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Extensions in the cohomology of Hilbert modular varieties

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Introduction

Let S be a Hilbert modular variety (uncompactified) defined over $\mathbb{Q}$ attached to a totally real field F. We assume S is nonsingular. The ℓ-adic cohomology of S carries a nontrivial weight filtration, and one may consider the possible extensions of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules thereby arising.

If $F = \mathbb{Q}$, then S is an open modular curve. The only possible cohomology where a nontrivial extension could arise is degree 1, and by the Manin–Drinfeld principle, the H^1 is in fact split.

In dimension greater than 1, the cohomology in each degree has at most two nonzero steps in the weight filtration. The Manin–Drinfeld principle still shows that cusp forms cannot give rise to nontrivial extensions in the cohomology of S, but there is the possibility that nontrivial extensions could arise between the boundary cohomology and the part of the cohomology coming from 1-dimensional automorphic representations. Caspar [2] investigated this in the case of Hilbert modular surfaces. He computed the extension classes that arise for the H^2, and showed that they are nontrivial, giving an explicit description via Kummer theory.

In this paper we consider the case of arbitrary F. We show (Theorems 2.1 and 2.3) that nontrivial extensions can occur only in degree $2r-2$, and that in this case the extensions which arise are nontrivial, and can again be described explicitly using Kummer theory.

One motivation for this work is the "plectic conjecture" of Nekovář and the second author [8]. A consequence of the results proved here is that the Galois action on $H^*(S)$ (for S now a $GL_2(F)$-Shimura variety) extends to the "plectic Galois group"; this completes the proof of [8, Proposition 6.6], as explained in

the last section. We also indicate how the same method gives a proof of the analogous statement [9, (3.3.11)] in Hodge theory.

After completing this paper we learnt of independent work by Silliman [11], proving results equivalent to Theorem 2.3 and its Hodge-theoretic analogue.

1. Hilbert modular varieties

Throught the paper, F will denote a fixed totally real number field of degree $r > 1$, $\mathfrak{o}_F$ its ring of integers, and $\Sigma = \mathrm{Hom}_{\mathbb{Q}-\mathrm{alg}}(F, \overline{\mathbb{Q}})$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. (We do not fix a preferred embedding of F into $\overline{\mathbb{Q}}$). For a field k, we write Γ_k for its absolute Galois group (for some algebraic closure, which will be clear from the context).

For any k-scheme X (where $k \subset \overline{\mathbb{Q}}$), we will generally write $H^*(X, \mathbb{Q}_\ell) = H^*_{\text{ét}}(X \otimes_k \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ and $H^*(X, \mathbb{Q}) = H^*(X(\mathbb{C}), \mathbb{Q})$, and similar for compact supports, or sheaves. (We make an exception to this convention in Proposition 2.5, where it would cause confusion.)

We let $G \subset R_{F/\mathbb{Q}}GL_2$ be the algebraic subgroup whose group of $\mathbb{Q}$-points is $G(\mathbb{Q}) = \{g \in GL_2(F) \mid \det g \in \mathbb{Q}^*\}$. Until the last section, S will be a Hilbert–Blumenthal modular variety over $\mathbb{Q}$ associated to some open subgroup $K \subset G(\mathbb{A}_{\mathbb{Q}}^\infty)$. We assume that K is sufficently small to ensure that S is smooth.

The minimal compactification. This is a compactification

$$S \xrightarrow{j} S^* \xleftarrow{i} S^\infty,$$

where S^* is normal and proper, and S^∞ is zero-dimensional. We have the long exact sequence of cohomology:

$$H^n_c(S, \mathbb{Q}_\ell) \to H^n(S, \mathbb{Q}_\ell) \to H^n(S^\infty, i^*Rj_*\mathbb{Q}_\ell) \to \cdots . \tag{1}$$

Write $H^n_!(S, \mathbb{Q}_\ell) = \mathrm{im}(H^n_c(S, \mathbb{Q}_\ell) \to H^n(S, \mathbb{Q}_\ell))$ for the interior cohomology, $H^n_\partial(S, \mathbb{Q}_\ell) = H^n(S^\infty, i^*Rj_*\mathbb{Q}_\ell)$ for the boundary cohomology. The exact sequence is auto-dual, via Poincaré duality between $H(S)$ and $H_c(S)$, and the duality on boundary cohomology $H^n_\partial(S, \mathbb{Q}_\ell)^\vee \simeq H^{2r-1-n}_\partial(S, \mathbb{Q}_\ell)(r)$.

