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Nontrivial homeomorphisms of Čech–Stone remainders

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Abstract. We study the group of automorphisms of certain corona C*-algebras. As a corollary of a more general C*-algebraic result, we show that, under the Continuum Hypothesis, $\beta X \setminus X$ has nontrivial homeomorphisms, whenever X is a noncompact locally compact metrizable manifold.

1. Introduction

If X is a locally compact, noncompact space we say that an homeomorphism ϕ of its Čech–Stone remainder $X^* = \beta X \setminus X$ is trivial if there are compact sets K_1, K_2 and an homeomorphism $f\colon (X \setminus K_1) \to (X \setminus K_2)$ with the property that $\phi = \beta f \restriction X^*$, where βf is the unique continuous extension of f to $\beta(X \setminus K_1)$. If X is Polish, there can be only $\mathfrak{c} = 2^{\aleph_0}$ trivial homeomorphisms of X^*. In general, the task of finding nontrivial homeomorphisms of Čech–Stone remainders is a challenging one. In fact, for a Polish X as above, the existence of nontrivial homeomorphisms of X^* is conjectured to be independent from the usual axioms of set theory (ZFC, the Zermelo–Fraenkel scheme plus the Axiom of Choice). Under the Continuum Hypothesis (CH from now on) it is conjectured that there are nontrivial homeomorphisms of X^*, while under the assumption of different set-theoretical axioms (like the Proper Forcing Axiom, PFA, or some of its consequences), it is conjectured that Homeo(X^*), the group of homeomorphisms of X^*, has only trivial elements.

The current state-of-the-art of the conjectures is as follow. Rudin's work [16] shows that, under CH, $\mathbb{N}^*$ has $2^{\mathfrak{c}}$-many homeomorphisms, and hence, since $\mathfrak{c} < 2^{\mathfrak{c}}$, nontrivial ones. The same result applies to X^* whenever X is locally compact noncompact Polish and zero-dimensional (in this case, under CH,

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X^* is homeomorphic to $\mathbb{N}^*$ by Parovičenko's theorem). On the other hand, for such an X, the results in [4, 7, 17, 18] showed that, under PFA (or some of its consequences), one can prove that X^* has only trivial homeomorphisms.

At a current stage, even though partial progress has been made (see, e.g., [8, Thm. 5.3]), all the spaces X for which PFA proves that all elements of $\mathrm{Homeo}(X^*)$ are trivial are zero-dimensional.

On the side of the conjecture dealing with constructing nontrivial homeomorphisms of X^* under CH, the first successful attempt to go beyond zero-dimensionality was Yu's. He used CH to construct a nontrivial homeomorphism of $\mathbb{R}^*$ (see [10, §9]). Currently, the sharpest results asserts that, if X is locally compact, noncompact and Polish and assuming CH, there are nontrivial elements of $\mathrm{Homeo}(X^*)$ if X has countably many clopen sets [3] or if X is an increasing union of compact sets K_n satisfying $\sup_n |K_n \cap \overline{(X \setminus K_n)}| < \infty$ (see [8, Thm. 2.5]). The latter result relies on countable saturation of the C*-algebra $C_b(X)/C_0(X)$ (see [6] and [8, Thm. 3.1]) and provided a different proof of Yu's result on the existence of nontrivial elements of $\mathrm{Homeo}(\mathbb{R}^*)$.

For $n \geq 2$, it was unknown if one could have a nontrivial homeomorphism of $(\mathbb{R}^n)^*$. The following appealing consequence of Theorem 3.1 below settles this uncertainty.

Theorem 1.1. *Assume CH and let X be a locally compact metrizable noncompact manifold. Then X^* has $2^{\mathfrak{c}}$-many homeomorphisms. In particular, X^* has nontrivial homeomorphisms.*

The proof of our main result combines both topological and C*-algebraic techniques and it is stated in terms of C*-algebras. The C*-algebraic approach is justified by the construction of the multiplier algebra of a nonunital C*-algebra, which is the noncommutative correspondent of the Čech–Stone compactification. Given a nonunital C*-algebra A one constructs its multiplier algebra, $\mathcal{M}(A)$, which is a nonseparable unital C*-algebra in which A sits as an ideal in a universal way. The quotient $\mathcal{M}(A)/A$ is called the corona of A. Multipliers and coronas are objects carrying many properties of broad interest (see [11, 14]).

In the commutative case, when $A = C_0(X)$ for some locally compact noncompact X, then $\mathcal{M}(A) = C_b(X) \cong C(\beta X)$ and $\mathcal{M}(A)/A \cong C(X^*)$. By duality, automorphisms of $\mathcal{M}(A)/A$ correspond bijectively to homeomorphisms of X^*.

The interest on the influence of set theory in automorphisms of coronas of noncommutative C*-algebras was motivated by the search for outer automorphisms of the Calkin algebra, the quotient of the bounded linear operators on $\ell^2(\mathbb{N})$ modulo the ideal of compact operators, see [2]. Phillips and Weaver [15] proved that under CH the Calkin algebra has $2^{\mathfrak{c}}$-automorphisms (and, therefore, outer ones), while Farah [5] showed that under the Open Coloring Axiom (a consequence of PFA) the Calkin algebra has only inner automorphisms.

