Münster J. of Math. **12** (2019), 239–410
DOI 10.17879/53149724705
urn:nbn:de:hbz:6-53149724977

Münster Journal of Mathematics

The term a_4 in the heat kernel expansion of noncommutative tori

Alain Connes and Farzad Fathizadeh

(Communicated by Joachim Cuntz)

Abstract. We consider the Laplacian associated with a general metric in the canonical conformal structure of the noncommutative two torus, and calculate a local expression for the term a_4 that appears in its corresponding small-time heat kernel expansion. The final formula involves one variable functions and lengthy two, three and four variable functions of the modular automorphism of the state that encodes the conformal perturbation of the flat metric. We confirm the validity of the calculated expressions by showing that they satisfy a family of conceptually predicted functional relations. By studying these functional relations abstractly, we derive a partial differential system which involves a natural action of cyclic groups of order 2, 3 and 4 and a flow in parameter space. We discover symmetries of the calculated expressions with respect to the action of the cyclic groups. In passing, we show that the main ingredients of our calculations, which come from a rearrangement lemma and relations between the derivatives up to order 4 of the conformal factor and those of its logarithm, can be derived by finite differences from the generating function of the Bernoulli numbers and its multiplicative inverse. We then shed light on the significance of exponential polynomials and their smooth fractions in understanding the general structure of the noncommutative geometric invariants appearing in the heat kernel expansion. As an application of our results we obtain the a_4 term for noncommutative four tori which split as products of two tori. These four tori are not conformally flat and the a_4 term gives a first hint of the Riemann curvature and the higher-dimensional modular structure.

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The second author is supported by the Marie Curie/SER Cymru II Cofund Research Fellowship 663830-SU-008.

1. Introduction

In noncommutative geometry the paradigm of a geometric space is given in spectral terms, and the local geometric invariants such as the Riemannian curvature are extracted from the functionals defined by the coefficients of heat kernel expansion. One of the new features of the theory which is entirely due to noncommutativity is the modular theory which associates to a state a one parameter group of automorphisms of the ambient von-Neumann algebra measuring to which extent the state fails to be a trace. When the state is associated to the volume form of a metric, a deep interplay arises between the local geometric invariants and the modular automorphism group. More specifically, the local geometric invariants of the noncommutative two torus $\mathbb{T}^2_\theta$ equipped with a curved metric are computed by calculating the coefficients that appear in the small-time heat kernel expansion of the Laplacian associated with the metric [15, 14, 20, 23]. The canonical flat metric of $\mathbb{T}^2_\theta$ can be perturbed conformally by means of a Weyl factor $e^h \in C^\infty(\mathbb{T}^2_\theta)$, where the dilaton h is a smooth selfadjoint element [15]. The effect of this perturbation is that the canonical trace or the flat volume form φ_0 of the noncommutative torus is replaced by a state φ, and the trace of the heat kernel $\exp(-t\triangle_\varphi)$ of the Laplacian $\triangle_\varphi$ of the curved metric has an asymptotic expansion with complicated coefficients.

That means there are unique elements $a_{2n} \in C^\infty(\mathbb{T}^2_\theta)$ such that for any $a \in C^\infty(\mathbb{T}^2_\theta)$, as the time $t \to 0^+$, there is an asymptotic expansion of the form[1]

$$\text{Trace}(a \exp(-t\triangle_\varphi)) \sim t^{-1}(\varphi_0(a\, a_0) + \varphi_0(a\, a_2)\, t + \varphi_0(a\, a_4)\, t^2 + \cdots). \quad (1)$$

In fact, each term a_{2n} is a curvature related invariant of the noncommutative torus $\mathbb{T}^2_\theta$ equipped with the curved metric.

[1] By the construction, since on a noncommutative torus of dimension m we use the normalized trace as the analog of the integration on the m-dimensional torus $\mathbb{T}^m = (\mathbb{R}/2\pi\mathbb{Z})^m$, we incorporate in our calculations the overall multiplicative factor $(2\pi)^m$ for the geometric invariants.

The exploration of the interplay between the local geometric invariants and the modular automorphism group has involved over the years alternating periods of hard calculations and conceptual understanding of their meaning [15, 14, 20, 22, 23]. So far one has reached a good understanding of the terms a_0 and a_2 in the heat expansion (1). The term a_0 is related to the volume, whose connection with the analog of Weyl's law and the trace theorem of [6] is studied in [19]. The term a_2, which is related to the analog of scalar curvature and the Gauss–Bonnet theorem for the noncommutative two torus [15, 18], is calculated and studied in [14, 20, 23]. The present paper investigates for the first time the hard calculation of the term $a_4 \in C^\infty(\mathbb{T}_\theta^2)$ appearing in (1). Due to the fact that the process of calculating this term involves exceedingly lengthy expressions and at times involves manipulations on a few hundred thousand terms, only the final outputs of the calculations are written in this paper.

Our key result is that we could confirm the validity of the lengthy calculations by checking that the final expressions satisfy a family of functional relations, conceptually proved along the lines of [14]. We derive a system of partial differential equations from the functional relations by specializing them to certain hyperplanes and study symmetries of certain combinations of our expressions with respect to a natural action of cyclic groups of order 2, 3 and 4 on the differential system. We also pay special attention to the general structure of the several variable functions that appear in the expression of the term a_4, which is closely related to the fact that the main ingredients of such calculations can be derived by finite differences from the generating function of the Bernoulli numbers and its inverse.

The results of this paper are presented in two main different parts. The first consists of Sections 2–12, where we mainly address the mathematically abstract work carried out for the calculation of the term $a_4 \in C^\infty(\mathbb{T}_\theta^2)$ and exploring its properties. The second part, which is roughly the last 50 pages, is formed by the appendices, where we present final outputs of our calculations that have lengthy expressions, in their simplified form. The aspect of this part of the paper shows that a far more suitable mean for communicating these lengthy expressions is to make them available in a mathematical program notebook, which is done in [8]. This in particular makes it easy to test further phenomena related to the analog of the Riemann curvature in noncommutative geometry.

In Section 2 we recall necessary preliminaries about the canonical translation invariant conformal structure on the noncommutative two torus $\mathbb{T}_\theta^2$, the Laplacian associated with a general metric in the conformal class, and the explicit formula for the term a_2 appearing in the expansion (1). In Section 3 we write the final expression for the term $a_4 \in C^\infty(\mathbb{T}_\theta^2)$ in the expansion, which involves one, two, three and four variable functions of a modular automorphism. We then present one of our main results, namely a family of functional relations that the several variable functions satisfy. As we shall see, it is quite interesting that finite differences of the inverse of the generating function of the Bernoulli numbers play an important role in the functional relations. In Section 4 we derive a partial differential system from the functional relations by specializing them to certain hyperplanes, and study symmetries of our finite difference expressions with respect to a natural action of cyclic groups of order 2, 3 and 4 on the system as well as a natural flow associated to the differential system. In Section 5, we explain the method of proving the functional relations, and prove a series of lemmas that will be used in Section 6 for calculating in terms of finite differences the gradient of the map that sends the dilaton $h = h^* \in C^\infty(\mathbb{T}_\theta^2)$ to the trace of the term a_4. Section 7 presents functional relations among functions that appear in the differential system. Some of the relations of this type, because of their lengthiness, are written in Appendix B.

In Section 8 we explain the details of our calculation of the term a_4, which is based on using the pseudodifferential calculus developed in [3], a rearrangement lemma, and a lemma proved in Section 6 that relates the derivates up to order 4 of the conformal factor $e^h \in C^\infty(\mathbb{T}_\theta^2)$ and those of the dilaton h. In particular, we show that all functions of several variables appearing in these lemmas can be constructed by finite differences from the generating function of the Bernoulli numbers and its inverse. By employing these tools and performing heavy calculations, we find the expression for the term a_4 and explicit formulas for the functions of one to four variables that appear in the final formula. The one and two variable functions are presented in Section 9, while, because of having algebraically lengthy expressions, the three and four variable functions are presented in Appendix C. We confirm the accuracy of our heavy calculations by explaining that the final functions check out the functional relations presented in Section 3 and in Appendix A. In Section 10, we explain why each local function appearing in the expression of the term a_4 is a rational function in variables s_i and $e^{s_i/2}$, whose denominator has a nice product formula that vanishes on certain hyperplanes. Moreover, we show that the coefficients of the numerator of each function satisfy a family of linear equations.

In Section 11 we explain how the explicit formulas for the term a_4 for $\mathbb{T}_\theta^2$ provide a first glimpse of the Riemann curvature beyond the conformally flat case. Indeed it yields the term a_4 for the four-dimensional noncommutative tori obtained as products of two noncommutative two tori and such spaces are generally not conformally flat. Moreover, they possess a natural two-dimensional modular structure, given by an action of $\mathbb{R}^2$ by automorphisms, obtained from the modular structure of the factors, while the measure theory given by the determinant part of the metric only involves the restriction of this action of $\mathbb{R}^2$ to the diagonal. This gives a strong motivation to develop conceptually the more general notion of twisting suggested in particular in [4] and which plays a fundamental role in the work of H. Moscovici and the first author [11] on the transverse geometry of foliations and the reduction by duality to the almost isometric case.

Our main results and conclusions are summarized in Section 12.

2. Preliminaries

We consider the noncommutative two torus $\mathbb{T}^2_\theta$, whose algebra $C(\mathbb{T}^2_\theta)$ is the universal C^*-algebra generated by two unitary elements U and V that satisfy the following commutation relation for a fixed irrational real number θ:

$$VU = e^{2\pi i\theta}\, UV.$$

We have a C^*-dynamical system by considering the following action α of the ordinary two torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ on the algebra $C(\mathbb{T}^2_\theta)$ of the noncommutative torus. For any $(s_1, s_2) \in \mathbb{T}^2$, one can define

$$\alpha_{(s_1,s_2)}(U^m V^n) = e^{i(ms_1+ns_2)} U^m V^n, \quad m, n \in \mathbb{Z}. \tag{2}$$

This definition extends to an automorphism of the C^*-algebra $C(\mathbb{T}^2_\theta)$, which is the noncommutative analog of translating a continuous function defined on the torus $\mathbb{T}^2$ by (s_1, s_2).

Associated with the above action α, there are two infinitesimal generators δ_1 and δ_2, which are derivations on the space of smooth elements $C^\infty(\mathbb{T}^2_\theta)$ in $C(\mathbb{T}^2_\theta)$. More precisely, $C^\infty(\mathbb{T}^2_\theta)$ consists of all elements x in the noncommutative torus such that the mapping $(s_1, s_2) \mapsto \alpha_{(s_1,s_2)}(x)$ from $\mathbb{T}^2$ to $C(\mathbb{T}^2_\theta)$ is a smooth map. Indeed, $C^\infty(\mathbb{T}^2_\theta)$ is a dense subalgebra of $C(\mathbb{T}^2_\theta)$, and it can alternatively be described as the space of all

$$x = \sum_{m,n\in\mathbb{Z}} a_{m,n}\, U^m V^n$$

such that the sequence of complex coefficients $(a_{m,n})$ is rapidly decaying in the sense that

$$\sup_{m,n\in Z} |a_{m,n}|(1 + |n| + |m|)^k < \infty,$$

for any non-negative integer k. Therefore, each of the derivations δ_1 and δ_2 on $C^\infty(\mathbb{T}^2_\theta)$ is characterized by its action on the generators U and V, which are in fact given by

$$\delta_1(U) = U, \quad \delta_1(V) = 0, \quad \delta_2(U) = 0, \quad \delta_2(V) = V.$$

While δ_1 and δ_2 are respectively the analogs of the differential operators $-i(\partial/\partial s_1)$ and $-i(\partial/\partial s_2)$ on the ordinary two torus, we also have an analog of the integration or volume form of the flat metric. The latter is provided by the unique normalized tracial state $\varphi_0 : C(\mathbb{T}^2_\theta) \to \mathbb{C}$, which is defined on the smooth dense subalgebra $C^\infty(\mathbb{T}^2_\theta)$ by

$$\varphi_0\Big(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n\Big) = a_{0,0}.$$

In fact, the uniqueness of this trace is due to the irrationality of θ. An important property of the trace φ_0 is its invariance under the action α, which yields

$$\varphi_0 \circ \delta_j = 0, \quad j = 1, 2,$$

hence the analog of integration by parts:

$$\varphi_0(x_1\delta_j(x_2)) = -\varphi_0(\delta_j(x_1)\,x_2), \quad x_1, x_2 \in C^\infty(\mathbb{T}^2_\theta).$$

Following [15], we consider a complex structure on the noncommutative two torus by setting the analog of the Dolbeault operators to be

$$\partial = \delta_1 - i\delta_2 : C^\infty(\mathbb{T}^2_\theta) \subset \mathcal{H}_0 \to \mathcal{H}^{(1,0)}, \quad \bar\partial = \delta_1 + i\delta_2 : C^\infty(\mathbb{T}^2_\theta) \subset \mathcal{H}_0 \to \mathcal{H}^{(0,1)},$$

where the Hilbert spaces $\mathcal{H}_0$, $\mathcal{H}^{(1,0)}$, $\mathcal{H}^{(0,1)}$ are defined as follows. The Hilbert space $\mathcal{H}_0$ is the completion of $C(\mathbb{T}^2_\theta)$ with respect to the inner product

$$\langle x, y\rangle = \varphi_0(y^*x), \quad x, y \in C(\mathbb{T}^2_\theta).$$

In order to define the Hilbert space $\mathcal{H}^{(1,0)}$, which is the analog of the space of $(1,0)$-forms, we need to consider the bimodule over $C^\infty(\mathbb{T}^2_\theta)$ of finite sums of the form $\sum a_i\partial(b_i)$, $a_i, b_i \in C^\infty(\mathbb{T}^2_\theta)$, and complete it with an inner product that comes from a positive Hochschild cocycle. The Hilbert space $\mathcal{H}^{(0,1)}$ is obtained similarly by a completion of the space of finite sums of the form $\sum a_i\bar\partial(b_i)$, $a_i, b_i \in C^\infty(\mathbb{T}^2_\theta)$. The positive Hochschild cocycle, which encodes the conformal structure of the metric [7] on $\mathbb{T}^2_\theta$, is defined by

$$\psi(x, y, z) = -\varphi_0(x\,\partial(y)\,\bar\partial(z)), \quad x, y, z \in C^\infty(\mathbb{T}^2_\theta),$$

and the inner product for $\mathcal{H}^{(1,0)}$ is given by

$$\langle x_1\partial(y_1), x_2\partial(y_2)\rangle = \psi(x_2^*x_1, y_1, y_2^*), \quad x_1, y_1, x_2, y_2 \in C^\infty(\mathbb{T}^2_\theta).$$

With this information, we can now calculate the Laplacian associated with the flat metric

$$\triangle := \partial^*\partial = \delta_1^2 + \delta_2^2,$$

which is an unbounded selfadjoint operator acting in the Hilbert space $\mathcal{H}_0$. By using a positive invertible element $e^h \in C^\infty(\mathbb{T}^2_\theta)$, where h is a smooth selfadjoint element, one can vary the metric inside the conformal structure. That is, one can consider the state $\varphi : C(\mathbb{T}^2_\theta) \to \mathbb{C}$ defined by

$$\varphi(x) = \varphi_0(xe^{-h}), \quad x \in C(\mathbb{T}^2_\theta), \tag{3}$$

which is the analog of the volume form of the conformal perturbation of the flat metric. We consider the Hilbert space $\mathcal{H}_\varphi$ obtained from completing $C(\mathbb{T}^2_\theta)$ with respect to the inner product

$$\langle x, y\rangle_\varphi = \varphi(y^*x), \quad x, y \in C(\mathbb{T}^2_\theta).$$

Clearly, the adjoint of the operator $\partial_\varphi = \partial : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}$ depends on the conformal factor e^h. It is shown in [15] that there is an anti-unitary equivalence between the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_\varphi$ that identifies the Laplacian

$$\triangle_\varphi := \partial_\varphi^*\partial_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi,$$

with the operator $e^{h/2}\triangle e^{h/2}$ acting in $\mathcal{H}_0$. Therefore, we make the identification

$$\triangle_\varphi = e^{h/2}\triangle e^{h/2} : \mathcal{H}_0 \to \mathcal{H}_0. \tag{4}$$

A purely noncommutative feature in the calculation of the terms $a_{2n} \in C^\infty(\mathbb{T}_\theta^2)$ in the small-time asymptotic expansion (1) of $\mathrm{Trace}(a \exp(-t\triangle_\varphi))$ is the appearance of the modular automorphism of the state φ in the final formulas. That is, the linear functional φ given by (3) is a KMS state that satisfies the condition

$$\varphi(xy) = \varphi(y\,\sigma_i(x)), \quad x, y \in C(\mathbb{T}_\theta^2),$$

for the 1-parameter group of automorphisms $\{\sigma_t\}_{t\in\mathbb{R}}$ defined by

$$\sigma_t(x) = e^{ith} x e^{-ith}, \quad x \in C(\mathbb{T}_\theta^2).$$

Clearly, the modular automorphism σ_i acts by conjugation with e^{-h}, and its logarithm is thereby given by

$$\nabla(x) := \log \sigma_i(x) = -\operatorname{ad}_h(x) = -hx + xh, \quad x \in C(\mathbb{T}_\theta^2).$$

The final formula for the second term in the expansion (1), which is calculated in [14, 20], is given by

$$a_2 = R_1(\nabla)(\delta_1^2(\ell) + \delta_2^2(\ell)) + R_2(\nabla,\nabla)(\delta_1(\ell)\cdot\delta_1(\ell) + \delta_2(\ell)\cdot\delta_2(\ell)), \tag{5}$$

where

$$\ell = \frac{h}{2} \in C^\infty(\mathbb{T}_\theta^2),$$

$$R_1(s_1) = \frac{4e^{\frac{s_1}{2}}\pi(2 + e^{s_1}(-2+s_1) + s_1)}{(-1+e^{s_1})^2 s_1},$$

$$\begin{aligned} R_2(s_1,s_2) = {} & -4\pi \frac{\cosh[s_2]s_1(s_1+s_2) - \cosh[s_1]s_2(s_1+s_2)}{\sinh[\frac{s_1}{2}]\sinh[\frac{s_2}{2}]\sinh^2[\frac{1}{2}(s_1+s_2)]s_1s_2(s_1+s_2)} \\ & + 4\pi \frac{(s_1-s_2)(\sinh[s_1] + \sinh[s_2] - \sinh[s_1+s_2] + s_1 + s_2)}{\sinh[\frac{s_1}{2}]\sinh[\frac{s_2}{2}]\sinh^2[\frac{1}{2}(s_1+s_2)]s_1s_2(s_1+s_2)}. \end{aligned}$$

Let us explain that, in general, given a rapidly decaying smooth function L defined on the Euclidean space $\mathbb{R}^n$ and elements $x_1, \dots, x_n$ in the algebra $C(\mathbb{T}_\theta^2)$ of the noncommutative torus, we consider the calculus defined by

$$L(\nabla,\dots,\nabla)(x_1\cdots x_n) = \int_{\mathbb{R}^n} \sigma_{t_1}(x_1)\cdots\sigma_{t_n}(x_n)\, g(t_1,\dots,t_n)\, dt_1\cdots dt_n,$$

where the function g is obtained by writing the function L as a Fourier transform:

$$L(s_1,\dots,s_n) = \int_{\mathbb{R}^n} e^{-i(t_1s_1+\cdots+t_ns_n)} g(t_1,\dots,t_n)\, dt_1\cdots dt_n.$$

3. THE TERM a_4 AND ITS FUNCTIONAL RELATIONS

We start this section by writing an expression for the term $a_4 \in C^\infty(\mathbb{T}_\theta^2)$ appearing in the small-time heat kernel expansion (1), which involves 20 functions of one to four variables that are denoted by $K_1, \dots, K_{20}$. We will then proceed to present one of our main results, namely a family of functional relations that are satisfied by the functions K_j, $j = 1, \ldots, 20$. Using the notations provided in Section 2, we have

(6)

$$\begin{aligned}
a_4 = -e^{2\ell}\Big(& K_1(\nabla)(\delta_1^2\delta_2^2(\ell)) + K_2(\nabla)(\delta_1^4(\ell) + \delta_2^4(\ell)) \\
&+ K_3(\nabla,\nabla)((\delta_1\delta_2(\ell))\cdot(\delta_1\delta_2(\ell))) + K_4(\nabla,\nabla)(\delta_1^2(\ell)\cdot\delta_2^2(\ell) + \delta_2^2(\ell)\cdot\delta_1^2(\ell)) \\
&+ K_5(\nabla,\nabla)(\delta_1^2(\ell)\cdot\delta_1^2(\ell) + \delta_2^2(\ell)\cdot\delta_2^2(\ell)) \\
&+ K_6(\nabla,\nabla)(\delta_1(\ell)\cdot\delta_1^3(\ell) + \delta_1(\ell)\cdot(\delta_1\delta_2^2(\ell)) + \delta_2(\ell)\cdot\delta_2^3(\ell) + \delta_2(\ell)\cdot(\delta_1^2\delta_2(\ell))) \\
&+ K_7(\nabla,\nabla)(\delta_1^3(\ell)\cdot\delta_1(\ell) + (\delta_1\delta_2^2(\ell))\cdot\delta_1(\ell) + \delta_2^3(\ell)\cdot\delta_2(\ell) + (\delta_1^2\delta_2(\ell))\cdot\delta_2(\ell)) \\
&+ K_8(\nabla,\nabla,\nabla)(\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_2^2(\ell) + \delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_1^2(\ell)) \\
&+ K_9(\nabla,\nabla,\nabla)(\delta_1(\ell)\cdot\delta_2(\ell)\cdot(\delta_1\delta_2(\ell)) + \delta_2(\ell)\cdot\delta_1(\ell)\cdot(\delta_1\delta_2(\ell))) \\
&+ K_{10}(\nabla,\nabla,\nabla)(\delta_1(\ell)\cdot(\delta_1\delta_2(\ell))\cdot\delta_2(\ell) + \delta_2(\ell)\cdot(\delta_1\delta_2(\ell))\cdot\delta_1(\ell)) \\
&+ K_{11}(\nabla,\nabla,\nabla)(\delta_1(\ell)\cdot\delta_2^2(\ell)\cdot\delta_1(\ell) + \delta_2(\ell)\cdot\delta_1^2(\ell)\cdot\delta_2(\ell)) \\
&+ K_{12}(\nabla,\nabla,\nabla)(\delta_1^2(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell) + \delta_2^2(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell)) \\
&+ K_{13}(\nabla,\nabla,\nabla)((\delta_1\delta_2(\ell))\cdot\delta_1(\ell)\cdot\delta_2(\ell) + (\delta_1\delta_2(\ell))\cdot\delta_2(\ell)\cdot\delta_1(\ell)) \\
&+ K_{14}(\nabla,\nabla,\nabla)(\delta_1^2(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell) + \delta_2^2(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell)) \\
&+ K_{15}(\nabla,\nabla,\nabla)(\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_1^2(\ell) + \delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_2^2(\ell)) \\
&+ K_{16}(\nabla,\nabla,\nabla)(\delta_1(\ell)\cdot\delta_1^2(\ell)\cdot\delta_1(\ell) + \delta_2(\ell)\cdot\delta_2^2(\ell)\cdot\delta_2(\ell)) \\
&+ K_{17}(\nabla,\nabla,\nabla,\nabla)(\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell) + \delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell)) \\
&+ K_{18}(\nabla,\nabla,\nabla,\nabla)(\delta_1(\ell)\cdot\delta_2(\ell)\cdot\delta_1(\ell)\cdot\delta_2(\ell) + \delta_2(\ell)\cdot\delta_1(\ell)\cdot\delta_2(\ell)\cdot\delta_1(\ell)) \\
&+ K_{19}(\nabla,\nabla,\nabla,\nabla)(\delta_1(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_1(\ell) + \delta_2(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_2(\ell)) \\
&+ K_{20}(\nabla,\nabla,\nabla,\nabla)(\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell) + \delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell))\Big).
\end{aligned}$$

Explicit formulas for the one and two variable functions $K_1, \ldots, K_7$ are given in Section 9, and lengthy formulas for the three and four variable functions $K_8, \ldots, K_{20}$ are provided in Appendix C.

In order to present the functional relations, it is convenient to define mild variants $\widetilde{K}_j$ of the functions K_j, $j = 1, 2, \ldots, 20$, which are given by

(7)

$$\begin{aligned}
\widetilde{K}_j(s_1) &= \frac{1}{2}\frac{\sinh(\frac{s_1}{2})}{\frac{s_1}{2}}K_j(s_1), && j = 1, 2, \\
\widetilde{K}_j(s_1, s_2) &= \frac{1}{2^2}\frac{\sinh(\frac{s_1+s_2}{2})}{\frac{s_1+s_2}{2}}K_j(s_1, s_2), && j = 3, 4, \ldots, 7, \\
\widetilde{K}_j(s_1, s_2, s_3) &= \frac{1}{2^3}\frac{\sinh(\frac{s_1+s_2+s_3}{2})}{\frac{s_1+s_2+s_3}{2}}K_j(s_1, s_2, s_3), && j = 8, 9, \ldots, 16, \\
\widetilde{K}_j(s_1, s_2, s_3, s_4) &= \frac{1}{2^4}\frac{\sinh(\frac{s_1+s_2+s_3+s_4}{2})}{\frac{s_1+s_2+s_3+s_4}{2}}K_j(s_1, s_2, s_3, s_4), && j = 17, 18, 19, 20.
\end{aligned}$$

We also need to introduce the following functions k_j, $j = 3, 4, \ldots, 20$, which are derived from the main functions by setting

$$\begin{aligned} k_j(s_1) &= K_j(s_1, -s_1), & j &= 3, \ldots, 7, \\ k_j(s_1, s_2) &= K_j(s_1, s_2, -s_1 - s_2), & j &= 8, 9, \ldots, 16, \\ k_j(s_1, s_2, s_3) &= K_j(s_1, s_2, s_3, -s_1 - s_2 - s_3), & j &= 17, 18, 19, 20. \end{aligned} \tag{8}$$

The following functions G_1, G_2, G_3, G_4, which are constructed in Lemmas 6.5, 6.6 and 6.7, play an important role as well in the functional relations. They are given explicitly by

$$\begin{aligned} G_1(s_1) &= \frac{e^{s_1} - 1}{s_1}, \\ G_2(s_1, s_2) &= \frac{e^{s_1}((e^{s_2} - 1)s_1 - s_2) + s_2}{s_1 s_2 (s_1 + s_2)}, \\ G_3(s_1, s_2, s_3) &= \\ &\frac{e^{s_1}(e^{s_2+s_3} s_1 s_2 (s_1+s_2) + (s_1+s_2+s_3)((s_1+s_2)s_3 - e^{s_2} s_1 (s_2+s_3))) - s_2 s_3 (s_2+s_3)}{s_1 s_2 (s_1+s_2) s_3 (s_2+s_3)(s_1+s_2+s_3)}, \\ G_4(s_1, s_2, s_3, s_4) &= \frac{e^{s_1+s_2}\left(\frac{1}{s_2(s_1+s_2)(s_3+s_4)} - \frac{e^{s_3}}{(s_2+s_3)(s_1+s_2+s_3)s_4}\right)}{s_3} \\ &+ \frac{\frac{1}{(s_1+s_2)(s_1+s_2+s_3)(s_1+s_2+s_3+s_4)} - \frac{e^{s_1}}{s_2(s_2+s_3)(s_2+s_3+s_4)}}{s_1} \\ &+ \frac{e^{s_1+s_2+s_3+s_4}}{s_4(s_3+s_4)(s_2+s_3+s_4)(s_1+s_2+s_3+s_4)}. \end{aligned} \tag{9}$$

At this stage, we are ready to present the functional relations explicitly. It is worth emphasizing that the proof of these relations requires a significant amount of work, which is carried out in Section 5 and Section 6. That is, the proof is based on calculating the gradient of the map that sends a general dilaton $h = h^* \in C^\infty(\mathbb{T}_\theta^2)$ to $\varphi_0(a_4)$, in two different ways: first, by using a fundamental identity proved in [14], and second, by using the Duhamel formula to find a formula in terms of finite differences.

Theorem 3.1. *Using the above notations, the functions of one to four variables $K_1, \ldots, K_{20}$ appearing in expression (6) for the term $a_4 \in C^\infty(\mathbb{T}_\theta^2)$ satisfy the functional relations presented in the following subsections and in Appendix A, in which each variant $\widetilde{K}_j$ of the function K_j is expressed as finite differences of the involved functions.*

Proof. The functional relations are derived by comparing the corresponding terms in the final formulas for the gradient

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \varphi_0(a_4(h + \varepsilon a)),$$

where $h, a \in C^\infty(\mathbb{T}_\theta^2)$ are selfadjoint elements, calculated in two different ways. The first method is explained in the beginning of Section 5, which gives rise to formula (52) where the variants $\widetilde{K}_j$ of the functions K_j appear. The second method is based on employing the lemmas proved in Section 5 and performing the calculations explained in Section 6 to calculate the above gradient in terms of finite differences. □

In each of the following subsections, the functional relations associated with the functions that depend on the same number of variables are given. We use the Mathematica computing system [8] to check that our calculated functions satisfy this family of highly nontrivial functional relations.

3.2. **The functions $\widetilde{K_1}, \widetilde{K_2}$.** In this subsection we present the functional relations associated with the one variable functions $\widetilde{K}_1$, $\widetilde{K}_2$ defined by (7).

3.2.1. *The function $\widetilde{K}_1$.* By using the identity (52) and considering the specific function of ∇ that acts on $\delta_1^2\delta_2^2(h)$ in the final formula for the gradient $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\varphi_0(a_4(h+\varepsilon a))$ as calculated in Section 6, we have

$$\begin{aligned}\widetilde{K}_1(s_1) = &-\frac{1}{15}\pi G_1(s_1)+\frac{1}{4}e^{s_1}k_3(-s_1)+\frac{1}{4}k_3(s_1)+\frac{1}{2}e^{s_1}k_4(-s_1)\\ &+\frac{1}{2}k_4(s_1)-\frac{1}{2}e^{s_1}k_6(-s_1)-\frac{1}{2}k_6(s_1)-\frac{1}{2}e^{s_1}k_7(-s_1)\\ &-\frac{1}{2}k_7(s_1)-\frac{\pi(e^{s_1}-1)}{15s_1}.\end{aligned} \tag{10}$$

3.2.2. *The function $\widetilde{K}_2$.* It is clear from the explicit formulas presented in Section 9 that the function $\widetilde{K}_2$ is a scalar multiple of the function $\widetilde{K}_1$. Therefore, it is interesting to see that the following functional relation, which has different ingredients compared to those for $\widetilde{K}_1$, gives the same function up to a scalar multiplication. For the second function we have

$$\begin{aligned}\widetilde{K}_2(s_1) = &-\frac{1}{30}\pi G_1(s_1)+\frac{1}{4}e^{s_1}k_5(-s_1)+\frac{1}{4}k_5(s_1)-\frac{1}{4}e^{s_1}k_6(-s_1)\\ &-\frac{1}{4}k_6(s_1)-\frac{1}{4}e^{s_1}k_7(-s_1)-\frac{1}{4}k_7(s_1)-\frac{\pi(e^{s_1}-1)}{30s_1}.\end{aligned} \tag{11}$$

3.3. **The functions $\widetilde{K_3},\dots,\widetilde{K_7}$.** In this subsection we present the functional relations associated with the two variable functions $\widetilde{K}_3,\dots,\widetilde{K}_7$ defined in (7).

3.3.1. *The function $\widetilde{K}_3$.* Similarly to the case of the one variable functions, by using the identity (52) and considering the specific function of ∇ that acts on $\delta_1\delta_2(h)\cdot\delta_1\delta_2(h)$ in the final formula for the gradient $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\varphi_0(a_4(h+\varepsilon a))$ as calculated in Section 6, we obtain the following functional relation for the function $\widetilde{K}_3$:

$$\begin{aligned}&\widetilde{K}_3(s_1,s_2)\\ &\quad=\frac{1}{15}(-4)\pi G_2(s_1,s_2)+\frac{1}{2}k_8(s_1,s_2)+\frac{1}{4}k_9(s_1,s_2)\\ &\quad\quad-\frac{1}{4}e^{s_1+s_2}k_9(-s_1-s_2,s_1)-\frac{1}{4}e^{s_1}k_9(s_2,-s_1-s_2)-\frac{1}{4}k_{10}(s_1,s_2)\end{aligned} \tag{12}$$

$$
\begin{aligned}
&- \frac{1}{4}e^{s_1+s_2}k_{10}(-s_1-s_2,s_1) + \frac{1}{4}e^{s_1}k_{10}(s_2,-s_1-s_2)\\
&+ \frac{1}{2}e^{s_1}k_{11}(s_2,-s_1-s_2) + \frac{1}{2}e^{s_1+s_2}k_{12}(-s_1-s_2,s_1) - \frac{1}{4}k_{13}(s_1,s_2)\\
&+ \frac{1}{4}e^{s_1+s_2}k_{13}(-s_1-s_2,s_1) - \frac{1}{4}e^{s_1}k_{13}(s_2,-s_1-s_2)\\
&+ \frac{1}{4}e^{s_2}G_1(s_1)k_3(-s_2) + \frac{1}{4}G_1(s_1)k_3(s_2) - G_1(s_1)k_6(s_2)\\
&- e^{s_2}G_1(s_1)k_7(-s_2) + \frac{(e^{s_1+s_2}-1)k_3(s_1)}{4(s_1+s_2)} + \frac{k_3(s_2)-k_3(s_1+s_2)}{4s_1}\\
&+ \frac{k_3(s_1+s_2)-k_3(s_1)}{4s_2} + \frac{k_6(s_1)-k_6(s_1+s_2)}{s_2} + \frac{k_6(s_1+s_2)-k_6(s_2)}{s_1}\\
&+ \frac{e^{s_1}(k_7(-s_1)-e^{s_2}k_7(-s_1-s_2))}{s_2} + \frac{e^{s_2}(e^{s_1}k_7(-s_1-s_2)-k_7(-s_2))}{s_1}\\
&- \frac{e^{s_2}(e^{s_1}k_3(-s_1-s_2)-k_3(-s_2))}{4s_1} - \frac{e^{s_1}(k_3(-s_1)-e^{s_2}k_3(-s_1-s_2))}{4s_2}\\
&- \frac{e^{s_1}(k_3(-s_1)+e^{s_2}k_3(s_1)-e^{s_2}k_3(-s_2)-k_3(s_2))}{4(s_1+s_2)}.
\end{aligned}
$$

3.3.2. *The function* $\widetilde{K}_4$. By making another comparison between the term containing the function $\widetilde{K}_4$ in (52), and the corresponding function coming out of the calculations in Section 6, we have

$$
\begin{aligned}
&\widetilde{K}_4(s_1,s_2) \qquad (13)\\
&= -\frac{1}{15}\pi G_2(s_1,s_2) - \frac{1}{8}e^{s_1+s_2}k_8(-s_1-s_2,s_1) - \frac{1}{8}e^{s_1}k_8(s_2,-s_1-s_2)\\
&\quad + \frac{1}{8}k_9(s_1,s_2) + \frac{1}{8}e^{s_1}k_{10}(s_2,-s_1-s_2) - \frac{1}{8}k_{11}(s_1,s_2)\\
&\quad - \frac{1}{8}e^{s_1+s_2}k_{11}(-s_1-s_2,s_1) - \frac{1}{8}k_{12}(s_1,s_2) - \frac{1}{8}e^{s_1}k_{12}(s_2,-s_1-s_2)\\
&\quad + \frac{1}{8}e^{s_1+s_2}k_{13}(-s_1-s_2,s_1) + \frac{1}{4}e^{s_2}G_1(s_1)k_4(-s_2) + \frac{1}{4}G_1(s_1)k_4(s_2)\\
&\quad - \frac{1}{4}G_1(s_1)k_6(s_2) - \frac{1}{4}e^{s_2}G_1(s_1)k_7(-s_2) + \frac{(e^{s_1+s_2}-1)k_4(s_1)}{4(s_1+s_2)}\\
&\quad + \frac{k_4(s_2)-k_4(s_1+s_2)}{4s_1} + \frac{k_4(s_1+s_2)-k_4(s_1)}{4s_2} + \frac{k_6(s_1)-k_6(s_1+s_2)}{4s_2}\\
&\quad + \frac{k_6(s_1+s_2)-k_6(s_2)}{4s_1} + \frac{e^{s_1}(k_7(-s_1)-e^{s_2}k_7(-s_1-s_2))}{4s_2}\\
&\quad + \frac{e^{s_2}(e^{s_1}k_7(-s_1-s_2)-k_7(-s_2))}{4s_1} - \frac{e^{s_2}(e^{s_1}k_4(-s_1-s_2)-k_4(-s_2))}{4s_1}\\
&\quad - \frac{e^{s_1}(k_4(-s_1)-e^{s_2}k_4(-s_1-s_2))}{4s_2} - \frac{e^{s_1+s_2}(k_4(s_1)-k_4(-s_2))}{4(s_1+s_2)}\\
&\quad - \frac{e^{s_1}(k_4(-s_1)-k_4(s_2))}{4(s_1+s_2)}.
\end{aligned}
$$

3.3.3. *The function* $\widetilde{K}_5$. By a similar comparison, we have

$$\begin{aligned}
(14)\quad & \widetilde{K}_5(s_1,s_2) \\
&= -\frac{1}{5}\pi G_2(s_1,s_2) - \frac{1}{8}k_{14}(s_1,s_2) + \frac{1}{4}e^{s_1+s_2}k_{14}(-s_1-s_2,s_1) \\
&\quad - \frac{1}{8}e^{s_1}k_{14}(s_2,-s_1-s_2) + \frac{1}{4}k_{15}(s_1,s_2) - \frac{1}{8}e^{s_1+s_2}k_{15}(-s_1-s_2,s_1) \\
&\quad - \frac{1}{8}e^{s_1}k_{15}(s_2,-s_1-s_2) - \frac{1}{8}k_{16}(s_1,s_2) - \frac{1}{8}e^{s_1+s_2}k_{16}(-s_1-s_2,s_1) \\
&\quad + \frac{1}{4}e^{s_1}k_{16}(s_2,-s_1-s_2) + \frac{1}{4}e^{s_2}G_1(s_1)k_5(-s_2) + \frac{1}{4}G_1(s_1)k_5(s_2) \\
&\quad - \frac{3}{4}G_1(s_1)k_6(s_2) - \frac{3}{4}e^{s_2}G_1(s_1)k_7(-s_2) + \frac{(e^{s_1+s_2}-1)k_5(s_1)}{4(s_1+s_2)} \\
&\quad + \frac{k_5(s_2)-k_5(s_1+s_2)}{4s_1} + \frac{k_5(s_1+s_2)-k_5(s_1)}{4s_2} + \frac{3(k_6(s_1)-k_6(s_1+s_2))}{4s_2} \\
&\quad + \frac{3e^{s_1}(k_7(-s_1)-e^{s_2}k_7(-s_1-s_2))}{4s_2} + \frac{3e^{s_2}(e^{s_1}k_7(-s_1-s_2)-k_7(-s_2))}{4s_1} \\
&\quad - \frac{e^{s_2}(e^{s_1}k_5(-s_1-s_2)-k_5(-s_2))}{4s_1} - \frac{3(k_6(s_2)-k_6(s_1+s_2))}{4s_1} \\
&\quad - \frac{e^{s_1}(k_5(-s_1)-e^{s_2}k_5(-s_1-s_2))}{4s_2} \\
&\quad - \frac{e^{s_1}(k_5(-s_1)+e^{s_2}k_5(s_1)-e^{s_2}k_5(-s_2)-k_5(s_2))}{4(s_1+s_2)}.
\end{aligned}$$

3.3.4. *The function* $\widetilde{K}_6$. In (52) we see that the operator $\widetilde{K}_6(\nabla,\nabla)$ acts on two different elements that are not the same up to switching δ_1 and δ_2, namely $\delta_1(h)\cdot\delta_1^3(h)$ and $\delta_1(h)\cdot\delta_1\delta_2^2(h)$. By looking at the corresponding finite difference expressions in the result of the second gradient calculation performed in Section 5 we find the following basic equations for $\widetilde{K}_6(s_1,s_2)$. From the expression associated with the term $\delta_1(h)\cdot\delta_1^3(h)$ we find that

$$\begin{aligned}
(15)\quad & \widetilde{K}_6(s_1,s_2) \\
&= \frac{1}{15}(-2)\pi G_2(s_1,s_2) + \frac{1}{8}e^{s_1+s_2}k_{14}(-s_1-s_2,s_1) - \frac{1}{8}e^{s_1}k_{14}(s_2,-s_1-s_2) \\
&\quad + \frac{1}{8}k_{15}(s_1,s_2) - \frac{1}{8}e^{s_1+s_2}k_{15}(-s_1-s_2,s_1) - \frac{1}{8}k_{16}(s_1,s_2) \\
&\quad + \frac{1}{8}e^{s_1}k_{16}(s_2,-s_1-s_2) + \frac{1}{2}e^{s_2}G_1(s_1)k_5(-s_2) + \frac{1}{2}G_1(s_1)k_5(s_2) \\
&\quad - \frac{1}{4}e^{s_2}G_1(s_1)k_6(-s_2) - \frac{3}{4}G_1(s_1)k_6(s_2) - \frac{3}{4}e^{s_2}G_1(s_1)k_7(-s_2) \\
&\quad - \frac{1}{4}G_1(s_1)k_7(s_2) + \frac{k_5(s_2)-k_5(s_1+s_2)}{2s_1} + \frac{(e^{s_1+s_2}-1)k_6(s_1)}{4(s_1+s_2)} \\
&\quad + \frac{e^{s_2}(e^{s_1}k_6(-s_1-s_2)-k_6(-s_2))}{4s_1} + \frac{k_6(s_1)-k_6(s_1+s_2)}{4s_2} \\
&\quad + \frac{e^{s_1}(k_7(-s_1)-e^{s_2}k_7(-s_1-s_2))}{4s_2} + \frac{3e^{s_2}(e^{s_1}k_7(-s_1-s_2)-k_7(-s_2))}{4s_1} \\
&\quad + \frac{k_7(s_1+s_2)-k_7(s_2)}{4s_1} - \frac{e^{s_2}(e^{s_1}k_5(-s_1-s_2)-k_5(-s_2))}{2s_1}
\end{aligned}$$

$$
- \frac{3(k_6(s_2) - k_6(s_1+s_2))}{4s_1} - \frac{e^{s_1+s_2}(k_6(s_1) - k_6(-s_2))}{4(s_1+s_2)}
- \frac{e^{s_1}(k_7(-s_1) - k_7(s_2))}{4(s_1+s_2)}.
$$

Moreover, the expression associated with the term $\delta_1(h) \cdot \delta_1\delta_2^2(h)$ as explained above yields:

$$
\begin{aligned}
(16) \qquad & \widetilde{K}_6(s_1, s_2) \\
&= \frac{1}{15}(-2)\pi G_2(s_1,s_2) + \frac{1}{4}e^{s_2}G_1(s_1)k_3(-s_2) + \frac{1}{4}G_1(s_1)k_3(s_2) \\
&+ \frac{1}{2}e^{s_2}G_1(s_1)k_4(-s_2) + \frac{1}{2}G_1(s_1)k_4(s_2) + \frac{(1-e^{s_1+s_2})k_6(s_1)}{4(-s_1-s_2)} \\
&- \frac{1}{4}e^{s_2}G_1(s_1)k_6(-s_2) + \frac{e^{s_1+s_2}(k_6(-s_2) - k_6(s_1))}{4(s_1+s_2)} \\
&+ \frac{e^{s_1+s_2}k_6(-s_1-s_2) - e^{s_2}k_6(-s_2)}{4s_1} - \frac{3}{4}G_1(s_1)k_6(s_2) \\
&+ \frac{3(k_6(s_1+s_2) - k_6(s_2))}{4s_1} - \frac{3}{4}e^{s_2}G_1(s_1)k_7(-s_2) \\
&+ \frac{3(e^{s_1+s_2}k_7(-s_1-s_2) - e^{s_2}k_7(-s_2))}{4s_1} - \frac{1}{4}G_1(s_1)k_7(s_2) \\
&+ \frac{e^{s_1}(k_7(s_2) - k_7(-s_1))}{4(s_1+s_2)} + \frac{k_7(s_1+s_2) - k_7(s_2)}{4s_1} + \frac{1}{8}k_8(s_1,s_2) \\
&- \frac{1}{8}e^{s_1+s_2}k_8(-s_1-s_2,s_1) + \frac{1}{8}k_9(s_1,s_2) - \frac{1}{8}e^{s_1+s_2}k_9(-s_1-s_2,s_1) \\
&- \frac{1}{8}k_{10}(s_1,s_2) + \frac{1}{8}e^{s_1}k_{10}(s_2,-s_1-s_2) - \frac{1}{8}k_{11}(s_1,s_2) \\
&+ \frac{1}{8}e^{s_1}k_{11}(s_2,-s_1-s_2) + \frac{1}{8}e^{s_1+s_2}k_{12}(-s_1-s_2,s_1) \\
&- \frac{1}{8}e^{s_1}k_{12}(s_2,-s_1-s_2) + \frac{1}{8}e^{s_1+s_2}k_{13}(-s_1-s_2,s_1) \\
&- \frac{1}{8}e^{s_1}k_{13}(s_2,-s_1-s_2) - \frac{e^{s_1+s_2}k_4(-s_1-s_2) - e^{s_2}k_4(-s_2)}{2s_1} \\
&- \frac{k_4(s_1+s_2) - k_4(s_2)}{2s_1} - \frac{e^{s_1+s_2}k_3(-s_1-s_2) - e^{s_2}k_3(-s_2)}{4s_1} \\
&- \frac{k_3(s_1+s_2) - k_3(s_2)}{4s_1} - \frac{k_6(s_1+s_2) - k_6(s_1)}{4s_2} \\
&- \frac{e^{s_1+s_2}k_7(-s_1-s_2) - e^{s_1}k_7(-s_1)}{4s_2}.
\end{aligned}
$$

3.3.5. *The function $\widetilde{K}_7$.* The situation for the last two variable function $\widetilde{K}_7$ is similar to that of $\widetilde{K}_6$ in the sense that the operator $\widetilde{K}_7(\nabla, \nabla)$ in (52) acts on two different types of elements that are different even modulo switching δ_1 and δ_2. By finding the finite difference expression of the modular operator that acts on $\delta_1^3(h)\cdot\delta_1(h)$ in the second gradient calculation of Section 5, we have

$$
\begin{aligned}
&\widetilde{K}_7(s_1,s_2) \\
&= \frac{1}{15}(-2)\pi G_2(s_1,s_2) - \frac{1}{8}k_{14}(s_1,s_2) + \frac{1}{8}e^{s_1+s_2}k_{14}(-s_1-s_2,s_1) \\
&\quad + \frac{1}{8}k_{15}(s_1,s_2) - \frac{1}{8}e^{s_1}k_{15}(s_2,-s_1-s_2) - \frac{1}{8}e^{s_1+s_2}k_{16}(-s_1-s_2,s_1) \\
&\quad + \frac{1}{8}e^{s_1}k_{16}(s_2,-s_1-s_2) - \frac{1}{4}G_1(s_1)k_6(s_2) - \frac{1}{4}e^{s_2}G_1(s_1)k_7(-s_2) \\
&\quad + \frac{k_5(s_1+s_2)-k_5(s_1)}{2s_2} + \frac{e^{s_1}(k_6(-s_1)-e^{s_2}k_6(-s_1-s_2))}{4s_2} \\
&\quad + \frac{3(k_6(s_1)-k_6(s_1+s_2))}{4s_2} + \frac{k_6(s_1+s_2)-k_6(s_2)}{4s_1} \\
&\quad + \frac{(e^{s_1+s_2}-1)k_7(s_1)}{4(s_1+s_2)} + \frac{3e^{s_1}(k_7(-s_1)-e^{s_2}k_7(-s_1-s_2))}{4s_2} \\
&\quad + \frac{e^{s_2}(e^{s_1}k_7(-s_1-s_2)-k_7(-s_2))}{4s_1} + \frac{k_7(s_1)-k_7(s_1+s_2)}{4s_2} \\
&\quad - \frac{e^{s_1}(k_5(-s_1)-e^{s_2}k_5(-s_1-s_2))}{2s_2} - \frac{e^{s_1}(k_6(-s_1)-k_6(s_2))}{4(s_1+s_2)} \\
&\quad - \frac{e^{s_1+s_2}(k_7(s_1)-k_7(-s_2))}{4(s_1+s_2)}.
\end{aligned} \tag{17}
$$

The finite difference expression associated with the element $\delta_1\delta_2^2(h)\cdot\delta_1(h)$ as explained above gives another basic identity:

$$
\begin{aligned}
&\widetilde{K}_7(s_1,s_2) \\
&= \frac{1}{15}(-2)\pi G_2(s_1,s_2) + \frac{e^{s_1+s_2}k_3(-s_1-s_2)-e^{s_1}k_3(-s_1)}{4s_2} \\
&\quad + \frac{k_3(s_1+s_2)-k_3(s_1)}{4s_2} + \frac{e^{s_1+s_2}k_4(-s_1-s_2)-e^{s_1}k_4(-s_1)}{2s_2} \\
&\quad + \frac{k_4(s_1+s_2)-k_4(s_1)}{2s_2} - \frac{1}{4}G_1(s_1)k_6(s_2) + \frac{e^{s_1}(k_6(s_2)-k_6(-s_1))}{4(s_1+s_2)} \\
&\quad + \frac{k_6(s_1+s_2)-k_6(s_2)}{4s_1} + \frac{(1-e^{s_1+s_2})k_7(s_1)}{4(-s_1-s_2)} - \frac{1}{4}e^{s_2}G_1(s_1)k_7(-s_2) \\
&\quad + \frac{e^{s_1+s_2}(k_7(-s_2)-k_7(s_1))}{4(s_1+s_2)} + \frac{e^{s_1+s_2}k_7(-s_1-s_2)-e^{s_2}k_7(-s_2)}{4s_1} \\
&\quad + \frac{1}{8}k_8(s_1,s_2) - \frac{1}{8}e^{s_1}k_8(s_2,-s_1-s_2) + \frac{1}{8}k_9(s_1,s_2) \\
&\quad - \frac{1}{8}e^{s_1}k_9(s_2,-s_1-s_2) - \frac{1}{8}e^{s_1+s_2}k_{10}(-s_1-s_2,s_1) \\
&\quad + \frac{1}{8}e^{s_1}k_{10}(s_2,-s_1-s_2) - \frac{1}{8}e^{s_1+s_2}k_{11}(-s_1-s_2,s_1) \\
&\quad + \frac{1}{8}e^{s_1}k_{11}(s_2,-s_1-s_2) - \frac{1}{8}k_{12}(s_1,s_2) \\
&\quad + \frac{1}{8}e^{s_1+s_2}k_{12}(-s_1-s_2,s_1) - \frac{1}{8}k_{13}(s_1,s_2) \\
&\quad + \frac{1}{8}e^{s_1+s_2}k_{13}(-s_1-s_2,s_1) - \frac{e^{s_1+s_2}k_6(-s_1-s_2)-e^{s_1}k_6(-s_1)}{4s_2}
\end{aligned} \tag{18}
$$

$$
- \frac{3(k_6(s_1+s_2)-k_6(s_1))}{4s_2} - \frac{3(e^{s_1+s_2}k_7(-s_1-s_2)-e^{s_1}k_7(-s_1))}{4s_2}
- \frac{k_7(s_1+s_2)-k_7(s_1)}{4s_2}.
$$

3.4. **The functions $\widetilde{K}_8, \dots, \widetilde{K}_{16}$.** In this subsection we present the functional relation associated with the three variable functions $\widetilde{K}_8$ defined by (7). Since the functional relations for the functions $\widetilde{K}_9, \dots, \widetilde{K}_{16}$ are similarly lengthy, for the sake of completeness, they are provided in Appendix A. Similarly to the previous functions, these functional relations are derived by using the identity (52) and by making a comparison with the corresponding terms in the result of the gradient calculation carried out in Section 6.

3.4.1. *The function $\widetilde{K}_8$.* We have

$$
\begin{aligned}
(19)\quad & \widetilde{K}_8(s_1,s_2,s_3) \\
&= \frac{1}{15}(-2)\pi G_3(s_1,s_2,s_3) + \frac{1}{2}e^{s_3}G_2(s_1,s_2)k_4(-s_3) \\
&\quad - \frac{e^{s_3}(e^{s_2}s_1k_4(-s_2-s_3)+e^{s_2}s_2k_4(-s_2-s_3)-e^{s_1+s_2}s_2k_4(-s_1-s_2-s_3)-s_1k_4(-s_3))}{2s_1s_2(s_1+s_2)} \\
&\quad + \frac{1}{2}G_2(s_1,s_2)k_4(s_3) + \frac{G_1(s_1)(k_4(s_3)-k_4(s_2+s_3))}{2s_2} \\
&\quad + \frac{s_1k_4(s_3)-s_1k_4(s_2+s_3)-s_2k_4(s_2+s_3)+s_2k_4(s_1+s_2+s_3)}{2s_1s_2(s_1+s_2)} \\
&\quad - \frac{1}{2}G_2(s_1,s_2)k_6(s_3) + \frac{G_1(s_1)(k_6(s_2)-k_6(s_2+s_3))}{4s_3} \\
&\quad + \frac{k_6(s_2)-k_6(s_1+s_2)-k_6(s_2+s_3)+k_6(s_1+s_2+s_3)}{4s_1s_3} \\
&\quad + \frac{-s_3k_6(s_1)+s_2k_6(s_1+s_2)+s_3k_6(s_1+s_2)-s_2k_6(s_1+s_2+s_3)}{4s_2s_3(s_2+s_3)} \\
&\quad + \frac{-s_1k_6(s_3)+s_1k_6(s_2+s_3)+s_2k_6(s_2+s_3)-s_2k_6(s_1+s_2+s_3)}{2s_1s_2(s_1+s_2)} \\
&\quad + \frac{e^{s_2}G_1(s_1)(k_7(-s_2)-e^{s_3}k_7(-s_2-s_3))}{4s_3} \\
&\quad - \frac{e^{s_1}(s_3k_7(-s_1)-e^{s_2}s_2k_7(-s_1-s_2)-e^{s_2}s_3k_7(-s_1-s_2)+e^{s_2+s_3}s_2k_7(-s_1-s_2-s_3))}{4s_2s_3(s_2+s_3)} \\
&\quad - \frac{e^{s_2}(e^{s_1}k_7(-s_1-s_2)-k_7(-s_2)+e^{s_3}k_7(-s_2-s_3)-e^{s_1+s_3}k_7(-s_1-s_2-s_3))}{4s_1s_3} \\
&\quad + \frac{e^{s_3}G_1(s_1)(e^{s_2}k_7(-s_2-s_3)-k_7(-s_3))}{2s_2} - \frac{1}{2}e^{s_3}G_2(s_1,s_2)k_7(-s_3) \\
&\quad + \frac{e^{s_3}(e^{s_2}s_1k_7(-s_2-s_3)+e^{s_2}s_2k_7(-s_2-s_3)-e^{s_1+s_2}s_2k_7(-s_1-s_2-s_3)-s_1k_7(-s_3))}{2s_1s_2(s_1+s_2)} \\
&\quad + \frac{(-1+e^{s_1+s_2+s_3})k_8(s_1,s_2)}{8(s_1+s_2+s_3)} + \frac{k_8(s_1,s_2+s_3)-k_8(s_1,s_2)}{8s_3} \\
&\quad - \frac{1}{8}e^{s_2+s_3}G_1(s_1)k_8(-s_2-s_3,s_2) \\
&\quad + \frac{e^{s_1+s_2+s_3}(k_8(-s_1-s_2-s_3,s_1)-k_8(-s_1-s_2-s_3,s_1+s_2))}{8s_2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{8}G_1(s_1)k_9(s_2,s_3)+\frac{k_9(s_2,s_3)-k_9(s_1+s_2,s_3)}{8s_1}\\
&+\frac{k_9(s_1+s_2,s_3)-k_9(s_1,s_2+s_3)}{8s_2}+\frac{1}{8}e^{s_2}G_1(s_1)k_{10}(s_3,-s_2-s_3)\\
&+\frac{e^{s_2}(k_{10}(s_3,-s_2-s_3)-e^{s_1}k_{10}(s_3,-s_1-s_2-s_3))}{8s_1}\\
&+\frac{e^{s_1}(e^{s_2}k_{10}(s_3,-s_1-s_2-s_3)-k_{10}(s_2+s_3,-s_1-s_2-s_3))}{8s_2}\\
&-\frac{1}{8}G_1(s_1)k_{11}(s_2,s_3)+\frac{k_{11}(s_1,s_2+s_3)-k_{11}(s_1+s_2,s_3)}{8s_2}\\
&+\frac{k_{11}(s_1+s_2,s_3)-k_{11}(s_2,s_3)}{8s_1}-\frac{1}{8}e^{s_2}G_1(s_1)k_{12}(s_3,-s_2-s_3)\\
&+\frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{13}(-s_2-s_3,s_2)\\
&+\frac{e^{s_2+s_3}(k_{13}(-s_2-s_3,s_2)-e^{s_1}k_{13}(-s_1-s_2-s_3,s_1+s_2))}{8s_1}\\
&-\frac{1}{16}k_{17}(s_1,s_2,s_3)-\frac{1}{16}e^{s_1+s_2}k_{17}(s_3,-s_1-s_2-s_3,s_1)\\
&-\frac{1}{16}e^{s_1}k_{19}(s_2,s_3,-s_1-s_2-s_3)-\frac{1}{16}e^{s_1+s_2+s_3}k_{19}(-s_1-s_2-s_3,s_1,s_2)\\
&-\frac{e^{s_2+s_3}(k_8(-s_2-s_3,s_2)-e^{s_1}k_8(-s_1-s_2-s_3,s_1+s_2))}{8s_1}\\
&-\frac{e^{s_2}(k_{12}(s_3,-s_2-s_3)-e^{s_1}k_{12}(s_3,-s_1-s_2-s_3))}{8s_1}\\
&-\frac{e^{s_3}G_1(s_1)(e^{s_2}k_4(-s_2-s_3)-k_4(-s_3))}{2s_2}-\frac{G_1(s_1)(k_6(s_3)-k_6(s_2+s_3))}{2s_2}\\
&-\frac{e^{s_1}(e^{s_2}k_{12}(s_3,-s_1-s_2-s_3)-k_{12}(s_2+s_3,-s_1-s_2-s_3))}{8s_2}\\
&-\frac{e^{s_1+s_2+s_3}(k_{13}(-s_1-s_2-s_3,s_1)-k_{13}(-s_1-s_2-s_3,s_1+s_2))}{8s_2}\\
&-\frac{e^{s_1}(k_{11}(s_2,-s_1-s_2)-k_{11}(s_2+s_3,-s_1-s_2-s_3))}{8s_3}\\
&-\frac{e^{s_1+s_2}(k_{12}(-s_1-s_2,s_1)-e^{s_3}k_{12}(-s_1-s_2-s_3,s_1))}{8s_3}\\
&-\frac{e^{s_1+s_2+s_3}(k_8(s_1,s_2)-k_8(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)}\\
&-\frac{e^{s_1}(k_{11}(s_2,-s_1-s_2)-k_{11}(s_2,s_3))}{8(s_1+s_2+s_3)}\\
&-\frac{e^{s_1+s_2}(k_{12}(-s_1-s_2,s_1)-k_{12}(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)}.
\end{aligned}
$$

3.5. **The functions $\widetilde{K}_{17},\dots,\widetilde{K}_{20}$.** In this subsection we present the functional relation associated with the four variable function $\widetilde{K}_{17}$, defined by (7). Like this functional relation, the corresponding basic equations for the functions $\widetilde{K}_{18},\widetilde{K}_{19},\widetilde{K}_{20}$, which are written in Appendix A, have quite lengthy expressions. These relations will in particular show that the four variable functions $K_{17},\dots,K_{20}$ appearing in expression (6) for the term a_4 can be constructed from the one and two variable functions given explicitly in Section 9 and the three variable functions provided explicitly in Appendix C, with the aid of the functions G_1, G_2, G_3, G_4 given by (9).