The boundary cohomology is independent of the choice of compactification; both it and the exact sequence (1) can be computed using singular cohomology of the Borel–Serre compactification, as was first done by Harder [3], who showed that for $n = 1$, $2r - 1$ one has $H^n_!(S, \mathbb{Q}) = 0$, and the sequence splits into short exact sequences:

- for $2 \le n < r$,

$$0 \to H^{n-1}_\partial(S, \mathbb{Q}) \to H^n_c(S, \mathbb{Q}) \to H^n_!(S, \mathbb{Q}) = H^n(S, \mathbb{Q}) \to 0, \tag{2}$$

- for $r < n \le 2r - 2$,

$$0 \to H^n_c(S, \mathbb{Q}) = H^n_!(S, \mathbb{Q}) \to H^n(S, \mathbb{Q}) \to H^n_\partial(S, \mathbb{Q}) \to 0,$$

- in middle degree,

$$0 \longrightarrow H^{r-1}_{\partial}(S,\mathbb{Q}) \longrightarrow H^r_c(S,\mathbb{Q}) \longrightarrow H^r(S,\mathbb{Q}) \longrightarrow H^r_{\partial}(S,\mathbb{Q}) \longrightarrow 0.$$
$$\searrow \quad \nearrow$$
$$H^r_!(S,\mathbb{Q}_\ell)$$

By the comparison isomorphism, the same holds for ℓ-adic cohomology.

It is also shown in [3] that for any $z \in S^\infty(\mathbb{C})$, the boundary cohomology at z satisfies

$$H^1((Rj_*\mathbb{Q})_z) = \mathrm{Hom}(\mathfrak{o}_F^*,\mathbb{Q}),$$
$$H^n((Rj_*\mathbb{Q})_z) = \bigwedge^n H^1((Rj_*\mathbb{Q})_z) \quad \text{for } 1 \le n \le r-1.$$

The toroidal compactification. This is a smooth, projective compactification $S \hookrightarrow \tilde{S}$ whose boundary $\tilde{S}^\infty$ is a divisor with strict normal crossings. It depends on a choice of admissible cone decomposition of the cone of totally positive elements $(F \otimes_{\mathbb{Q}} \mathbb{R})_+ \subset F \otimes_{\mathbb{Q}} \mathbb{R}$ (see [10, §4] or [4, §4.1.4]). The boundary component $\tilde{S}^\infty_y$ over $y \in S^\infty$ is the quotient Z_y/Δ_y, where Z_y is a reduced scheme locally of finite type over $k(y)$, whose irreducible components $Z_{y,\sigma}$ are smooth toric varieties, and $\Delta_y \subset \mathfrak{o}_F^*$ is a torsion-free subgroup of finite index.

The varieties $Z_{y,\sigma}$ have vanishing H^1. So by Meyer–Vietoris $H^1(Z_y)$ equals the H^1 of the nerve of the cover $\{Z_{y,\sigma}\}$. This nerve, being the simplicial complex associated to the cone decomposition of $(F \otimes_{\mathbb{Q}} \mathbb{R})_+$, is contractible, so Z_y has vanishing H^1. It follows that $H^1(\tilde{S}^\infty,\mathbb{Q}_\ell) = H^0(S^\infty,\mathbb{Q}_\ell) \otimes_{\mathbb{Q}} \mathrm{Hom}(\mathfrak{o}_F^*,\mathbb{Q})$ as $\Gamma_{\mathbb{Q}}$-modules, and that the natural homomorphism $H^1(\tilde{S}^\infty,\mathbb{Q}_\ell) \to H^1_{\partial}(S,\mathbb{Q}_\ell)$ is an isomorphism, giving isomorphisms

$$H^n_{\partial}(S,\mathbb{Q}_\ell) \simeq H^0(S^\infty,\mathbb{Q}_\ell) \otimes_{\mathbb{Q}} \bigwedge^n \mathrm{Hom}(\mathfrak{o}_F^*,\mathbb{Q})$$

for $1 \le n \le r-1$.

One also has $H^2_!(S,\mathbb{Q}_\ell) = \mathrm{im}(H^2(\tilde{S},\mathbb{Q}_\ell) \to H^2(S,\mathbb{Q}_\ell)) = W_2H^2(S,\mathbb{Q}_\ell)$ and so the exact sequence (2) for $n=2$ may be rewritten as

$$0 \to H^1(\tilde{S}^\infty,\mathbb{Q}_\ell) \to H^2_c(S,\mathbb{Q}_\ell) \to \mathrm{im}(H^2(\tilde{S},\mathbb{Q}_\ell) \to H^2(S,\mathbb{Q}_\ell)) \to 0. \tag{3}$$

Define, for any $y \in S^\infty(\overline{\mathbb{Q}})$, $\mathrm{Pic}^0\,\tilde{S}^\infty_y = \ker(\mathrm{Pic}\,\tilde{S}^\infty_y \to H^2(\tilde{S}^\infty_y,\mathbb{Q}_\ell(1))$.

