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An analogue of Raynaud's theorem: weak formal schemes and dagger spaces

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Abstract. We study the relationship between the categories of weak formal schemes and dagger spaces. We introduce the notion of weak formal blowups of weak formal schemes and show that they correspond to rational subdomains of the associated dagger spaces via the generic fiber functor. In analogy with Raynaud's theorem in formal and rigid geometry, we establish an equivalence of categories between the localized category of quasi-paracompact admissible weak formal schemes by weak formal blowups, and the category of quasi-paracompact quasi-separated dagger spaces.

1. Introduction

Let k be a field of characteristic $p > 0$ and K a completely discretely valued field of characteristic 0 with ring of integers R and residue field k. The objects that we study in this paper first arose in the search for a good p-adic cohomology theory. In [15] and [14], Monsky and Washnitzer defined a p-adic cohomology theory for affine and smooth varieties over k using the notion of weak completions of algebras. They showed that a lifting to a weakly complete algebra always exists for smooth algebras of finite type over k and hence established a good cohomology theory for affine and smooth varieties.

In Berthelot's rigid cohomology (which is defined for arbitrary varieties), Raynaud's generic fiber functor which associates a rigid analytic space to a formal scheme, plays a central role. To an affine formal scheme $P = \operatorname{Spf} A$ (with A an R-algebra of topologically finite type), the functor associates a rigid analytic space $P_K = \operatorname{Sp}(A \otimes K)$. The points of the rigid analytic space P_K are associated with integral formal subschemes of P which are finite flat over R. In fact, Raynaud proved a much stronger result ([16]), showing an equivalence of categories between quasi-paracompact (see 3.3 for the definition) admissible formal R-schemes localized by the class of admissible formal blowups and quasi-paracompact quasi-separated rigid analytic K-spaces.

Dagger spaces defined by Grosse-Klönne in [9], can be thought of as the overconvergent analogues of rigid analytic spaces. Grosse-Klönne gives another interpretation of rigid cohomology in terms of the de Rham cohomology of dagger spaces ([10]). This is done via an equivalence of categories between partially proper rigid analytic spaces and partially proper dagger spaces ([9, Thm. 2.27]).

Meredith introduced weak formal schemes in [13] whose underlying topological spaces are defined in the same way as formal schemes but whose structure sheaves are defined using weak completions of algebras.

The main idea of this paper is to establish an analogue of Raynaud's theory in the setting of weak formal schemes and dagger spaces as suggested in [9]. Analogous to the notion of formal blowups, we define weak formal blowups as weak completions of scheme theoretic blowups and establish some results about them. Our main result is:

Theorem 1.1. *There is an equivalence of categories between*

- *the category of quasi-paracompact admissible weak formal R-schemes, localized by the class S of weak formal blowups, and*
- *the category of quasi-separated quasi-paracompact K-dagger spaces.*

It may then be possible to redefine rigid cohomology by considering weak formal liftings instead of formal liftings. Of course, it would be necessary to prove an analogue of the fibration theorem for weak formal schemes and establish other such results in this setting.

Note: We use the usual multi-index notation for power series:

$$f = \sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n} X_1^{i_1} \ldots X_n^{i_n}$$

is often shortened to $\sum_I a_I X^I$ and $|I|$ is defined to be $i_1 + \ldots + i_n$.

2. Preliminaries

In this section we briefly review the notions of weak completions and weak formal schemes. We include here some definitions and technical results which we need later.

Let R be a noetherian ring and $\mathfrak{m} \subset R$ an arbitrary ideal of R.

Definition 2.1. Let A be an R-algebra and $\hat{A}$ denote its $\mathfrak{m}$-adic completion. The weak completion of A, denoted $A^\dagger$, is an R-subalgebra of $\hat{A}$ consisting of power series $z = \sum a_I x^I$ where $I = (i_1, \ldots, i_n)$, with coefficients in R and $x_1, \ldots, x_n \in A$ such that for some constant c and for all tuples $(i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$, the following condition (referred as MW condition) holds

$$\text{(MW)} \qquad c(\operatorname{ord}_{\mathfrak{m}}(a_{i_1 \ldots i_n}) + 1) \geq i_1 + \ldots + i_n.$$

An R-algebra A is said to be weakly complete if it is $\mathfrak{m}$-adically separated and if $A \to A^\dagger$ is a bijection.

Let S be a subset of a weakly complete algebra A. Its weak completion $S^\dagger$ is defined to be the subalgebra of A consisting of power series $z =$

$\sum a_{i_1\dots i_n} x_1^{i_1}\dots x_n^{i_n}$ with coefficients in R and $x_1,\dots,x_n \in S$, satisfying the MW condition. A weakly complete R-algebra A is called weakly complete finitely generated *(wcfg)* if $A = S^\dagger$ for some finite subset $S \subset A$. Elements of S are called weak generators of A.

Morphisms of wcfg R-algebras are defined as morphisms of R-algebras in the usual sense. Any such morphism of R-algebras is continuous with respect to the $\mathfrak{m}$-adic topology.

2.2. *Gauss norm on wcfg algebras:* We want to give an explicit description for weak completions of polynomial rings over wcfg algebras. For this purpose we make use of the Gauss norm on wcfg algebras introduced (in the dagger algebra context) in [5, 4.43].

Let $z = \sum_I a_I X^I$ be an element in $R[X]^\dagger$ where $X = X_1,\dots,X_m$ is a finite set of variables. For $\epsilon > 0$, define

$$\tilde{\gamma}_\epsilon(z) = \inf_{|I|}\{\mathrm{ord}_{\mathfrak{m}} a_I - \epsilon|I|\}.$$

The MW condition on z is equivalent to the existence of some $\epsilon > 0$ for which $\tilde{\gamma}_\epsilon(z) > -\infty$. For a fixed $\epsilon > 0$, the set of elements z in $R[X]^\dagger$ which satisfy $\tilde{\gamma}_\epsilon(z) > -\infty$ form an R-subalgebra of $R[X]^\dagger$ which we denote by $R[X]^\dagger_\epsilon$. Thus $\tilde{\gamma}_\epsilon$ defines a norm on $R[X]^\dagger_\epsilon$. For any wcfg algebra A, there exists a surjection $R[X_1,\dots,X_n]^\dagger \to A$ for some finite set of variables $X_1,\dots,X_n$ ([15, Thm. 2.2]). Denote by A_ϵ the image of $R[X_1,\dots,X_n]^\dagger_\epsilon$ under such a surjection and by γ_ϵ the quotient seminorm induced on A_ϵ by $\tilde{\gamma}_\epsilon$. Note that γ_ϵ are pseudo-valuations in the sense of [6, Def. 1.4] and hence give rise to a family of seminorms on A.

Let A be a wcfg algebra and let $A[\zeta]^\dagger$ be the weak completion of the polynomial algebra $A[\zeta]$ in n-variables for some $n \in \mathbb{N}$. Then, $A[\zeta]^\dagger$ consists of all power series $z \in A[\![\zeta]\!]$ where $z = \sum a_\nu \zeta^\nu$ such that there exists an $\epsilon > 0$ with $a_\nu \in A_\epsilon$ for all $\nu \in \mathbb{Z}^n_{\geq 0}$ and the following holds

$$\inf_{|\nu|}\{\gamma_\epsilon(a_\nu) - \epsilon|\nu|\} > -\infty.$$

Let A be a wcfg R-algebra, $f \in A$ and ζ a variable. Then we claim $A[f^{-1}]^\dagger = A[\zeta]^\dagger/(1 - f\zeta)$. First note that the canonical surjection $\phi : A[\zeta] \to A[f^{-1}]$ mapping ζ to f^{-1} can be extended to a morphism of wcfg algebras $\phi^\dagger : A[\zeta]^\dagger \to A[f^{-1}]^\dagger$ ([15, Thm. 1.5]). Since $\phi^\dagger(A[\zeta]^\dagger)$ is a weakly complete subalgebra of $A[f^{-1}]^\dagger$ (Corollary of [15, Thm. 2.1]) containing the weak generators of A and f^{-1}, it must be equal to $A[f^{-1}]^\dagger$. On the other hand, taking π-adic completions, the surjection $\hat{\phi} : A\langle\zeta\rangle \to A\langle f^{-1}\rangle$ has kernel $(1 - f\zeta)A\langle\zeta\rangle$. Therefore, $\ker\phi^\dagger = (1 - f\zeta)A[\zeta]^\dagger$ and the claim holds.

2.3. *Weak formal schemes:* With R as before, let A be a wcfg algebra and $\bar{A} = A/\mathfrak{m}A$.

Let M be an A-module. Let $\mathfrak{X} = \operatorname{Spec} \bar{A}$ be endowed with the Zariski topology. M induces a functor $\Gamma(\cdot\,, \tilde{M})$ on the principal open subsets $U = \mathfrak{X}_{\bar{f}}$ of $\mathfrak{X}$ defined by $\Gamma(U, \tilde{M}) = M \otimes_A {A_f}^\dagger$, where $f \in A$ is a preimage of $\bar{f} \in \bar{A}$. If $\mathfrak{X}_{\bar{f}} \supset \mathfrak{X}_{\bar{g}}$ for some $\bar{f}, \bar{g} \in \bar{A}$, there is a canonical A-homomorphism $\Gamma(\mathfrak{X}_{\bar{f}}, \tilde{M}) \to \Gamma(\mathfrak{X}_{\bar{g}}, \tilde{M})$. This defines a presheaf on principal open subsets of $\mathfrak{X}$, which is a sheaf—see [13, Thm. 8–14] for the proof for the case when M is finite and [12, II.3] for when M is an arbitrary A-module, A is an R-algebra and R is a complete discrete valuation ring. The functor $M \to \Gamma(U, \tilde{M})$ is exact, since for any wcfg algebra A and $f \in A$, $A_f^\dagger$ is flat over A ([13, 2.5]).

Definition 2.4. An affine weak formal R-scheme is a locally ringed space isomorphic to $(\operatorname{Spec} \bar{A}, \tilde{A})$ for some wcfg algebra A and will be denoted by $\operatorname{Spwf} A$. A weak formal R-scheme is a locally ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ such that every point of $\mathfrak{X}$ has a neighborhood isomorphic to an affine weak formal scheme.

Morphisms of weak formal R-schemes will mean morphisms of locally ringed spaces. Note that any weak formal scheme is automatically a topologically ringed space in the sense of [11, 0.4.1.1], as any affine subset derives this structure from the $\mathfrak{m}$-adic topology on the corresponding wcfg algebra.

As any wcfg algebra is noetherian ([7]), weak formal schemes are locally noetherian and hence quasi-separated.

Note: Henceforth, we work over a discrete valuation ring R with field of fractions K. We fix a uniformizer $\pi \in R - \{0\}$ such that the topology on R coincides with the π-adic topology and denote by k the residue field $R/(\pi)$. It will be assumed that R is complete and separated with respect to the π-adic topology.

2.5. *Quasi-coherent and coherent sheaves on weak formal schemes:* Let $\mathfrak{X}$ be a weak formal scheme and $\mathfrak{F}$ a quasi-coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$-modules (as defined for locally ringed spaces—see [11, 0.5.1.3]).

Then as in the scheme theoretic situation, it is easy to show that there exists an affine open covering of $\mathfrak{X}$ by subsets $U_i = \operatorname{Spwf} A_i$, such that for each i, there exists an A_i-module M_i with $\mathfrak{F}|_{U_i} \simeq \tilde{M}_i$. Note that in the situation where $\mathfrak{X} = \operatorname{Spwf} A$ is an affine weak formal scheme, it may not necessarily be true that $\mathfrak{F}$ is the sheafification of some A-module.

Coherent sheaves on an affine weak formal scheme $\mathfrak{X} = \operatorname{Spwf} A$ are in fact obtained as sheafifications of finite A-modules ([13, Thm. 3.3]). More precisely, the functor $M \to \tilde{M}$ is an equivalence of categories between the category of finite A-modules and the category of coherent $\mathcal{O}_{\mathfrak{X}}$-modules. Note also that the structure sheaf $\mathcal{O}_{\mathfrak{X}}$ of any weak formal scheme $\mathfrak{X}$ is a coherent sheaf (using the fact that the functor $M \mapsto \tilde{M}$ is exact and since weak formal schemes are locally noetherian).