3.5.1. *The function* $\widetilde{K}_{17}$. In this case also, by using the identity (52) and considering the specific function of ∇ that acts on $\delta_1(h)\cdot\delta_1(h)\cdot\delta_2(h)\cdot\delta_2(h)$ in the final formula for the gradient $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\varphi_0(a_4(h+\varepsilon a))$ as calculated in Section 6, we obtain the functional relation associated with the function $\widetilde{K}_{17}$. We have

(20)

$$
\begin{aligned}
&\widetilde{K}_{17}(s_1,s_2,s_3,s_4)\\
&= \frac{1}{15}(-4)\pi G_4(s_1,s_2,s_3,s_4) + \frac{s_3k_6(s_1)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&\quad + \frac{s_4k_6(s_1)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} + \frac{k_6(s_1+s_2)}{2s_1s_3(s_3+s_4)} + \frac{G_1(s_1)k_6(s_3)}{2s_2s_4} + \frac{G_2(s_1,s_2)k_6(s_3)}{2s_4}\\
&\quad + \frac{k_6(s_3)}{2s_2(s_1+s_2)s_4} + \frac{G_1(s_1)k_6(s_2+s_3)}{2s_3(s_3+s_4)} + \frac{G_1(s_1)k_6(s_2+s_3)}{2s_4(s_3+s_4)} + \frac{k_6(s_2+s_3)}{2s_1s_3(s_3+s_4)}\\
&\quad + \frac{k_6(s_2+s_3)}{2s_1s_4(s_3+s_4)} + \frac{k_6(s_1+s_2+s_3)}{2s_1(s_1+s_2)s_4} + \frac{k_6(s_1+s_2+s_3)}{(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&\quad + \frac{s_2k_6(s_1+s_2+s_3)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} + \frac{s_4k_6(s_1+s_2+s_3)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&\quad + \frac{s_2k_6(s_1+s_2+s_3)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)} + \frac{s_3k_6(s_1+s_2+s_3)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}\\
&\quad - \frac{1}{2}G_3(s_1,s_2,s_3)k_6(s_4) + \frac{G_1(s_1)k_6(s_3+s_4)}{2s_2(s_2+s_3)} + \frac{G_1(s_1)k_6(s_3+s_4)}{2s_3(s_2+s_3)} + \frac{G_2(s_1,s_2)k_6(s_3+s_4)}{2s_3}\\
&\quad + \frac{k_6(s_3+s_4)}{(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} + \frac{s_1k_6(s_3+s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&\quad + \frac{s_3k_6(s_3+s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} + \frac{s_1k_6(s_3+s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}\\
&\quad + \frac{s_2k_6(s_3+s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)} + \frac{G_1(s_1)k_6(s_2+s_3+s_4)}{2s_2s_4} + \frac{k_6(s_2+s_3+s_4)}{2s_1(s_1+s_2)s_4}\\
&\quad + \frac{k_6(s_2+s_3+s_4)}{2s_2(s_1+s_2)s_4} + \frac{s_2k_6(s_1+s_2+s_3+s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} + \frac{s_3k_6(s_1+s_2+s_3+s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&\quad + \frac{k_6(s_1+s_2+s_3+s_4)}{2s_1s_4(s_3+s_4)} + \frac{e^{s_1}s_3k_7(-s_1)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&\quad + \frac{e^{s_1}s_4k_7(-s_1)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} + \frac{e^{s_1+s_2}k_7(-s_1-s_2)}{2s_1s_3(s_3+s_4)} + \frac{e^{s_2+s_3}G_1(s_1)k_7(-s_2-s_3)}{2s_3(s_3+s_4)}\\
&\quad + \frac{e^{s_2+s_3}G_1(s_1)k_7(-s_2-s_3)}{2s_4(s_3+s_4)} + \frac{e^{s_2+s_3}k_7(-s_2-s_3)}{2s_1s_3(s_3+s_4)} + \frac{e^{s_2+s_3}k_7(-s_2-s_3)}{2s_1s_4(s_3+s_4)}\\
&\quad + \frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{2s_1(s_1+s_2)s_4} + \frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&\quad + \frac{e^{s_1+s_2+s_3}s_2k_7(-s_1-s_2-s_3)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} + \frac{e^{s_1+s_2+s_3}s_4k_7(-s_1-s_2-s_3)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&\quad + \frac{e^{s_1+s_2+s_3}s_2k_7(-s_1-s_2-s_3)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)} + \frac{e^{s_1+s_2+s_3}s_3k_7(-s_1-s_2-s_3)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}\\
&\quad + \frac{e^{s_3}G_1(s_1)k_7(-s_3)}{2s_2s_4} + \frac{e^{s_3}G_2(s_1,s_2)k_7(-s_3)}{2s_4} + \frac{e^{s_3}k_7(-s_3)}{2s_2(s_1+s_2)s_4} + \frac{e^{s_3+s_4}G_1(s_1)k_7(-s_3-s_4)}{2s_2(s_2+s_3)}\\
&\quad + \frac{e^{s_3+s_4}G_1(s_1)k_7(-s_3-s_4)}{2s_3(s_2+s_3)} + \frac{e^{s_3+s_4}G_2(s_1,s_2)k_7(-s_3-s_4)}{2s_3}\\
&\quad + \frac{e^{s_3+s_4}k_7(-s_3-s_4)}{(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} + \frac{e^{s_3+s_4}s_1k_7(-s_3-s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}
\end{aligned}
$$

$$+\frac{e^{s_3+s_4}s_3k_7(-s_3-s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}+\frac{e^{s_3+s_4}s_1k_7(-s_3-s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}$$
$$+\frac{e^{s_3+s_4}s_2k_7(-s_3-s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}+\frac{e^{s_2+s_3+s_4}G_1(s_1)k_7(-s_2-s_3-s_4)}{2s_2s_4}$$
$$+\frac{e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{2s_1(s_1+s_2)s_4}+\frac{e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{2s_2(s_1+s_2)s_4}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}s_2k_7(-s_1-s_2-s_3-s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}+\frac{e^{s_1+s_2+s_3+s_4}s_3k_7(-s_1-s_2-s_3-s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{2s_1s_4(s_3+s_4)}-\frac{1}{2}e^{s_4}G_3(s_1,s_2,s_3)k_7(-s_4)+\frac{k_8(s_1,s_2)}{4s_3(s_3+s_4)}$$
$$+\frac{k_8(s_1,s_2+s_3+s_4)}{4s_4(s_3+s_4)}+\frac{G_1(s_1)k_8(s_3,s_4)}{4s_2}+\frac{1}{4}G_2(s_1,s_2)k_8(s_3,s_4)+\frac{k_8(s_3,s_4)}{4s_2(s_1+s_2)}$$
$$+\frac{k_8(s_1+s_2+s_3,s_4)}{4s_1(s_1+s_2)}+\frac{k_9(s_1,s_2+s_3)}{8s_2s_4}+\frac{k_9(s_1,s_2+s_3+s_4)}{8s_2(s_2+s_3)}+\frac{G_1(s_1)k_9(s_2,s_3+s_4)}{8s_4}$$
$$+\frac{k_9(s_2,s_3+s_4)}{8s_1s_4}+\frac{k_9(s_1+s_2,s_3)}{8s_1s_4}+\frac{k_9(s_1+s_2,s_3+s_4)}{8s_1s_3}+\frac{k_9(s_1+s_2,s_3+s_4)}{8s_2s_4}$$
$$+\frac{G_1(s_1)k_9(s_2+s_3,s_4)}{8s_3}+\frac{k_9(s_2+s_3,s_4)}{8s_1s_3}+\frac{k_9(s_1+s_2+s_3,s_4)}{8s_3(s_2+s_3)}+\frac{e^{s_1+s_2}k_{10}(s_3,-s_1-s_2-s_3)}{8s_1s_4}$$
$$+\frac{e^{s_1}k_{10}(s_2+s_3,-s_1-s_2-s_3)}{8s_2s_4}+\frac{e^{s_2+s_3}G_1(s_1)k_{10}(s_4,-s_2-s_3-s_4)}{8s_3}$$
$$+\frac{e^{s_2+s_3}k_{10}(s_4,-s_2-s_3-s_4)}{8s_1s_3}+\frac{e^{s_1+s_2+s_3}k_{10}(s_4,-s_1-s_2-s_3-s_4)}{8s_3(s_2+s_3)}$$
$$+\frac{e^{s_2}G_1(s_1)k_{10}(s_3+s_4,-s_2-s_3-s_4)}{8s_4}+\frac{e^{s_2}k_{10}(s_3+s_4,-s_2-s_3-s_4)}{8s_1s_4}$$
$$+\frac{e^{s_1+s_2}k_{10}(s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_1s_3}+\frac{e^{s_1+s_2}k_{10}(s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_2s_4}$$
$$+\frac{e^{s_1}k_{10}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_2(s_2+s_3)}+\frac{e^{s_1}k_{11}(s_2,-s_1-s_2)}{4s_3(s_3+s_4)}$$
$$+\frac{e^{s_3}G_1(s_1)k_{11}(s_4,-s_3-s_4)}{4s_2}+\frac{1}{4}e^{s_3}G_2(s_1,s_2)k_{11}(s_4,-s_3-s_4)+\frac{e^{s_3}k_{11}(s_4,-s_3-s_4)}{4s_2(s_1+s_2)}$$
$$+\frac{e^{s_1+s_2+s_3}k_{11}(s_4,-s_1-s_2-s_3-s_4)}{4s_1(s_1+s_2)}+\frac{e^{s_1}k_{11}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_4(s_3+s_4)}$$
$$+\frac{e^{s_1+s_2}k_{12}(-s_1-s_2,s_1)}{4s_3(s_3+s_4)}+\frac{e^{s_3+s_4}G_1(s_1)k_{12}(-s_3-s_4,s_3)}{4s_2}$$
$$+\frac{1}{4}e^{s_3+s_4}G_2(s_1,s_2)k_{12}(-s_3-s_4,s_3)+\frac{e^{s_3+s_4}k_{12}(-s_3-s_4,s_3)}{4s_2(s_1+s_2)}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{12}(-s_1-s_2-s_3-s_4,s_1)}{4s_4(s_3+s_4)}+\frac{e^{s_1+s_2+s_3+s_4}k_{12}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{4s_1(s_1+s_2)}$$
$$+\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1)}{8s_2s_4}+\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_1s_4}$$
$$+\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2)}{8s_4}+\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2)}{8s_1s_4}$$
$$+\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_3}+\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_1s_3}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8s_2(s_2+s_3)}+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_1s_3}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_2s_4}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_3(s_2+s_3)}+\frac{k_{17}(s_1,s_2,s_3)}{16s_4}+\frac{k_{17}(s_1,s_2,s_3+s_4)}{16s_3}$$
$$+\frac{e^{s_1+s_2}k_{17}(s_3,-s_1-s_2-s_3,s_1)}{16s_4}-\frac{1}{16}e^{s_2}G_1(s_1)k_{17}(s_3,s_4,-s_2-s_3-s_4)$$

$$
\begin{aligned}
&+\frac{e^{s_1+s_2}k_{17}(s_3,s_4,-s_2-s_3-s_4)}{16(s_1+s_2+s_3+s_4)}+\frac{e^{s_1+s_2}k_{17}(s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_1}\\
&+\frac{e^{s_1}k_{17}(s_2+s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_2}-\frac{1}{16}e^{s_2+s_3+s_4}G_1(s_1)k_{17}(-s_2-s_3-s_4,s_2,s_3)\\
&+\frac{e^{s_1+s_2+s_3+s_4}k_{17}(-s_2-s_3-s_4,s_2,s_3)}{16(s_1+s_2+s_3+s_4)}+\frac{e^{s_1+s_2+s_3+s_4}k_{17}(-s_1-s_2-s_3-s_4,s_1,s_2+s_3)}{16s_2}\\
&+\frac{e^{s_1+s_2+s_3+s_4}k_{17}(-s_1-s_2-s_3-s_4,s_1+s_2,s_3)}{16s_1}+\frac{e^{s_1+s_2}k_{17}(s_3+s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_3}\\
&+\frac{k_{19}(s_1,s_2+s_3,s_4)}{16s_2}+\frac{e^{s_1}k_{19}(s_2,s_3,-s_1-s_2-s_3)}{16s_4}-\frac{1}{16}G_1(s_1)k_{19}(s_2,s_3,s_4)\\
&+\frac{e^{s_1}k_{19}(s_2,s_3,s_4)}{16(s_1+s_2+s_3+s_4)}+\frac{e^{s_1}k_{19}(s_2,s_3+s_4,-s_1-s_2-s_3-s_4)}{16s_3}+\frac{k_{19}(s_1+s_2,s_3,s_4)}{16s_1}\\
&+\frac{e^{s_1+s_2+s_3}k_{19}(-s_1-s_2-s_3,s_1,s_2)}{16s_4}+\frac{e^{s_1+s_2+s_3+s_4}k_{19}(-s_1-s_2-s_3-s_4,s_1,s_2)}{16s_3}\\
&-\frac{1}{16}e^{s_2+s_3}G_1(s_1)k_{19}(s_4,-s_2-s_3-s_4,s_2)+\frac{e^{s_1+s_2+s_3}k_{19}(s_4,-s_2-s_3-s_4,s_2)}{16(s_1+s_2+s_3+s_4)}\\
&+\frac{e^{s_1+s_2+s_3}k_{19}(s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_2}+\frac{e^{s_1+s_2+s_3}k_{19}(s_4,-s_1-s_2-s_3-s_4,s_1+s_2)}{16s_1}\\
&-\frac{e^{s_2}k_{17}(s_3,s_4,-s_2-s_3-s_4)}{16s_1}-\frac{e^{s_2+s_3+s_4}k_{17}(-s_2-s_3-s_4,s_2,s_3)}{16s_1}-\frac{k_{19}(s_2,s_3,s_4)}{16s_1}\\
&-\frac{e^{s_2+s_3}k_{19}(s_4,-s_2-s_3-s_4,s_2)}{16s_1}-\frac{G_1(s_1)k_8(s_2+s_3,s_4)}{4s_2}-\frac{e^{s_2+s_3}G_1(s_1)k_{11}(s_4,-s_2-s_3-s_4)}{4s_2}\\
&-\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{12}(-s_2-s_3-s_4,s_2+s_3)}{4s_2}-\frac{e^{s_1+s_2}k_{17}(s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_2}\\
&-\frac{e^{s_1+s_2+s_3+s_4}k_{17}(-s_1-s_2-s_3-s_4,s_1+s_2,s_3)}{16s_2}-\frac{k_{19}(s_1+s_2,s_3,s_4)}{16s_2}\\
&-\frac{e^{s_1+s_2+s_3}k_{19}(s_4,-s_1-s_2-s_3-s_4,s_1+s_2)}{16s_2}-\frac{k_8(s_2+s_3,s_4)}{4s_1(s_1+s_2)}-\frac{e^{s_2+s_3}k_{11}(s_4,-s_2-s_3-s_4)}{4s_1(s_1+s_2)}\\
&-\frac{e^{s_2+s_3+s_4}k_{12}(-s_2-s_3-s_4,s_2+s_3)}{4s_1(s_1+s_2)}-\frac{k_8(s_2+s_3,s_4)}{4s_2(s_1+s_2)}-\frac{e^{s_2+s_3}k_{11}(s_4,-s_2-s_3-s_4)}{4s_2(s_1+s_2)}\\
&-\frac{e^{s_2+s_3+s_4}k_{12}(-s_2-s_3-s_4,s_2+s_3)}{4s_2(s_1+s_2)}-\frac{G_2(s_1,s_2)k_6(s_4)}{2s_3}-\frac{e^{s_4}G_2(s_1,s_2)k_7(-s_4)}{2s_3}\\
&-\frac{G_1(s_1)k_9(s_2,s_3+s_4)}{8s_3}-\frac{e^{s_2}G_1(s_1)k_{10}(s_3+s_4,-s_2-s_3-s_4)}{8s_3}\\
&-\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2)}{8s_3}-\frac{k_{17}(s_1,s_2+s_3,s_4)}{16s_3}\\
&-\frac{e^{s_1+s_2+s_3}k_{17}(s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_3}-\frac{e^{s_1}k_{19}(s_2+s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_3}\\
&-\frac{e^{s_1+s_2+s_3+s_4}k_{19}(-s_1-s_2-s_3-s_4,s_1,s_2+s_3)}{16s_3}-\frac{k_9(s_2,s_3+s_4)}{8s_1s_3}-\frac{k_9(s_1+s_2+s_3,s_4)}{8s_1s_3}\\
&-\frac{e^{s_1+s_2+s_3}k_{10}(s_4,-s_1-s_2-s_3-s_4)}{8s_1s_3}-\frac{e^{s_2}k_{10}(s_3+s_4,-s_2-s_3-s_4)}{8s_1s_3}\\
&-\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2)}{8s_1s_3}-\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_1s_3}\\
&-\frac{G_1(s_1)k_6(s_2+s_3+s_4)}{2s_2(s_2+s_3)}-\frac{e^{s_2+s_3+s_4}G_1(s_1)k_7(-s_2-s_3-s_4)}{2s_2(s_2+s_3)}-\frac{k_9(s_1+s_2,s_3+s_4)}{8s_2(s_2+s_3)}\\
&-\frac{e^{s_1+s_2}k_{10}(s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_2(s_2+s_3)}-\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_2(s_2+s_3)}\\
&-\frac{G_1(s_1)k_6(s_4)}{2s_3(s_2+s_3)}-\frac{e^{s_4}G_1(s_1)k_7(-s_4)}{2s_3(s_2+s_3)}-\frac{k_9(s_1+s_2,s_3+s_4)}{8s_3(s_2+s_3)}\\
&-\frac{e^{s_1+s_2}k_{10}(s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_3(s_2+s_3)}-\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_3(s_2+s_3)}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{k_6(s_2+s_3+s_4)}{(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}-\frac{e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&-\frac{s_2k_6(s_2+s_3+s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}-\frac{s_3k_6(s_2+s_3+s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&-\frac{e^{s_2+s_3+s_4}s_2k_7(-s_2-s_3-s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}-\frac{e^{s_2+s_3+s_4}s_3k_7(-s_2-s_3-s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&-\frac{s_1k_6(s_2+s_3+s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}-\frac{s_3k_6(s_2+s_3+s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&-\frac{e^{s_2+s_3+s_4}s_1k_7(-s_2-s_3-s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}-\frac{e^{s_2+s_3+s_4}s_3k_7(-s_2-s_3-s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&-\frac{s_1k_6(s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}-\frac{s_2k_6(s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}\\
&-\frac{e^{s_4}s_1k_7(-s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}-\frac{e^{s_4}s_2k_7(-s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}\\
&-\frac{G_2(s_1,s_2)k_6(s_3+s_4)}{2s_4}-\frac{e^{s_3+s_4}G_2(s_1,s_2)k_7(-s_3-s_4)}{2s_4}-\frac{G_1(s_1)k_9(s_2,s_3)}{8s_4}\\
&-\frac{e^{s_2}G_1(s_1)k_{10}(s_3,-s_2-s_3)}{8s_4}-\frac{e^{s_2+s_3}G_1(s_1)k_{13}(-s_2-s_3,s_2)}{8s_4}-\frac{k_{17}(s_1,s_2,s_3+s_4)}{16s_4}\\
&-\frac{e^{s_1+s_2}k_{17}(s_3+s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_4}-\frac{e^{s_1}k_{19}(s_2,s_3+s_4,-s_1-s_2-s_3-s_4)}{16s_4}\\
&-\frac{e^{s_1+s_2+s_3+s_4}k_{19}(-s_1-s_2-s_3-s_4,s_1,s_2)}{16s_4}-\frac{k_9(s_2,s_3)}{8s_1s_4}-\frac{k_9(s_1+s_2,s_3+s_4)}{8s_1s_4}\\
&-\frac{e^{s_2}k_{10}(s_3,-s_2-s_3)}{8s_1s_4}-\frac{e^{s_1+s_2}k_{10}(s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_1s_4}-\frac{e^{s_2+s_3}k_{13}(-s_2-s_3,s_2)}{8s_1s_4}\\
&-\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_1s_4}-\frac{G_1(s_1)k_6(s_2+s_3)}{2s_2s_4}-\frac{G_1(s_1)k_6(s_3+s_4)}{2s_2s_4}\\
&-\frac{e^{s_2+s_3}G_1(s_1)k_7(-s_2-s_3)}{2s_2s_4}-\frac{e^{s_3+s_4}G_1(s_1)k_7(-s_3-s_4)}{2s_2s_4}-\frac{k_9(s_1,s_2+s_3+s_4)}{8s_2s_4}\\
&-\frac{k_9(s_1+s_2,s_3)}{8s_2s_4}-\frac{e^{s_1+s_2}k_{10}(s_3,-s_1-s_2-s_3)}{8s_2s_4}-\frac{e^{s_1}k_{10}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_2s_4}\\
&-\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_2s_4}-\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8s_2s_4}\\
&-\frac{k_6(s_2+s_3)}{2s_1(s_1+s_2)s_4}-\frac{k_6(s_1+s_2+s_3+s_4)}{2s_1(s_1+s_2)s_4}-\frac{e^{s_2+s_3}k_7(-s_2-s_3)}{2s_1(s_1+s_2)s_4}\\
&-\frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{2s_1(s_1+s_2)s_4}-\frac{k_6(s_2+s_3)}{2s_2(s_1+s_2)s_4}-\frac{k_6(s_3+s_4)}{2s_2(s_1+s_2)s_4}-\frac{e^{s_2+s_3}k_7(-s_2-s_3)}{2s_2(s_1+s_2)s_4}\\
&-\frac{e^{s_3+s_4}k_7(-s_3-s_4)}{2s_2(s_1+s_2)s_4}-\frac{G_1(s_1)k_6(s_2)}{2s_3(s_3+s_4)}-\frac{e^{s_2}G_1(s_1)k_7(-s_2)}{2s_3(s_3+s_4)}-\frac{k_8(s_1,s_2+s_3)}{4s_3(s_3+s_4)}\\
&-\frac{e^{s_1}k_{11}(s_2+s_3,-s_1-s_2-s_3)}{4s_3(s_3+s_4)}-\frac{e^{s_1+s_2+s_3}k_{12}(-s_1-s_2-s_3,s_1)}{4s_3(s_3+s_4)}-\frac{k_6(s_2)}{2s_1s_3(s_3+s_4)}\\
&-\frac{k_6(s_1+s_2+s_3)}{2s_1s_3(s_3+s_4)}-\frac{e^{s_2}k_7(-s_2)}{2s_1s_3(s_3+s_4)}-\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{2s_1s_3(s_3+s_4)}-\frac{G_1(s_1)k_6(s_2+s_3+s_4)}{2s_4(s_3+s_4)}\\
&-\frac{e^{s_2+s_3+s_4}G_1(s_1)k_7(-s_2-s_3-s_4)}{2s_4(s_3+s_4)}-\frac{k_8(s_1,s_2+s_3)}{4s_4(s_3+s_4)}-\frac{e^{s_1}k_{11}(s_2+s_3,-s_1-s_2-s_3)}{4s_4(s_3+s_4)}\\
&-\frac{e^{s_1+s_2+s_3}k_{12}(-s_1-s_2-s_3,s_1)}{4s_4(s_3+s_4)}-\frac{k_6(s_1+s_2+s_3)}{2s_1s_4(s_3+s_4)}-\frac{k_6(s_2+s_3+s_4)}{2s_1s_4(s_3+s_4)}\\
&-\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{2s_1s_4(s_3+s_4)}-\frac{e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{2s_1s_4(s_3+s_4)}-\frac{k_6(s_1+s_2)}{(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{e^{s_1+s_2}k_7(-s_1-s_2)}{(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}-\frac{s_3k_6(s_1+s_2)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{s_4k_6(s_1+s_2)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}-\frac{e^{s_1+s_2}s_3k_7(-s_1-s_2)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{e^{s_1+s_2}s_4k_7(-s_1-s_2)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}-\frac{s_2k_6(s_1+s_2)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{s_4k_6(s_1+s_2)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}-\frac{e^{s_1+s_2}s_2k_7(-s_1-s_2)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{e^{s_1+s_2}s_4k_7(-s_1-s_2)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}-\frac{s_2k_6(s_1+s_2+s_3+s_4)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{s_3k_6(s_1+s_2+s_3+s_4)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}-\frac{e^{s_1+s_2+s_3+s_4}s_2k_7(-s_1-s_2-s_3-s_4)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{e^{s_1+s_2+s_3+s_4}s_3k_7(-s_1-s_2-s_3-s_4)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}-\frac{k_{17}(s_1,s_2,s_3)}{16(s_1+s_2+s_3+s_4)}\\
&-\frac{e^{s_1+s_2}k_{17}(s_3,-s_1-s_2-s_3,s_1)}{16(s_1+s_2+s_3+s_4)}-\frac{e^{s_1}k_{19}(s_2,s_3,-s_1-s_2-s_3)}{16(s_1+s_2+s_3+s_4)}\\
&-\frac{e^{s_1+s_2+s_3}k_{19}(-s_1-s_2-s_3,s_1,s_2)}{16(s_1+s_2+s_3+s_4)}.
\end{aligned}
$$

As we mentioned earlier, the remaining basic functional relations of the above type, because of their similarly lengthy expressions, are recorded in Appendix A.

4. Differential system from functional relations and action of cyclic groups

In this section we present a system of differential equations that the functions $k_3,\dots,k_{20}$ satisfy. We recall that $k_3,\dots,k_{20}$, which are defined by (8), are derived from the two, three and four variable functions $K_3,\dots,K_{20}$ by specializing them to $s_1+\cdots+s_n=0$, where $n\in\{2,3,4\}$ is the number of variables that each K_j depends on. The functional relations stated in Theorem 3.1 and presented subsequently and in Appendix A allow us to abstractly use specific finite differences of the functions $k_3,\dots,k_{20}$ to express the functions

$$\widetilde{K}_j(s_1,\dots,s_n)=\frac{1}{2^n}\frac{\sinh((s_1+\cdots+s_n)/2)}{(s_1+\cdots+s_n)/2}K_j(s_1,\dots,s_n).$$

An interesting question that arises naturally here is whether one can extract more abstract information from these relations. We answer this question in this section.

4.1. System of differential equations for the functions $k_3,\dots,k_{20}$.

We start by noting that the factor that changes K_j to $\widetilde{K}_j$, namely

$$\frac{1}{2^n}\frac{\sinh((s_1+\cdots+s_n)/2)}{(s_1+\cdots+s_n)/2},$$

specializes to $\frac{1}{2^n}$ on $s_1+\cdots+s_n=0$, hence the left-hand side of each functional relation, under this specialization, behaves as follows:

$$\widetilde{K}_j(s_1,\dots,s_n)\to\frac{1}{2^n}k_j(s_1,\dots,s_{n-1}).$$

Now we should analyze the behavior of the right-hand side of each functional relation under the specialization to $s_1+\cdots+s_n=0$. We shall see shortly that on the right-hand side of each equation, most of the terms behave nicely with respect to a direct replacement of s_n by $-s_1 - \cdots - s_{n-1}$, except for few terms that have $s_1 + \cdots + s_n = 0$ in their denominators. We will specialize each of such terms by calculating a limit with the aid of an estimate of the form $s_1 + \cdots + s_n = \varepsilon \sim 0$, and will see that partial derivatives of the involved functions will appear. Hence, a system of differential equations will be derived.

We first demonstrate this process on the basic equation (12):

$$
\begin{aligned}
&\widetilde{K}_3(s_1, s_2)\\
&\quad = \frac{1}{15}(-4)\pi G_2(s_1, s_2) + \frac{1}{2}k_8(s_1, s_2) + \frac{1}{4}k_9(s_1, s_2) - \frac{1}{4}e^{s_1+s_2}k_9(-s_1 - s_2, s_1)\\
&\qquad - \frac{1}{4}e^{s_1}k_9(s_2, -s_1 - s_2) - \frac{1}{4}k_{10}(s_1, s_2) - \frac{1}{4}e^{s_1+s_2}k_{10}(-s_1 - s_2, s_1)\\
&\qquad + \frac{1}{4}e^{s_1}k_{10}(s_2, -s_1 - s_2) + \frac{1}{2}e^{s_1}k_{11}(s_2, -s_1 - s_2) + \frac{1}{2}e^{s_1+s_2}k_{12}(-s_1 - s_2, s_1)\\
&\qquad - \frac{1}{4}k_{13}(s_1, s_2) + \frac{1}{4}e^{s_1+s_2}k_{13}(-s_1 - s_2, s_1) - \frac{1}{4}e^{s_1}k_{13}(s_2, -s_1 - s_2)\\
&\qquad + \frac{1}{4}e^{s_2}G_1(s_1)k_3(-s_2) + \frac{1}{4}G_1(s_1)k_3(s_2) - G_1(s_1)k_6(s_2) - e^{s_2}G_1(s_1)k_7(-s_2)\\
&\qquad + \frac{(e^{s_1+s_2} - 1)k_3(s_1)}{4(s_1 + s_2)} + \frac{k_3(s_2) - k_3(s_1 + s_2)}{4s_1} + \frac{k_3(s_1 + s_2) - k_3(s_1)}{4s_2}\\
&\qquad + \frac{k_6(s_1) - k_6(s_1 + s_2)}{s_2} + \frac{k_6(s_1 + s_2) - k_6(s_2)}{s_1}\\
&\qquad + \frac{e^{s_1}(k_7(-s_1) - e^{s_2}k_7(-s_1 - s_2))}{s_2} + \frac{e^{s_2}(e^{s_1}k_7(-s_1 - s_2) - k_7(-s_2))}{s_1}\\
&\qquad - \frac{e^{s_2}(e^{s_1}k_3(-s_1 - s_2) - k_3(-s_2))}{4s_1} - \frac{e^{s_1}(k_3(-s_1) - e^{s_2}k_3(-s_1 - s_2))}{4s_2}\\
&\qquad - \frac{e^{s_1}(k_3(-s_1) + e^{s_2}k_3(s_1) - e^{s_2}k_3(-s_2) - k_3(s_2))}{4(s_1 + s_2)},
\end{aligned}
$$

which we want to specialize for $s_1 + s_2 = 0$. In that respect the term $\widetilde{K}_3(s_1, s_2)$ gives $\frac{1}{4}k_3(s_1)$ since the coefficient

$$\frac{1}{2^2}\frac{\sinh((s_1 + s_2)/2)}{(s_1 + s_2)/2}$$

is well behaved and gives $\frac{1}{4}$.

The two interesting terms in the above expression are those which have $s_1 + s_2$ in the denominator. The first one is

$$\frac{(e^{s_1+s_2} - 1)k_3(s_1)}{4(s_1 + s_2)},$$

which contributes exactly by $\frac{1}{4}k_3(s_1)$ and therefore cancels the previous term $\widetilde{K}_3(s_1, s_2)$ which also gives $\frac{1}{4}k_3(s_1)$. The second crucial term in the right-hand side of the basic equation is

$$-\frac{e^{s_1}(k_3(-s_1) + e^{s_2}k_3(s_1) - e^{s_2}k_3(-s_2) - k_3(s_2))}{4(s_1 + s_2)},$$

which can be treated directly using $s_1 + s_2 = \varepsilon \sim 0$ as

$$-\frac{e^{s_1}(k_3(-s_1) - k_3(-s_1+\varepsilon)) + e^{s_1+s_2}(k_3(s_1) - k_3(s_1-\varepsilon))}{4\varepsilon},$$

and it gives all the terms involving the derivative of k_3, namely

$$\frac{1}{4}e^{s_1}k_3'(-s_1) - \frac{1}{4}k_3'(s_1),$$

which corresponds to the differential equation (21) since all other terms are specialized without problem using $s_1 + s_2 = 0$.

As we mentioned above, this approach can be applied to each functional relation to derive a differential equation.

Theorem 4.2. *The functions $k_3, \ldots, k_{20}$ satisfy a system of partial differential equations as described below.*

Above, we explained the details of the derivation of the differential equation associated with the functional relation for the function $\widetilde{K}_3$. We can now write explicitly the differential equation, and will then perform a similar analysis on each functional relation to obtain explicitly a system of partial differential equations.

4.2.1. *Differential equation derived from the expression for $\widetilde{K}_3$.* We have

$$\begin{aligned}(21)\quad &\frac{1}{4}e^{s_1}k_3'(-s_1) - \frac{1}{4}k_3'(s_1)\\ &= \frac{1}{60s_1}\Big(16\pi s_1 G_2(s_1,-s_1) - 30s_1k_8(s_1,-s_1) + 15s_1k_9(0,s_1)\\ &\quad + 15e^{s_1}s_1k_9(-s_1,0) - 15s_1k_9(s_1,-s_1) + 15s_1k_{10}(0,s_1)\\ &\quad - 15e^{s_1}s_1k_{10}(-s_1,0) + 15s_1k_{10}(s_1,-s_1) - 30e^{s_1}s_1k_{11}(-s_1,0)\\ &\quad - 30s_1k_{12}(0,s_1) - 15s_1k_{13}(0,s_1) + 15e^{s_1}s_1k_{13}(-s_1,0)\\ &\quad + 15s_1k_{13}(s_1,-s_1) - 15s_1G_1(s_1)k_3(-s_1) - 15e^{-s_1}s_1G_1(s_1)k_3(s_1)\\ &\quad + 60s_1G_1(s_1)k_6(-s_1) + 60e^{-s_1}s_1G_1(s_1)k_7(s_1) - 15e^{s_1}k_3(-s_1)\\ &\quad - 15k_3(-s_1) - 15e^{-s_1}k_3(s_1) - 15k_3(s_1) + 60k_6(-s_1) + 60k_6(s_1)\\ &\quad + 60e^{s_1}k_7(-s_1) + 60e^{-s_1}k_7(s_1) + 60k_3(0) - 120k_6(0) - 120k_7(0)\Big).\end{aligned}$$

4.2.2. *Differential equation derived from the expression for $\widetilde{K}_4$.* In the basic equation (13) for $\widetilde{K}_4(s_1,s_2)$, the terms that need special care in order to specialize to $s_1 + s_2 = 0$ are

$$\frac{(e^{s_1+s_2}-1)k_4(s_1)}{4(s_1+s_2)}, \quad -\frac{e^{s_1+s_2}(k_4(s_1) - k_4(-s_2))}{4(s_1+s_2)}, \quad -\frac{e^{s_1}(k_4(-s_1) - k_4(s_2))}{4(s_1+s_2)}.$$

By writing $s_1+s_2 = \varepsilon \sim 0$, we find that the above terms respectively specialize to

$$\frac{1}{4}k_4(s_1), \quad -\frac{1}{4}k_4'(s_1), \quad \frac{1}{4}e^{s_1}k_4'(-s_1).$$

Therefore, by a direct replacement of s_2 by $-s_1$ in the rest of the terms, we find that

$$
\begin{aligned}
(22)\quad & \frac{1}{4}e^{s_1}k_4'(-s_1)-\frac{1}{4}k_4'(s_1)\\
&=\frac{1}{120s_1}\Big(8\pi s_1G_2(s_1,-s_1)+15s_1k_8(0,s_1)+15e^{s_1}s_1k_8(-s_1,0)\\
&\quad-15s_1k_9(s_1,-s_1)-15e^{s_1}s_1k_{10}(-s_1,0)+15s_1k_{11}(0,s_1)\\
&\quad+15s_1k_{11}(s_1,-s_1)+15e^{s_1}s_1k_{12}(-s_1,0)+15s_1k_{12}(s_1,-s_1)\\
&\quad-15s_1k_{13}(0,s_1)-30s_1G_1(s_1)k_4(-s_1)-30e^{-s_1}s_1G_1(s_1)k_4(s_1)\\
&\quad+30s_1G_1(s_1)k_6(-s_1)+30e^{-s_1}s_1G_1(s_1)k_7(s_1)-30e^{s_1}k_4(-s_1)\\
&\quad-30k_4(-s_1)-30e^{-s_1}k_4(s_1)-30k_4(s_1)+30k_6(-s_1)+30k_6(s_1)\\
&\quad+30e^{s_1}k_7(-s_1)+30e^{-s_1}k_7(s_1)+120k_4(0)-60k_6(0)-60k_7(0)\Big).
\end{aligned}
$$

4.2.3. *Differential equation derived from the expression for $\widetilde{K}_5$.* In the expression given in (14) for $\widetilde{K}_5$, the only terms that do not specialize to $s_1+s_2=0$ by a simple replacement of s_2 by $-s_1$ are

$$
\frac{(e^{s_1+s_2}-1)k_5(s_1)}{4(s_1+s_2)},\qquad -\frac{e^{s_1}(k_5(-s_1)+e^{s_2}k_5(s_1)-e^{s_2}k_5(-s_2)-k_5(s_2))}{4(s_1+s_2)}.
$$

Therefore, we use the estimate $s_1+s_2=\varepsilon\sim 0$ to find that the above terms respectively tend to the following functions on $s_1+s_2=0$:

$$
\frac{1}{4}k_5(s_1),\quad \frac{1}{4}e^{s_1}k_5'(-s_1)-\frac{1}{4}k_5'(s_1).
$$

Since the rest of the terms behave nicely when we replace s_2 by $-s_1$, we find the following identity:

$$
\begin{aligned}
(23)\quad & \frac{1}{4}e^{s_1}k_5'(-s_1)-\frac{1}{4}k_5'(s_1)\\
&=\frac{1}{40s_1}\Big(8\pi s_1G_2(s_1,-s_1)-10s_1k_{14}(0,s_1)+5e^{s_1}s_1k_{14}(-s_1,0)\\
&\quad+5s_1k_{14}(s_1,-s_1)+5s_1k_{15}(0,s_1)+5e^{s_1}s_1k_{15}(-s_1,0)\\
&\quad-10s_1k_{15}(s_1,-s_1)+5s_1k_{16}(0,s_1)-10e^{s_1}s_1k_{16}(-s_1,0)\\
&\quad+5s_1k_{16}(s_1,-s_1)-10s_1G_1(s_1)k_5(-s_1)-10e^{-s_1}s_1G_1(s_1)k_5(s_1)\\
&\quad+30s_1G_1(s_1)k_6(-s_1)+30e^{-s_1}s_1G_1(s_1)k_7(s_1)-10e^{s_1}k_5(-s_1)\\
&\quad-10k_5(-s_1)-10e^{-s_1}k_5(s_1)-10k_5(s_1)+30k_6(-s_1)+30k_6(s_1)\\
&\quad+30e^{s_1}k_7(-s_1)+30e^{-s_1}k_7(s_1)+40k_5(0)-60k_6(0)-60k_7(0)\Big).
\end{aligned}
$$

4.2.4. *Differential equations derived from the two expressions for $\widetilde{K}_6$.* As we explained in Section 3.3.4, since in formula (52) the operator associated with $\widetilde{K}_6$ acts on two different kinds of elements of $C^\infty(\mathbb{T}_\theta^2)$ that are not the same modulo switching δ_1 and δ_2, we have two different basic equations for K_6. In the first expression given by (15), the terms that do not specialize directly to $s_1+s_2=0$ are

$$\frac{(e^{s_1+s_2}-1)k_6(s_1)}{4(s_1+s_2)}, \quad -\frac{e^{s_1+s_2}(k_6(s_1)-k_6(-s_2))}{4(s_1+s_2)}, \quad -\frac{e^{s_1}(k_7(-s_1)-k_7(s_2))}{4(s_1+s_2)}.$$

By using the estimate $s_1+s_2=\varepsilon\sim 0$ and calculating a limit in each case, we find that the above functions respectively turn to the following functions on $s_1+s_2=0$:

$$\frac{1}{4}k_6(s_1), \quad -\frac{1}{4}k_6'(s_1), \quad \frac{1}{4}e^{s_1}k_7'(-s_1).$$

Then, by a simple replacement of s_2 by $-s_1$ in the rest of the terms, we find that

(24)

$$\begin{aligned}
&\frac{1}{4}e^{s_1}k_7'(-s_1)-\frac{1}{4}k_6'(s_1)\\
&\quad=\frac{1}{120s_1}\Big(16\pi s_1G_2(s_1,-s_1)-15s_1k_{14}(0,s_1)+15e^{s_1}s_1k_{14}(-s_1,0)\\
&\qquad+15s_1k_{15}(0,s_1)-15s_1k_{15}(s_1,-s_1)-15e^{s_1}s_1k_{16}(-s_1,0)\\
&\qquad+15s_1k_{16}(s_1,-s_1)-60s_1G_1(s_1)k_5(-s_1)-60e^{-s_1}s_1G_1(s_1)k_5(s_1)\\
&\qquad+90s_1G_1(s_1)k_6(-s_1)+30e^{-s_1}s_1G_1(s_1)k_6(s_1)+30s_1G_1(s_1)k_7(-s_1)\\
&\qquad+90e^{-s_1}s_1G_1(s_1)k_7(s_1)-60k_5(-s_1)-60e^{-s_1}k_5(s_1)+90k_6(-s_1)\\
&\qquad+30e^{-s_1}k_6(s_1)+30k_6(s_1)+30e^{s_1}k_7(-s_1)+30k_7(-s_1)+90e^{-s_1}k_7(s_1)\\
&\qquad+120k_5(0)-150k_6(0)-150k_7(0)\Big).
\end{aligned}$$

In the second expression for $\widetilde{K}_6$ given by (16), the terms that seemingly have singularity on $s_1+s_2=0$ are

$$\frac{(e^{s_1+s_2}-1)k_6(s_1)}{4(s_1+s_2)}, \quad -\frac{e^{s_1+s_2}(k_6(s_1)-k_6(-s_2))}{4(s_1+s_2)}, \quad -\frac{e^{s_1}(k_7(-s_1)-k_7(s_2))}{4(s_1+s_2)},$$

which are exactly the same as the seemingly singular terms in the first expression. Since we already know the restriction of these terms to $s_1+s_2=0$, we can derive the following identity by replacing s_2 by $-s_1$ in the remaining terms:

(25)

$$\begin{aligned}
&\frac{1}{4}e^{s_1}k_7'(-s_1)-\frac{1}{4}k_6'(s_1)\\
&\quad=\frac{1}{120s_1}\Big(16\pi s_1G_2(s_1,-s_1)+15s_1k_8(0,s_1)-15s_1k_8(s_1,-s_1)\\
&\qquad+15s_1k_9(0,s_1)-15s_1k_9(s_1,-s_1)-15e^{s_1}s_1k_{10}(-s_1,0)\\
&\qquad+15s_1k_{10}(s_1,-s_1)-15e^{s_1}s_1k_{11}(-s_1,0)+15s_1k_{11}(s_1,-s_1)\\
&\qquad-15s_1k_{12}(0,s_1)+15e^{s_1}s_1k_{12}(-s_1,0)-15s_1k_{13}(0,s_1)\\
&\qquad+15e^{s_1}s_1k_{13}(-s_1,0)-30s_1G_1(s_1)k_3(-s_1)-30e^{-s_1}s_1G_1(s_1)k_3(s_1)
\end{aligned}$$

$$
\begin{aligned}
&- 60s_1G_1(s_1)k_4(-s_1) - 60e^{-s_1}s_1G_1(s_1)k_4(s_1) + 90s_1G_1(s_1)k_6(-s_1)\\
&+ 30e^{-s_1}s_1G_1(s_1)k_6(s_1) + 30s_1G_1(s_1)k_7(-s_1) + 90e^{-s_1}s_1G_1(s_1)k_7(s_1)\\
&- 30k_3(-s_1) - 30e^{-s_1}k_3(s_1) - 60k_4(-s_1) - 60e^{-s_1}k_4(s_1) + 90k_6(-s_1)\\
&+ 30e^{-s_1}k_6(s_1) + 30k_6(s_1) + 30e^{s_1}k_7(-s_1) + 30k_7(-s_1) + 90e^{-s_1}k_7(s_1)\\
&+ 60k_3(0) + 120k_4(0) - 150k_6(0) - 150k_7(0)\Big).
\end{aligned}
$$

Since the left-hand side of equation (24) matches precisely with that of (25), we have the following functional relation.

Corollary 4.3. *We have*

$$
\begin{aligned}
&-s_1k_8(0,s_1) + s_1k_8(s_1,-s_1) - s_1k_9(0,s_1) + s_1k_9(s_1,-s_1) + e^{s_1}s_1k_{10}(-s_1,0)\\
&\quad - s_1k_{10}(s_1,-s_1) + e^{s_1}s_1k_{11}(-s_1,0) - s_1k_{11}(s_1,-s_1) + s_1k_{12}(0,s_1)\\
&\quad - e^{s_1}s_1k_{12}(-s_1,0) + s_1k_{13}(0,s_1) - e^{s_1}s_1k_{13}(-s_1,0) - s_1k_{14}(0,s_1)\\
&\quad + e^{s_1}s_1k_{14}(-s_1,0) + s_1k_{15}(0,s_1) - s_1k_{15}(s_1,-s_1) - e^{s_1}s_1k_{16}(-s_1,0)\\
&\quad + s_1k_{16}(s_1,-s_1) + 2s_1G_1(s_1)k_3(-s_1) + 2e^{-s_1}s_1G_1(s_1)k_3(s_1)\\
&\quad + 4s_1G_1(s_1)k_4(-s_1) + 4e^{-s_1}s_1G_1(s_1)k_4(s_1) - 4s_1G_1(s_1)k_5(-s_1)\\
&\quad - 4e^{-s_1}s_1G_1(s_1)k_5(s_1) + 2k_3(-s_1) + 2e^{-s_1}k_3(s_1) + 4k_4(-s_1)\\
&\quad + 4e^{-s_1}k_4(s_1) - 4k_5(-s_1) - 4e^{-s_1}k_5(s_1) - 4k_3(0) - 8k_4(0) + 8k_5(0)\\
&= 0.
\end{aligned}
$$

4.3.1. *Differential equations derived from the two expressions for $\widetilde{K}_7$.* Like the situation for $\widetilde{K}_6$, in (52) the operator associated with $\widetilde{K}_7$ acts on two different kinds of elements of $C^\infty(\mathbb{T}_\theta^2)$ that are not the same up to switching δ_1 and δ_2. We explained in Section 3.3.5 that we get two basic functional relations in this case as well. In the first expression for $\widetilde{K}_7$ given by (17), which is associated with the action of the corresponding operator on $\delta_1^3(\ell)\cdot\delta_1(\ell)$, the terms that do not specialize trivially to $s_1+s_2=0$ are

$$
-\frac{e^{s_1}(k_6(-s_1)-k_6(s_2))}{4(s_1+s_2)}, \quad \frac{(e^{s_1+s_2}-1)k_7(s_1)}{4(s_1+s_2)}, \quad -\frac{e^{s_1+s_2}(k_7(s_1)-k_7(-s_2))}{4(s_1+s_2)}.
$$

By taking a limit in each case with the aid of the estimation $s_1+s_2=\varepsilon\sim 0$, we find that the above terms respectively specialize on $s_1+s_2=0$ to

$$
\frac{1}{4}e^{s_1}k_6'(-s_1), \quad \frac{1}{4}k_7(s_1), \quad -\frac{1}{4}k_7'(s_1).
$$

We can then replace s_2 by $-s_1$ in the remaining terms and obtain the following identity:

$$
\begin{aligned}
(26)\qquad &\frac{1}{4}e^{s_1}k_6'(-s_1) - \frac{1}{4}k_7'(s_1)\\
&= \frac{1}{120s_1}\Big(16\pi s_1G_2(s_1,-s_1) - 15s_1k_{14}(0,s_1) + 15s_1k_{14}(s_1,-s_1)\\
&\quad + 15e^{s_1}s_1k_{15}(-s_1,0) - 15s_1k_{15}(s_1,-s_1) + 15s_1k_{16}(0,s_1)
\end{aligned}
$$

$$
\begin{aligned}
&- 15e^{s_1}s_1k_{16}(-s_1,0) + 30s_1G_1(s_1)k_6(-s_1)\\
&+ 30e^{-s_1}s_1G_1(s_1)k_7(s_1) - 60e^{s_1}k_5(-s_1) - 60k_5(s_1)\\
&+ 30e^{s_1}k_6(-s_1) + 30k_6(-s_1) + 90k_6(s_1) + 90e^{s_1}k_7(-s_1)\\
&+ 30e^{-s_1}k_7(s_1) + 30k_7(s_1) + 120k_5(0) - 150k_6(0) - 150k_7(0)\Big).
\end{aligned}
$$

In the second expression given by equation (18) for $\widetilde{K}_7$, which corresponds to $\delta_1\delta_2^2(\ell)\cdot\delta_1(\ell)$, the terms that do not behave nicely under the replacement of s_2 by $-s_1$ are

$$
-\frac{e^{s_1}(k_6(-s_1)-k_6(s_2))}{4(s_1+s_2)},\quad \frac{(e^{s_1+s_2}-1)k_7(s_1)}{4(s_1+s_2)},\quad -\frac{e^{s_1+s_2}(k_7(s_1)-k_7(-s_2))}{4(s_1+s_2)},
$$

which are exactly the same as those of the first expression for $\widetilde{K}_7$. Since we just worked out the restriction of these terms to $s_1+s_2=0$, we replace s_2 by $-s_1$ in the rest of the terms and obtain the following identity:

$$
\begin{aligned}
(27)\quad &\frac{1}{4}e^{s_1}k_6'(-s_1) - \frac{1}{4}k_7'(s_1)\\
&= \frac{1}{120s_1}\Big(16\pi s_1G_2(s_1,-s_1) + 15e^{s_1}s_1k_8(-s_1,0) - 15s_1k_8(s_1,-s_1)\\
&\quad + 15e^{s_1}s_1k_9(-s_1,0) - 15s_1k_9(s_1,-s_1) + 15s_1k_{10}(0,s_1)\\
&\quad - 15e^{s_1}s_1k_{10}(-s_1,0) + 15s_1k_{11}(0,s_1) - 15e^{s_1}s_1k_{11}(-s_1,0)\\
&\quad - 15s_1k_{12}(0,s_1) + 15s_1k_{12}(s_1,-s_1) - 15s_1k_{13}(0,s_1)\\
&\quad + 15s_1k_{13}(s_1,-s_1) + 30s_1G_1(s_1)k_6(-s_1) + 30e^{-s_1}s_1G_1(s_1)k_7(s_1)\\
&\quad - 30e^{s_1}k_3(-s_1) - 30k_3(s_1) - 60e^{s_1}k_4(-s_1) - 60k_4(s_1)\\
&\quad + 30e^{s_1}k_6(-s_1) + 30k_6(-s_1) + 90k_6(s_1) + 90e^{s_1}k_7(-s_1)\\
&\quad + 30e^{-s_1}k_7(s_1) + 30k_7(s_1) + 60k_3(0) + 120k_4(0) - 150k_6(0)\\
&\quad - 150k_7(0)\Big).
\end{aligned}
$$

Since the left-hand sides of (26) and (27) are identical, we find the following functional relation by equating their right-hand sides.

Corollary 4.4. *We have*

$$
\begin{aligned}
&-e^{s_1}s_1k_8(-s_1,0) + s_1k_8(s_1,-s_1) - e^{s_1}s_1k_9(-s_1,0) + s_1k_9(s_1,-s_1)\\
&\quad - s_1k_{10}(0,s_1) + e^{s_1}s_1k_{10}(-s_1,0) - s_1k_{11}(0,s_1) + e^{s_1}s_1k_{11}(-s_1,0)\\
&\quad + s_1k_{12}(0,s_1) - s_1k_{12}(s_1,-s_1) + s_1k_{13}(0,s_1) - s_1k_{13}(s_1,-s_1)\\
&\quad - s_1k_{14}(0,s_1) + s_1k_{14}(s_1,-s_1) + e^{s_1}s_1k_{15}(-s_1,0) - s_1k_{15}(s_1,-s_1)\\
&\quad + s_1k_{16}(0,s_1) - e^{s_1}s_1k_{16}(-s_1,0) + 2e^{s_1}k_3(-s_1) + 2k_3(s_1) + 4e^{s_1}k_4(-s_1)\\
&\quad + 4k_4(s_1) - 4e^{s_1}k_5(-s_1) - 4k_5(s_1) - 4k_3(0) - 8k_4(0) + 8k_5(0)\\
&= 0.
\end{aligned}
$$

4.4.1. *Differential equation derived from the expression for* $\widetilde{K}_8$. Let us start from the basic equation for the function

$$\widetilde{K}_8(s_1,s_2,s_3)=\frac{1}{2^3}\frac{\sinh((s_1+s_2+s_3)/2)}{(s_1+s_2+s_3)/2}K_8(s_1,s_2,s_3),$$

which is given by (19). We specialize the equation to $s_1+s_2+s_3=0$ as follows. On the left-hand side, $\widetilde{K}_8(s_1,s_2,s_3)$ specializes to $\frac{1}{8}k_8(s_1,s_2)$, and on the right-hand side most of the terms behave nicely except the following four terms. The first term that has $s_1+s_2+s_3$ in the denominator is

$$\frac{(e^{s_1+s_2+s_3}-1)k_8(s_1,s_2)}{8(s_1+s_2+s_3)},$$

which contributes $\frac{1}{8}k_8(s_1,s_2)$ when specialized to $s_1+s_2+s_3=0$. The second term is

$$-\frac{e^{s_1+s_2+s_3}(k_8(s_1,s_2)-k_8(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)},$$

which under the estimation $s_1+s_2+s_3=\varepsilon\sim 0$ turns to

$$-\frac{e^{\varepsilon}(k_8(s_1,s_2)-k_8(s_1-\varepsilon,s_2))}{8\varepsilon}\to\frac{-1}{8}\partial_1 k_8(s_1,s_2).$$

The third term is

$$-\frac{e^{s_1}(k_{11}(s_2,-s_1-s_2)-k_{11}(s_2,s_3))}{8(s_1+s_2+s_3)},$$

which under the above estimate specializes to

$$-\frac{e^{s_1}(k_{11}(s_2,-s_1-s_2)-k_{11}(s_2,-s_1-s_2+\varepsilon))}{8\varepsilon}\to\frac{1}{8}e^{s_1}\partial_2 k_{11}(s_2,-s_1-s_2).$$

The fourth term is

$$-\frac{e^{s_1+s_2}(k_{12}(-s_1-s_2,s_1)-k_{12}(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)},$$

which, under the estimate $s_1+s_2+s_3=\varepsilon\sim 0$, turns to

$$\begin{aligned}
&-e^{s_1+s_2}\frac{k_{12}(-s_1-s_2,s_1)-k_{12}(-s_1-s_2,s_1-\varepsilon)}{8\varepsilon}\\
&\quad-e^{s_1+s_2}\frac{k_{12}(-s_1-s_2,s_1-\varepsilon)-k_{12}(-s_1-s_2+\varepsilon,s_1-\varepsilon)}{8\varepsilon}\\
&\to-\frac{1}{8}e^{s_1+s_2}(\partial_2 k_{12}(-s_1-s_2,s_1)-\partial_1 k_{12}(-s_1-s_2,s_1))\\
&=-\frac{1}{8}e^{s_1+s_2}((\partial_2-\partial_1)k_{12}(-s_1-s_2,s_1)).
\end{aligned}$$

Putting together the above bad behaving terms, one can see that they add up to the following expression:

$$\begin{aligned}
&\widetilde{K}_{8,\mathrm{s}}(s_1,s_2,s_3)\\
&\quad=\frac{1}{8(s_1+s_2+s_3)}\big(-k_8(s_1,s_2)+e^{s_1+s_2+s_3}k_8(-s_2-s_3,s_2)
\end{aligned}$$

$$- e^{s_1} k_{11}(s_2, -s_1 - s_2) + e^{s_1} k_{11}(s_2, s_3) - e^{s_1+s_2} k_{12}(-s_1 - s_2, s_1) + e^{s_1+s_2} k_{12}(s_3, -s_2 - s_3)\big).$$

Therefore, by a simple replacement of s_3 by $-s_1 - s_2$ in the rest of the terms, we obtain the following identity:

$$\begin{aligned} \frac{1}{8} e^{s_1} \partial_2 k_{11}(s_2, -s_1 - s_2) &- \frac{1}{8} e^{s_1+s_2} \partial_2 k_{12}(-s_1 - s_2, s_1) \\ &- \frac{1}{8} \partial_1 k_8(s_1, s_2) + \frac{1}{8} e^{s_1+s_2} \partial_1 k_{12}(-s_1 - s_2, s_1) \\ &= -(\widetilde{K}_8(s_1, s_2, s_3) - \widetilde{K}_{8,\mathrm{s}}(s_1, s_2, s_3)) \big|_{s_3=-s_1-s_2} . \end{aligned} \tag{28}$$

4.4.2. *Differential equation derived from the expression for $\widetilde{K}_9$.* Now, similarly to the previous case, when we specialize the basic equation (62) for $\widetilde{K}_9$ to $s_1 + s_2 + s_3 = 0$, the left-hand side gives $\frac{1}{8} k_9(s_1, s_2)$. On the right-hand side, the following are the terms that have $s_1 + s_2 + s_3$ in their denominators. We write each of these terms accompanied with the limit that is calculated by using the estimate $s_1 + s_2 + s_3 = \varepsilon \sim 0$:

$$\begin{aligned} &\frac{(e^{s_1+s_2+s_3} - 1) k_9(s_1, s_2)}{8(s_1 + s_2 + s_3)} \to \frac{1}{8} k_9(s_1, s_2), \\ &-\frac{e^{s_1+s_2+s_3}(k_9(s_1, s_2) - k_9(-s_2 - s_3, s_2))}{8(s_1 + s_2 + s_3)} \to -\frac{1}{8} \partial_1 k_9(s_1, s_2), \\ &-\frac{e^{s_1}(k_{10}(s_2, -s_1 - s_2) - k_{10}(s_2, s_3))}{8(s_1 + s_2 + s_3)} \to \frac{1}{8} e^{s_1} \partial_2 k_{10}(s_2, -s_1 - s_2), \\ &-\frac{e^{s_1+s_2}(k_{13}(-s_1 - s_2, s_1) - k_{13}(s_3, -s_2 - s_3))}{8(s_1 + s_2 + s_3)} \\ &\quad \to -\frac{1}{8} e^{s_1+s_2}(\partial_2 k_{13}(-s_1 - s_2, s_1) - \partial_1 k_{13}(-s_1 - s_2, s_1)) \\ &\quad = -\frac{1}{8} e^{s_1+s_2}((\partial_2 - \partial_1) k_{13}(-s_1 - s_2, s_1)). \end{aligned}$$

One can see that the above bad behaving terms add up to the following expression:

$$\begin{aligned} &\widetilde{K}_{9,\mathrm{s}}(s_1, s_2, s_3) \\ &\quad = \frac{1}{8(s_1 + s_2 + s_3)} \big(-k_9(s_1, s_2) + e^{s_1+s_2+s_3} k_9(-s_2 - s_3, s_2) \\ &\quad\quad - e^{s_1} k_{10}(s_2, -s_1 - s_2) + e^{s_1} k_{10}(s_2, s_3) - e^{s_1+s_2} k_{13}(-s_1 - s_2, s_1) \\ &\quad\quad + e^{s_1+s_2} k_{13}(s_3, -s_2 - s_3)\big). \end{aligned}$$

Then, by simply replacing s_3 by $-s_1-s_2$ in the remaining terms, we obtain the following identity:

$$\begin{aligned}\tag{29} &\frac{1}{8}e^{s_1}\partial_2 k_{10}(s_2,-s_1-s_2)-\frac{1}{8}e^{s_1+s_2}\partial_2 k_{13}(-s_1-s_2,s_1)\\ &\quad-\frac{1}{8}\partial_1 k_9(s_1,s_2)+\frac{1}{8}e^{s_1+s_2}\partial_1 k_{13}(-s_1-s_2,s_1)\\ &\quad=-(\widetilde{K}_9(s_1,s_2,s_3)-\widetilde{K}_{9,\mathrm{s}}(s_1,s_2,s_3))\big|_{s_3=-s_1-s_2}.\end{aligned}$$

4.4.3. *Differential equation derived from the expression for $\widetilde{K}_{10}$.* In the expression given by (63) for $\widetilde{K}_{10}$, the following are the terms that have $s_1+s_2+s_3$ in their denominators, which are written along with their restrictions to

$$s_1+s_2+s_3=0$$

that are calculated by using the estimate $s_1+s_2+s_3=\varepsilon\sim 0$:

$$\begin{aligned}&-\frac{e^{s_1+s_2}(k_9(-s_1-s_2,s_1)-k_9(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)}\\ &\quad\to-\frac{1}{8}e^{s_1+s_2}(\partial_2 k_9(-s_1-s_2,s_1)-\partial_1 k_9(-s_1-s_2,s_1))\\ &\quad=-\frac{1}{8}e^{s_1+s_2}((\partial_2-\partial_1)k_9(-s_1-s_2,s_1)),\\ &\frac{(e^{s_1+s_2+s_3}-1)k_{10}(s_1,s_2)}{8(s_1+s_2+s_3)}\to\frac{1}{8}k_{10}(s_1,s_2),\\ &-\frac{e^{s_1+s_2+s_3}(k_{10}(s_1,s_2)-k_{10}(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)}\to-\frac{1}{8}\partial_1 k_{10}(s_1,s_2),\\ &-\frac{e^{s_1}(k_{13}(s_2,-s_1-s_2)-k_{13}(s_2,s_3))}{8(s_1+s_2+s_3)}\to\frac{1}{8}e^{s_1}\partial_2 k_{13}(s_2,-s_1-s_2).\end{aligned}$$

A simple calculation shows that the above bad behaving terms add up to the following expression:

$$\begin{aligned}&\widetilde{K}_{10,\mathrm{s}}(s_1,s_2,s_3)\\ &\quad=\frac{1}{8(s_1+s_2+s_3)}\big(-e^{s_1+s_2}k_9(-s_1-s_2,s_1)+e^{s_1+s_2}k_9(s_3,-s_2-s_3)\\ &\qquad-k_{10}(s_1,s_2)+e^{s_1+s_2+s_3}k_{10}(-s_2-s_3,s_2)-e^{s_1}k_{13}(s_2,-s_1-s_2)\\ &\qquad+e^{s_1}k_{13}(s_2,s_3)\big).\end{aligned}$$

Since $\widetilde{K}_{10}(s_1,s_2,s_3)$ specializes to $\frac{1}{8}k_{10}(s_1,s_2)$, and in the remaining terms we can simply replace s_3 by $-s_1-s_2$, we find the following identity:

$$\begin{aligned}\tag{30} &-\frac{1}{8}e^{s_1+s_2}\partial_2 k_9(-s_1-s_2,s_1)+\frac{1}{8}e^{s_1}\partial_2 k_{13}(s_2,-s_1-s_2)\\ &\quad+\frac{1}{8}e^{s_1+s_2}\partial_1 k_9(-s_1-s_2,s_1)-\frac{1}{8}\partial_1 k_{10}(s_1,s_2)\\ &\quad=-(\widetilde{K}_{10}(s_1,s_2,s_3)-\widetilde{K}_{10,\mathrm{s}}(s_1,s_2,s_3))\big|_{s_3=-s_1-s_2}.\end{aligned}$$

4.4.4. *Differential equation derived from the expression for $\widetilde{K}_{11}$.* We consider the basic equation (64) for $\widetilde{K}_{11}$ and specialize it to $s_1+s_2+s_3=0$, in which $\widetilde{K}_{11}(s_1,s_2,s_3)$ turns to $\frac{1}{8}k_{11}(s_1,s_2)$, and we treat the other side of the equation as follows. As in the previous cases, most of the terms on the other side behave nicely under the replacement of s_3 by $-s_1-s_2$, except the following terms which are written along with their restrictions to $s_1+s_2+s_3=0$ that are calculated by using the estimate $s_1+s_2+s_3=\varepsilon\sim 0$:

$$-\frac{e^{s_1+s_2}(k_8(-s_1-s_2,s_1)-k_8(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)}$$
$$\to -\frac{1}{8}e^{s_1+s_2}(\partial_2 k_8(-s_1-s_2,s_1)-\partial_1 k_8(-s_1-s_2,s_1))$$
$$= -\frac{1}{8}e^{s_1+s_2}((\partial_2-\partial_1)k_8(-s_1-s_2,s_1)),$$
$$\frac{(e^{s_1+s_2+s_3}-1)k_{11}(s_1,s_2)}{8(s_1+s_2+s_3)} \to \frac{1}{8}k_{11}(s_1,s_2),$$
$$-\frac{e^{s_1+s_2+s_3}(k_{11}(s_1,s_2)-k_{11}(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)} \to -\frac{1}{8}\partial_1 k_{11}(s_1,s_2),$$
$$-\frac{e^{s_1}(k_{12}(s_2,-s_1-s_2)-k_{12}(s_2,s_3))}{8(s_1+s_2+s_3)} \to \frac{1}{8}e^{s_1}\partial_2 k_{12}(s_2,-s_1-s_2).$$

Putting together the above bad behaving terms, one can see that their sum is equal to

$$\begin{aligned}&\widetilde{K}_{11,\mathrm{s}}(s_1,s_2,s_3)\\ &= \frac{1}{8(s_1+s_2+s_3)}\big(-e^{s_1+s_2}k_8(-s_1-s_2,s_1)+e^{s_1+s_2}k_8(s_3,-s_2-s_3)\\ &\quad - k_{11}(s_1,s_2)+e^{s_1+s_2+s_3}k_{11}(-s_2-s_3,s_2)-e^{s_1}k_{12}(s_2,-s_1-s_2)\\ &\quad + e^{s_1}k_{12}(s_2,s_3)\big).\end{aligned}$$

Therefore, we find the following identity:

$$\begin{aligned}(31)\qquad & -\frac{1}{8}e^{s_1+s_2}\partial_2 k_8(-s_1-s_2,s_1)+\frac{1}{8}e^{s_1}\partial_2 k_{12}(s_2,-s_1-s_2)\\ &\quad +\frac{1}{8}e^{s_1+s_2}\partial_1 k_8(-s_1-s_2,s_1)-\frac{1}{8}\partial_1 k_{11}(s_1,s_2)\\ &= -(\widetilde{K}_{11}(s_1,s_2,s_3)-\widetilde{K}_{11,\mathrm{s}}(s_1,s_2,s_3))\,\big|_{s_3=-s_1-s_2}.\end{aligned}$$

4.4.5. *Differential equation derived from the expression for $\widetilde{K}_{12}$.* We consider the basic equation (65) for $\widetilde{K}_{12}$, whose left-hand side specializes to $\frac{1}{8}k_{12}(s_1,s_2)$, and we treat the other side as follows. The following are the terms on the right-hand side of this equation that have $s_1+s_2+s_3$ in their denominators, which, by using the estimate $s_1+s_2+s_3=\varepsilon\sim 0$, their behavior on $s_1+s_2+s_3=0$

can be determined:

$$-\frac{e^{s_1}(k_8(s_2,-s_1-s_2)-k_8(s_2,s_3))}{8(s_1+s_2+s_3)} \to \frac{1}{8}e^{s_1}\partial_2 k_8(s_2,-s_1-s_2),$$

$$-\frac{e^{s_1+s_2}(k_{11}(-s_1-s_2,s_1)-k_{11}(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)}$$
$$\to -\frac{1}{8}e^{s_1+s_2}(\partial_2 k_{11}(-s_1-s_2,s_1)-\partial_1 k_{11}(-s_1-s_2,s_1))$$
$$= -\frac{1}{8}e^{s_1+s_2}((\partial_2-\partial_1)k_{11}(-s_1-s_2,s_1)),$$

$$\frac{(e^{s_1+s_2+s_3}-1)k_{12}(s_1,s_2)}{8(s_1+s_2+s_3)} \to \frac{1}{8}k_{12}(s_1,s_2),$$

$$-\frac{e^{s_1+s_2+s_3}(k_{12}(s_1,s_2)-k_{12}(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)} \to -\frac{1}{8}\partial_1 k_{12}(s_1,s_2).$$

The above bad behaving terms add up to the following expression:

$$\begin{aligned}
&\widetilde{K}_{12,\mathrm{s}}(s_1,s_2,s_3)\\
&= \frac{1}{8(s_1+s_2+s_3)}\big(-e^{s_1}k_8(s_2,-s_1-s_2)+e^{s_1}k_8(s_2,s_3)\\
&\quad - e^{s_1+s_2}k_{11}(-s_1-s_2,s_1)+e^{s_1+s_2}k_{11}(s_3,-s_2-s_3)-k_{12}(s_1,s_2)\\
&\quad + e^{s_1+s_2+s_3}k_{12}(-s_2-s_3,s_2)\big).
\end{aligned}$$

In the rest of the terms we can simply replace s_3 by $-s_1-s_2$ and obtain the following identity:

$$\begin{aligned}
(32)\qquad &\frac{1}{8}e^{s_1}\partial_2 k_8(s_2,-s_1-s_2)-\frac{1}{8}e^{s_1+s_2}\partial_2 k_{11}(-s_1-s_2,s_1)\\
&\quad + \frac{1}{8}e^{s_1+s_2}\partial_1 k_{11}(-s_1-s_2,s_1)-\frac{1}{8}\partial_1 k_{12}(s_1,s_2)\\
&\quad = -(\widetilde{K}_{12}(s_1,s_2,s_3)-\widetilde{K}_{12,\mathrm{s}}(s_1,s_2,s_3))\,\big|_{s_3=-s_1-s_2}\,.
\end{aligned}$$

4.4.6. *Differential equation derived from the expression for* $\widetilde{K}_{13}$. We use the basic identity given by (66), in which $\widetilde{K}_{13}(s_1,s_2,s_3)$ specializes to $\frac{1}{8}k_{13}(s_1,s_2)$ on $s_1+s_2+s_3=0$. On the right-hand side of this identity, the following are the terms that have $s_1+s_2+s_3$ in their denominators, which we specialize to $s_1+s_2+s_3=0$ by using the estimate $s_1+s_2+s_3=\varepsilon\sim 0$:

$$-\frac{e^{s_1}(k_9(s_2,-s_1-s_2)-k_9(s_2,s_3))}{8(s_1+s_2+s_3)} \to \frac{1}{8}e^{s_1}\partial_2 k_9(s_2,-s_1-s_2),$$

$$-\frac{e^{s_1+s_2}(k_{10}(-s_1-s_2,s_1)-k_{10}(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)}$$
$$\to -\frac{1}{8}e^{s_1+s_2}(\partial_2 k_{10}(-s_1-s_2,s_1)-\partial_1 k_{10}(-s_1-s_2,s_1))$$
$$= -\frac{1}{8}e^{s_1+s_2}((\partial_2-\partial_1)k_{10}(-s_1-s_2,s_1)),$$

$$\frac{(e^{s_1+s_2+s_3}-1)k_{13}(s_1,s_2)}{8(s_1+s_2+s_3)} \to \frac{1}{8}k_{13}(s_1,s_2),$$
$$-\frac{e^{s_1+s_2+s_3}(k_{13}(s_1,s_2)-k_{13}(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)} \to -\frac{1}{8}\partial_1 k_{13}(s_1,s_2).$$

Adding up the above bad behaving terms, one can see that their sum is the following expression:

$$\begin{aligned}&\widetilde{K}_{13,\mathrm{s}}(s_1,s_2,s_3)\\ &\quad=\frac{1}{8(s_1+s_2+s_3)}\big(-e^{s_1}k_9(s_2,-s_1-s_2)+e^{s_1}k_9(s_2,s_3)\\ &\quad\quad-e^{s_1+s_2}k_{10}(-s_1-s_2,s_1)+e^{s_1+s_2}k_{10}(s_3,-s_2-s_3)-k_{13}(s_1,s_2)\\ &\quad\quad+e^{s_1+s_2+s_3}k_{13}(-s_2-s_3,s_2)\big).\end{aligned}$$

Then, by a simple replacement of s_3 by $-s_1-s_2$ in the rest of the terms, we find that

$$\begin{aligned}\frac{1}{8}e^{s_1}\partial_2 k_9(s_2,-s_1-s_2)-\frac{1}{8}e^{s_1+s_2}\partial_2 k_{10}(-s_1-s_2,s_1)\\ +\frac{1}{8}e^{s_1+s_2}\partial_1 k_{10}(-s_1-s_2,s_1)-\frac{1}{8}\partial_1 k_{13}(s_1,s_2)\\ =-(\widetilde{K}_{13}(s_1,s_2,s_3)-\widetilde{K}_{13,\mathrm{s}}(s_1,s_2,s_3))\,\big|_{s_3=-s_1-s_2}.\end{aligned} \tag{33}$$

4.4.7. *Differential equation derived from the expression for $\widetilde{K}_{14}$.* In the basic equation written in (67), the function $\widetilde{K}_{14}(s_1,s_2,s_3)$ specializes to $\frac{1}{8}k_{14}(s_1,s_2)$ on $s_1+s_2+s_3=0$, and on the right-hand side we specialize by replacing s_3 by $-s_1-s_2$ in the terms that behave nicely, and the following are the terms that have $s_1+s_2+s_3$ in their denominators, for which we calculate their restriction to $s_1+s_2+s_3=0$ by using the estimate $s_1+s_2+s_3=\varepsilon\sim 0$:

$$\frac{(e^{s_1+s_2+s_3}-1)k_{14}(s_1,s_2)}{8(s_1+s_2+s_3)} \to \frac{1}{8}k_{14}(s_1,s_2),$$
$$-\frac{e^{s_1+s_2+s_3}(k_{14}(s_1,s_2)-k_{14}(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)} \to -\frac{1}{8}\partial_1 k_{14}(s_1,s_2),$$
$$-\frac{e^{s_1}(k_{15}(s_2,-s_1-s_2)-k_{15}(s_2,s_3))}{8(s_1+s_2+s_3)} \to \frac{1}{8}e^{s_1}\partial_2 k_{15}(s_2,-s_1-s_2),$$
$$\begin{aligned}&-\frac{e^{s_1+s_2}(k_{16}(-s_1-s_2,s_1)-k_{16}(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)}\\ &\quad\to-\frac{1}{8}e^{s_1+s_2}(\partial_2 k_{16}(-s_1-s_2,s_1)-\partial_1 k_{16}(-s_1-s_2,s_1))\\ &\quad=-\frac{1}{8}e^{s_1+s_2}((\partial_2-\partial_1)k_{16}(-s_1-s_2,s_1)).\end{aligned}$$

The above bad behaving terms simply add up to the following expression:

$$\begin{aligned}
&\widetilde{K}_{14,\mathrm{s}}(s_1,s_2,s_3)\\
&\quad=\frac{1}{8(s_1+s_2+s_3)}\big(-k_{14}(s_1,s_2)+e^{s_1+s_2+s_3}k_{14}(-s_2-s_3,s_2)\\
&\qquad-e^{s_1}k_{15}(s_2,-s_1-s_2)+e^{s_1}k_{15}(s_2,s_3)-e^{s_1+s_2}k_{16}(-s_1-s_2,s_1)\\
&\qquad+e^{s_1+s_2}k_{16}(s_3,-s_2-s_3)\big).
\end{aligned}$$

Therefore, we obtain the following identity:

$$\begin{aligned}
&\frac{1}{8}e^{s_1}\partial_2 k_{15}(s_2,-s_1-s_2)-\frac{1}{8}e^{s_1+s_2}\partial_2 k_{16}(-s_1-s_2,s_1)\\
&\qquad-\frac{1}{8}\partial_1 k_{14}(s_1,s_2)+\frac{1}{8}e^{s_1+s_2}\partial_1 k_{16}(-s_1-s_2,s_1)\\
&\qquad=-(\widetilde{K}_{14}(s_1,s_2,s_3)-\widetilde{K}_{14,\mathrm{s}}(s_1,s_2,s_3))\,\big|_{s_3=-s_1-s_2}\,.
\end{aligned}\tag{34}$$

4.4.8. *Differential equation derived from the expression for* $\widetilde{K}_{15}$. We start from the basic equation (68) for $\widetilde{K}_{15}$ and specialize it to $s_1+s_2+s_3=0$. In this process, $\widetilde{K}_{15}(s_1,s_2,s_3)$ specializes to $\frac{1}{8}k_{15}(s_1,s_2)$ and, except for the following terms, the rest of the terms behave nicely under the replacement of s_3 by $-s_1-s_2$. The terms that need special care with the aid of the estimate $s_1+s_2+s_3=\varepsilon\sim 0$ are

$$\begin{aligned}
&-\frac{e^{s_1+s_2}(k_{14}(-s_1-s_2,s_1)-k_{14}(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)}\\
&\quad\to-\frac{1}{8}e^{s_1+s_2}(\partial_2 k_{14}(-s_1-s_2,s_1)-\partial_1 k_{14}(-s_1-s_2,s_1))\\
&\quad=-\frac{1}{8}e^{s_1+s_2}((\partial_2-\partial_1)k_{14}(-s_1-s_2,s_1)),\\
&\frac{(e^{s_1+s_2+s_3}-1)k_{15}(s_1,s_2)}{8(s_1+s_2+s_3)}\to\frac{1}{8}k_{15}(s_1,s_2),\\
&-\frac{e^{s_1+s_2+s_3}(k_{15}(s_1,s_2)-k_{15}(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)}\to-\frac{1}{8}\partial_1 k_{15}(s_1,s_2),\\
&-\frac{e^{s_1}(k_{16}(s_2,-s_1-s_2)-k_{16}(s_2,s_3))}{8(s_1+s_2+s_3)}\to\frac{1}{8}\partial_2 e^{s_1}k_{16}(s_2,-s_1-s_2).
\end{aligned}$$

We put the bad behaving terms together and see that they add up to the following expression:

$$\begin{aligned}
&\widetilde{K}_{15,\mathrm{s}}(s_1,s_2,s_3)\\
&\quad=\frac{1}{8(s_1+s_2+s_3)}\big(-e^{s_1+s_2}k_{14}(-s_1-s_2,s_1)+e^{s_1+s_2}k_{14}(s_3,-s_2-s_3)\\
&\quad-k_{15}(s_1,s_2)+e^{s_1+s_2+s_3}k_{15}(-s_2-s_3,s_2)-e^{s_1}k_{16}(s_2,-s_1-s_2)\\
&\quad+e^{s_1}k_{16}(s_2,s_3)\big).
\end{aligned}$$

Thus, we find that the following identity holds:

$$(35)\qquad \begin{aligned} &-\frac{1}{8}e^{s_1+s_2}\partial_2 k_{14}(-s_1-s_2,s_1)+\frac{1}{8}e^{s_1}\partial_2 k_{16}(s_2,-s_1-s_2)\\ &\quad+\frac{1}{8}e^{s_1+s_2}\partial_1 k_{14}(-s_1-s_2,s_1)-\frac{1}{8}\partial_1 k_{15}(s_1,s_2)\\ &= -(\widetilde{K}_{15}(s_1,s_2,s_3)-\widetilde{K}_{15,\mathrm{s}}(s_1,s_2,s_3))\,\big|_{s_3=-s_1-s_2}\,. \end{aligned}$$