Lemma 1.1. *One has*

$$\mathrm{Pic}^0\,\tilde{S}^\infty_y = \ker(\mathrm{Pic}\,\tilde{S}^\infty_y \to \mathrm{Pic}\,Z_y) \simeq \mathrm{Hom}(\Delta_y,\overline{\mathbb{Q}}^*).$$

Proof. By the above discussion of the toroidal boundary, we have $H^0(Z_y,\mathbb{Q}_\ell) = \mathbb{Q}_\ell$ and $H^1(Z_y,\mathbb{Q}_\ell) = 0 = \mathrm{Pic}^0\,Z_y$. So from the Cartan spectral sequences for $Z_y \to \tilde{S}^\infty_y$ with coefficients in $\mathbb{G}_{\mathrm{m}}$, we obtain the exact rows of the commutative

diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(\Delta_y, \overline{\mathbb{Q}}^*) & \longrightarrow & \operatorname{Pic} \tilde{S}_y^\infty & \longrightarrow & (\operatorname{Pic} Z_y)^{\Delta_y} \\ & & \vdots & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^2(\Delta_y, \mathbb{Q}_\ell)(1) & \longrightarrow & H^2(\tilde{S}_y^\infty, \mathbb{Q}_\ell(1)) & \longrightarrow & H^2(Z_y, \mathbb{Q}_\ell(1))^{\Delta_y}, \end{array}$$

in which the right-hand vertical arrow is injective. If $x \in H^1(\Delta_y, \overline{\mathbb{Q}}^*)$, then its image in $H^2(\Delta_y, \mathbb{Q}_\ell)(1)$ is fixed by an open subgroup of $\Gamma_{\mathbb{Q}}$, hence the left-hand vertical map is zero. The result then follows by the snake lemma. □

The line bundles $\mathcal{L}_\tau$. For each $\tau\colon F \to L \subset \mathbb{C}$, there is an invertible sheaf $\mathcal{L}_\tau$ on $S \otimes L$, with the property that the sections of $\bigotimes \mathcal{L}_\tau^{k_\tau}$ are the modular forms of weight (k_τ) (see [10, 6.9 (b)]). If $L = \mathbb{C}$, this is the usual line bundle on $S(\mathbb{C})$ associated to the factor of automorphy $(\gamma, z) \mapsto (c_\tau z_\tau + d_\tau)$ (see [10, 6.15]), and extends to the toroidal compactification in a unique way such that the pullback to each Z_y is trivial (see [12, §II.7], which is for the case $r = 2$, but the general case is the same). By its very definition and the previous lemma, the restriction of $\mathcal{L}_\tau$ to $\tilde{S}_y^\infty$, $y \in S^\infty(\overline{\mathbb{Q}})$, lies in $\operatorname{Pic}^0 \tilde{S}_y^\infty$ and can be identified (up to a sign independent of τ) with the homomorphism $\tau \in H^1(\Delta_y, \overline{\mathbb{Q}}^*) \subset \operatorname{Hom}(F^*, \overline{\mathbb{Q}}^*)$.

By definition, the Galois action on the line bundles is given by $\sigma_* \mathcal{L}_\tau = \mathcal{L}_{\sigma\tau}$ for $\sigma \in \Gamma_{\mathbb{Q}}$.

Write $\eta_\tau \in H^2_!(S, \mathbb{Q}_\ell(1))$ for the cohomology class of $\mathcal{L}_\tau$. The classes η_τ are linearly independent and therefore generate a subspace isomorphic to the permutation representation $\mathbb{Q}_\ell[\Sigma]$.