Conjecturally (see [3, Conj. 1.2, 1.3]) if A is a nonunital separable C*-algebra then CH implies that $\mathcal{M}(A)/A$ has $2^{\mathfrak{c}}$-many automorphisms, while PFA implies that the structure of $\mathrm{Aut}(\mathcal{M}(A)/A)$ is as rigid as possible (for the definition of triviality in the noncommutative case see [3, Def. 1.1]). The conjectures have been verified for some classes of C*-algebras, even though a large amount of projections in A is usually needed. For more on this, see [3, 8, 9, 12] or the upcoming [13].

We are interested in C*-algebras of the form $C_0(X, A)$, the algebra of continuous functions from some locally compact space X to a C*-algebra A which vanish at infinity. If A is unital, the multiplier of $C_0(X, A)$ is (isomorphic to) $C_b(X, A)$, the algebra of bounded continuous functions from X to A. We denote $C_b(X, A)/C_0(X, A)$ by $Q(X, A)$. Our main result, Theorem 3.1, asserts that under the assumption of CH, if X is a space satisfying a technical condition ensuring some sort of flexibility (e.g., a manifold, see Definition 2.1), and A is a C*-algebra, then $Q(X, A)$ has $2^{\mathfrak{c}}$-many automorphisms. Note we do not ask for A to be unital. On the other hand, if A is nonunital, $Q(X, A)$ is not the corona of $C_0(X, A)$. In this case, we do not know whether a result similar to Theorem 3.1 applies to the corona of $C_0(X, A)$. For more on $\mathcal{M}(C_0(X, A))$ if A is nonunital, see [1].

The paper is structured as follows: Section 2 contains preliminaries and notation, and Section 3 is dedicated to the proof of our main result. Lastly, we provide an example of a space X for which it is not known whether $\mathrm{Homeo}(X^*)$ contains nontrivial elements, even under CH.

2. Preliminaries and notation

Unless stated differently, X is a locally compact noncompact Polish space with a metric d inducing its topology and A is a C*-algebra. Given a closed $Y \subseteq X$ we say that $\phi \in \mathrm{Homeo}(Y)$ *fixes the boundary* of Y if, whenever $y \in \mathrm{bd}_X(Y) = Y \cap \overline{(X \setminus Y)}$, then $\phi(y) = y$. We denote the set of all such homeomorphisms by $\mathrm{Homeo}_{\mathrm{bd}_X(Y)}(Y)$ (or $\mathrm{Homeo}_{\mathrm{bd}}(Y)$ if X is clear from the context). Every $\phi \in \mathrm{Homeo}_{\mathrm{bd}}(Y)$ can be extended in a canonical way to $\tilde{\phi} \in \mathrm{Homeo}(X)$ by

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & \text{if } x \in Y, \\ x & \text{otherwise.} \end{cases}$$

If $Y_n \subseteq X$, for $n \in \mathbb{N}$, are closed and disjoint sets with the property that no compact subset of X intersects infinitely many Y_n's, we have that $Y = \bigcup Y_n$ is closed. If $\phi_n \in \mathrm{Homeo}_{\mathrm{bd}}(Y_n)$ then $\phi = \bigcup \phi_n \in \mathrm{Homeo}_{\mathrm{bd}} Y$ is well defined. In this situation we abuse notation and say that $\tilde{\phi}$ as constructed above extends canonically $\{\phi_n\}$.

For a $\phi \in \mathrm{Homeo}(X)$ we will denote by $r(\phi)$ the *radius* of ϕ as

$$r(\phi) = \sup_{x \in X} d(x, \phi(x)).$$

If Y is compact and $\phi \in \mathrm{Homeo}_{\mathrm{bd}}(Y)$ we have that $r(\phi) < \infty$ and $r(\phi)$ is attained by some $y \in Y$. It can be easily verified that $r(\phi_0\phi_1) \leq r(\phi_0) + r(\phi_1)$ for $\phi_0, \phi_1 \in \mathrm{Homeo}(X)$.

Note that every $\tilde{\phi} \in \mathrm{Homeo}(X)$ determines uniquely a $\psi \in \mathrm{Aut}(C_b(X,A))$, which induces a $\tilde{\psi} \in \mathrm{Aut}(Q(X,A))$. If $Y_n \subseteq X$ are disjoint closed sets with the property that no compact $Z \subseteq X$ intersects infinitely many of them and $\phi_n \in \mathrm{Homeo}_{\mathrm{bd}}(Y_n)$, we will abuse of notation and say that ψ and $\tilde{\psi}$ are *canonically determined* by $\{\phi_n\}$.