4.4.9. *Differential equation derived from the expression for $\widetilde{K}_{16}$.* When we specialize the basic equation (69) for $\widetilde{K}_{16}$ to $s_1+s_2+s_3=0$, similarly to the previous cases, $\widetilde{K}_{16}(s_1,s_2,s_3)$ turns to $\frac{1}{8}k_{16}(s_1,s_2)$, and we have terms that behave nicely under the replacement of s_3 by $-s_1-s_2$, while the following are the terms that need to be specialized by using the estimate $s_1+s_2+s_3=\varepsilon\sim 0$:

$$-\frac{e^{s_1}(k_{14}(s_2,-s_1-s_2)-k_{14}(s_2,s_3))}{8(s_1+s_2+s_3)}\to\frac{1}{8}e^{s_1}\partial_2 k_{14}(s_2,-s_1-s_2),$$

$$\begin{aligned} &-\frac{e^{s_1+s_2}(k_{15}(-s_1-s_2,s_1)-k_{15}(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)}\\ &\quad\to -\frac{1}{8}e^{s_1+s_2}(\partial_2 k_{15}(-s_1-s_2,s_1)-\partial_1 k_{15}(-s_1-s_2,s_1))\\ &\quad= -\frac{1}{8}e^{s_1+s_2}((\partial_2-\partial_1)k_{15}(-s_1-s_2,s_1)), \end{aligned}$$

$$\frac{(e^{s_1+s_2+s_3}-1)k_{16}(s_1,s_2)}{8(s_1+s_2+s_3)}\to\frac{1}{8}k_{16}(s_1,s_2),$$

$$-\frac{e^{s_1+s_2+s_3}(k_{16}(s_1,s_2)-k_{16}(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)}\to -\frac{1}{8}\partial_1 k_{16}(s_1,s_2).$$

The sum of the above bad behaving terms is the following expression:

$$\begin{aligned} &\widetilde{K}_{16,\mathrm{s}}(s_1,s_2,s_3)\\ &\quad=\frac{1}{8(s_1+s_2+s_3)}\big(-e^{s_1}k_{14}(s_2,-s_1-s_2)+e^{s_1}k_{14}(s_2,s_3)\\ &\quad-e^{s_1+s_2}k_{15}(-s_1-s_2,s_1)+e^{s_1+s_2}k_{15}(s_3,-s_2-s_3)-k_{16}(s_1,s_2)\\ &\quad+e^{s_1+s_2+s_3}k_{16}(-s_2-s_3,s_2)\big). \end{aligned}$$

Therefore, the result of specializing the basic equation for $\widetilde{K}_{16}$ to $s_1+s_2+s_3=0$ is the following identity:

$$(36)\qquad \begin{aligned} &\frac{1}{8}e^{s_1}\partial_2 k_{14}(s_2,-s_1-s_2)-\frac{1}{8}e^{s_1+s_2}\partial_2 k_{15}(-s_1-s_2,s_1)\\ &\quad+\frac{1}{8}e^{s_1+s_2}\partial_1 k_{15}(-s_1-s_2,s_1)-\frac{1}{8}\partial_1 k_{16}(s_1,s_2)\\ &= -(\widetilde{K}_{16}(s_1,s_2,s_3)-\widetilde{K}_{16,\mathrm{s}}(s_1,s_2,s_3))\,\big|_{s_3=-s_1-s_2}\,. \end{aligned}$$

Remark 4.5. After replacing the lengthy expressions for $\widetilde{K}_8,\dots,\widetilde{K}_{16}$, given by (19) and (62)–(69) in the right-hand sides of equations (28)–(36), and after taking the common denominator and putting the terms together, in each case one finds a fraction whose numerator is a finite difference expression of the involved functions and its denominator is the expression

$$s_1 s_2 (s_1 + s_2).$$

4.5.1. *Differential equation derived from the expression for* $\widetilde{K}_{17}$. In order to derive a differential equation from the basic equation (20) for $\widetilde{K}_{17}$, we specialize it to $s_1 + s_2 + s_3 + s_4 = 0$. On the left-hand side of the equation, $\widetilde{K}_{17}(s_1, s_2, s_3, s_4)$ turns to $\frac{1}{16} k_{17}(s_1, s_2, s_3)$, and on the other side we have terms that behave nicely under the replacement of s_4 by $-s_1 - s_2 - s_3$, while we also have the following five terms which can be specialized with the aid of the estimate $s_1 + s_2 + s_3 + s_4 = \varepsilon \sim 0$. The first one is

$$\frac{(e^{s_1+s_2+s_3+s_4} - 1) k_{17}(s_1, s_2, s_3)}{16(s_1 + s_2 + s_3 + s_4)} \to \frac{1}{16} k_{17}(s_1, s_2, s_3).$$

The second term that needs to be treated with the above estimate is

$$\begin{aligned} &-\frac{e^{s_1+s_2}(k_{17}(s_3, -s_1 - s_2 - s_3, s_1) - k_{17}(s_3, s_4, -s_2 - s_3 - s_4))}{16(s_1 + s_2 + s_3 + s_4)} \\ &= -\frac{e^{s_1+s_2}(k_{17}(s_3, -s_1 - s_2 - s_3, s_1) - k_{17}(s_3, -s_1 - s_2 - s_3 + \varepsilon, s_1 - \varepsilon))}{16\varepsilon} \\ &\to -\frac{1}{16} e^{s_1+s_2}(\partial_3 k_{17}(s_3, -s_1 - s_2 - s_3, s_1) - \partial_2 k_{17}(s_3, -s_1 - s_2 - s_3, s_1)) \\ &= -\frac{1}{16} e^{s_1+s_2}((\partial_3 - \partial_2) k_{17}(s_3, -s_1 - s_2 - s_3, s_1)). \end{aligned}$$

For the third term of this kind we have

$$\begin{aligned} &-\frac{e^{s_1+s_2+s_3+s_4}(k_{17}(s_1, s_2, s_3) - k_{17}(-s_2 - s_3 - s_4, s_2, s_3))}{16(s_1 + s_2 + s_3 + s_4)} \\ &\quad = -\frac{e^{\varepsilon}(k_{17}(s_1, s_2, s_3) - k_{17}(s_1 - \varepsilon, s_2, s_3))}{16\varepsilon} \to -\frac{1}{16} \partial_1 k_{17}(s_1, s_2, s_3). \end{aligned}$$

For the fourth term we can write

$$\begin{aligned} &-\frac{e^{s_1}(k_{19}(s_2, s_3, -s_1 - s_2 - s_3) - k_{19}(s_2, s_3, s_4))}{16(s_1 + s_2 + s_3 + s_4)} \\ &\quad = -\frac{e^{s_1}(k_{19}(s_2, s_3, -s_1 - s_2 - s_3) - k_{19}(s_2, s_3, -s_1 - s_2 - s_3 + \varepsilon))}{16\varepsilon} \\ &\quad \to \frac{1}{16} e^{s_1} \partial_3 k_{19}(s_2, s_3, -s_1 - s_2 - s_3). \end{aligned}$$

Finally, we treat the last term that needs to be specialized to $s_1 + s_2 + s_3 + s_4 = 0$ by computing a limit:

$$\begin{aligned} &-\frac{e^{s_1+s_2+s_3}(k_{19}(-s_1 - s_2 - s_3, s_1, s_2) - k_{19}(s_4, -s_2 - s_3 - s_4, s_2))}{16(s_1 + s_2 + s_3 + s_4)} \\ &= -\frac{e^{s_1+s_2+s_3}(k_{19}(-s_1 - s_2 - s_3, s_1, s_2) - k_{19}(-s_1 - s_2 - s_3 + \varepsilon, s_1 - \varepsilon, s_2))}{16\varepsilon} \end{aligned}$$

$$\to -\frac{1}{16}e^{s_1+s_2+s_3}(\partial_2 k_{19}(-s_1-s_2-s_3,s_1,s_2)-\partial_1 k_{19}(-s_1-s_2-s_3,s_1,s_2))$$
$$= -\frac{1}{16}e^{s_1+s_2+s_3}((\partial_2-\partial_1)k_{19}(-s_1-s_2-s_3,s_1,s_2)).$$

Putting together the above bad behaving terms, one can see that they add up to the following expression:

$$\begin{aligned}
&\widetilde{K}_{17,\mathrm{s}}(s_1,s_2,s_3,s_4)\\
&\quad= \frac{1}{16(s_1+s_2+s_3+s_4)}\big(-k_{17}(s_1,s_2,s_3) - e^{s_1+s_2}k_{17}(s_3,-s_1-s_2-s_3,s_1)\\
&\qquad + e^{s_1+s_2}k_{17}(s_3,s_4,-s_2-s_3-s_4)\\
&\qquad + e^{s_1+s_2+s_3+s_4}k_{17}(-s_2-s_3-s_4,s_2,s_3)\\
&\qquad - e^{s_1}k_{19}(s_2,s_3,-s_1-s_2-s_3) + e^{s_1}k_{19}(s_2,s_3,s_4)\\
&\qquad -e^{s_1+s_2+s_3}k_{19}(-s_1-s_2-s_3,s_1,s_2)\\
&\qquad + e^{s_1+s_2+s_3}k_{19}(s_4,-s_2-s_3-s_4,s_2)\big).
\end{aligned}$$

Therefore, the above specialization of the basic equation for $\widetilde{K}_{17}$ yields the following identity:

(37)
$$\begin{aligned}
&-\frac{1}{16}e^{s_1+s_2}\partial_3 k_{17}(s_3,-s_1-s_2-s_3,s_1) + \frac{1}{16}e^{s_1}\partial_3 k_{19}(s_2,s_3,-s_1-s_2-s_3)\\
&\qquad + \frac{1}{16}e^{s_1+s_2}\partial_2 k_{17}(s_3,-s_1-s_2-s_3,s_1)\\
&\qquad - \frac{1}{16}e^{s_1+s_2+s_3}\partial_2 k_{19}(-s_1-s_2-s_3,s_1,s_2)\\
&\qquad - \frac{1}{16}\partial_1 k_{17}(s_1,s_2,s_3) + \frac{1}{16}e^{s_1+s_2+s_3}\partial_1 k_{19}(-s_1-s_2-s_3,s_1,s_2)\\
&\quad = -(\widetilde{K}_{17}(s_1,s_2,s_3,s_4) - \widetilde{K}_{17,\mathrm{s}}(s_1,s_2,s_3,s_4))\,\big|_{s_4=-s_1-s_2-s_3}\,.
\end{aligned}$$

4.5.2. *Differential equation derived from the expression for $\widetilde{K}_{18}$.* We specialize the basic equation (70) for $\widetilde{K}_{18}$ to $s_1+s_2+s_3+s_4=0$, which on the left-hand side turns to $\frac{1}{16}k_{18}(s_1,s_2,s_3)$, and the other side is treated as follows. The following are the terms on the right-hand side that have $s_1+s_2+s_3+s_4$ in their denominators and need to be treated with the estimate $s_1+s_2+s_3+s_4=\varepsilon\sim 0$:

$$\frac{(e^{s_1+s_2+s_3+s_4}-1)k_{18}(s_1,s_2,s_3)}{16(s_1+s_2+s_3+s_4)} \to \frac{1}{16}k_{18}(s_1,s_2,s_3),$$
$$-\frac{e^{s_1}(k_{18}(s_2,s_3,-s_1-s_2-s_3)-k_{18}(s_2,s_3,s_4))}{16(s_1+s_2+s_3+s_4)}$$
$$\to \frac{1}{16}e^{s_1}\partial_3 k_{18}(s_2,s_3,-s_1-s_2-s_3),$$
$$-\frac{e^{s_1+s_2}(k_{18}(s_3,-s_1-s_2-s_3,s_1)-k_{18}(s_3,s_4,-s_2-s_3-s_4))}{16(s_1+s_2+s_3+s_4)}$$

$$\to -\frac{1}{16}e^{s_1+s_2}(\partial_3 k_{18}(s_3,-s_1-s_2-s_3,s_1)-\partial_2 k_{18}(s_3,-s_1-s_2-s_3,s_1)),$$

$$-\frac{e^{s_1+s_2+s_3+s_4}(k_{18}(s_1,s_2,s_3)-k_{18}(-s_2-s_3-s_4,s_2,s_3))}{16(s_1+s_2+s_3+s_4)}$$

$$\to -\frac{1}{16}\partial_1 k_{18}(s_1,s_2,s_3),$$

$$-\frac{e^{s_1+s_2+s_3}(k_{18}(-s_1-s_2-s_3,s_1,s_2)-k_{18}(s_4,-s_2-s_3-s_4,s_2))}{16(s_1+s_2+s_3+s_4)}$$

$$\to -\frac{1}{16}e^{s_1+s_2+s_3}(\partial_2 k_{18}(-s_1-s_2-s_3,s_1,s_2) - \partial_1 k_{18}(-s_1-s_2-s_3,s_1,s_2)).$$

The above bad behaving terms add up to the following expression:

$$\begin{aligned}
&\widetilde{K}_{18,\mathrm{s}}(s_1,s_2,s_3,s_4)\\
&\quad= \frac{1}{16(s_1+s_2+s_3+s_4)}\big(-k_{18}(s_1,s_2,s_3)-e^{s_1}k_{18}(s_2,s_3,-s_1-s_2-s_3)\\
&\qquad + e^{s_1}k_{18}(s_2,s_3,s_4)-e^{s_1+s_2+s_3}k_{18}(-s_1-s_2-s_3,s_1,s_2)\\
&\qquad - e^{s_1+s_2}k_{18}(s_3,-s_1-s_2-s_3,s_1)+e^{s_1+s_2}k_{18}(s_3,s_4,-s_2-s_3-s_4)\\
&\qquad + e^{s_1+s_2+s_3+s_4}k_{18}(-s_2-s_3-s_4,s_2,s_3)\\
&\qquad + e^{s_1+s_2+s_3}k_{18}(s_4,-s_2-s_3-s_4,s_2)\big).
\end{aligned}$$

Note that in the rest of the terms we can directly replace s_4 by $-s_1-s_2-s_3$. Therefore, the result of this analysis, which allows us to specialize the basic equation (70) to $s_1+s_2+s_3+s_4=0$, is the following identity:

$$\begin{aligned}
(38)\qquad &\frac{1}{16}e^{s_1}\partial_3 k_{18}(s_2,s_3,-s_1-s_2-s_3)\\
&\quad -\frac{1}{16}e^{s_1+s_2}\partial_3 k_{18}(s_3,-s_1-s_2-s_3,s_1)\\
&\quad -\frac{1}{16}e^{s_1+s_2+s_3}\partial_2 k_{18}(-s_1-s_2-s_3,s_1,s_2)\\
&\quad +\frac{1}{16}e^{s_1+s_2}\partial_2 k_{18}(s_3,-s_1-s_2-s_3,s_1)\\
&\quad -\frac{1}{16}\partial_1 k_{18}(s_1,s_2,s_3)+\frac{1}{16}e^{s_1+s_2+s_3}\partial_1 k_{18}(-s_1-s_2-s_3,s_1,s_2)\\
&= -(\widetilde{K}_{18}(s_1,s_2,s_3,s_4)-\widetilde{K}_{18,\mathrm{s}}(s_1,s_2,s_3,s_4))\,\big|_{s_4=-s_1-s_2-s_3}\,.
\end{aligned}$$

4.5.3. *Differential equation derived from the expression for* $\widetilde{K}_{19}$. In a similar way, we can specialize the basic equation written in (71) for $\widetilde{K}_{19}$ to

$$s_1+s_2+s_3+s_4=0.$$

On the left-hand side we get $\frac{1}{16}k_{19}(s_1,s_2,s_3)$, and on the right-hand side we replace s_4 by $-s_1-s_2-s_3$ in the terms that behave nicely with respect to this replacement, while the bad behaving terms, which are listed below, can be treated using the estimate $s_1+s_2+s_3+s_4=\varepsilon\sim 0$ as follows:

$$-\frac{e^{s_1}(k_{17}(s_2,s_3,-s_1-s_2-s_3)-k_{17}(s_2,s_3,s_4))}{16(s_1+s_2+s_3+s_4)}$$
$$\to \frac{1}{16}e^{s_1}\partial_3 k_{17}(s_2,s_3,-s_1-s_2-s_3),$$
$$-\frac{e^{s_1+s_2+s_3}(k_{17}(-s_1-s_2-s_3,s_1,s_2)-k_{17}(s_4,-s_2-s_3-s_4,s_2))}{16(s_1+s_2+s_3+s_4)}$$
$$\to -\frac{1}{16}e^{s_1+s_2+s_3}(\partial_2 k_{17}(-s_1-s_2-s_3,s_1,s_2)$$
$$-\partial_1 k_{17}(-s_1-s_2-s_3,s_1,s_2)),$$
$$\frac{(e^{s_1+s_2+s_3+s_4}-1)k_{19}(s_1,s_2,s_3)}{16(s_1+s_2+s_3+s_4)}\to\frac{1}{16}k_{19}(s_1,s_2,s_3),$$
$$-\frac{e^{s_1+s_2}(k_{19}(s_3,-s_1-s_2-s_3,s_1)-k_{19}(s_3,s_4,-s_2-s_3-s_4))}{16(s_1+s_2+s_3+s_4)}$$
$$\to -\frac{1}{16}e^{s_1+s_2}(\partial_3 k_{19}(s_3,-s_1-s_2-s_3,s_1)-\partial_2 k_{19}(s_3,-s_1-s_2-s_3,s_1)),$$
$$-\frac{e^{s_1+s_2+s_3+s_4}(k_{19}(s_1,s_2,s_3)-k_{19}(-s_2-s_3-s_4,s_2,s_3))}{16(s_1+s_2+s_3+s_4)}$$
$$\to -\frac{1}{16}\partial_1 k_{19}(s_1,s_2,s_3).$$

One can see that the above bad behaving terms add up to the following expression:

$$\begin{aligned}
&\widetilde{K}_{19,\mathrm{s}}(s_1,s_2,s_3,s_4)\\
&\quad=\frac{1}{16(s_1+s_2+s_3+s_4)}\big(-e^{s_1}k_{17}(s_2,s_3,-s_1-s_2-s_3)+e^{s_1}k_{17}(s_2,s_3,s_4)\\
&\qquad-e^{s_1+s_2+s_3}k_{17}(-s_1-s_2-s_3,s_1,s_2)\\
&\qquad+e^{s_1+s_2+s_3}k_{17}(s_4,-s_2-s_3-s_4,s_2)\\
&\qquad-k_{19}(s_1,s_2,s_3)-e^{s_1+s_2}k_{19}(s_3,-s_1-s_2-s_3,s_1)\\
&\qquad+e^{s_1+s_2}k_{19}(s_3,s_4,-s_2-s_3-s_4)\\
&\qquad+e^{s_1+s_2+s_3+s_4}k_{19}(-s_2-s_3-s_4,s_2,s_3)\big).
\end{aligned}$$

Therefore, we obtain the following identity, in which $\widetilde{K}_{19}(s_1,s_2,s_3,s_4)$ denotes the right-hand side of equation (71), and the direct replacement of s_4 by $-s_1-s_2-s_3$ behaves nicely since, following the above discussion, we have apparently removed the bad behaving terms:

$$(39)\qquad \frac{1}{16}e^{s_1}\partial_3 k_{17}(s_2,s_3,-s_1-s_2-s_3)$$
$$-\frac{1}{16}e^{s_1+s_2}\partial_3 k_{19}(s_3,-s_1-s_2-s_3,s_1)$$

$$
\begin{aligned}
&- \frac{1}{16} e^{s_1+s_2+s_3} \partial_2 k_{17}(-s_1-s_2-s_3, s_1, s_2) \\
&+ \frac{1}{16} e^{s_1+s_2} \partial_2 k_{19}(s_3, -s_1-s_2-s_3, s_1) \\
&+ \frac{1}{16} e^{s_1+s_2+s_3} \partial_1 k_{17}(-s_1-s_2-s_3, s_1, s_2) - \frac{1}{16} \partial_1 k_{19}(s_1, s_2, s_3) \\
&\quad = -(\widetilde{K}_{19}(s_1, s_2, s_3, s_4) - \widetilde{K}_{19,\mathrm{s}}(s_1, s_2, s_3, s_4)) \big|_{s_4=-s_1-s_2-s_3} .
\end{aligned}
$$

4.5.4. *Differential equation derived from the expression for* $\widetilde{K}_{20}$. Finally, we specialize the basic identity given by (72) for $\widetilde{K}_{20}$ to $s_1+s_2+s_3+s_4=0$, and derive the last differential equation. On the left-hand side of the basic identity, $\widetilde{K}_{20}(s_1, s_2, s_3, s_4)$ specializes to $\frac{1}{16} k_{20}(s_1, s_2, s_3)$, and on the right-hand side we use the estimate $s_1+s_2+s_3+s_4=\varepsilon \sim 0$ to specialize the terms that do not behave nicely with respect to the replacement of s_4 by $-s_1-s_2-s_3$. In this respect, the bad behaving terms are

$$
\begin{aligned}
&\frac{(e^{s_1+s_2+s_3+s_4}-1) k_{20}(s_1, s_2, s_3)}{16(s_1+s_2+s_3+s_4)} \to \frac{1}{16} k_{20}(s_1, s_2, s_3), \\
&- \frac{e^{s_1}(k_{20}(s_2, s_3, -s_1-s_2-s_3) - k_{20}(s_2, s_3, s_4))}{16(s_1+s_2+s_3+s_4)} \\
&\quad \to \frac{1}{16} e^{s_1} \partial_3 k_{20}(s_2, s_3, -s_1-s_2-s_3), \\
&- \frac{e^{s_1+s_2}(k_{20}(s_3, -s_1-s_2-s_3, s_1) - k_{20}(s_3, s_4, -s_2-s_3-s_4))}{16(s_1+s_2+s_3+s_4)} \\
&\quad \to -\frac{1}{16} e^{s_1+s_2}(\partial_3 k_{20}(s_3, -s_1-s_2-s_3, s_1) - \partial_2 k_{20}(s_3, -s_1-s_2-s_3, s_1)), \\
&- \frac{e^{s_1+s_2+s_3+s_4}(k_{20}(s_1, s_2, s_3) - k_{20}(-s_2-s_3-s_4, s_2, s_3))}{16(s_1+s_2+s_3+s_4)} \\
&\quad \to -\frac{1}{16} \partial_1 k_{20}(s_1, s_2, s_3), \\
&- \frac{e^{s_1+s_2+s_3}(k_{20}(-s_1-s_2-s_3, s_1, s_2) - k_{20}(s_4, -s_2-s_3-s_4, s_2))}{16(s_1+s_2+s_3+s_4)} \\
&\quad \to -\frac{1}{16} e^{s_1+s_2+s_3}(\partial_2 k_{20}(-s_1-s_2-s_3, s_1, s_2) \\
&\qquad - \partial_1 k_{20}(-s_1-s_2-s_3, s_1, s_2)).
\end{aligned}
$$

Putting together the above bad behaving terms, one can see that their sum is the following expression:

$$
\begin{aligned}
&\widetilde{K}_{20,\mathrm{s}}(s_1, s_2, s_3, s_4) \\
&\quad = \frac{1}{16(s_1+s_2+s_3+s_4)} \big(-k_{20}(s_1, s_2, s_3) - e^{s_1} k_{20}(s_2, s_3, -s_1-s_2-s_3) \\
&\qquad + e^{s_1} k_{20}(s_2, s_3, s_4) - e^{s_1+s_2+s_3} k_{20}(-s_1-s_2-s_3, s_1, s_2) \\
&\qquad - e^{s_1+s_2} k_{20}(s_3, -s_1-s_2-s_3, s_1) + e^{s_1+s_2} k_{20}(s_3, s_4, -s_2-s_3-s_4)
\end{aligned}
$$

$$+ e^{s_1+s_2+s_3+s_4} k_{20}(-s_2 - s_3 - s_4, s_2, s_3)$$
$$+ e^{s_1+s_2+s_3} k_{20}(s_4, -s_2 - s_3 - s_4, s_2)\big).$$

Therefore, this analysis yields the following identity, in which $\widetilde{K}_{20}(s_1, s_2, s_3, s_4)$ denotes the right-hand side of the basic identity (72), and the direct replacement of s_4 by $-s_1-s_2-s_3$ behaves nicely since, following the above discussion, the terms that have $s_1 + s_2 + s_3 + s_4$ in their denominators are removed:

(40)
$$\begin{aligned} &\frac{1}{16} e^{s_1} \partial_3 k_{20}(s_2, s_3, -s_1 - s_2 - s_3) \\ &\quad - \frac{1}{16} e^{s_1+s_2} \partial_3 k_{20}(s_3, -s_1 - s_2 - s_3, s_1) \\ &\quad - \frac{1}{16} e^{s_1+s_2+s_3} \partial_2 k_{20}(-s_1 - s_2 - s_3, s_1, s_2) \\ &\quad + \frac{1}{16} e^{s_1+s_2} \partial_2 k_{20}(s_3, -s_1 - s_2 - s_3, s_1) \\ &\quad - \frac{1}{16} \partial_1 k_{20}(s_1, s_2, s_3) + \frac{1}{16} e^{s_1+s_2+s_3} \partial_1 k_{20}(-s_1 - s_2 - s_3, s_1, s_2) \\ &= -(\widetilde{K}_{20}(s_1, s_2, s_3, s_4) - \widetilde{K}_{20,\mathrm{s}}(s_1, s_2, s_3, s_4)) \big|_{s_4=-s_1-s_2-s_3} . \end{aligned}$$

In equations (21)–(27), we see s_1 as the denominator on the right-hand side of each equation. As we explained in Remark 4.5, for equations (28)–(36) the expression $s_1 s_2(s_1 + s_2)$ appears as the denominator on the right-hand side of the equations. Since we have similarly used a concise way of writing the right-hand side of each of equations (37)–(40), their denominators are not written explicitly and it is important to elaborate in the following remark on this matter.

Remark 4.6. After replacing the lengthy expressions for $\widetilde{K}_{17}, \ldots, \widetilde{K}_{20}$, given by (20) and (70)–(72), in the right-hand sides of equations (37)–(40), and after taking the common denominator and putting the terms together, in each case one finds that the denominator is the expression

$$s_1 s_2(s_1 + s_2)s_3(s_2 + s_3)(s_1 + s_2 + s_3).$$

4.7. **Cyclic groups involved in the differential system.** In the differential equations (21)–(27) it is apparent that the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{R}$ given by

$$T_2(s_1) = -s_1, \quad s_1 \in \mathbb{R},$$

is involved. Also on the right-hand side of each of these equations, the expression in the denominator is s_1, which up to sign is invariant under the above transformation. We shall see shortly that a similar situation holds for the remaining differential equations in the differential system presented in Section 4.1. First we present a result about a transformation property of the expressions appearing on either side of the differential equations (21)–(27) under the above action of $\mathbb{Z}/2\mathbb{Z}$. The left-hand side of each of these equations, up to an overall factor of $\frac{1}{4}$, is of the form

(41)
$$-k'_{j_0}(s_1) + e^{s_1} k'_{j_1}(-s_1) = -k'_{j_0}(s_1) + T_2(e^{-s_1} k'_{j_1})(s_1),$$

where

$$\begin{aligned}
&\text{for equation (21):} \quad (j_0, j_1) = (3,3),\\
&\text{for equation (22):} \quad (j_0, j_1) = (4,4),\\
&\text{for equation (23):} \quad (j_0, j_1) = (5,5),\\
&\text{for equation (24):} \quad (j_0, j_1) = (6,7),\\
&\text{for equation (25):} \quad (j_0, j_1) = (6,7),\\
&\text{for equation (26):} \quad (j_0, j_1) = (7,6),\\
&\text{for equation (27):} \quad (j_0, j_1) = (7,6).
\end{aligned}$$

Therefore, we can divide the above differential equations to the following four groups: the first three groups respectively consist of equations (21), (22) and (23) since they respectively involve only derivatives of k_3, k_4 and k_5, and the fourth group consists of equations (24)–(27) which involve the derivatives of k_6 and k_7. In fact, it will also be useful to use the following notation $f_{j_0,j_1}(s_1)$ to rewrite the differential equations (21)–(27) in a concise form:

$$-k'_{j_0}(s_1) + e^{s_1} k'_{j_1}(-s_1) = \frac{f_{j_0,j_1}(s_1)}{s_1}. \tag{42}$$

Apparently each $f_{j_0,j_1}(s_1)$ is given in terms of finite differences of the restriction to $s_1 + s_2 = 0$ of the functions k_j appearing on the right-hand sides of the equations, and note that using equations (24) and (25), we have two such expressions for $f_{6,7}(s_1)$. Similarly, in the case of the differential equations (26) and (27), we obtain two finite difference expressions for $f_{7,6}(s_1)$.

We can now start using the above group action to investigate invariance properties and symmetries of the expressions appearing in the differential equations.

Theorem 4.8. *The expression*

$$e^{-\frac{s_1}{2}}\big(-(k'_{j_0}(s_1) + k'_{j_1}(s_1)) + e^{s_1}(k'_{j_0}(-s_1) + k'_{j_1}(-s_1))\big)$$

is in the kernel of $1 + T_2$ *for any pair of integers* j_0, j_1 *in* $\{3,4,5,6,7\}$*. In particular, considering the differential equations* (21)–(27) *and the above discussion following equation* (41)*, we have obtained expressions that are in terms of finite differences of the* k_j *and are in the kernel of* $1 + T_2$*:*

(i) *When* $(j_0, j_1) = (3,3)$.
(ii) *When* $(j_0, j_1) = (4,4)$.
(iii) *When* $(j_0, j_1) = (5,5)$.
(iv) *When* $(j_0, j_1) = (6,7)$.

Proof. We multiply (41) by

$$\alpha_1(s_1) := e^{-\frac{s_1}{2}},$$

which yields

$$e^{-\frac{s_1}{2}}(-k'_{j_0}(s_1) + e^{s_1} k'_{j_1}(-s_1)) = -(\alpha_2 \cdot k'_{j_0})(s_1) + T_2(\alpha_2 \cdot k'_{j_1})(s_1).$$

By switching j_0 and j_1 in this equation, we obtain a new equation, and summing up the two equations, we get

$$\begin{aligned}
&e^{-\frac{s_1}{2}}(-(k'_{j_0}(s_1)+k'_{j_1}(s_1))+e^{s_1}(k'_{j_0}(-s_1)+k'_{j_1}(-s_1)))\\
&\quad=-(\alpha_1\cdot(k'_{j_0}+k'_{j_1}))(s_1)+T_2(\alpha_1\cdot(k'_{j_0}+k'_{j_1}))(s_1)\\
&\quad=(-1+T_2)(\alpha_1\cdot(k'_{j_0}+k'_{j_1}))(s_1),
\end{aligned}$$

which is in the kernel of $1+T_2$ since $T_2^2=1$. □

We elaborate on the second part of the statement of the above theorem. Using the notation introduced by equation (42) and the first part of the above theorem, we have proved that the following expressions are invariant under the transformation $T_2: s_1\mapsto -s_1$ on $\mathbb{R}$:

$$\begin{aligned}
&s_1e^{-\frac{s_1}{2}}(-k'_3(s_1)+e^{s_1}k'_3(-s_1))=e^{-\frac{s_1}{2}}f_{3,3}(s_1),\\
&s_1e^{-\frac{s_1}{2}}(-k'_4(s_1)+e^{s_1}k'_4(-s_1))=e^{-\frac{s_1}{2}}f_{4,4}(s_1),\\
&s_1e^{-\frac{s_1}{2}}(-k'_5(s_1)+e^{s_1}k'_5(-s_1))=e^{-\frac{s_1}{2}}f_{5,5}(s_1),\\
&s_1e^{-\frac{s_1}{2}}(-(k'_6(s_1)+k'_7(s_1))+e^{s_1}(k'_6(-s_1)+k'_7(-s_1)))\\
&\qquad=e^{-\frac{s_1}{2}}(f_{6,7}(s_1)+f_{7,6}(s_1)).
\end{aligned}$$

We now wish to investigate invariance properties of expressions that can be associated with the differential equations (28)–(36). In these equations, where partial derivatives of the functions $k_8(s_1,s_2),\dots,k_{16}(s_1,s_2)$ are involved, two transformations are apparently present, namely:

$$(s_1,s_2)\mapsto(-s_1-s_2,s_1),\quad (s_1,s_2)\mapsto(s_2,-s_1-s_2).$$

They correspond to the matrices

$$T_3=\begin{pmatrix}-1&-1\\1&0\end{pmatrix},\quad T_3^2=\begin{pmatrix}0&1\\-1&-1\end{pmatrix},$$

and together with the identity matrix they form a cyclic group of order 3. The partial differential system involves the values on the orbit of points for this group and hence these values should be considered simultaneously. Taking a close look on the left-hand side of equations (28)–(36), it is clear that, up to an overall factor of $\frac{1}{8}$, they are all of the form

(43)
$$\begin{aligned}
&-\partial_1k_{j_0}(s_1,s_2)-e^{s_1+s_2}(\partial_2-\partial_1)k_{j_1}(-s_1-s_2,s_1)+e^{s_1}\partial_2k_{j_2}(s_2,-s_1-s_2)\\
&\quad=-\partial_1k_{j_0}(s_1,s_2)-T_3(e^{-s_1}(\partial_2-\partial_1)k_{j_1})(s_1,s_2)+T_3^2(e^{-s_1-s_2}\partial_2k_{j_2})(s_1,s_2),
\end{aligned}$$

where

$$
(44)\qquad \begin{cases} \text{for equation (28):} & (j_0,j_1,j_2)=(8,12,11),\\ \text{for equation (29):} & (j_0,j_1,j_2)=(9,13,10),\\ \text{for equation (30):} & (j_0,j_1,j_2)=(10,9,13),\\ \text{for equation (31):} & (j_0,j_1,j_2)=(11,8,12),\\ \text{for equation (32):} & (j_0,j_1,j_2)=(12,11,8),\\ \text{for equation (33):} & (j_0,j_1,j_2)=(13,10,9),\\ \text{for equation (34):} & (j_0,j_1,j_2)=(14,16,15),\\ \text{for equation (35):} & (j_0,j_1,j_2)=(15,14,16),\\ \text{for equation (36):} & (j_0,j_1,j_2)=(16,15,14). \end{cases}
$$

We note that while ranging from equation (28) to equation (36), each j_0, j_1, j_2 attains each of the integers in $\{8, 9, \dots, 16\}$ exactly once. Moreover, we can put the equations in the following three groups. The first group consists of equations (28), (31) and (32), which involve partial derivatives of the functions k_8, k_{11}, k_{12}. We can put equations (29), (30) and (33) in the second group since they involve partial derivatives of only k_9, k_{10}, k_{13}. The last group consists of equations (34), (35) and (36), which involve partial derivatives of k_{14}, k_{15}, k_{16}.

It will be useful to introduce the following notation $f_{j_0,j_1,j_2}(s_1, s_2)$ to rewrite the differential equations (28)–(36) concisely:

(45)

$$
\begin{aligned} &-\partial_1 k_{j_0}(s_1,s_2) - e^{s_1+s_2}(\partial_2-\partial_1)k_{j_1}(-s_1-s_2,s_1) + e^{s_1}\partial_2 k_{j_2}(s_2,-s_1-s_2)\\ &\quad = \frac{f_{j_0,j_1,j_2}(s_1,s_2)}{s_1 s_2 (s_1+s_2)}, \end{aligned}
$$

where for each differential equation, (j_0, j_1, j_2) is as in (44). It is clear from the equations that each $f_{j_0,j_1,j_2}(s_1, s_2)$ is a finite difference expression of restriction of the involved functions k_j to $s_1+s_2+s_3 = 0$. Another important observation that we have made is that on the right-hand side of each of equations (28)–(36), the denominator is the expression $s_1s_2(s_1+s_2)$. We see in the following lemma that the above action of $\mathbb{Z}/3\mathbb{Z}$ leaves this expression invariant. The quadratic form written in this lemma, whose unit ball is given in Figure 1, can be obtained by averaging the standard quadratic form using the action of the cyclic group.

Lemma 4.9. *The following expressions are invariants of the action of* $\mathbb{Z}/3\mathbb{Z}$ *on* $\mathbb{R}^2$ *given by the above matrices:*

(i) *The positive definite quadratic form*

$$Q_2(s) = s_1^2 + s_2^2 + s_1 s_2, \quad s = (s_1, s_2) \in \mathbb{R}^2.$$

(ii) *The product* $s_1 s_2 (s_1 + s_2)$.

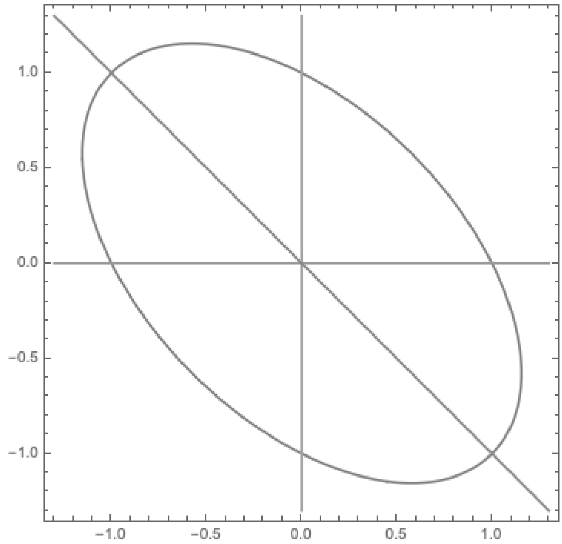

FIGURE 1. The ellipse is the unit ball for the quadratic form Q_2, and the straight lines are given by $s_1 = 0$, $s_2 = 0$, and $s_1 + s_2 = 0$.

We can now present expressions that are associated with the functions appearing in the differential equations and are in kernel of the transformation $1 + T_3 + T_3^2$, where T_3 is the above action of the cyclic group of order 3 on $\mathbb{R}^2$.

Theorem 4.10. *The expression*

$$\begin{aligned} e^{-\frac{2s_1}{3}-\frac{s_2}{3}} \Big(&-\partial_1(k_{j_0} + k_{j_1} + k_{j_2})(s_1, s_2) \\ &- e^{s_1+s_2}(\partial_2 - \partial_1)(k_{j_0} + k_{j_1} + k_{j_2})(-s_1 - s_2, s_1) \\ &+ e^{s_1}\partial_2(k_{j_0} + k_{j_1} + k_{j_2})(s_2, -s_1 - s_2)\Big) \end{aligned}$$

is in the kernel of $1 + T_3 + T_3^2$ *for any integers* j_0, j_1, j_2 *in* $\{8, 9, \dots, 16\}$. *In particular, considering the differential equations* (28)–(36) *and the above discussion following equation* (43), *in the following cases we have obtained expressions that are given by finite differences of the functions* k_j *and are in the kernel of* $1 + T_3 + T_3^2$*:*

(i) *When* $(j_0, j_1, j_2) = (8, 12, 11)$.
(ii) *When* $(j_0, j_1, j_2) = (9, 13, 10)$.
(iii) *When* $(j_0, j_1, j_2) = (14, 16, 15)$.

Proof. By multiplying (43) by

$$\alpha_2(s_1, s_2) := e^{-\frac{2s_1}{3}-\frac{s_2}{3}},$$

since

$$T_3\alpha_2(s_1,s_2) = e^{s_1+s_2}\alpha_2(s_1,s_2) = e^{\frac{1}{3}(s_1+2s_2)},$$
$$T_3^2\alpha_2(s_1,s_2) = e^{s_1}\alpha_2(s_1,s_2) = e^{\frac{1}{3}(s_1-s_2)},$$

we obtain

$$\begin{aligned}
&e^{-\frac{2s_1}{3}-\frac{s_2}{3}}(-\partial_1 k_{j_0}(s_1,s_2) - e^{s_1+s_2}(\partial_2-\partial_1)k_{j_1}(-s_1-s_2,s_1)\\
&\quad + e^{s_1}\partial_2 k_{j_2}(s_2,-s_1-s_2))\\
&= -e^{-\frac{2s_1}{3}-\frac{s_2}{3}}\partial_1 k_{j_0}(s_1,s_2) - e^{\frac{s_1}{3}+\frac{2s_2}{3}}(\partial_2-\partial_1)k_{j_1}(-s_1-s_2,s_1)\\
&\quad + e^{\frac{s_1}{3}-\frac{s_2}{3}}\partial_2 k_{j_2}(s_2,-s_1-s_2)\\
&= -\alpha_2(s_1,s_2)\partial_1 k_{j_0}(s_1,s_2) - T_3(\alpha_2\cdot\partial_2 k_{j_1})(s_1,s_2)\\
&\quad + T_3(\alpha_2\cdot\partial_1 k_{j_1})(s_1,s_2) + T_3^2(\alpha_2\cdot\partial_2 k_{j_2})(s_1,s_2)\\
&= -(\alpha_2\cdot\partial_1 k_{j_0}(s_1,s_2) + T_3(\alpha_2\cdot\partial_2 k_{j_1})(s_1,s_2))\\
&\quad + T_3(\alpha_2\cdot\partial_1 k_{j_1} + T_3(\alpha_2\cdot\partial_2 k_{j_2}))(s_1,s_2).
\end{aligned}$$

Thinking of this formula as being associated with the triple (j_0,j_1,j_2), we consider the formulas corresponding to the cyclic permutations (j_2,j_0,j_1) and (j_1,j_2,j_0) of (j_0,j_1,j_2) to get two more equations. Summing up the three equations, we have

$$\begin{aligned}
&e^{-\frac{2s_1}{3}-\frac{s_2}{3}}\big(-\partial_1(k_{j_0}+k_{j_1}+k_{j_2})(s_1,s_2)\\
&\quad - e^{s_1+s_2}(\partial_2-\partial_1)(k_{j_0}+k_{j_1}+k_{j_2})(-s_1-s_2,s_1)\\
&\quad + e^{s_1}\partial_2(k_{j_0}+k_{j_1}+k_{j_2})(s_2,-s_1-s_2)\big)\\
&= -\big(\alpha(s_1,s_2)\partial_1(k_{j_0}+k_{j_1}+k_{j_2})(s_1,s_2)\\
&\quad + T_3(\alpha_2\cdot\partial_2(k_{j_0}+k_{j_1}+k_{j_2}))(s_1,s_2)\big)\\
&\quad + T_3\big(\alpha_2\cdot\partial_1(k_{j_0}+k_{j_1}+k_{j_2}) + T_3(\alpha_2\cdot\partial_2(k_{j_0}+k_{j_1}+k_{j_2}))\big)(s_1,s_2)\\
&= (-1+T_3)\big(\alpha_2\cdot\partial_1(k_{j_0}+k_{j_1}+k_{j_2})\\
&\quad + T_3(\alpha_2\cdot\partial_2(k_{j_0}+k_{j_1}+k_{j_2}))\big)(s_1,s_2).
\end{aligned}$$

The latter is clearly in the kernel of $1+T_3+T_3^2$ since $T_3^3=1$. □

Let us elaborate on the second part of the statement of the above theorem. Using the notation (45), Lemma 4.9, and the above theorem, we have proved that the following expressions are in the kernel of the transformation $1+T_3+T_3^2$:

$$e^{-\frac{2s_1}{3}-\frac{s_2}{3}}(f_{8,12,11}(s_1,s_2) + f_{11,8,12}(s_1,s_2) + f_{12,11,8}(s_1,s_2)),$$
$$e^{-\frac{2s_1}{3}-\frac{s_2}{3}}(f_{9,13,10}(s_1,s_2) + f_{13,10,9}(s_1,s_2) + f_{10,9,13}(s_1,s_2)),$$
$$e^{-\frac{2s_1}{3}-\frac{s_2}{3}}(f_{14,15,16}(s_1,s_2) + f_{15,16,14}(s_1,s_2) + f_{16,14,15}(s_1,s_2)).$$

We now turn our focus to the functions $k_{17}(s_1,s_2,s_3),\dots,k_{20}(s_1,s_2,s_3)$. By considering the differential equations (37)–(40), one gets a cyclic group of order 4 which corresponds to the matrices

$$T_4 = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_4^2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_4^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}.$$

Together with the identity matrix, these matrices form a cyclic group of order 4. The transformations corresponding to the above matrices are

$$(s_1, s_2, s_3) \mapsto (-s_1 - s_2 - s_3, s_1, s_2),$$
$$(s_1, s_2, s_3) \mapsto (s_3, -s_1 - s_2 - s_3, s_1),$$
$$(s_1, s_2, s_3) \mapsto (s_2, s_3, -s_1 - s_2 - s_3).$$

We investigate the compatibility of these transformations with the denominators and the various functions involved.

In Remark 4.6, we pointed out that the denominator on the right-hand side of the differential equations (37)–(40) is the expression

$$s_1 s_2 (s_1 + s_2) s_3 (s_2 + s_3)(s_1 + s_2 + s_3).$$

The following lemma shows the invariance of this expression under the above transformations.

Lemma 4.11. *The action of $\mathbb{Z}/4\mathbb{Z}$ on $\mathbb{R}^3$ given by the above matrices leaves the following expressions invariant up to sign:*

(i) *The positive definite quadratic form*

$$Q_3(s) = s_1^2 + s_2^2 + s_3^2 + s_1 s_2 + s_2 s_3 + s_3 s_1, \quad s = (s_1, s_2, s_3) \in \mathbb{R}^3.$$

(ii) *The linear form $s_1 + s_3$ whose kernel defines a plane in which a generator of $\mathbb{Z}/4\mathbb{Z}$ acts by a rotation of angle $\frac{\pi}{2}$ for the metric induced by Q_3.*

(iii) *The product $(s_1 + s_2)(s_2 + s_3)$ and the line $\{s = (s_1, -s_1, s_1) \mid s_1 \in \mathbb{R}\}$ on which it acts as the symmetry $s \mapsto -s$.*

(iv) *The product $s_1 s_2 s_3 (s_1 + s_2 + s_3)$.*

This means that when using the positive definite quadratic form Q_3, whose unit ball is given in Figure 2, the generator of the action of $\mathbb{Z}/4\mathbb{Z}$ has the following form: it preserves the line $\{s = (s_1, -s_1, s_1) \mid s_1 \in \mathbb{R}\}$ on which it acts as the symmetry $s \mapsto -s$. The orthogonal of this line for the quadratic form Q_3 is the plane $s_1 + s_3 = 0$. On this plane the generator of the action of $\mathbb{Z}/4\mathbb{Z}$ is a rotation of angle $\frac{\pi}{2}$ for the metric induced by Q_3 which agrees with $s_1^2 + s_2^2$ since $s_1 + s_3 = 0$.

Note also that the natural lattice $L = \mathbb{Z}^3 \subset \mathbb{R}^3$ is contained in its dual $L^\perp$ with respect to the quadratic form Q_3 which is defined by

$$L^\perp = \{s = (s_1, s_2, s_3) \mid Q_3(s, t) \in \mathbb{Z} \text{ for all } t \in L\}.$$

One finds that the quotient $H = L^\perp / L$ is the group $\mathbb{Z}/4\mathbb{Z}$ generated by $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

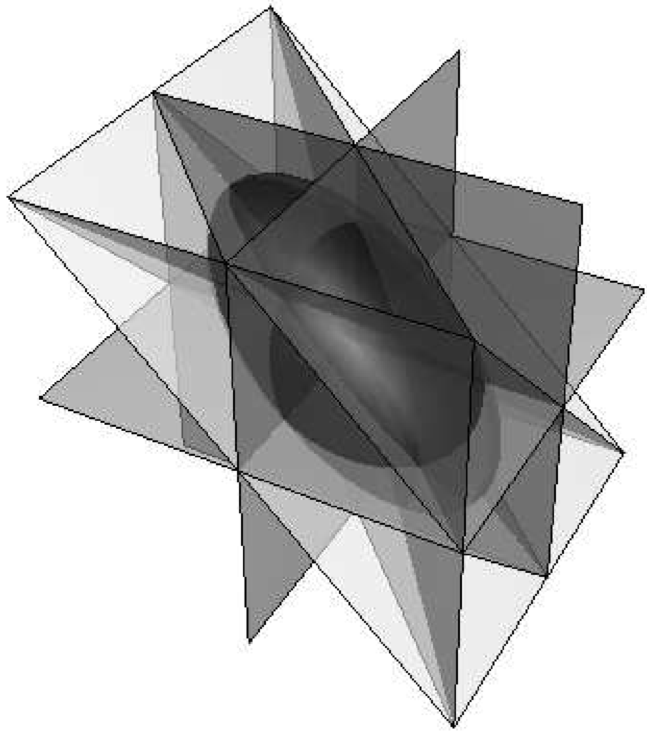

FIGURE 2. The ellipsoid is the unit ball for the quadratic form Q_3, and the various planes are given by $s_j = 0$, $s_1 + s_2 + s_3 = 0$, the plane $s_1 + s_3 = 0$ in which the generator of the action of $\mathbb{Z}/4\mathbb{Z}$ is a rotation of angle $\frac{\pi}{2}$, and the two planes corresponding to the product $(s_1 + s_2)(s_2 + s_3)$.

The left-hand side of the differential equations (37)–(40), up to multiplication by $\frac{1}{16}$, is of the form

$$\begin{aligned}
(46) \qquad & -\partial_1 k_{j_0}(s_1, s_2, s_3) - e^{s_1+s_2+s_3}(\partial_2 - \partial_1)k_{j_1}(-s_1 - s_2 - s_3, s_1, s_2) \\
& \quad - e^{s_1+s_2}(\partial_3 - \partial_2)k_{j_0}(s_3, -s_1 - s_2 - s_3, s_1) \\
& \quad + e^{s_1}\partial_3 k_{j_1}(s_2, s_3, -s_1 - s_2 - s_3) \\
& = -\partial_1 k_{j_0}(s_1, s_2, s_3) - T_4(e^{-s_1}(\partial_2 - \partial_1)k_{j_1})(s_1, s_2, s_3) \\
& \quad - T_4^2(e^{-s_1-s_2}(\partial_3 - \partial_2)k_{j_0})(s_1, s_2, s_3) \\
& \quad + T_4^3(e^{-s_1-s_2-s_3}k_{j_1})(s_1, s_2, s_3),
\end{aligned}$$

where

$$\text{(47)}\qquad \begin{cases} \text{for equation (37):} & (j_0, j_1) = (17, 19), \\ \text{for equation (38):} & (j_0, j_1) = (18, 18), \\ \text{for equation (39):} & (j_0, j_1) = (19, 17), \\ \text{for equation (40):} & (j_0, j_1) = (20, 20). \end{cases}$$

This suggests that we can put the differential equations (37)–(40) into three groups: one consisting of equations (37) and (39) which involve partial derivatives of only k_{17} and k_{19}, one group consisting of equation (38) since it involves partial derivatives of k_{18}, and finally we can consider (40) as the last group since it involves partial derivatives of only k_{20}. Also we will introduce the following concise notation $f_{j_0,j_1}(s_1, s_2, s_3)$ to rewrite the differential equations (37)–(40) as

$$\begin{aligned} \text{(48)}\qquad & -\partial_1 k_{j_0}(s_1, s_2, s_3) - e^{s_1+s_2+s_3}(\partial_2 - \partial_1) k_{j_1}(-s_1 - s_2 - s_3, s_1, s_2) \\ & - e^{s_1+s_2}(\partial_3 - \partial_2) k_{j_0}(s_3, -s_1 - s_2 - s_3, s_1) \\ & + e^{s_1} \partial_3 k_{j_1}(s_2, s_3, -s_1 - s_2 - s_3) \\ & = \frac{f_{j_0,j_1}(s_1, s_2, s_3)}{s_1 s_2 (s_1 + s_2) s_3 (s_2 + s_3)(s_1 + s_2 + s_3)}, \end{aligned}$$

where in each case (j_0, j_1) is given as in (47). Note that each $f_{j_0,j_1}(s_1, s_2, s_3)$ is written in terms of finite differences of restriction to $s_1 + s_2 + s_3 + s_4 = 0$ of the involved functions k_j.

We can now exploit the above action of the cyclic group of order 4 on $\mathbb{R}^3$ generated by T_4 to derive expressions from our differential equations that are in the kernel of the transformation $1 + T_4 + T_4^2 + T_4^3$.

Theorem 4.12. *The expression*

$$\begin{aligned} e^{-\frac{3s_1}{4} - \frac{s_2}{2} - \frac{s_3}{4}} \big(& -\partial_1 (k_{j_0} + k_{j_1})(s_1, s_2, s_3) \\ & - e^{s_1+s_2+s_3}(\partial_2 - \partial_1)(k_{j_0} + k_{j_1})(-s_1 - s_2 - s_3, s_1, s_2) \\ & - e^{s_1+s_2}(\partial_3 - \partial_2)(k_{j_0} + k_{j_1})(s_3, -s_1 - s_2 - s_3, s_1) \\ & + e^{s_1} \partial_3 (k_{j_0} + k_{j_1})(s_2, s_3, -s_1 - s_2 - s_3) \big) \end{aligned}$$

is in the kernel of $1 + T_4 + T_4^2 + T_4^3$ *for any pair of integers* j_0, j_1 *in the set* $\{17, 18, 19, 20\}$. *In particular, considering the differential equations* (37)–(40) *and the above discussion following equation* (46), *we have obtained in the following cases expressions that are given by finite differences of the* k_j *and are in the kernel of* $1 + T_4 + T_4^2 + T_4^3$*:*

(i) *When* $(j_0, j_1) = (17, 19)$.
(ii) *When* $(j_0, j_1) = (18, 18)$.
(iii) *When* $(j_0, j_1) = (20, 20)$.

Proof. We multiply (46) by

$$\alpha_3(s_1, s_2, s_3) := e^{-\frac{3s_1}{4} - \frac{s_2}{2} - \frac{s_3}{4}},$$

and since

$$\begin{aligned}
T_4\alpha_3(s_1,s_2,s_3) &= \alpha_3(s_1,s_2,s_3)\, e^{s_1+s_2+s_3} = e^{\frac{1}{4}(s_1+2s_2+3s_3)},\\
T_4^2\alpha_3(s_1,s_2,s_3) &= \alpha_3(s_1,s_2,s_3)\, e^{s_1+s_2} = e^{\frac{1}{4}(s_1+2s_2-s_3)},\\
T_4^3\alpha_3(s_1,s_2,s_3) &= \alpha_3(s_1,s_2,s_3)\, e^{s_1} = e^{\frac{1}{4}(s_1-2s_2-s_3)},
\end{aligned}$$

we find that

$$\begin{aligned}
&e^{-\frac{3s_1}{4}-\frac{s_2}{2}-\frac{s_3}{4}}\big(-\partial_1 k_{j_0}(s_1,s_2,s_3) - e^{s_1+s_2+s_3}(\partial_2-\partial_1)k_{j_1}(-s_1-s_2-s_3,s_1,s_2)\\
&\qquad - e^{s_1+s_2}(\partial_3-\partial_2)k_{j_0}(s_3,-s_1-s_2-s_3,s_1)\\
&\qquad + e^{s_1}\partial_3 k_{j_1}(s_2,s_3,-s_1-s_2-s_3)\big)\\
&\quad = -(\alpha_3\cdot\partial_1 k_{j_0})(s_1,s_2,s_3) - T_4(\alpha_3\cdot\partial_2 k_{j_1})(s_1,s_2,s_3)\\
&\qquad + T_4(\alpha_3\cdot\partial_1 k_{j_1})(s_1,s_2,s_3)\\
&\qquad - T_4^2(\alpha_3\cdot\partial_3 k_{j_0})(s_1,s_2,s_3) + T_4^2(\alpha_3\cdot\partial_2 k_{j_0})(s_1,s_2,s_3)\\
&\qquad + T_4^3(\alpha_3\cdot\partial_3 k_{j_1})(s_1,s_2,s_3).
\end{aligned}$$

We switch j_0 and j_1 in this equation to get a new one, and adding the two equations, we have

$$\begin{aligned}
&-(\alpha_3\cdot\partial_1(k_{j_0}+k_{j_1})) - T_4(\alpha_3\cdot\partial_2(k_{j_0}+k_{j_1})) + T_4(\alpha_3\cdot\partial_1(k_{j_0}+k_{j_1}))\\
&\qquad - T_4^2(\alpha_3\cdot\partial_3(k_{j_0}+k_{j_1})) + T_4^2(\alpha_3\cdot\partial_2(k_{j_0}+k_{j_1})) + T_4^3(\alpha_3\cdot\partial_3(k_{j_0}+k_{j_1}))\\
&\quad = -(\alpha_3\cdot\partial_1(k_{j_0}+k_{j_1}) + T_4(\alpha_3\cdot\partial_2(k_{j_0}+k_{j_1})) + T_4^2(\alpha_3\cdot\partial_3(k_{j_0}+k_{j_1})))\\
&\qquad + T_4(\alpha_3\cdot\partial_1(k_{j_0}+k_{j_1}) + T_4(\alpha_3\cdot\partial_2(k_{j_0}+k_{j_1})) + T_4^2(\alpha_3\cdot\partial_3(k_{j_0}+k_{j_1})))\\
&\quad = (-1+T_4)(\alpha_3\cdot\partial_1(k_{j_0}+k_{j_1}) + T_4(\alpha_3\cdot\partial_2(k_{j_0}+k_{j_1}))\\
&\qquad + T_4^2(\alpha_3\cdot\partial_3(k_{j_0}+k_{j_1}))),
\end{aligned}$$

which is in the kernel of $1+T_4+T_4^2+T_4^3$ since $T_4^4=1$. □

In order to explain the second statement in this theorem, it is important to mention that with the aid of the notation used in (48), Lemma 4.11, and the above theorem, we have shown that the following expressions are in the kernel of the transformation $1+T_4+T_4^2+T_4^3$:

$$\begin{aligned}
&e^{-\frac{3s_1}{4}-\frac{s_2}{2}-\frac{s_3}{4}}(f_{17,19}(s_1,s_2,s_3)+f_{19,17}(s_1,s_2,s_3)),\\
&e^{-\frac{3s_1}{4}-\frac{s_2}{2}-\frac{s_3}{4}} f_{18,18}(s_1,s_2,s_3),\\
&e^{-\frac{3s_1}{4}-\frac{s_2}{2}-\frac{s_3}{4}} f_{20,20}(s_1,s_2,s_3).
\end{aligned}$$

4.13. **Flow on the orbits and the differential system.** For any 1-variable function $k(s_1)$ let us consider the orbit $\mathcal{O}k$ of the function under the action of T_2 on $\mathbb{R}$:

$$\mathcal{O}k(s_1) = (k(s_1), k(-s_1)).$$

Since

$$\frac{d}{dt}\Big|_{t=0}\mathcal{O}k(s_1+t) = \frac{d}{dt}\Big|_{t=0}(k(s_1+t),k(-s_1-t)) = (k'(s_1),-k'(-s_1)),$$

the differential parts of the differential equations (21)–(27) when put together as in the statement of Theorem 4.8 are generated by the following flow when passed to the orbit using the action of T_2 on $\mathbb{R}$:

$$s_1 \mapsto s_1 + t.$$

On the other hand from the proof of Theorem 4.8, we can use the function $\alpha_1(s_1) = e^{-s_1/2}$ with $\mathcal{O}\alpha_1(s_1) = (e^{-s_1/2}, e^{s_1/2})$ to obtain the following statement, in which we use the notation given by (42).

Theorem 4.14. *Considering the cases for (j_0, j_1) listed below and by setting for each case*

$$k = k_{j_0} + k_{j_1}, \quad f = f_{j_0,j_1} + f_{j_1,j_0},$$

we have

$$\Big(\frac{d}{dt}\Big|_{t=0} \mathcal{O}k(s_1+t)\Big) \cdot (\mathcal{O}\alpha_1(s_1)) = -\frac{\alpha_1(s_1)\, f(s_1)}{s_1}.$$

The following are the cases:

(i) *When* $(j_0, j_1) = (3, 3)$.
(ii) *When* $(j_0, j_1) = (4, 4)$.
(iii) *When* $(j_0, j_1) = (5, 5)$.
(iv) *When* $(j_0, j_1) = (6, 7)$.

We note that Theorem 4.8 asserts that in each of the above cases the function $\alpha_1(s_1)\, f(s_1)/s_1$ is in the kernel of transformation $1 + T_2$, which sends via averaging any function defined on $\mathbb{R}$ to a function defined on the quotient space.

Now for any 2-variable function $k(s_1, s_2)$ we similarly define $\mathcal{O}k$ to be the orbit of the function k under the action of T_3 on $\mathbb{R}^2$:

$$\mathcal{O}k(s_1, s_2) = (k(s_1, s_2), k(-s_1 - s_2, s_1), k(s_2, -s_1 - s_2)).$$

It is easy to see that

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} \mathcal{O}k(s_1+t, s_2)\\
&\quad = \frac{d}{dt}\Big|_{t=0} (k(s_1+t, s_2), k(-s_1 - t - s_2, s_1 + t), k(s_2, -s_1 - t - s_2))\\
&\quad = (\partial_1 k(s_1, s_2),\, (\partial_2 - \partial_1)k(-s_1 - s_2, s_1),\, -\partial_2 k(s_2, -s_1 - s_2)).
\end{aligned}$$

This means that the flow induced on the orbit from the flow on $\mathbb{R}^2$ given by

$$(s_1, s_2) \mapsto (s_1 + t, s_2)$$

generates the differential components of the differential equations (28)–(36) when put together as in the statement of Theorem 4.10. Also we know from the proof of this theorem that the coefficients appearing on the left-hand sides of these equations are induced by the action of T_3 starting from the function

$$\alpha_2(s_1, s_2) = e^{-\frac{2s_1}{3} - \frac{s_2}{3}}$$

since

$$\mathcal{O}\alpha_2(s_1, s_2) = (e^{-\frac{2s_1}{3} - \frac{s_2}{3}}, e^{\frac{1}{3}(s_1 + 2s_2)}, e^{\frac{1}{3}(s_1 - s_2)}).$$

Therefore, using the notation introduced by (45), we have the following statement.

Theorem 4.15. *For each of the cases listed below we set*

$$k = k_{j_0} + k_{j_1} + k_{j_2}, \quad f = f_{j_0,j_1,j_2} + f_{j_2,j_0,j_1} + f_{j_1,j_2,j_0}.$$

Then we have

$$\Big(\frac{d}{dt}\Big|_{t=0} \mathcal{O}k(s_1+t,s_2)\Big)\cdot(\mathcal{O}\alpha_2(s_1,s_2)) = -\frac{\alpha_2(s_1,s_2)\,f(s_1,s_2)}{s_1s_2(s_1+s_2)},$$

where the following are the cases for (j_0,j_1,j_2)*:*

(i) *When* $(j_0,j_1,j_2) = (8,12,11)$.
(ii) *When* $(j_0,j_1,j_2) = (9,13,10)$.
(iii) *When* $(j_0,j_1,j_2) = (14,16,15)$.

We recall from Theorem 4.10 that in each of the cases mentioned in the above theorem, the function $\alpha_2(s_1,s_2)f(s_1,s_2)$ is in the kernel of the averaging operator $1+T_3+T_3^2$, which means that its average (defined on the quotient space) vanishes. We also note from Lemma 4.9 that the denominator $s_1s_2(s_1+s_2)$ is invariant under the action of T_3.

Similarly, for any 3-variable function $k(s_1,s_2,s_3)$ we define $\mathcal{O}k$ to be orbit of the function k under the action of T_4 on $\mathbb{R}^3$:

$$\mathcal{O}k(s_1,s_2,s_3) = (k(s_1,s_2,s_3),\, k(-s_1-s_2-s_3,s_1,s_2),\\ k(s_3,-s_1-s_2-s_3,s_1),\, k(s_2,s_3,-s_1-s_2-s_3)).$$

In this case as well the flow induced on the orbit from the flow on $\mathbb{R}^3$ given by

$$(s_1,s_2,s_3)\mapsto(s_1+t,s_2,s_3)$$

generates the differential components of the differential equations (37)–(40) when considered together as mentioned in the statement of Theorem 4.12. That is, one can see easily that

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} \mathcal{O}k(s_1+t,s_2,s_3)\\
&= \frac{d}{dt}\Big|_{t=0} \big(k(s_1+t,s_2,s_3), k(-s_1-t-s_2-s_3,s_1+t,s_2),\\
&\quad k(s_3,-s_1-t-s_2-s_3,s_1+t), k(s_2,s_3,-s_1-t-s_2-s_3)\big)\\
&= \big(\partial_1 k(s_1,s_2,s_3),\, (\partial_2-\partial_1)k(-s_1-s_2-s_3,s_1,s_2),\\
&\quad (\partial_3-\partial_2)k(s_3,-s_1-s_2-s_3,s_1),\, -\partial_3 k(s_2,s_3,-s_1-s_2-s_3)\big).
\end{aligned}$$

As we observed in the proof of Theorem 4.12 the coefficients appearing on the left-hand sides of these equations are induced by the transformation T_4 starting from the function

$$\alpha_3(s_1,s_2,s_3) = e^{-\frac{3s_1}{4}-\frac{s_2}{2}-\frac{s_3}{4}}$$

because we have

$$\mathcal{O}\alpha_3(s_1,s_2,s_3) = (e^{-\frac{3s_1}{4}-\frac{s_2}{2}-\frac{s_3}{4}}, e^{\frac{1}{4}(s_1+2s_2+3s_3)}, e^{\frac{1}{4}(s_1+2s_2-s_3)}, e^{\frac{1}{4}(s_1-2s_2-s_3)}).$$

Hence, using the notation introduced by (48), we have the following theorem.

Theorem 4.16. *For each case from the list given below set*

$$k = k_{j_0} + k_{j_1}, \quad f = f_{j_0,j_1} + f_{j_1,j_0}.$$

Then we have

$$\Big(\frac{d}{dt}\Big|_{t=0} \mathcal{O}k(s_1+t, s_2, s_3)\Big) \cdot (\mathcal{O}\alpha_3(s_1, s_2, s_3)) = -\frac{\alpha_3(s_1, s_2, s_3)\, f(s_1, s_2, s_3)}{s_1 s_2 (s_1+s_2) s_3 (s_2+s_3)(s_1+s_2+s_3)},$$

where the cases are the following:

(i) *When* $(j_0, j_1) = (17, 19)$.
(ii) *When* $(j_0, j_1) = (18, 18)$.
(iii) *When* $(j_0, j_1) = (20, 20)$.

We note that we proved in Theorem 4.12 that in each of the above cases the function $\alpha_3(s_1, s_2, s_3) f(s_1, s_2, s_3)$ is in the kernel of the averaging operator $1 + T_4 + T_4^2 + T_4^3$, hence its average (defined on the quotient space) vanishes. We also recall from Lemma 4.11 that the denominator

$$s_1 s_2 (s_1+s_2) s_3 (s_2+s_3)(s_1+s_2+s_3)$$

is invariant under the action of T_4.

5. The method and tools for proving the functional relations

In this section we provide the tools for calculating in two different ways the gradient of the map that sends a general dilaton $h = h^* \in C^\infty(\mathbb{T}_\theta^2)$ to $\varphi_0(a_4)$, which will eventually lead to the functional relations stated in Theorem 3.1 in Section 3. For the sake of clarity, let us write $a_4 = a_4(h)$ so that we can clearly view it as a function of the dilaton h. Also, we will use the following notation for the conformally perturbed Laplacian given by (4):

$$\triangle_h = e^{h/2} \triangle e^{h/2}.$$

The methods are in fact closely based on the techniques initiated in [14]. The first method employs a fundamental identity proved in [14] as follows. For a general element $a \in C^\infty(\mathbb{T}_\theta^2)$, consider the spectral zeta function defined by

$$\zeta_h(a, s) = \mathrm{Trace}(a \triangle_h^{-s}), \quad s \in \mathbb{C},\ \Re(s) \gg 0.$$

The zeta function is initially defined when the real part of the complex number s is large enough, and it has a meromorphic extension to the complex plane. It is proved in [14] that if h and a are smooth selfadjoint elements in $C(\mathbb{T}_\theta^2)$, then

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \zeta_{h+\varepsilon a}(1, s) = -\frac{s}{2} \zeta_h(\widetilde{a}, s), \tag{49}$$

where

$$\widetilde{a} = \int_{-1}^{1} e^{uh/2} a e^{-uh/2}\, du.$$

Using the Mellin transform and the asymptotic expansion (1), one can see that

$$\zeta_h(a,-1) = -\varphi_0(a\, a_4(h)), \quad a \in C^\infty(\mathbb{T}_\theta^2),\ h = h^* \in C^\infty(\mathbb{T}_\theta^2).$$

Therefore, it follows from (49) that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \varphi_0(a_4(h+\varepsilon a)) = -\frac{1}{2}\zeta_h(\widetilde{a},-1) = \frac{1}{2}\varphi_0(\widetilde{a}\, a_4(h)). \tag{50}$$

At this stage, following the method in [14], it is crucial to observe that, in general, given a smooth function L and elements $x_1,\dots,x_n \in C^\infty(\mathbb{T}_\theta^2)$, we have

(51)

$$\begin{aligned}
&\varphi_0(\widetilde{a} e^h L(\nabla,\dots,\nabla)(x_1\cdots x_n)) \\
&\quad = \varphi_0\Big(\int_{-1}^{1} e^{uh/2} a e^h e^{-uh/2}\, du\, L(\nabla,\dots,\nabla)(x_1\cdots x_n)\Big) \\
&\quad = \varphi_0\Big(a e^h \int_{-1}^{1} e^{-uh/2}\, L(\nabla,\dots,\nabla)(x_1\cdots x_n)\, e^{uh/2}\, du\Big) \\
&\quad = \varphi_0\Big(a e^h \Big(2\frac{\sinh(\nabla/2)}{\nabla/2}\Big)(L(\nabla,\dots,\nabla)(x_1\cdots x_n))\Big) \\
&\quad = \varphi_0\Big(a e^h \Big(2\frac{\sinh((s_1+\cdots+s_n)/2)}{(s_1+\cdots+s_n)/2} L(s_1,\dots,s_n)\Big)\Big|_{s_j=\nabla} (x_1\cdots x_n)\Big).
\end{aligned}$$

This justifies the reason for defining the variants $\widetilde{K}_j$ of the functions K_j as in (7).