2. The extension classes

For $I \subset \Sigma$, with $0 < \#I = m < r$, let $\eta_I = \bigwedge_{\tau \in I} \eta_\tau \in H^2_!(S, \mathbb{Q}_\ell(1))$, and let

$$H^{2m}_A(S, \mathbb{Q}_\ell) = \sum_{\#I = m} H^0(S, \mathbb{Q}_\ell) \cup \mathbb{Q}_\ell(-m)\eta_I \subset H^{2m}_!(S, \mathbb{Q}_\ell).$$

From [3] one has the following description of the interior cohomology:

- For $n \neq r$ odd or $n = 2r$, $H^n_!(S, \mathbb{Q}_\ell) = 0$.
- For $0 < n = 2m < 2r$, $n \neq r$, $H^{2m}_!(S, \mathbb{Q}_\ell) = H^{2m}_A(S, \mathbb{Q}_\ell)$.
- If $r = 2m$ is even, then $H^r_!(S, \mathbb{Q}_\ell) = H^r_A(S, \mathbb{Q}_\ell) \oplus H^r_{\mathrm{cusp}}(S, \mathbb{Q}_\ell)$, a direct sum of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules stable under the Hecke algebra.

Theorem 2.1. (i) *For $2 < n \leq r$, there is a unique splitting of* $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-modules: $H^n_c(S, \mathbb{Q}_\ell) = H^n_!(S, \mathbb{Q}_\ell) \oplus H^{n-1}_\partial(S, \mathbb{Q}_\ell)$.*

(ii) *For $r \leq n < 2r - 2$, there is a unique splitting of* $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-modules: $H^n(S, \mathbb{Q}_\ell) = H^n_!(S, \mathbb{Q}_\ell) \oplus H^n_\partial(S, \mathbb{Q}_\ell)$.*

Proof. As (i) and (ii) are equivalent by Poincaré duality, it is enough to prove (i). Because $H^{n-1}_\partial(S, \mathbb{Q}_\ell)$ is pure of weight 0, there is at most one splitting.

If $n < r$ is odd, there is nothing to prove as $H^n_! = 0$ for $n < r$.

If $n = r$ then by Manin–Drinfeld principle, the extension splits over H^r_{cusp}. So it is enough in every case to split the extension over $H^n_A \subset H^n_!$. Therefore (i) will follow from the next proposition. □

Proposition 2.2. *Let $1 < m \leq r/2$. Let $\cup^m H^2_c(S, \mathbb{Q}_\ell) \subset H^{2m}_c(S, \mathbb{Q}_\ell)$ be the image of $\bigotimes^m H^2_c(S, \mathbb{Q}_\ell)$ under the cup product. Then the composite*

$$\cup^m H^2_c(S, \mathbb{Q}_\ell) \hookrightarrow H^{2m}_c(S, \mathbb{Q}_\ell) \twoheadrightarrow H^{2m}_!(S, \mathbb{Q}_\ell)$$

is an isomorphism if $m < r/2$; for $r = 2m$ even, it gives an isomorphism

$$\cup^{r/2} H^2_c(S, \mathbb{Q}_\ell) \xrightarrow{\sim} H^r_A(S, \mathbb{Q}_\ell) \subset H^r_!(S, \mathbb{Q}_\ell).$$

Proof. It is enough to check that the composite

$$\bigotimes^m H^2_c(S, \mathbb{Q}_\ell) \xrightarrow{\cup} H^{2m}_c(S, \mathbb{Q}_\ell) \to H^{2m}(S, \mathbb{Q}_\ell)$$

has image H^{2m}_A, and is zero on elements $\otimes x_i$, where some x_i is in the image of the boundary cohomology. The first assertion is clear as the cup product map $\cup\colon \bigotimes^m H^2_A(S, \mathbb{Q}_\ell) \twoheadrightarrow H^{2m}_A(S, \mathbb{Q}_\ell)$ is surjective. As for the second, we have a commutative diagram:

$$\begin{array}{ccccc} H^1_\partial(S, \mathbb{Q}_\ell) \otimes H^2_c(S, \mathbb{Q}_\ell) & \xrightarrow{\partial^1 \otimes id} & H^2_c(S, \mathbb{Q}_\ell) \otimes H^2_c(S, \mathbb{Q}_\ell) & \xrightarrow{\cup} & H^4_c(S, \mathbb{Q}_\ell) \\ & \searrow^{0} & \downarrow & \nearrow_{\cup} & \\ & & H^2(S, \mathbb{Q}_\ell) \otimes H^2_c(S, \mathbb{Q}_\ell) & & \end{array}$$

and therefore the composite of the horizontal arrows $H^1_\partial \otimes H^2_c \to H^4_c$ is zero. Therefore, if $\eta^c_\tau \in H^2_c(\mathbb{Q}_\ell(1))$ are any classes lifting $\eta_\tau \in H^2_A(S, \mathbb{Q}_\ell(1))$, then

$$\cup^m H^2_c(S, \mathbb{Q}_\ell(1)) = \cup^m \langle \{\eta^c_\tau\} \rangle = \bigoplus_{\#I=m} (\textstyle\bigwedge_{\tau \in I} \eta^c_\tau)\mathbb{Q}_\ell \xrightarrow{\sim} H^{2m}_A(S, \mathbb{Q}_\ell(m)). \quad \square$$