2.6. *Weak completion of schemes:* In [13, §4], Meredith defines the weak completion of a scheme X of finite type over R, by setting $X^\dagger = \{x \in X \mid$

$\mathcal{O}_{X,x} \neq \pi\mathcal{O}_{X,x}\}$ along with the structure sheaf $\mathcal{O}_X^\dagger$ defined on affine open subsets $U \subset X$ by $\Gamma(U, \mathcal{O}_{X^\dagger}) = \Gamma(U, \mathcal{O}_X)^\dagger$. (Note that this indeed defines a sheaf on $X^\dagger$, as in 2.3 if X is affine and by a well-known extension property of sheaves for general X, see [11, 0.3.2].) For instance, when X is affine of the form Spec A, where A is an R-algebra of finite type, we obtain $X^\dagger = \operatorname{Spwf} A^\dagger$. Now let A be a wcfg algebra and B an A-algebra of finite type. Then using the definitions in [13, §4] we can define weak completions of affine schemes of the type Spec B as follows. Let $A[T_1, \dots, T_r] \twoheadrightarrow B$ be a presentation of B with kernel I, where $T_1, \dots, T_r$ are a finite set of variables. Then using the quotient seminorm of the Gauss norm on $A[T_1, \dots, T_r]^\dagger$ described in 2.2, it is clear that the weak completion $B^\dagger$ of B is isomorphic to $A[T_1, \dots, T_r]^\dagger/I$. Hence, $B^\dagger$ being the quotient of a wcfg algebra is also a wcfg algebra. Define the weak completion of Spec B to be Spwf $B^\dagger$. This is an affine weak formal scheme as defined in [13, §2]. Note that if A is an R-algebra of finite type, then the weak completions of Spec A and Spec $A^\dagger$ coincide.

We now define weak completions of arbitrary schemes of finite type over A where A is a wcfg algebra. We first state an easy lemma about glueing sheaves which we will use repeatedly in the construction of weak formal blowups.

Lemma 2.7. *Let X be a topological space with a sheaf $\mathcal{G}$ on it. Let $(V_j)_{j\in J}$ be an open covering of X along with a sheaf $\mathcal{G}_j$ defined on each open set V_j such that $\mathcal{G}_j$ is a subsheaf of $\mathcal{G}|_{V_j}$ and satisfying the condition $\mathcal{G}_j|_{V_j\cap V_{j'}} = \mathcal{G}_{j'}|_{V_j\cap V_{j'}}$ for all couples $j, j' \in J$. Consider a presheaf $\mathcal{F}$ defined as follows: For any open set $U \subset X$, let*

$$\mathcal{F}(U) = \{s \in \mathcal{G}(U) \mid s|_{V_j\cap U} \in \mathcal{G}_j(V_j \cap U) \text{ for all } j \in J\}$$

with morphisms given by restrictions of the morphisms of $\mathcal{G}$. Then, the presheaf $\mathcal{F}$ is a sheaf on X. □

Now suppose Y is a scheme of finite type over a wcfg algebra A. Let $(Y_i)_{i\in I}$ be a finite covering of Y by affine schemes $Y_i = \operatorname{Spec} A_i$, where A_i is of finite type over A. Define the weak completion $Y^\dagger$ of Y as follows. Denote by $\hat{Y}$ the formal completion of Y with respect to the ideal $\pi\mathcal{O}_Y$ and similarly let $\hat{Y}_i = \operatorname{Spf} \hat{A}_i$. Let the underlying topological space of $Y^\dagger$ be $\operatorname{Supp}(\mathcal{O}_Y/\pi\mathcal{O}_Y)$, which is the same as the topological space underlying $\hat{Y}$. Define a presheaf $\mathcal{O}_{Y^\dagger} \subset \mathcal{O}_{\hat{Y}}$ by setting for any arbitrary open subset $W \subset \hat{Y}$:

$$\mathcal{O}_{Y^\dagger}(W) = \{s \in \mathcal{O}_{\hat{Y}}(W) \mid s|_{\hat{Y}_i\cap W} \in \mathcal{O}_{Y_i^\dagger}(\hat{Y}_i \cap W)\}.$$

Using Lemma 2.7, $\mathcal{O}_{Y^\dagger}$ is a sheaf equipped with which $Y^\dagger$ is a weak formal scheme.

2.8. *Admissible weak formal schemes:* We call a wcfg algebra admissible if it has no π-torsion. We say that an affine weak formal scheme Spwf A is admissible if A is an admissible wcfg algebra. We first show that admissibility is a local condition.

Lemma 2.9. *Let $\phi : A \to B$ be a morphism of wcfg algebras and suppose B is flat as an A-module. Then B is faithfully flat over A if and only if $B/\pi B$ is faithfully flat over $A/\pi A$.*

Proof. One direction of the assertion follows from [4, Prop. 3.3.5]. Now, suppose $B/\pi B$ is a faithfully flat $A/\pi A$-module and N is a finite A-module such that $B \otimes_A N = 0$. This implies that $B/\pi B \otimes_{A/\pi A} N/\pi N = 0$ and so $N/\pi N = 0$ or $N = \pi N$. Using [15, Thm. 1.6], πA is contained in the Jacobson radical of A and hence, $N = 0$. □

Lemma 2.10. *Let A be a wcfg algebra. If $f_0, \ldots, f_r$ are elements in A generating the unit ideal in $A/\pi A$, then they generate the unit ideal in A. Hence, if $\mathfrak{X} = \mathrm{Spwf}\, A$ is an affine weak formal scheme with $f_0, \ldots, f_r \in A$, then the basic open subsets $(\mathrm{Spwf}\, A[f_i^{-1}]^\dagger)_i$ cover $\mathfrak{X}$ if and only if the elements $f_0, \ldots, f_r$ generate the unit ideal in A.*

Proof. Let $\bar{a}_i \in A/\pi A$ such that $\sum_{i=0}^{r} \bar{a}_i f_i = 1$. Let $a_i \in A$ be in the preimage of $\bar{a}_i$. Then, $\sum \bar{a}_i f_i = 1$ implies that $u = \sum a_i f_i = 1 - \pi c$ for some $c \in A$. This is a unit since πA is contained in the Jacobson radical of A ([15, Thm. 1.6]). Setting $a_i' = u^{-1} a_i$, we see that $\sum a_i' f_i = 1$. □

Proposition 2.11. *Let $\mathfrak{X} = \mathrm{Spwf}\, A$ be an affine weak formal scheme and $(\mathrm{Spwf}\, B_i)_{i \in J}$ be an affine open covering. A is an admissible wcfg algebra if and only if B_i is an admissible wcfg algebra for each $i \in J$.*

Proof. Using Lemma 2.10 we can assume that $B_i = A[f_i^{-1}]^\dagger$ where $(f_i)_{i \in J}$ generate the unit ideal in A and also that i varies over a finite index $1 \leq i \leq r$. If A has no π-torsion, the map $A \to A[\pi^{-1}]$ is injective. Since $A[f_i^{-1}]^\dagger$ is flat over A ([13, Cor. 1.4]), tensoring the above map with it, yields an injective map $A[f_i^{-1}]^\dagger \to (A[f_i^{-1}]^\dagger)[\pi^{-1}]$. Therefore, $A[f_i^{-1}]^\dagger$ is admissible.

Before proving the other direction, note that $A \to \prod_{i=1}^r A[f_i^{-1}]^\dagger$ is faithfully flat using Lemma 2.9 and the corresponding fact for ordinary localization. Combining this with the fact that the map $\prod_{i=1}^r A[f_i^{-1}]^\dagger \to \prod_{i=1}^r A[f_i^{-1}]^\dagger[\pi^{-1}]$ is injective, we see that $A \to A[\pi^{-1}]$ is injective. □

Using the above proposition, we can extend the notion of admissibility to arbitrary weak formal schemes:

Definition 2.12. A weak formal scheme is said to be admissible if it has an affine open covering $(\mathrm{Spwf}\, A_i)_{i \in J}$ where each A_i is an admissible wcfg algebra.

Let $\mathfrak{X}$ be a weak formal scheme and consider the ideal $\mathfrak{I} \subset \mathcal{O}_{\mathfrak{X}}$ defined as follows: For any open set $U \subset \mathfrak{X}$, $\mathfrak{I}(U)$ consists of all the sections $f \in \mathcal{O}_{\mathfrak{X}}(U)$ such that there is an affine open covering $(U_i)_{i \in J}$ satisfying the condition that each restriction $f|_{U_i}$ is killed by π^n, for some $n \in \mathbb{N}$. Then $(\mathrm{Supp}\ (\mathcal{O}_{\mathfrak{X}}/\mathfrak{I}), \mathcal{O}_{\mathfrak{X}}/\mathfrak{I})$ is a weak formal scheme whose structure sheaf does not have π-torsion.

It is called the admissible weak formal scheme induced from $\mathfrak{X}$ and is denoted by $\mathfrak{X}_{\mathrm{ad}}$.

2.13. *Flat morphisms of weak formal schemes:* A morphism of weak formal schemes $f : \mathfrak{X} \to \mathfrak{Y}$ is said to be flat if it is flat as a morphism of locally ringed spaces. That is, if the induced map on the stalks $f_x : \mathcal{O}_{\mathfrak{Y},f(x)} \to \mathcal{O}_{\mathfrak{X},x}$ at each point $x \in \mathfrak{X}$ is a flat morphism ([11, §6.7]). Now consider the case where f is a morphism of admissible weak formal schemes and let $\hat{f} : \hat{\mathfrak{X}} \to \hat{\mathfrak{Y}}$ be the associated morphism of formal schemes. The maps on stalks $\hat{f}_x$ modulo π coincide with the stalks f_x modulo π and are therefore flat. Moreover the maps $\hat{f}_x$ are flat after tensoring with K because the stalks on an affinoid dagger space coincide with the stalks on its completion (see [9, Rem. 2.4]). Using the flatness criterion in [1, Lemma 2.6.1], we see that the maps $\hat{f}_x$ are flat.

If the formal R-schemes $\hat{\mathfrak{X}}$ and $\hat{\mathfrak{Y}}$ are of finite type, then this is equivalent to the condition that for every affine open $U \subseteq \hat{\mathfrak{Y}}$ and every affine open $V \subseteq f^{-1}(U)$, the morphism $V \to U$ corresponds to a flat ring homomorphism $\mathcal{O}_{\hat{\mathfrak{Y}}}(U) \to \mathcal{O}_{\hat{\mathfrak{X}}}(V)$. Since for any wcfg algebra A, the morphism $A \to \hat{A}$ to its π-adic completion is flat (Corollaire of Proposition 5.4.3, [4, Chap. III]), the analogous statement is true for admissible weak formal schemes.

2.14. *Fiber product:* The tensor product $A \otimes_R^\dagger B$ of wcfg algebras A and B is defined to be the weak completion of the ordinary tensor product $A \otimes_R B$. This is a wcfg algebra with weak generators provided by the tensor product of the weak generators of A and B. It defines a product of A and B in the category of wcfg R-algebras.

Fiber products exist in the category of weak formal schemes. The product of weak formal schemes Spwf A and Spwf B is Spwf $(A \otimes_R^\dagger B) = (\mathrm{Spec}\,(A \otimes_R B))^\dagger$. The fiber product in the category of admissible weak formal schemes of two admissible weak formal schemes X and Y is the admissible weak formal scheme $(X \times Y)_{\mathrm{ad}}$ induced from the fiber product $X \times Y$ in the category of weak formal schemes.

3. Weak formal blowups

We now define weak formal blowing-up, which is a universal construction with respect to making an ideal in a weak formal scheme invertible. Let us first consider the case when $\mathfrak{X} = \mathrm{Spwf}\, A$ is an affine weak formal scheme. Let $\mathfrak{I} \subset \mathcal{O}_{\mathfrak{X}}$ be a coherent open ideal and $J = \Gamma(\mathrm{Spec}\, A/\pi A, \mathfrak{I})$ be the finitely generated open ideal in A associated to it ([13, Thm. 3.3]). Consider the scheme theoretic blowup of J on $X = \mathrm{Spec}\, A$ given by $P = \mathrm{Proj}\left(\bigoplus_{d \geq 0} J^d\right)$. The weak formal blowup of $\mathfrak{I}$ on $\mathfrak{X}$ is defined to be the weak completion $P^\dagger$ of P (see 2.6).

