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Ordinal Proof Theory of Kripke-Platek Set Theory Augmented by Strong Reflection Principles

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Ordinal Proof Theory of Kripke-Platek Set Theory Augmented by Strong Reflection Principles

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Contents

Preface

This thesis belongs to the area of mathematical logic, more precisely to the area of proof theory. One of the topics of proof theory arose from the famous talk given by David Hilbert in 1900 at Paris. Shattered by the paradoxes of set theory discovered at that time[1] he suggested a research programme, based on two pillars: At first he demanded an axiomatization of the existing mathematics and at second he proposed the establishment of metamathematics (proof theory) to secure the consistency of this axiomatization by pure finitist means.

While the first pillar of this research programme is a fruitful and standardized method ("the axiomatic method") in nowadays mathematics, the second demand of Hilbert suffered a severe setback in 1936 by the famous second incompleteness theorem of Kurt Gödel. Gödel could show in [Goe31] that it is impossible to prove the consistency of a theory T, which provides a coding-machinery[2] by pure means of T. Thereby, e.g., it is impossible to prove the consistency of analysis by pure finitist means.

Nonetheless, at the same time Gerhard Gentzen proved in [Gen36] the consistency of Peano Arithmetic by pure finitist means plus the principle of transfinite induction up to the ordinal ε_0.

Although this consistency proof cannot be a proof by finitist means (due to Gödel's theorem) it still provides an exact encapsulation of the transfinite content of Peano Arithmetic to the principle of transfinite induction up to the ordinal ε_0.

The result of Gentzen was the date of founding of ordinal proof theory. Inspired by this result the proof-theoretic ordinal $\|T\|$ of a theory T[‡] is defined by

$$\|T\| \;:=\; \sup\{\mathrm{otyp}(\prec) \mid \prec \text{ is a primitive-recursive well-ordering } \& \ T \vdash TI(\prec)\},$$

where $TI(\prec)$ denotes the principle of transfinite induction for $\prec$.

An ordinal analysis of a theory T denotes the endeavor of computing the proof-theoretic ordinal of T.[3] An ordinal analysis of T not only provides a consistency proof of T, but also provides proof-theoretic reductions and a characterization of the provable recursive functions of T.

As soon as the first big obstacle of ordinal proof theory, the extension of the methods of Gentzen to impredicative theories, was mastered by Howard in [How72] continuous progress was achieved in this area of proof theory, culminating in ordinal analyses of

[1] Most prominently the Russel paradox of the set of all sets that are not members of themselves.

[2] Already a fragment of Peano Arithmetic provides this.

[‡] Which comprises arithmetic and proves Π^1_1-sentences.

[3] In terms of a "natural" well-ordering. Allowing, e.g. a well-ordering which is defined by use of a formalized proof-predicate for T, this task becomes trivial, see [Rat99].

Stability and parameter-free Π^1_2-comprehension by Michael Rathjen in [Rat05b] and [Rat05a].

The original plan of my dissertation thesis was elaborating a characterization of the provable recursive functions of Stability by extending the methods developed by Andreas Weiermann and Benjamin Blankertz in [BW99] and [Bla97]. This was supposed to be achieved by recoursing to the ordinal analysis of Stability given in [Rat05b].

However, as things turned out the ordinal analysis of Stability presented in [Rat05b] contains some errors and incompletions.[4] Because of this, I decided to take (at first) a step back:

An ordinal analysis of Stability requires a proof-theoretic treatment of ordinals κ, which are $\kappa + \theta$-stable, where $\theta < \kappa$. Therefore it seems natural to start with investigating the case $\theta = 1$. A theory whose ordinal analysis provides techniques to handle an ordinal κ which is $\kappa + 1$-stable is the subsystem of set theory KP (Kripke-Platek set theory) augmented by a first order reflection scheme. From now on denoted by Π_ω-Ref.

An ordinal analysis of Π_ω-Ref requires a transfinite iteration of techniques needed for an ordinal analysis of Π_3-Ref as published in [Rat94b]. The ordinal analysis of Π_4-Ref given by Christoph Duchhardt in [Duc08] reveals that even a finite iteration of these techniques is far from being trivial.[5] Therefore an ordinal analysis of Π_ω-Ref seems to be an adequate intermediate step towards a proof-theoretic treatment of Stability.

The present thesis divides therefore into four parts.

The first part of this thesis contains an ordinal analysis[6] of Π_ω-Ref. To overcome the imperfections of [Rat05b] an in depth analysis of the collapsing hierarchies which give rise to the ordinal notation system is required (cf. chapter 3).

Beyond that the ordinal analysis presented here features two improvements of the techniques developed in [Rat94b]. First, the semi-formal calculus is just equipped with (cofinal many) Π_n-reflection rules instead of being equipped with pseudo-Σ_{n+1}-reflection rules. Thereby a "Strengthening Reflection Theorem" is not necessary anymore. Secondly, the Reflection Elimination Theorem (a.k.a. Impredicative Cut Elimination Theorem) is stated with a minimum of provisos, i.e. there is no need for the informally called "pancake-conditions". This conduces to a better readability of the theorem's proof.

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[4] Contrary to the claim in [Rat05b] a property similar to that in (15.14) (on page 139 in this thesis) does in general not hold for the reflections instances in [Rat05b]. Moreover this article lacks a Theorem like 15.2.1, although it is definitely needed there, too. Theses deficiencies seem to be profound because already the fixing of the first issue requires a change of the definition of the collapsing hierarchies. This makes the proof of a theorem corresponding to Theorem 15.2.1 (of this thesis) more awkward than it would be for the collapsing hierarchies actually given in [Rat05b]. Since the ordinal analysis given in [Rat05a] can be regarded as an extension of [Rat05b] the same remarks also apply to [Rat05a].

[5] He uses an approach different to ours.

[6] An ordinal analysis of a theory T comprises two parts in general. At first the elaboration of an ordinal notation system and by its means a characterization of an upper-bound of the proof-theoretic ordinal of T is required. In a second step it is proved that this upper-bound is sharp by giving well-ordering proofs for all ordinals below this upper-bound within T.

Throughout this thesis we use the term "ordinal analysis" to denote the first part of an ordinal analysis. I.e. in this thesis no well-ordering proofs are given.

In the second part of the present thesis a characterization of the provable recursive functions of Π_ω-Ref is given by applying an improved version of the methods of A. Weiermann and B. Blankertz to the ordinal analysis given in the first part of this thesis.

The crucial new idea employed here is to work with relativized subrecursive hierarchies, e.g. subrecursive hierarchies defined on proper subsets of the ordinal notation system instead of working with one fixed subrecursive hierarchy defined on the whole ordinal notation system. Thereby subrecursive hierarchies become "collapsible" and thus a refined version of the Reflection Elimination Theorem is obtained nearly for free.

In the third part of this dissertation the methods developed in the first part are extended (transfinitely iterated) to obtain an ordinal analysis of Stability. In essence the required enhancements are of technical nature.

The Reflection Elimination Theorem of this ordinal analysis differs from that of the first part. Here $\Sigma_n(\kappa+\theta)$-sentences have to be collapsed, with parameters below κ and parameters of stages in the interval $[\kappa, \kappa+\theta]$. I.e. a collapsing of intervals is taking place. However, since θ is always less than κ this new issue is quite easy to handle.

Finally in the fourth and last part of this thesis the results of the second part are applied to the ordinal analysis given in the third part to achieve a characterization of the provable recursive functions of Stability.

In addition there is a review-part following the first part, in which the theories Π_n-Ref are discussed. This part, in cooperation with the first part, might serve as a good starting point to the reader how is familiar with the ordinal analysis of Π_3-Ref given in [Rat94b].

Acknowledgment

I'm deeply indebted to my thesis advisor Professor Dr. Wolfram Pohlers. He suggested this interesting and challenging dissertation undertaking to me. Many fruitful discussions with him on the topic of proof theory always inspired and motivate me.

Part I.

An Ordinal Analysis of Π_ω-Ref

1. Introduction

In this first part of the thesis we want to elaborate an ordinal analysis of Π_ω-Ref.

By introducing an infinite verification calculus for (pseudo-)Π^1_1-sentences, which is $\mathbb{N}$-correct and $\mathbb{N}$-complete it is possible to assign a uniquely determined ordinal, called truth-complexity (tc), to every (pseudo-)Π^1_1-sentence. Moreover this can be done in a way that for a primitive-recursive well-ordering $\prec$ it holds otyp$(\prec)$ =tc$(TI(\prec))$ if otyp$(\prec) \in$ Lim (cf. [Poh09], Theorem 6.7.2). Thereby it follows that the ordinal

$$\|T\|_{\Pi^1_1} \;:=\; \sup\{\mathrm{tc}(F)+1 \,|\, F \text{ is a pseudo } \Pi^1_1\text{-sentence } \& \; T \vdash F\}$$

is greater than or equal to the proof-theoretic ordinal of T. In addition, it holds already for a conservative second-order extension T of Peano Arithmetic that $\|T\|_{\Pi^1_1} \leq \|T\|$ (cf. [Poh09], Theorem 6.7.4). Thereby for most theories computing the proof-theoretic ordinal is the same as computing the Π^1_1 ordinal.

Thus it follows by the Spector-Gandy Theorem that for subsystems of set theory, which are $\mathbb{L}$-correct and in which $\mathbb{L}_{\omega_1^{CK}}$ is definable, the ordinal

$$\|T\|_{\Sigma_1^{\omega_1^{CK}}} \;:=\; \min\{\alpha \,|\, \mathbb{L}_\alpha \models F \text{ for every } \Sigma_1^{\omega_1^{CK}}\text{-sentence } F \text{ such that } T \vdash F\},$$

is an upper-bound of $\|T\|$. Since Π_ω-Ref meets these requirements we therefore compute $\|\Pi_\omega\text{-Ref}\|_{\Sigma_1^{\omega_1^{CK}}}$ in this first part of the thesis.

The theory Π_ω-Ref denotes the subsystem KP of set theory (Kripke-Platek Set-Theory) augmented by reflection schemes for first-order formulae. The theory KP was established by S. Kripke and R. Platek in the 1960's as an axiomatization of generalized recursion-theory on sets.

The first crucial step in the proof-theoretic treatment of KP augmented by first-order reflection schemes was taken by M. Rathjen in [Rat94b], in which he achieved an ordinal analysis of Π_3-Ref (KP augmented by a reflection scheme for Π_3-formulae).

In the following we assume that the reader is familiar with [Rat94b]. Since our ordinal analysis of Π_ω-Ref reads as easy as [Rat94b], as soon as the interplay between collapsing-hierarchies, reflection instances and the elimination of reflection-rules is understood and the fine structure of the collapsing hierarchies is elaborated, we first outline the underlying ideas of these concepts.

Collapsing Hierarchies, Reflection Instances, and the Elimination of Reflection-Rules

In this explanation we confine to the treatment of Π_n-Ref for $n \geq 4$. To keep it simple we argue completely semantically, i.e. we ignore any recursiveness-conditions. So let us assume, a derivation is given in which applications of Π_n-reflection rules occur and let us also assume there exists a collapsing hierarchy with reflection degree (rdh) $n-1$, i.e. a hierarchy $\langle \mathfrak{M}_\alpha \mid \alpha \in \mathsf{T} \rangle$, such that the elements of $\mathfrak{M}_\alpha$ are Π_{n-1}-reflecting on $\mathfrak{M}_\xi$ for every $\xi < \alpha$ (as we build up our collapsing hierarchies on the cardinal-analogues of Π_n-reflecting ordinals there is always a 2-shift between the notion of the reflection-strength of ordinals in context of collapsing hierarchies and in context of the actual reflection-rules).

The idea of "stationary collapsing", as established in [Rat94b], is to transform the given derivation (with supposed derivation-length α), in which Π_n-reflection rules might occur, into card($\mathfrak{M}^{\hat{\alpha}}$)-many derivations, in which the applications of the Π_n-reflection rules are replaced by applications of $\mathfrak{M}^{\hat{\xi}}$-Π_{n-1}-reflection rules. Such a Reflection Elimination Theorem states (in our approach, i.e. employing (cofinal-many) Π_{n-1}-reflection rules) as follows: Suppose $\Gamma \subseteq \Sigma_{n-1}(\pi)$, with parameters in L_σ for some $\sigma < \pi$, then

$$\mathcal{H}[\mathcal{A}] \Big|\frac{\alpha}{\bar{\mu}}\, \Gamma \quad \Rightarrow \quad \mathcal{H}'[\mathcal{A}, \kappa] \Big|\frac{\Psi^{\hat{\alpha} \oplus \kappa}}{\cdot}\, \Gamma^{(\pi,\kappa)} \quad \text{for all } \sigma < \kappa \in \mathfrak{M}^{\hat{\alpha}}.$$

In the proof of this theorem, which runs by induction on α, we face the following situation: In (the crucial) case that the last inference was an application of the Π_n-reflection rule, there is an $F \in \Pi_n(\pi)$, such that

$$\mathcal{H}[\mathcal{A}] \Big|\frac{\alpha_0}{\bar{\mu}}\, \Gamma, F.$$

By an application of the derivable rule ($\forall$-Inv) we obtain

$$\mathcal{H}[\mathcal{A}, t] \Big|\frac{\alpha_0}{\bar{\mu}}\, \Gamma, F'(t), \quad \text{for all } t \in \mathcal{T}_\pi.$$

An application of the induction hypothesis yields

$$\mathcal{H}'[\mathcal{A}, t] \Big|\frac{\Psi^{\hat{\alpha_0} \oplus \lambda}}{\cdot}\, \Gamma^{(\pi,\lambda)}, F'(t)^{(\pi,\lambda)} \quad \text{for all } \sigma, t < \lambda \in \mathfrak{M}^{\hat{\alpha_0}}.$$

By use of a $\forall$-inference we obtain

$$\mathcal{H}'[\mathcal{A}] \Big|\frac{\Psi^{\hat{\alpha_0} \oplus \lambda}+1}{\cdot}\, \Gamma^{(\pi,\lambda)}, F^{(\pi,\lambda)} \quad \text{for all } \sigma < \lambda \in \mathfrak{M}^{\hat{\alpha_0}}.$$

Now we fix a $\sigma < \kappa \in \mathfrak{M}^{\hat{\alpha}}$. Then it holds $\{\lambda \mid \sigma < \lambda \in \mathfrak{M}^{\hat{\alpha_0}}\} \cap \kappa \neq \emptyset$ and we obtain by an application of an $\exists$-inference

$$\mathcal{H}'[\mathcal{A}] \Big|\frac{\Psi^{\hat{\alpha_0} \oplus \lambda}+2}{\cdot}\, \Gamma^{(\pi,\lambda)}, \exists z^\kappa (z \models F) \quad \text{for all } \sigma < \lambda \in \mathfrak{M}^{\hat{\alpha_0}} \cap \kappa. \qquad \text{(A)}$$

To finish this case we have to transform $\Gamma^{(\pi,\lambda)}$ to $\Gamma^{(\pi,\kappa)}$. Therefore let ${}_\sigma M^{\hat{\alpha_0}}$ be a formalization of the predicate $\{\lambda \mid \sigma < \lambda \in \mathfrak{M}^{\hat{\alpha_0}}\}$. Then we obtain from (A)

$$\mathcal{H}'[\mathcal{A}] \overset{\Psi^{\hat{\alpha_0}\oplus\kappa}}{\underset{\bullet}{\vdash}} \forall x^\kappa\big({}_\sigma M^{\hat{\alpha_0}}(x) \to \bigvee \Gamma^{(\pi,x)}\big), \exists z^\kappa(z \models F).$$

Moreover we have

$$\mathcal{H}'[\mathcal{A}] \overset{\Psi^{\hat{\alpha_0}\oplus\kappa}}{\underset{\bullet}{\vdash}} \neg(\bigvee \Gamma^{(\pi,\kappa)}), \bigvee \Gamma^{(\pi,\kappa)},$$

and $\neg\big(\bigvee \Gamma^{(\pi,\kappa)}\big)$ is a $\Pi_{n-1}(\pi)$-sentence (if the class of Π_{n-1}-sentences is closed under the Boolean connectives $\vee$ and $\wedge$). If every $\sigma < \kappa \in \mathfrak{M}^{\hat{\alpha}}$ is equipped with a ${}_\sigma M^{\hat{\alpha_0}}$-$\Pi_{n-1}(\kappa)$-reflection rule, we obtain

$$\mathcal{H}'[\mathcal{A}] \overset{\Psi^{\hat{\alpha_0}\oplus\kappa}}{\underset{\bullet}{\vdash}} \exists x^\kappa\big({}_\sigma M^{\hat{\alpha_0}}(x) \wedge \neg(\bigvee \Gamma^{(\pi,x)})\big), \bigvee \Gamma^{(\pi,\kappa)}, \tag{B}$$

and we get the desired result by a cut.

Now let us have a look at Definition 5.2.7 on page 68 and Definition 4.2.8 to see their relevance for fulfilling the requirements of the reflection elimination process outlined above. By clause 0.1 of Definition 5.2.7 and the fourth clause of Definition 4.2.8 π is equipped with a Π_n-reflection rule. The collapsing hierarchy of π is defined by means of clause 1. of Definition 5.2.7. By subclause 1.3 and the fifth clause of Definition 4.2.8 the defined elements of the collapsing hierarchy of π are equipped with the reflection rules required to attain (B).

Therefore we are able to eliminate the Π_n-reflection rule of π at the cost of introducing new elements κ_α, which are equipped with $\mathfrak{M}^\xi$-Π_{n-1}-reflection rules for all $\xi < \alpha$. In the following we denote this property of κ_α by $\kappa_\alpha \models \mathsf{M}^{<\alpha}$-$\mathsf{P}_{n-1}$.

Now we want to iterate the above outlined elimination process to get rid of the reflection rules for κ_α. First we observe that it is impossible to iterate this process by introducing only one collapsing hierarchy for κ_α. The reason is simple: Let us assume the above outlined case with π replaced by κ_α and the last inference was an application of a ${}_\sigma M^\xi$-$\Pi_{n-1}(\kappa_\alpha)$-reflection rule, for some $\xi < \alpha$. Then we have to derive $\exists z^\kappa\big({}_\sigma M^\xi(z) \wedge z \models F\big)$ instead of $\exists z^\kappa(z \models F)$ in (A). Therefore all the sets of the collapsing hierarchy $\langle \mathfrak{M}^\beta_{\kappa_\alpha} \mid \beta \in \mathsf{T}\rangle$ of κ_α have to be subsets of $\mathfrak{M}^\xi$. However, since that must hold for any $\xi < \alpha$ it would follow that any element of the collapsing hierarchy of κ_α is already $\mathfrak{M}^\xi$-Π_{n-1}-reflecting for every $\xi < \alpha$, if $\alpha \in \mathrm{Lim}$. As we want the elements of the collapsing hierarchy of κ_α to be below κ_α (to secure that the reflection-rule elimination process terminates at all) this is absurd, since κ_α could itself be the minimal ordinal with the property $\mathfrak{M}^\xi$-Π_{n-1}-reflecting for every $\xi < \alpha$.

Therefore we have to introduce for every $\xi < \alpha$ a collapsing hierarchy $\langle \mathfrak{M}^\beta_{\kappa_\alpha,\xi} \mid \beta \in \mathsf{T}\rangle$ whose sets are subsets of $\mathfrak{M}^\xi$. Moreover to obtain equation (B) in the above scenario every $\kappa \in \mathfrak{M}^{\hat{\beta}}_{\kappa_\alpha,\xi}$ has to be equipped with a ${}_\sigma M^{\hat{\beta_0}}_{\kappa_\alpha,\zeta}$-$\Pi_{n-2}(\kappa)$-reflection-rule, for every $\zeta < \alpha$, since the last inference of the given derivation could have been a M^ζ-Π_{n-1}-(κ_α)-reflection inference even if we are proving the theorem for $\kappa \in \mathfrak{M}^{\hat{\beta}}_{\kappa_\alpha,\xi}$. Thus every

$\kappa \in \mathfrak{M}^{\beta}_{\kappa_\alpha,\xi}$ has to be $\mathfrak{M}^{\beta_0}_{\kappa_\alpha,\zeta}$-$\Pi_{n-2}$-reflecting for every $\beta_0 < \beta$ and every $\zeta < \alpha$. Therefore the collapsing hierarchies $\langle \mathfrak{M}^{\beta}_{\kappa_\alpha,\xi} \,|\, \beta \in \mathsf{T}\rangle$ have to be defined simultaneously for all $\xi < \alpha$.

A closer look at Definition 5.2.7 reveals that this process actually takes place in clause 2. (with $\Psi^{\delta}_{\mathbb{Z}} = \kappa_\alpha$ in our example). Therefore a reflection configuration "splits" a block $\mathsf{M}^{<\alpha}$-P_{n-1} of reflection rules into reflection instances "M^{ξ}-P_{n-1}" with $\xi < \alpha$, which have to be eliminated simultaneously.

At this point it becomes clear how "slow" and intricate the reflection elimination process actually is. In the above outlined second elimination step we have transformed a derivation of a set of $\Sigma_{n-1}(\kappa_\alpha)$-sentences Γ^{κ_α}, with $\kappa_\alpha \models \mathsf{M}^{<\alpha}$-$\mathsf{P}_{n-1}$, into derivations of Γ^{κ_β}, where $\kappa_\beta \models \mathsf{M}^{<\xi}$-$\mathsf{P}_{n-1}$ with $\xi < \alpha$ and $\kappa_\beta \models \mathsf{M}^{<\beta}_{\kappa_\alpha,\zeta}$-$\mathsf{P}_{n-2}$ for all $\zeta < \alpha$.

We proceed and define a collapsing hierarchy for κ_β. Since κ_β is equipped with Π_{n-1}-reflection rules we realize that the reflection degree of this hierarchy must be $n-2$ (cf. equation (B)). In this step we cannot eliminate the ${}_{\sigma}M^{\beta_0}_{\kappa_\alpha,\zeta}$-$\Pi_{n-2}$-$(\kappa_\beta)$-reflection rules, which κ_β also bears, as otherwise we would have to define a collapsing hierarchy of κ_β with reflection degree $n-2$ whose sets are subsets of $\mathfrak{M}^{\beta_0}_{\kappa_\alpha,\zeta}$ (to obtain equation (A)). However, this is impossible in general as $\mathfrak{M}^{\beta_0}_{\kappa_\alpha,\zeta}$ is itself a member of a hierarchy with reflection degree $n-2$. Therefore we can just eliminate the M^{ζ}-Π_{n-1}-(κ_β)-reflection rules and have to inherit all other rules (of reflection degree $n-2$) to the elements of the collapsing hierarchy of κ_β.

The iteration of the above outlined reflection elimination process leads to elements which bear more and more reflection rules.

At this point the reader should toy around with Definition 5.2.7 and continue the above given example. Thereby (s)he should become aware that there are sooner or later elements such that the length of $\vec{R}$ (cf. 5.2.7) increases up to $n-3$. Moreover it should become clear that the vector $\vec{R}$ is just the tip of the iceberg with respect to the reflection rules an element of a collapsing hierarchy can bear. In general the particular entries of $\vec{R}$ have to be regarded as stacks of entries of the form $\mathsf{M}^{<\alpha}$-P_{n-1} (cf. 3.1.2 – 3.1.4).

Carrying on with defining collapsing hierarchies until elements are reached that do not bear any reflection rule (cf. clause 1. of 5.2.7 with $m = 0$) seems to be fine at first glance. Actually such elements are reached since ON is well-founded. However, the big problem in the whole definition process is that we are a bit too careless at each step at which we define a new collapsing hierarchy for π. We just secure that the newly defined elements are equipped with all the reflection rules that are needed to eliminate the reflection rules of π in the way outlined above.[1] However, this does not rule out the possibility that some of these elements are also part of another collapsing hierarchy. This causes a problem since for the latter hierarchy the outlined proof scheme would fail at step (B) as these elements might not be equipped with the needed reflection rules.[2]

[1] Although, even that is not totally clear. E.g. we cut off $\vec{R}_{\Psi^{\xi}_{\vec{\nu}}}$ at m in clause 2. of Definition 5.2.7. That we do not loose any information in doing so follows by the Correctness Lemma 3.1.6.

[2] In [Duc08] C. Duchhardt elaborates an ordinal analysis of Π_4-reflection and his concept differs from ours exactly in this point. To avoid the just mentioned difficulties he defines the collapsing

To handle this problem we need an in depth analysis about the allocation of elements of the form $\Psi_{\mathbb{X}}^{\alpha}$ over the different collapsing hierarchies. This analysis is carried out in chapter 3. There we show that all collapsing hierarchies containing $\Psi_{\mathbb{X}}^{\alpha}$ can be decoded out of $\vec{R}_{\Psi_{\mathbb{X}}^{\alpha}}$ (cf. Theorem 3.2.4). By using this result we can easily show that all elements of a given collapsing hierarchy are equipped with the required reflection rules to perform the reflection elimination process set out above (cf. Theorem 5.2.1).

hierarchies more carefully. He secures by definition that there are no unwanted overlappings. Although there is no need for a fine structure analysis of his collapsing hierarchies we consider his concept as more cumbersome than ours from a technical point of view (cf. e.g. the $<$-Comparison theorems in his paper). Moreover it seems to be unclear how his concept can be generalized to an ordinal analysis of $\mathsf{\Pi}_{\omega}$-Ref.

2. Ordinal Theory

In this chapter we define the ordinal notation system $\mathsf{T}(\Xi)$ by which means we are able to give an upper-bound of the proof-theoretic ordinal of Π_ω-Ref. By utilizing cardinal-analogues of the required recursive ordinals many complexity considerations narrow down to simple cardinality arguments. Thereby we employ these cardinal-analogues. However, a well-ordering proof, which we do not give in this thesis, but which can be carried out by the methods of [Sch93] or [Rat94a], requires the use of the recursive ordinals.

Since the cardinal-analogue of a Π_{n+2}-reflecting ordinal is a Π^1_n-indescribable cardinal there is always a $+2$-shift by switching from notational-based terms, e.g. reflection instances, to the corresponding reflection-rules (cf. e.g. Definition 4.2.8).

2.1. Π^1_n-Indescribable Cardinals

Notation. In the following we denote the class of ordinal numbers by ON, the class of successor ordinals by Succ, the class of limit ordinals, i.e. $\mathrm{ON}\setminus(\mathrm{Succ}\cup\{0\})$, by Lim, the class of transfinite cardinal numbers by Card and the class of regular cardinals by Reg. We denote the cardinal successor of a cardinal κ by κ^+ and $\mathrm{card}(S)$ denotes the cardinality of the set S.

As usual we denote the binary Veblen function by φ, i.e. $\varphi(\alpha,\beta)$ is defined by transfinite recursion on α as the enumerating function of the class $\{\omega^\beta \mid \beta \in \mathrm{ON} \wedge \forall\xi\in\alpha\,(\varphi(\xi,\beta)=\omega^\beta)\}$. Throughout the thesis we write ω^β for $\varphi(0,\beta)$. In addition we denote the class of strongly critical ordinals by SC, i.e. it holds $\gamma\in\mathrm{SC}$ iff $\varphi(\alpha,\beta)<\gamma$ for all $\alpha,\beta<\gamma$.

We refer to the n-time Cartesian product of an ordinal class by adding the exponent n to the denotation of the respecting class, e.g. $\mathrm{ON}^2 := \mathrm{ON}\times\mathrm{ON}$. Moreover we use the following notation:

$$\forall^i_1 k\ F(k)\ :\equiv \forall k\big((1\le k\le i)\to F(k)\big).$$

Definition 2.1.1. A cardinal π is called Π^1_n-indescribable if for every $P_1,\dots,P_k\subseteq V_\pi$ and for all Π^1_n-sentences F of the language of $\langle V_\pi,\in,P_1,\dots,P_k\rangle$ such that

$$\langle V_\pi,\in,P_1,\dots,P_k\rangle \models F,$$

there exists a $0<\kappa<\pi$ such that

$$\langle V_\kappa,\in,P_1\cap V_\kappa,\dots,P_k\cap V_\kappa\rangle \models F.$$

The cardinal π is called $\mathfrak{M}$-Π^1_n-indescribable if in the above situation a $0 < \kappa \in \pi \cap \mathfrak{M}$ can be found.

A cardinal π is called Π^2_0-indescribable if it is Π^1_n-indescribable for every $n \in \omega$.

Lemma 2.1.2. *π is Π^1_0-indescribable iff π is strongly inaccessible.*

Proof. See [Dra74], Ch.9, Theorem 1.3. □

Theorem 2.1.3. *For every $n < \omega$ there is a Π^1_n formula $\psi_n(X_1, \dots, X_k, x)$, which is universal for Π^1_n-sentences, i.e. for every Π^1_n-sentence ϕ, for every limit ordinal $\alpha > \omega$ and any $P_1, \dots, P_k \subseteq V_\alpha$ we have*

$$\langle V_\alpha, \in, P_1, \dots, P_k \rangle \models \phi \leftrightarrow \psi_n(P_1, \dots, P_k, \ulcorner\phi\urcorner),$$

where $\ulcorner\phi\urcorner$ is a Gödel-set for ϕ.

Proof. For $0 < n$ see [Dra74], Ch 9, §1, Lemma 1.9. The proof of this Lemma also shows the existence of a Δ^1_0-formula ψ which is universal for Π^1_0-sentences. □

Corollary 2.1.4. *Since V_π is closed under pairing if π is Π^1_n-indescribable w.l.o.g. we may assume $k = 1$ in the above definition. Thus π is $\mathfrak{M}$-Π^1_n-indescribable iff*

$$\langle V_\pi, \in, \mathfrak{M} \rangle \models \forall X\, \forall x \left(\psi_n(X, x) \to \exists \kappa {\in} \mathfrak{M} \left(\kappa \neq \emptyset \wedge \mathrm{Tran}(\kappa) \wedge \psi^\kappa_n(X \cap V_\kappa, x)\right)\right),$$

i.e. the $\mathfrak{M}$-Π^1_n-indescribableness of π is describable by a Π^1_{n+1}-statement (in the parameter $\mathfrak{M}$).

2.2. Collapsing Hierarchies

To obtain an ordinal notation system for Π_ω-Ref we define by simultaneous recursion on α the sets $C(\alpha, \beta)$ (the αth Skolem closure of β), reflection instances $\mathbb{X}$, collapsing hierarchies $\mathfrak{M}^\alpha_{\mathbb{X}}$ and collapsing functions $\Psi_{\mathbb{X}}$.

Reflection instances are strings consisting of ordinals, parentheses and the symbols ; , -, ϵ, M and P. More exactly they are quintuplets of the form $(\pi; \mathsf{P}_m; \vec{R}; \mathbb{Z}; \alpha)$ or of the form $(\pi; \mathsf{M}^\xi_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m; \vec{R}; \mathbb{Z}; \alpha)$. Reflection instances are intended as arrays, providing information of the reflection strength of π (second and third component), and record their own recursive development in the fourth component. The fifth component denotes the ordinal of the reflection instance, i.e. the step in the recursive process in which the reflection configuration is generated.

To be able to talk easily about related reflection instances we also introduce reflection configurations as functions which map finite sequences of ordinals to reflection instances.

In the following we denote the reflection instance $(\Xi; \mathsf{P}_m; \epsilon; \epsilon; \omega)$ by $\mathbb{A}(m)$ for $m \in (0, \omega)$, i.e. we use $\mathbb{A}$ as a name for the reflection configuration which maps an element $m \in \omega$ to $(\Xi; \mathsf{P}_m; \epsilon; \epsilon; 0)$. Moreover we use $\mathbb{U}, \mathbb{V}, \mathbb{W}, \mathbb{X}, \mathbb{Y}, \mathbb{Z}$ as variables for reflection instances and 0-ary reflection configurations and $\mathbb{F}, \mathbb{E}, \mathbb{G}, \mathbb{H}, \mathbb{K}, \mathbb{M}, \mathbb{R}, \mathbb{S}$ as variables for reflection configurations. We denote the αth collapsing hierarchy of a reflection instance $\mathbb{X}$ by $\mathfrak{M}^{\alpha}_{\mathbb{X}}$.

The following definitions have to be read simultaneously to Definition 2.2.4.

Definition 2.2.1. Let $\mathbb{F}(\vec{\eta}) = \mathbb{X} = (\pi; \ldots; \mathbb{Z}; \delta)$ be a reflection instance with predecessor reflection instance $\mathbb{Z} = \mathbb{G}(\vec{\nu})$. Then we refer to $\mathbb{F}$ as the reflection configuration of $\mathbb{X}$ and vice versa we call $\mathbb{X}$ a reflection instance of $\mathbb{F}$, i.e. $\mathbb{V}$ is a reflection instance of $\mathbb{E}$ iff there is a $\vec{\nu} \in \mathrm{dom}(\mathbb{E})$, such that $\mathbb{V} = \mathbb{E}(\vec{\nu})$ and $\mathbb{E}$ is then referred to as the reflection configuration of $\mathbb{V}$. Moreover we define

$$\begin{aligned}
\mathrm{o}(\mathbb{X}) := \mathrm{o}(\mathbb{F}) &:= \delta, \\
\mathrm{i}(\mathbb{X}) := \mathrm{i}(\mathbb{F}) &:= \pi, \\
\mathrm{Prinst}(\epsilon) := \mathrm{Prcnfg}(\epsilon) &:= \emptyset, \\
\mathrm{Prinst}(\mathbb{X}) := \mathrm{Prinst}(\mathbb{F}) &:= \{\mathbb{Z}\} \cup \mathrm{Prinst}(\mathbb{Z}), \\
\overline{\mathrm{Prinst}}(\mathbb{X}) &:= \mathrm{Prinst}(\mathbb{X}) \cup \{\mathbb{X}\}, \\
\mathrm{Prcnfg}(\mathbb{X}) := \mathrm{Prcnfg}(\mathbb{F}) &:= \{\mathbb{G}\} \cup \mathrm{Prcnfg}(\mathbb{Z}), \\
\overline{\mathrm{Prcnfg}}(\mathbb{X}) := \overline{\mathrm{Prcnfg}}(\mathbb{F}) &:= \mathrm{Prcnfg}(\mathbb{F}) \cup \{\mathbb{F}\}.
\end{aligned}$$

Notation. Let κ be an ordinal, $\vec{\eta} = (\eta_1, \ldots, \eta_n)$ be a vector of ordinals and $M \subseteq \mathrm{ON}^n$. Then we use the following abbreviations[1]

$$\begin{aligned}
\vec{\eta} \in C(\kappa) \quad &:\Leftrightarrow \quad \forall^n_1 k \ \big(\eta_k \in C(\eta_k, \kappa)\big), \\
\vec{\eta} \in M_{C(\kappa)} \quad &:\Leftrightarrow \quad \vec{\eta} \in M \cap C(\kappa).
\end{aligned}$$

Definition 2.2.2 (M-P-Expressions)**.** In the following we assume that we have for every reflection configuration $\mathbb{F}$ a symbol $\mathsf{M}_{\mathbb{F}}$. Let α be an ordinal and $n \in \omega$ or $n = -1$. Then we refer to expressions of the form $\mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_n$ as M-P-expressions. For technical convenience we also define ϵ as an M-P-expression and as a finite sequence of M-P-expressions with zero length.

Let $\mathbb{U}$ be an 0-ary reflection configuration, $\mathbb{G}$ a non 0-ary reflection configuration and $\mathbb{M}$ an arbitrary reflection configuration. Let $\xi > \mathrm{o}(\mathbb{U})$, $\xi' > \mathrm{o}(\mathbb{G})$, and $\gamma \geq \mathrm{o}(\mathbb{M})$, plus $\vec{R} = (\mathsf{M}^{<\xi_1}_{\mathbb{R}_1}\text{-}\mathsf{P}_{m_1}, \ldots, \mathsf{M}^{<\xi_i}_{\mathbb{R}_i}\text{-}\mathsf{P}_{m_i})$, with $m_1 > \ldots > m_i$. Then we define

$$\begin{aligned}
\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{-1} &:= \epsilon, \\
\widetilde{\mathsf{M}}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m &:= \begin{cases} (\vec{R}_{\mathrm{i}(\mathbb{M})})_m & \text{if } \gamma = \mathrm{o}(\mathbb{M}) \text{ and } \mathbb{M} \neq \mathbb{A}, \\ \mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m & \text{otherwise,} \end{cases}
\end{aligned}$$

[1]Note, that $C(\kappa)$ and $M_{C(\kappa)}$ are not well-defined terms.

and

$$
\begin{aligned}
\kappa \models \epsilon &:\Leftrightarrow \emptyset \notin \emptyset \\
\kappa \models \mathsf{M}_{\mathbb{M}}^{<\mathrm{o}(\mathbb{M})}\text{-}\mathsf{P}_m &:\Leftrightarrow \kappa \text{ is } \Pi^1_m\text{-indescribable}, \\
\kappa \models \mathsf{M}_{\mathbb{U}}^{<\xi}\text{-}\mathsf{P}_m &:\Leftrightarrow \forall \zeta \in [\mathrm{o}(\mathbb{U}), \xi)_{C(\kappa)} \ (\kappa \text{ is } \mathfrak{M}^{\zeta}_{\mathbb{U}}\text{-}\Pi^1_m\text{-indescribable,}) \\
\kappa \models \mathsf{M}_{\mathbb{G}}^{<\xi'}\text{-}\mathsf{P}_m &:\Leftrightarrow \forall (\zeta, \vec{\eta}) \in [\mathrm{o}(\mathbb{G}), \xi')_{C(\kappa)} \times \mathrm{dom}(\mathbb{G})_{C(\kappa)} \\
&\qquad (\kappa \text{ is } \mathfrak{M}^{\zeta}_{\mathbb{G}(\vec{\eta})}\text{-}\Pi^1_m\text{-indescribable}), \\
\kappa \models \vec{R} &:\Leftrightarrow \forall^i_1 k \ (\kappa \models \mathsf{M}_{\mathbb{R}_k}^{<\xi_k}\text{-}\mathsf{P}_{m_k}).
\end{aligned}
$$

Moreover we define the following substrings of $\vec{R}$

$$
\begin{aligned}
\vec{R}_k &:= \begin{cases} (\mathsf{M}_{\mathbb{R}_l}^{<\xi_l}\text{-}\mathsf{P}_{m_l}) & \text{if } m_l = k \text{ for some } 1 \le l \le i, \\ \epsilon & \text{otherwise.} \end{cases} \\
\vec{R}_{<k} &:= (\vec{R}_{k-1}, \dots, \vec{R}_0), \\
\vec{R}_{>k} &:= (\vec{R}_{m_1}, \dots, \vec{R}_{k+1}), \\
\vec{R}_{(l,k)} &:= (\vec{R}_l, \vec{R}_{l-1}, \dots, \vec{R}_{k+1}) \quad \text{for } l \ge k.
\end{aligned}
$$

Analogously we define the substrings $\vec{R}_{\le k}$, $\vec{R}_{\ge k}$ and $\vec{R}_{(l,k]}$.

Definition 2.2.3. Let $\mathbb{X}$ be a reflection instance. We define by recursion on $\mathrm{o}(\mathbb{X})$

$$
\begin{aligned}
\mathrm{par}\, \mathbb{A}(m) &:= \{m\}, \\
\mathrm{par}(\kappa^+; \dots; 0) &:= \{\kappa\}, \\
\mathrm{par}(\Psi^{\delta}_{\mathbb{Z}}; \mathsf{P}_m; \dots) &:= \{\delta\} \cup \mathrm{par}\, \mathbb{Z}, \\
\mathrm{par}(\Psi^{\delta}_{\mathbb{Z}}; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m; \dots) &:= \{\delta, \xi, \vec{\nu}\} \cup \mathrm{par}\, \mathbb{Z}.
\end{aligned}
$$

In the following we write $\mathbb{X} \in C(\alpha, \pi)$ iff $\mathrm{par}\, \mathbb{X} \subseteq C(\alpha, \pi)$ and $\mathbb{X} \in C(\kappa)$ iff $\mathrm{par}\, \mathbb{X} \subseteq C(\kappa)$.

Definition 2.2.4 (Collapsing Hierarchies)**.** [2] Let Ξ be a Π^2_0-indescribable cardinal number. By simultaneous recursion on α we define the sets $C(\alpha, \pi)$, reflection instances $\mathbb{X}$, reflection configurations $\mathbb{F}$, collapsing hierarchies $\mathfrak{M}$, finite sequences of M-P-expressions $\vec{R}_\Psi$ and (partial) collapsing functions Ψ (all these where appropriate with arguments and indices).

The class of reflection instances is partitioned in three types enumerated by $1., 2., 3.$ Whenever we define a new reflection instance we indicate by $\to p.$, with $p \in \{1, 2, 3\}$

[2] In this Definition we define finite sequences of M-P-expressions, which we denote by $\vec{R}_\pi$ for $\pi > \omega$. The reader should be aware that $\vec{R}_k$ with $k < \omega$ denotes an M-P-expression, while $\vec{R}_\pi$ with $\pi > \omega$ denotes a finite sequence of M-P-expressions, cf. Definition 2.2.2.

of which type this reflection instance is. Clause $p.$ of the definition then specifies how to proceed with reflection instances of type $p.$ in the recursive definition process.

$$
\begin{aligned}
C(\alpha,\pi) &:= \bigcup_{n<\omega} C^n(\alpha,\pi), \quad \text{where} \\
C^0(\alpha,\pi) &:= \pi \cup \{0,\Xi\}, \quad \text{and} \\
C^{n+1}(\alpha,\pi) &:= \begin{cases} C^n(\alpha,\pi) \cup \\ \{\gamma+\omega^\delta \mid \gamma,\delta \in C^n(\alpha,\pi) \wedge \gamma =_{\text{NF}} \omega^{\gamma_1}+\ldots+\omega^{\gamma_m} \wedge \gamma_m \geq \delta\}^\dagger \cup \\ \{\varphi(\xi,\eta) \mid \xi,\eta \in C^n(\alpha,\pi)\} \cup \\ \{\kappa^+ \mid \kappa \in C^n(\alpha,\pi) \cap \mathrm{Card} \cap \Xi\} \cup \\ \{\Psi^\gamma_{\mathbb{X}} \mid \mathbb{X},\gamma \in C^n(\alpha,\pi) \wedge \gamma < \alpha \wedge \Psi^\gamma_{\mathbb{X}} \text{ is well-defined}\}. \end{cases}
\end{aligned}
$$

0.1. We define the reflection configuration $\mathbb{A}$ with $\mathrm{dom}(\mathbb{A}) = (0,\omega)$ and reflection instances[3]

$$\mathbb{A}(m) = (\Xi, \mathsf{P}_m; \epsilon; \epsilon; \omega) \qquad \to 1.$$

For technical convenience we also define $\vec{R}_\Xi := \epsilon$.

0.2. For every cardinal $\omega \leq \kappa < \Xi$ we define the 0-ary reflection configuration and reflection instance

$$(\kappa^+; \mathsf{P}_0; \epsilon; \epsilon; 0) \qquad \to 2.$$

For technical convenience we also define $\vec{R}_{\kappa^+} := (\mathsf{M}^{<\omega}_{\mathbb{A}}\text{-}\mathsf{P}_0)$.

1. Let $\mathbb{X} := \mathbb{A}(m+1)$ be a reflection instance of the form

$$(\Xi; \mathsf{P}_{m+1}; \epsilon; \epsilon; \omega).$$

For $\alpha \geq \omega$ we define $\mathfrak{M}^\alpha_{\mathbb{X}}$ as the set of all ordinals $\kappa < \Xi$ satisfying

1. $C(\alpha,\kappa) \cap \Xi = \kappa$,
2. $\mathbb{X}, \alpha \in C(\kappa)$,
3. κ is Π^1_m-indescribable,
4. $\kappa \models \mathsf{M}^{<\alpha}_{\mathbb{A}}\text{-}\mathsf{P}_m$.

From now on we assume $\mathfrak{M}^\alpha_{\mathbb{X}} \neq \emptyset$ and by $\Psi^\alpha_{\mathbb{X}}$ we denote the least element of $\mathfrak{M}^\alpha_{\mathbb{X}}$. Moreover we define $\vec{R}_{\Psi^\alpha_{\mathbb{X}}} := (\mathsf{M}^{<\alpha}_{\mathbb{A}}\text{-}\mathsf{P}_m, \mathsf{M}^{<\alpha}_{\mathbb{A}}\text{-}\mathsf{P}_{m-1}, \ldots, \mathsf{M}^{<\alpha}_{\mathbb{A}}\text{-}\mathsf{P}_0)$.

---

†The formulation of closure under $+$ seems to be unnecessarily complicated, but we want $+$ to be injective.

[3]Here we could also proceed with $\mathrm{o}(\mathbb{A}(m)) := 0$. However, it is more convenient to define $\mathrm{o}(\mathbb{A}(m)) = \omega$, as otherwise we would have to differentiate between finite and transfinite arguments of reflection configurations in Lemma 2.3.2 to obtain ① (e). Moreover in the treatment of part two of this thesis the parameter m becomes a relevant parameter of $\mathrm{par}\,\mathbb{A}$ and therefore at least then we have to choose $\mathrm{o}(\mathbb{A}(m)) = \omega$.

1.1. Let $\alpha = \omega$. Then we define the 0-ary reflection configuration and reflection instance

$$(\Psi_{\mathbb{X}}^{\alpha}; \mathsf{P}_m; (\mathsf{M}_{\mathbb{A}}^{<\alpha}\text{-}\mathsf{P}_{m-1}, \dots, \mathsf{M}_{\mathbb{A}}^{<\alpha}\text{-}\mathsf{P}_0); \mathbb{X}; \omega + 1) \qquad \to 2.$$

1.2. Let $\alpha > \omega$. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) := [\omega, \alpha)_{C(\Psi_{\mathbb{X}}^{\alpha})} \times (m, \omega)$ and reflection instances

$$\mathbb{G}(\zeta, n) := (\Psi_{\mathbb{X}}^{\alpha}; \mathsf{M}_{\mathbb{A}(n)}^{\zeta}\text{-}\mathsf{P}_m; (\mathsf{M}_{\mathbb{A}}^{<\alpha}\text{-}\mathsf{P}_{m-1}, \dots, \mathsf{M}_{\mathbb{A}}^{<\alpha}\text{-}\mathsf{P}_0); \mathbb{X}; \alpha+1) \qquad \to 3.$$

2. Let $\mathbb{X} = \mathbb{F}$ be a 0-ary reflection configuration and a reflection instance of the form

$$(\pi; \mathsf{P}_m; \vec{R}; \mathbb{Z}; \delta)$$

Then we either have $\delta = 0 = m$ and $\pi = \bar{\kappa}^+$ for some cardinal $\bar{\kappa}$, or $\delta = \delta_0 + 1$ and $\pi = \Psi_{\mathbb{Z}}^{\delta_0}$. In any case we have $\vec{R}_\pi = (\mathsf{M}_{\mathbb{A}}^{<\omega}\text{-}\mathsf{P}_m, \vec{R})$.

For $\alpha \geq \delta$ we define $\mathfrak{M}_{\mathbb{X}}^{\alpha}$ as the set consisting of all ordinals $\kappa < \pi$ satisfying

1. $C(\alpha, \kappa) \cap \pi = \kappa$,
2. $\mathbb{X}, \alpha \in C(\kappa)$,
3. if $m > 0$: $\kappa \models \vec{R}$,
4. if $m > 0$: $\kappa \models \mathsf{M}_{\mathbb{F}}^{<\alpha}\text{-}\mathsf{P}_{m-1}$.

From now on we assume $\mathfrak{M}_{\mathbb{X}}^{\alpha} \neq \emptyset$ and by $\Psi_{\mathbb{X}}^{\alpha}$ we denote the least element of $\mathfrak{M}_{\mathbb{X}}^{\alpha}$. If $m > 0$ we also define $\vec{R}_{\Psi_{\mathbb{F}}^{\alpha}} := (\widetilde{\mathsf{M}}_{\mathbb{F}}^{<\alpha}\text{-}\mathsf{P}_{m-1}, \vec{R}_{<m-1})$.

For the following 2.· subclauses we suppose $m = m_0 + 1 > 0$. If $m = 0$ we do not equip $\Psi_{\mathbb{X}}^{\alpha}$ with any reflection configurations or instances.

2.1. Let $\alpha = \delta$ and $\vec{R}_{m_0} = (\mathsf{M}_{\mathbb{R}_1}^{<\xi_1}\text{-}\mathsf{P}_{m_0})$ with $\mathrm{o}(\mathbb{R}_1) = \xi_1$. Due to the $\widetilde{\cdot}$-operator we then have $\mathbb{R}_1 = \mathbb{A}$, i.e. $\xi_1 = \omega$ and we define the 0-ary reflection configuration and reflection instance

$$(\Psi_{\mathbb{X}}^{\alpha}; \mathsf{P}_{m_0}; \vec{R}_{<m_0}; \mathbb{X}; \alpha + 1) \qquad \to 2.$$

2.2 Let $\alpha = \delta$ and $\vec{R}_{m_0} = (\mathsf{M}_{\mathbb{A}}^{<\xi_1}\text{-}\mathsf{P}_{m_0})$ with $\omega < \xi_1$. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) := [\omega, \xi_1)_{C(\Psi_{\mathbb{X}}^{\alpha})} \times (m_0, \omega)$ and reflection instances

$$\mathbb{G}(\zeta, n) := (\Psi_{\mathbb{X}}^{\alpha}; \mathsf{M}_{\mathbb{A}(n)}^{\zeta}\text{-}\mathsf{P}_{m_0}; \vec{R}_{<m_0}; \mathbb{X}; \alpha + 1) \qquad \to 3.$$

2.3 Let $\alpha = \delta$ and $\vec{R}_{m_0} = (\mathsf{M}_{\mathbb{R}_1}^{<\xi_1}\text{-}\mathsf{P}_{m_0})$ with $\mathbb{R}_1 \neq \mathbb{A}$ and $\mathrm{o}(\mathbb{R}_1) < \xi_1$. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) := [\mathrm{o}(\mathbb{R}_1), \xi_1)_{C(\Psi_{\mathbb{X}}^{\alpha})} \times \mathrm{dom}(\mathbb{R}_1)_{C(\Psi_{\mathbb{X}}^{\alpha})}$ and reflection instances

$$\mathbb{G}(\zeta, \vec{\eta}) := (\Psi_{\mathbb{X}}^{\alpha}; \mathsf{M}_{\mathbb{R}_1(\vec{\eta})}^{\zeta}\text{-}\mathsf{P}_{m_0}; \vec{R}_{<m_0}; \mathbb{X}; \alpha + 1) \qquad \to 3.$$

2.4. Let $\alpha > \delta$. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) := [\delta, \alpha)_{C(\Psi^{\alpha}_{\mathbb{X}})}$ and reflection instances

$$\mathbb{G}(\zeta) := (\Psi^{\alpha}_{\mathbb{X}}; \mathsf{M}^{\zeta}_{\mathbb{F}}\text{-}\mathsf{P}_{m_0}; \vec{R}_{<m_0}; \mathbb{X}; \alpha + 1) \qquad \to 3.$$

3. Let $\mathbb{X} := \mathbb{F}(\xi, \vec{\nu})$ be a reflection configuration of the form[4]

$$(\Psi^{\delta}_{\mathbb{Z}}; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m; \vec{R}; \mathbb{Z}; \delta + 1).$$

Then it holds $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m, \vec{R})$ for some $\gamma > \xi$ and $\mathbb{M} \in \mathrm{Prcnfg}(\mathbb{X})$.

For $\alpha \geq \delta + 1$ we define $\mathfrak{M}^{\alpha}_{\mathbb{X}}$ as the set consisting of all ordinals $\kappa \in \mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})} \cap \Psi^{\delta}_{\mathbb{Z}}$ satisfying

1. $C(\alpha, \kappa) \cap \Psi^{\delta}_{\mathbb{Z}} = \kappa$,
2. $\mathbb{X}, \alpha \in C(\kappa)$,
3. if $m > 0$: $\kappa \models \vec{R}$,
4. if $m > 0$: $\kappa \models \mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}$.

From now on we assume $\mathfrak{M}^{\alpha}_{\mathbb{X}} \neq \emptyset$ and by $\Psi^{\alpha}_{\mathbb{X}}$ we denote the least element of $\mathfrak{M}^{\alpha}_{\mathbb{X}}$. Moreover we define $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} := ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{\geq m}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}, \vec{R}_{<m-1})$.

Let $\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}} = (\mathsf{M}^{<\sigma_1}_{\mathbb{S}_1}\text{-}\mathsf{P}_{s_1}, \ldots)$.

3.1. Let $\sigma_1 = \mathrm{o}(\mathbb{S}_1)$. Due to the $\widetilde{\cdot}$-operator we then have $\mathbb{S}_1 = \mathbb{A}$, i.e. $\sigma_1 = \omega$. Then we define the 0-ary reflection configuration and reflection instance

$$(\Psi^{\alpha}_{\mathbb{X}}; \mathsf{P}_{s_1}; ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{(s_1,m]}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}, \vec{R}_{<m-1}); \mathbb{X}; \alpha + 1) \qquad \to 2.$$

3.2. Let $\sigma_1 > \mathrm{o}(\mathbb{S}_1)$ and $\mathbb{S}_1 = \mathbb{A}$ Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) := [\omega, \sigma_1)_{C(\Psi^{\alpha}_{\mathbb{X}})} \times (s_1, \omega)$ and reflection instances

$$\mathbb{G}(\zeta, n) := (\Psi^{\alpha}_{\mathbb{X}}; \mathsf{M}^{\zeta}_{\mathbb{A}(n)}\text{-}\mathsf{P}_{s_1};$$
$$((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{(s_1,m]}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}, \vec{R}_{<m-1}); \mathbb{X}; \alpha + 1) \qquad \to 3.$$

3.3. Let $\sigma_1 > \mathrm{o}(\mathbb{S}_1)$ and $\mathbb{S}_1 \neq \mathbb{A}$. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) = [\mathrm{o}(\mathbb{S}_1), \sigma_1)_{C(\Psi^{\alpha}_{\mathbb{X}})} \times \mathrm{dom}(\mathbb{S}_1)_{C(\Psi^{\alpha}_{\mathbb{X}})}$ and reflection instances

$$\mathbb{G}(\zeta, \vec{\eta}) := (\Psi^{\alpha}_{\mathbb{X}}; \mathsf{M}^{\zeta}_{\mathbb{S}_1(\vec{\eta})}\text{-}\mathsf{P}_{s_1};$$
$$((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{(s_1,m]}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}, \vec{R}_{<m-1}); \mathbb{X}; \alpha + 1) \qquad \to 3.$$

[4]Note, that $\vec{\nu}$ can also be a vector of zero length, e.g. if $\mathbb{X}$ is defined by subclause 2.4

2.3. Structure Theory

In this section we show that $\mathfrak{M}^\alpha_\mathbb{X} \neq \emptyset$ if $\alpha \in C(\mathrm{i}(\mathbb{X}))$ and give criteria for $<$ -comparisons like $\Psi^\alpha_\mathbb{X} < \Psi^\beta_\mathbb{Y}$.

Definition 2.3.1. Let $\mathbb{X} = (\pi; \mathsf{P}_m; \vec{R}; \ldots)$ or $\mathbb{X} = (\pi; \mathsf{M}^\xi_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m; \vec{R}; \ldots)$ be a reflection instance with reflection configuration $\mathbb{F}$, and $\vec{S} = (\mathsf{M}^{<\sigma_1}_{\mathbb{S}_1}\text{-}\mathsf{P}_{s_1}, \ldots)$ be a finite sequence of M-P-expressions. Then we define.

$$\begin{aligned}
\vec{R}_\mathbb{X} := \vec{R}_\mathbb{F} &:= \vec{R}\\
\mathrm{rdh}(\mathbb{A}(m+1)) &:= m,\\
\mathrm{rdh}(\mathbb{X}) &:= m-1,\\
\mathrm{Rdh}(\mathbb{A}) &:= \omega,\\
\mathrm{Rdh}(\mathbb{F}) &:= \{m-1\}, \quad \text{if } \mathbb{F} \neq \mathbb{A},\\
\mathrm{rd}(\vec{S}) &:= s_1,\\
\mathrm{rd}(\epsilon) &:= -1,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{initl}(\mathbb{F}) &:= \begin{cases} \mathbb{F} & \text{if } \mathbb{X} = (\kappa^+; \ldots; 0) \text{ for some } \kappa,\\ \mathbb{A} & \text{otherwise,}\end{cases}\\
\mathrm{ran}^\alpha_\mathbb{X}(\mathbb{Z}) := \mathrm{ran}^\alpha_\mathbb{F}(\mathbb{Z}) &:= \begin{cases} \delta & \text{if there is a refl. inst. } (\Psi^\delta_\mathbb{Z}; \ldots) \in \overline{\mathrm{Prinst}}(\mathbb{X}),\\ \alpha & \text{otherwise,}\end{cases}\\
\mathrm{ran}^\alpha_\mathbb{X}(\mathbb{E}) := \mathrm{ran}^\alpha_\mathbb{F}(\mathbb{E}) &:= \begin{cases} \delta & \text{if there is a refl. inst. } (\Psi^\delta_\mathbb{Z}; \ldots) \in \overline{\mathrm{Prinst}}(\mathbb{X})\\ & \text{and } \mathbb{E} \text{ is the refl. config. of } \mathbb{Z},\\ \alpha & \text{otherwise.}\end{cases}
\end{aligned}$$

Remark. It follows by induction on $\mathrm{o}(\mathbb{X})$ that $\mathrm{ran}^\alpha_\mathbb{X}(\mathbb{Z})$ and $\mathrm{ran}^\alpha_\mathbb{X}(\mathbb{E})$ are well-defined, since we have $(\Psi^\delta_\mathbb{Z}; \ldots) = (\Psi^\delta_\mathbb{Z}; \ldots; \mathbb{Z}; \delta+1)$ and $\mathbb{Z} \notin \mathrm{Prinst}(\mathbb{Z})$.

Notation. Let $M \subseteq \mathrm{ON}^n$ be a set of vectors of ordinals and let α be an ordinal. Then we write $\alpha \geq M \;:\Leftrightarrow \forall(\eta_1, \ldots, \eta_n) \in M\ \forall^n_1 i\ (\alpha \geq \eta_i)$.

Lemma 2.3.2 (Well-Definedness of Definition 2.2.4)**.** *Let $\mathbb{X} = (\Psi^\delta_\mathbb{Z}; \ldots; \vec{R}; \mathbb{Z}; \delta+1)$ be a reflection instance with reflection configuration $\mathbb{F}$. Then it holds:*

❶ (a) $\mathrm{o}(\mathbb{Z}) < \mathrm{o}(\mathbb{X}) = \delta + 1$,

(b) $\vec{R}_{\Psi^\delta_\mathbb{Z}} = \begin{cases} (\mathsf{M}^{<\omega}_\mathbb{A}\text{-}\mathsf{P}_m, \vec{R}) & \text{if } \mathbb{X} = (\pi; \mathsf{P}_m; \ldots),\\ (\mathsf{M}^{<\gamma}_\mathbb{M}\text{-}\mathsf{P}_m, \vec{R}) \text{ for some } \gamma > \xi \geq \mathrm{o}(\mathbb{M}), & \text{if } \mathbb{X} = (\pi; \mathsf{M}^\xi_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m; \ldots),\end{cases}$

(c) $\mathrm{rdh}(\mathbb{M}(\vec{\nu})) \geq m > \mathrm{rdh}(\mathbb{X})$ *if* $\mathbb{X} = (\pi; \mathsf{M}^\xi_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m; \ldots)$,

(d) $\mathbb{X} \in C(\Psi^\delta_\mathbb{Z})$,

(e) $\operatorname{par}\mathbb{X} < \mathrm{o}(\mathbb{X}) = \delta + 1$.

❷ $\forall\kappa\in\mathfrak{M}^{\alpha}_{\mathbb{X}}\big(\kappa \models \vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} \;\wedge\; \kappa \models (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_{\leq \mathrm{rdh}(\mathbb{X})}\big)$.

❸ *Let* $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = (\mathsf{M}^{<\xi_1}_{\mathbb{R}_1}\text{-}\mathsf{P}_{r_1}, \ldots, \mathsf{M}^{<\xi_k}_{\mathbb{R}_k}\text{-}\mathsf{P}_{r_k})$. *Then it holds for all* $1 \leq i \leq k$*:*

(a) $\mathbb{R}_i \in \overline{\mathrm{Prcnfg}}(\mathbb{X})$ *and* $\mathrm{dom}(\mathbb{R}_i) \leq \alpha$,

(b) $\xi_i \leq \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{R}_i)$ *and* $\big(\xi_i = \mathrm{o}(\mathbb{R}_i) \;\Rightarrow\; \mathbb{R}_i = \mathbb{A}$, *i.e.* $\xi_i = \omega\big)$,

(c) $\xi_i \in \operatorname{par}\mathbb{X} \cup \{\alpha\}$,

(d) $r_1 \geq \mathrm{rdh}(\mathbb{X})$ *and* $(r_1, \ldots, r_k) = (k-1, k-2, \ldots, 1, 0)$,

(e) $\exists\nu_i\in\mathrm{dom}(\mathbb{R}_i) \cap \operatorname{par}\mathbb{X}\,\big(\mathrm{rdh}(\mathbb{R}_i(\vec{\nu}_i) = r_i\big)$,

(f) $(\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq \mathrm{rdh}(\mathbb{X})} = (\widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathrm{rdh}(\mathbb{X})}, (\vec{R}_{\mathbb{F}})_{<\mathrm{rdh}(\mathbb{X})})$

(g) $\forall\vec{\nu}_i\in\mathrm{dom}(\mathbb{R}_i)\,\big(\mathrm{rdh}(\mathbb{R}_i(\vec{\nu}_i)) = r_i = \mathrm{rd}(\vec{R}_{\mathbb{R}_i})\big)$ *if* $\mathbb{R}_i \neq \mathbb{A}$.

Corollary 2.3.3. *The claims made in Definition 2.2.4, e.g. "* $\mathbb{M} \in \mathrm{Prcnfg}(\mathbb{X})$*" in clause 3., are true.*

Moreover for every reflection instance $\mathbb{Z} \neq (\mathrm{i}(\mathbb{Z}); \mathsf{P}_0; \ldots)$ *and every* δ*, such that* $\mathfrak{M}^{\delta}_{\mathbb{Z}} \neq \emptyset$ *there is a reflection instance* $\mathbb{X}$ *with* $\mathrm{i}(\mathbb{X}) = \Psi^{\delta}_{\mathbb{Z}}$.

Now we turn to the proof of Lemma 2.3.2.

Proof. At first we observe that for $\mathbb{X} = \mathbb{A}(m+1)$ with $m \in \omega$ the propositions of item ③, except for ③ (f), do hold. Done this we show the claim by induction on $\mathrm{o}(\mathbb{X})$:

Case 1, $\mathbb{X} = (\Psi^{\delta}_{\mathbb{Z}}; \mathsf{P}_m; \vec{R}; \mathbb{Z}; \delta + 1)$: ① Then the reflection instance $\mathbb{X}$ is defined by means of the subclause 1.1 or 2.1 or 3.1 of Definition 2.2.4. The propositions of ① hold in any of these cases owing to the following arguments:

(a) $\mathrm{o}(\mathbb{Z}) \leq \delta$, since $\mathfrak{M}^{\delta}_{\mathbb{Z}}$ is only defined for $\delta \geq \mathrm{o}(\mathbb{Z})$;

(b) we have $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\omega}_{\mathbb{A}}\text{-}\mathsf{P}_m, \vec{R})$ by means of the induction hypothesis ③ (b);

(c) here is nothing to show;

(d) the definition of $\mathfrak{M}^{\delta}_{\mathbb{Z}}$ implies that $\operatorname{par}\mathbb{Z} \cup \{\delta\} = \operatorname{par}\mathbb{X} \in C(\Psi^{\delta}_{\mathbb{Z}})$;

(e) by means of the induction hypothesis we have $\operatorname{par}\mathbb{Z} < \mathrm{o}(\mathbb{Z}) < \delta + 1$ and thus $\operatorname{par}\mathbb{X} = \operatorname{par}\mathbb{Z} \cup \{\delta\} < \delta + 1 = \mathrm{o}(\mathbb{X})$.

② Follows by definition of $\mathfrak{M}^{\alpha}_{\mathbb{X}}$.

③ If $\alpha = \delta + 1$ we have $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_{<m}$ and the claim follows by means of induction hypothesis ③ and the following arguments:

(a) $\overline{\mathrm{Prcnfg}}(\mathbb{Z}) \subseteq \mathrm{Prcnfg}(\mathbb{X})$, $\delta < \alpha$;

(b) $\mathrm{ran}^{\delta}_{\mathbb{Z}}(\mathbb{R}_i) = \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{R}_i)$, if $\mathbb{R}_i \in \overline{\mathrm{Prcnfg}}(\mathbb{Z})$;

(c) $\operatorname{par}\mathbb{Z} \cup \{\delta\} = \operatorname{par}\mathbb{X}$;

(d) $m - 1 \geq \mathrm{rdh}(\mathbb{X})$;

(e) $\operatorname{par}\mathbb{Z} \subseteq \operatorname{par}\mathbb{X}$;,

(f) follows directly by definition;

(g) here is no extra argument needed.

If $\alpha > \delta + 1$ we have $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = (\mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}, (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_{<m-1})$. For $i = 1$ all propositions follow directly by definition and for $i > 1$ by means of the induction hypothesis as before in the case $\alpha = \delta + 1$.

Case 2, $\mathbb{X} = (\Psi^{\delta}_{\mathbb{Z}}; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m; \ldots)$*:* ① Then the reflection instance $\mathbb{X}$ is defined by means of one of the subclauses 1.2, 2.2, 2.3, 2.4, 3.2 or 3.3 of Definition 2.2.4. The propositions of ① hold in any of these cases owing to the following arguments:

(a) cf. the first case;

(b) follows directly by means of the subclause which defines $\mathbb{X}$;

(c) in case of $\mathbb{M} = \mathbb{A}$ the claim follows by means of the subclause which defines $\mathbb{X}$. If $\mathbb{M} \neq \mathbb{A}$ it follows due to the induction hypothesis ③ (g).

(d) It follows by definition of $\mathfrak{M}^{\delta}_{\mathbb{Z}}$, that $\operatorname{par}\mathbb{Z} \cup \{\delta\} \in C(\Psi^{\delta}_{\mathbb{Z}})$. In addition the subclauses 1.2, 2.2, 2.3, 2.4, 3.2 and 3.3 secure that $\xi, \vec{\nu} \in C(\Psi^{\delta}_{\mathbb{Z}})$. Thereby we have $\operatorname{par}\mathbb{Z} \cup \{\delta, \xi, \vec{\nu}\} = \operatorname{par}\mathbb{X} \in C(\Psi^{\delta}_{\mathbb{Z}})$.

(e) By means of the induction hypothesis we have $\operatorname{par}\mathbb{Z} < \mathrm{o}(\mathbb{Z}) < \delta + 1$. To also prove $\xi, \vec{\nu} < \delta + 1$ we differentiate between the clauses, which define $\mathbb{X}$.

If $\mathbb{X}$ is defined by clause 1.2 it holds $(\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m, \vec{R}) = (\mathsf{M}^{<\delta}_{\mathbb{A}}\text{-}\mathsf{P}_m, \ldots, \mathsf{M}^{<\delta}_{\mathbb{A}}\text{-}\mathsf{P}_0)$ and thus $\xi < \delta$. Moreover it holds $\vec{\nu} \in \mathrm{dom}(\mathbb{M}) = \mathrm{dom}(\mathbb{A}) = \omega \leq \delta$ since $\delta \geq \mathrm{o}(\mathbb{A}) = \omega$. Thus we have $\operatorname{par}\mathbb{X} < \delta + 1$.

If $\mathbb{X}$ is defined by clause 2.2 or 2.3 we have $(\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m, \vec{R}) = (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_{\leq m}$ and therefore it follows due to the induction hypothesis ③ (b) that $\xi < \gamma \leq \mathrm{ran}^{\delta}_{\mathbb{Z}}(\mathbb{M}) = \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{M}) < \mathrm{o}(\mathbb{X}) = \delta + 1$ and owing to ③ (a) that $\vec{\nu} \in \mathrm{dom}(\mathbb{M}) \leq \delta < \delta + 1$.

If $\mathbb{X}$ is defined by clause 2.4 then $\vec{\nu}$ is a vector of zero length and $\xi < \delta < \delta + 1$.

If $\mathbb{X}$ is defined by clause 3.2 or 3.3 then $\mathbb{Z}$ has a predecessor reflection instance $\mathbb{Z}'$ and for $\mathrm{o}(\mathbb{Z}) = \delta' + 1$ we have $\vec{R}_{\Psi^{\delta'}_{\mathbb{Z}'}} = (\mathsf{M}^{<\gamma'}_{\mathbb{M}'}\text{-}\mathsf{P}_{m'}, \ldots)$ for some $\mathbb{M}' \in \mathrm{Prcnfg}(\mathbb{Z}')$. Moreover there are a $\xi' < \gamma'$ and a $\vec{\nu}' \in \mathrm{dom}(\mathbb{M}')$ such that $\vec{R}_{\Psi^{\xi'}_{\mathbb{M}'(\vec{\nu}')}} = (\mathsf{M}^{<\sigma_1}_{\mathbb{S}_1}\text{-}\mathsf{P}_{s_1}, \ldots)$ and $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m, \vec{R}) = (\mathsf{M}^{<\sigma_1}_{\mathbb{S}_1}\text{-}\mathsf{P}_{s_1}, \ldots)$. Thereby we obtain by means of the induction hypothesis ③ (b) that

$$\begin{aligned}\xi < \sigma_1 &\leq \mathrm{ran}^{\xi'}_{\mathbb{M}'(\vec{\nu}')}(\mathbb{S}_1) \leq \max\{\xi', \mathrm{o}\,\mathbb{M}'\} \leq \max\{\gamma', \delta\}\\ &\leq \max\{\mathrm{ran}^{\delta'}_{\mathbb{Z}'}(\mathbb{M}'), \delta\} \leq \max\{\delta', \mathrm{o}(\mathbb{Z}'), \delta\} = \delta < \delta + 1\end{aligned}$$

and due to ③ (a) $\vec{\nu} \in \mathrm{dom}(\mathbb{S}_1) \leq \delta' < \delta + 1$.

Thus we have shown $\{\delta, \xi, \vec{\nu}\} \cup \operatorname{par}\mathbb{Z} = \operatorname{par}\mathbb{X} < \delta + 1$.

② We have just shown that $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m, \ldots)$. Thereby it follows by means of induction hypothesis ③ that $\mathbb{M} \in \overline{\mathrm{Prcnfg}}(\mathbb{Z}) \subseteq \mathrm{Prcnfg}(\mathbb{X})$. Thus we obtain $\forall \kappa \in \mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\, (\kappa \models \vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})$ by employing the induction hypothesis ②. Taking also into account the definition of $\mathfrak{M}^{\alpha}_{\mathbb{X}}$ claim ② follows.

③ For $r_i \geq m$ the claims follow by an application of the induction hypothesis ③ to $\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}}$, analogously to the first case. For $r_i < m$ the claims follow by an application of the induction hypothesis ③ to $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}}$ or since $\mathbb{R}_i = \mathbb{F}$. We also have $r_1 \geq \mathrm{rdh}(\mathbb{M}(\vec{\nu}) \geq m > m - 1 = \mathrm{rdh}(\mathbb{X})$. □

Lemma 2.3.4. *It holds*

- $\alpha \le \beta \wedge \kappa \le \pi \quad \Rightarrow \quad C(\alpha,\kappa) \subseteq C(\beta,\pi)$,
- $\mathrm{card}\big(C(\alpha,\kappa)\big) = \max\{\aleph_0, \mathrm{card}(\kappa)\}$,
- $\lambda \in \mathrm{Lim} \quad \Rightarrow \quad C(\lambda,\kappa) = \bigcup_{\xi<\lambda} C(\xi,\kappa) \;\&\; C(\alpha,\lambda) = \bigcup_{\xi<\lambda} C(\alpha,\xi)$.

Proof. Folklore. □

Lemma 2.3.5. *Let $\omega < \pi$ be a regular cardinal and $\mathbb{X}$ be a reflection instance. Then it holds*

❶ $C^\alpha_\pi := \{\kappa < \pi \,|\, C(\alpha,\kappa) \cap \pi = \kappa\}$ *is club in* π.

❷ *If* $\mathrm{par}(\mathbb{X}) \le \alpha$ *and* $\mathbb{X}, \alpha \in C(\pi)$ *then*

$$C^\alpha_{\mathbb{X},\pi} := \{\kappa < \pi \,|\, C(\alpha,\kappa) \cap \pi = \kappa \wedge \mathbb{X}, \alpha \in C(\kappa)\} \text{ is club in } \pi.$$

Proof. For the former claim see Lemma 4.3 in [Rat05b].
Lemma 4.3 also implies that

$$\tilde{C}^\rho_\pi := \{\kappa < \pi \,|\, C(\rho,\kappa) \cap \pi = \kappa \wedge \rho \in C(\rho,\kappa)\}$$

is club in π for all $\rho \in \mathrm{par}(\mathbb{X}) \cup \{\alpha\}$. Since $\mathrm{card}(\mathrm{par}(\mathbb{X})) < \aleph_0$ and $\omega < \pi$ is a regular cardinal we have $\bigcap_{\rho\in\mathrm{par}(\mathbb{X})\cup\{\alpha\}} \tilde{C}^\rho_\pi = C^\alpha_{\mathbb{X},\pi}$ is club in π. □

Definition 2.3.6. Let $\mathbb{X}$ be a reflection instance. We define

$$\vec{\mathbb{X}} := \begin{cases} (m) & \text{if } \mathbb{X} = \mathbb{A}(m), \\ (\kappa) & \text{if } \mathbb{X} = (\kappa^+; \mathsf{P}_0; \ldots), \\ (\delta, \vec{\mathbb{Z}}) & \text{if } \mathbb{X} = (\Psi^\delta_{\mathbb{Z}}; \mathsf{P}_m; \ldots), \\ (\delta, \xi, \vec{\nu}, \vec{\mathbb{Z}}) & \text{if } \mathbb{X} = (\Psi^\delta_{\mathbb{Z}}; \mathsf{M}^\xi_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m; \ldots). \end{cases}$$

Remark. It follows by induction on $\mathrm{o}(\mathbb{X})$ that $\vec{\mathbb{X}} = \vec{\mathbb{Y}} \;\Leftrightarrow\; \mathbb{X} = \mathbb{Y}$.

Definition 2.3.7 (Coding of $C(\alpha,\pi)$)**.** Let $\pi \in \mathrm{Reg}$. For $\beta \in C(\alpha,\pi)$ we define the set of codes $Cd_{\alpha,\pi}(\beta)$ for β as follows (we write $\ulcorner\gamma\urcorner \in Cd_{\alpha,\pi}(\cdot)$ for $\ulcorner\gamma\urcorner \in Cd_{\alpha,\pi}(\gamma)$):

$$\begin{aligned}
\beta \in{}& Cd_{\alpha,\pi}(\beta) && \text{if } \beta < \pi, \\
\{1\} \in{}& Cd_{\alpha,\pi}(\beta) && \text{if } \beta = \Xi, \\
\langle 1, \ulcorner\gamma\urcorner, \ulcorner\delta\urcorner\rangle \in{}& Cd_{\alpha,\pi}(\beta) && \text{if } \beta \underset{\mathrm{NF}}{=} \gamma + \omega^\delta \text{ and } \ulcorner\gamma\urcorner, \ulcorner\delta\urcorner \in Cd_{\alpha,\pi}(\cdot), \\
\langle 2, \ulcorner\xi\urcorner, \ulcorner\eta\urcorner\rangle \in{}& Cd_{\alpha,\pi}(\beta) && \text{if } \beta \underset{\mathrm{NF}}{=} \varphi(\xi,\eta) \text{ and } \ulcorner\xi\urcorner, \ulcorner\eta\urcorner \in Cd_{\alpha,\pi}(\cdot), \\
\langle 3, \ulcorner\kappa\urcorner\rangle \in{}& Cd_{\alpha,\pi}(\beta) && \text{if } \beta = \kappa^+ \text{ and } \ulcorner\kappa\urcorner \in Cd_{\alpha,\pi}(\kappa), \\
\langle 4, \ulcorner\gamma\urcorner, \ulcorner\xi_1\urcorner, \ldots, \ulcorner\xi_p\urcorner\rangle \in{}& Cd_{\alpha,\pi}(\beta) && \text{if } \beta = \Psi^\gamma_{\mathbb{X}} \text{ and } \vec{\mathbb{X}} = (\xi_1, \ldots, \xi_p) \text{ plus} \\
& && \ulcorner\gamma\urcorner, \ulcorner\xi_1\urcorner, \ldots, , \ulcorner\xi_p\urcorner \in Cd_{\alpha,\pi}(\cdot).
\end{aligned}$$

Furthermore we define $U_{\alpha,\pi} := \{Cd_{\alpha,\pi}(\beta) \,|\, \beta \in C(\alpha,\pi)\}$.

Definition 2.3.8 (Coding of $\mathfrak{M}_{\mathbb{F}}^{<\xi}$)**.** Let $\mathbb{F}$ be a reflection configuration with $\mathrm{i}(\mathbb{F}) = \pi$ and $\mathrm{o}(\mathbb{F}) \leq \xi \leq \alpha$. Suppose further that $\forall(\zeta,\vec{\eta})\in[\mathrm{o}(\mathbb{F}),\xi)_{C(\pi)} \times \mathrm{dom}(\mathbb{F})_{C(\pi)}\,(\mathfrak{M}^{\zeta}_{\mathbb{F}(\vec{\eta})} \neq \emptyset)$. Then we define

$$\ulcorner\mathfrak{M}_{\mathbb{F}}^{<\xi}\urcorner_{\alpha,\pi} := \big\{Cd_{\alpha,\pi}(\zeta) \times Cd_{\alpha,\pi}(\eta_1) \times \ldots \times Cd_{\alpha,\pi}(\eta_k) \times \mathfrak{M}^{\zeta}_{\mathbb{F}(\vec{\eta})} \;\big|\; (\zeta,\vec{\eta}) \in [\mathrm{o}(\mathbb{F}),\xi)_{C(\pi)} \times \mathrm{dom}(\mathbb{F})_{C(\pi)},\ \text{where } \vec{\eta} = (\eta_1,\ldots,\eta_k)\big\}$$

Corollary 2.3.9. *Let $\kappa,\pi \in \mathrm{Reg}$ with $\kappa < \pi$. Then it holds*

- $U_{\alpha,\pi} \cap V_\kappa = U_{\alpha,\kappa}$*,*
- $U_{\alpha,\pi} \subseteq \mathbb{L}_\pi$*.*
- *If $\ulcorner\mathfrak{M}_{\mathbb{F}}^{<\xi}\urcorner_{\alpha,\pi}$ is defined, it holds $\ulcorner\mathfrak{M}_{\mathbb{F}}^{<\xi}\urcorner_{\alpha,\pi} \cap V_\kappa = \ulcorner\mathfrak{M}_{\mathbb{F}}^{<\xi}\urcorner_{\alpha,\kappa}$.*

Lemma 2.3.10. *Let $\mathbb{F}$ be a reflection configuration with $\mathrm{i}(\mathbb{F}) = \pi$ and let $\pi \geq \kappa \in \mathrm{Reg}$. If $\ulcorner\mathfrak{M}_{\mathbb{F}}^{<\xi}\urcorner_{\alpha,\pi}$ is defined, then there is a Π^1_{m+1}-sentence $\psi^m_{\mathbb{F}}(\ulcorner\mathfrak{M}_{\mathbb{F}}^{<\xi}\urcorner_{\alpha,\pi})$, such that*

$$\kappa \models \mathsf{M}_{\mathbb{F}}^{<\xi}\text{-}\mathsf{P}_m \quad\Leftrightarrow\quad \langle V_\kappa, \in, \ulcorner\mathfrak{M}_{\mathbb{F}}^{<\xi}\urcorner_{\alpha,\pi}\rangle \models \psi^m_{\mathbb{F}}(\ulcorner\mathfrak{M}_{\mathbb{F}}^{<\xi}\urcorner_{\alpha,\pi}).$$

Proof. We define

$$\psi^m_{\mathbb{F}}(Y) \;:\equiv \forall x_0 \ldots \forall x_k \Big(\exists y\,\big((x_0,\ldots,x_k,y) \in Y\big) \rightarrow \forall X\,\forall z\,\Big(\psi_m(X,z) \rightarrow \exists\kappa_0\,\big((x_0,\ldots,x_k,\kappa_0) \in Y \wedge \psi_m(X\cap V_{\kappa_0},z)^{\kappa_0}\big)\Big)\Big),$$

where ψ_m denotes the formula of Corollary 2.1.4. □

Theorem 2.3.11 (Existence)**.** *Let $\mathbb{X} = (\pi;\ldots;\delta)$ be a reflection instance and $\delta \leq \alpha \in C(\pi)$. Then $\mathfrak{M}^{\alpha}_{\mathbb{X}} \neq \emptyset$.*

Proof. By (main)induction on α. Let $\mathbb{X} = \mathbb{F} = (\pi;\mathsf{P}_m;\vec{R};\ldots;\delta)$ ($\mathbb{X} = \mathbb{F}(\xi,\vec{\nu}) = (\pi;\mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m;\vec{R};\ldots;\delta)$, resp.) plus $\vec{R}_\pi = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m,\vec{R})$ if $\delta > 0$, and suppose the claim holds for all $\alpha' < \alpha$.

