shown in [1].
- In both cases we have functoriality with respect to arbitrary group homomorphisms, not just injective ones. The injection $G \rightarrowtail A_1$ of the first factor, the projection $A_1 \twoheadrightarrow G$ onto the first factor, and the naturality of the assembly map together allow us to conclude that R-IC for the group

A_1 implies R-IC for G. (In the case of the reduced C^*-algebra, it is unknown in general whether the projection $A_1 \twoheadrightarrow G$ defines an appropriate element of $KK(C_r^*(A_1), C_r^*(G))$. If it does, then the stronger conclusions of Theorems 2.2 and 2.3 would apply as well to Theorem 2.1.)

We next consider the smaller class $\mathcal{FF}$. In order to duplicate the above argument, the construction of the acyclic envelope requires modification, as Block's construction does not preserve this class. Instead (as in [11]), we use Leary's metric refinement of the Kan–Thurston construction [14]. To any complex X Leary associates a locally CAT(0) cubical complex $C(X)$ together with a map $p_X : C(X) \to X$ which is an epimorphism on π_1 and an isomorphism in homology. The association $X \mapsto (C(X), p_X)$ is functorial in X; moreover if X is finite, so is $C(X)$.

Let $G \in \mathcal{FF}$, and fix a finite basepointed complex X_G with $X_G \simeq BG$. Let $\widehat{X_G}$ denote the cone on X_G; then the canonical inclusion $X_G \hookrightarrow \widehat{X_G}$ is covered by an inclusion of locally CAT(0) cubical complexes $C(X_G) \hookrightarrow C(\widehat{X_G})$. Define the groups A_i, $1 \leq i \leq 3$ by

$$A_1 := G \times \pi_1(C(\widehat{X_G})), \quad A_2 := \pi_1(C(\widehat{X_G})), \quad A_3 := A_1 *_{\pi_1(C(X_G))} A_2,$$

where $\pi_1(C(X_G)) \hookrightarrow \pi_1(C(\widehat{X_G}))$ is the inclusion of CAT(0)-groups corresponding to the inclusion $X_G \hookrightarrow \widehat{X_G}$ and the inclusion $\pi_1(C(X_G)) \hookrightarrow A_1$ is similar to the inclusion $T(G) \hookrightarrow A_1$ defined in the first paragraph of this proof. (Leary shows that for any inclusion of complexes $X \hookrightarrow Y$, the resulting inclusion $C(X) \hookrightarrow C(Y)$ is isometric and that the image is a totally geodesic subcomplex of $C(Y)$, implying injectivity on π_1.) Writing $L_*^{\langle -\infty \rangle}(\mathbb{Z}[H])$ as $L_*(\mathbb{Z}[H])$ and $H_*(BH; \mathbb{L}(\mathbb{Z}))$ simply as $HL_*(BH)$, one has as before a commuting diagram of long-exact sequences with the horizontal maps induced by assembly:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
HL_{n+1}(BA_3) & \xrightarrow{\psi^3_{n+1}} & L_{n+1}(\mathbb{Z}[A_3]) \\
\downarrow \partial & & \downarrow \partial \\
HL_n(B\pi_1(C(X_G))) & \xrightarrow{\psi^C_n} & L_n(\mathbb{Z}[\pi_1(C(X_G))]) \\
\downarrow & & \downarrow \\
HL_n(BA_1) \oplus HL_n(BA_2) & \xrightarrow{\psi^1_n \oplus \psi^2_n} & L_n(\mathbb{Z}[A_1]) \oplus L_n(\mathbb{Z}[A_2]) \\
\downarrow & & \downarrow \\
HL_n(BA_3) & \xrightarrow{\psi^3_n} & L_n(\mathbb{Z}[A_3]) \\
\downarrow \partial & & \downarrow \partial \\
HL_{n-1}(B\pi_1(C(X_G))) & \xrightarrow{\psi^C_{n-1}} & L_{n-1}(\mathbb{Z}[\pi_1(C(X_G))]) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