Now, suppose $\mathfrak{X}$ is an arbitrary weak formal scheme and $\mathfrak{I}$ a coherent open ideal on $\mathfrak{X}$. Let $\hat{\mathfrak{X}}$ and $\hat{\mathfrak{I}}$ be the formal completions with respect to the ideal $\pi\mathcal{O}_{\mathfrak{X}}$ of $\mathfrak{X}$ and $\mathfrak{I}$ respectively. The formal blowup ([3, §2]) of $\hat{\mathfrak{I}}$ on $\hat{\mathfrak{X}}$ is given by

$$\hat{\mathfrak{X}}_{\hat{\mathfrak{I}}} = \varinjlim_{n} \mathrm{Proj}\left(\bigoplus_{d \geq 0} \hat{\mathfrak{I}}^d \otimes_{\mathcal{O}_{\hat{\mathfrak{X}}}} \mathcal{O}_{\hat{\mathfrak{X}}}/\pi^n \mathcal{O}_{\hat{\mathfrak{X}}}\right)$$

together with the canonical projection map $\hat{\mathfrak{X}}_{\hat{\mathfrak{J}}} \to \hat{\mathfrak{X}}$.

Consider an affine open covering $\{\mathcal{U}_i\}_{i\in I}$ of $\mathfrak{X}$, where $\mathcal{U}_i = \operatorname{Spwf} A_i$ where each A_i is a wcfg algebra. Denote by $\hat{\mathcal{U}}_i$ the corresponding affine open formal subschemes $\operatorname{Spf} \hat{A}_i$ of $\hat{\mathfrak{X}}$. Let ${P_i}^\dagger$ denote the weak formal blowup of $\mathfrak{J}|_{\mathcal{U}_i}$ on $\mathcal{U}_i$, as constructed above. Note that the restriction $\hat{\mathfrak{X}}_{\hat{\mathfrak{J}}} \times_{\hat{\mathfrak{X}}} \hat{\mathcal{U}}_i$ of the formal blowup $\hat{\mathfrak{X}}_{\hat{\mathfrak{J}}}$ to $\hat{\mathcal{U}}_i$ denoted by $\hat{P}_i$, coincides with the formal blowup of $\hat{\mathfrak{J}}|_{\hat{\mathcal{U}}_i}$ on $\hat{\mathcal{U}}_i$.

Let $V \subset \hat{\mathfrak{X}}_{\hat{\mathfrak{J}}}$ be an arbitrary open set. Define a subpresheaf of $\mathcal{O}_{\hat{\mathfrak{X}}_{\hat{\mathfrak{J}}}}$, denoted by $\mathcal{O}_{\mathfrak{X}_\mathfrak{J}}$, as follows:

$$\mathcal{O}_{\mathfrak{X}_\mathfrak{J}}(V) = \{s \in \mathcal{O}_{\hat{\mathfrak{X}}_{\hat{\mathfrak{J}}}}(V) \ \mid \ s|_{\hat{P}_i \cap V} \in \mathcal{O}_{P_i^\dagger}(\hat{P}_i \cap V) \text{ for all } i \in I\}.$$

If $U \subset V \subset \hat{\mathfrak{X}}_{\hat{\mathfrak{J}}}$ are two open sets, let the morphism $\mathcal{O}_{\mathfrak{X}_\mathfrak{J}}(V) \to \mathcal{O}_{\mathfrak{X}_\mathfrak{J}}(U)$ be given by the restriction of the morphism $\mathcal{O}_{\hat{\mathfrak{X}}_{\hat{\mathfrak{J}}}}(V) \to \mathcal{O}_{\hat{\mathfrak{X}}_{\hat{\mathfrak{J}}}}(U)$ to $\mathcal{O}_{\mathfrak{X}_\mathfrak{J}}(V)$. Again using Lemma 2.7, we see that this is a sheaf on the topological space underlying $\hat{\mathfrak{X}}_{\hat{\mathfrak{J}}}$. Denote by $\mathfrak{X}_\mathfrak{J}$, the locally ringed space whose topological space is the same as that underlying $\hat{\mathfrak{X}}_{\hat{\mathfrak{J}}}$, equipped with the sheaf $\mathcal{O}_{\mathfrak{X}_\mathfrak{J}}$. Clearly, the construction does not depend on the choice of covering.

Definition 3.1. With notation as above, the weak formal blowup of $\mathfrak{J}$ on $\mathfrak{X}$ is the weak formal R-scheme $\mathfrak{X}_\mathfrak{J}$ defined as above, along with the canonical projection map $\mathfrak{X}_\mathfrak{J} \to \mathfrak{X}$.

Proposition 3.2. *We can immediately deduce the following properties of weak formal blowups:*

(a) Let $\mathfrak{X} = \operatorname{Spwf} A$ be an affine admissible weak formal scheme and $\mathfrak{J} \subset \mathcal{O}_\mathfrak{X}$ be a coherent open sheaf of ideals. Then the ideal $\mathfrak{J}\mathcal{O}_{\mathfrak{X}_\mathfrak{J}} \subset \mathcal{O}_{\mathfrak{X}_\mathfrak{J}}$ is invertible.

Let $I = \Gamma(\operatorname{Spec} A/\pi A, \mathfrak{J}) \subset A$ be the finitely generated ideal corresponding to $\mathfrak{J}$ ([13, Thm. 3.3]*) with generators $f_0, \dots, f_r$. The locus in $\mathfrak{X}_\mathfrak{J}$ where $\mathfrak{J}\mathcal{O}_{\mathfrak{X}_\mathfrak{J}}$ is generated by f_i is given by* $\operatorname{Spwf} A_i$*, where $A_i = A_i'/(\pi - \mathit{torsion})$ and*

$$A_i' = A\left[\frac{f_j}{f_i}\ ;\ j \neq i\right]^\dagger \subseteq {A_{f_i}}^\dagger.$$

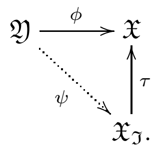
Hence, the weak formal blowup $\mathfrak{X}_\mathfrak{J}$ is an admissible weak formal scheme. (Note that this corresponds to [1, Prop. 2.6.7]*.)*

Proof. Let $\tilde{A}_i' = A\left[\frac{f_j}{f_i}\ ;\ j \neq i\right]$ and $\tilde{A}_i = \tilde{A}_i'/(f_i - \text{torsion})$. The scheme-theoretic blowup X of I on the ordinary scheme $\operatorname{Spec} A$ has an affine open covering by $\operatorname{Spec} \tilde{A}_i$ where i varies between $0, \dots, r$. The analogous assertions hold in the scheme-theoretic situation: the ideal $I\mathcal{O}_X$ is invertible on X; The locus in X where $I\mathcal{O}_X$ is generated by f_i is given by $\operatorname{Spec} \tilde{A}_i$. Since I contains a power of π (being an open ideal), $(f_i - \text{torsion})_{\tilde{A}_i'} \subset (\pi - \text{torsion})_{\tilde{A}_i'}$. On the other hand, since A is admissible, $\tilde{A}_i$ has no π-torsion and therefore $(\pi - \text{torsion})_{\tilde{A}_i'} = (f_i - \text{torsion})_{\tilde{A}_i'}$. Therefore, we get $\tilde{A}_i = \tilde{A}_i'/(\pi - \text{torsion})$. It follows from the definition that the weak formal blowup $\mathfrak{X}_\mathfrak{J}$ has an affine open covering $(\operatorname{Spwf} A_i)_{i=0,\dots,r}$. Note that since $\tilde{A}_i'$ is of finite type over A (which is a wcfg algebra), its completion A'_i is a wcfg algebra and hence is flat over $\tilde{A}'_i$ ([7] and [13, Cor. 1.4]). Similarly, A'_i is flat over $\tilde{A}'_i$ and the first assertion is proved.

Since A'_i is flat over $\tilde{A}'_i$, we get $(\pi - \text{torsion})_{A'_i} = (\pi - \text{torsion})_{\tilde{A}'_i} \otimes A'_i$ and similarly for $(f_i - \text{torsion})_{A'_i}$ (as in the proof of [1, Prop. 2.6.7]). From this it follows that the weak completions of $\tilde{A}_i$ and $\tilde{A}'_i$ are A_i and A'_i respectively and the second assertion holds. □

(b) Universal property of weak formal blowups: *(Note that this corresponds to* [1, Prop. 2.6.9]*.) Let $\mathfrak{X}$ be an admissible weak formal scheme and $\mathfrak{I} \subset \mathcal{O}_{\mathfrak{X}}$ be a coherent open ideal. Then $\mathfrak{X}_{\mathfrak{I}}$, the weak formal blowup of $\mathfrak{I}$ on $\mathfrak{X}$, satisfies the following universal property:*

Any morphism of weak formal schemes $\phi : \mathfrak{Y} \to \mathfrak{X}$ satisfying the property that $\mathfrak{I}\mathcal{O}_{\mathfrak{Y}}$ is invertible in $\mathcal{O}_{\mathfrak{Y}}$, factorizes uniquely through $\mathfrak{X}_{\mathfrak{I}}$, that is, there exists a unique morphism $\psi : \mathfrak{Y} \to \mathfrak{X}_{\mathfrak{I}}$ which makes the following diagram commutative:



Proof. Suppose there exists a morphism of weak formal schemes $\phi : \mathfrak{Y} \to \mathfrak{X}$ such that $\mathfrak{I}\mathcal{O}_{\mathfrak{Y}}$ is invertible in $\mathcal{O}_{\mathfrak{Y}}$. Since weak formal blowing up is a local construction we may assume that $\mathfrak{X}$ is affine, say $\mathfrak{X} = \operatorname{Spwf} A$. The ideal $\mathfrak{I} \subset \mathcal{O}_{\mathfrak{X}}$ is associated to a finitely generated ideal $I = (f_0, \ldots, f_r) \subset A$. Further, we can assume that $\mathfrak{Y}$ is also an affine weak formal scheme, $\mathfrak{Y} = \operatorname{Spwf} B$. Consider the morphism of ordinary schemes $\operatorname{Spec} B \to \operatorname{Spec} A$. Since I is invertible on $\operatorname{Spec} B$, by the universal property of (scheme-theoretic) blowups for the blowup P of I on $\operatorname{Spec} A$, this map factorizes uniquely via P, so that we get the morphism of schemes $\operatorname{Spec} B \to P$. Using [15, Thm. 1.5], we can extend this to get a unique morphism of weak formal schemes $\operatorname{Spwf} B \to P^\dagger$ as desired. □

(c) Let $\mathfrak{X}$ be an admissible weak formal scheme, $\mathcal{A}, \mathcal{B} \subset \mathcal{O}_{\mathfrak{X}}$ be coherent open ideals on $\mathfrak{X}$ and $\mathfrak{X}_{\mathcal{A}}$ be the weak formal blowup of $\mathcal{A}$ on $\mathfrak{X}$. Let $\mathcal{B}' := \mathcal{B}\mathcal{O}_{\mathfrak{X}_{\mathcal{A}}} \subset \mathcal{O}_{\mathfrak{X}_{\mathcal{A}}}$ and denote by $(\mathfrak{X}_{\mathcal{A}})_{\mathcal{B}'}$ the weak formal blowup of $\mathcal{B}'$ on $\mathfrak{X}_{\mathcal{A}}$. Then, the composition of the weak formal blowup of $\mathcal{B}'$ on $\mathfrak{X}_{\mathcal{A}}$ with the weak formal blowup of $\mathcal{A}$ on $\mathfrak{X}$, given by

$$(\mathfrak{X}_{\mathcal{A}})_{\mathcal{B}'} \to \mathfrak{X}_{\mathcal{A}} \to \mathfrak{X}$$

is isomorphic to the weak formal blowup of $\mathcal{A}\mathcal{B}$ on $\mathfrak{X}$.