By using the identities (50) and (51), and considering expression (6) for the term $a_4(h)$ with $\ell = \frac{h}{2}$, we can conclude that under the trace φ_0 we have

(52)

$$\begin{aligned}
&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \varphi_0(a_4(h+\varepsilon a)) \\
&\quad = -a e^h \Big(\widetilde{K}_1(\nabla)(\delta_1^2\delta_2^2(h)) + \widetilde{K}_2(\nabla)(\delta_1^4(h) + \delta_2^4(h)) \\
&\qquad + \widetilde{K}_3(\nabla,\nabla)((\delta_1\delta_2(h))\cdot(\delta_1\delta_2(h))) + \widetilde{K}_4(\nabla,\nabla)(\delta_1^2(h)\cdot\delta_2^2(h) + \delta_2^2(h)\cdot\delta_1^2(h)) \\
&\qquad + \widetilde{K}_5(\nabla,\nabla)(\delta_1^2(h)\cdot\delta_1^2(h) + \delta_2^2(h)\cdot\delta_2^2(h)) \\
&\qquad + \widetilde{K}_6(\nabla,\nabla)(\delta_1(h)\cdot\delta_1^3(h) + \delta_1(h)\cdot(\delta_1\delta_2^2(h)) + \delta_2(h)\cdot\delta_2^3(h) + \delta_2(h)\cdot(\delta_1^2\delta_2(h))) \\
&\qquad + \widetilde{K}_7(\nabla,\nabla)(\delta_1^3(h)\cdot\delta_1(h) + (\delta_1\delta_2^2(h))\cdot\delta_1(h) + \delta_2^3(h)\cdot\delta_2(h) + (\delta_1^2\delta_2(h))\cdot\delta_2(h)) \\
&\qquad + \widetilde{K}_8(\nabla,\nabla,\nabla)(\delta_1(h)\cdot\delta_1(h)\cdot\delta_2^2(h) + \delta_2(h)\cdot\delta_2(h)\cdot\delta_1^2(h)) \\
&\qquad + \widetilde{K}_9(\nabla,\nabla,\nabla)(\delta_1(h)\cdot\delta_2(h)\cdot(\delta_1\delta_2(h)) + \delta_2(h)\cdot\delta_1(h)\cdot(\delta_1\delta_2(h))) \\
&\qquad + \widetilde{K}_{10}(\nabla,\nabla,\nabla)(\delta_1(h)\cdot(\delta_1\delta_2(h))\cdot\delta_2(h) + \delta_2(h)\cdot(\delta_1\delta_2(h))\cdot\delta_1(h))
\end{aligned}$$

$$
\begin{aligned}
&+ \widetilde{K}_{11}(\nabla,\nabla,\nabla)(\delta_1(h)\cdot\delta_2^2(h)\cdot\delta_1(h)+\delta_2(h)\cdot\delta_1^2(h)\cdot\delta_2(h))\\
&+ \widetilde{K}_{12}(\nabla,\nabla,\nabla)(\delta_1^2(h)\cdot\delta_2(h)\cdot\delta_2(h)+\delta_2^2(h)\cdot\delta_1(h)\cdot\delta_1(h))\\
&+ \widetilde{K}_{13}(\nabla,\nabla,\nabla)((\delta_1\delta_2(h))\cdot\delta_1(h)\cdot\delta_2(h)+(\delta_1\delta_2(h))\cdot\delta_2(h)\cdot\delta_1(h))\\
&+ \widetilde{K}_{14}(\nabla,\nabla,\nabla)(\delta_1^2(h)\cdot\delta_1(h)\cdot\delta_1(h)+\delta_2^2(h)\cdot\delta_2(h)\cdot\delta_2(h))\\
&+ \widetilde{K}_{15}(\nabla,\nabla,\nabla)(\delta_1(h)\cdot\delta_1(h)\cdot\delta_1^2(h)+\delta_2(h)\cdot\delta_2(h)\cdot\delta_2^2(h))\\
&+ \widetilde{K}_{16}(\nabla,\nabla,\nabla)(\delta_1(h)\cdot\delta_1^2(h)\cdot\delta_1(h)+\delta_2(h)\cdot\delta_2^2(h)\cdot\delta_2(h))\\
&+ \widetilde{K}_{17}(\nabla,\nabla,\nabla,\nabla)(\delta_1(h)\cdot\delta_1(h)\cdot\delta_2(h)\cdot\delta_2(h)+\delta_2(h)\cdot\delta_2(h)\cdot\delta_1(h)\cdot\delta_1(h))\\
&+ \widetilde{K}_{18}(\nabla,\nabla,\nabla,\nabla)(\delta_1(h)\cdot\delta_2(h)\cdot\delta_1(h)\cdot\delta_2(h)+\delta_2(h)\cdot\delta_1(h)\cdot\delta_2(h)\cdot\delta_1(h))\\
&+ \widetilde{K}_{19}(\nabla,\nabla,\nabla,\nabla)(\delta_1(h)\cdot\delta_2(h)\cdot\delta_2(h)\cdot\delta_1(h)+\delta_2(h)\cdot\delta_1(h)\cdot\delta_1(h)\cdot\delta_2(h))\\
&+ \widetilde{K}_{20}(\nabla,\nabla,\nabla,\nabla)(\delta_1(h)\cdot\delta_1(h)\cdot\delta_1(h)\cdot\delta_1(h)+\delta_2(h)\cdot\delta_2(h)\cdot\delta_2(h)\cdot\delta_2(h))\Big).
\end{aligned}
$$

For our second calculation of the above gradient, we use Duhamel's formula in a crucial way, which, given a family of operators A_s, allows one to write

$$\frac{d}{ds}e^{-A_s} = -\int_0^1 e^{-uA_s}\frac{dA_s}{ds}e^{-(1-u)A_s}\,ds.$$

This method will give rise to a formula for the desired gradient in terms of finite differences, which is described by performing explicit calculation in Section 6. However, first, we need to prove a series of lemmas, which are given in the following subsections.

5.1. **Gradients of functional calculi with ∇.** In this subsection, we find explicit formulas for certain gradients of functions of the modular automorphism acting on elements of the noncommutative torus. For convenience, we will use the notation

$$\nabla_\varepsilon = \nabla_{h+\varepsilon a} = \mathrm{ad}_{-h-\varepsilon a} = \nabla - \varepsilon\,\mathrm{ad}_a,$$

where ε is a real number, and h and a are selfadjoint elements in $C^\infty(\mathbb{T}_\theta^2)$.

Lemma 5.2. *Let $L(s_1)$ be a smooth function and x_1, x_2 be elements of the algebra $C(\mathbb{T}_\theta^2)$ of the noncommutative torus. Under the trace φ_0, we can write*

$$e^h\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L(\nabla_\varepsilon)(x_1)\Big)x_2 = ae^hL_{1,1}^\varepsilon(\nabla,\nabla)(x_1\cdot x_2) + ae^hL_{1,2}^\varepsilon(\nabla,\nabla)(x_2\cdot x_1),$$

where

$$
\begin{aligned}
L_{1,1}^\varepsilon(s_1,s_2) &= e^{s_1+s_2}\frac{L(-s_2)-L(s_1)}{s_1+s_2},\\
L_{1,2}^\varepsilon(s_1,s_2) &= e^{s_1}\frac{L(s_2)-L(-s_1)}{s_1+s_2}.
\end{aligned}
$$

Proof. Writing L as a Fourier transform $L(v) = \int e^{-itv} g(t)\,dt$, and using Duhamel's formula and trace property of φ_0, we can write under the latter

$$\begin{aligned} &e^h\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L(\nabla_\varepsilon)(x_1)\Big)x_2 \\ &= e^h \int\int_0^1 it\,\sigma_{ut}\,\mathrm{ad}_a\,\sigma_{(1-u)t}(x_1)\,g(t)\,x_2\,du\,dt \\ &= e^h \int\int_0^1 it\,a\,\sigma_{t-ut}(x_1)\sigma_{-ut}(x_2)\,g(t)\,du\,dt \\ &\quad - e^h \int\int_0^1 it\,\sigma_{t-ut}(x_1)\,a\,\sigma_{-ut}(x_2)\,g(t)\,du\,dt \\ &= aL_1(\nabla,\nabla)(x_1x_2e^h) + aL_2(\nabla,\nabla)(x_2e^hx_1), \end{aligned}$$

where

$$\begin{aligned} L_1(s_1,s_2) &:= \int_0^1\int ite^{-i(t-ut)s_1+iuts_2}g(t)\,dt\,du = \frac{G(-s_2)-G(s_1)}{s_1+s_2}, \\ L_2(s_1,s_2) &:= -\int_0^1\int ite^{iuts_1-i(t-ut)s_2}g(t)\,dt\,du = \frac{G(s_2)-G(-s_1)}{s_1+s_2}. \qquad \square \end{aligned}$$

We need a version of the above lemma that involves functions of higher number of variables involved. In the following we treat the two and three variable cases, which will suffice for our purposes in Section 6.

Lemma 5.3. *Let $L(s_1, s_2)$ be a smooth function and let x_1, x_2, x_3 be elements in $C(\mathbb{T}_\theta^2)$. Under the trace φ_0, we have*

$$\begin{aligned} &e^h\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L(\nabla_\varepsilon,\nabla_\varepsilon)(x_1\cdot x_2)\Big)x_3 \\ &\quad= ae^hL^\varepsilon_{2,1}(\nabla,\nabla,\nabla)(x_1\cdot x_2\cdot x_3) + ae^hL^\varepsilon_{2,2}(\nabla,\nabla,\nabla)(x_2\cdot x_3\cdot x_1) \\ &\qquad + ae^hL^\varepsilon_{2,3}(\nabla,\nabla,\nabla)(x_3\cdot x_1\cdot x_2), \end{aligned}$$

where

$$\begin{aligned} L^\varepsilon_{2,1}(s_1,s_2,s_3) &:= e^{s_1+s_2+s_3}\,\frac{L(-s_2-s_3,s_2)-L(s_1,s_2)}{s_1+s_2+s_3}, \\ L^\varepsilon_{2,2}(s_1,s_2,s_3) &:= e^{s_1+s_2}\,\frac{L(s_3,-s_2-s_3)-L(-s_1-s_2,s_1)}{s_1+s_2+s_3}, \\ L^\varepsilon_{2,3}(s_1,s_2,s_3) &:= e^{s_1}\,\frac{L(s_2,s_3)-L(s_2,-s_1-s_2)}{s_1+s_2+s_3}. \end{aligned}$$

Proof. Writing

$$L(s_1,s_2) = \int e^{-it_1s_1-it_2s_2}g(t_1,t_2)\,dt_1\,dt_2,$$

we have

$$L(\nabla_\varepsilon,\nabla_\varepsilon)(x_1x_2) = \int e^{-it_1\nabla_\varepsilon}(x_1)e^{-it_2\nabla_\varepsilon}(x_2)g(t_1,t_2)\,dt_1\,dt_2.$$

We can then use the Duhamel formula to write the following equalities under the trace φ_0:

$$\begin{aligned}
&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L(\nabla_\varepsilon,\nabla_\varepsilon)(x_1x_2)\\
&= \int \Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e^{-it_1\nabla_\varepsilon}(x_1)\Big)e^{-it_2\nabla_\varepsilon}(x_2)g(t_1,t_2)\,dt_1\,dt_2\\
&\quad+\int e^{-it_1\nabla_\varepsilon}(x_1)\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e^{-it_2\nabla_\varepsilon}(x_2)\Big)g(t_1,t_2)\,dt_1\,dt_2\\
&= \int\int_0^1 it_1\sigma_{ut_1}\operatorname{ad}_a\sigma_{(1-u)t_1}(x_1)\,\sigma_{t_2}(x_2)g(t_1,t_2)\,du\,dt_1\,dt_2\\
&\quad+\int\int_0^1 it_2\sigma_{t_1}(x_1)\sigma_{ut_2}\operatorname{ad}_a\sigma_{(1-u)t_2}(x_2)g(t_1,t_2)\,du\,dt_1\,dt_2\\
&= \int_0^1\int it_1\sigma_{ut_1}(a)\sigma_{t_1}(x_1)\sigma_{t_2}(x_2)g(t_1,t_2)\,dt_1\,dt_2\,du\\
&\quad-\int_0^1\int it_1\sigma_{t_1}(x_1)\sigma_{ut_1}(a)\sigma_{t_2}(x_2)g(t_1,t_2)\,dt_1\,dt_2\,du\\
&\quad+\int_0^1\int it_2\sigma_{t_1}(x_1)\sigma_{ut_2}(a)\sigma_{t_2}(x_2)g(t_1,t_2)\,dt_1\,dt_2\,du\\
&\quad-\int_0^1\int it_2\sigma_{t_1}(x_1)\sigma_{t_2}(x_2)\sigma_{ut_2}(a)g(t_1,t_2)\,dt_1\,dt_2\,du\\
&= L_1(\nabla,\nabla,\nabla)(ax_1x_2)+L_2(\nabla,\nabla,\nabla)(x_1ax_2)\\
&\quad+L_3(\nabla,\nabla,\nabla)(x_1ax_2)+L_4(\nabla,\nabla,\nabla)(x_1x_2a),
\end{aligned}$$

where

$$\begin{aligned}
L_1(s_1,s_2,s_3) &= \int_0^1\int it_1e^{-iut_1s_1-it_1s_2-it_2s_3}g(t_1,t_2)\,dt_1\,dt_2\,du\\
&= -\frac{L(s_1+s_2,s_3)-L(s_2,s_3)}{s_1}.
\end{aligned}$$

With similar calculations we have

$$\begin{aligned}
L_2(s_1,s_2,s_3) &= \frac{L(s_1+s_2,s_3)-L(s_1,s_3)}{s_2},\\
L_3(s_1,s_2,s_3) &= -\frac{L(s_1,s_2+s_3)-L(s_1,s_3)}{s_2},\\
L_4(s_1,s_2,s_3) &= \frac{L(s_1,s_2+s_3)-L(s_1,s_2)}{s_3}.
\end{aligned}$$

The statement of the lemma then follows if one uses the trace property of φ_0 to move the element a cyclically to the very left, and then uses the modular automorphism. □

We also need a version of the above lemmas for the case when L is a function of three variables, which can be proved in a similar way.

Lemma 5.4. *Let $L(s_1, s_2, s_3)$ be a smooth function and let x_1, x_2, x_3, x_4 be elements in $C(\mathbb{T}_\theta^2)$. Under the trace φ_0, we have*

$$\begin{aligned} e^h\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L(\nabla_\varepsilon, \nabla_\varepsilon, \nabla_\varepsilon)(x_1 \cdot x_2 \cdot x_3)\Big)x_4 &\\ = ae^h L^\varepsilon_{3,1}(\nabla,\nabla,\nabla,\nabla)(x_1\cdot x_2\cdot x_3\cdot x_4)&\\ + ae^h L^\varepsilon_{3,2}(\nabla,\nabla,\nabla,\nabla)(x_2\cdot x_3\cdot x_4\cdot x_1)&\\ + ae^h L^\varepsilon_{3,3}(\nabla,\nabla,\nabla,\nabla)(x_3\cdot x_4\cdot x_1\cdot x_2)&\\ + ae^h L^\varepsilon_{3,4}(\nabla,\nabla,\nabla,\nabla)(x_4\cdot x_1\cdot x_2\cdot x_3),& \end{aligned}$$

where

$$L^\varepsilon_{3,1}(s_1,s_2,s_3,s_4) := e^{s_1+s_2+s_3+s_4}\frac{L(-s_2-s_3-s_4,s_2,s_3)-L(s_1,s_2,s_3)}{s_1+s_2+s_3+s_4},$$

$$L^\varepsilon_{3,2}(s_1,s_2,s_3,s_4) := \frac{L(s_4,-s_2-s_3-s_4,s_2)-L(-s_1-s_2-s_3,s_1,s_2)}{e^{-s_1-s_2-s_3}(s_1+s_2+s_3+s_4)},$$

$$L^\varepsilon_{3,3}(s_1,s_2,s_3,s_4) := \frac{L(s_3,s_4,-s_2-s_3-s_4)-L(s_3,-s_1-s_2-s_3,s_1)}{e^{-s_1-s_2}(s_1+s_2+s_3+s_4)},$$

$$L^\varepsilon_{3,4}(s_1,s_2,s_3,s_4) := e^{s_1}\frac{L(s_2,s_3,s_4)-L(s_2,s_3,-s_1-s_2-s_3)}{s_1+s_2+s_3+s_4}.$$

It should be noted that the reason for considering the particular expressions in the statement of the above lemmas is that we wish to prepare the ground for making the comparison between the gradients calculated in the beginning of this section and in Section 6, and derive the functional relations presented in Section 3.

5.5. **Derivatives $\boldsymbol{\delta_j}$ of functional calculi with $\boldsymbol{\nabla}$.** For our purposes, given a function of the modular automorphism acting on an element of $C(\mathbb{T}_\theta^2)$, it is important to have an explicit formula for the derivative δ_j, $j = 1, 2$, of such a term. In this subsection we work out the necessary explicit formulas. The one variable case, which is given in the following lemma, was also proved in [14] and it is given here for the sake of explaining the idea in the simplest case. We present the cases when two and three variable functions are involved in Lemma 5.7 and Lemma 5.8.

Lemma 5.6. *Let $L(s_1)$ be a smooth function and let x_1 be an element of the algebra $C(\mathbb{T}_\theta^2)$ of the noncommutative torus. We have*

$$\begin{aligned}&\delta_j(L(\nabla)(x_1))\\ &\quad = L(\nabla)(\delta_j(x_1)) + L^\delta_{1,1}(\nabla,\nabla)(\delta_j(h)\cdot x_1) + L^\delta_{1,2}(\nabla,\nabla)(x_1\cdot\delta_j(h)),\end{aligned}$$

where

$$L^\delta_{1,1}(s_1,s_2) := \frac{L(s_2)-L(s_1+s_2)}{s_1},$$

$$L^\delta_{1,2}(s_1,s_2) := \frac{L(s_1+s_2)-L(s_1)}{s_2}.$$

Proof. One can start by writing

$$\begin{aligned}\delta_j\sigma_t(x_1) - \sigma_t\delta_j(x_1) &= -\int_0^1 e^{-iut\nabla}\,[\delta_j, it\nabla]\,e^{-i(1-u)t\nabla}(x_1)\,du\\ &= it\int_0^1 \sigma_{ut}\,\mathrm{ad}_{\delta_j(h)}\,\sigma_{(1-u)t}(x_1)\,du.\end{aligned}$$

Therefore, after writing $L(s_1) = \int e^{-its_1}g(t)\,dt$, one gets

$$\begin{aligned}&\delta_j(L(\nabla)(x_1)) - L(\nabla)(\delta_j(x_1))\\ &\quad= \int_0^1\int it\sigma_{ut}(\delta_j(h))\,\sigma_t(x_1)\,g(t)\,dt\,du - \int_0^1\int it\,\sigma_t(x_1)\,\sigma_{ut}(\delta_j(h))\,g(t)\,dt\,du\\ &\quad= L^\delta_{1,1}(\nabla,\nabla)(\delta_j(h)\cdot x_1) + L^\delta_{1,2}(\nabla,\nabla)(x_1\cdot\delta_j(h)),\end{aligned}$$

where

$$\begin{aligned}L^\delta_{1,1}(s_1,s_2) &= \int_0^1\int ite^{-iuts_1-itS_2}g(t)\,dt\,du = \frac{L(s_2)-L(s_1+s_2)}{s_1},\\ L^\delta_{1,2}(s_1,s_2) &= -\int_0^1\int ite^{-its_1-iutS_2}g(t)\,dt\,du = \frac{L(s_1+s_2)-L(s_1)}{s_2}.\end{aligned}$$ □

Now we prove a generalization of the above lemma for the case when the function L depends smoothly on two variables.

Lemma 5.7. *Let $L(s_1,s_2)$ be a smooth function and let x_1,x_2 be elements of $C(\mathbb{T}^2_\theta)$. We have*

$$\begin{aligned}&\delta_j(L(\nabla,\nabla)(x_1\cdot x_2))\\ &\quad= L(\nabla,\nabla)(\delta_j(x_1)\cdot x_2) + L(\nabla,\nabla)(x_1\cdot\delta_j(x_2))\\ &\qquad+ L^\delta_{2,1}(\nabla,\nabla,\nabla)(\delta_j(h)\cdot x_1\cdot x_2) + L^\delta_{2,2}(\nabla,\nabla,\nabla)(x_1\cdot\delta_j(h)\cdot x_2)\\ &\qquad+ L^\delta_{2,3}(\nabla,\nabla,\nabla)(x_1\cdot x_2\cdot\delta_j(h)),\end{aligned}$$

where

$$\begin{aligned}L^\delta_{2,1}(s_1,s_2,s_3) &:= \frac{L(s_2,s_3)-L(s_1+s_2,s_3)}{s_1},\\ L^\delta_{2,2}(s_1,s_2,s_3) &:= \frac{L(s_1+s_2,s_3)-L(s_1,s_2+s_3)}{s_2},\\ L^\delta_{2,3}(s_1,s_2,s_3) &:= \frac{L(s_1,s_2+s_3)-L(s_1,s_2)}{s_3}.\end{aligned}$$

Proof. We start by considering the identity

$$\begin{aligned}&\delta_j(\sigma_{t_1}(x_1)\sigma_{t_2}(x_2)) - \sigma_{t_1}(\delta_j(x_1))\sigma_{t_2}(x_2) - \sigma_{t_1}(x_1)\sigma_{t_2}(\delta_j(x_2))\\ &\quad= [\delta_j,\sigma_{t_1}](x_1)\,\sigma_{t_2}(x_2) + \sigma_{t_1}(x_1)\,[\delta_j,\sigma_{t_2}](x_2).\end{aligned}$$

Therefore, by writing

$$L(s_1,s_2) = \int e^{-it_1s_1-it_2s_2}g(t_1,t_2)\,dt_1\,dt_2,$$

we have

$$
\begin{aligned}
&\delta_j L(\nabla,\nabla)(x_1x_2) - L(\nabla,\nabla)(\delta_j(x_1)x_2) - L(\nabla,\nabla)(x_1\delta_j(x_2))\\
&\quad = \int it_1 \int_0^1 \sigma_{ut_1}\, \mathrm{ad}_{\delta_j(h)}\, \sigma_{(1-u)t_1}(x_1)\, du\, \sigma_{t_2}(x_2)\, g(t_1,t_2)\, dt_1\, dt_2\\
&\qquad + \int \sigma_{t_1}(x_1) it_2 \int_0^1 \sigma_{ut_2}\, \mathrm{ad}_{\delta_j(h)}\, \sigma_{(1-u)t_2}(x_2)\, du\, g(t_1,t_2)\, dt_1\, dt_2\\
&\quad = \int_0^1 \int it_1 \sigma_{ut_1}(\delta_j(h))\sigma_{t_1}(x_1)\sigma_{t_2}(x_2) g(t_1,t_2)\, dt_1\, dt_2\, du\\
&\qquad - \int_0^1 \int it_1 \sigma_{t_1}(x_1)\sigma_{ut_1}(\delta_j(h))\sigma_{t_2}(x_2) g(t_1,t_2)\, dt_1\, dt_2\, du\\
&\qquad + \int_0^1 \int it_2 \sigma_{t_1}(x_1)\sigma_{ut_2}(\delta_j(h))\sigma_{t_2}(x_2) g(t_1,t_2)\, dt_1\, dt_2\, du\\
&\qquad - \int_0^1 \int it_2 \sigma_{t_1}(x_1)\sigma_{t_2}(x_2)\sigma_{ut_2}(\delta_j(h)) g(t_1,t_2)\, dt_1\, dt_2\, du\\
&\quad = L^{\delta}_{2,1}(\nabla,\nabla,\nabla)(\delta_j(h)\cdot x_1\cdot x_2) + L^{\delta}_{2,2}(\nabla,\nabla,\nabla)(x_1\cdot \delta_j(h)\cdot x_2)\\
&\qquad + L^{\delta}_{2,3}(\nabla,\nabla,\nabla)(x_1\cdot x_2\cdot \delta_j(h)),
\end{aligned}
$$

where

$$
\begin{aligned}
L^{\delta}_{2,1}(s_1,s_2,s_3) &= \int_0^1 \int it_1 e^{-iut_1s_1 - it_1s_2 - it_2s_3} g(t_1,t_2)\, dt_1\, dt_2\, du\\
&= \frac{L(s_2,s_3) - L(s_1+s_2,s_3)}{s_1},\\
L^{\delta}_{2,2}(s_1,s_2,s_3) &= -\int_0^1 \int it_1 e^{-it_1s_1 - iut_1s_2 - it_2s_3} g(t_1,t_2)\, dt_1\, dt_2\, du\\
&\quad + \int_0^1 \int it_2 e^{-it_1s_1 - iut_2s_2 - it_2s_3} g(t_1,t_2)\, dt_1\, dt_2\, du\\
&= \frac{L(s_1+s_2,s_3) - L(s_1,s_3)}{s_2} - \frac{L(s_1,s_2+s_3) - L(s_1,s_3)}{s_2}\\
&= \frac{L(s_1+s_2,s_3) - L(s_1,s_2+s_3)}{s_2},\\
L^{\delta}_{2,3}(s_1,s_2,s_3) &= -\int_0^1 \int it_2 e^{-it_1s_1 - it_2s_2 - iut_2s_3} g(t_1,t_2)\, dt_1\, dt_2\, du\\
&= \frac{L(s_1,s_2+s_3) - L(s_1,s_2)}{s_3}. \qquad \square
\end{aligned}
$$

Finally, we treat the case when a smooth three variable function is involved, and find an explicit formula for the derivative δ_j, $j=1,2$, of a general associated element of the noncommutative torus.

Lemma 5.8. *Let $L(s_1,s_2,s_3)$ be a smooth function and let x_1,x_2,x_3 be elements of $C(\mathbb{T}_\theta^2)$. We have*

$$\begin{aligned}
&\delta_j(L(\nabla,\nabla,\nabla)(x_1\cdot x_2\cdot x_3))\\
&\quad= L(\nabla,\nabla,\nabla)(\delta_j(x_1)\cdot x_2\cdot x_3)+L(\nabla,\nabla,\nabla)(x_1\cdot\delta_j(x_2)\cdot x_3)\\
&\qquad+L(\nabla,\nabla,\nabla)(x_1\cdot x_2\cdot\delta_j(x_3))+L^{\delta}_{3,1}(\nabla,\nabla,\nabla,\nabla)(\delta_j(h)\cdot x_1\cdot x_2\cdot x_3)\\
&\qquad+L^{\delta}_{3,2}(\nabla,\nabla,\nabla,\nabla)(x_1\cdot\delta_j(h)\cdot x_2\cdot x_3)\\
&\qquad+L^{\delta}_{3,3}(\nabla,\nabla,\nabla,\nabla)(x_1\cdot x_2\cdot\delta_j(h)\cdot x_3)\\
&\qquad+L^{\delta}_{3,4}(\nabla,\nabla,\nabla,\nabla)(x_1\cdot x_2\cdot x_3\cdot\delta_j(h)),
\end{aligned}$$

where

$$\begin{aligned}
L^{\delta}_{3,1}(s_1,s_2,s_3,s_4)&:=\frac{L(s_2,s_3,s_4)-L(s_1+s_2,s_3,s_4)}{s_1},\\
L^{\delta}_{3,2}(s_1,s_2,s_3,s_4)&:=\frac{L(s_1+s_2,s_3,s_4)-L(s_1,s_2+s_3,s_4)}{s_2},\\
L^{\delta}_{3,3}(s_1,s_2,s_3,s_4)&:=\frac{L(s_1,s_2+s_3,s_4)-L(s_1,s_2,s_3+s_4)}{s_3},\\
L^{\delta}_{3,4}(s_1,s_2,s_3,s_4)&:=\frac{L(s_1,s_2,s_3+s_4)-L(s_1,s_2,s_3)}{s_4}.
\end{aligned}$$

Proof. One can prove this lemma by starting with the following identity and by continuing as in the proof of Lemma 5.7:

$$\begin{aligned}
&\delta_j(\sigma_{t_1}(x_1)\sigma_{t_2}(x_2)\sigma_{t_3}(x_3))-\sigma_{t_1}(\delta_j(x_1))\sigma_{t_2}(x_2)\sigma_{t_3}(x_3)\\
&\qquad-\sigma_{t_1}(x_1)\sigma_{t_2}(\delta_j(x_2))\sigma_{t_3}(x_3)-\sigma_{t_1}(x_1)\sigma_{t_2}(x_2)\sigma_{t_3}(\delta_j(x_3))\\
&\quad=[\delta_j,\sigma_{t_1}](x_1)\,\sigma_{t_2}(x_2)\sigma_{t_3}(x_3)+\sigma_{t_1}(x_1)\,[\delta_j,\sigma_{t_2}](x_2)\sigma_{t_3}(x_3)\\
&\qquad+\sigma_{t_1}(x_1)\,\sigma_{t_2}(x_2)[\delta_j,\sigma_{t_3}](x_3).
\end{aligned}$$

□

It is interesting that in the explicit formulas derived in this subsection, certain finite differences of an original function determine the final formulas.

6. Calculation of the gradient of $h\mapsto\varphi_0(a_4)$ in terms of finite differences

In the beginning of Section 5, we explained how, for selfadjoint elements $h,a\in C^\infty(\mathbb{T}_\theta^2)$, the following gradient can be calculated by using an important identity proved in [14]:

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\,\varphi_0(a_4(h+\varepsilon a)).$$

In this section, we demonstrate a second way of calculating the above gradient, which is based on using the Duhamel formula and the lemmas proved in the subsections of Section 5. This method gives rise to expressions that involve finite differences, and by comparing the final outcome with the first formula (52) derived for the above gradient, we find the functional relations presented in Theorem 3.1 in Section 3. Then we confirm the accuracy of the lengthy formulas presented in Section 9 and in the appendices for the functions $K_1, \dots, K_{20}$ appearing in formula (6) for the a_4, by checking that they satisfy the expected functional relations. Before starting the second method of calculating the desired gradient, we need some lemmas.

6.1. **Cyclic permutations and functional calculi with $\boldsymbol{\nabla}$.** The first type of lemmas that we will need concern exploiting the trace and invariance property of φ_0 to prepare the outcome of the second calculation of the mentioned gradient for comparison with the first formula given by (52).

Lemma 6.2. *Let K be a smooth function of n variables and let $x_1, \dots, x_n$ belong to the noncommutative torus $C(\mathbb{T}^2_\theta)$. If $n = 1$, under the trace φ_0 we have*

$$K(\nabla)(x_1) = K(0)\, x_1,$$

and if $n > 1$, under φ_0 we have

$$K(\nabla, \dots, \nabla)(x_1 \cdots x_n) = L(\nabla, \dots, \nabla)(x_1 \cdots x_{n-1})\, x_n,$$

where

$$L(s_1, \dots, s_{n-1}) := K(s_1, \dots, s_{n-1}, -s_1 - \cdots - s_{n-1}).$$

Proof. Writing $K(s_1, \dots, s_n) = \int e^{-it_1 s_1 - \cdots - it_n s_n} g(t_1, \dots, t_n)\, dt_1 \cdots dt_n$, we have

$$K(\nabla, \dots, \nabla)(x_1 \cdots x_n) = \int \sigma_{t_1}(x_1) \cdots \sigma_{t_n}(x_n) g(t_1, \dots, t_n)\, dt_1 \cdots dt_n.$$

Under the trace φ_0, the latter is equal to

$$\begin{aligned}\int \sigma_{t_1 - t_n}(x_1) \cdots \sigma_{t_{n-1} - t_n}(x_{n-1}) g(t_1, \dots, t_n)\, dt_1 \cdots dt_n \cdot x_n \\ = L(\nabla, \dots, \nabla)(x_1 \cdots x_{n-1})\, x_n,\end{aligned}$$

where

$$\begin{aligned}L(s_1, \dots, s_{n-1}) &= \int e^{-i(t_1 - t_n)s_1 - \cdots - i(t_{n-1} - t_n)s_n} g(t_1, \dots, t_n)\, dt_1 \cdots dt_n \\ &= K(s_1, \dots, s_{n-1}, -s_1 - \cdots - s_{n-1}).\end{aligned}$$

□

The following lemma allows us to cyclically permute elements of $C(\mathbb{T}^2_\theta)$ by varying appropriately the function of the modular automorphism that is involved in each calculation.

Lemma 6.3. *Let L be a smooth function of $n-1$ variables and let $x_1, \dots, x_n$ belong to $C(\mathbb{T}^2_\theta)$. For any $1 \le j \le n-1$, under the trace φ_0 we have*

$$L(\nabla, \dots, \nabla)(x_1 \cdots x_{n-1})\, x_n = x_j\, L^c_j(\nabla, \dots, \nabla)(x_{j+1} \cdots x_n \cdot x_1 \cdots x_{j-1}),$$

where

$$L^c_j(s_1, \dots, s_{n-1}) := L(s_{j+2}, \dots, s_{n-1}, -s_1 - s_2 - \cdots - s_{n-1}, s_1, s_2, \dots, s_j).$$

Proof. Writing

$$L(s_1,\ldots,s_{n-1}) = \int e^{-it_1s_1-\cdots-it_{n-1}s_{n-1}} g(t_1,\ldots,t_{n-1})\,dt_1\cdots dt_{n-1},$$

and using the trace property and the invariance of φ_0, under the latter we can write

$$\begin{aligned}
&L(\nabla,\ldots,\nabla)(x_1\cdots x_{n-1})\,x_n\\
&\quad= \int \sigma_{t_1}(x_1)\cdots\sigma_{t_j}(x_j)\cdots\sigma_{t_{n-1}}(x_{n-1})\,x_n\,g(t_1,\ldots,t_{n-1})\,dt_1\cdots dt_j\cdots dt_{n-1}\\
&\quad= \int \sigma_{t_j}(x_j)\cdots\sigma_{t_{n-1}}(x_{n-1})x_n\sigma_{t_1}(x_1)\cdots\sigma_{t_{j-1}}(x_{j-1})g(t_1,\ldots,t_{n-1})dt_1\cdots dt_{n-1}\\
&\quad= x_j\int \sigma_{t_{j+1}-t_j}(x_{j+1})\cdots\sigma_{t_{n-1}-t_j}(x_{n-1})\,\sigma_{-t_j}(x_n)\,\sigma_{t_1-t_j}(x_1)\cdots\sigma_{t_{j-1}-t_j}(x_{j-1})\\
&\qquad\qquad g(t_1,\ldots,t_{n-1})\,dt_1\cdots dt_{n-1}\\
&\quad= x_j\,L_j^c(\nabla,\ldots,\nabla)(x_{j+1}\cdots x_n\cdot x_1\cdots x_{j-1}),
\end{aligned}$$

where

$$\begin{aligned}
&L_j^c(s_1,\ldots,s_{n-1})\\
&\quad= \int e^{-i(t_{j+1}-t_j)s_1-\cdots-i(t_{n-1}-t_j)s_{n-1-j}+it_js_{n-j}-i(t_1-t_j)s_{n-j+1}-\cdots-i(t_{j-1}-t_j)s_{n-1}}\\
&\qquad\qquad g(t_1,\ldots,t_{n-1})\,dt_1\cdots dt_{n-1}\\
&\quad= L(s_{j+2},\ldots,s_{n-1},-s_1-s_2-\cdots-s_{n-1},s_1,s_2,\ldots,s_j),
\end{aligned}$$

as desired. □

6.4. **Finite differences of $G_1(\nabla)$ relating derivatives of h and e^h.** The second type of lemmas that we shall need in the remainder of this section concerns writing the derivatives up to order 4 of the conformal factor $e^h \in C^\infty(\mathbb{T}^2_\theta)$ in terms of the derivatives of the dilaton h. In [14, 20], for the purpose of writing the final formula for the scalar curvature in terms of derivatives of h, the functions $G_1(s_1)$ and $G_2(s_1,s_2)$ were found such that for $j_1,j_2\in\{1,2\}$:

$$\begin{aligned}
e^{-h}\delta_{j_1}(e^h) &= G_1(\nabla)(\delta_{j_1}(h)),\\
e^{-h}\delta_{j_1}\delta_{j_2}(e^h) &= G_1(\nabla)(\delta_{j_1}\delta_{j_2}(h)) + G_2(\nabla,\nabla)(\delta_{j_1}(h)\cdot\delta_{j_2}(h)+\delta_{j_2}(h)\cdot\delta_{j_1}(h)).
\end{aligned}$$

For our purposes in this paper, in the sequel we will need to treat terms of the form $e^{-h}\delta_{j_1}\delta_{j_2}\delta_{j_3}(e^h)$ and of the form $e^{-h}\delta_{j_1}\delta_{j_2}\delta_{j_3}\delta_{j_4}(e^h)$. This is done in the following lemmas, where we first give a general form of these expressions in terms of derivatives of h and will then describe the functions G_1,G_2,G_3,G_4 in Lemma 6.6 and Lemma 6.7.

Lemma 6.5. *Let j_1,j_2,j_3,j_4 be either equal to 1 or 2. Using the functions G_1,G_2,G_3,G_4 presented in Lemma 6.6 and Lemma 6.7, we can write:*

$$\begin{aligned}
&e^{-h}\delta_{j_1}\delta_{j_2}\delta_{j_3}(e^h)\\
&\quad= G_1(\nabla)(\square_{3,1}(h)) + G_2(\nabla,\nabla)(\square_{3,2}(h)) + G_3(\nabla,\nabla,\nabla)(\square_{3,3}(h)),
\end{aligned}$$

where

$$\begin{aligned}
\square_{3,1}(h) &:= \delta_{j_1}\delta_{j_2}\delta_{j_3}(h),\\
\square_{3,2}(h) &:= \delta_{j_1}(h)\cdot(\delta_{j_2}\delta_{j_3})(h)+\delta_{j_2}(h)\cdot(\delta_{j_1}\delta_{j_3})(h)+(\delta_{j_1}\delta_{j_2})(h)\cdot\delta_{j_3}(h)\\
&\quad+\delta_{j_3}(h)\cdot(\delta_{j_1}\delta_{j_2})(h)+(\delta_{j_1}\delta_{j_3})(h)\cdot\delta_{j_2}(h)+(\delta_{j_2}\delta_{j_3})(h)\cdot\delta_{j_1}(h),\\
\square_{3,3}(h) &:= \delta_{j_1}(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_3}(h)+\delta_{j_1}(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_2}(h)\\
&\quad+\delta_{j_2}(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_3}(h)+\delta_{j_2}(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_1}(h)\\
&\quad+\delta_{j_3}(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_2}(h)+\delta_{j_3}(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_1}(h).
\end{aligned}$$

Moreover, for the case when the order of differentiation is 4, we have

$$\begin{aligned}
e^{-h}\delta_{j_1}\delta_{j_2}\delta_{j_3}\delta_{j_4}(e^h) &= G_1(\nabla)(\square_{4,1}(h))+G_2(\nabla,\nabla)(\square_{4,2}(h))\\
&\quad+G_3(\nabla,\nabla,\nabla)(\square_{4,3}(h))+G_4(\nabla,\nabla,\nabla,\nabla)(\square_{4,4}(h)),
\end{aligned}$$

where

$$\begin{aligned}
\square_{4,1}(h) &:= (\delta_{j_1}\delta_{j_2}\delta_{j_3}\delta_{j_4})(h),\\
\square_{4,2}(h) &:= \delta_{j_1}(h)\cdot(\delta_{j_2}\delta_{j_3}\delta_{j_4})(h)+\delta_{j_2}(h)\cdot(\delta_{j_1}\delta_{j_3}\delta_{j_4})(h)+(\delta_{j_1}\delta_{j_2})(h)\cdot(\delta_{j_3}\delta_{j_4})(h)\\
&\quad+\delta_{j_3}(h)\cdot(\delta_{j_1}\delta_{j_2}\delta_{j_4})(h)+(\delta_{j_1}\delta_{j_3})(h)\cdot(\delta_{j_2}\delta_{j_4})(h)+(\delta_{j_2}\delta_{j_3})(h)\cdot(\delta_{j_1}\delta_{j_4})(h)\\
&\quad+(\delta_{j_1}\delta_{j_2}\delta_{j_3})(h)\cdot\delta_{j_4}(h)+\delta_{j_4}(h)\cdot(\delta_{j_1}\delta_{j_2}\delta_{j_3})(h)+(\delta_{j_1}\delta_{j_4})(h)\cdot(\delta_{j_2}\delta_{j_3})(h)\\
&\quad+(\delta_{j_2}\delta_{j_4})(h)\cdot(\delta_{j_1}\delta_{j_3})(h)+(\delta_{j_1}\delta_{j_2}\delta_{j_4})(h)\cdot\delta_{j_3}(h)+(\delta_{j_3}\delta_{j_4})(h)\cdot(\delta_{j_1}\delta_{j_2})(h)\\
&\quad+(\delta_{j_1}\delta_{j_3}\delta_{j_4})(h)\cdot\delta_{j_2}(h)+(\delta_{j_2}\delta_{j_3}\delta_{j_4})(h)\cdot\delta_{j_1}(h),\\
\square_{4,3}(h) &:= \delta_{j_1}(h)\cdot\delta_{j_2}(h)\cdot(\delta_{j_3}\delta_{j_4})(h)+\delta_{j_1}(h)\cdot\delta_{j_3}(h)\cdot(\delta_{j_2}\delta_{j_4})(h)+\delta_{j_1}(h)\cdot(\delta_{j_2}\delta_{j_3})(h)\cdot\delta_{j_4}(h)\\
&\quad+\delta_{j_1}(h)\cdot\delta_{j_4}(h)\cdot(\delta_{j_2}\delta_{j_3})(h)+\delta_{j_1}(h)\cdot(\delta_{j_2}\delta_{j_4})(h)\cdot\delta_{j_3}(h)+\delta_{j_1}(h)\cdot(\delta_{j_3}\delta_{j_4})(h)\cdot\delta_{j_2}(h)\\
&\quad+\delta_{j_2}(h)\cdot\delta_{j_1}(h)\cdot(\delta_{j_3}\delta_{j_4})(h)+\delta_{j_2}(h)\cdot\delta_{j_3}(h)\cdot(\delta_{j_1}\delta_{j_4})(h)+\delta_{j_2}(h)\cdot(\delta_{j_1}\delta_{j_3})(h)\cdot\delta_{j_4}(h)\\
&\quad+\delta_{j_2}(h)\cdot\delta_{j_4}(h)\cdot(\delta_{j_1}\delta_{j_3})(h)+\delta_{j_2}(h)\cdot(\delta_{j_1}\delta_{j_4})(h)\cdot\delta_{j_3}(h)+\delta_{j_2}(h)\cdot(\delta_{j_3}\delta_{j_4})(h)\cdot\delta_{j_1}(h)\\
&\quad+(\delta_{j_1}\delta_{j_2})(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_4}(h)+(\delta_{j_1}\delta_{j_2})(h)\cdot\delta_{j_4}(h)\cdot\delta_{j_3}(h)+\delta_{j_3}(h)\cdot\delta_{j_1}(h)\cdot(\delta_{j_2}\delta_{j_4})(h)\\
&\quad+\delta_{j_3}(h)\cdot\delta_{j_2}(h)\cdot(\delta_{j_1}\delta_{j_4})(h)+\delta_{j_3}(h)\cdot(\delta_{j_1}\delta_{j_2})(h)\cdot\delta_{j_4}(h)+\delta_{j_3}(h)\cdot\delta_{j_4}(h)\cdot(\delta_{j_1}\delta_{j_2})(h)\\
&\quad+\delta_{j_3}(h)\cdot(\delta_{j_1}\delta_{j_4})(h)\cdot\delta_{j_2}(h)+\delta_{j_3}(h)\cdot(\delta_{j_2}\delta_{j_4})(h)\cdot\delta_{j_1}(h)+(\delta_{j_1}\delta_{j_3})(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_4}(h)\\
&\quad+(\delta_{j_1}\delta_{j_3})(h)\cdot\delta_{j_4}(h)\cdot\delta_{j_2}(h)+(\delta_{j_2}\delta_{j_3})(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_4}(h)+(\delta_{j_2}\delta_{j_3})(h)\cdot\delta_{j_4}(h)\cdot\delta_{j_1}(h)\\
&\quad+\delta_{j_4}(h)\cdot\delta_{j_1}(h)\cdot(\delta_{j_2}\delta_{j_3})(h)+\delta_{j_4}(h)\cdot\delta_{j_2}(h)\cdot(\delta_{j_1}\delta_{j_3})(h)+\delta_{j_4}(h)\cdot(\delta_{j_1}\delta_{j_2})(h)\cdot\delta_{j_3}(h)\\
&\quad+\delta_{j_4}(h)\cdot\delta_{j_3}(h)\cdot(\delta_{j_1}\delta_{j_2})(h)+\delta_{j_4}(h)\cdot(\delta_{j_1}\delta_{j_3})(h)\cdot\delta_{j_2}(h)+\delta_{j_4}(h)\cdot(\delta_{j_2}\delta_{j_3})(h)\cdot\delta_{j_1}(h)\\
&\quad+(\delta_{j_1}\delta_{j_4})(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_3}(h)+(\delta_{j_1}\delta_{j_4})(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_2}(h)+(\delta_{j_2}\delta_{j_4})(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_3}(h)\\
&\quad+(\delta_{j_2}\delta_{j_4})(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_1}(h)+(\delta_{j_3}\delta_{j_4})(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_2}(h)+(\delta_{j_3}\delta_{j_4})(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_1}(h),\\
\square_{4,4}(h) &:= \delta_{j_1}(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_4}(h)+\delta_{j_1}(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_4}(h)\cdot\delta_{j_3}(h)\\
&\quad+\delta_{j_1}(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_4}(h)+\delta_{j_1}(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_4}(h)\cdot\delta_{j_2}(h)\\
&\quad+\delta_{j_1}(h)\cdot\delta_{j_4}(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_3}(h)+\delta_{j_1}(h)\cdot\delta_{j_4}(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_2}(h)\\
&\quad+\delta_{j_2}(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_4}(h)+\delta_{j_2}(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_4}(h)\cdot\delta_{j_3}(h)\\
&\quad+\delta_{j_2}(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_4}(h)+\delta_{j_2}(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_4}(h)\cdot\delta_{j_1}(h)\\
&\quad+\delta_{j_2}(h)\cdot\delta_{j_4}(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_3}(h)+\delta_{j_2}(h)\cdot\delta_{j_4}(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_1}(h)\\
&\quad+\delta_{j_3}(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_4}(h)+\delta_{j_3}(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_4}(h)\cdot\delta_{j_2}(h)\\
&\quad+\delta_{j_3}(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_4}(h)+\delta_{j_3}(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_4}(h)\cdot\delta_{j_1}(h)\\
&\quad+\delta_{j_3}(h)\cdot\delta_{j_4}(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_2}(h)+\delta_{j_3}(h)\cdot\delta_{j_4}(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_1}(h)\\
&\quad+\delta_{j_4}(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_3}(h)+\delta_{j_4}(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_2}(h)\\
&\quad+\delta_{j_4}(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_3}(h)+\delta_{j_4}(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_1}(h)\\
&\quad+\delta_{j_4}(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_1}(h)\cdot\delta_{j_2}(h)+\delta_{j_4}(h)\cdot\delta_{j_3}(h)\cdot\delta_{j_2}(h)\cdot\delta_{j_1}(h).
\end{aligned}$$

The above lemma can be proved by writing an expansional formula as in Section 6.1 of [14] (or the method used in [15, 18]). This method will also show that the functions G_1, G_2, G_3, G_4 can be constructed as follows.

Lemma 6.6. *The functions G_1, G_2, G_3, G_4 can be constructed recursively by setting*

$$G_0 = 1,$$

and by writing

$$G_n(s_1, \ldots, s_n) = \int_0^1 r^{n-1} e^{s_1 r}\, G_{n-1}(rs_2, rs_3, \ldots, rs_n)\, dr.$$

Therefore, we have

$$G_1(s_1) = \frac{e^{s_1} - 1}{s_1}$$

and

$$\begin{aligned} &G_n(s_1, \ldots, s_n) \\ &\quad = \int_0^1 \int_0^1 \cdots \int_0^1 r_1^{n-1} r_2^{n-2} \cdots r_{n-1}\, e^{s_1 r_1 + s_2 r_1 r_2 + \cdots + s_{n-1} r_1 r_2 \cdots r_{n-1}} \\ &\qquad \times G_1(r_1 r_2 \cdots r_{n-1} s_n)\, dr_1\, dr_2 \cdots dr_{n-1} \\ &\quad = \int_0^1 \int_0^1 \cdots \int_0^1 \prod_{j=1}^{n-1} (r_j^{n-j}\, e^{s_j \prod_{i=1}^{j} r_i}) G_1(r_1 r_2 \cdots r_{n-1} s_n)\, dr_1\, dr_2 \cdots dr_{n-1}. \end{aligned}$$

Now, we can explain how the functions G_2, G_3, G_4 can be obtained by finite differences of the function G_1.

Lemma 6.7. *Starting from the function*

$$G_1(s_1) = \frac{e^{s_1} - 1}{s_1},$$

we have

$$\begin{aligned} &G_2(s_1, s_2) = \frac{1}{s_2}(G_1(s_1 + s_2) - G_1(s_1)), \\ &G_3(s_1, s_2, s_3) = \frac{1}{s_3}\Big(\frac{G_1(s_1 + s_2 + s_3) - G_1(s_1)}{s_2 + s_3} - \frac{G_1(s_1 + s_2) - G_1(s_1)}{s_2}\Big), \\ &G_4(s_1, s_2, s_3, s_4) \\ &\quad = \frac{1}{s_4}\frac{1}{s_3 + s_4}\Big(\frac{G_1(s_1 + s_2 + s_3 + s_4) - G_1(s_1)}{s_2 + s_3 + s_4} - \frac{G_1(s_1 + s_2) - G_1(s_1)}{s_2}\Big) \\ &\qquad - \frac{1}{s_4}\frac{1}{s_3}\Big(\frac{G_1(s_1 + s_2 + s_3) - G_1(s_1)}{s_2 + s_3} - \frac{G_1(s_1 + s_2) - G_1(s_1)}{s_2}\Big). \end{aligned}$$

We recall that we provided the explicit formulas for the functions G_1, G_2, G_3, G_4 by (9) since they play an important role in the functional relations stated in Theorem 3.1 in Section 3.

Now we are absolutely ready to start the second calculation of the gradient

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \varphi_0(a_4(h + \varepsilon a)),$$

where $h, a \in C^\infty(\mathbb{T}_\theta^2)$ are selfadjoint elements, by computing the gradient corresponding to each term appearing in expression (6) for the term a_4. In fact, the terms that have functions of the modular automorphism with the same number of variables involved give rise to, although lengthy and different, but somewhat similar calculations. Therefore, we have chosen four different terms from (6) that respectively involve one, two, three and four variable functions, and will demonstrate the gradient calculation for each individual case in the following subsections.

6.8. **Gradient of $\frac{1}{2}\varphi_0(e^h K_1(\nabla)(\delta_1^2\delta_2^2(h)))$.** In order to compute

$$\frac{d}{d\varepsilon}\,\Big|_{\varepsilon=0}\,\varphi_0(e^{h+\varepsilon a}K_1(\nabla_\varepsilon)(\delta_1^2\delta_2^2(h+\varepsilon a)),$$

first we use Lemma 6.2 to write

$$\varphi_0(e^h K_1(\nabla)(\delta_1^2\delta_2^2(h))) = \frac{-2\pi}{15}\,\varphi_0(e^h\delta_1^2\delta_2^2(h)).$$

Therefore, it suffices to calculate

$$\frac{d}{d\varepsilon}\,\Big|_{\varepsilon=0}\,\Omega(h+\varepsilon a),$$

where

$$\Omega(h) = \varphi_0(e^h\delta_1^2\delta_2^2(h)).$$

Since

$$\Omega(h+\varepsilon a) = \varphi_0(e^{h+\varepsilon a}\delta_1^2\delta_2^2(h)) + \varepsilon\varphi_0(e^{h+\varepsilon a}\delta_1^2\delta_2^2(a)),$$

under the trace φ_0 we have

$$\begin{aligned}\frac{d}{d\varepsilon}\,\Big|_{\varepsilon=0}\,\Omega(h+\varepsilon a) &= \frac{d}{d\varepsilon}\,\Big|_{\varepsilon=0}\,e^{h+\varepsilon a}\,\delta_1^2\delta_2^2(h) + e^h\delta_1^2\delta_2^2(a)\\ &= \frac{1-e^{-\nabla}}{\nabla}(a)e^h\delta_1^2\delta_2^2(h) + e^h\delta_1^2\delta_2^2(a)\\ &= ae^h\frac{1-e^{\nabla}}{-\nabla}(\delta_1^2\delta_2^2(h)) + ae^he^{-h}\delta_1^2\delta_2^2(e^h).\end{aligned}$$

In the last expression, the term $e^{-h}\delta_1^2\delta_2^2(e^h)$ can then be expanded using Lemma 6.5.

Putting everything together, up to multiplying the right-hand side of the following equality by ae^h, under the trace φ_0, we have

$$\begin{aligned}&\frac{d}{d\varepsilon}\,\Big|_{\varepsilon=0}\,\frac{1}{2}\,\varphi_0(e^{h+\varepsilon a}K_1(\nabla)(\delta_1^2\delta_2^2(h+\varepsilon a)))\\ &\quad= \Big(\frac{(1-e^{s_1})\pi}{15s_1} - \frac{1}{15}\pi G_1(s_1)\Big)\,\Big|_{s_j=\nabla}\,(\delta_1^2\delta_2^2(h))\\ &\qquad+ \Big(-\frac{1}{15}\pi G_2(s_1,s_2)\Big)\,\Big|_{s_j=\nabla}\,\delta_1^2(h)\delta_2^2(h)\\ &\qquad+ \Big(-\frac{1}{15}\pi G_2(s_1,s_2)\Big)\,\Big|_{s_j=\nabla}\,\delta_2^2(h)\delta_1^2(h)\\ &\qquad+ \Big(\frac{1}{15}(-2)\pi G_2(s_1,s_2)\Big)\,\Big|_{s_j=\nabla}\,\delta_1(h)(\delta_1\delta_2^2(h))\end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{1}{15}(-2)\pi G_2(s_1,s_2)\Big)\Big|_{s_j=\nabla}\ \delta_2(h)(\delta_1^2\delta_2(h))\\
&+\Big(\frac{1}{15}(-2)\pi G_2(s_1,s_2)\Big)\Big|_{s_j=\nabla}\ (\delta_1^2\delta_2(h))\delta_2(h)\\
&+\Big(\frac{1}{15}(-2)\pi G_2(s_1,s_2)\Big)\Big|_{s_j=\nabla}\ (\delta_1\delta_2^2(h))\delta_1(h)\\
&+\Big(\frac{1}{15}(-4)\pi G_2(s_1,s_2)\Big)\Big|_{s_j=\nabla}\ (\delta_1\delta_2(h))(\delta_1\delta_2(h))\\
&+\Big(\frac{1}{15}(-2)\pi G_3(s_1,s_2,s_3)\Big)\Big|_{s_j=\nabla}\ \delta_1(h)\delta_1(h)\delta_2^2(h)\\
&+\Big(\frac{1}{15}(-2)\pi G_3(s_1,s_2,s_3)\Big)\Big|_{s_j=\nabla}\ \delta_1(h)\delta_2^2(h)\delta_1(h)\\
&+\Big(\frac{1}{15}(-2)\pi G_3(s_1,s_2,s_3)\Big)\Big|_{s_j=\nabla}\ \delta_1^2(h)\delta_2(h)\delta_2(h)\\
&+\Big(\frac{1}{15}(-2)\pi G_3(s_1,s_2,s_3)\Big)\Big|_{s_j=\nabla}\ \delta_2(h)\delta_1^2(h)\delta_2(h)\\
&+\Big(\frac{1}{15}(-2)\pi G_3(s_1,s_2,s_3)\Big)\Big|_{s_j=\nabla}\ \delta_2(h)\delta_2(h)\delta_1^2(h)\\
&+\Big(\frac{1}{15}(-2)\pi G_3(s_1,s_2,s_3)\Big)\Big|_{s_j=\nabla}\ \delta_2^2(h)\delta_1(h)\delta_1(h)\\
&+\Big(\frac{1}{15}(-4)\pi G_3(s_1,s_2,s_3)\Big)\Big|_{s_j=\nabla}\ \delta_1(h)\delta_2(h)(\delta_1\delta_2(h))\\
&+\Big(\frac{1}{15}(-4)\pi G_3(s_1,s_2,s_3)\Big)\Big|_{s_j=\nabla}\ \delta_1(h)(\delta_1\delta_2(h))\delta_2(h)\\
&+\Big(\frac{1}{15}(-4)\pi G_3(s_1,s_2,s_3)\Big)\Big|_{s_j=\nabla}\ \delta_2(h)\delta_1(h)(\delta_1\delta_2(h))\\
&+\Big(\frac{1}{15}(-4)\pi G_3(s_1,s_2,s_3)\Big)\Big|_{s_j=\nabla}\ \delta_2(h)(\delta_1\delta_2(h))\delta_1(h)\\
&+\Big(\frac{1}{15}(-4)\pi G_3(s_1,s_2,s_3)\Big)\Big|_{s_j=\nabla}\ (\delta_1\delta_2(h))\delta_1(h)\delta_2(h)\\
&+\Big(\frac{1}{15}(-4)\pi G_3(s_1,s_2,s_3)\Big)\Big|_{s_j=\nabla}\ (\delta_1\delta_2(h))\delta_2(h)\delta_1(h)\\
&+\Big(\frac{1}{15}(-4)\pi G_4(s_1,s_2,s_3,s_4)\Big)\Big|_{s_j=\nabla}\ \delta_1(h)\delta_1(h)\delta_2(h)\delta_2(h)\\
&+\Big(\frac{1}{15}(-4)\pi G_4(s_1,s_2,s_3,s_4)\Big)\Big|_{s_j=\nabla}\ \delta_1(h)\delta_2(h)\delta_1(h)\delta_2(h)\\
&+\Big(\frac{1}{15}(-4)\pi G_4(s_1,s_2,s_3,s_4)\Big)\Big|_{s_j=\nabla}\ \delta_1(h)\delta_2(h)\delta_2(h)\delta_1(h)\\
&+\Big(\frac{1}{15}(-4)\pi G_4(s_1,s_2,s_3,s_4)\Big)\Big|_{s_j=\nabla}\ \delta_2(h)\delta_1(h)\delta_1(h)\delta_2(h)\\
&+\Big(\frac{1}{15}(-4)\pi G_4(s_1,s_2,s_3,s_4)\Big)\Big|_{s_j=\nabla}\ \delta_2(h)\delta_1(h)\delta_2(h)\delta_1(h)\\
&+\Big(\frac{1}{15}(-4)\pi G_4(s_1,s_2,s_3,s_4)\Big)\Big|_{s_j=\nabla}\ \delta_2(h)\delta_2(h)\delta_1(h)\delta_1(h).
\end{aligned}
$$

6.9. **Gradient of $\frac{1}{4}\varphi_0(e^h K_3(\nabla,\nabla)((\delta_1\delta_2(h))\cdot(\delta_1\delta_2(h))))$.** In order to compute

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\,\Omega(h+\varepsilon a),$$

where

$$\Omega(h)=\varphi_0\big(e^h K_3(\nabla,\nabla)((\delta_1\delta_2(h))\cdot(\delta_1\delta_2(h)))\big),$$

first we use Lemma 6.2 to write

$$\Omega(h)=\varphi_0(e^h K(\nabla)(\delta_1\delta_2(h))\delta_1\delta_2(h)),$$

where the function K is defined by

$$K(s_1)=K_3(s_1,-s_1).$$

Therefore, under the trace φ_0,

$$\begin{aligned}\Omega(h+\varepsilon a)&=e^{h+\varepsilon a}K(\nabla_\varepsilon)(\delta_1\delta_2(h))\,\delta_1\delta_2(h)+\varepsilon e^{h+\varepsilon a}K(\nabla_\varepsilon)(\delta_1\delta_2(a))\,\delta_1\delta_2(h)\\&\quad+\varepsilon e^{h+\varepsilon a}K(\nabla_\varepsilon)(\delta_1\delta_2(h))\,\delta_1\delta_2(a)+\varepsilon^2(\cdots).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\,\Omega(h+\varepsilon a)&=\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\,e^{h+\varepsilon a}\cdot K(\nabla)(\delta_1\delta_2(h))\,\delta_1\delta_2(h)\\&\quad+e^h\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\,K(\nabla_\varepsilon)(\delta_1\delta_2(h))\,\delta_1\delta_2(h)\\&\quad+e^hK(\nabla)(\delta_1\delta_2(a))\,\delta_1\delta_2(h)\\&\quad+e^hK(\nabla)(\delta_1\delta_2(h))\,\delta_1\delta_2(a).\end{aligned}\tag{53}$$

The first term in (53), under φ_0, is equal to

$$\begin{aligned}&\frac{1-e^{-\nabla}}{\nabla}(a)\,e^hK(\nabla)(\delta_1\delta_2(h))\,\delta_1\delta_2(h)\\&\quad=ae^h\frac{1-e^{-\nabla}}{\nabla}(K(\nabla)(\delta_1\delta_2(h))\,\delta_1\delta_2(h))\\&\quad=ae^h\Big(\frac{1-e^{s_1+s_2}}{-(s_1+s_2)}K(s_1)\Big)\Big|_{s_1=\nabla,s_2=\nabla}\,(\delta_1\delta_2(h)\,\delta_1\delta_2(h)).\end{aligned}$$

Using Lemma 5.2, for the second term in (53), under φ_0, we have

$$\begin{aligned}&e^h\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\,K(\nabla_\varepsilon)(\delta_1\delta_2(h))\,\delta_1\delta_2(h)\\&\quad=ae^h\Big(e^{s_1+s_2}\frac{K(-s_2)-K(s_1)}{s_1+s_2}\Big)\Big|_{s_1,s_2=\nabla}\,(\delta_1\delta_2(h)\,\delta_1\delta_2(h))\\&\qquad+ae^h\Big(e^{s_1}\frac{K(s_2)-K(-s_1)}{s_1+s_2}\Big)\Big|_{s_1,s_2=\nabla}\,(\delta_1\delta_2(h)\,\delta_1\delta_2(h)).\end{aligned}$$

For the third term in (53), under φ_0, we can write

$$\begin{aligned}
&e^h K(\nabla)(\delta_1\delta_2(a))\,\delta_1\delta_2(h)\\
&\quad= \delta_1\delta_2(a) e^h K(-\nabla) e^{\nabla}(\delta_1\delta_2(h))\\
&\quad= a e^h e^{-h}\delta_1\delta_2(e^h) K_v(\nabla)(\delta_1\delta_2(h)) + a e^h e^{-h}\delta_1(e^h)\delta_2(K_v(\nabla)(\delta_1\delta_2(h)))\\
&\qquad + a e^h e^{-h}\delta_2(e^h)\,\delta_1(K_v(\nabla)(\delta_1\delta_2(h))) + a e^h \delta_1\delta_2(K_v(\nabla)(\delta_1\delta_2(h))),
\end{aligned}$$

where

$$K_v(s_1) = K(-s_1)e^{s_1}.$$

In the above expression, terms of the form $e^{-h}\delta_{j_1}(e^h)$ and $e^{-h}\delta_{j_1}\delta_{j_2}(e^h)$ can then be expanded using Lemma 6.5. Furthermore, we use Lemma 5.6 and Lemma 5.7 to expand the terms of the form

$$\delta_2(K_v(\nabla)(\delta_1\delta_2(h))) \quad \text{and} \quad \delta_1\delta_2(K_v(\nabla)(\delta_1\delta_2(h))).$$

Similarly, for the fourth term in (53), under φ_0, we can write

$$\begin{aligned}
&e^h K(\nabla)(\delta_1\delta_2(h))\,\delta_1\delta_2(a)\\
&\quad= a\delta_1\delta_2(e^h K(\nabla)(\delta_1\delta_2(h)))\\
&\quad= a\, e^h e^{-h}\delta_1\delta_2(e^h) K(\nabla)(\delta_1\delta_2(h)) + a e^h e^{-h}\delta_1(e^h)\delta_2(K(\nabla)(\delta_1\delta_2(h)))\\
&\qquad + a e^h e^{-h}\delta_2(e^h)\,\delta_1(K(\nabla)(\delta_1\delta_2(h))) + a e^h \delta_1\delta_2(K(\nabla)(\delta_1\delta_2(h))).
\end{aligned}$$

In this expression also, terms of the form $e^{-h}\delta_{j_1}(e^h)$ and $e^{-h}\delta_{j_1}\delta_{j_2}(e^h)$ can then be expanded using Lemma 6.5, and the terms of the form $\delta_2(K(\nabla)(\delta_1\delta_2(h)))$ and $\delta_1\delta_2(K(\nabla)(\delta_1\delta_2(h)))$ need to be expanded using Lemmas 5.6 and 5.7.