We are interested in a particular class of topological spaces:

Definition 2.1. A locally compact noncompact Polish space (X,d) is *flexible* if there are disjoint sets $Y_n \subseteq X$ and $\phi_{n,m} \in \mathrm{Homeo}_{\mathrm{bd}}(Y_n)$ with the following properties:

(i) every Y_n is a compact subset of X and there is no compact $Z \subseteq X$ that intersects infinitely many Y_n's, and
(ii) for all n, $r(\phi_{n,m})$ is a decreasing sequence tending to 0 as $m \to \infty$, with $r(\phi_{n,m}) \neq 0$ whenever $n,m \in \mathbb{N}$.

The sets Y_n and the homeomorphisms $\phi_{n,m}$ are said to witness that X is flexible.

Remark 2.2. We do not know whether condition (ii) is equivalent to having a sequence of disjoint Y_n's satisfying (i) for which $\mathrm{Homeo}_{\mathrm{bd}}(Y_n)$ has a continuous path. This condition is clearly stronger than (ii). In fact, being $\mathrm{Homeo}_{\mathrm{bd}}(Y_n)$ a group, if it contains a path, then there is a path $a(t) \subseteq \mathrm{Homeo}_{\mathrm{bd}}(Y_n)$ with $a(0) = \mathrm{id}$ and $a(t) \neq a(0)$ if $t \neq 0$. By continuity, if a path exists, it can be chosen so that $s < t$ implies $r(a(s)) < r(a(t))$. Since any closed ball in $\mathbb{R}^n$ has this property, a typical example of a flexible space is a manifold.

We should also note that if X is a locally compact Polish space for which there is a closed discrete sequence x_n and a sequence of open sets U_n with $U_i \cap U_j = \varnothing$ if $i \neq j$, $x_n \in U_n$, and such that each U_n is a manifold, then X is flexible. In particular, if X has a connected component which is a noncompact Polish manifold, then X is flexible.

Lastly, if X is flexible and Y has a compact clopen Z, then $\mathrm{bd}(Y_n \times Z) = \mathrm{bd}(Y_n) \times Z$, therefore $Z_n = Y_n \times Z$ and $\rho_{m,n} = \psi_{n,m} \times \mathrm{id}$ witness the flexibility of $X \times Y$. In particular, if Y is compact and metrizable, $X \times Y$ is flexible.

By $\mathbb{N}^{\mathbb{N}\uparrow}$ we denote the set of all increasing sequences of natural numbers, where $f(n) > 0$ for all n. If $f_1, f_2 \in \mathbb{N}^{\mathbb{N}\uparrow}$ we write $f_1 \leq^* f_2$ if

$$\forall^\infty n (f_1(n) \leq f_2(n)).$$

($\forall^\infty n$ stands for $\exists n_0 \forall n \geq n_0$. Similarly, the notation $\exists^\infty n$ means *there are infinitely many* n.) If $Y_n \subseteq X$ are compact sets with the property that no compact $Z \subseteq X$ intersects infinitely many Y_n, we can associate to every $f \in \mathbb{N}^{\mathbb{N}\uparrow}$ a subalgebra of $C_b(X,A)$ as

$$D_f(X,A,Y_n) = \Big\{ g \in C_b(X,A) \mid \forall \epsilon > 0 \ \forall^\infty n \ \forall x,y \in Y_n \\ \Big(d(x,y) < \frac{1}{f(n)} \Rightarrow \|g(x) - g(y)\| < \epsilon \Big) \Big\}.$$

We denote by $C_f(X,A,Y_n)$ the image of $D_f(X,A,Y_n)$ under the quotient map $\pi\colon C_b(X,A)\to Q(X,A)$. (If X, Y_n and A are clear from the context, we simply write D_f and C_f.)

The following proposition clarifies the structure of the D_f's and the C_f's.

Proposition 2.3. *Let (X,d) be a locally compact noncompact Polish space and let A be a* C*-*algebra. Let $Y_n\subset X$ be infinite compact disjoint sets such that no compact subset of X intersects infinitely many Y_n's. Then:*

(i) *For all $f\in\mathbb{N}^{\mathbb{N}\uparrow}$ we have that D_f is a* C*-*subalgebra of $C_b(X,A)$. If A is unital, so is D_f.*
(ii) *If $f_1\leq^* f_2$ then $C_{f_1}\subseteq C_{f_2}$.*
(iii) *$C_b(X,A)=\bigcup_{f\in\mathbb{N}^{\mathbb{N}\uparrow}}D_f$.*
(iv) *For all $f\colon\mathbb{N}\to\mathbb{N}$ there is $g\in C_b(X,A)$ such that $\pi(g)\notin C_f$.*

Proof. (1) and (2) follow directly from the definition of D_f and C_f. For (3), take $g\in C_b(X,A)$. Since each Y_n is compact and metric, we have that $g\restriction Y_n$ is uniformly continuous. In particular, there is $\delta_n>0$ such that $d(x,y)<\delta_n$ implies $\|g(x)-g(y)\|<2^{-n}$ for all $x,y\in Y_n$. Fix m_n such that $1/m_n<\delta_n$ and let $f(n)=m_n$. Then $g\in D_f$.