Both A_2 and $\pi_1(C(X_G))$ are fundamental groups of finite locally CAT(0) cubical complexes; it follows from the results of [2] that the assembly maps ψ_*^C and ψ_*^2 are isomorphisms. Moreover, $HL_*(BA_1) \cong HL_*(BG)$, and so as before one has an identification of kernels

$$\ker(\psi_*^1) \cong \ker(HL_*(BG) \to L_*(\mathbb{Z}[G]))$$

which, together with the injectivity of ψ_*^3, yields an injection

$$\ker(HL_*(BG) \to L_*(\mathbb{Z}[G])) \cong \ker(\psi_*^1) \hookrightarrow \operatorname{coker}(\psi_{*+1}^3).$$

As all of the groups in the above diagram are objects in the category $\mathcal{FF}$, we arrive at the same conclusion as before. This completes the proof of Theorem 2.2. In the case of the reduced group C^*-algebra, the same argument for torsion-free groups applies, given that groups acting properly on cubical CAT(0)-complexes satisfy the Haagerup property [17], and thus satisfy the strong BC conjecture by the work of Higson–Kasparov [10], which completes the proof of Theorem 2.1.

Next we consider the statement of the third theorem when $\mathcal{C} = \mathcal{FF}$. For brevity, we say that G satisfies condition $\mathcal{F}$CAT if it acts properly, isometrically and cocompactly on a finite-dimensional CAT(0)-space. (More precisely, one only needs G to be in the class $\mathcal{B}$ as given in [2, Def. 1] for Lemma 3.1 to apply.)

Lemma 3.1. *Suppose G satisfies $\mathcal{F}$CAT. Then the natural transformation of spectrum-valued functors $\mathbb{K}(-) \to \mathbb{KH}(-)$ from algebraic to homotopy K-theory induces a weak equivalence*

$$\mathbb{K}(\mathbb{Z}[G]) \xrightarrow{\simeq} \mathbb{KH}(\mathbb{Z}[G]).$$

Proof. For an arbitrary ring A, there exists a right half-plane spectral sequence (cp. [22, Thm 1.3]):

$$E^1_{pq} := N^pK_q(A) \Rightarrow KH_{p+q}(A), \quad p \geq 0,\ q \in \mathbb{Z}.$$

For the group ring $A = \mathbb{Z}[G]$ and the integer $p > 0$, the groups $N^pK_*(\mathbb{Z}[G])$ are summands of $K_*(\mathbb{Z}[G \times \mathbb{Z}^p])$. But if G satisfies $\mathcal{F}$CAT, so does $G \times \mathbb{Z}^p$ for all $p \geq 0$. Again, by the main result of [2, 21], these summands identify isomorphically with the corresponding summands in the domain of the Farrell–Jones assembly map, where they vanish. Thus for such groups, $N^pK_*(\mathbb{Z}[G]) = 0$ for all $p > 0$, yielding the required isomorphism on homotopy groups in all degrees. □

Thus the Farrell–Jones assembly map for $KH(-)$ (which for torsion-free groups agrees with the classical assembly map $H_*(BG; \mathbb{K}(\mathbb{Z}) \to KH_*(\mathbb{Z}[G]))$ is an isomorphism for G satisfying $\mathcal{F}$CAT (cp. [21]). With this additional fact in hand, the proof of Theorem 2.3 is complete.

Unlike the reduced C^*-algebra, the full (or maximal) group C^*-algebra is functorial with respect to arbitrary group homomorphisms. The methods of the previous two theorems imply the following.

Corollary 3.2. *There exist acyclic groups G for which the assembly map*
$$H_*^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbb{K}^t) \to K_*^t(C^*(G))$$
fails to be an isomorphism, even rationally.

Proof. Suppose that for all acyclic groups the assembly maps are isomorphisms. A proof similar to those of Theorem 2.3 and 2.4 gives that the assembly map
$$H_*^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbb{K}^t) \to K_*^t(C^*(G))$$
is isomorphic for any G. Lafforgue [13] proved that for some infinite group K with Kazhdan's property (T), the Baum–Connes assembly map
$$H_*^G(E_{\mathcal{F}\mathrm{in}}(K);\mathbb{K}^t) \to K_*^t(C_r^*(K))$$
is an isomorphism. Since the latter map factors through the former (cp. [16, p. 83]), we have an isomorphism
$$K_*^t(C^*(K)) \cong K_*^t(C_r^*(K)).$$
However, it is well known that for any infinite group with property (T) these groups are not isomorphic, even rationally (cp. [12, Cor. 3.1] and its proof). This gives a contradiction. ☐