Finally, putting everything together, up to multiplying the right-hand side of the following equality by ae^h, under the trace φ_0, we have

$$\begin{aligned}
&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \frac{1}{4}\,\varphi_0(e^{h+\varepsilon a} K_3(\nabla_\varepsilon,\nabla_\varepsilon)((\delta_1\delta_2(h+\varepsilon a))\cdot(\delta_1\delta_2(h+\varepsilon a))))\\
&= \Big(\frac{1}{4}e^{s_1}k_3(-s_1) + \frac{1}{4}k_3(s_1)\Big)\Big|_{s_j=\nabla} (\delta_1^2\delta_2^2(h)) + \Big(-\frac{e^{s_1+s_2}k_3(-s_1-s_2)}{4s_1} + \frac{1}{4}e^{s_2}G_1(s_1)k_3(-s_2)\\
&\quad + \frac{e^{s_2}k_3(-s_2)}{4s_1} + \frac{1}{4}G_1(s_1)k_3(s_2) + \frac{k_3(s_2)}{4s_1} - \frac{k_3(s_1+s_2)}{4s_1}\Big)\Big|_{s_j=\nabla} \delta_1(h)(\delta_1\delta_2^2(h))\\
&\quad + \Big(-\frac{e^{s_1+s_2}k_3(-s_1-s_2)}{4s_1} + \frac{1}{4}e^{s_2}G_1(s_1)k_3(-s_2) + \frac{e^{s_2}k_3(-s_2)}{4s_1}\\
&\quad + \frac{1}{4}G_1(s_1)k_3(s_2) + \frac{k_3(s_2)}{4s_1} - \frac{k_3(s_1+s_2)}{4s_1}\Big)\Big|_{s_j=\nabla} \delta_2(h)(\delta_1^2\delta_2(h))\\
&\quad + \Big(-\frac{e^{s_1}k_3(-s_1)}{4s_2} + \frac{e^{s_1+s_2}k_3(-s_1-s_2)}{4s_2} + \frac{k_3(s_1+s_2)}{4s_2} - \frac{k_3(s_1)}{4s_2}\Big)\Big|_{s_j=\nabla} (\delta_1^2\delta_2(h))\delta_2(h)\\
&\quad + \Big(-\frac{e^{s_1}k_3(-s_1)}{4s_2} + \frac{e^{s_1+s_2}k_3(-s_1-s_2)}{4s_2} + \frac{k_3(s_1+s_2)}{4s_2} - \frac{k_3(s_1)}{4s_2}\Big)\Big|_{s_j=\nabla} (\delta_1\delta_2^2(h))\delta_1(h)\\
&\quad + \Big(-\frac{e^{s_1}k_3(-s_1)}{2(s_1+s_2)} - \frac{e^{s_1}s_1k_3(-s_1)}{4s_2(s_1+s_2)} + \frac{e^{s_1+s_2}s_1k_3(-s_1-s_2)}{4s_2(s_1+s_2)} + \frac{e^{s_2}s_1G_1(s_1)k_3(-s_2)}{4(s_1+s_2)}\\
&\quad + \frac{e^{s_2}s_2G_1(s_1)k_3(-s_2)}{4(s_1+s_2)} + \frac{e^{s_2}k_3(-s_2)}{4(s_1+s_2)} + \frac{e^{s_1+s_2}k_3(-s_2)}{4(s_1+s_2)} + \frac{e^{s_2}s_2k_3(-s_2)}{4s_1(s_1+s_2)} + \frac{s_1G_1(s_1)k_3(s_2)}{4(s_1+s_2)}\\
&\quad + \frac{s_2G_1(s_1)k_3(s_2)}{4(s_1+s_2)} + \frac{e^{s_1}k_3(s_2)}{4(s_1+s_2)} + \frac{k_3(s_2)}{4(s_1+s_2)} + \frac{s_2k_3(s_2)}{4s_1(s_1+s_2)} + \frac{s_1k_3(s_1+s_2)}{4s_2(s_1+s_2)} - \frac{k_3(s_1)}{2(s_1+s_2)}\\
&\quad - \frac{e^{s_1+s_2}s_2k_3(-s_1-s_2)}{4s_1(s_1+s_2)} - \frac{s_2k_3(s_1+s_2)}{4s_1(s_1+s_2)} - \frac{s_1k_3(s_1)}{4s_2(s_1+s_2)}\Big)\Big|_{s_j=\nabla} (\delta_1\delta_2(h))(\delta_1\delta_2(h))
\end{aligned}$$

$$
\begin{aligned}
&+\Big(-\frac{e^{s_2+s_3}G_1(s_1)k_3(-s_2-s_3)}{4(s_1+s_2)}-\frac{e^{s_2+s_3}k_3(-s_2-s_3)}{4s_1(s_1+s_2)}-\frac{e^{s_2+s_3}k_3(-s_2-s_3)}{4s_2(s_1+s_2)}\\
&-\frac{e^{s_2+s_3}s_1G_1(s_1)k_3(-s_2-s_3)}{4s_2(s_1+s_2)}+\frac{e^{s_1+s_2+s_3}k_3(-s_1-s_2-s_3)}{4s_1(s_1+s_2)}+\frac{e^{s_3}G_1(s_1)k_3(-s_3)}{4(s_1+s_2)}\\
&+\frac{e^{s_3}s_1G_1(s_1)k_3(-s_3)}{4s_2(s_1+s_2)}+\frac{e^{s_3}s_1G_2(s_1,s_2)k_3(-s_3)}{4(s_1+s_2)}+\frac{e^{s_3}s_2G_2(s_1,s_2)k_3(-s_3)}{4(s_1+s_2)}+\frac{e^{s_3}k_3(-s_3)}{4s_2(s_1+s_2)}\\
&+\frac{G_1(s_1)k_3(s_3)}{4(s_1+s_2)}+\frac{s_1G_1(s_1)k_3(s_3)}{4s_2(s_1+s_2)}+\frac{s_1G_2(s_1,s_2)k_3(s_3)}{4(s_1+s_2)}+\frac{s_2G_2(s_1,s_2)k_3(s_3)}{4(s_1+s_2)}+\frac{k_3(s_3)}{4s_2(s_1+s_2)}\\
&+\frac{k_3(s_1+s_2+s_3)}{4s_1(s_1+s_2)}-\frac{G_1(s_1)k_3(s_2+s_3)}{4(s_1+s_2)}-\frac{k_3(s_2+s_3)}{4s_1(s_1+s_2)}-\frac{s_1G_1(s_1)k_3(s_2+s_3)}{4s_2(s_1+s_2)}\\
&-\frac{k_3(s_2+s_3)}{4s_2(s_1+s_2)}\Big)\Big|_{s_j=\nabla}\ \delta_1(h)\delta_2(h)(\delta_1\delta_2(h))\\
&+\Big(-\frac{e^{s_2+s_3}G_1(s_1)k_3(-s_2-s_3)}{4(s_1+s_2)}-\frac{e^{s_2+s_3}k_3(-s_2-s_3)}{4s_1(s_1+s_2)}-\frac{e^{s_2+s_3}k_3(-s_2-s_3)}{4s_2(s_1+s_2)}\\
&-\frac{e^{s_2+s_3}s_1G_1(s_1)k_3(-s_2-s_3)}{4s_2(s_1+s_2)}+\frac{e^{s_1+s_2+s_3}k_3(-s_1-s_2-s_3)}{4s_1(s_1+s_2)}+\frac{e^{s_3}G_1(s_1)k_3(-s_3)}{4(s_1+s_2)}\\
&+\frac{e^{s_3}s_1G_1(s_1)k_3(-s_3)}{4s_2(s_1+s_2)}+\frac{e^{s_3}s_1G_2(s_1,s_2)k_3(-s_3)}{4(s_1+s_2)}+\frac{e^{s_3}s_2G_2(s_1,s_2)k_3(-s_3)}{4(s_1+s_2)}+\frac{e^{s_3}k_3(-s_3)}{4s_2(s_1+s_2)}\\
&+\frac{G_1(s_1)k_3(s_3)}{4(s_1+s_2)}+\frac{s_1G_1(s_1)k_3(s_3)}{4s_2(s_1+s_2)}+\frac{s_1G_2(s_1,s_2)k_3(s_3)}{4(s_1+s_2)}+\frac{s_2G_2(s_1,s_2)k_3(s_3)}{4(s_1+s_2)}\frac{k_3(s_3)}{4s_2(s_1+s_2)}\\
&+\frac{k_3(s_1+s_2+s_3)}{4s_1(s_1+s_2)}-\frac{G_1(s_1)k_3(s_2+s_3)}{4(s_1+s_2)}-\frac{k_3(s_2+s_3)}{4s_1(s_1+s_2)}-\frac{s_1G_1(s_1)k_3(s_2+s_3)}{4s_2(s_1+s_2)}\\
&-\frac{k_3(s_2+s_3)}{4s_2(s_1+s_2)}\Big)\Big|_{s_j=\nabla}\ \delta_2(h)\delta_1(h)(\delta_1\delta_2(h))\\
&+\Big(\frac{e^{s_1+s_2}k_3(-s_1-s_2)}{4s_1s_3}+\frac{k_3(s_1+s_2)}{4s_1s_3}+\frac{e^{s_2+s_3}G_1(s_1)k_3(-s_2-s_3)}{4s_3}+\frac{e^{s_2+s_3}k_3(-s_2-s_3)}{4s_1s_3}\\
&+\frac{G_1(s_1)k_3(s_2+s_3)}{4s_3}+\frac{k_3(s_2+s_3)}{4s_1s_3}-\frac{e^{s_2}G_1(s_1)k_3(-s_2)}{4s_3}-\frac{G_1(s_1)k_3(s_2)}{4s_3}-\frac{e^{s_2}k_3(-s_2)}{4s_1s_3}\\
&-\frac{k_3(s_2)}{4s_1s_3}-\frac{e^{s_1+s_2+s_3}k_3(-s_1-s_2-s_3)}{4s_1s_3}-\frac{k_3(s_1+s_2+s_3)}{4s_1s_3}\Big)\Big|_{s_j=\nabla}\ \delta_1(h)(\delta_1\delta_2(h))\delta_2(h)\\
&+\Big(\frac{e^{s_1+s_2}k_3(-s_1-s_2)}{4s_1s_3}+\frac{k_3(s_1+s_2)}{4s_1s_3}+\frac{e^{s_2+s_3}G_1(s_1)k_3(-s_2-s_3)}{4s_3}+\frac{e^{s_2+s_3}k_3(-s_2-s_3)}{4s_1s_3}\\
&+\frac{G_1(s_1)k_3(s_2+s_3)}{4s_3}+\frac{k_3(s_2+s_3)}{4s_1s_3}-\frac{e^{s_2}G_1(s_1)k_3(-s_2)}{4s_3}-\frac{G_1(s_1)k_3(s_2)}{4s_3}-\frac{e^{s_2}k_3(-s_2)}{4s_1s_3}\\
&-\frac{k_3(s_2)}{4s_1s_3}-\frac{e^{s_1+s_2+s_3}k_3(-s_1-s_2-s_3)}{4s_1s_3}-\frac{k_3(s_1+s_2+s_3)}{4s_1s_3}\Big)\Big|_{s_j=\nabla}\ \delta_2(h)(\delta_1\delta_2(h))\delta_1(h)\\
&+\Big(\frac{e^{s_1}k_3(-s_1)}{4s_2(s_2+s_3)}+\frac{k_3(s_1)}{4s_2(s_2+s_3)}+\frac{e^{s_1+s_2+s_3}k_3(-s_1-s_2-s_3)}{4s_3(s_2+s_3)}+\frac{k_3(s_1+s_2+s_3)}{4s_3(s_2+s_3)}\\
&-\frac{e^{s_1+s_2}k_3(-s_1-s_2)}{4s_2(s_2+s_3)}-\frac{k_3(s_1+s_2)}{4s_2(s_2+s_3)}-\frac{e^{s_1+s_2}k_3(-s_1-s_2)}{4s_3(s_2+s_3)}\\
&-\frac{k_3(s_1+s_2)}{4s_3(s_2+s_3)}\Big)\Big|_{s_j=\nabla}\ (\delta_1\delta_2(h))\delta_1(h)\delta_2(h)\\
&+\Big(\frac{e^{s_1}k_3(-s_1)}{4s_2(s_2+s_3)}+\frac{k_3(s_1)}{4s_2(s_2+s_3)}+\frac{e^{s_1+s_2+s_3}k_3(-s_1-s_2-s_3)}{4s_3(s_2+s_3)}+\frac{k_3(s_1+s_2+s_3)}{4s_3(s_2+s_3)}\\
&-\frac{e^{s_1+s_2}k_3(-s_1-s_2)}{4s_2(s_2+s_3)}-\frac{k_3(s_1+s_2)}{4s_2(s_2+s_3)}-\frac{e^{s_1+s_2}k_3(-s_1-s_2)}{4s_3(s_2+s_3)}\\
&-\frac{k_3(s_1+s_2)}{4s_3(s_2+s_3)}\Big)\Big|_{s_j=\nabla}\ (\delta_1\delta_2(h))\delta_2(h)\delta_1(h).
\end{aligned}
$$

6.10. **Gradient of $\frac{1}{8}\varphi_0(e^h K_{13}(\nabla,\nabla,\nabla)((\delta_1\delta_2(h))\cdot\delta_1(h)\cdot\delta_2(h)))$.** Now, we use Lemma 6.2 to write

$$\begin{aligned}\Omega(h) &= \varphi_0\big(e^h K_{13}(\nabla,\nabla,\nabla)((\delta_1\delta_2(h))\cdot\delta_1(h)\cdot\delta_2(h))\big)\\ &= \varphi_0\big(e^h K(\nabla,\nabla)((\delta_1\delta_2(h))\cdot\delta_1(h))\cdot\delta_2(h)\big),\end{aligned}$$

where the function K is defined by

$$K(s_1,s_2) = K_{13}(s_1,s_2,-s_1-s_2).$$

Therefore, under the trace φ_0, we have

$$\begin{aligned}\Omega(h+\varepsilon a) = e^{h+\varepsilon a}&K(\nabla_\varepsilon,\nabla_\varepsilon)(\delta_1\delta_2(h)\,\delta_1(h))\,\delta_2(h)\\ &+\varepsilon e^{h+\varepsilon a}K(\nabla_\varepsilon,\nabla_\varepsilon)(\delta_1\delta_2(h)\,\delta_1(h))\,\delta_2(a)\\ &+\varepsilon e^{h+\varepsilon a}K(\nabla_\varepsilon,\nabla_\varepsilon)(\delta_1\delta_2(h)\,\delta_1(a))\,\delta_2(h)\\ &+\varepsilon e^{h+\varepsilon a}K(\nabla_\varepsilon,\nabla_\varepsilon)(\delta_1\delta_2(a)\,\delta_1(h))\,\delta_2(h)+\varepsilon^2(\cdots)+\varepsilon^3(\cdots).\end{aligned}$$

Under φ_0, we can thus continue to write

$$\begin{aligned}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Omega(h+\varepsilon a) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}& e^{h+\varepsilon a}K(\nabla,\nabla)(\delta_1\delta_2(h)\,\delta_1(h))\,\delta_2(h)\\ &+e^h\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}K(\nabla_\varepsilon,\nabla_\varepsilon)(\delta_1\delta_2(h)\,\delta_1(h))\,\delta_2(h)\\ &+e^h K(\nabla,\nabla)(\delta_1\delta_2(h)\,\delta_1(h))\,\delta_2(a)\\ &+e^h K(\nabla,\nabla)(\delta_1\delta_2(h)\,\delta_1(a))\,\delta_2(h)\\ &+e^h K(\nabla,\nabla)(\delta_1\delta_2(a)\,\delta_1(h))\,\delta_2(h).\end{aligned}\tag{54}$$

The first term in the above expression, under φ_0, is equal to

$$\begin{aligned}&\frac{1-e^{-\nabla}}{\nabla}(a)\,e^h\,K(\nabla,\nabla)(\delta_1\delta_2(h)\,\delta_1(h))\,\delta_2(h)\\ &\quad= ae^h\frac{1-e^{\nabla}}{-\nabla}(K(\nabla,\nabla)(\delta_1\delta_2(h)\,\delta_1(h)))\\ &\quad= ae^h\Big(\frac{1-e^{s_1+s_2+s_3}}{-s_1-s_2-s_3}K(s_1,s_2)\Big)\Big|_{s_1,s_2,s_3=\nabla}(\delta_1\delta_2(h)\,\delta_1(h)\,\delta_2(h)).\end{aligned}$$

Using Lemma 5.3, for the second term in (54), under φ_0, we have

$$\begin{aligned}&e^h\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}K(\nabla_\varepsilon,\nabla_\varepsilon)(\delta_1\delta_2(h)\,\delta_1(h))\,\delta_2(h)\\ &\quad= ae^h\Big(e^{s_1+s_2+s_3}\frac{K(s_1,s_2)-K(-s_2-s_3,s_2)}{-s_1-s_2-s_3}\Big)\Big|_{s_j=\nabla}(\delta_1\delta_2(h)\delta_1(h)\delta_2(h))\\ &\qquad+ae^h\Big(e^{s_1+s_2}\frac{K(-s_1-s_2,s_2)-K(s_3,-s_2-s_3)}{-s_1-s_2-s_3}\Big)\Big|_{s_j=\nabla}(\delta_1(h)\delta_2(h)\delta_1\delta_2(h))\\ &\qquad+ae^h\Big(e^{s_1}\frac{K(s_2,-s_1-s_2)-K(s_2,s_3)}{-s_1-s_2-s_3}\Big)\Big|_{s_j=\nabla}(\delta_2(h)\delta_1\delta_2(h)\delta_1(h)).\end{aligned}$$

For the third term in (54), under φ_0, we can write

$$\begin{aligned}
&e^h\, K(\nabla,\nabla)(\delta_1\delta_2(h)\,\delta_1(h))\,\delta_2(a)\\
&\quad= \delta_2(a)e^h\, K(\nabla,\nabla)(\delta_1\delta_2(h)\,\delta_1(h))\\
&\quad= -ae^he^{-h}\delta_2(e^h)\, K(\nabla,\nabla)(\delta_1\delta_2(h)\,\delta_1(h))\\
&\qquad - ae^h\,\delta_2(K(\nabla,\nabla)(\delta_1\delta_2(h)\,\delta_1(h))).
\end{aligned}$$

The fourth term in (54), under φ_0 and by using Lemma 6.3, can be written as

$$\begin{aligned}
&e^h\, K(\nabla,\nabla)(\delta_1\delta_2(h)\,\delta_1(a))\,\delta_2(h)\\
&\quad= \delta_1(a)K_v(\nabla,\nabla)(\delta_2(h)\cdot e^h\delta_1\delta_2(h))\\
&\quad= \delta_1(a)e^hK_{vv}(\nabla,\nabla)(\delta_2(h)\,\delta_1\delta_2(h))\\
&\quad= -ae^he^{-h}\delta_1(e^h)K_{vv}(\nabla,\nabla)(\delta_2(h)\,\delta_1\delta_2(h))\\
&\qquad - ae^h\delta_1(K_{vv}(\nabla,\nabla)(\delta_2(h)\,\delta_1\delta_2(h))),
\end{aligned}$$

where

$$K_v(s_1,s_2)=K(s_2,-s_1-s_2),\quad K_{vv}(s_1,s_2)=e^{s_1}K_v(s_1,s_2).$$

In the above expression, the term of the form $e^{-h}\delta_{j_1}(e^h)$ can be expanded using Lemma 6.5. We use Lemma 5.7 to expand $\delta_1(K_{vv}(\nabla,\nabla)(\delta_2(h)\,\delta_1\delta_2(h)))$.

Similarly, for the fifth term in (54), under φ_0 and using Lemma 6.3, we write

$$\begin{aligned}
&e^h\, K(\nabla,\nabla)(\delta_1\delta_2(a)\,\delta_1(h))\,\delta_2(h)\\
&\quad= e^h\delta_1\delta_2(a)K_w(\nabla,\nabla)(\delta_1(h)\,\delta_2(h))\\
&\quad= \delta_1\delta_2(a)e^hK_{ww}(\nabla,\nabla)(\delta_1(h)\,\delta_2(h))\\
&\quad= ae^he^{-h}\delta_1\delta_2(e^h)K_{ww}(\nabla,\nabla)(\delta_1(h)\,\delta_2(h))\\
&\qquad + ae^he^{-h}\delta_1(e^h)\,\delta_2(K_{ww}(\nabla,\nabla)(\delta_1(h)\,\delta_2(h)))\\
&\qquad + ae^he^{-h}\delta_2(e^h)\,\delta_1(K_{ww}(\nabla,\nabla)(\delta_1(h)\,\delta_2(h)))\\
&\qquad + ae^h\,\delta_1\delta_2(K_{ww}(\nabla,\nabla)(\delta_1(h)\,\delta_2(h))),
\end{aligned}$$

where

$$K_w(s_1,s_2)=K(-s_1-s_2,s_1),\quad K_{ww}(s_1,s_2)=e^{s_1+s_2}K_w(s_1,s_2).$$

Here, one can use Lemma 6.5 for expanding the terms of the form $e^{-h}\delta_{j_1}(e^h)$ and $e^{-h}\delta_{j_1}\delta_{j_2}(e^h)$, Lemma 5.7 can be used for $\delta_{j_1}(K_{ww}(\nabla,\nabla)(\delta_1(h)\,\delta_2(h)))$, and Lemma 5.7 and Lemma 5.8 allow one to expand the term

$$\delta_1\delta_2(K_{ww}(\nabla,\nabla)(\delta_1(h)\,\delta_2(h))).$$

Putting everything together, up to multiplying the right-hand side of the following equality by ae^h, under the trace φ_0, we have

$$\begin{aligned}
&\frac{d}{d\varepsilon}\,\Big|_{\varepsilon=0}\,\frac{1}{8}\,\varphi_0\big(e^{h+\varepsilon a}K_{13}(\nabla_\varepsilon,\nabla_\varepsilon,\nabla_\varepsilon)((\delta_1\delta_2(h+\varepsilon a))\cdot\delta_1(h+\varepsilon a)\cdot\delta_2(h+\varepsilon a))\big)\\
&\quad= \Big(-\frac{1}{8}k_{13}(s_1,s_2)\Big)\,\Big|_{s_j=\nabla}\,(\delta_1\delta_2^2(h))\delta_1(h) + \Big(\frac{1}{8}e^{s_1+s_2}k_{13}(-s_1-s_2,s_1)\Big)\,\Big|_{s_j=\nabla}\,\delta_1(h)(\delta_1\delta_2^2(h))
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{1}{8}e^{s_1+s_2}k_{13}(-s_1-s_2,s_1)\Big)\Big|_{s_j=\nabla}\ \delta_1^2(h)\delta_2^2(h)\\
&+\Big(\frac{1}{8}e^{s_1+s_2}k_{13}(-s_1-s_2,s_1)\Big)\Big|_{s_j=\nabla}\ (\delta_1^2\delta_2(h))\delta_2(h)\\
&+\Big(-\frac{1}{8}e^{s_1}k_{13}(s_2,-s_1-s_2)\Big)\Big|_{s_j=\nabla}\ \delta_2(h)(\delta_1^2\delta_2(h))\\
&+\Big(-\frac{1}{8}k_{13}(s_1,s_2)+\frac{1}{8}e^{s_1+s_2}k_{13}(-s_1-s_2,s_1)\\
&-\frac{1}{8}e^{s_1}k_{13}(s_2,-s_1-s_2)\Big)\Big|_{s_j=\nabla}\ (\delta_1\delta_2(h))(\delta_1\delta_2(h))\\
&+\Big(\frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{13}(-s_2-s_3,s_2)+\frac{e^{s_2+s_3}k_{13}(-s_2-s_3,s_2)}{8s_1}\\
&-\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_1}\Big)\Big|_{s_j=\nabla}\ \delta_2(h)\delta_1^2(h)\delta_2(h)\\
&+\Big(\frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{13}(-s_2-s_3,s_2)+\frac{e^{s_2+s_3}k_{13}(-s_2-s_3,s_2)}{8s_1}\\
&+\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_2}-\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_1}\\
&-\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1)}{8s_2}\Big)\Big|_{s_j=\nabla}\ \delta_1(h)\delta_1(h)\delta_2^2(h)\\
&+\Big(-\frac{e^{s_1+s_2}k_{13}(-s_1-s_2,s_1)}{8s_3}+\frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{13}(-s_2-s_3,s_2)+\frac{e^{s_2+s_3}k_{13}(-s_2-s_3,s_2)}{8s_1}\\
&+\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1)}{8s_3}+\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_2}\\
&-\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_1}\\
&-\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1)}{8s_2}\Big)\Big|_{s_j=\nabla}\ \delta_1(h)(\delta_1\delta_2(h))\delta_2(h)\\
&+\Big(-\frac{e^{s_1+s_2}k_{13}(-s_1-s_2,s_1)}{8s_3}+\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1)}{8s_3}\\
&+\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_2}\\
&-\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1)}{8s_2}\Big)\Big|_{s_j=\nabla}\ \delta_1^2(h)\delta_2(h)\delta_2(h)\\
&+\Big(\frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{13}(-s_2-s_3,s_2)+\frac{e^{s_2+s_3}k_{13}(-s_2-s_3,s_2)}{8s_1}\\
&+\frac{e^{s_1}k_{13}(s_2+s_3,-s_1-s_2-s_3)}{8s_2}-\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_1}\\
&-\frac{e^{s_1+s_2}k_{13}(s_3,-s_1-s_2-s_3)}{8s_2}\Big)\Big|_{s_j=\nabla}\ \delta_2(h)\delta_1(h)(\delta_1\delta_2(h))\\
&+\Big(\frac{k_{13}(s_1,s_2+s_3)}{8s_2}+\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1)}{8s_3}-\frac{k_{13}(s_1+s_2,s_3)}{8s_2}\\
&-\frac{e^{s_1+s_2}k_{13}(-s_1-s_2,s_1)}{8s_3}\Big)\Big|_{s_j=\nabla}\ (\delta_1\delta_2(h))\delta_2(h)\delta_1(h)\\
&+\Big(\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1)}{8s_3}-\frac{e^{s_1+s_2}k_{13}(-s_1-s_2,s_1)}{8s_3}\Big)\Big|_{s_j=\nabla}\ \delta_1(h)\delta_2^2(h)\delta_1(h)\\
&+\Big(-\frac{e^{s_1+s_2}k_{13}(-s_1-s_2,s_1)}{4(s_1+s_2+s_3)}-\frac{e^{s_1+s_2}s_1k_{13}(-s_1-s_2,s_1)}{8s_3(s_1+s_2+s_3)}-\frac{e^{s_1+s_2}s_2k_{13}(-s_1-s_2,s_1)}{8s_3(s_1+s_2+s_3)}\\
&+\frac{e^{s_1+s_2+s_3}s_1k_{13}(-s_1-s_2-s_3,s_1)}{8s_3(s_1+s_2+s_3)}+\frac{e^{s_1+s_2+s_3}s_2k_{13}(-s_1-s_2-s_3,s_1)}{8s_3(s_1+s_2+s_3)}\\
&+\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8(s_1+s_2+s_3)}+\frac{e^{s_1+s_2+s_3}s_1k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_2(s_1+s_2+s_3)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{e^{s_1+s_2+s_3} s_3 k_{13}(-s_1-s_2-s_3, s_1+s_2)}{8s_2(s_1+s_2+s_3)} + \frac{e^{s_1+s_2} k_{13}(s_3, -s_2-s_3)}{8(s_1+s_2+s_3)} \\
&+ \frac{e^{s_1+s_2} k_{13}(s_3, -s_1-s_2-s_3)}{8(s_1+s_2+s_3)} + \frac{e^{s_1+s_2} s_2 k_{13}(s_3, -s_1-s_2-s_3)}{8s_1(s_1+s_2+s_3)} \\
&+ \frac{e^{s_1+s_2} s_3 k_{13}(s_3, -s_1-s_2-s_3)}{8s_1(s_1+s_2+s_3)} - \frac{e^{s_2} k_{13}(s_3, -s_2-s_3)}{8(s_1+s_2+s_3)} - \frac{e^{s_2} s_1 G_1(s_1) k_{13}(s_3, -s_2-s_3)}{8(s_1+s_2+s_3)} \\
&- \frac{e^{s_2} s_2 G_1(s_1) k_{13}(s_3, -s_2-s_3)}{8(s_1+s_2+s_3)} - \frac{e^{s_2} s_3 G_1(s_1) k_{13}(s_3, -s_2-s_3)}{8(s_1+s_2+s_3)} - \frac{e^{s_2} s_2 k_{13}(s_3, -s_2-s_3)}{8s_1(s_1+s_2+s_3)} \\
&- \frac{e^{s_2} s_3 k_{13}(s_3, -s_2-s_3)}{8s_1(s_1+s_2+s_3)} - \frac{e^{s_1+s_2+s_3} s_1 k_{13}(-s_1-s_2-s_3, s_1)}{8s_2(s_1+s_2+s_3)} \\
&- \frac{e^{s_1+s_2+s_3} s_3 k_{13}(-s_1-s_2-s_3, s_1)}{8s_2(s_1+s_2+s_3)} \Big) \big|_{s_j=\nabla}\ \delta_1(h)\delta_2(h)(\delta_1\delta_2(h)) \\
&+ \Big(\frac{s_1 k_{13}(s_1, s_2)}{8s_3(s_1+s_2+s_3)} + \frac{s_2 k_{13}(s_1, s_2)}{8s_3(s_1+s_2+s_3)} + \frac{e^{s_2+s_3} s_1 G_1(s_1) k_{13}(-s_2-s_3, s_2)}{8(s_1+s_2+s_3)} \\
&+ \frac{e^{s_2+s_3} s_2 G_1(s_1) k_{13}(-s_2-s_3, s_2)}{8(s_1+s_2+s_3)} + \frac{e^{s_2+s_3} s_3 G_1(s_1) k_{13}(-s_2-s_3, s_2)}{8(s_1+s_2+s_3)} \\
&+ \frac{e^{s_2+s_3} k_{13}(-s_2-s_3, s_2)}{8(s_1+s_2+s_3)} + \frac{e^{s_1+s_2+s_3} k_{13}(-s_2-s_3, s_2)}{8(s_1+s_2+s_3)} + \frac{e^{s_2+s_3} s_2 k_{13}(-s_2-s_3, s_2)}{8s_1(s_1+s_2+s_3)} \\
&+ \frac{e^{s_2+s_3} s_3 k_{13}(-s_2-s_3, s_2)}{8s_1(s_1+s_2+s_3)} + \frac{e^{s_1+s_2+s_3} s_1 k_{13}(-s_1-s_2-s_3, s_1+s_2)}{8s_2(s_1+s_2+s_3)} \\
&+ \frac{e^{s_1+s_2+s_3} s_3 k_{13}(-s_1-s_2-s_3, s_1+s_2)}{8s_2(s_1+s_2+s_3)} - \frac{k_{13}(s_1, s_2+s_3)}{8(s_1+s_2+s_3)} \\
&- \frac{e^{s_1+s_2+s_3} k_{13}(-s_1-s_2-s_3, s_1)}{8(s_1+s_2+s_3)} - \frac{e^{s_1+s_2+s_3} s_2 k_{13}(-s_1-s_2-s_3, s_1+s_2)}{8s_1(s_1+s_2+s_3)} \\
&- \frac{e^{s_1+s_2+s_3} s_3 k_{13}(-s_1-s_2-s_3, s_1+s_2)}{8s_1(s_1+s_2+s_3)} - \frac{e^{s_1+s_2+s_3} s_1 k_{13}(-s_1-s_2-s_3, s_1)}{8s_2(s_1+s_2+s_3)} \\
&- \frac{e^{s_1+s_2+s_3} s_3 k_{13}(-s_1-s_2-s_3, s_1)}{8s_2(s_1+s_2+s_3)} - \frac{s_1 k_{13}(s_1, s_2+s_3)}{8s_3(s_1+s_2+s_3)} \\
&- \frac{s_2 k_{13}(s_1, s_2+s_3)}{8s_3(s_1+s_2+s_3)} \Big) \big|_{s_j=\nabla}\ (\delta_1\delta_2(h))\delta_1(h)\delta_2(h) \\
&+ \Big(\frac{e^{s_1} s_1 k_{13}(s_2, -s_1-s_2)}{8s_3(s_1+s_2+s_3)} + \frac{e^{s_1} s_2 k_{13}(s_2, -s_1-s_2)}{8s_3(s_1+s_2+s_3)} + \frac{e^{s_1} k_{13}(s_2, s_3)}{8(s_1+s_2+s_3)} + \frac{k_{13}(s_1+s_2, s_3)}{8(s_1+s_2+s_3)} \\
&+ \frac{s_2 k_{13}(s_1+s_2, s_3)}{8s_1(s_1+s_2+s_3)} + \frac{s_3 k_{13}(s_1+s_2, s_3)}{8s_1(s_1+s_2+s_3)} - \frac{s_1 G_1(s_1) k_{13}(s_2, s_3)}{8(s_1+s_2+s_3)} - \frac{s_2 G_1(s_1) k_{13}(s_2, s_3)}{8(s_1+s_2+s_3)} \\
&- \frac{s_3 G_1(s_1) k_{13}(s_2, s_3)}{8(s_1+s_2+s_3)} - \frac{k_{13}(s_2, s_3)}{8(s_1+s_2+s_3)} - \frac{e^{s_1} k_{13}(s_2+s_3, -s_1-s_2-s_3)}{8(s_1+s_2+s_3)} \\
&- \frac{s_2 k_{13}(s_2, s_3)}{8s_1(s_1+s_2+s_3)} - \frac{s_3 k_{13}(s_2, s_3)}{8s_1(s_1+s_2+s_3)} - \frac{e^{s_1} s_1 k_{13}(s_2+s_3, -s_1-s_2-s_3)}{8s_3(s_1+s_2+s_3)} \\
&- \frac{e^{s_1} s_2 k_{13}(s_2+s_3, -s_1-s_2-s_3)}{8s_3(s_1+s_2+s_3)} \Big) \big|_{s_j=\nabla}\ \delta_2(h)(\delta_1\delta_2(h))\delta_1(h) \\
&+ \Big(\frac{e^{s_3+s_4} G_1(s_1) k_{13}(-s_3-s_4, s_3)}{8(s_1+s_2)} + \frac{e^{s_3+s_4} s_1 G_1(s_1) k_{13}(-s_3-s_4, s_3)}{8s_2(s_1+s_2)} \\
&+ \frac{e^{s_3+s_4} s_1 G_2(s_1, s_2) k_{13}(-s_3-s_4, s_3)}{8(s_1+s_2)} + \frac{e^{s_3+s_4} s_2 G_2(s_1, s_2) k_{13}(-s_3-s_4, s_3)}{8(s_1+s_2)} \\
&+ \frac{e^{s_3+s_4} k_{13}(-s_3-s_4, s_3)}{8s_2(s_1+s_2)} + \frac{e^{s_2+s_3+s_4} s_1 G_1(s_1) k_{13}(-s_2-s_3-s_4, s_2+s_3)}{8(s_1+s_2)s_3} \\
&+ \frac{e^{s_2+s_3+s_4} s_2 G_1(s_1) k_{13}(-s_2-s_3-s_4, s_2+s_3)}{8(s_1+s_2)s_3} + \frac{e^{s_2+s_3+s_4} k_{13}(-s_2-s_3-s_4, s_2+s_3)}{8(s_1+s_2)s_3} \\
&+ \frac{e^{s_2+s_3+s_4} s_2 k_{13}(-s_2-s_3-s_4, s_2+s_3)}{8s_1(s_1+s_2)s_3} + \frac{e^{s_1+s_2+s_3+s_4} k_{13}(-s_1-s_2-s_3-s_4, s_1+s_2)}{8(s_1+s_2)s_3}
\end{aligned}
$$

$$+\frac{e^{s_1+s_2+s_3+s_4}s_2k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_1(s_1+s_2)s_3}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_1(s_1+s_2)}$$
$$-\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8(s_1+s_2)}$$
$$-\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_1(s_1+s_2)}-\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_2(s_1+s_2)}$$
$$-\frac{e^{s_2+s_3+s_4}s_1G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_2(s_1+s_2)}-\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2)}{8(s_1+s_2)s_3}$$
$$-\frac{e^{s_2+s_3+s_4}s_1G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2)}{8(s_1+s_2)s_3}-\frac{e^{s_2+s_3+s_4}s_2G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2)}{8(s_1+s_2)s_3}$$
$$-\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8(s_1+s_2)s_3}-\frac{e^{s_2+s_3+s_4}s_2k_{13}(-s_2-s_3-s_4,s_2)}{8s_1(s_1+s_2)s_3}$$
$$-\frac{e^{s_1+s_2+s_3+s_4}s_2k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_1(s_1+s_2)s_3}\Big)\Big|_{s_j=\nabla}\delta_2(h)\delta_1(h)\delta_1(h)\delta_2(h)$$
$$+\Big(-\frac{e^{s_2+s_3}s_2G_1(s_1)k_{13}(-s_2-s_3,s_2)}{8(s_2+s_3)s_4}-\frac{e^{s_2+s_3}s_3G_1(s_1)k_{13}(-s_2-s_3,s_2)}{8(s_2+s_3)s_4}$$
$$-\frac{e^{s_2+s_3}s_2k_{13}(-s_2-s_3,s_2)}{8s_1(s_2+s_3)s_4}-\frac{e^{s_2+s_3}s_3k_{13}(-s_2-s_3,s_2)}{8s_1(s_2+s_3)s_4}+\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1)}{8(s_2+s_3)s_4}$$
$$+\frac{e^{s_1+s_2+s_3}s_3k_{13}(-s_1-s_2-s_3,s_1)}{8s_2(s_2+s_3)s_4}+\frac{e^{s_1+s_2+s_3}s_2k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_1(s_2+s_3)s_4}$$
$$+\frac{e^{s_1+s_2+s_3}s_3k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_1(s_2+s_3)s_4}+\frac{e^{s_2+s_3+s_4}s_2G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2)}{8(s_2+s_3)s_4}$$
$$+\frac{e^{s_2+s_3+s_4}s_3G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2)}{8(s_2+s_3)s_4}+\frac{e^{s_2+s_3+s_4}s_2k_{13}(-s_2-s_3-s_4,s_2)}{8s_1(s_2+s_3)s_4}$$
$$+\frac{e^{s_2+s_3+s_4}s_3k_{13}(-s_2-s_3-s_4,s_2)}{8s_1(s_2+s_3)s_4}+\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8(s_2+s_3)}$$
$$+\frac{e^{s_2+s_3+s_4}s_2G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_3(s_2+s_3)}+\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_1(s_2+s_3)}$$
$$+\frac{e^{s_2+s_3+s_4}s_2k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_1s_3(s_2+s_3)}+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8s_2(s_2+s_3)}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_1(s_2+s_3)}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}s_2k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_1s_3(s_2+s_3)}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8(s_2+s_3)s_4}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}s_3k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_2(s_2+s_3)s_4}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_3(s_2+s_3)}-\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2)}{8(s_2+s_3)}$$
$$-\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2)}{8s_1(s_2+s_3)}-\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_1(s_2+s_3)}$$
$$-\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_2(s_2+s_3)}-\frac{e^{s_2+s_3+s_4}s_2G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2)}{8s_3(s_2+s_3)}$$
$$-\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_3(s_2+s_3)}-\frac{e^{s_2+s_3+s_4}s_2k_{13}(-s_2-s_3-s_4,s_2)}{8s_1s_3(s_2+s_3)}$$
$$-\frac{e^{s_1+s_2+s_3+s_4}s_2k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_1s_3(s_2+s_3)}-\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8(s_2+s_3)s_4}$$

$$
\begin{aligned}
&- \frac{e^{s_1+s_2+s_3+s_4} k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8(s_2+s_3)s_4} - \frac{e^{s_1+s_2+s_3+s_4} s_2 k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_1(s_2+s_3)s_4} \\
&- \frac{e^{s_1+s_2+s_3+s_4} s_3 k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_1(s_2+s_3)s_4} - \frac{e^{s_1+s_2+s_3} s_3 k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_2(s_2+s_3)s_4} \\
&- \frac{e^{s_1+s_2+s_3+s_4} s_3 k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8s_2(s_2+s_3)s_4}\Big) \big|_{s_j=\nabla}\ \delta_1(h)\delta_1(h)\delta_2(h)\delta_2(h) \\
&+ \Big(- \frac{e^{s_2+s_3} G_1(s_1) k_{13}(-s_2-s_3,s_2)}{8s_4} - \frac{e^{s_2+s_3} k_{13}(-s_2-s_3,s_2)}{8s_1s_4} \\
&+ \frac{e^{s_1+s_2+s_3} k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_1s_4} + \frac{e^{s_2+s_3+s_4} G_1(s_1) k_{13}(-s_2-s_3-s_4,s_2)}{8s_4} \\
&+ \frac{e^{s_2+s_3+s_4} k_{13}(-s_2-s_3-s_4,s_2)}{8s_1s_4} \\
&- \frac{e^{s_1+s_2+s_3+s_4} k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_1s_4}\Big) \big|_{s_j=\nabla}\ \delta_2(h)\delta_1(h)\delta_2(h)\delta_1(h) \\
&+ \Big(\frac{e^{s_1+s_2} k_{13}(-s_1-s_2,s_1) s_2^2}{8(s_1+s_2)s_3(s_2+s_3)(s_3+s_4)} + \frac{e^{s_3+s_4} s_3 G_2(s_1,s_2) k_{13}(-s_3-s_4,s_3) s_2^2}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)} \\
&+ \frac{e^{s_3+s_4} s_4 G_2(s_1,s_2) k_{13}(-s_3-s_4,s_3) s_2^2}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)} + \frac{e^{s_1+s_2+s_3+s_4} k_{13}(-s_1-s_2-s_3-s_4,s_1) s_2^2}{8(s_1+s_2)(s_2+s_3)s_4(s_3+s_4)} \\
&- \frac{e^{s_1+s_2+s_3} k_{13}(-s_1-s_2-s_3,s_1) s_2^2}{8(s_1+s_2)s_3(s_2+s_3)(s_3+s_4)} - \frac{e^{s_1+s_2+s_3} k_{13}(-s_1-s_2-s_3,s_1) s_2^2}{8(s_1+s_2)(s_2+s_3)s_4(s_3+s_4)} \\
&+ \frac{e^{s_1+s_2} k_{13}(-s_1-s_2,s_1) s_2}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)} + \frac{e^{s_1+s_2} s_1 k_{13}(-s_1-s_2,s_1) s_2}{8(s_1+s_2)s_3(s_2+s_3)(s_3+s_4)} \\
&+ \frac{e^{s_3+s_4} s_3 G_1(s_1) k_{13}(-s_3-s_4,s_3) s_2}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)} + \frac{e^{s_3+s_4} s_4 G_1(s_1) k_{13}(-s_3-s_4,s_3) s_2}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)} \\
&+ \frac{e^{s_3+s_4} s_3^2 G_2(s_1,s_2) k_{13}(-s_3-s_4,s_3) s_2}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)} + \frac{e^{s_3+s_4} s_1 s_3 G_2(s_1,s_2) k_{13}(-s_3-s_4,s_3) s_2}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)} \\
&+ \frac{e^{s_3+s_4} s_1 s_4 G_2(s_1,s_2) k_{13}(-s_3-s_4,s_3) s_2}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)} + \frac{e^{s_3+s_4} s_3 s_4 G_2(s_1,s_2) k_{13}(-s_3-s_4,s_3) s_2}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)} \\
&+ \frac{e^{s_1+s_2+s_3+s_4} s_1 k_{13}(-s_1-s_2-s_3-s_4,s_1) s_2}{8(s_1+s_2)(s_2+s_3)s_4(s_3+s_4)} + \frac{e^{s_1+s_2+s_3+s_4} s_3 k_{13}(-s_1-s_2-s_3-s_4,s_1) s_2}{8(s_1+s_2)(s_2+s_3)s_4(s_3+s_4)} \\
&+ \frac{e^{s_1+s_2+s_3+s_4} k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3) s_2}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)} \\
&+ \frac{e^{s_1+s_2+s_3+s_4} s_3 k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3) s_2}{8s_1(s_1+s_2)(s_2+s_3)(s_3+s_4)} \\
&+ \frac{e^{s_1+s_2+s_3+s_4} s_4 k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3) s_2}{8s_1(s_1+s_2)(s_2+s_3)(s_3+s_4)} \\
&+ \frac{e^{s_1+s_2+s_3+s_4} s_4 k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3) s_2}{8(s_1+s_2)s_3(s_2+s_3)(s_3+s_4)} \\
&- \frac{e^{s_1+s_2+s_3} k_{13}(-s_1-s_2-s_3,s_1) s_2}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)} - \frac{e^{s_2+s_3+s_4} s_3 G_1(s_1) k_{13}(-s_2-s_3-s_4,s_2+s_3) s_2}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)} \\
&- \frac{e^{s_2+s_3+s_4} s_4 G_1(s_1) k_{13}(-s_2-s_3-s_4,s_2+s_3) s_2}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)} \\
&- \frac{e^{s_1+s_2+s_3+s_4} k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2) s_2}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)} - \frac{e^{s_2+s_3+s_4} s_3 k_{13}(-s_2-s_3-s_4,s_2+s_3) s_2}{8s_1(s_1+s_2)(s_2+s_3)(s_3+s_4)} \\
&- \frac{e^{s_2+s_3+s_4} s_4 k_{13}(-s_2-s_3-s_4,s_2+s_3) s_2}{8s_1(s_1+s_2)(s_2+s_3)(s_3+s_4)} - \frac{e^{s_1+s_2+s_3} s_1 k_{13}(-s_1-s_2-s_3,s_1) s_2}{8(s_1+s_2)s_3(s_2+s_3)(s_3+s_4)} \\
&- \frac{e^{s_1+s_2+s_3+s_4} s_4 k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2) s_2}{8(s_1+s_2)s_3(s_2+s_3)(s_3+s_4)} - \frac{e^{s_1+s_2+s_3} s_1 k_{13}(-s_1-s_2-s_3,s_1) s_2}{8(s_1+s_2)(s_2+s_3)s_4(s_3+s_4)} \\
&- \frac{e^{s_1+s_2+s_3} s_3 k_{13}(-s_1-s_2-s_3,s_1) s_2}{8(s_1+s_2)(s_2+s_3)s_4(s_3+s_4)} + \frac{e^{s_1+s_2} s_1 k_{13}(-s_1-s_2,s_1)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}
\end{aligned}
$$

$$+\frac{e^{s_3+s_4}s_3^2G_1(s_1)k_{13}(-s_3-s_4,s_3)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}+\frac{e^{s_3+s_4}s_1s_3G_1(s_1)k_{13}(-s_3-s_4,s_3)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$+\frac{e^{s_3+s_4}s_1s_4G_1(s_1)k_{13}(-s_3-s_4,s_3)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}+\frac{e^{s_3+s_4}s_3s_4G_1(s_1)k_{13}(-s_3-s_4,s_3)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$+\frac{e^{s_3+s_4}s_1s_3^2G_1(s_1)k_{13}(-s_3-s_4,s_3)}{8s_2(s_1+s_2)(s_2+s_3)(s_3+s_4)}+\frac{e^{s_3+s_4}s_1s_3s_4G_1(s_1)k_{13}(-s_3-s_4,s_3)}{8s_2(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$+\frac{e^{s_3+s_4}s_1s_3^2G_2(s_1,s_2)k_{13}(-s_3-s_4,s_3)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}+\frac{e^{s_3+s_4}s_1s_3s_4G_2(s_1,s_2)k_{13}(-s_3-s_4,s_3)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$+\frac{e^{s_3+s_4}s_3k_{13}(-s_3-s_4,s_3)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}+\frac{e^{s_3+s_4}s_4k_{13}(-s_3-s_4,s_3)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$+\frac{e^{s_3+s_4}s_3^2k_{13}(-s_3-s_4,s_3)}{8s_2(s_1+s_2)(s_2+s_3)(s_3+s_4)}+\frac{e^{s_3+s_4}s_3s_4k_{13}(-s_3-s_4,s_3)}{8s_2(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}s_3k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}+\frac{e^{s_1+s_2+s_3+s_4}s_4k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}s_1s_3k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8s_2(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}s_1s_4k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8s_2(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}s_1s_3k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8(s_1+s_2)(s_2+s_3)s_4(s_3+s_4)}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}s_1k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}s_3^2k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_1(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}s_3s_4k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_1(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}s_1s_4k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8(s_1+s_2)s_3(s_2+s_3)(s_3+s_4)}$$
$$-\frac{e^{s_1+s_2+s_3}s_1k_{13}(-s_1-s_2-s_3,s_1)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}-\frac{e^{s_2+s_3+s_4}s_3k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$-\frac{e^{s_2+s_3+s_4}s_4k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}-\frac{e^{s_2+s_3+s_4}s_3^2G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$-\frac{e^{s_2+s_3+s_4}s_1s_3G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$-\frac{e^{s_2+s_3+s_4}s_1s_4G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$-\frac{e^{s_2+s_3+s_4}s_3s_4G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$-\frac{e^{s_1+s_2+s_3+s_4}s_1k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$-\frac{e^{s_1+s_2+s_3+s_4}s_3k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$-\frac{e^{s_1+s_2+s_3+s_4}s_4k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$-\frac{e^{s_2+s_3+s_4}s_3^2k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_1(s_1+s_2)(s_2+s_3)(s_3+s_4)}-\frac{e^{s_2+s_3+s_4}s_3s_4k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_1(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$-\frac{e^{s_2+s_3+s_4}s_3^2k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_2(s_1+s_2)(s_2+s_3)(s_3+s_4)}-\frac{e^{s_2+s_3+s_4}s_3s_4k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_2(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$-\frac{e^{s_2+s_3+s_4}s_1s_3^2G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_2(s_1+s_2)(s_2+s_3)(s_3+s_4)}$$

$$- \frac{e^{s_2+s_3+s_4} s_1 s_3 s_4 G_1(s_1) k_{13}(-s_2-s_3-s_4, s_2+s_3)}{8 s_2 (s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$- \frac{e^{s_1+s_2+s_3+s_4} s_1 s_3 k_{13}(-s_1-s_2-s_3-s_4, s_1+s_2)}{8 s_2 (s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$- \frac{e^{s_1+s_2+s_3+s_4} s_1 s_4 k_{13}(-s_1-s_2-s_3-s_4, s_1+s_2)}{8 s_2 (s_1+s_2)(s_2+s_3)(s_3+s_4)}$$
$$- \frac{e^{s_1+s_2+s_3+s_4} s_1 s_4 k_{13}(-s_1-s_2-s_3-s_4, s_1+s_2)}{8 (s_1+s_2) s_3 (s_2+s_3)(s_3+s_4)}$$
$$- \frac{e^{s_1+s_2+s_3} s_1 s_3 k_{13}(-s_1-s_2-s_3, s_1)}{8 (s_1+s_2)(s_2+s_3) s_4 (s_3+s_4)} \Big) \Big|_{s_j=\nabla} \delta_1(h)\delta_2(h)\delta_1(h)\delta_2(h)$$
$$+ \Big(\frac{e^{s_1+s_2} k_{13}(-s_1-s_2, s_1)}{8 s_3 (s_3+s_4)} + \frac{e^{s_1+s_2+s_3} k_{13}(-s_1-s_2-s_3, s_1)}{8 s_2 (s_3+s_4)}$$
$$+ \frac{e^{s_1+s_2+s_3} s_3 k_{13}(-s_1-s_2-s_3, s_1)}{8 s_2 s_4 (s_3+s_4)} + \frac{e^{s_1+s_2+s_3+s_4} k_{13}(-s_1-s_2-s_3-s_4, s_1)}{8 s_4 (s_3+s_4)}$$
$$+ \frac{e^{s_1+s_2+s_3+s_4} k_{13}(-s_1-s_2-s_3-s_4, s_1+s_2)}{8 s_2 (s_3+s_4)}$$
$$+ \frac{e^{s_1+s_2+s_3+s_4} s_3 k_{13}(-s_1-s_2-s_3-s_4, s_1+s_2)}{8 s_2 s_4 (s_3+s_4)}$$
$$- \frac{e^{s_1+s_2+s_3} k_{13}(-s_1-s_2-s_3, s_1+s_2)}{8 s_2 (s_3+s_4)} - \frac{e^{s_1+s_2+s_3+s_4} k_{13}(-s_1-s_2-s_3-s_4, s_1)}{8 s_2 (s_3+s_4)}$$
$$- \frac{e^{s_1+s_2+s_3} k_{13}(-s_1-s_2-s_3, s_1)}{8 s_3 (s_3+s_4)} - \frac{e^{s_1+s_2+s_3} k_{13}(-s_1-s_2-s_3, s_1)}{8 s_4 (s_3+s_4)}$$
$$- \frac{e^{s_1+s_2+s_3} s_3 k_{13}(-s_1-s_2-s_3, s_1+s_2)}{8 s_2 s_4 (s_3+s_4)}$$
$$- \frac{e^{s_1+s_2+s_3+s_4} s_3 k_{13}(-s_1-s_2-s_3-s_4, s_1)}{8 s_2 s_4 (s_3+s_4)} \Big) \Big|_{s_j=\nabla} \delta_1(h)\delta_2(h)\delta_2(h)\delta_1(h).$$

6.11. Gradient of $\frac{1}{16}\varphi_0(e^h K_{18}(\nabla,\nabla,\nabla,\nabla)(\delta_1(h)\cdot\delta_2(h)\cdot\delta_1(h)\cdot\delta_2(h)))$.

We need to calculate

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Omega(h+\varepsilon a),$$

where

$$\Omega(h) = \varphi_0(e^h K_{18}(\nabla,\nabla,\nabla,\nabla)(\delta_1(h)\cdot\delta_2(h)\cdot\delta_1(h)\cdot\delta_2(h))).$$

Using Lemma 6.2, we have

$$\Omega(h) = \varphi_0(e^h K(\nabla,\nabla,\nabla)(\delta_1(h)\cdot\delta_2(h)\cdot\delta_1(h)))\,\delta_2(h),$$

where the function K is defined by

$$K(s_1,s_2,s_3) = K_{18}(s_1,s_2,s_3,-s_1-s_2-s_3).$$

Therefore, under the trace φ_0, we have

$$\begin{aligned}\Omega(h+\varepsilon a) = {} & e^{h+\varepsilon a} K(\nabla_\varepsilon,\nabla_\varepsilon,\nabla_\varepsilon)(\delta_1(h)\,\delta_2(h)\,\delta_1(h))\,\delta_2(h) \\ & + \varepsilon e^{h+\varepsilon a} K(\nabla_\varepsilon,\nabla_\varepsilon,\nabla_\varepsilon)(\delta_1(a)\,\delta_2(h)\,\delta_1(h))\,\delta_2(h) \\ & + \varepsilon e^{h+\varepsilon a} K(\nabla_\varepsilon,\nabla_\varepsilon,\nabla_\varepsilon)(\delta_1(h)\,\delta_2(a)\,\delta_1(h))\,\delta_2(h) \\ & + \varepsilon e^{h+\varepsilon a} K(\nabla_\varepsilon,\nabla_\varepsilon,\nabla_\varepsilon)(\delta_1(h)\,\delta_2(h)\,\delta_1(a))\,\delta_2(h) \\ & + \varepsilon e^{h+\varepsilon a} K(\nabla_\varepsilon,\nabla_\varepsilon,\nabla_\varepsilon)(\delta_1(h)\,\delta_2(h)\,\delta_1(h))\,\delta_2(a) \\ & + \varepsilon^2(\cdots) + \varepsilon^3(\cdots) + \varepsilon^4(\cdots).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Omega(h+\varepsilon a) &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e^{h+\varepsilon a}\, K(\nabla,\nabla,\nabla)(\delta_1(h)\,\delta_2(h)\,\delta_1(h))\,\delta_2(h)\\ &\quad + e^h \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} K(\nabla_\varepsilon,\nabla_\varepsilon,\nabla_\varepsilon)(\delta_1(h)\,\delta_2(h)\,\delta_1(h))\,\delta_2(h)\\ &\quad + e^h K(\nabla,\nabla,\nabla)(\delta_1(a)\,\delta_2(h)\,\delta_1(h))\,\delta_2(h)\\ &\quad + e^h K(\nabla,\nabla,\nabla)(\delta_1(h)\,\delta_2(a)\,\delta_1(h))\,\delta_2(h)\\ &\quad + e^h K(\nabla,\nabla,\nabla)(\delta_1(h)\,\delta_2(h)\,\delta_1(a))\,\delta_2(h)\\ &\quad + e^h K(\nabla,\nabla,\nabla)(\delta_1(h)\,\delta_2(h)\,\delta_1(h))\,\delta_2(a). \qquad (55)\end{aligned}$$

The first term in (55), under the trace φ_0, is equal to

$$\begin{aligned}&\frac{1-e^{-\nabla}}{\nabla}(a)\, e^h\, K(\nabla,\nabla,\nabla)(\delta_1(h)\,\delta_2(h)\,\delta_1(h))\,\delta_2(h)\\ &\quad = ae^h\Big(\frac{1-e^{s_1+s_2+s_3+s_4}}{-s_1-s_2-s_3-s_4}K(s_1,s_2,s_3)\Big)\Big|_{s_1,s_2,s_3,s_4=\nabla}(\delta_1(h)\,\delta_2(h)\,\delta_1(h)\,\delta_2(h)).\end{aligned}$$

Using Lemma 5.4, for the second term in (55), under φ_0, we have

$$\begin{aligned}&e^h\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} K(\nabla_\varepsilon,\nabla_\varepsilon,\nabla_\varepsilon)(\delta_1(h)\,\delta_2(h)\,\delta_1(h))\,\delta_2(h)\\ &\quad = ae^h\Big\{\Big(\frac{K(s_1,s_2,s_3)-K(-s_2-s_3-s_4,s_2,s_3)}{e^{-s_1-s_2-s_3-s_4}(-s_1-s_2-s_3-s_4)}\Big)\Big|_{s_j=\nabla}(\delta_1(h)\delta_2(h)\delta_1(h)\delta_2(h))\\ &\qquad + \Big(\frac{K(-s_1-s_2-s_3,s_1,s_2)-K(s_4,-s_2-s_3-s_4,s_2)}{e^{-s_1-s_2-s_3}(-s_1-s_2-s_3-s_4)}\Big)\Big|_{s_j=\nabla}(\delta_2(h)\delta_1(h)\delta_2(h)\delta_1(h))\\ &\qquad + \Big(\frac{K(s_3,-s_1-s_2-s_3,s_1)-K(s_3,s_4,-s_2-s_3-s_4)}{e^{-s_1-s_2}(-s_1-s_2-s_3-s_4)}\Big)\Big|_{s_j=\nabla}(\delta_1(h)\delta_2(h)\delta_1(h)\delta_2(h))\\ &\qquad + \Big(\frac{K(s_2,s_3,-s_1-s_2-s_3)-K(s_2,s_3,s_4)}{e^{-s_1}(-s_1-s_2-s_3-s_4)}\Big)\Big|_{s_j=\nabla}(\delta_2(h)\delta_1(h)\delta_2(h)\delta_1(h))\Big\}.\end{aligned}$$

For the third term in (55), using Lemma 6.3, under the trace φ_0, we write

$$\begin{aligned}&e^h K(\nabla,\nabla,\nabla)(\delta_1(a)\,\delta_2(h)\,\delta_1(h))\,\delta_2(h)\\ &\quad = e^h\delta_1(a)K_v(\nabla,\nabla,\nabla)(\delta_2(h)\,\delta_1(h)\,\delta_2(h))\\ &\quad = \delta_1(a)e^h K_{vv}(\nabla,\nabla,\nabla)(\delta_2(h)\,\delta_1(h)\,\delta_2(h))\\ &\quad = -ae^h e^{-h}\delta_1(e^h)K_{vv}(\nabla,\nabla,\nabla)(\delta_2(h)\,\delta_1(h)\,\delta_2(h))\\ &\qquad - ae^h\delta_1(K_{vv}(\nabla,\nabla,\nabla)(\delta_2(h)\,\delta_1(h)\,\delta_2(h))),\end{aligned}$$

where the functions K_v and K_{vv} are defined by

$$\begin{aligned}K_v(s_1,s_2,s_3) &= K(-s_1-s_2-s_3,s_1,s_2),\\ K_{vv}(s_1,s_2,s_3) &= e^{s_1+s_2+s_3}K_v(s_1,s_2,s_3).\end{aligned}$$

Here, $e^{-h}\delta_1(e^h)$ can be expanded using Lemma 6.5, and

$$\delta_1(K_{vv}(\nabla,\nabla,\nabla)(\delta_2(h)\,\delta_1(h)\,\delta_2(h)))$$

can be expanded using Lemma 5.8.

For the fourth term in (55), using Lemma 6.3, under the trace φ_0, we write

$$\begin{aligned} &e^h K(\nabla,\nabla,\nabla)(\delta_1(h)\,\delta_2(a)\,\delta_1(h))\,\delta_2(h) \\ &\quad = \delta_2(a)K_w(\nabla,\nabla,\nabla)(\delta_1(h)\cdot\delta_2(h)\cdot e^h\delta_1(h)) \\ &\quad = \delta_2(a)e^h K_{ww}(\nabla,\nabla,\nabla)(\delta_1(h)\,\delta_2(h)\,\delta_1(h)) \\ &\quad = -ae^h e^{-h}\delta_2(e^h)K_{ww}(\nabla,\nabla,\nabla)(\delta_1(h)\,\delta_2(h)\,\delta_1(h)) \\ &\qquad - ae^h\delta_2(K_{ww}(\nabla,\nabla,\nabla)(\delta_1(h)\,\delta_2(h)\,\delta_1(h))), \end{aligned}$$

where the functions K_w and K_{ww} are given by

$$\begin{aligned} K_w(s_1,s_2,s_3) &= K(s_3,-s_1-s_2-s_3,s_1), \\ K_{ww}(s_1,s_2,s_3) &= e^{s_1+s_2}K_w(s_1,s_2,s_3). \end{aligned}$$

Here also, $e^{-h}\delta_2(e^h)$ can be expanded using Lemma 6.5, and

$$\delta_2(K_{ww}(\nabla,\nabla,\nabla)(\delta_1(h)\,\delta_2(h)\,\delta_1(h)))$$

can be expanded using Lemma 5.8.

For the fifth term in (55), using Lemma 6.3, under the trace φ_0, we have

$$\begin{aligned} &e^h K(\nabla,\nabla,\nabla)(\delta_1(h)\,\delta_2(h)\,\delta_1(a))\,\delta_2(h) \\ &\quad = \delta_1(a)K_z(\nabla,\nabla,\nabla)(\delta_2(h)\cdot e^h\delta_1(h)\cdot\delta_2(h)) \\ &\quad = \delta_1(a)e^h K_{zz}(\nabla,\nabla,\nabla)(\delta_2(h)\,\delta_1(h)\,\delta_2(h)) \\ &\quad = -ae^h e^{-h}\delta_1(e^h)K_{zz}(\nabla,\nabla,\nabla)(\delta_2(h)\,\delta_1(h)\,\delta_2(h)) \\ &\qquad - ae^h\delta_1(K_{zz}(\nabla,\nabla,\nabla)(\delta_2(h)\,\delta_1(h)\,\delta_2(h))), \end{aligned}$$

where

$$K_z(s_1,s_2,s_3) = K(s_2,s_3,-s_1-s_2-s_3), \quad K_{zz}(s_1,s_2,s_3) = e^{s_1}K_z(s_1,s_2,s_3).$$

Similar to the previous cases, Lemma 6.5 and Lemma 5.8 can be used for further expansions of the above expression.

Finally, for the sixth term in (55), under φ_0, we have

$$\begin{aligned} &e^h K(\nabla,\nabla,\nabla)(\delta_1(h)\,\delta_2(h)\,\delta_1(h))\,\delta_2(a) \\ &\quad = \delta_2(a)e^h K(\nabla,\nabla,\nabla)(\delta_1(h)\,\delta_2(h)\,\delta_1(h)) \\ &\quad = -ae^h e^{-h}\delta_2(e^h)K(\nabla,\nabla,\nabla)(\delta_1(h)\,\delta_2(h)\,\delta_1(h)) \\ &\qquad - ae^h\delta_2(K(\nabla,\nabla,\nabla)(\delta_1(h)\,\delta_2(h)\,\delta_1(h))). \end{aligned}$$

Similarly to the previous cases, we apply Lemma 6.5 and Lemma 5.8 for expanding the latter further.

Putting everything together, up to multiplying the right-hand side of the following equality by ae^h, under the trace φ_0, we have

$$\begin{aligned} &\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\frac{1}{16}\varphi_0(e^{h+\varepsilon a}K_{18}(\nabla_\varepsilon,\nabla_\varepsilon,\nabla_\varepsilon,\nabla_\varepsilon)(\delta_1(h+\varepsilon a)\cdot\delta_2(h+\varepsilon a)\cdot\delta_1(h+\varepsilon a)\cdot\delta_2(h+\varepsilon a))) \\ &\quad = \Big(-\frac{1}{16}e^{s_1}k_{18}(s_2,s_3,-s_1-s_2-s_3) \\ &\qquad -\frac{1}{16}e^{s_1+s_2+s_3}k_{18}(-s_1-s_2-s_3,s_1,s_2)\Big)\Big|_{s_j=\nabla}\delta_2(h)\delta_1(h)(\delta_1\delta_2(h)) \end{aligned}$$

$$
\begin{aligned}
&+ \Big(-\frac{1}{16}e^{s_1}k_{18}(s_2,s_3,-s_1-s_2-s_3) \\
&- \frac{1}{16}e^{s_1+s_2+s_3}k_{18}(-s_1-s_2-s_3,s_1,s_2)\Big)\big|_{s_j=\nabla}\ \delta_2(h)\delta_1^2(h)\delta_2(h) \\
&+ \Big(-\frac{1}{16}e^{s_1}k_{18}(s_2,s_3,-s_1-s_2-s_3) \\
&- \frac{1}{16}e^{s_1+s_2+s_3}k_{18}(-s_1-s_2-s_3,s_1,s_2)\Big)\big|_{s_j=\nabla}\ (\delta_1\delta_2(h))\delta_1(h)\delta_2(h) \\
&+ \Big(-\frac{1}{16}k_{18}(s_1,s_2,s_3) - \frac{1}{16}e^{s_1+s_2}k_{18}(s_3,-s_1-s_2-s_3,s_1)\Big)\big|_{s_j=\nabla}\ \delta_1(h)\delta_2(h)(\delta_1\delta_2(h)) \\
&+ \Big(-\frac{1}{16}k_{18}(s_1,s_2,s_3) - \frac{1}{16}e^{s_1+s_2}k_{18}(s_3,-s_1-s_2-s_3,s_1)\Big)\big|_{s_j=\nabla}\ \delta_1(h)\delta_2^2(h)\delta_1(h) \\
&+ \Big(-\frac{1}{16}k_{18}(s_1,s_2,s_3) - \frac{1}{16}e^{s_1+s_2}k_{18}(s_3,-s_1-s_2-s_3,s_1)\Big)\big|_{s_j=\nabla}\ (\delta_1\delta_2(h))\delta_2(h)\delta_1(h) \\
&+ \Big(-\frac{e^{s_1}k_{18}(s_2+s_3,s_4,-s_1-s_2-s_3-s_4) - e^{s_1}k_{18}(s_2,s_3+s_4,-s_1-s_2-s_3-s_4)}{16s_3} \\
&- \frac{e^{s_1+s_2}k_{18}(s_3,s_4,-s_1-s_2-s_3-s_4) - e^{s_1}k_{18}(s_2+s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_2} \\
&- \frac{e^{s_1+s_2+s_3+s_4}k_{18}(-s_1-s_2-s_3-s_4,s_1,s_2+s_3) - e^{s_1+s_2+s_3+s_4}k_{18}(-s_1-s_2-s_3-s_4,s_1,s_2)}{16s_3} \\
&- \frac{e^{s_1+s_2+s_3+s_4}k_{18}(-s_1-s_2-s_3-s_4,s_1+s_2,s_3) - e^{s_1+s_2+s_3+s_4}k_{18}(-s_1-s_2-s_3-s_4,s_1,s_2+s_3)}{16s_2}\Big) \\
&\quad\big|_{s_j=\nabla}\ \delta_2(h)\delta_1(h)\delta_1(h)\delta_2(h) \\
&+ \Big(-\frac{1}{16}G_1(s_1)k_{18}(s_2,s_3,s_4) + \frac{e^{s_1}(k_{18}(s_2,s_3,s_4) - k_{18}(s_2,s_3,-s_1-s_2-s_3))}{16(s_1+s_2+s_3+s_4)} \\
&+ \frac{k_{18}(s_1+s_2,s_3,s_4) - k_{18}(s_2,s_3,s_4)}{16s_1} \\
&- \frac{e^{s_1+s_2+s_3+s_4}k_{18}(-s_1-s_2-s_3-s_4,s_1,s_2) - e^{s_1+s_2+s_3}k_{18}(-s_1-s_2-s_3,s_1,s_2)}{16s_4} \\
&- \frac{1}{16}e^{s_2+s_3}G_1(s_1)k_{18}(s_4,-s_2-s_3-s_4,s_2) \\
&+ \frac{e^{s_1+s_2+s_3}(k_{18}(s_4,-s_2-s_3-s_4,s_2) - k_{18}(-s_1-s_2-s_3,s_1,s_2))}{16(s_1+s_2+s_3+s_4)} \\
&+ \frac{e^{s_1+s_2+s_3}k_{18}(s_4,-s_1-s_2-s_3-s_4,s_1+s_2) - e^{s_2+s_3}k_{18}(s_4,-s_2-s_3-s_4,s_2)}{16s_1} \\
&- \frac{e^{s_1}k_{18}(s_2,s_3+s_4,-s_1-s_2-s_3-s_4) - e^{s_1}k_{18}(s_2,s_3,-s_1-s_2-s_3)}{16s_4}\Big)\big|_{s_j=\nabla}\ \delta_2(h)\delta_1(h)\delta_2(h)\delta_1(h) \\
&+ \Big(\frac{(1-e^{s_1+s_2+s_3+s_4})k_{18}(s_1,s_2,s_3)}{16(-s_1-s_2-s_3-s_4)} - \frac{1}{16}e^{s_2}G_1(s_1)k_{18}(s_3,s_4,-s_2-s_3-s_4) \\
&+ \frac{e^{s_1+s_2}(k_{18}(s_3,s_4,-s_2-s_3-s_4) - k_{18}(s_3,-s_1-s_2-s_3,s_1))}{16(s_1+s_2+s_3+s_4)} \\
&+ \frac{e^{s_1+s_2}k_{18}(s_3,s_4,-s_1-s_2-s_3-s_4) - e^{s_2}k_{18}(s_3,s_4,-s_2-s_3-s_4)}{16s_1} \\
&- \frac{1}{16}e^{s_2+s_3+s_4}G_1(s_1)k_{18}(-s_2-s_3-s_4,s_2,s_3) \\
&+ \frac{e^{s_1+s_2+s_3+s_4}(k_{18}(-s_2-s_3-s_4,s_2,s_3) - k_{18}(s_1,s_2,s_3))}{16(s_1+s_2+s_3+s_4)} \\
&+ \frac{e^{s_1+s_2+s_3+s_4}k_{18}(-s_1-s_2-s_3-s_4,s_1+s_2,s_3) - e^{s_2+s_3+s_4}k_{18}(-s_2-s_3-s_4,s_2,s_3)}{16s_1} \\
&- \frac{e^{s_1+s_2}k_{18}(s_3+s_4,-s_1-s_2-s_3-s_4,s_1) - e^{s_1+s_2}k_{18}(s_3,-s_1-s_2-s_3,s_1)}{16s_4} \\
&- \frac{k_{18}(s_1,s_2,s_3+s_4) - k_{18}(s_1,s_2,s_3)}{16s_4}\Big)\big|_{s_j=\nabla}\ \delta_1(h)\delta_2(h)\delta_1(h)\delta_2(h) \\
&+ \Big(-\frac{k_{18}(s_1,s_2+s_3,s_4) - k_{18}(s_1,s_2,s_3+s_4)}{16s_3}
\end{aligned}
$$

$$- \frac{e^{s_1+s_2+s_3}k_{18}(s_4,-s_1-s_2-s_3-s_4,s_1+s_2) - e^{s_1+s_2+s_3}k_{18}(s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_2}$$
$$- \frac{e^{s_1+s_2+s_3}k_{18}(s_4,-s_1-s_2-s_3-s_4,s_1) - e^{s_1+s_2}k_{18}(s_3+s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_3}$$
$$- \frac{k_{18}(s_1+s_2,s_3,s_4) - k_{18}(s_1,s_2+s_3,s_4)}{16s_2}\Big)\Big|_{s_j=\nabla} \delta_1(h)\delta_2(h)\delta_2(h)\delta_1(h).$$

7. Functional relations among the functions $k_3, \dots, k_{20}$

In the results presented thus far, there are indications about functional relations that involve only the functions $k_3, \dots, k_{20}$. For example, the basic functional relations presented in Section 3 and in Appendix A are derived by equating a function $\widetilde{K}_j$ that appears in the first calculation of the gradient in Section 5 with the corresponding term in the outcome of the second calculation of the same gradient in terms of finite differences of the functions k_j in Section 6. Since the operator associated with a function $\widetilde{K}_j$ acts on different elements of $C^\infty(\mathbb{T}^2_\theta)$, this raises the question whether there are different finite difference expressions for the $\widetilde{K}_j$, which would provide a functional relation that involves only the functions k_j.