For (4), fix $f\in\mathbb{N}^{\mathbb{N}\uparrow}$ and $x_n\neq y_n\in Y_n$ with $d(y_n,z_n)<1/f(n)$. Since no compact set intersects infinitely many Y_n's, both $Y'=\{y_n\}_n$ and $Z'=\{z_n\}_n$ are closed in X. Pick any $a\in A$ with $\|a\|=1$ and let g be a bounded continuous function such that $g(Y')=0$ and $g(Z')=a$. It is easy to see that $g\notin C_f$. □

The following lemma represents the connections between the filtration we obtained and an automorphism of $Q(X,A)$.

Lemma 2.4. *Let (X,d), A, and Y_n be as in Proposition 2.3 and suppose that $\phi_n\in\mathrm{Homeo}_{\mathrm{bd}}(Y_n)$. Let $\tilde\phi\in\mathrm{Homeo}(X)$ and $\tilde\psi\in\mathrm{Aut}(Q(X,A))$ be canonically determined by $\{\phi_n\}$ and $f\in\mathbb{N}^{\mathbb{N}\uparrow}$. Then:*

(i) *If there are k,n_0 such that for all $n\geq n_0$ the inequality*
$$r(\phi_n)\leq\frac{k}{f(n)}$$
holds, then $\tilde\psi(g)=g$ for all $g\in C_f$.
(ii) *If for infinitely many n the inequality*
$$r(\phi_n)\geq\frac{n}{f(n)}$$
holds, then there is $g\in C_f$ such that $\tilde\psi(g)\neq g$.

Proof. Note that, if $g\in C_b(X,A)$ and $\tilde\psi$ is as above, we have $\tilde\psi(g)=g$ if and only if $g-\psi(g)\in C_0(X,A)$ where $\psi\in\mathrm{Aut}(C_b(X,A))$ is canonically determined by $\{\phi_n\}$.

To prove (1), let k,n_0 be as above. Fix $\epsilon>0$ and $n_1>n_0$ such that whenever $n\geq n_1$ we have that $k/f(n)<\epsilon$ and if $x,y\in Y_n$ with $d(x,y)<$

$1/f(n)$ then $\|g(x) - g(y)\| < \epsilon/k$. Such an n_1 can be found, since $g \in D_f$. Let now $x \notin \bigcup_{i \leq n_1} Y_i$. Since $g(x) - \psi(g(x)) = g(x) - g(\tilde{\phi}(x))$, if $x \notin \bigcup Y_n$ we have $\tilde{\phi}(x) = x$ and so $g(x) - \psi(g)(x) = 0$. If $x \in Y_n$ for $n \geq n_1$, we have $d(x, \phi_n(x)) < r(\phi_n) \leq k/f(n)$ and by our choice of n_1,

$$\|g(x) - \psi(g)(x)\| = \|g(x) - g(\phi_n(x))\| \leq \epsilon.$$

Since $\bigcup_{i \leq n_1} Y_i$ is compact, we have that $g - \psi(g) \in C_0(X, A)$, and (1) follows.

For (2), let $k(n)$ be a sequence of natural numbers such that

$$r(\phi_{k(n)}) \geq \frac{k(n)}{f(k(n))}.$$

We will construct $h \in D_f$ and show that $h - \psi(h) \notin C_0(X, A)$. Fix some $a \in A$ with $\|a\| = 1$. If $m \neq k(n)$ for all n, set $h(Y_m) = 0$. If $m = k(n)$, let $r = r(\phi_m)$ and pick $x_0 = x_0(m)$ such that $d(x_0, \phi_m(x_0)) = r$. Set $x_1 = x_1(m) = \phi_m(x_0)$ and, for $i = 0, 1$, let

$$Z_i = \{z \in Y_m \mid d(z, x_i) \leq r/2\}.$$

If $z \in Z_0$ define

$$h(z) = \Big(\frac{d(z, x_0)}{r}\Big)a$$

and if $z \in Z_1$ let

$$h(z) = \Big(1 - \frac{d(z, x_1)}{r}\Big)a,$$

while for $z \in Y_m \setminus (Z_0 \cup Z_1)$ let $h(z) = a/2$. Let $h' \in C_b(X, A)$ be any function such that $h'(x) = h(x)$ whenever $x \in \bigcup Y_i$. Note that we have that $h' \in D_f$, as this only depends on its values on $\bigcup Y_i$. We want to show that $h' - \psi(h') \notin C_0(X, A)$. To see this, note that if $m = k(n)$ for some n we have

$$h'(x_0(m)) = 0$$

and

$$\psi(h')(x_0(m)) = h'(\psi_m(x_0(m))) = h'(x_1(m)) = a.$$

Since $\{x_0(m)\}_{m \in \mathbb{N}}$ is not contained in any compact subsets of X, we have the thesis. □

We are ready to introduce our main concept.