Proof. We use the notation of Proposition 3.2(a). The statement follows from the universal property of weak formal blowups (Proposition 3.2(b)) and the fact that the ideal $\mathcal{A}$ is invertible on $(\mathfrak{X}_{\mathcal{A}})_{\mathcal{B}'}$ since it has no f_i-torsion. Compare with the proof in case of formal schemes ([1, Rem. 2.6.10]). $\square$

(d) *In the category of admissible weak formal schemes, weak formal blowups commute with flat base change.*

Proof. It is enough to prove this for a flat morphism of affine weak formal schemes $\phi : \mathfrak{Y} \to \mathfrak{X}$ where $\mathfrak{X} = \operatorname{Spwf} A$ and $\mathfrak{Y} = \operatorname{Spwf} B$. Suppose $\mathcal{A} \subset \mathcal{O}_{\mathfrak{X}}$ be a coherent open ideal, and $\mathfrak{X}_{\mathcal{A}} \to \mathfrak{X}$ is the weak formal blowup on $\mathfrak{X}$ with respect to $\mathcal{A}$. Let $\mathcal{A}$ be associated to the finitely generated ideal $\mathfrak{a} \subset A$ and $S = \bigoplus_{d\geq 0} \mathfrak{a}^d$. Then $\mathfrak{X}_{\mathcal{A}} = (\operatorname{Proj} S)^\dagger$ and admits a covering by affine weak formal schemes $\operatorname{Spwf} S^\dagger_{(f_i)}$ (where $S^\dagger_{(f_i)}$ is isomorphic to A_i, in the notation of Proposition 3.2(a)). Locally $\mathfrak{a}S^\dagger_{(f_i)}$ is generated by f_i and is invertible in $S^\dagger_{(f_i)}$. Let $S' = \bigoplus_{d\geq 0}(\mathfrak{a}B)^d$. Then, $\mathfrak{a}{S'}^\dagger_{(f_i)} = f_i {S'}^\dagger_{(f_i)}$. Since B is flat over A, $(f_i - \text{torsion})_B = 0$, and f_i is not a zero-divisor in ${S'}^\dagger_{(f_i)}$. Hence $\mathfrak{a}{S'}^\dagger_{(f_i)}$ is invertible in ${S'}^\dagger_{(f_i)}$ and we obtain the required result. $\square$

Definition 3.3. A topological space X is said to be quasi-paracompact if there exists a covering $\{X_i\}_{i\in J}$ by quasi-compact open subspaces X_i which is moreover a covering of finite type, i.e. for any index $i \in J$, the intersection $X_i \cap X_j$ is nonempty for at most finitely many $j \in J$.

We say that a weak formal scheme $\mathfrak{X}$ is quasi-paracompact, if its underlying topological space is quasi-paracompact.

Proposition 3.4. *Let $\mathfrak{X}$ be an admissible quasi-paracompact weak formal scheme. Let $\mathcal{U} \subset \mathfrak{X}$ be a quasi-compact open subscheme. Then any coherent open ideal $\mathcal{F}_{\mathcal{U}}$ on $\mathcal{U}$ extends to a coherent open ideal $\mathcal{F} \subset \mathcal{O}_{\mathfrak{X}}$ on $\mathfrak{X}$, which moreover satisfies the condition that for any open subscheme $V \subset \mathfrak{X}$ disjoint from $\mathcal{U}$, $\mathcal{F}|_V$ coincides with $\mathcal{O}_{\mathfrak{X}}|_V$. (Note that this corresponds to* [1, Prop. 2.6.13]*).*

Proof. Let us first consider the case when $\mathfrak{X}$ is an affine weak formal scheme. We use the idea for the construction in the analogous result in the scheme-theoretic situation ([11, 9.4]). Let $i : \mathcal{U} \to \mathfrak{X}$ be the canonical morphism and consider the induced morphism of structure sheaves $j : \mathcal{O}_{\mathfrak{X}} \to i_*\mathcal{O}_{\mathcal{U}}$. Let $\mathcal{F} = j^{-1}(i_*\mathcal{F}_{\mathcal{U}})$. We first show that $i_*\mathcal{F}_{\mathcal{U}}$ is quasi-coherent. Indeed, using the methods in the proof of the analogous scheme-theoretic result ([11, Prop. 9.2.1]), we can reduce it to the case of an open immersion of a basic open subscheme into $\mathfrak{X}$, in which case the result is clear. Then, $\mathcal{F}$ being the preimage of a quasi-coherent sheaf is itself quasi-coherent ([11, 0.5.1.4]). Using the fact that any wcfg algebra is noetherian and since $\mathcal{F} \subset \mathcal{O}_{\mathfrak{X}}$ is a subsheaf of the coherent sheaf $\mathcal{O}_{\mathfrak{X}}$, we see that $\mathcal{F}$ is a coherent open ideal satisfying the required properties.

Now, let $\mathfrak{X}$ be a quasi-paracompact weak formal scheme and $\mathcal{U} \subset \mathfrak{X}$ be a quasi-compact open subscheme with a coherent open ideal $\mathcal{F}_{\mathcal{U}}$ on $\mathcal{U}$. Let $\{\mathfrak{X}_i\}_{i\in J}$ be an affine open covering of $\mathfrak{X}$ and denote by $\mathcal{F}_{\mathcal{U}_i}$ the restriction of $\mathcal{F}_{\mathcal{U}}$ to $\mathfrak{X}_i \cap \mathcal{U}$. Using the result in the affine case, we can extend $\mathcal{F}_{\mathcal{U}_i}$ to a coherent open ideal $\mathcal{F}_i$ on $\mathfrak{X}_i$. To construct a coherent sheaf of ideals $\mathcal{F}$ on $\mathfrak{X}$, we use the same idea as in the construction of weak formal blowups. Consider the formal completion $\hat{\mathfrak{X}}$ of $\mathfrak{X}$ along the subscheme defined by the ideal $\pi\mathcal{O}_{\mathfrak{X}}$. Using [1, Prop. 2.6.13], we can extend the formal completion $\hat{\mathcal{F}_{\mathcal{U}}}$ of $\mathcal{F}_{\mathcal{U}}$ on $\mathcal{U}$ to a coherent sheaf $\hat{\mathcal{F}}$ on $\hat{\mathfrak{X}}$. Then using Lemma 2.7, we can define a subsheaf $\mathcal{F}$ of $\hat{\mathcal{F}}$ on $\mathfrak{X}$ as follows: for any open $W \subset \mathfrak{X}$, let

$$\mathcal{F}(W) = \{s \in \hat{\mathcal{F}}(W) \ \mid \ s|_{W\cap\mathfrak{X}_i} \in \mathcal{F}_i(W \cap \mathfrak{X}_i)\}.$$

This is a coherent open ideal by construction, which moreover satisfies the required properties. □

Proposition 3.5. *Let $\mathfrak{X}$ be an admissible quasi-compact weak formal scheme and let $\phi : \mathfrak{X}' \to \mathfrak{X}$ and $\phi' : \mathfrak{X}'' \to \mathfrak{X}'$ be weak formal blowups. Then, the composition $\phi \circ \phi' : \mathfrak{X}'' \to \mathfrak{X}$ is also a weak formal blowup.*

Proof. We follow the proof in [3, Prop. 2.5] and use the corresponding fact in the scheme-theoretic situation ([17, 5.1.4]). Let $\phi : \mathfrak{X}' \to \mathfrak{X}$ and $\phi' : \mathfrak{X}'' \to \mathfrak{X}'$ be the weak formal blowups of $\mathcal{A} \subset \mathcal{O}_{\mathfrak{X}}$ on $\mathfrak{X}$ and $\mathcal{A}' \subset \mathcal{O}_{\mathfrak{X}'}$ on $\mathfrak{X}'$ respectively. First consider the case when $\mathfrak{X}$ is affine, say $\mathfrak{X} = \operatorname{Spwf} A$. Let $\mathcal{A}$ be associated to the coherent open ideal $\mathfrak{a} \subset A$. Let $\tilde{\mathfrak{X}} = \operatorname{Spec} A$ and $\tilde{\phi} : \tilde{\mathfrak{X}}' \to \tilde{\mathfrak{X}}$ be the scheme theoretic blowups of $\mathfrak{a}$ on $\tilde{\mathfrak{X}}$. Let $j : \mathfrak{X}' \to \tilde{\mathfrak{X}}'$ be the associated morphism of ringed spaces and $u : \mathcal{O}_{\tilde{\mathfrak{X}}'} \to j_*\mathcal{O}_{\mathfrak{X}'}$ be the corresponding morphism of sheaves.

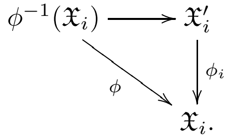
Using the arguments in the proof of Proposition 3.4, $\tilde{\mathcal{A}}' = u^{-1}(j_*\mathcal{A}') \subset \mathcal{O}_{\tilde{\mathfrak{X}}'}$ is a coherent open ideal and such that $j^*\tilde{\mathcal{A}}'$ generates $\mathcal{A}' \subset \mathcal{O}_{\mathfrak{X}'}$.

Let $\tilde{\phi}' : \tilde{\mathfrak{X}}'' \to \tilde{\mathfrak{X}}'$ be the scheme theoretic blowup of $\tilde{\mathcal{A}}'$ on $\tilde{\mathfrak{X}}'$. Using [17, 5.1.4], there exists an admissible open ideal $\mathfrak{b} \subset A$ such that $\mathfrak{b}\mathcal{O}_{\tilde{\mathfrak{X}}'} = \mathcal{A}^m \tilde{\mathcal{A}'}^n \mathcal{O}_{\tilde{\mathfrak{X}}'}$ for some integers m, n and $\tilde{\phi} \circ \tilde{\phi}' : \tilde{\mathfrak{X}}'' \to \tilde{\mathfrak{X}}$ is the scheme theoretic blowup on $\tilde{\mathfrak{X}}$ of the ideal in $\mathcal{O}_{\tilde{\mathfrak{X}}}$ associated to $\mathfrak{a}\mathfrak{b}$. Let $\iota : \mathcal{O}_{\tilde{\mathfrak{X}}} \to \tilde{\phi}_*\mathcal{O}_{\tilde{\mathfrak{X}}'}$ be the morphism of sheaves induced from $\tilde{\phi}$. The inverse image $\mathfrak{b}' = \iota^{-1}(\tilde{\phi}_*(\mathcal{A}^m \tilde{\mathcal{A}'}^n \mathcal{O}_{\tilde{\mathfrak{X}}'}))$ is a coherent open ideal in $\tilde{\mathfrak{X}}$ since it is noetherian. Therefore, $\phi \circ \phi' : \mathfrak{X}'' \to \mathfrak{X}$ is the weak formal blowup of $\mathfrak{a}\mathfrak{b}'$ in $\mathfrak{X}$ which settles the assertion in the affine case.

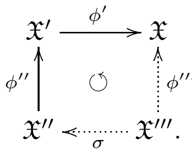
Since weak formal blowing-up is compatible with flat base change (Proposition 3.2(d)), we can use the arguments in [1, Prop. 2.6.11] to globalize the argument. □

3.6. Using Proposition 3.4 and Proposition 3.5, it is easy to derive the following corollaries (see [1, Prop. 2.6.14 and 2.6.15]).

(a) Let $\mathfrak{X}$ be a quasi-paracompact admissible weak formal scheme. Consider a covering $\mathfrak{X} = \bigcup_{i\in J} \mathfrak{X}_i$ of finite type by quasi-compact open subschemes $\mathfrak{X}_i \subset \mathfrak{X}$ and weak formal blowups $\phi_i : \mathfrak{X}_i' \to \mathfrak{X}_i$, $i \in J$. Then there is a weak formal blowup $\phi : \mathfrak{X}' \to \mathfrak{X}$ such that for each $i \in J$, there exists a unique morphism $\phi^{-1}(\mathfrak{X}_i) \to \mathfrak{X}'_i$ and for all $i \in J$ the following diagram commutes:



(b) Let $\mathfrak{X}$ be a quasi-paracompact admissible weak formal scheme. Let $\phi' : \mathfrak{X}' \to \mathfrak{X}$ and $\phi'' : \mathfrak{X}'' \to \mathfrak{X}'$ be weak formal blowups. Then, there exists a weak formal blowup $\phi''' : \mathfrak{X}''' \to \mathfrak{X}$ along with a morphism $\sigma : \mathfrak{X}''' \to \mathfrak{X}''$ such that the following diagram commutes:



4. Establishing an equivalence of categories

Dagger spaces can be interpreted as generic fibers of weak formal schemes just as rigid analytic spaces can be thought of as generic fibers of formal schemes. We make this relationship precise in this section and prove the result stated at the beginning.

Defining the functor:

Lemma 4.1. *Consider $R[X_1, \ldots, X_n]^\dagger$, the weak completion of the polynomial ring in n-variables. Then,*

$$R[X_1, \ldots, X_n]^\dagger \otimes_R K = K\langle X_1, \ldots, X_n\rangle^\dagger,$$

where $K\langle X_1, \ldots, X_n\rangle^\dagger$ is the Monsky–Washnitzer algebra also denoted by W_n (as in [9, 1.2]*) consisting of power series over K, convergent over a radius larger than one.*

Proof. For convenience, we include here the easy and well-known proof.