By studying the result of the second gradient calculation from this perspective, we find that if in formula (52) the operator associated with a function $\widetilde{K}_j$ acts on two elements of $C^\infty(\mathbb{T}^2_\theta)$ that are the same modulo switching δ_1 and δ_2, then the corresponding finite difference expressions in the second calculation are precisely the same. However, there are two cases, namely for $\widetilde{K}_6$ and $\widetilde{K}_7$, where the corresponding operators act on different kinds of elements of $C^\infty(\mathbb{T}^2_\theta)$ that are not the same up to switching δ_1 and δ_2. It turns out that in these cases, the corresponding finite difference expressions have different ingredients, hence functional relations of the desired type. We should also recall that in a closely related manner, in Corollary 4.3 and Corollary 4.4 we presented functional relations that involve restrictions of the functions k_j. We will see shortly that there are more functional relations between the k_j, which are derived from the fact that the final explicit expressions for some of the functions K_j are identical up to multiplication by scalars. We should note that the explicit expressions for K_j are given in Section 9 and in Appendix C.

Theorem 7.1. *There are functional relations that involve only the functions* $k_3, \dots, k_{20}$*, as presented in the following subsections and in Appendix B.*

We now start writing the new functional relations and will explain more specifically how each relation is derived.

7.2. **Functional relation between $k_3, k_4, k_5, k_8, k_9, \dots, k_{16}$.** In (52), the operator $\widetilde{K}_6(\nabla,\nabla)$ acts on two kinds of elements, namely $\delta_1(h)\cdot\delta_1^3(h)$ and $\delta_1(h)\cdot\delta_1\delta_2^2(h)$, from which one can obtain the other two elements by switching δ_1 and δ_2. On the other hand, in the outcome of the second calculation of the same gradient the finite difference expressions of ∇ acting on these elements have different ingredients. Hence we have two different finite different expressions for $\widetilde{K}_6(\nabla,\nabla)$ and thereby obtain the following functional relation:

$$\begin{aligned}
&-s_1k_8(s_1,s_2)+e^{s_1+s_2}s_1k_8(-s_1-s_2,s_1)-s_1k_9(s_1,s_2)\\
&\quad+e^{s_1+s_2}s_1k_9(-s_1-s_2,s_1)+s_1k_{10}(s_1,s_2)-e^{s_1}s_1k_{10}(s_2,-s_1-s_2)\\
&\quad+s_1k_{11}(s_1,s_2)-e^{s_1}s_1k_{11}(s_2,-s_1-s_2)-e^{s_1+s_2}s_1k_{12}(-s_1-s_2,s_1)\\
&\quad+e^{s_1}s_1k_{12}(s_2,-s_1-s_2)-e^{s_1+s_2}s_1k_{13}(-s_1-s_2,s_1)\\
&\quad+e^{s_1}s_1k_{13}(s_2,-s_1-s_2)+e^{s_1+s_2}s_1k_{14}(-s_1-s_2,s_1)\\
&\quad-e^{s_1}s_1k_{14}(s_2,-s_1-s_2)+s_1k_{15}(s_1,s_2)-e^{s_1+s_2}s_1k_{15}(-s_1-s_2,s_1)\\
&\quad-s_1k_{16}(s_1,s_2)+e^{s_1}s_1k_{16}(s_2,-s_1-s_2)-2e^{s_2}(s_1G_1(s_1)+1)k_3(-s_2)\\
&\quad-2s_1G_1(s_1)k_3(s_2)-4e^{s_2}s_1G_1(s_1)k_4(-s_2)-4s_1G_1(s_1)k_4(s_2)\\
&\quad+4e^{s_2}s_1G_1(s_1)k_5(-s_2)+4s_1G_1(s_1)k_5(s_2)+2e^{s_1+s_2}k_3(-s_1-s_2)\\
&\quad-2k_3(s_2)+2k_3(s_1+s_2)+4e^{s_1+s_2}k_4(-s_1-s_2)-4e^{s_2}k_4(-s_2)\\
&\quad-4k_4(s_2)+4k_4(s_1+s_2)-4e^{s_1+s_2}k_5(-s_1-s_2)+4e^{s_2}k_5(-s_2)\\
&\quad+4k_5(s_2)-4k_5(s_1+s_2)\\
&=0.
\end{aligned}$$

7.3. **Another functional relation between $k_3, k_4, k_5, k_8, k_9, \ldots, k_{16}$.** The situation in formula (52) for $\widetilde{K}_7(\nabla,\nabla)$ is similar to the case of its preceding term with $\widetilde{K}_6(\nabla,\nabla)$ involved: there are two kinds of elements that $\widetilde{K}_7(\nabla,\nabla)$ acts on, namely $\delta_1^3(h)\cdot\delta_1(h)$ and $\delta_1\delta_2^2(h)\cdot\delta_1(h)$, that are not the same up to switching δ_1 and δ_2, and in the second gradient calculation there are correspondingly two different finite difference expressions for $\widetilde{K}_7$. This yields the following functional relation:

$$\begin{aligned}
&-s_2k_8(s_1,s_2)+e^{s_1}s_2k_8(s_2,-s_1-s_2)-s_2k_9(s_1,s_2)+e^{s_1}s_2k_9(s_2,-s_1-s_2)\\
&\quad+e^{s_1+s_2}s_2k_{10}(-s_1-s_2,s_1)-e^{s_1}s_2k_{10}(s_2,-s_1-s_2)\\
&\quad+e^{s_1+s_2}s_2k_{11}(-s_1-s_2,s_1)-e^{s_1}s_2k_{11}(s_2,-s_1-s_2)\\
&\quad+s_2k_{12}(s_1,s_2)-e^{s_1+s_2}s_2k_{12}(-s_1-s_2,s_1)+s_2k_{13}(s_1,s_2)\\
&\quad-e^{s_1+s_2}s_2k_{13}(-s_1-s_2,s_1)-s_2k_{14}(s_1,s_2)+e^{s_1+s_2}s_2k_{14}(-s_1-s_2,s_1)\\
&\quad+s_2k_{15}(s_1,s_2)-e^{s_1}s_2k_{15}(s_2,-s_1-s_2)-e^{s_1+s_2}s_2k_{16}(-s_1-s_2,s_1)\\
&\quad+e^{s_1}s_2k_{16}(s_2,-s_1-s_2)+2e^{s_1}k_3(-s_1)+2k_3(s_1)-2e^{s_1+s_2}k_3(-s_1-s_2)\\
&\quad-2k_3(s_1+s_2)+4e^{s_1}k_4(-s_1)+4k_4(s_1)-4e^{s_1+s_2}k_4(-s_1-s_2)\\
&\quad-4k_4(s_1+s_2)-4e^{s_1}k_5(-s_1)-4k_5(s_1)+4e^{s_1+s_2}k_5(-s_1-s_2)\\
&\quad+4k_5(s_1+s_2)\\
&=0.
\end{aligned}$$

7.4. **Functional relation between k_3, k_4, k_5.** The explicit final formulas for the functions $K_1,\ldots,K_{20}$ are provided in Section 9 and in Appendix C. One can see by using these explicit formulas that some of these functions are scalar multiples of each other. For example, we have $2K_2(s_1) = K_1(s_1)$, which clearly implies that $2\widetilde{K}_2 = \widetilde{K}_1$. However, the functional relations given by (10) and (11) for $\widetilde{K}_1$ and $\widetilde{K}_2$ apparently have different ingredients, which yields the following functional relation:

$$-e^{s_1}k_3(-s_1) - k_3(s_1) + 2(-e^{s_1}k_4(-s_1) - k_4(s_1) + e^{s_1}k_5(-s_1) + k_5(s_1)) = 0.$$

7.5. **Functional relation between $k_3, k_4, k_6, k_7, \dots, k_{13}, k_{17}, k_{18}, k_{19}$.** In a very similar manner, one can use the final explicit formulas given in Appendix C to see that $2\widetilde{K}_{11} = \widetilde{K}_{10}$. Therefore, one can then use the finite difference expressions given by (63) and (64) to obtain the following functional relation:

$$\begin{aligned}
&\frac{e^{s_1+s_2}k_3(-s_1-s_2)}{4s_1s_3} + \frac{k_3(s_1+s_2)}{4s_1s_3} + \frac{e^{s_2+s_3}G_1(s_1)k_3(-s_2-s_3)}{4s_3} + \frac{e^{s_2+s_3}k_3(-s_2-s_3)}{4s_1s_3} \\
&\quad + \frac{G_1(s_1)k_3(s_2+s_3)}{4s_3} + \frac{k_3(s_2+s_3)}{4s_1s_3} + \frac{e^{s_2}G_1(s_1)k_4(-s_2)}{s_3} + \frac{e^{s_2}k_4(-s_2)}{s_1s_3} + \frac{G_1(s_1)k_4(s_2)}{s_3} \\
&\quad + \frac{k_4(s_2)}{s_1s_3} + \frac{e^{s_1+s_2+s_3}k_4(-s_1-s_2-s_3)}{s_1s_3} + \frac{k_4(s_1+s_2+s_3)}{s_1s_3} + \frac{k_6(s_1)}{2s_2(s_2+s_3)} + \frac{k_6(s_1)}{2s_3(s_2+s_3)} \\
&\quad + \frac{k_6(s_3)}{4s_1(s_1+s_2)} + \frac{k_6(s_3)}{4s_2(s_1+s_2)} + \frac{k_6(s_1+s_2+s_3)}{4s_1s_2} + \frac{k_6(s_1+s_2+s_3)}{2s_2s_3} + \frac{e^{s_1}k_7(-s_1)}{2s_2(s_2+s_3)} \\
&\quad + \frac{e^{s_1}k_7(-s_1)}{2s_3(s_2+s_3)} + \frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{4s_1s_2} + \frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{2s_2s_3} + \frac{e^{s_3}k_7(-s_3)}{4s_1(s_1+s_2)} \\
&\quad + \frac{e^{s_3}k_7(-s_3)}{4s_2(s_1+s_2)} + \frac{k_8(s_1,s_2+s_3)}{4s_2} + \frac{k_8(s_1,s_2+s_3)}{4s_3} + \frac{e^{s_1+s_2}k_8(-s_1-s_2,s_1)}{4(s_1+s_2+s_3)} \\
&\quad + \frac{1}{4}G_1(s_1)k_8(s_2,s_3) + \frac{k_8(s_2,s_3)}{4s_1} + \frac{e^{s_1+s_2+s_3}k_8(-s_1-s_2-s_3,s_1)}{4s_3} \\
&\quad + \frac{1}{4}e^{s_2}G_1(s_1)k_8(s_3,-s_2-s_3) + \frac{e^{s_2}k_8(s_3,-s_2-s_3)}{4s_1} + \frac{k_9(s_1,s_2)}{8s_3} \\
&\quad + \frac{e^{s_1+s_2}k_9(-s_1-s_2,s_1)}{8s_3} - \frac{1}{8}G_1(s_1)k_9(s_2,s_3) + \frac{k_9(s_1+s_2,s_3)}{8s_1} + \frac{k_9(s_1+s_2,s_3)}{8s_2} \\
&\quad - \frac{1}{8}e^{s_2}G_1(s_1)k_9(s_3,-s_2-s_3) + \frac{e^{s_1+s_2}k_9(s_3,-s_2-s_3)}{8(s_1+s_2+s_3)} + \frac{e^{s_1+s_2}k_9(s_3,-s_1-s_2-s_3)}{8s_1} \\
&\quad + \frac{k_{10}(s_1,s_2)}{8(-s_1-s_2-s_3)} + \frac{k_{10}(s_1,s_2)}{8s_3} + \frac{e^{s_1}k_{10}(s_2,-s_1-s_2)}{8s_3} - \frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{10}(-s_2-s_3,s_2) \\
&\quad + \frac{e^{s_1+s_2+s_3}k_{10}(-s_2-s_3,s_2)}{8(s_1+s_2+s_3)} + \frac{e^{s_1+s_2+s_3}k_{10}(-s_1-s_2-s_3,s_1+s_2)}{8s_1} \\
&\quad - \frac{1}{8}e^{s_2}G_1(s_1)k_{10}(s_3,-s_2-s_3) + \frac{e^{s_1+s_2}k_{10}(s_3,-s_1-s_2-s_3)}{8s_1} \\
&\quad + \frac{e^{s_1+s_2}k_{10}(s_3,-s_1-s_2-s_3)}{8s_2} + \frac{e^{s_1+s_2+s_3}k_{11}(s_1,s_2)}{4(-s_1-s_2-s_3)} + \frac{e^{s_1+s_2+s_3}k_{11}(s_1,s_2)}{4(s_1+s_2+s_3)} \\
&\quad + \frac{k_{11}(s_1,s_2+s_3)}{4s_3} + \frac{1}{4}e^{s_2+s_3}G_1(s_1)k_{11}(-s_2-s_3,s_2) + \frac{e^{s_2+s_3}k_{11}(-s_2-s_3,s_2)}{4s_1} \\
&\quad + \frac{1}{4}e^{s_2}G_1(s_1)k_{11}(s_3,-s_2-s_3) + \frac{e^{s_2}k_{11}(s_3,-s_2-s_3)}{4s_1} + \frac{e^{s_1}k_{11}(s_2+s_3,-s_1-s_2-s_3)}{4s_2} \\
&\quad + \frac{e^{s_1}k_{11}(s_2+s_3,-s_1-s_2-s_3)}{4s_3} + \frac{e^{s_1}k_{12}(s_2,-s_1-s_2)}{4(s_1+s_2+s_3)} + \frac{1}{4}G_1(s_1)k_{12}(s_2,s_3) + \frac{k_{12}(s_2,s_3)}{4s_1} \\
&\quad + \frac{1}{4}e^{s_2+s_3}G_1(s_1)k_{12}(-s_2-s_3,s_2) + \frac{e^{s_2+s_3}k_{12}(-s_2-s_3,s_2)}{4s_1} \\
&\quad + \frac{e^{s_1+s_2+s_3}k_{12}(-s_1-s_2-s_3,s_1)}{4s_2} + \frac{e^{s_1+s_2+s_3}k_{12}(-s_1-s_2-s_3,s_1)}{4s_3} \\
&\quad + \frac{e^{s_1}k_{12}(s_2+s_3,-s_1-s_2-s_3)}{4s_3} + \frac{e^{s_1+s_2}k_{13}(-s_1-s_2,s_1)}{8s_3} + \frac{e^{s_1}k_{13}(s_2,-s_1-s_2)}{8s_3}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{8}G_1(s_1)k_{13}(s_2,s_3)+\frac{e^{s_1}k_{13}(s_2,s_3)}{8(s_1+s_2+s_3)}+\frac{k_{13}(s_1+s_2,s_3)}{8s_1}-\frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{13}(-s_2-s_3,s_2)\\
&+\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_1}+\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_2}\\
&-\frac{1}{16}k_{17}(s_1,s_2,s_3)-\frac{1}{16}e^{s_1}k_{17}(s_2,s_3,-s_1-s_2-s_3)-\frac{1}{16}e^{s_1+s_2+s_3}k_{17}(-s_1-s_2-s_3,s_1,s_2)\\
&-\frac{1}{16}e^{s_1+s_2}k_{17}(s_3,-s_1-s_2-s_3,s_1)+\frac{1}{8}k_{18}(s_1,s_2,s_3)+\frac{1}{8}e^{s_1}k_{18}(s_2,s_3,-s_1-s_2-s_3)\\
&+\frac{1}{8}e^{s_1+s_2+s_3}k_{18}(-s_1-s_2-s_3,s_1,s_2)+\frac{1}{8}e^{s_1+s_2}k_{18}(s_3,-s_1-s_2-s_3,s_1)\\
&-\frac{1}{16}k_{19}(s_1,s_2,s_3)-\frac{1}{16}e^{s_1}k_{19}(s_2,s_3,-s_1-s_2-s_3)-\frac{1}{16}e^{s_1+s_2+s_3}k_{19}(-s_1-s_2-s_3,s_1,s_2)\\
&-\frac{1}{16}e^{s_1+s_2}k_{19}(s_3,-s_1-s_2-s_3,s_1)-\frac{k_8(s_1+s_2,s_3)}{4s_1}-\frac{e^{s_1+s_2}k_8(s_3,-s_1-s_2-s_3)}{4s_1}\\
&-\frac{e^{s_1+s_2+s_3}k_{11}(-s_1-s_2-s_3,s_1+s_2)}{4s_1}-\frac{e^{s_1+s_2}k_{11}(s_3,-s_1-s_2-s_3)}{4s_1}-\frac{k_{12}(s_1+s_2,s_3)}{4s_1}\\
&-\frac{e^{s_1+s_2+s_3}k_{12}(-s_1-s_2-s_3,s_1+s_2)}{4s_1}-\frac{k_9(s_2,s_3)}{8s_1}-\frac{e^{s_2}k_9(s_3,-s_2-s_3)}{8s_1}\\
&-\frac{e^{s_2+s_3}k_{10}(-s_2-s_3,s_2)}{8s_1}-\frac{e^{s_2}k_{10}(s_3,-s_2-s_3)}{8s_1}-\frac{k_{13}(s_2,s_3)}{8s_1}-\frac{e^{s_2+s_3}k_{13}(-s_2-s_3,s_2)}{8s_1}\\
&-\frac{k_8(s_1+s_2,s_3)}{4s_2}-\frac{e^{s_1+s_2}k_{11}(s_3,-s_1-s_2-s_3)}{4s_2}-\frac{e^{s_1+s_2+s_3}k_{12}(-s_1-s_2-s_3,s_1+s_2)}{4s_2}\\
&-\frac{k_9(s_1,s_2+s_3)}{8s_2}-\frac{e^{s_1}k_{10}(s_2+s_3,-s_1-s_2-s_3)}{8s_2}-\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1)}{8s_2}\\
&-\frac{k_6(s_3)}{4s_1s_2}-\frac{e^{s_3}k_7(-s_3)}{4s_1s_2}-\frac{k_6(s_1+s_2+s_3)}{4s_1(s_1+s_2)}-\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{4s_1(s_1+s_2)}-\frac{k_6(s_1+s_2+s_3)}{4s_2(s_1+s_2)}\\
&-\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{4s_2(s_1+s_2)}-\frac{k_{11}(s_1,s_2)}{4(-s_1-s_2-s_3)}-\frac{e^{s_1+s_2+s_3}k_{10}(s_1,s_2)}{8(-s_1-s_2-s_3)}\\
&-\frac{e^{s_2+s_3}G_1(s_1)k_4(-s_2-s_3)}{s_3}-\frac{G_1(s_1)k_4(s_2+s_3)}{s_3}-\frac{e^{s_2}G_1(s_1)k_3(-s_2)}{4s_3}-\frac{G_1(s_1)k_3(s_2)}{4s_3}\\
&-\frac{k_8(s_1,s_2)}{4s_3}-\frac{e^{s_1+s_2}k_8(-s_1-s_2,s_1)}{4s_3}-\frac{k_{11}(s_1,s_2)}{4s_3}-\frac{e^{s_1}k_{11}(s_2,-s_1-s_2)}{4s_3}\\
&-\frac{e^{s_1+s_2}k_{12}(-s_1-s_2,s_1)}{4s_3}-\frac{e^{s_1}k_{12}(s_2,-s_1-s_2)}{4s_3}-\frac{k_9(s_1,s_2+s_3)}{8s_3}\\
&-\frac{e^{s_1+s_2+s_3}k_9(-s_1-s_2-s_3,s_1)}{8s_3}-\frac{k_{10}(s_1,s_2+s_3)}{8s_3}-\frac{e^{s_1}k_{10}(s_2+s_3,-s_1-s_2-s_3)}{8s_3}\\
&-\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1)}{8s_3}-\frac{e^{s_1}k_{13}(s_2+s_3,-s_1-s_2-s_3)}{8s_3}-\frac{e^{s_1+s_2}k_4(-s_1-s_2)}{s_1s_3}\\
&-\frac{k_4(s_1+s_2)}{s_1s_3}-\frac{e^{s_2+s_3}k_4(-s_2-s_3)}{s_1s_3}-\frac{k_4(s_2+s_3)}{s_1s_3}-\frac{e^{s_2}k_3(-s_2)}{4s_1s_3}-\frac{k_3(s_2)}{4s_1s_3}\\
&-\frac{e^{s_1+s_2+s_3}k_3(-s_1-s_2-s_3)}{4s_1s_3}-\frac{k_3(s_1+s_2+s_3)}{4s_1s_3}-\frac{k_6(s_1)}{2s_2s_3}-\frac{e^{s_1}k_7(-s_1)}{2s_2s_3}-\frac{k_6(s_1+s_2+s_3)}{2s_2(s_2+s_3)}\\
&-\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{2s_2(s_2+s_3)}-\frac{k_6(s_1+s_2+s_3)}{2s_3(s_2+s_3)}-\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{2s_3(s_2+s_3)}\\
&-\frac{e^{s_1+s_2}k_8(s_3,-s_2-s_3)}{4(s_1+s_2+s_3)}-\frac{e^{s_1+s_2+s_3}k_{11}(-s_2-s_3,s_2)}{4(s_1+s_2+s_3)}-\frac{e^{s_1}k_{12}(s_2,s_3)}{4(s_1+s_2+s_3)}\\
&-\frac{e^{s_1+s_2}k_9(-s_1-s_2,s_1)}{8(s_1+s_2+s_3)}-\frac{e^{s_1+s_2+s_3}k_{10}(s_1,s_2)}{8(s_1+s_2+s_3)}-\frac{e^{s_1}k_{13}(s_2,-s_1-s_2)}{8(s_1+s_2+s_3)}\\
&=0.
\end{aligned}
$$

There are more functional relations of this type, which are written in Appendix B because of their algebraically lengthy expressions.

8. Calculation of the term a_4

In this section we provide the details of the calculation of the term a_4, whose final formula is given by (6). In order to derive the small-time asymptotic expansion (1), one can start by using the Cauchy integral formula to write

$$e^{-t\triangle_\varphi} = \int_\gamma e^{-t\lambda}(\triangle_\varphi - \lambda)^{-1}\, d\lambda,$$

where the contour γ goes clockwise around the non-negative real axis, where the eigenvalues of $\triangle_\varphi$ are located. Then one can use the pseudodifferential calculus developed in [3] for C^*-dynamical systems to approximate the parametrix of $\triangle_\varphi - \lambda$. We elaborate on this procedure in the following subsection. This approach is indeed an implementation of the heat kernel method explained in [21] in a noncommutative setting.

8.1. **Heat expansion.** The pseudodifferential calculus that was developed in [3] reduces to the following case for the noncommutative two torus $\mathbb{T}_\theta^2$. The symbols are smooth maps $\rho : \mathbb{R}^2 \to C^\infty(\mathbb{T}_\theta^2)$ such that the norm of $\rho(\xi)$ and its derivatives satisfy certain growth conditions depending on the order of the symbol, see [3] for details. For example, the symbol of a differential operator $\sum a_{i,j}\delta_1^i\delta_2^j$, where $a_{i,j} \in C^\infty(\mathbb{T}_\theta^2)$, is the polynomial $\sum a_{i,j}\xi_1^i\xi_2^j$ whose coefficients apparently belong to $C^\infty(\mathbb{T}_\theta^2)$. The correspondence between a general symbol ρ and its associated pseudodifferential operator P_ρ acting on $C^\infty(\mathbb{T}_\theta^2)$ is established by the formula

$$P_\rho(x) = (2\pi)^{-2}\int\int e^{-is\cdot\xi}\,\rho(\xi)\,\alpha_s(x)\,ds\,d\xi, \quad x \in C^\infty(\mathbb{T}_\theta^2),$$

where the dynamics α_s is given by (2). Given two symbols ρ_1 and ρ_2, the symbol of the composition of their corresponding pseudodifferential operators is asymptotically given by

$$\rho_1 \circ \rho_2 \sim \sum_{\alpha_1,\alpha_2\in\mathbb{Z}_{\geq 0}} \frac{1}{\alpha_1!\alpha_2!}\Big(\frac{\partial^{\alpha_1+\alpha_2}}{\partial\xi_1^{\alpha_1}\partial\xi_2^{\alpha_2}}\rho_1(\xi)\Big)\,\delta_1^{\alpha_1}\delta_2^{\alpha_2}(\rho_2(\xi)).$$

Using this calculus, the parametrix R_λ of $\triangle_\varphi - \lambda$ can be approximated by a recursive procedure. That is, one can start by assuming that the symbol $\sigma(R_\lambda)$ of R_λ has an expansion of the form $\sum_{j=0}^\infty r_j(\xi,\lambda)$, where each r_j is homogeneous of order $-2-j$ in the sense that for any $t > 0$:

$$r_j(t\xi, t^2\lambda) = t^{-2-j}r_j(\xi,\lambda).$$

The reason for starting from order -2 is that $\triangle_\varphi - \lambda$ is of order 2 whose symbol is of the form $(p_2(\xi)-\lambda) + p_1(\xi) + p_0(\xi)$, where as calculated in [15], we have

$$\begin{aligned}
p_2(\xi) &= (\xi_1^2+\xi_2^2)e^h, \\
p_1(\xi) &= 2\xi_1 e^{h/2}\delta_1(e^{h/2}) + 2\xi_2 e^{h/2}\delta_2(e^{h/2}), \\
p_0(\xi) &= e^{h/2}\delta_1^2(e^{h/2}) + e^{h/2}\delta_2^2(e^{h/2}).
\end{aligned} \tag{56}$$

As is explained in detail in [18], using the calculus of symbols mentioned above, one can solve the following equation recursively:

$$\left(\sum_{j=0}^{\infty} r_j(\xi,\lambda)\right) \circ ((p_2(\xi)-\lambda)+p_1(\xi)+p_0(\xi)) \sim 1.$$

In fact, one finds that

$$r_0(\xi,\lambda) = (p_2(\xi)-\lambda)^{-1},$$

and for each $n > 1$:

$$r_n(\xi,\lambda) = -\sum \frac{1}{\alpha_1!\alpha_2!} \frac{\partial^{\alpha_1+\alpha_2} r_j(\xi,\lambda)}{\partial\xi_1^{\alpha_1}\partial\xi_2^{\alpha_2}} \delta_1^{\alpha_1}\delta_2^{\alpha_2}(p_k(\xi))\, r_0(\xi,\lambda),$$

where the summation is over all $\alpha_1, \alpha_2 \in \mathbb{Z}_{\geq 0}$, $j \in \{1,\dots,n-1\}$ and $k \in \{0,1,2\}$ such that $2+j+\alpha_1+\alpha_2-k = n$.

Finally, after finding each r_n recursively, one can show that each $a_{2n} \in C^\infty(\mathbb{T}_\theta^2)$ appearing in the small-time heat kernel expansion (1) can be written as

$$a_{2n} = \frac{1}{2\pi i}\int_{\mathbb{R}^2}\int_\gamma e^{-\lambda} r_{2n}(\xi,\lambda)\, d\lambda\, d\xi. \tag{57}$$

By using a homogeneity argument and classical complex analysis, one can explicitly calculate the integrals involved in the above expression, except for a final part that has a purely noncommutative obstruction. That is, one has to calculate the integral of certain $C^\infty(\mathbb{T}_\theta^2)$-valued functions defined over the positive real line (due to passing to the polar coordinates and the necessity to integrate in the radial direction). This task can be accomplished by the so-called rearrangement lemmas, see [15, 1, 14, 20, 22]. We discuss the necessary rearrangement lemma for the calculation of the term a_4 in the following subsection.

8.2. **Rearrangement lemma.** As we explained above, the appearance of the modular automorphism $\sigma_i(x) = e^{\nabla}(x) = e^{-h}xe^h$, $x \in C(\mathbb{T}_\theta^2)$, in the calculation of the term a_4 comes from the fact that in the process of applying the method explained in the previous subsection, one encounters integrals of $C^\infty(\mathbb{T}_\theta^2)$-valued functions defined on the positive real line. This type of integrals can be worked out by means of the so-called rearrangement lemmas. In Lemma 8.5 we present the rearrangement lemma that is needed for the calculation of the term a_4. However, since we wish to explain a recursive procedure for constructing the functions appearing in this lemma, it is useful to first recall the rearrangement lemma that was used in [14, 20] for the calculation of the term a_2.

Lemma 8.3. *For any*

$$m = (m_0, m_1, \dots, m_n) \in \mathbb{Z}_{>0}^{n+1} \quad \textit{and} \quad \rho_1,\dots,\rho_n \in C^\infty(\mathbb{T}_\theta^2),$$

one has

$$\int_0^\infty (e^h u+1)^{-m_0} \prod_{j=1}^n \rho_j (e^h u+1)^{-m_j} u^{|m|-2}\, du$$
$$= e^{-(|m|-1)h} F_m(\sigma_i, \ldots, \sigma_i)(\rho_1 \cdots \rho_n),$$

where the function F_m is defined by

$$F_m(u_1, \ldots, u_n) = \int_0^\infty \frac{x^{|m|-2}}{(x+1)^{m_0}} \prod_{j=1}^n \Big(x \prod_{\nu=1}^j u_\nu + 1 \Big)^{-m_j} dx.$$

In the following proposition we present functional relations that allow us to construct recursively any function $F_m(u_1, \ldots, u_n)$ appearing in the previous lemma, essentially starting from the generating function of the Bernoulli numbers.

Proposition 8.4. *Considering the functions F_m appearing in the statement of Lemma 8.3, we have*

$$F_{(1,1)}(u_1) = \frac{\log u_1}{u_1 - 1}.$$

If $m_0, m_1 \geq 2$ and $m_3, \ldots, m_n \geq 1$, then

$$F_{(m_0,m_1,\ldots,m_n)}(u_1, u_2, \ldots, u_n) = \frac{1}{u_1-1} F_{(m_0, m_1-1, m_2, \ldots, m_n)}(u_1, \ldots, u_n) - \frac{1}{u_1-1} F_{(m_0-1, m_1, \ldots, m_n)}(u_1, \ldots, u_n).$$

If $m_1 \geq 2$ and $m_2, \ldots, m_n \geq 1$, then

$$F_{(1,m_1,m_2,\ldots,m_n)}(u_1, u_2, \ldots, u_n)$$
$$= \frac{1}{u_1-1} F_{(1,m_1-1,m_2,\ldots,m_n)}(u_1, \ldots, u_n) - \frac{u_1^{-m_1-m_2-\cdots-m_n+1}}{u_1-1} F_{(m_1,m_2,\ldots,m_n)}(u_2, u_3, \ldots, u_n).$$

If $m_0 \geq 2$ and $m_2, \ldots, m_n \geq 1$, then

$$F_{(m_0,1,m_2,\ldots,m_n)}(u_1, u_2, \ldots, u_n)$$
$$= \frac{1}{u_1-1} F_{(m_0,m_2,\ldots,m_n)}(u_1 u_2, u_3, \ldots, u_n) - \frac{1}{u_1-1} F_{(m_0-1,1,m_2,\ldots,m_n)}(u_1, u_2, \ldots, u_n).$$

If $m_2, \ldots, m_n \geq 1$, then

$$F_{(1,1,m_2,\ldots,m_n)}(u_1, u_2, \ldots, u_n)$$
$$= \frac{1}{u_1-1} F_{(1,m_2,\ldots,m_n)}(u_1 u_2, u_3, \ldots, u_n) - \frac{u_1^{-m_2-m_3-\cdots-m_n}}{u_1-1} F_{(m_0-1,1,m_2,\ldots,m_n)}(u_1, u_2, \ldots, u_n).$$

Proof. It follows from considering the definition of the functions and writing a partial fraction decomposition for $\frac{1}{(x+1)(xu+1)}$. ☐

For the calculation of a_4, since the pseudodifferential symbol that we have to use has a different homogeneity order than the one for a_2, we need a variant of Lemma 8.3, which is given in the following lemma.

Lemma 8.5. *For any*

$$m = (m_0, m_1, \dots, m_n) \in \mathbb{Z}_{>0}^{n+1} \quad \textit{and} \quad \rho_1, \dots, \rho_n \in C^\infty(\mathbb{T}_\theta^2),$$

we have

$$\int_0^\infty (e^h u + 1)^{-m_0} \prod_{j=1}^n \rho_j (e^h u + 1)^{-m_j} u^{|m|-3}\, du$$
$$= e^{-(|m|-2)h} F_m^v(\sigma_i, \dots, \sigma_i)(\rho_1 \cdots \rho_n),$$

where the function F_m^v is defined by

$$F_m^v(u_1, \dots, u_n) = \int_0^\infty \frac{x^{|m|-3}}{(x+1)^{m_0}} \prod_{j=1}^n \Big(x \prod_{\nu=1}^j u_\nu + 1 \Big)^{-m_j} dx.$$

Proof. The proof of [14, Lem. 6.2] works verbatim, by choosing the positive numbers α_j such that they add up to 2 instead of 1. ☐

In fact, the functions F_m^v appearing in the statement of Lemma 8.5 are closely related to those appearing in Lemma 8.3.

Proposition 8.6. *If $m_0, m_1 \geq 1$ and $m_3, \dots, m_n \geq 0$, then*

$$F^v_{(m_0, m_1, \dots, m_n)}(u_1, \dots, u_n)$$
$$= \frac{1}{u_1 - 1} F_{(m_0, m_1 - 1, \dots, m_n)}(u_1, \dots, u_n)$$
$$+ \frac{u_1}{u_1 - 1} F_{(m_0 - 1, m_1 \dots, m_n)}(u_1, \dots, u_n).$$

Hence, all functions F_m^v that are necessary for the calculation of the term a_4 can be constructed recursively, as explained in this subsection, from

$$F_{(1,1)}(u_1) = \frac{\log u_1}{u_1 - 1}.$$

The methods and the tools that we have discussed so far in this section allow us to calculate an expression for the term $a_4 \in C^\infty(\mathbb{T}_\theta^2)$ that involves functions of the operator ∇ acting on terms that involve derivatives of the conformal factor $e^h \in C^\infty(\mathbb{T}_\theta^2)$. However, it is important to write the final expression in terms of derivatives of h. This task can be achieved by using Lemma 6.5 to employ the functions G_1, G_2, G_3, G_4, which are constructed in Lemma 6.6 and Lemma 6.7, to find explicitly an expression of the form (6) for the term a_4. We note that such a procedure was performed for the final formula for the term a_2 as well in [14, 20], where making use of the functions G_1 and G_2 was sufficient.

9. Explicit formulas for the local functions describing a_4

After performing the calculation of the term $a_4 \in C^\infty(\mathbb{T}^2_\theta)$ as explained in Section 8, which involved heavy calculations, we find explicit formulas for the functions $K_1, \dots, K_{20}$ that appear in expression (6). In this section we present the explicit formulas for the one and two variable functions $K_1, \dots, K_7$. The lengthy final expressions for the three and four variable functions $K_8, \dots, K_{20}$ are given in Appendix C. The explicit expressions for $K_1, \dots, K_{20}$ are also made available in a Mathematica notebook [8]. Although the final formulas are quite lengthy, it should be noted that they are the results of enormous amount of cancellations and simplifications that typically occur in this type of calculations; cp. [14, 20, 16]. More importantly, in order to confirm the accuracy of the final formulas, we have checked that they satisfy the functional relations stated in Theorem 3.1.

9.1. **The functions of one variable.** The function K_1 has a simple expression:

$$K_1(s_1) = -\frac{4\pi e^{\frac{3s_1}{2}}((4e^{s_1} + e^{2s_1} + 1)s_1 - 3e^{2s_1} + 3)}{(e^{s_1} - 1)^4 s_1}. \tag{58}$$

The graph of this function is given in Figure 3.

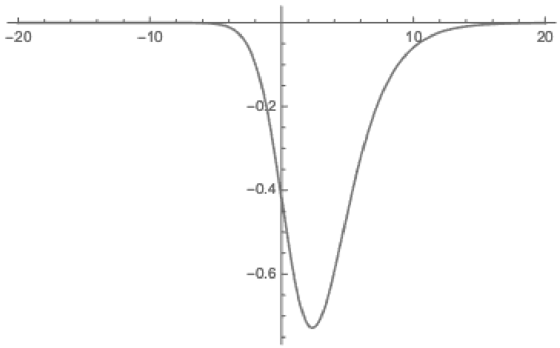

Figure 3. Graph of K_1.

The function K_2 turns out to be a scalar multiple of K_1:

$$K_2(s_1) = -\frac{2\pi e^{\frac{3s_1}{2}}((4e^{s_1} + e^{2s_1} + 1)s_1 - 3e^{2s_1} + 3)}{(e^{s_1} - 1)^4 s_1} = \frac{1}{2} K_1(s_1).$$

9.2. **The functions of two variables.** The functions $K_3, \dots, K_7$ in expression (6) are the two variable functions.

The function K_3 is of the form

$$K_3(s_1, s_2) = \frac{K_3^{\text{num}}(s_1, s_2)}{(e^{s_1} - 1)^2 (e^{s_2} - 1)^2 (e^{s_1+s_2} - 1)^4 s_1 s_2 (s_1 + s_2)}, \tag{59}$$

where the function in the numerator is given by

$$\begin{aligned}
&K_3^{\mathrm{num}}(s_1, s_2)\\
&= 16\, e^{\frac{3s_1}{2}+\frac{3s_2}{2}}\pi\Big[(e^{s_1}-1)(e^{s_2}-1)(e^{s_1+s_2}-1)\\
&\quad\times\big\{(-5e^{s_1}-e^{s_2}+6e^{s_1+s_2}-e^{2s_1+s_2}-5e^{s_1+2s_2}+3e^{2s_1+2s_2}+3)s_1\\
&\quad+(e^{s_1}+5e^{s_2}-6e^{s_1+s_2}+5e^{2s_1+s_2}+e^{s_1+2s_2}-3e^{2s_1+2s_2}-3)s_2\big\}\\
&\quad-2(e^{s_1}-e^{s_2})(e^{s_1+s_2}-1)\\
&\quad\times(-e^{s_1}-e^{s_2}-e^{2s_1+s_2}-e^{s_1+2s_2}+2e^{2s_1+2s_2}+2)s_1s_2\\
&\quad+2e^{s_1}(e^{s_2}-1)^3(e^{s_1}-e^{s_1+s_2}+2e^{2s_1+s_2}-2)s_1^2\\
&\quad-2e^{s_2}(e^{s_1}-1)^3(e^{s_2}-e^{s_1+s_2}+2e^{s_1+2s_2}-2)s_2^2\Big].
\end{aligned}$$

Clearly, the function K_3^{num} is a polynomial in s_1, s_2, $e^{s_1/2}$, $e^{s_2/2}$. By plotting the points (i,j) and (m,n) such that $s_1^i s_2^j e^{ms_1/2}e^{ns_2/2}$ appears in K_3^{num}, we find that these exponents appear in an orderly fashion. This can be seen in Figure 4, where the points (i,j) are the blue dots on the lower left corner, and the points (m,n) are the yellow dots on the upper right-hand side. The graph of the function K_3 is provided in Figure 5.

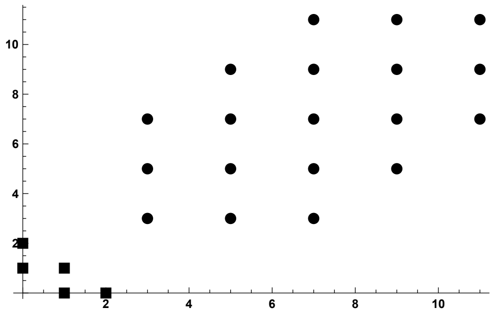

FIGURE 4. The points (i,j) marked by squares and (m,n) marked by circles such that $s_1^i\, s_2^j\, e^{ms_1/2}\, e^{ns_2/2}$ appears in the expression for K_3^{num}.

The function K_4 is given by

$$K_4(s_1,s_2) = \frac{K_4^{\mathrm{num}}(s_1,s_2)}{(e^{s_1}-1)^2(e^{s_2}-1)^2(e^{s_1+s_2}-1)^4 s_1 s_2 (s_1+s_2)},$$

where

$$\begin{aligned}
&K_4^{\mathrm{num}}(s_1,s_2)\\
&= 4e^{\frac{3s_1}{2}+\frac{s_2}{2}}\pi\big[(e^{s_1}-1)(e^{s_2}-1)(e^{s_1+s_2}-1)\big\{\big(5e^{s_2}-3e^{2s_2}-e^{s_1+s_2}\\
&\quad-5e^{2(s_1+s_2)}+2e^{3(s_1+s_2)}+6e^{s_1+2s_2}-3e^{s_1+3s_2}+e^{2s_1+3s_2}-2\big)s_1+
\end{aligned}$$

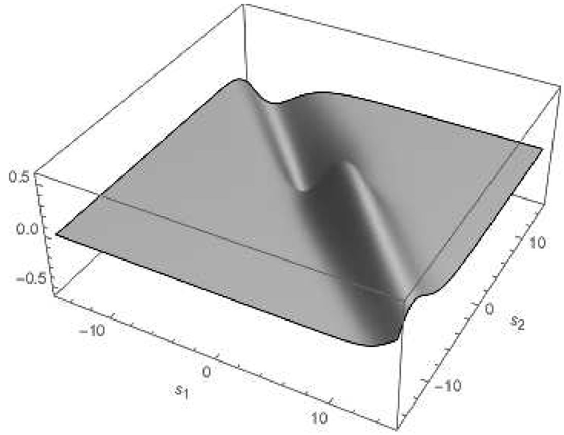

Figure 5. Graph of K_3.

$$\big(-e^{s_2} + 3e^{2s_2} + 5e^{s_1+s_2} - 6e^{s_1+2s_2} + e^{2s_1+2s_2} + 3e^{s_1+3s_2} - 5e^{2s_1+3s_2} + 2e^{3s_1+3s_2} - 2\big)s_2\big\} - (e^{s_1+s_2} - 1)(-2e^{s_2} + e^{s_1+s_2} + 1)\big(e^{s_1} - e^{s_2} + 3e^{s_1+s_2} + 3e^{2(s_1+s_2)} + e^{3(s_1+s_2)} - 6e^{2s_1+s_2} - 2e^{s_1+2s_2} + e^{3s_1+2s_2} - e^{2s_1+3s_2} + 1\big)s_1s_2 + (e^{s_2} - 1)^3(-e^{s_1} - 4e^{s_1+s_2} - e^{2(s_1+s_2)} + 6e^{2s_1+s_2} + e^{3s_1+2s_2} - 1)s_1^2 - e^{2s_2}(e^{s_1} - 1)^3(e^{s_2} - 6e^{s_1+s_2} + e^{2(s_1+s_2)} + 4e^{s_1+2s_2} + e^{2s_1+3s_2} - 1)s_2^2\big].$$

Clearly, the function K_4^{num} is also a polynomial in $s_1, s_2, e^{s_1/2}, e^{s_2/2}$. In Figure 6, the points (i,j) and (m,n) such that $s_1^i s_2^j e^{ms_1/2} e^{ns_2/2}$ appears in K_4^{num} are plotted. The graph of K_4 is given in Figure 7.

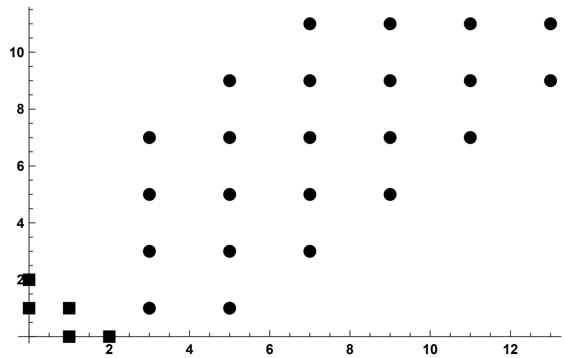

Figure 6. The points (i,j) marked by squares and (m,n) marked by circles such that $s_1^i\, s_2^j\, e^{ms_1/2}\, e^{ns_2/2}$ appears in the expressions for K_4^{num} and K_5^{num}.

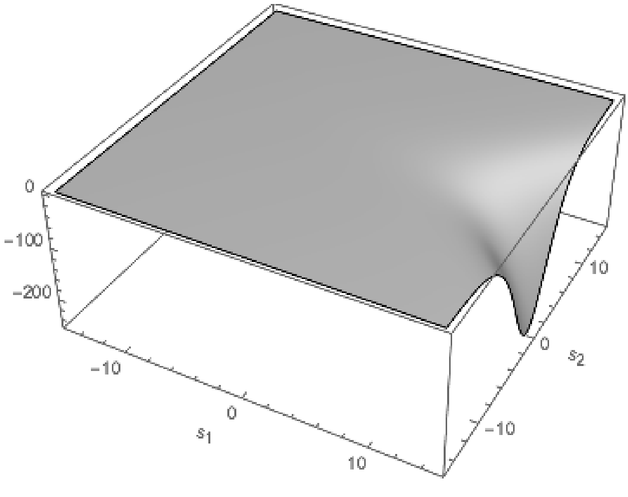

Figure 7. Graph of K_4.

The function K_5 is given by the quotient

$$K_5(s_1, s_2) = \frac{K_5^{\mathrm{num}}(s_1, s_2)}{(e^{s_1}-1)^2(e^{s_2}-1)^2(e^{s_1+s_2}-1)^4 s_1 s_2 (s_1+s_2)},$$

where the function K_5^{num} is given by

$$\begin{aligned}
&K_5^{\mathrm{num}}(s_1, s_2)\\
&\quad = 4e^{\frac{3s_1}{2}+\frac{s_2}{2}}\pi\Big[(e^{s_1}-1)(e^{s_2}-1)(e^{s_1+s_2}-1)\Big\{\big(11e^{s_2}-5e^{2s_2}-11e^{s_1+s_2}\\
&\qquad -7e^{2(s_1+s_2)}+2e^{3(s_1+s_2)}+18e^{s_1+2s_2}-13e^{s_1+3s_2}+7e^{2s_1+3s_2}-2\big)s_1\\
&\qquad +\big(-7e^{s_2}+13e^{2s_2}+7e^{s_1+s_2}-18e^{s_1+2s_2}+11e^{2s_1+2s_2}+5e^{s_1+3s_2}-11e^{2s_1+3s_2}\\
&\qquad +2e^{3s_1+3s_2}-2\big)s_2\Big\}(e^{s_1+s_2}-1)\big(-e^{s_1}+3e^{s_2}+6e^{2s_2}-4e^{3s_2}-10e^{s_1+s_2}\\
&\qquad +9e^{2s_1+s_2}+9e^{s_1+2s_2}-18e^{2s_1+2s_2}+9e^{3s_1+2s_2}-4e^{s_1+3s_2}+9e^{2s_1+3s_2}\\
&\qquad -10e^{3s_1+3s_2}-e^{4s_1+3s_2}-4e^{s_1+4s_2}+6e^{2s_1+4s_2}+3e^{3s_1+4s_2}-e^{4s_1+4s_2}-1\big)s_1 s_2\\
&\qquad +(e^{s_2}-1)^3(-e^{s_1}-12e^{s_1+s_2}+10e^{2s_1+s_2}-5e^{2s_1+2s_2}+9e^{3s_1+2s_2}-1)s_1^2\\
&\qquad e^{2s_2}(e^{s_1}-1)^3(-5e^{s_2}+10e^{s_1+s_2}-12e^{s_1+2s_2}-e^{2s_1+2s_2}-e^{2s_1+3s_2}+9)s_2^2\Big].
\end{aligned}$$

The points (i, j) and (m, n) such that $s_1^i s_2^j e^{ms_1/2} e^{ns_2/2}$ appears in K_5^{num} are the same as those for K_4^{num}, given in Figure 6. The graph of K_5 is given in Figure 8. We note that although the functions K_4 and K_5 are quite different, they however match precisely on the diagonal $s_1 = s_2$.

The function K_6 is of the form

$$K_6(s_1, s_2) = \frac{K_6^{\mathrm{num}}(s_1, s_2)}{(e^{s_1}-1)(e^{s_2}-1)^3(e^{s_1+s_2}-1)^4 s_1 s_2 (s_1+s_2)},$$

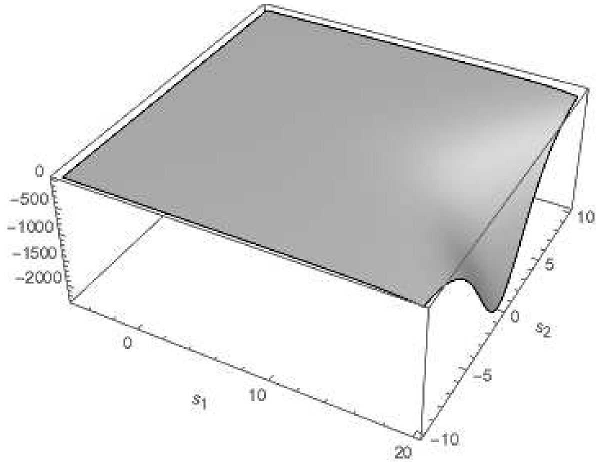

FIGURE 8. Graph of K_5.

where

$$\begin{aligned} &K_6^{\text{num}}(s_1,s_2)\\ &= 4e^{\frac{3s_1}{2}+\frac{3s_2}{2}}\pi\Big[(e^{s_1}-1)(e^{s_2}-1)(e^{s_1+s_2}-1)\big\{(-3e^{s_2}+e^{2s_2}-3e^{s_1+s_2}\\ &\quad -14e^{s_1+2s_2}+e^{2s_1+2s_2}+5e^{s_1+3s_2}+5e^{2s_1+3s_2}+8)s_1+\big(21e^{s_2}-11e^{2s_2}\\ &\quad -15e^{s_1+s_2}+10e^{s_1+2s_2}+e^{2s_1+2s_2}-7e^{s_1+3s_2}+5e^{2s_1+3s_2}-4\big)s_2\big\}\\ &\quad -2\big(-2e^{s_1}+7e^{s_2}-6e^{2s_2}+2e^{3s_2}-6e^{s_1+s_2}+6e^{2s_1+s_2}+2e^{s_1+2s_2}\\ &\quad -6e^{2s_1+2s_2}+2e^{3s_1+2s_2}-10e^{s_1+3s_2}+24e^{2s_1+3s_2}-14e^{3s_1+3s_2}+4e^{s_1+4s_2}\\ &\quad -6e^{2s_1+4s_2}+2e^{3s_1+4s_2}+2e^{4s_1+4s_2}-2e^{3s_1+5s_2}+e^{4s_1+5s_2}\big)s_1s_2\\ &\quad +2e^{s_1}(e^{s_2}-1)^4(e^{s_1+s_2}+2)s_1^2e^{2s_2}(e^{s_1}-1)^2\big(6e^{s_2}-2e^{2s_2}-2e^{s_1+s_2}\\ &\quad +14e^{s_1+2s_2}-6e^{s_1+3s_2}-2e^{2s_1+3s_2}-e^{2s_1+4s_2}-7\big)s_2^2\Big]. \end{aligned}$$

The points (i,j) and (m,n) such that $s_1^i s_2^j e^{ms_1/2}e^{ns_2/2}$ appears in K_6^{num} are presented in Figure 9, and the graph of K_6 is given in Figure 10.

The function K_7 is also a fraction of the form

$$K_7(s_1,s_2)=\frac{K_7^{\text{num}}(s_1,s_2)}{(e^{s_1}-1)^3(e^{s_2}-1)(e^{s_1+s_2}-1)^4s_1s_2(s_1+s_2)},$$

where

$$\begin{aligned} &K_7^{\text{num}}(s_1,s_2)\\ &= 4\pi e^{\frac{3s_1}{2}+\frac{s_2}{2}}\Big[(e^{s_1}-1)(e^{s_2}-1)(e^{s_1+s_2}-1)\big\{\big(-e^{s_1}+7e^{s_2}-10e^{s_1+s_2}\\ &\quad +15e^{2s_1+s_2}+11e^{s_1+2s_2}-21e^{2s_1+2s_2}+4e^{3s_1+2s_2}-5\big)s_1+\big(-e^{s_1}-5e^{s_2} \end{aligned}$$

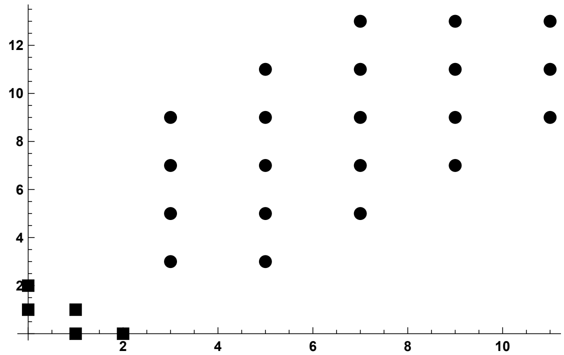

FIGURE 9. The points (i, j) marked by squares and (m, n) marked by circles such that $s_1^i\, s_2^j\, e^{ms_1/2}\, e^{ns_2/2}$ appears in the expression for K_6^{num}.

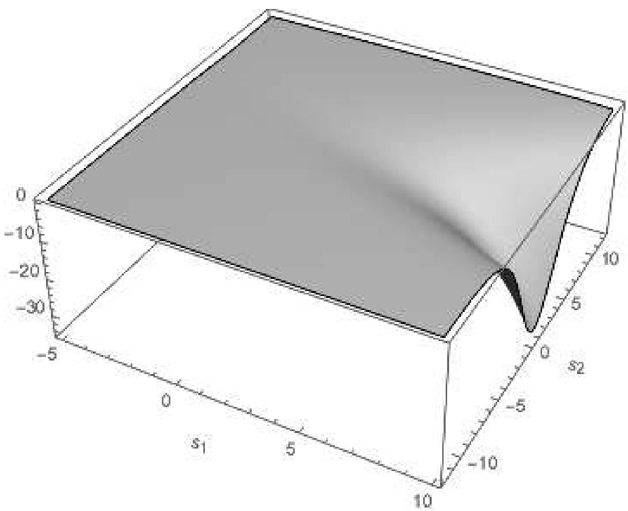

FIGURE 10. Graph of K_6.

$$
\begin{aligned}
&+ 14e^{s_1+s_2} + 3e^{2s_1+s_2} - e^{s_1+2s_2} + 3e^{2s_1+2s_2} - 8e^{3s_1+2s_2} - 5\big)s_2\big\} \\
&- 2\big(-2e^{s_1} + 2e^{s_2} - 2e^{s_1+s_2} - 24e^{2(s_1+s_2)} - 2e^{3(s_1+s_2)} - 7e^{4(s_1+s_2)} \\
&+ 14e^{2s_1+s_2} - 2e^{3s_1+s_2} + 6e^{s_1+2s_2} + 6e^{3s_1+2s_2} - 6e^{4s_1+2s_2} - 4e^{s_1+3s_2} \\
&+ 10e^{2s_1+3s_2} + 6e^{4s_1+3s_2} + 2e^{5s_1+3s_2} - 2e^{2s_1+4s_2} + 6e^{3s_1+4s_2} - 1\big)s_1s_2 \\
&+ 2(e^{s_2} - 1)^2\big(2e^{s_1} + 6e^{s_1+s_2} - 14e^{2s_1+s_2} + 2e^{3s_1+s_2} + 2e^{2s_1+2s_2} \\
&- 6e^{3s_1+2s_2} + 7e^{4s_1+2s_2} + 1\big)s_1^2 - 2e^{2s_2}(e^{s_1} - 1)^4(2e^{s_1+s_2} + 1)s_2^2\Big].
\end{aligned}
$$

The points (i,j) and (m,n) such that $s_1^i s_2^j e^{ms_1/2} e^{ns_2/2}$ appears in $K_7^{\rm num}$ are plotted in Figure 11 and the graph of K_7 is given in Figure 12.

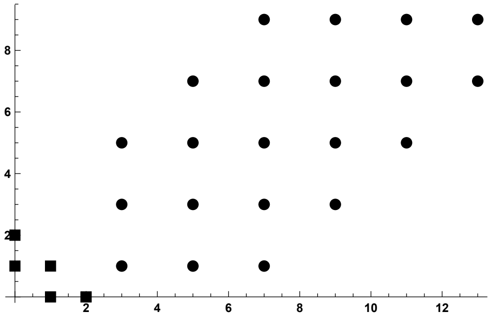

FIGURE 11. The points (i,j) marked by squares and (m,n) marked by circles such that $s_1^i\, s_2^j\, e^{ms_1/2}\, e^{ns_2/2}$ appears in the expression for $K_7^{\rm num}$.

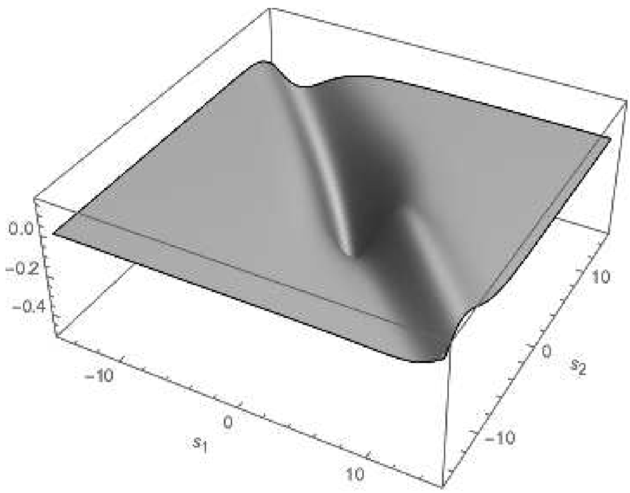

FIGURE 12. Graph of K_7.

10. General structure of the noncommutative local invariants

In Section 9 and in Appendix C, we have presented the explicit formulas for the functions $K_1, \dots, K_{20}$ that appear in the local expression (6) for the term $a_4 \in C^\infty(\mathbb{T}_\theta^2)$. Also, we observed that each function K_j, $j = 1, \dots, 20$, is a rational function in $s_1, \dots, s_n, e^{s_1/2}, \dots, e^{s_n/2}$, where $n \in \{1,2,3,4\}$ is the number of variables that each function depends on. This fact can indeed be justified by using Lemma 6.7 and our results in Section 8 where we construct the main ingredients of our calculations with finite differences from the generating function of the Bernoulli numbers and its inverse $G_1(s_1) = (e^{s_1} - 1)/s_1$. Moreover, these considerations explain why the denominators of the functions K_j have nice product formulas. At the first glance, it might seem odd that the fractions $s_i/2$ appear in the exponents of the calculated functions since finite differences of $G_1(s_1)$ and its inverse yield rational functions in s_i and e^{s_i}. The reason for the appearance of the $e^{s_i/2}$ in the final formulas is that the pseudodifferential symbol of the Laplacian $\triangle_\varphi$, which is given by (56), has $e^{h/2}$ and its derivatives in its expression. Therefore, when we apply the rearrangement lemma, namely Lemma 8.5, to calculate the integral in formula (57) in the case of a_4, the elements ρ_j in the statement of the lemma might be of the form $e^{-h/2}\delta_1(h/2)e^{h/2} = e^{\nabla/2}(\delta_1(h/2))$ or $e^{-3h/2}\delta_1\delta_2(h/2)e^{3h/2} = e^{3\nabla/2}(\delta_1\delta_2(h/2))$, hence the appearance of the fractions $s_i/2$ in the exponents of the final formulas.

This discussion justifies that each function K_j, $j = 1, \ldots, 20$, is a smooth rational function in variables s_i and $e^{s_i/2}$, whose denominator has a concise product formula of a general type; see for example the explicit formulas (58), (59), (73) and (75) presented in Section 9 and in Appendix C. We now wish to argue that the coefficients of the numerator of each function K_j, considering a monomial ordering, belong to a linear space that can potentially be of low dimension. Therefore, up to multiplication by a constant, the concise denominator and the exponents appearing in the numerator can have significant information about each function K_j.

Let us first analyze the function K_1 from this perspective. We have

$$\begin{aligned} K_1(s_1) &= -\frac{4\pi e^{\frac{3s_1}{2}}\left((4e^{s_1} + e^{2s_1} + 1)s_1 - 3e^{2s_1} + 3\right)}{(e^{s_1} - 1)^4 s_1} \\ &= \frac{-4\pi e^{\frac{3s_1}{2}}s_1 - 16\pi e^{\frac{5s_1}{2}}s_1 - 4\pi e^{\frac{7s_1}{2}}s_1 - 12\pi e^{\frac{3s_1}{2}} + 12\pi e^{\frac{7s_1}{2}}}{(e^{s_1} - 1)^4 s_1}. \end{aligned}$$

By considering the monomial ordering of the numerator inherited from the dictionary ordering of the pairs (i, j) such that $s_1^i e^{js_1/2}$ appears in the numerator, we replace the specific coefficients by $c_1, \ldots, c_5$ and write

$$K_1(s_1) = \frac{c_1 e^{\frac{3s_1}{2}} + c_2 e^{\frac{7s_1}{2}} + c_3 e^{\frac{3s_1}{2}}s_1 + c_4 e^{\frac{5s_1}{2}}s_1 + c_5 e^{\frac{7s_1}{2}}s_1}{(e^{s_1} - 1)^4 s_1}.$$

Since for the denominator we have $K_1^{\mathrm{den}}(s_1) = (e^{s_1} - 1)^4 s_1 = O(s_1^5)$, and the function K_1 is smooth, we conclude that

$$c_1 e^{\frac{3s_1}{2}} + c_2 e^{\frac{7s_1}{2}} + c_3 e^{\frac{3s_1}{2}}s_1 + c_4 e^{\frac{5s_1}{2}}s_1 + c_5 e^{\frac{7s_1}{2}}s_1 = O(s_1^5).$$

Therefore, by writing the Taylor series for each term of the left-hand side of the above expression, we have

$$c_1\Big(\frac{27s_1^4}{128}+\frac{9s_1^3}{16}+\frac{9s_1^2}{8}+\frac{3s_1}{2}+1\Big)+c_2\Big(\frac{2401s_1^4}{384}+\frac{343s_1^3}{48}+\frac{49s_1^2}{8}+\frac{7s_1}{2}+1\Big)$$
$$+c_3\Big(\frac{9s_1^4}{16}+\frac{9s_1^3}{8}+\frac{3s_1^2}{2}+s_1\Big)+c_4\Big(\frac{125s_1^4}{48}+\frac{25s_1^3}{8}+\frac{5s_1^2}{2}+s_1\Big)$$
$$+c_5\Big(\frac{343s_1^4}{48}+\frac{49s_1^3}{8}+\frac{7s_1^2}{2}+s_1\Big)=0.$$

This implies that the vector $(c_1, c_2, c_3, c_4, c_5)$ belongs to the kernel of the 5×5 matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 3/2 & 7/2 & 1 & 1 & 1 \\ 9/8 & 49/8 & 3/2 & 5/2 & 7/2 \\ 9/16 & 343/48 & 9/8 & 25/8 & 49/8 \\ 27/128 & 2401/384 & 9/16 & 125/48 & 343/48 \end{pmatrix},$$

whose rank is 4. Therefore, the vector of the coefficients $(c_1, c_2, c_3, c_4, c_5)$ of the numerator of the function K_1 belongs to a one-dimensional linear space, and up to multiplication by a constant, the numerator of K_1 is determined by the exponents appearing in its numerator, which are plotted in Figure 13.

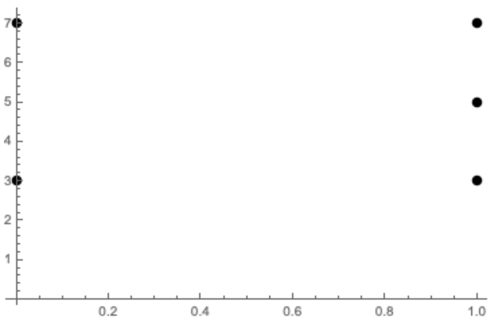

FIGURE 13. The points (i, j) such that $s_1^i e^{js_1/2}$ appears in the numerator of K_1.

A similar analysis can in fact be performed for all functions K_j, $j = 1, \ldots, 20$, as far as each monomial in the numerator cannot afford to provide a non-zero smooth quotient by itself. We just need to confirm that, chosen a number of such monomials for the numerator, a system of linear equations for their coefficients will be equivalent to smoothness of the quotient. This is easy to see as we discuss it for example for the function K_3. This function is given by

$$K_3(s_1, s_2) = \frac{K_3^{\text{num}}(s_1, s_2)}{K_3^{\text{den}}(s_1, s_2)},$$

where

$$K_3^{\mathrm{den}}(s_1,s_2)=(e^{s_1}-1)^2(e^{s_2}-1)^2(e^{s_1+s_2}-1)^4 s_1 s_2(s_1+s_2),$$

and K_3^{num} is the sum of 74 different monomials in $s_1, s_2, e^{s_1/2}e^{s_2/2}$:

$$\begin{aligned}
&K_3^{\mathrm{num}}(s_1,s_2)\\
&\quad=64e^{\frac{5s_1}{2}+\frac{3s_2}{2}}\pi s_1^2-32e^{\frac{7s_1}{2}+\frac{3s_2}{2}}\pi s_1^2-192e^{\frac{5s_1}{2}+\frac{5s_2}{2}}\pi s_1^2+128e^{\frac{7s_1}{2}+\frac{5s_2}{2}}\pi s_1^2\\
&\qquad-64e^{\frac{9s_1}{2}+\frac{5s_2}{2}}\pi s_1^2+192e^{\frac{5s_1}{2}+\frac{7s_2}{2}}\pi s_1^2-192e^{\frac{7s_1}{2}+\frac{7s_2}{2}}\pi s_1^2+192e^{\frac{9s_1}{2}+\frac{7s_2}{2}}\pi s_1^2\\
&\qquad-64e^{\frac{5s_1}{2}+\frac{9s_2}{2}}\pi s_1^2+128e^{\frac{7s_1}{2}+\frac{9s_2}{2}}\pi s_1^2-192e^{\frac{9s_1}{2}+\frac{9s_2}{2}}\pi s_1^2-32e^{\frac{7s_1}{2}+\frac{11s_2}{2}}\pi s_1^2\\
&\qquad+64e^{\frac{9s_1}{2}+\frac{11s_2}{2}}\pi s_1^2+64e^{\frac{5s_1}{2}+\frac{3s_2}{2}}\pi s_2 s_1-32e^{\frac{7s_1}{2}+\frac{3s_2}{2}}\pi s_2 s_1\\
&\qquad-64e^{\frac{3s_1}{2}+\frac{5s_2}{2}}\pi s_2 s_1-64e^{\frac{7s_1}{2}+\frac{5s_2}{2}}\pi s_2 s_1+32e^{\frac{3s_1}{2}+\frac{7s_2}{2}}\pi s_2 s_1\\
&\qquad+64e^{\frac{5s_1}{2}+\frac{7s_2}{2}}\pi s_2 s_1+64e^{\frac{9s_1}{2}+\frac{7s_2}{2}}\pi s_2 s_1+32e^{\frac{11s_1}{2}+\frac{7s_2}{2}}\pi s_2 s_1\\
&\qquad-64e^{\frac{7s_1}{2}+\frac{9s_2}{2}}\pi s_2 s_1-64e^{\frac{11s_1}{2}+\frac{9s_2}{2}}\pi s_2 s_1-32e^{\frac{7s_1}{2}+\frac{11s_2}{2}}\pi s_2 s_1\\
&\qquad+64e^{\frac{9s_1}{2}+\frac{11s_2}{2}}\pi s_2 s_1-48e^{\frac{3s_1}{2}+\frac{3s_2}{2}}\pi s_1+128e^{\frac{5s_1}{2}+\frac{3s_2}{2}}\pi s_1-80e^{\frac{7s_1}{2}+\frac{3s_2}{2}}\pi s_1\\
&\qquad+64e^{\frac{3s_1}{2}+\frac{5s_2}{2}}\pi s_1-192e^{\frac{5s_1}{2}+\frac{5s_2}{2}}\pi s_1+64e^{\frac{7s_1}{2}+\frac{5s_2}{2}}\pi s_1+64e^{\frac{9s_1}{2}+\frac{5s_2}{2}}\pi s_1\\
&\qquad-16e^{\frac{3s_1}{2}+\frac{7s_2}{2}}\pi s_1+128e^{\frac{5s_1}{2}+\frac{7s_2}{2}}\pi s_1-128e^{\frac{9s_1}{2}+\frac{7s_2}{2}}\pi s_1+16e^{\frac{11s_1}{2}+\frac{7s_2}{2}}\pi s_1\\
&\qquad-64e^{\frac{5s_1}{2}+\frac{9s_2}{2}}\pi s_1-64e^{\frac{7s_1}{2}+\frac{9s_2}{2}}\pi s_1+192e^{\frac{9s_1}{2}+\frac{9s_2}{2}}\pi s_1-64e^{\frac{11s_1}{2}+\frac{9s_2}{2}}\pi s_1\\
&\qquad+80e^{\frac{7s_1}{2}+\frac{11s_2}{2}}\pi s_1-128e^{\frac{9s_1}{2}+\frac{11s_2}{2}}\pi s_1+48e^{\frac{11s_1}{2}+\frac{11s_2}{2}}\pi s_1-64e^{\frac{3s_1}{2}+\frac{5s_2}{2}}\pi s_2^2\\
&\qquad+192e^{\frac{5s_1}{2}+\frac{5s_2}{2}}\pi s_2^2-192e^{\frac{7s_1}{2}+\frac{5s_2}{2}}\pi s_2^2+64e^{\frac{9s_1}{2}+\frac{5s_2}{2}}\pi s_2^2+32e^{\frac{3s_1}{2}+\frac{7s_2}{2}}\pi s_2^2\\
&\qquad-128e^{\frac{5s_1}{2}+\frac{7s_2}{2}}\pi s_2^2+192e^{\frac{7s_1}{2}+\frac{7s_2}{2}}\pi s_2^2-128e^{\frac{9s_1}{2}+\frac{7s_2}{2}}\pi s_2^2+32e^{\frac{11s_1}{2}+\frac{7s_2}{2}}\pi s_2^2\\
&\qquad+64e^{\frac{5s_1}{2}+\frac{9s_2}{2}}\pi s_2^2-192e^{\frac{7s_1}{2}+\frac{9s_2}{2}}\pi s_2^2+192e^{\frac{9s_1}{2}+\frac{9s_2}{2}}\pi s_2^2-64e^{\frac{11s_1}{2}+\frac{9s_2}{2}}\pi s_2^2\\
&\qquad+48e^{\frac{3s_1}{2}+\frac{3s_2}{2}}\pi s_2-64e^{\frac{5s_1}{2}+\frac{3s_2}{2}}\pi s_2+16e^{\frac{7s_1}{2}+\frac{3s_2}{2}}\pi s_2-128e^{\frac{3s_1}{2}+\frac{5s_2}{2}}\pi s_2\\
&\qquad+192e^{\frac{5s_1}{2}+\frac{5s_2}{2}}\pi s_2-128e^{\frac{7s_1}{2}+\frac{5s_2}{2}}\pi s_2+64e^{\frac{9s_1}{2}+\frac{5s_2}{2}}\pi s_2+80e^{\frac{3s_1}{2}+\frac{7s_2}{2}}\pi s_2\\
&\qquad-64e^{\frac{5s_1}{2}+\frac{7s_2}{2}}\pi s_2+64e^{\frac{9s_1}{2}+\frac{7s_2}{2}}\pi s_2-80e^{\frac{11s_1}{2}+\frac{7s_2}{2}}\pi s_2-64e^{\frac{5s_1}{2}+\frac{9s_2}{2}}\pi s_2\\
&\qquad+128e^{\frac{7s_1}{2}+\frac{9s_2}{2}}\pi s_2-192e^{\frac{9s_1}{2}+\frac{9s_2}{2}}\pi s_2+128e^{\frac{11s_1}{2}+\frac{9s_2}{2}}\pi s_2\\
&\qquad-16e^{\frac{7s_1}{2}+\frac{11s_2}{2}}\pi s_2+64e^{\frac{9s_1}{2}+\frac{11s_2}{2}}\pi s_2-48e^{\frac{11s_1}{2}+\frac{11s_2}{2}}\pi s_2.
\end{aligned}$$

By considering the monomial ordering of the numerator K_3^{num} inherited from the dictionary ordering of the 4-tuples $(\alpha_i,\beta_i,\gamma_i,\nu_i)$ such that

$$s_1^{\alpha_i}s_2^{\beta_i}e^{\gamma_i s_1/2}e^{\nu_i s_2/2}$$

appears in the numerator, let us replace the specific coefficients by $c_1, \ldots, c_{74}$ and write

$$K_3^{\mathrm{num}}(s_1, s_2) = \sum_{i=1}^{74} c_i \, s_1^{\alpha_i} \, s_2^{\beta_i} \, e^{\gamma_i s_1/2} \, e^{\nu_i s_2/2}.$$

By passing to the polar coordinates

$$s_1 = r\cos\theta, \quad s_2 = r\sin\theta,$$

for the denominator of K_3 we have

$$K_3^{\mathrm{den}}(s_1, s_2) = (e^{s_1} - 1)^2 (e^{s_2} - 1)^2 (e^{s_1+s_2} - 1)^4 s_1 s_2 (s_1 + s_2) = O(r^{11}),$$

which, combined with the smoothness of the function K_3 at the origin, implies that

$$K_3^{\mathrm{num}}(s_1, s_2) = \sum_{i=1}^{74} c_i \, s_1^{\alpha_i} \, s_2^{\beta_i} \, e^{\gamma_i s_1/2} \, e^{\nu_i s_2/2} = O(r^{11}).$$

By considering the Taylor series of each term in the middle expression in the latter, we obtain an 11×74 matrix, whose entries are trigonometric functions of θ, and the vector $(c_1, \ldots, c_{74})$ of the coefficients of $K_3^{\mathrm{num}}(s_1, s_2)$ belongs to the kernel of this matrix for each θ. These linear equations only correspond to smoothness of the following mapping at the origin:

$$(s_1, s_2) \mapsto \frac{\sum_{i=1}^{74} c_i \, s_1^{\alpha_i} \, s_2^{\beta_i} \, e^{\gamma_i s_1/2} \, e^{\nu_i s_2/2}}{K_3^{\mathrm{den}}(s_1, s_2)}. \tag{60}$$

Since the zeros of $K_3^{\mathrm{den}}(s_1, s_2)$ are located on the lines $s_1 = 0$, $s_2 = 0$, $s_1 + s_2 = 0$, we still need to observe that there are further linear equations that guarantee smoothness of the mapping given by (60). This can also be seen as follows. Since in the Taylor series of $K_3^{\mathrm{den}}(s_1, s_2)$, the coefficients of $s_1^p s_2^q$ are zero for $p, q \in \{0, 1, 2, 3, 4, 5\}$, the corresponding coefficients are required to be zero in the Taylor series of $\sum_{i=1}^{74} c_i \, s_1^{\alpha_i} \, s_2^{\beta_i} \, e^{\gamma_i s_1/2} \, e^{\nu_i s_2/2}$ in order to have a smooth quotient, which settles the fact that smoothness of the quotient on $s_1 = 0$ and $s_2 = 0$ is equivalent to a system of linear equations for the coefficients $c_1, \ldots, c_{74}$. Also, in order to treat the smoothness of the quotient (60) on the line $s_1 + s_2 = 0$, we note that

$$\partial_1^j K_3^{\mathrm{den}}(-s_2, s_2) = \partial_2^j K_3^{\mathrm{den}}(s_1, -s_1) = 0, \quad 0 \le j \le 4.$$

Therefore, the limit of the map given by (60) as $s_1 \to -s_2$ or $s_2 \to -s_1$ exists if and only if for any $0 \le j \le 4$:

$$\begin{aligned} &\partial_1^j \Big(\sum_{i=1}^{74} c_i \, s_1^{\alpha_i} \, s_2^{\beta_i} \, e^{\gamma_i s_1/2} \, e^{\nu_i s_2/2} \Big) \Big|_{s_1=-s_2} \\ &= \partial_2^j \Big(\sum_{i=1}^{74} c_i \, s_1^{\alpha_i} \, s_2^{\beta_i} \, e^{\gamma_i s_1/2} \, e^{\nu_i s_2/2} \Big) \Big|_{s_2=-s_1} \\ &= 0. \end{aligned}$$

This finishes the justification of our claim that the smoothness of the quotient map given by (60) is equivalent to a system of linear equations for the coefficients $c_1, \dots, c_{74}$.