Finally, we consider the statement of Theorem 2.4. Here the results of Waldhausen [20] produce a Mayer–Vietoris type of long-exact sequence which appears as the right column in the following commuting diagram:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
HFS_{n+1}(A_3) & \xrightarrow{\phi_{n+1}^3} & K_{n+1}(S[A_3]) \\
\downarrow \partial & & \downarrow \partial \\
HFS_n(T(G)) & \xrightarrow{\phi_n^T} & K_n(S[T(G)]) \oplus \mathrm{Nil}_n(T(G),A_1,A_2) \\
\downarrow & & \downarrow \\
HFS_n(A_1) \oplus HFS_n(A_2) & \xrightarrow{\phi_n^1 \oplus \phi_n^2} & K_n(S[A_1]) \oplus K_n(S[A_2]) \\
\downarrow & & \downarrow \\
HFS_n(A_3) & \xrightarrow{\phi_n^3} & K_n(S[A_3]) \\
\downarrow \partial & & \downarrow \partial \\
HFS_{n-1}(T(G)) & \xrightarrow{\phi_{n-1}^T} & K_{n-1}(S[T(G)]) \oplus \mathrm{Nil}_n(T(G),A_1,A_2) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

Assume that the Farrell–Jones assembly map is an isomorphism for any acyclic group. Then ϕ_*^2 and ϕ_*^3 are isomorphisms. When either $\mathbb{Q} \subset S$ or $\mathbb{K}$ represents rationalized algebraic K-theory with $S = \mathbb{Z}$, the Farrell–Jones assembly map is injective for any group in the Waldhausen–Cappell class $\mathfrak{C}$ [1]. With ϕ_*^T injective, the map ϕ_n^1 is injective by a diagram chase. This shows that the kernel

$$\ker(H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbb{K}) \to K_n(S[G])) \subset \ker(\phi_n^1)$$

is trivial. When $G \in \mathcal{FF}$, we produce A_1, A_2 and A_3 using locally CAT(0) cubical complexes as before. The group $\pi_1(C(X_G)$ acts properly and cocompactly on the universal cover of $C(X_G)$, which is a CAT(0) cubical complex. According to [21], the Farrell–Jones conjecture is true for $\pi_1(C(X_G))$ with any coefficients. Using a similar diagram chasing, we see that

$$\ker(H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbb{K}) \to K_n(S[G])) = 0$$

in case (iii) of Theorem 2.4. The rational algebraic K-theory with $R = \mathbb{Z}$ is proved similarly, completing the proof of Theorem 2.4.

For a torsion-free acyclic group A, there are isomorphisms

$$H_n^A(E_{\mathcal{F}\mathrm{in}}(A);\mathbb{F}) \cong H_n(\mathrm{B}A;\mathbb{F}(A/e)) \cong H_n(\mathrm{pt};\mathbb{F}(A/e)),$$

where e denotes the trivial subgroup of A. This implies that the assembly map is injective for a torsion-free acyclic group. Therefore we have the following.

Corollary 3.3. *Following Theorems 2.1 and 2.4:*

(i) *The Baum–Connes conjecture is true for every torsion-free group if and only if the Baum–Connes assembly map is an epimorphism for every torsion-free group.*

(ii) *Let S be a regular ring with $\mathbb{Q} \subset S$. The Farrell–Jones conjecture with coefficients in S (resp. the rational Farrell–Jones conjecture with coefficients in $\mathbb{Z}$) holds for every torsion-free group if and only if the integral (resp. rational) assembly map is an epimorphism for every torsion-free group.*

(iii) *Let S be a regular ring. The Farrell–Jones conjecture is true for every FF group (with coefficients in S) if and only if the assembly map is an epimorphism for every FF group (with coefficients in S).*

Remark 3.4. It is currently unknown whether the original Baum–Connes conjecture holds for CAT(0)-groups of the type considered by Bartels and Lück in [2]. However, based on the results of [6, 23], it seems plausible that similar results as those above can be obtained for the coarse Baum–Connes conjecture. We hope to address these issues more completely in future work.