Case 1, $m = 0$: If $\mathbb{X} = (\pi;\mathsf{P}_0;\ldots)$ then we either have $\pi = \kappa^+$ for some κ (and $\delta = 0$) or π is Π^1_0-indescribable by 2.3.2 ②. Thus $\omega < \pi$ is a regular cardinal. Therefore $C^{\alpha}_{\mathbb{X},\pi}$ is club in π by means of Lemma 2.3.5 and thereby $\mathfrak{M}^{\alpha}_{\mathbb{X}} \neq \emptyset$.

If $\mathbb{X} = (\pi;\mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_0;\ldots)$ then again $C^{\alpha}_{\mathbb{X},\pi}$ is club in π and by Lemma 2.3.2 it follows that π is $\mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})}$-$\Pi^1_0$-indescribable. Since "$C^{\alpha}_{\mathbb{X},\pi}$ is unbounded" is expressible by a Π^1_0-sentence in the parameter $C^{\alpha}_{\mathbb{X},\pi}$, there is a $\kappa \in \mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})}$, such that $C^{\alpha}_{\mathbb{X},\pi}$ is unbounded in V_κ. Therefore $\mathbb{X},\alpha \in C(\kappa)$ and $C(\alpha,\kappa) \cap \pi = \kappa$. Hence $\kappa \in \mathfrak{M}^{\alpha}_{\mathbb{X}}$.

Case 2, $m > 0$: We show by subsidiary induction on β:

$$\forall\zeta \in [\mathrm{o}(\mathbb{X}),\beta]_{C(\pi)}\,(\pi \text{ is } \mathfrak{M}^{\zeta}_{\mathbb{F}}\text{-}\Pi^1_m\text{-indescribable}) \tag{2.1}$$

$(\forall(\zeta,\xi',\vec{\nu}') \in [\mathrm{o}(\mathbb{X}),\beta]_{C(\pi)} \times \mathrm{dom}(\mathbb{F})_{C(\pi)}\,(\pi \text{ is } \mathfrak{M}^{\zeta}_{\mathbb{F}(\xi',\vec{\nu}')}\text{-}\Pi^1_m\text{-indescribable})$, resp.).

Let $\beta \in C(\pi)$ and suppose (2.1) holds for all $(\zeta, \vec{\eta}) \in [\mathrm{o}(\mathbb{X}), \beta)_{C(\pi)} \times \mathrm{dom}(\mathbb{F})_{C(\pi)}$. By means of Lemma 2.3.2 we have $\omega < \pi$ is a regular cardinal and $\mathbb{X} \in C(\pi)$. Therefore

1. & 2. $C^{\alpha}_{\mathbb{X},\pi}$ is club in π.

due to Lemma 2.3.5. By means of Lemma 2.3.2 ② we have $\pi \models \vec{R}_\pi$. Thereby π is Π^1_m-indescribable and $\pi \models \vec{R}$. Thus we have

3. $\pi \models \vec{Q}$,

where $\vec{Q} := (\mathsf{M}^{<\xi'_1}_{\mathbb{R}'_1}\text{-}\mathsf{P}_{r'_1}, \ldots, \mathsf{M}^{<\xi'_n}_{\mathbb{R}'_n}\text{-}\mathsf{P}_{r'_n})$ denotes the sequence of M-P-expressions of the third proviso in the definition of $\mathfrak{M}^{\alpha}_{\mathbb{X}}$. Furthermore the subsidiary induction hypothesis provides

4. $\pi \models \mathsf{M}^{<\beta}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}$.

By means of Lemma 2.3.2 ③ and the main induction hypothesis it follows the well-definedness of $\ulcorner\mathfrak{M}^{<\beta}_{\mathbb{F}}\urcorner_{\beta,\pi}$ and $\ulcorner\mathfrak{M}^{<\xi'_k}_{\mathbb{R}'_k}\urcorner_{\beta,\pi}$ for all $1 \le k \le n$. Thus we have

$$V_\pi \models \text{“}C^{\beta}_{\mathbb{X},\pi} \text{ is unbounded ”} \wedge \bigwedge_{k=1}^{n} \psi^{r_k}_{\mathbb{R}'_k}(\ulcorner\mathfrak{M}^{<\xi'_k}_{\mathbb{R}'_k}\urcorner_{\beta,\pi}) \wedge \psi^{m-1}_{\mathbb{F}}(\ulcorner\mathfrak{M}^{<\beta}_{\mathbb{F}}\urcorner_{\beta,\pi}).$$

Due to Lemma 2.3.10 this is a Π^1_m-sentence in the parameters $C^{\beta}_{\mathbb{X},\pi}$, $\ulcorner\mathfrak{M}^{<\xi'_1}_{\mathbb{R}'_1}\urcorner_{\beta,\pi}$, $\ldots$, $\ulcorner\mathfrak{M}^{<\xi'_n}_{\mathbb{R}'_n}\urcorner_{\beta,\pi}$ and $\ulcorner\mathfrak{M}^{<\beta}_{\mathbb{F}}\urcorner_{\beta,\pi}$. Suppose now $\langle V_\pi, \in, P\rangle \models F(P)$ for an arbitrary Π^1_m-sentence F and an arbitrary parameter P. Since π is Π^1_m-indescribable (π is $\mathfrak{M}^{\xi'}_{\mathbb{M}(\vec{\nu}')}$-$\Pi^1_m$-indescribable, for $(\xi', \vec{\nu}') \in \mathrm{dom}(\mathbb{F})$, resp.) there exists a $0 < \kappa < \pi$ ($0 < \kappa \in \mathfrak{M}^{\xi'}_{\mathbb{M}(\vec{\nu}')} \cap \pi$, resp.) such that

$$V_\kappa \models F(P \cap V_\kappa) \wedge \text{“}C^{\beta}_{\mathbb{X},\pi} \cap V_\kappa \text{ is unbounded”} \wedge \\ \bigwedge_{k=1}^{n} \psi^{r_k}_{\mathbb{R}'_k}(\ulcorner\mathfrak{M}^{<\xi'_k}_{\mathbb{R}'_k}\urcorner_{\beta,\pi} \cap V_\kappa) \wedge \psi^{m-1}_{\mathbb{F}}(\ulcorner\mathfrak{M}^{<\beta}_{\mathbb{F}}\urcorner_{\beta,\pi} \cap V_\kappa).$$

Thus we have

1. $C(\beta, \kappa) \cap \pi = \kappa$,
2. $\beta, \mathbb{X} \in C(\kappa)$,
3. $\forall^n_1 k\, \big(V_\kappa \models \psi^{r_k}_{\mathbb{R}'_k}(\ulcorner\mathfrak{M}^{<\xi'_k}_{\mathbb{R}'_k}\urcorner_{\beta,\pi} \cap V_\kappa)\big)$,
4. $V_\kappa \models \psi^{m-1}_{\mathbb{F}}(\ulcorner\mathfrak{M}^{<\beta}_{\mathbb{F}}\urcorner_{\beta,\pi} \cap V_\kappa)$.

In addition it also holds

$$\forall^n_1 k\, \big(V_\kappa \models \psi^{r_k}_{\mathbb{R}'_k}(\ulcorner\mathfrak{M}^{<\xi'_k}_{\mathbb{R}'_k}\urcorner_{\beta,\pi} \cap V_\kappa) \;\Leftrightarrow\; V_\kappa \models \psi^{r_k}_{\mathbb{R}'_k}(\ulcorner\mathfrak{M}^{<\xi'_k}_{\mathbb{R}'_k}\urcorner_{\beta,\kappa}) \;\Leftrightarrow\; \kappa \models \mathsf{M}^{<\xi'_k}_{\mathbb{R}'_k}\text{-}\mathsf{P}_{r'_k}\big),$$

and

$$V_\kappa \models \psi_{\mathbb{F}}^{m-1}(\ulcorner \mathfrak{M}_{\mathbb{F}}^{<\beta} \urcorner_{\beta,\pi} \cap V_\kappa) \;\Leftrightarrow\; V_\kappa \models \psi_{\mathbb{F}}^{m-1}(\ulcorner \mathfrak{M}_{\mathbb{F}}^{<\beta} \urcorner_{\beta,\kappa}) \;\Leftrightarrow\; \kappa \models \mathsf{M}_{\mathbb{F}}^{<\beta}\text{-}\mathsf{P}_{m-1}.$$

Thus it follows $\kappa \in \mathfrak{M}_{\mathbb{F}}^{\beta}$ ($\kappa \in \mathfrak{M}_{\mathbb{F}(\xi',\vec{\nu}')}^{\beta}$, resp.) and $\langle V_\kappa, \in P \cap V_\kappa \rangle \models F(P \cap V_\kappa)$. Thereby π is $\mathfrak{M}_{\mathbb{X}}^{\beta}$-$\Pi_m^1$-indescribable ($\mathfrak{M}_{\mathbb{F}(\xi',\vec{\nu}')}^{\beta}$-$\Pi_m^1$-indescribable, resp.) and hence (2.1) follows which implies $\mathfrak{M}_{\mathbb{X}}^{\alpha} \neq \emptyset$ if $\alpha \in C(\pi)$. □

Corollary 2.3.12. *The ordinal* $\Psi_{\mathbb{X}}^{\alpha}$ *is well-defined iff* $\mathrm{o}(\mathbb{X}) \leq \alpha \in C(\mathrm{i}(\mathbb{X}))$.

Notation. Let $\vec{\alpha}, \vec{\beta} \in \mathrm{ON}^n$. Then we denote by $\vec{\alpha} <_{\mathrm{lex}} \vec{\beta}$ that $\vec{\alpha}$ is less than $\vec{\beta}$ with respect to the lexicographic ordering on ON^n.

Lemma 2.3.13. *Let* $\mathbb{F}$ *be a reflection configuration with* $\mathrm{rdh}(\mathbb{F}(\vec{\eta})) = m \geq 0$ *and let* κ *be an ordinal. Suppose* $\mathrm{o}(\mathbb{F}) \leq \beta \in (\alpha+1)_{C(\kappa)}$ *and* $\vec{\eta}, \vec{\nu} \in \mathrm{dom}(\mathbb{F})_{C(\kappa)}$. *Then it holds*

$$\kappa \in \mathfrak{M}_{\mathbb{F}(\vec{\eta})}^{\alpha} \;\&\; (\alpha, \vec{\eta}) >_{\mathrm{lex}} (\beta, \vec{\nu}) \quad \Rightarrow \quad \kappa \textit{ is } \mathfrak{M}_{\mathbb{F}(\vec{\nu})}^{\beta}\text{-}\Pi_m^1\textit{-indescribable.}$$

Proof. We proceed by induction on $\mathrm{o}(\mathbb{F})$. Let $\vec{\eta} = (\eta_1, \ldots, \eta_n)$ and $\vec{\nu} = (\nu_1, \ldots, \nu_n)$.

Case 1, $\alpha > \beta$: Since $\kappa \in \mathfrak{M}_{\mathbb{F}(\vec{\eta})}^{\alpha}$ we have $\kappa \models \mathsf{M}_{\mathbb{F}}^{<\alpha}\text{-}\mathsf{P}_m$ by the fourth proviso of the definition of the collapsing hierarchy $\mathfrak{M}_{\mathbb{F}(\vec{\eta})}^{\alpha}$. Thus κ is $\mathfrak{M}_{\mathbb{F}(\vec{\nu})}^{\beta}$-$\Pi_m^1$-indescribable.

Case 2, $(\alpha, \eta_1, \ldots, \eta_{i-1}) = (\beta, \nu_1, \ldots, \nu_{i-1})$ *and* $\eta_i > \nu_i$ *for some* $1 \leq i \leq n$*:* Then $\mathbb{F}(\vec{\eta}) = (\mathrm{i}(\mathbb{F}); \mathsf{M}_{\mathbb{M}(\eta_2,\ldots,\eta_n)}^{\eta_1}\text{-}\mathsf{P}_{m+1}; \vec{R}; \ldots)$ and $\mathfrak{M}_{\mathbb{F}(\vec{\eta})}^{\alpha} \subseteq \mathfrak{M}_{\mathbb{M}(\eta_2,\ldots,\eta_n)}^{\eta_1}$ for some reflection configuration $\mathbb{M} \in \mathrm{Prcnfg}(\mathbb{F})$. Since we have $\mathrm{o}(\mathbb{M}) < \mathrm{o}(\mathbb{F})$, $\kappa \in \mathfrak{M}_{\mathbb{M}(\eta_2,\ldots,\eta_n)}^{\eta_1}$ and $\mathrm{rdh}(\mathbb{M}_{\eta_2,\ldots,\eta_n}) \geq m+1$ by Lemma 2.3.2 ① (c) the induction hypothesis provides that κ is $\mathfrak{M}_{\mathbb{M}(\nu_2,\ldots,\nu_n)}^{\nu_1}$-$\Pi_{m+1}^1$-indescribable. Furthermore we have

1. & 2. "$C_{\mathbb{F}(\vec{\nu}),\mathrm{i}(\mathbb{F})}^{\alpha}$ is unbounded in κ",
3. $\kappa \models \vec{R}$,
4. $\kappa \models \mathsf{M}_{\mathbb{F}}^{<\alpha}\text{-}\mathsf{P}_m$.

By coding $C(\alpha, \kappa)$ in $\mathbb{L}_\kappa$ (cf. Definition 2.3.7) these statements are expressible by a Π_{m+1}^1-sentence Φ in some parameters $\vec{Q}$ of $\mathbb{L}_\kappa$.

Suppose now $\langle V_\kappa, \in, P \rangle \models F$ for an arbitrary Π_m^1-sentence F. By employing the $\mathfrak{M}_{\mathbb{M}(\nu_2,\ldots,\nu_n)}^{\nu_1}$-$\Pi_{m+1}^1$-indescribability of κ we obtain a $0 < \kappa_0 \in \mathfrak{M}_{\mathbb{M}(\nu_2,\ldots,\nu_n)}^{\nu_1} \cap \kappa$ such that $\langle V_{\kappa_0}, \in, \vec{Q} \cap V_{\kappa_0}, P \cap V_{\kappa_0} \rangle \models (\Phi \wedge F)$. Therefore it holds

1. $C(\alpha, \kappa_0) \cap \mathrm{i}(\mathbb{F}) = \kappa_0$,
2. $\mathbb{F}(\vec{\nu}), \alpha \in C(\kappa_0)$,
3. $\kappa_0 \models \vec{R}$,
4. $\kappa_0 \models \mathsf{M}_{\mathbb{F}}^{<\alpha}\text{-}\mathsf{P}_m$.

Hence $\kappa_0 \in \mathfrak{M}_{\mathbb{F}(\vec{\nu})}^{\alpha} = \mathfrak{M}_{\mathbb{F}(\vec{\nu})}^{\beta}$ and thereby κ is $\mathfrak{M}_{\mathbb{F}(\vec{\nu})}^{\beta}$-$\Pi_m^1$-indescribable. □

Theorem 2.3.14 (<-Comparison)**.** *Let* $\mathbb{X} = \mathbb{F}(\vec{\eta})$ *plus* $\mathbb{Y} = \mathbb{G}(\vec{\nu})$ *be reflection instances and suppose* $\kappa := \Psi^{\alpha}_{\mathbb{X}}$ *plus* $\pi := \Psi^{\beta}_{\mathbb{Y}}$ *are well-defined. Then it holds* $\kappa < \pi$ *iff*

$$\pi \geq \mathrm{i}(\mathbb{X}) \tag{a}$$

$$\textit{or} \quad \kappa < \mathrm{i}(\mathbb{Y}) \;\wedge\; \Big(\alpha < \beta \vee \big(\mathbb{F} = \mathbb{G} \wedge (\alpha, \vec{\eta}) <_{\mathrm{lex}} (\beta, \vec{\nu})\big)\Big) \;\wedge\; \mathbb{X}, \alpha \in C(\pi) \tag{b}$$

$$\textit{or} \quad \alpha \geq \beta \;\wedge\; \neg\big(\mathbb{Y}, \beta \in C(\kappa)\big) \tag{c}$$

Proof. Obviously the claim follows if (a) is true. So let us assume ¬(a), i.e. $\pi < \mathrm{i}(\mathbb{X})$, in the following cases.

Case 1, $\alpha < \beta$: *Suppose* $\kappa < \pi$*:* We have to show (b) or (c). It holds $\mathbb{X}, \alpha \in C(\kappa)$ and $\kappa < \pi < \mathrm{i}(\mathbb{Y})$. Thus $\mathbb{X}, \alpha \in C(\pi)$ and $\kappa < \mathrm{i}(\mathbb{Y})$. Therefore we have (b).

Suppose $\kappa \geq \pi$. We have to show ¬(b) and ¬(c). Since $\alpha < \beta$ we have ¬(c). Now assume (b). Then $\mathbb{X}, \alpha \in C(\pi)$ and $\kappa < \mathrm{i}(\mathbb{Y})$. Therefore it holds $\kappa \in C(\beta, \pi) \cap \mathrm{i}(\mathbb{Y}) = \pi$, i.e. $\kappa < \pi$. Contradiction! So we have ¬(b), too.

Case 2, $\beta < \alpha$*:* Suppose $\kappa < \pi$. We have to show (b) or (c). Since $\mathbb{Y}, \beta \in C(\kappa)$ would imply $\pi \in C(\alpha, \kappa) \cap \mathrm{i}(\mathbb{X}) = \kappa$, i.e. $\pi < \kappa$ we have (c).

Suppose $\kappa \geq \pi$. We have to show ¬(b) and ¬(c). We have ¬(b) since $\beta < \alpha$. Moreover we have ¬(c) since $\mathbb{Y}, \beta \in C(\pi) \subseteq C(\kappa)$.

Case 3, $\alpha = \beta$*:*

Subcase 3.1, $\mathrm{i}(\mathbb{X}) < \mathrm{i}(\mathbb{Y})$*:* Since we assume ¬(a) we cannot have $\kappa < \pi$, since otherwise we would have $\mathrm{i}(\mathbb{X}) \in C(\alpha, \kappa) \cap \mathrm{i}(\mathbb{Y}) \subseteq C(\beta, \pi) \cap \mathrm{i}(\mathbb{Y}) = \pi$ contradicting $\pi < \mathrm{i}(\mathbb{X})$. So we have $\kappa \geq \pi$ and we have to show ¬(b) and ¬(c). We have $\neg(\alpha < \beta)$ and $\mathbb{F} \neq \mathbb{G}$ since $\mathrm{i}(\mathbb{X}) \neq \mathrm{i}(\mathbb{Y})$, hence ¬(b). Moreover we have ¬(c) since $\mathbb{Y}, \beta \in C(\pi) \subseteq C(\kappa)$.

Subcase 3.2, $\mathrm{i}(\mathbb{Y}) < \mathrm{i}(\mathbb{X})$*:* Suppose $\kappa < \pi$. Then we have to show (b) or (c). We have (c) since $\mathbb{Y} \notin C(\kappa)$, because otherwise we would have $\pi < \mathrm{i}(\mathbb{Y}) \in C(\alpha, \kappa) \cap \mathrm{i}(\mathbb{X}) = \kappa$.

Suppose $\kappa \geq \pi$. Then we have to show ¬(b) and ¬(c). Since $\neg(\alpha < \beta)$ and $\mathbb{F} \neq \mathbb{G}$ we have ¬(b). Moreover we have ¬(c) since $\mathbb{Y}, \beta \in C(\pi) \subseteq C(\kappa)$.

Subcase 3.3, $\mathrm{i}(\mathbb{X}) = \mathrm{i}(\mathbb{Y})$*, i.e.* $\mathbb{F} = \mathbb{G}$*:* Suppose $\kappa < \pi$. We have to show (b) or (c).

Assume ¬(b), i.e. $(\beta, \vec{\nu}) <_{\mathrm{lex}} (\alpha, \vec{\eta})$ since $\mathbb{X}, \alpha \in C(\kappa) \subseteq C(\pi)$, $\kappa < \pi < \mathrm{i}(\mathbb{Y})$ and $(\beta, \vec{\nu}) = (\alpha, \vec{\eta})$ would contradict $\kappa < \pi$. We have to show (c), i.e. $\mathbb{Y} \notin C(\kappa) \vee \beta \notin C(\kappa)$.

If $\mathbb{Y}, \beta \in C(\kappa)$ and $\mathrm{rdh}(\mathbb{G}(\vec{\eta})) \geq 0$ then κ would be $\mathfrak{M}^{\beta}_{\mathbb{G}(\vec{\nu})}$-$\Pi^1_{\mathrm{rdh}(\mathbb{G}(\vec{\eta}))}$-indescribable by Lemma 2.3.13 since $\kappa \in \mathfrak{M}^{\alpha}_{\mathbb{G}(\vec{\eta})}$. Thus $\Psi^{\beta}_{\mathbb{G}(\vec{\nu})} = \pi < \kappa$. Contradiction!

If $\mathbb{Y}, \beta \in C(\kappa)$ and $\mathbb{Y} = (\mathrm{i}(\mathbb{Y}); \mathsf{P}_0; \ldots) = \mathbb{X}$ then it follows by means of Lemma 2.3.5 ② that $\Psi^{\beta}_{\mathbb{G}(\vec{\nu})} = \pi < \kappa$. Contradiction!

If $\mathbb{Y}, \beta \in C(\kappa)$ and $\mathbb{Y} = \mathbb{F}(\zeta, \vec{\nu}') = (\mathrm{i}(\mathbb{X}); \mathsf{M}^{\xi'}_{\mathbb{M}(\vec{\nu}')}\text{-}\mathsf{P}_0; \ldots)$ and $\beta < \alpha$ we obtain the contradiction $\pi \in C(\alpha, \kappa) \cap \mathrm{i}(\mathbb{Y}) = \kappa$. So we must have $\alpha = \beta$ and $(\zeta, \vec{\nu}') <_{\mathrm{lex}} (\xi, \vec{\eta}')$, where $\mathbb{X} = \mathbb{F}(\xi, \vec{\eta}')$. However, then it follows by means of Lemma 2.3.13 that κ is $\mathfrak{M}^{\zeta}_{\mathbb{M}(\vec{\nu}')}$-$\Pi^1_0$-indescribable. Taking also into account Lemma 2.3.5 ② we obtain again the contradiction $\pi < \kappa$. So in any case it holds (c).

Now let us assume ¬(c). We have to show (b). It holds $\mathbb{F} = \mathbb{G}$, $\mathbb{X}, \alpha \in C(\kappa) \subseteq C(\pi)$ and $\kappa < \mathrm{i}(\mathbb{X}) = \mathrm{i}(\mathbb{Y})$. Thus it remains to show $(\alpha, \vec{\eta}) <_{\mathrm{lex}} (\beta, \vec{\nu})$. Suppose $(\beta, \vec{\nu}) \leq_{\mathrm{lex}} (\alpha, \vec{\eta})$. Since $\kappa \neq \pi$ this assumption implies $(\beta, \vec{\nu}) <_{\mathrm{lex}} (\alpha, \vec{\eta})$. Since we assume ¬(c) we have $\mathbb{Y}, \beta \in C(\kappa)$. Thereby it follows as above in the case of the assumption $\neg$(b) that $\pi < \kappa$. Contradiction! So we must have $(\alpha, \vec{\eta}) <_{\text{lex}} (\beta, \vec{\nu})$ and thereby (b).

Suppose $\kappa \geq \pi$. Then we have to show $\neg$(b) and $\neg$(c). The validity of (b) would lead to the contradiction $\kappa < \pi$ in the same way as we obtained the contradiction $\pi < \kappa$ under the assumption $\kappa < \pi$ above. Moreover we have (c) since $\mathbb{Y}, \beta \in C(\pi) \subseteq C(\kappa)$. ☐

Corollary 2.3.15. *Suppose $\Psi^{\alpha}_{\mathbb{X}}$ and $\Psi^{\beta}_{\mathbb{Y}}$ are well-defined. Then*

$$\Psi^{\alpha}_{\mathbb{X}} = \Psi^{\beta}_{\mathbb{Y}} \quad \Leftrightarrow \quad \alpha = \beta \ \wedge \ \mathbb{X} = \mathbb{Y}.$$

Proof. Let $\mathbb{X} = \mathbb{F}(\vec{\eta})$, $\mathbb{Y} = \mathbb{G}(\vec{\nu})$ and let $\kappa := \Psi^{\alpha}_{\mathbb{X}}$ plus $\pi := \Psi^{\beta}_{\mathbb{Y}}$.

The implication from right to left is trivial. So assume $\kappa = \pi$. If $\alpha \neq \beta$ we would have (b) of Theorem 2.3.14. Contradiction! Moreover the assumption $\mathrm{i}(\mathbb{X}) < \mathrm{i}(\mathbb{Y})$ ($\mathrm{i}(\mathbb{Y}) < \mathrm{i}(\mathbb{X})$, resp.) leads to the contradiction $\mathrm{i}(\mathbb{X}) \in C(\beta, \pi) \cap \mathrm{i}(\mathbb{Y}) = \pi = \kappa$, i.e. $\mathrm{i}(\mathbb{X}) < \kappa$ ($\mathrm{i}(\mathbb{Y}) \in C(\alpha, \kappa) \cap \mathrm{i}(\mathbb{X}) = \kappa = \pi$, resp.). Therefore we have $\mathbb{F} = \mathbb{G}$. The assumption $\vec{\eta} \neq \vec{\nu}$ then again implies (b) of Theorem 2.3.14 and leads again to the contradictions $\kappa < \pi$ or $\pi < \kappa$. ☐

Notation. As usual we define the normal form for ordinals as follows

$$\alpha \underset{\text{NF}}{=} \omega^{\alpha_1} + \ldots + \omega^{\alpha_m} \quad :\Leftrightarrow \quad \alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_m} \ \wedge \ \alpha_1 \geq \ldots \geq \alpha_m$$

$$\alpha \underset{\text{NF}}{=} \varphi(\eta, \zeta) \quad :\Leftrightarrow \quad \alpha = \varphi(\eta, \zeta) \ \wedge \ \eta, \zeta < \alpha.$$

Let $\alpha \underset{\text{NF}}{=} \omega^{\alpha_1} + \ldots + \omega^{\alpha_m}$, $\beta \underset{\text{NF}}{=} \omega^{\alpha_{m+1}} + \ldots + \omega^{\alpha_n}$ and $\sigma = (\sigma(1), \ldots, \sigma(n))$ be a permutation of $1, \ldots, n$, such that $\forall_1^{n-1} i \ (\alpha_{\sigma(i)} \geq \alpha_{\sigma(i+1)})$. Then we define the natural sum by means of

$$\alpha \oplus \beta \ := \ \omega^{\alpha_{\sigma(1)}} + \ldots + \omega^{\alpha_{\sigma(n)}}.$$

Lemma 2.3.16. *It holds*

❶ $\alpha \underset{\text{NF}}{=} \omega^{\alpha_1} + \ldots + \omega^{\alpha_m} \ \Rightarrow \ \big(\alpha \in C(\beta, \pi) \ \Leftrightarrow \ \alpha_1, \ldots, \alpha_m \in C(\beta, \pi)\big)$,

❷ $\alpha \underset{\text{NF}}{=} \varphi(\eta, \zeta) \ \Rightarrow \ \big(\alpha \in C(\beta, \pi) \ \Leftrightarrow \ \eta, \zeta \in C(\beta, \pi)\big)$,

❸ $\kappa \in \mathrm{Card} \cap \Xi \ \Rightarrow \ \big(\kappa \in C(\beta, \pi) \ \Leftrightarrow \ \kappa^+ \in C(\beta, \pi)\big)$,

❹ $\Psi^{\xi}_{\mathbb{X}}$ *is well-defined and* $\pi \leq \Psi^{\xi}_{\mathbb{X}} \ \Rightarrow \ \big(\Psi^{\xi}_{\mathbb{X}} \in C(\beta, \pi) \ \Leftrightarrow \ \xi < \beta \ \wedge \ \xi, \mathbb{X} \in C(\beta, \pi)\big)$.

❺ $\alpha \oplus \beta \in C(\gamma, \pi) \ \Leftrightarrow \ \alpha, \beta \in C(\gamma, \pi)$.

Proof. We have $C(\beta, \pi) = \bigcup_{n<\omega} C^n(\beta, \pi)$ and the propositions follow by induction on n. Claim ④ follows by utilizing Corollary 2.3.15 and the fact that $\Psi^{\xi}_{\mathbb{X}}$ cannot enter $C^{n+1}(\beta, \pi)$ by means of $+, \varphi$ or the cardinal successor function, since $C(\xi, \Psi^{\xi}_{\mathbb{X}}) \cap \mathrm{i}(\mathbb{X}) = \Psi^{\xi}_{\mathbb{X}}$ and $C(\xi, \Psi^{\xi}_{\mathbb{X}})$ is closed under these respective functions. ☐

2.4. The Ordinal Notation System $\mathsf{T}(\Xi)$

In this section we define the (primitive) recursive ordinal notation system $\mathsf{T}(\Xi)$.

Definition 2.4.1. The set of ordinal notations $\mathsf{T}(\Xi)$ is inductively defined as follows:

- $0, \Xi \in \mathsf{T}(\Xi)$,
- if $\alpha \underset{\text{NF}}{=} \omega^{\alpha_1} + \ldots + \omega^{\alpha_m}$ and $\alpha_1, \ldots, \alpha_m \in \mathsf{T}(\Xi)$ plus $m > 1$, then $\alpha \in \mathsf{T}(\Xi)$,
- if $\alpha \underset{\text{NF}}{=} \varphi(\eta, \zeta)$ and $\eta, \zeta \in \mathsf{T}(\Xi)$, then $\alpha \in \mathsf{T}(\Xi)$,
- if $\kappa \in \mathsf{T}(\Xi) \cap \mathrm{Card} \cap \Xi$, then $\kappa^+ \in \mathsf{T}(\Xi)$,
- if $\mathbb{X}, \xi \in \mathsf{T}(\Xi)$ and $\mathrm{o}(\mathbb{X}) \leq \xi \in C(\xi, \mathrm{i}(\mathbb{X}))$, then $\Psi^{\xi}_{\mathbb{X}} \in \mathsf{T}(\Xi)$.

Remark. Obviously we have $\mathsf{T}(\Xi) = C(\Gamma_{\Xi+1}, 0)$, where $\Gamma_{\Xi+1}$ denotes the first strongly critical $\gamma > \Xi$.

For every β it follows easily by induction on β that $C(\beta, 0) \cap \omega^+$ is transitive, hence $\mathsf{T}(\Xi) \cap \omega^+$ is transitive. For details see [Duc08], Lemma 4.1

It follows by Lemma 2.3.16 that different ordinal notations denote different ordinals in $\mathsf{T}(\Xi)$. To conceive $\langle \mathsf{T}(\Xi), < \rangle$ as a recursive ordinal notation system we need to be able to determine if $\kappa \in \mathsf{T}(\Xi)$ and to decide if $\kappa < \kappa'$, for arbitrary κ, κ', by reducing these questions to proper subterms of κ, κ', resp. Thus, taking into account Theorem 2.3.14 and for other $<$-comparisons the closure properties of the C sets plus the fact $C(\alpha, \Psi^{\alpha}_{\mathbb{X}}) \cap \mathrm{i}(\mathbb{X}) = \Psi^{\alpha}_{\mathbb{X}}$, it only remains to show that $\alpha \in C(\alpha, \pi)$ can be determined in a recursive way.[5]

Obviously we have $\alpha \in C(\delta, \pi)$ if there is no subterm $t \equiv \Psi^{\beta}_{\mathbb{Y}}$ of α, with $t \geq \pi$ and $\beta \geq \delta$. Therefore we can determine $\alpha \in C(\delta, \pi)$ if we know the arguments of Ψ greater or equal than δ in subterms of α. Therefore we define simultaneously to Definition 2.4.1 by induction on the complexity of $\alpha \in \mathsf{T}(\Xi)$:

Definition 2.4.2. The set $\mathrm{K}_\pi(\alpha)$:

$$\begin{aligned}
\mathrm{K}_\pi(0) := \mathrm{K}_\pi(\Xi) &:= \emptyset, && \\
\mathrm{K}_\pi(\alpha) &:= \bigcup_{i=1}^{m} \mathrm{K}_\pi(\alpha_i) && \text{if } \alpha \underset{\text{NF}}{=} \omega^{\alpha_1} + \ldots + \omega^{\alpha_m}, \\
\mathrm{K}_\pi(\alpha) &:= \mathrm{K}_\pi(\eta) \cup \mathrm{K}_\pi(\zeta) && \text{if } \alpha \underset{\text{NF}}{=} \varphi(\eta, \zeta), \\
\mathrm{K}_\pi(\alpha) &:= \mathrm{K}_\pi(\kappa) && \text{if } \alpha = \kappa^+, \\
\mathrm{K}_\pi(\Psi^{\xi}_{\mathbb{X}}) &:= \emptyset && \text{if } \Psi^{\xi}_{\mathbb{X}} < \pi, \\
\mathrm{K}_\pi(\Psi^{\xi}_{\mathbb{X}}) &:= \bigcup_{\rho \in \mathrm{par}(\mathbb{X})} \mathrm{K}_\pi(\rho) \cup \mathrm{K}_\pi(\xi) \cup \{\xi\} && \text{if } \Psi^{\xi}_{\mathbb{X}} \geq \pi.
\end{aligned}$$

[5]To be able to determine if $\mathbb{X} \in \mathsf{T}(\Xi)$, it is also necessary to encode $\vec{R}_{\Psi^{\xi}_{\mathbb{X}}}$ for every $\Psi^{\xi}_{\mathbb{X}} \in \mathsf{T}(\Xi)$.

Lemma 2.4.3. *For ordinals $\alpha, \delta, \pi \in \mathsf{T}(\Xi)$ it holds*

$$\alpha \in C(\delta, \pi) \quad \Leftrightarrow \quad \mathrm{K}_\pi(\alpha) < \delta.$$

Proof. By induction on α. □

Theorem 2.4.4. $\langle \mathsf{T}(\Xi), < \rangle$ *is a (primitive) recursive ordinal notation system.*

Proof. This follows by remark 2.4 and Lemma 2.4.3. □

From now on small Greek letters denote ordinals from $\mathsf{T}(\Xi)$.

3. The Fine Structure of the Collapsing Hierarchies

"The introduction of suitable abstractions is our only
mental aid to organize and master complexity."
Edsger W. Dijkstra

Up to now, we hardly know the sets $\mathfrak{M}^{\alpha}_{\mathbb{X}} \cap \mathsf{T}(\Xi)$. Of course we have by definition $\{\Psi^{\gamma}_{\mathbb{X}} \,|\, \gamma \geq \alpha\} \subseteq \mathfrak{M}^{\alpha}_{\mathbb{X}}$, but we cannot exclude that different elements of $\mathsf{T}(\Xi)$ like $\Psi^{\beta}_{\mathbb{Y}}$, for some $\mathbb{X} \neq \mathbb{Y}$, also enter $\mathfrak{M}^{\alpha}_{\mathbb{X}}$ (and of course for some $\mathbb{Y}$ they do). However, to be able to perform the proof strategy of stationary collapsing, i.e. replacing a main reflection rule of $\mathrm{i}(\mathbb{X})$ in a derivation of a set Γ of sentences with derivation length α by transforming this derivation into derivations of Γ^{κ} for all $\kappa \in \mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}$ we have to secure that every $\kappa \in \mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}$ is equipped with the required reflection rules (cf. equation (B) in the introduction of this part of the thesis).

In Definition 4.2.8 we define the reflection rules of an element π in essence by making recourse to $\vec{R}_{\pi}$. Therefore we have to prove that for every $\pi \in \mathsf{T}(\Xi) \cap \mathrm{SC}$ the collapsing hierarchies to which π belongs can be decoded from $\vec{R}_{\pi}$.

We pursue this non-trivial endeavor in the present chapter.

The proof breaks down into three steps. At first we show that an element $\Psi^{\alpha}_{\mathbb{X}}$ is at most an element of collapsing hierarchies of reflection instances $\mathbb{Y} = \mathbb{G}(\vec{\eta})$, with $\mathbb{G} \in \mathrm{Prcnfg}(\mathbb{X})$ (Path-Fidelity Theorem 3.1.1).

In a second step we show some kind of inversion of Lemma 2.3.2 ②, i.e. if $C(\alpha, \kappa) \cap \mathrm{i}(\mathbb{X}) = \kappa$ plus $\mathbb{X}, \alpha \in C(\kappa)$ and $\kappa \models \vec{R}_{\Psi^{\alpha}_{\mathbb{X}}}$ then it follows $\kappa \in \mathfrak{M}^{\alpha}_{\mathbb{X}}$ (Correctness Lemma 3.1.6).

By use of this result we show in a third step that every $\kappa < \mathrm{i}(\mathbb{X})$ which satisfies "more" than $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}}$ cannot be the minimal element of $\mathfrak{M}^{\alpha}_{\mathbb{X}}$, i.e. $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}}$ is an exact characterization of the reflection strength of $\Psi^{\alpha}_{\mathbb{X}}$.

Summing up these results we are able to prove that every collapsing hierarchy which contains π can be decoded out of $\vec{R}_{\pi}$ (Domination Theorem 3.2.4).

3.1. Path Fidelity and Correctness

Theorem 3.1.1 (Path Fidelity)**.** *Let $\mathbb{X}$ be a reflection instance and suppose $\Psi^{\alpha}_{\mathbb{X}}$ is well-defined. Moreover let $\mathbb{Y} \neq (\mathrm{i}(\mathbb{Y}); \mathsf{P}_0; \ldots)$ be a reflection instance with reflection configuration $\mathbb{G}$. Then it holds*

$$\Psi^{\alpha}_{\mathbb{X}} \in \mathfrak{M}^{\beta}_{\mathbb{Y}} \quad \Rightarrow \quad \mathbb{G} \in \overline{\mathrm{Prcnfg}}(\mathbb{X}) \ \ \& \ \ \beta \leq \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{G}) \ \ \& \ \ \Psi^{\beta}_{\mathbb{Y}} \leq \Psi^{\alpha}_{\mathbb{X}} < \mathrm{i}(\mathbb{X}) \leq \mathrm{i}(\mathbb{Y}).$$

Proof. Let $\kappa := \Psi^{\alpha}_{\mathbb{X}}$. At first we observe, that we must have $\mathrm{o}(\mathbb{X}) > 0$ by means of Lemma 2.3.5 ①, since $\kappa \in \mathfrak{M}^{\beta}_{\mathbb{Y}}$ and $\mathbb{Y} \neq (\mathrm{i}(\mathbb{Y}); \mathsf{P}_0; \ldots)$ and thus κ is at least Π^1_0-indescribable, i.e. a regular cardinal. Thus we have $\mathbb{X}, \alpha \in C(\kappa)$ and $\alpha > \rho$ for all $\rho \in \mathrm{par}(\mathbb{X})$ and thereby $\mathbb{X}, \alpha \in C(\alpha, \kappa)$. Moreover we have $C(\beta, \kappa) \cap \mathrm{i}(\mathbb{Y}) = \kappa$ and thus $\beta \leq \alpha$ since otherwise we would have $\kappa = \Psi^{\alpha}_{\mathbb{X}} \in C(\beta, \kappa)$.

Therefore we have $\kappa < \mathrm{i}(\mathbb{Y}) \in C(\beta, \kappa) \subseteq C(\alpha, \kappa)$ plus $C(\alpha, \kappa) \cap \mathrm{i}(\mathbb{X}) = \kappa$ and thus $\mathrm{i}(\mathbb{X}) \leq \mathrm{i}(\mathbb{Y})$.

Let $\mathbb{F}$ be the reflection configuration of $\mathbb{X}$. It remains to show $\mathbb{G} \in \overline{\mathrm{Prcnfg}}(\mathbb{F})$ and $\beta \leq \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{G})$.

We show the former by contradiction. Let us assume $\mathbb{G} \notin \overline{\mathrm{Prcnfg}}(\mathbb{F})$.

Case 1, it holds $\mathrm{initl}(\mathbb{F}) \neq \mathrm{initl}(\mathbb{G})$*:* Since $\mathbb{Y} \neq (\mathrm{i}(\mathbb{Y}); \mathsf{P}_0; \ldots)$ we must have $\mathbb{Y} = \mathbb{A}(m)$ for some m and thus $\mathbb{X} = (\pi^+; \mathsf{P}_0; \ldots)$ for some cardinal $\omega \leq \pi < \Xi$. However, above we already have observed that $\mathrm{o}(\mathbb{X}) > 0$. Contradiction!

Case 2: There is an $\mathbb{H} \in \overline{\mathrm{Prcnfg}}(\mathbb{F}) \cap \mathrm{Prcnfg}(\mathbb{G})$ with $\mathbb{X}' = \mathbb{H}(\xi', \vec{\nu}') \in \overline{\mathrm{Prinst}}(\mathbb{X})$ and $\mathbb{Y}' = \mathbb{H}(\zeta', \vec{\eta}') \in \mathrm{Prinst}(\mathbb{Y})$ plus $\alpha' := \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{X}')$ and $\beta' := \mathrm{ran}^{\beta}_{\mathbb{Y}}(\mathbb{Y}')$, where $(\alpha', \xi', \vec{\nu}') \neq (\beta', \zeta', \vec{\eta}')$, i.e. $\Psi^{\alpha'}_{\mathbb{X}'} \neq \Psi^{\beta'}_{\mathbb{Y}'}$.†

Subcase 2.1, $\Psi^{\beta'}_{\mathbb{Y}'} < \Psi^{\alpha'}_{\mathbb{X}'}$*:* Since $\kappa \in \mathfrak{M}^{\beta}_{\mathbb{Y}} \subseteq \Psi^{\beta'}_{\mathbb{Y}'}$ there must be an $\mathbb{X}'' \in \overline{\mathrm{Prinst}}(\mathbb{X})$ with $\mathrm{i}(\mathbb{X}'') > \Psi^{\beta'}_{\mathbb{Y}'}$ and $\Psi^{\beta}_{\mathbb{Y}} < \Psi^{\alpha''}_{\mathbb{X}''} \leq \Psi^{\beta'}_{\mathbb{Y}'}$, where $\alpha'' := \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{X}'')$. A visualization of this situation is given in figure 3.1 (where dotted segments might be of zero length). Since $\Psi^{\alpha''}_{\mathbb{X}''} < \Psi^{\alpha'}_{\mathbb{X}'}$ we must have $\mathbb{X}' \in \mathrm{Prinst}(\mathbb{X}'')$ and thus $\alpha'' > \mathrm{par}(\mathbb{X}'') \supseteq \mathrm{par}(\mathbb{X}') \cup \{\alpha'\}$ and thereby $\alpha'' > \alpha'$.

Subcase 2.1.1, $\beta' \leq \alpha'$*:* We have $\beta' > \mathrm{par}(\mathbb{Y}')$ and $\mathbb{Y} \in C(\Psi^{\beta}_{\mathbb{Y}})$. Thus $\beta', \mathbb{Y}' \in C(\beta', \Psi^{\beta}_{\mathbb{Y}})$. Moreover we have $\beta' \leq \alpha' < \alpha''$ and thereby $\Psi^{\alpha''}_{\mathbb{X}''} \leq \Psi^{\beta'}_{\mathbb{Y}'} \in C(\beta' + 1, \Psi^{\beta}_{\mathbb{Y}}) \cap \mathrm{i}(\mathbb{X}'') \subseteq C(\alpha'', \Psi^{\alpha''}_{\mathbb{X}''}) \cap \mathrm{i}(\mathbb{X}'') = \Psi^{\alpha''}_{\mathbb{X}''}$. Contradiction!

Subcase 2.1.2, $\alpha' < \beta'$*:* We have $\alpha' > \mathrm{par}(\mathbb{X}')$ and $\mathbb{X}'' \in C(\Psi^{\alpha''}_{\mathbb{X}''})$. Thus $\alpha', \mathbb{X}' \in C(\alpha' + 1, \Psi^{\alpha''}_{\mathbb{X}''}) \subseteq C(\beta', \Psi^{\beta'}_{\mathbb{Y}'})$. Therefore it holds $\Psi^{\beta'}_{\mathbb{Y}'} < \Psi^{\alpha'}_{\mathbb{X}'} \in C(\beta', \Psi^{\beta'}_{\mathbb{Y}'}) \cap \mathrm{i}(\mathbb{H}) = \Psi^{\beta'}_{\mathbb{Y}'}$. Contradiction!

Subcase 2.2, $\Psi^{\alpha'}_{\mathbb{X}'} < \Psi^{\beta'}_{\mathbb{Y}'}$*:* Since $\kappa \leq \Psi^{\alpha'}_{\mathbb{X}'}$ there must be a $\mathbb{Y}'' \in \overline{\mathrm{Prinst}}(\mathbb{Y})$ with $\Psi^{\beta''}_{\mathbb{Y}''} \leq \Psi^{\alpha'}_{\mathbb{X}'} < \mathrm{i}(\mathbb{Y}'')$ where $\beta'' := \mathrm{ran}^{\beta}_{\mathbb{Y}}(\mathbb{Y}'')$.

Since $\Psi^{\beta''}_{\mathbb{Y}''} < \Psi^{\beta'}_{\mathbb{Y}'}$ we must have $\mathbb{Y}' \in \mathrm{Prinst}(\mathbb{Y}'')$ and thus $\beta'' > \mathrm{par}(\mathbb{Y}'') \supseteq \mathrm{par}(\mathbb{Y}') \cup \{\beta'\}$ and $\beta'' > \beta'$.

Subcase 2.2.1, $\beta' < \alpha'$*:* Since $\mathbb{Y}'' \in C(\Psi^{\beta''}_{\mathbb{Y}''})$ we have $\beta', \mathbb{Y}' \in C(\beta' + 1, \Psi^{\beta''}_{\mathbb{Y}''}) \subseteq C(\alpha', \Psi^{\alpha'}_{\mathbb{X}'})$. Therefore it holds $\Psi^{\alpha'}_{\mathbb{X}'} < \Psi^{\beta'}_{\mathbb{Y}'} \in C(\alpha', \Psi^{\alpha'}_{\mathbb{X}'}) \cap \mathrm{i}(\mathbb{H}) = \Psi^{\alpha'}_{\mathbb{X}'}$. Contradiction!

Subcase 2.2.2, $\alpha' \leq \beta'$*:* We have $\alpha' > \mathrm{par}(\mathbb{X}')$ and $\mathrm{par}(\mathbb{X}) \supseteq \mathrm{par}(\mathbb{X}') \cup \{\alpha'\}$. Since $\mathbb{X} \in C(\kappa)$ we thus have $\alpha', \mathbb{X}' \in C(\alpha' + 1, \kappa) \subseteq C(\beta'', \kappa)$

Subcase 2.2.2.1, $\kappa = \Psi^{\alpha}_{\mathbb{X}} \leq \Psi^{\beta''}_{\mathbb{Y}''}$*:* We have $\Psi^{\beta''}_{\mathbb{Y}''} \leq \Psi^{\alpha'}_{\mathbb{X}'} \in C(\beta'', \kappa) \cap \mathrm{i}(\mathbb{Y}'') \subseteq C(\beta'', \Psi^{\beta''}_{\mathbb{Y}''}) \cap \mathrm{i}(\mathbb{Y}'') = \Psi^{\beta''}_{\mathbb{Y}''}$. Contradiction!

Subcase 2.2.2.2, $\Psi^{\beta''}_{\mathbb{Y}''} < \kappa$*:* Then we have $\beta = \beta''$ and $\mathbb{Y}'' = \mathbb{Y}$ since otherwise

†In the following the reader should be aware that for a reflection instance $\mathbb{Z}$ and $\mathbb{Z}' \in \mathrm{Prinst}(\mathbb{Z})$ plus $\xi' = \mathrm{ran}^{\xi}_{\mathbb{Z}}(\mathbb{Z}')$ it does hold $\mathfrak{M}^{\xi}_{\mathbb{Z}} \subseteq \Psi^{\xi'}_{\mathbb{Z}'}$ for all $\xi \geq \mathrm{o}(\mathbb{Z})$. This follows easily by induction on $\mathrm{card}(\mathrm{Prinst}(\mathbb{Z}) \setminus \mathrm{Prinst}(\mathbb{Z}'))$.

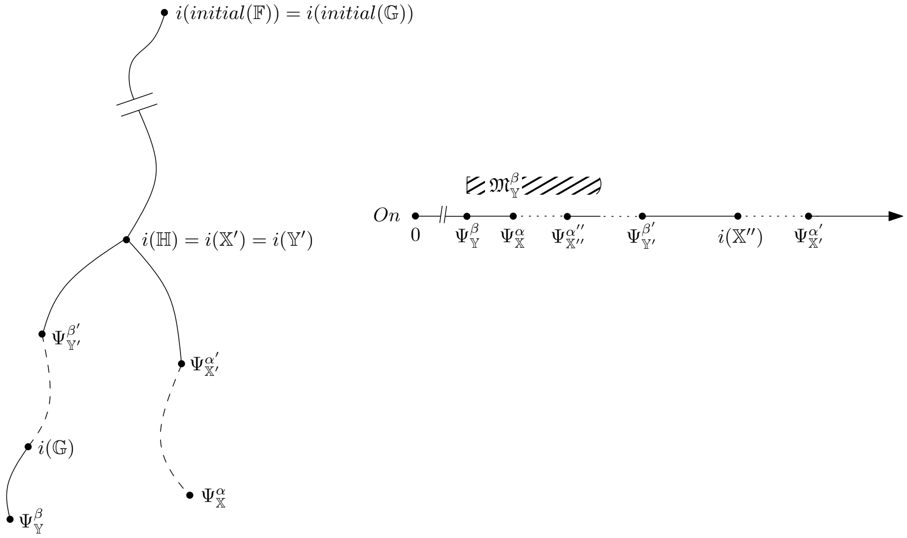

Figure 3.1.: A visualization of the situation in Subcase 2.1.

we would have $\mathfrak{M}^{\beta}_{\mathbb{Y}} \subseteq \Psi^{\beta''}_{\mathbb{Y}''}$ contradicting $\kappa \in \mathfrak{M}^{\beta}_{\mathbb{Y}}$. Therefore we have $\kappa \leq \Psi^{\alpha'}_{\mathbb{X}'} \in C(\beta'', \kappa) \cap \mathrm{i}(\mathbb{Y}'') = C(\beta, \kappa) \cap \mathrm{i}(\mathbb{Y}) = \kappa$. Contradiction!

Case 3, it holds $\mathbb{F} \in \mathrm{Prcnfg}(\mathbb{G})$*:* Let $\mathbb{Y}' := \mathbb{F}(\zeta'\vec{\eta}',) \in \mathrm{Prinst}(\mathbb{Y})$ and $\beta' := \mathrm{ran}^{\beta}_{\mathbb{Y}}(\mathbb{Y}')$. Then we have $\mathrm{i}(\mathbb{Y}) \leq \Psi^{\beta'}_{\mathbb{Y}'} < \mathrm{i}(\mathbb{F}) = \mathrm{i}(\mathbb{X})$. Contradiction, since we have already shown $\mathrm{i}(\mathbb{X}) \leq \mathrm{i}(\mathbb{Y})$!

Since the three treated cases provide an exhausting case differentiation (imagine the nodes in the $\overline{\mathrm{Prcnfg}}(\mathbb{F})$ path, where the development of $\mathbb{G}$ could branch out!) it remains to show $\beta \leq \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{G})$ if $\mathbb{G} \neq \mathbb{F}$, i.e. if $\mathbb{G} \in \mathrm{Prcnfg}(\mathbb{F})$.

Suppose $\mathbb{G} \in \mathrm{Prcnfg}(\mathbb{F})$ and let $\mathbb{Y}' := \mathbb{G}(\zeta', \vec{\eta}') \in \mathrm{Prinst}(\mathbb{X})$ plus $\beta' := \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{G})$. Then we have $\beta' > \mathrm{par}(\mathbb{Y}')$ and $\mathrm{par}(\mathbb{Y}') \cup \{\beta'\} \subseteq \mathrm{par}(\mathbb{X})$. Since $\mathbb{X} \in C(\kappa)$ we obtain thereby $\beta', \mathbb{Y}' \in C(\beta', \kappa)$. As the assumption $\beta' < \beta$ yields the contradiction $\kappa < \Psi^{\beta'}_{\mathbb{Y}'} \in C(\beta, \kappa) \cap \mathrm{i}(\mathbb{Y}') = C(\beta, \kappa) \cap \mathrm{i}(\mathbb{Y}) = \kappa$ the claim follows. □

Definition 3.1.2. Let $\mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m$ be an M-P-expression. Then we define by recursion on $\mathrm{o}(\mathbb{M})$

$$\mathrm{Tc}(\mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m) := \begin{cases} \{\mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m\} & \text{if } \mathrm{o}(\mathbb{M}) \notin \mathrm{Succ}, \\ \{\mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m\} \cup \mathrm{Tc}((\vec{R}_{\mathrm{i}(\mathbb{M})})_m) & \text{otherwise.} \end{cases}$$

Definition 3.1.3. On M-P-expressions we define the binary relation $\preceq$ by

$$\begin{aligned} \epsilon &\preceq \mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m \quad \text{for any M-P-expression } \mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m, \\ \mathsf{M}^{<\zeta}_{\mathbb{G}}\text{-}\mathsf{P}_m &\preceq \mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m \quad :\Leftrightarrow \quad \exists \gamma \left(\mathrm{o}(\mathbb{G}) \leq \zeta \leq \gamma \ \wedge \ \mathsf{M}^{<\gamma}_{\mathbb{G}}\text{-}\mathsf{P}_m \in \mathrm{Tc}(\mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m) \right). \end{aligned}$$

We extend this relation to finite sequences of M-P-expressions $\vec{R}$ and $\vec{S}$ as follows

$$\vec{R} \preceq \vec{S} \quad :\Leftrightarrow \forall m \in \omega\, (\vec{R}_m \preceq \vec{S}_m).$$

Remark. Due to Lemma 2.3.2 ③ (a) Tc is well-defined. Moreover $\preceq$ is transitive and $\prec$, the strict part of $\preceq$, is well-founded.

Lemma 3.1.4. *Let $\mathbb{X}$ be a reflection instance with reflection configuration $\mathbb{F}$ and $\mathrm{rdh}(\mathbb{X}) \geq m$. Let κ be an ordinal satisfying $\mathbb{X} \in C(\kappa)$. Then*

$$\kappa \models \widetilde{\mathsf{M}}_{\mathbb{F}}^{<\alpha}\text{-}\mathsf{P}_m \;\&\; \mathsf{M}_{\mathbb{M}}^{<\zeta}\text{-}\mathsf{P}_m \preceq \widetilde{\mathsf{M}}_{\mathbb{F}}^{<\alpha}\text{-}\mathsf{P}_m \quad \Rightarrow \quad \kappa \models \mathsf{M}_{\mathbb{M}}^{<\zeta}\text{-}\mathsf{P}_m.$$

Proof. We proceed by induction on $\mathrm{o}(\mathbb{F})$. If $\mathrm{o}(\mathbb{F}) \notin \mathrm{Succ}$ the claim is trivial. If $\mathrm{o}(\mathbb{F}) = \delta+1$ and $\alpha = \delta+1$ it follows by definition that $\widetilde{\mathsf{M}}_{\mathbb{F}}^{<\alpha}\text{-}\mathsf{P}_m = (\vec{R}_{\mathrm{i}(\mathbb{F})})_m$. Therefore the claim follows by the induction hypothesis.

If $\mathrm{o}(\mathbb{F}) = \delta+1$ and $\alpha > \delta+1$ we either have $\mathsf{M}_{\mathbb{M}}^{<\alpha}\text{-}\mathsf{P}_m = \mathsf{M}_{\mathbb{F}}^{<\alpha}\text{-}\mathsf{P}_m$ and $\mathrm{o}(\mathbb{F}) \leq \zeta \leq \alpha$ or $\mathsf{M}_{\mathbb{M}}^{<\zeta}\text{-}\mathsf{P}_m \preceq (\vec{R}_{\mathrm{i}(\mathbb{F})})_m = (\vec{R}_{\mathrm{i}(\mathbb{X})})_m$. In the first case the claim is trivial. In the latter case we have to show, that for every $(\gamma, \vec{\eta}) \in [\mathrm{o}(\mathbb{M}), \zeta)_{C(\kappa)} \times \mathrm{dom}(\mathbb{M})_{C(\kappa)}$ and every Π^1_m-sentence F in parameters $\vec{P} \subseteq V_\kappa$ such that $\langle V_\kappa, \vec{P}\rangle \models F$ we can find a $\kappa_0 \in \mathfrak{M}^\gamma_{\mathbb{M}(\vec{\eta})}$ such that $\langle V_{\kappa_0}, \vec{P} \cap V_{\kappa_0}\rangle \models F$.

By the provisos $\kappa \models \widetilde{\mathsf{M}}_{\mathbb{F}}^{<\alpha}\text{-}\mathsf{P}_m$ and $\mathbb{X} \in C(\kappa)$ it follows that κ is $\mathfrak{M}^{\mathrm{o}(\mathbb{X})}_{\mathbb{X}}$-$\Pi^1_m$- indescribable. Proceeding as in the proof of Theorem 2.3.11 we can find a $\kappa_0' \in \mathfrak{M}^{\mathrm{o}(\mathbb{X})}_{\mathbb{X}}$ such that $\langle V_{\kappa_0'}, \vec{P} \cap V_{\kappa_0'}\rangle \models F$ and $\mathrm{par}\,\mathbb{X} \cup \{\gamma, \vec{\eta}\} \subseteq C(\kappa_0')$. By means of Lemma 2.3.2 ② it follows that $\kappa_0' \models (\vec{R}_{\mathrm{i}(\mathbb{X})})_m$. Let $(\vec{R}_{\mathrm{i}(\mathbb{X})})_m = \mathsf{M}_{\mathbb{E}}^{<\varepsilon}\text{-}\mathsf{P}_m$. Due to Lemma 2.3.2 ③ (e) there exists a $\vec{\nu} \in \mathrm{dom}(\mathbb{E}) \cap \mathrm{par}\,\mathbb{X}$ such that $\mathrm{rdh}(\mathbb{E}(\vec{\nu})) = m$. Therefore it follows by the induction hypothesis applied to $\mathbb{E}(\vec{\nu})$ that $\kappa_0' \models \mathsf{M}_{\mathbb{M}}^{<\zeta}\text{-}\mathsf{P}_m$. Since $\gamma, \vec{\eta} \in C(\kappa_0')$ this implies that κ_0' is $\mathfrak{M}^\gamma_{\mathbb{M}(\vec{\eta})}$-$\Pi^1_m$-indescribable. Thus there is a $\kappa_0 \in \mathfrak{M}^\gamma_{\mathbb{M}(\vec{\eta})}$, such that $\langle V_{\kappa_0}, \vec{P} \cap V_{\kappa_0}\rangle \models F$. □

Lemma 3.1.5. *Let $\mathbb{X} = (\pi; \ldots; \delta)$ be a reflection instance, $\kappa \leq \pi$ and $\alpha \geq \delta$. Suppose $C(\alpha, \kappa) \cap \pi = \kappa$ and $\mathbb{X}, \alpha \in C(\kappa)$. Then it holds for any $\mathbb{Z} \in \mathrm{Prinst}(\mathbb{X})$:*

❶ $C(\mathrm{ran}^\alpha_{\mathbb{X}}(\mathbb{Z}), \kappa) \cap \mathrm{i}(\mathbb{Z}) = \kappa \quad \& \quad \mathbb{Z}, \mathrm{ran}^\alpha_{\mathbb{X}}(\mathbb{Z}) \in C(\kappa)$.

❷ $\mathbb{E} \in \mathrm{Prcnfg}(\mathbb{X}) \quad \& \quad \forall \vec{\eta}'\big(\vec{\eta}' \in \mathrm{dom}(\mathbb{E}) \cap \mathrm{par}\,\mathbb{X} \;\Rightarrow\; \mathbb{E}(\vec{\eta}') \in C(\kappa)\big)$.

Proof. ① Let $\mathbb{X}_0 := \mathbb{X}$, $\alpha_0 := \alpha$, $\pi_0 := \mathrm{i}(\mathbb{X}) = \pi$ and $\mathbb{X}_{i+1} := \mathbb{Z}_i$ if $\mathbb{X}_i = (\ldots; \mathbb{Z}_i; \ldots)$, i.e. $\mathbb{X}_{i+1}$ is the predecessor of $\mathbb{X}_i$ plus $\alpha_{i+1} := \mathrm{ran}^\alpha_{\mathbb{X}}(\mathbb{X}_{i+1})$ and $\pi_{i+1} := \mathrm{i}(\mathbb{X}_{i+1})$.

Obviously we have for all i:

$$\kappa \leq \pi_i < \pi_{i+1} \;\&\; \alpha_i > \alpha_{i+1} \;\&\; C(\alpha_{i+1}, \pi_i) \cap \pi_{i+1} = \pi_i.$$

We show by induction on i that $C(\alpha_i, \kappa) \cap \pi_i = \kappa$.

In the initial case $i = 0$ there is nothing to show. In the successor case $i+1 > 0$ we have

$$\begin{aligned} & C(\alpha_{i+1}, \pi_i) \cap \pi_{i+1} = \pi_i \quad \Rightarrow \quad C(\alpha_{i+1}, \kappa) \cap \pi_{i+1} \subseteq \pi_i \\ \Rightarrow \quad & C(\alpha_{i+1}, \kappa) \cap \pi_{i+1} = C(\alpha_{i+1}, \kappa) \cap \pi_i \subseteq C(\alpha_i, \kappa) \cap \pi_i \overset{\text{ind.hyp.}}{=} \kappa. \end{aligned}$$

Therefore $C(\alpha_{i+1}, \kappa) \cap \pi_{i+1} \subseteq \kappa$ and by definition it holds $C(\alpha_{i+1}, \kappa) \cap \pi_{i+1} \supseteq \kappa$. Thus we have shown for $\mathbb{Z} \in \mathrm{Prinst}(\mathbb{X})$ that $C(\mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{Z}), \kappa) \cap \mathrm{i}(\mathbb{Z}) = \kappa$. Moreover we have $\mathrm{par}(\mathbb{Z}) \cup \{\mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{Z})\} \subseteq \mathrm{par}(\mathbb{X}) \subseteq C(\kappa)$.

② Let $\mathbb{E} \in \mathrm{Prcnfg}(\mathbb{X})$. Then there is a $\vec{\eta}'' \in \mathrm{dom}(\mathbb{E})$ such that $\mathbb{E}(\vec{\eta}'') \in \mathrm{Prinst}(\mathbb{X})$. Thus we have $\mathrm{par}(\mathbb{E}(\vec{\eta}'')) \subseteq \mathrm{par}(\mathbb{X}) \subseteq C(\kappa)$ and therefore $\mathbb{E}(\vec{\eta}') \in C(\kappa)$ since $\vec{\eta}' \in \mathbb{X}$ by assumption. □

Lemma 3.1.6 (Correctness)**.** *Let $\mathbb{X} \neq (\mathrm{i}(\mathbb{X}); \mathsf{P}_0; \ldots)$ be a reflection instance and suppose $\Psi^{\alpha}_{\mathbb{X}}$ is well-defined. Then it holds:*

❶ *For any reflection configuration $\mathbb{E}$, any $\mathrm{o}(\mathbb{E}) \leq \varepsilon$, and any $e \in \mathrm{Rdh}(\mathbb{E})$*

$$\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e \preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_e = \mathsf{M}^{<\rho}_{\mathbb{E}}\text{-}\mathsf{P}_e \quad \Rightarrow \quad \vec{R}^{(\varepsilon,e)}_{\mathbb{E}} \preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq e},$$

where $\vec{R}^{(\varepsilon,e)}_{\mathbb{E}} := \vec{R}_{\mathbb{E}}$ if $\mathbb{E} \neq \mathbb{A}$ and $\vec{R}^{(\varepsilon,e)}_{\mathbb{E}} := (\mathsf{M}^{<\varepsilon}_{\mathbb{A}}\text{-}\mathsf{P}_e, \ldots, \mathsf{M}^{<\varepsilon}_{\mathbb{A}}\text{-}\mathsf{P}_0)$ otherwise.

❷ *For any κ, such that $C(\alpha, \kappa) \cap \mathrm{i}(\mathbb{X}) = \kappa$ and $\mathbb{X}, \alpha \in C(\kappa)$*

$$\kappa \models \vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} \quad \Rightarrow \quad \kappa \in \mathfrak{M}^{\alpha}_{\mathbb{X}}.$$

Proof. We proceed by induction on $\mathrm{o}(\mathbb{X})$. In the following let $\mathbb{F}$ be the reflection configuration of $\mathbb{X}$.

① *Case 1, $\mathbb{X} = \mathbb{A}(m+1)$ for some $m \in \omega$:* Then $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = (\mathsf{M}^{<\alpha}_{\mathbb{A}}\text{-}\mathsf{P}_m, \ldots, \mathsf{M}^{<\alpha}_{\mathbb{A}}\text{-}\mathsf{P}_0)$. Thus we have $\mathbb{E} = \mathbb{A}$ and $\varepsilon \leq \alpha$ plus $e \leq m$. Thereby the claim follows.

Case 2, $\mathbb{X} = (\Psi^{\delta}_{\mathbb{Z}}; \mathsf{P}_m; \vec{R}; \ldots)$ with $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\omega}_{\mathbb{A}}\text{-}\mathsf{P}_m, \vec{R})$: Then it holds $m > 0$ and $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = (\widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}, \vec{R}_{<m-1})$.

If $\mathbb{E} = \mathbb{F}$ then we have $e = m-1$ and the claim holds, since $\vec{R}_{\mathbb{E}} = \vec{R} \preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq e}$.

If $\mathbb{E} \neq \mathbb{F}$ and $e = m-1$ we have $(\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{m-1} = \mathsf{M}^{<\rho}_{\mathbb{E}}\text{-}\mathsf{P}_{m-1}$. Thus we must have $\alpha = \delta + 1$, i.e. $(\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{m-1} = (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_{m-1}$. Therefore we have $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e \preceq \mathsf{M}^{<\rho}_{\mathbb{E}}\text{-}\mathsf{P}_{m-1} = (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{m-1} = (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_{m-1}$. Thus we obtain by means of the induction hypothesis $\vec{R}^{(\varepsilon,e)}_{\mathbb{E}} \preceq (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_{\leq e} \preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq e}$.

If $e < m-1$ we also have $(\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{m-1} = (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_{m-1}$ and the claim follows by means of the induction hypothesis, too.

Case 3, $\mathbb{X} = (\Psi^{\delta}_{\mathbb{Z}}; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m; \vec{R}; \ldots)$ with $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m, \vec{R})$: Then it holds $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{\geq m}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}, \vec{R}_{<m-1})$ and $\xi < \gamma$. If $e < m$ the claim follows as in the second case. So let us assume $e \geq m$. Then we have $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e \preceq \mathsf{M}^{<\rho}_{\mathbb{E}}\text{-}\mathsf{P}_e = (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_e = (\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_e$. If $\mathbb{Z} = \mathbb{A}(n)$ for some $n \in \omega$ it follows by definition that $\mathbb{M} = \mathbb{A}$ and thus

$o(\mathbb{M}) < o(\mathbb{X})$. If $o(\mathbb{Z}) > \omega$ it follows by means of Lemma 2.3.2 ③(a) that $o(\mathbb{M}) < o(\mathbb{X})$. Therefore we obtain by the induction hypothesis applied to $\mathbb{M}(\vec{\nu})$

$$\vec{R}_{\mathbb{E}}^{(\varepsilon,e)} \preceq (\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{\leq e}. \tag{3.1}$$

In addition it holds $\widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m \preceq \mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m = (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_m$. Moreover we have $m \in \mathrm{Rdh}(\mathbb{M})$ either by definition if $\mathbb{Z} = \mathbb{A}(n)$ for some $n \in \omega$ (and then $\mathbb{M} = \mathbb{A}$), or by Lemma 2.3.2 ③(e) otherwise. Thus we obtain by the induction hypothesis applied to $\mathbb{Z}$

$$(\vec{R}_{\mathbb{M}}^{(\xi,m)})_{<m} \preceq (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_{<m} = \vec{R} \preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{<m}. \tag{3.2}$$

Since it holds $(\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{\leq e} = (\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{[e,m]}, (\vec{R}_{\mathbb{M}}^{(\xi,m)})_{<m}$ either by definition if $\mathbb{M} = \mathbb{A}$ or by Lemma 2.3.2 ③(f) otherwise, it follows from (3.1) and (3.2) plus the transitivity of $\preceq$ that $\vec{R}_{\mathbb{E}}^{(\varepsilon,e)} \preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq e}$.

② If $\mathbb{X} = (\pi; \mathsf{P}_m; \ldots)$ the claim is trivial. If $\mathbb{X} = \mathbb{A}(m+1)$ for some $m \in \omega$ the claim follows by use of Lemma 3.1.4. So let us assume that $\mathbb{X} = (\Psi^{\delta}_{\mathbb{Z}}; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m; \vec{R}; \ldots)$ with $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m, \vec{R})$. Then it holds $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{\geq m}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}, \vec{R}_{<m-1})$. Since we have $\widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m \preceq \mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m = (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_m$ we obtain by means of ① and Lemma 2.3.2 ③(e),(f) that

$$(\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{<m} = (\vec{R}_{\mathbb{M}}^{(\xi,m)})_{<m} \preceq (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_{<m} = \vec{R} \preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{<m}.$$

Therefore the assumption $\kappa \models \vec{R}_{\Psi^{\alpha}_{\mathbb{X}}}$ in collaboration with Lemma 3.1.4 implies $\kappa \models ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{\geq m}, (\vec{R}_{\mathbb{M}}^{(\xi,m)})_{<m})$, i.e. $\kappa \models \vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}}$.

Since $\xi, \vec{\nu} \in \mathrm{par}\,\mathbb{X}$, $\mathbb{X} \in C(\kappa)$, $\mathbb{M} \in \mathrm{Prcnfg}(\mathbb{X})$ and $\xi < \gamma \leq \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{M})$ it follows by means of Lemma 3.1.5 that $C(\xi,\kappa) \cap i(\mathbb{M}) = \kappa$ and $\xi, \mathbb{M}(\vec{\nu}) \in C(\kappa)$. Therefore we obtain by use of the induction hypothesis that $\kappa \in \mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})}$. Combining this with the provisos $C(\alpha,\kappa) \cap \mathrm{i}(\mathbb{X}) = \kappa$ and $\mathbb{X}, \alpha \in C(\kappa)$ plus $\kappa \models (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{<m}$ it follows $\kappa \in \mathfrak{M}^{\alpha}_{\mathbb{X}}$. □

3.2. The Domination Theorem

Definition 3.2.1. Let $\mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m$ be an M-P-expression, $\mathbb{D}$ a reflection configuration, and suppose $\vec{R} = (\mathsf{M}^{<\rho_1}_{\mathbb{R}_1}\text{-}\mathsf{P}_{r_1}, \ldots, \mathsf{M}^{<\rho_j}_{\mathbb{R}_j}\text{-}\mathsf{P}_{r_j})$ is a finite sequence of M-P-expressions. Then we define

$$(\mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m)^{\mathbb{D}} := \begin{cases} \mathsf{M}^{<\zeta}_{\mathbb{D}}\text{-}\mathsf{P}_m & \text{if } \mathsf{M}^{<\zeta}_{\mathbb{D}}\text{-}\mathsf{P}_m \in \mathrm{Tc}(\mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m) \\ \epsilon & \text{otherwise,} \end{cases}$$
$$(\vec{R})^{\mathbb{D}} := ((\mathsf{M}^{<\rho_1}_{\mathbb{R}_1}\text{-}\mathsf{P}_{r_1})^{\mathbb{D}}, \ldots, (\mathsf{M}^{<\rho_j}_{\mathbb{R}_j}\text{-}\mathsf{P}_{r_j})^{\mathbb{D}}).$$

Remark. It follows by induction on $o(\mathbb{M})$ that $(\mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m)^{\mathbb{D}}$ is well-defined.

Lemma 3.2.2. *Let* $\mathbb{X} \neq (\mathrm{i}(\mathbb{X}); \mathsf{P}_0; \ldots)$ *be a reflection instance,* $\mathbb{E}$ *be a reflection configuration, such that* $\mathbb{E} \in \overline{\mathrm{Prcnfg}}(\mathbb{X})$ *and suppose* $\Psi^{\alpha}_{\mathbb{X}}$ *is well-defined. Then it holds for any* $g \leq \min\{\mathrm{rd}(\vec{R}_{\mathbb{E}}), \mathrm{rd}(\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})\}$ *and any* $\mathbb{D} \in \mathrm{Prcnfg}(\mathbb{E})$*:*

$$(\vec{R}_{\mathbb{E}})_{\leq g} \preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq g} \quad \Rightarrow \quad (\vec{R}_{\mathbb{E}})^{\mathbb{D}}_{\leq g} = (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})^{\mathbb{D}}_{\leq g}.$$

Proof. We proceed by induction on $\mathrm{o}(\mathbb{X})$. In the following let $\mathbb{F}$ be the reflection configuration of $\mathbb{X}$.

Case 1, $\mathbb{X} = \mathbb{A}(m+1)$ *for some* $m < \omega$*:* Then we have nothing to show, since there is not any $\mathbb{D} \in \mathrm{Prcnfg}(\mathbb{A})$.

Case 2, $\mathbb{X} = (\Psi^{\delta}_{\mathbb{Z}}; \mathsf{P}_m; \vec{R}; \ldots)$ *with* $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\omega}_{\mathbb{A}}\text{-}\mathsf{P}_m, \vec{R})$*:* Then it holds $m > 0$ and $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = (\widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}, \vec{R}_{<m-1})$. If $\mathbb{E} = \mathbb{F}$ then the claim holds since

$$(\vec{R}_{\mathbb{E}})^{\mathbb{D}}_{\leq g} = (\vec{R}_{\mathbb{F}})^{\mathbb{D}}_{\leq g} = (\vec{R})^{\mathbb{D}}_{\leq g} \overset{\mathbb{D}\neq\mathbb{F}}{=} (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})^{\mathbb{D}}_{\leq g}.$$

So let us assume $\mathbb{E} \neq \mathbb{F}$. Then it follows $\mathbb{E} \in \overline{\mathrm{Prcnfg}}(\mathbb{Z})$ and the proviso $(\vec{R}_{\mathbb{E}})_{\leq g} \preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq g}$ implies $(\vec{R}_{\mathbb{E}})_{\leq g} \preceq (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_{\leq g}$, since $\mathbb{E} \neq \mathbb{F}$. Therefore we obtain

$$(\vec{R}_{\mathbb{E}})^{\mathbb{D}}_{\leq g} \overset{\text{ind.hyp.}}{=} (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})^{\mathbb{D}}_{\leq g} = (\vec{R})^{\mathbb{D}}_{\leq g} \overset{\mathbb{D}\neq\mathbb{F}}{=} (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})^{\mathbb{D}}_{\leq g}.$$

Case 3, $\mathbb{X} = \mathbb{F}(\xi, \vec{\nu}) = (\Psi^{\delta}_{\mathbb{Z}}; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m; \vec{R}; \ldots)$ *with* $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m, \vec{R})$ *for some* $\gamma > \xi$*:* Then it holds $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{\geq m}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}, \vec{R}_{<m-1})$.

Subcase 3.1, $g < m$*:* Then the claim follows by the same considerations as in the second case.

Subcase 3.2, $g \geq m$*:* Then we must have $\mathbb{E} \neq \mathbb{F}$, since $\mathrm{rd}(\vec{R}_{\mathbb{F}}) = m - 1$, i.e. $\mathbb{E} \in \overline{\mathrm{Prcnfg}}(\mathbb{Z})$. The assumption $\mathbb{M} \in \mathrm{Prcnfg}(\mathbb{E})$ leads to the following contradiction: By proviso we have $(\vec{R}_{\mathbb{E}})_{\leq m} \preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq m}$. Since $\mathrm{o}(\mathbb{E}) < \mathrm{o}(\mathbb{F})$ this implies $(\vec{R}_{\mathbb{E}})_{\leq m} \preceq (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_{\leq m}$ and therefore we obtain by the induction hypothesis (with $\mathbb{D} = \mathbb{M}$)

$$\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m = (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})^{\mathbb{M}}_m \overset{\text{ind.hyp.}}{=} (\vec{R}_{\mathbb{E}})^{\mathbb{M}}_m \preceq (\vec{R}_{\mathbb{E}})_m \overset{\text{assump.}}{\preceq} (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_m = \mathsf{M}^{<\xi}_{\mathbb{A}}\text{-}\mathsf{P}_m.$$

However this contradicts $\xi < \gamma$ and thus we must have $\mathbb{M} \notin \mathrm{Prcnfg}(\mathbb{E})$, i.e. $\mathrm{o}(\mathbb{D}) < \mathrm{o}(\mathbb{E}) \leq \mathrm{o}(\mathbb{M}) < \mathrm{o}(\mathbb{F})$. If $\mathbb{M} = \mathbb{A}$ we have nothing to show, since there is not any $\mathbb{D} \in \mathrm{Prcnfg}(\mathbb{A})$. So it remains the case $g \geq m$ with $\mathbb{A} \neq \mathbb{M} \notin \mathrm{Prcnfg}(\mathbb{E})$.

We want to apply the induction hypothesis to $\mathbb{M}(\vec{\nu})$. Therefore we have to show that $(\vec{R}_{\mathbb{E}})_{<m} \preceq (\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{<m} = (\vec{R}_{\mathbb{M}})_{<m}$. By the provisos we have $(\vec{R}_{\mathbb{E}})_{<m} \preceq \vec{R} = (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_{<m}$ since $\mathrm{o}(\mathbb{E}) < \mathrm{o}(\mathbb{F})$. Thus the induction hypothesis applied to $\mathbb{Z}$ provides

$$(\vec{R}_{\mathbb{E}})^{\mathbb{D}}_{<m} = (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})^{\mathbb{D}}_{<m} \quad \text{for every } \mathbb{D} \in \mathrm{Prcnfg}(\mathbb{E}). \tag{3.3}$$

In addition we have $\widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m \preceq \mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m = (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_m$. Thereby it follows $(\vec{R}_{\mathbb{M}})_{<m} \preceq (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_{<m}$ by Lemma 3.1.6 ①. Thus the induction hypothesis applied to $\mathbb{Z}$ with $\mathbb{E} = \mathbb{M}$ provides

$$(\vec{R}_{\mathbb{M}})^{\mathbb{D}}_{<m} = (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})^{\mathbb{D}}_{<m} \quad \text{for every } \mathbb{D} \in \mathrm{Prcnfg}(\mathbb{M}). \tag{3.4}$$

Since $\mathrm{Prcnfg}(\mathbb{E}) \subseteq \mathrm{Prcnfg}(\mathbb{M})$ it follows by means of (3.3) and (3.4) plus taking into account Lemma 2.3.2 ③(a) that $(\vec{R}_{\mathbb{E}})_{<m} \preceq (\vec{R}_{\mathbb{M}})_{<m}$. Therefore we have $(\vec{R}_{\mathbb{E}})_{\leq g} \preceq (\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{[g,m]}, (\vec{R}_{\mathbb{M}})_{<m} = (\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{\leq g}$. Thus the induction hypothesis applied to $\mathbb{M}(\vec{\nu})$ yields

$$(\vec{R}_{\mathbb{E}})^{\mathbb{D}}_{[g,m]} = (\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})^{\mathbb{D}}_{[g,m]} = (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})^{\mathbb{D}}_{[g,m]} \quad \text{for every } \mathbb{D} \in \mathrm{Prcnfg}(\mathbb{E}). \tag{3.5}$$

Moreover it holds $(\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})^{\mathbb{D}}_{<m} = \vec{R}^{\mathbb{D}} = (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})^{\mathbb{D}}_{<m}$ for every $\mathbb{D} \in \mathrm{Prcnfg}(\mathbb{E})$. Thereby the claim follows by composing the equations (3.5) and (3.3). □

Lemma 3.2.3. *Let* $\mathbb{X} \neq (\mathrm{i}(\mathbb{X}); \mathsf{P}_0; \ldots)$ *be a reflection instance and suppose* $\Psi^{\alpha}_{\mathbb{X}}$ *is well-defined and* $\mathrm{rdh}^*(\mathbb{X}) := \max\{0, \mathrm{rdh}(\mathbb{X})\}$. *Let* $\mathbb{E} \in \overline{\mathrm{Prcnfg}}(\mathbb{X})$ *be a reflection configuration and* $\mathrm{o}(\mathbb{E}) \leq \varepsilon \leq \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{E})$ *plus* $e \in \mathrm{Rdh}(\mathbb{E})$. *Then it holds for any* κ, *such that* $C(\alpha, \kappa) \cap \mathrm{i}(\mathbb{X}) = \kappa$ *and* $\mathbb{X}, \alpha \in C(\kappa)$

$$\kappa \models \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e, (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{<e} \;\&\; \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e \not\preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_e \quad \Rightarrow \quad \kappa \textit{ is } \mathfrak{M}^{\alpha}_{\mathbb{X}}\text{-}\Pi^1_{\mathrm{rdh}^*(\mathbb{X})}\textit{-indescribable.}$$

Proof. We proceed by induction on $\mathrm{o}(\mathbb{X})$. In the following let $\mathbb{F}$ be the reflection configuration of $\mathbb{X}$ and let F be a $\Pi^1_{\mathrm{rdh}^*(\mathbb{X})}$-sentence in parameters $\vec{P} \subseteq V_\kappa$, such that $\langle V_\kappa, \vec{P}\rangle \models F$.