For the second result we need some more notation. Consider the Kummer homomorphism $\kappa_F\colon F^* \to H^1(\Gamma_F, \mathbb{Q}_\ell(1))$. Composing with the isomorphism given by Shapiro's lemma:

$$H^1(\Gamma_F, \mathbb{Q}_\ell(1)) \xrightarrow{\sim} H^1(\Gamma_\mathbb{Q}, \mathbb{Q}^\Sigma_\ell(1))$$

(which does not depend on a choice of embedding $F \subset \overline{\mathbb{Q}}$), we obtain a homomorphism

$$\kappa'_F\colon F^* \to H^1(\Gamma_\mathbb{Q}, \mathbb{Q}^\Sigma_\ell(1))$$

inducing an isomorphism between the completed tensor product $F^* \hat{\otimes} \mathbb{Q}_\ell$ and the right-hand side.

The morphism of 0-dimensional schemes $\varepsilon\colon S^\infty \to \pi_0(S)$ gives the $\Gamma_\mathbb{Q}$-equivariant map

$$\varepsilon^*\colon H^0(S, \mathbb{Q}_\ell) \to H^0(S^\infty, \mathbb{Q}_\ell).$$

Theorem 2.3. *Assume $r > 2$. Consider the extension*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_\partial(S,\mathbb{Q}_\ell) & \longrightarrow & H^2_c(S,\mathbb{Q}_\ell) & \longrightarrow & H^2_!(S,\mathbb{Q}_\ell) & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & \mathrm{Hom}(\mathfrak{o}_F^*, H^0(S^\infty,\mathbb{Q}_\ell)) & & & & H^0(S,\mathbb{Q}_\ell)\otimes\mathbb{Q}_\ell[\Sigma](-1). & & \end{array}$$

Its class is the image of $\varepsilon^*\otimes\kappa'_F$ under the map

$$\begin{array}{c} \mathrm{Hom}_{\Gamma_\mathbb{Q}}(H^0(S,\mathbb{Q}_\ell), H^0(S^\infty,\mathbb{Q}_\ell))\otimes \mathrm{Hom}(\mathfrak{o}_F^*, H^1(\Gamma_\mathbb{Q}, \mathbb{Q}_\ell^\Sigma(1))) \\ \downarrow \\ \mathrm{Hom}(\mathfrak{o}_F^*, H^1(\Gamma_\mathbb{Q}, \mathrm{Hom}(H^0(S,\mathbb{Q}_\ell), H^0(S^\infty,\mathbb{Q}_\ell)\otimes\mathbb{Q}_\ell^\Sigma(1))) \\ \| \\ \mathrm{Ext}^1_{\Gamma_\mathbb{Q}}(H^2_!(S,\mathbb{Q}_\ell), H^1_\partial(S,\mathbb{Q}_\ell))\,. \end{array}$$

Remarks 2.4. (i) By duality, the same class classifies the extension in cohomology without support:

$$0 \to H^{2r-2}_!(S,\mathbb{Q}_\ell) \to H^{2r-2}(S,\mathbb{Q}_\ell) \to H^{2r-2}_\partial(S,\mathbb{Q}_\ell) \to 0.$$

(ii) The analogous result for $r = 2$ is proved in [2] by a different method; the same proof as given below also works in this case with minor modification.

Proof. The extension class is determined by its restriction to any open subgroup of $\Gamma_\mathbb{Q}$. Let $k \subset \overline{\mathbb{Q}}$ be a number field containing a Galois closure of F, for which Γ_k acts trivially on $\pi_0(S\otimes\overline{\mathbb{Q}})$ and $S^\infty(\overline{\mathbb{Q}})$. For each connected component $S' \subset S\otimes_\mathbb{Q} k$ and for each $\tau\in\Sigma = \mathrm{Hom}_{\mathbb{Q}-\mathrm{alg}}(F,k)$, consider the pullback $E(S',\eta)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_\partial(S',\mathbb{Q}_\ell) & \longrightarrow & H^2_c(S',\mathbb{Q}_\ell) & \longrightarrow & H^2_!(S',\mathbb{Q}_\ell) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(\mathfrak{o}_F^*, H^0(S'^\infty,\mathbb{Q}_\ell)) & \longrightarrow & E(S',\eta) & \longrightarrow & \mathbb{Q}_\ell(-1)\eta_\tau & \longrightarrow & 0. \end{array}$$