$R[X_1, \ldots, X_n]^\dagger \otimes_R K$ consists of power series of the form $f = \sum_\nu a_\nu X^\nu$ with coefficients $a_\nu \in K$ along with a constant c such that for all $\nu \in \mathbb{Z}^n_{\geq 0}$ the following condition holds:

$$|\nu| \leq c[\mathrm{ord}_\pi(a_\nu) + 1]. \tag{1}$$

We can choose constants c_1, c_2, with $c_1 > 0$ so that $|\nu| \leq c_1 \mathrm{ord}_\pi(a_\nu) + c_2$.

We want to show that f converges on a polydisc with radius strictly larger than 1. Suppose $|X_i| \leq 1 + \epsilon$ for some $\epsilon > 0$ and $0 \leq i \leq n$. Then,

$$|a_{\nu_1,\ldots,\nu_n} X_1^{\nu_1} \ldots X_n^{\nu_n}| = |a_\nu|\, |X_1|^{\nu_1} \ldots |X_n|^{\nu_n} = |a_\nu|(1+\epsilon)^{|\nu|} \leq e^{\frac{c_2}{c_1}}$$

if we choose c_1 so that $e^{\frac{1}{c_1}} = 1 + \epsilon$. Now, by increasing c_1 if necessary, we can find an $M > 0$ such that $|a_{\nu_1 \dots \nu_n} X_1^{\nu_1} \dots X_n^{\nu_n}| \leq M$ for sufficiently large ν. Thus the growth condition (1) implies that $f = \sum_{0 \leq \nu_1, \dots, \nu_n} a_{\nu_1 \dots \nu_n} X_1^{\nu_1} \dots X_n^{\nu_n}$ is overconvergent. We can reverse the above argument to prove the opposite direction as well. $\square$

4.2. Given a wcfg algebra A, we can interpret $A \otimes_R K$ as the localization $A \otimes_R K = A \otimes_R (S^{-1}R) = S^{-1}A$ where $S = R \setminus \{0\}$. Since any wcfg R-algebra is a homomorphic image of the weak completion of $R[X_1 \dots X_n]$ for some variables $X = X_1, \dots, X_n$ ([15, Thm. 2.2]), we can write A in the form $R[X]^\dagger / \mathfrak{a}$ for some ideal $\mathfrak{a} \subset R[X]^\dagger$. But then using the above lemma, $A \otimes_R K = K\langle X \rangle^\dagger / (\mathfrak{a})$ is a K-dagger algebra. Thus, we can define a functor $\mathrm{rig}^\dagger$ from the category of affine weak formal schemes to the category of affinoid dagger spaces, associating to Spwf A, the affinoid dagger space $\mathrm{Sp}\,(A \otimes_R K)$.

The π-adic completion of $R[X]^\dagger / \mathfrak{a}$ is an R-algebra of topologically finite type, $R\langle X \rangle / (\mathfrak{a})$. On tensoring it with K, we obtain the affinoid algebra $K\langle X \rangle / (\mathfrak{a})$ which coincides with the completion of the dagger algebra $K\langle X \rangle^\dagger / (\mathfrak{a})$ ([9, Prop. 1.6]). Let rig denote Raynaud's functor from the category of formal R-schemes, locally of topologically finite type, to the category of rigid analytic K-spaces ([3, §4]). Then, we obtain the following diagram at the affine level:

$$\begin{array}{ccc} \mathrm{Spwf}\, A & \xmapsto{\mathrm{rig}^\dagger} & \mathrm{Sp}\,(A \otimes_R K) \\ \downarrow & & \downarrow \\ \mathrm{Spf}\, \hat{A} & \xmapsto{\mathrm{rig}} & \mathrm{Sp}\,(\hat{A} \otimes_R K). \end{array}$$

Let $f \in A$. Using 2.2, we know that $A[f^{-1}]^\dagger \otimes_R K = A[\zeta]^\dagger / (1 - f\zeta) \otimes_R K$, where ζ is a variable. It follows from the description of $A[\zeta]^\dagger$ using Gauss norms explained in 2.2 and [5, 4.43] that $A[\zeta]^\dagger \otimes_R K$ is the dagger algebra associated to the product of the affinoid variety $\mathrm{Sp}\,(A \otimes_R K)$ with the closed dagger disc $\mathrm{Sp}\, K\langle \zeta \rangle^\dagger$. But this dagger algebra coincides with $(A \otimes_R K)[\zeta]^\dagger = (A \otimes_R K) \otimes_K^\dagger K\langle \zeta \rangle^\dagger$ (compare [9, 1.17]). Tensoring with K the exact sequence

$$0 \longrightarrow (1 - f\zeta) \longrightarrow A[\zeta]^\dagger \longrightarrow A[\zeta]^\dagger / (1 - f\zeta) \longrightarrow 0$$

yields the equality $A[\zeta]^\dagger / (1 - f\zeta) \otimes_R K = (A \otimes_R K)\langle \zeta \rangle^\dagger / (1 - f\zeta)$. Thus, the functor $\mathrm{rig}^\dagger$ maps $A[f^{-1}]^\dagger$ to $(A \otimes_R K)\langle f^{-1} \rangle^\dagger$.

Therefore, we obtain the following diagram showing that a basic open subscheme of Spwf A is mapped to a rational subdomain of $\mathrm{Sp}\,(A \otimes_R K)$:

$$\begin{array}{ccc} \mathrm{Spwf}\, A & \longmapsto & \mathrm{Sp}\,(A \otimes_R K) \\ \uparrow & & \uparrow \\ \mathrm{Spwf}\, A[f^{-1}]^\dagger & \longmapsto & \mathrm{Sp}\,(A \otimes_R K)\langle f^{-1} \rangle^\dagger. \end{array}$$

It follows that any open weak formal subscheme of $\mathrm{Spwf}\, A$ (being quasi-compact) is mapped to an admissible open in $\mathrm{Sp}\,(A \otimes_R K)$ under the functor $\mathrm{rig}^\dagger$. To globalize the construction, we can follow the proof in [1, Prop. 2.4.3]. If $\mathfrak{X}$ is an arbitrary weak formal scheme, we can fix a covering by affine weak formal schemes $(U_i)_{i \in I}$. From the above diagram, it is clear that the functor $\mathrm{rig}^\dagger$ maps any open subscheme of U_i to an admissible open in $U_{i,\mathrm{rig}^\dagger}$. Since weak formal schemes are quasi-separated, the intersection $U_i \cap U_j$ is quasi-compact for each pair $i, j \in I$, and hence is mapped to admissible open subsets in $U_{i,\mathrm{rig}^\dagger}$ and $U_{j,\mathrm{rig}^\dagger}$ respectively. We can glue the affinoid dagger spaces $U_{i,\mathrm{rig}^\dagger}$ along these admissible opens ([2, §9.1.3]) to obtain the dagger space $\mathfrak{X}_{\mathrm{rig}^\dagger}$. Thereby, we obtain the following commutative diagram of functors:

$$\begin{array}{ccc} \text{Admissible weak formal } R\text{-schemes} & \xrightarrow{\mathrm{rig}^\dagger} & K\text{-dagger spaces} \\ \mathcal{F}\big\downarrow & & \mathcal{F}'\big\downarrow \\ \text{Admissible formal } R\text{-schemes} & \xrightarrow{\mathrm{rig}} & \text{Rigid } K\text{-analytic spaces.} \end{array}$$

Here, $\mathcal{F}$ is the formal completion functor and $\mathcal{F}'$ is the faithful functor as defined in [9, 2.19].

4.3. Now observe that the functor $\mathrm{rig}^\dagger$ factors via the functor $\mathfrak{X} \mapsto \mathfrak{X}_{\mathrm{ad}}$ through the category of admissible weak formal R-schemes, since tensoring with K kills π-torsion. Therefore the generic fiber of a weak formal scheme $\mathfrak{X}$ and that of the induced admissible weak formal scheme $\mathfrak{X}_{\mathrm{ad}}$ coincide.

We claim that $\mathrm{rig}^\dagger$ takes values in the category of quasi-separated K-dagger spaces. We can see this as follows. First note that, given affinoid dagger spaces $\mathrm{Sp}\,(W_m/\mathfrak{a})$ and $\mathrm{Sp}\,(W_n/\mathfrak{b})$, the associated affinoid rigid spaces under the functor $\mathcal{F}'$ (as in the above diagram of functors) are given by $\mathrm{Sp}\,(T_m/\mathfrak{a})$ and $\mathrm{Sp}\,(T_n/\mathfrak{b})$ respectively, and their fibered product is $\mathrm{Sp}\,(T_{n+m}/(\mathfrak{a}+\mathfrak{b}))$. On the other hand, the fibered product of the dagger spaces is given by $\mathrm{Sp}\,(W_{m+n}/(\mathfrak{a}+\mathfrak{b}))$ (refer [9, 1.16]) and hence we see that taking fibered product commutes with the functor $\mathcal{F}'$.

Now, given an admissible weak formal scheme, its associated admissible formal scheme is locally topologically of finite type and hence quasi-separated, with quasi-separated generic fiber which is the image under the functor rig (refer above diagram of functors). But this rigid space is G-topologically isomorphic to the dagger space associated to the given weak formal scheme under the functor $\mathrm{rig}^\dagger$ ([9, Thm. 2.19]). Further, since the functor $\mathcal{F}'$ commutes with fibered products, we see that the dagger space obtained as image under $\mathrm{rig}^\dagger$ is also quasi-separated. Hence we obtain:

Proposition 4.4. *The functor $A \mapsto A \otimes_R K$ from admissible wcfg R-algebras to K-dagger algebras induces a functor* $\mathrm{rig}^\dagger : \mathfrak{X} \mapsto \mathfrak{X}_{\mathrm{rig}^\dagger}$ *from the category of admissible weak formal R-schemes to the category of quasi-separated K-dagger spaces.*

4.5. *Specialization map:* As in the formal geometric situation, there exists a bijective correspondence between closed immersions of weak formal schemes of the form $\mathrm{Spwf}\, B \to \mathrm{Spwf}\, A$ with B a local integral domain of dimension 1 which is finite over R, and points of the dagger space $\mathrm{Sp}\,(A \otimes_R K)$. Given such a closed immersion, the kernel $\mathfrak{p}$ of the associated surjective morphism $\phi^\# : A \to B$ is a prime ideal generating a maximal ideal in $A \otimes_R K$. On the other hand, given a maximal ideal $\mathfrak{m} \subset A \otimes_R K$, the field $K' = (A \otimes_R K)/\mathfrak{m}$ is finite over K ([9, 1.4.2]). The image of A in K' given by $B = A/(\mathfrak{m} \cap A)$ is a wcfg algebra ([15, Thm. 2.1]) which is a local ring of dimension 1, finite over R. If we let $\mathfrak{p} = \mathfrak{m} \cap A$, then we see that any point in $\mathrm{Sp}\,(A \otimes_R K)$ corresponds to a closed immersion $\mathrm{Spwf}\, A/\mathfrak{p} \to \mathrm{Spwf}\, A$ as required.

We can now define a specialization map at the level of underlying points,

$$\mathrm{sp} : |\mathfrak{X}_{\mathrm{rig}^\dagger}| \to |\mathfrak{X}_k|$$

by associating to a point in $\mathfrak{X}_{\mathrm{rig}^\dagger}$ the image of the closed immersion $\mathrm{Spec}\, B/\pi B \to \mathrm{Spec}\, A/\pi A$ (using the notation in the previous paragraph). Note that if $\mathfrak{X} = \mathrm{Spwf}\, A$ is an affine weak formal scheme with A some wcfg algebra, then $|\mathfrak{X}_{\mathrm{rig}^\dagger}| = |\hat{\mathfrak{X}}_{\mathrm{rig}}|$ since $\hat{A} \otimes K$ is the completion of $A \otimes K$ and using [9, Thm. 1.7]. Hence the specialization map defined above coincides with Raynaud's specialization map ([1, Prop. 2.7.7]). Therefore, we can deduce from [1, Prop. 2.7.8] that it is surjective onto the set of closed points of $\mathfrak{X}_k$.

Localization of categories: There exists a one-to-one correspondence between weak formal blowups of a weak formal scheme and rational subdomains of the corresponding dagger space obtained as its generic fiber. One way of this correspondence is shown in the following result. We remark here that for any category $\mathfrak{C}$ and a class of morphisms S in $\mathfrak{C}$, the localization $\mathfrak{C}_S$ of $\mathfrak{C}$ by S always exists (refer [8, III.2]). Hence we can localize the category of weak formal schemes by weak formal blowups.