Case 1, $\mathbb{X} = \mathbb{A}(m+1)$ *for some* $m < \omega$: Then $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = (\mathsf{M}^{<\alpha}_{\mathbb{A}}\text{-}\mathsf{P}_m, \ldots, \mathsf{M}^{<\alpha}_{\mathbb{A}}\text{-}\mathsf{P}_m)$. Since $\mathbb{E} \in \overline{\mathrm{Prcnfg}}(\mathbb{X})$ and $\varepsilon \leq \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{E})$ it holds $\mathbb{E} = \mathbb{A}$ and $\varepsilon \leq \alpha$. Due to $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e \not\preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_e$ we must have $e > m$. Therefore $\kappa \models \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e, (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{<e}$ implies that κ is Π^1_e-indescribable and $\kappa \models (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{<e}$. Proceeding as in the proof of Theorem 2.3.11 it follows the existence of a $\kappa_0 < \kappa$, such that $C(\alpha, \kappa_0) \cap \mathrm{i}(\mathbb{X}) = \kappa_0$, $\mathbb{X}, \alpha \in C(\kappa_0)$, κ_0 is Π^1_m-indescribable, and $\kappa_0 \models \mathsf{M}^{<\alpha}_{\mathbb{A}}\text{-}\mathsf{P}_m$ plus $\langle V_{\kappa_0}, \vec{P} \cap V_{\kappa_0}\rangle \models F$. Thus κ is $\mathfrak{M}^{\alpha}_{\mathbb{X}}\text{-}\Pi^1_{\mathrm{rdh}^*(\mathbb{X})}$-indescribable.

Case 2, $\mathbb{X} = (\Psi^{\delta}_{\mathbb{Z}}; \mathsf{P}_m; \vec{R}; \ldots)$ *with* $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\omega}_{\mathbb{A}}\text{-}\mathsf{P}_m, \vec{R})$: Then it holds $m > 0$ and $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = (\widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}, \vec{R}_{<m-1})$.

Subcase 2.1, $e < m$: If $\mathbb{E} = \mathbb{F}$ we have $e = m-1$ and $\varepsilon \leq \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{F}) = \alpha$. However this leads to the contradiction $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e = \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1} \preceq \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1} = (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_e$.

Therefore we must have $\mathbb{E} \neq \mathbb{F}$. Then it holds $\mathbb{E} \in \overline{\mathrm{Prcnfg}}(\mathbb{Z})$ and by means of Lemma 3.1.5 we have $C(\delta, \kappa) \cap \mathrm{i}(\mathbb{Z}) = \kappa$ and $\mathbb{Z}, \delta \in C(\kappa)$. Moreover it holds $\mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{E}) = \mathrm{ran}^{\delta}_{\mathbb{Z}}(\mathbb{E})$. The provisos $\kappa \models \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e, (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{<e}$ and $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e \not\preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_e$ imply $\kappa \models \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e, (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_{<e}$ and $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e \not\preceq (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_e$. Thus the induction hypothesis provides that κ is $\mathfrak{M}^{\delta}_{\mathbb{Z}}\text{-}\Pi^1_{\mathrm{rdh}^*(\mathbb{Z})}$-indescribable, but this is absurd, since the proviso $C(\alpha, \kappa) \cap \mathrm{i}(\mathbb{X}) = \kappa$ implies $\kappa \leq \mathrm{i}(\mathbb{X}) = \Psi^{\delta}_{\mathbb{Z}}$. Therefore we must $e \geq m$.

Subcase 2.2, $e \geq m$: Then the proviso $\kappa \models \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e, (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{<e}$ implies that κ is Π^1_m-indescribable and $\kappa \models \vec{R}_{\Psi^{\alpha}_{\mathbb{X}}}$. Thereby there is a $\kappa_0 < \kappa$ such that $C(\alpha, \kappa_0) \cap \mathrm{i}(\mathbb{X}) = \kappa$, $\mathbb{X}, \alpha \in C(\kappa_0)$, $\kappa_0 \models \vec{R}_{\Psi^{\alpha}_{\mathbb{X}}}$ and $\langle V_{\kappa_0}, \vec{P} \cap V_{\kappa_0}\rangle \models F$. Thus Theorem 3.1.6 ② provides that κ is $\mathfrak{M}^{\alpha}_{\mathbb{X}}\text{-}\Pi^1_{\mathrm{rdh}^*(\mathbb{X})}$-indescribable.

Case3, $\mathbb{X} = \mathbb{F}(\xi, \vec{\nu}) = (\Psi^{\delta}_{\mathbb{Z}}; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m; \vec{R}; \ldots)$ *with* $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m, \vec{R})$ *for some* $\gamma > \xi$*:* Then it holds $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{>m}, \widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}, \vec{R}_{<m-1})$.

Subcase 3.1, $e < m$*:* Then we have $e \geq 0$, since $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{-1} = \epsilon \preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{-1} = \epsilon$. Thus the claim follows by the same considerations as in the second case.

Subcase 3.2, $e = m$*:* Since $\mathrm{Rdh}(\mathbb{F}) = \{m-1\}$ and $e \in \mathrm{Rdh}(\mathbb{E})$ it follows $\mathbb{E} \neq \mathbb{F}$. Thus we cannot have $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e \not\preceq (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_e = \widetilde{\mathsf{M}}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m$, as otherwise we would obtain a contradiction as in subcase 2.1. Therefore we have $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e \not\preceq \widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m$, but $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e \preceq \mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m$. Thus it follows $\xi < \varepsilon \leq \gamma$ and $\mathbb{E} = \mathbb{M}$. Thereby κ is $\mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\Pi^1_m$-indescribable and $\kappa \models (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{<m}$. Therefore there exists a $\kappa_0 < \mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})} \cap \kappa$, such that $C(\alpha, \kappa_0) \cap \mathrm{i}(\mathbb{X}) = \kappa_0$, $\mathbb{X}, \alpha \in C(\kappa_0)$, $\kappa_0 \models \vec{R}$ and $\kappa_0 \models \mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}$ plus $\langle V_{\kappa_0}, \vec{P} \cap V_{\kappa_0}\rangle \models F$. Thus κ is $\mathfrak{M}^{\alpha}_{\mathbb{X}}\text{-}\Pi^1_{\mathrm{rdh}^*(\mathbb{X})}$-indescribable.

Subcase 3.3, $m < e \leq \mathrm{rd}(\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})$*:* Let $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e = \mathsf{M}^{<\varepsilon_0}_{\mathbb{E}_0}\text{-}\mathsf{P}_e$. If $\varepsilon_0 = \mathrm{o}(\mathbb{E}_0)$ we have $\mathbb{E}_0 = \mathbb{A}$ and $\varepsilon_0 = \omega$. However, then we obtain the contradiction $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e \preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_e$. Thus we must have $\varepsilon_0 > \mathrm{o}(\mathbb{E}_0)$. Since $m-1 < e \in \mathrm{Rdh}(\mathbb{E}_0)$ we also have $(\mathrm{i}(\mathbb{E}_0); \mathsf{P}_0; \ldots) \neq \mathbb{E}_0 \neq \mathbb{F}$.

Subcase 3.3.1, $\mathbb{M} \notin \mathrm{Prcnfg}(\mathbb{E}_0)$*:* Then it holds $\mathbb{E}_0 \in \overline{\mathrm{Prcnfg}}(\mathbb{M})$ and $\mathbb{M} \neq \mathbb{A}$. By the provisos we have $\kappa \models (\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{(e,m]}, (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{<m}$. Moreover it holds $\widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_m \preceq \mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_m = (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_m$. Thus Theorem 3.1.6 ① provides that $(\vec{R}_{\mathbb{M}})_{<m} \preceq (\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}})_{<m} \preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{<m}$. Therefore $\kappa \models (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{<m}$ implies $\kappa \models (\vec{R}_{\mathbb{M}})_{<m}$ due to Lemma 3.1.4. Since it holds $(\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{<e} = (\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{(e,m]}, (\vec{R}_{\mathbb{M}})_{<m}$ we have

$$\kappa \models \mathsf{M}^{<\varepsilon_0}_{\mathbb{E}_0}\text{-}\mathsf{P}_e, (\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{<e} \quad \& \quad \mathsf{M}^{<\varepsilon_0}_{\mathbb{E}_0}\text{-}\mathsf{P}_e \not\preceq (\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_e, \tag{3.6}$$

and $\mathbb{E}_0 \in \overline{\mathrm{Prcnfg}}(\mathbb{M})$ plus $\varepsilon_0 \leq \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{E}_0)$ by means of Lemma 2.3.2 ③(b).

If $\mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{E}_0) \geq \varepsilon_0 > \mathrm{ran}^{\xi}_{\mathbb{M}(\vec{\nu})}(\mathbb{E}_0)$ it holds $\mathbb{E}_0 = \mathbb{M}$ and $\varepsilon_0 > \xi$. Thus κ is $\mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\Pi^1_m$-indescribable, since $e > m$.

If $\varepsilon_0 \leq \mathrm{ran}^{\xi}_{\mathbb{M}(\vec{\nu})}(\mathbb{E}_0)$ we also have $C(\xi, \kappa) \cap \mathrm{i}(\mathbb{M}(\vec{\nu})) = \kappa$ and $\xi, \mathbb{M}(\vec{\nu}) \in C(\kappa)$ by means of Lemma 3.1.5 ①. Thus we obtain from (3.6) and the induction hypothesis that κ is $\mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\Pi^1_m$-indescribable, as $\mathrm{rdh}(\mathbb{M}(\vec{\nu})) \geq m$.

In addition it holds $C(\alpha, \kappa) \cap \mathrm{i}(\mathbb{X}) = \kappa$, $\mathbb{X}, \alpha \in C(\kappa)$, $\kappa \models \vec{R}$, $\kappa \models \mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}$ and $\langle V_{\kappa}, \vec{P}\rangle \models F$. All these statements are expressible by a Π^1_m-sentence. Thereby κ is $\mathfrak{M}^{\alpha}_{\mathbb{X}}\text{-}\Pi^1_{\mathrm{rdh}^*(\mathbb{X})}$-indescribable.

Subcase 3.3.2, $\mathbb{M} \in \mathrm{Prcnfg}(\mathbb{E}_0)$*:* At first we observe that $\mathbb{E}_0 \neq \mathbb{A}$, as otherwise we would have $\mathbb{M} \notin \mathrm{Prcnfg}(\mathbb{E}_0)$. Thus we have $e = \mathrm{rd}(\vec{R}_{\mathbb{E}_0})$ due to Lemma 2.3.2 ③(g).

We proceed by subsidiary induction on $d := e - m$. The initial step of the subsidiary induction is treated in subcase 3.2. In the successor step we have $e = m + d_0 + 1 > m$. Since we assume $\varepsilon_0 > \mathrm{o}(\mathbb{E}_0)$ the proviso $\kappa \models \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_e, (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{<e}$ implies that κ is $\mathfrak{M}^{\mathrm{o}(\mathbb{E}_0)}_{\mathbb{E}_0(\vec{\eta})}\text{-}\Pi^1_e$-indescribable (for some $\vec{\eta} \in \mathrm{dom}(\mathbb{E}_0) \cap \mathbb{X}$—which does exist due to Lemma

2.3.2 since $\mathbb{E}_0 \in \text{Prcnfg}(\mathbb{X})$) and thereby $\kappa \models \vec{R}_{\mathbb{E}_0}$, as any element of $\mathfrak{M}^{\text{o}(\mathbb{E}_0)}_{\mathbb{E}_0(\vec{\eta})}$ satisfies $\vec{R}_{\mathbb{E}_0}$ and $\text{rd}(\vec{R}_{\mathbb{E}_0}) = e$ by Definition 2.2.4.

Let us assume that $(\vec{R}_{\mathbb{E}_0})_{<e} \preceq (\vec{R}_{\Psi^\alpha_\mathbb{X}})_{<e}$. Since $\text{o}(\mathbb{E}_0) < \text{o}(\mathbb{F})$ this implies $(\vec{R}_{\mathbb{E}_0})_{\leq m} \preceq (\vec{R}_{\Psi^\delta_\mathbb{Z}})_{\leq m}$. Moreover we have $\mathbb{M} \in \text{Prcnfg}(\mathbb{E}_0)$, $\mathbb{E}_0 \in \overline{\text{Prcnfg}}(\mathbb{Z})$, since $\mathbb{E} \in \text{Prcnfg}(\mathbb{F})$, and $m \leq \min\{\text{rd}(\vec{R}_{\mathbb{E}_0}), \text{rd}(\vec{R}_{\Psi^\delta_\mathbb{Z}})\}$. Thus we obtain the following contradiction by means of Lemma 3.2.2

$$(\vec{R}_{\mathbb{E}_0})^\mathbb{M}_m \preceq (\vec{R}_{\Psi^\alpha_\mathbb{X}})_m = \mathsf{M}^{<\xi}_\mathbb{M}\text{-}\mathsf{P}_m \prec \mathsf{M}^{<\gamma}_\mathbb{M}\text{-}\mathsf{P}_m = (\vec{R}_{\Psi^\delta_\mathbb{Z}})^\mathbb{M}_m \overset{3.2.2}{=} (\vec{R}_{\mathbb{E}_0})^\mathbb{M}_m.$$

Therefore we must have $(\vec{R}_{\mathbb{E}_0})_{<e} \not\preceq (\vec{R}_{\Psi^\alpha_\mathbb{X}})_{<e}$. Let $\vec{R}_{\mathbb{E}_0} = (\mathsf{M}^{<\varepsilon_1}_{\mathbb{E}_1}\text{-}\mathsf{P}_{e_1}, \ldots, \mathsf{M}^{<\varepsilon_l}_{\mathbb{E}_l}\text{-}\mathsf{P}_{e_l})$. Then there exists a $1 < j \leq l$ such that $\mathsf{M}^{<\varepsilon_j}_{\mathbb{E}_j}\text{-}\mathsf{P}_{e_j} \not\preceq (\vec{R}_{\Psi^\alpha_\mathbb{X}})_{e_j}$ and furthermore $\kappa \models \mathsf{M}^{<\varepsilon_j}_{\mathbb{E}_j}\text{-}\mathsf{P}_{e_j}, (\vec{R}_{\Psi^\alpha_\mathbb{X}})_{<e_j}$. In addition it holds by Lemma 2.3.2 that $\mathbb{E}_j \in \text{Prcnfg}(\mathbb{E}_0) \subseteq \text{Prcnfg}(\mathbb{F})$ and $\varepsilon_j < \text{ran}^\alpha_{\mathbb{E}_0}(\mathbb{E}_j) = \text{ran}^\alpha_\mathbb{X}(\mathbb{E}_j)$. Thus κ is $\mathfrak{M}^\alpha_\mathbb{X}$-$\Pi^1_{\text{rdh}^*(\mathbb{X})}$-indescribable either by subcase 3.1 if $e_j < m$, or by subcase 3.2 if $e_j = m$, or by subcase 3.3.1, if $m \leq e_j < e$ and $\mathbb{M} \notin \text{Prcnfg}(\mathbb{E}_j)$, or by the subsidiary induction hypothesis, if $m \leq e_j < e$ and $\mathbb{M} \in \text{Prcnfg}(\mathbb{E}_j)$.

Subcase 3.4, $e > \text{rd}(\vec{R}_{\Psi^\xi_{\mathbb{M}(\vec{\nu})}})$: Then the claim follows just like in the first case. □

Theorem 3.2.4 (Domination). *Let $\mathbb{X}$ and $\mathbb{Y} \neq (\text{i}(\mathbb{Y}); \mathsf{P}_0; \ldots)$ be reflection instances and suppose $\Psi^\alpha_\mathbb{X}$ is well-defined plus $\beta \in \text{ON}$. Then it holds*

$$\Psi^\alpha_\mathbb{X} \in \mathfrak{M}^\beta_\mathbb{Y} \quad \Rightarrow \quad \vec{R}_{\Psi^\beta_\mathbb{Y}} \preceq \vec{R}_{\Psi^\alpha_\mathbb{X}}.$$

Proof. Let us at first assume that $\mathbb{X} = (\text{i}(\mathbb{X}); \mathsf{P}_0; \ldots)$. Since $\Psi^\alpha_\mathbb{X} \in \mathfrak{M}^\beta_\mathbb{Y}$ it follows that $\Psi^\alpha_\mathbb{X}$ is Π^1_0-indescribable, and thereby $\Psi^\alpha_\mathbb{X}$ is a regular cardinal by means of Lemma 2.1.2. Since $\mathbb{X}, \alpha \in C(\Psi^\alpha_\mathbb{X})$ Lemma 2.3.5 ② provides a $\kappa_0 < \Psi^\alpha_\mathbb{X}$, such that $C(\alpha, \kappa_0) \cap \text{i}(\mathbb{X}) = \kappa_0$ and $\mathbb{X}, \alpha \in C(\kappa_0)$, i.e. $\kappa_0 \in \mathfrak{M}^\alpha_\mathbb{X} \cap \Psi^\alpha_\mathbb{X}$. However, this contradicts the minimality of $\Psi^\alpha_\mathbb{X}$.

Thus we must have $\mathbb{X} \neq (\text{i}(\mathbb{X}); \mathsf{P}_0; \ldots)$. Let $\mathbb{G}$ be the reflection configuration of $\mathbb{Y}$. It follows by Theorem 3.1.1 that $\mathbb{G} \in \overline{\text{Prcnfg}}(\mathbb{X})$ and $\beta \leq \text{ran}^\alpha_\mathbb{X}(\mathbb{G})$. Let $\vec{R}_{\Psi^\beta_\mathbb{Y}} = (\mathsf{M}^{<\xi_1}_{\mathbb{R}_1}\text{-}\mathsf{P}_{r_1}, \ldots, \mathsf{M}^{<\xi_k}_{\mathbb{R}_k}\text{-}\mathsf{P}_{r_k})$. Then it follows by means of Lemma 2.3.2 for all $1 \leq i \leq k$ that $\mathbb{R}_i \in \overline{\text{Prcnfg}}(\mathbb{G}) \subseteq \overline{\text{Prcnfg}}(\mathbb{X})$ and $\xi_i \leq \text{ran}^\beta_\mathbb{G}(\mathbb{R}_i) \leq \text{ran}^\alpha_\mathbb{X}(\mathbb{R}_i)$. To obtain a proof by contradiction let us assume $\vec{R}_{\Psi^\beta_\mathbb{Y}} \not\preceq \vec{R}_{\Psi^\alpha_\mathbb{X}}$. Then there exists a $1 \leq j \leq k$ such that

$$\mathsf{M}^{<\xi_j}_{\mathbb{R}_j}\text{-}\mathsf{P}_{r_j} \not\preceq (\vec{R}_{\Psi^\alpha_\mathbb{X}})_{r_j}. \tag{3.7}$$

Let $\kappa := \Psi^\alpha_\mathbb{X}$. Then it holds $C(\alpha, \kappa) \cap \text{i}(\mathbb{X}) = \kappa$, $\mathbb{X}, \alpha \in C(\kappa)$ and $\kappa \models \vec{R}_{\Psi^\alpha_\mathbb{X}}$, since $\kappa \in \mathfrak{M}^\alpha_\mathbb{X}$. Moreover we have $\kappa \models \vec{R}_{\Psi^\beta_\mathbb{Y}}$ as $\kappa \in \mathfrak{M}^\beta_\mathbb{Y}$. Therefore it holds

$$\kappa \models \mathsf{M}^{<\xi_j}_{\mathbb{R}_j}\text{-}\mathsf{P}_{r_j}, (\vec{R}_{\Psi^\alpha_\mathbb{X}})_{<r_j}. \tag{3.8}$$

Thus it follows from (3.7) and (3.8) and Lemma 3.2.3 that there is a $\kappa_0 \in \mathfrak{M}^\alpha_\mathbb{X} \cap \kappa$. Since this contradicts the minimality of $\kappa = \Psi^\alpha_\mathbb{X}$, we must have $\vec{R}_{\Psi^\beta_\mathbb{Y}} \preceq \vec{R}_{\Psi^\alpha_\mathbb{X}}$. □

4. A semi-formal Calculus for Π_ω-Ref

In this chapter we augment the Tait-language $\mathcal{L}_\in^T$ of set theory by terms, representing elements of the constructible hierarchy $\mathbb{L}$, whose $\mathbb{L}$-rank is an ordinal of $\mathsf{T}(\Xi)$. Moreover we augment $\mathcal{L}_\in^T$ by predicates $M_{\mathbb{X}}^\xi$, which are intended to represent the sets $\{\mathbb{L}_\kappa \mid \kappa \in \mathfrak{M}_{\mathbb{X}}^\xi\}$.

Straight forwardly it is possible to define an infinitary derivation calculus for this augmented language, which is correct (respecting $\mathbb{L}$) for cut-free derivations of formulae whose "ordinal parameters" belong to the transitive part of $\mathsf{T}(\Xi)$. Moreover by employing a reflection rule, such a calculus is strong enough to derive all axioms of Π_ω-Ref.

Inspired by this we define an infinitary calculus which features the mentioned items, but also allows derivations restricted to subsets of $\mathsf{T}(\Xi)$, i.e. all derivations-lengths and all ordinal parameters of the formulae of such derivations have to belong to a certain subset of $\mathsf{T}(\Xi)$. This restriction is arranged in a way, that if a finite set Γ of sentences is derivable on a subset $\mathcal{H}_1$ and it holds $\mathcal{H}_1 \subseteq \mathcal{H}_2$, then Γ is also derivable on $\mathcal{H}_2$.

4.1. Ramified Set Theory

Definition 4.1.1. We augment the Tait language $\mathcal{L}_\in^T$ of set theory (for which we regard equality as defined, i.e. $x = y$ is an abbreviation for $\forall z \in x\ (z \in y) \land \forall z \in y\ (z \in x)$) by new unary predicate symbols $(\neg)_\tau M_{\mathbb{X}}^\alpha$ for every reflection instance $\mathbb{X}$ and every $\alpha \geq \mathrm{o}(\mathbb{X})$ and every $\tau < \mathrm{i}(\mathbb{X})$. The augmented language is denoted by $\mathcal{L}_{M(\Xi)}$.

Notation. We write $\forall x \in z\, F(x)$ for $\forall x\, (x \in z \to F(x))$ and $\exists x \in z\, F(x)$ for $\exists x\, (x \in z \land F(x))$ and call such quantifiers restricted. Moreover we write $F(x)^z$ for the $\mathcal{L}_{M(\Xi)}$-formula $F(x)$ in which every unrestricted quantifier is restricted to z.

By $\neg F$ we denote the $\mathcal{L}_{M(\Xi)}$-formula which arises from F by putting $\neg$ in front of each atomic subformula (by use of the De Morgan's laws) and then dropping double negations and replacing $\neg(s \in t)$ by $s \notin t$.

Definition 4.1.2 (The Language $\mathcal{L}_{RS(\Xi)}$)**.** The $\mathcal{L}_{RS(\Xi)}$-terms and their stages are defined as follows:

- for each $\alpha \leq \Xi$, L_α is an $\mathcal{L}_{RS(\Xi)}$-term of stage α.
- the formal expression $\{x \in L_\alpha \mid F(x, s_1, \ldots, s_n)^{L_\alpha}\}$ is an $\mathcal{L}_{RS(\Xi)}$-term of stage $\alpha < \Xi$, if $F(x, y_1, \ldots, y_n)$ is an $\mathcal{L}_{M(\Xi)}$-formula and $s_1, \ldots, s_n$ are $\mathcal{L}_{RS(\Xi)}$-terms with stages less than α.

By $\mathcal{T}_\alpha$ we denote the set of $\mathcal{L}_{RS(\Xi)}$-terms with stages less than α. The $\mathcal{L}_{RS(\Xi)}$-sentences are the expressions of the form $F(s_1,\dots,s_n)^{L_\Xi}$, where $F(x_1,\dots,x_n)$ is an $\mathcal{L}_{M(\Xi)}$-formula which contains at most the shown free variables $x_1,\dots,x_n$ and $s_1,\dots,s_n$ are $\mathcal{L}_{RS(\Xi)}$-terms.

We define $\mathrm{lh}(F(s_1,\dots,s_n)^{L_\Xi})$ as the number of occurrences of the connectives $\vee$ and $\wedge$ in $F(x_1,\dots,x_n)^z$.

Notation. In the sequel, $\mathcal{L}_{RS(\Xi)}$-sentences are referred to as sentences. The same usage applies to $\mathcal{L}_{RS(\Xi)}$-terms.

Definition 4.1.3 (The transfinite content of ordinals, terms and sentences)**.** For ordinals we define[1]

$$\begin{aligned} \mathrm{k}(0) := \mathrm{k}(1) &:= \emptyset, \\ \mathrm{k}(\alpha) &:= \{\alpha\}, \text{ if } 0 \neq \alpha \neq 1. \end{aligned}$$

For terms we define by recursion on α:

$$\begin{aligned} \mathrm{k}(L_\alpha) &:= \{\alpha\}, \\ \mathrm{k}(\{x \in L_\alpha \mid F(x, s_1, \dots, s_n)^{L_\alpha}\}) &:= \{\alpha\} \cup \bigcup_{i=1}^{n} \mathrm{k}(s_i). \end{aligned}$$

For a sentence F we define by recursion on the complexity of F:

$$\begin{aligned} \mathrm{k}(s \in t) &:= \mathrm{k}(s) \cup \mathrm{k}(t), \\ \mathrm{k}({}_\tau M^\alpha_{\mathbb{X}}(s)) &:= \{\tau, \alpha\} \cup \mathrm{par}\, \mathbb{X} \cup \mathrm{k}(s), \\ \mathrm{k}(F_0 \vee F_1) &:= \mathrm{k}(F_0) \cup \mathrm{k}(F_1), \\ \mathrm{k}(\exists x \in t\; G(x)) &:= \mathrm{k}(t) \cup \mathrm{k}(G(L_0)), \\ \mathrm{k}(\neg G) &:= \mathrm{k}(G). \end{aligned}$$

If ϕ is a term or a sentence, we set $|\phi| := \max(\mathrm{k}(\phi))$.

If $\mathcal{A}$ is a finite set consisting of ordinals, terms and sentences, we put

$$\mathrm{k}(\mathcal{A}) := \bigcup_{\phi \in \mathcal{A}} \mathrm{k}(\phi) \quad \text{and} \quad |\mathcal{A}| := \sup\{|\phi| \mid \phi \in \mathcal{A}\}.$$

Since we want to apply reflection rules to sentences of the shape $\bigwedge \Gamma$, where Γ is a finite set of Π_n-sentences, we close the set of Π_n-sentences under the Boolean connectives $\wedge$ and $\vee$.

[1] We define the transfinite content of ordinals just for technical convenience.

Definition 4.1.4. A sentence is said to be $\Delta_0(\pi)$ if it contains only terms with stages less than π. A sentence F is elementary-$\Pi_n(\pi)$ if it has the form

$$\forall x_1 \in L_\pi \dots Q_n x_n \in L_\pi \; F(x_1, \dots, x_n),$$

where the n quantifiers in front are alternate and $F(L_0, \dots, L_0)$ is $\Delta_0(\pi)$. Analogously we define elementary-$\Sigma_n(\pi)$-sentences.

The set of $\Pi_n(\pi)$-sentences ($\Sigma_n(\pi)$-sentences, resp.) is the smallest set of $\mathcal{L}_{RS(\Xi)}$-sentences which contains the elementary-$\Pi_k(\pi)$-sentences (elementary-$\Sigma_k(\pi)$-sentences, resp.) for $k \leq n$ plus the elementary-$\Sigma_j(\pi)$-sentences (elementary-$\Pi_j(\pi)$-sentences, resp.) for $j < n$ and is closed under the connectives $\wedge$ and $\vee$.

By $\Delta^1_0(\pi)$ we denote the smallest class of sentences, which comprises the class $\Pi_n(\pi)$, for any $n \in \omega$.

Notation. In the following we refer to a $\Pi_{n+1}(\pi)$ sentence F as $F(F_1, \dots, F_m)$, where $F_1, \dots, F_m$ denote exactly the elementary-$\Pi_{n+1}(\pi)$ and elementary-$\Pi_n(\pi)$ sub-sentences of F and for $1 \leq i \leq m$ it holds $F_i \equiv \forall x_i \in L_\pi \; F'_i(x_i)$. Thus we have $F(F'_1(t_1), \dots F'_m(t_m)) \in \Sigma_n(\pi)$ for all $\vec{t} = (t_1, \dots, t_m) \in \mathcal{T}^m_\pi$.

Given a sentence F and terms s, t, we denote by $F^{(s,t)}$ the sentence which arises from F by replacing all restricted quantifiers $\forall x \in s$ and $\exists x \in s$ by $\forall x \in t$ and $\exists x \in t$, respectively. We also write $F^{(s,\pi)}$ for $F^{(s,L_\pi)}$ and $\forall x^\pi$ for $\forall x \in L_\pi$.

If $F(s_1, \dots, s_r) \in \Pi_n(\pi)$, where $s_1, \dots, s_r$ denote all terms of F of levels less than π we define

$$z \models F(s_1, \dots, s_r) \quad \Leftrightarrow \quad z \neq \emptyset \wedge \mathrm{Tran}(z) \wedge \bigwedge_{i=1}^{r} (s_i \in z) \wedge F^{(\pi,z)}.$$

4.2. Semi-formal Derivations on Hull-Sets of $\mathsf{T}(\Xi)$

Now we define an infinitary derivation calculus for sentences of $\mathcal{L}_{RS(\Xi)}$, which "acts" on subsets of $\mathsf{T}(\Xi)$. As we want the following persistency property

$$(\mathsf{Str}) \quad \mathcal{H}_1 \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \;\&\; \mathcal{H}_1 \subseteq \mathcal{H}_2 \quad \Rightarrow \quad \mathcal{H}_2 \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma$$

we have to relativize the parameter restriction in the $(\forall)$-rule.

It will transpire that the axioms of Π_ω-Ref are already derivable on sets which satisfy minimal closure conditions. In the following we denote sets, which meet such closure conditions as hull-sets.

Definition 4.2.1. In the following we call a (partial) function $f : (\mathsf{T}(\Xi))^n \longrightarrow_{\mathrm{p}} \mathsf{T}(\Xi)$ or a (partial) function $g : (\mathsf{T}(\Xi))^{<\omega} \longrightarrow_{\mathrm{p}} \mathsf{T}(\Xi)$ an ordinal function.

A hull-set is a triple $\langle \mathcal{H}, \mathcal{H}_0, (f_i)_{i \in I} \rangle$, such that $\mathcal{H}, \mathcal{H}_0 \subseteq \mathsf{T}(\Xi)$, $0 \in \mathcal{H}_0$, $(f_i)_{i \in I}$ is a finite family of ordinal functions, $\alpha, \beta \mapsto \alpha + \beta$, $\gamma \mapsto \omega^\gamma \in (f_i)_{i \in I}$ and $\mathcal{H}$ is the closure of $\mathcal{H}_0$ under $(f_i)_{i \in I}$.

Henceforth we often omit the second and third component of a hull-set, i.e. we refer to $\langle \mathcal{H}, \mathcal{H}_0, (f_i)_{i \in I} \rangle$ simply by $\mathcal{H}$.

Let $\langle \mathcal{H}, \mathcal{H}_0, (f_i)_{i\in I}\rangle$ be a hull-set and $\mathcal{A}$ be a finite set containing ordinals, terms and sentences. The we denote by $\mathcal{H}[\mathcal{A}]$ the hull-set $\langle \mathcal{G}, \mathcal{H}_0 \cup \mathrm{k}(\mathcal{A}), (f_i)_{i\in I}\rangle = \langle \mathcal{G}, \mathcal{H} \cup \mathrm{k}(\mathcal{A}), (f_i)_{i\in I}\rangle$.†

Example. For every α and π the set $C(\alpha,\pi)$ can be regarded as the hull-set $\langle C(\alpha,\pi), \pi \cup \{0,\Xi\}, (+,\varphi,\cdot^+,\tilde{\Psi})\rangle$, where $\tilde{\Psi} : (\mathsf{T}(\Xi))^{<\omega} \longrightarrow_{\mathrm{p}} \mathsf{T}(\Xi)$ and

$$\tilde{\Psi}(\gamma,\vec{\eta}) := \begin{cases} \Psi^{\gamma}_{\mathbb{X}} & \text{if } \vec{\eta} = \vec{\mathbb{X}} \text{ for some refl. inst. } \mathbb{X} \text{ and } \mathrm{o}(\mathbb{X}) \le \gamma \in C(\mathrm{i}(\mathbb{X})) \cap \alpha, \\ \uparrow & \text{otherwise.} \end{cases}$$

Corollary 4.2.2. *It holds*

- $\mathcal{A} \subseteq \mathcal{H} \quad \Rightarrow \quad \mathcal{H}[\mathcal{A}] = \mathcal{H}$.
- *Let ϕ be an ordinal, a term or a sentence. Then $\mathcal{H}[\mathcal{A}][\phi] = \mathcal{H}[\mathcal{A},\phi]$.*

Definition 4.2.3. We use $\equiv$ to mean syntactical identity. For terms s,t with $|s| < |t|$ we set

$$s \dot{\in} t \equiv \begin{cases} F(s) & \text{if } t \equiv \{x \in L_\alpha \mid F(x)\} \\ \top & \text{if } t \equiv L_\alpha \end{cases}$$

where $\top$ is not considered as a formula, but we define $\top \wedge F :\equiv F$.‡

Definition 4.2.4. To each sentence F we assign (a possibly infinite) disjunction $\bigvee(F_t)_{t\in T}$ or conjunction $\bigwedge(F_t)_{t\in T}$ of sentences. This assignment is indicated by $F \cong \bigvee(F_t)_{t\in T}$ and $F \cong \bigwedge(F_t)_{t\in T}$, respectively.

$$\begin{aligned} r \in s &\cong \bigvee(t \dot{\in} s \wedge t = r)_{t \in \mathcal{T}_{|s|}}, \\ {}_\tau M^{\alpha}_{\mathbb{X}}(s) &\cong \bigvee(t = s)_{t\in T} \text{ where } T := \{L_\beta \mid \beta \in \mathfrak{M}^{\alpha}_{\mathbb{X}} \ \wedge \ \tau < \beta \le |s|\}, \\ \exists x \in s\, G(x) &\cong \bigvee(t \dot{\in} s \wedge G(t))_{t\in \mathcal{T}_{|s|}}, \\ F_0 \vee F_1 &\cong \bigvee(F_t)_{t\in\{0,1\}}, \\ \neg F &\cong \bigwedge(\neg F_t)_{t\in T}, \text{ if } F \cong \bigvee(F_t)_{t\in T}. \end{aligned}$$

Notation. We use the following notation

$$F_s \in \mathrm{CS}(F) \quad :\Leftrightarrow \quad F \cong \bigvee(F_t)_{t\in T} \ \text{ or } \ F \cong \bigwedge(F_t)_{t\in T} \ \text{ and } \ s \in T.$$

†The reader who is familiar to Buchholz concept of operator-controlled derivations should have realized, that derivations on hull-sets will be the same as operator controlled derivations, controlled by a uniform kind of finite operators, since every hull-set induces a finite operator via $\mathcal{A} \mapsto \mathcal{H}[\mathcal{A}]$.
The notion of "derivations on subsets" simplifies the proof of the Reflection Elimination Theorem. In this proof we directly employ the hull-set $C(\gamma,0)$ instead of defining an operator $\mathcal{H}_\delta$, as e.g. in [Rat94b].

‡This improvement is due to [Buc01] and has the advantage that $\exists x^\pi\, G(x) \cong \bigvee(G(t))_{t\in\mathcal{T}_\pi}$ in contrast to the "usual" setting $s \dot{\in} L_\alpha \equiv s \notin L_0$.

Definition 4.2.5. For terms and sentences we define the rank as follows:

$$\begin{aligned}
\mathrm{rnk}(L_\alpha) &:= \omega\cdot\alpha,\\
\mathrm{rnk}(\{x\in L_\alpha \mid F(x,s_1,\dots,s_n)^{L_\alpha}\}) &:= \max\{\omega\cdot\alpha+1, \mathrm{rnk}(F(L_0))+2\},\\
\mathrm{rnk}(s\in t) &:= \max\{\mathrm{rnk}(s)+6, \mathrm{rnk}(t)+1\},\\
\mathrm{rnk}({}_\tau M^\alpha_{\mathbb{X}}(s)) &:= \mathrm{rnk}(s)+5,\\
\mathrm{rnk}(F_0\vee F_1) &:= \max\{\mathrm{rnk}(F_0),\mathrm{rnk}(F_1)\}+1,\\
\mathrm{rnk}(\exists x\!\in\! t\; G(x)) &:= \max\{\mathrm{rnk}(t), \mathrm{rnk}(F(L_0))+2\},\\
\mathrm{rnk}(\neg G) &:= \mathrm{rnk}(G).
\end{aligned}$$

Lemma 4.2.6. *Let F and G be sentences and s,t be terms. Then the following holds*

- $\mathrm{rnk}(F_t) < \mathrm{rnk}(F)$ *for all* $F_t \in \mathrm{CS}(F)$,
- $\mathrm{k}(t) \subseteq \mathrm{k}(F_t) \subseteq \mathrm{k}(F)\cup\mathrm{k}(t)$ *for all* $t\in T$ *and* $F_t\in\mathrm{CS}(F)$,
- $\mathrm{rnk}(F) = \omega\cdot|F| + n$ *for some* $n<\omega$,
- $\mathrm{rnk}(s) = \omega\cdot|s| + m$ *for some* $m<\omega$,
- $|F| < |G| \;\Rightarrow\; \mathrm{rnk}(F) < \mathrm{rnk}(G)$,
- $|s| < |t| \;\Rightarrow\; \mathrm{rnk}(s) < \mathrm{rnk}(t)$.

Proof. See [Buc93], Lemma 1.9 □

Notation. Let $\mathbb{F}$ be a reflection configuration, then we define

$$\mathrm{dom}(\mathbb{F})^{\geq m} := \{\vec{\eta}\in\mathrm{dom}(\mathbb{F}) \mid \mathrm{rdh}(\mathbb{F}(\vec{\eta}))\geq m\}.$$

Definition 4.2.7. Let $\vec{R} = (\mathsf{M}^{<\xi_1}_{\mathbb{R}_1}\text{-}\mathsf{P}_{r_1},\dots)$ be a finite sequence of M-P-expressions. Then we define

$$\vec{R}' := ((\vec{R}_{\mathrm{i}(\mathbb{R}_1)})_{r_1}, \vec{R}_{<r_1}).$$

Definition 4.2.8. Let $\mathcal{H}$ be a hull-set and let Γ, Δ be finite sets of $\mathcal{L}_{RS(\Xi)}$-sentences. Then we define $\mathcal{H}\,|\!\frac{\alpha}{\rho}\,\Gamma$ by recursion on α via

$$\{\alpha\}\cup\mathrm{k}(\Gamma)\subseteq\mathcal{H}$$

and the following inductive clauses:

(⋁) $$\frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Delta, F_{t_0} \quad \text{for some } t_0 \in T}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, \bigvee(F_t)_{t\in T}} \qquad \alpha_0 < \alpha$$

(⋀) $$\frac{\mathcal{H}[t] \overset{\alpha_t}{\underset{\rho}{\vdash}} \Delta, F_t \quad \text{for all } t \in T}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, \bigwedge(F_t)_{t\in T}} \qquad \alpha_t < \alpha$$

(Cut) $$\frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Delta, F \qquad \mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Delta, \neg F}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta} \qquad \begin{array}{r}\alpha_0 < \alpha \\ \mathrm{rnk}(F) < \rho\end{array}$$

($\Pi_{m+2}(\pi)$-Ref) $$\frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Delta, F}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, \exists z^\pi (z \models F)} \qquad \begin{array}{r}\alpha_0 < \alpha \\ F \in \Pi_{m+2}(\pi)\end{array}$$

if there is a reflection instance $(\pi; \mathsf{P}_m; \ldots)$

(${}_\tau M^\xi_{\mathbb{M}(\vec{\nu})}$-$\Pi_{m+2}(\pi)$-Ref) $$\frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Delta, F}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, \exists z^\pi \big({}_\tau M^\xi_{\mathbb{M}(\vec{\nu})}(z) \wedge z \models F\big)} \qquad \begin{array}{r}\alpha_0 < \alpha \\ F \in \Pi_{m+2}(\pi)\end{array}$$

for all $\tau < \pi$ if there is a reflection instance $(\pi; \mathsf{M}^\xi_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m; \ldots)$

(${}_\tau M^\zeta_{\mathbb{K}(\vec{\eta})}$-$\Pi_{k+2}(\pi)$-Ref) $$\frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Delta, F}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, \exists z^\pi \big({}_\tau M^\zeta_{\mathbb{K}(\vec{\eta})}(z) \wedge z \models F\big)} \qquad \begin{array}{r}\alpha_0 < \alpha \\ F \in \Pi_{k+2}(\pi) \\ \zeta \in C(\pi) \\ \vec{\eta} \in \mathrm{dom}(\mathbb{K})^{\geq k}_{C(\pi)}\end{array}$$

for all $\tau < \pi$, $\mathbb{K}$ and k if there is a refl. inst. $(\pi; \ldots)$, such that $\mathsf{M}^{<\zeta+1}_{\mathbb{K}}\text{-}\mathsf{P}_k \preceq (\vec{R}'_\pi)_k$

Notation. To the last but two and the last but one rule we refer to as “main reflection rules of π”, while the last rule we call a “subsidiary reflection rule of π”.

Notation. Let $\mathcal{H}$ and $\mathcal{H}'$ be hull-sets. By $\mathcal{H} \sqsubseteq \mathcal{H}'$ we denote that $\mathcal{H} \subseteq \mathcal{H}'$ and that for any ordinal function f, $\mathcal{H}'$ is closed under f, if $\mathcal{H}$ is closed under f.

Lemma 4.2.9 (Derived rules of semi-formal derivations on hull-sets of $\mathsf{T}(\Xi)$)**.**

(Hull) $\mathcal{H}[\mathcal{A}] \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \;\&\; \mathcal{A} \subseteq \mathcal{H} \quad \Rightarrow \quad \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma,$

(Str) $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \;\&\; \mathcal{H} \sqsubseteq \mathcal{H}' \;\&\; \alpha < \alpha' \;\&\; \rho < \rho' \;\&\; \Gamma \subseteq \Gamma' \quad \Rightarrow \quad \mathcal{H}'[\alpha', \Gamma'] \overset{\alpha'}{\underset{\rho'}{\vdash}} \Gamma',$

(∨-Ex) $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, F \vee G \quad \Rightarrow \quad \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, F, G,$

(∧-Ex) $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, F \wedge G \quad \Rightarrow \quad \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, F \;\&\; \mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, G,$

(Up-Per) $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, \exists x^\pi F(x) \;\&\; F(L_0) \in \Delta(\pi) \;\&\; \pi < \rho, \gamma \;\&\; \gamma \in \mathrm{SC}$

$\Rightarrow \quad \mathcal{H}[\gamma] \overset{(\alpha+\gamma)\cdot 2}{\underset{\rho}{\vdash}} \Gamma, \exists x^\gamma F(x),$

(∀-Inv) $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, \bigwedge(F_t)_{t\in T} \quad \Rightarrow \quad \forall t \in T\; \mathcal{H}[t] \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, F_t.$

Let F be a $\Pi_{n+1}(\pi)$-sentence, then:

$$\begin{array}{lll}
(\mathsf{E}\text{-}\forall) & \forall \vec{t} \in \mathcal{T}_\pi^m \ \mathcal{H}[\vec{t}] \,|\!\frac{\alpha}{\rho}\, \Gamma, F(F_1'(t_1), \dots, F_m'(t_m)) \ \Rightarrow \ \mathcal{H}[\pi] \,\Big|\!\frac{\alpha+2\cdot\mathrm{lh}(F)}{\rho}\, \Gamma, F, \\
(\mathsf{E}\text{-}\forall\text{-}\mathsf{Inv}) & \mathcal{H} \,|\!\frac{\alpha}{\rho}\, \Gamma, F \quad \Rightarrow \quad \forall \vec{t} \in \mathcal{T}_\pi^m \ \mathcal{H}[\vec{t}] \,|\!\frac{\alpha}{\rho}\, \Gamma, F(F_1'(t_1), \dots, F_m'(t_m)), \\
(\mathsf{E}\text{-}\mathsf{Up}\text{-}\mathsf{Per}) & \mathcal{H} \,|\!\frac{\alpha}{\rho}\, \Gamma, \neg F \,\&\, n = 0 \,\&\, \pi < \rho, \gamma \,\&\, \gamma \in \mathrm{SC} \ \Rightarrow \ \mathcal{H}[\gamma] \,\Big|\!\frac{(\alpha+\gamma)\cdot 2}{\rho}\, \Gamma, \neg F^{(\pi,\gamma)}.
\end{array}$$

Proof. (Hull) holds since $\mathcal{H}[\mathcal{A}] = \mathcal{H}$.

We show (Up-Per) by induction on α. If $\exists x^\pi F(x)$ is not the principal formula of the last inference the claim follows by use of the induction hypothesis and the last inference. If $\exists x^\pi F(x)$ is the principal formula of the last inference and the last inference is ($\bigvee$) then the claim follows by a ($\bigvee$)-inference, since the premise of this inference is also contained in $\mathrm{CS}(\exists x^\gamma F(x))$. Now suppose the last inference is a (Ref)-inference with principal formula $\exists x^\pi F(x)$. Then we have $\mathrm{rnk}(\exists x^\pi F(x)) = \pi$ since $\pi \in \mathrm{SC}$. For an arbitrary sentence G it follows by induction on $\mathrm{rnk}(G)$ that $\mathcal{H}[G] \,\Big|\!\frac{\omega^{\mathrm{rnk}(G)}\cdot 2}{0}\, G, \neg G$. Moreover it follows by induction on α that $\mathcal{H} \,|\!\frac{\alpha}{\rho}\, \forall x^\gamma G(x)$ implies $\mathcal{H}[\pi] \,|\!\frac{\alpha}{\rho}\, \forall x^\pi G(x)$ for all $\pi \leq \gamma$. Thereby we obtain $\mathcal{H}[\gamma] \,|\!\frac{\gamma\cdot 2}{0}\, \exists x^\gamma F(x), \neg(\exists x^\pi F(x))$. Moreover we have $\mathcal{H} \,|\!\frac{\alpha}{\rho}\, \Gamma, \exists x^\pi F(x)$. Thus the claim follows by a cut, since $\pi < \rho$.

The claim (E-$\forall$) follows by induction on the complexity of F by use of ($\vee$-Ex) and ($\wedge$-Ex).

All remaining propositions follow by induction on α. □

4.3. Embedding of Π_ω-Ref

In this section we show that the axioms of Π_ω-Ref and logically valid sentences are derivable on arbitrary hull-sets.

To avoid bothering with derivation lengths we follow the concept of [Buc93] and introduce an intermediate proof calculus which operates on multisets.[2] Derivations of these intermediate calculus are finally easily transformable into derivations on hull-sets by use of the formula rank, extended to finite sets of formulae, as derivation lengths.

Definition 4.3.1 (Axioms of Π_ω-Ref)**.** The language of Π_ω-Ref is the language $\mathcal{L}_\in$ of set theory (with identity).

[2]A finite multiset is the same as a finite family, i.e. a finite set, whose members can have more than one (but at most finitely many) occurrences in this.

The axioms of $\mathsf{\Pi}_\omega$-Ref comprise the following sentences and schemes:

(Ext)	$\forall x\,\forall y\,\forall z\,\big(x = y \to (x \in z \to y \in z)\big)$
(Found)	$\forall \vec{z}\big(\forall x\,(\forall y\!\in\! x\, F(y,\vec{z}) \to F(x,\vec{z})) \to \forall x F(x,\vec{z})\big)$
(Nullset)	$\exists x\,\forall y\,(y \notin x)$
(Pair)	$\forall x\,\forall y\,\exists z\,(x \in z \wedge y \in z)$
(Union)	$\forall x\,\exists z\,\forall y\!\in\! x\,\forall u\!\in\! y\,(u \in z)$
(Δ_0-Sep)	$\forall \vec{z}\,\forall w\,\exists y\,\big(\forall x\!\in\! y(x \in w \wedge F(x,\vec{z})) \wedge \forall x\!\in\! w(F(x,\vec{z}) \to x \in y)\big)$ $(F \in \Delta_0)$
(Refl)	$\forall \vec{z}\,\big(F(\vec{z}) \to \exists x(x \models F(\vec{z}))\big)$ $\quad (F \in \Pi_n,\ n < \omega)$

Remark. The two omitted axioms of $\mathsf{KP}\omega$

(Inf)	$\exists x\,\big(\exists y\,(y \in x) \wedge \forall y\!\in\! x\,\exists z\!\in\! x\,(y \in z)\big)$
(Δ_0-Col)	$\forall \vec{z}\,\forall w\,\big(\forall x\!\in\! w\,\exists y\,F(x,y,\vec{z}) \to \exists w_1\,\forall x\!\in\! w\,\exists y\!\in\! w_1 F(x,y,\vec{z})\big)$ $\quad (F \in \Delta_0)$

are easily derivable by use of (Nullset), (Pair), (Union) and (Refl). For details confer [Poh09], section 11.8. Therefore $\mathsf{KP}\omega$ is a subtheory of $\mathsf{\Pi}_\omega$-Ref.

Definition 4.3.2 (The calculus $RS^\star$)**.** We define $RS^\star$ as the collection of all derivations of finite multisets of $\mathcal{L}_{RS(\Xi)}$-sentences generated by the following two inference rules

$$(\bigvee)^\star \qquad \frac{\Lambda, F_{t_1}, \ldots F_{t_n}}{\Lambda, \bigvee(F_t)_{t\in T}} \quad \text{if } t_1,\ldots,t_n \in T \text{ and } \mathrm{k}(t_1,\ldots,t_n) \subseteq \mathrm{k}(\Lambda, \bigvee(F_t)_{t\in T})^\star$$

$$(\bigwedge)^\star \qquad \frac{\Lambda, F_t \quad \text{for all } t \in T}{\Lambda, \bigwedge(F_t)_{t\in T}}$$

where for $S \subseteq \mathrm{ON}$ we set $S^\star := S \cup \{\xi + 1 \mid \xi \in S\} \cup \{\omega\}$.

We denote by $\overset{\star}{\vdash} \Lambda$ that there is an $RS^\star$-derivation of Λ.

Notation. In the following we write $s \subseteq t$ for the sentence $\forall x\!\in\! s\,(x \in t)$.

Lemma 4.3.3. *Let s, t be terms and F a sentence. Then*

❶ $\overset{\star}{\vdash} F, \neg F$

❷ $\overset{\star}{\vdash} s \notin s$

❸ $\overset{\star}{\vdash} s \subseteq s$

❹ $\overset{\star}{\vdash} s \dot{\notin} t, s \dot{\in} t$ *for* $s \in \mathcal{T}_{|t|}$

❺ $\overset{\star}{\vdash} s \neq t, t = s$

❻ $\overset{\star}{\vdash} s \dot{\in} L_\tau$ *and* $\overset{\star}{\vdash} s \in L_\tau$*, if* $|s| < \tau$

❼ $\stackrel{\star}{\vdash} \mathrm{Tran}(L_\alpha)$ *for all* α

❽ $\stackrel{\star}{\vdash} \vec{s} \neq \vec{t}, \neg F(\vec{s}), F(\vec{t})$

Proof. See [Buc93], Lemma 2.4, Lemma 2.5. and Lemma 2.7 plus the Corollary to 2.7. □

Theorem 4.3.4 ((Ext), (Nullset), (Pair), (Union), (Δ_0-Sep))**.** *For every limit ordinal* λ *we have*

$$\stackrel{\star}{\vdash} (\mathrm{Ext})^\lambda \wedge (\mathrm{Nullset})^\lambda \wedge (\mathrm{Pair})^\lambda \wedge (\mathrm{Union})^\lambda \wedge (\Delta_0\text{-Sep})^\lambda.$$

Proof. We obtain (Nullset) as follows:
$\stackrel{\star}{\vdash} s \notin L_0$ for all $s \in \mathcal{T}_\lambda$ and thus $\stackrel{\star}{\vdash} \forall y^\lambda\, (y \notin L_0)$ and hence $\stackrel{\star}{\vdash} \exists x^\lambda\, \forall y^\lambda\, (y \notin x)$.
For the remaining statements consult the proof of [Buc93], Theorem 2.9. □

Definition 4.3.5. For a multiset $\Lambda = \{F_1, \ldots, F_n\}$ let $\|\Lambda\| := \omega^{\mathrm{rnk}(F_1)} \oplus \ldots \oplus \omega^{\mathrm{rnk}(F_n)}$. For a set Γ of sentences we use the following abbreviation

$$\Vdash \Gamma \quad :\Leftrightarrow \quad \mathcal{H}[\Gamma] \,\frac{\;\|\Gamma\|\;}{0}\, \Gamma \quad \text{for every hull-set } \mathcal{H}.$$

Lemma 4.3.6 (Embedding of $RS^\star$)**.** *It holds*

$$\stackrel{\star}{\vdash} \Gamma \quad \Rightarrow \quad \Vdash \Gamma.$$

Proof. See [Buc93], Lemma 3.10. □

Lemma 4.3.7 (Found)**.** *Let* $F(L_0)$ *be a sentence. Then it holds for every* α

$$\Vdash \forall x^\alpha \big(\forall y \in x\, F(y) \to F(x)\big) \to \forall x^\alpha F(x).$$

Proof. See [Rat94b], Lemma 8.5. or Lemma 8.2.2 of the second part of this thesis. □

Lemma 4.3.8 (Refl)**.** *Let* $F \in \Pi_n(\Xi)$*. Then*

$$\Vdash F \to \exists z^\Xi (z \models F).$$

Proof. Choose $m < \omega$, such that $F \in \Pi_{m+2}(\Xi)$. By Lemma 4.3.3 ① we have $\stackrel{\star}{\vdash} \neg F, F$. By use of the $RS^\star$-embedding and an application of ($\Pi_{m+2}(\Xi)$-Ref) plus a (V)-inference we obtain the claim. □

Lemma 4.3.9 (Embedding of Logic)**.** *Let* $\lambda \in \mathrm{Lim}$*. If* $\Gamma(\vec{u})$ *is a logically valid set of* $\mathcal{L}_{M(\Xi)}$*-formulae, then there is an* $m \in \omega$ *such that*

$$\mathcal{H}[\vec{s}, \lambda] \,\frac{\;\omega^{\omega\cdot\lambda+m}\;}{\omega^\lambda}\, \Gamma^\lambda(\vec{s}) \qquad \textit{for all } \vec{s} \in \mathcal{T}_\lambda.$$

Proof. See [Buc93], Lemma 3.11. □

Since $\mathcal{L}_\in$ is a language with identity, but in $\mathcal{L}_{RS(\Xi)}$ we regard identity as defined, we also have to secure that the identity axioms

$$\begin{array}{ll}\text{(Iden)} & \forall x\,(x = x), \\ & \forall x\,\forall y\,(x = y \to y = x), \\ & \forall x\,\forall y\,\forall z\,\big((x = y \wedge y = z) \to x = z\big), \\ & \forall x\,\forall y\,\forall u\,\forall v\,\big(x = y \wedge u = v \to (x \in u \to y \in v)\big),\end{array}$$

are derivable on hull-sets.

Lemma 4.3.10. *It holds* $\vdash^{\star} \bigwedge(\mathrm{Iden})$.

Proof. Follows directly from Lemma 4.3.3 ③, ⑤ and ⑧. □

Theorem 4.3.11 (Embedding of Π_ω-Ref)**.** *Let F be a theorem of Π_ω-Ref. Then there is an $m \in \omega$ such that*

$$\mathcal{H}[\Xi] \,\Big|\frac{\omega^{\Xi+m}}{\Xi+m}\, F^{\Xi}.$$

Proof. Follows by the Embedding of Logic and the above given derivations of the axioms of Π_ω-Ref plus the derivability of (Iden) via some cuts. □

5. Cut and Reflection Elimination Theorems

The aim of this chapter is to transform a given derivation on a hull-set, in which reflection rules may occur, into a derivation, in which no reflection rules occur by use of the scheme of stationary collapsing.

5.1. Predicative Cut Elimination

Lemma 5.1.1. *It holds*

- $\varphi(\alpha, \beta)$ *is closed under* $+$ *and* $\oplus$,
- $\beta_0 < \beta \quad \Rightarrow \quad \varphi(\alpha, \beta_0) < \varphi(\alpha, \beta)$,
- $\alpha_0 < \alpha \quad \Rightarrow \quad \varphi(\alpha_0, (\varphi(\alpha, \beta)) = \varphi(\alpha, \beta)$.

Proof. Folklore. □

Lemma 5.1.2 (Reduction Lemma)**.** *Let* $F \cong \bigvee(F_t)_{t \in T}$ *and* $\rho := \mathrm{rnk}(F)$ *not be regular. Then*

$$\mathcal{H} \Big|\frac{\alpha}{\rho}\, \Lambda, \neg F \;\;\&\;\; \mathcal{H} \Big|\frac{\beta}{\rho}\, \Gamma, F \quad \Rightarrow \quad \mathcal{H} \Big|\frac{\alpha+\beta}{\rho}\, \Lambda, \Gamma.$$

Proof. Use induction on β. For details see [Buc93], Lemma 3.14. □

Theorem 5.1.3 (Predicative Cut Elimination)**.** *Let* $\mathcal{H}$ *be closed under* φ *and* $[\rho, \rho + \omega^\alpha) \cap \mathrm{Reg} = \emptyset$ *plus* $\alpha \in \mathcal{H}$. *Then*

$$\mathcal{H} \Big|\frac{\beta}{\rho+\omega^\alpha}\, \Gamma \quad \Rightarrow \quad \mathcal{H} \Big|\frac{\varphi(\alpha,\beta)}{\rho}\, \Gamma.$$

Proof. By main induction on α and subsidiary induction on β. For details see [Buc93], Theorem 3.16. □

Corollary 5.1.4. $\mathcal{H} \Big|\frac{\beta}{\rho+1}\, \Gamma \;\;\&\;\; \rho \notin \mathrm{Reg} \;\Rightarrow\; \mathcal{H} \Big|\frac{\omega^\beta}{\rho}\, \Gamma.$

5.2. Reflection Elimination

Theorem 5.2.1 (Existence of Reflection Rules). *Let* $\mathbb{X}$ *be a reflection instance with reflection configuration* $\mathbb{F}$ *and* $\mathrm{rdh}(\mathbb{X}) = n \geq 0$. *Let* $\kappa \in \mathfrak{M}^{\alpha}_{\mathbb{X}}$. *Then*

❶ $\forall \tau < \kappa\ \forall \alpha_0 \in [o(\mathbb{X}), \alpha)_{C(\kappa)}\ \forall \vec{\eta} \in \mathrm{dom}(\mathbb{F})^{\geq n}_{C(\kappa)}\ \big(({}_{\tau}M^{\alpha_0}_{\mathbb{F}(\vec{\eta})}\text{-}\Pi_{n+2}(\kappa)\text{-}\mathsf{Ref})$ *exists*$\big)$.

❷ $\forall \tau < \kappa\ \forall \zeta \in C(\kappa)\ \forall \mathbb{K}\ \forall \vec{\eta} \in \mathrm{dom}(\mathbb{K})_{C(\kappa)}\ \big(({}_{\tau}M^{\zeta}_{\mathbb{K}(\vec{\eta})}\text{-}\Pi_{k+2}(\mathrm{i}(\mathbb{X}))\text{-}\mathsf{Ref})$ *is a subsidiary reflection rule of* $\mathrm{i}(\mathbb{X}) \quad \Rightarrow \quad ({}_{\tau}M^{\zeta}_{\mathbb{K}(\vec{\eta})}\text{-}\Pi_{k+2}(\kappa)\text{-}\mathsf{Ref})$ *exists*$\big)$.

Proof. ① Let us assume, that $\alpha > \mathrm{o}(\mathbb{X})$, as otherwise we have nothing to show. As κ is at least Π^1_0-indescribable there must be a reflection instance $\mathbb{Y}$ and a β, so that $\kappa = \Psi^{\beta}_{\mathbb{Y}}$. By Theorem 3.2.4 it follows $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} \preceq \vec{R}_{\kappa}$. If $(\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_n = \mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_n \preceq (\vec{R}'_{\kappa})_n$ it follows by Definition 4.2.8 that $({}_{\tau}M^{\alpha_0}_{\mathbb{F}(\vec{\eta})}\text{-}\Pi_{n+2}(\kappa)\text{-}\mathsf{Ref})$ is a (subsidiary) reflection rule of κ. If $(\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_n \not\preceq (\vec{R}'_{\kappa})_n$ then it must hold $\vec{R}_{\kappa} = (\mathsf{M}^{<\tilde{\alpha}}_{\mathbb{F}}\text{-}\mathsf{P}_n, \ldots)$, for some $\alpha \leq \tilde{\alpha}$. Thus for $\alpha_0 \in [o(\mathbb{X}), \tilde{\alpha})_{C(\kappa)}$ and $\vec{\eta} \in \mathrm{dom}(\mathbb{F})^{\geq n}_{C(\kappa)}$ Definition 2.2.4 provides the existence of the reflection instance $(\kappa; \mathsf{M}^{\alpha_0}_{\mathbb{F}(\vec{\eta})}\text{-}\mathsf{P}_n; \ldots)$ and by Definition 4.2.8 $({}_{\tau}M^{\alpha_0}_{\mathbb{F}(\vec{\eta})}\text{-}\Pi_{n+2}(\kappa)\text{-}\mathsf{Ref})$ is a (main) reflection rule of κ.

② Let $({}_{\tau}M^{\zeta}_{\mathbb{K}(\vec{\eta})}\text{-}\Pi_{k+2}(\mathrm{i}(\mathbb{X}))\text{-}\mathsf{Ref})$ be a subsidiary reflection rule of $\mathrm{i}(\mathbb{X})$. Then $\mathbb{F} \neq \mathbb{A}$ and a run through the cases of Definition 2.2.4 yields $\vec{R}'_{\mathrm{i}(\mathbb{X})} \preceq \vec{R}_{\Psi^{\alpha}_{\mathbb{X}}}$. Therefore the claim follows analogously to proposition ①, since $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} \preceq \vec{R}_{\kappa}$. □

Lemma 5.2.2. *Let* $\beta \leq \alpha$ *and* $\kappa \leq \pi$. *Then*

$$\rho \in C(\alpha, \kappa) \cap C(\beta, \pi) \text{ and } C(\alpha, \kappa) \cap \pi = \kappa \ \Rightarrow\ \rho \in C(\beta, \kappa).$$

Proof. Taking into account Definition 2.3.6 and Definition 2.3.7 we sloppily refer to $C(\gamma, \nu)$ as $\bigcup_{n<\omega} C^n(\gamma, \nu)$, where $C^0(\gamma, \nu) := \nu \cup \{0, \Xi\}$ and $C^{n+1} := \{f_i(\vec{\alpha}) \mid 1 \leq i \leq 4 \ \&\ \alpha_j \in C^n(\gamma, \nu)\}$, not specifying the arities of the f_i. However f_1 is intended to be $+$, $f_2 = \varphi$, f_3 the cardinal successor function and f_4 is the partial function Ψ, where all f_i come with the provisos given in the actual definition of $C(\gamma, \nu)$, i.e. every f_i is injective (for f_4 this follows by Theorem 2.3.15).

We show the claim by induction on $\mathrm{st}_{\beta,\pi}(\rho)$, where $\mathrm{st}_{\gamma,\nu}(\rho) := \min(n \mid \rho \in C^n(\gamma, \nu))$.
If $\mathrm{st}_{\beta,\pi}(\rho) = 0$ it holds $\rho \in \kappa \cup \{0, \Xi\}$ since $C(\alpha, \kappa) \cap \pi = \kappa$. Thus $\rho \in C(\beta, \kappa)$.
Let $\mathrm{st}_{\beta,\pi}(\rho) = n + 1$. Then

$$\rho \underset{\text{NF}}{=} f_i(\alpha_1, \ldots, \alpha_m) \text{ for some } \alpha_1, \ldots, \alpha_m \in C^n(\beta, \pi).$$

Since $\kappa \leq \pi$ we cannot have $\rho \in C^0(\alpha, \kappa)$. Thus $\mathrm{st}_{\alpha,\kappa}(\rho) > 0$ and therefore

$$\rho \underset{\text{NF}}{=} f_j(\alpha'_1, \ldots, \alpha'_{m'}), \text{ for some } \alpha'_1, \ldots, \alpha'_{m'} \in C(\alpha, \kappa).$$

Since the f_k, $1 \leq k \leq 4$, have pairwise disjoint ranges on normal form conditions we must have $f_i = f_j$. Due to the injectivity of the f_k it follows $m = m'$ and $\alpha_k = \alpha'_k$ for all $1 \leq k \leq m$. Considering the induction hypothesis we have $\alpha_1 \ldots, \alpha_m \in C(\beta, \kappa)$. Thus $\rho = f_i(\alpha_1, \ldots, \alpha_m) \in C(\beta, \kappa)$. □

Notation. Henceforth we use the following notation

$$\mathcal{C}_\gamma[\mathcal{A}] := C(\gamma+1,0)[\mathcal{A}],$$
$$_\sigma\mathfrak{M}^\xi_{\mathbb{X}} := \{\kappa \in \mathfrak{M}^\xi_{\mathbb{X}} \mid \sigma < \kappa\}.$$

Remark. It holds

$$\mathcal{C}_\gamma[\mathcal{A}] = C(\gamma+1,0)[\mathcal{A}] = \bigcup_{n<\omega} C^n(\gamma+1,\mathrm{k}(\mathcal{A})),$$

where $C^n(\alpha,\mathrm{k}(\mathcal{A}))$ is defined analogously to $C^n(\alpha,\pi)$, i.e. we set

$$C^0(\alpha,\mathrm{k}(\mathcal{A})) := \mathrm{k}(\mathcal{A}) \cup \{0,\Xi\}, \quad \text{and}$$

$$C^{n+1}(\alpha,\mathrm{k}(\mathcal{A})) := \begin{cases} C^n(\alpha,\mathrm{k}(\mathcal{A})) \cup \\ \{\gamma+\omega^\delta \mid \gamma,\delta \in C^n(\alpha,\mathrm{k}(\mathcal{A})) \wedge \gamma \underset{\text{NF}}{=} \omega^{\gamma_1}+\ldots+\omega^{\gamma_m} \wedge \gamma_m \geq \delta\} \cup \\ \{\varphi(\xi,\eta) \mid \xi,\eta \in C^n(\alpha,\mathrm{k}(\mathcal{A}))\} \cup \\ \{\kappa^+ \mid \kappa \in C^n(\alpha,\mathrm{k}(\mathcal{A})) \cap \mathrm{Card}\} \cup \\ \{\Psi^\gamma_{\mathbb{X}} \mid \mathbb{X},\gamma \in C^n(\alpha,\mathrm{k}(\mathcal{A})) \wedge \gamma < \alpha \wedge \Psi^\gamma_{\mathbb{X}} \text{ is well-defined}\}. \end{cases}$$

Lemma 5.2.3. *Let $\mathbb{X} = \mathbb{F}(\vec{\nu}) = (\pi;\ldots;\delta)$ be a reflection instance and $\gamma,\mathbb{X},\mu \in \mathcal{C}_\gamma[\mathcal{A}]$, where $\omega,\delta \leq \gamma+1$ and $\sigma := |\mathcal{A}| < \pi \leq \mu \in \mathrm{Card}$. Let $\hat{\alpha} := \gamma \oplus \omega^{\alpha\oplus\mu}$. Then the following hold:*

❶ *If $\alpha_0,\alpha \in \mathcal{C}_\gamma[\mathcal{A}]$ and $\alpha_0 < \alpha$ then $\emptyset \neq {_\sigma\mathfrak{M}}^{\hat{\alpha}}_{\mathbb{X}} \subseteq {_\sigma\mathfrak{M}}^{\hat{\alpha_0}}_{\mathbb{X}}$. Moreover for every $\kappa \in {_\sigma\mathfrak{M}}^{\hat{\alpha}}_{\mathbb{X}}$ and every $\mathbb{Y} := \mathbb{F}(\vec{\eta})$ with $\vec{\eta} \in \mathrm{dom}(\mathbb{F}) \cap \mathcal{C}_\gamma[\mathcal{A}]$ the ordinals $\Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}}$ and $\Psi^{\hat{\alpha_0}\oplus\kappa}_{\mathbb{Y}}$ are well-defined and it holds $\Psi^{\hat{\alpha_0}\oplus\kappa}_{\mathbb{Y}} < \Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}} \in \mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A},\kappa]$*

❷ *It holds $\pi \in \mathcal{C}_\gamma[\mathcal{A}]$. Moreover for $\kappa \in \mathrm{Card}$ and $\rho \in \mathcal{C}_\gamma[\mathcal{A}]$, such that $\pi \leq \kappa \leq \rho \leq \kappa^+$ we have $\kappa,\kappa^+ \in \mathcal{C}_\gamma[\mathcal{A}]$.*

❸ *Let $\pi \leq \mu \in \mathrm{Reg} \cap \mathcal{C}_\gamma[\mathcal{A}]$. Then there exists a reflection instance $\mathbb{Z}$ with $\mathrm{i}(\mathbb{Z}) = \mu$, $\mathrm{o}(\mathbb{Z}) \leq \gamma+1$ and $\mathbb{Z} \in \mathcal{C}_\gamma[\mathcal{A}]$.*

Proof. ① Since $\gamma,\alpha,\mu \in \mathcal{C}_\gamma[\mathcal{A}] \subseteq C(\gamma+1,\sigma+1)$ and $\sigma < \pi$ we have $\hat{\alpha}\oplus\sigma \in C(\pi)$ and thereby $\kappa_\sigma := \Psi^{\hat{\alpha}\oplus\sigma}_{\mathbb{X}}$ is well-defined. This implies $\sigma \in C(\hat{\alpha}\oplus\sigma,\kappa_\sigma) \cap \pi = \kappa_\sigma$ and hence $\sigma < \kappa_\sigma$. Moreover we have $\hat{\alpha} < \hat{\alpha}\oplus\sigma$ and $\gamma,\alpha,\mu \in C(\gamma+1,\sigma+1) \subseteq C(\hat{\alpha},\kappa_\sigma)$ and thereby $\kappa_\sigma \in {_\sigma\mathfrak{M}}^{\hat{\alpha}}_{\mathbb{X}}$.

Now let $\kappa \in {_\sigma\mathfrak{M}}^{\hat{\alpha}}_{\mathbb{X}}$. Then it holds $\hat{\alpha_0} < \hat{\alpha}$ and $\gamma,\alpha_0,\mu \in C(\gamma+1,\sigma+1) \subseteq C(\hat{\alpha_0},\kappa)$, hence $\hat{\alpha_0} \in C(\kappa)$ and thus $\kappa \in {_\sigma\mathfrak{M}}^{\hat{\alpha_0}}_{\mathbb{X}}$.

Obviously we have $\hat{\alpha}\oplus\kappa \in C(\pi)$, thus $\Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}}$ is well-defined and $\Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}} \in \mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A},\kappa]$.

Now let $\mathbb{Y} := \mathbb{F}(\vec{\eta})$ for some $\vec{\eta} \in \mathrm{dom}(\mathbb{F}) \cap \mathcal{C}_\gamma[\mathcal{A}]$. As shown above for $\mathbb{X}$ it follows that $\kappa_{\mathbb{Y}} := \Psi^{\hat{\alpha_0}\oplus\kappa}_{\mathbb{Y}}$ is well-defined. If $\vec{\eta} = \epsilon$ it holds $\mathbb{Y} = \mathbb{X}$ and it follows by Theorem 2.3.14 that $\kappa_{\mathbb{Y}} < \kappa_{\mathbb{X}} := \Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}}$. If $\vec{\eta} \neq \epsilon$ let us assume $\kappa_{\mathbb{Y}} \not< \kappa_{\mathbb{X}}$. Since $\hat{\alpha_0}\oplus\kappa \neq \hat{\alpha}\oplus\kappa$ this implies by Theorem 2.3.14 $\kappa_{\mathbb{X}} < \kappa_{\mathbb{Y}}$. As we have $\mathbb{X},\hat{\alpha_0}\oplus\kappa \in C(\kappa_{\mathbb{X}})$ there must be a component η_i of $\vec{\eta}$, such that $\eta_i \notin C(\kappa_{\mathbb{X}})$. By proviso we have $\vec{\eta} \in \mathcal{C}_\gamma[\mathcal{A}] \subseteq$

$C(\gamma+1,\sigma+1) \subseteq C(\hat{\alpha}\oplus\kappa,\kappa_{\mathbb{X}})$ and thereby $\eta_i < \hat{\alpha}\oplus\kappa$. Moreover we have $\eta_i \in C(\kappa_{\mathbb{Y}})$ by definition and $C(\hat{\alpha}\oplus\kappa,\kappa_{\mathbb{X}}) \cap \pi = \kappa_{\mathbb{X}}$, i.e. $C(\hat{\alpha}\oplus\kappa,\kappa_{\mathbb{X}}) \cap \kappa_{\mathbb{Y}} = \kappa_{\mathbb{X}}$. However, this implies the contradiction $\eta_i \in C(\kappa_{\mathbb{X}})$ by means of Lemma 5.2.2. Therefore we must have $\kappa_{\mathbb{Y}} < \kappa_{\mathbb{X}}$.

② If $\pi = \Xi$ or π is a successor cardinal the claim follows since $\operatorname{par}\mathbb{X} \in \mathcal{C}_\gamma[\mathcal{A}]$. Otherwise we have $\mathbb{X} = (\Psi^{\delta-1}_{\mathbb{V}};\ldots;\mathbb{V};\ldots;\delta)$. Then it holds $\delta-1 < \gamma+1$ and $\{\delta\} \cup \operatorname{par}\mathbb{V} \subseteq \operatorname{par}\mathbb{X}$. Thus $\pi \in \mathcal{C}_\gamma[\mathcal{A}]$.

By definition we have $\mathcal{C}_\gamma[\mathcal{A}] = \bigcup_{n\in\omega} C^n(\gamma+1,\mathrm{k}(\mathcal{A}))$. We show by induction on n, if $\rho \in C^n(\gamma+1,\mathrm{k}(\mathcal{A}))$ then $\kappa,\kappa^+ \in C(\gamma+1,\mathrm{k}(\mathcal{A}))$. The interesting case is that $\kappa < \rho = \Psi^{\xi}_{\mathbb{W}} < \kappa^+$. Then we have $\xi \leq \gamma$ and $\operatorname{par}\mathbb{W} \subseteq C^n(\gamma+1,\mathrm{k}(\mathcal{A}))$ as $\rho \notin \mathrm{k}(\mathcal{A})$. Since $C(\xi,\rho)\cap \mathrm{i}(\mathbb{W}) = \rho$ and $C(\xi,\rho)$ is closed under the cardinal successor function we must have $\mathrm{i}(\mathbb{W}) = \kappa^+$ and hence $\kappa,\kappa^+ \in C(\gamma,\mathrm{k}(\mathcal{A}))$.

③ If $\mu = \Xi$ or $\mu = \nu^+$ for some $\nu \in \mathrm{Card}\cap\Xi$ the claim is trivial. Otherwise we have $\mu = \Psi^{\zeta}_{\mathbb{V}}$ for some ζ and a reflection instance $\mathbb{V}$. By Lemma 2.3.16 it follows $\zeta,\mathbb{V} \in \mathcal{C}_\gamma[\mathcal{A}] \subseteq C(\gamma+1,\sigma+1)$. Thus $\zeta \leq \gamma$. Assuming $\mathbb{V} = (\mathrm{i}(\mathbb{V});\mathsf{P}_0;\ldots)$ we obtain that an element $\kappa \in \mathfrak{M}^{\zeta}_{\mathbb{V}}$ only has to satisfy $C(\zeta,\kappa)\cap\mathrm{i}(\mathbb{V}) = \kappa$ and $\mathbb{V},\zeta \in C(\kappa)$. However, since μ is regular, these elements form a club in μ by Lemma 2.3.5② in contradiction to the fact that μ is the least element of $\mathfrak{M}^{\zeta}_{\mathbb{V}}$. So $\mathbb{V} \neq (\mathrm{i}(\mathbb{V});\mathsf{P}_0;\ldots)$ and the claim follows from Corollary 2.3.3.

If the arity of $\mathbb{G}$ is zero, we can choose $\mathbb{Z} = \mathbb{G} \in \mathcal{C}_\gamma[\mathcal{A}]$. If $\operatorname{dom}(\mathbb{G}) = [\mathrm{o}(\mathbb{M}),\xi)_{C(\mu)} \times \operatorname{dom}(\mathbb{M})_{C(\mu)}$ with $\mathbb{G}(\zeta',\vec{\nu}) = (\mu;\mathsf{M}^{\zeta'}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m;\ldots)$, we have $\mathbb{M} \in \mathrm{Prcnfg}(\mathbb{G})$, i.e. it exists a $\vec{\eta} \in \operatorname{dom}(\mathbb{M})_{C(\mu)}$ such that $\mathbb{M}(\vec{\eta}) \in \mathrm{Prinst}(\mathbb{G}) = \overline{\mathrm{Prinst}}(\mathbb{V})$. Thus we have $\mathbb{Z} := \mathbb{G}(\mathrm{o}(\mathbb{M}),\vec{\eta}) \in \mathcal{C}_\gamma[\mathcal{A}]$. □

Definition 5.2.4. For $\mu \in \mathrm{Card}$ we define

$$\bar{\mu} := \begin{cases} \mu+1 & \text{if } \mu \in \mathrm{Reg} \\ \mu & \text{otherwise.} \end{cases}$$

Theorem 5.2.5 (Reflection Elimination)**.** *Let $\mathbb{X} = (\pi;\ldots;\delta)$ be a reflection instance with $\mathrm{rdh}(\mathbb{X}) = m-1$ and $\gamma,\mathbb{X},\mu \in \mathcal{C}_\gamma[\mathcal{A}]$, where $\omega,\delta \leq \gamma+1$ and $\sigma := |\mathcal{A}| < \pi \leq \mu \in \mathrm{Card}$. Let $\Gamma \subseteq \Sigma_{m+1}(\pi)$ and $\hat{\alpha} := \gamma \oplus \omega^{\alpha\oplus\mu}$. Then*

$$\mathcal{C}_\gamma[\mathcal{A}] \,\Big|\!\frac{\alpha}{\bar{\mu}}\; \Gamma \quad\Rightarrow\quad \mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A},\kappa] \,\Big|\!\frac{\Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}}}{\cdot}\; \Gamma^{(\pi,\kappa)} \quad \textit{for all } \kappa \in {}_{\sigma}\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}.$$

Proof. We proceed by main induction on μ and subsidiary induction on α.

Case 1: The last inference is $(\bigvee)$ with principal formula $F \cong \bigvee(F_t)_{t\in T} \in \Gamma$. Thus

$$\mathcal{C}_\gamma[\mathcal{A}] \,\Big|\!\frac{\alpha_0}{\bar{\mu}}\; \Gamma, F_{t_0},$$

for some $\alpha_0 < \alpha$ and some $t_0 \in T$. By use of the subsidiary induction hypothesis we obtain

$$\mathcal{C}_{\hat{\alpha_0}\oplus\kappa}[\mathcal{A},\kappa] \,\Big|\!\frac{\Psi^{\hat{\alpha_0}\oplus\kappa}_{\mathbb{X}}}{\cdot}\; \Gamma^{(\pi,\kappa)}, F^{(\pi,\kappa)}_{t_0} \quad \text{for all } \kappa \in {}_{\sigma}\mathfrak{M}^{\hat{\alpha_0}}_{\mathbb{X}}. \tag{5.1}$$

By Lemma 5.2.3 ① we have ${}_\sigma\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}} \subseteq \mathfrak{M}^{\hat{\alpha_0}}_{\mathbb{X}}$ and $\Psi^{\hat{\alpha_0}\oplus\kappa}_{\mathbb{X}} < \Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}} \in \mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A},\kappa]$. Moreover it holds $F^{(\pi,\kappa)} \cong \bigvee(F_t^{(\pi,\kappa)})_{t\in T\cap\mathcal{T}_\kappa}$ and $|t_0| \in \mathcal{C}_\gamma[\mathcal{A}] \cap \pi \subseteq C(\gamma+1,\sigma+1)\cap\pi \subseteq C(\hat{\alpha},\kappa)\cap\pi = \kappa$ for $\kappa \in {}_\sigma\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}$. Thus we obtain from (5.1) the desired result by means of (Str) and ($\bigvee$).

Case 2: The last inference is ($\bigwedge$) with principal formula $F \cong \bigwedge(F_t)_{t\in T} \in \Gamma$. Then for all $t \in T$, there exists an $\alpha_t < \alpha$ such that

$$\mathcal{C}_\gamma[\mathcal{A},t] \,\Big|\frac{\alpha_t}{\bar{\mu}}\, \Gamma, F_t. \tag{5.2}$$

Since $F \in \Gamma$ we have $F_t \in \Sigma_{m+1}(\pi)$ and $|t| < \pi$ for all $t \in T$. Thus we may apply the subsidiary induction hypothesis to (5.2) and obtain for all $t \in T$

$$\mathcal{C}_{\hat{\alpha_t}\oplus\lambda}[\mathcal{A},t,\lambda] \,\Big|\frac{\Psi^{\hat{\alpha_t}\oplus\lambda}_{\mathbb{X}}}{\cdot}\, \Gamma^{(\pi,\lambda)}, F_t^{(\pi,\lambda)} \quad \text{for all } \lambda \in {}_{\sigma_t}\mathfrak{M}^{\hat{\alpha_t}}_{\mathbb{X}}, \tag{5.3}$$

where $\sigma_t := |\mathcal{A},t|$.