It is then enough to check that the extension class of each $E(S',\eta)$ in

$$\mathrm{Ext}^1_{\Gamma_k}(\mathbb{Q}_\ell(-1), \mathrm{Hom}(\mathfrak{o}_F^*, H^0(S'^\infty,\mathbb{Q}_\ell))) = \mathrm{Hom}\big(\mathfrak{o}_F^*, H^1(\Gamma_k,\mathbb{Q}_\ell(1))^{S'^\infty}\big)$$

is (up to sign independent of S' and η) given by the composite homomorphism

$$\mathfrak{o}_F^* \xrightarrow{\tau_*\circ\kappa_F} H^1(\Gamma_k,\mathbb{Q}_\ell(1)) \xhookrightarrow{\mathrm{diag}} H^1(\Gamma_k,\mathbb{Q}_\ell(1))^{S'^\infty}.$$

For this, we use the alternative description (3) of the extension, which then puts us in the following general situation. Let k be any field of characteristic different from ℓ, X/k smooth and proper, $i\colon Y \hookrightarrow X$ the inclusion of a reduced divisor, and $U = X\setminus Y$. To avoid ambiguity, we temporarily change notation in order to distinguish between the ℓ-adic cohomology of $\overline{X} = X\otimes_k \bar{k}$ and that of X, and likewise for Y.

Let $\mathcal{L} \in \operatorname{Pic} X$ such that $0 = cl_{\overline{Y}}(i^*\mathcal{L}) \in H^2(\overline{Y}, \mathbb{Q}_\ell(1))$. We then obtain by pullback an extension $E_{\mathcal{L}}$ of Γ_k-modules:

$$\begin{array}{ccccccccc}
0 \to \operatorname{coker}\big(H^1(\overline{X},\mathbb{Q}_\ell) \to H^1(\overline{Y},\mathbb{Q}_\ell)\big) & \to & H^2_c(\overline{U},\mathbb{Q}_\ell) & \to & H^2(\overline{X},\mathbb{Q}_\ell) & \xrightarrow{i^*} & H^2(\overline{Y},\mathbb{Q}_\ell) \\
\| & & \vdots & & \uparrow{\scriptstyle cl_{\overline{X}}(\mathcal{L})} & \nearrow{\scriptstyle cl_{\overline{Y}}(\mathcal{L})=0} & \\
0 \to \operatorname{coker}\big(H^1(\overline{X},\mathbb{Q}_\ell) \to H^1(\overline{Y},\mathbb{Q}_\ell)\big) & \longrightarrow & E_{\mathcal{L}} & \longrightarrow & \mathbb{Q}_\ell(-1) & \longrightarrow & 0,
\end{array}$$

and thus an extension class $e_{\mathcal{L}} \in H^1(\Gamma_k, \operatorname{coker}(H^1(\overline{X},\mathbb{Q}_\ell) \to H^1(\overline{Y},\mathbb{Q}_\ell))(1))$.

Proposition 2.5. *$e_{\mathcal{L}}$ equals the image of $i^*\mathcal{L}$ under the composite map*

$$\begin{aligned}\operatorname{Pic}^0 Y &\xrightarrow{AJ_Y} H^1(\Gamma_k, H^1(\overline{Y},\mathbb{Q}_\ell)(1)) \\ &\to H^1(\Gamma_k, \operatorname{coker}(H^1(\overline{X},\mathbb{Q}_\ell) \to H^1(\overline{Y},\mathbb{Q}_\ell))(1)),\end{aligned}$$

where AJ_Y is the ℓ-adic Abel-Jacobi map.

Recall that AJ_Y is defined to be the composite of the following two maps:

- the Chern class
$$\operatorname{Pic}^0 Y \to \operatorname{Fil}^1 H^2(Y,\mathbb{Q}_\ell(1)) = \ker\big(H^2(Y,\mathbb{Q}_\ell(1)) \to H^2(\overline{Y},\mathbb{Q}_\ell(1))^{\Gamma_k}\big),$$
- the map
$$\operatorname{Fil}^1 H^2(Y,\mathbb{Q}_\ell(1)) \to H^1(\Gamma_k, H^1(\overline{Y},\mathbb{Q}_\ell)(1))$$
coming from the Hochschild–Serre spectral sequence in continuous ℓ-adic cohomology.