Definition 2.5. Let (X, d), Y_n and A be as in Proposition 2.3 and let κ be uncountable. Let $\{f_\alpha\}_{\alpha < \kappa} \subseteq \mathbb{N}^{\mathbb{N}\uparrow}$ be a $\leq^*$-increasing sequence of functions and $\{\phi_n^\alpha\}_{\alpha < \kappa}$ be such that for all α and n,

$$\phi_n^\alpha \in \mathrm{Homeo}_{\mathrm{bd}}(Y_n).$$

The sequence $\{\phi_n^\alpha\}$ is said to be *coherent with respect to* $\{f_\alpha\}$ if

$$\alpha < \beta \;\Rightarrow\; \exists k\, \forall^\infty n \,\Big(r\big(\phi_n^\alpha(\phi_n^\beta)^{-1}\big) \leq \frac{k}{f_\alpha(n)}\Big).$$

If γ is countable, $\{\phi_n^\alpha\}_{\alpha\leq\gamma}$ is coherent with respect to $\{f_\alpha\}_{\alpha\leq\gamma} \subseteq \mathbb{N}^{\mathbb{N}\uparrow}$ if for all $\alpha < \beta \leq \gamma$ we have that

$$\exists k\, \forall^\infty n \left(r\big(\phi_n^\alpha(\phi_n^\beta)^{-1}\big) \leq \frac{k}{f_\alpha(n)} \right).$$

Remark 2.6. Definition 2.5 is stated in great generality. We do not ask for the sequence $\{f_\alpha\}_{\alpha<\kappa}$ to have particular properties (e.g., being cofinal) or for the space X to be flexible, even though Definition 2.5 will be used in such context.

Note that if $\{\phi_n^\alpha\}_{\alpha<\omega_1}$ is such that $\{\phi_n^\alpha\}_{\alpha\leq\gamma}$ is coherent with respect to $\{f_\alpha\}_{\alpha\leq\gamma}$ for all $\gamma < \omega_1$, then $\{\phi_n^\alpha\}_{\alpha<\omega_1}$ is coherent with respect to $\{f_\alpha\}_{\alpha<\omega_1}$.

Recalling that $\mathfrak{d}$ denotes the smallest cardinality of a $\leq^*$-cofinal family in $\mathbb{N}^{\mathbb{N}\uparrow}$, we say that a $\leq^*$ increasing and cofinal sequence $\{f_\alpha\}_{\alpha<\kappa} \subseteq \mathbb{N}^{\mathbb{N}\uparrow}$, for some $\kappa > \mathfrak{d}$, is *fast* if for all α and n,

$$n f_\alpha(n) \leq f_{\alpha+1}(n).$$

If $\{f_\alpha\}_{\alpha<\mathfrak{d}}$ is fast, the same argument as in Proposition 2.3 shows that

$$Q(X, A) = \bigcup_\alpha C_{f_\alpha}.$$

The following lemma is going to be key for our construction. Its proof follows almost immediately from the definitions above, but we sketch it for convenience.

Lemma 2.7. *Let (X, d), A and Y_n be fixed as in Proposition 2.3. Let $\{f_\alpha\}$ be a fast sequence and suppose that $\{\phi_n^\alpha\}$ is a coherent sequence with respect to $\{f_\alpha\}$. Let $\tilde{\psi}_\alpha \in \mathrm{Aut}(Q(X, A))$ be canonically determined by $\{\phi_n^\alpha\}_n$. Then there is a unique $\tilde{\Psi} \in \mathrm{Aut}(Q(X, A))$ with the property that*

$$\tilde{\Psi}(g) = \tilde{\psi}_\alpha(g), \quad g \in C_{f_\alpha}.$$

Proof. We define $\tilde{\Psi}(g) = \tilde{\psi}_\alpha(g)$ for $g \in C_{f_\alpha}$. If $\alpha < \beta$, we define $\tilde{\psi}_{\alpha\beta} = \tilde{\psi}_\alpha(\tilde{\psi}_\beta)^{-1}$. As $\tilde{\psi}_{\alpha\beta}$ is canonically determined by $\{\psi_n^\alpha(\phi_n^\beta)^{-1}\}_n$, and by coherence, there are $k, n_0 \in \mathbb{N}$ such that whenever $n > n_0$ we have

$$r(\phi_n^\alpha(\phi_n^\beta)^{-1}) < \frac{k}{f_\alpha(n)}.$$

By condition (1) of Lemma 2.4 we therefore have that $\tilde{\psi}_{\alpha\beta}(g) = g$ whenever $g \in C_{f_\alpha}$, and in this case $\tilde{\psi}_\alpha(g) = \tilde{\psi}_\beta(g)$, so $\tilde{\Psi}$ is a well-defined morphism of $Q(X, A)$ into itself. Let $\tilde{\psi}'_\alpha \in \mathrm{Aut}(Q(X, A))$ be canonically determined by $\{(\psi_n^\alpha)^{-1}\}_n$. Since $\{\phi_n^\alpha\}$ is coherent with respect to $\{f_\alpha\}$, so is $\{(\phi_n^\alpha)^{-1}\}$. In particular, if we let $\tilde{\Psi}'$ be defined by $\tilde{\Psi}'(g) = \tilde{\psi}'_\alpha(g)$ for $g \in C_{f_\alpha}$, we have that $\tilde{\psi}'$ is a well-defined morphism from $Q(X, A)$ into itself, with the property that $\tilde{\Psi}'\tilde{\Psi} = \tilde{\Psi}\tilde{\Psi}' = \mathrm{id}$, hence $\tilde{\Psi}$ is an automorphism. This concludes the proof. $\square$