Let $\kappa \in {}_\sigma\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}$. Then we have $\kappa \in {}_{\sigma_t}\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}$ for all $t \in T\cap\mathcal{T}_\kappa$. Thus Lemma 5.2.3 ① (with $\mathcal{A}$ replaced by $\mathcal{A},t$) provides $\kappa \in {}_{\sigma_t}\mathfrak{M}^{\hat{\alpha_t}}_{\mathbb{X}}$ and $\Psi^{\hat{\alpha_t}\oplus\kappa}_{\mathbb{X}} < \Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}}$. Therefore the claim follows from (5.3) by use of (Str) and a ($\bigwedge$)-inference, since $F^{(\pi,\kappa)} \cong \bigwedge(F_t^{(\pi,\kappa)})_{t\in T\cap\mathcal{T}_\kappa}$ and obviously $\Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}} \in \mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A},\kappa]$.

Case 3: The last inference is (Cut) Then for some $\alpha_0 < \alpha$ it holds

$$\mathcal{C}_\gamma[\mathcal{A}] \,\Big|\frac{\alpha_0}{\bar{\mu}}\, \Gamma, (\neg)F, \tag{5.4}$$

where $\mathrm{rnk}(F) < \bar{\mu}$.

Subcase 3.1, $\mathrm{rnk}(F) < \pi$*:* Then we have $(\neg)F \in \Sigma_{m+1}(\pi)$. Thus we are allowed to apply the subsidiary induction hypothesis to (5.4) and obtain

$$\mathcal{C}_{\hat{\alpha_0}\oplus\kappa}[\mathcal{A},\kappa] \,\Big|\frac{\Psi^{\hat{\alpha_0}\oplus\kappa}_{\mathbb{X}}}{\cdot}\, \Gamma^{(\pi,\kappa)}, (\neg)F^{(\pi,\kappa)} \quad \text{for all } \kappa \in {}_\sigma\mathfrak{M}^{\hat{\alpha_0}}_{\mathbb{X}}.$$

By similar considerations as in the first case it follows that $\mathrm{rnk}(F^{(\pi,\kappa)}) = \mathrm{rnk}(F) < \kappa < \Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}}$ for $\kappa \in {}_\sigma\mathfrak{M}^{\hat{\alpha_0}}_{\mathbb{X}}$. Thus the claim follows by a (Cut) and (Str) taking into account Lemma 5.2.3 ①.

Subcase 3.2, $\pi \le \mathrm{rnk}(F) \le \mu$*:* If $\mathrm{rnk}(F) < \mu$ then it holds $\pi \le \mu_0 := \sup\{\kappa \in \mathrm{Card} \mid \kappa \le \mathrm{rnk}(F)\} \le \mathrm{rnk}(F) < \mu_0^+ \le \mu$ and by Lemma 5.2.3 ② it follows $(\mu_0^+;\mathsf{P}_0;\epsilon;\epsilon;0) \in \mathcal{C}_\gamma[\mathcal{A}]$. If $\mathrm{rnk}(F) = \mu$ we must have $\mu \in \mathrm{Reg}$ and by Lemma 5.2.3 ③ it follows the existence of a reflection instance $\mathbb{Z} \in C_\gamma[\mathcal{A}]$ with $\mathrm{i}(\mathbb{Z}) = \mu$ and $\mathrm{o}(\mathbb{Z}) \le \gamma+1$. Thus in all cases there is a reflection instance $\mathbb{Y} \in \mathcal{C}_\gamma[\mathcal{A}]$ with $\pi \le \mathrm{rnk}(F) \le \mu_1 := \mathrm{i}(\mathbb{Y}) \le \mu$ and $\mathrm{o}(\mathbb{Y}) \le \gamma+1$. In the following we choose $\mathbb{Y} = \mathbb{X}$ if $\pi = \mu$.

W.l.o.g. F is an elementary $\Sigma_1(\mu_1)$-sentence and $\neg F \equiv \forall x^{\mu_1} G(x)$ an elementary $\Pi_1(\mu_1)$-sentence. Therefore applying the subsidiary induction hypothesis to (5.4) we obtain

$$\mathcal{C}_{\hat{\alpha_0}\oplus\lambda}[\mathcal{A},\lambda] \,\Big|\frac{\Psi^{\hat{\alpha_0}\oplus\lambda}_{\mathbb{Y}}}{\cdot}\, \Gamma^{(\mu_1,\lambda)}, F^{(\mu_1,\lambda)} \quad \text{for all } \lambda \in {}_\sigma\mathfrak{M}^{\hat{\alpha_0}}_{\mathbb{Y}} \tag{5.5}$$

and by use of (5.4) and ($\forall$-Inv) plus the subsidiary induction hypothesis we obtain for all $t \in \mathcal{T}_{\mu_1}$

$$\mathcal{C}_{\hat{\alpha_0}\oplus\lambda}[\mathcal{A}, t, \lambda] \left|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\lambda}}{\cdot}\right. \Gamma^{(\mu_1,\lambda)}, G(t) \quad \text{for all } \lambda \in {}_{\sigma_t}\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}, \tag{5.6}$$

where $\sigma_t := |\mathcal{A}, t|$. Let $\lambda \in {}_{\sigma}\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}$. Then it follows $\lambda \in {}_{\sigma_t}\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}$ for all $t \in \mathcal{T}_\lambda$. Thus by an application of ($\bigwedge$) to (5.6) we obtain

$$\mathcal{C}_{\hat{\alpha_0}\oplus\lambda}[\mathcal{A}, \lambda] \left|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\lambda}+1}{\cdot}\right. \Gamma^{(\mu_1,\lambda)}, \neg F^{(\mu_1,\lambda)} \quad \text{for all } \lambda \in {}_{\sigma}\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}, \tag{5.7}$$

Since $\mathrm{k}(F^{(\mu_1,\lambda)}) \in \mathcal{C}_\gamma[\mathcal{A}, \lambda] \cap \mu_1 \subseteq C(\hat{\alpha_0} \oplus \lambda, \Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\lambda}) \cap \mu_1 = \Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\lambda}$ if $\lambda \in {}_{\sigma}\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}$ it follows $\mathrm{rnk}(F^{(\mu_1,\lambda)}) < \Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\lambda}$. Thus a (Cut) applied to (5.5) and(5.7) yields

$$\mathcal{C}_{\hat{\alpha_0}\oplus\lambda+1}[\mathcal{A}, \lambda] \left|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\lambda+1}}{\cdot}\right. \Gamma^{(\mu_1,\lambda)} \quad \text{for all } \lambda \in {}_{\sigma}\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}. \tag{5.8}$$

If $\mathbb{Y} = \mathbb{X}$ Lemma 5.2.3 ① provides that the claim follows from (5.8) by use of (Str).

If $\mathbb{Y} \neq \mathbb{X}$ it follows $\pi < \mu_1 \leq \mu$. Let $\lambda_0 := \Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\sigma}$ and $\eta := \Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\lambda_0+1}$. Then it follows as shown in the proof of Lemma 5.2.3 ① $\lambda_0 \in {}_{\sigma}\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}$ and $\pi < \eta < \mu_1$ since $\pi \in C(\gamma+1, \sigma+1) \subseteq C(\hat{\alpha_0} \oplus \sigma, \lambda_0) \cap \mu_1 = \lambda_0$. Hence $\Gamma^{(\pi,\lambda_0)} = \Gamma$ and (5.8) plus (Hull) provides

$$\mathcal{C}_{\hat{\alpha_0}\oplus\lambda_0+1}[\mathcal{A}] \left|\frac{\eta}{\cdot}\right. \Gamma. \tag{5.9}$$

If $\eta \in \mathrm{Card}$ we apply the main induction hypothesis to (5.9) and obtain

$$\mathcal{C}_{\nu\oplus\kappa}[\mathcal{A}, \kappa] \left|\frac{\Psi_{\mathbb{X}}^{\nu\oplus\kappa}}{\cdot}\right. \Gamma^{(\pi,\kappa)} \quad \text{for all } \kappa \in {}_{\sigma}\mathfrak{M}_{\mathbb{X}}^{\nu}, \tag{5.10}$$

where $\nu := (\hat{\alpha_0} \oplus \lambda_0 \oplus 1) \oplus \omega^{\eta\oplus\bar{\eta}}$. Since $\alpha_0 \oplus \mu, \lambda_0+1, \eta\oplus\bar{\eta} < \alpha\oplus\mu$ it follows $\nu < \hat{\alpha}$ and since $\gamma, \alpha_0, \mu, \mathbb{Y} \in C(\gamma+1, \sigma+1)$ it follows successively $\lambda_0, \eta, \nu, \nu\oplus\kappa \in C(\nu\oplus\kappa, \Psi_{\mathbb{X}}^{\hat{\alpha}\oplus\kappa})$ for $\kappa \in {}_{\sigma}\mathfrak{M}_{\mathbb{X}}^{\nu}$ and thus $\Psi_{\mathbb{X}}^{\nu\oplus\kappa} < \Psi_{\mathbb{X}}^{\hat{\alpha}\oplus\kappa}$ for all $\kappa \in {}_{\sigma}\mathfrak{M}_{\mathbb{X}}^{\nu}$ by Theorem 2.3.14. In the same vein it follows $\nu \in C(\nu, \kappa)$ for every $\kappa \in {}_{\sigma}\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha}}$ and hence $\kappa \in {}_{\sigma}\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha}}$ implies $\kappa \in {}_{\sigma}\mathfrak{M}_{\mathbb{X}}^{\nu}$. Thereby the claim follows from (5.10) by use of (Str).

If $\pi < \mu_1 \leq \mu$ and $\eta \notin \mathrm{Card}$ then it holds $\pi \leq \mu_0 < \eta < \mu_0^+ = \mu_1$. Through the use of predicative cut elimination, (5.9) yields

$$\mathcal{C}_{\hat{\alpha_0}\oplus\lambda_0+1}[\mathcal{A}] \left|\frac{\varphi(\eta,\eta)}{\bar{\mu_0}}\right. \Gamma. \tag{5.11}$$

Since $\mu_1 \in \mathcal{C}_\gamma[\mathcal{A}]$ implies $\mu_0 \in \mathcal{C}_\gamma[\mathcal{A}]$ we may apply the main induction to (5.11) and obtain

$$\mathcal{C}_{\hat{\nu}\oplus\kappa}[\mathcal{A}, \kappa] \left|\frac{\Psi_{\mathbb{X}}^{\hat{\nu}\oplus\kappa}}{\cdot}\right. \Gamma^{(\pi,\kappa)} \quad \text{for all } \kappa \in {}_{\sigma}\mathfrak{M}_{\mathbb{X}}^{\hat{\nu}},$$

where $\hat{\nu} := (\hat{\alpha_0} \oplus \lambda_0 \oplus 1) \oplus \omega^{\varphi(\eta,\eta)\oplus\bar{\mu_0}}$. By analogue considerations as in the case $\eta \in \mathrm{Card}$ the claim follows by (Str).

Case 4, the last inference is by a main reflexion rule of π:

Subcase 4.1, it holds $\mathbb{X} = \mathbb{A}(m)$ (i.e. $\pi = \Xi$) and the last inference is ($\Pi_{n+2}(\pi)$-Ref) *with principal formula $\exists z^\pi(z \models F)$:* Let $n' := \max(m, n)$ and $\mathbb{Y} := \mathbb{A}(n')$.[†] Then we have

$$\mathcal{C}_\gamma[\mathcal{A}] \left|\frac{\alpha_0}{\mu}\right. \Gamma, F$$

for some $\alpha_0 < \alpha$ and $F \in \Pi_{n'+2}(\pi)$. Thus by an application of (E-$\forall$-Inv) we obtain for all $\vec{t} \in \mathcal{T}_\pi$

$$\mathcal{C}_\gamma[\mathcal{A}, \vec{t}\,] \left|\frac{\alpha_0}{\mu}\right. \Gamma, F(F_1'(t_1), \dots, F_r'(t_r)). \tag{5.12}$$

Since $\mathbb{Y} \in \mathcal{C}_\gamma[\mathcal{A}]$ and $\Gamma, F(F_1'(t_1), \dots, F_r'(t_r)) \in \Sigma_{n'+1}(\pi)$ for all $\vec{t} \in \mathcal{T}_\pi$ we may apply the subsidiary induction hypothesis and obtain for all $\vec{t} \in \mathcal{T}_\pi$

$$\mathcal{C}_{\hat{\alpha_0}\oplus\lambda}[\mathcal{A}, \vec{t}, \lambda] \left|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\lambda}}{\cdot}\right. \Gamma^{(\pi,\lambda)}, F^{(\pi,\lambda)}(F_1'^{(\pi,\lambda)}(t_1), \dots, F_r'^{(\pi,\lambda)}(t_r)) \quad \text{for all } \lambda \in {}_{\sigma_{\vec{t}}}\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}, \tag{5.13}$$

where $\sigma_{\vec{t}} := |\mathcal{A}, \vec{t}\,|$. Let $\lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}$. Then we have $\lambda \in {}_{\sigma_{\vec{t}}}\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}$ for all $\vec{t} \in \mathcal{T}_\lambda$. Thus we obtain from (5.13) by means of (E-$\forall$) for some $l < \omega$

$$\mathcal{C}_{\hat{\alpha_0}\oplus\lambda}[\mathcal{A}, \lambda] \left|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\lambda}+l}{\cdot}\right. \Gamma^{(\pi,\lambda)}, F^{(\pi,\lambda)} \quad \text{for all } \sigma < \lambda \in \mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}. \tag{5.14}$$

In the sequel, we fix a $\kappa \in {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha}}$.

It holds $\sigma < \Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\sigma} \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}$. Since $\gamma, \alpha_0, \mu, \sigma, \mathbb{Y} \in C(\hat{\alpha}, \kappa)$ and $\hat{\alpha_0} \oplus \sigma < \hat{\alpha}$ it follows $\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\sigma} \in C(\hat{\alpha}, \kappa) \cap \pi = \kappa$. Therefore we have ${}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}} \cap \kappa \neq \emptyset$.

For $\lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}} \cap \kappa$ we have $\Vdash L_\lambda \neq \emptyset \wedge \mathrm{Tran}(L_\lambda) \wedge \bigwedge_{i=1}^p a_i \in L_\lambda$, where $a_1, \dots, a_p$ denote the terms of F less than π. Thus we obtain from (5.14) by means of ($\bigwedge$) and ($\bigvee$)

$$\mathcal{C}_{\hat{\alpha_0}\oplus\lambda}[\mathcal{A}, \lambda, \kappa] \left|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\lambda}+\omega}{\cdot}\right. \bigvee \Gamma^{(\pi,\lambda)}, \exists z^\kappa(z \models F) \quad \text{for all } \lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}} \cap \kappa. \tag{5.15}$$

In the following let $G := \exists z^\kappa(z \models F)$. Now let $s \in \mathcal{T}_\kappa$. By Lemma 4.3.3 ⑧ we get

$$\Vdash L_\lambda \neq s, \bigwedge \neg\Gamma^{(\pi,\lambda)}, \bigvee \Gamma^{(\pi,s)}. \tag{5.16}$$

Applying (Cut) to (5.15) and (5.16) we obtain

$$\mathcal{C}_{\hat{\alpha_0}\oplus\kappa}[\mathcal{A}, \lambda, \kappa, s] \left|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus|s|}+\omega+1}{\cdot}\right. L_\lambda \neq s, \bigvee \Gamma^{(\pi,s)}, G \quad \text{for all } \lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}} \cap |s|+1.$$

[†]Here we could also define $n' := \max\{m, n\} + 1$. Then we could directly apply the subsidiary induction hypothesis to the derivation of Γ, F. However by choosing $n' := \max\{m, n\}$ the treatment of subcase 4.1 works also fine for subcase 4.2.

Since $\neg_\sigma M^{\hat{\alpha_0}}_{\mathbb{Y}}(s) \cong \bigwedge(L_\tau \neq s)_{\tau \in {}_\sigma\mathfrak{M}^{\hat{\alpha_0}}_{\mathbb{Y}} \cap |s|+1}$ we obtain by a ($\bigwedge$)- and a ($\bigvee$)-inference

$$\mathcal{C}_{\hat{\alpha_0}\oplus\kappa}[\mathcal{A},\kappa,s] \overset{\Psi^{\hat{\alpha_0}\oplus|s|}_{\mathbb{Y}}+\omega+3}{\underset{\cdot}{\Bigg|}} \neg_\sigma M^{\hat{\alpha_0}}_{\mathbb{Y}}(s) \vee s=\emptyset \vee \neg\,\mathrm{Tran}(s) \vee \bigvee_{i=1}^{q} b_i \notin s, \bigvee \Gamma^{(\pi,s)}, G,$$

for $s \in \mathcal{T}_\kappa$, where $b_1,\dots,b_q$ denotes the terms of Γ less than κ. By means of ($\bigvee$) and ($\bigwedge$), we arrive at

$$\mathcal{C}_{\hat{\alpha_0}\oplus\kappa}[\mathcal{A},\kappa] \overset{\Psi^{\hat{\alpha_0}\oplus\kappa}_{\mathbb{Y}}}{\underset{\cdot}{\Bigg|}} \forall x^\kappa\big(\neg_\sigma M^{\hat{\alpha_0}}_{\mathbb{Y}}(x) \vee x \not\models \bigwedge \neg\Gamma^{(\pi,x)}\big), G. \tag{5.17}$$

Since $\mathrm{rdh}(\mathbb{X}) = m-1 \geq 0$ (note that $\mathrm{dom}(\mathbb{A}) = (0,\omega)$), $\mathrm{k}(\Gamma) \subseteq \mathcal{C}_\gamma[\mathcal{A}]$ and $\kappa \in {}_\sigma\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}$ Theorem 5.2.1 provides the existence of the reflection rule (${}_\sigma M^{\hat{\alpha_0}}_{\mathbb{Y}}$-$\Pi_{m+1}(\kappa)$-Ref). As $\bigwedge\neg\Gamma^{(\pi,\kappa)}$ is a $\Pi_{m+1}(\kappa)$ formula we obtain by Lemma 4.3.3 ⑧, an application of this rule and (Str)

$$\mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A},\kappa] \overset{\Psi^{\hat{\alpha_0}\oplus\kappa}_{\mathbb{Y}}}{\underset{\cdot}{\Bigg|}} \exists x^\kappa\big({}_\sigma M^{\hat{\alpha_0}}_{\mathbb{Y}}(x) \wedge x \models \bigwedge \neg\Gamma^{(\pi,x)}\big), \bigvee \Gamma^{(\pi,\kappa)}. \tag{5.18}$$

Therefore we obtain the desired derivation by a (Cut) applied to (5.17) and (5.18) plus ($\vee$-Ex) and (Str).

Subcase 4.2, it holds $\mathbb{X} = (\pi;\mathsf{P}_m;\vec{R};\dots) \neq \mathbb{A}(m)$ *and the last inference is by the main reflection rule* ($\Pi_{m+2}(\pi)$-Ref)*:* Then the claim follows by simplifying (i.e. setting $\mathbb{Y} := \mathbb{X}$) the considerations of the previous case. If $m=0$ then $\bigvee\Gamma^{(\pi,\lambda)} \in \Sigma_1(\lambda)$ and the claim follows directly from (5.15), with $\lambda := \Psi^{\hat{\alpha_0}\oplus\sigma}_{\mathbb{Y}}$ and by use of (E-Up-Per) plus (Hull) and (Str).

Subcase 4.3, it holds $\mathbb{X} = \mathbb{F}(\xi,\vec{\nu}) = (\pi;\mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m;\dots)$ *and the last inference is by a main reflection rule* (${}_\tau M^{\zeta}_{\mathbb{M}(\vec{\eta})}$-$\Pi_{m+2}(\pi)$-Ref) *of* π *with* $(\zeta,\vec{\eta}) \in \mathrm{dom}(\mathbb{F})$ *and principal formula* $\exists z^\pi\big({}_\tau M^{\zeta}_{\mathbb{M}(\vec{\eta})}(z) \wedge z \models F\big)$*:* Let $\mathbb{Y} := \mathbb{F}(\zeta,\vec{\eta})$. Then we have

$$\mathcal{C}_\gamma[\mathcal{A}] \,\big|\!\frac{\alpha_0}{\bar{\mu}}\, \Gamma, F$$

for some $\alpha_0 < \alpha$ and $F \in \Pi_{m+2}(\pi)$. Since $\mathbb{Y} \in \mathcal{C}_\gamma[\mathcal{A}]$ we may proceed analogously to case 4.1 and obtain by use of (E-$\forall$-Inv), the subsidiary induction hypothesis and (E-$\forall$) for some $l < \omega$

$$\mathcal{C}_{\hat{\alpha_0}\oplus\lambda}[\mathcal{A},\lambda] \overset{\Psi^{\hat{\alpha_0}\oplus\lambda}_{\mathbb{Y}}+l}{\underset{\cdot}{\Bigg|}} \Gamma^{(\pi,\lambda)}, F^{(\pi,\lambda)} \quad \text{for all } \lambda \in {}_\sigma\mathfrak{M}^{\hat{\alpha_0}}_{\mathbb{Y}}. \tag{5.19}$$

In the sequel, we fix a $\kappa \in {}_\sigma\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}$. As shown in subcase 4.1 it follows ${}_\sigma\mathfrak{M}^{\hat{\alpha_0}}_{\mathbb{Y}} \cap \kappa \neq \emptyset$.

Since by Definition 2.2.4 it holds $\mathfrak{M}^{\hat{\alpha_0}}_{\mathbb{Y}} \subseteq \mathfrak{M}^{\zeta}_{\mathbb{M}(\vec{\eta})}$ and moreover $\tau \in \mathcal{C}_\gamma[\mathcal{A}] \cap \pi \subseteq C(\gamma+1,\sigma+1) \cap \pi \subseteq C(\hat{\alpha_0},\lambda) \cap \pi = \lambda$ for $\lambda \in {}_\sigma\mathfrak{M}^{\hat{\alpha_0}}_{\mathbb{Y}}$ we have

$$\Vdash {}_\tau M^{\zeta}_{\mathbb{M}(\vec{\eta})}(L_\lambda) \wedge L_\lambda \neq \emptyset \wedge \mathrm{Tran}(L_\lambda) \wedge \bigwedge_{i=1}^{p} a_i \in L_\lambda \quad \text{for all } \lambda \in {}_\sigma\mathfrak{M}^{\hat{\alpha_0}}_{\mathbb{Y}},$$

where $a_1, \ldots, a_p$ denotes the terms of F less than κ. Therefore we obtain from (5.19) by means of ($\bigwedge$) and ($\bigvee$)

$$\mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A}, \lambda, \kappa] \overset{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\lambda}+\omega}{\underset{\bullet}{\Big|\!\!\text{—}}} \bigvee\Gamma^{(\pi,\lambda)}, \exists z^\kappa\big({}_\tau M^\zeta_{\mathbb{M}(\vec{\eta})}(z) \wedge z \models F\big) \text{ for all } \lambda \in {}_\sigma\mathfrak{M}^{\hat{\alpha_0}}_{\mathbb{Y}} \cap \kappa. \tag{5.20}$$

If $m = 0$, we choose $\lambda = \Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\sigma}$. Since $\bigvee\Gamma^{(\pi,\lambda)} \in \Sigma_1(\lambda)$ the claim follows from (5.20) by use of (E-Up-Per) plus (Hull) and (Str).

If $m > 0$ let $G := \exists z^\kappa\big({}_\tau M^\zeta_{\mathbb{M}(\vec{\eta})}(z) \wedge z \models F\big)$. Analogously to subcase 4.1 we obtain

$$\mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A}, \kappa] \overset{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\kappa}}{\underset{\bullet}{\Big|\!\!\text{—}}} \forall x^\kappa\big(\neg_\tau M^{\hat{\alpha_0}}_{\mathbb{Y}}(x) \vee x \not\models \bigwedge\neg\Gamma^{(\pi,x)}\big), G. \tag{5.21}$$

Since $\mathrm{rdh}(\mathbb{X}) = m - 1 \geq 0$ and $\tau < \kappa \in {}_\sigma\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}$ Theorem 5.2.1 provides the existence of the reflection rule (${}_\tau M^{\hat{\alpha_0}}_{\mathbb{Y}}$-$\Pi_{m+1}(\kappa)$-Ref). Since $\bigwedge\neg\Gamma^{(\pi,\kappa)} \in \Pi_{m+1}(\kappa)$ we obtain by use of Lemma 4.3.3 ⑧ and this reflection rule

$$\mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A}, \kappa] \overset{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\kappa}}{\underset{\bullet}{\Big|\!\!\text{—}}} \exists x^\kappa\big({}_\tau M^{\hat{\alpha_0}}_{\mathbb{Y}}(x) \wedge x \models \bigwedge\neg\Gamma^{(\pi,x)}\big), \bigvee\Gamma^{(\pi,\kappa)}. \tag{5.22}$$

Since $(\zeta, \vec{\eta}) \in \mathrm{dom}(\mathbb{F}) \cap \mathcal{C}_\gamma[\mathcal{A}]$ Lemma 5.2.3① provides $\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\kappa} < \Psi_{\mathbb{X}}^{\hat{\alpha}\oplus\kappa}$ and a (Cut) applied to (5.21) and (5.22) yields the desired derivation.

Case 5, the last inference is by a subsidiary reflection rule of π *of the form* (${}_\tau M^\zeta_{\mathbb{K}(\vec{\eta})}$-$\Pi_{k+2}(\pi)$-Ref)*, i.e.* $\mathsf{M}^{<\zeta+1}_{\mathbb{K}}$-$\mathsf{P}_k \preceq \vec{R}'_\pi$*, and principal formula* $\exists z^\pi\big({}_\tau M^\zeta_{\mathbb{K}(\vec{\eta})}(z) \wedge z \models F\big)$*:* We have

$$\mathcal{C}_\gamma[\mathcal{A}] \overset{\alpha_0}{\underset{\bar{\mu}}{\big|\!\text{—}}} \Gamma, F$$

for some $\alpha_0 < \alpha$, $\zeta \in \mathcal{C}_\gamma[\mathcal{A}]_{C(\pi)}$, $\vec{\eta} \in \mathcal{C}_\gamma[\mathcal{A}] \cap \mathrm{dom}(\mathbb{K})^{\geq k}_{C(\pi)}$ and $\Gamma, F \in \Pi_{m+2}(\pi)$, since $k \leq m$. Proceeding as in subcase 4.1, i.e. applying (E-$\forall$-Inv), the subsidiary induction hypothesis, (E-$\forall$) and taking into account ${}_\sigma\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}} \subseteq {}_\sigma\mathfrak{M}^{\hat{\alpha_0}}_{\mathbb{X}}$ we obtain

$$\mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A}, \kappa] \overset{\Psi_{\mathbb{X}}^{\hat{\alpha_0}\oplus\kappa}}{\underset{\bullet}{\Big|\!\!\text{—}}} \Gamma^{(\pi,\kappa)}, F^{(\pi,\kappa)} \qquad \text{for all } \kappa \in {}_\sigma\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}. \tag{5.23}$$

Let $\vec{\eta} = (\eta_1, \ldots, \eta_h)$. Since we have $\forall_1^h i\,(\zeta, \eta_i \leq \gamma)$ by Lemma 2.3.2, $\zeta, \vec{\eta} \in C(\pi)$ and $\zeta, \vec{\eta} \in C(\gamma+1, \sigma+1) \subseteq C(\gamma+1, \kappa)$ plus $C(\gamma+1, \kappa) \cap \pi = \kappa$ it follows by Lemma 5.2.2 $\zeta, \vec{\eta} \in C(\kappa)$ for any $\kappa \in {}_\sigma\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}$. Moreover we have $\tau \in \mathcal{C}_\gamma[\mathcal{A}] \cap \pi \subseteq C(\gamma+1, \sigma+1) \cap \pi \subseteq C(\hat{\alpha}, \kappa) \cap \pi = \kappa$ for $\kappa \in {}_\sigma\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}$. Thus the reflection rule (${}_\tau M^\zeta_{\mathbb{K}(\vec{\eta})}$-$\Pi_{k+2}(\kappa)$-Ref) exists by means of Theorem 5.2.1. Applying this rule to (5.23) and taking into account Lemma 5.2.3 ① plus (Str) we obtain the desired derivation.

Case 6, the last inference is a (${}_\tau M^\zeta_{\mathbb{G}(\vec{\eta})}$-$\Pi_{g+2}(\pi_0)$-Ref) *inference with* $\tau < \pi_0 < \pi$ *and principal formula* $\exists z^{\pi_0}\big({}_\tau M^\zeta_{\mathbb{G}(\vec{\eta})}(z) \wedge z \models F\big)$*:* Thus we have

$$\mathcal{C}_\gamma[\mathcal{A}] \overset{\alpha_0}{\underset{\bar{\mu}}{\big|\!\text{—}}} \Gamma, F \tag{5.24}$$

for some $\alpha_0 < \alpha$. Since $\pi_0 < \pi$ we have $\Gamma, F \in \Sigma_{m+1}(\pi)$. Thus we may apply the subsidiary induction hypothesis to (5.24) and obtain

$$\mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A},\kappa] \left|\frac{\Psi_{\mathbb{X}}^{\hat{\alpha_0}\oplus\kappa}}{\cdot}\right. \Gamma^{(\pi,\kappa)}, F^{(\pi,\kappa)} \quad \text{for all } \kappa \in {}_{\sigma}\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha_0}}. \tag{5.25}$$

Since $\pi_0 \in \mathcal{C}_\gamma[\mathcal{A}] \cap \pi \subseteq C(\gamma+1, \sigma+1) \cap \pi \subseteq C(\hat{\alpha_0}, \kappa) \cap \pi = \kappa$ we have $F^{(\pi,\kappa)} \equiv F$ if $\kappa \in {}_{\sigma}\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha_0}}$. Thus by means of $({}_{\tau}M^{\zeta}_{\mathbb{G}(\vec{\eta})}\text{-}\Pi_{g+2}(\pi_0)\text{-}\mathsf{Ref})$ and (Str) plus taking into account ${}_{\sigma}\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha}} \subseteq {}_{\sigma}\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha_0}}$ the claim follows from (5.25). □

Theorem 5.2.6. *In the following let $\alpha_0 := 1$ and $\alpha_{n+1} := \Xi^{\alpha_n}$.*†

The property of being an admissible set above ω can be expressed by a $\mathcal{L}(\epsilon)$-Δ_0-formula $\mathrm{Ad}(x)$. For definiteness let $\mathrm{Ad}(x)$ be the formula given in [RA74]. If F is a Σ_1-sentence and

$$\Pi_\omega\text{-}\mathsf{Ref} \vdash \forall x\big(\mathrm{Ad}(x) \to F^x\big),$$

then there is a $k < \omega$ such that

$$\mathcal{C}_{\alpha_k} \left|\frac{\Psi_{\mathbb{X}}^{\alpha_k \oplus \Psi_{\mathbb{X}}^{\alpha_k}}}{\cdot}\right. F^{\Psi_{\mathbb{X}}^{\alpha_k}},$$

where $\mathbb{X} = (\omega^+; \epsilon; \epsilon; \epsilon; 0)$. Thus at $\mathbb{L}_{\Psi_{\mathbb{X}}^{\varepsilon_{\Xi+1}}}$ all $\Sigma_1^{\omega_1^{ck}}$-sentences of Π_ω-Ref are true, i.e. $|\Pi_\omega\text{-}\mathsf{Ref}|_{\Sigma_1^{\omega_1^{ck}}} \leq \Psi_{\mathbb{X}}^{\varepsilon_{\Xi+1}}$.

Proof. By the embedding theorem there exists an m such that

$$\mathcal{C}_0 \left|\frac{\omega^{\Xi+m}}{\Xi+m}\right. \forall x^{\Xi}\big(\mathrm{Ad}(x) \to F^x\big).$$

By ($\forall$-Inv) and (V-Ex) we obtain

$$\mathcal{C}_0 \left|\frac{\omega^{\Xi+m}}{\Xi+m}\right. \neg\,\mathrm{Ad}(L_{\omega^+}), F^{\omega^+}.$$

Moreover we have[1]

$$\mathcal{C}_0 \left|\frac{\omega^{\omega^+ +1}}{\omega^+ + \omega}\right. \mathrm{Ad}(L_{\omega^+}).$$

†Note that for $\alpha'_0 := 0$ and $\alpha'_{n+1} := \omega^{\Xi+\alpha'_n}$ we have $\alpha_n = \omega^{\alpha'_n}$. Thus $\alpha_n \in \mathsf{T}(\Xi) \cap \varepsilon_{\Xi+1}$.

[1] This is a bit cheated, but there is an abstract argument, which justifies that we do not bother about an exact proof for this: Let $\mathcal{L}_{Ad}(\in)$ be the language $\mathcal{L}(\in)$ augmented by an unary predicate symbol Ad and let Π_ω-Ref^* be the extension by definitions of Π_ω-Ref formalized in the language $\mathcal{L}_{Ad}(\in)$ plus the axiom (Ad.1) $\forall x\big(Ad(x) \to (\mathrm{Tran}(x) \wedge x \models \mathsf{KP}_\omega)\big)$. Defining $Ad(s) :\cong (L_\kappa = s)_{\kappa \in T}$ and $T := \{\kappa \in \mathrm{Reg} \cap |s|+1\}$ we easily obtain a derivation of $(\mathrm{Ad.1})^{\Xi}$ on hull-sets (for L_κ the axiom (Δ_0-Coll) is derivable by use of ($\Pi_2(\kappa)$-Ref)). Moreover it holds $\Pi_\omega\text{-}\mathsf{Ref}^* \vdash \big(\forall x(\mathrm{Ad}(x) \to F^x)\big) \leftrightarrow \big(\forall x(Ad(x) \to F^x)\big)$, if F is a Σ_1-sentence, and thus $\Pi_\omega\text{-}\mathsf{Ref} \vdash \forall x\big(\mathrm{Ad}(x) \to F^x\big)$ implies $\Pi_\omega\text{-}\mathsf{Ref}^* \vdash \forall x\big(Ad(x) \to F^x\big)$. Thereby we do not need a formal derivation of $\mathrm{Ad}(L_{\omega^+})$.

Thus by (Cut) it follows

$$\mathcal{C}_0 \left|\frac{\alpha_3}{\Xi+m}\right. F^{\omega^+}.$$

Applying predicative cut elimination $m-1$ times we obtain

$$\mathcal{C}_0 \left|\frac{\alpha_{m+2}}{\Xi+1}\right. F^{\omega^+}.$$

Since $0, \mathbb{X}, \Xi \in \mathcal{C}_\omega = C(\omega, 0)$, $0, \omega \leq \omega < \omega^+ \leq \Xi$ and $F^{\omega^+} \in \Sigma_1(\omega^+)$ we can apply Theorem 5.2.5 plus (Hull) and obtain

$$\mathcal{C}_{\alpha_{m+5}} \left|\frac{\Psi_{\mathbb{X}}^{\alpha_{m+4}} \oplus \Psi_{\mathbb{X}}^{\alpha_{m+4}}}{\cdot}\right. F^{\Psi_{\mathbb{X}}^{\alpha_{m+4}}}. \tag{5.26}$$

Semi-formal derivations with countable cut rank of $\mathcal{L}_{RS(\Xi)}$-sentences containing only parameters less than ω^+ are correct, since $\mathsf{T}(\Xi) \cap \omega^+$ is transitive. Thus, (5.26) implies $\mathbb{L}_{\Psi_{\mathbb{X}}^{\alpha_{m+4}}} \models F$. □

Review: Ordinal Analysis of Π_n-Ref

Ordinal Analysis of Π_n-Ref

The treatment of Π_ω-Ref which we present in the first part of this thesis is highly cumulative, i.e. by some minor modifications we obtain ordinal analyses of Π_n-Ref for every $n < \omega$.

Although these modifications consist essentially of (trivial) cutbacks of Definition 2.2.4 we give them here, as they provide a good access point to an understanding of our methods to the reader who is familiar with an ordinal analysis of $\mathsf{KP}\omega$ or Π_3-Ref.

Beyond that we outline which amount of Structure and Fine Structure Theory is needed in the different cases. Thereby it becomes clear that the increase of effort needed for a treatment of the different cases can be visualized as follows:

$$\Pi_2\text{-Ref} \prec \Pi_3\text{-Ref} \prec \Pi_4\text{-Ref} = \Pi_5\text{-Ref} = \ldots = \Pi_n\text{-Ref} \prec \Pi_\omega\text{-Ref}.$$

Singular Collapsing: Π_2-Ref

The proof-theoretic ordinal of Π_2-Ref equals that of $\mathsf{KP}\omega$ and ID_1, the theory of non-iterated inductive definitions. It is the well-known Howard-Bachmann-Ordinal. The theory ID_1 can be viewed as a paradigmatic example of an impredicative theory, since the definition of a fix-point of an operator "from above" is one of the obvious examples of an impredicative definition.

To make our approach work for a treatment of Π_2-Ref we need the following:

Definition.

$$\begin{aligned}
C(\alpha,\pi) &:= \bigcup_{n<\omega} C^n(\alpha,\pi), \quad \text{where} \\
C^0(\alpha,\pi) &:= \pi \cup \{0,\omega^+\}, \quad \text{and} \\
C^{n+1}(\alpha,\pi) &:= \begin{cases} C^n(\alpha,\pi) \cup \\ \{\gamma+\omega^\delta \,|\, \gamma,\delta \in C^n(\alpha,\pi) \wedge \gamma =_{\text{NF}} \omega^{\gamma_1}+\ldots+\omega^{\gamma_m} \wedge \gamma_m \geq \delta\} \cup \\ \{\varphi(\xi,\eta) \,|\, \xi,\eta \in C^n(\alpha,\pi)\} \cup \\ \{\Psi^\gamma_{\mathbb{X}} \,|\, \mathbb{X},\gamma \in C^n(\alpha,\pi) \wedge \gamma < \alpha \wedge \Psi^\gamma_{\mathbb{X}} \text{ is well-defined}\}. \end{cases}
\end{aligned}$$

For ω^+ we define the following 0-ary reflection configuration and a reflection instance

$(\omega^+; \mathsf{P}_0; \epsilon; \epsilon; 0)$.

For $\alpha \geq 0$ we define $\mathfrak{M}_{\mathbb{X}}^{\alpha}$ as the set consisting of all ordinals $\kappa < \omega^+$ satisfying

1. $C(\alpha, \kappa) \cap \omega^+ = \kappa$,

2. $\mathbb{X}, \alpha \in C(\kappa)$.

If $\mathfrak{M}_{\mathbb{X}}^{\alpha} \neq \emptyset$ we denote the least element of $\mathfrak{M}_{\mathbb{X}}^{\alpha}$ by $\Psi_{\mathbb{X}}^{\alpha}$.

Disscusion. Of course this definition is oversized, as it would also be enough to just define the minimal elements $\Psi_{\mathbb{X}}^{\alpha}$ for all $\alpha \in C(\Gamma_{\omega^+ +1}, 0)$ of the hierarchies $\mathfrak{M}_{\mathbb{X}}^{\alpha}$.

The existence of $\Psi_{\mathbb{X}}^{\alpha}$ follows directly by means of Lemma 2.3.5. The proof of the <-comparison Theorem 2.3.14 shrinks down to a part of subcase 3.3. Moreover the Reflection Elimination Theorem 5.2.5 has just to be proved for $m = 0$ and $\mu = \omega^+$. Therefore we can suspend with the consideration $\mathbb{X} \neq \mathbb{Y}$ in subcase 3.2 in the proof of this Theorem and in case four of this proof we just have to consider the subcase 4.2 with $m = 0$. Moreover the cases five and six are not needed at all. Thus there is not any need for a fine structure analysis of the collapsing hierarchies.

Before turning to the theory Π_3-Ref we want to emphasize that it was (historically) a big step from an ordinal-analysis of Π_2-Ref towards an ordinal-analysis of Π_3-Ref. Thereby there are a lot of theories in strength between Π_2-Ref and Π_3-Ref which have been investigated proof-theoretically. Most prominent the theories KPl, KPi and KPM.

The first one axiomatizes a universe, which is a union of admissible sets and the second one an admissible union of admissible sets. The fundamentals of a proof-theoretic treatment of these theories goes back to the investigations of W. Pohlers and W. Buchholz on theories of iterated-inductive-definability (see. e.g [WBS81]) and the work of G. Jäger, mainly [Jae86]. For an excellent exposition of the historical development of impredicative proof-theory consult [Fef10]. Ordinal-analyses of these theories can be found in [Buc93] and [Poh98]. The latter also gives a good survey of proof-theoretic investigations of a hole zoo of theories about the strength of KPl and KPi.

The theory KPM axiomatizes a recursive Mahlo-Universe. An ordinal-analysis of this theory was at first obtained by M. Rathjen in [Rat91].

By simple extensions of the above given definition of the collapsing-hierarchies for Π_2-Ref we obtain collapsing-hierarchies for the theories KPl, KPi and KPM. In technical aspects the proof of the Reflection Elimination Theorem for this theories differs from that of Π_2-Ref in the point, that we have to consider the case $\mathbb{Y} \neq \mathbb{X}$ in subcase 3.2.

Simultaneous Collapsing: Π_3-Ref

The theory Π_3-Ref is the simplest theory whose proof-theoretic treatment actually requires a definition of collapsing hierarchies instead of just defining collapsing functions.

An ordinal analysis of this theory was at first achieved by M. Rathjen in [Rat94b]. To make our approach work for a treatment of Π_3-Ref we define:

Definition. Let Ξ be a Π^1_1-indescribable cardinal.

$$
\begin{aligned}
C(\alpha,\pi) &:= \bigcup_{m<\omega} C^m(\alpha,\pi), \quad \text{where} \\
C^0(\alpha,\pi) &:= \pi \cup \{0,\Xi\}, \quad \text{and} \\
C^{m+1}(\alpha,\pi) &:= \begin{cases} C^m(\alpha,\pi)\cup \\ \{\gamma+\omega^\delta \mid \gamma,\delta \in C^m(\alpha,\pi) \wedge \gamma \underset{\text{NF}}{=} \omega^{\gamma_1}+\ldots+\omega^{\gamma_m} \wedge \gamma_m \geq \delta\} \cup \\ \{\varphi(\xi,\eta) \mid \xi,\eta \in C^m(\alpha,\pi)\} \cup \\ \{\kappa^+ \mid \kappa \in C^m(\alpha,\pi) \cap \mathrm{Card} \cap \Xi\} \cup \\ \{\Psi^\gamma_{\mathbb{X}} \mid \mathbb{X},\gamma \in C^m(\alpha,\pi) \wedge \gamma<\alpha \wedge \Psi^\gamma_{\mathbb{X}} \text{ is well-defined}\}. \end{cases}
\end{aligned}
$$

0.1. We define the 0-ary reflection configuration and reflection instance

$$\mathbb{A} := (\Xi; \mathsf{P}_1; \mathsf{M}^{<0}_{\mathbb{A}}\text{-}\mathsf{P}_0; \epsilon; 0) \qquad \to 1.$$

For technical convenience we also define $\vec{R}_\Xi := (\mathsf{M}^{<0}_{\mathbb{A}}\text{-}\mathsf{P}_1, \mathsf{M}^{<0}_{\mathbb{A}}\text{-}\mathsf{P}_0)$.

0.2. For every cardinal $\omega \leq \kappa < \Xi$ we define the 0-ary reflection configuration and reflection instance

$$(\kappa^+; \mathsf{P}_0; \epsilon; \epsilon; 0) \qquad \to 1.$$

For technical convenience we also define $\vec{R}_{\kappa^+} := (\mathsf{M}^{<0}_{\mathbb{A}}\text{-}\mathsf{P}_0)$.

1. Let $\mathbb{X} = \mathbb{F}$ be a 0-ary reflection configuration and a reflection instance of the form

$$(\pi; \mathsf{P}_m; \epsilon; \mathbb{Z}; \delta)$$

Then we either have $\delta = 0 = m$ and $\pi = \bar{\kappa}^+$ for some cardinal $\bar{\kappa}$, or $\delta = \delta_0 + 1$, $m = 1$ and $\pi = \Psi^{\delta_0}_{\mathbb{Z}}$.

For $\alpha \geq \delta$ we define $\mathfrak{M}^\alpha_{\mathbb{X}}$ as the set consisting of all ordinals $\kappa < \pi$ satisfying

1. $C(\alpha,\kappa) \cap \pi = \kappa$,
2. $\mathbb{X}, \alpha \in C(\kappa)$,
3. if $m = 1$: $\kappa \models \epsilon$,
4. if $m = 1$: $\kappa \models \mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_0$.

From now on we assume $\mathfrak{M}^\alpha_{\mathbb{X}} \neq \emptyset$ and by $\Psi^\alpha_{\mathbb{X}}$ we denote the least element of $\mathfrak{M}^\alpha_{\mathbb{X}}$. If $m = 1$ we also define $\vec{R}_{\Psi^\alpha_{\mathbb{F}}} := (\widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_0)$.

For the following 1.$\cdot$ subclauses we suppose $m = 1$. If $m = 0$ we do not equip $\Psi^\alpha_{\mathbb{X}}$ with any reflection configurations or instances.

1.1. Let $\alpha = \delta$. Then we define the 0-ary reflection configuration and reflection instance

$$(\Psi^\alpha_{\mathbb{X}}; \mathsf{P}_0; \epsilon; \mathbb{X}; \alpha+1) \qquad \to 1.$$

1.2. Let $\alpha > \delta$. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) := [\delta, \alpha)_{C(\Psi^{\alpha}_{\mathbb{X}})}$ and reflection instances

$$\mathbb{G}(\zeta) := (\Psi^{\alpha}_{\mathbb{X}}; \mathsf{M}^{\zeta}_{\mathbb{F}}\text{-}\mathsf{P}_0; \epsilon; \alpha + 1) \qquad \rightarrow 2.$$

2. Let $\mathbb{X} := \mathbb{F}(\xi)$ be a reflection configuration of the form

$$(\Psi^{\delta}_{\mathbb{Z}}; \mathsf{M}^{\xi}_{\mathbb{M}}\text{-}\mathsf{P}_0; \epsilon; \mathbb{Z}; \delta + 1).$$

Then it holds $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_0)$ for some $\gamma > \xi$ and $\mathbb{M} = \mathbb{A}$.

For $\alpha \geq \delta + 1$ we define $\mathfrak{M}^{\alpha}_{\mathbb{X}}$ as the set consisting of all ordinals $\kappa \in \mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})} \cap \Psi^{\delta}_{\mathbb{Z}}$ satisfying

1. $C(\alpha, \kappa) \cap \Psi^{\delta}_{\mathbb{Z}} = \kappa$,
2. $\mathbb{X}, \alpha \in C(\kappa)$.

From now on we assume $\mathfrak{M}^{\alpha}_{\mathbb{X}} \neq \emptyset$ and by $\Psi^{\alpha}_{\mathbb{X}}$ we denote the least element of $\mathfrak{M}^{\alpha}_{\mathbb{X}}$. Moreover we define $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} := (\widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_0)$.

2.1. Let $\xi = 0$. Then we define the 0-ary reflection configuration and reflection instance

$$(\Psi^{\alpha}_{\mathbb{X}}; \mathsf{P}_0; \epsilon; \mathbb{X}; \alpha + 1) \qquad \rightarrow 1.$$

2.2. Let $\xi > 0$. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) = [0, \xi)_{C(\Psi^{\alpha}_{\mathbb{X}})}$ and reflection instances

$$\mathbb{G}(\zeta) := (\Psi^{\alpha}_{\mathbb{X}}; \mathsf{M}^{\zeta}_{\mathbb{M}}\text{-}\mathsf{P}_0; \epsilon; \mathbb{X}; \alpha + 1) \qquad \rightarrow 2.$$

Disscusion. Here we need in essence the full Structure Theory presented in section 2.3.

With respect to the Fine Structure Theory we just have to secure that for every $\kappa \in \mathsf{T}(\Xi) \cap \mathfrak{M}^{\alpha}_{\mathbb{A}}$ there are reflection instances $(\kappa; \mathsf{M}^{\xi}_{\mathbb{A}}\text{-}\mathsf{P}_0; \ldots)$ for every $\xi \in [0, \alpha)_{C(\kappa)}$. This holds since if $\kappa \in \mathfrak{M}^{\alpha}_{\mathbb{A}}$ there must be a reflection instance $\mathbb{X}$ and an ζ such that $\kappa = \Psi^{\zeta}_{\mathbb{X}}$ as $C(\alpha, \kappa_0)$ is closed under $+, \varphi$ and $\cdot^{+}$ for every κ_0.

If $\mathbb{X} = \mathbb{A}$ we must have $\zeta \geq \alpha$ as otherwise we would obtain the contradiction $\kappa \in C(\alpha, \kappa) \cap \Xi = \kappa$. Therefore we have $\vec{R}_{\kappa} = (\mathsf{M}^{<\zeta}_{\mathbb{A}}\text{-}\mathsf{P}_0)$ for some $\zeta \geq \alpha$ and thus the desired reflection instances exists.

The case $\mathbb{X} = (\pi; \mathsf{P}_0; \ldots,)$ cannot occur, since $\kappa \in \mathfrak{M}^{\alpha}_{\mathbb{A}}$ and therefore κ is Π^1_0-indescribable. This would contradict the minimality of $\kappa \in \mathfrak{M}^{\zeta}_{\mathbb{X}}$.

If $\mathbb{X} = (\pi; \mathsf{M}^{\gamma}_{\mathbb{A}}\text{-}\mathsf{P}_0; \ldots)$ we have $\vec{R}_{\kappa} = (\widetilde{\mathsf{M}}^{<\gamma}_{\mathbb{A}}\text{-}\mathsf{P}_0)$. If we would have $\alpha > \gamma$ then κ would be $\mathfrak{M}^{\gamma}_{\mathbb{A}}\text{-}\Pi^1_0$-indescribable, but then there would be a $\kappa_0 < \kappa$ with the defining properties of κ. Thereby we must have $\alpha \leq \gamma$ and the desired reflection instances exist.

Iterated Simultaneous Collapsing: $\mathsf{\Pi}_n$-Ref

An ordinal-analysis of $\mathsf{\Pi}_n$-Ref cannot be obtained from an ordinal-analysis of $\mathsf{\Pi}_3$-Ref by simple iteration-arguments as indicated in [Rat94b].

Even in a treatment of $\mathsf{\Pi}_4$-Ref we are confronted with the handling of ordinals which are Π_2-reflecting on a set of Π_3-reflecting ordinals. Such "schizophrenic" ordinals, as C. Duchhardt calls them, do not occur in an ordinal-analysis of $\mathsf{\Pi}_3$-Ref. A proof-theoretic management of such ordinals requires either a painstaking definition of the collapsing hierarchies or already the Fine Structure Theory of Chapter 3. The first option was chosen by C. Duchhardt in [Duc08] to obtain an ordinal analysis of $\mathsf{\Pi}_4$-Ref.

In the following we want to present some modifications of the first part of this thesis which lead to an ordinal analysis of $\mathsf{\Pi}_n$-Ref. It will transpire that the case $n = 4$ is the generic case for a treatment of $\mathsf{\Pi}_n$-Ref for arbitrary $n < \omega$.

Definition 5.2.7. Suppose that $n > 1$ and let Ξ be a Π^1_n-indescribable cardinal.

$$\begin{aligned} C(\alpha,\pi) &:= \bigcup_{m<\omega} C^m(\alpha,\pi), \quad \text{where} \\ C^0(\alpha,\pi) &:= \pi \cup \{0,\Xi\}, \quad \text{and} \\ C^{m+1}(\alpha,\pi) &:= \begin{cases} C^m(\alpha,\pi) \cup \\ \{\gamma+\omega^\delta \mid \gamma,\delta \in C^m(\alpha,\pi) \wedge \gamma \underset{\mathrm{NF}}{=} \omega^{\gamma_1}+\ldots+\omega^{\gamma_m} \wedge \gamma_m \geq \delta\} \cup \\ \{\varphi(\xi,\eta) \mid \xi,\eta \in C^m(\alpha,\pi)\} \cup \\ \{\kappa^+ \mid \kappa \in C^m(\alpha,\pi) \cap \mathrm{Card} \cap \Xi\} \cup \\ \{\Psi^\gamma_{\mathbb{X}} \mid \mathbb{X},\gamma \in C^m(\alpha,\pi) \wedge \gamma < \alpha \wedge \Psi^\gamma_{\mathbb{X}} \text{ is well-defined}\}. \end{cases} \end{aligned}$$

0.1. We define the 0-ary reflection configuration and reflection instance

$$\mathbb{A} := (\Xi; \mathsf{P}_n; (\mathsf{M}^{<0}_{\mathbb{A}}\text{-}\mathsf{P}_{n-1},\ldots,\mathsf{M}^{<0}_{\mathbb{A}}\text{-}\mathsf{P}_0); \epsilon; 0) \qquad \to 1.$$

For technical convenience we also define $\vec{R}_\Xi := (\mathsf{M}^{<0}_{\mathbb{A}}\text{-}\mathsf{P}_n,\ldots,\mathsf{M}^{<0}_{\mathbb{A}}\text{-}\mathsf{P}_0)$.

0.2. For every cardinal $\omega \leq \kappa < \Xi$ we define the 0-ary reflection configuration and reflection instance

$$(\kappa^+; \mathsf{P}_0; \epsilon; \epsilon; 0) \qquad \to 1.$$

For technical convenience we also define $\vec{R}_{\kappa^+} := (\mathsf{M}^{<0}_{\mathbb{A}}\text{-}\mathsf{P}_0)$.

1. Let $\mathbb{X} = \mathbb{F}$ be a 0-ary reflection configuration and a reflection instance of the form

$$(\pi; \mathsf{P}_m; \vec{R}; \mathbb{Z}; \delta)$$

Then we either have $\delta = 0 = m$ and $\pi = \bar{\kappa}^+$ for some cardinal $\bar{\kappa}$, or $\delta = \delta_0 + 1$ and $\pi = \Psi^{\delta_0}_{\mathbb{Z}}$. In any case we have $\vec{R}_\pi = (\mathsf{M}^{<0}_{\mathbb{A}}\text{-}\mathsf{P}_m, \vec{R})$.

For $\alpha \geq \delta$ we define $\mathfrak{M}_{\mathbb{X}}^{\alpha}$ as the set consisting of all ordinals $\kappa < \pi$ satisfying

1. $C(\alpha, \kappa) \cap \pi = \kappa$,
2. $\mathbb{X}, \alpha \in C(\kappa)$,
3. if $m > 0$: $\kappa \models \vec{R}$,
4. if $m > 0$: $\kappa \models \mathsf{M}_{\mathbb{F}}^{<\alpha}\text{-}\mathsf{P}_{m-1}$.

From now on we assume $\mathfrak{M}_{\mathbb{X}}^{\alpha} \neq \emptyset$ and by $\Psi_{\mathbb{X}}^{\alpha}$ we denote the least element of $\mathfrak{M}_{\mathbb{X}}^{\alpha}$. If $m > 0$ we also define $\vec{R}_{\Psi_{\mathbb{F}}^{\alpha}} := (\widetilde{\mathsf{M}}_{\mathbb{F}}^{<\alpha}\text{-}\mathsf{P}_{m-1}, \vec{R}_{<m-1})$.

For the following 1.· subclauses we suppose $m = m_0 + 1 > 0$. If $m = 0$ we do not equip $\Psi_{\mathbb{X}}^{\alpha}$ with any reflection configurations or instances.

1.1. Let $\alpha = \delta$ and $\vec{R}_{m_0} = (\mathsf{M}_{\mathbb{R}_1}^{<\xi_1}\text{-}\mathsf{P}_{m_0})$ with $\mathrm{o}(\mathbb{R}_1) = \xi_1$. Due to the $\widetilde{\cdot}$-operator we then have $\xi_1 = 0$ and $\mathbb{R}_1 = \mathbb{A}$. Then we define the 0-ary reflection configuration and reflection instance

$$(\Psi_{\mathbb{X}}^{\alpha}; \mathsf{P}_{m_0}; \vec{R}_{<m_0}; \mathbb{X}; \alpha + 1) \qquad \to 1.$$

1.2. Let $\alpha = \delta$ and $\vec{R}_{m_0} = (\mathsf{M}_{\mathbb{R}_1}^{<\xi_1}\text{-}\mathsf{P}_{m_0})$ with $\mathrm{o}(\mathbb{R}_1) < \xi_1$. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) := [\mathrm{o}(\mathbb{R}_1), \xi_1)_{C(\Psi_{\mathbb{X}}^{\alpha})} \times \mathrm{dom}(\mathbb{R}_1)_{C(\Psi_{\mathbb{X}}^{\alpha})}$ and reflection instances

$$\mathbb{G}(\zeta, \vec{\eta}) := (\Psi_{\mathbb{X}}^{\alpha}; \mathsf{M}_{\mathbb{R}_1(\vec{\eta})}^{\zeta}\text{-}\mathsf{P}_{m_0}; \vec{R}_{<m_0}; \mathbb{X}; \alpha + 1) \qquad \to 2.$$

1.3. Let $\alpha > \delta$. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) := [\delta, \alpha)_{C(\Psi_{\mathbb{X}}^{\alpha})}$ and reflection instances

$$\mathbb{G}(\zeta) := (\Psi_{\mathbb{X}}^{\alpha}; \mathsf{M}_{\mathbb{F}}^{\zeta}\text{-}\mathsf{P}_{m_0}; \vec{R}_{<m_0}; \mathbb{X}; \alpha + 1) \qquad \to 2.$$

2. Let $\mathbb{X} := \mathbb{F}(\xi, \vec{\nu})$ be a reflection configuration of the form[2]

$$(\Psi_{\mathbb{Z}}^{\delta}; \mathsf{M}_{\mathbb{M}(\vec{\nu})}^{\xi}\text{-}\mathsf{P}_m; \vec{R}; \mathbb{Z}; \delta + 1).$$

Then it holds $\vec{R}_{\Psi_{\mathbb{Z}}^{\delta}} = (\mathsf{M}_{\mathbb{M}}^{<\gamma}\text{-}\mathsf{P}_m, \vec{R})$ for some $\gamma > \xi$ and $\mathbb{M} \in \mathrm{Prcnfg}(\mathbb{X})$.

For $\alpha \geq \delta + 1$ we define $\mathfrak{M}_{\mathbb{X}}^{\alpha}$ as the set consisting of all ordinals $\kappa \in \mathfrak{M}_{\mathbb{M}(\vec{\nu})}^{\xi} \cap \Psi_{\mathbb{Z}}^{\delta}$ satisfying

1. $C(\alpha, \kappa) \cap \Psi_{\mathbb{Z}}^{\delta} = \kappa$,
2. $\mathbb{X}, \alpha \in C(\kappa)$,
3. if $m > 0$: $\kappa \models \vec{R}$,
4. if $m > 0$: $\kappa \models \mathsf{M}_{\mathbb{F}}^{<\alpha}\text{-}\mathsf{P}_{m-1}$.

[2]Note, that $\vec{\nu}$ can also be a vector of zero length, cf. 1.3

From now on we assume $\mathfrak{M}^{\alpha}_{\mathbb{X}} \neq \emptyset$ and by $\Psi^{\alpha}_{\mathbb{X}}$ we denote the least element of $\mathfrak{M}^{\alpha}_{\mathbb{X}}$. Moreover we define $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} := ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{\geq m}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}, \vec{R}_{<m-1})$.

Let $\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}} = (\mathsf{M}^{<\sigma_1}_{\mathbb{S}_1}\text{-}\mathsf{P}_{s_1}, \ldots)$.

2.1. Let $\sigma_1 = \mathrm{o}(\mathbb{S}_1)$. Due to the $\widetilde{\cdot}$-operator we then have $\sigma_1 = 0$ and $\mathbb{S}_1 = \mathbb{A}$. Then we define the 0-ary reflection configuration and reflection instance

$$(\Psi^{\alpha}_{\mathbb{X}}; \mathsf{P}_{s_1}; ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{(s_1,m]}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}, \vec{R}_{<m-1}); \mathbb{X}; \alpha+1) \qquad \to 1.$$

2.2. Let $\sigma_1 > \mathrm{o}(\mathbb{S}_1)$. Then we define the reflection configuration $\mathbb{G}$, defined on $\mathrm{dom}(\mathbb{G}) := [\mathrm{o}(\mathbb{S}_1), \sigma_1)_{C(\Psi^{\alpha}_{\mathbb{X}})} \times \mathrm{dom}(\mathbb{S}_1)_{C(\Psi^{\alpha}_{\mathbb{X}})}$ and reflection instances

$$\begin{aligned}\mathbb{G}(\zeta, \vec{\eta}) := (&\Psi^{\alpha}_{\mathbb{X}}; \mathsf{M}^{\zeta}_{\mathbb{S}_1(\vec{\eta})}\text{-}\mathsf{P}_{s_1};\\ &((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{(s_1,m]}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{m-1}, \vec{R}_{<m-1}); \mathbb{X}; \alpha+1) \qquad \to 2.\end{aligned}$$

Disscusion

Here we need the full amount of Structure Theory and also the entire Chapter 3 of Fine Structure Theory.

Thus a treatment of Π_ω-Ref differs only in the presence of the reflection configuration $\mathbb{A}$, as defined in clause 0.1 and 1 on page 17 from a treatment of Π_n-Ref. Since this is a reflection configuration with variable reflection degree the vector $\vec{R}_{\Psi^{\alpha}_{\mathbb{A}(m+1)}}$ has to be defined as $(\mathsf{M}^{<\alpha}_{\mathbb{A}}\text{-}\mathsf{P}_m, \ldots, \mathsf{M}^{<\alpha}_{\mathbb{A}}\text{-}\mathsf{P}_0)$ instead of just $(\mathsf{M}^{<\alpha}_{\mathbb{A}}\text{-}\mathsf{P}_m)$ (note that $\kappa \in \mathfrak{M}^{\alpha}_{\mathbb{A}(m+1)}$ implies $\kappa \in \mathfrak{M}^{\alpha}_{\mathbb{A}(k+1)}$ for all $k < m$).[†]

Moreover to enable a Fine Structure Theory some non-canonical (with respect to a straight forward extension of Definition 5.2.7) restrictions of domains (cf. subclauses 1.2, 2.2, 3.2 of Definition 2.2.4) have to be made.

These peculiarities of reflection configurations with variable reflection degrees become even more awkward in a proof-theoretical treatment of the Theory Stability, which is given in part three of this thesis.

[†]The entry $(\mathsf{M}^{<0}_{\mathbb{A}}\text{-}\mathsf{P}_{n-1}, \ldots, \mathsf{M}^{<0}_{\mathbb{A}}\text{-}\mathsf{P}_0)$ in clause 0.1 on page 68 is of pure technical nature to secure that for all $i \leq \mathrm{rdh}(\mathbb{X})$ the component $(\vec{R}_{\mathbb{X}})_i \neq \epsilon$, but could also be defined as ϵ.

Part II.

The Provable Recursive Functions of Π_ω-Ref

6. Introduction

In this second part of the thesis we want to achieve a characterization of the provable recursive functions of Π_ω-Ref. There are several ways to extract a classification of the provable recursive functions of a theory T out of an ordinal-analysis of T. For a brief overview see [Rat99]. The most perspicuous one seems to be the method of A. Weiermann, developed in [Wei96].

Roughly sketched this method runs as follows: Define a subrecursive hierarchy $\langle f_\alpha \mid \alpha \in \mathsf{T}\rangle$ on the given ordinal notation system T by use of fundamental sequences defined via a norm on the notation system (e.g. a canonical norm is given by letting $\mathrm{N}(\alpha)$ be the number of symbols occurring in the term for α) as established in [CBW94]. Define a semi-formal derivation calculus which acts on fragmented hull-sets instead of hull-sets, where the fragmentation of a hull-set $\mathcal{H}[\mathcal{A}]$ to the index γ is defined as the set of all $\alpha \in \mathcal{H}[\mathcal{A}]$ with $\mathrm{N}(\alpha) < f_\gamma(N(\mathcal{A}))$, in the following denoted by $\mathcal{F}_\gamma[\mathcal{A}]$.

Then it holds for a derivation of a $\Sigma_1(\omega)$-sentence $\exists z^\omega F(z)$ on a fragmented hull-set $\mathcal{F}_\gamma[\mathcal{A}]$ that there exists an $s < f_\gamma(\mathrm{N}(\mathcal{A}))$ which satisfies F. Therefore we are able to dominate the provable recursive functions of a theory T by functions of $\langle f_\alpha \mid \alpha \in \mathsf{T}\rangle$ as soon as we are able to transform the embedding and (im)-predicative-cut-elimination theorems given by an ordinal-analysis of T to embedding and (im)-predicative-cut-elimination theorems on fragmented hull-sets. Moreover this domination of the provable recursive functions of T provides a characterization of these functions, if the utilized domination-functions f_γ are itself provable recursive in T, i.e. if the index γ is less than the proof-theoretic ordinal of T.

By this method we obtain straight forwardly a characterization of the provable recursive functions of Peano Arithmetic, PA, from an ordinal-analysis of PA. We obtain the embedding theorems and the predicative cut-elimination theorem of such an ordinal-analysis also on fragmented hull-sets by choosing the fragmentation index γ (more or less) equal to the derivation lengths occurring in these theorems.

In case of KP_ω (i.e. Π_2-Ref) we can nearly proceed in the same way, but instead of choosing γ equal to the derivation length α we have to set $\gamma \approx \Psi(\alpha)$. With this setting it is possible to transform the impredicative-cut-elimination theorem on fragmented hull-sets as follows:

$$\mathcal{F}_{\Psi(\gamma)}[\mathcal{A}] \left|\frac{\alpha}{\Omega+1}\right. \Gamma \quad \Rightarrow \quad \mathcal{F}_{\Psi(\hat{\alpha}_\gamma)}[\mathcal{A}] \left|\frac{\Psi(\hat{\alpha}_\gamma)}{\cdot}\right. \Gamma, \quad \text{if } \Gamma \subseteq \Sigma_1(\Omega).$$

For details see e.g. [Ste06].

However, in case of theories whose ordinal-analysis requires an iterated impredicative cut-elimination theorem the parameter γ cannot be handled in such a trivial manner. An iterated application of the collapsing function Ψ to γ fails, since in general it does not hold $f_\gamma(x) < f_{\Psi(\gamma)}(x)$ for almost all x.

In [BW99] and [Bla97] A. Weiermann and B. Blankertz refined the controlling conditions of the parameter γ, by which means even a treatment of theories whose ordinal-analyses require an iterated impredicative-cut-elimination theorem can be carried out. The drawbacks of such a refined parameter γ are a proper amount of (often trivial but tedious) extra calculations.

In the following we want to introduce a slightly modificated version of the Weiermannian method, by which means it is possible to obtain a characterization of the provable recursive functions from an ordinal analysis in an easy way. Even for theories whose proof-theoretic handling requires an iterated cut-elimination theorem.

The main idea is to employ not only the subrecursive hierarchy $\langle f_\alpha \,|\, \alpha \in \mathsf{T}\rangle$, but to utilize also subrecursive hierarchies $\langle f_\alpha \,|\, \alpha \in \mathcal{H}\rangle$ (uniformly) defined on proper subsets $\mathcal{H}$ of T. Henceforth we want to indicate by a supscript $\mathcal{H}$ that $f_\gamma^{\mathcal{H}}$ is defined with respect to the underlying set $\mathcal{H}$. Such relativized defined subrecursive hierarchies are "collapsible" in the following sense: Let $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathsf{T}$ and suppose $\Psi : \mathcal{H}_1 \to \mathcal{H}_2$ is an order-preserving function (which behaves "good" with respect to the norm). Then it holds $f_\gamma^{\mathcal{H}_1}(x) \leq f_{\Psi(\gamma)}^{\mathcal{H}_2}(x)$ for almost all x. Thereby we do not have to care about the parameter γ in the Reflection Elimination Theorem, but we can choose it equally to the "complexity degree" (i.e. the parameter γ in Theorem 5.2.5) of this theorem, if there eventually is an order-preserving function which collapses γ to the transitive part of T. However, such a function can easily be given in this situation. E.g. suppose we have to "collapse" $f_\gamma^{C(\gamma,0)}$. This function is equal to $f_\gamma^{C(\gamma,0)\cap(\gamma+1)}$ and

$$\lambda\xi.\Psi_{\mathbb{X}}^{\gamma\oplus\xi} : C(\gamma,0) \cap \gamma+1 \longrightarrow C(\gamma\cdot 2+1,0),$$

is a total, order-preserving function, if $\gamma, \mathbb{X} \in C(\gamma,0)$. Therefore we have

$$f_\gamma^{C(\gamma,0)}(x) \leq f_{\Psi_{\mathbb{X}}^{\gamma\cdot 2}}^{C(\gamma\cdot 2+1,0)}(x) \quad \text{for almost all } x.$$

To be able to chose the fragmentation index γ in the embedding theorems on fragmented hull-sets equal to the derivation length we also modify (compared to [BW99] and [Bla97]) the definition of the norm $|t|_{\mathrm{N}}$ of an $\mathcal{L}_{RS(\Xi)}$-term t. By this modification we lose the property $|t|_{\mathrm{N}} = |t|$ for terms t of finite stage, but even in our setting there exists for every $u \in \mathbb{L}_\omega$ a canonical term $\tilde{u}$, such that $|\tilde{u}|_{\mathrm{N}}$ and $|\tilde{u}|$ are related in a primitive recursive manner.

Finally we also save the trouble of giving a derivation (on a fragmented hull-set) of a formula "$x = \mathbb{L}_\omega$". Instead we add a unary predicate symbol Ad_0 by which means it is possible to characterize $\mathbb{L}_\omega$ as the amenable set, whose elements are all finite.

7. Subrecursive Hierarchies

In this chapter we define the subrecursive hierarchies (relativized on subsets of $\mathsf{T}(\Xi)$) by means of the methods developed in [CBW94]. I.e. we utilize a norm on $\mathsf{T}(\Xi)$, which is directly given by the inductive definition of $\mathsf{T}(\Xi)$, by which means we define fundamental sequences which give rise to a subrecursive hierarchy (cf. Definition 7.3.1).[1]

7.1. The Theory Π_ω-Ref*

Since there are no primitive recursive function symbols in $\mathcal{L}_\in$ a coding of a computation of a partial recursive function can only be carried out in $\mathbb{L}_\omega$ but not in ω. So to be able to talk about recursive functions in $\mathcal{L}_\in$ we either need an $\mathcal{L}_\in$-formula $\ell(u)$ which defines $\mathbb{L}_\omega$ or we assume that there is a unary predicate Ad_0, which applies exactly to $\mathbb{L}_\omega$. As the second approach is proof-theoretically a bit easier to handle, we follow this way.

Definition 7.1.1. Let $\mathcal{L}_{Ad_0}(\in)$ be the language $\mathcal{L}(\in)$ augmented by a unary predicate symbol Ad_0. Let Π_ω-Ref* be the theory Π_ω-Ref formalized in the language $\mathcal{L}_{Ad_0}(\in)$ and extended by the axioms and scheme:

$$(\mathrm{Ad}_0.1)\quad \forall x\big(Ad_0(x) \to ((\mathrm{Tran})^x \wedge (\mathrm{Nullset})^x \wedge (\mathrm{Pair})^x \wedge (\mathrm{Union})^x)\big),$$

$$(\mathrm{Ad}_0.2)\quad \forall x\big(Ad_0(x) \to (\Delta_0\text{-Sep})^x\big),$$

$$(\mathrm{Ad}_0.3)\quad \forall x\big(Ad_0(x) \to \forall y \exists j \big(\exists f \in j\, \exists n \in j\, (\mathrm{Fun}(f) \wedge \mathrm{Natno}(n) \wedge f : y \xrightarrow{1-1} n)\big)^x\big),$$

where $\mathrm{Fun}(x)$, $\mathrm{Natno}(x)$ and $f : y \xrightarrow{1-1} n$ are $\mathcal{L}(\in)$-Δ_0-formulae describing that x is a function, x is a finite ordinal and f is a one-to-one function from y to n, respectively.

Corollary 7.1.2. *The theory* Π_ω-Ref* *is a conservative extension* Π_ω-Ref.

Henceforth we also assume that the language $\mathcal{L}_{RS(\Xi)}$ is defined with respect to the language $\mathcal{L}_{Ad_0}(\in)$ instead of $\mathcal{L}(\in)$. Thereby we obtain new primitive $\mathcal{L}_{RS(\Xi)}$ formulae $Ad_0(s)$. For these we define:

Definition 7.1.3.

$$\begin{aligned} \mathrm{k}(Ad_0(s)) &:= \mathrm{k}(s), \\ Ad_0(s) &:\cong (L_\omega = s), \\ \mathrm{rnk}(Ad_0(s)) &:= \max\{\mathrm{rnk}(L_\omega), \mathrm{rnk}(s)\} + 5. \end{aligned}$$

[1] We do not define these fundamental sequences explicitly, but employ them implicitly in the definition of the subrecursive hierarchy.

7.2. The Finite Content of Ordinals, Terms and Sentences

Definition 7.2.1. $\mathrm{N}(\alpha) := \min\{n \in \omega \mid \alpha \in C^n(\Gamma_{\Xi+1}, 0)\}$.

Lemma 7.2.2. N *is a well-defined norm on* $\mathsf{T}(\Xi)$, *i.e. it holds* $\mathrm{N} : \mathsf{T}(\Xi) \to \omega$ *and* $\forall n < \omega\, (\mathrm{card}\{\xi \mid \mathrm{N}(\xi) \leq n\} < \omega)$. *Moreover for all* α *it holds* $\mathrm{N}(\alpha+1) = \mathrm{N}(\alpha) + 1$.

Convention. From now on we assume that par is defined as follows

$$\begin{aligned}
\mathrm{par}\,\mathbb{A}(m) &:= \{m\},\\
\mathrm{par}(\kappa^+; \ldots; 0) &:= \{\kappa, \mathrm{N}(\kappa)+1\},\\
\mathrm{par}(\Psi^\delta_{\mathbb{Z}}; \mathsf{P}_m; \ldots) &:= \mathrm{par}\,\mathbb{Z} \cup \{\delta, \max\{\mathrm{N}(\zeta) \mid \zeta \in \mathrm{par}\,\mathbb{Z}\} + 1\},\\
\mathrm{par}(\Psi^\delta_{\mathbb{Z}}; \mathsf{M}^\xi_{\mathbb{M}(\vec{\nu})}; \ldots) &:= \mathrm{par}\,\mathbb{Z} \cup \{\xi, \vec{\nu}, \delta, \max\{\mathrm{N}(\zeta) \mid \zeta \in \mathrm{par}\,\mathbb{Z}\} + 1\}.
\end{aligned}$$

All propositions of the foregoing chapters also hold with this modificated definition of par, since ω is a subset of every hull-set and we just add finite parameters to the usual par sets.

Definition 7.2.3 (The finite content of ordinals, terms and sentences)**.**

$$\begin{aligned}
|\alpha|_{\mathrm{N}} &:= \mathrm{N}(\alpha),\\
|L_\alpha|_{\mathrm{N}} &:= 2 \cdot \mathrm{N}(\alpha),\\
|\{x \in L_\alpha \mid F(x, s_1, \ldots, s_n)^{L_\alpha}\}|_{\mathrm{N}} &:= \max\{2 \cdot \mathrm{N}(\alpha) + 1, |F(L_0)|_{\mathrm{N}} + 2\},\\
|s \in t|_{\mathrm{N}} &:= \max\{|s|_{\mathrm{N}} + 6, |t|_{\mathrm{N}} + 1\},\\
|Ad_0(s)|_{\mathrm{N}} &:= \max\{9, |s|_{\mathrm{N}} + 5\},\\
|{}_\tau M^\alpha_{\mathbb{X}}(s)|_{\mathrm{N}} &:= \max\{\mathrm{N}(\tau), \mathrm{N}(\alpha), |s|_{\mathrm{N}} + 5, \mathrm{N}(\xi) \mid \xi \in \mathrm{par}\,\mathbb{X}\},\\
|F_0 \vee F_1|_{\mathrm{N}} &:= \max\{|F_0|_{\mathrm{N}}, |F_1|_{\mathrm{N}}\} + 1,\\
|\exists x \in t\, G(x)|_{\mathrm{N}} &:= \max\{|t|_{\mathrm{N}}, |G(L_0)|_{\mathrm{N}} + 2\},\\
|\neg G|_{\mathrm{N}} &:= |G|_{\mathrm{N}}.
\end{aligned}$$

If $\mathcal{A}$ is a finite set consisting of ordinals, terms and sentences we put

$$|\mathcal{A}|_{\mathrm{N}} := \sup\{\mathrm{card}(\mathcal{A}), |\phi|_{\mathrm{N}} \mid \phi \in \mathcal{A}\},$$

and

$$|\vec{t}|_{\mathrm{N}} = |(t_1, \ldots, t_n)|_{\mathrm{N}} := |\{t_1, \ldots, t_n\}|_{\mathrm{N}}.$$

7.3. Subrecursive Hierarchies on Hull-Sets of $\mathsf{T}(\Xi)$

Definition 7.3.1 (Subrecursive Hierarchies on $\mathcal{H}$)**.** Let $\mathcal{H}$ be a hull-set and let $\Phi(0) := 1$ and $\Phi(n+1) := 2^{\Phi(n)}$ for $n \in \omega$. By recursion on $\alpha \in \mathcal{H}$ we define

$$f^{\mathcal{H}}_\alpha(x) := \max\big(\{2x+1\} \cup \{f^{\mathcal{H}}_\beta(f^{\mathcal{H}}_\beta(x)) \mid \beta \in \mathcal{H} \cap \alpha \;\&\; \mathrm{N}(\beta) \leq \Phi^2(\mathrm{N}(\alpha) + x)\}\big),$$

where for a number theoretic function g we set $g^2 := g \circ g$.

Lemma 7.3.2. *Let $\mathcal{H}$,$\mathcal{H}_1$,$\mathcal{H}_2$ be hull-sets and $\alpha \in \mathcal{H},\mathcal{H}_1$. Then it holds*

- $f^{\mathcal{H}_1}_\alpha(x) \le f^{\mathcal{H}_2}_\alpha(x)$ *for all* $x \in \omega$, *if* $\mathcal{H}_1 \subseteq \mathcal{H}_2$,
- $\mathrm{N}(\alpha) < f^{\mathcal{H}}_\alpha(x)$ *for all* $\alpha \in \mathcal{H}$ *and* $x \in \omega$,
- $\Phi^2(\mathrm{N}(\alpha) + x) < f^{\mathcal{H}}_\alpha(x)$ *if* $\omega \le \alpha$,
- $f^{\mathcal{H}}_\alpha(x + y) < f^{\mathcal{H}}_{\alpha+y}(x)$ *for all* $\alpha \in \mathcal{H}$ *and* $x, y \in \omega$,
- $x \cdot y < \Phi(x + y + 1)$,
- $8 \cdot (\Phi^2(x))^2 < \Phi^2(x + 1)$ *if* $x \ge 2$.

Proof. The first four statements follow by induction on α.
The fifth statement follows since $x < 2^x$.
The last item holds, since for $x \ge 1$ we have

$$\Phi(x) + 2 < \Phi(x + 1) \quad \text{and} \quad 2 \cdot \Phi(x) + 3 < \Phi(x + 2).$$

Therefore it holds for $x = x' + 1 \ge 2$

$$8 \cdot (\Phi(x))^2 = 2^3 \cdot (2^{\Phi(x')})^2 = 2^{2\cdot\Phi(x')+3} < 2^{\Phi(x'+2)} = \Phi(x + 2).$$

Thus it holds for $x \ge 2$

$$8 \cdot (\Phi^2(x))^2 < \Phi(\Phi(x) + 2) < \Phi^2(x + 1).$$ □

Notation. Let $\mathcal{A}$ be a finite set of ordinals, terms and sentences. Then we use the abbreviation $f^{\mathcal{H}}_\alpha(\mathcal{A}) := f^{\mathcal{H}}_\alpha(|\mathcal{A}|_{\mathrm{N}})$. Moreover we dispense with the index $\mathcal{H}$ of f in inequalities, i.e. we define

$$f_\alpha(x) < f_\beta(x) \text{ over } \mathcal{H} \quad :\Leftrightarrow \quad f^{\mathcal{H}}_\alpha(x) < f^{\mathcal{H}}_\beta(y).$$

Lemma 7.3.3. *Let ϕ be a term or a sentence, s be a term and F_t plus F be sentences, such that $F_t \in \mathrm{CS}(F)$. Then it holds*

❶ $2 \cdot \mathrm{lh}(F) < \Phi^2(F)$,

❷ $\mathrm{N}(|\phi|) \le \mathrm{N}(\mathrm{rnk}(\phi)) \le |\phi|_{\mathrm{N}} < \omega$,

❸ $|s|_{\mathrm{N}} \le |F(s)|_{\mathrm{N}} \le |F(L_0)|_{\mathrm{N}} + |s|_{\mathrm{N}}$,

❹ $|F_t|_{\mathrm{N}} < |F|_{\mathrm{N}} + |t|_{\mathrm{N}} + 7$.