Since the condition and the arguments used above apply to all functions appearing in expression (6) for the term a_4, and based on the generality of the arguments given for proving Lemma 6.7 and the results in Section 8, it is natural to expect a similar phenomena to occur for the following terms in the expansion (1). That is, we predict that each term $a_{2n} \in C^\infty(\mathbb{T}^2_\theta)$ appearing in this expansion is described by smooth rational functions of variables $s_1, \dots, s_{2n}, e^{s_1/2}, \dots, e^{s_{2n}/2}$, whose denominators have concise product formulas, which vanish on certain hyperplanes, and the coefficients in the numerator of each function satisfy a family of linear equations.

11. The term a_4 for certain noncommutative four tori

As a corollary of the main calculation of the present paper one obtains, for noncommutative four tori with curved metrics of the product form $\mathbb{T}^2_{\theta'} \times \mathbb{T}^2_{\theta''}$, the explicit form of the local geometric invariant given by the term a_4. As explained below, this required the computation of the term a_4 for each component of the product. The reason why this gives a first hint of the analog of the Riemann curvature in the general noncommutative twisted case is that product metrics of the above form are in general not conformally flat while in the traditional Riemannian case the term a_4 already involves complicated expressions [21, Thm. 4.8.18], in terms of the curvature tensor. More explicitly the classical (commutative) version of the product metric that we consider is the following. Using the local coordinates $(x_1, y_1, x_2, y_2) \in \mathbb{T}^4 = (\mathbb{R}/2\pi\mathbb{Z})^4$ on the four torus, the metric is written as

$$g = e^{-h_1(x_1,y_1)}(dx_1^2 + dy_1^2) + e^{-h_2(x_2,y_2)}(dx_2^2 + dy_2^2),$$

where h_1 and h_2 are smooth real-valued functions. The following non-vanishing components of the Weyl curvature tensor of this metric determine its Weyl curvature:

$$\begin{aligned}
C_{1212} &= \frac{1}{6}e^{-h_1(x_1,y_1)}\partial_{y_1}^2 h_1(x_1,y_1) + \frac{1}{6}e^{h_2(x_2,y_2)-2h_1(x_1,y_1)}\partial_{y_2}^2 h_2(x_2,y_2) \\
&\quad + \frac{1}{6}e^{-h_1(x_1,y_1)}\partial_{x_1}^2 h_1(x_1,y_1) + \frac{1}{6}e^{h_2(x_2,y_2)-2h_1(x_1,y_1)}\partial_{x_2}^2 h_2(x_2,y_2), \\
C_{1313} &= -\frac{1}{2}e^{-h_2(x_2,y_2)+h_1(x_1,y_1)}C_{1212}, \\
C_{2424} &= C_{2323} = C_{1414} = C_{1313}, \\
C_{3434} &= e^{-2h_2(x_2,y_2)+2h_1(x_1,y_1)}C_{1212}.
\end{aligned}$$

In the noncommutative case for the product metric on $\mathbb{T}^2_{\theta'} \times \mathbb{T}^2_{\theta''}$, the modular automorphism groups of both factors combine to give an action of $\mathbb{R}^2$ which defines a more refined twisting than the classical determinant twisting of spectral triples, which itself corresponds to the restriction of the above action of $\mathbb{R}^2$ to the diagonal $\mathbb{R} \subset \mathbb{R}^2$. This begs to investigate the more general notion of twisting suggested in particular in [4] and which plays a fundamental role in the work of H. Moscovici and the first author [11] on the transverse geometry of foliations and the reduction by duality to the almost isometric case.

For now we simply explain how to derive the a_4 term for the product metric on $\mathbb{T}^2_{\theta'} \times \mathbb{T}^2_{\theta''}$. That is, let us consider a noncommutative four torus of the form $\mathbb{T}^2_{\theta'} \times \mathbb{T}^2_{\theta''}$, whose algebra has four unitary generators U_1, V_1, U_2, V_2 such that each element of the pair (U_1, V_1) commutes with each element of the pair (U_2, V_2), and we have the following commutation relations for fixed irrational real numbers θ' and θ'':

$$V_1\, U_1 = e^{2\pi i\theta'}\, U_1\, V_1, \quad V_2\, U_2 = e^{2\pi i\theta''}\, U_2\, V_2.$$

By conformally perturbing the flat metric on each two torus factor of such a noncommutative four torus, one obtains the Laplacian associated with a curved metric on this noncommutative space. That is, by using conformal factors $e^{-h'}$ and $e^{-h''}$, where h' and h'' are respectively selfadjoint elements in $C^\infty(\mathbb{T}^2_{\theta'})$ and $C^\infty(\mathbb{T}^2_{\theta''})$, we can consider the corresponding perturbed Laplacians $\triangle_{\varphi'}$ and $\triangle_{\varphi''}$ of the form given by (4), and form the following Laplacian on the noncommutative four torus:

$$\triangle_{\varphi',\varphi''} = \triangle_{\varphi'} \otimes 1 + 1 \otimes \triangle_{\varphi''}.$$

As the notation suggests, φ' and φ'' are respectively the states on $C(\mathbb{T}^2_{\theta'})$ and $C(\mathbb{T}^2_{\theta''})$ obtained from the corresponding canonical traces φ_0' and φ_0'', using the conformal factors $e^{-h'}$ and $e^{-h''}$. There are unique elements

$$a_{2n} \in C^\infty(\mathbb{T}^2_{\theta'} \times \mathbb{T}^2_{\theta''})$$

such that for any $a \in C^\infty(\mathbb{T}^2_{\theta'} \times \mathbb{T}^2_{\theta''})$, as $t \to 0^+$, we have

(61)
$$\begin{aligned}&\mathrm{Trace}(a \exp(-t\triangle_{\varphi',\varphi''})) \\ &\quad\sim t^{-2}((\varphi_0' \otimes \varphi_0'')(a\, a_0) + (\varphi_0' \otimes \varphi_0'')(a\, a_2)\, t + (\varphi_0' \otimes \varphi_0'')(a\, a_4)\, t^2 + \cdots).\end{aligned}$$

The term a_4 is the most fundamental and desirable term in this expansion since it is the first term in which the analog of the Riemann curvature tensor manifests itself, whereas a_0 and a_2 are only the analogs of the volume form and the scalar curvature, respectively.

An extremely difficult way of calculating the desired a_4 appearing in the expansion (61) is to use the method described in Section 8, now in a four-dimensional case. However, the expansion (1) confirms the existence of the unique elements $a_{2n}' \in C^\infty(\mathbb{T}^2_{\theta'})$ and $a_{2n}'' \in C^\infty(\mathbb{T}^2_{\theta''})$ such that for any $a' \in C^\infty(\mathbb{T}^2_{\theta'})$ and $a'' \in C^\infty(\mathbb{T}^2_{\theta''})$, we have the following small-time expansions:

$$\mathrm{Trace}(a' \exp(-t\triangle_{\varphi'})) \sim t^{-1}(\varphi_0'(a'\, a_0') + \varphi_0'(a'\, a_2')\, t + \varphi_0'(a'\, a_4')\, t^2 + \cdots),$$
$$\mathrm{Trace}(a'' \exp(-t\triangle_{\varphi''})) \sim t^{-1}(\varphi_0''(a''\, a_0'') + \varphi_0''(a''\, a_2'')t + \varphi_0''(a''\, a_4'')t^2 + \cdots).$$

Therefore, the uniqueness of the terms a_{2n} appearing in the expansion (61), combined with these expansions and making use of simple tensors as test elements, readily implies that we have

$$a_{2n} = \sum_{i=0}^{n} a'_{2i} \otimes a''_{2(n-i)} \in C^\infty(\mathbb{T}^2_{\theta'} \times \mathbb{T}^2_{\theta''}).$$

We recall from [19] that the terms a'_0 and a''_0 are quite easy to calculate, namely $a'_0 = \pi e^{-h'}$ and $a''_0 = \pi e^{-h''}$. The terms a'_2 and a''_2 are given by (5) which was calculated in [14, 20], and, most importantly, the a'_4 and a''_4 are now given explicitly by formula (6), emphasizing that we have achieved explicit formulas for all components of the latter. Hence, we have available an explicit formula for the term

$$a_4 = a'_0 \otimes a''_4 + a'_2 \otimes a''_2 + a'_4 \otimes a''_0 \in C^\infty(\mathbb{T}^2_{\theta'} \times \mathbb{T}^2_{\theta''}),$$

which appears in the heat kernel expansion (61) associated with a curved noncommutative geometry of spectral dimension four, hence explicit information about the analog of the Riemann curvature tensor in the noncommutative setting.

12. Conclusions

The local geometric invariants of the noncommutative two torus $\mathbb{T}^2_\theta$ equipped with a conformally flat metric have complicated dependence upon the modular automorphism of the state that encodes the volume form of the metric and plays the role of the Weyl factor. This dependence involves lengthy several variable functions of the modular automorphism of the state. In this paper we have calculated this dependence for the invariant $a_4 \in C^\infty(\mathbb{T}^2_\theta)$, which determines the third term in the small-time asymptotic expansion of the trace of the heat kernel of the Laplacian associated with the conformally flat metric. After performing heavy calculations and simplifying the result, we have confirmed the accuracy of the final functions appearing in the expression for the term a_4 by checking that they satisfy a family of functional relations which were abstractly predicted before performing the computation. The derivation of these functional relations is based on using a fundamental spectral identity proved in [14] for the gradient of a functional, and by calculating the same gradient with finite differences. This method was indeed used in [14] for checking the validity of the term a_2, which is related to the scalar curvature for $\mathbb{T}^2_\theta$. However, the tools needed for performing this check for the term a_4 and the consequent functional relations are far more involved and complicated, compared to the case of the term a_2. Also by studying the functional relations abstractly, we have derived a partial differential system, on which the cyclic groups of order 2, 3 and 4 act naturally and we have studied invariance properties and symmetries of the calculated expressions with respect to these actions. Moreover, we have found a natural flow that is associated with the differential system.

We stress that the basic functional relations for the functions $\widetilde{K}_1, \ldots, \widetilde{K}_{20}$ stated in Theorem 3.1 were conceptually predicted, and lead to the differential system and the discovery of symmetric expressions, when studied abstractly. However, the functional relations among the functions $k_3, \ldots, k_{20}$ stated in Theorem 7.1 were found after comparing the final explicit formulas for the functions $K_3, \ldots, K_{20}$ and it is an open question to understand them a priori in a conceptual manner.

We have shown that the main ingredients of our calculations can be derived by finite differences from the generating function of the Bernoulli numbers and its inverse, and we have paid special attention to the general structure of the final functions that describe the term a_4. That is, due to noncommutativity, several variable functions of the modular automorphism are needed for integrating certain $C^\infty(\mathbb{T}_\theta^2)$-valued functions defined on the positive real line, and for writing the final formula for the term a_4 in terms of the derivatives of the logarithm of the conformal factor. We have argued that each of the final functions is a rational function of variables s_i and $e^{s_i/2}$, whose denominator has a concise product formula, and the coefficients of its numerator satisfy a family of linear equations, which, given a monomial ordering, restrict them to belong to a potentially low-dimensional linear space. This begs for bringing to the front scene the algebraic geometry, familiar in transcendence theory [26], of exponential polynomials and their smooth fractions in understanding the general structure of the noncommutative local invariants.

It is also important to stress that, because of their geometric nature, the terms a_{2n} in (61) are local geometric invariants of the curved metric on the noncommutative four torus in the following sense. In noncommutative geometry the notion of locality is less obvious than the classical naive notion. This notion nevertheless makes sense based on the Fourier transform, and what it means concretely in the above context is that only the high frequency contributions are relevant in the computation of the coefficients appearing in the expansion of the trace of the heat kernel. This illuminates the locality of the terms $a_{2n} \in C^\infty(\mathbb{T}_\theta^2)$ since when one considers using the pseudodifferential calculus built on a phase space with a noncommutative configuration space [3], the high frequencies appear preponderantly in the derivation of the heat expansion (61). This locality principle is already at the core of the local index formula of [10] and also plays an important simplifying role in the case of quantum groups.

As is well known in the case of Riemannian manifolds, the local geometric invariants appearing in the heat expansion are certain expressions in terms of the Riemann curvature tensor and its contractions and covariant derivatives. These expressions are very complicated. However, because of their local nature, one can identify them using invariance theory [21], noting that only a few terms have in practice been identified due to the rapid growth in complexity of the formulas. As indicated earlier, the first two terms are given by the volume form and the scalar curvature, and the third term, the analog of which in our noncommutative setting is the term a_4, is the first place where the Riemann curvature tensor manifests itself beyond the curvature scalar. Thus, we view the results achieved in this paper as an important step towards the exploration of further properties of the analog of the Riemann curvature tensor in noncommutative geometry as explained briefly in Section 11 for four tori. This paper should be viewed as providing a vast reservoir of concrete data, in the form of the obtained explicit formulas, which can be exploited further for testing ideas. In fact, research in this area of noncommutative geometry possesses an experimental nature, and we share in a Mathematica notebook [8] the invaluable resource of data that we have obtained for the a_4 term of the heat expansion. This will pave the way for researchers to discover many more and far different phenomena in noncommutative differential geometry [5, 7, 10], awaiting to be discovered in our data. Our final formulas are securely tested since they satisfy a highly nontrivial family of conceptually predicted functional relations, derived by comparing the outcomes of two different abstract calculations of a gradient. This method, first introduced in [14], plays in our setting the role of invariance theory in the classical case for performing a check on a calculated expression. The symmetries that we have discovered in the calculated expressions are quite striking and are closely related to the appearance of the action of the cyclic groups in the differential system. Note also that considering the work carried out in [24], our calculations for noncommutative tori are universal in the sense that they extend to noncommutative toric manifolds [9].

We end our conclusions by pointing out that while the results of this paper have a very concrete aspect, they suggest, as explained above in Section 11 for four tori, the need to explore at the conceptual level the more general twisting of spectral triples already arising in the transverse geometry of foliations, which goes well beyond the non-tracial type III nature of the measure theory since the latter is limited to the behavior of the determinant of the metric. The theory of twisted spectral triples has reached a satisfactory status [13, 2, 15, 25, 17]. The general case appears as an open land where the case of the transverse geometry of foliations provides a wealth of examples and where general principles such as the reduction to the almost isometric case [4, 11] should be valid in general and allow one to apply the local geometric methods of [5, 10, 12].

Appendix A. Lengthy functional relations

We explained in Section 3 that like the basic functional relations given respectively by equations (19) and (20) for the functions $\widetilde{K}_9$ and $\widetilde{K}_{17}$, the remaining three and four variable functions have algebraically lengthy basic functional identities. We present these lengthy expressions in this appendix.

A.1. **Functional relations for $\widetilde{K}_9, \dots, \widetilde{K}_{16}$.** First we cover the remaining three variable functions.

A.1.1. *The function $\widetilde{K}_9$.* We have

$$\begin{aligned}\widetilde{K}_9(s_1,s_2,s_3) \\ &= \frac{1}{15}(-4)\pi G_3(s_1,s_2,s_3)+\frac{1}{4}e^{s_3}G_2(s_1,s_2)k_3(-s_3)\end{aligned} \tag{62}$$

$$
\begin{aligned}
&- \frac{e^{s_3}(e^{s_2}s_1k_3(-s_2-s_3)+e^{s_2}s_2k_3(-s_2-s_3)-e^{s_1+s_2}s_2k_3(-s_1-s_2-s_3)-s_1k_3(-s_3))}{4s_1s_2(s_1+s_2)}\\
&+ \frac{1}{4}G_2(s_1,s_2)k_3(s_3)+\frac{G_1(s_1)(k_3(s_3)-k_3(s_2+s_3))}{4s_2}\\
&+ \frac{s_1k_3(s_3)-s_1k_3(s_2+s_3)-s_2k_3(s_2+s_3)+s_2k_3(s_1+s_2+s_3)}{4s_1s_2(s_1+s_2)}-G_2(s_1,s_2)k_6(s_3)\\
&+ \frac{G_1(s_1)(k_6(s_2)-k_6(s_2+s_3))}{2s_3}+\frac{k_6(s_2)-k_6(s_1+s_2)-k_6(s_2+s_3)+k_6(s_1+s_2+s_3)}{2s_1s_3}\\
&+ \frac{-s_3k_6(s_1)+s_2k_6(s_1+s_2)+s_3k_6(s_1+s_2)-s_2k_6(s_1+s_2+s_3)}{2s_2s_3(s_2+s_3)}\\
&+ \frac{-s_1k_6(s_3)+s_1k_6(s_2+s_3)+s_2k_6(s_2+s_3)-s_2k_6(s_1+s_2+s_3)}{s_1s_2(s_1+s_2)}\\
&+ \frac{e^{s_2}G_1(s_1)(k_7(-s_2)-e^{s_3}k_7(-s_2-s_3))}{2s_3}\\
&+ \frac{e^{s_2}(-e^{s_1}k_7(-s_1-s_2)+k_7(-s_2)-e^{s_3}k_7(-s_2-s_3)+e^{s_1+s_3}k_7(-s_1-s_2-s_3))}{4s_1s_3}\\
&- \frac{e^{s_1}(s_3k_7(-s_1)-e^{s_2}s_2k_7(-s_1-s_2)-e^{s_2}s_3k_7(-s_1-s_2)+e^{s_2+s_3}s_2k_7(-s_1-s_2-s_3))}{2s_2s_3(s_2+s_3)}\\
&- \frac{e^{s_2}(e^{s_1}k_7(-s_1-s_2)-k_7(-s_2)+e^{s_3}k_7(-s_2-s_3)-e^{s_1+s_3}k_7(-s_1-s_2-s_3))}{4s_1s_3}\\
&+ \frac{e^{s_3}G_1(s_1)(e^{s_2}k_7(-s_2-s_3)-k_7(-s_3))}{s_2}-e^{s_3}G_2(s_1,s_2)k_7(-s_3)\\
&+ \frac{e^{s_3}(e^{s_2}s_1k_7(-s_2-s_3)+e^{s_2}s_2k_7(-s_2-s_3)-e^{s_1+s_2}s_2k_7(-s_1-s_2-s_3)-s_1k_7(-s_3))}{s_1s_2(s_1+s_2)}\\
&+ \frac{1}{4}G_1(s_1)k_8(s_2,s_3)+\frac{k_8(s_2,s_3)-k_8(s_1+s_2,s_3)}{4s_1}+\frac{k_8(s_1+s_2,s_3)-k_8(s_1,s_2+s_3)}{4s_2}\\
&+ \frac{(-1+e^{s_1+s_2+s_3})k_9(s_1,s_2)}{8(s_1+s_2+s_3)}+\frac{k_9(s_1,s_2+s_3)-k_9(s_1,s_2)}{8s_3}+\frac{1}{8}G_1(s_1)k_9(s_2,s_3)\\
&+ \frac{k_9(s_2,s_3)-k_9(s_1+s_2,s_3)}{8s_1}+\frac{k_9(s_1+s_2,s_3)-k_9(s_1,s_2+s_3)}{8s_2}-\frac{1}{8}e^{s_2+s_3}G_1(s_1)k_9(-s_2-s_3,s_2)\\
&+ \frac{e^{s_1+s_2+s_3}(k_9(-s_1-s_2-s_3,s_1)-k_9(-s_1-s_2-s_3,s_1+s_2))}{8s_2}-\frac{1}{8}G_1(s_1)k_{10}(s_2,s_3)\\
&+ \frac{k_{10}(s_1,s_2+s_3)-k_{10}(s_1+s_2,s_3)}{8s_2}+\frac{k_{10}(s_1+s_2,s_3)-k_{10}(s_2,s_3)}{8s_1}\\
&+ \frac{1}{8}e^{s_2}G_1(s_1)k_{10}(s_3,-s_2-s_3)+\frac{e^{s_2}(k_{10}(s_3,-s_2-s_3)-e^{s_1}k_{10}(s_3,-s_1-s_2-s_3))}{8s_1}\\
&+ \frac{e^{s_1}(e^{s_2}k_{10}(s_3,-s_1-s_2-s_3)-k_{10}(s_2+s_3,-s_1-s_2-s_3))}{8s_2}+\frac{1}{4}e^{s_2}G_1(s_1)k_{11}(s_3,-s_2-s_3)\\
&+ \frac{e^{s_2}(k_{11}(s_3,-s_2-s_3)-e^{s_1}k_{11}(s_3,-s_1-s_2-s_3))}{4s_1}\\
&+ \frac{e^{s_1}(e^{s_2}k_{11}(s_3,-s_1-s_2-s_3)-k_{11}(s_2+s_3,-s_1-s_2-s_3))}{4s_2}+\frac{1}{4}e^{s_2+s_3}G_1(s_1)k_{12}(-s_2-s_3,s_2)\\
&+ \frac{e^{s_2+s_3}(k_{12}(-s_2-s_3,s_2)-e^{s_1}k_{12}(-s_1-s_2-s_3,s_1+s_2))}{4s_1}+\frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{13}(-s_2-s_3,s_2)\\
&+ \frac{e^{s_2+s_3}(k_{13}(-s_2-s_3,s_2)-e^{s_1}k_{13}(-s_1-s_2-s_3,s_1+s_2))}{8s_1}-\frac{1}{8}e^{s_2}G_1(s_1)k_{13}(s_3,-s_2-s_3)\\
&- \frac{1}{16}e^{s_1}k_{17}(s_2,s_3,-s_1-s_2-s_3)-\frac{1}{16}e^{s_1+s_2+s_3}k_{17}(-s_1-s_2-s_3,s_1,s_2)\\
&- \frac{1}{16}k_{18}(s_1,s_2,s_3)-\frac{1}{16}e^{s_1}k_{18}(s_2,s_3,-s_1-s_2-s_3)-\frac{1}{16}e^{s_1+s_2+s_3}k_{18}(-s_1-s_2-s_3,s_1,s_2)\\
&- \frac{1}{16}e^{s_1+s_2}k_{18}(s_3,-s_1-s_2-s_3,s_1)-\frac{1}{16}k_{19}(s_1,s_2,s_3)-\frac{1}{16}e^{s_1+s_2}k_{19}(s_3,-s_1-s_2-s_3,s_1)\\
&- \frac{e^{s_2+s_3}(k_9(-s_2-s_3,s_2)-e^{s_1}k_9(-s_1-s_2-s_3,s_1+s_2))}{8s_1}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{e^{s_2}(k_{13}(s_3,-s_2-s_3)-e^{s_1}k_{13}(s_3,-s_1-s_2-s_3))}{8s_1} - \frac{G_1(s_1)(k_6(s_3)-k_6(s_2+s_3))}{s_2}\\
&- \frac{e^{s_3}G_1(s_1)(e^{s_2}k_3(-s_2-s_3)-k_3(-s_3))}{4s_2}\\
&- \frac{e^{s_1+s_2+s_3}(k_{12}(-s_1-s_2-s_3,s_1)-k_{12}(-s_1-s_2-s_3,s_1+s_2))}{4s_2}\\
&- \frac{e^{s_1+s_2+s_3}(k_{13}(-s_1-s_2-s_3,s_1)-k_{13}(-s_1-s_2-s_3,s_1+s_2))}{8s_2}\\
&- \frac{e^{s_1}(e^{s_2}k_{13}(s_3,-s_1-s_2-s_3)-k_{13}(s_2+s_3,-s_1-s_2-s_3))}{8s_2}\\
&- \frac{e^{s_1}(k_{10}(s_2,-s_1-s_2)-k_{10}(s_2+s_3,-s_1-s_2-s_3))}{8s_3}\\
&- \frac{e^{s_1+s_2}(k_{13}(-s_1-s_2,s_1)-e^{s_3}k_{13}(-s_1-s_2-s_3,s_1))}{8s_3}\\
&- \frac{e^{s_1+s_2+s_3}(k_9(s_1,s_2)-k_9(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)} - \frac{e^{s_1}(k_{10}(s_2,-s_1-s_2)-k_{10}(s_2,s_3))}{8(s_1+s_2+s_3)}\\
&- \frac{e^{s_1+s_2}(k_{13}(-s_1-s_2,s_1)-k_{13}(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)}.
\end{aligned}
$$

A.1.2. *The function $\widetilde{K}_{10}$.* We have

(63)

$$
\begin{aligned}
&\widetilde{K}_{10}(s_1,s_2,s_3)\\
&\quad = \frac{1}{15}(-4)\pi G_3(s_1,s_2,s_3)\\
&\qquad + \frac{e^{s_2}(e^{s_1}k_3(-s_1-s_2)-k_3(-s_2)+e^{s_3}k_3(-s_2-s_3)-e^{s_1+s_3}k_3(-s_1-s_2-s_3))}{4s_1s_3}\\
&\qquad + \frac{-k_3(s_2)+k_3(s_1+s_2)+k_3(s_2+s_3)-k_3(s_1+s_2+s_3)}{4s_1s_3} - \frac{1}{2}G_2(s_1,s_2)k_6(s_3)\\
&\qquad + \frac{G_1(s_1)(k_6(s_2)-k_6(s_2+s_3))}{s_3} + \frac{k_6(s_2)-k_6(s_1+s_2)-k_6(s_2+s_3)+k_6(s_1+s_2+s_3)}{s_1s_3}\\
&\qquad + \frac{-s_3k_6(s_1)+s_2k_6(s_1+s_2)+s_3k_6(s_1+s_2)-s_2k_6(s_1+s_2+s_3)}{2s_2s_3(s_2+s_3)}\\
&\qquad + \frac{-s_1k_6(s_3)+s_1k_6(s_2+s_3)+s_2k_6(s_2+s_3)-s_2k_6(s_1+s_2+s_3)}{2s_1s_2(s_1+s_2)}\\
&\qquad + \frac{e^{s_2}G_1(s_1)(k_7(-s_2)-e^{s_3}k_7(-s_2-s_3))}{s_3}\\
&\qquad - \frac{e^{s_1}(s_3k_7(-s_1)-e^{s_2}s_2k_7(-s_1-s_2)-e^{s_2}s_3k_7(-s_1-s_2)+e^{s_2+s_3}s_2k_7(-s_1-s_2-s_3))}{2s_2s_3(s_2+s_3)}\\
&\qquad - \frac{e^{s_2}(e^{s_1}k_7(-s_1-s_2)-k_7(-s_2)+e^{s_3}k_7(-s_2-s_3)-e^{s_1+s_3}k_7(-s_1-s_2-s_3))}{s_1s_3}\\
&\qquad + \frac{e^{s_3}G_1(s_1)(e^{s_2}k_7(-s_2-s_3)-k_7(-s_3))}{2s_2} - \frac{1}{2}e^{s_3}G_2(s_1,s_2)k_7(-s_3)\\
&\qquad + \frac{e^{s_3}(e^{s_2}s_1k_7(-s_2-s_3)+e^{s_2}s_2k_7(-s_2-s_3)-e^{s_1+s_2}s_2k_7(-s_1-s_2-s_3)-s_1k_7(-s_3))}{2s_1s_2(s_1+s_2)}\\
&\qquad + \frac{k_8(s_1,s_2+s_3)-k_8(s_1,s_2)}{4s_3} + \frac{1}{4}G_1(s_1)k_8(s_2,s_3) + \frac{k_8(s_2,s_3)-k_8(s_1+s_2,s_3)}{4s_1}\\
&\qquad + \frac{k_9(s_1,s_2+s_3)-k_9(s_1,s_2)}{8s_3} + \frac{1}{8}G_1(s_1)k_9(s_2,s_3) + \frac{k_9(s_2,s_3)-k_9(s_1+s_2,s_3)}{8s_1}\\
&\qquad + \frac{k_9(s_1+s_2,s_3)-k_9(s_1,s_2+s_3)}{8s_2} + \frac{e^{s_1+s_2}(k_9(-s_1-s_2,s_1)-e^{s_3}k_9(-s_1-s_2-s_3,s_1))}{8s_3}\\
&\qquad - \frac{1}{8}e^{s_2}G_1(s_1)k_9(s_3,-s_2-s_3) + \frac{(-1+e^{s_1+s_2+s_3})k_{10}(s_1,s_2)}{8(s_1+s_2+s_3)} + \frac{k_{10}(s_1,s_2)-k_{10}(s_1,s_2+s_3)}{8s_3}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{10}(-s_2-s_3,s_2)+\frac{1}{8}e^{s_2}G_1(s_1)k_{10}(s_3,-s_2-s_3)\\
&+\frac{e^{s_2}(k_{10}(s_3,-s_2-s_3)-e^{s_1}k_{10}(s_3,-s_1-s_2-s_3))}{8s_1}\\
&+\frac{e^{s_1}(e^{s_2}k_{10}(s_3,-s_1-s_2-s_3)-k_{10}(s_2+s_3,-s_1-s_2-s_3))}{8s_2}+\frac{1}{4}e^{s_2}G_1(s_1)k_{11}(s_3,-s_2-s_3)\\
&+\frac{e^{s_2}(k_{11}(s_3,-s_2-s_3)-e^{s_1}k_{11}(s_3,-s_1-s_2-s_3))}{4s_1}+\frac{1}{4}e^{s_2+s_3}G_1(s_1)k_{12}(-s_2-s_3,s_2)\\
&+\frac{e^{s_2+s_3}(k_{12}(-s_2-s_3,s_2)-e^{s_1}k_{12}(-s_1-s_2-s_3,s_1+s_2))}{4s_1}-\frac{1}{8}G_1(s_1)k_{13}(s_2,s_3)\\
&+\frac{k_{13}(s_1+s_2,s_3)-k_{13}(s_2,s_3)}{8s_1}+\frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{13}(-s_2-s_3,s_2)\\
&+\frac{e^{s_2+s_3}(k_{13}(-s_2-s_3,s_2)-e^{s_1}k_{13}(-s_1-s_2-s_3,s_1+s_2))}{8s_1}\\
&+\frac{e^{s_1}(k_{13}(s_2,-s_1-s_2)-k_{13}(s_2+s_3,-s_1-s_2-s_3))}{8s_3}-\frac{1}{16}k_{17}(s_1,s_2,s_3)\\
&-\frac{1}{16}e^{s_1}k_{17}(s_2,s_3,-s_1-s_2-s_3)-\frac{1}{16}e^{s_1+s_2+s_3}k_{17}(-s_1-s_2-s_3,s_1,s_2)\\
&-\frac{1}{16}e^{s_1+s_2}k_{17}(s_3,-s_1-s_2-s_3,s_1)-\frac{1}{16}k_{19}(s_1,s_2,s_3)-\frac{1}{16}e^{s_1}k_{19}(s_2,s_3,-s_1-s_2-s_3)\\
&-\frac{1}{16}e^{s_1+s_2+s_3}k_{19}(-s_1-s_2-s_3,s_1,s_2)-\frac{1}{16}e^{s_1+s_2}k_{19}(s_3,-s_1-s_2-s_3,s_1)\\
&-\frac{e^{s_2}(k_9(s_3,-s_2-s_3)-e^{s_1}k_9(s_3,-s_1-s_2-s_3))}{8s_1}\\
&-\frac{e^{s_2+s_3}(k_{10}(-s_2-s_3,s_2)-e^{s_1}k_{10}(-s_1-s_2-s_3,s_1+s_2))}{8s_1}-\frac{G_1(s_1)(k_6(s_3)-k_6(s_2+s_3))}{2s_2}\\
&-\frac{e^{s_1+s_2+s_3}(k_{13}(-s_1-s_2-s_3,s_1)-k_{13}(-s_1-s_2-s_3,s_1+s_2))}{8s_2}\\
&-\frac{e^{s_2}G_1(s_1)(k_3(-s_2)-e^{s_3}k_3(-s_2-s_3))}{4s_3}-\frac{G_1(s_1)(k_3(s_2)-k_3(s_2+s_3))}{4s_3}\\
&-\frac{e^{s_1}(k_{11}(s_2,-s_1-s_2)-k_{11}(s_2+s_3,-s_1-s_2-s_3))}{4s_3}\\
&-\frac{e^{s_1+s_2}(k_{12}(-s_1-s_2,s_1)-e^{s_3}k_{12}(-s_1-s_2-s_3,s_1))}{4s_3}\\
&-\frac{e^{s_1}(k_{10}(s_2,-s_1-s_2)-k_{10}(s_2+s_3,-s_1-s_2-s_3))}{8s_3}\\
&-\frac{e^{s_1+s_2}(k_{13}(-s_1-s_2,s_1)-e^{s_3}k_{13}(-s_1-s_2-s_3,s_1))}{8s_3}\\
&-\frac{e^{s_1+s_2}(k_9(-s_1-s_2,s_1)-k_9(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)}-\frac{e^{s_1+s_2+s_3}(k_{10}(s_1,s_2)-k_{10}(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)}\\
&-\frac{e^{s_1}(k_{13}(s_2,-s_1-s_2)-k_{13}(s_2,s_3))}{8(s_1+s_2+s_3)}.
\end{aligned}
$$

A.1.3. *The function* $\widetilde{K}_{11}$. We have

(64)
$$
\begin{aligned}
&\widetilde{K}_{11}(s_1,s_2,s_3)\\
&\quad=\frac{1}{15}(-2)\pi G_3(s_1,s_2,s_3)\\
&\qquad-\frac{e^{s_2}(-e^{s_1}k_4(-s_1-s_2)+k_4(-s_2)-e^{s_3}k_4(-s_2-s_3)+e^{s_1+s_3}k_4(-s_1-s_2-s_3))}{4s_1s_3}\\
&\qquad+\frac{e^{s_2}(e^{s_1}k_4(-s_1-s_2)-k_4(-s_2)+e^{s_3}k_4(-s_2-s_3)-e^{s_1+s_3}k_4(-s_1-s_2-s_3))}{4s_1s_3}\\
&\qquad+\frac{-k_4(s_2)+k_4(s_1+s_2)+k_4(s_2+s_3)-k_4(s_1+s_2+s_3)}{2s_1s_3}-\frac{1}{4}G_2(s_1,s_2)k_6(s_3)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{G_1(s_1)(k_6(s_2)-k_6(s_2+s_3))}{2s_3}+\frac{k_6(s_2)-k_6(s_1+s_2)-k_6(s_2+s_3)+k_6(s_1+s_2+s_3)}{2s_1s_3}\\
&+\frac{-s_3k_6(s_1)+s_2k_6(s_1+s_2)+s_3k_6(s_1+s_2)-s_2k_6(s_1+s_2+s_3)}{4s_2s_3(s_2+s_3)}\\
&+\frac{-s_1k_6(s_3)+s_1k_6(s_2+s_3)+s_2k_6(s_2+s_3)-s_2k_6(s_1+s_2+s_3)}{4s_1s_2(s_1+s_2)}\\
&+\frac{e^{s_2}G_1(s_1)(k_7(-s_2)-e^{s_3}k_7(-s_2-s_3))}{2s_3}\\
&+\frac{e^{s_2}(-e^{s_1}k_7(-s_1-s_2)+k_7(-s_2)-e^{s_3}k_7(-s_2-s_3)+e^{s_1+s_3}k_7(-s_1-s_2-s_3))}{4s_1s_3}\\
&-\frac{e^{s_1}(s_3k_7(-s_1)-e^{s_2}s_2k_7(-s_1-s_2)-e^{s_2}s_3k_7(-s_1-s_2)+e^{s_2+s_3}s_2k_7(-s_1-s_2-s_3))}{4s_2s_3(s_2+s_3)}\\
&-\frac{e^{s_2}(e^{s_1}k_7(-s_1-s_2)-k_7(-s_2)+e^{s_3}k_7(-s_2-s_3)-e^{s_1+s_3}k_7(-s_1-s_2-s_3))}{4s_1s_3}\\
&+\frac{e^{s_3}G_1(s_1)(e^{s_2}k_7(-s_2-s_3)-k_7(-s_3))}{4s_2}-\frac{1}{4}e^{s_3}G_2(s_1,s_2)k_7(-s_3)\\
&+\frac{e^{s_3}(e^{s_2}s_1k_7(-s_2-s_3)+e^{s_2}s_2k_7(-s_2-s_3)-e^{s_1+s_2}s_2k_7(-s_1-s_2-s_3)-s_1k_7(-s_3))}{4s_1s_2(s_1+s_2)}\\
&+\frac{k_8(s_1+s_2,s_3)-k_8(s_1,s_2+s_3)}{8s_2}+\frac{e^{s_1+s_2}(k_8(-s_1-s_2,s_1)-e^{s_3}k_8(-s_1-s_2-s_3,s_1))}{8s_3}\\
&-\frac{1}{8}e^{s_2}G_1(s_1)k_8(s_3,-s_2-s_3)+\frac{k_9(s_1,s_2+s_3)-k_9(s_1,s_2)}{8s_3}+\frac{1}{8}G_1(s_1)k_9(s_2,s_3)\\
&+\frac{k_9(s_2,s_3)-k_9(s_1+s_2,s_3)}{8s_1}+\frac{1}{8}e^{s_2}G_1(s_1)k_{10}(s_3,-s_2-s_3)\\
&+\frac{e^{s_2}(k_{10}(s_3,-s_2-s_3)-e^{s_1}k_{10}(s_3,-s_1-s_2-s_3))}{8s_1}+\frac{(-1+e^{s_1+s_2+s_3})k_{11}(s_1,s_2)}{8(s_1+s_2+s_3)}\\
&+\frac{k_{11}(s_1,s_2)-k_{11}(s_1,s_2+s_3)}{8s_3}-\frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{11}(-s_2-s_3,s_2)\\
&+\frac{e^{s_1}(e^{s_2}k_{11}(s_3,-s_1-s_2-s_3)-k_{11}(s_2+s_3,-s_1-s_2-s_3))}{8s_2}-\frac{1}{8}G_1(s_1)k_{12}(s_2,s_3)\\
&+\frac{k_{12}(s_1+s_2,s_3)-k_{12}(s_2,s_3)}{8s_1}+\frac{e^{s_1}(k_{12}(s_2,-s_1-s_2)-k_{12}(s_2+s_3,-s_1-s_2-s_3))}{8s_3}\\
&+\frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{13}(-s_2-s_3,s_2)\\
&+\frac{e^{s_2+s_3}(k_{13}(-s_2-s_3,s_2)-e^{s_1}k_{13}(-s_1-s_2-s_3,s_1+s_2))}{8s_1}\\
&-\frac{1}{16}k_{18}(s_1,s_2,s_3)-\frac{1}{16}e^{s_1}k_{18}(s_2,s_3,-s_1-s_2-s_3)-\frac{1}{16}e^{s_1+s_2+s_3}k_{18}(-s_1-s_2-s_3,s_1,s_2)\\
&-\frac{1}{16}e^{s_1+s_2}k_{18}(s_3,-s_1-s_2-s_3,s_1)-\frac{e^{s_2}(k_8(s_3,-s_2-s_3)-e^{s_1}k_8(s_3,-s_1-s_2-s_3))}{8s_1}\\
&-\frac{e^{s_2+s_3}(k_{11}(-s_2-s_3,s_2)-e^{s_1}k_{11}(-s_1-s_2-s_3,s_1+s_2))}{8s_1}-\frac{G_1(s_1)(k_6(s_3)-k_6(s_2+s_3))}{4s_2}\\
&-\frac{e^{s_1+s_2+s_3}(k_{12}(-s_1-s_2-s_3,s_1)-k_{12}(-s_1-s_2-s_3,s_1+s_2))}{8s_2}\\
&-\frac{e^{s_2}G_1(s_1)(k_4(-s_2)-e^{s_3}k_4(-s_2-s_3))}{2s_3}-\frac{G_1(s_1)(k_4(s_2)-k_4(s_2+s_3))}{2s_3}\\
&-\frac{e^{s_1}(k_{10}(s_2,-s_1-s_2)-k_{10}(s_2+s_3,-s_1-s_2-s_3))}{8s_3}\\
&-\frac{e^{s_1+s_2}(k_{13}(-s_1-s_2,s_1)-e^{s_3}k_{13}(-s_1-s_2-s_3,s_1))}{8s_3}
\end{aligned}
$$

$$- \frac{e^{s_1+s_2}(k_8(-s_1-s_2,s_1)-k_8(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)} - \frac{e^{s_1+s_2+s_3}(k_{11}(s_1,s_2)-k_{11}(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)}$$
$$- \frac{e^{s_1}(k_{12}(s_2,-s_1-s_2)-k_{12}(s_2,s_3))}{8(s_1+s_2+s_3)}.$$

A.1.4. *The function* $\widetilde{K}_{12}$. We have

(65)

$$\widetilde{K}_{12}(s_1,s_2,s_3)$$
$$= \frac{1}{15}(-2)\pi G_3(s_1,s_2,s_3)$$
$$+ \frac{e^{s_1}(s_3k_4(-s_1)-e^{s_2}s_2k_4(-s_1-s_2)-e^{s_2}s_3k_4(-s_1-s_2)+e^{s_2+s_3}s_2k_4(-s_1-s_2-s_3))}{2s_2s_3(s_2+s_3)}$$
$$+ \frac{s_3k_4(s_1)-s_2k_4(s_1+s_2)-s_3k_4(s_1+s_2)+s_2k_4(s_1+s_2+s_3)}{2s_2s_3(s_2+s_3)} - \frac{1}{4}G_2(s_1,s_2)k_6(s_3)$$
$$+ \frac{G_1(s_1)(k_6(s_2)-k_6(s_2+s_3))}{4s_3} + \frac{k_6(s_2)-k_6(s_1+s_2)-k_6(s_2+s_3)+k_6(s_1+s_2+s_3)}{4s_1s_3}$$
$$+ \frac{-s_3k_6(s_1)+s_2k_6(s_1+s_2)+s_3k_6(s_1+s_2)-s_2k_6(s_1+s_2+s_3)}{2s_2s_3(s_2+s_3)}$$
$$+ \frac{-s_1k_6(s_3)+s_1k_6(s_2+s_3)+s_2k_6(s_2+s_3)-s_2k_6(s_1+s_2+s_3)}{4s_1s_2(s_1+s_2)}$$
$$+ \frac{e^{s_2}G_1(s_1)(k_7(-s_2)-e^{s_3}k_7(-s_2-s_3))}{4s_3}$$
$$- \frac{e^{s_1}(s_3k_7(-s_1)-e^{s_2}s_2k_7(-s_1-s_2)-e^{s_2}s_3k_7(-s_1-s_2)+e^{s_2+s_3}s_2k_7(-s_1-s_2-s_3))}{2s_2s_3(s_2+s_3)}$$
$$- \frac{e^{s_2}(e^{s_1}k_7(-s_1-s_2)-k_7(-s_2)+e^{s_3}k_7(-s_2-s_3)-e^{s_1+s_3}k_7(-s_1-s_2-s_3))}{4s_1s_3}$$
$$+ \frac{e^{s_3}G_1(s_1)(e^{s_2}k_7(-s_2-s_3)-k_7(-s_3))}{4s_2} - \frac{1}{4}e^{s_3}G_2(s_1,s_2)k_7(-s_3)$$
$$+ \frac{e^{s_3}(e^{s_2}s_1k_7(-s_2-s_3)+e^{s_2}s_2k_7(-s_2-s_3)-e^{s_1+s_2}s_2k_7(-s_1-s_2-s_3)-s_1k_7(-s_3))}{4s_1s_2(s_1+s_2)}$$
$$+ \frac{1}{8}G_1(s_1)k_8(s_2,s_3) + \frac{k_8(s_2,s_3)-k_8(s_1+s_2,s_3)}{8s_1}$$
$$+ \frac{e^{s_1}(k_8(s_2,-s_1-s_2)-k_8(s_2+s_3,-s_1-s_2-s_3))}{8s_3} + \frac{k_9(s_1,s_2+s_3)-k_9(s_1,s_2)}{8s_3}$$
$$+ \frac{k_9(s_1+s_2,s_3)-k_9(s_1,s_2+s_3)}{8s_2}$$
$$+ \frac{e^{s_1}(e^{s_2}k_{10}(s_3,-s_1-s_2-s_3)-k_{10}(s_2+s_3,-s_1-s_2-s_3))}{8s_2}$$
$$+ \frac{e^{s_1+s_2}(k_{11}(-s_1-s_2,s_1)-e^{s_3}k_{11}(-s_1-s_2-s_3,s_1))}{8s_3}$$
$$+ \frac{e^{s_1+s_2+s_3}(k_{11}(-s_1-s_2-s_3,s_1)-k_{11}(-s_1-s_2-s_3,s_1+s_2))}{8s_2}$$
$$+ \frac{1}{8}e^{s_2}G_1(s_1)k_{11}(s_3,-s_2-s_3) + \frac{e^{s_2}(k_{11}(s_3,-s_2-s_3)-e^{s_1}k_{11}(s_3,-s_1-s_2-s_3))}{8s_1}$$
$$+ \frac{(-1+e^{s_1+s_2+s_3})k_{12}(s_1,s_2)}{8(s_1+s_2+s_3)} + \frac{k_{12}(s_1,s_2)-k_{12}(s_1,s_2+s_3)}{8s_3} + \frac{k_{12}(s_1,s_2+s_3)-k_{12}(s_1+s_2,s_3)}{8s_2}$$
$$+ \frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{12}(-s_2-s_3,s_2) + \frac{e^{s_2+s_3}(k_{12}(-s_2-s_3,s_2)-e^{s_1}k_{12}(-s_1-s_2-s_3,s_1+s_2))}{8s_1}$$
$$- \frac{1}{16}e^{s_1}k_{17}(s_2,s_3,-s_1-s_2-s_3) - \frac{1}{16}e^{s_1+s_2+s_3}k_{17}(-s_1-s_2-s_3,s_1,s_2) - \frac{1}{16}k_{19}(s_1,s_2,s_3)$$
$$- \frac{1}{16}e^{s_1+s_2}k_{19}(s_3,-s_1-s_2-s_3,s_1) - \frac{G_1(s_1)(k_6(s_3)-k_6(s_2+s_3))}{4s_2}$$

$$
\begin{aligned}
&- \frac{e^{s_1}(e^{s_2}k_8(s_3,-s_1-s_2-s_3)-k_8(s_2+s_3,-s_1-s_2-s_3))}{8s_2}\\
&- \frac{e^{s_1+s_2+s_3}(k_{13}(-s_1-s_2-s_3,s_1)-k_{13}(-s_1-s_2-s_3,s_1+s_2))}{8s_2}\\
&- \frac{e^{s_1}(k_{10}(s_2,-s_1-s_2)-k_{10}(s_2+s_3,-s_1-s_2-s_3))}{8s_3}\\
&- \frac{e^{s_1+s_2}(k_{13}(-s_1-s_2,s_1)-e^{s_3}k_{13}(-s_1-s_2-s_3,s_1))}{8s_3} - \frac{e^{s_1}(k_8(s_2,-s_1-s_2)-k_8(s_2,s_3))}{8(s_1+s_2+s_3)}\\
&- \frac{e^{s_1+s_2}(k_{11}(-s_1-s_2,s_1)-k_{11}(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)} - \frac{e^{s_1+s_2+s_3}(k_{12}(s_1,s_2)-k_{12}(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)}.
\end{aligned}
$$

A.1.5. *The function $\widetilde{K}_{13}$.* We have

(66)

$$
\begin{aligned}
&\widetilde{K}_{13}(s_1,s_2,s_3)\\
&\quad = \frac{1}{15}(-4)\pi G_3(s_1,s_2,s_3)\\
&\qquad + \frac{e^{s_1}(s_3k_3(-s_1)-e^{s_2}s_2k_3(-s_1-s_2)-e^{s_2}s_3k_3(-s_1-s_2)+e^{s_2+s_3}s_2k_3(-s_1-s_2-s_3))}{4s_2s_3(s_2+s_3)}\\
&\qquad + \frac{s_3k_3(s_1)-s_2k_3(s_1+s_2)-s_3k_3(s_1+s_2)+s_2k_3(s_1+s_2+s_3)}{4s_2s_3(s_2+s_3)} - \frac{1}{2}G_2(s_1,s_2)k_6(s_3)\\
&\qquad + \frac{G_1(s_1)(k_6(s_2)-k_6(s_2+s_3))}{2s_3} + \frac{k_6(s_2)-k_6(s_1+s_2)-k_6(s_2+s_3)+k_6(s_1+s_2+s_3)}{2s_1s_3}\\
&\qquad + \frac{-s_3k_6(s_1)+s_2k_6(s_1+s_2)+s_3k_6(s_1+s_2)-s_2k_6(s_1+s_2+s_3)}{s_2s_3(s_2+s_3)}\\
&\qquad + \frac{-s_1k_6(s_3)+s_1k_6(s_2+s_3)+s_2k_6(s_2+s_3)-s_2k_6(s_1+s_2+s_3)}{2s_1s_2(s_1+s_2)}\\
&\qquad + \frac{e^{s_2}G_1(s_1)(k_7(-s_2)-e^{s_3}k_7(-s_2-s_3))}{2s_3}\\
&\qquad - \frac{e^{s_1}(s_3k_7(-s_1)-e^{s_2}s_2k_7(-s_1-s_2)-e^{s_2}s_3k_7(-s_1-s_2)+e^{s_2+s_3}s_2k_7(-s_1-s_2-s_3))}{s_2s_3(s_2+s_3)}\\
&\qquad - \frac{e^{s_2}(e^{s_1}k_7(-s_1-s_2)-k_7(-s_2)+e^{s_3}k_7(-s_2-s_3)-e^{s_1+s_3}k_7(-s_1-s_2-s_3))}{2s_1s_3}\\
&\qquad + \frac{e^{s_3}G_1(s_1)(e^{s_2}k_7(-s_2-s_3)-k_7(-s_3))}{2s_2} - \frac{1}{2}e^{s_3}G_2(s_1,s_2)k_7(-s_3)\\
&\qquad + \frac{e^{s_3}(e^{s_2}s_1k_7(-s_2-s_3)+e^{s_2}s_2k_7(-s_2-s_3)-e^{s_1+s_2}s_2k_7(-s_1-s_2-s_3)-s_1k_7(-s_3))}{2s_1s_2(s_1+s_2)}\\
&\qquad + \frac{k_8(s_1,s_2+s_3)-k_8(s_1,s_2)}{4s_3} + \frac{k_8(s_1+s_2,s_3)-k_8(s_1,s_2+s_3)}{4s_2} + \frac{k_9(s_1,s_2+s_3)-k_9(s_1,s_2)}{8s_3}\\
&\qquad + \frac{1}{8}G_1(s_1)k_9(s_2,s_3) + \frac{k_9(s_2,s_3)-k_9(s_1+s_2,s_3)}{8s_1} + \frac{k_9(s_1+s_2,s_3)-k_9(s_1,s_2+s_3)}{8s_2}\\
&\qquad + \frac{e^{s_1}(k_9(s_2,-s_1-s_2)-k_9(s_2+s_3,-s_1-s_2-s_3))}{8s_3}\\
&\qquad + \frac{e^{s_1+s_2}(k_{10}(-s_1-s_2,s_1)-e^{s_3}k_{10}(-s_1-s_2-s_3,s_1))}{8s_3}\\
&\qquad + \frac{e^{s_1+s_2+s_3}(k_{10}(-s_1-s_2-s_3,s_1)-k_{10}(-s_1-s_2-s_3,s_1+s_2))}{8s_2} + \frac{1}{8}e^{s_2}G_1(s_1)k_{10}(s_3,-s_2-s_3)\\
&\qquad + \frac{e^{s_2}(k_{10}(s_3,-s_2-s_3)-e^{s_1}k_{10}(s_3,-s_1-s_2-s_3))}{8s_1}\\
&\qquad + \frac{e^{s_1}(e^{s_2}k_{10}(s_3,-s_1-s_2-s_3)-k_{10}(s_2+s_3,-s_1-s_2-s_3))}{8s_2}\\
&\qquad + \frac{e^{s_1}(e^{s_2}k_{11}(s_3,-s_1-s_2-s_3)-k_{11}(s_2+s_3,-s_1-s_2-s_3))}{4s_2} + \frac{(-1+e^{s_1+s_2+s_3})k_{13}(s_1,s_2)}{8(s_1+s_2+s_3)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{k_{13}(s_1,s_2)-k_{13}(s_1,s_2+s_3)}{8s_3} + \frac{k_{13}(s_1,s_2+s_3)-k_{13}(s_1+s_2,s_3)}{8s_2}\\
&+ \frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{13}(-s_2-s_3,s_2) + \frac{e^{s_2+s_3}(k_{13}(-s_2-s_3,s_2)-e^{s_1}k_{13}(-s_1-s_2-s_3,s_1+s_2))}{8s_1}\\
&- \frac{1}{16}k_{17}(s_1,s_2,s_3) - \frac{1}{16}e^{s_1+s_2}k_{17}(s_3,-s_1-s_2-s_3,s_1) - \frac{1}{16}k_{18}(s_1,s_2,s_3)\\
&- \frac{1}{16}e^{s_1}k_{18}(s_2,s_3,-s_1-s_2-s_3) - \frac{1}{16}e^{s_1+s_2+s_3}k_{18}(-s_1-s_2-s_3,s_1,s_2)\\
&- \frac{1}{16}e^{s_1+s_2}k_{18}(s_3,-s_1-s_2-s_3,s_1) - \frac{1}{16}e^{s_1}k_{19}(s_2,s_3,-s_1-s_2-s_3)\\
&- \frac{1}{16}e^{s_1+s_2+s_3}k_{19}(-s_1-s_2-s_3,s_1,s_2) - \frac{G_1(s_1)(k_6(s_3)-k_6(s_2+s_3))}{2s_2}\\
&- \frac{e^{s_1+s_2+s_3}(k_{12}(-s_1-s_2-s_3,s_1)-k_{12}(-s_1-s_2-s_3,s_1+s_2))}{4s_2}\\
&- \frac{e^{s_1}(e^{s_2}k_9(s_3,-s_1-s_2-s_3)-k_9(s_2+s_3,-s_1-s_2-s_3))}{8s_2}\\
&- \frac{e^{s_1+s_2+s_3}(k_{13}(-s_1-s_2-s_3,s_1)-k_{13}(-s_1-s_2-s_3,s_1+s_2))}{8s_2}\\
&- \frac{e^{s_1}(k_{11}(s_2,-s_1-s_2)-k_{11}(s_2+s_3,-s_1-s_2-s_3))}{4s_3}\\
&- \frac{e^{s_1+s_2}(k_{12}(-s_1-s_2,s_1)-e^{s_3}k_{12}(-s_1-s_2-s_3,s_1))}{4s_3}\\
&- \frac{e^{s_1}(k_{10}(s_2,-s_1-s_2)-k_{10}(s_2+s_3,-s_1-s_2-s_3))}{8s_3}\\
&- \frac{e^{s_1+s_2}(k_{13}(-s_1-s_2,s_1)-e^{s_3}k_{13}(-s_1-s_2-s_3,s_1))}{8s_3} - \frac{e^{s_1}(k_9(s_2,-s_1-s_2)-k_9(s_2,s_3))}{8(s_1+s_2+s_3)}\\
&- \frac{e^{s_1+s_2}(k_{10}(-s_1-s_2,s_1)-k_{10}(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)} - \frac{e^{s_1+s_2+s_3}(k_{13}(s_1,s_2)-k_{13}(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)}.
\end{aligned}
$$

A.1.6. *The function* $\widetilde{K}_{14}$. We have

(67)

$$
\begin{aligned}
&\widetilde{K}_{14}(s_1,s_2,s_3)\\
&\quad= \frac{1}{5}(-2)\pi G_3(s_1,s_2,s_3)\\
&\qquad+ \frac{e^{s_1}(s_3k_5(-s_1)-e^{s_2}s_2k_5(-s_1-s_2)-e^{s_2}s_3k_5(-s_1-s_2)+e^{s_2+s_3}s_2k_5(-s_1-s_2-s_3))}{2s_2s_3(s_2+s_3)}\\
&\qquad+ \frac{s_3k_5(s_1)-s_2k_5(s_1+s_2)-s_3k_5(s_1+s_2)+s_2k_5(s_1+s_2+s_3)}{2s_2s_3(s_2+s_3)} - \frac{3}{4}G_2(s_1,s_2)k_6(s_3)\\
&\qquad+ \frac{3G_1(s_1)(k_6(s_2)-k_6(s_2+s_3))}{4s_3} + \frac{3(k_6(s_2)-k_6(s_1+s_2)-k_6(s_2+s_3)+k_6(s_1+s_2+s_3))}{4s_1s_3}\\
&\qquad+ \frac{3(-s_1k_6(s_3)+s_1k_6(s_2+s_3)+s_2k_6(s_2+s_3)-s_2k_6(s_1+s_2+s_3))}{4s_1s_2(s_1+s_2)}\\
&\qquad+ \frac{3e^{s_2}G_1(s_1)(k_7(-s_2)-e^{s_3}k_7(-s_2-s_3))}{4s_3}\\
&\qquad+ \frac{e^{s_2}(-e^{s_1}k_7(-s_1-s_2)+k_7(-s_2)-e^{s_3}k_7(-s_2-s_3)+e^{s_1+s_3}k_7(-s_1-s_2-s_3))}{4s_1s_3}\\
&\qquad- \frac{3e^{s_1}(s_3k_7(-s_1)-e^{s_2}s_2k_7(-s_1-s_2)-e^{s_2}s_3k_7(-s_1-s_2)+e^{s_2+s_3}s_2k_7(-s_1-s_2-s_3))}{2s_2s_3(s_2+s_3)}\\
&\qquad- \frac{e^{s_2}(e^{s_1}k_7(-s_1-s_2)-k_7(-s_2)+e^{s_3}k_7(-s_2-s_3)-e^{s_1+s_3}k_7(-s_1-s_2-s_3))}{2s_1s_3}\\
&\qquad+ \frac{3e^{s_3}G_1(s_1)(e^{s_2}k_7(-s_2-s_3)-k_7(-s_3))}{4s_2} - \frac{3}{4}e^{s_3}G_2(s_1,s_2)k_7(-s_3)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3e^{s_3}(e^{s_2}s_1k_7(-s_2-s_3)+e^{s_2}s_2k_7(-s_2-s_3)-e^{s_1+s_2}s_2k_7(-s_1-s_2-s_3)-s_1k_7(-s_3))}{4s_1s_2(s_1+s_2)}\\
&+\frac{(-1+e^{s_1+s_2+s_3})k_{14}(s_1,s_2)}{8(s_1+s_2+s_3)}+\frac{k_{14}(s_1,s_2)-k_{14}(s_1,s_2+s_3)}{8s_3}\\
&+\frac{k_{14}(s_1,s_2+s_3)-k_{14}(s_1+s_2,s_3)}{8s_2}+\frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{14}(-s_2-s_3,s_2)\\
&+\frac{e^{s_2+s_3}(k_{14}(-s_2-s_3,s_2)-e^{s_1}k_{14}(-s_1-s_2-s_3,s_1+s_2))}{8s_1}+\frac{k_{15}(s_1,s_2+s_3)-k_{15}(s_1,s_2)}{4s_3}\\
&+\frac{1}{8}G_1(s_1)k_{15}(s_2,s_3)+\frac{k_{15}(s_2,s_3)-k_{15}(s_1+s_2,s_3)}{8s_1}+\frac{k_{15}(s_1+s_2,s_3)-k_{15}(s_1,s_2+s_3)}{4s_2}\\
&+\frac{e^{s_1}(k_{15}(s_2,-s_1-s_2)-k_{15}(s_2+s_3,-s_1-s_2-s_3))}{8s_3}\\
&+\frac{e^{s_1+s_2}(k_{16}(-s_1-s_2,s_1)-e^{s_3}k_{16}(-s_1-s_2-s_3,s_1))}{8s_3}\\
&+\frac{e^{s_1+s_2+s_3}(k_{16}(-s_1-s_2-s_3,s_1)-k_{16}(-s_1-s_2-s_3,s_1+s_2))}{8s_2}\\
&+\frac{1}{8}e^{s_2}G_1(s_1)k_{16}(s_3,-s_2-s_3)+\frac{e^{s_2}(k_{16}(s_3,-s_2-s_3)-e^{s_1}k_{16}(s_3,-s_1-s_2-s_3))}{8s_1}\\
&+\frac{e^{s_1}(e^{s_2}k_{16}(s_3,-s_1-s_2-s_3)-k_{16}(s_2+s_3,-s_1-s_2-s_3))}{4s_2}-\frac{1}{16}k_{20}(s_1,s_2,s_3)\\
&-\frac{1}{16}e^{s_1}k_{20}(s_2,s_3,-s_1-s_2-s_3)-\frac{1}{16}e^{s_1+s_2+s_3}k_{20}(-s_1-s_2-s_3,s_1,s_2)\\
&-\frac{1}{16}e^{s_1+s_2}k_{20}(s_3,-s_1-s_2-s_3,s_1)-\frac{3G_1(s_1)(k_6(s_3)-k_6(s_2+s_3))}{4s_2}\\
&-\frac{e^{s_1+s_2+s_3}(k_{14}(-s_1-s_2-s_3,s_1)-k_{14}(-s_1-s_2-s_3,s_1+s_2))}{4s_2}\\
&-\frac{e^{s_1}(e^{s_2}k_{15}(s_3,-s_1-s_2-s_3)-k_{15}(s_2+s_3,-s_1-s_2-s_3))}{8s_2}\\
&-\frac{e^{s_1+s_2}(k_{14}(-s_1-s_2,s_1)-e^{s_3}k_{14}(-s_1-s_2-s_3,s_1))}{4s_3}\\
&-\frac{e^{s_1}(k_{16}(s_2,-s_1-s_2)-k_{16}(s_2+s_3,-s_1-s_2-s_3))}{4s_3}\\
&-\frac{3(s_3k_6(s_1)-s_2k_6(s_1+s_2)-s_3k_6(s_1+s_2)+s_2k_6(s_1+s_2+s_3))}{2s_2s_3(s_2+s_3)}\\
&-\frac{e^{s_1+s_2+s_3}(k_{14}(s_1,s_2)-k_{14}(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)}-\frac{e^{s_1}(k_{15}(s_2,-s_1-s_2)-k_{15}(s_2,s_3))}{8(s_1+s_2+s_3)}\\
&-\frac{e^{s_1+s_2}(k_{16}(-s_1-s_2,s_1)-k_{16}(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)}.
\end{aligned}
$$