3. The construction

This section is dedicated to prove our main result:

Theorem 3.1. *Let X be flexible and let A be a* C^**-algebra. Suppose that* $\mathfrak{d} = \omega_1$ *and* $2^{\aleph_0} < 2^{\aleph_1}$*. Then* $Q(X, A)$ *has* $2^{\aleph_1}$*-many automorphisms. In particular, under CH, there are* $2^{\mathfrak{c}}$*-many automorphisms of* $Q(X, A)$*.*

Proof. Fix d, Y_n and $\phi_{n,m} \in \mathrm{Homeo}_{\mathrm{bd}} Y_n$ witnessing that X is flexible.

We have to give a technical restriction (see Remark 3.4) on the kind of elements of $\mathbb{N}^{\mathbb{N}\uparrow}$ we are allowed to use. This restriction depends strongly on the choice of d, on the witnesses Y_n and on the $\phi_{n,m}$'s. We define

$$A_n = \big\{k \in \mathbb{N} \mid \exists m \left(r(\phi_{n,m}) \in [1/(k+1), 1/k]\right)\big\}$$

As $r(\phi_{n,m}) \to 0$ for $m \to \infty$, the set A_n is always infinite. We define

$$\mathbb{N}^{\mathbb{N}\uparrow}(X) = \{f \in \mathbb{N}^{\mathbb{N}\uparrow} \mid f(n) \in A_n\} \subseteq \mathbb{N}^{\mathbb{N}\uparrow}.$$

Since each A_n is infinite, $\mathbb{N}^{\mathbb{N}\uparrow}(X)$ is cofinal in $\mathbb{N}^{\mathbb{N}\uparrow}$.

As $\mathfrak{d} = \omega_1$, we can fix a fast sequence $\{f_\alpha\}_{\alpha\in\omega_1} \subseteq \mathbb{N}^{\mathbb{N}\uparrow}(X)$. Let $C_\alpha := C_{f_\alpha}$. Finally, fix, for each limit ordinal $\beta < \omega_1$, a sequence $\alpha_{\beta,n}$ that is strictly increasing and cofinal in β.

We will make use of Lemma 2.7 and construct, for each $p \in 2^{\omega_1}$, a sequence $\phi_n^\alpha(p)$ that is coherent with respect to $\{f_\alpha\}$. For simplicity we write ϕ_n^α for $\phi_n^\alpha(p)$. Let $\phi_n^0 = \mathrm{id}$. Once ϕ_n^α has been constructed, let

$$\phi_n^{\alpha+1} = \phi_{n,m}\phi_n^\alpha$$

if $p(\alpha) = 1$, where m is the smallest integer such that

$$r(\phi_{n,m}) \in [1/(f_\alpha(n)+1), 1/f_\alpha(n)],$$

and

$$\phi_n^{\alpha+1} = \phi_n^\alpha$$

otherwise.

Claim 3.2. If $\{\phi_n^\gamma\}_{\gamma\le\alpha}$ is coherent with respect to $\{f_\gamma\}_{\gamma\le\alpha}$ then $\{\phi_n^\gamma\}_{\gamma\le\alpha+1}$ is coherent with respect to $\{f_\gamma\}_{\gamma\le\alpha+1}$.

Proof. We want to show that whenever $\gamma < \alpha$ there is k such that

$$\forall^\infty n \left(r\big(\phi_n^\gamma(\phi_n^{\alpha+1})^{-1}\big) \le \frac{k}{f_\gamma(n)}\right).$$

If $p(\alpha) = 0$ this is clear, so suppose that $p(\alpha) = 1$. Note that

$$\phi_n^\gamma(\phi_n^{\alpha+1})^{-1} = \phi_n^\gamma(\phi_n^\alpha)^{-1}\phi_{n,m}^{-1},$$

where m was chosen as above, and so

$$r\big(\phi_n^\gamma(\phi_n^{\alpha+1})^{-1}\big) \le r\big(\phi_n^\gamma(\phi_n^\alpha)^{-1}\big) + r(\phi_{n,m}) \le \frac{k}{f_\gamma(n)} + \frac{1}{f_\alpha(n)}$$

for some k (and eventually after a certain n_0). Since $f_\alpha(n) \ge f_\gamma(n)$ (again, eventually after a certain n_1), the conclusion follows. □

We are left with the limit step. Suppose then that ϕ_n^α has be defined whenever $\alpha < \beta$. For shortness, let $\alpha_i = \alpha_{i,\beta}$.