Proof. ① follows by induction on the build-up of F by use of Lemma 7.3.2.
② follows readily by definition, since $\mathrm{N}(\omega \cdot \alpha) \le 2 \cdot \mathrm{N}(\alpha)$.
③ Follows by induction on the build-up of F.
④ Follows by an exhausting run through the different cases by use of $|s = t|_{\mathrm{N}} = \max\{9, |s|_{\mathrm{N}} + 4, |t|_{\mathrm{N}} + 4\}$ and ③. □

8. A refined semi-formal Calculus for Π_ω-Ref*

This chapter is the analogue of chapter 4 elaborated on fragmented hull-sets. At first we give the definition of the fragmentation of a hull-set. Of course the paradigm of a hull-set is not a transitive set and in this point of view already a hull-set is fragmented in general. However, a fragmented hull-set (in the sense defined below) is a set of ordinals which consist only of sections of finite "length". Notably the fragmentation of a hull-set is a set which is so strongly fragmented that it is not a hull-set at all.

8.1. Semi-formal Derivations on Fragmented Hull-Sets of $\mathsf{T}(\Xi)$

Definition 8.1.1 (Fragmentation of a hull-set)**.** Let $\mathcal{H}$ be a hull-set and $\mathcal{A}$ a finite set of ordinals, terms and sentences. For $\gamma \in \mathcal{H}[\mathcal{A}]$ we define

$$\mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}] := \big\{\alpha \in \mathcal{H}[\mathcal{A}] \;\big|\; \mathrm{N}(\alpha) < f_\gamma(\mathcal{A}) \text{ over } \mathcal{H}[\mathcal{A}]\big\}.$$

Notation. In the following we use the abbreviation $\mathcal{F}_\gamma^{\mathcal{H}} := \mathcal{F}_\gamma^{\mathcal{H}}[\emptyset]$.

Definition 8.1.2. Let $\mathcal{H}$ be a hull-set, $\mathcal{A}$ be a finite set of ordinals, terms and sentences and suppose $\gamma \in \mathcal{H}[\mathcal{A}]$. For a finite set Γ of sentences we define the semi-formal derivability relation $\mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}] \left|\frac{\alpha}{\rho}\right. \Gamma$ by recursion on α via

$$\{\alpha\} \cup k(\Gamma) \subseteq \mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}] \quad \& \quad |\Gamma|_{\mathrm{N}} \in \mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}],$$

plus the inference-rules of Definition 4.2.8 with $\mathcal{H}$ replaced by $\mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}]$ and the convention $\mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}][t] := \mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}, t]$.

A main difference between hull-sets and fragmented hull-sets is that former feature the hull property, i.e. $\mathcal{A} \subset \mathcal{H} \;\Rightarrow\; \mathcal{H}[\mathcal{A}] = \mathcal{H}$, which does not hold for fragmented hull-sets.

Thereby we have to argue more carefully in the following as in the foregoing chapters, were we often make tacitly use of the hull property. However, as we employ the iteration of f_β in the definition of the subrecursive hierarchies we are able to handle this issue without bothering too much on adequate iterations of the fragmentation index.

Lemma 8.1.3 (Derived rules of semi-formal derivations on fragmented hull-sets).

(Inc) $\mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}] \,\big|\!\frac{\alpha}{\rho}\, \Gamma \;\&\; \alpha < \alpha' \;\&\; \rho < \rho' \;\&\; \Gamma \subseteq \Gamma' \;\&\; \mathrm{k}(\alpha', \rho', \Gamma') \subseteq \mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}]$
$\;\&\; |\{\alpha', \rho', \Gamma'\}|_{\mathrm{N}} \in \mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}] \quad\Rightarrow\quad \mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}] \,\big|\!\frac{\alpha'}{\rho'}\, \Gamma'$,

(Str) $\gamma \in \mathcal{H}[\mathcal{A}],\, \delta \in \mathcal{H}[\mathcal{B}] \;\&\; \mathcal{A} \subseteq \mathcal{H}[\mathcal{B}] \;\&\; f_{\gamma}(\mathcal{A}) \leq f_{\delta}(\mathcal{B})$ *over* $\mathcal{H}[\mathcal{B}]$,
then $\mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}] \,\big|\!\frac{\alpha}{\rho}\, \Gamma \quad\Rightarrow\quad \mathcal{F}^{\mathcal{H}}_{\delta}[\mathcal{B}] \,\big|\!\frac{\alpha}{\rho}\, \Gamma$,

($\vee$-Ex) $\mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}] \,\big|\!\frac{\alpha}{\rho}\, \Gamma, F \vee G \quad\Rightarrow\quad \mathcal{F}^{\mathcal{H}}_{\gamma+1}[\mathcal{A}] \,\big|\!\frac{\alpha}{\rho}\, \Gamma, F, G$,

($\wedge$-Ex) $\mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}] \,\big|\!\frac{\alpha}{\rho}\, \Gamma, F \wedge G \quad\Rightarrow\quad \mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}] \,\big|\!\frac{\alpha}{\rho}\, \Gamma, F \;\&\; \mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}] \,\big|\!\frac{\alpha}{\rho}\, \Gamma, G$,

(Up-Per) $\mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}] \,\big|\!\frac{\alpha}{\rho}\, \Gamma,\, \exists x^{\kappa}\, F(x) \;\&\; \kappa < \pi, \rho \;\&\; \pi \in \mathrm{SC} \;\&\; \pi \cdot 2 < \gamma$
$\Rightarrow\; \mathcal{F}^{\mathcal{H}}_{\gamma+1}[\mathcal{A}, \pi] \,\big|\!\frac{(\alpha+\pi)\cdot 2}{\rho}\, \Gamma,\, \exists x^{\pi}\, F(x)$,

($\forall$-Inv) $\mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}] \,\big|\!\frac{\alpha}{\rho}\, \Gamma, \bigwedge (F_t)_{t \in T} \quad\Rightarrow\quad \forall t \in T\; \mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}, t] \,\big|\!\frac{\alpha}{\rho}\, \Gamma, F_t$.

Let F *be a* $\Pi_{n+1}(\pi)$*-sentence, then:*

(E-$\forall$) $\forall \vec{t} \in \mathcal{T}^{m}_{\pi}\; \mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}, \vec{t}\,] \,\big|\!\frac{\alpha}{\rho}\, \Gamma, F(F'_1(t_1), \ldots, F'_m(t_m)) \;\&\; \pi \cdot 2 \in \mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}]$
$\Rightarrow\quad \mathcal{F}^{\mathcal{H}}_{\gamma+\Phi^2(F)}[\mathcal{A}] \,\big|\!\frac{\alpha+\Phi^2(F)}{\rho}\, \Gamma, F$,

(E-$\forall$-Inv) $\mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}] \,\big|\!\frac{\alpha}{\rho}\, \Gamma, F \;\Rightarrow\; \forall \vec{t} \in \mathcal{T}^{m}_{\pi}\; \mathcal{F}^{\mathcal{H}}_{\gamma+1}[\mathcal{A}, \vec{t}\,] \,\big|\!\frac{\alpha}{\rho}\, \Gamma, F(F'_1(t_1), \ldots, F'_m(t_m))$,

(E-Up-Per) $\mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}] \,\big|\!\frac{\alpha}{\rho}\, \Gamma, \neg F \;\&\; n = 0 \;\&\; \pi < \lambda, \rho \;\&\; \lambda \in \mathrm{SC} \;\&\; \lambda \cdot 2 < \gamma$
$\Rightarrow\quad \mathcal{F}^{\mathcal{H}}_{\gamma+1}[\mathcal{A}, \lambda] \,\big|\!\frac{(\alpha+\lambda)\cdot 2}{\rho}\, \Gamma, \neg F^{(\pi,\lambda)}$.

Proof. The rule (E-Up-Per) follows as in the proof of Lemma 4.2.9 supplemented with the following extra considerations: In the case that the last inference is a (Ref)-inference with principal formula $\exists x^{\kappa} F(x)$ it follows by induction on the build-up of F that

$$\mathcal{F}_{\pi\cdot 2}[\exists x^{\pi} F(x), \forall x^{\kappa} F(x)] \,\big|\!\frac{\pi\cdot 2}{0}\, \exists x^{\pi} F(x), \forall x^{\kappa} F(x).$$

Since $\pi \cdot 2 < \gamma$ and it holds by assumption $\mathcal{F}_{\gamma}[\mathcal{A}] \,\big|\!\frac{\alpha_0}{\rho}\, \Gamma, \exists x^{\kappa} F(x)$ this leads to

$$\mathcal{F}_{\gamma+1}[\mathcal{A}, \pi] \,\big|\!\frac{\pi\cdot 2}{0}\, \exists x^{\pi} F(x),\, \forall x^{\kappa} F(x),$$

and the claim follows by a cut.

Propositions (Inc) – ($\forall$-Inv) follow by induction on α. Note that in case of ($\vee$-Ex) the index γ has to be increased by one since the cardinality of a finite set of sentences is also a parameter of its finite content.

To prove (E-$\forall$) we proceed by induction on the build-up of F: If F is an elementary $\Pi_{n+1}(\pi)$-sentence the claim follows by an application of ($\bigwedge$). If $F(F_1, \ldots, F_m) \equiv G_0(F_1, \ldots, F_k) \vee G_1(F_{k+1}, \ldots, F_m)$, with $G_0, G_1 \in \Pi_{n+1}(\pi)$ we obtain by use of ($\vee$-Ex)

$$\forall \vec{t} \in \mathcal{T}^{m}_{\pi} \quad \mathcal{F}^{\mathcal{H}}_{\gamma+1}[\mathcal{A}, \vec{t}\,] \,\big|\!\frac{\alpha}{\rho}\, \Gamma, G_0(F'_1(t_1), \ldots, F'_k(t_k)), G_1(F'_{k+1}(t_{k+1}), \ldots, F'_m(t_m)).$$

Applying the induction hypothesis twice we get

$$\mathcal{F}^{\mathcal{H}}_{\gamma+1+\Phi^2(G_0)+\Phi^2(G_1)}[\mathcal{A}] \Big|\frac{\alpha+\Phi^2(G_0)+\Phi^2(G_1)}{\rho}\, \Gamma, G_0, G_1,$$

and the claim follows by two (V)-inferences and use of (Inc) and (Str). If we have $F(F_1,\ldots,F_m) \equiv G_0(F_1,\ldots,F_k) \wedge G_1(F_{k+1},\ldots,F_m)$ the claim follows analogously.

The claim $(\mathsf{E}\text{-}\forall\text{-}\mathsf{Inv})$ follows by induction on α. If $F \equiv G_0 \vee G_1$ and F is the principal formula of the last inference the claim follows since $|F(F_1'(t_1),\ldots,F_m'(t_m))|_{\mathrm{N}} \leq |F|_{\mathrm{N}} + |\vec{t}|_{\mathrm{N}} < \mathcal{F}^{\mathcal{H}}_{\gamma+1}[\mathcal{A},\vec{t}]$. □

8.2. Embedding of Π_ω-Ref*

Notation. In the following we extend the dot-notation as follows

$$\mathcal{F}^{\mathcal{H}}_{\cdot}[\mathcal{A}] \,\Big|\frac{\alpha}{\rho}\, \Gamma \quad :\Leftrightarrow \quad \mathcal{F}^{\mathcal{H}}_{\alpha}[\mathcal{A}] \,\Big|\frac{\alpha}{\rho}\, \Gamma.$$

Moreover we define

$$\Big\|\!\frac{}{\mathrm{N}}\, \Gamma \quad :\Leftrightarrow \quad \mathcal{F}^{\mathcal{H}}_{\cdot}[\Gamma] \,\Big|\frac{\|\Gamma\|}{0}\, \Gamma \text{ for every hull-set } \mathcal{H}.$$

Remark. Without loosing any propositions of Lemma 4.3.3 and Theorem 4.3.4 we can add the proviso $|\{t_1,\ldots,t_n\}|_{\mathrm{N}} \leq \Phi^2(\Gamma)$ to the $(\bigvee)^\star$-rule of Definition 4.3.2.[†]

Lemma 8.2.1 (Embedding of $RS^\star$)**.** *It holds* $\quad \Big|\!\frac{\star}{}\, \Gamma \quad \Rightarrow \quad \Big\|\!\frac{}{\mathrm{N}}\, \Gamma.$

Proof. We proceed by induction on the $RS^\star$-derivation.

Case 1, the last inference is by the $(\mathsf{V})^\star$*-rule:* Then we have $\Gamma = \Gamma', \bigvee(F_t)_{t\in T}$ and there are $t_1,\ldots,t_n \in T$, such that

$$\frac{\Gamma', F_{t_1},\ldots,F_{t_n}}{\Gamma', \bigvee(F_t)_{t\in T}} \quad \text{where } k\{t_1,\ldots,t_n\} \subseteq \mathrm{k}(\Gamma)^\star \text{ and } |\{t_1,\ldots,t_n\}|_{\mathrm{N}} \leq \Phi^2(\Gamma). \tag{8.1}$$

Let $\Gamma_0 := \Gamma', F_{t_1},\ldots,F_{t_n}$ and $\alpha_0 := \|\Gamma_0\|$ plus $\alpha := \|\Gamma\|$. The induction hypothesis provides

$$\mathcal{F}^{\mathcal{H}}_{\cdot}[\Gamma_0] \,\Big|\frac{\alpha_0}{0}\, \Gamma_0. \tag{8.2}$$

[†]Only in the proof of the Corollary to Lemma 2.7 in [Buc93] there is an inference which makes use of more than one (but less or equal then $\mathrm{lh}(\bigvee(F_t)_{t\in T})$) premises of $\bigvee(F_t)_{t\in T}$ and thus we come through all these proofs with $n \leq \Phi^2(\Gamma)$. With respect to $|t_i|_{\mathrm{N}} < \Phi^2(\Gamma)$ there is just the non-trivial case of d in the $RS^\star$-derivation of (Δ_0-Sep). However, there we have $|d|_{\mathrm{N}} = \max\{2\,\mathrm{N}(|a|+3), 2\,\mathrm{N}(|c_1|)+3,\ldots,2\,\mathrm{N}(|c_n|)+3, |L_0 \in a \wedge \phi(L_0,\vec{c})|_{\mathrm{N}}+2\} < \Phi(|L_0 \in a \wedge \phi(L_0,\vec{c})|_{\mathrm{N}}) < \Phi(|\exists y \in L_\lambda\, (\psi_1(y,a,\vec{c}) \wedge \psi_2(y,a,\vec{c}))|_{\mathrm{N}})$.

Moreover we have

$$\begin{aligned}
\mathrm{N}(\alpha_0) &\leq \mathrm{N}(\alpha) + \sum_{i=1}^{n}\big(\mathrm{N}(\mathrm{rnk}(F_{t_i})) + 1\big) \leq \mathrm{N}(\alpha) + \sum_{i=1}^{n}\big(|F_{t_i}|_{\mathrm{N}} + 1\big) \\
&< \mathrm{N}(\alpha) + \sum_{i=1}^{n}\big(\big|\bigvee(F_t)_{t\in T}\big|_{\mathrm{N}} + |t_i|_{\mathrm{N}} + 8\big) \\
&< \mathrm{N}(\alpha) + \Phi^2(\Gamma)\cdot(|\Gamma|_{\mathrm{N}} + \Phi^2(\Gamma) + 8) < \mathrm{N}(\alpha) + 3(\Phi^2(\Gamma))^2 \\
&< \mathrm{N}(\alpha) + \Phi^2(|\Gamma|_{\mathrm{N}} + 1), \qquad (8.3)
\end{aligned}$$

and by nearly the same considerations

$$|\Gamma_0|_{\mathrm{N}} < \Phi^2(|\Gamma|_{\mathrm{N}} + 1). \qquad (8.4)$$

From (8.1) it follows $\mathrm{k}(\Gamma_0) \subseteq \mathrm{k}(\{\Gamma', \bigvee(F_t)_{t\in T}, t_1, \ldots, t_n\}) \subseteq \mathrm{k}(\Gamma^\star) \subseteq \mathcal{H}[\mathcal{A}]$, $\alpha_0 < \alpha$, $\alpha_0 \in \mathcal{H}[\Gamma_0]$ and (8.3) plus (8.4) provide

$$f_{\alpha_0}(\Gamma_0) < f_{\alpha_0}(f_{|\Gamma_0|_{\mathrm{N}}}(\Gamma)) < f_{\alpha_0+|\Gamma_0|_{\mathrm{N}}+1}(\Gamma) < f_\alpha(\Gamma) \quad \text{over } \mathcal{H}[\Gamma].$$

Thereby we obtain from (8.2) and by means of (Str)

$$\mathcal{F}^{\mathcal{H}}_{\alpha}[\Gamma] \,\Big|\!\frac{\alpha_0}{0}\, \Gamma', F_{t_1}, \ldots, F_{t_n}. \qquad (8.5)$$

Taking into account (8.3), $n \leq \Phi^2(\Gamma)$ and $\omega \leq \alpha$ it follows $\alpha_0 + n \in \mathcal{F}^{\mathcal{H}}_{\alpha}[\Gamma]$. Moreover we have $|\Gamma, F_{t_1}, \ldots, F_{t_n}|_{\mathrm{N}} < \Phi^2(|\Gamma|_{\mathrm{N}} + 1) < f^{\mathcal{H}[\Gamma]}_{\alpha}(\Gamma)$. Thus we may perform n ($\bigvee$)-inferences to obtain the claim from (8.5).

Case 2, the last inference is by the ($\bigwedge$)**-rule:* Then we have $\Gamma = \Gamma', \bigwedge(F_t)_{t\in T}$ and

$$\frac{\Gamma', F_t \quad \text{for all } t \in T}{\Gamma', \bigwedge(F_t)_{t\in T}}$$

Let $\Gamma_t := \Gamma', F_t$ and $\alpha_t := \|\Gamma_t\|$ plus $\alpha := \|\Gamma\|$. By use of the induction hypothesis we obtain

$$\mathcal{F}^{\mathcal{H}}_{\cdot}[\Gamma_t] \,\Big|\!\frac{\alpha_t}{0}\, \Gamma_t \quad \text{for all } t \in T. \qquad (8.6)$$

Just like in the first case it follows

$$\mathrm{N}(\alpha_t) < \mathrm{N}(\alpha) + \Phi^2(|\Gamma, t|_{\mathrm{N}} + 1) \quad \text{and} \quad |\Gamma_t|_{\mathrm{N}} < \Phi^2(|\Gamma, t|_{\mathrm{N}} + 1). \qquad (8.7)$$

Moreover we have $\mathrm{k}(\Gamma_t) \subseteq \mathcal{H}[\Gamma, t]$, $\alpha_t < \alpha$ and $\alpha_t \in \mathcal{H}[\Gamma, t]$, $\alpha \in \mathcal{H}[\Gamma]$. By use (8.7) we obtain

$$f_{\alpha_t}(\Gamma_t) < f_{\alpha_t}(f_{|\Gamma_t|_{\mathrm{N}}}(\Gamma, t)) < f_{\alpha_t+|\Gamma_t|_{\mathrm{N}}+1}(\Gamma, t) < f_\alpha(\Gamma, t) \quad \text{over } \mathcal{H}[\Gamma, t] \text{ for all } t \in T.$$

Therefore we obtain by (8.6) and taking into account (Str)

$$\mathcal{F}^{\mathcal{H}}_{\alpha}[\Gamma, t] \,\Big|\!\frac{\alpha_t}{0}\, \Gamma', F_t \quad \text{for all } t \in T.$$

Thus the claim follows by an application of ($\bigwedge$). □

Lemma 8.2.2 (Found)**.** *Let $F(L_0)$ be a sentence. Then it holds for every ξ*

$$\Vdash_{\mathrm{N}} \forall x^{\xi}\big(\forall y \in x F(y) \to F(x)\big) \to \forall x^{\xi} F(x).$$

Proof. Let $\mathcal{H}$ be an arbitrary hull-set and let $G :\equiv \forall x^{\xi}\big(\forall y \in xF(y) \to F(x)\big)$ plus $\alpha_s := \omega^{\mathrm{rnk}(G)} \oplus \omega^{|s|+1}$. At first we show by induction on $|s|$

$$\mathcal{F}^{\mathcal{H}}_{\cdot}[G,s] \Big|\frac{\alpha_s}{0}\, \neg G, F(s) \quad \text{for all } s \in \mathcal{T}_{\xi}. \tag{8.8}$$

Equation (8.8) holds for all $t \in \mathcal{T}_{|s|}$ by the induction hypothesis. Let $G_s \in \mathrm{CS}(G)$. Then $\neg\big(\forall y \in sF(y)\big)$ is a subformula of G_s. Thus we have

$$|\forall y \in sF(y)|_{\mathrm{N}} < |G|_{\mathrm{N}} + |s|_{\mathrm{N}} + 7 \le 2 \cdot |G,s|_{\mathrm{N}} + 7 \quad \text{and} \tag{8.9}$$

$$\begin{aligned} |t \dot{\in} s \to F(t)|_{\mathrm{N}} &< |\forall y \in sF(y)|_{\mathrm{N}} + |t|_{\mathrm{N}} + 7 < 2 \cdot |G,s|_{\mathrm{N}} + 7 + |t|_{\mathrm{N}} + 7 \\ &\le 3 \cdot |G,s,t|_{\mathrm{N}} + 14 \end{aligned} \tag{8.10}$$

This implies $|\neg G, t \dot{\in} s \to F(t)|_{\mathrm{N}} < f_{\alpha_t+1}(G,s,t)$ over $\mathcal{H}[G,s,t]$, and thereby we obtain by means of the induction hypothesis and an application of (⋁)

$$\mathcal{F}^{\mathcal{H}}_{\cdot}[G,s,t] \Big|\frac{\alpha_t+1}{0}\, \neg G, t \dot{\in} s \to F(t) \quad \text{for all } t \in \mathcal{T}_{|s|}. \tag{8.11}$$

Let $\beta_s := \omega^{\mathrm{rnk}(G)} \oplus \omega^{|s|}$. Since $f_{\alpha_t+1}(G,s,t) < f_{\beta_s+2}(G,s,t)$ over $\mathcal{H}[G,s,t]$ for all $t \in \mathcal{T}_{|s|}$ we obtain from (8.11) by use of (Str) and (Inc), a (⋀)-inference and taking into account (8.9)

$$\mathcal{F}^{\mathcal{H}}_{\cdot}[G,s] \Big|\frac{\beta_s+2}{0}\, \neg G,\, \forall x \in sF(x). \tag{8.12}$$

By Lemma 4.3.3 ① and the embedding of $RS^{\star}$ we have $\Vdash_{\mathrm{N}} \neg F(s), F(s)$. Moreover it holds $\mathrm{N}(\mathrm{rnk}(F(s))) \le |F(s)|_{\mathrm{N}} \le |s \dot{\in} s \to F(s)|_{\mathrm{N}} \le 3 \cdot |G,s|_{\mathrm{N}} + 14$ by (8.10), and thus

$$f_{\omega^{\mathrm{rnk}(F(s))}\cdot 2}(\neg F(s), F(s)) < f_{\beta_s}(f_{\beta_s}(G,s)) < f_{\beta_s+2}(G,s) \quad \text{over } \mathcal{H}[G,s].$$

Thereby we have

$$\mathcal{F}^{\mathcal{H}}_{\cdot}[G,s] \Big|\frac{\beta_s+2}{0}\, \neg F(s), F(s).$$

By use of (8.12) and an (⋀)-inference we obtain

$$\mathcal{F}^{\mathcal{H}}_{\cdot}[G,s] \Big|\frac{\beta_s+3}{0}\, \neg G,\, \forall y \in sF(y) \wedge \neg F(s), F(s),$$

and therefore

$$\mathcal{F}^{\mathcal{H}}_{\cdot}[G,s] \Big|\frac{\beta_s+4}{0}\, \neg G, \exists x^{\xi}\big(\forall y \in xF(y) \wedge \neg F(x)\big), F(s),$$

via (⋁). This shows (8.8).

Since $\alpha_s < \omega^{\mathrm{rnk}(G)} \oplus \omega^{\mathrm{rnk}(\forall x^\xi F(x))} < \|G \to \forall x^\xi F(x)\|$ for all $s \in \mathcal{T}_\xi$ and $f_{\alpha_s}(G, s, \xi) < f_{\|G \to \forall x^\xi F(x)\|}(G, s, \xi)$ over $\mathcal{H}[G, s, \xi]$ we obtain by (Str) and (Inc) plus a ($\bigwedge$)-inference from (8.8)

$$\mathcal{F}^{\mathcal{H}}_{\|G \to \forall x^\xi F(x)\|}[G, \xi] \left|\frac{\omega^{\mathrm{rnk}(G)} \oplus \omega^{\mathrm{rnk}(\forall x^\xi F(x))}}{0}\right. G, \forall x^\xi F(x).$$

From this the claim follows by use of ($\bigvee$) and (Str). □

Lemma 8.2.3 (Refl)**.** *Let $F \in \Pi_n(\Xi)$. Then*

$$\Vdash_{\mathrm{N}} F \to \exists z^\Xi (z \models F).$$

Proof. Let $\mathcal{H}$ be an arbitrary hull-set. Choose $m < \omega$, such that $F \in \Pi_{m+2}(\Xi)$. By Lemma 4.3.3 ① and the $RS^\star$-embedding we have

$$\mathcal{F}^{\mathcal{H}}_{\cdot}[\neg F, F] \left|\frac{\omega^{\mathrm{rnk}(F)} \cdot 2}{0}\right. \neg F, F.$$

By an application of ($\Pi_{m+2}(\Xi)$-Ref) plus a ($\bigvee$)-inference we obtain

$$\mathcal{F}^{\mathcal{H}}_{\cdot}[\neg F, F, \Xi] \left|\frac{\omega^{\mathrm{rnk}(F)} \cdot 2 + 2}{0}\right. F \to \exists z^\Xi (z \models F). \tag{8.13}$$

Since $\mathrm{rnk}(F) < \mathrm{rnk}(F \to \exists z^\Xi (z \models F))$ it follows

$$\omega^{\mathrm{rnk}(F)} \cdot 2 + 2 < \|F \to \exists z^\Xi (z \models F)\| \quad \text{and}$$
$$f_{\omega^{\mathrm{rnk}(F)} \cdot 2+2}(\neg F, F, \Xi) < f_{\|F \to \exists z^\Xi (z \models F)\|}(F \to \exists z^\Xi (z \models F))$$
$$\text{over } \mathcal{H}[\neg F, F, \Xi] = \mathcal{H}[F \to \exists z^\Xi (z \models F)].$$

Therefore the claim follows by use of (Str) and (Inc) from (8.13). □

Lemma 8.2.4. *Let $\Gamma(\vec{x})$ be a finite set of $\mathcal{L}_{M(\Xi)}$-formulae and $\lambda \in \mathcal{H}$. Then there is an $m \in \omega$, such that for all $\vec{s} = (s_1, \ldots, s_n)$ with $s_i \in \mathcal{T}_\lambda$ it holds*

$$\Vdash_{\mathrm{N}} \Gamma^\lambda(\vec{s}) \quad \Rightarrow \quad \mathcal{F}^{\mathcal{H}}_{\cdot}[s_1, \ldots, s_n] \left|\frac{\omega^{\omega\lambda+m}}{0}\right. \Gamma^\lambda(\vec{s}).$$

Proof. Suppose

$$\mathcal{F}^{\mathcal{H}}_{\cdot}[\Gamma^\lambda(\vec{s})] \left|\frac{\|\Gamma^\lambda(\vec{s})\|}{0}\right. \Gamma^\lambda(\vec{s}). \tag{8.14}$$

Lemma 4.2.6 implies, that there is an m_1 such that $\|\Gamma^\lambda(\vec{s})\| < \omega^{\omega\lambda+m_1}$ and for $F \in \Gamma$ we have

$$\begin{aligned}
|F^\lambda(\vec{s})|_{\mathrm{N}} &\le |F^{L_0}(\vec{L_0})|_{\mathrm{N}} + |L_\lambda|_{\mathrm{N}} + \sum_{i=1}^{n} |s_i|_{\mathrm{N}} \\
&\le |F^{L_0}(\vec{L_0})|_{\mathrm{N}} + |L_\lambda|_{\mathrm{N}} + |F^{L_0}(\vec{L_0})|_{\mathrm{N}} \cdot |\vec{s}|_{\mathrm{N}} \\
&\le |\Gamma^{L_0}(\vec{L_0})|_{\mathrm{N}} + |L_\lambda|_{\mathrm{N}} + |\Gamma^{L_0}(\vec{L_0})|_{\mathrm{N}} \cdot |\vec{s}|_{\mathrm{N}}
\end{aligned}$$

Thus there is an $m_2 \in \omega$ such that for all $F \in \Gamma$ we have $|F^\lambda(\vec{s})|_N < m_2 \cdot |\vec{s}|_N$. Since $N(\text{rnk}(F^\lambda(\vec{s})) < |F^\lambda(\vec{s})|_N$ there is an $m_3 \in \omega$ such that $|\Gamma^\lambda(\vec{s})|_N < m_3 \cdot |\vec{s}|_N$ and $N(\|\Gamma^\lambda(\vec{s})\|) < m_3 \cdot |\vec{s}|_N$ for all $\vec{s} \in \mathcal{T}_\lambda$.

Thereby we obtain for $m := \max\{m_1, m_2, m_3\}$ and $\gamma := \|\Gamma^\lambda(\vec{s})\|$ that

$$f_\gamma(\Gamma^\lambda(\vec{s})) < f_\gamma(m \cdot |\vec{s}|_N) < f_\gamma(f_{\gamma+m}(\vec{s})) < f_{\gamma+m+1}(\vec{s}) < f_{\omega^{\omega\lambda+m}}(\vec{s}) \quad \text{over } \mathcal{H}[\Gamma^\lambda(\vec{s})] = \mathcal{H}[s_1, \ldots, s_n].$$

Thus the claim follows from (8.14) by use of (Inc) and (Str). □

Lemma 8.2.5 (Embedding of Logic)**.** *Let $\lambda \in \mathcal{H}$. If $\Gamma(\vec{u})$ is a logically valid set of $\mathcal{L}_{M(\Xi)}$-formulae, then there is an $m \in \omega$ such that*

$$\mathcal{F}^{\mathcal{H}}_{\cdot}[s_1, \ldots, s_n] \left|\frac{\omega^{\omega\cdot\lambda+m}}{\omega^\lambda}\right. \Gamma^\lambda(\vec{s}) \qquad \textit{for all } \vec{s} = (s_1, \ldots, s_n), \textit{ with } s_i \in \mathcal{T}_\lambda.$$

Proof. Analogously to [Buc93], Lemma 3.11. by use of Lemma 8.2.4. □

8.2.1. Embedding of $(\mathbf{Ad}_0.1) - (\mathbf{Ad}_0.3)$

Lemma 8.2.6. *There exists an $m < \omega$, such that for every hull-set $\mathcal{H}$ with $\Xi \in \mathcal{H}$ it holds*

$$\mathcal{F}^{\mathcal{H}}_{\cdot} \left|\frac{\omega^{\Xi+m}}{\Xi}\right. (\text{Ad}_0.i)^\Xi \quad \textit{for } i \in \{1, 2\}.$$

Proof. By use of Lemma 4.3.3 ⑧ and a $(\bigwedge)^\star$-, a $(\bigvee)^\star$- plus again a $(\bigwedge)^\star$-inference we obtain

$$\left|\frac{\star}{}\right. (\text{Ad}_0.1)^\Xi, \neg\big((\text{Tran})^\omega \wedge (\text{Nullset})^\omega \wedge (\text{Pair})^\omega \wedge (\text{Union})^\omega\big),$$

and by Theorem 4.3.4 we have

$$\left|\frac{\star}{}\right. (\text{Tran})^\omega \wedge (\text{Nullset})^\omega \wedge (\text{Pair})^\omega \wedge (\text{Union})^\omega.$$

Thereby the claim follows for $i = 1$ by use of the $RS^\star$-Embedding Lemma 8.2.1 and Lemma 8.2.4 plus a (Cut), taking into account that $|L_\omega|_N = 4$.

In the same vein the claim follows for all instances of the scheme $(\text{Ad}_0.2)$. □

To obtain Lemma 8.2.6 also for $i = 3$ we have to prove

$$\left|\frac{\star}{}\right. \forall y^\omega \exists j^\omega \big(\exists f \in j\, \exists n \in j\, (\text{Fun}(f) \wedge \text{Natno}(n) \wedge f : y \xrightarrow{1-1} n)\big). \tag{8.15}$$

Since we cannot expect completeness for $\Pi_2(\omega)$-sentences derived on fragmented-hull-sets we have to do some extra work to yield (8.15).

Notation. By $\mathbb{L}$ we denote the constructible hierarchy. For $u \in \mathbb{L}$ we define $|u|_{\mathbb{L}} := \min\{\alpha \mid u \in \mathbb{L}_\alpha\}$.

Definition 8.2.7 (The Canonical Interpretation of $\mathcal{L}_{RS(\Xi)}$)**.** To obtain an interpretation of $\mathcal{L}_{RS(\Xi)}$ in $\mathbb{L}$ we interpret $\in$ as the standard membership relation of $\mathbb{L}$, Ad_0 as the set $\{\mathbb{L}_\omega\}$ and the predicates ${}_\tau M^\xi_{\mathbb{X}}$ as the sets $\{\mathbb{L}_\zeta \,|\, \zeta \in {}_\tau\mathfrak{M}^\xi_{\mathbb{X}}\}$.

We define the interpretation of the $\mathcal{L}_{RS(\Xi)}$-terms by recursion on α as follows

$$\begin{aligned} L_\alpha^{\mathbb{L}} &:= \mathbb{L}_\alpha, \\ \{x \in L_\alpha \,|\, F(x, s_1, \dots, s_n)^{L_\alpha}\}^{\mathbb{L}} &:= \{u \in \mathbb{L}_\alpha \,|\, \mathbb{L}_\alpha \models F(u, s_1^{\mathbb{L}}, \dots, s_n^{\mathbb{L}})\}. \end{aligned}$$

In general the finite content of an $\mathcal{L}_{RS(\Xi)}$-term $t \in \mathcal{T}_\omega$ contains too much syntactical information (the length of t) to decode the stage of t out of $|t|_{\mathrm{N}}$. Nevertheless for every $u \in \mathbb{L}_\omega$ there is a canonical term $\tilde{t}$, such that $\mathbb{L} \models u = \tilde{t}^{\mathbb{L}}$ and $|\tilde{t}|_{\mathrm{N}} \leq \Phi(|\tilde{t}| + 1)$.

Definition 8.2.8 (Canonical Terms for $\mathbb{L}_\omega$)**.** By recursion on $n \in \omega$ we define the sets $\widetilde{\mathcal{T}}_n \subseteq \mathcal{T}_n$ as follows:

$$\begin{aligned} \widetilde{\mathcal{T}}_0 &:= \emptyset, \\ \widetilde{\mathcal{T}}_{n+1} &:= \{L_n\} \cup \big\{\{x \in L_n \;\big|\; E(x, \tilde{t_1}, \dots, \tilde{t_k})^{L_n}\} \,| \\ &\qquad E(x, \tilde{t_1}, \dots, \tilde{t_k}) \equiv x = \tilde{t_1} \vee \ldots \vee x = \tilde{t_k} \;\&\; \forall_1^k i\, (\tilde{t_i} \in \widetilde{\mathcal{T}}_n) \;\&\; k \leq \Phi(n)\big\}, \\ \widetilde{\mathcal{T}}_\omega &:= \bigcup_{n \in \omega} \widetilde{\mathcal{T}}_n. \end{aligned}$$

Lemma 8.2.9. *Let $s \in \mathbb{L}_\omega$. Then there exists an $\tilde{s} \in \widetilde{\mathcal{T}}_\omega$, such that $\mathbb{L} \models s = \tilde{s}^{\mathbb{L}}$ and $|s|_{\mathbb{L}} = |\tilde{s}|$ plus $|\tilde{s}|_{\mathrm{N}} < \Phi(|\tilde{s}|_{\mathbb{L}} + 1)$.*

Moreover for $t \in \mathcal{T}_n$ there is a $\tilde{t} \in \widetilde{\mathcal{T}}_n$, such that $\mathbb{L} \models t^{\mathbb{L}} = \tilde{t}^{\mathbb{L}}$.

Proof. We show the first claim by induction on $m := |s|_{\mathbb{L}}$. If $m = 1$ the claim holds with $\tilde{s} = L_0$. If $m = j + 2$ then there is a $k < \Phi(j+1)$ and $s_1, \dots, s_k \in \mathbb{L}_{j+1}$ such that $s = \{s_1, \dots, s_k\}$ since $\mathrm{card}(\mathbb{L}_{j+1}) < \Phi(j+1)$. Thus it follows by the induction hypothesis that there are $\tilde{s}_1, \dots, \tilde{s}_k \in \widetilde{\mathcal{T}}_\omega$ such that for all $1 \leq i \leq k$ it holds $\mathbb{L} \models s_i^{\mathbb{L}} = \tilde{s}_i^{\mathbb{L}}$ and $|s_i|_{\mathbb{L}} = |\tilde{s}_i|$, i.e. $\tilde{s}_i \in \widetilde{\mathcal{T}}_{j+1}$. Therefore we have for $\tilde{s} := \{x \in L_{j+1} \,|\, E(x, \tilde{s}_1, \dots, \tilde{s}_k)^{L_{j+1}}\} \in \widetilde{\mathcal{T}}_\omega$ that $\mathbb{L} \models s = \tilde{s}^{\mathbb{L}}$ and $|\tilde{s}| = j + 2$. Moreover we have $|L_0 = \tilde{s}_i|_{\mathrm{N}} \leq \max\{9, |L_0|_{\mathrm{N}} + 4, |\tilde{s}_i|_{\mathrm{N}} + 4\}$ and hence $|E(L_0, \tilde{s}_1, \dots, \tilde{s}_k)|_{\mathrm{N}} \leq \max\{9, |\tilde{s}_1|_{\mathrm{N}} + 4, \dots, |\tilde{s}_k|_{\mathrm{N}} + 4, \Phi(j+1)\} \leq \Phi(j+2) + 5$. Thereby it follows

$$|\tilde{s}|_{\mathrm{N}} := \max\{2\,\mathrm{N}(j+1) + 1, |E(L_0, \tilde{s}_1, \dots, \tilde{s}_k)|_{\mathrm{N}} + 2\} < \Phi(j+3) = \Phi(|\tilde{s}| + 1).$$

Now let $t \in \mathcal{T}_n$. Then $|t^{\mathbb{L}}|_{\mathbb{L}} \leq n$ and thus there is a $\tilde{t} \in \widetilde{\mathcal{T}}_n$ with $\mathbb{L} \models t^{\mathbb{L}} = \tilde{t}^{\mathbb{L}}$. ☐

Lemma 8.2.10. *Let $F \equiv \forall x^\omega\, \exists y^\omega\, G(x, y)$ be an elementary $\Pi_2(\omega)$-sentence, such that $\mathbb{L} \models F^{\mathbb{L}}$ and for every $s \in \mathbb{L}_\omega$ there is a $t \in \mathbb{L}_\omega$ with $|t|_{\mathbb{L}} < 2 \cdot \Phi(|s|_{\mathbb{L}})$ and $\mathbb{L} \models G(s, t)^{\mathbb{L}}$. Then it holds $\overset{\star}{\models} F$.*

Proof. Let H be a true $\Delta_0(\omega)$-sentence, i.e. $\mathbb{L} \models H^{\mathbb{L}}$. Then it follows by a straight forward induction on $\mathrm{rnk}(H)$ (by use of canonical terms) that $\overset{\star}{\models} H$.

Now let $s \in \mathcal{T}_\omega$. Then there is a $t \in \mathbb{L}_\omega$ such that $\mathbb{L} \models G(s^{\mathbb{L}}, t)^{\mathbb{L}}$ with $|t|_{\mathbb{L}} < 2 \cdot \Phi(|s^{\mathbb{L}}|_{\mathbb{L}})$. Therefore Lemma 8.2.9 provides a $\tilde{t} \in \widetilde{\mathcal{T}}_\omega$ such that

$$\stackrel{\star}{\vdash} G(s, \tilde{t}),$$

and

$$|\tilde{t}|_{\mathrm{N}} \leq \Phi(|\tilde{t}|_{\mathbb{L}} + 1) < \Phi(2 \cdot \Phi(|s^{\mathbb{L}}|_{\mathbb{L}}) + 1) \leq \Phi(2 \cdot \Phi(|s|) + 1) < \Phi^2(|s|) < \Phi^2(|s|_{\mathrm{N}}).$$

Thus it follows $\stackrel{\star}{\vdash} F$ by means of $(\mathsf{V})^\star$ plus $(\mathsf{\Lambda})^\star$. □

Corollary 8.2.11. *There exists an $m < \omega$, such that for every hull-set $\mathcal{H}$ with $\Xi \in \mathcal{H}$ it holds*

$$\mathcal{F}^{\mathcal{H}}_{\cdot} \left|\frac{\omega^{\Xi+m}}{\Xi}\right. (\mathrm{Ad}_0.3)^\Xi.$$

Proof. For every $u \in \mathbb{L}_\omega$ there is an $f : u \xrightarrow{1\text{-}1} \mathrm{card}(u)$, such that $\{f, \mathrm{card}(u)\} \in \mathbb{L}_{\Phi(2 \cdot |u|_{\mathbb{L}})}$ since $\mathrm{card}(\mathbb{L}_n) < \Phi(n)$. Therefore we obtain (8.15) by means of Lemma 8.2.10 and the claim follows analogously to the proof of Lemma 8.2.6. □

Theorem 8.2.12 (Embedding of $\mathsf{\Pi}_\omega$-Ref^*)**.** *Let F be a theorem of $\mathsf{\Pi}_\omega$-Ref and $\Xi \in \mathcal{H}$. Then there is an $m \in \omega$ such that*

$$\mathcal{F}^{\mathcal{H}}_{\cdot} \left|\frac{\omega^{\Xi+m}}{\Xi+m}\right. F^\Xi.$$

Proof. Follows by the Embedding of Logic, Theorem 4.3.4, the $RS^\star$-derivability of (Iden), the embedding of $RS^\star$-derivations and the above given derivations of (Found), (Ref) and $(\mathrm{Ad}_0.1)$-$(\mathrm{Ad}_0.3)$ via some cuts. □

9. Cut and Reflection Elimination Theorems for Refined Derivations

In this chapter we prove the Reflection Elimination Theorem for derivations on fragmented hull-sets. It will turn out that we just have to do some minimal extra considerations compared to the proof of Theorem 5.2.5, which are (in essence) explained by the loss of the hull-property.

9.1. Predicative Cut Elimination for Infinitary Derivations on Fragmented Hull-Sets

Lemma 9.1.1 (Reduction Lemma)**.** *Let $F \cong \bigvee(F_t)_{t\in T}$ and $\rho := \mathrm{rnk}(F)$ not be regular. Then*

$$\mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}] \,\Big|\frac{\alpha}{\rho}\, \Delta, \neg F \;\;\&\;\; \mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}] \,\Big|\frac{\beta}{\rho}\, \Gamma, F \quad\Rightarrow\quad \mathcal{F}^{\mathcal{H}}_{\gamma+1}[\mathcal{A}] \,\Big|\frac{\alpha+\beta}{\rho}\, \Delta, \Gamma.$$

Proof. Analogue to [Buc93], Lemma 3.14 by induction on β. Taking also into account that $\mathrm{k}(\iota_0) \subseteq \mathrm{k}(C_{\iota_0}) \subseteq \mathcal{H}[\mathcal{A}]$ and $|\iota_0|_{\mathrm{N}} \le |C_{\iota_0}|_{\mathrm{N}} < f_\gamma(\mathcal{A})$ imply $f_\gamma(\mathcal{A},\iota) < f_\gamma(\mathcal{A}, f_\gamma(\mathcal{A})) \le f_{\gamma+1}(\mathcal{A})$ over $\mathcal{H}[\mathcal{A}]$ and therefore $\mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A},\iota_0] \subseteq \mathcal{F}^{\mathcal{H}}_{\gamma+1}[\mathcal{A}]$.

Moreover have in mind, that $2 \cdot f_\gamma(\mathcal{A}) < f_{\gamma+1}(\mathcal{A})$ and therefore $\alpha+\beta \in \mathcal{F}^{\mathcal{H}}_{\gamma+1}[\mathcal{A}]$ if $\alpha,\beta \in \mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}]$ and $|\Gamma',\Gamma|_{\mathrm{N}} \in \mathcal{F}^{\mathcal{H}}_{\gamma+1}[\mathcal{A}]$ if $|\Gamma'|_{\mathrm{N}}, |\Gamma|_{\mathrm{N}} \in \mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}]$. □

Theorem 9.1.2 (Predicative Cut Elimination[1])**.** *Let $\mathcal{H}$ be closed under φ and $[\rho,\rho+\alpha) \cap \mathrm{Reg} = \emptyset$ plus $\alpha \in \mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}]$. Then*

$$\mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A}] \,\Big|\frac{\beta}{\rho+\alpha}\, \Gamma \quad\Rightarrow\quad \mathcal{F}^{\mathcal{H}}_{\gamma\oplus\varphi(\alpha,\beta)+1}[\mathcal{A}] \,\Big|\frac{\varphi(\alpha,\beta)}{\rho}\, \Gamma.$$

Proof. We proceed by main induction on α and subsidiary induction on β. We only treat the non-trivial cases, that the last inference is $(\bigwedge)$ or (Cut).

In the former case we have $\Gamma = \Gamma', \bigwedge(F_t)_{t\in T}$ and

$$\mathcal{F}^{\mathcal{H}}_{\gamma}[\mathcal{A},t] \,\Big|\frac{\beta_t}{\rho+\alpha}\, \Gamma', F_t,$$

for all $t \in T$, with $\beta_t < \beta$. The induction hypothesis provides

$$\mathcal{F}^{\mathcal{H}}_{\gamma\oplus\varphi(\alpha,\beta_t)+1}[\mathcal{A},t] \,\Big|\frac{\varphi(\alpha,\beta_t)}{\rho}\, \Gamma', F_t, \quad \text{for all } t \in T. \tag{9.1}$$

[1] This theorem also holds with $\rho+\alpha$ replaced by $\rho+\omega^\alpha$ in the premise. However, in impredicative proof theory we do not need this stronger version.

Since $\mathrm{N}(\beta_t) < f_\gamma^{\mathcal{H}[\mathcal{A},t]}(\mathcal{A},t)$ it holds

$$f_{\gamma\oplus\varphi(\alpha,\beta_t)+1}(\mathcal{A},t) < f_{\gamma\oplus\varphi(\alpha,\beta)}(f_\gamma(\mathcal{A},t)) < f_{\gamma\oplus\varphi(\alpha,\beta)+1}(\mathcal{A},t) \quad \text{over } \mathcal{H}[\mathcal{A},t],$$

for all $t \in T$. Thus the claim follows from (9.1) by use of (Str) and ($\bigwedge$).

If the last inference is a (Cut) we have

$$\mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}] \left|\frac{\beta_0}{\rho+\alpha}\right. \Gamma, (\neg)F,$$

for some $\beta_0 < \beta$ and $\sigma := \mathrm{rnk}(F) < \rho + \alpha$. Therefore the subsidiary induction hypothesis provides

$$\mathcal{F}^{\mathcal{H}}_{\gamma\oplus\varphi(\alpha,\beta_0)+1}[\mathcal{A}] \left|\frac{\varphi(\alpha,\beta_0)}{\rho}\right. \Gamma, (\neg)F. \tag{9.2}$$

Case 1, it holds $\sigma < \rho$: Since $\beta_0 < \beta$ and $\alpha, \beta_0, \beta \in \mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}]$ it follows $\gamma\oplus\varphi(\alpha,\beta_0)+1 < \gamma \oplus \varphi(\alpha,\beta) + 1$ and

$$f_{\gamma\oplus\varphi(\alpha,\beta_0)+1}(\mathcal{A}) < f_{\gamma\oplus\varphi(\alpha,\beta)}(f_\gamma(\mathcal{A})) < f_{\gamma\oplus\varphi(\alpha,\beta)+1}(\mathcal{A}) \quad \text{over } \mathcal{H}[\mathcal{A}].$$

Thus the claim follows from (9.2) by use of (Str), (Inc) and a (Cut).

Case 2, it holds $\rho \leq \sigma = \rho + \alpha_0$, for some $0 \leq \alpha_0 < \alpha$: Since $\sigma \notin \mathrm{Reg}$ we may apply the Reduction Lemma to (9.2) and obtain

$$\mathcal{F}^{\mathcal{H}}_{\gamma\oplus\varphi(\alpha,\beta_0)+2}[\mathcal{A}] \left|\frac{\varphi(\alpha,\beta_0)\cdot 2}{\rho+\alpha_0}\right. \Gamma. \tag{9.3}$$

As we have $\mathrm{k}(F) \subseteq \mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}]$ and $|F|_{\mathrm{N}} \in \mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}]$ it follows $\sigma \in \mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}]$ and thereby $\alpha_0 \in \mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}]$. Therefore an application of the main induction hypothesis yields

$$\mathcal{F}_\eta^{\mathcal{H}}[\mathcal{A}] \left|\frac{\varphi(\alpha_0,\varphi(\alpha,\beta_0)\cdot 2)}{\rho}\right. \Gamma, \tag{9.4}$$

with $\eta := \gamma \oplus \varphi(\alpha,\beta_0) + 2 \oplus \varphi(\alpha_0, \varphi(\alpha,\beta_0)\cdot 2) + 1$.

Since $\varphi(\alpha_0,\varphi(\alpha,\beta_0)\cdot 2) < \varphi(\alpha_0,\varphi(\alpha,\beta_0+1)) = \varphi(\alpha,\beta_0+1) \leq \varphi(\alpha,\beta)$ we obtain the claim from (9.4) by means of (Inc) and (Str) if we can show $f_\eta(\mathcal{A}) < f_{\gamma\oplus\varphi(\alpha,\beta)+1}(\mathcal{A})$ over $\mathcal{H}[\mathcal{A}]$.

It holds $\eta < \varphi(\alpha,\beta) + 1$. Moreover we have

$$\mathrm{N}(\eta) \leq \mathrm{N}(\gamma) + 3N(\alpha) + \mathrm{N}(\alpha_0) + 3\,\mathrm{N}(\beta_0) + 5 < 6 \cdot f_\gamma(\mathcal{A}).$$

Thus we obtain

$$f_\eta(\mathcal{A}) < f_{\gamma\oplus\varphi(\alpha,\beta)}(f_\gamma(\mathcal{A})) < f_{\gamma\oplus\varphi(\alpha,\beta)+1}(\mathcal{A}) \quad \text{over } \mathcal{H}[\mathcal{A}].$$

□

9.2. Reflection Elimination for Infinitary Derivations on Fragmented Hull-Sets

Notation. Suppose $\gamma \in C(\gamma+1,0)$. Then we use the following notation

$$\mathcal{N}_\gamma[\mathcal{A}] := \mathcal{F}^{\mathcal{C}_\gamma}_{\gamma+1}[\mathcal{A}] = \mathcal{F}^{C(\gamma+1,0)}_{\gamma+1}[\mathcal{A}].$$

Lemma 9.2.1 ($\mathcal{N}$-version of Lemma 5.2.3)**.** *Let $\mathbb{X} = \mathbb{F}(\vec{\nu}) = (\pi; \dots; \delta)$ be a reflection instance and $\gamma, \mathbb{X}, \mu \in \mathcal{N}_\gamma[\mathcal{A}]$, where $\omega, \delta \leq \gamma+1$ and $\sigma := |\mathcal{A}| < \pi \leq \mu \in$* Card. *Let $\hat{\alpha} := \gamma \oplus \omega^{\alpha\oplus\mu}$. Then the following holds:*

- ❶ *If $\alpha_0, \alpha \in \mathcal{N}_\gamma[\mathcal{A}]$ and $\alpha_0 < \alpha$ then $\emptyset \neq {}_\sigma\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}} \subseteq {}_\sigma\mathfrak{M}^{\hat{\alpha_0}}_{\mathbb{X}}$. For every $\kappa \in {}_\sigma\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}$ and every $j < \Phi^2(f_\gamma(\mathcal{A},\kappa))$ it holds $\mathcal{N}_{\hat{\alpha_0}\oplus\kappa+j}[\mathcal{A},\kappa] \subseteq \mathcal{N}_{\hat{\alpha}\oplus\kappa}[\mathcal{A},\kappa]$. Moreover for every $\mathbb{Y} := \mathbb{F}(\vec{\eta})$ with $\vec{\eta} \in \mathrm{dom}(\mathbb{F}) \cap \mathcal{N}_\gamma[\mathcal{A}]$ the ordinals $\Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}}$ and $\Psi^{\hat{\alpha_0}\oplus\kappa}_{\mathbb{Y}}$ are well-defined and it holds $\Psi^{\hat{\alpha_0}\oplus\kappa}_{\mathbb{Y}} < \Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}} + j \in \mathcal{N}_{\hat{\alpha}\oplus\kappa}[\mathcal{A},\kappa]$.*
- ❷ *It holds $\pi \in \mathcal{N}_\gamma[\mathcal{A}]$. In addition for $\kappa \in$ Card and $\rho \in \mathcal{N}_\gamma[\mathcal{A}]$, such that $\pi \leq \kappa \leq \rho \leq \kappa^+$ it holds $\kappa, \kappa^+ \in \mathcal{N}_\gamma[\mathcal{A}]$.*
- ❸ *Let $\pi \leq \mu \in \mathrm{Reg} \cap \mathcal{N}_\gamma[\mathcal{A}]$. Then there exists a reflection instance $\mathbb{Z}$ with $\mathrm{i}(\mathbb{Z}) = \mu$, $\mathrm{o}(\mathbb{Z}) \leq \gamma+1$ and $\mathbb{Z}, \mathrm{par}\,\mathbb{Z} \in \mathcal{N}_\gamma[\mathcal{A}]$.*

Proof. We just show those parts of the propositions which go beyond the statements of Lemma 5.2.3.

① Let $j < \Phi^2(f_\gamma(\mathcal{A},\kappa))$. We have $\hat{\alpha_0} + j < \hat{\alpha}$ and since $\mathrm{card}(\mathcal{A},\kappa) \geq 1$ it follows $f_\gamma(\mathcal{A},\kappa) \geq 3$. Therefore we have

$$\mathrm{N}(\hat{\alpha_0} \oplus \kappa) + j + 1 < 8 \cdot \Phi^2(f_\gamma(\mathcal{A},\kappa)) < \Phi^2(f_\gamma(\mathcal{A},\kappa)+1) < \Phi^2(\mathrm{N}(\hat{\alpha}\oplus\kappa) + f_\gamma(\mathcal{A},\kappa))$$

and thereby

$$\begin{aligned} f_{\hat{\alpha_0}\oplus\kappa+j+1}(\mathcal{A},\kappa) &< f_{\hat{\alpha_0}\oplus\kappa+j+1}(f_\gamma(\mathcal{A},\kappa)) \\ &< f_{\hat{\alpha}\oplus\kappa}(f_\gamma(\mathcal{A},\kappa)) < f_{\hat{\alpha}\oplus\kappa+1}(\mathcal{A},\kappa) \quad \text{over } \mathcal{C}_\gamma[\mathcal{A},\kappa]. \end{aligned}$$

Hence $\mathcal{N}_{\hat{\alpha_0}\oplus\kappa+j}[\mathcal{A},\kappa] \subseteq \mathcal{N}_{\hat{\alpha}\oplus\kappa}[\mathcal{A},\kappa]$.

It holds

$$\begin{aligned} \mathrm{N}(\Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}}) + j &< 8 \cdot \Phi^2(f_\gamma(\mathcal{A},\kappa)) < \Phi^2(f_\gamma(\mathcal{A},\kappa)+1) \\ &< f_{\hat{\alpha}}(f_{\hat{\alpha}}(\mathcal{A},\kappa)) < f_{\hat{\alpha}\oplus\kappa+1}(\mathcal{A},\kappa) \quad \text{over } \mathcal{C}_\gamma[\mathcal{A},\kappa], \end{aligned}$$

and thereby $\Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}} + j \in \mathcal{N}_{\hat{\alpha}\oplus\kappa}[\mathcal{A},\kappa]$.

② If $\pi = \Xi$ or π is a successor cardinal the claim follows since $\mathrm{par}\,\mathbb{X} \in \mathcal{N}_\gamma[\mathcal{A}]$. Otherwise we have $\mathbb{X} = (\Psi^{\delta-1}_{\mathbb{V}}; \dots; \mathbb{V}; \dots; \delta)$. Then it holds $\delta - 1 < \gamma+1$, $\mathrm{par}\,\mathbb{V} \subseteq \mathbb{X}$ and $\mathrm{N}(\pi) = \max(\{\mathrm{N}(\delta)\} \cup \{\mathrm{N}(\xi)+1 \mid \xi \in \mathrm{par}\,\mathbb{V}\}) \leq \max\{\mathrm{N}(\zeta) \mid \zeta \in \mathrm{par}\,\mathbb{X}\}$ and thereby $\pi \in \mathcal{N}_\gamma[\mathcal{A}]$.

The second claim follows as in Lemma 5.2.3.

③ If $\mu = \Xi$ or $\mu = \nu^+$ for some $\nu \in \mathrm{Card} \cap \Xi$ the claim is trivial. Otherwise we have $\mu = \Psi^{\zeta}_{\mathbb{V}}$ for some ζ and a reflection instance $\mathbb{V}$. Since $\sigma < \pi \le \mu$ it follows by Lemma 2.3.16 that $\zeta, \mathbb{V} \in \mathcal{N}_\gamma[\mathcal{A}] \subseteq C(\gamma+1, \sigma+1)$. Thus $\zeta \le \gamma$ and as $\mu \in \mathrm{Reg}$ it follows the existence of a reflection configuration $\mathbb{G} = (\mu; \ldots; \mathbb{V}; \zeta+1)^{\dagger}$.

If the arity of $\mathbb{G}$ is zero, we can choose $\mathbb{Z} = \mathbb{G} \in \mathcal{N}_\gamma[\mathcal{A}]$, since $\max\{\mathrm{N}(\varrho) \mid \varrho \in \mathrm{par}\,\mathbb{G}\} \le \mathrm{N}(\mu)$. If $\mathrm{dom}(\mathbb{G}) = [\mathrm{o}(\mathbb{M}), \xi)_{C(\mu)} \times \mathrm{dom}(\mathbb{M})_{C(\mu)}$ with $\mathbb{G}(\zeta', \vec{\nu}) = (\mu; \mathsf{M}^{\zeta'}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_m; \ldots)$, we have $\mathbb{M} \in \mathrm{Prcnfg}(\mathbb{G})$, i.e. it exists a $\vec{\eta} \in \mathrm{dom}(\mathbb{M})_{C(\mu)}$ such that $\mathbb{M}(\vec{\eta}) \in \mathrm{Prinst}(\mathbb{G}) = \overline{\mathrm{Prinst}}(\mathbb{V})$. Thus we have $\mathbb{Z} := \mathbb{G}(\mathrm{o}(\mathbb{M}), \vec{\eta}) \in \mathcal{N}_\gamma[\mathcal{A}]$. □

Theorem 9.2.2 (Reflection Elimination on fragmented Hull-Sets)**.** *Let* $\mathbb{X} = (\pi; \ldots; \delta)$ *be a reflection instance with* $\mathrm{rdh}(\mathbb{X}) = m-1$*,* $\mathcal{N}_\gamma[\mathcal{A}]$ *be defined and* $\gamma, \mathbb{X}, \mu \in \mathcal{N}_\gamma[\mathcal{A}]$*, where* $\omega, \delta \le \gamma+1$ *and* $\sigma := |\mathcal{A}| < \pi \le \mu \in \mathrm{Card}$*. Let* $\Gamma \in \Sigma_{m+1}(\pi)$ *and* $\hat{\alpha} := \gamma \oplus \omega^{\alpha \oplus \mu}$*. Then*

$$\mathcal{N}_\gamma[\mathcal{A}] \left|\frac{\alpha}{\mu}\right. \Gamma \quad \Rightarrow \quad \mathcal{N}_{\hat{\alpha}\oplus\kappa}[\mathcal{A}, \kappa] \left|\frac{\Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}}}{\cdot}\right. \Gamma^{(\pi,\kappa)} \quad \text{for all } \kappa \in {}_{\sigma}\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}.$$

Proof. The proof is similar to the proof of Theorem 5.2.5[‡]. Consequently we list the extra considerations, which are necessary to come through the proof of the $\mathcal{N}$-version:

- $|\Gamma^{(\pi,\kappa)}|_{\mathrm{N}} \le |\Gamma|_{\mathrm{N}} + |L_\kappa|_{\mathrm{N}}$ and hence $|\Gamma^{(\pi,\kappa)}|_{\mathrm{N}} < f_{\hat{\alpha}\oplus\kappa}(\mathcal{A}, \kappa)$ since $|\Gamma|_{\mathrm{N}} < f_\gamma(\mathcal{A})$ over $\mathcal{C}_\gamma[\mathcal{A}, \kappa]$.

Case 1 up to Subcase 3.2 with $\pi = \mu$:

- Follow by use of Lemma 9.2.1

Subcase 3.2, with $\pi < \mu$:

- Instead of (5.9) we only get $\mathcal{N}_{\hat{\alpha_0}\oplus\lambda_0+2}[\mathcal{A}] \left|\frac{\eta}{\cdot}\right. \Gamma$, with $\eta := \Psi^{\hat{\alpha_0}\oplus\lambda_0+2}_{\mathbb{Y}}$. Therefore we have to replace "1" by "2" in ν, but we still can argue as in the proof of Theorem 5.2.5. $\mathcal{N}_{\nu\oplus\kappa}[\mathcal{A}, \kappa] \subseteq \mathcal{N}_{\hat{\alpha}\oplus\kappa}[\mathcal{A}, \kappa]$ follows by the same considerations as in the proof of Lemma 9.2.1 ①, since $\mathrm{N}(\nu) < \Phi(f_\gamma(\mathcal{A}, \kappa))$.

- If $\eta \notin \mathrm{Card}$ statement (5.11) modifies to $\mathcal{N}_{\hat{\alpha_0}\oplus\lambda_0\oplus\varphi(\eta,\eta)+1}[\mathcal{A}] \left|\frac{\varphi(\eta,\eta)}{\bar{\mu_0}}\right. \Gamma$. Thus in the following we have to set $\hat{\nu} := (\hat{\alpha_0} \oplus \lambda_0 \oplus \varphi(\eta,\eta) + 1) \oplus \omega^{\varphi(\eta,\eta)\oplus\bar{\mu_0}}$.

Case 4, Subcase 4.1:

- We only get equation (5.12) with γ replaced by $\gamma+1$, but this modification does not affect the applicability of the induction hypothesis it only necessitates to replace $\mathcal{N}_{\hat{\alpha_0}\oplus\lambda}$ by $\mathcal{N}_{\hat{\alpha_0}\oplus\lambda+1}$ and $\Psi^{\hat{\alpha_0}\oplus\lambda}_{\mathbb{Y}}$ by $\Psi^{\hat{\alpha_0}\oplus\lambda+1}_{\mathbb{Y}}$ in (5.13).

[†]For a more detailed argumentation confer the proof of Lemma 5.2.3 on page 54.

[‡]In the proof of Theorem 5.2.5 we tacitly replaced Γ' by Γ, where $\Gamma = \Gamma', F$ and F was the principal formula of the last inference. We were allowed to do so by means of (Str). However, for a proof of the $\mathcal{N}$-version of Theorem 5.2.5 we have to work with Γ' instead of Γ.

- The application of (E-∀) forces us to replace the index $\hat{\alpha_0}$ of $\mathcal{N}$ by $\hat{\alpha_0} + 1 + \Phi^2(f_\gamma(\mathcal{A}, \kappa))$ in (5.14) – (5.17). Moreover l has to be replaced by $\Phi^2(f_\gamma(\mathcal{A}, \kappa))$ in (5.14).

- Let $H :\equiv L_\lambda \neq \emptyset \wedge \mathrm{Tran}(L_\lambda) \wedge \bigwedge_{i=1}^{p} a_i \in L_\lambda$. Then $\Vert\!\!\frac{}{\mathrm{N}}\, H$ implies $\mathcal{N}_{\|H\|}[H] \left|\frac{\|H\|}{0}\right. H$. It follows easily that $|H|_\mathrm{N} = \max\{12, |L_\lambda|_\mathrm{N}{+}p{+}7, |a_i|_\mathrm{N}{+}p{+}8\} < \Phi(f_\gamma(\mathcal{A}, \lambda))$ since $|F|_\mathrm{N} < f_\gamma(\mathcal{A})$. Hence $\mathcal{N}_{\|H\|}[H] \subseteq \mathcal{N}_{\hat{\alpha_0} \oplus \lambda}[\mathcal{A}, \lambda, \kappa]$.

- To obtain (5.15) we have to perform $\mathrm{card}(\Gamma) - 1$ many (⋁)-inferences. Thus we can replace ω by $3 \cdot f_\gamma(\mathcal{A})$ in (5.15), since $\mathrm{card}(\Gamma) \leq |\Gamma|_\mathrm{N} < f_\gamma(\mathcal{A})$.

- Let $H' :\equiv L_\lambda \neq s, \bigwedge \neg\Gamma^{(\pi,\lambda)}, \bigvee \Gamma^{(\pi,s)}$. Since $|L_\lambda \neq s|_\mathrm{N} = \max\{9, |L_\lambda|_\mathrm{N}{+}4, |s|_\mathrm{N}{+}4\}$ and $|\Gamma|_\mathrm{N} < f_\gamma(\mathcal{A})$ it follows $|H'|_\mathrm{N} < \Phi(f_\gamma(\mathcal{A}, \lambda, s))$ and thereby $\mathcal{N}_{\|H'\|}[H'] \subseteq \mathcal{N}_{\hat{\alpha_0} \oplus \kappa}[\mathcal{A}, \lambda, \kappa, s]$. Therefore we are able to perform the (Cut) to (5.15) and (5.16).

- $|\neg_\sigma M_{\mathbb{Y}}^{\hat{\alpha_0}}(s) \vee = \emptyset \vee \neg\,\mathrm{Tran}(s) \vee (\bigwedge_{i=1}^{q} b_i \in s), \bigvee \Gamma^{(\pi,s)}, G|_\mathrm{N} < \Phi(f_\gamma(\mathcal{A}, \kappa, s)) < f_{\hat{\alpha_0} \oplus \kappa}[\mathcal{A}, \kappa, s]$ over $\mathcal{C}_{\hat{\alpha_0} \oplus \kappa}[\mathcal{A}, \kappa, s]$.

- It also follows easily that the finite contents of the formulae in (5.17) and (5.18) are in $\mathcal{N}_{\hat{\alpha_0} \oplus \kappa}[\mathcal{A}, \kappa]$, since $|\Gamma, F|_\mathrm{N} \in \mathcal{N}_\gamma[\mathcal{A}]$.

Subcase 4.2 up to Case 6:

- The proof of this cases is made by the same extra considerations as stated above. □

10. A Characterization of the Provable Recursive Functions of Π_ω-Ref

In this final chapter we utilize the "collapsibility" of relativized defined subrecursive hierarchies to achieve a characterization of the provable recursive functions of Π_ω-Ref.

10.1. Collapsing and the Witnessing Theorem

The growth rate of $f_\alpha^{\mathcal{H}}$ only depends on the otyp of $< \restriction \mathcal{H} \cap \alpha$ and the norm on $\mathcal{H} \cap \alpha$, but not on the actual size of α. Thereby, given a hull-set $\mathcal{H}'$ and a $\beta < \alpha$ such that otyp($< \restriction \mathcal{H} \cap \alpha$) <otyp($< \restriction \mathcal{H}' \cap \beta$) and assumed that the norm on $\mathcal{H} \cap \alpha$ and $\mathcal{H}' \cap \beta$ is "similar" it follows that $f_\alpha^{\mathcal{H}}(x) < f_\beta^{\mathcal{H}'}(x)$ for all but finitely many $x \in \omega$.

Thereby a subrecursive hierarchy defined on a hull-set $\mathcal{H}$ is "collapsible" by replacing $\mathcal{H}$ by an appropriate hull-set $\mathcal{H}'$.

Notation. Let S_1, S_2 be subsets of $\mathsf{T}(\Xi)$ and $\Psi : S_1 \longrightarrow S_2$. Then we declare the following notation

$$\begin{aligned}\Psi : S_1 \xrightarrow[\text{N-stable}]{<\text{-stable}} S_2 \quad :\Leftrightarrow \quad & \forall \alpha, \beta \in \mathrm{dom}(\Psi)\Big(\big(\alpha < \beta \ \rightarrow \ \Psi(\alpha) < \Psi(\beta)\big) \wedge \\ & \forall x \in \omega\big(\mathrm{N}(\alpha) \leq \Phi^2(\mathrm{N}(\beta) + x) \ \Rightarrow \\ & \qquad \mathrm{N}(\Psi(\alpha)) \leq \Phi^2(\mathrm{N}(\Psi(\beta)) + x)\big)\Big).\end{aligned}$$

Theorem 10.1.1 (Collapsing[1])**.** *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be hull-sets, $\mathcal{A}$ be a finite set of ordinals, terms and sentences and $\gamma \in \mathcal{H}_1[\mathcal{A}] \subseteq \mathcal{H}_2[\mathcal{A}]$. Then*

$$\Psi : \mathcal{H}_1[\mathcal{A}] \cap (\gamma + 1) \xrightarrow[\text{N-stable}]{<\text{-stable}} \mathcal{H}_2[\mathcal{A}], \quad \Rightarrow \quad \mathcal{F}_\gamma^{\mathcal{H}_1}[\mathcal{A}] \subseteq \mathcal{F}_{\Psi(\gamma)}^{\mathcal{H}_2}[\mathcal{A}].$$

Proof. Since $\mathcal{H}_1[\mathcal{A}] \subseteq \mathcal{H}_2[\mathcal{A}]$ we just have to prove $f_\gamma^{\mathcal{H}_1[\mathcal{A}]}(\mathcal{A}) \leq f_{\Psi(\gamma)}^{\mathcal{H}_2[\mathcal{A}]}(\mathcal{A})$.

We show by induction on $\alpha \leq \gamma$ that $f_\alpha^{\mathcal{H}_1[\mathcal{A}]}(x) \leq f_{\Psi(\alpha)}^{\mathcal{H}_2[\mathcal{A}]}(x)$ for all $x \in \omega$.

[1] The name of this theorem is a bit pointless without specifying Ψ as a "collapsing-function".

If $\alpha = 0$ the claim is trivial. For $\alpha > 0$ it holds

$$\begin{aligned}
f_\alpha^{\mathcal{H}_1[\mathcal{A}]}(x) &= \max\{f_\beta^{\mathcal{H}_1[\mathcal{A}]^2}(x) \mid \beta \in \mathcal{H}_1[\mathcal{A}] \cap \alpha \;\&\; \mathrm{N}(\beta) \le \Phi^2(\mathrm{N}(\alpha)+x)\} \\
&\le \max\{f_{\Psi(\beta)}^{\mathcal{H}_2[\mathcal{A}]^2}(x) \mid \beta \in \mathcal{H}_1[\mathcal{A}] \cap \alpha \;\&\; \mathrm{N}(\beta) \le \Phi^2(\mathrm{N}(\alpha)+x)\} \\
&\le \max\{f_\delta^{\mathcal{H}_2[\mathcal{A}]^2}(x) \mid \delta \in \mathcal{H}_2[\mathcal{A}] \cap \Psi(\alpha) \;\&\; \mathrm{N}(\delta) \le \Phi^2(\mathrm{N}(\Psi(\alpha))+x)\} \\
&= f_{\Psi(\alpha)}^{\mathcal{H}_2[\mathcal{A}]}(x).
\end{aligned}$$

□

Lemma 10.1.2 (Detachment)**.** *Suppose $D \in \Delta_0(\omega)$ and $\mathbb{L} \models \neg D^{\mathbb{L}}$. Then*

$$\mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}] \Big|\!\frac{\alpha}{0}\, \Gamma, D \quad \Rightarrow \quad \mathcal{F}_{\gamma \oplus \omega^{\alpha+1}}^{\mathcal{H}}[\mathcal{A}] \Big|\!\frac{\alpha}{0}\, \Gamma.$$

Proof. We proceed by induction on α.

Case 1, the principal formula of the last inference is $D \cong \bigvee(D_t)_{t\in T}$: Then we have

$$\mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}] \Big|\!\frac{\alpha_0}{0}\, \Gamma, D_{t_0},$$

for some $\alpha_0 < \alpha$ and some $t_0 \in T$. Since $\mathbb{L} \models \neg D^{\mathbb{L}}$ we also have $\mathbb{L} \models \neg D_{t_0}^{\mathbb{L}}$. Therefore the claim follows by an application of the induction hypothesis and means of (Str) since $f_{\gamma\oplus\omega^{\alpha_0+1}}(\mathcal{A}) < f_{\gamma\oplus\omega^{\alpha+1}}(\mathcal{A})$ over $\mathcal{H}[\mathcal{A}]$.

Case 2, the principal formula of the last inference is $D \cong \bigwedge(D_t)_{t\in T}$:

Subcase 2.1, it holds $T = \mathcal{T}_{|s|}$ for some $s \in \mathcal{T}_\omega$: Then we have

$$\mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}, t] \Big|\!\frac{\alpha_t}{0}\, \Gamma, D_t \quad \text{for all } t \in \mathcal{T}_{|s|}, \tag{10.1}$$

with $\alpha_t < \alpha$. Since $\mathbb{L} \models \neg D^{\mathbb{L}}$ there exists a $t_0 \in \mathcal{T}_{|s|}$ such that $\mathbb{L} \models \neg D_{t_0}^{\mathbb{L}}$. By Lemma 8.2.9 we can choose a $\tilde{t}_0 \in \widetilde{\mathcal{T}}_\omega$ with $|t_0| = |\tilde{t}_0|$ and $\mathbb{L} \models t_0^{\mathbb{L}} = \tilde{t}_0^{\mathbb{L}}$ plus $|\tilde{t}_0|_{\mathrm{N}} < \Phi(|t_0|+1) \le \Phi(|s|) < \Phi(D) < \Phi(f_\gamma(\mathcal{A}))$ over $\mathcal{H}[\mathcal{A}]$. Thus we have

$$\mathcal{F}_\gamma^{\mathcal{H}}[\mathcal{A}, \tilde{t}_0] \Big|\!\frac{\alpha_{\tilde{t}_0}}{0}\, \Gamma, D_{\tilde{t}_0} \quad \& \quad \mathbb{L}_\omega \models \neg D_{\tilde{t}_0}^{\mathbb{L}}. \tag{10.2}$$

Since $\mathcal{H}[\mathcal{A}, \tilde{t}_0] = \mathcal{H}[\mathcal{A}]$ and $f_\gamma(\mathcal{A}, \tilde{t}_0) < f_\gamma(\Phi(f_\gamma(\mathcal{A}))) < f_{\gamma\oplus\omega+2}(\mathcal{A})$ over $\mathcal{H}[\mathcal{A}]$ we obtain from (10.2)

$$\mathcal{F}_{\gamma\oplus\omega+2}[\mathcal{A}] \Big|\!\frac{\alpha_{\tilde{t}_0}}{0}\, \Gamma, D_{\tilde{t}_0}. \tag{10.3}$$

An application of the induction hypothesis yields

$$\mathcal{F}_{\gamma\oplus\omega+2\oplus\omega^{\alpha_{\tilde{t}_0}+1}}[\mathcal{A}] \Big|\!\frac{\alpha_{\tilde{t}_0}}{0}\, \Gamma. \tag{10.4}$$

Since $\mathrm{N}(\alpha_{\tilde{t}_0}) < f_\gamma(\mathcal{A}, \tilde{t}_0) < f_{\gamma\oplus\omega+2}(\mathcal{A})$ over $\mathcal{H}[\mathcal{A}]$ it holds

$$\begin{aligned}
f_{\gamma\oplus\omega^{\alpha_{\tilde{t}_0}+1}+\omega+2}(\mathcal{A}) &< f_{\gamma\oplus\omega^{\alpha_{\tilde{t}_0}+1}+\omega+2}(f_{\gamma\oplus\omega+2}(\mathcal{A})) < f_{\gamma\oplus\omega^\alpha+\omega+3}(f_{\gamma\oplus\omega+2}(\mathcal{A})) \\
&< f_{\gamma\oplus\omega^\alpha+\omega+4}(\mathcal{A}) < f_{\gamma\oplus\omega^{\alpha+1}}(\mathcal{A}) \quad \text{over } \mathcal{H}[\mathcal{A}].
\end{aligned}$$

Thereby the claim follows from (10.4) by use of (Str).

Subcase 2.2, it holds $T = \{0,1\}$: Then the claim follows analogously to the first case.

Case 3, the principal formula $F \cong \bigwedge(F_t)_{t\in\mathcal{T}_{|s|}} \in \Gamma$: Let $\Gamma = \Gamma', F$. Then we have

$$\mathcal{F}_\gamma[\mathcal{A},t] \left|\frac{\alpha_t}{0}\right. \Gamma', F_t, D \quad \text{for all } t \in \mathcal{T}_{|s|}, \tag{10.5}$$

with $\alpha_t < \alpha$. The induction hypothesis provides

$$\mathcal{F}_{\gamma\oplus\omega^{\alpha_t+1}}[\mathcal{A},t] \left|\frac{\alpha_t}{0}\right. \Gamma', F_t \quad \text{for all } t \in \mathcal{T}_{|s|}. \tag{10.6}$$

Since $\mathcal{H}[\mathcal{A},t] = \mathcal{H}[\mathcal{A}]$ and $\mathrm{N}(\alpha_t) < f_\gamma(\mathcal{A},t)$ over $\mathcal{H}[\mathcal{A}]$ we have

$$f_{\gamma\oplus\omega^{\alpha_t+1}}(\mathcal{A},t) < f_{\gamma\oplus\omega^\alpha+1}(f_\gamma(\mathcal{A},t)) < f_{\gamma\oplus\omega^\alpha+2}(\mathcal{A},t) < f_{\gamma\oplus\omega^{\alpha+1}}(\mathcal{A},t) \text{ over } \mathcal{H}[\mathcal{A}].$$

Thereby the claim follows from (10.6) by use of (Str) and $(\bigwedge)$.

Case 4, the principal formula F of the last inference belongs to Γ but is not of the form $(\bigwedge(F_t)_{t\in\mathcal{T}_{|s|}}$: Then the claim follows by use of the induction hypothesis and the last inference. □

Theorem 10.1.3 (Witnessing Theorem)**.** *Let $G(L_0) \in \Delta_0(\omega)$. Then*

$$\mathcal{F}^{\mathcal{H}}_\gamma[\mathcal{A}] \left|\frac{\alpha}{0}\right. \exists x^\omega G(x) \quad \Rightarrow \quad \mathbb{L} \models \exists x^m\, G(x)^{\mathbb{L}}, \;\; \textit{where } m := f^{\mathcal{H}[\mathcal{A}]}_{\gamma\oplus\omega^{\alpha+1}}(\mathcal{A}).$$

Proof. We proceed by induction on α. We do not know, if $\exists x^\omega G(x)$ already occurs in the premise of the last inference, but by use of (Str) we have in either case

$$\mathcal{F}^{\mathcal{H}}_{\gamma+1}[\mathcal{A}] \left|\frac{\alpha_0}{0}\right. \exists x^\omega G(x), G(t_0), \tag{10.7}$$

for some $\alpha_0 < \alpha$ and some $t_0 \in \mathcal{T}_\omega$. If $\mathbb{L} \models G(t_0)^{\mathbb{L}}$ the claim follows, since $|t_0^{\mathbb{L}}|_{\mathbb{L}} \leq |t_0| \leq |t_0|_{\mathrm{N}} < m$ as $|t_0|_{\mathrm{N}} \in \mathcal{F}^{\mathcal{H}}_\gamma[\mathcal{A}]$.

If $\mathbb{L} \models \neg G(t_0)^{\mathbb{L}}$ we apply the Detachment Lemma 10.1.2 to (10.7) and obtain

$$\mathcal{F}^{\mathcal{H}}_{\gamma+1\oplus\omega^{\alpha_0+1}}[\mathcal{A}] \left|\frac{\alpha_0}{0}\right. \exists x^\omega G(x). \tag{10.8}$$

The induction hypothesis provides $\mathbb{L} \models \exists x^{m_0} G(x)^{\mathbb{L}}$ with $m_0 := f^{\mathcal{H}[\mathcal{A}]}_{\gamma+1\oplus\omega^{\alpha_0+1}\cdot 2}(\mathcal{A})$. Since $\mathrm{N}(\alpha_0) < f^{\mathcal{H}[\mathcal{A}]}_\gamma(\mathcal{A})$ it holds

$$m_0 < f_{\gamma+2\oplus\omega^\alpha\cdot 2}(f_\gamma(\mathcal{A})) < f_{\gamma+3\oplus\omega^\alpha\cdot 2}(\mathcal{A}) < m \quad \text{over } \mathcal{H}[\mathcal{A}].$$

Since $G(L_0) \in \Delta_0(\omega)$ the claim follows by the upwards persistency of Σ-sentences. □

10.2. A Subrecursive Hierarchy Dominating the Provable Recursive Functions of Π_ω-Ref

It is an elementary result of recursion theory that for every partial recursive function f there is a primitive recursive predicate T_f such that

$$f(x)) \simeq y \ \Leftrightarrow \ \exists z\, T_f(x,y,z).$$

Since we can code arbitrary n-tuples in $\mathbb{L}_\omega$ via $\langle x,y\rangle := \{\{x\},\{x,y\}\}$ there is an $\mathcal{L}_\in$-Δ_0-formula $T'_f(x,y,z)$ such that $f(x)\simeq y \ \Leftrightarrow\ \mathbb{L}_\omega \models \exists z\, T'_f(x,y,z)$. Coding also y and z into one set we obtain an $\mathcal{L}_\in$-Δ_0-formula $G_f(x,z)$, such that f is recursive iff $\mathbb{L}_\omega \models \forall x\,\exists z\, G_f(x,z)$.