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References

[1] A. Bartels and W. Lück, Isomorphism conjecture for homotopy K-theory and groups acting on trees, J. Pure Appl. Algebra **205** (2006), no. 3, 660–696. MR2210223

[2] A. Bartels and W. Lück, The Borel conjecture for hyperbolic and CAT(0)-groups, Ann. of Math. (2) **175** (2012), no. 2, 631–689. MR2993750

[3] A. Bartels, W. Lück and H. Reich, On the Farrell–Jones conjecture and its applications, J. Topol. **1** (2008), no. 1, 57–86. MR2365652

[4] A. J. Berrick, I. Chatterji and G. Mislin, From acyclic groups to the Bass conjecture for amenable groups, Math. Ann. **329** (2004), no. 4, 597–621. MR2076678

[5] J. Block, Some remarks concerning the Baum–Connes conjecture, Comm. Pure Appl. Math. **50** (1997), no. 9, 813–820. MR1459588

[6] J. Brodzki et al., Property A and CAT(0) cube complexes, J. Funct. Anal. **256** (2009), no. 5, 1408–1431. MR2490224

[7] S. E. Cappell, A splitting theorem for manifolds, Invent. Math. **33** (1976), no. 2, 69–170. MR0438359

[8] S. E. Cappell, On homotopy invariance of higher signatures, Invent. Math. **33** (1976), no. 2, 171–179. MR0438349

[9] J. F. Davis and W. Lück, Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory, K-Theory **15** (1998), no. 3, 201–252. MR1659969

[10] N. Higson and G. Kasparov, E-theory and KK-theory for groups which act properly and isometrically on Hilbert space, Invent. Math. **144** (2001), no. 1, 23–74. MR1821144

[11] R. Ji, C. Ogle and B. Ramsey, $\mathcal{B}$-bounded cohomology and applications, Internat. J. Algebra Comput. **23** (2013), no. 1, 147–204. MR3040807

[12] P. Julg and A. Valette, K-theoretic amenability for $\mathrm{SL}_2(\mathbf{Q}_p)$, and the action on the associated tree, J. Funct. Anal. **58** (1984), no. 2, 194–215. MR0757995

[13] V. Lafforgue, Une démonstration de la conjecture de Baum–Connes pour les groupes réductifs sur un corps p-adique et pour certains groupes discrets possédant la propriété (T), C. R. Acad. Sci. Paris Sér. I Math. **327** (1998), no. 5, 439–444. MR1652538

[14] I. J. Leary, A metric Kan–Thurston theorem, J. Topol. **6** (2013), no. 1, 251–284. MR3029427

[15] W. Lück and H. Reich, The Baum–Connes and the Farrell–Jones conjectures in K- and L-theory, in *Handbook of K-theory. Vol. 1, 2*, 703–842, Springer, Berlin, 2005. MR2181833

[16] G. Mislin and A. Valette, *Proper group actions and the Baum–Connes conjecture*, Adv. Courses Math. CRM Barcelona, Birkhäuser, Basel, 2003. MR2027168

[17] G. A. Niblo and M. A. Roller, Groups acting on cubes and Kazhdan's property (T), Proc. Amer. Math. Soc. **126** (1998), no. 3, 693–699. MR1459140

[18] H. Oyono-Oyono, Baum–Connes conjecture and group actions on trees, K-Theory **24** (2001), no. 2, 115–134. MR1869625

[19] M. V. Pimsner, KK-groups of crossed products by groups acting on trees, Invent. Math. **86** (1986), no. 3, 603–634. MR0860685

[20] F. Waldhausen, Algebraic K-theory of generalized free products. I, II, Ann. of Math. (2) **108** (1978), no. 1, 135–204. MR0498807

[21] C. Wegner, The K-theoretic Farrell–Jones conjecture for CAT(0)-groups, Proc. Amer. Math. Soc. **140** (2012), no. 3, 779–793. MR2869063

[22] C. A. Weibel, Homotopy algebraic K-theory, in *Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987)*, 461–488, Contemp. Math., 83, Amer. Math. Soc., Providence, 1989. MR0991991

[23] G. L. Yu, Coarse Baum–Connes conjecture, K-Theory **9** (1995), no. 3, 199–221. MR1344138

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