Claim 3.3. For all $i \in \mathbb{N}$ there is k_i such that whenever $j \geq i$ there exists $n_{i,j}$ such that

$$r\big(\phi_n^{\alpha_i}(\phi_n^{\alpha_j})^{-1}\big) \leq \frac{k_i}{f_{\alpha_i(n)}},$$

whenever $n \geq n_{i,j}$

Proof. Fix $i \in \mathbb{N}$. By coherence there are $\bar{k} < \bar{n}$ such that whenever $n \geq \bar{n}$ we have

$$r\big(\phi_n^{\alpha_i}(\phi_n^{\alpha_{i+1}})^{-1}\big) < \frac{\bar{k}}{f_{\alpha_i}(n)}.$$

Let $j > i$ and $n'(j) > k'(j) > \bar{n}$ such that if $n \geq n'(j)$ then

$$r\big(\phi_n^{\alpha_{i+1}}(\phi_n^{\alpha_j})^{-1}\big) < \frac{k'(j)}{f_{\alpha_{i+1}}(n)}$$

and

$$f_{\alpha_{i+1}}(n) \geq n f_{\alpha_i}(n).$$

Fix $k_i = \bar{k} + 1$ and $n_{i,j} = n'(j)$. Then, for $n \geq n_{i,j}$,

$$\begin{aligned} r\big(\phi_n^{\alpha_i}(\phi_n^{\alpha_j})^{-1}\big) &\leq r\big(\phi_n^{\alpha_i}(\phi_n^{\alpha_{i+1}})^{-1}\big) + r\big(\phi_n^{\alpha_{i+1}}(\phi_n^{\alpha_j})^{-1}\big) \\ &\leq \frac{k'(j)}{f_{\alpha_{i+1}}(n)} + \frac{\bar{k}}{f_{\alpha_i}(n)} \\ &\leq \frac{n}{f_{\alpha_{i+1}}(n)} + \frac{\bar{k}}{f_{\alpha_i}(n)} \\ &\leq \frac{k_i}{f_{\alpha_i}(n)}. \end{aligned}$$

□

Fix a sequence of k_i as provided by the claim. Let $m_0 = 0$ and let m_{i+1} be the least natural above m_i such that if $n \geq m_i$ and $j > i \geq l$ then

$$r\big(\phi_n^{\alpha_l}(\phi_n^{\alpha_j})^{-1}\big) < \frac{k_l}{f_{\alpha_l}(n)}.$$

Defining $\phi_n^\beta = \phi_n^{\alpha_i}$ whenever $n \in [m_{i-1}, m_i)$, we have that coherence is preserved, that is, $\{\phi_n^\alpha\}_{\alpha \leq \beta}$ is coherent with respect to $\{f_\alpha\}_{\alpha \leq \beta}$. We just proved that we can define ϕ_n^α for every countable ordinal.

By Remark 2.6 we have that the sequence $\{\phi_n^\alpha\}_{\alpha < \omega_1}$ is coherent with respect to $\{f_\alpha\}_{\alpha < \omega_1}$. By Lemma 2.7 there is a unique $\tilde{\Psi} = \tilde{\Psi}_p \in \mathrm{Aut}(Q(X,A))$ determined by $\{\phi_n^\alpha(p)\}_{\alpha < \omega_1}$.

To conclude the proof, we claim that if $p \neq q$ we have $\tilde{\Phi}_p \neq \tilde{\Phi}_q$. Let α be minimum such that $p(\alpha) \neq q(\alpha)$, and suppose $p(\alpha) = 1$. Then

$$\phi_n^\alpha(p) = \phi_n^\alpha(q)$$

so

$$\phi_n^{\alpha+1}(p) = \phi_{n,m}\phi_n^\alpha(p) = \phi_{n,m}\phi_n^\alpha(q),$$

where m is the smallest integer for which $r(\phi_{n,m}) \in [1/(f_\alpha(n)+1), 1/f_\alpha(n)]$. In particular, eventually after a certain n_0,

$$r\big(\phi_n^{\alpha+1}(q)\phi_n^{\alpha+1}(p)^{-1}\big) = r(\phi_{n,m}) \geq \frac{1}{f_\alpha(n)+1} \geq \frac{n}{f_{\alpha+1}(n)}.$$

By Lemma 2.4, if $\tilde{\psi}_{\alpha+1}(p) \in \mathrm{Aut}(Q(X,A))$ is determined by $\phi_n^{\alpha+1}(p)$, there is $g \in C_{\alpha+1}$ such that

$$\tilde{\psi}_{\alpha+1}(p)(g) \neq \tilde{\psi}_{\alpha+1}(q)(g).$$

As Lemma 2.7 states that

$$\tilde{\Psi}_p(g) = \tilde{\psi}_{\alpha+1}(p)(g)$$

whenever $g \in C_{\alpha+1}$, we have that

$$\tilde{\Psi}_p(g) = \tilde{\psi}_{\alpha+1}(p)(g) \neq \tilde{\psi}_{\alpha+1}(q)(g) = \tilde{\Psi}_q(g),$$

which completes the proof of Theorem 3.1. □

Remark 3.4. The requirement of using $\mathbb{N}^{\mathbb{N}\uparrow}(X)$ instead of $\mathbb{N}^{\mathbb{N}\uparrow}$ is purely technical. If it is possible to choose Y_n so that $\mathrm{Homeo}_{\mathrm{bd}}(Y_n)$ has a path (e.g., if X is a manifold) then, following Remark 2.2, we can pick $\{\phi_{n,m}\}$ in order to have $A_n = \mathbb{N}$ (eventually truncating a finite set).