Therefore an $\mathbb{L}$-correct subsystem T of set theory, in which $\mathbb{L}_\omega$ is definable, proves the recursiveness of f iff $T \vdash \forall x\big(\text{“}x = L_\omega\text{”}\to \forall y^x \exists z^x G_f(y,z)\big)$. Since $\mathbb{L}_\omega \models G_f(s,t)$ also implies $f(s) < |t|_{\mathbb{L}}$ we are able to outvote the provable recursive functions of T, if we are able to give an upper bound (depending on y) for the witness z in the above formula.

Lemma 10.2.1. *Let $\ell(u)$ be a $\Delta_1^{\mathsf{KP}\omega}$-formula, which defines $\mathbb{L}_\omega$. Then it holds*

$$\Pi_\omega\text{-}\mathsf{Ref}^* \vdash \forall x\big(\ell(x) \leftrightarrow Ad_0(x)\big).$$

Proof. We argue in an arbitrary model of Π_ω-Ref^*. The direction from left to right is trivial. So let us assume $Ad_0(u)$ for some set u. We have to show $u = \mathbb{L}_\omega$.

At first we show by induction on n that $\mathbb{L}_n \subseteq u$ for all $n \in \omega$. So assume $\mathbb{L}_n \subseteq u$. If $n = 0$ it follows by $(\text{Nullset})^u$ and $(\Delta_0\text{-Sep})^u$ that $\mathbb{L}_0 \in u$. If $n > 0$ it follows by iterated use of $(\text{Pair})^u$ and $(\text{Union})^u$ that $\mathbb{L}_n \in u$, as $\mathbb{L}_n$ is finite. Therefore we obtain by use of $(\Delta_0\text{-Sep})^u$ that $\mathbb{L}_{n+1} \subseteq u$. Thus we have shown $\mathbb{L}_\omega \subseteq u$.

In addition it follows for every $z \in u$ by induction on $\text{rnk}_{\mathbb{V}}(z)$ that $z \in \mathbb{L}_\omega$ since z is finite due to $(\text{Ad}_0.3)$. Therefore we also have $u \subseteq \mathbb{L}_\omega$. □

Notation. Let $\mathbb{X} := (\omega^+; \mathsf{P}_0; \epsilon; \epsilon; 0)$ and $\mathcal{H} := \Psi_{\mathbb{X}}^{\varepsilon_{\Xi+1}}$. Then we just write $f_\alpha(x)$ instead of $f_\alpha^{\mathcal{H}}(x)$ for $\alpha < \Psi_{\mathbb{X}}^{\varepsilon_{\Xi+1}}$.

Theorem 10.2.2. *Let $\ell(u)$ be a $\Delta_1^{\mathsf{KP}\omega}$-formula, which defines $\mathbb{L}_\omega$. Suppose $\mathbb{X} := (\omega^+; \mathsf{P}_0; \epsilon; \epsilon; 0)$ and $F \equiv \forall x\,\exists y\, G(x,y)$ is an elementary $\mathcal{L}_\in$-Π_2^0-sentence, such that*

$$\Pi_\omega\text{-}\mathsf{Ref} \vdash \forall z\big(\ell(z) \to F^z\big).$$

Then there exists an $\alpha < \Psi_{\mathbb{X}}^{\varepsilon_{\Xi+1}}$, satisfying

$$\mathbb{L}_\omega \models \forall x\,\exists y^{f_\alpha(|x|_{\mathbb{L}})}\, G(x,y).$$

Proof. By Lemma 10.2.1 and Theorem 8.2.12 there is an $m \in \omega$ such that

$$\mathcal{N}.\Big|\frac{\omega^{\Xi+m}}{\Xi+m}\ \forall z^{\Xi}\big(Ad_0(z) \to F^z\big).$$

By ($\forall$-Inv), (Str) and ($\vee$-Ex) we obtain

$$\mathcal{N}.\Big|\frac{\omega^{\Xi+m}+2}{\Xi+m}\ \neg Ad_0(L_\omega) \to F^\omega. \tag{10.9}$$

Due to Lemma 4.3.3 ③, Lemma 8.2.1 and Lemma 8.2.4 there is an m' such that

$$\mathcal{N}.\Big|\frac{\omega^{\Xi+m'}}{0}\ Ad_0(L_\omega). \tag{10.10}$$

Let $\alpha_0 := \omega^{\Xi+m} + 2 \oplus \omega^{\Xi+m'}$. By a (Cut) applied to (10.9) and (10.10) we obtain

$$\mathcal{N}.\Big|\frac{\alpha_0}{\Xi+m}\ F^\omega.$$

For $1 \le i < m$ let $\alpha_{i+1} := \alpha_i \oplus \omega^{\alpha_i} + 1$. By iterated application of predicative cut elimination we obtain

$$\mathcal{N}.\Big|\frac{\alpha_{m-1}}{\Xi+1}\ F^\omega.$$

Let $\hat{\alpha}_{m-1} := \alpha_{m-1} \oplus \omega^{\alpha_{m-1}\oplus\Xi}$ and $\alpha_m := \hat{\alpha}_{m-1} \oplus \Psi_{\mathbb{X}}^{\hat{\alpha}_{m-1}} + 1$. Then we obtain by Theorem 9.2.2 and (Str)

$$\mathcal{N}_{\alpha_m}\Big|\frac{\Psi_{\mathbb{X}}^{\alpha_m}}{\cdot}\ F^\omega. \tag{10.11}$$

Moreover it holds

$$\lambda\xi.\Psi_{\mathbb{X}}^{\alpha_m+1\oplus\xi} : \mathcal{C}_{\alpha_m} \cap (\alpha_m + 2) \xrightarrow[\text{N-stable}]{\text{<-stable}} \mathcal{C}_{(\alpha_m+1)\cdot 2}$$

Let $\mathcal{H} := \mathcal{C}_{(\alpha_m+1)\cdot 2}$. Thus it follows by Theorem 10.1.1 that $\mathcal{N}_{\alpha_m} \subseteq \mathcal{F}^{\mathcal{H}}_{\Psi_{\mathbb{X}}^{(\alpha_m+1)\cdot 2}}$. Thereby we obtain from (10.11) and use of (Str)

$$\mathcal{F}^{\mathcal{H}}_{\Psi_{\mathbb{X}}^{(\alpha_m+1)\cdot 2}}\Big|\frac{\Psi_{\mathbb{X}}^{\alpha_m}}{\cdot}\ F^\omega.$$

Let $\alpha := \varphi(\Psi_{\mathbb{X}}^{\alpha_m}, \Psi_{\mathbb{X}}^{\alpha_m})$ and $\gamma := \Psi_{\mathbb{X}}^{(\alpha_m+1)\cdot 2} \oplus \alpha + 1$. By predicative cut elimination we obtain

$$\mathcal{F}^{\mathcal{H}}_\gamma \Big|\frac{\alpha}{0}\ F^\omega.$$

By means of ($\forall$-Inv) we get

$$\mathcal{F}^{\mathcal{H}}_\gamma[\tilde{s}] \Big|\frac{\alpha}{0}\ \exists y^\omega G(\tilde{s}, y) \quad \text{for all } \tilde{s} \in \widetilde{\mathcal{T}}_\omega.$$

By the Witnessing Theorem 10.1.3 it follows

$$\mathbb{L} \models \exists y^m G(\tilde{s}, y)^{\mathbb{L}}, \text{ where } m := f^{\mathcal{H}[\tilde{s}]}_{\gamma\oplus\omega^{\alpha+1}}(\tilde{s}) \quad \text{for all } \tilde{s} \in \widetilde{\mathcal{T}}_\omega. \tag{10.12}$$

A closer look at the build-up of α and γ reveals that $\alpha, \gamma < \Psi_{\mathbb{X}}^{\varepsilon_{\Xi+1}}$. Since $\mathcal{H}[\tilde{s}] = \mathcal{H}$ and $\mathcal{H} \cap \Psi_{\mathbb{X}}^{\varepsilon_{\Xi+1}} \subseteq \Psi_{\mathbb{X}}^{\varepsilon_{\Xi+1}}$ it follows $f^{\mathcal{H}}_\delta(x) \le f_\delta(x)$ for all $\delta \in \mathcal{H} \cap \Psi_{\mathbb{X}}^{\varepsilon_{\Xi+1}}$ and $x \in \omega$. Moreover we have $f_\delta(\Phi(x) + 1) < f_{\delta+1}(x)$ if $\delta \ge \omega$. Thus we obtain from (10.12) and Lemma 8.2.9

$$\mathbb{L}_\omega \models \forall x\, \exists y^m\, G(x, y), \quad \text{where } m := f_{\gamma\oplus\omega^{\alpha+2}}(|x|_{\mathbb{L}}).$$ □

As a consequence of the last theorem we obtain:

Theorem 10.2.3. *The provable recursive functions of* Π_ω-Ref *are contained in the class* $\mathfrak{F}$*, where* $\mathfrak{F}$ *is the smallest class of number theoretic functions, which contains* $S, C^n_k, P^n_k, (f_\alpha)_{\alpha \in \Psi^{\varepsilon(\Xi+1)}_{\mathbb{X}}}$ *and is closed under substitution and primitive recursion.*

Part III.

An Ordinal Analysis of Stability

11. Introduction

A transitive set which is a model of KP is called admissible. The theory KPi comprises the theory KP plus an axiom (Lim), which postulates that every set is contained in an admissible set. The theory Stability is the theory KPi augmented by the axiom $\forall\alpha\,\exists\kappa \geq \alpha\,(L_\kappa \preceq_1 L_{\kappa+\alpha})$, where $L_\kappa \preceq_1 L_{\kappa+\alpha}$ denotes that L_κ is a Σ_1-elementary substructure of $L_{\kappa+\alpha}$.[1]

Obviously every $\kappa+1$-stable ordinal is already Π_n reflecting, for all $n<\omega$. In this sense the notion of an α-stable ordinal can be regarded as a transfinite extension of the notion of a Π_n-reflecting ordinal. Therefore one might informally refer to the theory Stability as Aut-Π-Ref. Thus a treatment of Stability marks in some sense a completion of ordinal analyses of theories of the form KP augmented by a (first order) reflection principle.

However, in view of ordinal-analyses of even stronger theories, i.e. KP + Σ_1- separation a proof-theoretic treatment of Stability features the simplest case of a new paradigm: The collapsing of intervals.

To achieve a reflection-elimination theorem for the theory Π_ω-Ref we have to "collapse" the derivation of a finite set of $\Sigma_n(\pi)$-sentences Γ^π with parameters in L_σ for some $\sigma<\pi$ to a derivation of Γ^κ, for $\kappa<\pi$. In contrast to that in case of Stability we have to employ a collapsing-procedure for $\Sigma_n(\pi+\theta)$-sentences, i.e. sentences with parameters in some L_σ for $\sigma<\pi$ and in $L_{\pi+\theta}\backslash L_\pi$, as visualized in figure 11.1.

However, in case of Stability this new issue is not that problematic, since it always holds $\theta<\sigma$ and thereby we are able to collapse the interval $[\pi,\pi+\theta]$ by a term-shift-down procedure $t\mapsto t^{\pi\mapsto\kappa}$, where $t^{\pi\mapsto\kappa}$ is obtained from $t\in\mathcal{T}_{\pi+\theta}$ by replacing all occurrences of $L_{\pi+\delta}$, with $\delta<\theta$ in t (viewed as a string of $\mathcal{L}_\in$-symbols plus the symbols $\{L_{\pi+\delta}\,|\,\delta<\theta\}$ and terms of $\mathcal{T}_\pi$) by $L_{\kappa+\delta}$.

With respect to the Ordinal Theory and the fine structure theory of the collapsing hierarchies we just have to face the problem of keeping the notion of a reflection instance finite. A one-to-one adaption of Definition 2.2.4 to Stability would quickly lead to reflection instances of infinite length. To avoid this we introduce the notion of the closure $\vec{R}^{\,\mathrm{cl}}$ of a vector $\vec{R}$ of M-P-expressions. The required modifications in the analogue of chapter 3 are due to this notion.

[1]The denotation "Stability" is explained by the fact that an ordinal κ such that $L_\kappa \preceq_1 L_{\kappa+\alpha}$ is called $\kappa+\alpha$-stable.

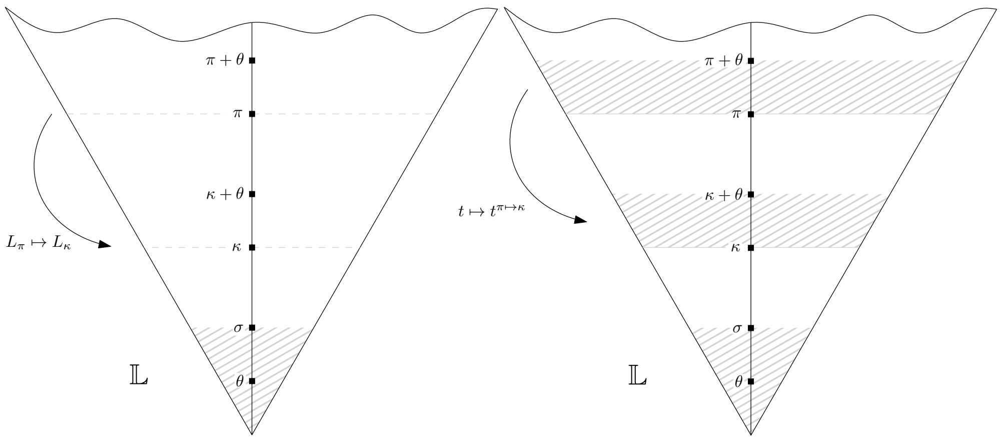

Figure 11.1.: Parameter allocation in case of single point- and interval-collapsing.

12. Ordinal Theory for Stability

Just like in the treatment of Π_ω-Ref we employ cardinal-analogues of the required recursive ordinals to define the collapsing hierarchies which give rise to the ordinal notation system $\mathsf{T}(\Upsilon)$.

12.1. θ-Indescribable Cardinals

Definition 12.1.1. Let $\theta > 0$. A cardinal π is called θ-Π^1_n-indescribable if for any $P_1, \ldots, P_k \subseteq V_\pi$ and for all Π^1_n-sentences $F(x_1, \ldots, x_k)$ in the language of set-theory $\mathcal{L}_\in$, whenever

$$V_{\pi+\theta} \models F(P_1, \ldots, P_k)$$

then there exists a $0 < \kappa < \pi$ such that

$$V_{\kappa+\theta} \models F(P_1 \cap V_\kappa, \ldots, P_k \cap V_\kappa).$$

The cardinal π is called $\mathfrak{M}$-θ-Π^1_n-indescribable if in the above situation a $0 < \kappa \in \pi \cap \mathfrak{M}$ can be found.

Remark. The notation of θ-indescribability can be regarded as a transfinite extension of the concept of Π^m_n-indescribability as it holds:

$$\begin{aligned} \pi \text{ is } 1\text{-}\Pi^1_0\text{-indescribable } &\Leftrightarrow \pi \text{ is } \Pi^1_n\text{-indescribable for all } n \in \omega, \\ & \text{ i.e. } \pi \text{ is } \Pi^2_0\text{-indescribable}, \\ \pi \text{ is } \omega\text{-}\Pi^1_0\text{-indescribable } &\Leftrightarrow \pi \text{ is } \Pi^m_n\text{-indescribable for all } m, n \in \omega, \\ & \text{ i.e. } \pi \text{ is totally indescribable.} \end{aligned}$$

Notation. To assimilate notations we refer to cardinals, which are Π^1_n-indescribable as 0-Π^1_n-indescribable cardinals.

Theorem 12.1.2.

❶ *Let $\tau < \theta < \pi \in \mathrm{Lim}$. Then there is a Δ_{n+1}-formula $\phi_n(x_1, \ldots, x_{k+3})$ such that*

$$V_{\pi+\theta} \models \big(\phi_n(P_1, \ldots, P_k, V_\pi, \theta, \ulcorner F \urcorner) \leftrightarrow V_{\pi+\tau} \models F(P_1, \ldots, P_n)\big),$$

for every set-theoretic Π^1_n-formula $F(x_1, \ldots, x_k)$ and any $P_1, \ldots, P_k \subseteq V_\pi$. Where $x_1, \ldots, x_k$ are all free variables occurring in F and $\ulcorner F \urcorner$ is a Gödel-set for F.

❷ *Let $n < \omega$ and $0 < \theta < \pi \in \mathrm{Lim}$. Then there is a Π^1_n-formula $\Phi_n(x_1, \ldots, x_{k+1})$, which is universal for the Π^1_n-formulae, i.e. for every set-theoretic Π^1_n-formula $F(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k$ are all free variables occurring in F, and any $P_1, \ldots, P_k \subseteq V_\pi$ it holds*

$$V_{\pi+\theta} \models F(P_1, \ldots, P_k) \leftrightarrow \Phi_n(P_1, \ldots, P_k, \ulcorner F \urcorner),$$

where $\ulcorner F \urcorner$ is a Gödel-set for F.

Proof. ① By use of the parameters V_π and θ plus a flat pairing function, which does not increase the rank (cf. [Dra74], Ch 2, §3, 3.11(10)) we can describe $V_{\pi+\xi}$ for all $0 < \xi < \theta$ within $V_{\pi+\theta}$. Therefore a Π^1_n-sentences F holding in $V_{\pi+\tau}$ can be expressed equally well as a Π^0_n-sentences holding in $V_{\pi+\theta}$ by replacing first oder quantifiers $\mathsf{Q}x$ by $\mathsf{Q}x \in V_{\pi+\tau}$ and replacing second order quantifiers by first order quantifiers, if $\tau+1 = \theta$, and by $\mathsf{Q}x \in V_{\pi+\tau+1}$ if $\tau + 1 < \theta$. Thereby the claim follows by use of a Π_{n+1}-, Σ_{n+1}-respectively, satisfaction relation for Π^0_n-formulae.

②Analogue to [Dra74], Ch 9, §1, Lemma 1.9. □

12.2. Collapsing Hierarchies based on θ-Indescribable Cardinals

Definition 12.2.1. From now on we denote by Υ the minimal cardinal, such that[1]

$$\forall\theta < \Upsilon\ \exists\kappa < \Upsilon \quad \text{“}\kappa \text{ is } \theta\text{-indescribable”},$$
$$\forall\theta < \Upsilon\ \forall\kappa < \Upsilon \quad (\text{“}\kappa \text{ is } \theta\text{-indescribable”} \to\ \theta < \kappa).$$

Moreover we define $\Theta(\theta)$ as the least ordinal, which is θ-indescribable.

Remark. If $\theta < \Upsilon$ it follows $\Theta(\theta) < \Upsilon$.

In the following small fraktur letters denote elements of $(\Upsilon \times \omega) \cup \{(0, -1)\}$. Since a confusion with elements of Υ (which are denoted by small Greek letters) can be excluded we use the symbol $<$ not only to denote the usual ordering of Υ but also to denote the lexicographic-ordering on $\Upsilon \times \omega$. In addition we use the predicates Succ and Lim also in the context of $\langle \Upsilon \times \omega, < \rangle$ and define $0 := (0, 0)$, $-1 := (0, -1)$ and $(\theta, m) + 1 := (\theta, m + 1)$. Moreover we extend the relation $<$ on $\Upsilon \times \omega$ by -1 with the convention $-1 < \mathfrak{x}$ for all $\mathfrak{x} \in \Upsilon \times \omega$. Finally we define $(\theta, m) \in C(\alpha, \pi) :\Leftrightarrow \theta \in C(\alpha, \pi)$.

The following Definition is the (canonical) extension of M-P-expressions to elements of $\Upsilon \times \omega$.

Definition 12.2.2 (M-P-Expressions)**.** Henceforth we assume that we have for every reflection configuration $\mathbb{F}$ a symbol $\mathsf{M}_{\mathbb{F}}$. We refer to expressions of the form $\mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{n}}$ as M-P-expressions. For technical convenience we also define ϵ and $\mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{n}}$ as M-P-expressions. Moreover ϵ is also a finite sequence of M-P-expressions with zero length.

[1] The existence of Υ follows from the existence of a subtle cardinal, for details see [Rat05b] Corollary 2.9.

Let $\mathbb{U}$ be an 0-ary reflection configuration, $\mathbb{G}$ a non 0-ary reflection configuration and $\mathbb{M}$ an arbitrary reflection configuration. Let $\xi > \mathrm{o}(\mathbb{U})$, $\xi' > \mathrm{o}(\mathbb{G})$, and $\gamma \geq \mathrm{o}(\mathbb{M})$, plus $\vec{R} = (\mathsf{M}^{<\xi_1}_{(\mathbb{R}_1)}\text{-}\mathsf{P}_{\mathfrak{m}_1}, \ldots, \mathsf{M}^{<\xi_i}_{(\mathbb{R}_i)}\text{-}\mathsf{P}_{\mathfrak{m}_i})$ with $\mathfrak{m}_1 \geq \ldots \geq \mathfrak{m}_i$. Then we define

$$\begin{aligned}
\vec{R}_{\mathfrak{k}} &:= \begin{cases} (\mathsf{M}^{<\xi_l}_{(\mathbb{R}_l)}\text{-}\mathsf{P}_{\mathfrak{m}_l}) & \text{if } \mathfrak{m}_l = \mathfrak{k} \text{ for some } 1 \leq l \leq i, \\ \epsilon & \text{otherwise.} \end{cases} \\
\vec{R}_{<\mathfrak{k}} &:= (\mathsf{M}^{<\xi_k}_{(\mathbb{R}_k)}\text{-}\mathsf{P}_{\mathfrak{m}_k}, \ldots, \mathsf{M}^{<\xi_i}_{(\mathbb{R}_i)}\text{-}\mathsf{P}_{\mathfrak{m}_i}), \quad \mathfrak{m}_k := \max\{\mathfrak{m}_j \mid 1 \leq j \leq i \wedge \mathfrak{m}_j < \mathfrak{k}\}
\end{aligned}$$

Analogously we define the finite substrings $\vec{R}_{\leq\mathfrak{k}}$, $\vec{R}_{<\mathfrak{k}}, \vec{R}_{\geq\mathfrak{k}}$ and for $\mathfrak{l} \geq \mathfrak{k}$ the finite sections $\vec{R}_{(\mathfrak{l},\mathfrak{k})}$ plus $\vec{R}_{(\mathfrak{l},\mathfrak{k}]}$. Moreover we define the (possibly infinite) sets

$$\begin{aligned}
\vec{R}^{\,\mathrm{cl}} &:= \left\{\mathsf{M}^{<\xi_j}_{(\mathbb{R}_j)}\text{-}\mathsf{P}_{\mathfrak{n}} \;\middle|\; 1 \leq j \leq i \;\wedge\; \mathfrak{m}_j \geq \mathfrak{n} > \mathfrak{m}_{j+1}, \text{ where } \mathfrak{m}_{i+1} := -1\right\}, \\
(\vec{R}^{\,\mathrm{cl}})_{\mathfrak{k}} &:= \begin{cases} (\mathsf{M}^{<\xi_l}_{(\mathbb{R}_l)}\text{-}\mathsf{P}_{\mathfrak{k}}) & \text{if for some } 1 \leq l \leq i \text{ it holds } \mathfrak{m}_l \geq \mathfrak{k} > \mathfrak{m}_{l+1}, \\ \epsilon & \text{otherwise,} \end{cases} \\
(\vec{R}^{\,\mathrm{cl}})_{<\mathfrak{k}} &:= \left\{\mathsf{M}^{<\xi_j}_{(\mathbb{R}_j)}\text{-}\mathsf{P}_{\mathfrak{n}} \;\middle|\; \mathsf{M}^{<\xi_j}_{(\mathbb{R}_j)}\text{-}\mathsf{P}_{\mathfrak{n}} \in \vec{R}^{\,\mathrm{cl}} \;\wedge\; \mathfrak{n} < \mathfrak{k}\right\}.
\end{aligned}$$

Analogously to $\vec{R}_{(\mathfrak{l},\mathfrak{k})}$ and $(\vec{R}^{\,\mathrm{cl}})_{<\mathfrak{k}}$ we also use the notation $(\vec{R}^{\,\mathrm{cl}})_{(\mathfrak{l},\mathfrak{k})}$.

The meaning of the M-P-expressions is given by

$$\begin{aligned}
\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{-1} &:= \epsilon, \\
\widetilde{\mathsf{M}}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}} &:= \begin{cases} (\vec{R}^{\,\mathrm{cl}}_{\mathrm{i}(\mathbb{M})})_{\mathfrak{m}} & \text{if } \gamma = \mathrm{o}(\mathbb{M}) \\ \mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}} & \text{otherwise,} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\kappa \models \epsilon &:\Leftrightarrow \emptyset \notin \emptyset \\
\kappa \models \mathsf{M}^{<0}\text{-}\mathsf{P}_{(\theta,m)} &:\Leftrightarrow \kappa \text{ is } \theta\text{-}\Pi^1_m\text{-indescribable}, \\
\kappa \models \mathsf{M}^{<\mathrm{o}(\mathbb{M})}_{\mathbb{M}}\text{-}\mathsf{P}_{(\theta,m)} &:\Leftrightarrow \kappa \text{ is } \theta\text{-}\Pi^1_m\text{-indescribable}, \\
\kappa \models \mathsf{M}^{<\xi}_{\mathbb{U}}\text{-}\mathsf{P}_{(\theta,m)} &:\Leftrightarrow \forall\zeta \in [\mathrm{o}(\mathbb{U}), \xi)_{C(\kappa)}\, (\kappa \text{ is } \mathfrak{M}^{\zeta}_{\mathbb{U}}\text{-}\theta\text{-}\Pi^1_m\text{-indescribable,}) \\
\kappa \models \mathsf{M}^{<\xi'}_{\mathbb{G}}\text{-}\mathsf{P}_{(\theta,m)} &:\Leftrightarrow \forall(\zeta, \vec{\eta}) \in [\mathrm{o}(\mathbb{G}), \xi')_{C(\kappa)} \times \mathrm{dom}(\mathbb{G})_{C(\kappa)} \\
&\qquad (\kappa \text{ is } \mathfrak{M}^{\zeta}_{\mathbb{G}(\vec{\eta})}\text{-}\theta\text{-}\Pi^1_m\text{-indescribable}), \\
\kappa \models \vec{R} &:\Leftrightarrow \forall^i_1 k\, (\kappa \models \mathsf{M}^{<\xi_k}_{(\mathbb{R}_k)}\text{-}\mathsf{P}_{\mathfrak{m}_k}), \\
\kappa \models \vec{R}^{\,\mathrm{cl}} &:\Leftrightarrow \forall \mathsf{M}^{<\xi_j}_{(\mathbb{R}_j)}\text{-}\mathsf{P}_{\mathfrak{n}} \in \vec{R}^{\,\mathrm{cl}}\, (\kappa \models \mathsf{M}^{<\xi_j}_{(\mathbb{R}_j)}\text{-}\mathsf{P}_{\mathfrak{n}}).
\end{aligned}$$

Definition 12.2.3. Let $\mathbb{X}$ be a refl. instance. Then we define by recursion on $\mathrm{o}(\mathbb{X})$

$$\begin{aligned}
\mathrm{par}(\Theta(\rho); \mathsf{P}_{\mathfrak{m}}; \ldots) &:= \{\rho, \mathfrak{m}\}, \\
\mathrm{par}(\kappa^+; \ldots) &:= \{\kappa\}, \\
\mathrm{par}(\Psi^{\delta}_{\mathbb{Z}}; \mathsf{P}_{\mathfrak{m}}; \ldots) &:= \{\delta, \mathfrak{m}\} \cup \mathrm{par}\,\mathbb{Z}, \\
\mathrm{par}(\Psi^{\delta}_{\mathbb{Z}}; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_{\mathfrak{m}}; \ldots) &:= \{\delta, \xi, \vec{\nu}, \mathfrak{m}\} \cup \mathrm{par}\,\mathbb{Z}.
\end{aligned}$$

Moreover we define $\mathbb{X} \in C(\alpha, \pi) :\Leftrightarrow \operatorname{par}\mathbb{X} \subseteq C(\alpha, \pi)$ and $\mathbb{X} \in C(\kappa) :\Leftrightarrow \operatorname{par}\mathbb{X} \subseteq C(\kappa)$.

Definition 12.2.4. By simultaneous recursion on α we define the sets $C(\alpha, \pi)$, reflection instances $\mathbb{X}$, reflection configurations $\mathbb{F}$, collapsing hierarchies $\mathfrak{M}$, finite sequences of M-P-expressions $\vec{R}_\Psi$ and collapsing functions Ψ (all these where appropriate with arguments and indices).

The class of reflection instances is partitioned in four types enumerated by 1. – 4. Whenever we define a new reflection instance we indicate by $\to p.$, with $p \in \{1, \ldots, 4\}$ of which type this reflection instance is. Clause p of the definition then specifies how to proceed with reflection instances of type $p.$ in the recursive definition process.

There are reflection configurations, whose first argument is an element of $\Upsilon \times \omega$. To these reflection configurations we refer to as “reflection configurations with variable reflection degree”, while we refer to reflection configurations whose first argument is an ordinal or which are constant reflection configurations as “reflection configurations with constant reflection degree”. Reflection configurations of the former type are treated in the clauses 1. and 3., while reflection configurations of the second type are treated in the clauses 2. and 4.

$$C(\alpha, \pi) := \bigcup_{n<\omega} C^n(\alpha, \pi), \quad \text{where}$$

$$C^0(\alpha, \pi) := \pi \cup \{0, \Upsilon\}, \quad \text{and}$$

$$C^{n+1}(\alpha, \pi) := \begin{cases} C^n(\alpha, \pi) \cup \\ \{\gamma + \omega^\delta \mid \gamma, \delta \in C^n(\alpha, \pi) \wedge \gamma \underset{\mathrm{NF}}{=} \omega^{\gamma_1} + \ldots + \omega^{\gamma_m} \wedge \gamma_m \geq \delta\}^\dagger \cup \\ \{\varphi(\xi, \eta) \mid \xi, \eta \in C^n(\alpha, \pi)\} \cup \\ \{\kappa^+ \mid \kappa \in C^n(\alpha, \pi) \cap \mathrm{Card} \cap \Upsilon\} \cup \\ \{\Theta(\lambda) \mid \lambda \in C^n(\alpha, \pi) \cap \Upsilon \cap \mathrm{Lim}\} \cup \\ \{\Psi^\gamma_{\mathbb{X}} \mid \mathbb{X}, \gamma \in C^n(\alpha, \pi) \wedge \gamma < \alpha \wedge \Psi^\gamma_{\mathbb{X}} \text{ is well-defined}\}. \end{cases}$$

0.1. For any $\lambda \in \mathrm{Lim} \cap \Upsilon$ we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) = (0, \mathfrak{l})_{C(\Theta(\lambda))}$, where $\mathfrak{l} := (\lambda, 0)$ and reflection instances

$$\mathbb{G}(\mathfrak{m}) := (\Theta(\lambda); \mathsf{P}_\mathfrak{m}; \epsilon; \epsilon; \Upsilon) \qquad \to 1.$$

For technical convenience we also define $\vec{R}_{\Theta(\lambda)} := (\mathsf{M}^{<0}\text{-}\mathsf{P}_\mathfrak{l})$.

0.2. For any cardinal $\omega \leq \kappa < \Upsilon$ we define the 0-ary reflection configuration and reflection instance

$$(\kappa^+; \mathsf{P}_0; \epsilon; \epsilon; 0) \qquad \to 2.$$

For technical convenience we also define $\vec{R}_{\kappa^+} := (\mathsf{M}^{<0}\text{-}\mathsf{P}_0)$.

For the remainder of this definition we refer to the nth component of $\vec{R}$ as $\mathsf{M}^{<\xi_n}_{\mathbb{R}_n}\text{-}\mathsf{P}_{\mathfrak{r}_n}$.

†The formulation of closure under + seems to be unnecessarily complicated, but we want + to be injective.

1. Let $\mathbb{X} := \mathbb{F}(\mathfrak{m})$ be a reflection instance of the form

$$(\pi; \mathsf{P}_{\mathfrak{m}}; \vec{R}; \mathbb{Z}; \delta).$$

Then we have $\vec{R}_\pi = (\mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{l}}, \vec{R})$ with $\mathfrak{m} < \mathfrak{l} \in \mathrm{Lim}$ and $\mathfrak{m} > 0$ plus $\mathfrak{m} > \mathfrak{r}_1$ if $\vec{R} \neq \epsilon$.

For $\alpha \geq \delta$ we define $\mathfrak{M}^\alpha_{\mathbb{X}}$ as the set of all ordinals $\kappa < \pi$ satisfying

1. $C(\alpha, \kappa) \cap \pi = \kappa$
2. $\mathbb{X}, \alpha \in C(\kappa)$
3. $\kappa \models (\vec{R}^{\mathrm{cl}}_\pi)_{\mathfrak{m}}, \vec{R}$
4. $\kappa \models \mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}}$

From now on we assume $\mathfrak{M}^\alpha_{\mathbb{X}} \neq \emptyset$ and by $\Psi^\alpha_{\mathbb{X}}$ we denote the least element of $\mathfrak{M}^\alpha_{\mathbb{X}}$. Moreover we define $\vec{R}_{\Psi^\alpha_{\mathbb{X}}} := (\widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}}, \vec{R})$.

In the following subclauses replace $\mathfrak{r}_1$ by 0 if $\vec{R} = \epsilon$.

1.1. Suppose that $\alpha = \delta$ and $\mathfrak{m} \in \mathrm{Lim}$. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) := (\mathfrak{r}_1, \mathfrak{m})_{C(\Psi^\alpha_{\mathbb{X}})}$ and reflection instances

$$\mathbb{G}(\mathfrak{n}) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{P}_{\mathfrak{n}}; \vec{R}; \mathbb{X}; \alpha + 1) \qquad \to 1.$$

1.2. Suppose that $\alpha = \delta$ and $\mathfrak{m} \in \mathrm{Succ}$. Then we define the 0-ary reflection configuration and reflection instance

$$(\Psi^\alpha_{\mathbb{X}}; \mathsf{P}_{\mathfrak{m}}; \vec{R}; \mathbb{X}; \alpha + 1) \qquad \to 2.$$

1.3. Suppose that $\alpha > \delta$ and $\mathfrak{m} \in \mathrm{Lim}$. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) := \{(\mathfrak{n}, \zeta, \mathfrak{r}) \in (\mathfrak{r}_1, \mathfrak{m})_{C(\Psi^\alpha_{\mathbb{X}})} \times [\delta, \alpha)_{C(\Psi^\alpha_{\mathbb{X}})} \times \mathrm{dom}(\mathbb{F})_{C(\Psi^\alpha_{\mathbb{X}})} \mid \mathfrak{r} > \mathfrak{n}\}$ and reflection instances

$$\mathbb{G}(\mathfrak{n}, \zeta, \mathfrak{r}) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{M}^\zeta_{\mathbb{F}(\mathfrak{r})}\text{-}\mathsf{P}_{\mathfrak{n}}; \vec{R}; \mathbb{X}; \alpha + 1) \qquad \to 3.$$

1.4. Suppose that $\alpha > \delta$ and $\mathfrak{m} \in \mathrm{Succ}$. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) := [\delta, \alpha)_{C(\Psi^\alpha_{\mathbb{X}})} \times (\mathfrak{m}, \mathfrak{l})_{C(\Psi^\alpha_{\mathbb{X}})}$ and reflection instances

$$\mathbb{G}(\zeta, \mathfrak{r}) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{M}^\zeta_{\mathbb{F}(\mathfrak{r})}\text{-}\mathsf{P}_{\mathfrak{m}}; \vec{R}; \mathbb{X}; \alpha + 1) \qquad \to 4.$$

2. Let $\mathbb{X}$ be a reflection instance of the form

$$(\pi; \mathsf{P}_{\mathfrak{m}}; \vec{R}; \ldots; \delta).$$

Then it holds $\vec{R}_\pi = (\mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{m}}, \vec{R})$ and $\mathfrak{m} \notin \mathrm{Lim}$. Moreover we have $\mathfrak{m} > \mathfrak{r}_1$ if $\vec{R} \neq \epsilon$.

For $\alpha \geq \delta$ we define the set $\mathfrak{M}^\alpha_{\mathbb{X}}$ as the set consisting of all ordinals $\kappa < \pi$ satisfying

1. $C(\alpha, \kappa) \cap \Psi^\delta_{\mathbb{Z}} = \kappa$
2. $\mathbb{X}, \alpha \in C(\kappa)$
3. if $\mathfrak{m} > 0$: $\kappa \models (\vec{R}^{\mathrm{cl}}_\pi)_{\mathfrak{m}-1}, \vec{R}_{<\mathfrak{m}-1}$
4. if $\mathfrak{m} > 0$: $\kappa \models \mathsf{M}^{<\alpha}_{\mathbb{X}}\text{-}\mathsf{P}_{\mathfrak{m}-1}$

From now on we assume $\mathfrak{M}^\alpha_{\mathbb{X}} \neq \emptyset$ and by $\Psi^\alpha_{\mathbb{X}}$ we denote the least element of $\mathfrak{M}^\alpha_{\mathbb{X}}$. Moreover we define $\vec{R}_{\Psi^\alpha_{\mathbb{X}}} := (\widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}-1}, \vec{R}_{<\mathfrak{m}-1})$.

For the following 2.$\cdot$ subclauses we suppose $\mathfrak{m} = \mathfrak{m}_0 + 1 > 0$. If $\mathfrak{m} = 0$ we do not equip $\Psi^\alpha_{\mathbb{X}}$ with any reflection configurations and instances. In the following let $\mathfrak{r}_0 := \mathfrak{r}_1$ if $\mathfrak{m}_0 > \mathfrak{r}_1$ and $\mathfrak{r}_0 := \mathfrak{r}_2$ otherwise.

2.1. Suppose that $\alpha = \delta$ plus $(\vec{R}^{\mathrm{cl}}_\pi)_{\mathfrak{m}_0} = \mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{m}_0}$ and $\mathfrak{m}_0 \in \mathrm{Lim}$. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) = (\mathfrak{r}_0, \mathfrak{m}_0)_{C(\Psi^\alpha_{\mathbb{X}})}$ and reflection instances

$$\mathbb{G}(\mathfrak{n}) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{P}_{\mathfrak{n}}; \vec{R}_{<\mathfrak{m}_0}; \mathbb{X}; \alpha + 1) \qquad \to 1.$$

2.2. Suppose that $\alpha = \delta$ plus $(\vec{R}^{\mathrm{cl}}_\pi)_{\mathfrak{m}_0} = \mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{m}_0}$ and $\mathfrak{m}_0 \notin \mathrm{Lim}$. Then we define the 0-ary reflection configuration and reflection instance

$$(\Psi^\alpha_{\mathbb{X}}; \mathsf{P}_{\mathfrak{m}_0}; \vec{R}_{<\mathfrak{m}_0}; \mathbb{X}; \alpha + 1) \qquad \to 2.$$

2.3. Suppose that $\alpha = \delta$ plus $\vec{R}_{\mathfrak{m}_0} = \mathsf{M}^{<\xi_1}_{\mathbb{R}_1}\text{-}\mathsf{P}_{\mathfrak{m}_0}$ with $\mathfrak{m}_0 \in \mathrm{Lim}$ and $\mathbb{R}_1$ is a reflection configuration with variable reflection degree. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) = \{(\mathfrak{n}, \zeta, \mathfrak{r}, \vec{\eta}) \in (\mathfrak{r}_2, \mathfrak{m}_0)_{C(\Psi^\alpha_{\mathbb{X}})} \times [\mathrm{o}(\mathbb{R}_1), \xi_1)_{C(\Psi^\alpha_{\mathbb{X}})} \times \mathrm{dom}(\mathbb{R}_1)_{C(\Psi^\alpha_{\mathbb{X}})} \,|\, \mathfrak{r} > \mathfrak{n}\}$ and reflection instances

$$\mathbb{G}(\mathfrak{n}, \zeta, \mathfrak{r}, \vec{\eta}) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{M}^\zeta_{\mathbb{R}_1(\mathfrak{r}, \vec{\eta})}\text{-}\mathsf{P}_{\mathfrak{n}}; \vec{R}_{<\mathfrak{m}_0}; \mathbb{X}; \alpha + 1) \qquad \to 3.$$

2.4. Suppose that $\alpha = \delta$ plus $\vec{R}_{\mathfrak{m}_0} = \mathsf{M}^{<\xi_1}_{\mathbb{R}_1}\text{-}\mathsf{P}_{\mathfrak{m}_0}$ with $\mathfrak{m}_0 \in \mathrm{Lim}$ and $\mathbb{R}_1$ is a reflection configuration with constant reflection degree. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) = (\mathfrak{r}_2, \mathfrak{m}_0)_{C(\Psi^\alpha_{\mathbb{X}})} \times [\mathrm{o}(\mathbb{R}_1), \xi_1)_{C(\Psi^\alpha_{\mathbb{X}})} \times \mathrm{dom}(\mathbb{R}_1)_{C(\Psi^\alpha_{\mathbb{X}})}$ and reflection instances

$$\mathbb{G}(\mathfrak{n}, \zeta, \vec{\eta}) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{M}^\zeta_{\mathbb{R}_1(\vec{\eta})}\text{-}\mathsf{P}_{\mathfrak{n}}; \vec{R}_{<\mathfrak{m}_0}; \mathbb{X}; \alpha + 1) \qquad \to 3.$$

2.5. Suppose that $\alpha = \delta$ plus $\vec{R}_{\mathfrak{m}_0} = \mathsf{M}^{<\xi_1}_{\mathbb{R}_1}\text{-}\mathsf{P}_{\mathfrak{m}_0}$ with $\mathfrak{m}_0 \notin \mathrm{Lim}$ and $\mathbb{R}_1$ is a reflection configuration with variable reflection degree. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) = \{(\zeta, \mathfrak{r}, \vec{\eta}) \in [\mathrm{o}(\mathbb{R}_1), \xi_1)_{C(\Psi^\alpha_{\mathbb{X}})} \times \mathrm{dom}(\mathbb{R}_1)_{C(\Psi^\alpha_{\mathbb{X}})} \,|\, \mathfrak{r} > \mathfrak{m}_0\}$ and reflection instances

$$\mathbb{G}(\zeta, \mathfrak{r}, \vec{\eta}) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{M}^\zeta_{\mathbb{R}_1(\mathfrak{r}, \vec{\eta})}\text{-}\mathsf{P}_{\mathfrak{m}_0}; \vec{R}_{<\mathfrak{m}_0}; \mathbb{X}; \alpha + 1) \qquad \to 4.$$

2.6. Suppose that $\alpha = \delta$ plus $\vec{R}_{\mathfrak{m}_0} = \mathsf{M}^{<\xi_1}_{\mathbb{R}_1}\text{-}\mathsf{P}_{\mathfrak{m}_0}$ with $\mathfrak{m}_0 \notin \mathrm{Lim}$ and $\mathbb{R}_1$ is a reflection configuration with constant reflection degree. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) = [\mathrm{o}(\mathbb{R}_1), \xi_1)_{C(\Psi^\alpha_{\mathbb{X}})} \times \mathrm{dom}(\mathbb{R}_1)_{C(\Psi^\alpha_{\mathbb{X}})}$ and reflection instances

$$\mathbb{G}(\zeta, \vec{\eta}) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{M}^{\zeta}_{\mathbb{R}_1(\vec{\eta})}\text{-}\mathsf{P}_{\mathfrak{m}_0}; \vec{R}_{<\mathfrak{m}_0}; \mathbb{X}; \alpha+1) \qquad \to 4.$$

2.7. Suppose that $\alpha > \delta$ and $\mathfrak{m}_0 \in \mathrm{Lim}$. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) := (\mathfrak{r}_0, \mathfrak{m}_0)_{C(\Psi^\alpha_{\mathbb{X}})} \times [\delta, \alpha)_{C(\Psi^\alpha_{\mathbb{X}})}$ and reflection instances

$$\mathbb{G}(\mathfrak{n}, \zeta) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{M}^{\zeta}_{\mathbb{X}}\text{-}\mathsf{P}_{\mathfrak{n}}; \vec{R}_{<\mathfrak{m}_0}; \mathbb{X}; \alpha+1) \qquad \to 3.$$

2.8. Suppose that $\alpha > \delta$ and $\mathfrak{m}_0 \notin \mathrm{Lim}$. Then we define the 0-ary reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) := [\delta, \alpha)_{C(\Psi^\alpha_{\mathbb{X}})}$ and reflection instances

$$\mathbb{G}(\zeta) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{M}^{\zeta}_{\mathbb{X}}\text{-}\mathsf{P}_{\mathfrak{m}_0}; \vec{R}_{<\mathfrak{m}_0}; \mathbb{X}; \alpha+1) \qquad \to 4.$$

3. Let $\mathbb{X} := \mathbb{F}(\mathfrak{m}, \xi, \vec{\nu})$ be a reflection instance of the form

$$(\Psi^\delta_{\mathbb{Z}}; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_{\mathfrak{m}}; \vec{R}; \mathbb{Z}; \delta+1).$$

Then we have $\vec{R}_{\Psi^\delta_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{l}}, \vec{R})$, for some $\mathfrak{m} < \mathfrak{l} \in \mathrm{Lim}$ and $\xi < \gamma$. Moreover it holds $\mathbb{M} \in \mathrm{Prcnfg}(\mathbb{X})$ and $\mathfrak{m} > 0$, plus $\mathfrak{m} > \mathfrak{r}_1$ if $\vec{R} \neq \epsilon$.

For $\alpha \geq \delta + 1$ we define $\mathfrak{M}^\alpha_{\mathbb{X}}$ as the set of all ordinals $\kappa \in \mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})} \cap \Psi^\delta_{\mathbb{Z}}$ satisfying

1. $C(\alpha, \kappa) \cap \Psi^\delta_{\mathbb{Z}} = \kappa$
2. $\mathbb{X}, \alpha \in C(\kappa)$
3. $\kappa \models (\vec{R}^{\mathrm{cl}}_\pi)_{\mathfrak{m}}, \vec{R}$
4. $\kappa \models \mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}}$

From now on we assume $\mathfrak{M}^\alpha_{\mathbb{X}} \neq \emptyset$ and by $\Psi^\alpha_{\mathbb{X}}$ we denote the least element of $\mathfrak{M}^\alpha_{\mathbb{X}}$. Moreover we define $\vec{R}_{\Psi^\alpha_{\mathbb{X}}} := ((\vec{R}_{\Psi^\xi_{\mathbb{M}(\vec{\nu})}})_{>\mathfrak{m}}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}}, \vec{R})$.

Let $\vec{R}_{\Psi^\xi_{\mathbb{M}(\vec{\nu})}} = (\mathsf{M}^{<\sigma_1}_{\mathbb{S}_1}\text{-}\mathsf{P}_{\mathfrak{s}_1}, \ldots, \mathsf{M}^{<\sigma_j}_{\mathbb{S}_j}\text{-}\mathsf{P}_{\mathfrak{s}_j})$. In the following replace $\mathfrak{s}_2$ by $\mathfrak{m}$ if $j = 1$.

3.1. Suppose that $\sigma_1 = 0$ and $\mathfrak{s}_1 \in \mathrm{Lim}$. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) = (\mathfrak{s}_2, \mathfrak{s}_1)_{C(\Psi^\alpha_{\mathbb{X}})}$ and reflection instances

$$\mathbb{G}(\mathfrak{n}) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{P}_{\mathfrak{n}}; ((\vec{R}_{\Psi^\xi_{\mathbb{M}(\vec{\nu})}})_{(\mathfrak{s}_1, \mathfrak{m})}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}}, \vec{R}); \mathbb{X}; \alpha+1) \qquad \to 1.$$

3.2. Suppose that $\sigma_1 = 0$ and $\mathfrak{s}_1 \notin \mathrm{Lim}$. Then we define the 0-ary reflection configuration and reflection instance

$$(\Psi^\alpha_{\mathbb{X}}; \mathsf{P}_{\mathfrak{m}}; ((\vec{R}_{\Psi^\xi_{\mathbb{M}(\vec{\nu})}})_{(\mathfrak{s}_1, \mathfrak{m})}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}}, \vec{R}); \mathbb{X}; \alpha+1) \qquad \to 2.$$

3.3. Suppose that $\sigma_1 > \mathrm{o}(\mathbb{S}_1)$, $\mathfrak{s}_1 \in \mathrm{Lim}$ and $\mathbb{S}_1$ is a reflection configuration with variable reflection degree. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) = \{(\mathfrak{n}, \zeta, \mathfrak{r}, \vec{\eta}) \in (\mathfrak{s}_2, \mathfrak{s}_1)_{C(\Psi^\alpha_{\mathbb{X}})} \times [\mathrm{o}(\mathbb{S}_1), \alpha_1)_{C(\Psi^\alpha_{\mathbb{X}})} \times \mathrm{dom}(\mathbb{S}_1)_{C(\Psi^\alpha_{\mathbb{X}})} \,|\, \mathfrak{r} > \mathfrak{n}\}$ and reflection instances

$$\mathbb{G}(\mathfrak{n}, \zeta, \mathfrak{r}, \vec{\eta}) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{M}^\zeta_{\mathbb{S}_1(\mathfrak{r},\vec{\eta})}\text{-}\mathsf{P}_{\mathfrak{n}}; ((\vec{R}_{\Psi^\xi_{\mathbb{M}(\vec{\nu})}})_{(\mathfrak{s}_1,\mathfrak{m})}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}}, \vec{R}); \mathbb{X}; \alpha + 1) \qquad \to 3.$$

3.4. Suppose that $\sigma_1 > \mathrm{o}(\mathbb{S}_1)$, $\mathfrak{s}_1 \in \mathrm{Lim}$ and $\mathbb{S}_1$ is a reflection configuration with constant reflection degree. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) = (\mathfrak{s}_2, \mathfrak{s}_1)_{C(\Psi^\alpha_{\mathbb{X}})} \times [\mathrm{o}(\mathbb{S}_1), \alpha_1)_{C(\Psi^\alpha_{\mathbb{X}})} \times \mathrm{dom}(\mathbb{R}_1)_{C(\Psi^\alpha_{\mathbb{X}})}$ and reflection instances

$$\mathbb{G}(\mathfrak{n}, \zeta, \vec{\eta}) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{M}^\zeta_{\mathbb{S}_1(\vec{\eta})}\text{-}\mathsf{P}_{\mathfrak{n}}; ((\vec{R}_{\Psi^\xi_{\mathbb{M}(\vec{\nu})}})_{(\mathfrak{s}_1,\mathfrak{m})}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}}, \vec{R}); \mathbb{X}; \alpha + 1) \qquad \to 3.$$

3.5. Suppose that $\sigma_1 > \mathrm{o}(\mathbb{S}_1)$, $\mathfrak{s}_1 \notin \mathrm{Lim}$ and $\mathbb{S}_1$ is a reflection configuration with variable reflection degree. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) = \{(\zeta, \mathfrak{r}, \vec{\eta}) \in [\mathrm{o}(\mathbb{S}_1), \alpha_1)_{C(\Psi^\alpha_{\mathbb{X}})} \times \mathrm{dom}(\mathbb{S}_1)_{C(\Psi^\alpha_{\mathbb{X}})} \,|\, \mathfrak{r} > \mathfrak{s}_1\}$ and reflection instances

$$\mathbb{G}(\zeta, \mathfrak{r}, \vec{\eta}) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{M}^\zeta_{\mathbb{S}_1(\mathfrak{r},\vec{\eta})}\text{-}\mathsf{P}_{\mathfrak{s}_1}; ((\vec{R}_{\Psi^\xi_{\mathbb{M}(\vec{\nu})}})_{(\mathfrak{s}_1,\mathfrak{m})}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}}, \vec{R}); \mathbb{X}; \alpha + 1) \qquad \to 4.$$

3.6. Suppose that $\sigma_1 > \mathrm{o}(\mathbb{S}_1)$, $\mathfrak{s}_1 \notin \mathrm{Lim}$ and $\mathbb{S}_1$ is a reflection configuration with constant reflection degree. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) = [\mathrm{o}(\mathbb{S}_1), \alpha_1)_{C(\Psi^\alpha_{\mathbb{X}})} \times \mathrm{dom}(\mathbb{S}_1)_{C(\Psi^\alpha_{\mathbb{X}})}$ and reflection instances

$$\mathbb{G}(\zeta, \vec{\eta}) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{M}^\zeta_{\mathbb{S}_1(\vec{\eta})}\text{-}\mathsf{P}_{\mathfrak{s}_1}; ((\vec{R}_{\Psi^\xi_{\mathbb{M}(\vec{\nu})}})_{(\mathfrak{s}_1,\mathfrak{m})}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}}, \vec{R}); \mathbb{X}; \alpha + 1) \qquad \to 4.$$

4. Let $\mathbb{X} := \mathbb{F}(\xi, \vec{\nu})$ be a reflection instance of the form

$$(\Psi^\delta_{\mathbb{Z}}; \mathsf{M}^\xi_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_{\mathfrak{m}}; \vec{R}; \mathbb{Z}; \delta + 1).$$

Then we have $\vec{R}_{\Psi^\delta_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}, \vec{R})$ for some $\xi < \gamma$ plus $\mathfrak{m} \notin \mathrm{Lim}$ and $\mathbb{M} \in \mathrm{Prcnfg}(\mathbb{X})$.

For $\alpha \geq \delta+1$ we define $\mathfrak{M}^{\alpha}_{\mathbb{X}}$ as the set of all ordinals $\kappa \in \mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})} \cap \Psi^{\delta}_{\mathbb{Z}}$ satisfying

1. $C(\alpha,\kappa) \cap \Psi^{\delta}_{\mathbb{Z}} = \kappa$
2. $\mathbb{X}, \alpha \in C(\kappa)$
3. if $\mathfrak{m} > 0$: $\kappa \models (\vec{R}^{\text{cl}}_{\pi})_{\mathfrak{m}-1}, \vec{R}_{<\mathfrak{m}-1}$
4. if $\mathfrak{m} > 0$: $\kappa \models \mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}-1}$

From now on we assume $\mathfrak{M}^{\alpha}_{\mathbb{X}} \neq \emptyset$ and by $\Psi^{\alpha}_{\mathbb{X}}$ we denote the least element of $\mathfrak{M}^{\alpha}_{\mathbb{X}}$. Moreover we define $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} := ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{\geq \mathfrak{m}}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}-1}, \vec{R}_{<\mathfrak{m}-1})$.

Let $\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}} = (\mathsf{M}^{<\sigma_1}_{\mathbb{S}_1}\text{-}\mathsf{P}_{\mathfrak{s}_1}, \ldots, \mathsf{M}^{<\sigma_j}_{\mathbb{S}_j}\text{-}\mathsf{P}_{\mathfrak{s}_j})$. In the following replace $\mathfrak{s}_2$ by $\mathfrak{m}$ if $j = 1$.

4.1. Suppose that $\sigma_1 = 0$ and $\mathfrak{s}_1 \in \text{Lim}$. Then we define the reflection configuration $\mathbb{G}$ with $\text{dom}(\mathbb{G}) = (\mathfrak{s}_2, \mathfrak{s}_1)_{C(\Psi^{\alpha}_{\mathbb{X}})}$ and reflection instances

$$\mathbb{G}(\mathfrak{n}) := (\Psi^{\alpha}_{\mathbb{X}}; \mathsf{P}_{\mathfrak{n}}; ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{(\mathfrak{s}_1,\mathfrak{m}]}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}-1}, \vec{R}_{<\mathfrak{m}-1}); \mathbb{X}; \alpha+1) \qquad \to 1.$$

4.2. Suppose that $\sigma_1 = 0$ and $\mathfrak{s}_1 \notin \text{Lim}$. Then we define the 0-ary reflection configuration and reflection instance

$$(\Psi^{\alpha}_{\mathbb{X}}; \mathsf{P}_{\mathfrak{m}}; ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{(\mathfrak{s}_1,\mathfrak{m}]}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}-1}, \vec{R}_{<\mathfrak{m}-1}); \mathbb{X}; \alpha+1) \qquad \to 2.$$

4.3. Suppose that $\sigma_1 > \text{o}(\mathbb{S}_1)$, $\mathfrak{s}_1 \in \text{Lim}$ and $\mathbb{S}_1$ is a reflection configuration with variable reflection degree. Then we define the reflection configuration $\mathbb{G}$ with $\text{dom}(\mathbb{G}) = \{(\mathfrak{n}, \zeta, \mathfrak{r}, \vec{\eta}) \in (\mathfrak{s}_2, \mathfrak{s}_1)_{C(\Psi^{\alpha}_{\mathbb{X}})} \times [\text{o}(\mathbb{S}_1), \alpha_1)_{C(\Psi^{\alpha}_{\mathbb{X}})} \times \text{dom}(\mathbb{S}_1)_{C(\Psi^{\alpha}_{\mathbb{X}})} \mid \mathfrak{r} > \mathfrak{n}\}$ and reflection instances

$$\begin{aligned}\mathbb{G}(\mathfrak{n}, \zeta, \mathfrak{r}, \vec{\eta}) := (\Psi^{\alpha}_{\mathbb{X}}; \mathsf{M}^{\zeta}_{\mathbb{S}_1(\mathfrak{r},\vec{\eta})}\text{-}\mathsf{P}_{\mathfrak{n}};& \\ ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{(\mathfrak{s}_1,\mathfrak{m}]}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}-1}, \vec{R}_{<\mathfrak{m}-1}); \mathbb{X}; \alpha+1) \qquad &\to 3.\end{aligned}$$

4.4. Suppose that $\sigma_1 > \text{o}(\mathbb{S}_1)$, $\mathfrak{s}_1 \in \text{Lim}$ and $\mathbb{S}_1$ is a reflection configuration with constant reflection degree. Then we define the reflection configuration $\mathbb{G}$ with $\text{dom}(\mathbb{G}) = (\mathfrak{s}_2, \mathfrak{s}_1)_{C(\Psi^{\alpha}_{\mathbb{X}})} \times [\text{o}(\mathbb{S}_1), \alpha_1)_{C(\Psi^{\alpha}_{\mathbb{X}})} \times \text{dom}(\mathbb{R}_1)_{C(\Psi^{\alpha}_{\mathbb{X}})}$ and reflection instances

$$\begin{aligned}\mathbb{G}(\mathfrak{n}, \zeta, \vec{\eta}) := (\Psi^{\alpha}_{\mathbb{X}}; \mathsf{M}^{\zeta}_{\mathbb{S}_1(\vec{\eta})}\text{-}\mathsf{P}_{\mathfrak{n}};& \\ ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{(\mathfrak{s}_1,\mathfrak{m}]}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}-1}, \vec{R}_{<\mathfrak{m}-1}); \mathbb{X}; \alpha+1) \qquad &\to 3.\end{aligned}$$

4.5. Suppose that $\sigma_1 > \mathrm{o}(\mathbb{S}_1)$, $\mathfrak{s}_1 \notin \mathrm{Lim}$ and $\mathbb{S}_1$ is a reflection configuration with variable reflection degree. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) = \{(\zeta, \mathfrak{r}, \vec{\eta}) \in [\mathrm{o}(\mathbb{S}_1), \alpha_1)_{C(\Psi^\alpha_{\mathbb{X}})} \times \mathrm{dom}(\mathbb{S}_1)_{C(\Psi^\alpha_{\mathbb{X}})} \,|\, \mathfrak{r} > \mathfrak{s}_1\}$ and reflection instances

$$\mathbb{G}(\zeta, \mathfrak{r}, \vec{\eta}) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{M}^{\zeta}_{\mathbb{S}_1(\mathfrak{r},\vec{\eta})}\text{-}\mathsf{P}_{\mathfrak{s}_1}; \\ ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{(\mathfrak{s}_1,\mathfrak{m}]}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}-1}, \vec{R}_{<\mathfrak{m}-1}); \mathbb{X}; \alpha+1) \qquad \to 4.$$

4.6. Suppose that $\sigma_1 > \mathrm{o}(\mathbb{S}_1)$, $\mathfrak{s}_1 \notin \mathrm{Lim}$ and $\mathbb{S}_1$ is a reflection configuration with constant reflection degree. Then we define the reflection configuration $\mathbb{G}$ with $\mathrm{dom}(\mathbb{G}) = [\mathrm{o}(\mathbb{S}_1), \alpha_1)_{C(\Psi^\alpha_{\mathbb{X}})} \times \mathrm{dom}(\mathbb{S}_1)_{C(\Psi^\alpha_{\mathbb{X}})}$ and reflection instances

$$\mathbb{G}(\zeta, \vec{\eta}) := (\Psi^\alpha_{\mathbb{X}}; \mathsf{M}^{\zeta}_{\mathbb{S}_1(\vec{\eta})}\text{-}\mathsf{P}_{\mathfrak{s}_1}; \\ ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{(\mathfrak{s}_1,\mathfrak{m}]}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}-1}, \vec{R}_{<\mathfrak{m}-1}); \mathbb{X}; \alpha+1) \qquad \to 4.$$

12.3. Structure Theory of Stability

In this section we show that $\mathfrak{M}^\alpha_{\mathbb{X}} \neq \emptyset$ if $\alpha \in C(\mathrm{i}(\mathbb{X}))$ and give criteria for $<$ -comparisons like $\Psi^\alpha_{\mathbb{X}} < \Psi^\beta_{\mathbb{Y}}$.

The existence of the universal formulae of Theorem 12.1.2 enables us to express $\kappa \models \mathsf{M}^{<\gamma}_{\mathbb{F}}\text{-}\mathsf{P}_{(\tau,m)}$ by a Π^0_{m+1}-formula in $V_{\kappa+\theta}$ for every $\theta > \tau$ and by a Π^1_{m+1}-formula in $V_{\kappa+\tau}$.

In the following we assume a coding of $C(\alpha, \pi)$ analogue to Definitions 2.3.7 and 2.3.8.

Lemma 12.3.1. *Let $\mathbb{F}$ be a reflection configuration with $\mathrm{i}(\mathbb{F}) = \pi$ and let $\pi \geq \kappa \in \mathrm{Reg}$ plus $0 < \theta < \kappa$. Suppose $\ulcorner \mathfrak{M}^{<\xi}_{\mathbb{F}} \urcorner_{\alpha,\pi}$ is defined. Then it holds:*

❶ *For every $(\tau, m) < (\theta, 0)$ there is a Π^0_{m+1}-formula $\phi_{\mathbb{F}}(x)$ such that*

$$\kappa \models \mathsf{M}^{<\xi}_{\mathbb{F}}\text{-}\mathsf{P}_{(\tau,m)} \quad \Leftrightarrow \quad V_{\kappa+\theta} \models \phi_{\mathbb{F}}(\ulcorner \mathfrak{M}^{<\xi}_{\mathbb{F}} \urcorner_{\alpha,\pi} \cap V_\kappa).$$

❷ *For every $m < \omega$ there is a Π^1_{m+1}-formula $\Phi_{\mathbb{F}}(x)$ such that*

$$\kappa \models \mathsf{M}^{<\xi}_{\mathbb{F}}\text{-}\mathsf{P}_{(\theta,m)} \quad \Leftrightarrow \quad V_{\kappa+\theta} \models \Phi_{\mathbb{F}}(\ulcorner \mathfrak{M}^{<\xi}_{\mathbb{F}} \urcorner_{\alpha,\pi} \cap V_\kappa).$$

Proof. Assume w.l.o.g. that there is only one parameter in the reflected formulae and proceed as in the proof of Lemma 2.3.10, but employ the universal formulae of Theorem 12.1.2 instead of the universal formula of Theorem 2.1.3. □

Corollary 12.3.2. *Let $\theta < \kappa$ and $(\theta, m) \geq (\tau, n)$ then it holds*

- κ *is* $(\mathfrak{M})$-θ-Π^1_m*-indescribable* $\quad \Rightarrow \quad$ κ *is* $(\mathfrak{M})$-τ-Π^1_n*-indescribable,*
- $\kappa \models \vec{R} \quad \Rightarrow \quad \kappa \models \vec{R}^{\,\mathrm{cl}}$.

Definition 12.3.3. Let $\mathbb{X} = \mathbb{F} = (\pi; \mathsf{P}_{\mathfrak{m}}; \vec{R}; \ldots)$ or $\mathbb{X} = \mathbb{F}(\xi, \vec{\nu}) = (\pi; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_{\mathfrak{m}}; \vec{R}; \ldots)$, and $\mathbb{Y} = \mathbb{G}(\mathfrak{n}) = (\pi; \mathsf{P}_{\mathfrak{n}}; \vec{R}; \ldots)$ or $\mathbb{Y} = \mathbb{G}(\mathfrak{n}, \xi', \vec{\nu}') = (\pi; \mathsf{M}^{\xi'}_{\mathbb{M}(\vec{\nu}')}\text{-}\mathsf{P}_{\mathfrak{n}}; \vec{R}; \ldots)$.† Moreover let $\mathbb{V}$ be a reflection instance and $\vec{S} = (\mathsf{M}^{<\sigma_1}_{\mathbb{S}_1}\text{-}\mathsf{P}_{\mathfrak{s}_1}, \ldots)$ be a finite sequence of M-P-expressions. Then we define.

$$
\begin{aligned}
\mathrm{rd}(\vec{S}) &:= \mathfrak{s}_1, \\
\mathrm{rd}(\epsilon) &:= -1, \\
\vec{R}_{\mathbb{F}} := \vec{R}_{\mathbb{X}} := \vec{R}_{\mathbb{G}} := \vec{R}_{\mathbb{Y}} &:= \vec{R} \\
\mathrm{Rdh}(\mathbb{F}) &:= \{\mathfrak{m} - 1\}, \\
\mathrm{Rdh}^{\mathrm{cl}}(\mathbb{F}) &:= \{\mathfrak{m} - 1\} \cup \{\mathfrak{r} \,|\, \mathfrak{m} > \mathfrak{r} > \mathrm{rd}(\vec{R}_{<\mathfrak{m}-1})\}, \\
\mathrm{Rdh}(\mathbb{G}) := \mathrm{Rdh}^{\mathrm{cl}}(\mathbb{G}) &:= \{\mathfrak{r} \,|\, \exists \vec{\eta}\,\big((\mathfrak{r}, \vec{\eta}) \in \mathrm{dom}(\mathbb{G})\big)\}, \\
\mathrm{rdh}(\mathbb{X}) &:= \mathfrak{m} - 1, \\
\mathrm{rdh}(\mathbb{Y}) &:= \mathfrak{n},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{initl}(\mathbb{V}) &:= \mathbb{E}, \ \text{where } \mathbb{E} \in \overline{\mathrm{Prcnfg}}(\mathbb{V}) \text{ and } \mathrm{o}(\mathbb{E}) \notin \mathrm{Succ}, \\
\mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{Z}) := \mathrm{ran}^{\alpha}_{\mathbb{F}}(\mathbb{Z}) &:= \begin{cases} \delta & \text{if there is a refl. inst. } (\Psi^{\delta}_{\mathbb{Z}}; \ldots) \in \overline{\mathrm{Prinst}}(\mathbb{X}), \\ \alpha & \text{otherwise,} \end{cases} \\
\mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{E}) := \mathrm{ran}^{\alpha}_{\mathbb{F}}(\mathbb{E}) &:= \begin{cases} \delta & \text{if there is a refl. inst. } (\Psi^{\delta}_{\mathbb{Z}}; \ldots) \in \overline{\mathrm{Prinst}}(\mathbb{X}) \\ & \text{and } \mathbb{E} \text{ is the refl. config. of } \mathbb{Z}, \\ \alpha & \text{otherwise.} \end{cases}
\end{aligned}
$$

Remark. $\mathrm{initl}(\mathbb{V})$ is-well defined, since we either have $\mathbb{V} = (\kappa^+; \ldots; 0)$ or there is a uniquely determined $\mathbb{E}(\lambda) = (\Theta(\lambda); \ldots; \Upsilon) \in \overline{\mathrm{Prinst}}(\mathbb{X})$.

Lemma 12.3.4 (Well-Definedness of Definition 12.2.4)**.** *Let $\mathbb{X} = (\pi; \ldots; \vec{R}; \mathbb{Z}; \delta + 1)$ be a reflection instance with reflection configuration $\mathbb{F}$. Then it holds:*

❶ (a) $\mathrm{o}(\mathbb{Z}) < \mathrm{o}(\mathbb{X}) = \delta + 1$,

(b) $\vec{R}_{\pi} = \begin{cases} (\mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{m}}, \vec{R}) & \textit{if } \mathbb{X} = (\pi; \mathsf{P}_{\mathfrak{m}}; \vec{R}; \ldots), \\ (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}, \vec{R}) \textit{ for some } \gamma > \xi \geq \mathrm{o}(\mathbb{M}), & \textit{if } \mathbb{X} = (\pi; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_{\mathfrak{m}}; \vec{R}; \ldots), \end{cases}$

(c) $\mathrm{rdh}(\mathbb{M}(\vec{\nu})) \geq \mathfrak{m} \geq \mathrm{rdh}(\mathbb{X})$ *and* $\mathrm{rdh}(\mathbb{M}(\vec{\nu})) > \mathrm{rdh}(\mathbb{X})$ *if* $\mathbb{X} = (\pi; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_{\mathfrak{m}}; \ldots)$,

(d) $\mathbb{X} \in C(\pi)$,

(e) $\mathrm{par}\,\mathbb{X} < \mathrm{o}(\mathbb{X}) = \delta + 1$.

†I.e. $\mathbb{F}$ is supposed to be a reflection configuration with constant reflection degree and $\mathbb{G}$ is supposed to be a reflection configuration with variable reflection degree.

❷ $\forall\kappa\in\mathfrak{M}^{\alpha}_{\mathbb{X}}\big(\kappa\models\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}}\ \wedge\ \kappa\models(\vec{R}^{\,\mathrm{cl}}_{\pi})_{\leq\mathrm{rdh}(\mathbb{X})}\big)$.

❸ *Let* $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}}=(\mathsf{M}^{<\xi_1}_{(\mathbb{R}_1)}\text{-}\mathsf{P}_{\mathfrak{r}_1},\ldots,\mathsf{M}^{<\xi_k}_{(\mathbb{R}_k)}\text{-}\mathsf{P}_{\mathfrak{r}_k})$*. Then it holds for all* $1\leq i\leq k$*:*

(a) $\mathbb{R}_i\in\overline{\mathrm{Prcnfg}}(\mathbb{X})$ *and* $\mathrm{dom}(\mathbb{R}_i)\leq\alpha$*,*

(b) $\xi_i\leq\mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{R}_i)$ *and* $\big(\xi_i>\mathrm{o}(\mathbb{R}_i)\ \vee\ \mathsf{M}^{<\xi_i}_{\mathbb{R}_i}\text{-}\mathsf{P}_{\mathfrak{r}_i}\equiv\mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{r}_i}\ \big)$*,*

(c) $\xi_i,\mathfrak{r}_i\in\mathrm{par}\,\mathbb{X}\cup\{\alpha\}$*,*

(d) $\mathfrak{r}_1\geq\mathrm{rdh}(\mathbb{X})$ *and* $\mathfrak{r}_1>\ldots>\mathfrak{r}_k\geq0$ *plus* $\forall(\mathfrak{r}_i\geq\mathfrak{r}'>\mathfrak{r}_{i+1})\,\exists\varepsilon\exists\mathbb{E}\,\big(\mathfrak{r}'\in\mathrm{Rdh}^{\mathrm{cl}}(\mathbb{R}_i)\wedge(\vec{R}^{\,\mathrm{cl}}_{\mathrm{i}(\mathbb{R}_i)})_{\mathfrak{r}_i}=\mathsf{M}^{\leq\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{r}_i}\wedge(\vec{R}^{\,\mathrm{cl}}_{\mathrm{i}(\mathbb{R}_i)})_{\mathfrak{r}'}=\mathsf{M}^{\leq\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{r}'}\big)$*,*

(e) $\exists\nu_i\in\mathrm{dom}(\mathbb{R}_i)\cap\mathrm{par}\,\mathbb{X}\ \big(\mathrm{rdh}(\mathbb{R}_i(\vec{\nu}_i)\geq\mathfrak{r}_i\big)$*,*

(f) $(\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq\mathrm{rdh}(\mathbb{X})}=(\widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathrm{rdh}(\mathbb{X})},(\vec{R}_{\mathbb{F}})_{<\mathrm{rdh}(\mathbb{X})})$

(g) $\forall\vec{\nu}_i\in\mathrm{dom}\,\mathbb{R}_i\ \big(\mathrm{rdh}(\mathbb{R}_i(\vec{\nu}_i))\geq\mathfrak{r}_i\geq\mathrm{rd}(\vec{R}_{\mathbb{R}_i})\big)$ *if* $\mathbb{R}_i$ *is a reflection configuration with constant reflection degree,*

Proof. The proof of Lemma 2.3.2 provides an adequate instruction how to proceed here. □

Corollary 12.3.5. *The claims in Definition 12.2.4, e.g. "*$\mathbb{M}\in\mathrm{Prcnfg}(\mathbb{X})$*" in clause 3., are true.*

Moreover for every reflection instance $\mathbb{Z}\neq(\mathrm{i}(\mathbb{Z});\mathsf{P}_0;\ldots)$ *and every* δ*, such that* $\mathfrak{M}^{\delta}_{\mathbb{Z}}\neq\emptyset$ *there is a reflection instance* $\mathbb{X}$ *with* $\mathrm{i}(\mathbb{X})=\Psi^{\delta}_{\mathbb{Z}}$*.*

Theorem 12.3.6 (Existence)**.** *Let* $\mathbb{X}=(\pi;\ldots;\delta)$ *be a reflection instance and* $\delta\leq\alpha\in C(\pi)$*. Then* $\mathfrak{M}^{\alpha}_{\mathbb{X}}\neq\emptyset$*.*

Proof. The claim follows analogously to Theorem 2.3.11. If $\mathrm{rd}(\vec{R}_{\pi})\in\mathrm{Lim}$ we have to make use of Theorem 12.1.2 ① and Lemma 12.3.1 ① instead of Theorem 2.1.3 and Lemma 2.3.10. If $\mathrm{rd}(\vec{R}_{\pi})=(\theta,m)$ with $\theta>0<m$ the claim follows by use of Theorem 12.1.2 ② and Lemma 12.3.1 ②. □

Lemma 12.3.7. *Let* $\mathbb{F}$ *be a reflection configuration with* $\mathrm{rdh}(\mathbb{F}(\vec{\eta}))=\mathfrak{m}=(\theta,m)\geq0$ *and suppose* κ *is an ordinal. Let* $\mathrm{o}(\mathbb{F})\leq\beta\in(\alpha+1)_{C(\kappa)}$ *and* $\vec{\eta},\vec{\nu}\in\mathrm{dom}(\mathbb{F})_{C(\kappa)}$*. Then it holds*

$$\kappa\in\mathfrak{M}^{\alpha}_{\mathbb{F}(\vec{\eta})}\ \&\ (\alpha,\vec{\eta})>_{\mathrm{lex}}(\beta,\vec{\nu})\quad\Rightarrow\quad\kappa\textit{ is }\mathfrak{M}^{\beta}_{\mathbb{F}(\vec{\nu})}\text{-}\theta\text{-}\Pi^1_m\textit{-indescribable.}$$

Proof. We proceed by induction on $\mathrm{o}(\mathbb{F})$. Let $\vec{\eta}=(\eta_1,\ldots,\eta_n)$ and $\vec{\nu}=(\nu_1,\ldots,\nu_n)$.

Case 1, $\alpha>\beta$*:* Since $\kappa\in\mathfrak{M}^{\alpha}_{\mathbb{F}(\vec{\eta})}$ we have $\kappa\models\mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}}$ by the fourth proviso of the definition of the collapsing hierarchy $\mathfrak{M}_{\mathbb{F}(\vec{\eta})}$. Thus κ is $\mathfrak{M}^{\beta}_{\mathbb{F}(\vec{\nu})}$-$\theta$-$\Pi^1_m$-indescribable.

Case 2, $(\alpha,\eta_1,\ldots,\eta_{i-1})=(\beta,\nu_1,\ldots,\nu_{i-1})$ *and* $\eta_i>\nu_i$ *for some* $1\leq i\leq n$*:*

Subcase 2.1, $\mathbb{F}(\vec{\eta})=\mathbb{F}(\mathfrak{m})=(\pi;\mathsf{P}_{\mathfrak{m}};\ldots)$ *for some* $\mathfrak{m}$*, i.e.* $n=1$*:* Then we have $\vec{\nu}=\mathfrak{n}$ for some $\mathfrak{n}<\mathfrak{m}$ and $\mathfrak{m}=(\theta,m)$. The assumption $\kappa\in\mathfrak{M}^{\alpha}_{\mathbb{F}(\vec{\eta})}$ implies

$$\text{“}C^{\alpha}_{\mathbb{F}(\mathfrak{n}),\pi}\text{ is unbounded in }\kappa\text{”}\ \wedge\ \kappa\models(\vec{R}^{\,\mathrm{cl}}_{\pi})_{\mathfrak{n}},\vec{R}_{<\mathfrak{n}}\ \wedge\ \kappa\models\mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{n}}.\tag{12.1}$$

Let $F(P)$ be a Π^1_m-sentence in parameter $P \subseteq V_\kappa$ such that

$$V_{\kappa+\theta} \models F(P). \tag{12.2}$$

Owing to Lemma 12.3.1 and the θ-Π^1_m-indescribableness of κ there exists a $\kappa_0 < \kappa$, which satisfies the conjunction of (12.1) and (12.2). Hence $\kappa_0 \in \mathfrak{M}^\beta_{\mathbb{F}(\vec{\nu})} \cap \kappa$ and thereby κ is $\mathfrak{M}^\beta_{\mathbb{F}(\vec{\nu})}$-$\theta$-$\Pi^1_m$-indescribable.

Subcase 2.2, $\mathbb{F}(\vec{\eta}) = \mathbb{F}(\mathfrak{m}, \xi, \vec{\eta}\,') = (\Psi^\delta_{\mathbb{Z}}; \mathsf{M}^\xi_{\mathbb{M}(\vec{\eta}\,')}\text{-}\mathsf{P}_\mathfrak{m}; \ldots)$ *and* $\vec{R}_{\Psi^\delta_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_\mathfrak{l}, \ldots)$ *for some* $\mathfrak{m} < \mathfrak{l} \in \mathrm{Lim}$*:* Then we have $\mathfrak{m} = (\theta, m)$. In the following let $\vec{\nu} = (\zeta, \mathfrak{n}, \vec{\nu}\,')$.

Subcase 2.2.1, $\mathfrak{m} > \mathfrak{n}$*:* The proviso $\kappa \in \mathfrak{M}^\alpha_{\mathbb{F}(\vec{\eta})}$ implies that $\kappa \models \mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_\mathfrak{m}$ due to the third clause of the definition of $\mathfrak{M}^\alpha_{\mathbb{F}(\vec{\eta})}$. Since we have $\vec{\nu} \in \mathrm{dom}(\mathbb{F})_{C(\kappa)}$ it follows thereby that κ is $\mathfrak{M}^\zeta_{\mathbb{M}(\vec{\nu}\,')}$-$\theta$-$\Pi^1_m$-indescribable. In addition we have $C^\alpha_{\mathbb{F}(\mathfrak{n}),\pi}$ is unbounded in κ, $\kappa \models (\vec{R}^{\mathrm{cl}}_\pi)_\mathfrak{n}, \vec{R}_{<\mathfrak{n}}$ and $\kappa \models \mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_\mathfrak{n}$. Thus it follows analogously to subcase 2.1 that κ is $\mathfrak{M}^\beta_{\mathbb{F}(\vec{\nu})}$-$\theta$-$\Pi^1_m$-indescribable.

Subcase 2.2.2, $\mathfrak{m} = \mathfrak{n}$*:* The proviso $\kappa \in \mathfrak{M}^\alpha_{\mathbb{F}(\vec{\eta})}$ implies that $\kappa \in \mathfrak{M}^\xi_{\mathbb{M}(\vec{\eta}\,')}$, since it holds by definition that $\mathfrak{M}^\alpha_{\mathbb{F}(\vec{\eta})} \subseteq \mathfrak{M}^\xi_{\mathbb{M}(\vec{\eta}\,')}$. Moreover we have $\mathrm{o}(\mathbb{M}) < \mathrm{o}(\mathbb{F})$, $\mathrm{rdh}(\mathbb{M}(\vec{\eta}\,') > \mathfrak{m}$ by Lemma 12.3.4 ①(c) and $(\xi, \vec{\eta}\,') >_{\mathrm{lex}} (\zeta, \vec{\nu}\,')$. Thus it follows by means of the induction hypothesis and Corollary 12.3.2 that κ is $\mathfrak{M}^\zeta_{\mathbb{M}(\vec{\nu}\,')}$-$\theta$-$\Pi^1_{m+1}$-indescribable. As in the subcases before we also have $C^\alpha_{\mathbb{F}(\mathfrak{n}),\pi}$ is unbounded in κ, $\kappa \models (\vec{R}^{\mathrm{cl}}_\pi)_\mathfrak{n}, \vec{R}_{<\mathfrak{n}}$ and $\kappa \models \mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_\mathfrak{n}$. Thereby it follows that κ is $\mathfrak{M}^\beta_{\mathbb{F}(\vec{\nu})}$-$\theta$-$\Pi^1_m$-indescribable.