Proposition 4.6. *The functor* $\mathrm{rig}^\dagger$ *transforms weak formal blowups into isomorphisms. Therefore,* $\mathrm{rig}^\dagger$ *factors through the localization of admissible weak formal schemes by the class of weak formal blowups.*

Proof. Using the notation of Proposition 3.2(a), in the affine situation the weak formal blowup $\mathfrak{X}_{\mathfrak{I}}$ of $\mathfrak{I}$ on $\mathfrak{X} = \mathrm{Spwf}\, A$ has a covering by $(\mathrm{Spwf}\, A_i)_i$, $i = 0, \dots, r$ where

$$\mathrm{Spwf}\, A_i = A\left[\frac{f_j}{f_i}\,;\, j \neq i\right]^\dagger /(\pi - \text{torsion}).$$

Since I is open in A, it contain a power of π and hence $f_0, \dots, f_r$ generate the unit ideal in $A \otimes_R K$. But this is equivalent to the fact that $f_0, \dots, f_r$ have no common zeroes on $\mathfrak{X}_{\mathrm{rig}^\dagger}$. Hence $\mathrm{Spwf}\, A_i$ is mapped under $\mathrm{rig}^\dagger$ to rational subdomains in $\mathfrak{X}_{\mathrm{rig}^\dagger}$ and as i varies through $0, \dots, r$ we obtain a covering of $\mathfrak{X}_{\mathrm{rig}^\dagger}$. Therefore, the map $\mathfrak{X}_{\mathfrak{I}} \to \mathfrak{X}$ is mapped to the isomorphism $\mathfrak{X}_{\mathfrak{I},\mathrm{rig}^\dagger} \xrightarrow{\sim} \mathfrak{X}_{\mathrm{rig}^\dagger}$. This settles the assertion in the affine case.

The result in the general case follows since weak formal blowing up is a local construction. □

The following lemma establishes the other direction.

Lemma 4.7. *Let $\mathfrak{X}$ be a quasi-paracompact admissible weak formal scheme and $\mathfrak{X}_{\mathrm{rig}^\dagger}$ the associated dagger space. Suppose $\mathfrak{U}_{\mathrm{rig}^\dagger}$ is an admissible covering of finite type of $\mathfrak{X}_{\mathrm{rig}^\dagger}$ by quasi-compact open subspaces. Then there exists a weak formal blowing up $\mathfrak{X}' \to \mathfrak{X}$ along with an open covering $\mathfrak{U}'$ of $\mathfrak{X}'$ such that the associated family of subspaces ${\mathfrak{U}'}_{\mathrm{rig}^\dagger}$ of $\mathfrak{X}_{\mathrm{rig}^\dagger}$ coincides with $\mathfrak{U}_{\mathrm{rig}^\dagger}$. (Note that this corresponds to* [1, Lemma 2.8.4]*.)*

Proof. Let us first consider the case where $\mathfrak{X} = \mathrm{Spwf}\, A$ is an affine weak formal scheme, so that $\mathfrak{U}_{\mathrm{rig}^\dagger}$ can be taken to be a finite covering. By the analogue of the theorem of Gerritzen–Grauert for dagger spaces ([9, 2.8]), each $U_{\mathrm{rig}^\dagger} \in \mathfrak{U}_{\mathrm{rig}^\dagger}$ has a finite covering by rational subdomains. Therefore, we can take each $U_{\mathrm{rig}^\dagger}$ to be of the form

$$U_{\mathrm{rig}^\dagger} = \mathrm{Sp}\,(A \otimes_R K) \left\langle \frac{f_1}{f_0}, \dots, \frac{f_r}{f_0} \right\rangle^\dagger$$

for some global sections $f_0, \dots, f_r$ of $\mathcal{O}_{\mathfrak{X}_{\mathrm{rig}^\dagger}}$ having no common zeros. This implies that they generate the unit ideal in $A \otimes_R K$ and hence that there is a relation of the form $c_0 f_0 + \ldots + c_r f_r = 1$ for some $c_i \in A \otimes_R K$. Let n be sufficiently large so that $\pi^n c_i$ and $\pi^n f_i$ are in A for all i. Then the ideal $(\pi^n f_0, \dots, \pi^n f_r)$ in A is admissible since it contains π^{2n}. Consider the associated coherent open ideal $\mathcal{A} \subset \mathcal{O}_{\mathfrak{X}}$ and the corresponding weak formal blowup $\mathfrak{X}_{\mathcal{A}} \to \mathfrak{X}$. From the proof of Proposition 3.2(a), we know that the locus in $\mathfrak{X}_{\mathcal{A}}$ where $\mathcal{A}\mathcal{O}_{\mathfrak{X}_{\mathcal{A}}}$ is generated by f_0, is given by $U = \mathrm{Spwf}\, A_0$ with $A_0 = A\big[\frac{f_j}{f_0} | j \neq 0\big]^\dagger / (\pi - \text{torsion})$ and clearly, $\mathrm{rig}^\dagger$ maps U to $U_{\mathrm{rig}^\dagger}$.

Consider the product $\mathcal{A}$ of the coherent open ideals associated to each $U_{\mathrm{rig}^\dagger} \in \mathfrak{U}_{\mathrm{rig}^\dagger}$ and its weak formal blowup $\mathfrak{X}' \to \mathfrak{X}$. Denote the corresponding collection of open subschemes of $\mathfrak{X}'$ by $\mathfrak{U}'$. To see that $\mathfrak{U}'$ covers $\mathfrak{X}'$, note that any closed point $x \in \mathfrak{X}'$ induces a closed point in its special fiber $\mathfrak{X}_k$ which corresponds to a point of $\mathfrak{X}_{\mathrm{rig}^\dagger}$ (see 4.5) and hence lies in some $U_{\mathrm{rig}^\dagger}$ in $\mathfrak{U}_{\mathrm{rig}^\dagger}$. So $x \in U' \in \mathfrak{U}'$ which implies that the closed part $\mathfrak{X} - \bigcup_{U' \in \mathfrak{U}'} U'$ is empty.

In the general case, consider an affine open covering $(\mathfrak{X}_j)_{j \in J}$ of $\mathfrak{X}$ of finite type. Since $\mathfrak{X}_{\mathrm{rig}^\dagger}$ is quasi-separated, $\mathfrak{U}_{\mathrm{rig}^\dagger}$ restricts to an admissible covering by quasi-compact open subspaces on $\mathfrak{X}_{j,\mathrm{rig}^\dagger}$ of finite type. Using the assertion in the affine case, we can construct suitable coherent open ideals $\mathcal{A}_j \subset \mathcal{O}_{\mathfrak{X}_j}$. Using Proposition 3.4, we can extend each $\mathcal{A}_j$ to a coherent open ideal in $\mathcal{O}_{\mathfrak{X}}$ and then consider their product $\mathcal{A} \subset \mathcal{O}_{\mathfrak{X}}$. The weak formal blowup $\mathfrak{X}_{\mathcal{A}}$ of $\mathcal{A}$ on $\mathfrak{X}$ admits a system of open formal subschemes $\mathfrak{U}'$ which induces the family $\mathfrak{U}_{\mathrm{rig}^\dagger}$. □

A technical digression: We need the following technical results to prove the fullness of $\mathrm{rig}^\dagger$. We deduce them from the analogous statements in the rigid analytic context ([1, Lemmata 1.4.12 and 1.4.13]).

Lemma 4.8.

(i) Let $W_d \to A$ be a finite monomorphism where A is a K-algebra which is torsion-free as a W_d-module. Suppose $f \in A$. Then there exists a unique monic polynomial $p_f = \zeta^r + a_1\zeta^{r-1} + \cdots + a_r \in W_d[\zeta]$ of minimal degree such that $p_f(f) = 0$. Furthermore, p_f generates the kernel of the W_d-homomorphism

$$W_d[\zeta] \longrightarrow A; \qquad \zeta \mapsto f$$

(ii) Keep the same notation as above. Suppose $x \in \operatorname{Max} W_d$ and let $y_1, \ldots, y_s$ be the maximal ideals in A lying over x.
Then $\max_{i=1,\ldots,s} |f(y_i)| = \max_{j=1,\ldots,r} |a_j(x)|^{1/j}$. Hence $|f|_{\sup} = \max_{j=1,\ldots,r} |a_j|^{1/j}$.

(iii) Let $\phi : B \to A$ be a finite morphism of dagger algebras. Then, for any $f \in A$ there is an integral monic equation

$$f^r + b_1 f^{r-1} + \cdots + b_r = 0$$

with coefficients $b_i \in B$ such that $|f|_{\sup} = \max_{i=1,\ldots,r} |b_i|_{\sup}^{1/i}$.

Proof. (i) Consider the subalgebra $W_d[f]$ of A generated over W_d by f. This is still finite over W_d, since W_d is noetherian. Consider the surjection $\sigma : W_d[\zeta] \to W_d[f]$ given by $\zeta \mapsto f$. Let the kernel of this surjection be I. The above surjection can be extended to a surjection (also denoted by σ) $\sigma : W_{d+1} = W_d\langle\zeta\rangle^\dagger \to W_d[f]$ (see [9, 1.9]) so that we obtain an isomorphism $W_{d+1}/I \simeq W_d[f]$. Taking completions we get isomorphisms

$$T_{d+1}/(I) \simeq T_d \langle\zeta\rangle /(I) \simeq T_d \langle\zeta\rangle /(p_f(\zeta)) \simeq T_d[f]$$

where $p_f(\zeta) \in T_d[\zeta]$ is the unique monic polynomial of minimal degree given by [1, Lemma 1.4.12].

For $\rho > 1$, consider $T_d(\rho)$ (along with its Gauss norm) which consists of $z \in T_d$ such that $|z|_\rho < \infty$. Let $\delta > 0$ corresponding to ρ be such that, $\gamma_\delta^{(d)}$ gives the corresponding pseudo-valuation on $T_d(\rho)$ (i.e. $T_d(\rho) = \{z \in T_d \mid \gamma_\delta^{(d)}(z) > -\infty\}$). We can extend $\gamma_\delta^{(d)}$ to $\gamma_\delta^{(d+1)}$ on $T_d \langle\zeta\rangle$ by setting:

$$\gamma_\delta^{(d+1)}\Big(\sum_j a_j \zeta^j\Big) = \min_j\{\gamma_\delta^{(d)}(a_j) - \delta \cdot j\}$$

([5, 4.43]). By construction, there exists $\rho > 1$ such that f is contained in the image of $T_{d+1}(\rho) \to T_d[f]$, and hence contained in $T_d(\rho)[f]$. Let $\delta > 0$ corresponding to ρ be defined as above and let $\tilde{\gamma_\delta}$ be the quotient seminorm of $\gamma_\delta^{(d+1)}$ on $T_d[f]$. Then, $\tilde{\gamma_\delta}(f) > -\infty$. Note that $T_d[f]$ is finite free as a T_d-module, and is of degree s, where s is the degree of $p_f(\zeta)$. Let $p_f(\zeta) = \zeta^s + a_{s-1}\zeta^{s-1} + \ldots + a_1\zeta + a_0$ with coefficients $a_i \in T_d$, and hence $f^s = -a_{s-1}f^{s-1} - \ldots - a_1 f - a_0$. On $T_d^s \simeq T_d[f]$ (isomorphic as T_d-modules), consider the order function $\mu_\delta(t_1, \cdots, t_s) = \min_i\{\gamma_\delta^{(d)}(t_i)\}$ where $(t_1, \ldots, t_s) \in$

T_d^s. Using [6, Lemma 1.10], the pseudo-valuation $\tilde{\gamma_\delta}$ is linearly equivalent to μ_δ in the sense of [6, Def. 1.7]. Hence

$$\tilde{\gamma_\delta}(f) > -\infty \Longleftrightarrow \min_i \{\gamma_\delta^{(d)}(a_i)\} > -\infty.$$

This shows that $p_f(\zeta) \in W_d[\zeta]$ as required.

(ii) This follows from the previous statement, that $\operatorname{Max} A = \operatorname{Max} A'$ ([9, Thm. 1.7]) and the corresponding statement in the rigid analytic context ([1, Lemma 1.4.12]).