As promised, we are ready to give a proof of Theorem 1.1, which in particular shows that, whenever $n \geq 1$, $(\mathbb{R}^n)^*$ has plenty of nontrivial homeomorphisms under CH.

Corollary 3.5. *Assume CH. Let X be a locally compact noncompact metrizable manifold. Then there are $2^{\mathfrak{c}}$-many nontrivial homeomorphisms of X^*. Suppose moreover that Y is a locally compact space with a compact connected component. Then $(X \times Y)^*$ has $2^{\mathfrak{c}}$-many nontrivial homeomorphisms.*

Proof. Manifolds are flexible thanks to Remark 2.2, and homeomorphisms of X^* correspond to automorphisms of $Q(X, \mathbb{C})$. Since there can be only $\mathfrak{c}$-many trivial homeomorphisms of X^*, the first assertion is proved. The second assertion follows similarly from Remark 2.2, because if X is flexible and Y has a compact connected component, then $X \times Y$ is flexible. □

Even though Theorem 3.1 does not apply to the corona of $C_0(X, A)$ whenever A is nonunital, we can still say something in a particular case. Along the same lines as in the proof of Corollary 3.5, if A is a C*-algebra that has a nonzero central projection[1] then it is possible to prove that under CH the corona of $C_0(X, A)$ has $2^{\mathfrak{c}}$-many automorphisms.

To conclude the paper, we show the existence of a one-dimensional space X for which it is still unknown whether CH implies the existence of a nontrivial element of $\mathrm{Homeo}(X^*)$. In fact, this space is not flexible (according to Definition 2.1), it is connected (so the main result of [3] does not apply), and it does not have an increasing sequence of compact subsets K_n for which

[1] A projection p is central if $pa = ap$ for all $a \in A$.

$\sup_n |K_n \cap (\overline{X \setminus K_n})| < \infty$ (and therefore it does not satisfy the hypothesis of [8, Thm. 2.5]). The space X is a modification of a construction of Kuperberg that appeared in the introduction of [15].

The construction of X takes place in the plane $\mathbb{R}^2$, and it goes as follows: take a copy of interval $a(t)$, $t \in [0,1]$ and let $x_0 = a(0)$. At the midpoint of a, we attach a copy of the interval. We now have three copies of the interval attached to each other. At the midpoint of the first one, we attach two copies of the interval, at the midpoint of the second one we attach three of them, and four to the third. We order the new midpoints and attach five intervals to the first one, six to the second, and so forth. We repeat this construction infinitely many times, making sure that at every step of the construction the length of new intervals attached is short enough to satisfy the following three conditions:

- the construction takes place in a prescribed big enough compact subset of $\mathbb{R}^2$ containing a;
- for every new interval attached at the n-th stage the only point of intersection with the construction at stage $n-1$ is the midpoint to which the new interval was attached. Similarly, two new intervals intersect if and only if they are attached to the same point of the construction at stage $n-1$;
- the length of every interval attached at stage n is less than 2^{-n}.

Let Y be the closure of this iterated construction. For $n \geq 1$, let $y_n = a(1/2^n)$, let a_n be one of the intervals attached to y_n, and let x_n be the endpoint of a_n not belonging to a. Let $X = Y \setminus \{x_n\}_{n\geq 0}$. Since $y_n \to x_0$ and the length of a_n goes to 0 when $n \to \infty$, we have $x_n \to x_0$. In particular, X is locally compact. By construction, the set

$$\bigcup_{n\geq 3} \{x \mid X \setminus \{x\} \text{ has } n\text{-many connected components}\}$$

is dense in X, and for every $n \geq 3$ there is a unique point $x \in X$ such that $X \setminus \{x\}$ has exactly n-many connected components. Therefore X has no homeomorphism other than the identity. Since X is a connected space, it is not flexible and it does not satisfy [3, Hyp. 4.1].

We are left to show that we cannot apply [8, Thm. 2.5], that is, we show that if $X = \bigcup K_n$ for some compact sets K_n, then $\sup_n |K_n \cap (\overline{X \cap K_n})| = \infty$. Note that $a_k \setminus \{x_k\}$ cannot be contained in a compact set of X and all a_k's are disjoint. In particular, if K is compact and $K \cap a_k \neq \varnothing$ then there is $y \in K \cap a_k$ such that $y \in \overline{X \setminus K}$. If $\bigcup K_n = X$ for some compact sets K_n, for all k there is n such that $K_n \cap a_i \neq \varnothing$ for all $i \leq k$. Therefore $|K_n \cap (\overline{X \setminus K_n})| \geq k$.

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