Subcase 2.3, $\mathbb{F}(\vec{\eta}) = \mathbb{F}(\xi, \vec{\eta}\,') = (\Psi^\delta_{\mathbb{Z}}; \mathsf{M}^\xi_{\mathbb{M}(\vec{\eta}\,')}\text{-}\mathsf{P}_{\mathfrak{m}+1}; \ldots)$ *and* $\vec{R}_{\Psi^\delta_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}+1}, \ldots)$*:* Then the claim follows literally as in subcase 2.2.2 by use of the induction hypothesis. □

Theorem 12.3.8 (<-Comparison)**.** *Let* $\mathbb{X} = \mathbb{F}(\vec{\eta})$ *and* $\mathbb{Y} = \mathbb{G}(\vec{\nu})$ *be reflection instances and suppose* $\kappa := \Psi^\alpha_{\mathbb{X}}$ *and* $\pi := \Psi^\beta_{\mathbb{Y}}$ *are well-defined. Then it holds* $\kappa < \pi$ *iff*

$$\pi \geq \mathrm{i}(\mathbb{X}) \tag{a}$$

$$\text{or} \quad \kappa < \mathrm{i}(\mathbb{Y}) \;\wedge\; \Big(\alpha < \beta \vee \big(\mathbb{F} = \mathbb{G} \wedge (\alpha, \vec{\eta}) <_{\mathrm{lex}} (\beta, \vec{\nu})\big)\Big) \;\wedge\; \mathbb{X}, \alpha \in C(\pi) \tag{b}$$

$$\text{or} \quad \alpha \geq \beta \;\wedge\; \neg\big(\mathbb{Y}, \beta \in C(\kappa)\big) \tag{c}$$

Proof. The proof is literally the same as the proof of Theorem 2.3.14. Instead of Lemma 2.3.13 the Lemma 12.3.7 has to be employed. □

Corollary 12.3.9. *It holds:*

❶ $\Psi^\alpha_{\mathbb{X}}$ *and* $\Psi^\beta_{\mathbb{Y}}$ *are well-defined* $\;\Rightarrow\; \big(\Psi^\alpha_{\mathbb{X}} = \Psi^\beta_{\mathbb{Y}} \;\Leftrightarrow\; \alpha = \beta \wedge \mathbb{X} = \mathbb{Y}\big)$,

❷ $\alpha \underset{\mathrm{NF}}{=} \omega^{\alpha_1} + \ldots + \omega^{\alpha_m} \;\Rightarrow\; \big(\alpha \in C(\beta, \pi) \;\Leftrightarrow\; \alpha_1, \ldots, \alpha_m \in C(\beta, \pi)\big)$,

❸ $\alpha \underset{\mathrm{NF}}{=} \varphi(\eta, \zeta) \;\Rightarrow\; \big(\alpha \in C(\beta, \pi) \;\Leftrightarrow\; \eta, \zeta \in C(\beta, \pi)\big)$, ❹ $\kappa \in \mathrm{Card} \cap \Upsilon \;\Rightarrow\; \big(\kappa \in C(\beta,\pi) \;\Leftrightarrow\; \kappa^+ \in C(\beta,\pi)\big)$,

❺ $\pi \leq \Psi^{\xi}_{\mathbb{X}}$ *and* $\Psi^{\xi}_{\mathbb{X}}$ *is well-defined* $\Rightarrow\; \big(\Psi^{\xi}_{\mathbb{X}} \in C(\beta,\pi) \;\Leftrightarrow\; \xi < \beta \wedge \xi, \mathbb{X} \in C(\beta,\pi)\big)$.

❻ $\alpha \oplus \beta \in C(\gamma,\pi) \;\Leftrightarrow\; \alpha, \beta \in C(\gamma,\pi)$.

12.4. The Ordinal Notation System $\mathsf{T}(\Upsilon)$

Analogue to the Definition of $\mathsf{T}(\Xi)$ we define the ordinal notation system $\mathsf{T}(\Upsilon)$.

Definition 12.4.1. The set of ordinal notations $\mathsf{T}(\Upsilon)$ is inductively defined as follows:

- $0, \Upsilon \in \mathsf{T}(\Upsilon)$,
- if $\alpha \underset{\mathrm{NF}}{=} \omega^{\alpha_1} + \ldots + \omega^{\alpha_m}$ and $\alpha_1, \ldots, \alpha_m \in \mathsf{T}(\Upsilon)$ plus $m > 1$, then $\alpha \in \mathsf{T}(\Upsilon)$,
- if $\alpha \underset{\mathrm{NF}}{=} \varphi(\eta, \zeta)$ and $\eta, \zeta \in \mathsf{T}(\Upsilon)$, then $\alpha \in \mathsf{T}(\Upsilon)$,
- if $\kappa \in \mathsf{T}(\Upsilon) \cap \mathrm{Card} \cap \Upsilon$, then $\kappa^+ \in \mathsf{T}(\Upsilon)$,
- if $\rho \in \mathsf{T}(\Upsilon) \cap \mathrm{Lim} \cap \Upsilon$, then $\Theta(\rho) \in \mathsf{T}(\Upsilon)$,
- if $\mathbb{X}, \xi \in \mathsf{T}(\Upsilon)$ and $\mathrm{o}(\mathbb{X}) \leq \xi \in C(\xi, \mathrm{i}(\mathbb{X}))$, then $\Psi^{\xi}_{\mathbb{X}} \in \mathsf{T}(\Upsilon)$.

Defining simultaneously the set $\mathrm{K}_\pi(\alpha)$ analogously to Definition 2.4.2 with the extra clause $\mathrm{K}_\pi(\Theta(\rho)) := \mathrm{K}_\pi(\rho))$ we get:

Theorem 12.4.2. $\langle \mathsf{T}(\Upsilon), < \rangle$ *is a (primitive) recursive ordinal notation system.*

13. The Fine Structure Theory

Theorem 13.0.3 (Path Fidelity)**.** *Let* $\mathbb{X}$ *be a reflection instance and suppose* $\Psi^{\alpha}_{\mathbb{X}}$ *is well-defined. Moreover let* $\mathbb{Y} \neq (\mathrm{i}(\mathbb{Y}); 0\text{-}\mathsf{P}_0; \ldots)$ *be a reflection instance with reflection configuration* $\mathbb{G}$*. Then it holds*

$$\Psi^{\alpha}_{\mathbb{X}} \in \mathfrak{M}^{\beta}_{\mathbb{Y}} \quad \Rightarrow \quad \mathbb{G} \in \overline{\mathrm{Prcnfg}}(\mathbb{X}) \;\&\; \beta \leq \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{G}) \;\&\; \Psi^{\beta}_{\mathbb{Y}} \leq \Psi^{\alpha}_{\mathbb{X}} < \mathrm{i}(\mathbb{X}) \leq \mathrm{i}(\mathbb{Y}).$$

Proof. The proof is essentially the same as the proof of Theorem 3.1.1. There is just one new subcase in the first case, in which we lead the assumption $\mathrm{initl}(\mathbb{F}) \neq \mathrm{initl}(\mathbb{G})$ to a contradiction.

As a new subcase we have to treat the case $\mathbb{X} = (\Theta(\rho_1); \mathsf{P}_{(\rho_1,0)}; \epsilon; \epsilon; \Upsilon)$ and $\mathbb{Y} = (\Theta(\rho_2); \mathsf{P}_{(\rho_2,0)}; \epsilon; \epsilon; \Upsilon)$ with $\rho_1 \neq \rho_2$. We obtain a contradiction by the following argumentation: Since $\Psi^{\alpha}_{\mathbb{X}} \in \mathfrak{M}^{\alpha}_{\mathbb{X}} \cap \mathfrak{M}^{\beta}_{\mathbb{Y}}$ we have $\Psi^{\alpha}_{\mathbb{X}} < \Theta(\rho_1), \Theta(\rho_2)$ and $\rho_1, \rho_2 \in C(\Psi^{\alpha}_{\mathbb{X}})$. Thus the assumption $\rho_2 < \rho_1$ implies the contradiction $\Theta(\rho_2) \in C(\alpha, \Psi^{\alpha}_{\mathbb{X}}) \cap \Theta(\rho_1) = \Psi^{\alpha}_{\mathbb{X}}$, since $\rho_2 \in C(\rho_2, \Psi^{\alpha}_{\mathbb{X}}) \subseteq C(\alpha, \Psi^{\alpha}_{\mathbb{X}})$ and $\rho_2 < \rho_1$ implies $\Theta(\rho_2) < \Theta(\rho_1)$. The assumption $\rho_1 < \rho_2$ implies the contradiction $\Theta(\rho_1) \in C(\beta, \Psi^{\alpha}_{\mathbb{X}}) \cap \Theta(\rho_2) = \Psi^{\alpha}_{\mathbb{X}}$, since $\rho_1 \in C(\rho_1, \Psi^{\alpha}_{\mathbb{X}}) \subseteq C(\alpha, \Psi^{\alpha}_{\mathbb{X}})$ and $\rho_1 < \rho_2$ implies $\Theta(\rho_1) < \Theta(\rho_2)$. Therefore we obtain the desired contradiction for this new case, too. □

Definition 13.0.4. We define by recursion on $\mathrm{o}(\mathbb{M})$

$$\begin{aligned} \mathrm{Tc}(\mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{m}}) &:= \{\mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{m}}\}, \\ \mathrm{Tc}(\mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}) &:= \{\mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}\} \cup \mathrm{Tc}((\vec{R}^{\mathrm{cl}}_{\mathrm{i}(\mathbb{M})})_{\mathfrak{m}}). \end{aligned}$$

Definition 13.0.5. On M-P-expressions we define the binary relation $\preceq$ by

$$\begin{aligned} \epsilon &\preceq \mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}} \quad \text{for any M-P-expression } \mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}, \\ \mathsf{M}^{<\zeta}_{\mathbb{G}}\text{-}\mathsf{P}_{\mathfrak{m}} &\preceq \mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}} \quad :\Leftrightarrow \quad \exists \gamma \left(\mathrm{o}(\mathbb{G}) \leq \zeta \leq \gamma \;\wedge\; \mathsf{M}^{<\gamma}_{\mathbb{G}}\text{-}\mathsf{P}_{\mathfrak{m}} \in \mathrm{Tc}(\mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}})\right). \end{aligned}$$

We extend this relation to the closures of finite sequences of M-P- expressions $\vec{R}$ and $\vec{S}$ as follows

$$\vec{R}^{\mathrm{cl}} \preceq \vec{S}^{\mathrm{cl}} \qquad :\Leftrightarrow \; \forall \mathfrak{m} \left((\vec{R}^{\mathrm{cl}})_{\mathfrak{m}} \preceq (\vec{S}^{\mathrm{cl}})_{\mathfrak{m}}\right).$$

Remark. Due to Lemma 12.3.4 ③(a),(b) and since $\vec{R}_{\Theta(\lambda)} = (\mathsf{M}^{<0}\text{-}\mathsf{P}_{(\lambda,0)})$ the operator Tc is well-defined, if $\mathfrak{m} \in \mathrm{Rdh}^{\mathrm{cl}}(\mathbb{M})$. Moreover $\preceq$ is transitive.

Lemma 13.0.6. *Let* $\mathbb{X}$ *be a reflection instance with reflection configuration* $\mathbb{F}$ *and* $\mathrm{rdh}(\mathbb{X}) \geq \mathfrak{m}$*. Let* κ *be an ordinal satisfying* $\mathbb{X} \in C(\kappa)$*. Then*

$$\kappa \models \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}} \;\&\; \mathsf{M}^{<\zeta}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}} \preceq \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}} \quad \Rightarrow \quad \kappa \models \mathsf{M}^{<\zeta}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}.$$

Proof. Follows analogously to the proof of Lemma 3.1.4 by taking into account Lemma 12.3.4 ②, ③ (e) instead of Lemma 2.3.2. □

Remark. Lemma 3.1.5 also holds in the context of reflection instances defined by means of Definition 12.2.4.

Lemma 13.0.7. *Let* $\mathbb{X} = (\Psi^{\delta}_{\mathbb{Z}}; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_{\mathfrak{m}}; \vec{R}; \ldots)$ *be a reflection instance with* $\mathfrak{m}_0 := \mathrm{rdh}(\mathbb{X})$*. Then it holds*

$$\vec{R}^{\mathrm{cl}}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}} = (\vec{R}^{\mathrm{cl}}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{>\mathfrak{m}_0} \;\cup\; ((\widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}, (\vec{R}_{\mathbb{M}})_{<\mathfrak{m}})^{\mathrm{cl}})_{\leq\mathfrak{m}_0}.$$

Proof. Let $\mathfrak{n} := \mathrm{rdh}(\mathbb{M}(\vec{\nu}))$. Then it holds

$$\vec{R}^{\mathrm{cl}}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}} = (\vec{R}^{\mathrm{cl}}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{>\mathfrak{n}} \;\cup\; ((\widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{n}}, (\vec{R}_{\mathbb{M}})_{<\mathfrak{n}})^{\mathrm{cl}}),$$

either by definition, if $\mathbb{M}$ is an initial reflection configuration, or by means of Lemma 12.3.4 ③(f) otherwise. Thus the claim follows if we can prove

$$(\vec{R}_{\mathbb{M}})_{<\mathfrak{n}} = (\vec{R}_{\mathbb{M}})_{<\mathfrak{m}} \quad \& \quad \mathfrak{n} \geq \mathfrak{m} \geq \mathfrak{m}_0. \tag{13.1}$$

We have $\mathfrak{n}, \mathfrak{m} \in \mathrm{Rdh}^{\mathrm{cl}}(\mathbb{M})$ by means of Lemma 12.3.4 ③(d) since $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{l}}, \vec{R})$ with $\mathfrak{l} > \mathfrak{m} > \mathrm{rd}(\vec{R})$ if $\mathbb{X}$ is a reflection instance with variable reflection degree, or $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}, \vec{R})$ if $\mathbb{X}$ is a reflection instance with constant reflection degree. Thus we have $(\vec{R}_{\mathbb{M}})_{<\mathfrak{n}} = (\vec{R}_{\mathbb{M}})_{<\mathfrak{m}}$ since we either have $\mathfrak{n}, \mathfrak{m} > \mathrm{rd}(\vec{R}_{\mathbb{M}})$ if $\mathbb{M}$ is a reflection configuration with variable reflection degree, or $\mathfrak{n} \geq \mathfrak{n}, \mathfrak{m} > \mathrm{rd}((\vec{R}_{\mathbb{M}})_{<\mathfrak{n}})$ otherwise (cf. Definition 12.3.3).

Finally we have $\mathfrak{n} \geq \mathfrak{m} \geq \mathfrak{m}_0$ by Lemma 12.3.4 ①(c). Thereby it follows (13.1) and thus the claim. □

Theorem 13.0.8 (Correctness)**.** *Let* $\mathbb{X} \neq (\mathrm{i}(\mathbb{X}); \mathsf{P}_0; \ldots)$ *be a reflection instance and suppose* $\Psi^{\alpha}_{\mathbb{X}}$ *is well-defined. Then it holds:*

❶ *For any* κ*, any reflection configuration* $\mathbb{E}$*, any* $\mathrm{o}(\mathbb{E}) \leq \varepsilon$*, and any* $\mathfrak{e} \in \mathrm{Rdh}^{\mathrm{cl}}(\mathbb{E})$

$$\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} \preceq (\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\mathfrak{e}} = \mathsf{M}^{<\rho}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} \quad \Rightarrow \quad (\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}_{\mathbb{E}})_{<\mathfrak{e}})^{\mathrm{cl}} \preceq (\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq\mathfrak{e}}.$$

❷ *For any* κ*, such that* $C(\alpha,\kappa) \cap \mathrm{i}(\mathbb{X}) = \kappa$ *and* $\mathbb{X}, \alpha \in C(\kappa)$

$$\kappa \models \vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} \quad \Rightarrow \quad \kappa \in \mathfrak{M}^{\alpha}_{\mathbb{X}}.$$

Proof. We proceed by induction on $\mathrm{o}(\mathbb{X})$. In the following let $\mathbb{F}$ be the reflection configuration of $\mathbb{X}$.

① *Case 1,* $\mathbb{X} = (\pi; \mathsf{P}_{\mathfrak{m}}; \vec{R}; \ldots)$ *with* $\vec{R}_{\pi} = (\mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{l}^*}, \vec{R})$*, where either* $\mathfrak{m} < \mathfrak{l}^* \in \mathrm{Lim}$ *or* $\mathfrak{m} = \mathfrak{l}^* \in \mathrm{Succ}$*:* Then it holds $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = (\widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}_0}, \vec{R}_{<\mathfrak{m}_0})$, where $\mathrm{rdh}(\mathbb{X}) = \mathfrak{m}_0 = \mathfrak{m}$ if $\mathfrak{m} < \mathfrak{l}^* \in \mathrm{Lim}$ and $\mathfrak{m}_0 + 1 = \mathfrak{m}$ if $\mathfrak{m} = \mathfrak{l}^* \in \mathrm{Succ}$.

If $\mathbb{E} = \mathbb{F}$ we must have $\mathfrak{e} \leq \mathfrak{m}_0$ and $\varepsilon \leq \alpha$. Moreover it holds $(\vec{R}_{\mathbb{E}})_{<\mathfrak{e}} = (\vec{R}_{\mathbb{F}})_{<\mathfrak{m}_0} = \vec{R}_{<\mathfrak{m}_0}$ since $\mathfrak{e}, \mathfrak{m}_0 \in \mathrm{Rdh}^{\mathrm{cl}}(\mathbb{F})$ and if $\mathbb{F}$ is a reflection configuration with variable reflection degree it holds $\mathfrak{e}, \mathfrak{m}_0 > \mathrm{rd}(\vec{R}_{\mathbb{F}})$ and otherwise $\mathfrak{m}_0 \geq \mathfrak{m}_0, \mathfrak{e} > \mathrm{rd}((\vec{R}_{\mathbb{F}})_{<\mathfrak{m}_0})$. Thereby the claim follows.

If $\mathbb{E} \neq \mathbb{F}$ there exist a δ and a $\mathbb{Z}$ such that $\pi = \Psi^{\delta}_{\mathbb{Z}}$, since $\mathbb{E} \in \mathrm{Prcnfg}(\mathbb{F})$. Moreover the proviso $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} \preceq \mathsf{M}^{<\rho}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} = (\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\mathfrak{e}}$ then implies $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} \preceq \mathsf{M}^{<\rho}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} = (\vec{R}^{\mathrm{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})_{\mathfrak{e}}$. Therefore we obtain by means of the induction hypothesis

$$(\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}_{\mathbb{E}})_{<\mathfrak{e}})^{\mathrm{cl}} \preceq (\vec{R}^{\mathrm{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})_{\leq \mathfrak{e}} \preceq (\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq \mathfrak{e}}.$$

Case 2, $\mathbb{X} = (\Psi^{\delta}_{\mathbb{Z}}; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_{\mathfrak{m}}; \vec{R}; \ldots)$ *with* $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{l}}, \vec{R})$*, where* $\xi < \gamma$ *and either* $\mathfrak{m} < \mathfrak{l} \in \mathrm{Lim}$ *or* $\mathfrak{m} = \mathfrak{l} \notin \mathrm{Lim}$*:* Let $\mathfrak{m}_0 := \mathrm{rdh}(\mathbb{X})$. Then it holds $\mathfrak{m}_0 = \mathfrak{m}$ if $\mathfrak{l} \in \mathrm{Lim}$ and $\mathfrak{m}_0 + 1 = \mathfrak{m}$ elsewise. In addition we have $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{>\mathfrak{m}_0}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}_0}, \vec{R}_{<\mathfrak{m}_0})$.

If $\mathfrak{e} \leq \mathfrak{m}_0$ the claim follows as in the first case. So let us assume $\mathfrak{e} > \mathfrak{m}_0$. Then we have $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} \preceq \mathsf{M}^{<\rho}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} = (\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\mathfrak{e}} = (\vec{R}^{\mathrm{cl}}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{\mathfrak{e}}$. Thus we obtain by means of the induction hypothesis applied to $\mathbb{M}(\vec{\nu})$ and Lemma 13.0.7

$$(\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}_{\mathbb{E}})_{<\mathfrak{e}})^{\mathrm{cl}} \preceq (\vec{R}^{\mathrm{cl}}_{\Psi^{\xi}_{\mathbb{M}}(\vec{\nu})})_{\leq \mathfrak{e}} = (\vec{R}^{\mathrm{cl}}_{\Psi^{\xi}_{\mathbb{M}}(\vec{\nu})})_{[\mathfrak{e},\mathfrak{m}_0)} \cup ((\widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}, (\vec{R}_{\mathbb{M}})_{<\mathfrak{m}})^{\mathrm{cl}})_{\leq \mathfrak{m}_0}. \tag{13.2}$$

Moreover we have $\widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}} \preceq \mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}} = (\vec{R}^{\mathrm{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})_{\mathfrak{m}}$ and $\mathfrak{m} \in \mathrm{Rdh}^{\mathrm{cl}}(\mathbb{M})$ by Lemma 12.3.4 ③(d). Thereby we obtain by the induction hypothesis applied to $\mathbb{Z}$

$$((\widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}, (\vec{R}_{\mathbb{M}})_{<\mathfrak{m}})^{\mathrm{cl}})_{\leq \mathfrak{m}_0} \preceq (\vec{R}^{\mathrm{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})_{\leq \mathfrak{m}_0} \preceq (\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq \mathfrak{m}_0}. \tag{13.3}$$

Combining (13.2) and (13.3) the claim follows by means of the transitivity of $\preceq$.

② If $\mathbb{X} = (\pi; \mathsf{P}_{\mathfrak{m}}; \ldots)$ the claim follows by use of Lemma 13.0.6.

If $\mathbb{X} = (\Psi^{\delta}_{\mathbb{Z}}; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_{\mathfrak{m}}; \vec{R}; \ldots)$ we show at first

$$\kappa \models \vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} \quad \Rightarrow \quad \kappa \in \mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})}. \tag{13.4}$$

By definition we have $\kappa \models \vec{R}_{\Psi^{\alpha}_{\mathbb{X}}}$, which implies $\kappa \models (\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{>\mathfrak{m}_0}$. Moreover it follows by means of ① plus taking into account Lemma 13.0.6 that $\kappa \models ((\widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}, (\vec{R}_{\mathbb{M}})_{<\mathfrak{m}})^{\mathrm{cl}})_{\leq \mathfrak{m}_0}$ (for more details also consult the proof of Lemma 3.1.6 ② on page 36). Thus we have $\kappa \models \vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}}$ by means of Lemma 13.0.7.

In addition the provisos $C(\alpha, \kappa) \cap \mathrm{i}(\mathbb{X}) = \kappa$ and $\mathbb{X}, \alpha \in C(\kappa)$ imply $C(\xi, \kappa) \cap \mathrm{i}(\mathbb{M}) = \kappa$ and $\mathbb{M}(\vec{\nu}), \xi \in C(\kappa)$ by means of Lemma 3.1.5. Thus we obtain $\kappa \in \mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})}$ by use of the induction hypothesis and therefore we have proved (13.4).

As $\kappa \models \vec{R}_{\Psi^{\alpha}_{\mathbb{X}}}$ also implies that $\kappa \models (\vec{R}^{\mathrm{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})_{\mathfrak{m}_0}, \vec{R}_{<\mathfrak{m}_0}$ and $\kappa \models \mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}_0}$ it follows that $\kappa \in \mathfrak{M}^{\alpha}_{\mathbb{X}}$. □

13.1. The Domination Theorem

Definition 13.1.1. Let $\mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}$ be an M-P-expression, $\vec{R} = (\mathsf{M}^{<\rho_1}_{\mathbb{R}_1}\text{-}\mathsf{P}_{\mathfrak{r}_1}, \dots, \mathsf{M}^{<\rho_j}_{\mathbb{R}_j}\text{-}\mathsf{P}_{\mathfrak{r}_j})$ be a finite sequence of M-P-expressions and $\mathbb{D}$ be a reflection configuration. Then we define

$$(\mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}})^{\mathbb{D}} := \begin{cases} \mathsf{M}^{<\zeta}_{\mathbb{D}}\text{-}\mathsf{P}_{\mathfrak{m}} & \text{if } \mathsf{M}^{<\zeta}_{\mathbb{D}}\text{-}\mathsf{P}_{\mathfrak{m}} \in \mathrm{Tc}(\mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}) \\ \epsilon & \text{otherwise,} \end{cases}$$

$$(\vec{R})^{\mathbb{D}} := ((\mathsf{M}^{<\rho_1}_{\mathbb{R}_1}\text{-}\mathsf{P}_{\mathfrak{r}_1})^{\mathbb{D}}, \dots, (\mathsf{M}^{<\rho_j}_{\mathbb{R}_j}\text{-}\mathsf{P}_{\mathfrak{r}_j})^{\mathbb{D}}).$$

Lemma 13.1.2. *Let $\mathbb{X} \neq (\mathrm{i}(\mathbb{X}); \mathsf{P}_0; \dots)$ be a reflection instance and suppose $\Psi^{\alpha}_{\mathbb{X}}$ is well-defined. Let $\mathbb{E} \in \overline{\mathrm{Prcnfg}}(\mathbb{X})$ be a reflection configuration and $\varepsilon \geq \mathrm{o}(\mathbb{E})$ plus $\mathfrak{e} \in \mathrm{Rdh}^{\mathrm{cl}}(\mathbb{E})$. Then it holds for any $\mathfrak{g} \leq \min\{\mathrm{rd}(\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}}), \mathfrak{e}\}$ and any $\mathbb{D} \in \mathrm{Prcnfg}(\mathbb{E})$*

$$\big((\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}_{\mathbb{E}})_{<\mathfrak{e}})^{\mathrm{cl}}\big)_{\leq\mathfrak{g}} \preceq (\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq\mathfrak{g}} \quad\Rightarrow\quad \big((\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}_{\mathbb{E}})_{<\mathfrak{e}})^{\mathrm{cl}}\big)^{\mathbb{D}}_{\leq\mathfrak{g}} = (\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})^{\mathbb{D}}_{\leq\mathfrak{g}}.$$

Proof. We proceed by induction on $\mathrm{o}(\mathbb{X})$. In the following let $\mathbb{F}$ be the reflection configuration of $\mathbb{X}$ and $\mathfrak{P} := (\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}_{\mathbb{E}})_{<\mathfrak{e}})^{\mathrm{cl}}$.

If $\mathbb{X}$ is an initial reflection instance, i.e. $\mathrm{o}(\mathbb{X}) = 0$ or $\mathrm{o}(\mathbb{X}) = \Upsilon$ then we have nothing to show, since there is not any $\mathbb{D} \in \mathrm{Prcnfg}(\mathbb{X})$.

Case 1, $\mathbb{X} = (\Psi^{\delta}_{\mathbb{Z}}; \mathsf{P}_{\mathfrak{m}}; \vec{R}; \dots)$: It holds $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{l}}, \vec{R})$, where either $\mathfrak{m} < \mathfrak{l} \in \mathrm{Lim}$ or $\mathfrak{m} = \mathfrak{l} \notin \mathrm{Lim}$. Then we have $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = (\widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}_0}, \vec{R}_{<\mathfrak{m}_0})$ with $\mathrm{rdh}(\mathbb{X}) = \mathfrak{m}_0 = \mathfrak{m}$ if $\mathfrak{l} \in \mathrm{Lim}$ and $\mathfrak{m}_0 + 1 = \mathfrak{m}$ otherwise.

If $\mathbb{E} = \mathbb{F}$ we must have $\mathfrak{e} \leq \mathfrak{m}_0$ and $\varepsilon \leq \alpha$. Moreover it holds $(\vec{R}_{\mathbb{E}})_{<\mathfrak{e}} = (\vec{R}_{\mathbb{F}})_{<\mathfrak{m}_0} = \vec{R}_{<\mathfrak{m}_0}$ since $\mathfrak{e}, \mathfrak{m}_0 \in \mathrm{Rdh}^{\mathrm{cl}}(\mathbb{F})$ and if $\mathbb{F}$ is a reflection configuration with variable reflection degree it holds $\mathfrak{e}, \mathfrak{m}_0 > \mathrm{rd}(\vec{R}_{\mathbb{F}})$ and otherwise $\mathfrak{m}_0 \geq \mathfrak{m}_0, \mathfrak{e} > \mathrm{rd}((\vec{R}_{\mathbb{F}})_{<\mathfrak{m}_0})$. Thereby the claim follows.

If $\mathbb{E} \neq \mathbb{F}$ then it follows $\mathbb{E} \in \overline{\mathrm{Prcnfg}}(\mathbb{Z})$ and the proviso $\mathfrak{P}_{\leq\mathfrak{g}} \preceq (\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq\mathfrak{g}}$ implies $\mathfrak{P}_{\leq\mathfrak{g}} \preceq (\vec{R}^{\mathrm{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})_{\leq\mathfrak{g}}$ since $\mathrm{o}(\mathbb{E}) < \mathrm{o}(\mathbb{F})$. Therefore we obtain by means of the induction hypothesis applied to $\mathbb{Z}$

$$\mathfrak{P}^{\mathbb{D}}_{\leq\mathfrak{g}} = (\vec{R}^{\mathrm{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})^{\mathbb{D}}_{\leq\mathfrak{g}} \overset{\mathbb{D}\neq\mathbb{F}}{=} (\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})^{\mathbb{D}}_{\leq\mathfrak{g}}.$$

Case 2, $\mathbb{X} = (\Psi^{\delta}_{\mathbb{Z}}; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_{\mathfrak{m}}; \vec{R}; \dots)$ with $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{l}}, \vec{R})$, where $\xi < \gamma$ and either $\mathfrak{m} < \mathfrak{l} \in \mathrm{Lim}$ or $\mathfrak{m} = \mathfrak{l} \notin \mathrm{Lim}$: Let $\mathfrak{m}_0 := \mathrm{rdh}(\mathbb{X})$. Then it holds $\mathfrak{m}_0 = \mathfrak{m}$ if $\mathfrak{l} \in \mathrm{Lim}$ and $\mathfrak{m}_0 + 1 = \mathfrak{m}$ otherwise. In addition we have $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{>\mathfrak{m}_0}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}_0}, \vec{R}_{<\mathfrak{m}_0})$.

If $\mathfrak{g} \leq \mathfrak{m}_0$ the claim follows as in the first case. So let us assume $\mathfrak{e} \geq \mathfrak{g} > \mathfrak{m}_0$. At first we observe that we cannot have $\mathbb{E} = \mathbb{F}$, because the assumption $\mathbb{E} = \mathbb{F}$ leads to the following contradiction:

$$\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}_0+1} = (\vec{R}^{\mathrm{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})_{\mathfrak{m}_0+1} \overset{\text{Def.}}{\preceq} \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}_0+1} = \mathfrak{P}_{\mathfrak{m}_0+1} \overset{\text{Prov.}}{\preceq} (\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\mathfrak{m}_0+1} = \widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}_0+1}.$$

Thus we have $\mathbb{E} \in \overline{\mathrm{Prcnfg}}(\mathbb{Z})$. Now we show that then we also must have $\mathbb{E} \in \overline{\mathrm{Prcnfg}}(\mathbb{M})$. The assumption $\mathbb{E} \notin \overline{\mathrm{Prcnfg}}(\mathbb{M})$ implies $\mathbb{M} \in \mathrm{Prcnfg}(\mathbb{E})$ since

$\mathbb{E}, \mathbb{M} \in \overline{\text{Prcnfg}}(\mathbb{Z})$. Thus we obtain by use of the induction hypothesis applied to $\mathbb{Z}$—note that $\mathfrak{P}_{\leq \mathfrak{g}} \preceq (\vec{R}^{\text{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq \mathfrak{g}}$ implies $\mathfrak{P}_{\leq \mathfrak{m}} \preceq (\vec{R}^{\text{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})_{\leq \mathfrak{m}}$—the following contradiction:

$$\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}} = (\vec{R}^{\text{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})^{\mathbb{M}}_{\mathfrak{m}} \overset{\text{ind.hyp.}}{=} \mathfrak{P}^{\mathbb{M}}_{\mathfrak{m}} \overset{\text{Prov.}}{\preceq} (\vec{R}^{\text{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\mathfrak{m}} = \widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}.$$

Thereby we have $\mathfrak{g} > \mathfrak{m}_0$ and $\mathbb{E} \in \overline{\text{Prcnfg}}(\mathbb{M})$. To obtain the claim we want to apply the induction hypothesis to $\mathbb{M}(\vec{\nu})$. In the following let $\mathfrak{R} := (\widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}, (\vec{R}_{\mathbb{M}})_{<\mathfrak{m}})^{\text{cl}}$. We have

$$\mathfrak{P}_{\leq \mathfrak{g}} \preceq (\vec{R}^{\text{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq \mathfrak{g}} = (\vec{R}^{\text{cl}}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{[\mathfrak{g}, \mathfrak{m}_0)} \cup (\vec{R}^{\text{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\geq \mathfrak{m}_0}, \tag{13.5}$$

and

$$\vec{R}^{\text{cl}}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}} = (\vec{R}^{\text{cl}}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{>\mathfrak{m}_0} \cup \mathfrak{R}_{\leq \mathfrak{m}_0}, \tag{13.6}$$

by Lemma 13.0.7. Thereby we have to prove

$$\mathfrak{P}_{\leq \mathfrak{m}_0} \preceq \mathfrak{R}_{\leq \mathfrak{m}_0}. \tag{13.7}$$

The provisos imply

$$\mathfrak{P}_{\leq \mathfrak{m}} \preceq (\vec{R}^{\text{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})_{\leq \mathfrak{m}}. \tag{13.8}$$

Thus we obtain by the induction hypothesis

$$\mathfrak{P}^{\mathbb{D}}_{\leq \mathfrak{m}} = (\vec{R}^{\text{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})^{\mathbb{D}}_{\leq \mathfrak{m}} \quad \text{for any } \mathbb{D} \in \text{Prcnfg}(\mathbb{E}). \tag{13.9}$$

Moreover we have $\widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}} \preceq \mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}} = (\vec{R}^{\text{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})_{\mathfrak{m}}$ and $\mathbb{M} \in \overline{\text{Prcnfg}}(\mathbb{Z})$ plus $\mathfrak{m} \in \text{Rdh}^{\text{cl}}(\mathbb{M})$. Thus it follows by means of Theorem 13.0.8 ① that $\mathfrak{R} \preceq (\vec{R}^{\text{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})_{\leq \mathfrak{m}}$. Thereby we obtain by the induction hypothesis applied to $\mathbb{Z}$

$$\mathfrak{R}^{\mathbb{D}} = (\vec{R}^{\text{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})^{\mathbb{D}}_{\leq \mathfrak{m}} \quad \text{for any } \mathbb{D} \in \text{Prcnfg}(\mathbb{M}). \tag{13.10}$$

To prove (13.7), let $(\vec{R}_{\mathbb{E}})_{<\mathfrak{e}} = (\mathsf{M}^{<\varepsilon_1}_{(\mathbb{E}_1)}\text{-}\mathsf{P}_{\mathfrak{e}_1}, \ldots, \mathsf{M}^{<\varepsilon_l}_{(\mathbb{E}_l)}\text{-}\mathsf{P}_{\mathfrak{e}_l})$. Now let $\mathfrak{m}_0 \geq \mathfrak{k} > \mathfrak{e}_1$. If $\mathbb{E} = \mathbb{M}$ it holds $\text{rd}((\vec{R}_{\mathbb{E}})_{<\mathfrak{e}}) = \text{rd}((\vec{R}_{\mathbb{M}})_{<\mathfrak{m}}) < \mathfrak{m}$ since $\mathfrak{e}, \mathfrak{m} \in \text{Rdh}^{\text{cl}}(\mathbb{M})$ and thus we have $\mathfrak{e}, \mathfrak{m} > \text{rd}(\vec{R}_{\mathbb{M}})$ if $\mathbb{M}$ is a reflection configuration with variable reflection degree or $\text{rdh}(\mathbb{M}) \geq \mathfrak{e}, \mathfrak{m} > \text{rd}((\vec{R}_{\mathbb{M}})_{<\text{rdh}(\mathbb{M})})$ otherwise. Thus we have $\mathfrak{P}_{\mathfrak{m}} = \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}} \preceq (\vec{R}^{\text{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\mathfrak{m}} = \widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}$ and thereby $\varepsilon \leq \xi$. Thus it follows

$$\mathfrak{P}_{\mathfrak{k}} = \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{k}} \preceq \widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{k}} = \mathfrak{R}_{\mathfrak{k}}. \tag{13.11}$$

If $\mathbb{E} \neq \mathbb{M}$ we must have $\mathbb{E} \in \text{Prcnfg}(\mathbb{M})$. Then it holds

$$\mathfrak{P}_{\mathfrak{k}} = \widetilde{\mathsf{M}}^{<\xi}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{k}} =: \mathsf{M}^{<\varepsilon_0}_{\mathbb{E}_0}\text{-}\mathsf{P}_{\mathfrak{k}} \overset{(13.8)}{\preceq} (\vec{R}^{\text{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})^{\mathbb{E}_0}_{\mathfrak{k}} \overset{(13.10)}{=} \mathfrak{R}^{\mathbb{E}_0}_{\mathfrak{k}} \preceq \mathfrak{R}_{\mathfrak{k}}, \tag{13.12}$$

since $\mathbb{E}_0 \in \overline{\text{Prcnfg}}(\mathbb{E}) \subseteq \text{Prcnfg}(\mathbb{M})$.

Now suppose $\mathfrak{m}_0 \geq \mathfrak{k}$ and $\mathfrak{e}_j \geq \mathfrak{k} > \mathfrak{e}_{j+1}$ for some $1 \leq j \leq l$, where $\mathfrak{e}_{l+1} := 0$. If $\mathsf{M}^{<\varepsilon_j}_{(\mathbb{E}_j)}\text{-}\mathsf{P}_{\mathfrak{e}_j} \equiv \mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{e}_j}$ it holds $\mathfrak{P}_{\mathfrak{k}} = \mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{k}} \preceq \mathfrak{R}_{\mathfrak{k}}$. If $\mathsf{M}^{<\varepsilon_j}_{(\mathbb{E}_j)}\text{-}\mathsf{P}_{\mathfrak{e}_j} \not\equiv \mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{e}_j}$ we have

$$\mathfrak{P}_{\mathfrak{k}} = \mathsf{M}^{<\varepsilon_j}_{\mathbb{E}_j}\text{-}\mathsf{P}_{\mathfrak{k}} = \mathfrak{P}^{\mathbb{E}_j}_{\mathfrak{k}} \stackrel{(13.9)}{=} (\vec{R}^{\text{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})^{\mathbb{E}_j}_{\mathfrak{k}} \stackrel{(13.10)}{=} \mathfrak{R}^{\mathbb{E}_j}_{\mathfrak{k}} \preceq \mathfrak{R}_{\mathfrak{k}}, \tag{13.13}$$

since $\mathbb{E}_j \in \text{Prcnfg}(\mathbb{E})$ by Lemma 12.3.4 ③(a).

By means of (13.11), (13.12) and (13.13) it follows (13.7), which in collaboration with (13.5) and (13.6) provides $\mathfrak{P}_{\leq \mathfrak{g}} \preceq (\vec{R}^{\text{cl}}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{\leq \mathfrak{g}}$. Since $\mathbb{E} \in \overline{\text{Prcnfg}}(\mathbb{M})$ we obtain by means of the induction hypothesis applied to $\mathbb{M}(\vec{\nu})$

$$\mathfrak{P}^{\mathbb{D}}_{[\mathfrak{g},\mathfrak{m}_0)} = (\vec{R}^{\text{cl}}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})^{\mathbb{D}}_{[\mathfrak{g},\mathfrak{m}_0)} \quad \text{for any } \mathbb{D} \in \text{Prcnfg}(\mathbb{E}). \tag{13.14}$$

In addition (13.8) and the induction hypothesis applied to $\mathbb{Z}$ provide

$$\mathfrak{P}^{\mathbb{D}}_{\leq \mathfrak{m}_0} = (\vec{R}^{\text{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})^{\mathbb{D}}_{\leq \mathfrak{m}_0} \stackrel{\mathbb{D}\neq\mathbb{F}}{=} (\vec{R}^{\text{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})^{\mathbb{D}}_{\leq \mathfrak{m}_0} \quad \text{for any } \mathbb{D} \in \text{Prcnfg}(\mathbb{E}). \tag{13.15}$$

Combining (13.14) and (13.15) the claim follows. □

Lemma 13.1.3. *Let* $\mathbb{X} \neq (\text{i}(\mathbb{X}); \mathsf{P}_0; \ldots)$ *be a reflection instance with* $\text{rdh}(\mathbb{X}) = \mathfrak{m}_0 = (\theta, m_0)$ *and suppose that* $\Psi^{\alpha}_{\mathbb{X}}$ *is well-defined plus* $m^*_0 := \max\{0, m_0\}$*. Let* $\mathbb{E} \in \overline{\text{Prcnfg}}(\mathbb{X})$ *be a reflection configuration and* $\text{o}(\mathbb{E}) \leq \varepsilon \leq \text{ran}^{\alpha}_{\mathbb{X}}(\mathbb{E})$ *plus* $\mathfrak{e} \in \text{Rdh}^{\text{cl}}(\mathbb{E})$*. Then it holds for any* κ*, such that* $C(\alpha,\kappa) \cap \text{i}(\mathbb{X}) = \kappa$ *and* $\mathbb{X}, \alpha \in C(\kappa)$

$$\kappa \models \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}^{\text{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{<\mathfrak{e}} \;\&\; \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} \not\preceq (\vec{R}^{\text{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\mathfrak{e}} \quad \Rightarrow \quad \kappa \textit{ is } \mathfrak{M}^{\alpha}_{\mathbb{X}}\text{-}\theta\text{-}\Pi^1_{m^*_0}\textit{-indescribable.}$$

Proof. We proceed by induction on $\text{o}(\mathbb{X})$. In the following let $\mathbb{F}$ be the reflection configuration of $\mathbb{X}$ and let $F(\vec{P})$ be a $\Pi^1_{m^*_0}$-sentence in parameters $\vec{P} \subseteq V_\kappa$, such that $V_{\kappa+\theta} \models F(\vec{P})$.

Case 1, $\mathbb{X} = (\pi; \mathsf{P}_{\mathfrak{m}}; \vec{R}; \ldots)$*:* It holds $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{l}}, \vec{R})$, where either $0 < \mathfrak{m} < \mathfrak{l} \in \text{Lim}$ or $0 < \mathfrak{m} = \mathfrak{l} \notin \text{Lim}$. Then we have $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = (\widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}_0}, \vec{R}_{<\mathfrak{m}_0})$ with $\mathfrak{m}_0 = \mathfrak{m}$ if $\mathfrak{l} \in \text{Lim}$ and $\mathfrak{m}_0 + 1 = \mathfrak{m}$ otherwise.

Subcase 1.1, $\mathfrak{e} \leq \mathfrak{m}_0$*:* Then we must have $\mathbb{E} \neq \mathbb{F}$ as otherwise we would have $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} \preceq (\vec{R}^{\text{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\mathfrak{e}} = \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{e}}$, since $\mathfrak{e} \in \text{Rdh}^{\text{cl}}(\mathbb{X})$ and $\varepsilon \leq \text{ran}^{\alpha}_{\mathbb{X}}(\mathbb{E})$.

Therefore we have $\text{o}(\mathbb{E}) < \text{o}(\mathbb{F})$ and thereby $\text{o}(\mathbb{F}) \in \text{Succ}$. Thus there is a reflection instance $\mathbb{Z}$ and a δ such that $\pi = \Psi^{\delta}_{\mathbb{Z}}$. Let $\text{rdh}(\mathbb{Z}) = (\theta_0, n_0)$. By means of the provisos $\kappa \models \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}^{\text{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{<\mathfrak{e}}$ and $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} \not\preceq (\vec{R}^{\text{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\mathfrak{e}}$ we also have $\kappa \models \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}^{\text{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})_{<\mathfrak{e}}$ and $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} \not\preceq (\vec{R}^{\text{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})_{\mathfrak{e}}$. In addition it follows from Lemma 3.1.5 that $C(\delta,\kappa) \cap \text{i}(\mathbb{Z}) = \kappa$ and $\mathbb{Z}, \delta \in C(\kappa)$. Moreover we are allowed to apply the induction hypothesis, but thereby we obtain that κ is $\mathfrak{M}^{\delta}_{\mathbb{Z}}\text{-}\theta_0\text{-}\Pi^1_{n^*_0}$-indescribable, which is absurd since $\kappa \leq \text{i}(\mathbb{X}) = \Psi^{\delta}_{\mathbb{Z}}$. Thus the case $\mathfrak{e} \leq \mathfrak{m}_0$ cannot occur.

Subcase 1.2, $\mathfrak{e} > \mathfrak{m}_0$*:* Then $\kappa \models \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}^{\,\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{<\mathfrak{e}}$ implies that κ is θ-$\Pi^1_{m_0+1}$-indescribable. Moreover we have

$$C(\alpha,\kappa) \cap \mathrm{i}(\mathbb{X}) = \kappa, \quad \mathbb{X}, \alpha \in C(\kappa), \quad \kappa \models (\vec{R}^{\,\mathrm{cl}}_{\pi})_{\mathfrak{m}_0}, \vec{R}_{<\mathfrak{m}_0},$$
$$\kappa \models \mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}_0} \quad \text{and} \quad V_{\kappa+\theta} \models F(\vec{P}). \tag{13.16}$$

Lemma 12.3.1 ① in combination with Lemma 12.1.2 ① shows, that we are able to express the statements of (13.16) by a $\Pi^1_{m_0+1}$-sentence in $V_{\kappa+\theta}$. Thus it follows by means of the θ-$\Pi^1_{m_0+1}$-indescribability of κ, that there is a $\kappa_0 < \kappa$ which features (13.16). Therefore κ is $\mathfrak{M}^{\alpha}_{\mathbb{X}}$-$\theta$-$\Pi^1_{m_0^*}$-indescribable by means of Theorem 13.0.8 ②.

Case 2, $\mathbb{X} = (\Psi^{\delta}_{\mathbb{Z}}; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_{\mathfrak{m}}; \vec{R}; \dots)$ *with* $\vec{R}_{\Psi^{\delta}_{\mathbb{Z}}} = (\mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{l}}, \vec{R})$*, where* $\xi < \gamma$ *and either* $\mathfrak{m} < \mathfrak{l} \in \mathrm{Lim}$ *or* $\mathfrak{m} = \mathfrak{l} \notin \mathrm{Lim}$*:* Let $\mathfrak{m}_0 := \mathrm{rdh}(\mathbb{X})$. Then it holds $\mathfrak{m}_0 = \mathfrak{m}$ if $\mathfrak{l} \in \mathrm{Lim}$ and $\mathfrak{m}_0 + 1 = \mathfrak{m}$ otherwise. In addition we have $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}} = ((\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{>\mathfrak{m}_0}, \widetilde{\mathsf{M}}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}_0}, \vec{R}_{<\mathfrak{m}_0})$.

Subcase 2.1, $\mathfrak{e} \leq \mathfrak{m}_0$*:* Then we must have $\mathfrak{m}_0 \geq 0$, since $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{-1} = \epsilon \preceq (\vec{R}^{\,\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{-1}$, and the claim follows by the same considerations as in subcase 1.1.

Subcase 2.2, $\mathfrak{e} = \mathfrak{m}_0 + 1$*:* If $\mathbb{E} = \mathbb{F}$ then $\mathbb{F}$ is a reflection configuration with variable reflection degree and $\kappa \models \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}^{\,\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{<\mathfrak{e}}$ implies that $\kappa \models \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}_0+1}$. Since $\mathbb{F}(\mathfrak{m}_0+1, \xi, \vec{\nu}) \in C(\kappa)$ and $\mathrm{rdh}(\mathbb{F}(\mathfrak{m}_0+1, \xi, \vec{\nu})) = \mathfrak{m}_0 + 1$ it follows by Lemma 13.0.6 that $\kappa \models (\vec{R}^{\,\mathrm{cl}}_{\pi})_{\mathfrak{m}_0+1}$. Thus κ is $\mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})}$-$\theta$-$\Pi^1_{m_0+1}$-indescribable since $(\vec{R}^{\,\mathrm{cl}}_{\pi})_{\mathfrak{m}_0+1} = \mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}_0+1}$. Therefore the claim follows analogously to subcase 1.2.

If $\mathbb{E} \neq \mathbb{F}$ we have $\kappa \models \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}^{\,\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{<\mathfrak{e}}$ and $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} \not\preceq (\vec{R}^{\,\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\mathfrak{e}} = (\vec{R}^{\,\mathrm{cl}}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{\mathfrak{e}} = \widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{e}}$, either by definition if $\mathbb{M}$ is an initial reflection configuration or by means of Lemma 12.3.4 ③(f) and Lemma 13.0.7. Thus we cannot have $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} \not\preceq \mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{e}}$ since otherwise we would obtain a contradiction as in subcase 1.1 by use of the induction hypothesis. Therefore we have $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} \not\preceq \mathsf{M}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{e}}$ but $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} \preceq \mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{e}}$. Thus it follows that $\mathbb{E} = \mathbb{M}$ and $\xi < \varepsilon \leq \gamma$. Thereby $\kappa \models \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}^{\,\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{<\mathfrak{e}}$ implies that κ is $\mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})}$-$\theta$-$\Pi^1_{m_0+1}$-indescribable and the claim follows analogously to subcase 1.2.

Subcase 2.3, $\mathfrak{m}_0 + 1 < \mathfrak{e} \leq \mathrm{rd}(\vec{R}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})$*:* Let $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} = \mathsf{M}^{<\varepsilon_0}_{\mathbb{E}_0}\text{-}\mathsf{P}_{\mathfrak{e}}$. We must have $\mathsf{M}^{<\varepsilon_0}_{\mathbb{E}_0}\text{-}\mathsf{P}_{\mathfrak{e}} \neq \mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{e}}$ (and thus $\varepsilon_0 > \mathrm{o}(\mathbb{E}_0)$) since otherwise we would have $\widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\mathsf{P}_{\mathfrak{e}} \preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{\mathfrak{e}}$. If $\mathbb{E}_0 = \mathbb{F}$ then the claim follows readily as in subcase 2.2. Therefore let us assume $\mathbb{E}_0 \neq \mathbb{F}$ in the following.

Subcase 2.3.1, $\mathbb{M} \notin \mathrm{Prcnfg}(\mathbb{E}_0)$*, i.e.* $\mathbb{E}_0 \in \overline{\mathrm{Prcnfg}}(\mathbb{M})$*:* By the provisos we have $\kappa \models (\vec{R}^{\,\mathrm{cl}}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{(\mathfrak{e},\mathfrak{m}_0)}, (\vec{R}^{\,\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq \mathfrak{m}_0}$. Moreover it holds $\widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}} \preceq \mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}} = (\vec{R}^{\,\mathrm{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})_{\mathfrak{m}}$, since $\mathfrak{m} > \mathrm{rd}(\vec{R})$. Thus Theorem 13.0.8 ① provides that $((\widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}, (\vec{R}_{\mathbb{M}})_{<\mathfrak{m}})^{\mathrm{cl}})_{\leq \mathfrak{m}_0} \preceq (\vec{R}^{\,\mathrm{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})_{\leq \mathfrak{m}_0} \preceq (\vec{R}^{\,\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq \mathfrak{m}_0}$. Thereby $\kappa \models (\vec{R}^{\,\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{<\mathfrak{m}_0}$ implies $\kappa \models ((\widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}}, (\vec{R}_{\mathbb{M}})_{<\mathfrak{m}})^{\mathrm{cl}})_{\leq \mathfrak{m}_0}$ due to Lemma 13.0.6. By means of Lemma 13.0.7 it follows

$$\kappa \models \mathsf{M}^{<\varepsilon_0}_{\mathbb{E}_0}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}^{\,\mathrm{cl}}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{<\mathfrak{e}} \quad \& \quad \mathsf{M}^{<\varepsilon_0}_{\mathbb{E}_0}\text{-}\mathsf{P}_{\mathfrak{e}} \not\preceq (\vec{R}^{\,\mathrm{cl}}_{\Psi^{\xi}_{\mathbb{M}(\vec{\nu})}})_{\mathfrak{e}}.$$

Moreover we have by means of Lemma 3.1.5 that $C(\xi, \kappa) \cap \mathrm{i}(\mathbb{M}(\vec{\nu})) = \kappa$ and

$\xi, \mathbb{M}(\vec{\nu}) \in C(\kappa)$. By proviso we have $\varepsilon_0 \leq \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{E}_0)$. If $\mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{E}_0) \neq \mathrm{ran}^{\xi}_{\mathbb{M}(\vec{\nu})}(\mathbb{E}_0)$ we must have $\mathbb{E}_0 = \mathbb{M}$ and $\varepsilon_0 > \xi$. Then the claim follows as in subcase 1.2. If $\mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{E}_0) = \mathrm{ran}^{\xi}_{\mathbb{M}(\vec{\nu})}(\mathbb{E}_0)$ we are allowed to apply the induction hypothesis to $\mathbb{M}(\vec{\nu})$ and obtain that κ is $\mathfrak{M}^{\xi}_{\mathbb{M}(\vec{\nu})}$-$\theta$-$\Pi^1_{m_0+1}$-indescribable, since $\mathrm{rdh}(\mathbb{M}(\vec{\nu})) > \mathfrak{m}_0$ due to Lemma 12.3.4 ①(c). Therefore the claim follows analogously to subcase 1.2.

Subcase 2.3.2, $\mathbb{M} \in \mathrm{Prcnfg}(\mathbb{E}_0)$: Suppose $\mathfrak{e} = \mathfrak{m}_0 + 1 + \mathfrak{d}$. We proceed by subsidiary induction on $\mathfrak{d}$. So, let us assume the claim holds for all $\mathfrak{e}' = \mathfrak{m}_0 + 1 + \mathfrak{d}'$ with $\mathfrak{d}' < \mathfrak{d}$.

The proviso $\kappa \models \mathsf{M}^{<\varepsilon_0}_{\mathbb{E}_0}$-$\mathsf{P}_{\mathfrak{e}}$ implies $\kappa \models (\vec{R}_{\mathbb{E}_0})_{<\mathfrak{e}}$ since $\varepsilon_0 > \mathrm{o}(\mathbb{E})$.[1]

At first we consider the case that $((\mathsf{M}^{<\varepsilon_0}_{\mathbb{E}_0}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}_{\mathbb{E}_0})_{<\mathfrak{e}})^{\mathrm{cl}})_{<\mathfrak{e}} \preceq (\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{<\mathfrak{e}}$. This implies $((\mathsf{M}^{<\varepsilon_0}_{\mathbb{E}_0}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}_{\mathbb{E}_0})_{<\mathfrak{e}})^{\mathrm{cl}})_{\leq \mathfrak{m}_0+1} \preceq (\vec{R}^{\mathrm{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})_{\leq \mathfrak{m}_0+1}$ since $\mathrm{o}(\mathbb{E}_0) < \mathrm{o}(\mathbb{F})$. Moreover we have $\mathbb{M} \in \mathrm{Prcnfg}(\mathbb{E}_0)$ and $\mathbb{E}_0 \in \overline{\mathrm{Prcnfg}}(\mathbb{Z})$. However, then we obtain by means of Lemma 13.1.2 the following contradiction

$$\begin{aligned}
((\mathsf{M}^{<\varepsilon_0}_{\mathbb{E}_0}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}_{\mathbb{E}_0})_{<\mathfrak{e}})^{\mathrm{cl}})^{\mathbb{M}}_{\mathfrak{m}_0+1} \preceq ((\mathsf{M}^{<\varepsilon_0}_{\mathbb{E}_0}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}_{\mathbb{E}_0})_{<\mathfrak{e}})^{\mathrm{cl}})_{\mathfrak{m}_0+1} &\overset{\text{assump.}}{\preceq} (\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\mathfrak{m}_0+1} \\
= \widetilde{\mathsf{M}}^{<\xi}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}_0+1} \prec \mathsf{M}^{<\gamma}_{\mathbb{M}}\text{-}\mathsf{P}_{\mathfrak{m}_0+1}& \\
= (\vec{R}^{\mathrm{cl}}_{\Psi^{\delta}_{\mathbb{Z}}})^{\mathbb{M}}_{\mathfrak{m}_0+1} \overset{13.1.2}{=} ((\mathsf{M}^{<\varepsilon_0}_{\mathbb{E}_0}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}_{\mathbb{E}_0})_{<\mathfrak{e}})^{\mathrm{cl}})^{\mathbb{M}}_{\mathfrak{m}_0+1}&
\end{aligned}$$

Therefore we must have $((\mathsf{M}^{<\varepsilon_0}_{\mathbb{E}_0}\text{-}\mathsf{P}_{\mathfrak{e}}, (\vec{R}_{\mathbb{E}_0})_{<\mathfrak{e}})^{\mathrm{cl}})_{<\mathfrak{e}} \not\preceq (\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{<\mathfrak{e}}$. Let $(\vec{R}_{\mathbb{E}_0})_{<\mathfrak{e}} = (\mathsf{M}^{<\varepsilon_1}_{\mathbb{E}_1}\text{-}\mathsf{P}_{\mathfrak{e}_1}, \ldots, \mathsf{M}^{<\varepsilon_l}_{\mathbb{E}_l}\text{-}\mathsf{P}_{\mathfrak{e}_l})$. Then there exist a $0 \leq j \leq l$ and an $\mathfrak{e} > \mathfrak{e}'_j \in \mathrm{Rdh}^{\mathrm{cl}}(\mathbb{E}_j)$ such that $\mathsf{M}^{<\varepsilon_j}_{\mathbb{E}_j}\text{-}\mathsf{P}_{\mathfrak{e}'_j} \not\preceq (\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\mathfrak{e}'_j}$ and $\kappa \models \mathsf{M}^{<\varepsilon_j}_{\mathbb{E}_j}\text{-}\mathsf{P}_{\mathfrak{e}'_j}, (\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{<\mathfrak{e}'_j}$. Furthermore it holds by Lemma 12.3.4 that $\mathbb{E}_j \in \overline{\mathrm{Prcnfg}}(\mathbb{E}) \subseteq \mathrm{Prcnfg}(\mathbb{F})$ and $\varepsilon_j < \mathrm{ran}^{\hat{\alpha}}_{\mathbb{E}}(\mathbb{E}_j) = \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{E}_j)$, where $\hat{\alpha} := \mathrm{ran}^{\alpha}_{\mathbb{X}}(\mathbb{E}_0)$.

Thus κ is $\mathfrak{M}^{\alpha}_{\mathbb{X}}$-$\theta$-$\Pi^1_{m^*_0}$-indescribable either by subcase 2.1, if $\mathfrak{e}'_j \leq \mathfrak{m}_0$, or by subcase 2.2.2, if $\mathfrak{m}_0 + 1 = \mathfrak{e}'_j$ or by subcase 2.3.1, if $\mathfrak{m}_0 + 1 < \mathfrak{e}'_j$ and $\mathbb{M} \notin \mathrm{Prcnfg}(\mathbb{E}_j)$ or by the subsidiary induction hypothesis, if $\mathfrak{m}_0 + 1 < \mathfrak{e}'_j$ and $\mathbb{M} \in \mathrm{Prcnfg}(\mathbb{E}_j)$.

Subcase 2.4, $(\tau, e) > (\theta_1, s_1)$: Then the assumption $\kappa \models \widetilde{\mathsf{M}}^{<\varepsilon}_{\mathbb{E}}\text{-}\tau\text{-}\mathsf{P}_e, (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{<(\tau,e)}$ implies that κ is θ_1-$\Pi^1_{s_1+1}$-indescribable and the claim follows just like in subcase 1.2 with $\vec{R}$ replaced by $\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}}$. □

Theorem 13.1.4 (Domination for $\mathsf{Stability}$)**.** *Let $\mathbb{X}$ and $\mathbb{Y} \neq (\mathrm{i}(\mathbb{Y}); \mathsf{P}_0; \ldots)$ be reflection instances and suppose that $\Psi^{\alpha}_{\mathbb{X}}$ is well-defined plus $\beta \in On$. Then it holds*

$$\Psi^{\alpha}_{\mathbb{X}} \in \mathfrak{M}^{\beta}_{\mathbb{Y}} \quad \Rightarrow \quad \vec{R}^{\mathrm{cl}}_{\Psi^{\beta}_{\mathbb{Y}}} \preceq \vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}}.$$

Proof. This proof runs in the same vein as the proof of Theorem 3.2.4 by use of Lemma 13.1.3 instead of Lemma 3.2.3. □

[1] For a more detailed argumentation consult the proof of Lemma 3.2.3, subcase 3.3.2. on page 39.

14. A semi-formal Calculus for Stability

A look at the Reflection Elimination Theorem 15.2.4 which we want to prove, reveals that this theorem becomes false, if the sentence $\neg\, {}^{\pi}_{0}M^{\mathrm{o}(\mathbb{X})}_{\mathbb{X}}(\pi)$ occurs in Γ, because it must hold ${}^{\pi}_{0}M^{\mathrm{o}(\mathbb{X})}_{\mathbb{X}}(\kappa)$ if $\kappa \in {}_{\sigma}\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}$ for any reasonable formalization of ${}^{\pi}_{0}M^{\mathrm{o}(\mathbb{X})}_{\mathbb{X}}$.

To ban such sentences from an appearing in the Reflection Elimination Theorem we define the compound language $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$.

14.1. The Compound Language $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$

Definition 14.1.1 (The Languages $\mathcal{L}_{RS(\Upsilon)}$, $\mathcal{L}^{*}_{RS(\Upsilon)}$, and $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$)**.** We define the language $\mathcal{L}_{RS(\Upsilon)}$ analogously to the language $\mathcal{L}_{RS(\Xi)}$ and the $\mathcal{L}_{RS(\Upsilon)}$-terms analogously to the $\mathcal{L}_{RS(\Xi)}$-terms, but with the proviso that $\mathcal{L}_{M(\Xi)}$ is replaced by $\mathcal{L}^{T}_{\in}$ in the definition. Therefore the only predicate-symbols of $\mathcal{L}_{RS(\Upsilon)}$ are $\in$ and $\notin$.

We define the classes of $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-terms and $\mathcal{L}^{*}_{RS(\Upsilon)}$-terms as the class of $\mathcal{L}_{RS(\Upsilon)}$-terms.

The class of $\mathcal{L}^{*}_{RS(\Upsilon)}$-sentences is the smallest class of formulae which contains the $\mathcal{L}_{RS(\Upsilon)}$-sentences and for every $\kappa \in \mathsf{T}(\Upsilon)$ with $\vec{R}_{\kappa} = (\mathsf{M}^{<0}\text{-}\mathsf{P}_{(\theta,m)}, \ldots)$, any $n < \omega$, any $\vec{t} \in (\mathcal{T}_{\kappa+\theta} \cup \{L_{\kappa+\theta}\})^{n}$ and any $\mathcal{L}^{*}_{RS(\Upsilon)}$-terms $s_1, \ldots, s_n$ the primitive formulae $(\neg)^{\vec{t}}M_{\kappa}(s_1, \ldots, s_n)$ and is closed under the boolean operations plus bounded (by an $\mathcal{L}^{*}_{RS(\Upsilon)}$-term) quantification.

In the same vein we obtain the class of $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-sentences from the class of $\mathcal{L}^{*}_{RS(\Upsilon)}$-sentences by adding for any reflection instance $\mathbb{X}$ with $\mathrm{i}(\mathbb{X}) = [\pi, \pi{+}\theta]$ for any $\alpha \geq \mathrm{o}(\mathbb{X})$, any $\tau < \pi$, any $n < \omega$ and any $\vec{t} = (t_1, \ldots, t_n) \in (\mathcal{T}_{\pi+\theta} \cup \{L_{\pi+\theta}\})^{n}$ the n-ary predicate symbols $(\neg)^{\vec{t}}_{\tau}M^{\alpha}_{\mathbb{X}}$.

Remark. Obviously we have $\mathcal{L}_{RS(\Upsilon)} \subseteq \mathcal{L}^{*}_{RS(\Upsilon)} \subseteq \mathcal{L}^{\circledast}_{RS(\Upsilon)}$. In the following we refer to $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-sentences as sentences and to $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-terms as terms.

Definition 14.1.2 (Term Shift Down)**.** Let $\kappa < \pi$ and $t \in \mathcal{T}_{\pi+\pi} \cup \{0, 1\}$. Then we define $t^{\pi\mapsto\kappa}$ by recursion on the build-up of t by means of the following clauses

$$\begin{aligned} t^{\pi\mapsto\kappa} &:= t \quad \text{if } |t| < \pi \text{ or } t \in \{0,1\}, \\ (L_{\pi+\delta})^{\pi\mapsto\kappa} &:= L_{\kappa+\delta} \quad \text{for all } \delta < \pi, \\ \big\{x \in L_{\pi+\delta} \;\big|\; F(x, \vec{s})^{L_{\pi+\delta}}\big\}^{\pi\mapsto\kappa} &:= \big\{x \in L_{\kappa+\delta} \;\big|\; F(x, \vec{s}^{\pi\mapsto\kappa})^{L_{\kappa+\delta}}\big\}. \end{aligned}$$

Let $T \subseteq \mathcal{T}_{\pi+\kappa}$ be a set of terms. Then we define $T^{\pi\mapsto\kappa} := \{t^{\pi\mapsto\kappa} \,|\, t \in T\}$.

Definition 14.1.3. We define the transfinite content of ordinals, terms and sentences as in the context of $\mathsf{T}(\Xi)$. The same applies for the definition of the characteristic sequences and the definition of the term and formula rank of sentences of $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$. Let $\vec{t} = (t_1, \ldots, t_n)$ and $\vec{r} = (r_1, \ldots, r_n)$. Then we define for the new predicates

$$\begin{aligned}
\mathrm{k}({}^{\vec{t}}M_\pi(\vec{r})) &:= \mathrm{k}(\vec{t}) \cup \{\pi\} \cup \mathrm{k}(\vec{r}),\\
\mathrm{k}({}^{\vec{t}}_{\tau}M^{\alpha}_{\mathbb{X}}(\vec{r})) &:= \mathrm{k}(\vec{t}) \cup \{\tau, \alpha\} \cup \mathrm{par}\,\mathbb{X} \cup \mathrm{k}(\vec{r}),\\
{}^{\vec{t}}M_\pi\,(\vec{r}) &:\cong \bigvee \big(\vec{t}^{\,\pi\mapsto\kappa} = \vec{r}\big)_{\kappa\in\,\mathrm{SC}\,\cap(|\vec{r}|+1)}{}^{\dagger},\\
{}^{\vec{t}}_{\tau}M^{\alpha}_{\mathbb{X}}\,(\vec{r}) &:\cong \bigvee \big(\vec{t}^{\,\pi\mapsto\kappa} = \vec{r}\big)_{\kappa\in{}_{\tau}\mathfrak{M}^{\alpha}_{\mathbb{X}}\cap|\vec{r}|+1},\\
\mathrm{rnk}({}^{\vec{t}}M_\pi(\vec{r})) &:= \max\{\mathrm{rnk}(r_i)\,|\,1 \le i \le n\} + \max\{\mathrm{rnk}(t_j^{\pi\mapsto 0})\,|\,1 \le j \le n\} + n + 5,\\
\mathrm{rnk}({}^{\vec{t}}_{\tau}M^{\alpha}_{\mathbb{X}}(\vec{r})) &:= \max\{\mathrm{rnk}(r_i)\,|\,1 \le i \le n\} + \max\{\mathrm{rnk}(t_j^{\pi\mapsto 0})\,|\,1 \le j \le n\} + n + 5.
\end{aligned}$$

Notation. In the context of the language $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$ we slightly modify the complexity classes of sentences. More exactly, we extend the class of elementary-$\Pi_n(\pi)$-sentences (cf. Definition 4.1.4) as follows:

A sentence F is elementary-$\Pi_n(\pi)$ if it has the form

$$(\forall x_{1_1} \in L_\pi \ldots \forall x_{1_{k_1}} \in L_\pi) \ldots (Q_n x_{n_1} \in L_\pi, \ldots, Q_n x_{k_n} \in L_\pi)\; F(x_{1_1}, \ldots, x_{1_{k_1}}, \ldots, x_{n_1}, \ldots, x_{n_{k_n}}),$$

where the n quantifier blocks in front are alternate and $F(L_0, \ldots, L_0)$ is $\Delta_0(\pi)$. Analogously we define the elementary-$\Sigma_n(\pi)$-sentences.

We redefine the set of $\Pi_n(\pi)$-, $\Sigma_n(\pi)$- and Δ^1_0-sentences by use of the above extended version of elementary-$\Pi_n(\pi)$- and elementary-$\Sigma_n(\pi)$-sentences.

In the following we refer to a $\Pi_{n+1}(\pi)$ sentence F as $F(F_1, \ldots, F_m)$. In this denotation the $F_1, \ldots, F_m$ represent exactly the elementary-$\Pi_{n+1}(\pi)$- and elementary-$\Pi_n(\pi)$-subsentences of F and for $1 \le i \le m$ it holds $F_i \equiv (\forall x_{i_1} \in L_\pi \ldots \forall x_{i_{k_i}} \in L_\pi) F'_i(x_{i_1}, \ldots, x_{i_{k_i}})$. Therefore we have $F(F'_1(\vec{t}_1), \ldots, F'_m(\vec{t}_m)) \in \Sigma_n(\pi)$ for all $\vec{t} = (\vec{t}_1, \ldots, \vec{t}_m) \in (\mathcal{T}_\pi)^{k_1} \times \ldots \times (\mathcal{T}_\pi)^{k_n}$.

Definition 14.1.4. We let $\Pi_n(\kappa{+}0^*) := \Pi_n(\kappa)$ and for $0 < \theta < \kappa$ we define $\Pi_n(\kappa{+}\theta^*)$ as the smallest class of $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-sentences which contains the $\Delta^1_0(\kappa)$-$\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-sentences plus the $\Pi_n(\kappa + \theta)$-$\mathcal{L}^{*}_{RS(\Upsilon)}$-sentences and is closed under the boolean connectives $\vee$ and $\wedge$.

Analogously we define the class $\Sigma_n(\kappa + \theta^*)$.

† Here and in the following we write $\vec{t}^{\,\pi\mapsto\kappa} = \vec{r}$ for the sentence $\bigwedge_{i=1}^{n}(t_i^{\pi\mapsto\kappa} = r_i)$ and we let $|\vec{r}| := |\{r_1, \ldots, r_n\}|$.

14.2. Semi-formal Derivations on Hull-Sets of $\mathsf{T}(\Upsilon)$

Notation. To be able to define the reflection rules for $\mathsf{Stability}$ in a uniform way we introduce the following notation

$$\Pi_{m\dot{+}2}(\pi+\theta) \;:=\; \begin{cases} \Pi_{m+2}(\pi+\theta) & \text{if } \theta = 0, \\ \Pi_m(\pi+\theta) & \text{otherwise.} \end{cases}$$

Notation. Let $\mathbb{F}$ be a reflection configuration, then we define

$$\operatorname{dom}(\mathbb{F})^{>(\theta,m)} := \begin{cases} \{(\mathfrak{n},\vec{\eta}) \in \operatorname{dom}(\mathbb{F}) \mid \mathfrak{n} > (\theta,m)\} & \text{if } \mathbb{F} \text{ is a refl. config. with variable reflection degree,} \\ \operatorname{dom}(\mathbb{F}) & \text{otherwise.} \end{cases}$$

Definition 14.2.1. Let $\vec{R}$ be a finite sequence of M-P-expressions. Then we define

$$\vec{R}^{\,\mathrm{cl}'} \;:=\; \begin{cases} (\vec{R}^{\,\mathrm{cl}})_{<\mathfrak{r}_1} & \text{if } \vec{R} = (\mathsf{M}^{<0}\text{-}\mathsf{P}_{\mathfrak{r}_1},\ldots), \\ ((\vec{R}^{\,\mathrm{cl}}_{\mathrm{i}(\mathbb{R}_1)})_{\mathfrak{r}_1}, (\vec{R}^{\,\mathrm{cl}})_{<\mathfrak{r}_1}) & \text{if } \vec{R} = (\mathsf{M}^{<\xi_1}_{\mathbb{R}_1}\text{-}\mathsf{P}_{\mathfrak{r}_1},\ldots). \end{cases}$$

Definition 14.2.2. Let $\mathcal{H}$ be a hull-set and Γ, Δ be finite sets of $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-sentences. Suppose F is a $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-sentence, such that $\vec{s}$ denotes exactly all terms occurring in F of stage greater than or equal to π. Then we define $\mathcal{H} \,|\!\frac{\alpha}{\rho}\, \Gamma$ by recursion on α via

$$\{\alpha\} \cup \mathrm{k}(\Gamma) \subseteq \mathcal{H}$$

and the following inductive clauses:

(⋁) $$\frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Delta, F_{t_0} \quad \text{for some } t_0 \in T}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, \bigvee(F_t)_{t\in T}} \qquad \alpha_0 < \alpha$$

(⋀) $$\frac{\mathcal{H}[t] \overset{\alpha_t}{\underset{\rho}{\vdash}} \Delta, F_t \quad \text{for all } t \in T}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, \bigwedge(F_t)_{t\in T}} \qquad \alpha_t < \alpha$$

(Cut) $$\frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Delta, F \qquad \mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Delta, \neg F}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta} \qquad \begin{array}{r}\alpha_0 < \alpha\\ \mathrm{rnk}(F) < \rho\end{array}$$

($\Pi_{m\dot{+}2}(\pi+\theta)$-Ref) $$\frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Delta, F(\vec{s})}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, \exists \vec{z}^{\pi} \big({}^{\vec{s}}M_{\pi}(\vec{z}) \wedge F(\vec{z})\big)} \qquad \begin{array}{r}\alpha_0 < \alpha\\ F \in \Pi_{m\dot{+}2}(\pi+\theta^*)\end{array}$$

if there is a reflection instance $(\pi; \mathsf{P}_{(\theta,m)}; \ldots)$

(${}_{\tau}M^{\xi}_{\mathbb{M}(\vec{\nu})}$-$\Pi_{m\dot{+}2}(\pi+\theta)$-Ref)

$$\frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Delta, F(\vec{s})}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, \exists \vec{z}^{\pi} \big({}^{\vec{s}}_{\tau}M^{\xi}_{\mathbb{M}(\vec{\nu})}(\vec{z}) \wedge F(\vec{z})\big)} \qquad \begin{array}{r}\alpha_0 < \alpha\\ F \in \Pi_{m\dot{+}2}(\pi+\theta^*)\end{array}$$

for all $\tau < \pi$, if there is a reflection instance $(\pi; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_{(\theta,m)}; \ldots)$

(${}_{\tau}M^{\zeta}_{\mathbb{K}(\vec{\eta})}$-$\Pi_{k\dot{+}2}(\pi+\sigma)$-Ref)

$$\frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Delta, F(\vec{s})}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Delta, \exists \vec{z}^{\pi} \big({}^{\vec{s}}_{\tau}M^{\zeta}_{\mathbb{K}(\vec{\eta})}(\vec{z}) \wedge F(\vec{z})\big)} \qquad \begin{array}{r}\alpha_0 < \alpha\\ F \in \Pi_{k\dot{+}2}(\pi+\sigma^*)\\ \zeta \in C(\pi)\\ \vec{\eta} \in \mathrm{dom}(\mathbb{K})^{>(\sigma,k)}_{C(\pi)}\end{array}$$

for all $\tau < \pi$, $\mathbb{K}$ and $(\sigma,k) \notin \mathrm{Lim}$ if there is a reflection instance $(\pi; \ldots)$, such that $\mathsf{M}^{<\zeta+1}_{\mathbb{K}}\text{-}\mathsf{P}_{(\sigma,k)} \preceq (\vec{R}^{\mathrm{cl}'}_{\pi})_{(\sigma,k)}$

Remark. The propositions of Lemma 4.2.6 and Lemma 4.2.9 also hold in the context of $\mathsf{T}(\Upsilon)$.

14.3. Embedding of Stability

Instead of embedding Stability it is more convenient to embed the theory KP augmented by the axiom (Stab), where

(Stab) $\quad \forall x\, \exists \alpha \in \mathrm{Lim}\, \exists \kappa \in \mathrm{Lim}\, \big(\alpha \le \kappa \wedge x \in L_\alpha \wedge \forall y \in L_\kappa \big(Sat_1(y)^{L_{\kappa+\alpha}} \to Sat_1(y)^{L_\kappa}\big)\big).$

Here $Sat_1(z)$ denotes a Σ_1-formula, such that for all transitive, rudimentary closed sets A_1, A_2 it holds

$$A_1 \preceq_1 A_2 \quad \Leftrightarrow \quad \forall z \in A_1 \big(Sat_1(z)^{A_2} \to Sat_1(z)^{A_1}\big).$$

Such a formula exists, see e.g. [Dev84], Ch.VI, Lemma 1.15.

Theorem 14.3.1. *The theories* Stability *and* KP+(Stab) *have the same proof-theoretic ordinal.*

Proof. At first we show that Stability $+ V = L$ and KP+(Stab) are equal in the sense that every axiom of the first one is a theorem of the latter one and vice versa. Therefore we have to show that in every model of Stability$+V = L$ it holds (Stab) and contrariwise that in every model of KP+(Stab) it holds $V = L$, (Lim) and $\forall\alpha\,\exists\kappa\geq\alpha\,(L_\kappa \preceq_1 L_{\kappa+\alpha})$.

To obtain (Stab) in any model of Stability $+ V = L$ choose to a given x a $\beta > \omega$ such that $x \in L_\beta$ and let $\alpha := \beta+\omega$. Such β and α exist since we assume $V = L$ and argue in a model of KPω. Thus we obtain a $\kappa \geq \alpha$ such that $L_\kappa \preceq_1 L_{\kappa+\alpha}$. By [Bar75], Ch.V, Theorem 7.5 it holds that every stable ordinal is admissible. Thereby κ is admissible and thereby we have $\kappa \in \mathrm{Lim}$. Thus it follows (Stab).

Vice versa it is obvious that any model of KP+(Stab) satisfies $V = L$. To proof $\forall\alpha\,\exists\kappa\geq\alpha\,(L_\kappa \preceq_1 L_{\kappa+\alpha})$ let α be given. By (Stab) there exists a $\lambda \in \mathrm{Lim}$ such that $\alpha \in L_\lambda$ and a $\lambda \leq \kappa \in \mathrm{Lim}$ such that $\forall y{\in}L_\kappa\big(Sat_1(y)^{L_{\kappa+\lambda}} \to Sat_1(y)^{L_\kappa}\big)$. Since $L_{\kappa+\lambda}$ and L_λ are rudimentary closed it follows $L_\kappa \preceq_1 L_{\kappa+\lambda}$ and thus it follows by [Bar75], Ch.V, Proposition 7.4 that $L_\kappa \preceq_1 L_{\kappa+\alpha}$.

To show (Lim) choose by (Stab) to any given x an α, such that $x \in L_\alpha$. Due to (Stab) there exists an admissible $\kappa \geq \alpha$. Thus we have found an admissible κ such that $x \in L_\kappa$.

To finish the proof we observe that Stability $\vdash$ Stability^L and therefore the theories Stability and Stability $+ V = L$ have the same proof-theoretic ordinal by Theorem 7.10 of [Sch93]. □

Unfortunately the axiomatization of KP+(Stab) makes use of the Σ-function symbols $+$ and $\lambda\xi.L_\xi$. Thus an embedding of KP+(Stab) requires $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-derivations of all instances of these functions, i.e. let $F(x, y)$ be the Σ-formula used in (Stab) to describe the Σ-function $\lambda\xi.L_\xi$, such that $F(\alpha, x) \Leftrightarrow x = \mathbb{L}_\alpha$. Then we have to proof a theorem like $\Vert\!\frac{}{\;\Upsilon\;}\, F(\alpha^\bullet, L_\alpha)^\Upsilon$, for every $\alpha < \Upsilon$, where $\alpha^\bullet$ denotes a (canonical) $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-term for α.

To proof such a theorem in full detail is a tedious and extensive task. As we are not intrinsically interested in the theory Stability, but rather use it to present proof-theoretic techniques (fine structure of the collapsing hierarchies, Reflection Elimination Theorem) required for a treatment of even stronger theories (set-theoretic analogues of (parameter-free) Π^1_2-comprehension), whose axiomatization does not make use of Σ-function symbols, we just outline how an embedding of these Σ-function symbols can be obtained, but omit most of the proofs.