Proof. Apply [5, 9.5] to the triangle

$$R\Gamma(\overline{X},\mathbb{Q}_\ell) \xrightarrow{i^*} R\Gamma(\overline{Y},\mathbb{Q}_\ell) \to R\Gamma_c(\overline{U},\mathbb{Q}_\ell)[1] \to R\Gamma(\overline{X},\mathbb{Q}_\ell)[1]$$

to get the commutative pentagon:

$$\begin{array}{ccc}
H^0(\Gamma_k, \ker H^2(i^*)(1)) & \longrightarrow & H^1(\Gamma_k, \operatorname{coker} H^1(i^*)(1)) \\
\uparrow & & \uparrow \\
\ker\big(H^2(X,\mathbb{Q}_\ell(1)) \to H^2(\overline{Y},\mathbb{Q}_\ell(1))^{\Gamma_k}\big) & \longrightarrow & H^1(\Gamma_k, H^1(\overline{Y},\mathbb{Q}_\ell)(1)) \\
& {\scriptstyle i^*}\searrow \quad \nearrow & \\
& \ker\big(H^2(Y,\mathbb{Q}_\ell(1)) \to H^2(\overline{Y},\mathbb{Q}_\ell(1))^{\Gamma_k}\big), &
\end{array}$$

under which the various cohomology classes of $\mathcal{L}$ are mapped as follows:

$$\begin{array}{ccc}
cl_{\overline{X}}(\mathcal{L}) & \longmapsto & e_{\mathcal{L}} \\
\uparrow & & \\
cl_X(\mathcal{L}) & \longmapsto & AJ_Y(i^*\mathcal{L}).
\end{array}$$

The commutativity then gives the desired result. □

To apply this in our situation, take k as above, $X = \tilde{S}' \otimes_{\mathbb{Q}} k$, $Y = \tilde{S}'^{\infty} \otimes_{\mathbb{Q}} k$, and $\mathcal{L} = \mathcal{L}_\tau$. We have seen that for each $y \in S^\infty(k)$, the restriction $\mathcal{L}_\tau|_{\tilde{S}_y^\infty} \in \mathrm{Pic}^0(\tilde{S}_y^\infty) = \mathrm{Hom}(\Delta_y, k^*)$ (using the isomorphism of Lemma 1.1) is the map τ (up to a sign independent of y and τ). The result then follows from the commutative diagram

$$\begin{array}{ccc}
\mathrm{Pic}^0(\tilde{S}_y^\infty) & \xrightarrow{AJ} & \mathrm{Fil}^1 H^2(\tilde{S}_y^\infty, \mathbb{Q}_\ell(1)) \subset H^2(\tilde{S}_y^\infty, \mathbb{Q}_\ell(1)) \\
\uparrow \wr & & \uparrow \wr \\
H^1(\Delta_y, H^0(Z_y, \mathbb{G}_{\mathrm{m}}) & & H^1(\Delta_y, H^1(Z_y, \mathbb{Q}_\ell(1)) \\
\| & & \| \\
H^1(\Delta_y, k^*) & \xrightarrow{H^1(\kappa_k)} & H^1(\Delta_y, H^1(\Gamma_k, \mathbb{Q}_\ell(1)),
\end{array}$$

where the right-hand vertical isomorphism comes from the Cartan spectral sequence for $Z_y \to \tilde{S}_y^\infty$. □

3. Further remarks

We may perform the same computations in Hodge theory. The proof of the splitting in Theorem 2.1 carries through without change. For the proof of Theorem 2.3, one should replace absolute ℓ-adic cohomology with absolute Hodge cohomology [1]. Then the Kummer homomorphism κ'_F is replaced by the archimedean regulator map

$$\begin{aligned}
\kappa'_{F,\mathcal{H}} \colon F^* &\to H^1_{\mathcal{H}}(\mathrm{Spec}\,\mathbb{C}, \mathbb{R}(1)^\Sigma) = \mathbb{R}^\Sigma, \\
x &\mapsto (\log|\tau(x)|)_{\tau\in\Sigma}.
\end{aligned}$$

This gives a proof of [9, Theorem 3.3.11]. An alternative approach is to use explicit formulae for Eisenstein cohomology, as done in the case $r = 2$ in [2]; details will appear elsewhere.