(iii) Let A' and B' denote the completions of A and B respectively. Consider the following diagram:

$$\begin{array}{ccc} B' & \xrightarrow{\phi'} & A' \\ \uparrow & & \uparrow \\ B & \xrightarrow{\phi} & A. \end{array}$$

Using Noether normalization for dagger algebras ([9, 1.4]), there is a morphism $W_d \to B$ which induces a finite monomorphism $W_d \hookrightarrow B/\ker\phi$, so that $W_d \to A$ is a finite monomorphism. Then $T_d \to B'$ induces a finite monomorphism $T_d \to A'$ ([9, 1.12]). As in the proof of [1, Lemma 1.4.13], we see that there is an integral equation

$$f^r + a_1 f^{r-1} + \cdots + a_r = 0$$

with coefficients $a_j \in T_d$ whose images $b_j \in B'$ satisfy $|f|_{\sup} = \max_{j=1,\ldots,r} |b_j|_{\sup}^{1/j}$. Now using the arguments in the proof of i), we see that the coefficients a_j are in fact in W_d. Therefore, replacing a_j by its image $b_j \in B$, we obtain an integral equation $f^r + b_1 f^{r-1} + \cdots + b_r = 0$ with coefficients $b_j \in B$ for $j = 1, \ldots, r$.

Further, since $\operatorname{Max} A = \operatorname{Max} A'$ we obtain

$$|f|_{\sup} = \max_{x \in \operatorname{Max} A} |f(x)| = \max_{x \in \operatorname{Max} A'} |f(x)| = \max_{j=1,\ldots,r} |b_j|^{1/j}. \qquad \square$$

As an aside, we prove Noether normalization for wcfg algebras.

Proposition 4.9. *Let A be a wcfg algebra. Then A is integral over $R[\zeta_1, \ldots, \zeta_d]^\dagger$ for some finite set of variables $\zeta_1, \ldots, \zeta_d$.*

Proof. Let $A_{\mathrm{rig}^\dagger} = A \otimes_R K$ and consider the inclusion $A \hookrightarrow A_{\mathrm{rig}^\dagger}$. Since A is a wcfg algebra, we can find a surjection $\alpha : R[\zeta_1, \ldots, \zeta_n]^\dagger \to A$ for some $n \in \mathbb{N}$. On tensoring with K, we obtain the surjection $\alpha_K : K\langle \zeta_1, \ldots, \zeta_n\rangle^\dagger \to A_{\mathrm{rig}^\dagger}$. By Noether normalization for dagger algebras ([9, 1.4(2)]), there exists a finite injection $K\langle \zeta_1, \ldots, \zeta_d\rangle^\dagger \to K\langle \zeta_1, \ldots, \zeta_n\rangle^\dagger \to A_{\mathrm{rig}^\dagger}$ for some $d \in \mathbb{N}$, so that we can consider the following diagram:

$$\begin{array}{ccccc} K\langle \zeta_1, \ldots, \zeta_d\rangle^\dagger & \longrightarrow & K\langle \zeta_1, \ldots, \zeta_n\rangle^\dagger & \xrightarrow{\alpha_K} & A_{\mathrm{rig}^\dagger} \\ \uparrow & & \uparrow & & \uparrow \\ R[\zeta_1, \ldots, \zeta_d]^\dagger & \longrightarrow & R[\zeta_1, \ldots, \zeta_n]^\dagger & \xrightarrow{\alpha} & A. \end{array}$$

Now, note that A consists of all those elements $f \in A_{\mathrm{rig}^\dagger}$ which satisfy $|f|_{\alpha_K} \leq 1$. This implies that any such element $f \in A$ also satisfies $|f|_{\sup} \leq 1$. Using Lemma 4.8(iii), f satisfies an integral equation with coefficients a_j in $K\langle \zeta_1, \ldots, \zeta_d \rangle^\dagger$ satisfying $|f|_{\sup} = \max_j |a_j|_{\sup}^{1/j}$. Since $|f|_{\sup} \leq 1$, we get $|a_j|_{\sup} = |a_j| \leq 1$. Therefore, $a_j \in R[\zeta_1, \ldots, \zeta_d]^\dagger$ and hence A is integral over $R[\zeta_1, \ldots, \zeta_d]^\dagger$. □

Proof of the main theorem:

Lemma 4.10. *Let A be an admissible wcfg algebra and $A_{\mathrm{rig}^\dagger}$ be the corresponding dagger algebra. Suppose $f_1, \ldots, f_n$ are elements in $A_{\mathrm{rig}^\dagger}$ such that $|f_i|_{sup} \leq 1$ for $i = 1, \ldots, n$. Then $A' = A[f_1, \ldots, f_n]$ is an admissible wcfg algebra which is finite over A. Further, the canonical morphism $\tau : \operatorname{Spwf} A' \to \operatorname{Spwf} A$ coincides with the weak formal blowup of the coherent open ideal $\mathfrak{a} = (\pi^r, \pi^r f_1, \ldots, \pi^r f_n) \subset A$ for some r chosen so that $\pi^r f_i \in A$ for all i. (Note that this corresponds to* [1, Lemma 2.8.5]*.)*

Proof. Let $R[\zeta_1, \ldots, \zeta_n]^\dagger \to A$ be a surjective homomorphism with $\zeta_1, \ldots, \zeta_n$ some variables and let $\alpha : K\langle \zeta_1, \ldots, \zeta_n \rangle^\dagger \to A_{\mathrm{rig}^\dagger}$ be the induced surjection.

By Noether normalization for dagger algebras ([9, 1.4.2]), $A_{\mathrm{rig}^\dagger}$ admits a finite injection $K\langle \zeta_1, \ldots, \zeta_d \rangle^\dagger \hookrightarrow K\langle \zeta_1, \ldots, \zeta_n \rangle^\dagger \to A_{\mathrm{rig}^\dagger}$ for some $d \in \mathbb{N}$. Using Lemma 4.8(iii), f_i (for some $1 \leq i \leq n$) satisfies an integral equation with coefficients $a_j \in K\langle \zeta_1, \ldots, \zeta_d \rangle^\dagger$ satisfying $|f_i|_{\sup} = \max_j |a_j|_{\sup}^{1/j}$. Since $|f_i|_{\sup} \leq 1$, the coefficients of the integral equation satisfy $|a_j|_{\sup} = |a_j| \leq 1$.

The images $\bar{a}_j \in A_{\mathrm{rig}^\dagger}$ of these coefficients satisfy $|\bar{a}_j|_\alpha \leq 1$ and hence lie in A. So A' is integral and of finite type over A, hence finite over A. This implies that A' is an admissible wcfg algebra with associated dagger algebra ${A'}_{\mathrm{rig}^\dagger} = A_{\mathrm{rig}^\dagger}$.

Consider the scheme theoretic blowup $Y \to \operatorname{Spec} A$ of the ideal

$$\mathfrak{a} = (\pi^r, \pi^r f_1, \ldots, \pi^r f_n) \subset A$$

on $\operatorname{Spec} A$. The ideal $\mathfrak{a}A' \subset A'$ is generated by π^r and hence is invertible. Therefore, using Proposition 3.2(b), $\operatorname{Spec} A' \to \operatorname{Spec} A$ factorizes via Y.

The locus V in Y, where $\mathfrak{a}\mathcal{O}_Y$ is generated by π^r, coincides with the image of $\operatorname{Spec} A'$. We claim that $V = Y$. Y is covered by $(\operatorname{Spec} A_i)_{i=0,\ldots,r}$ where

$$A_i = A\left[\frac{\pi^r}{\pi^r f_i}, \frac{\pi^r f_1}{\pi^r f_i}, \ldots, \frac{\pi^r f_n}{\pi^r f_i}\right] / (\pi - \text{torsion}).$$

To prove the claim we show that $\mathfrak{a}A_i$ is generated by π^r. Suppose $\mathfrak{a}A_i$ is generated by $\pi^r f_i$ for some i. Consider the inclusions $A_i \hookrightarrow A_i[f_i] \hookrightarrow A_{\mathrm{rig}^\dagger}$. Since $\pi^r \in (\pi^r f_i)A_i$, and π^r is not a zero divisor in A_i, f_i is invertible in $A_i[f_i]$ with inverse $f_i^{-1} \in A_i$. Let $f_i^s + a_1 f_i^{s-1} + \ldots + a_s = 0$ be an integral equation for f_i over A_i. Multiplying the above equation with $f_i^{-s+1} \in A_i$ we get $f_i + a_1 + \ldots + a_s f_i^{-s+1} = 0$. Therefore, $f_i \in A_i$ and hence is a unit in A_i, which implies that, $\mathfrak{a}A_i$ is generated by π^r. So, $Y \to \operatorname{Spec} A$ coincides with

Spec $A' \to$ Spec A. Taking weak completions and using [15, Thm. 1.5] we see that Spwf $A' \to$ Spwf A is the weak formal blowup of $\mathfrak{a}$ on Spwf A. □

We can now prove the main result announced at the beginning and show that $\mathrm{rig}^\dagger$ induces an equivalence of categories:

Theorem 4.11. *There is an equivalence of categories between*

- *the category* $(\mathrm{Wf\,Sch}\,/R)_S$ *of quasi-paracompact admissible weak formal R-schemes, localized by the class S of weak formal blowups, and*
- *the category* Dag/K *of quasi-separated quasi-paracompact K-dagger spaces.*

We prove that the functor $\mathrm{rig}^\dagger$ from the category of quasi-paracompact admissible weak formal R-schemes to the category of quasi-separated quasi-paracompact K-dagger spaces satisfies the condition of localization by the class S of weak formal blowups, i.e. we show that $\mathrm{rig}^\dagger$ is universal with respect to functors from quasi-paracompact admissible weak formal schemes to a category $\mathfrak{D}$ such that weak formal blowups are mapped to isomorphisms. To show this we can follow the proof in [1, Thm. 2.8.3], using Proposition 4.6 and the following results (i, ii, iii and iv). In particular, the induced functor from the category $(\mathrm{Wf\,Sch}\,/R)_S$ to Dag/K remains faithful.

(i) Two morphisms of admissible weak formal R-schemes, $\phi, \psi : \mathfrak{X} \to \mathfrak{Y}$ coincide if the associated morphisms $\phi_{\mathrm{rig}^\dagger}, \psi_{\mathrm{rig}^\dagger} : \mathfrak{X}_{\mathrm{rig}^\dagger} \to \mathfrak{Y}_{\mathrm{rig}^\dagger}$ coincide.

Proof. Consider the following commutative diagram of functors:

$$\begin{array}{ccc} \text{Admissible weak formal } R\text{-schemes} & \xrightarrow[\text{Formal completion}]{\mathcal{F}} & \text{Admissible formal } R\text{-schemes} \\ \mathrm{rig}^\dagger \big\downarrow & & \mathrm{rig} \big\downarrow \\ K\text{-dagger spaces} & \xrightarrow{\mathcal{F}'} & \text{Rigid } K\text{-analytic spaces.} \end{array}$$

Then, from [9, Thm. 2.19], the functor $\mathcal{F}'$ is faithful. From the proof of Raynaud's theorem (as in [1, Thm. 2.8.3b]), the functor rig is faithful. Using [11, 10.6.10], the formal completion functor $\mathcal{F}$ is faithful. Therefore, $\mathrm{rig}^\dagger$ is also faithful. □

(ii) Let $\mathfrak{X}, \mathfrak{Y}$ be quasi-paracompact admissible weak formal schemes with a morphism $\Phi : \mathfrak{X}_{\mathrm{rig}^\dagger} \to \mathfrak{Y}_{\mathrm{rig}^\dagger}$ between the associated dagger spaces. Then there exists a weak formal blowup $\tau : \mathfrak{X}' \to \mathfrak{X}$ and a morphism $\phi : \mathfrak{X}' \to \mathfrak{Y}$ of admissible weak formal schemes such that $\phi_{\mathrm{rig}^\dagger} = \Phi \circ \tau_{\mathrm{rig}^\dagger}$.

Proof. Let us first consider the affine situation where $\mathfrak{X} =$ Spwf A and $\mathfrak{Y} =$ Spwf B and let $\Phi^\# : B \otimes_R K = B_{\mathrm{rig}^\dagger} \to A_{\mathrm{rig}^\dagger} = A \otimes_R K$ be the morphism induced from Φ. Choose a surjection $\alpha : R[\zeta_1, \ldots, \zeta_n]^\dagger \to B$ with $(\zeta_1, \ldots, \zeta_n)$ some variables and let $\alpha_K : K\langle\zeta_1, \ldots, \zeta_n\rangle^\dagger \to B_{\mathrm{rig}^\dagger}$ be the induced surjection. Note that $|b|_{\sup} \leq 1$ for all $b \in B$.