14.3.1. Embedding of Ordinal-Addition and $\lambda\xi.L_\xi$

Notation. In the following we use $\mathrm{Ord}(x)$, $\mathrm{Succ}(x)$, $\mathrm{Lim}(x)$, $\mathrm{Reln}(x)$, $1^{\mathrm{st}}(x)$ as abbreviations for $\mathcal{L}(\in)$-formulae, which describe that x is an ordinal, a successor ordinal, a limit ordinal, a binary relation or the first argument of an ordered pair, respectively. Moreover we write $\mathrm{S}(x)$ for the successor set of x, i.e. $x \cup \{x\}$ and we use small Greek letters to denote variables of $\mathcal{L}(\in)$ which range over ordinals. In addition we extend the notation of Definition 4.3.5 in the following way:

$$\Vdash_{\Upsilon} \Gamma \quad :\Leftrightarrow \quad \mathcal{H}[\Gamma] \left|\frac{\|\Gamma\|}{\Upsilon}\right. \Gamma \quad \text{for every hull-set } \mathcal{H}.$$

Definition 14.3.2. Let α be an ordinal. Then we define the $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-term $\alpha^{\bullet}$ as follows:

$$\alpha^{\bullet} := \begin{cases} L_0 & \text{if } \alpha = 0, \\ \{x \in L_\alpha \mid \mathrm{Ord}(x) \vee \big(\forall z{\in}x\ \mathrm{Ord}(z) \wedge \forall z(\mathrm{Ord}(z) \to z \in x)\big)\} & \text{if } \alpha \in \mathrm{Succ}, \\ \{x \in L_\alpha \mid \mathrm{Ord}(x)\} & \text{if } \alpha \in \mathrm{Lim}\,. \end{cases}$$

Corollary 14.3.3. *Let α be an ordinal. Then it holds*

$$\left|\frac{\star}{\ }\right. \mathrm{Ord}(\alpha^{\bullet}), \quad \left|\frac{\star}{\ }\right. \mathrm{Succ}(\alpha^{\bullet}) \text{ if } \alpha \in \mathrm{Succ}, \quad \left|\frac{\star}{\ }\right. \mathrm{Lim}(\alpha^{\bullet}) \text{ if } \alpha \in \mathrm{Lim}$$

$$\text{and} \quad \left|\frac{\star}{\ }\right. (\alpha+1)^{\bullet} = \mathrm{S}(\alpha^{\bullet}).$$

To keep the RS-embedding of the ordinal-addition and $\lambda\xi.L_\xi$ as simple as possible, it is convenient to assume that the defining $\mathcal{L}(\in)$-formulae of these functions make use of binary relations (which are indeed functions) instead of functions. Thereby we do not have to give an RS-proof of the uniqueness of these functions. Therefore we put:

Definition 14.3.4.

$$\begin{aligned} R_+(r,\kappa,\alpha) \ :=\ & \mathrm{Reln}(r) \ \wedge\ \forall x{\in}r\big(1^{\mathrm{st}}(x) \in \mathrm{S}(\alpha)\big) \\ & \wedge\ (0,\kappa) \in r \ \wedge\ \forall z\big((o,z) \in r \to z = \kappa\big) \\ & \wedge\ \forall(\xi,z){\in}r\bigg(\Big(\xi \in \mathrm{Succ} \to \exists y\big((\xi-1,y) \in r \wedge z = \mathrm{S}(y)\big)\Big) \wedge \\ & \qquad \Big(\xi \in \mathrm{Lim} \to \Big(\forall z_0{\in}z\ \exists\xi_0{\in}\xi\ \exists y\big((\xi_0,y) \in r \wedge z_0 \in y\big) \\ & \qquad\qquad \wedge \forall\xi_0{\in}\xi\ \exists y\big((\xi_0,y) \in r \wedge y \in z\big)\Big)\Big)\bigg). \end{aligned}$$

By use of this formula we define the $\mathcal{L}(\in)$-formula

$$(\kappa + \alpha = x)^z \ :=\ \exists r{\in}z\big(R_+(r,\kappa,\alpha) \ \wedge\ (\alpha,x) \in r\big).$$

Remark. Obviously in every model of KP it holds $\kappa+\alpha = x$ iff x is the ordinal $\kappa+\alpha$.

Lemma 14.3.5. *For every ordinal α it holds*

$$\Vdash_{\Upsilon} \big(\kappa^{\bullet} + \alpha^{\bullet} = (\kappa+\alpha)^{\bullet}\big)^{L_{\kappa+\alpha+4}}.$$

Proof. The claim follows by induction on α by employing Corollary 14.3.3 and making use of the $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-terms:

$$
\begin{aligned}
t_{r,0} &:= \Big\{x \in L_{\kappa+1} \;\Big|\; x = (0, \kappa^{\bullet})\Big\},\\
t_{r,\beta+1} &:= \Big\{x \in L_{\kappa+\beta+4} \;\Big|\; \exists r\Big(R_{+}(r, \kappa^{\bullet}, \beta^{\bullet}) \wedge \\
&\qquad \Big(x \in r \;\vee\; \exists\xi\big((\beta^{\bullet}, \xi) \in r \;\wedge\; x = ((\beta+1)^{\bullet}, \mathrm{S}(\xi))\big)\Big)\Big)\Big\},\\
t_{r,\lambda} &:= \Big\{x \in L_{\kappa+\lambda+3} \;\Big|\; \exists\xi\in\lambda^{\bullet}\,\exists r\big(R_{+}(r, \kappa^{\bullet}, \xi) \;\wedge\; x \in r\big)\\
&\qquad \vee \exists z\Big(x = (\lambda^{\bullet}, z) \wedge \\
&\qquad\quad \forall z_0 \in z\; \exists \xi \in \lambda^{\bullet}\; \exists y\big(y = \kappa^{\bullet} + \xi \wedge z_0 \in y\big) \wedge \\
&\qquad\quad \forall \xi \in \lambda^{\bullet}\; \exists y\big(y = \kappa^{\bullet} + \xi \wedge y \in z\big)\Big)\Big\},
\end{aligned}
$$

as witnesses for r, if $\alpha = 0$, $\alpha = \beta + 1$ or $\alpha = \lambda \in \mathrm{Lim}$, respectively. □

Definition 14.3.6. Let $z = \mathrm{Def}(y)$ be an $\mathcal{L}(\in)$-formula which states that z is the set of all subsets of y which are y-definable, i.e. the set of all subsets a of y, such that there is an $\mathcal{L}(\in)$-formula $F(x, \vec{z})$ and $s_1, \ldots, s_n \in y$ such that $a = \{x \in y \,|\, y \models F(x, s_1, \ldots, s_n)\}$. Then we define

$$
\begin{aligned}
R_{\ell}(r, \alpha) \;:=\; & \mathrm{Reln}(r) \;\wedge\; \forall x\in r\big(1^{\mathrm{st}}(x) \in \mathrm{S}(\alpha)\big)\\
& \wedge\; (0, \emptyset) \in r \;\wedge\; \forall z\big((0, z) \in r \to z = \emptyset\big)\\
& \wedge\; \forall (\xi, z)\in r\bigg(\Big(\xi \in \mathrm{Succ} \to \exists y\big((\xi - 1, y) \in r \wedge z = \mathrm{Def}(y)\big)\Big) \wedge\\
& \qquad \Big(\xi \in \mathrm{Lim} \to \Big(\forall z_0 \in z\; \exists \xi_0 \in \xi\; \exists y\big((\xi_0, y) \in r \wedge z_0 \in y\big)\\
& \qquad\qquad \wedge \forall \xi_0 \in \xi\; \exists y\big((\xi_0, y) \in r \wedge y \in z\big)\Big)\Big)\bigg).
\end{aligned}
$$

By use of this formula we define the $\mathcal{L}(\in)$-formula

$$\ell(\alpha, x)^{z} := \exists r\in z\big(R_{\ell}(r, \alpha) \;\wedge\; (\alpha, x) \in r\big).$$

Lemma 14.3.7. *For every ordinal α it holds*

$$\Vdash_{\Upsilon} \ell(\alpha^{\bullet}, L_{\alpha})^{L_{\alpha+5}}.$$

Proof. This proof runs in the same vein as the proof of Lemma 14.3.5, but instead of employing Corollary 14.3.3 we also have to prove

$$\Vdash_{\Upsilon} L_{\alpha+1} = \mathrm{Def}(L_{\alpha})^{L_{\alpha+4}} \quad \text{for every } \alpha \in \mathrm{ON}. \tag{14.1}$$

An economic way to prove (14.1) is to assume that $z = \mathrm{Def}(y)$ is exactly the formula given in [Dev84] on page 67, and to elaborate an RS-Version of Ch.I,9. of [Dev84] up to Lemma 9.11. By this we mean to choose canonical $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-terms for the sets given there as codes for the language $\mathcal{L}(\in)$ (in [Dev84] denoted by LST) and to prove RS-versions of the lemmata given there. E.g. assuming an enumeration $\langle v_0, v_1 \ldots, v_i, \ldots \mid i \in \omega\rangle$ of the free variables of $\mathcal{L}(\in)$ the (canonical) $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-term for the code of the variable v_i is $(v_i)^{\bullet} := \{x \in L_{i+1} \mid x = \{2^{\bullet}\} \vee x = \{2^{\bullet}, i^{\bullet}\}\}$ and the (canonical) $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-term for the code of an $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-term $t = \{x \in L_\alpha \mid F(x)\}$ is $\mathring{t} := \{x \in L_{\alpha+1} \mid x = \{3^{\bullet}\} \vee x = \{3^{\bullet}, t\}\}$. Proceeding in this vein the RS-version of Lemma 9.1 of [Dev84] reads as

$$\models^{\star} \mathrm{Vbl}((v_i)^{\bullet}), \quad \models^{\star} \mathrm{Const}(\mathring{t}), \quad \models^{\star} \mathrm{PFml}(P^{\bullet}),$$

if $(v_i)^{\bullet}$ is the canonical term for the code of v_i, $\mathring{t}$ is the canonical term for the code of t, and $P^{\bullet}$ is the canonical term for the code of a primitive formula.

More generally all lemmata of Ch.I,9. of [Dev84] of the form "A, B, C *are* Σ_0" have an RS-analogon like that of Lemma 9.1. Moreover lemmata of the form "F(x) *is* Δ_1^{BS}" provide propositions of the form $\models^{\star} F_\Sigma(t^{\bullet})^{L_\lambda}$ and $\models^{\star} F_\Pi(t^{\bullet})^{L_\lambda}$ for all appropriate terms $t^{\bullet}$ and every $\lambda \in \mathrm{Lim}$ with $\omega, |t^{\bullet}| < \lambda$, where F_Σ, F_Π are the provided Σ- and Π-formulae for F. E.g. Lemma 9.6 "*The* LST *formula* Fml(x) *is* Δ_1^{BS}." turns into $\models^{\star} \mathrm{Fml}(F^{\bullet})^{\lambda}$ for every $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-term $F^{\bullet}$, which represents the code of a formula and every $\omega, |F^{\bullet}| < \lambda \in \mathrm{Lim}$.

By use of this RS-versions up to Lemma 9.11 we obtain for every α and every $\mathcal{L}(\in)$-formula $F(v_0, \ldots, v_n)$ plus the term $f((v_0)^{\bullet}, \ldots, (v_0)^{\bullet})^{\bullet}$ of the code of this formula, that

$$\Vdash_{\Upsilon} \forall x_0 \in L_\alpha \ldots \forall x_n \in L_\alpha \big(F(x_0, \ldots, x_n)^{L_\alpha} \leftrightarrow \mathrm{Sat}(L_\alpha, f(\mathring{x}_0, \ldots, \mathring{x}_n)^{\bullet})\big)^{L_{\alpha+4}}.$$

Finally we are able to prove (14.1) by employing this proposition. □

14.3.2. Embedding of the Axiom (Stab)

Since we have already proved embedding theorems for all axioms of KP and for pure logic in section 4.3 we just have to care about the axiom (Stab) to obtain an embedding of KP+(Stab).

Lemma 14.3.8 (Stab)**.** *It holds*

$$\Vdash_{\Upsilon} \forall x\, \exists \alpha \in \mathrm{Lim}\ \exists \kappa \in \mathrm{Lim}\ \big(\alpha \leq \kappa \wedge x \in L_\alpha \wedge\ \forall y \in L_\kappa \big(Sat_1(y)^{L_{\kappa+\alpha}} \to Sat_1(y)^{L_\kappa}\big)\big)^{\Upsilon}.$$

Proof. Let us assume given is a term s. In the following our strategy is to employ $|s| + \omega$ as a witness for α and $\Theta(|s| + \omega)$ as a witness for κ.

Let $Sat_1(z) = \exists y F(y,z)$. Then the claim renders to

$$\Vdash_{\Upsilon} \forall x\, \exists y\, \exists z\, \exists w\, \exists l_1\, \exists l_2\, \exists l_3 \Big(\mathrm{Lim}(y) \;\wedge\; \mathrm{Lim}(z) \;\wedge\; (y \in z \vee y = z) \;\wedge$$
$$\ell(y,l_1) \;\wedge\; x \in l_1 \;\wedge\; \ell(z,l_2) \;\wedge\; z+y=w \;\wedge\; \ell(w,l_3) \;\wedge$$
$$\forall z_0 \in l_1 \big(\forall y_0 \in l_3 \neg F(y_0,z_0) \;\vee\; \exists y_1 \in l_2 F(y_1,z_0)\big)\Big)^{\Upsilon}.$$

Let s be an $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-term and let $\lambda := |s|+\omega$. Moreover let $s_0 \in \mathcal{T}_{\Theta(\lambda)+\lambda}$ and $t_0 \in \mathcal{T}_{\Theta(\lambda)}$. By use of 4.3.3 and 4.3.6 we obtain for every hull-set $\mathcal{H}$

$$\mathcal{H}[s_0,t_0] \,\Big|\frac{\omega^{\mathrm{rnk}(F(s_0,t_0))}\cdot 2}{0}\; \neg F(s_0,t_0), F(s_0,t_0).$$

Let $\sigma_0 := 0$ if $|s_0| \le \Theta(\lambda)$ and choose σ_0 such that $\Theta(\lambda)+\sigma_0 = |s_0|$ elsewise.

As there exists the reflection instance $\mathbb{G}((\sigma_0+1,0)) = (\Theta(\lambda); \mathsf{P}_{(|\sigma_0|+1,0)}; \epsilon; \epsilon; \Upsilon)$ there exists also the reflection-rule $(\Pi_0(\Theta(\lambda)+\sigma_0+1)$-$\mathsf{Ref})$. An application of this rule provides

$$\mathcal{H}[s,s_0,t_0] \,\Big|\frac{\omega^{\mathrm{rnk}(F(s_0,t_0))}\cdot 2+1}{0}\; \neg F(s_0,t_0), \exists y_1^{\Theta(\lambda)} \big({}^{s_0}M_{\Theta(\lambda)}(y_1) \wedge F(y_1,t_0)\big).$$

Due to Lemma 4.3.3, (Str), two $(\bigwedge)$-inferences and a $(\bigwedge)$-inference we also obtain

$$\mathcal{H}[s,s_0,t_0] \,\Big|\frac{\Theta(\lambda)}{0}\; \forall y_1^{\Theta(\lambda)} \big(\neg^{s_0} M_{\Theta(\lambda)}(y_1) \vee \neg F(y_1,t_0)\big), \exists y_1^{\Theta(\lambda)} F(y_1,t_0).$$

An application of (Cut) yields

$$\mathcal{H}[s,s_0,t_0] \,\Big|\frac{\Theta(\lambda)+\omega^{\mathrm{rnk}(F(s_0,t_0))}\cdot 2+2}{\Upsilon}\; \neg F(s_0,t_0), \exists y_1^{\Theta(\lambda)} F(y_1,t_0),$$

and by use of $(\bigwedge)$ and $(\bigvee)$ we obtain

$$\mathcal{H}[s,t_0] \,\Big|\frac{(\Theta(\lambda)+\lambda)\cdot 3}{\Upsilon}\; \exists y_0^{\Theta(\lambda)+\lambda} F(y_0,t_0) \to \exists y_1^{\Theta(\lambda)} F(y_1,t_0).$$

By another application of $(\bigwedge)$ we get

$$\mathcal{H}[s] \,\Big|\frac{\Theta(\lambda)+\lambda)\cdot 3+1}{\Upsilon}\; \forall z_0^{\Theta(\lambda)} \big(\exists y_0^{\Theta(\lambda)+\lambda} F(y_0,z_0) \to \exists y_1^{\Theta(\lambda)} F(y_1,z_0)\big).$$

The embedding of all other subformulae of (Stab) follows directly by the results of section 14.3.1. Finally the claim follows by six $(\bigvee)$-inferences and a $(\bigwedge)$-inference. □

Theorem 14.3.9 (Embedding of KP+(Stab))**.** *Let F be a theorem of* KP+(Stab). *Then there is an $m \in \omega$ such that*

$$\mathcal{H}[\Upsilon] \,\Big|\frac{\omega^{\Upsilon+m}}{\Upsilon+m}\; F^{\Upsilon}.$$

Proof. This follows by the results of section 4.3 and the above given derivation of (Stab). □

15. Elimination of Reflection Rules

In this chapter we prove the Reflection Elimination Theorem for Stability. In contrast to Π_ω-Ref now we have to collapse intervals instead of just single points. To be able to handle this we need the following preparations.

15.1. Useful Properties of $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$

Notation. Let $\kappa < \pi$ and F be an $\Delta_0(\pi+\pi)$-$\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-sentence. Then we denote by $F^{\pi\mapsto\kappa}$ the formula F in which every term t is replaced by $t^{\pi\mapsto\kappa}$. If F_t and $F(s)$ meet the conditions of F we also use the following notations

$$F_t^{\pi\mapsto\kappa} := (F_t)^{\pi\mapsto\kappa} \quad \text{and} \quad F(s)^{\pi\mapsto\kappa} := (F(s))^{\pi\mapsto\kappa}.$$

Lemma 15.1.1. *Let $\theta < \kappa < \pi$ and $\kappa, \pi \in \mathrm{SC}$ plus $n \in \omega$. Suppose $F \in \Sigma_n(\pi+\theta^*)$ (or $F \in \Pi_n(\pi+\theta^*)$) with $F \cong \bigvee(F_t)_{t\in T}$ and $\mathrm{k}(F)\cap\pi \subseteq \kappa$. Then*

$$F^{\pi\mapsto\kappa} \cong \bigvee(F_t^{\pi\mapsto\kappa})_{t\in(T\cap\mathcal{T}_\kappa)\cup(T\setminus\mathcal{T}_\pi)}.$$

The analogue statement holds for $F \cong \bigwedge(F_t)_{t\in T}$.

Remark. The Lemma does not hold for $F \in \Sigma_n(\pi+\theta)$ in general; e.g. $F \equiv {}^{t}_{\tau}M^{\alpha}_{\mathbb{X}}(\pi)$.

Proof. To prove the Lemma we proceed by induction on the build-up of F. At first assume that $F \in \Delta^1_0(\pi)$-$\mathcal{L}^{\circledast}_{RS(\Upsilon)}$. Then $F^{\pi\mapsto\kappa} \equiv F^{(\pi,\kappa)}$ and a closer look at the possible characteristic sequences of F reveals the claim.

Now suppose $F \in \Sigma_n(\pi+\theta)$-$\mathcal{L}^{*}_{RS(\Upsilon)}$. If $F \equiv (\neg)(r \in s)$ or $F \equiv (\neg)(\exists x \in s\, G(x))$ then we have $T = \mathcal{T}_{|s|}$ and it holds $\mathcal{T}_{|s^{\pi\mapsto\kappa}|} = (T\cap\mathcal{T}_\kappa)\cup(T\setminus\mathcal{T}_\pi)^{\pi\mapsto\kappa}$. If $F \equiv (\neg)^{\vec{t}}M_{\pi'}(\vec{r})$ then it holds $T = \mathrm{SC}\cap(|\vec{r}|+1)$ and we have $\mathrm{SC}\cap(|\vec{r}^{\,\pi\mapsto\kappa}|+1) = (T\cap\mathcal{T}_\kappa)\cup(T\setminus\mathcal{T}_\pi)^{\pi\mapsto\kappa}$ since $\kappa,\pi \in \mathrm{SC}$. If $F \equiv (\neg)(F_0 \vee F_1)$ then $T = \{0,1\}$ and the claim holds since $(\{0,1\}\setminus\mathcal{T}_\pi)^{\pi\mapsto\kappa} = \{0,1\}$.

Finally, if F is a Boolean composition of $\Sigma_n(\pi+\theta^*)$-sentences the claim holds, since $T = \{0,1\}$. $\square$

Let F be a $\Delta_0(\pi+1)$-sentence. Then we can find a $\Delta^1_0(\pi)$-sentence $\tilde{F}$ such that $\mathbb{L} \models F \leftrightarrow \tilde{F}$. We simply obtain $\tilde{F}$ from F by resolving all terms $t = \{x \in L_\pi \mid H(x,\vec{s})^{L_\pi}\}$ of stage π occurring in F.

In the next Lemma we show that this equivalence does also hold for derivations on hull-sets.

Definition 15.1.2. Let F be a $\Delta_0(\pi{+}(\theta{+}1)^*)$-sentence. Then we define F' recursively by means of the following clauses:

$$(r \in s)' \;:=\; \begin{cases} r \in s & \text{if } |r| < \pi + \theta, \\ \exists z \in s\,(z = r) & \text{otherwise,} \end{cases}$$

and

$$\big(M(s)\big)' := M(s),\ \big(\exists x \in s\, G(x)\big)' := \exists x \in s\,(G(x))',\ (F_0 \vee F_1)' := F_0' \vee F_1',\ (\neg F)' := \neg(F)',$$

where M represents an arbitrary $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$ $\mathfrak{M}$-predicate.

Now let $G \in \Delta_0(\pi + (\theta+1)^*)$ be a sentence without any occurrences of subformulae of the form $r \in s$ with $|r| \geq \pi + \theta$. Then we define:

$$(r \in s)'' \;:=\; \begin{cases} r \in s & \text{if } |s| < \pi + \theta, \\ \exists z \in L_{\pi+\theta}\,(z = r) & \text{if } s \equiv L_{\pi+\theta}, \\ \exists z \in L_{\pi+\theta}\big(H(z) \wedge z = r\big) & \text{if } s \equiv \{x \in L_{\pi+\theta} \mid H(x)\} \end{cases}$$

$$\big(\exists x \in s\, G(x)\big)'' \;:=\; \begin{cases} \exists x \in s\, G(x)'' & \text{if } |s| < \pi + \theta, \\ \exists x \in L_{\pi+\theta}\big(H(x) \wedge G(x)''\big) & \text{if } s \equiv \{x \in L_{\pi+\theta} \mid H(x)\}. \end{cases}$$

and

$$M(s)'' := M(s), \qquad (G_0 \vee G_1)'' := G_0'' \vee G_1'', \qquad (\neg G)'' := \neg(G'').$$

Finally we define $\tilde{F} := (F')''$.

Lemma 15.1.3. *Let $\theta < \kappa < \pi$ and let $F \in \Delta_0(\pi + (\theta+1)^*)$ plus Γ be a finite set of $\Delta^1_0(\pi)$-$\mathcal{L}^{\circledast}_{RS(\Upsilon)}$ sentences. Then it holds*

- $\tilde{F} \in \Delta^1_0(\pi + \theta^*)$,
- $\mathcal{H} \,|\!\frac{\alpha}{\rho}\, \Gamma, F \quad \Rightarrow \quad \mathcal{H} \,|\!\frac{\alpha}{\rho}\, \Gamma, \tilde{F}$.
- $(\tilde{F})^{\pi \mapsto \kappa} \equiv \widetilde{F^{\pi \mapsto \kappa}}$.

Proof. At first we observe that $\tilde{F}$ is well-defined, since there are not any subformulae in F' of the form $r \in s$ with $|r| \geq \pi + \theta$.

The first and third claims follow by induction on the build-up of F'.

The second claim holds, since it follows by induction on α that $\mathcal{H} \,|\!\frac{\alpha}{\rho}\, \Gamma, F \Leftrightarrow \mathcal{H} \,|\!\frac{\alpha}{\rho}\, \Gamma, F'$, as it holds $\mathrm{CS}(r \in s) = \mathrm{CS}(\exists z \in s\,(z = r))$ if $|r| \geq \pi + \theta$ and we must have $F = F'$ if F is the principal formula of a reflection inference.

Referring to these arguments we can also prove that $\mathcal{H} \,|\!\frac{\alpha}{\rho}\, \Gamma, F' \Leftrightarrow \mathcal{H} \,|\!\frac{\alpha}{\rho}\, \Gamma, \tilde{F}$. □

15.2. Reflection Elimination for Stability

Theorem 15.2.1 (Existence of Reflection-Rules). *Let* $\mathbb{X} = (\pi; \ldots)$ *be a reflection instance with reflection configuration* $\mathbb{F}$ *plus* $\mathrm{rdh}(\mathbb{X}) = \mathfrak{m} \geq 0$*, and let* $\kappa \in \mathfrak{M}^{\alpha}_{\mathbb{X}}$*.*

❶ *It holds:*

$$\exists \mathfrak{S} \subseteq \Upsilon \times \omega \quad \forall (\sigma, s) \in \mathfrak{S} \quad \forall \tau < \kappa \quad \forall \alpha_0 \in [o(\mathbb{X}), \alpha)_{C(\kappa)}$$
$$\forall \vec{\eta} \in \mathrm{dom}(\mathbb{F})^{>(\sigma,s)}_{C(\kappa)} \quad \big(\sup(\mathfrak{S}) = (\mathfrak{m}) \wedge ({}_{\tau}M^{\alpha_0}_{\mathbb{F}(\vec{\eta})}\text{-}\Pi_{s\dot{+}2}(\kappa{+}\sigma)\text{-}\mathsf{Ref}) \text{ exists}\big).$$

❷ *For every* (ϑ, k)*, such that* $(\vartheta, k) \notin \mathrm{Rdh}(\mathbb{F})$ *or* $\mathfrak{m} \geq (\vartheta, k)$*, it holds:*

$$\forall \tau < \kappa \quad \forall \zeta \in C(\kappa) \quad \forall \mathbb{K} \quad \forall \vec{\eta} \in \mathrm{dom}(\mathbb{K})_{C(\kappa)}$$
$$\Big(({}_{\tau}M^{\zeta}_{\mathbb{K}(\vec{\eta})}\text{-}\Pi_{k\dot{+}2}(\pi + \vartheta)\text{-}\mathsf{Ref}) \text{ is a subsidiary reflection rule of } \pi \quad \Rightarrow$$
$$\big(({}_{\tau}M^{\zeta}_{\mathbb{K}(\vec{\eta})}\text{-}\Pi_{k\dot{+}2}(\kappa + \vartheta)\text{-}\mathsf{Ref}) \text{ exists}\big)\Big).$$

Proof. At first we observe, that there must be a reflection instance $\mathbb{Y}$ and a β, such that $\kappa = \Psi^{\beta}_{\mathbb{Y}}$, since κ is at least Π^1_0-indescribable.

① If $\alpha = \mathrm{o}(\mathbb{X})$ we have nothing to show. So let us assume $\alpha > \mathrm{o}(\mathbb{X})$. Due to the Domination Theorem 13.1.4 it holds $\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}} \preceq \vec{R}^{\mathrm{cl}}_{\kappa}$. If we have $(\vec{R}^{\mathrm{cl}}_{\Psi^{\alpha}_{\mathbb{X}}})_{\mathfrak{m}} = \mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}} \preceq (\vec{R}^{\mathrm{cl}'}_{\kappa})_{\mathfrak{m}}$ and $\mathfrak{m} \in \mathrm{Lim}$ the claim follows taking into account Lemma 2.3.2 ③ (g) and by means of Definition 14.2.2. If $\mathfrak{m} \notin \mathrm{Lim}$ the claim follows with $\mathfrak{S} := \{\mathfrak{m}\}$ by means of Definition 14.2.2. In both cases the reflection rules are subsidiary reflection rules of κ.

Now suppose $\mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}} \not\preceq (\vec{R}^{\mathrm{cl}'}_{\kappa})_{\mathfrak{m}}$. Then we must have $\vec{R}_{\kappa} = (\mathsf{M}^{<\xi}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}}, \ldots)$ for some $\xi \geq \alpha$ since $\mathsf{M}^{<\alpha}_{\mathbb{F}}\text{-}\mathsf{P}_{\mathfrak{m}} \preceq (\vec{R}_{\kappa})_{\mathfrak{m}}$. Thus Definition 12.2.4 provides that there are appropriate reflection instances and an $\mathfrak{S}$ with $\sup(\mathfrak{S}) = \mathfrak{m}$ such that Definition 14.2.2 provides the desired reflection rules (as main reflection rules of κ).

② Suppose $({}_{\tau}M^{\zeta}_{\mathbb{K}(\vec{\eta})}\text{-}\Pi_{k\dot{+}2}(\pi + \vartheta)\text{-}\mathsf{Ref})$ is a subsidiary reflection rule of π and let $\mathfrak{k} := (\vartheta, k)$. A run through the four cases of Definition 12.2.4 reveals that it holds $(\vec{R}^{\mathrm{cl}'}_{\pi})_{\leq\mathfrak{k}} \preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{\leq\mathfrak{k}}$, if $\mathfrak{k} \notin \mathrm{Rdh}(\mathbb{F})$ or $\mathfrak{m} \geq \mathfrak{k}$. Thereby we have $\mathsf{M}^{<\zeta+1}_{\mathbb{K}}\text{-}\mathsf{P}_{\mathfrak{k}} \preceq (\vec{R}^{\mathrm{cl}'}_{\pi})_{\mathfrak{k}} \preceq (\vec{R}_{\Psi^{\alpha}_{\mathbb{X}}})_{\mathfrak{k}}$ and the claim follows by the same considerations as in the proof of ① taking also into account that $\mathfrak{k} \notin \mathrm{Lim}$ by means of Definition 14.2.2.

□

Definition 15.2.2. Let $\vec{\nu} = (\nu_1, \ldots, \nu_n)$ and $\vec{\eta} = (\eta_1, \ldots, \eta_n)$ be vectors of ordinals of the same length. Then we define

$$\mathrm{pmax}\{\vec{\nu}, \vec{\eta}\} \; := \; (\mu_1, \ldots, \mu_n), \quad \text{where for all } 1 \leq i \leq n \;\; \mu_i := \max\{\nu_i, \eta_i\}.$$

Lemma 15.2.3. *Let* $\mathbb{F}$ *be a reflection configuration and* $\vec{\nu}, \vec{\eta} \in \mathrm{dom}(\mathbb{F})$ *and* $\mathrm{o}(\mathbb{F}) \leq \alpha, \beta \in C(\mathrm{i}(\mathbb{F}))$*. Then it holds*

- $\mathrm{pmax}\{\vec{\nu}, \vec{\eta}\} \in \mathrm{dom}(\mathbb{F})$*,*

- $(\beta, \vec{\eta}) = \mathrm{pmax}\{(\alpha, \vec{\nu}), (\beta, \vec{\eta})\} \quad \Rightarrow \quad \mathfrak{M}^{\beta}_{\mathbb{F}(\vec{\eta})} \subseteq \mathfrak{M}^{\alpha}_{\mathbb{F}(\vec{\nu})}$.

Proof. This follows by a straight forward induction on $\mathrm{o}(\mathbb{F})$. □

Remark. The propositions of Lemma 5.2.3 (with $\Upsilon, \delta \leq \gamma+1$) also hold in the context of Stability. In the proof of proposition ③ of this Lemma in the case that $\mu = \Psi^{\zeta}_{\mathbb{V}}$ we can extract an appropriate reflection degree for the desired reflection instance $\mathbb{Z} \in \mathcal{C}_{\gamma}[\mathcal{A}]$ out of $\vec{R}_{\Psi^{\zeta}_{\mathbb{V}}}$.

Theorem 15.2.4 (Reflection Elimination)**.** *Let $\mathbb{X} = (\pi; \ldots; \delta)$ be a reflection instance with $\mathrm{rdh}(\mathbb{X}) = (\theta, m)$ and $\gamma, \mathbb{X}, \mu \in \mathcal{C}_{\gamma}[\mathcal{A}]$, where $\Upsilon, \delta \leq \gamma + 1$ and $\theta < \sigma := |\mathcal{A}| < \pi \leq \mu \in \mathrm{Card}$. Let $\Gamma \subseteq \Sigma_{m\dot{+}2}(\pi + \theta^*)$ and $\hat{\alpha} := \gamma \oplus \omega^{\alpha \oplus \mu}$. Then*

$$\mathcal{C}_{\gamma}[\mathcal{A}] \overset{\alpha}{\underset{\mu}{\vdash}} \Gamma \quad \Rightarrow \quad \mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A}, \kappa] \overset{\Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}}}{\underset{\cdot}{\vdash}} \Gamma^{\pi\mapsto\kappa} \quad \textit{for all } \kappa \in {}_{\sigma}\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}.$$

Proof. We proceed by main induction on μ and subsidiary induction on α.

Case 1, the last inference is $(\bigvee)$ *with principal formula $F \cong \bigvee(F_t)_{t\in T} \in \Gamma$:* Thus

$$\mathcal{C}_{\gamma}[\mathcal{A}] \overset{\alpha_0}{\underset{\mu}{\vdash}} \Gamma, F_{t_0},$$

for some $\alpha_0 < \alpha$ and some $t_0 \in T$. By use of the subsidiary induction hypothesis we obtain

$$\mathcal{C}_{\hat{\alpha_0}\oplus\kappa}[\mathcal{A}, \kappa] \overset{\Psi^{\hat{\alpha_0}\oplus\kappa}_{\mathbb{X}}}{\underset{\cdot}{\vdash}} \Gamma^{\pi\mapsto\kappa}, F^{\pi\mapsto\kappa}_{t_0} \quad \text{for all } \kappa \in {}_{\sigma}\mathfrak{M}^{\hat{\alpha_0}}_{\mathbb{X}}. \tag{15.1}$$

By Lemma 5.2.3 ① we have ${}_{\sigma}\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}} \subseteq \mathfrak{M}^{\hat{\alpha_0}}_{\mathbb{X}}$ and $\Psi^{\hat{\alpha_0}\oplus\kappa}_{\mathbb{X}} < \Psi^{\hat{\alpha}\oplus\kappa}_{\mathbb{X}} \in \mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A}, \kappa]$. Moreover it holds $F^{\pi\mapsto\kappa} \cong \bigvee(F^{\pi\mapsto\kappa}_t)_{t\in(T\cap\mathcal{T}_{\kappa})\cup(T\setminus\mathcal{T}_{\pi})}$ by Lemma 15.1.1. We either have $t_0 \in T \cap \mathcal{T}_{\pi}$ and then $|t_0| \in \mathcal{C}_{\gamma}[\mathcal{A}] \cap \pi \subseteq C(\gamma+1, \sigma+1) \cap \pi \subseteq C(\hat{\alpha}, \kappa) \cap \pi = \kappa$, i.e. $t_0 \in T \cap \mathcal{T}_{\kappa}$, or $t_0 \in T \setminus \mathcal{T}_{\pi}$. Thereby it follows $F^{\pi\mapsto\kappa}_{t_0} \in \mathrm{CS}(F^{\pi\mapsto\kappa})$. Thus we obtain the desired result by means of (Str) and $(\bigvee)$ from (15.1).

Case 2, the last inference is $(\bigwedge)$ *with principal formula $F \cong \bigwedge(F_t)_{t\in T} \in \Gamma$:* Then for all $t \in T$ there exists an $\alpha_t < \alpha$ such that

$$\mathcal{C}_{\gamma}[\mathcal{A}, t] \overset{\alpha_t}{\underset{\mu}{\vdash}} \Gamma, F_t.$$

Since $\pi \in \mathcal{C}_{\gamma}[\mathcal{A}]$ it holds $\mathcal{C}_{\gamma}[\mathcal{A}, t] = \mathcal{C}_{\gamma}[\mathcal{A}, t^{\pi\mapsto 0}]$. Therefore we obtain by use of (Str)

$$\mathcal{C}_{\gamma}[\mathcal{A}, t^{\pi\mapsto 0}] \overset{\alpha_t}{\underset{\mu}{\vdash}} \Gamma, F_t, \tag{15.2}$$

for all $t \in T$. Since $F \in \Gamma$ we have $F_t \in \Sigma_{m\dot{+}2}(\pi + \theta^*)$. Moreover it holds $\sigma_{t_0} := |\mathcal{A}, t^{\pi\mapsto 0}| < \pi$ for all $t \in T$. Thus we may apply the subsidiary induction hypothesis to (15.2) and obtain for all $t \in T$

$$\mathcal{C}_{\hat{\alpha_t}\oplus\lambda}[\mathcal{A}, t^{\pi\mapsto 0}, \lambda] \overset{\Psi^{\hat{\alpha_t}\oplus\lambda}_{\mathbb{X}}}{\underset{\cdot}{\vdash}} \Gamma^{\pi\mapsto\lambda}, F^{\pi\mapsto\lambda}_t \quad \text{for all } \lambda \in {}_{\sigma_{t_0}}\mathfrak{M}^{\hat{\alpha_t}}_{\mathbb{X}}. \tag{15.3}$$

Now let $\kappa \in {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha}}$. For $t \in T \cap \mathcal{T}_\kappa$ we have $\kappa \in {}_{\sigma_{t_0}}\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha}}$. For $t \in (T \backslash \mathcal{T}_\pi)$ it holds $|t^{\pi \mapsto 0}| < \sigma$ since $\theta < \sigma$ and thus $\sigma_{t_0} = \sigma$ and hence $\kappa \in {}_{\sigma_{t_0}}\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha}}$. Making also use of Lemma 5.2.3 ① (with $\mathcal{A}$ replaced by $\mathcal{A}, t$) it follows for every $t \in (T \cap \mathcal{T}_\kappa) \cup (T \backslash \mathcal{T}_\pi)$ that $\kappa \in {}_{\sigma_{t_0}}\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha_t}}$. Since we also have $\mathcal{C}_\gamma[\mathcal{A}, t^{\pi \mapsto 0}, \kappa] = \mathcal{C}_\gamma[\mathcal{A}, t^{\pi \mapsto \kappa}, \kappa]$ we obtain by use of (Str) from (15.3)

$$\mathcal{C}_{\hat{\alpha_t} \oplus \kappa}[\mathcal{A}, t^{\pi \mapsto \kappa}, \kappa] \left| \frac{\Psi_{\mathbb{X}}^{\hat{\alpha_t} \oplus \kappa}}{\cdot} \right. \Gamma^{\pi \mapsto \kappa}, F_t^{\pi \mapsto \kappa} \quad \text{for all } t \in (T \cap \mathcal{T}_\kappa) \cup (T \backslash \mathcal{T}_\pi). \tag{15.4}$$

Thereby the claim follows by a $(\bigwedge)$-inference, taking into account Lemma 15.1.1.

Case 3, the last inference is (Cut)*:* Then for some $\alpha_0 < \alpha$ it holds

$$\mathcal{C}_\gamma[\mathcal{A}] \left| \frac{\alpha_0}{\bar{\mu}} \right. \Gamma, (\neg)F, \tag{15.5}$$

where $\mathrm{rnk}(F) < \bar{\mu}$.

Subcase 3.1, $\mathrm{rnk}(F) < \pi$*:* Then we have $(\neg)F \in \Sigma_0(\pi + \theta^*)$. Therefore we are allowed to apply the subsidiary induction hypothesis to (15.5) and obtain

$$\mathcal{C}_{\hat{\alpha_0} \oplus \kappa}[\mathcal{A}, \kappa] \left| \frac{\Psi_{\mathbb{X}}^{\hat{\alpha_0} \oplus \kappa}}{\cdot} \right. \Gamma^{\pi \mapsto \kappa}, (\neg)F^{\pi \mapsto \kappa} \quad \text{for all } \kappa \in {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha_0}}.$$

By similar considerations as in the first case it follows that $\mathrm{rnk}(F^{\pi \mapsto \kappa}) = \mathrm{rnk}(F) < \kappa < \Psi_{\mathbb{X}}^{\hat{\alpha} \oplus \kappa}$ for $\kappa \in {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha_0}}$. Thus the claim follows by a (Cut) and (Str) taking into account Lemma 5.2.3 ①.

Subcase 3.2, $\pi \le \mathrm{rnk}(F) \le \mu$*:* If $\mathrm{rnk}(F) < \mu$ then it holds $\pi \le \mu_0 := \sup\{\kappa \in \mathrm{Card} \mid \kappa \le \mathrm{rnk}(F)\} \le \mathrm{rnk}(F) < \mu_0^+ \le \mu$ and due to Lemma 5.2.3 ② it follows $(\mu_0^+; \mathsf{P}_0; \epsilon; \epsilon; 0) \in \mathcal{C}_\gamma[\mathcal{A}]$. If $\mathrm{rnk}(F) = \mu$ we must have $\mu \in \mathrm{Reg}$ and by Lemma 5.2.3 ③ it follows the existence of a reflection instance $\mathbb{Z} \in \mathcal{C}_\gamma[\mathcal{A}]$ with $\mathrm{i}(\mathbb{Z}) = \mu$ and $\mathrm{o}(\mathbb{Z}) \le \gamma + 1$. Thus in all cases there is a reflection instance $\mathbb{Y} \in \mathcal{C}_\gamma[\mathcal{A}]$ with $\mathrm{i}(\mathbb{Y}) = \mu_1$ and $\mathrm{o}(\mathbb{Y}) \le \gamma + 1$ such that $\pi \le \mathrm{rnk}(F) \le \mu_1 \le \mu$. Let $(\vartheta, n) := \mathrm{rdh}(\mathbb{Y})$. In the following we choose $\mathbb{Y} = \mathbb{X}$ if $\pi = \mu$.

It holds $\Gamma \in \Sigma_{n \dot{+} 2}(\mu_1 + \vartheta^*)$ and w.l.o.g. F is an elementary $\Sigma_1(\mu_1)$-sentence and $\neg F \equiv \forall x^{\mu_1} G(x)$ an elementary $\Pi_1(\mu_1)$-sentence since $\mathrm{rnk}(F) \le \mu_1$. As we also have $\Sigma_1(\mu_1) \subseteq \Sigma_{n \dot{+} 2}(\mu_1 + \vartheta^*)$ we are allowed to apply the subsidiary induction hypothesis to (15.5) and obtain

$$\mathcal{C}_{\hat{\alpha_0} \oplus \lambda}[\mathcal{A}, \lambda] \left| \frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0} \oplus \lambda}}{\cdot} \right. \Gamma^{\mu_1 \mapsto \lambda}, F^{\mu_1 \mapsto \lambda} \quad \text{for all } \lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}} \tag{15.6}$$

By use of $(\forall\text{-}\mathsf{Inv})$, the subsidiary induction hypothesis and an $(\bigwedge)$-inference the derivation of $\Gamma, \neg F$ of (15.5) is transformable to

$$\mathcal{C}_{\hat{\alpha_0} \oplus \lambda}[\mathcal{A}, \lambda] \left| \frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0} \oplus \lambda} + 1}{\cdot} \right. \Gamma^{\mu_1 \mapsto \lambda}, \neg F^{\mu_1 \mapsto \lambda} \quad \text{for all } \lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}. \tag{15.7}$$

Since $\mathrm{k}(F^{\mu_1 \mapsto \lambda}) \in \mathcal{C}_\gamma[\mathcal{A}, \lambda] \cap \mu_1 \subseteq C(\hat{\alpha_0} \oplus \lambda, \Psi_{\mathbb{Y}}^{\hat{\alpha_0} \oplus \lambda}) \cap \mu_1 = \Psi_{\mathbb{Y}}^{\hat{\alpha_0} \oplus \lambda}$ if $\lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}$ it follows $\mathrm{rnk}(F^{\mu_1 \mapsto \lambda}) < \Psi_{\mathbb{Y}}^{\hat{\alpha_0} \oplus \lambda}$. Thus a (Cut) applied to (15.6) and(15.7) yields

$$\mathcal{C}_{\hat{\alpha_0} \oplus \lambda + 1}[\mathcal{A}, \lambda] \Big|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0} \oplus \lambda + 1}}{\cdot} \Gamma^{\mu_1 \mapsto \lambda} \quad \text{for all } \lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}. \tag{15.8}$$

If $\mathbb{Y} = \mathbb{X}$ Lemma 5.2.3 ① provides that the claim follows from (15.8) by use of (Str).

If $\mathbb{Y} \neq \mathbb{X}$ we have $\pi + \theta < \mu_1 \leq \mu$ and therefore $\Gamma^{\mu_1 \mapsto \lambda} \equiv \Gamma$. Let $\lambda_0 := \Psi_{\mathbb{Y}}^{\hat{\alpha_0} \oplus \sigma}$ and $\eta := \Psi_{\mathbb{Y}}^{\hat{\alpha_0} \oplus \lambda_0 + 1}$. Then it follows as shown in the proof of Lemma 5.2.3 ① $\lambda_0 \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}$ and $\pi < \eta < \mu_1$ since $\pi \in C(\gamma + 1, \sigma + 1) \subseteq C(\hat{\alpha_0} \oplus \sigma, \lambda_0) \cap \mu_1 = \lambda_0$. Thus (15.8) plus (Hull) provides

$$\mathcal{C}_{\hat{\alpha_0} \oplus \lambda_0 + 1}[\mathcal{A}] \Big|\frac{\eta}{\cdot} \Gamma. \tag{15.9}$$

If $\eta \in \mathrm{Card}$ we apply the main induction hypothesis to (15.9) and obtain

$$\mathcal{C}_{\nu \oplus \kappa}[\mathcal{A}, \kappa] \Big|\frac{\Psi_{\mathbb{X}}^{\nu \oplus \kappa}}{\cdot} \Gamma^{\pi \mapsto \kappa} \quad \text{for all } \kappa \in {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\nu}, \tag{15.10}$$

where $\nu := (\hat{\alpha_0} \oplus \lambda_0 \oplus 1) \oplus \omega^{\eta \oplus \bar{\eta}}$. Since $\alpha_0 \oplus \mu, \lambda_0 + 1, \eta \oplus \bar{\eta} < \alpha \oplus \mu$ it follows $\nu < \hat{\alpha}$ and since $\gamma, \alpha_0, \mu, \mathbb{Y} \in C(\gamma + 1, \sigma + 1)$ it follows successively $\lambda_0, \eta, \nu, \nu \oplus \kappa \in C(\nu \oplus \kappa, \Psi_{\mathbb{X}}^{\hat{\alpha} \oplus \kappa})$ for $\kappa \in {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\nu}$ and hence $\Psi_{\mathbb{X}}^{\nu \oplus \kappa} < \Psi_{\mathbb{X}}^{\hat{\alpha} \oplus \kappa}$ for all $\kappa \in {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\nu}$ by Theorem 2.3.14. In the same vein it follows $\nu \in C(\nu, \kappa)$ for every $\kappa \in {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha}}$ and hence $\kappa \in {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha}}$ implies $\kappa \in \mathfrak{M}_{\mathbb{X}}^{\nu}$. Thereby the claim follows from (15.10) by use of (Str).

If $\pi < \mu_1 \leq \mu$ and $\eta \notin \mathrm{Card}$ then it holds $\pi \leq \mu_0 < \eta < \mu_0^+ = \mu_1$. Through the use of predicative cut elimination, (15.9) yields

$$\mathcal{C}_{\hat{\alpha_0} \oplus \lambda_0 + 1}[\mathcal{A}] \Big|\frac{\varphi(\eta, \eta)}{\bar{\mu_0}} \Gamma. \tag{15.11}$$

Since $\mu_1 \in \mathcal{C}_\gamma[\mathcal{A}]$ implies $\mu_0 \in \mathcal{C}_\gamma[\mathcal{A}]$ we may apply the main induction to (15.11) and obtain

$$\mathcal{C}_{\hat{\nu} \oplus \kappa}[\mathcal{A}, \kappa] \Big|\frac{\Psi_{\mathbb{X}}^{\hat{\nu} \oplus \kappa}}{\cdot} \Gamma^{\pi \mapsto \kappa} \quad \text{for all } \kappa \in {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\hat{\nu}},$$

where $\hat{\nu} := (\hat{\alpha_0} \oplus \lambda_0 \oplus 1) \oplus \omega^{\varphi(\eta, \eta) \oplus \bar{\mu_0}}$. By analogue considerations as in the case $\eta \in \mathrm{Card}$ the claim follows by (Str).

Case 4, the last inference is by a main reflexion rule of π:

Subcase 4.1, the reflection configuration $\mathbb{F}$ of $\mathbb{X}$ is a reflection configuration with variable reflection degree, i.e. it holds $\mathbb{X} = \mathbb{F}(\mathfrak{m}) = (\pi; \mathsf{P}_{\mathfrak{m}}; \ldots)$ or $\mathbb{X} = \mathbb{F}(\mathfrak{m}, \xi, \vec{\nu}) = (\pi; \mathsf{M}_{\mathbb{M}(\vec{\nu})}^{\xi}\text{-}\mathsf{P}_{\mathfrak{m}}; \ldots)$, with $\mathfrak{m} = (\theta, m) > 0$:

Let us at first treat the latter case, i.e. $\mathbb{X} = \mathbb{F}(\mathfrak{m}, \xi, \vec{\nu})$ and the last inference is by a main reflection rule $({}_\tau M_{\mathbb{M}(\vec{\eta})}^{\zeta}\text{-}\Pi_{n \dot{+} 2}(\pi + \vartheta)\text{-Ref})$ of π with $((\vartheta, n), \zeta, \vec{\eta}) \in \mathrm{dom}(\mathbb{F})$ and principal formula $G := \exists \vec{z}^{\pi} \big({}_\tau^{\vec{s}} M_{\mathbb{M}(\vec{\eta})}^{\zeta}(\vec{z}) \wedge F(\vec{z}) \big)$. Let $\mathfrak{n} := (\vartheta, n)$. We define $\mathfrak{n}' := (\vartheta', n') := \max\{\mathfrak{m} + 1, \mathfrak{n} + 1\}$ and $(\zeta', \vec{\eta}') := \mathrm{pmax}\{(\xi, \vec{\nu}), (\zeta, \vec{\eta})\}$. It follows by means of Lemma 15.2.3 and since $\mathfrak{m}, \mathfrak{n} < \mathrm{rd}(\vec{R}_\pi) \in \mathrm{Lim}$ that $(\mathfrak{n}', \zeta', \vec{\eta}') \in \mathrm{dom}(\mathbb{F})$. Therefore we can define $\mathbb{Y} := \mathbb{F}(\mathfrak{n}', \zeta', \vec{\eta}')$. Then we have

$$\mathcal{C}_\gamma[\mathcal{A}] \left|\frac{\alpha_0}{\bar{\mu}}\right. \Gamma, F(\vec{s}), \tag{15.12}$$

for some $\alpha_0 < \alpha$ and $\Gamma, F(\vec{s}) \subseteq \Sigma_0(\pi + \vartheta'^*)$, where $\vec{s}$ denotes all terms occurring in F with stage greater than or equal to π. Thereby we are allowed to apply the subsidiary induction hypothesis to (15.12) and obtain

$$\mathcal{C}_{\hat{\alpha_0} \oplus \lambda}[\mathcal{A}, \lambda] \left|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0} \oplus \lambda}}{\cdot}\right. \Gamma^{\pi \mapsto \lambda}, F(\vec{s})^{\pi \mapsto \lambda} \text{ for all } \lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}. \tag{15.13}$$

In the sequel, we fix a $\kappa \in {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha}}$. It holds $\sigma < \Psi_{\mathbb{Y}}^{\hat{\alpha_0} \oplus \sigma} \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}$. Since $\gamma, \alpha_0, \mu, \sigma, \mathbb{Y} \in C(\hat{\alpha}, \kappa)$ and $\hat{\alpha_0} \oplus \sigma < \hat{\alpha}$ it follows $\Psi_{\mathbb{Y}}^{\hat{\alpha_0} \oplus \sigma} \in C(\hat{\alpha}, \kappa) \cap \pi = \kappa$. Therefore we have ${}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}} \cap \kappa \neq \emptyset$.

Due to Definition 12.2.4 and Lemma 15.2.3 it holds $\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}} \subseteq \mathfrak{M}_{\mathbb{M}(\vec{\eta}')}^{\zeta'} \subseteq \mathfrak{M}_{\mathbb{M}(\vec{\eta})}^{\zeta}$ and moreover $\tau \in \mathcal{C}_\gamma[\mathcal{A}] \cap \pi \subseteq C(\gamma + 1, \sigma + 1) \subseteq C(\hat{\alpha_0}, \lambda) \cap \pi = \lambda$ for $\lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}$. Thus we have

$$\Vdash {}_\tau^{\vec{s}}M_{\mathbb{M}(\vec{\eta})}^{\zeta}(\vec{s}^{\pi \mapsto \lambda}) \quad \text{for all } \lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}. \tag{15.14}$$

Thus we obtain from (15.13) by means of $(\bigwedge)$ and $(\bigvee)$

$$\mathcal{C}_{\hat{\alpha_0} \oplus \lambda}[\mathcal{A}, \lambda, \kappa] \left|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0} \oplus \lambda} + \omega}{\cdot}\right. \bigvee \Gamma^{\pi \mapsto \lambda}, \exists \vec{z}^{\kappa} \big({}_\tau^{\vec{s}}M_{\mathbb{M}(\vec{\eta})}^{\zeta}(\vec{z}) \wedge F(\vec{z}) \big) \\ \text{for all } \lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}} \cap \kappa. \tag{15.15}$$

Now we define $\bar{\Gamma} := \tilde{\Gamma}$ if $\mathfrak{m} = (\theta_0 + 1, 0)$ for some θ_0, and $\bar{\Gamma} := \Gamma$ otherwise. Let $(\overline{15.15})$ be (15.15) with Γ replaced by $\bar{\Gamma}$. Due to Lemma 15.1.3 the derivability of (15.15) implies $(\overline{15.15})$. Now let $\vec{c}$ be a list of all the terms occurring in $\bar{\Gamma}$ with stage greater than or equal to π and $\vec{a} \in (\mathcal{T}_\kappa^{\cdot})^p$, where p is the arity of $\vec{c}$. By Lemma 4.3.3 (viii) we get

$$\Vdash \vec{c}^{\pi \mapsto \lambda} \neq \vec{a}, \neg \bigvee \bar{\Gamma}(\vec{c})^{\pi \mapsto \lambda}, \bigvee \bar{\Gamma}(\vec{a}). \tag{15.16}$$

Applying (Cut) to $(\overline{15.15})$ and (15.16) we obtain

$$\mathcal{C}_{\hat{\alpha_0} \oplus \kappa}[\mathcal{A}, \lambda, \kappa, \vec{a}] \left|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0} \oplus |\vec{a}|} + \omega + 1}{\cdot}\right. \vec{c}^{\pi \mapsto \lambda} \neq \vec{a}, \bigvee \bar{\Gamma}(\vec{a}), G^{(\pi, \kappa)} \quad \text{for all } \lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}} \cap |\vec{a}| + 1.$$

Since $\neg {}_\sigma^{\vec{c}}M_{\mathbb{Y}}^{\hat{\alpha_0}}(\vec{a}) \cong \bigwedge(\vec{c}^{\pi \mapsto \lambda} \neq \vec{a})_{\lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}} \cap |\vec{a}| + 1}$ we obtain by a $(\bigwedge)$- and a $(\bigvee)$-inference

$$\mathcal{C}_{\hat{\alpha_0} \oplus \kappa}[\mathcal{A}, \kappa, \vec{a}] \left|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0} \oplus |\vec{a}|} + \omega + 3}{\cdot}\right. \neg {}_\sigma^{\vec{c}}M_{\mathbb{Y}}^{\hat{\alpha_0}}(\vec{a}) \vee \bigvee \bar{\Gamma}(\vec{a}), G^{(\pi, \kappa)},$$

for every $\vec{a} \in (\mathcal{T}_\kappa)^p$. By means of p ($\bigwedge$)-inferences, we arrive at

$$\mathcal{C}_{\hat{\alpha_0}\oplus\kappa}[\mathcal{A},\kappa] \overset{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\kappa}}{\underset{\cdot}{\Big|}} \forall\vec{x}^\kappa\big(\neg{}^{\vec{c}}_{\sigma}M_{\mathbb{Y}}^{\hat{\alpha_0}}(\vec{x}) \vee \bigvee\bar{\Gamma}(\vec{x})\big), G^{(\pi,\kappa)}. \tag{15.17}$$

Due to Lemma 15.2.1 ① there exists an $\mathfrak{S}$ with $\sup(\mathfrak{S}) = \mathfrak{m}$, such that for every $(\tau, s) \in \mathfrak{S}$ there exists the reflection rule (${}^{\vec{c}}_{\sigma}M_{\mathbb{Y}}^{\hat{\alpha_0}}$-$\Pi_{s\dot{+}2}(\kappa+\tau)$-Ref). If $\mathfrak{m} \notin \mathrm{Lim}$ we must have $\mathfrak{m} \in \mathfrak{S}$. If $\theta \in \mathrm{Lim}$ and $m = 0$ it holds that $\neg\bigvee\bar{\Gamma}^{\pi\mapsto\kappa}$ is a $\Delta_0(\kappa+\theta^*)$-sentence, i.e. there exists a $\tau < \theta$ such that $\neg\bigvee\bar{\Gamma}^{\pi\mapsto\kappa}$ is a $\Delta_0(\kappa+\tau^*)$-sentence. If $\mathfrak{m} = (\theta_0+1, 0)$ then $\neg\bigvee\Gamma^{\pi\mapsto\kappa}$ is a $\Delta_0(\kappa+(\theta_0+1)^*)$-sentence and hence $\neg\bigvee\bar{\Gamma}^{\pi\mapsto\kappa}$ is a $\Delta_0^1(\kappa+\theta_0^*)$-sentence, i.e. there exists a $(\theta_0, s) < \mathfrak{m}$ such that $\neg\bigvee\bar{\Gamma}^{\pi\mapsto\kappa}$ is a $\Pi_{s\dot{+}2}(\kappa+\theta^*)$-sentence. Thus in any case we can find a $(\tau, s) \in \mathfrak{S}$, such that $\neg\bigvee\bar{\Gamma}^{\pi\mapsto\kappa}$ is a $\Pi_{s\dot{+}2}(\kappa+\tau^*)$-sentence.

Moreover we have $\mathrm{k}(\exists\vec{x}^\kappa({}^{\vec{c}}_{\sigma}M_{\mathbb{Y}}^{\hat{\alpha_0}}(\vec{x}) \wedge \neg\bigvee\bar{\Gamma}(\vec{x})), \bigvee\bar{\Gamma}^{\pi\mapsto\kappa}) \subseteq \mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A},\kappa]$. Therefore we obtain by means of Lemma 4.3.3 ①, Lemma 4.3.6 and Lemma 15.1.3 plus an application of (${}^{\vec{c}}_{\sigma}M_{\mathbb{Y}}^{\hat{\alpha_0}}$-$\Pi_{s\dot{+}2}(\kappa+\tau)$-Ref)

$$\mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A},\kappa] \overset{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\kappa}}{\underset{\cdot}{\Big|}} \exists\vec{x}^\kappa\big({}^{\vec{c}}_{\sigma}M_{\mathbb{Y}}^{\hat{\alpha_0}}(\vec{x}) \wedge \neg\bigvee\bar{\Gamma}(\vec{x})\big), \bigvee\Gamma^{\pi\mapsto\kappa}. \tag{15.18}$$

Thereby we obtain the desired derivation by a (Cut) applied to (15.17) and (15.18) plus ($\vee$-Ex) and (Str).

If it holds $\mathbb{X} = \mathbb{F}(\mathfrak{m})$ then the proof is literally the same, with $\mathbb{Y} := \mathbb{F}(\mathfrak{n}')$ and ${}^{\vec{s}}_{\tau}M^{\zeta}_{\mathbb{M}(\vec{\eta})}$ replaced by ${}^{\vec{s}}M_\pi$.

Subcase 4.2, the reflection configuration $\mathbb{F}$ of $\mathbb{X}$ is a reflection configuration with constant reflection degree, i.e. it holds $\mathbb{X} = \mathbb{F} = (\pi; \mathsf{P}_{\mathfrak{m}+1}; \ldots)$ or $\mathbb{X} = \mathbb{F}(\xi, \vec{\nu}) = (\pi; \mathsf{M}^{\xi}_{\mathbb{M}(\vec{\nu})}\text{-}\mathsf{P}_{\mathfrak{m}+1}; \ldots)$ with $\mathfrak{m} \geq -1$:

Let us at first treat the latter case, i.e. $\mathbb{X} = \mathbb{F}(\xi, \vec{\nu})$ and the last inference is by a main reflection rule (${}_{\tau}M^{\zeta}_{\mathbb{M}(\vec{\eta})}$-$\Pi_{m\dot{+}2+1}(\pi+\theta)$-Ref) of π with $(\zeta, \vec{\eta}) \in \mathrm{dom}(\mathbb{F})$ and principal formula $G :\equiv \exists\vec{z}^\pi\big({}^{\vec{s}}_{\tau}M^{\zeta}_{\mathbb{M}(\vec{\eta})}(\vec{z}) \wedge F(\vec{z})\big)$. Let $\mathbb{Y} := \mathbb{F}(\zeta, \vec{\eta})$. Then we have

$$\mathcal{C}_\gamma[\mathcal{A}] \overset{\alpha_0}{\underset{\bar{\mu}}{|}} \Gamma, F(\vec{s}), \tag{15.19}$$

for some $\alpha_0 < \alpha$ and $F(\vec{s}) \equiv F(\vec{s}, F_1(\vec{s}), \ldots, F_q(\vec{s})) \in \Pi_{m\dot{+}2+1}(\pi+\theta^*)$, where $\vec{s}$ lists all the terms occurring in F with stage greater than or equal to π. Thus by an application of (E-$\forall$-Inv) we obtain for all $\vec{t} = (\vec{t_1}, \ldots, \vec{t_q}) \in (\mathcal{T}_{\pi+\theta})^{k_1} \times \ldots \times (\mathcal{T}_{\pi+\theta})^{k_q} =: (\mathcal{T}_{\pi+\theta})^{\vec{q}}$

$$\mathcal{C}_\gamma[\mathcal{A}, \vec{t}\,] \overset{\alpha_0}{\underset{\bar{\mu}}{|}} \Gamma, F(\vec{s}, F_1'(\vec{t_1}, \vec{s}), \ldots, F_q'(\vec{t_q}, \vec{s})). \tag{15.20}$$

Since $\mathcal{C}_\gamma[\mathcal{A}, \vec{t}\,] = \mathcal{C}_\gamma[\mathcal{A}, \vec{t}^{\,\pi\mapsto 0}]$ and $\mathbb{Y} \in \mathcal{C}_\gamma[\mathcal{A}]$ plus $\Gamma, F(\vec{s}, F_1'(\vec{t_1}, \vec{s}), \ldots, F_q'(\vec{t_q}, \vec{s})) \in \Sigma_{m\dot{+}2}(\pi+\theta^*)$ we may apply the subsidiary induction hypothesis to (15.20) and obtain for all $\vec{t} \in (\mathcal{T}_{\pi+\theta})^{\vec{q}}$

$$\mathcal{C}_\gamma[\mathcal{A}, \vec{t}^{\,\pi\mapsto 0}, \lambda] \overset{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\lambda}}{\underset{\cdot}{\Big|}} \Gamma^{\pi\mapsto\lambda}, F(\vec{s}, F_1'(\vec{t_1}, \vec{s}), \ldots, F_q'(\vec{t_q}, \vec{s}))^{\pi\mapsto\lambda}$$

$$\text{for all } \lambda \in {}_{\sigma_{\vec{\mathfrak{t}}_\circ}}\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}, \tag{15.21}$$

where $\sigma_{\vec{t}_0} := |\mathcal{A}, \vec{t}^{\pi\mapsto 0}|$. For $\lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}$ and $\vec{t} \in (\mathcal{T}_\lambda \cup (\mathcal{T}_{\pi+\theta} \backslash \mathcal{T}_\pi))^{\vec{q}}$ it holds $\lambda \in {}_{\sigma_{\vec{t}_0}}\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}$. Moreover we have $\mathcal{C}_\gamma[\mathcal{A}, \vec{t}^{\pi\mapsto 0}, \lambda] = \mathcal{C}_\gamma[\mathcal{A}, \vec{t}^{\pi\mapsto\lambda}, \lambda]$. Thus by use of (Str) and (E-$\forall$) applied to (15.21) we obtain for some $l < \omega$

$$\mathcal{C}_\gamma[\mathcal{A}, \lambda] \Big|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\lambda}+l}{\cdot} \Gamma^{\pi\mapsto\lambda}, F(\vec{s})^{\pi\mapsto\lambda} \quad \text{for all } \lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}. \tag{15.22}$$

Now fix a $\kappa \in {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha}}$. Analogue to case 4.1 we obtain from (15.22)

$$\mathcal{C}_\gamma[\mathcal{A}, \lambda] \Big|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\lambda}+\omega}{\cdot} \Gamma^{\pi\mapsto\lambda}, G^{(\pi,\kappa)} \quad \text{for all } \lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}. \tag{15.23}$$

If $\mathfrak{m} = -1$ it holds $\Gamma^{\pi\mapsto\lambda} \subseteq \Sigma_1(\lambda)$. We choose $\lambda = \Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\sigma} \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}} \cap \kappa$ and the claim follows from (15.23) by use of (E-Up-Per) plus (Inc) and (Str).

If $\mathfrak{m} > -1$ we proceed as in subcase 4.1 (cf. (15.15)–(15.17) and obtain

$$\mathcal{C}_{\hat{\alpha_0}\oplus\kappa}[\mathcal{A}, \kappa] \Big|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\kappa}}{\cdot} \forall\vec{x}^\kappa\big(\neg{}_\sigma^{\vec{c}}M_{\mathbb{Y}}^{\hat{\alpha_0}}(\vec{x}) \vee \bigvee\bar{\Gamma}(\vec{x})\big), G^{(\pi,\kappa)}, \tag{15.24}$$

where $\bar{\Gamma} := \tilde{\Gamma}$ if $\mathfrak{m} = (\theta_0 + 1, 0)$ and $\bar{\Gamma} := \Gamma$ otherwise.

By the same argumentation as in subcase 4.1 it follows the existence of a set $\mathfrak{S}$, such that $\sup(\mathfrak{S}) = \mathfrak{m}$ and there is a $(\sigma, s) \in \mathfrak{S}$ such that $\neg\bigvee\bar{\Gamma}^{\pi\mapsto\kappa}$ is a $\Pi_{s\dot{+}2}(\kappa+\sigma^*)$-sentence and the reflection rule $({}_\sigma^{\vec{c}}M_{\mathbb{Y}}^{\hat{\alpha_0}}\text{-}\Pi_{s\dot{+}2}(\kappa+\sigma)\text{-Ref})$ exists. Therefore we obtain

$$\mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A}, \kappa] \Big|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\kappa}}{\cdot} \exists\vec{x}^\kappa\big({}_\sigma^{\vec{c}}M_{\mathbb{Y}}^{\hat{\alpha_0}}(\vec{x}) \wedge \neg\bigvee\bar{\Gamma}(\vec{x})\big), \bigvee\Gamma^{\pi\mapsto\kappa} \tag{15.25}$$

and the desired derivation follows by a (Cut) applied to (15.24) and (15.25) plus ($\vee$-Ex) and (Str).

If it holds $\mathbb{X} = \mathbb{F}$ then the proof is literally the same, with $\mathbb{Y} := \mathbb{X}$ and ${}_\tau^{\vec{s}}M_{\mathbb{M}(\vec{\eta})}^\zeta$ replaced by ${}^{\vec{s}}M_\pi$.

Case 5, the last inference is by a subsidiary reflection rule of π of the form $({}_\tau^{\vec{s}}M_{\mathbb{K}(\vec{\eta})}^\zeta\text{-}\Pi_{k\dot{+}2}(\pi+\vartheta)\text{-Ref})$ with $\mathsf{M}_{\mathbb{K}}^{<\zeta+1}\text{-}\vartheta\text{-}\mathsf{P}_k \preceq (\vec{R}_\pi^{\mathrm{cl}'})_{(\vartheta,k)}$ and principal formula $G := \exists\vec{z}^\pi\big({}_\tau^{\vec{s}}M_{\mathbb{K}(\vec{\eta})}^\zeta(\vec{z}) \wedge F(\vec{z})\big)$: Let $\vec{s}$ denote all the terms occurring in F with stage greater than or equal to π. Then we have

$$\mathcal{C}_\gamma[\mathcal{A}] \Big|\frac{\alpha_0}{\mu} \Gamma, F(\vec{s}) \tag{15.26}$$

for some $\alpha_0 < \alpha$ and $\zeta \in \mathcal{C}_\gamma[\mathcal{A}] \cap C(\pi)$ plus $\vec{\eta} \in \mathcal{C}_\gamma[\mathcal{A}] \cap \mathrm{dom}(\mathbb{K})_{C(\pi)}$.

Subcase 5.1, $\mathbb{X} = \mathbb{F}(\mathfrak{m}, \vec{\nu})$ is a reflection configuration with variable reflection degree: Then we have $\mathfrak{k} := (\vartheta, k) \in \mathrm{Rdh}(\mathbb{F})$ or $\mathfrak{m} > \mathfrak{k}$. Let $\mathfrak{m}' = (\theta', m') := \max\{(\vartheta, k)+1, \mathfrak{m}\}$. Then it holds $(\mathfrak{m}', \vec{\nu}) \in \mathrm{dom}(\mathbb{F})$ and we set $\mathbb{Y} := \mathbb{F}(\mathfrak{m}', \vec{\nu})$. In the following we fix a $\kappa \in {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha}}$. Just like in subcase 4.1 it holds ${}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}} \cap \kappa \neq \emptyset$. Since $\Gamma, F \subseteq \Sigma_{m'\dot{+}2}(\pi + \theta'^*)$ we obtain by means of the subsidiary induction hypothesis

$$\mathcal{C}_{\hat{\alpha}\oplus\lambda}[\mathcal{A}, \lambda] \Big|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\lambda}}{\cdot} \Gamma^{\pi\mapsto\lambda}, F(\vec{s})^{\pi\mapsto\lambda} \quad \text{for all } \lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}} \cap \kappa. \tag{15.27}$$

Let $\vec{\eta} = (\eta_1, \ldots, \eta_h)$. Since we have $\forall_1^h i\,(\zeta, \eta_i \leq \gamma)$ by Lemma 12.3.4, $\zeta, \vec{\eta} \in C(\pi)$ and $\zeta, \vec{\eta} \in C(\gamma + 1, \sigma + 1) \subseteq C(\gamma + 1, \lambda)$ plus $C(\gamma + 1, \lambda) \cap \pi = \lambda$ it follows by Lemma 5.2.2 $\zeta, \vec{\eta} \in C(\lambda)$ for $\lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}$. Moreover we have for such a λ that $\tau \in \mathcal{C}_\gamma[\mathcal{A}] \cap \pi \subseteq C(\gamma+1, \sigma+1) \cap \pi \subseteq C(\hat{\alpha}, \lambda) \cap \pi = \lambda$. Thus due to Lemma 15.2.1 ② for every $\lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}}$ there exists the reflection rule $({}^{\vec{s}}_{\tau}M^{\zeta}_{\mathbb{K}(\vec{\eta})}\text{-}\Pi_{k\dot{+}2}(\lambda + \vartheta)\text{-}\mathsf{Ref})$. By use of these reflection rules plus $(\mathsf{Up\text{-}Per})$ we obtain from 15.27

$$\mathcal{C}_{\hat{\alpha}\oplus\lambda}[\mathcal{A}, \lambda, \kappa] \Big|\frac{\Psi_{\mathbb{Y}}^{\hat{\alpha_0}\oplus\lambda}+1}{\cdot}\; \Gamma^{\pi\mapsto\lambda}, G^{\pi\mapsto\kappa} \quad \text{for all } \lambda \in {}_\sigma\mathfrak{M}_{\mathbb{Y}}^{\hat{\alpha_0}} \cap \kappa. \tag{15.28}$$

Proceeding now as in subcase 4.1 (cf. equations (15.15)–(15.18)) the claim follows.

Subcase 5.2, it holds $\mathbb{F}$ is a reflection configuration with constant reflection degree: Then we either have $\mathfrak{k} = \mathfrak{m} + 1 \notin \mathrm{Rdh}(\mathbb{F})$ or $\mathfrak{m} \geq \mathfrak{k}$. If $\mathfrak{m} > \mathfrak{k}$ we are allowed to apply the subsidiary induction hypothesis directly to (15.26). If $\mathfrak{m} \leq \mathfrak{k}$ we proceed just like in subcase 4.2, i.e. we apply $(\mathsf{E\text{-}\forall\text{-}Inv})$ to F, make use of the subsidiary induction hypothesis and employ $(\mathsf{E\text{-}\forall})$. In either case we obtain

$$\mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A}, \kappa] \Big|\frac{\Psi_{\mathbb{X}}^{\hat{\alpha_0}\oplus\kappa}}{\cdot}\; \Gamma^{\pi\mapsto\kappa}, F(\vec{s})^{\pi\mapsto\kappa} \quad \text{for all } \kappa \in {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha}}, \tag{15.29}$$

since ${}_\sigma\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha}} \subseteq {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha_0}}$.

As in subcase 5.1 it follows by means of Theorem 15.2.1 ② the existence of the reflection rule $({}^{\vec{s}}_{\tau}M^{\zeta}_{\mathbb{K}(\vec{\eta})}\text{-}\Pi_{k\dot{+}2}(\kappa + \vartheta)\text{-}\mathsf{Ref})$. Applying this rule to (15.29) and taking into account Lemma 5.2.3 ① plus (Str) we obtain the desired derivation.

Case 6, the last inference is a $({}^{\vec{s}}_{\tau}M^{\zeta}_{\mathbb{G}(\vec{\eta})}\text{-}\Pi_{g\dot{+}2}(\pi_0+\theta_0)\text{-}\mathsf{Ref})$ inference with $\tau < \pi_0 < \pi$ and principal formula $\exists \vec{z}^{\pi_0}\big({}^{\vec{s}}_{\tau}M^{\zeta}_{\mathbb{G}(\vec{\eta})}(\vec{z}) \wedge F(\vec{z})\big)$: Thus we have

$$\mathcal{C}_\gamma[\mathcal{A}] \Big|\frac{\alpha_0}{\bar{\mu}}\; \Gamma, F(\vec{s}) \tag{15.30}$$

for some $\alpha_0 < \alpha$. Since $\pi_0 < \pi$ we also have $\pi_0 + \theta_0 < \pi$ and thereby it holds $\Gamma, F \in \Sigma_{m\dot{+}2}(\pi + \theta^*)$. Therefore we may apply the subsidiary induction hypothesis to (15.30) and obtain

$$\mathcal{C}_{\hat{\alpha}\oplus\kappa}[\mathcal{A}, \kappa] \Big|\frac{\Psi_{\mathbb{X}}^{\hat{\alpha_0}\oplus\kappa}}{\cdot}\; \Gamma^{\pi\mapsto\kappa}, F(\vec{s}) \quad \text{for all } \kappa \in {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha_0}}. \tag{15.31}$$

Thus the claim follows from (15.31) by means of $({}^{\vec{s}}_{\tau}M^{\zeta}_{\mathbb{G}(\vec{\eta})}\text{-}\Pi_{g\dot{+}2}(\pi_0+\theta_0)\text{-}\mathsf{Ref})$ and (Str) plus taking into account ${}_\sigma\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha}} \subseteq {}_\sigma\mathfrak{M}_{\mathbb{X}}^{\hat{\alpha_0}}$. ☐

Theorem 15.2.5. *Let $\alpha_0 := 1$ and $\alpha_{n+1} := \Upsilon^{\alpha_n}$ The property of being an admissible set above ω can be expressed by a $\mathcal{L}(\epsilon)$-Δ_0-formula $\mathrm{Ad}(x)$. If F is a Σ_1-sentence and*

$$\mathsf{KP} + (\mathrm{Stab}) \vdash \forall x\,\big(\mathrm{Ad}(x) \to F^x\big),$$

then there is a $k < \omega$ such that

$$\mathcal{C}_{\alpha_k} \Big|\frac{\Psi_{\mathbb{X}}^{\alpha_k \oplus \Psi_{\mathbb{X}}^{\alpha_k}}}{\cdot}\; F^{\Psi_{\mathbb{X}}^{\alpha_k}},$$

where $\mathbb{X} = (\omega^+; \mathsf{P}_0; \epsilon; \epsilon; 0)$*. Thus at* $\mathbb{L}_{\Psi_{\mathbb{X}}^{\varepsilon\Upsilon+1}}$ *all* $\Sigma_1^{\omega_1^{ck}}$*-sentences of* Stability *are true, i.e.* $|\mathsf{Stability}|_{\Sigma_1^{\omega_1^{ck}}} \leq \Psi_{\mathbb{X}}^{\varepsilon\Upsilon+1}$.

Proof. Analogue to the proof of Theorem 5.2.6 by use of Theorem 15.2.4 and Theorem 14.3.1. □

Part IV.

The Provable Recursive Functions of Stability

16. A Characterization of the Provable Recursive Functions of Stability

In this short final part we want to apply the methods developed in Part II to the theory Stability.

We extend all definitions and modifications of Part II to the theory Stability. Moreover we define

$$(\theta, m) \in \mathcal{F}_\gamma[\mathcal{A}] \ :\Leftrightarrow \ \theta, m \in \mathcal{F}_\gamma[\mathcal{A}].$$

It follows by simultaneous induction on the build-up of terms and sentences that it holds $|t^{\pi\mapsto\kappa}|_{\mathrm{N}} \leq |t|_{\mathrm{N}} + \kappa$ for every $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-term t and $|F^{\pi\mapsto\kappa}|_{\mathrm{N}} \leq |F^{\pi\mapsto\kappa}|_{\mathrm{N}} + \kappa$ for every $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-sentence F if $\kappa \in \mathrm{Lim}$.

A run through the cases of Definition 15.1.2 yields that for every $\mathcal{L}^{\circledast}_{RS(\Upsilon)}$-sentence we have $|F|_{\mathrm{N}} \geq |\tilde{F}|_{\mathrm{N}}$ and thereby we have a $\mathcal{N}$-version of Lemma 15.1.3, too.

Theorem 16.0.6 (Reflection Elimination)**.** *Let $\mathbb{X} = (\pi; \ldots; \delta)$ be a reflection instance with $\mathrm{rdh}(\mathbb{X}) = (\theta, m)$ and $\gamma, \mathbb{X}, \mu \in \mathcal{N}_\gamma[\mathcal{A}]$, where $\Upsilon, \delta \leq \gamma + 1$ and $\theta < \sigma := |\mathcal{A}| < \pi \leq \mu \in \mathrm{Card}$. Let $\Gamma \subseteq \Sigma_{m+2}(\pi + \theta^*)$ and $\hat{\alpha} := \gamma \oplus \omega^{\alpha \oplus \mu}$. Then*

$$\mathcal{N}_\gamma[\mathcal{A}] \,\Big|\frac{\alpha}{\mu}\, \Gamma \quad \Rightarrow \quad \mathcal{N}_{\hat{\alpha}\oplus\kappa}[\mathcal{A}, \kappa] \,\Bigg|\frac{\Psi_{\mathbb{X}}^{\hat{\alpha}\oplus\kappa}}{\cdot}\, \Gamma^{\pi\mapsto\kappa} \quad \textit{for all } \kappa \in {}_\sigma\mathfrak{M}^{\hat{\alpha}}_{\mathbb{X}}.$$

Proof. The proof follows by an application of the modifications given in the proof of Theorem 9.2.2 to the proof of Theorem 15.2.4.

Even the application of the rule (${}^{\vec{c}}_{\sigma}M^{\hat{\alpha}_0}_{\mathbb{Y}}$-$\Pi_{s+2}(\kappa+\tau)$-Ref) to obtain equation (15.18) on page 138 is not an obstacle, since the parameter (τ, m) does not have to be controlled. ☐

By use of the results of Section 10 and Section 8.2 plus a refined embedding of the axiom (Stab), which is a Sisyphean challenge, we obtain:

Theorem 16.0.7. *Let $\ell(u)$ be a $\Delta_1^{\mathsf{KP}\omega}$-formula, which defines $\mathbb{L}_\omega$. Suppose $\mathbb{X} := (\omega^+; \mathsf{P}_0; \epsilon; \epsilon; 0)$ and $F \equiv \forall x\, \exists y\, G(x, y)$ is an elementary $\mathcal{L}_\in$-Π^0_2-sentence, such that*

$$\mathsf{KP} + (\mathrm{Stab}) \vdash \forall z \big(\ell(z) \rightarrow F^z\big).$$

Then there exists an $\alpha < \Psi_{\mathbb{X}}^{\varepsilon_{\Upsilon+1}}$, satisfying

$$\mathbb{L}_\omega \models \forall x\, \exists y^{f_\alpha(|x|_{\mathbb{L}})}\, G(x, y).$$

Thereby we obtain:

Theorem 16.0.8. *The provable recursive functions of the theory* Stability *are contained in the class* $\mathfrak{F}$*, where* $\mathfrak{F}$ *is the smallest class of number theoretic functions, which contains* $S, C^n_k, P^n_k, (f_\alpha)_{\alpha \in \Psi^{\varepsilon(\Upsilon+1)}_{\mathbb{X}}}$ *and is closed under substitution and primitive recursion.*

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Index of Notation Part I

Ξ denotes a Π^2_0-indescribable cardinal. V_α denotes the αth stage in the von Neumann hierarchy of sets and $\mathbb{L}_\alpha$ denotes the αth stage in the constructible hierarchy. $\mathbb{A}(m) := (\Xi; \mathsf{P}_m; \epsilon; \epsilon; \omega)$ and $\mathbb{U}, \mathbb{V}, \mathbb{W}, \mathbb{X}, \mathbb{Y}, \mathbb{Z}$ denote variables for reflection instances and 0-ary reflection configurations and $\mathbb{F}, \mathbb{E}, \mathbb{G}, \mathbb{H}, \mathbb{K}, \mathbb{M}, \mathbb{R}, \mathbb{S}$ denote variables for reflection configurations. Small Greek letters denote ordinals and small Roman letters denote finite ordinals.

Index of Notation Part II

Items not listed here can be found in the Index of Notation Part I.

Index of Notation Part III

Items not listed here can be found in the Index of Notation Part I. Small fraktur letters denote elements of $\Upsilon \times \omega \cup \{(0,-1)\}$.