Suppose now that $S = S_K$ is a Shimura variety for the full group GL_2/F, where $K \subset GL_2(\mathbb{A}_F^\infty)$ is a sufficiently small open compact subgroup. Theorems 2.1 and 2.3 are equally valid in this setting. We can now complete the proof of the relevant part of [8, Proposition 6.6]:

Corollary 3.1. *There exists an action of Γ_F^{pl} on $H^*(S, \overline{\mathbb{Q}}_\ell)$, extending the action of $\Gamma_{\mathbb{Q}}$.*

Proof. We recall some definitions and facts from [8] concerning the "plectic Galois group", which is the group

$$\Gamma_F^{pl} = \mathrm{Aut}(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}/F).$$

It canonically contains $\Gamma_{\mathbb{Q}}$ as a subgroup. After fixing embeddings $\bar{\tau} \colon \overline{F} \to \overline{\mathbb{Q}}$ extending $\tau \in \Sigma$, one obtains an isomorphism with the wreath product

$$\Gamma_F^{pl} \xrightarrow{\sim} \Gamma_F^\Sigma \ltimes \mathrm{Sym}(\Sigma).$$

The homomorphism $\Gamma_F^{\Sigma} \ltimes \mathrm{Sym}(\Sigma) \to \Gamma_F^{\mathrm{ab}}$, which is trivial on the symmetric group and on each copy of Γ_F is the obvious quotient, defines a homomorphism $\Gamma_F^{pl} \to G_F^{\mathrm{ab}}$ which does not depend on choices, and whose restriction to $\Gamma_{\mathbb{Q}}$ is the transfer homomorphism $\mathrm{Ver}\colon \Gamma_{\mathbb{Q}} \to \Gamma_F^{\mathrm{ab}}$.

The action on $\Gamma_{\mathbb{Q}}$ on both $\pi_0(S \otimes \overline{\mathbb{Q}})$ and $S^{\infty}(\overline{\mathbb{Q}})$ factors[1] through Ver, and so extends to Γ_F^{pl}. The subspace of $H_!^2(S, \mathbb{Q}_\ell)$ spanned by the classes η_τ is the induced representation $\mathrm{Ind}_{\Gamma_F}^{\Gamma_{\mathbb{Q}}} \mathbb{Q}_\ell(-1) = \mathbb{Q}_\ell(-1)^{\Sigma}$, and more generally the subspace of $H_!^{2m}(S, \mathbb{Q}_\ell)$ spanned by the products η_I is the degree m part of the tensor induction $(\mathbb{Q}_\ell(0) \oplus \mathbb{Q}_\ell(-1))^{\otimes \Sigma}$, with $\mathbb{Q}_\ell(-i)$ in degree i, so extends (canonically) to a representation of Γ_F^{pl}. It follows from all of this that there is a canonical action of the plectic Galois group on $H_\partial^*(S, \mathbb{Q}_\ell)$ and $H_A^*(S, \mathbb{Q}_\ell)$.

The main result of [7] shows that $H^r_{\mathrm{cusp}}(S, \overline{\mathbb{Q}}_\ell)$ is a sum of tensor inductions of 2-dimensional representations of Γ_F, and therefore carries a (noncanonical) action of the plectic Galois group extending that of $\Gamma_{\mathbb{Q}}$. To complete the proof, in view of Theorems 2.1 and 2.3 it is therefore enough to show that the action of $\Gamma_{\mathbb{Q}}$ on $H_!^2(S, \mathbb{Q}_\ell)$ can be extended to the plectic Galois group. Since ε^* is Γ_F^{pl}-equivariant, this follows from the next lemma:

Lemma 3.2. *The restriction homomorphism*

$$H^1(\Gamma_F^{pl}, \mathbb{Q}_\ell(1)^{\Sigma}) \to H^1(\Gamma_{\mathbb{Q}}, \mathbb{Q}_\ell(1)^{\Sigma}) = H^1(\Gamma_F, \mathbb{Q}_\ell(1))$$

is an isomorphism.

This is a consequence of the Künneth formula:

$$\begin{aligned} H^1(\Gamma_F^{pl}, \mathbb{Q}_\ell(1)^{\Sigma}) &\simeq H^1(\Gamma_F^{\Sigma}, \mathbb{Q}_\ell(1)^{\Sigma})^{\mathrm{Sym}(\Sigma)} \\ &= \left(H^1(\Gamma_F, \mathbb{Q}_\ell(1))^{\Sigma}\right)^{\mathrm{Sym}(\Sigma)} \\ &= H^1(\Gamma_F, \mathbb{Q}_\ell(1)). \end{aligned}$$

□

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[1] It is here that we use the fact that S is a GL_2-Shimura variety. For the varieties considered earlier, this is false; see [6, (0.3)].

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