Now, consider the composition of maps

$$R[\zeta_1,\ldots,\zeta_n]^\dagger \stackrel{\alpha}{\to} B \hookrightarrow B_{\mathrm{rig}^\dagger} \stackrel{\Phi^\#}{\to} A_{\mathrm{rig}^\dagger}.$$

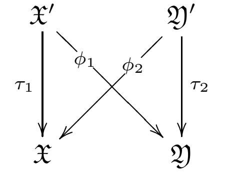
If f_i denotes the image of ζ_i in $A_{\mathrm{rig}^\dagger}$, then $|f_i|_{\sup} \leq 1$, since $\Phi^\#$ is contractive. Using Lemma 4.10, $A' = A[f_1,\ldots,f_n]$ is a wcfg algebra satisfying ${A'}_{\mathrm{rig}^\dagger} = A_{\mathrm{rig}^\dagger}$ such that $\tau : \operatorname{Spwf} A' \to \operatorname{Spwf} A$ is a weak formal blowup. Further, $\Phi^\#$ restricts to a morphism $B \to A'$ inducing a morphism $\phi : \operatorname{Spwf} A' \to \operatorname{Spwf} B$ which satisfies $\phi_{\mathrm{rig}^\dagger} = \Phi \circ \tau_{\mathrm{rig}^\dagger}$. This settles the assertion in the affine case.

Consider the general case, where $\mathfrak{X}$ and $\mathfrak{Y}$ are quasi-paracompact admissible weak formal schemes. Fix affine open coverings of finite type $\mathfrak{U}$ and $\mathfrak{V}$ of $\mathfrak{X}$ and $\mathfrak{Y}$ respectively and let $\mathfrak{U}_{\mathrm{rig}^\dagger}$ and $\mathfrak{V}_{\mathrm{rig}^\dagger}$ be the induced admissible coverings of $\mathfrak{X}_{\mathrm{rig}^\dagger}$ and $\mathfrak{Y}_{\mathrm{rig}^\dagger}$. Then $\Phi^{-1}(\mathfrak{V}_{\mathrm{rig}^\dagger})$ is an admissible covering of $\mathfrak{X}_{\mathrm{rig}^\dagger}$. Writing each $U_{\mathrm{rig}^\dagger} \in \mathfrak{U}_{\mathrm{rig}^\dagger}$ as a union of elements in $\Phi^{-1}(\mathfrak{V}_{\mathrm{rig}^\dagger})$, we obtain a refinement $\mathfrak{W}_{\mathrm{rig}^\dagger}$ of $\mathfrak{U}_{\mathrm{rig}^\dagger}$ which is an admissible covering of finite type and such that each element of $\mathfrak{W}_{\mathrm{rig}^\dagger}$ is mapped by Φ into some element in $\mathfrak{V}_{\mathrm{rig}^\dagger}$. Using Lemma 4.7 for the covering $\mathfrak{W}_{\mathrm{rig}^\dagger}$ of $\mathfrak{X}_{\mathrm{rig}^\dagger}$, we obtain a weak formal blowup $\rho : \mathfrak{X}' \to \mathfrak{X}$ and an affine open covering $\mathfrak{U}'$ of $\mathfrak{X}'$ such that ${\mathfrak{U}'}_{\mathrm{rig}^\dagger} = \mathfrak{W}_{\mathrm{rig}^\dagger}$. Since for any $U' \in \mathfrak{U}'$, the associated open subspace $U'_{\mathrm{rig}^\dagger}$ of $\mathfrak{X}_{\mathrm{rig}^\dagger}$ is mapped to some $V_{\mathrm{rig}^\dagger} \in \mathfrak{V}_{\mathrm{rig}^\dagger}$, using the result in the affine case, we can find a weak formal blowup $\tau_{U'} : U'' \to U'$ and a morphism of weak formal schemes $\phi_{U'} : U'' \to V \hookrightarrow Y$ such that $\phi_{U',\mathrm{rig}^\dagger} = \Phi|_{{U'}_{\mathrm{rig}^\dagger}} \circ \tau_{U',\mathrm{rig}^\dagger}$. Further, using 3.6 (a), there exists a weak formal blowup $\tau : \mathfrak{X}'' \to \mathfrak{X}'$ satisfying the property that for each $U' \in \mathfrak{U}'$ there exists a corresponding unique morphism $\sigma_{U'} : \tau^{-1}(U') \to U''$ such that $\tau|_{\tau^{-1}(U')} = \tau_{U'} \circ \sigma_{U'}$. The morphisms $\phi_{U'} \circ \sigma_{U'}$ can be glued to obtain a well defined morphism $\phi : \mathfrak{X}'' \to \mathfrak{Y}$. Further, the induced maps under $\mathrm{rig}^\dagger$ satisfy $\phi_{U',\mathrm{rig}^\dagger} \circ \sigma_{U',\mathrm{rig}^\dagger} = \Phi|_{U'_{\mathrm{rig}^\dagger}} \circ \tau|_{\tau^{-1}(U'),\mathrm{rig}^\dagger}$. Now, using the faithfulness of $\mathrm{rig}^\dagger$ (4.11(i)) we see that the morphism ϕ constructed above satisfies $\phi_{\mathrm{rig}^\dagger} = \Phi \circ \tau_{\mathrm{rig}^\dagger}$. □

We can in fact extend this result to show:

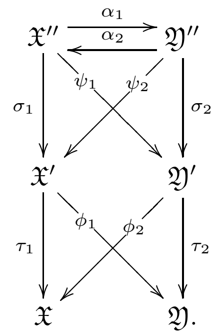
(iii) Let $\mathfrak{X}$ and $\mathfrak{Y}$ be quasi-compact admissible weak formal schemes such that the morphism between the associated dagger spaces $\Phi : \mathfrak{X}_{\mathrm{rig}^\dagger} \xrightarrow{\sim} \mathfrak{Y}_{\mathrm{rig}^\dagger}$ is an isomorphism. Then, there exist weak formal blowups $\tau : \mathfrak{X}' \to \mathfrak{X}$ and $\phi : \mathfrak{X}' \to \mathfrak{Y}$ satisfying $\phi_{\mathrm{rig}^\dagger} = \Phi \circ \tau_{\mathrm{rig}^\dagger}$.

Proof. Using 4.11(ii) for the morphisms Φ and Φ^{-1}, we obtain the following diagram:



where τ_1, τ_2 are weak formal blowups of coherent open ideals $\mathcal{A} \subset \mathcal{O}_{\mathfrak{X}}$, $\mathcal{B} \subset \mathcal{O}_{\mathfrak{Y}}$ respectively and ϕ_1, ϕ_2 are morphisms of weak formal schemes.

Now, consider the weak formal blowing-up σ_1 of $\mathcal{B}\mathcal{O}_{\mathfrak{X}'}$ on $\mathfrak{X}'$ and similarly, the weak formal blowing-up σ_2 of $\mathcal{A}\mathcal{O}_{\mathfrak{Y}'}$ on $\mathfrak{Y}'$. Due to the universal property of blowups (Proposition 3.2(b)), $\phi_1 \circ \sigma_1$ factors through $\mathfrak{Y}'$ via a morphism ψ_1 and similarly, $\phi_2 \circ \sigma_2$ factors through $\mathfrak{X}'$ via a morphism ψ_2. Further, since $\mathcal{A} \subset \mathcal{O}_{\mathfrak{X}}$ remains invertible in $\mathcal{O}_{\mathfrak{X}''}$, using the universal property of blowups for σ_2, we see that ψ_1 factors through $\mathfrak{Y}''$ via a morphism α_1 and similarly ψ_2 factors through $\mathfrak{X}''$ via a morphism α_2. Thereby, we obtain the following diagram



Applying $\mathrm{rig}^\dagger$ to the above diagram, we obtain a diagram which commutes with Φ and Φ^{-1} since all the blowups become isomorphisms under $\mathrm{rig}^\dagger$ (Proposition 4.6). Further, because the functor $\mathrm{rig}^\dagger$ is faithful (Theorem 4.11(i)), we see that the above diagram is also commutative and so α_1 and α_2 are inverse to each other. Then, the assertion follows from the fact that the composition of two weak formal blowups is again a weak formal blowup (Proposition 3.5). □

(iv) Let $\mathfrak{X}_K$ be a dagger space which is quasi-paracompact and quasi-separated. Then, there exists a quasi-paracompact admissible weak formal scheme $\mathfrak{X}$ such that $\mathfrak{X}_{\mathrm{rig}^\dagger} \simeq \mathfrak{X}_K$.

Proof. Fix an admissible covering $(\mathfrak{X}_{i,K})_{i \in J}$ of finite type consisting of quasi-compact open subspaces $\mathfrak{X}_{i,K}$ of $\mathfrak{X}_K$ and consider first the case when J is finite. We may assume that $\mathfrak{X}_{i,K}$ is affinoid for all $i \in J$ ([9, Prop. 2.8]). We prove the assertion by induction on the number of elements in the covering of $\mathfrak{X}_K$. For the case when the covering consists of a single element, $\mathfrak{X}_K = \operatorname{Sp} A_K$ where A_K is a dagger algebra, fix a surjection $K\langle \zeta_1, \ldots, \zeta_n \rangle^\dagger \to A_K$. The image of $R[\zeta_1, \ldots, \zeta_n]^\dagger$ under this surjection is a wcfg algebra, denote this by A. Then $\operatorname{Spwf} A$ is a weak formal model of $\operatorname{Sp} A_K$. Now, assume that $\mathfrak{X}_K = \mathfrak{X}_{1,K} \cup \mathfrak{X}_{2,K}$ where $\mathfrak{X}_{1,K}$ and $\mathfrak{X}_{2,K}$ are quasi-compact open subspaces of $\mathfrak{X}_K$ admitting weak formal models $\mathfrak{X}_1$ and $\mathfrak{X}_2$ respectively. Since $\mathfrak{X}_K$ is quasi-separated, $U_K = \mathfrak{X}_{1,K} \cap \mathfrak{X}_{2,K}$ is quasi-compact. We can now choose a finite covering of $\mathfrak{X}_{1,K}$ by affinoid subdomains and restrict it to U_K to get a finite admissible covering of U_K consisting of affinoid subdomains. Using Lemma 4.7 we obtain a weak formal blowup $\mathfrak{X}'_1 \to \mathfrak{X}_1$ such that the open immersion $U_K \to \mathfrak{X}_{1,K}$ is induced by an open immersion $U_1 \to \mathfrak{X}'_1$, where $U_{1,\mathrm{rig}^\dagger} = U_K$. Similarly, using Lemma 4.7 for $\mathfrak{X}_{2,K}$ we obtain another weak formal model U_2 for U_K. Applying Theorem 4.11(iii), we obtain weak formal blowups $U \to U_1$ and $U \to U_2$ which can be extended to weak formal blowups $\mathfrak{X}''_1 \to \mathfrak{X}_1$ and $\mathfrak{X}''_2 \to \mathfrak{X}_2$ (Proposition 3.4). Now, $\mathfrak{X}''_1$ and $\mathfrak{X}''_2$ can be glued along U to obtain a weak formal model $\mathfrak{X}$ such that $\mathfrak{X}_{\mathrm{rig}^\dagger} = \mathfrak{X}_K$. Following the proof in [1, Prop. 2.8.3e] and using Proposition 3.4, we can extend this construction to the general case. □

This completes the proof of Theorem 4.11.

Corollary 4.12. *Let $\mathfrak{X}$ be a separated K-scheme of finite type. Then the dagger space $\mathfrak{X}^{\mathrm{rig}^\dagger}$ obtained via the dagger analytification functor (*[9, 3.3]*) is a separated and quasi-paracompact dagger space and hence admits a weak formal model X.*

Proof. The rigid analytification functor ([1, §1.13]) associates a partially proper rigid analytic space $\mathfrak{X}^{\mathrm{rig}}$ to $\mathfrak{X}$. Furthermore, $\mathfrak{X}^{\mathrm{rig}}$ is separated and quasi-paracompact ([1, Prop. 2.8.6]). Using [9, 2.27], one can associate a partially proper dagger space $\mathfrak{X}^{\mathrm{rig}^\dagger}$ to $\mathfrak{X}^{\mathrm{rig}}$. It is clearly quasi-paracompact, since they have the same underlying topological spaces. Further, using [9, 2.19.4], we see that $\mathfrak{X}^{\mathrm{rig}}$ is separated, so that $\mathfrak{X}^{\mathrm{rig}^\dagger}$ is separated as well. □

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