A.1.7. *The function* $\widetilde{K}_{15}$. We have

(68)

$$
\begin{aligned}
&\widetilde{K}_{15}(s_1,s_2,s_3)\\
&\quad=\frac{1}{5}(-2)\pi G_3(s_1,s_2,s_3)+\frac{1}{2}e^{s_3}G_2(s_1,s_2)k_5(-s_3)\\
&\qquad-\frac{e^{s_3}(e^{s_2}s_1k_5(-s_2-s_3)+e^{s_2}s_2k_5(-s_2-s_3)-e^{s_1+s_2}s_2k_5(-s_1-s_2-s_3)-s_1k_5(-s_3))}{2s_1s_2(s_1+s_2)}\\
&\qquad+\frac{1}{2}G_2(s_1,s_2)k_5(s_3)+\frac{G_1(s_1)(k_5(s_3)-k_5(s_2+s_3))}{2s_2}\\
&\qquad+\frac{s_1k_5(s_3)-s_1k_5(s_2+s_3)-s_2k_5(s_2+s_3)+s_2k_5(s_1+s_2+s_3)}{2s_1s_2(s_1+s_2)}-\frac{3}{2}G_2(s_1,s_2)k_6(s_3)\\
&\qquad+\frac{3G_1(s_1)(k_6(s_2)-k_6(s_2+s_3))}{4s_3}+\frac{3(k_6(s_2)-k_6(s_1+s_2)-k_6(s_2+s_3)+k_6(s_1+s_2+s_3))}{4s_1s_3}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3(-s_3k_6(s_1)+s_2k_6(s_1+s_2)+s_3k_6(s_1+s_2)-s_2k_6(s_1+s_2+s_3))}{4s_2s_3(s_2+s_3)}\\
&+\frac{3e^{s_2}G_1(s_1)(k_7(-s_2)-e^{s_3}k_7(-s_2-s_3))}{4s_3}\\
&+\frac{e^{s_2}(-e^{s_1}k_7(-s_1-s_2)+k_7(-s_2)-e^{s_3}k_7(-s_2-s_3)+e^{s_1+s_3}k_7(-s_1-s_2-s_3))}{2s_1s_3}\\
&-\frac{3e^{s_1}(s_3k_7(-s_1)-e^{s_2}s_2k_7(-s_1-s_2)-e^{s_2}s_3k_7(-s_1-s_2)+e^{s_2+s_3}s_2k_7(-s_1-s_2-s_3))}{4s_2s_3(s_2+s_3)}\\
&-\frac{e^{s_2}(e^{s_1}k_7(-s_1-s_2)-k_7(-s_2)+e^{s_3}k_7(-s_2-s_3)-e^{s_1+s_3}k_7(-s_1-s_2-s_3))}{4s_1s_3}\\
&+\frac{3e^{s_3}G_1(s_1)(e^{s_2}k_7(-s_2-s_3)-k_7(-s_3))}{2s_2}-\frac{3}{2}e^{s_3}G_2(s_1,s_2)k_7(-s_3)\\
&+\frac{3e^{s_3}(e^{s_2}s_1k_7(-s_2-s_3)+e^{s_2}s_2k_7(-s_2-s_3)-e^{s_1+s_2}s_2k_7(-s_1-s_2-s_3)-s_1k_7(-s_3))}{2s_1s_2(s_1+s_2)}\\
&+\frac{1}{4}e^{s_2+s_3}G_1(s_1)k_{14}(-s_2-s_3,s_2)\\
&+\frac{e^{s_2+s_3}(k_{14}(-s_2-s_3,s_2)-e^{s_1}k_{14}(-s_1-s_2-s_3,s_1+s_2))}{4s_1}\\
&-\frac{1}{8}e^{s_2}G_1(s_1)k_{14}(s_3,-s_2-s_3)+\frac{(-1+e^{s_1+s_2+s_3})k_{15}(s_1,s_2)}{8(s_1+s_2+s_3)}+\frac{k_{15}(s_1,s_2+s_3)-k_{15}(s_1,s_2)}{8s_3}\\
&+\frac{1}{4}G_1(s_1)k_{15}(s_2,s_3)+\frac{k_{15}(s_2,s_3)-k_{15}(s_1+s_2,s_3)}{4s_1}+\frac{k_{15}(s_1+s_2,s_3)-k_{15}(s_1,s_2+s_3)}{4s_2}\\
&-\frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{15}(-s_2-s_3,s_2)\\
&+\frac{e^{s_1+s_2+s_3}(k_{15}(-s_1-s_2-s_3,s_1)-k_{15}(-s_1-s_2-s_3,s_1+s_2))}{8s_2}-\frac{1}{8}G_1(s_1)k_{16}(s_2,s_3)\\
&+\frac{k_{16}(s_1,s_2+s_3)-k_{16}(s_1+s_2,s_3)}{8s_2}+\frac{k_{16}(s_1+s_2,s_3)-k_{16}(s_2,s_3)}{8s_1}+\frac{1}{4}e^{s_2}G_1(s_1)k_{16}(s_3,-s_2-s_3)\\
&+\frac{e^{s_2}(k_{16}(s_3,-s_2-s_3)-e^{s_1}k_{16}(s_3,-s_1-s_2-s_3))}{4s_1}\\
&+\frac{e^{s_1}(e^{s_2}k_{16}(s_3,-s_1-s_2-s_3)-k_{16}(s_2+s_3,-s_1-s_2-s_3))}{4s_2}-\frac{1}{16}k_{20}(s_1,s_2,s_3)\\
&-\frac{1}{16}e^{s_1}k_{20}(s_2,s_3,-s_1-s_2-s_3)-\frac{1}{16}e^{s_1+s_2+s_3}k_{20}(-s_1-s_2-s_3,s_1,s_2)\\
&-\frac{1}{16}e^{s_1+s_2}k_{20}(s_3,-s_1-s_2-s_3,s_1)-\frac{e^{s_2}(k_{14}(s_3,-s_2-s_3)-e^{s_1}k_{14}(s_3,-s_1-s_2-s_3))}{8s_1}\\
&-\frac{e^{s_2+s_3}(k_{15}(-s_2-s_3,s_2)-e^{s_1}k_{15}(-s_1-s_2-s_3,s_1+s_2))}{8s_1}\\
&-\frac{e^{s_3}G_1(s_1)(e^{s_2}k_5(-s_2-s_3)-k_5(-s_3))}{2s_2}-\frac{3G_1(s_1)(k_6(s_3)-k_6(s_2+s_3))}{2s_2}\\
&-\frac{e^{s_1+s_2+s_3}(k_{14}(-s_1-s_2-s_3,s_1)-k_{14}(-s_1-s_2-s_3,s_1+s_2))}{4s_2}\\
&-\frac{e^{s_1}(e^{s_2}k_{14}(s_3,-s_1-s_2-s_3)-k_{14}(s_2+s_3,-s_1-s_2-s_3))}{8s_2}\\
&-\frac{3(s_1k_6(s_3)-s_1k_6(s_2+s_3)-s_2k_6(s_2+s_3)+s_2k_6(s_1+s_2+s_3))}{2s_1s_2(s_1+s_2)}\\
&-\frac{e^{s_1+s_2}(k_{14}(-s_1-s_2,s_1)-e^{s_3}k_{14}(-s_1-s_2-s_3,s_1))}{8s_3}\\
&-\frac{e^{s_1}(k_{16}(s_2,-s_1-s_2)-k_{16}(s_2+s_3,-s_1-s_2-s_3))}{8s_3}\\
&-\frac{e^{s_1+s_2}(k_{14}(-s_1-s_2,s_1)-k_{14}(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)}-\frac{e^{s_1+s_2+s_3}(k_{15}(s_1,s_2)-k_{15}(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)}\\
&-\frac{e^{s_1}(k_{16}(s_2,-s_1-s_2)-k_{16}(s_2,s_3))}{8(s_1+s_2+s_3)}.
\end{aligned}
$$

A.1.8. *The function* $\widetilde{K}_{16}$. We have

(69)

$$
\begin{aligned}
&\widetilde{K}_{16}(s_1,s_2,s_3)\\
&\quad=\frac{1}{5}(-2)\pi G_3(s_1,s_2,s_3)\\
&\quad-\frac{e^{s_2}(-e^{s_1}k_5(-s_1-s_2)+k_5(-s_2)-e^{s_3}k_5(-s_2-s_3)+e^{s_1+s_3}k_5(-s_1-s_2-s_3))}{4s_1s_3}\\
&\quad+\frac{e^{s_2}(e^{s_1}k_5(-s_1-s_2)-k_5(-s_2)+e^{s_3}k_5(-s_2-s_3)-e^{s_1+s_3}k_5(-s_1-s_2-s_3))}{4s_1s_3}\\
&\quad+\frac{-k_5(s_2)+k_5(s_1+s_2)+k_5(s_2+s_3)-k_5(s_1+s_2+s_3)}{2s_1s_3}-\frac{3}{4}G_2(s_1,s_2)k_6(s_3)\\
&\quad+\frac{3G_1(s_1)(k_6(s_2)-k_6(s_2+s_3))}{2s_3}+\frac{3(k_6(s_2)-k_6(s_1+s_2)-k_6(s_2+s_3)+k_6(s_1+s_2+s_3))}{2s_1s_3}\\
&\quad+\frac{3(-s_3k_6(s_1)+s_2k_6(s_1+s_2)+s_3k_6(s_1+s_2)-s_2k_6(s_1+s_2+s_3))}{4s_2s_3(s_2+s_3)}\\
&\quad+\frac{3(-s_1k_6(s_3)+s_1k_6(s_2+s_3)+s_2k_6(s_2+s_3)-s_2k_6(s_1+s_2+s_3))}{4s_1s_2(s_1+s_2)}\\
&\quad+\frac{3e^{s_2}G_1(s_1)(k_7(-s_2)-e^{s_3}k_7(-s_2-s_3))}{2s_3}\\
&\quad+\frac{3e^{s_2}(-e^{s_1}k_7(-s_1-s_2)+k_7(-s_2)-e^{s_3}k_7(-s_2-s_3)+e^{s_1+s_3}k_7(-s_1-s_2-s_3))}{4s_1s_3}\\
&\quad-\frac{3e^{s_1}(s_3k_7(-s_1)-e^{s_2}s_2k_7(-s_1-s_2)-e^{s_2}s_3k_7(-s_1-s_2)+e^{s_2+s_3}s_2k_7(-s_1-s_2-s_3))}{4s_2s_3(s_2+s_3)}\\
&\quad-\frac{3e^{s_2}(e^{s_1}k_7(-s_1-s_2)-k_7(-s_2)+e^{s_3}k_7(-s_2-s_3)-e^{s_1+s_3}k_7(-s_1-s_2-s_3))}{4s_1s_3}\\
&\quad+\frac{3e^{s_3}G_1(s_1)(e^{s_2}k_7(-s_2-s_3)-k_7(-s_3))}{4s_2}-\frac{3}{4}e^{s_3}G_2(s_1,s_2)k_7(-s_3)\\
&\quad+\frac{3e^{s_3}(e^{s_2}s_1k_7(-s_2-s_3)+e^{s_2}s_2k_7(-s_2-s_3)-e^{s_1+s_2}s_2k_7(-s_1-s_2-s_3)-s_1k_7(-s_3))}{4s_1s_2(s_1+s_2)}\\
&\quad-\frac{1}{8}G_1(s_1)k_{14}(s_2,s_3)+\frac{k_{14}(s_1+s_2,s_3)-k_{14}(s_2,s_3)}{8s_1}+\frac{1}{4}e^{s_2+s_3}G_1(s_1)k_{14}(-s_2-s_3,s_2)\\
&\quad+\frac{e^{s_2+s_3}(k_{14}(-s_2-s_3,s_2)-e^{s_1}k_{14}(-s_1-s_2-s_3,s_1+s_2))}{4s_1}\\
&\quad+\frac{e^{s_1}(k_{14}(s_2,-s_1-s_2)-k_{14}(s_2+s_3,-s_1-s_2-s_3))}{8s_3}+\frac{k_{15}(s_1,s_2+s_3)-k_{15}(s_1,s_2)}{4s_3}\\
&\quad+\frac{1}{4}G_1(s_1)k_{15}(s_2,s_3)+\frac{k_{15}(s_2,s_3)-k_{15}(s_1+s_2,s_3)}{4s_1}+\frac{k_{15}(s_1+s_2,s_3)-k_{15}(s_1,s_2+s_3)}{8s_2}\\
&\quad+\frac{e^{s_1+s_2}(k_{15}(-s_1-s_2,s_1)-e^{s_3}k_{15}(-s_1-s_2-s_3,s_1))}{8s_3}-\frac{1}{8}e^{s_2}G_1(s_1)k_{15}(s_3,-s_2-s_3)\\
&\quad+\frac{(-1+e^{s_1+s_2+s_3})k_{16}(s_1,s_2)}{8(s_1+s_2+s_3)}+\frac{k_{16}(s_1,s_2)-k_{16}(s_1,s_2+s_3)}{8s_3}\\
&\quad-\frac{1}{8}e^{s_2+s_3}G_1(s_1)k_{16}(-s_2-s_3,s_2)+\frac{1}{4}e^{s_2}G_1(s_1)k_{16}(s_3,-s_2-s_3)\\
&\quad+\frac{e^{s_2}(k_{16}(s_3,-s_2-s_3)-e^{s_1}k_{16}(s_3,-s_1-s_2-s_3))}{4s_1}\\
&\quad+\frac{e^{s_1}(e^{s_2}k_{16}(s_3,-s_1-s_2-s_3)-k_{16}(s_2+s_3,-s_1-s_2-s_3))}{8s_2}-\frac{1}{16}k_{20}(s_1,s_2,s_3)\\
&\quad-\frac{1}{16}e^{s_1}k_{20}(s_2,s_3,-s_1-s_2-s_3)-\frac{1}{16}e^{s_1+s_2+s_3}k_{20}(-s_1-s_2-s_3,s_1,s_2)\\
&\quad-\frac{1}{16}e^{s_1+s_2}k_{20}(s_3,-s_1-s_2-s_3,s_1)-\frac{e^{s_2}(k_{15}(s_3,-s_2-s_3)-e^{s_1}k_{15}(s_3,-s_1-s_2-s_3))}{8s_1}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{e^{s_2+s_3}(k_{16}(-s_2-s_3,s_2)-e^{s_1}k_{16}(-s_1-s_2-s_3,s_1+s_2))}{8s_1}-\frac{3G_1(s_1)(k_6(s_3)-k_6(s_2+s_3))}{4s_2}\\
&-\frac{e^{s_1+s_2+s_3}(k_{14}(-s_1-s_2-s_3,s_1)-k_{14}(-s_1-s_2-s_3,s_1+s_2))}{8s_2}\\
&-\frac{e^{s_2}G_1(s_1)(k_5(-s_2)-e^{s_3}k_5(-s_2-s_3))}{2s_3}-\frac{G_1(s_1)(k_5(s_2)-k_5(s_2+s_3))}{2s_3}\\
&-\frac{e^{s_1+s_2}(k_{14}(-s_1-s_2,s_1)-e^{s_3}k_{14}(-s_1-s_2-s_3,s_1))}{4s_3}\\
&-\frac{e^{s_1}(k_{16}(s_2,-s_1-s_2)-k_{16}(s_2+s_3,-s_1-s_2-s_3))}{4s_3}\\
&-\frac{e^{s_1}(k_{14}(s_2,-s_1-s_2)-k_{14}(s_2,s_3))}{8(s_1+s_2+s_3)}-\frac{e^{s_1+s_2}(k_{15}(-s_1-s_2,s_1)-k_{15}(s_3,-s_2-s_3))}{8(s_1+s_2+s_3)}\\
&-\frac{e^{s_1+s_2+s_3}(k_{16}(s_1,s_2)-k_{16}(-s_2-s_3,s_2))}{8(s_1+s_2+s_3)}.
\end{aligned}
$$

A.2. **Functional relations for $\widetilde{K}_{18}, \widetilde{K}_{19}, \widetilde{K}_{20}$.** Now we present the remaining basic functional relations associated with the four variable functions.

A.2.1. *The function* $\widetilde{K}_{18}$. We have

(70)

$$
\begin{aligned}
&\widetilde{K}_{18}(s_1,s_2,s_3,s_4)\\
&=\frac{1}{15}(-4)\pi G_4(s_1,s_2,s_3,s_4)+\frac{s_3k_6(s_1)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&\quad+\frac{s_4k_6(s_1)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}+\frac{k_6(s_1+s_2)}{2s_1s_3(s_3+s_4)}+\frac{G_1(s_1)k_6(s_3)}{2s_2s_4}+\frac{G_2(s_1,s_2)k_6(s_3)}{2s_4}\\
&\quad+\frac{k_6(s_3)}{2s_2(s_1+s_2)s_4}+\frac{G_1(s_1)k_6(s_2+s_3)}{2s_3(s_3+s_4)}+\frac{G_1(s_1)k_6(s_2+s_3)}{2s_4(s_3+s_4)}+\frac{k_6(s_2+s_3)}{2s_1s_3(s_3+s_4)}+\frac{k_6(s_2+s_3)}{2s_1s_4(s_3+s_4)}\\
&\quad+\frac{k_6(s_1+s_2+s_3)}{2s_1(s_1+s_2)s_4}+\frac{k_6(s_1+s_2+s_3)}{(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}+\frac{s_2k_6(s_1+s_2+s_3)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&\quad+\frac{s_4k_6(s_1+s_2+s_3)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}+\frac{s_2k_6(s_1+s_2+s_3)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}\\
&\quad+\frac{s_3k_6(s_1+s_2+s_3)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}-\frac{1}{2}G_3(s_1,s_2,s_3)k_6(s_4)+\frac{G_1(s_1)k_6(s_3+s_4)}{2s_2(s_2+s_3)}\\
&\quad+\frac{G_1(s_1)k_6(s_3+s_4)}{2s_3(s_2+s_3)}+\frac{G_2(s_1,s_2)k_6(s_3+s_4)}{2s_3}+\frac{k_6(s_3+s_4)}{(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&\quad+\frac{s_1k_6(s_3+s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}+\frac{s_3k_6(s_3+s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&\quad+\frac{s_1k_6(s_3+s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}+\frac{s_2k_6(s_3+s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}\\
&\quad+\frac{G_1(s_1)k_6(s_2+s_3+s_4)}{2s_2s_4}+\frac{k_6(s_2+s_3+s_4)}{2s_1(s_1+s_2)s_4}+\frac{k_6(s_2+s_3+s_4)}{2s_2(s_1+s_2)s_4}\\
&\quad+\frac{s_2k_6(s_1+s_2+s_3+s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}+\frac{s_3k_6(s_1+s_2+s_3+s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&\quad+\frac{k_6(s_1+s_2+s_3+s_4)}{2s_1s_4(s_3+s_4)}+\frac{e^{s_1}s_3k_7(-s_1)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&\quad+\frac{e^{s_1}s_4k_7(-s_1)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}+\frac{e^{s_1+s_2}k_7(-s_1-s_2)}{2s_1s_3(s_3+s_4)}+\frac{e^{s_2+s_3}G_1(s_1)k_7(-s_2-s_3)}{2s_3(s_3+s_4)}\\
&\quad+\frac{e^{s_2+s_3}G_1(s_1)k_7(-s_2-s_3)}{2s_4(s_3+s_4)}+\frac{e^{s_2+s_3}k_7(-s_2-s_3)}{2s_1s_3(s_3+s_4)}+\frac{e^{s_2+s_3}k_7(-s_2-s_3)}{2s_1s_4(s_3+s_4)}\\
&\quad+\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{2s_1(s_1+s_2)s_4}+\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{e^{s_1+s_2+s_3} s_2 k_7(-s_1-s_2-s_3)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} + \frac{e^{s_1+s_2+s_3} s_4 k_7(-s_1-s_2-s_3)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} \\
&+ \frac{e^{s_1+s_2+s_3} s_2 k_7(-s_1-s_2-s_3)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)} + \frac{e^{s_1+s_2+s_3} s_3 k_7(-s_1-s_2-s_3)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)} \\
&+ \frac{e^{s_3} G_1(s_1) k_7(-s_3)}{2s_2 s_4} + \frac{e^{s_3} G_2(s_1,s_2) k_7(-s_3)}{2s_4} + \frac{e^{s_3} k_7(-s_3)}{2s_2(s_1+s_2)s_4} \\
&+ \frac{e^{s_3+s_4} G_1(s_1) k_7(-s_3-s_4)}{2s_2(s_2+s_3)} + \frac{e^{s_3+s_4} G_1(s_1) k_7(-s_3-s_4)}{2s_3(s_2+s_3)} + \frac{e^{s_3+s_4} G_2(s_1,s_2) k_7(-s_3-s_4)}{2s_3} \\
&+ \frac{e^{s_3+s_4} k_7(-s_3-s_4)}{(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} + \frac{e^{s_3+s_4} s_1 k_7(-s_3-s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} \\
&+ \frac{e^{s_3+s_4} s_3 k_7(-s_3-s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} + \frac{e^{s_3+s_4} s_1 k_7(-s_3-s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)} \\
&+ \frac{e^{s_3+s_4} s_2 k_7(-s_3-s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)} + \frac{e^{s_2+s_3+s_4} G_1(s_1) k_7(-s_2-s_3-s_4)}{2s_2 s_4} \\
&+ \frac{e^{s_2+s_3+s_4} k_7(-s_2-s_3-s_4)}{2s_1(s_1+s_2)s_4} + \frac{e^{s_2+s_3+s_4} k_7(-s_2-s_3-s_4)}{2s_2(s_1+s_2)s_4} \\
&+ \frac{e^{s_1+s_2+s_3+s_4} s_2 k_7(-s_1-s_2-s_3-s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} + \frac{e^{s_1+s_2+s_3+s_4} s_3 k_7(-s_1-s_2-s_3-s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} \\
&+ \frac{e^{s_1+s_2+s_3+s_4} k_7(-s_1-s_2-s_3-s_4)}{2s_1 s_4(s_3+s_4)} - \frac{1}{2} e^{s_4} G_3(s_1,s_2,s_3) k_7(-s_4) + \frac{k_8(s_1,s_2+s_3)}{4s_2 s_4} \\
&+ \frac{k_8(s_1+s_2,s_3+s_4)}{4s_1 s_3} + \frac{k_8(s_1+s_2,s_3+s_4)}{4s_2 s_4} + \frac{G_1(s_1) k_8(s_2+s_3,s_4)}{4s_3} + \frac{k_8(s_2+s_3,s_4)}{4s_1 s_3} \\
&+ \frac{k_9(s_1,s_2)}{8s_3(s_3+s_4)} + \frac{k_9(s_1,s_2+s_3+s_4)}{8s_2(s_2+s_3)} + \frac{k_9(s_1,s_2+s_3+s_4)}{8s_4(s_3+s_4)} + \frac{G_1(s_1) k_9(s_2,s_3+s_4)}{8s_4} \\
&+ \frac{k_9(s_2,s_3+s_4)}{8s_1 s_4} + \frac{k_9(s_1+s_2,s_3)}{8s_1 s_4} + \frac{G_1(s_1) k_9(s_3,s_4)}{8s_2} + \frac{1}{8} G_2(s_1,s_2) k_9(s_3,s_4) \\
&+ \frac{k_9(s_3,s_4)}{8s_2(s_1+s_2)} + \frac{k_9(s_1+s_2+s_3,s_4)}{8s_1(s_1+s_2)} + \frac{k_9(s_1+s_2+s_3,s_4)}{8s_3(s_2+s_3)} + \frac{e^{s_1} k_{10}(s_2,-s_1-s_2)}{8s_3(s_3+s_4)} \\
&+ \frac{e^{s_1+s_2} k_{10}(s_3,-s_1-s_2-s_3)}{8s_1 s_4} + \frac{e^{s_3} G_1(s_1) k_{10}(s_4,-s_3-s_4)}{8s_2} + \frac{1}{8} e^{s_3} G_2(s_1,s_2) k_{10}(s_4,-s_3-s_4) \\
&+ \frac{e^{s_3} k_{10}(s_4,-s_3-s_4)}{8s_2(s_1+s_2)} + \frac{e^{s_1+s_2+s_3} k_{10}(s_4,-s_1-s_2-s_3-s_4)}{8s_1(s_1+s_2)} \\
&+ \frac{e^{s_1+s_2+s_3} k_{10}(s_4,-s_1-s_2-s_3-s_4)}{8s_3(s_2+s_3)} + \frac{e^{s_2} G_1(s_1) k_{10}(s_3+s_4,-s_2-s_3-s_4)}{8s_4} \\
&+ \frac{e^{s_2} k_{10}(s_3+s_4,-s_2-s_3-s_4)}{8s_1 s_4} + \frac{e^{s_1} k_{10}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_2(s_2+s_3)} \\
&+ \frac{e^{s_1} k_{10}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_4(s_3+s_4)} + \frac{e^{s_1} k_{11}(s_2+s_3,-s_1-s_2-s_3)}{4s_2 s_4} \\
&+ \frac{e^{s_2+s_3} G_1(s_1) k_{11}(s_4,-s_2-s_3-s_4)}{4s_3} + \frac{e^{s_2+s_3} k_{11}(s_4,-s_2-s_3-s_4)}{4s_1 s_3} \\
&+ \frac{e^{s_1+s_2} k_{11}(s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_1 s_3} + \frac{e^{s_1+s_2} k_{11}(s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_2 s_4} \\
&+ \frac{e^{s_1+s_2+s_3} k_{12}(-s_1-s_2-s_3,s_1)}{4s_2 s_4} + \frac{e^{s_2+s_3+s_4} G_1(s_1) k_{12}(-s_2-s_3-s_4,s_2+s_3)}{4s_3} \\
&+ \frac{e^{s_2+s_3+s_4} k_{12}(-s_2-s_3-s_4,s_2+s_3)}{4s_1 s_3} + \frac{e^{s_1+s_2+s_3+s_4} k_{12}(-s_1-s_2-s_3-s_4,s_1+s_2)}{4s_1 s_3} \\
&+ \frac{e^{s_1+s_2+s_3+s_4} k_{12}(-s_1-s_2-s_3-s_4,s_1+s_2)}{4s_2 s_4} + \frac{e^{s_1+s_2} k_{13}(-s_1-s_2,s_1)}{8s_3(s_3+s_4)} \\
&+ \frac{e^{s_1+s_2+s_3} k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_1 s_4} + \frac{e^{s_3+s_4} G_1(s_1) k_{13}(-s_3-s_4,s_3)}{8s_2} \\
&+ \frac{1}{8} e^{s_3+s_4} G_2(s_1,s_2) k_{13}(-s_3-s_4,s_3) + \frac{e^{s_3+s_4} k_{13}(-s_3-s_4,s_3)}{8s_2(s_1+s_2)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2)}{8s_4}+\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2)}{8s_1s_4}\\
&+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8s_2(s_2+s_3)}+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8s_4(s_3+s_4)}\\
&+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_1(s_1+s_2)}\\
&+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_3(s_2+s_3)}+\frac{k_{17}(s_1,s_2+s_3,s_4)}{16s_2}\\
&+\frac{e^{s_1}k_{17}(s_2,s_3+s_4,-s_1-s_2-s_3-s_4)}{16s_3}+\frac{e^{s_1+s_2+s_3+s_4}k_{17}(-s_1-s_2-s_3-s_4,s_1,s_2)}{16s_3}\\
&+\frac{e^{s_1+s_2+s_3}k_{17}(s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_2}+\frac{k_{18}(s_1,s_2,s_3)}{16s_4}+\frac{e^{s_1}k_{18}(s_2,s_3,-s_1-s_2-s_3)}{16s_4}\\
&-\frac{1}{16}G_1(s_1)k_{18}(s_2,s_3,s_4)+\frac{e^{s_1}k_{18}(s_2,s_3,s_4)}{16(s_1+s_2+s_3+s_4)}+\frac{k_{18}(s_1+s_2,s_3,s_4)}{16s_1}\\
&+\frac{e^{s_1+s_2+s_3}k_{18}(-s_1-s_2-s_3,s_1,s_2)}{16s_4}+\frac{e^{s_1+s_2}k_{18}(s_3,-s_1-s_2-s_3,s_1)}{16s_4}\\
&-\frac{1}{16}e^{s_2}G_1(s_1)k_{18}(s_3,s_4,-s_2-s_3-s_4)+\frac{e^{s_1+s_2}k_{18}(s_3,s_4,-s_2-s_3-s_4)}{16(s_1+s_2+s_3+s_4)}\\
&+\frac{e^{s_1+s_2}k_{18}(s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_1}-\frac{1}{16}e^{s_2+s_3+s_4}G_1(s_1)k_{18}(-s_2-s_3-s_4,s_2,s_3)\\
&+\frac{e^{s_1+s_2+s_3+s_4}k_{18}(-s_2-s_3-s_4,s_2,s_3)}{16(s_1+s_2+s_3+s_4)}+\frac{e^{s_1+s_2+s_3+s_4}k_{18}(-s_1-s_2-s_3-s_4,s_1+s_2,s_3)}{16s_1}\\
&-\frac{1}{16}e^{s_2+s_3}G_1(s_1)k_{18}(s_4,-s_2-s_3-s_4,s_2)+\frac{e^{s_1+s_2+s_3}k_{18}(s_4,-s_2-s_3-s_4,s_2)}{16(s_1+s_2+s_3+s_4)}\\
&+\frac{e^{s_1+s_2+s_3}k_{18}(s_4,-s_1-s_2-s_3-s_4,s_1+s_2)}{16s_1}+\frac{k_{19}(s_1,s_2,s_3+s_4)}{16s_3}\\
&+\frac{e^{s_1}k_{19}(s_2+s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_2}+\frac{e^{s_1+s_2+s_3+s_4}k_{19}(-s_1-s_2-s_3-s_4,s_1,s_2+s_3)}{16s_2}\\
&+\frac{e^{s_1+s_2}k_{19}(s_3+s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_3}-\frac{k_{18}(s_2,s_3,s_4)}{16s_1}-\frac{e^{s_2}k_{18}(s_3,s_4,-s_2-s_3-s_4)}{16s_1}\\
&-\frac{e^{s_2+s_3+s_4}k_{18}(-s_2-s_3-s_4,s_2,s_3)}{16s_1}-\frac{e^{s_2+s_3}k_{18}(s_4,-s_2-s_3-s_4,s_2)}{16s_1}-\frac{G_1(s_1)k_9(s_2+s_3,s_4)}{8s_2}\\
&-\frac{e^{s_2+s_3}G_1(s_1)k_{10}(s_4,-s_2-s_3-s_4)}{8s_2}-\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_2}\\
&-\frac{k_{17}(s_1+s_2,s_3,s_4)}{16s_2}-\frac{e^{s_1+s_2+s_3}k_{17}(s_4,-s_1-s_2-s_3-s_4,s_1+s_2)}{16s_2}\\
&-\frac{e^{s_1+s_2}k_{19}(s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_2}-\frac{e^{s_1+s_2+s_3+s_4}k_{19}(-s_1-s_2-s_3-s_4,s_1+s_2,s_3)}{16s_2}\\
&-\frac{k_9(s_2+s_3,s_4)}{8s_1(s_1+s_2)}-\frac{e^{s_2+s_3}k_{10}(s_4,-s_2-s_3-s_4)}{8s_1(s_1+s_2)}-\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_1(s_1+s_2)}\\
&-\frac{k_9(s_2+s_3,s_4)}{8s_2(s_1+s_2)}-\frac{e^{s_2+s_3}k_{10}(s_4,-s_2-s_3-s_4)}{8s_2(s_1+s_2)}-\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_2(s_1+s_2)}\\
&-\frac{G_2(s_1,s_2)k_6(s_4)}{2s_3}-\frac{e^{s_4}G_2(s_1,s_2)k_7(-s_4)}{2s_3}-\frac{G_1(s_1)k_8(s_2,s_3+s_4)}{4s_3}\\
&-\frac{e^{s_2}G_1(s_1)k_{11}(s_3+s_4,-s_2-s_3-s_4)}{4s_3}-\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{12}(-s_2-s_3-s_4,s_2)}{4s_3}\\
&-\frac{e^{s_1}k_{17}(s_2+s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_3}-\frac{e^{s_1+s_2+s_3+s_4}k_{17}(-s_1-s_2-s_3-s_4,s_1,s_2+s_3)}{16s_3}\\
&-\frac{k_{19}(s_1,s_2+s_3,s_4)}{16s_3}-\frac{e^{s_1+s_2+s_3}k_{19}(s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_3}-\frac{k_8(s_2,s_3+s_4)}{4s_1s_3}
\end{aligned}
$$

$$- \frac{k_8(s_1+s_2+s_3,s_4)}{4s_1s_3} - \frac{e^{s_1+s_2+s_3}k_{11}(s_4,-s_1-s_2-s_3-s_4)}{4s_1s_3} - \frac{e^{s_2}k_{11}(s_3+s_4,-s_2-s_3-s_4)}{4s_1s_3}$$
$$- \frac{e^{s_2+s_3+s_4}k_{12}(-s_2-s_3-s_4,s_2)}{4s_1s_3} - \frac{e^{s_1+s_2+s_3+s_4}k_{12}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{4s_1s_3}$$
$$- \frac{G_1(s_1)k_6(s_2+s_3+s_4)}{2s_2(s_2+s_3)} - \frac{e^{s_2+s_3+s_4}G_1(s_1)k_7(-s_2-s_3-s_4)}{2s_2(s_2+s_3)}$$
$$- \frac{k_9(s_1+s_2,s_3+s_4)}{8s_2(s_2+s_3)} - \frac{e^{s_1+s_2}k_{10}(s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_2(s_2+s_3)}$$
$$- \frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_2(s_2+s_3)} - \frac{G_1(s_1)k_6(s_4)}{2s_3(s_2+s_3)} - \frac{e^{s_4}G_1(s_1)k_7(-s_4)}{2s_3(s_2+s_3)}$$
$$- \frac{k_9(s_1+s_2,s_3+s_4)}{8s_3(s_2+s_3)} - \frac{e^{s_1+s_2}k_{10}(s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_3(s_2+s_3)}$$
$$- \frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_3(s_2+s_3)} - \frac{k_6(s_2+s_3+s_4)}{(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}$$
$$- \frac{e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} - \frac{s_2k_6(s_2+s_3+s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}$$
$$- \frac{s_3k_6(s_2+s_3+s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} - \frac{e^{s_2+s_3+s_4}s_2k_7(-s_2-s_3-s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}$$
$$- \frac{e^{s_2+s_3+s_4}s_3k_7(-s_2-s_3-s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} - \frac{s_1k_6(s_2+s_3+s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}$$
$$- \frac{s_3k_6(s_2+s_3+s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} - \frac{e^{s_2+s_3+s_4}s_1k_7(-s_2-s_3-s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}$$
$$- \frac{e^{s_2+s_3+s_4}s_3k_7(-s_2-s_3-s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} - \frac{s_1k_6(s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}$$
$$- \frac{s_2k_6(s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)} - \frac{e^{s_4}s_1k_7(-s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}$$
$$- \frac{e^{s_4}s_2k_7(-s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)} - \frac{G_2(s_1,s_2)k_6(s_3+s_4)}{2s_4} - \frac{e^{s_3+s_4}G_2(s_1,s_2)k_7(-s_3-s_4)}{2s_4}$$
$$- \frac{G_1(s_1)k_9(s_2,s_3)}{8s_4} - \frac{e^{s_2}G_1(s_1)k_{10}(s_3,-s_2-s_3)}{8s_4} - \frac{e^{s_2+s_3}G_1(s_1)k_{13}(-s_2-s_3,s_2)}{8s_4}$$
$$- \frac{k_{18}(s_1,s_2,s_3+s_4)}{16s_4} - \frac{e^{s_1}k_{18}(s_2,s_3+s_4,-s_1-s_2-s_3-s_4)}{16s_4}$$
$$- \frac{e^{s_1+s_2+s_3+s_4}k_{18}(-s_1-s_2-s_3-s_4,s_1,s_2)}{16s_4} - \frac{e^{s_1+s_2}k_{18}(s_3+s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_4}$$
$$- \frac{k_9(s_2,s_3)}{8s_1s_4} - \frac{k_9(s_1+s_2,s_3+s_4)}{8s_1s_4} - \frac{e^{s_2}k_{10}(s_3,-s_2-s_3)}{8s_1s_4}$$
$$- \frac{e^{s_1+s_2}k_{10}(s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_1s_4} - \frac{e^{s_2+s_3}k_{13}(-s_2-s_3,s_2)}{8s_1s_4}$$
$$- \frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_1s_4} - \frac{G_1(s_1)k_6(s_2+s_3)}{2s_2s_4} - \frac{G_1(s_1)k_6(s_3+s_4)}{2s_2s_4}$$
$$- \frac{e^{s_2+s_3}G_1(s_1)k_7(-s_2-s_3)}{2s_2s_4} - \frac{e^{s_3+s_4}G_1(s_1)k_7(-s_3-s_4)}{2s_2s_4} - \frac{k_8(s_1,s_2+s_3+s_4)}{4s_2s_4}$$
$$- \frac{k_8(s_1+s_2,s_3)}{4s_2s_4} - \frac{e^{s_1+s_2}k_{11}(s_3,-s_1-s_2-s_3)}{4s_2s_4} - \frac{e^{s_1}k_{11}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_2s_4}$$
$$- \frac{e^{s_1+s_2+s_3}k_{12}(-s_1-s_2-s_3,s_1+s_2)}{4s_2s_4} - \frac{e^{s_1+s_2+s_3+s_4}k_{12}(-s_1-s_2-s_3-s_4,s_1)}{4s_2s_4}$$
$$- \frac{k_6(s_2+s_3)}{2s_1(s_1+s_2)s_4} - \frac{k_6(s_1+s_2+s_3+s_4)}{2s_1(s_1+s_2)s_4} - \frac{e^{s_2+s_3}k_7(-s_2-s_3)}{2s_1(s_1+s_2)s_4}$$
$$- \frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{2s_1(s_1+s_2)s_4} - \frac{k_6(s_2+s_3)}{2s_2(s_1+s_2)s_4} - \frac{k_6(s_3+s_4)}{2s_2(s_1+s_2)s_4} - \frac{e^{s_2+s_3}k_7(-s_2-s_3)}{2s_2(s_1+s_2)s_4}$$

$$
\begin{aligned}
&-\frac{e^{s_3+s_4}k_7(-s_3-s_4)}{2s_2(s_1+s_2)s_4}-\frac{G_1(s_1)k_6(s_2)}{2s_3(s_3+s_4)}-\frac{e^{s_2}G_1(s_1)k_7(-s_2)}{2s_3(s_3+s_4)}-\frac{k_9(s_1,s_2+s_3)}{8s_3(s_3+s_4)}\\
&-\frac{e^{s_1}k_{10}(s_2+s_3,-s_1-s_2-s_3)}{8s_3(s_3+s_4)}-\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1)}{8s_3(s_3+s_4)}-\frac{k_6(s_2)}{2s_1s_3(s_3+s_4)}\\
&-\frac{k_6(s_1+s_2+s_3)}{2s_1s_3(s_3+s_4)}-\frac{e^{s_2}k_7(-s_2)}{2s_1s_3(s_3+s_4)}-\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{2s_1s_3(s_3+s_4)}-\frac{G_1(s_1)k_6(s_2+s_3+s_4)}{2s_4(s_3+s_4)}\\
&-\frac{e^{s_2+s_3+s_4}G_1(s_1)k_7(-s_2-s_3-s_4)}{2s_4(s_3+s_4)}-\frac{k_9(s_1,s_2+s_3)}{8s_4(s_3+s_4)}-\frac{e^{s_1}k_{10}(s_2+s_3,-s_1-s_2-s_3)}{8s_4(s_3+s_4)}\\
&-\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1)}{8s_4(s_3+s_4)}-\frac{k_6(s_1+s_2+s_3)}{2s_1s_4(s_3+s_4)}-\frac{k_6(s_2+s_3+s_4)}{2s_1s_4(s_3+s_4)}\\
&-\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{2s_1s_4(s_3+s_4)}-\frac{e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{2s_1s_4(s_3+s_4)}-\frac{k_6(s_1+s_2)}{(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{e^{s_1+s_2}k_7(-s_1-s_2)}{(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}-\frac{s_3k_6(s_1+s_2)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{s_4k_6(s_1+s_2)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}-\frac{e^{s_1+s_2}s_3k_7(-s_1-s_2)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{e^{s_1+s_2}s_4k_7(-s_1-s_2)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}-\frac{s_2k_6(s_1+s_2)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{s_4k_6(s_1+s_2)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}-\frac{e^{s_1+s_2}s_2k_7(-s_1-s_2)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{e^{s_1+s_2}s_4k_7(-s_1-s_2)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}-\frac{s_2k_6(s_1+s_2+s_3+s_4)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{s_3k_6(s_1+s_2+s_3+s_4)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}-\frac{e^{s_1+s_2+s_3+s_4}s_2k_7(-s_1-s_2-s_3-s_4)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{e^{s_1+s_2+s_3+s_4}s_3k_7(-s_1-s_2-s_3-s_4)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}-\frac{k_{18}(s_1,s_2,s_3)}{16(s_1+s_2+s_3+s_4)}-\frac{e^{s_1}k_{18}(s_2,s_3,-s_1-s_2-s_3)}{16(s_1+s_2+s_3+s_4)}\\
&-\frac{e^{s_1+s_2+s_3}k_{18}(-s_1-s_2-s_3,s_1,s_2)}{16(s_1+s_2+s_3+s_4)}-\frac{e^{s_1+s_2}k_{18}(s_3,-s_1-s_2-s_3,s_1)}{16(s_1+s_2+s_3+s_4)}.
\end{aligned}
$$

A.2.2. *The function* $\widetilde{K}_{19}$. We have

(71)

$$
\begin{aligned}
&\widetilde{K}_{19}(s_1,s_2,s_3,s_4)\\
&\quad=\frac{1}{15}(-4)\pi G_4(s_1,s_2,s_3,s_4)+\frac{s_3k_6(s_1)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&\qquad+\frac{s_4k_6(s_1)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}+\frac{k_6(s_1+s_2)}{2s_1s_3(s_3+s_4)}+\frac{G_1(s_1)k_6(s_3)}{2s_2s_4}+\frac{G_2(s_1,s_2)k_6(s_3)}{2s_4}\\
&\qquad+\frac{k_6(s_3)}{2s_2(s_1+s_2)s_4}+\frac{G_1(s_1)k_6(s_2+s_3)}{2s_3(s_3+s_4)}+\frac{G_1(s_1)k_6(s_2+s_3)}{2s_4(s_3+s_4)}+\frac{k_6(s_2+s_3)}{2s_1s_3(s_3+s_4)}+\frac{k_6(s_2+s_3)}{2s_1s_4(s_3+s_4)}\\
&\qquad+\frac{k_6(s_1+s_2+s_3)}{2s_1(s_1+s_2)s_4}+\frac{k_6(s_1+s_2+s_3)}{(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}+\frac{s_2k_6(s_1+s_2+s_3)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&\qquad+\frac{s_4k_6(s_1+s_2+s_3)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}+\frac{s_2k_6(s_1+s_2+s_3)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}\\
&\qquad+\frac{s_3k_6(s_1+s_2+s_3)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}-\frac{1}{2}G_3(s_1,s_2,s_3)k_6(s_4)+\frac{G_1(s_1)k_6(s_3+s_4)}{2s_2(s_2+s_3)}\\
&\qquad+\frac{G_1(s_1)k_6(s_3+s_4)}{2s_3(s_2+s_3)}+\frac{G_2(s_1,s_2)k_6(s_3+s_4)}{2s_3}+\frac{k_6(s_3+s_4)}{(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&\qquad+\frac{s_1k_6(s_3+s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}+\frac{s_3k_6(s_3+s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&\qquad+\frac{s_1k_6(s_3+s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}+\frac{s_2k_6(s_3+s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{G_1(s_1)k_6(s_2+s_3+s_4)}{2s_2s_4}+\frac{k_6(s_2+s_3+s_4)}{2s_1(s_1+s_2)s_4}+\frac{k_6(s_2+s_3+s_4)}{2s_2(s_1+s_2)s_4}\\
&+\frac{s_2k_6(s_1+s_2+s_3+s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}+\frac{s_3k_6(s_1+s_2+s_3+s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&+\frac{k_6(s_1+s_2+s_3+s_4)}{2s_1s_4(s_3+s_4)}+\frac{e^{s_1}s_3k_7(-s_1)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&+\frac{e^{s_1}s_4k_7(-s_1)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}+\frac{e^{s_1+s_2}k_7(-s_1-s_2)}{2s_1s_3(s_3+s_4)}+\frac{e^{s_2+s_3}G_1(s_1)k_7(-s_2-s_3)}{2s_3(s_3+s_4)}\\
&+\frac{e^{s_2+s_3}G_1(s_1)k_7(-s_2-s_3)}{2s_4(s_3+s_4)}+\frac{e^{s_2+s_3}k_7(-s_2-s_3)}{2s_1s_3(s_3+s_4)}+\frac{e^{s_2+s_3}k_7(-s_2-s_3)}{2s_1s_4(s_3+s_4)}\\
&+\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{2s_1(s_1+s_2)s_4}+\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&+\frac{e^{s_1+s_2+s_3}s_2k_7(-s_1-s_2-s_3)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}+\frac{e^{s_1+s_2+s_3}s_4k_7(-s_1-s_2-s_3)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&+\frac{e^{s_1+s_2+s_3}s_2k_7(-s_1-s_2-s_3)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}+\frac{e^{s_1+s_2+s_3}s_3k_7(-s_1-s_2-s_3)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}\\
&+\frac{e^{s_3}G_1(s_1)k_7(-s_3)}{2s_2s_4}+\frac{e^{s_3}G_2(s_1,s_2)k_7(-s_3)}{2s_4}+\frac{e^{s_3}k_7(-s_3)}{2s_2(s_1+s_2)s_4}+\frac{e^{s_3+s_4}G_1(s_1)k_7(-s_3-s_4)}{2s_2(s_2+s_3)}\\
&+\frac{e^{s_3+s_4}G_1(s_1)k_7(-s_3-s_4)}{2s_3(s_2+s_3)}+\frac{e^{s_3+s_4}G_2(s_1,s_2)k_7(-s_3-s_4)}{2s_3}+\frac{e^{s_3+s_4}k_7(-s_3-s_4)}{(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&+\frac{e^{s_3+s_4}s_1k_7(-s_3-s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}+\frac{e^{s_3+s_4}s_3k_7(-s_3-s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&+\frac{e^{s_3+s_4}s_1k_7(-s_3-s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}+\frac{e^{s_3+s_4}s_2k_7(-s_3-s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}\\
&+\frac{e^{s_2+s_3+s_4}G_1(s_1)k_7(-s_2-s_3-s_4)}{2s_2s_4}+\frac{e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{2s_1(s_1+s_2)s_4}\\
&+\frac{e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{2s_2(s_1+s_2)s_4}+\frac{e^{s_1+s_2+s_3+s_4}s_2k_7(-s_1-s_2-s_3-s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&+\frac{e^{s_1+s_2+s_3+s_4}s_3k_7(-s_1-s_2-s_3-s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}+\frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{2s_1s_4(s_3+s_4)}\\
&-\frac{1}{2}e^{s_4}G_3(s_1,s_2,s_3)k_7(-s_4)+\frac{k_8(s_1,s_2+s_3+s_4)}{4s_2(s_2+s_3)}+\frac{G_1(s_1)k_8(s_2,s_3+s_4)}{4s_4}+\frac{k_8(s_2,s_3+s_4)}{4s_1s_4}\\
&+\frac{k_8(s_1+s_2,s_3)}{4s_1s_4}+\frac{k_8(s_1+s_2+s_3,s_4)}{4s_3(s_2+s_3)}+\frac{k_9(s_1,s_2)}{8s_3(s_3+s_4)}+\frac{k_9(s_1,s_2+s_3)}{8s_2s_4}+\frac{k_9(s_1,s_2+s_3+s_4)}{8s_4(s_3+s_4)}\\
&+\frac{k_9(s_1+s_2,s_3+s_4)}{8s_1s_3}+\frac{k_9(s_1+s_2,s_3+s_4)}{8s_2s_4}+\frac{G_1(s_1)k_9(s_3,s_4)}{8s_2}+\frac{1}{8}G_2(s_1,s_2)k_9(s_3,s_4)\\
&+\frac{k_9(s_3,s_4)}{8s_2(s_1+s_2)}+\frac{G_1(s_1)k_9(s_2+s_3,s_4)}{8s_3}+\frac{k_9(s_2+s_3,s_4)}{8s_1s_3}+\frac{k_9(s_1+s_2+s_3,s_4)}{8s_1(s_1+s_2)}\\
&+\frac{e^{s_1}k_{10}(s_2,-s_1-s_2)}{8s_3(s_3+s_4)}+\frac{e^{s_1}k_{10}(s_2+s_3,-s_1-s_2-s_3)}{8s_2s_4}+\frac{e^{s_3}G_1(s_1)k_{10}(s_4,-s_3-s_4)}{8s_2}\\
&+\frac{1}{8}e^{s_3}G_2(s_1,s_2)k_{10}(s_4,-s_3-s_4)+\frac{e^{s_3}k_{10}(s_4,-s_3-s_4)}{8s_2(s_1+s_2)}+\frac{e^{s_2+s_3}G_1(s_1)k_{10}(s_4,-s_2-s_3-s_4)}{8s_3}\\
&+\frac{e^{s_2+s_3}k_{10}(s_4,-s_2-s_3-s_4)}{8s_1s_3}+\frac{e^{s_1+s_2+s_3}k_{10}(s_4,-s_1-s_2-s_3-s_4)}{8s_1(s_1+s_2)}\\
&+\frac{e^{s_1+s_2}k_{10}(s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_1s_3}+\frac{e^{s_1+s_2}k_{10}(s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_2s_4}\\
&+\frac{e^{s_1}k_{10}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_4(s_3+s_4)}+\frac{e^{s_1+s_2}k_{11}(s_3,-s_1-s_2-s_3)}{4s_1s_4}\\
&+\frac{e^{s_1+s_2+s_3}k_{11}(s_4,-s_1-s_2-s_3-s_4)}{4s_3(s_2+s_3)}+\frac{e^{s_2}G_1(s_1)k_{11}(s_3+s_4,-s_2-s_3-s_4)}{4s_4}\\
&+\frac{e^{s_2}k_{11}(s_3+s_4,-s_2-s_3-s_4)}{4s_1s_4}+\frac{e^{s_1}k_{11}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_2(s_2+s_3)}
\end{aligned}
$$

$$+\frac{e^{s_1+s_2+s_3}k_{12}(-s_1-s_2-s_3,s_1+s_2)}{4s_1s_4}+\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{12}(-s_2-s_3-s_4,s_2)}{4s_4}$$
$$+\frac{e^{s_2+s_3+s_4}k_{12}(-s_2-s_3-s_4,s_2)}{4s_1s_4}+\frac{e^{s_1+s_2+s_3+s_4}k_{12}(-s_1-s_2-s_3-s_4,s_1)}{4s_2(s_2+s_3)}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{12}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{4s_3(s_2+s_3)}+\frac{e^{s_1+s_2}k_{13}(-s_1-s_2,s_1)}{8s_3(s_3+s_4)}$$
$$+\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1)}{8s_2s_4}+\frac{e^{s_3+s_4}G_1(s_1)k_{13}(-s_3-s_4,s_3)}{8s_2}$$
$$+\frac{1}{8}e^{s_3+s_4}G_2(s_1,s_2)k_{13}(-s_3-s_4,s_3)+\frac{e^{s_3+s_4}k_{13}(-s_3-s_4,s_3)}{8s_2(s_1+s_2)}$$
$$+\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_3}+\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_1s_3}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8s_4(s_3+s_4)}+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_1s_3}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_2s_4}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_1(s_1+s_2)}+\frac{e^{s_1}k_{17}(s_2,s_3,-s_1-s_2-s_3)}{16s_4}$$
$$-\frac{1}{16}G_1(s_1)k_{17}(s_2,s_3,s_4)+\frac{e^{s_1}k_{17}(s_2,s_3,s_4)}{16(s_1+s_2+s_3+s_4)}+\frac{k_{17}(s_1+s_2,s_3,s_4)}{16s_1}$$
$$+\frac{e^{s_1+s_2+s_3}k_{17}(-s_1-s_2-s_3,s_1,s_2)}{16s_4}-\frac{1}{16}e^{s_2+s_3}G_1(s_1)k_{17}(s_4,-s_2-s_3-s_4,s_2)$$
$$+\frac{e^{s_1+s_2+s_3}k_{17}(s_4,-s_2-s_3-s_4,s_2)}{16(s_1+s_2+s_3+s_4)}+\frac{e^{s_1+s_2+s_3}k_{17}(s_4,-s_1-s_2-s_3-s_4,s_1+s_2)}{16s_1}$$
$$+\frac{k_{18}(s_1,s_2,s_3+s_4)}{16s_3}+\frac{k_{18}(s_1,s_2+s_3,s_4)}{16s_2}+\frac{e^{s_1}k_{18}(s_2,s_3+s_4,-s_1-s_2-s_3-s_4)}{16s_3}$$
$$+\frac{e^{s_1}k_{18}(s_2+s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_2}+\frac{e^{s_1+s_2+s_3+s_4}k_{18}(-s_1-s_2-s_3-s_4,s_1,s_2)}{16s_3}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{18}(-s_1-s_2-s_3-s_4,s_1,s_2+s_3)}{16s_2}+\frac{e^{s_1+s_2+s_3}k_{18}(s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_2}$$
$$+\frac{e^{s_1+s_2}k_{18}(s_3+s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_3}+\frac{k_{19}(s_1,s_2,s_3)}{16s_4}+\frac{e^{s_1+s_2}k_{19}(s_3,-s_1-s_2-s_3,s_1)}{16s_4}$$
$$-\frac{1}{16}e^{s_2}G_1(s_1)k_{19}(s_3,s_4,-s_2-s_3-s_4)+\frac{e^{s_1+s_2}k_{19}(s_3,s_4,-s_2-s_3-s_4)}{16(s_1+s_2+s_3+s_4)}$$
$$+\frac{e^{s_1+s_2}k_{19}(s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_1}-\frac{1}{16}e^{s_2+s_3+s_4}G_1(s_1)k_{19}(-s_2-s_3-s_4,s_2,s_3)$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{19}(-s_2-s_3-s_4,s_2,s_3)}{16(s_1+s_2+s_3+s_4)}+\frac{e^{s_1+s_2+s_3+s_4}k_{19}(-s_1-s_2-s_3-s_4,s_1+s_2,s_3)}{16s_1}$$
$$-\frac{k_{17}(s_2,s_3,s_4)}{16s_1}-\frac{e^{s_2+s_3}k_{17}(s_4,-s_2-s_3-s_4,s_2)}{16s_1}-\frac{e^{s_2}k_{19}(s_3,s_4,-s_2-s_3-s_4)}{16s_1}$$
$$-\frac{e^{s_2+s_3+s_4}k_{19}(-s_2-s_3-s_4,s_2,s_3)}{16s_1}-\frac{G_1(s_1)k_9(s_2+s_3,s_4)}{8s_2}$$
$$-\frac{e^{s_2+s_3}G_1(s_1)k_{10}(s_4,-s_2-s_3-s_4)}{8s_2}-\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_2}$$
$$-\frac{k_{18}(s_1+s_2,s_3,s_4)}{16s_2}-\frac{e^{s_1+s_2}k_{18}(s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_2}$$
$$-\frac{e^{s_1+s_2+s_3+s_4}k_{18}(-s_1-s_2-s_3-s_4,s_1+s_2,s_3)}{16s_2}$$
$$-\frac{e^{s_1+s_2+s_3}k_{18}(s_4,-s_1-s_2-s_3-s_4,s_1+s_2)}{16s_2}-\frac{k_9(s_2+s_3,s_4)}{8s_1(s_1+s_2)}-\frac{e^{s_2+s_3}k_{10}(s_4,-s_2-s_3-s_4)}{8s_1(s_1+s_2)}$$

$$
\begin{aligned}
&-\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_1(s_1+s_2)}-\frac{k_9(s_2+s_3,s_4)}{8s_2(s_1+s_2)}-\frac{e^{s_2+s_3}k_{10}(s_4,-s_2-s_3-s_4)}{8s_2(s_1+s_2)}\\
&-\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2+s_3)}{8s_2(s_1+s_2)}-\frac{G_2(s_1,s_2)k_6(s_4)}{2s_3}-\frac{e^{s_4}G_2(s_1,s_2)k_7(-s_4)}{2s_3}\\
&-\frac{G_1(s_1)k_9(s_2,s_3+s_4)}{8s_3}-\frac{e^{s_2}G_1(s_1)k_{10}(s_3+s_4,-s_2-s_3-s_4)}{8s_3}\\
&-\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2)}{8s_3}-\frac{k_{18}(s_1,s_2+s_3,s_4)}{16s_3}\\
&-\frac{e^{s_1}k_{18}(s_2+s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_3}-\frac{e^{s_1+s_2+s_3+s_4}k_{18}(-s_1-s_2-s_3-s_4,s_1,s_2+s_3)}{16s_3}\\
&-\frac{e^{s_1+s_2+s_3}k_{18}(s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_3}-\frac{k_9(s_2,s_3+s_4)}{8s_1s_3}-\frac{k_9(s_1+s_2+s_3,s_4)}{8s_1s_3}\\
&-\frac{e^{s_1+s_2+s_3}k_{10}(s_4,-s_1-s_2-s_3-s_4)}{8s_1s_3}-\frac{e^{s_2}k_{10}(s_3+s_4,-s_2-s_3-s_4)}{8s_1s_3}\\
&-\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2)}{8s_1s_3}-\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_1s_3}\\
&-\frac{G_1(s_1)k_6(s_2+s_3+s_4)}{2s_2(s_2+s_3)}-\frac{e^{s_2+s_3+s_4}G_1(s_1)k_7(-s_2-s_3-s_4)}{2s_2(s_2+s_3)}-\frac{k_8(s_1+s_2,s_3+s_4)}{4s_2(s_2+s_3)}\\
&-\frac{e^{s_1+s_2}k_{11}(s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_2(s_2+s_3)}-\frac{e^{s_1+s_2+s_3+s_4}k_{12}(-s_1-s_2-s_3-s_4,s_1+s_2)}{4s_2(s_2+s_3)}\\
&-\frac{G_1(s_1)k_6(s_4)}{2s_3(s_2+s_3)}-\frac{e^{s_4}G_1(s_1)k_7(-s_4)}{2s_3(s_2+s_3)}-\frac{k_8(s_1+s_2,s_3+s_4)}{4s_3(s_2+s_3)}\\
&-\frac{e^{s_1+s_2}k_{11}(s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_3(s_2+s_3)}-\frac{e^{s_1+s_2+s_3+s_4}k_{12}(-s_1-s_2-s_3-s_4,s_1+s_2)}{4s_3(s_2+s_3)}\\
&-\frac{k_6(s_2+s_3+s_4)}{(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}-\frac{e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&-\frac{s_2k_6(s_2+s_3+s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}-\frac{s_3k_6(s_2+s_3+s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&-\frac{e^{s_2+s_3+s_4}s_2k_7(-s_2-s_3-s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}-\frac{e^{s_2+s_3+s_4}s_3k_7(-s_2-s_3-s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&-\frac{s_1k_6(s_2+s_3+s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}-\frac{s_3k_6(s_2+s_3+s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&-\frac{e^{s_2+s_3+s_4}s_1k_7(-s_2-s_3-s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}-\frac{e^{s_2+s_3+s_4}s_3k_7(-s_2-s_3-s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&-\frac{s_1k_6(s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}-\frac{s_2k_6(s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}\\
&-\frac{e^{s_4}s_1k_7(-s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}-\frac{e^{s_4}s_2k_7(-s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}\\
&-\frac{G_2(s_1,s_2)k_6(s_3+s_4)}{2s_4}-\frac{e^{s_3+s_4}G_2(s_1,s_2)k_7(-s_3-s_4)}{2s_4}-\frac{G_1(s_1)k_8(s_2,s_3)}{4s_4}\\
&-\frac{e^{s_2}G_1(s_1)k_{11}(s_3,-s_2-s_3)}{4s_4}-\frac{e^{s_2+s_3}G_1(s_1)k_{12}(-s_2-s_3,s_2)}{4s_4}\\
&-\frac{e^{s_1}k_{17}(s_2,s_3+s_4,-s_1-s_2-s_3-s_4)}{16s_4}-\frac{e^{s_1+s_2+s_3+s_4}k_{17}(-s_1-s_2-s_3-s_4,s_1,s_2)}{16s_4}\\
&-\frac{k_{19}(s_1,s_2,s_3+s_4)}{16s_4}-\frac{e^{s_1+s_2}k_{19}(s_3+s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_4}-\frac{k_8(s_2,s_3)}{4s_1s_4}\\
&-\frac{k_8(s_1+s_2,s_3+s_4)}{4s_1s_4}-\frac{e^{s_2}k_{11}(s_3,-s_2-s_3)}{4s_1s_4}-\frac{e^{s_1+s_2}k_{11}(s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_1s_4}\\
&-\frac{e^{s_2+s_3}k_{12}(-s_2-s_3,s_2)}{4s_1s_4}-\frac{e^{s_1+s_2+s_3+s_4}k_{12}(-s_1-s_2-s_3-s_4,s_1+s_2)}{4s_1s_4}\\
&-\frac{G_1(s_1)k_6(s_2+s_3)}{2s_2s_4}-\frac{G_1(s_1)k_6(s_3+s_4)}{2s_2s_4}-\frac{e^{s_2+s_3}G_1(s_1)k_7(-s_2-s_3)}{2s_2s_4}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{e^{s_3+s_4} G_1(s_1)k_7(-s_3-s_4)}{2s_2s_4} - \frac{k_9(s_1,s_2+s_3+s_4)}{8s_2s_4} - \frac{k_9(s_1+s_2,s_3)}{8s_2s_4}\\
&- \frac{e^{s_1+s_2} k_{10}(s_3,-s_1-s_2-s_3)}{8s_2s_4} - \frac{e^{s_1} k_{10}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_2s_4}\\
&- \frac{e^{s_1+s_2+s_3} k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_2s_4} - \frac{e^{s_1+s_2+s_3+s_4} k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8s_2s_4}\\
&- \frac{k_6(s_2+s_3)}{2s_1(s_1+s_2)s_4} - \frac{k_6(s_1+s_2+s_3+s_4)}{2s_1(s_1+s_2)s_4} - \frac{e^{s_2+s_3} k_7(-s_2-s_3)}{2s_1(s_1+s_2)s_4}\\
&- \frac{e^{s_1+s_2+s_3+s_4} k_7(-s_1-s_2-s_3-s_4)}{2s_1(s_1+s_2)s_4} - \frac{k_6(s_2+s_3)}{2s_2(s_1+s_2)s_4} - \frac{k_6(s_3+s_4)}{2s_2(s_1+s_2)s_4}\\
&- \frac{e^{s_2+s_3} k_7(-s_2-s_3)}{2s_2(s_1+s_2)s_4} - \frac{e^{s_3+s_4} k_7(-s_3-s_4)}{2s_2(s_1+s_2)s_4} - \frac{G_1(s_1)k_6(s_2)}{2s_3(s_3+s_4)} - \frac{e^{s_2} G_1(s_1)k_7(-s_2)}{2s_3(s_3+s_4)}\\
&- \frac{k_9(s_1,s_2+s_3)}{8s_3(s_3+s_4)} - \frac{e^{s_1} k_{10}(s_2+s_3,-s_1-s_2-s_3)}{8s_3(s_3+s_4)} - \frac{e^{s_1+s_2+s_3} k_{13}(-s_1-s_2-s_3,s_1)}{8s_3(s_3+s_4)}\\
&- \frac{k_6(s_2)}{2s_1s_3(s_3+s_4)} - \frac{k_6(s_1+s_2+s_3)}{2s_1s_3(s_3+s_4)} - \frac{e^{s_2} k_7(-s_2)}{2s_1s_3(s_3+s_4)} - \frac{e^{s_1+s_2+s_3} k_7(-s_1-s_2-s_3)}{2s_1s_3(s_3+s_4)}\\
&- \frac{G_1(s_1)k_6(s_2+s_3+s_4)}{2s_4(s_3+s_4)} - \frac{e^{s_2+s_3+s_4} G_1(s_1)k_7(-s_2-s_3-s_4)}{2s_4(s_3+s_4)} - \frac{k_9(s_1,s_2+s_3)}{8s_4(s_3+s_4)}\\
&- \frac{e^{s_1} k_{10}(s_2+s_3,-s_1-s_2-s_3)}{8s_4(s_3+s_4)} - \frac{e^{s_1+s_2+s_3} k_{13}(-s_1-s_2-s_3,s_1)}{8s_4(s_3+s_4)} - \frac{k_6(s_1+s_2+s_3)}{2s_1s_4(s_3+s_4)}\\
&- \frac{k_6(s_2+s_3+s_4)}{2s_1s_4(s_3+s_4)} - \frac{e^{s_1+s_2+s_3} k_7(-s_1-s_2-s_3)}{2s_1s_4(s_3+s_4)} - \frac{e^{s_2+s_3+s_4} k_7(-s_2-s_3-s_4)}{2s_1s_4(s_3+s_4)}\\
&- \frac{k_6(s_1+s_2)}{(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} - \frac{e^{s_1+s_2} k_7(-s_1-s_2)}{(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&- \frac{s_3k_6(s_1+s_2)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} - \frac{s_4k_6(s_1+s_2)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&- \frac{e^{s_1+s_2} s_3k_7(-s_1-s_2)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} - \frac{e^{s_1+s_2} s_4k_7(-s_1-s_2)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&- \frac{s_2k_6(s_1+s_2)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} - \frac{s_4k_6(s_1+s_2)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&- \frac{e^{s_1+s_2} s_2k_7(-s_1-s_2)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} - \frac{e^{s_1+s_2} s_4k_7(-s_1-s_2)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&- \frac{s_2k_6(s_1+s_2+s_3+s_4)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)} - \frac{s_3k_6(s_1+s_2+s_3+s_4)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}\\
&- \frac{e^{s_1+s_2+s_3+s_4} s_2k_7(-s_1-s_2-s_3-s_4)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)} - \frac{e^{s_1+s_2+s_3+s_4} s_3k_7(-s_1-s_2-s_3-s_4)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}\\
&- \frac{e^{s_1} k_{17}(s_2,s_3,-s_1-s_2-s_3)}{16(s_1+s_2+s_3+s_4)} - \frac{e^{s_1+s_2+s_3} k_{17}(-s_1-s_2-s_3,s_1,s_2)}{16(s_1+s_2+s_3+s_4)} - \frac{k_{19}(s_1,s_2,s_3)}{16(s_1+s_2+s_3+s_4)}\\
&- \frac{e^{s_1+s_2} k_{19}(s_3,-s_1-s_2-s_3,s_1)}{16(s_1+s_2+s_3+s_4)}.
\end{aligned}
$$

A.2.3. *The function* $\widetilde{K}_{20}$. Finally, for the last four variable function we have

(72)

$$
\begin{aligned}
&\widetilde{K}_{20}(s_1,s_2,s_3,s_4)\\
&\quad = \frac{1}{5}(-4)\pi G_4(s_1,s_2,s_3,s_4) + \frac{3s_3k_6(s_1)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&\qquad + \frac{3s_4k_6(s_1)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} + \frac{3k_6(s_1+s_2)}{2s_1s_3(s_3+s_4)} + \frac{3G_1(s_1)k_6(s_3)}{2s_2s_4} + \frac{3G_2(s_1,s_2)k_6(s_3)}{2s_4}\\
&\qquad + \frac{3k_6(s_3)}{2s_2(s_1+s_2)s_4} + \frac{3G_1(s_1)k_6(s_2+s_3)}{2s_3(s_3+s_4)} + \frac{3G_1(s_1)k_6(s_2+s_3)}{2s_4(s_3+s_4)} + \frac{3k_6(s_2+s_3)}{2s_1s_3(s_3+s_4)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{3k_6(s_2+s_3)}{2s_1s_4(s_3+s_4)} + \frac{3k_6(s_1+s_2+s_3)}{2s_1(s_1+s_2)s_4} + \frac{3k_6(s_1+s_2+s_3)}{(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} \\
&+ \frac{3s_2k_6(s_1+s_2+s_3)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} + \frac{3s_4k_6(s_1+s_2+s_3)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} \\
&+ \frac{3s_2k_6(s_1+s_2+s_3)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)} + \frac{3s_3k_6(s_1+s_2+s_3)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)} \\
&- \frac{3}{2}G_3(s_1,s_2,s_3)k_6(s_4) + \frac{3G_1(s_1)k_6(s_3+s_4)}{2s_2(s_2+s_3)} + \frac{3G_1(s_1)k_6(s_3+s_4)}{2s_3(s_2+s_3)} + \frac{3G_2(s_1,s_2)k_6(s_3+s_4)}{2s_3} \\
&+ \frac{3k_6(s_3+s_4)}{(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} + \frac{3s_1k_6(s_3+s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} \\
&+ \frac{3s_3k_6(s_3+s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} + \frac{3s_1k_6(s_3+s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)} \\
&+ \frac{3s_2k_6(s_3+s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)} + \frac{3G_1(s_1)k_6(s_2+s_3+s_4)}{2s_2s_4} + \frac{3k_6(s_2+s_3+s_4)}{2s_1(s_1+s_2)s_4} \\
&+ \frac{3k_6(s_2+s_3+s_4)}{2s_2(s_1+s_2)s_4} + \frac{3s_2k_6(s_1+s_2+s_3+s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} + \frac{3s_3k_6(s_1+s_2+s_3+s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} \\
&+ \frac{3k_6(s_1+s_2+s_3+s_4)}{2s_1s_4(s_3+s_4)} + \frac{3e^{s_1}s_3k_7(-s_1)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} \\
&+ \frac{3e^{s_1}s_4k_7(-s_1)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} + \frac{3e^{s_1+s_2}k_7(-s_1-s_2)}{2s_1s_3(s_3+s_4)} + \frac{3e^{s_2+s_3}G_1(s_1)k_7(-s_2-s_3)}{2s_3(s_3+s_4)} \\
&+ \frac{3e^{s_2+s_3}G_1(s_1)k_7(-s_2-s_3)}{2s_4(s_3+s_4)} + \frac{3e^{s_2+s_3}k_7(-s_2-s_3)}{2s_1s_3(s_3+s_4)} + \frac{3e^{s_2+s_3}k_7(-s_2-s_3)}{2s_1s_4(s_3+s_4)} \\
&+ \frac{3e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{2s_1(s_1+s_2)s_4} + \frac{3e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} \\
&+ \frac{3e^{s_1+s_2+s_3}s_2k_7(-s_1-s_2-s_3)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} + \frac{3e^{s_1+s_2+s_3}s_4k_7(-s_1-s_2-s_3)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)} \\
&+ \frac{3e^{s_1+s_2+s_3}s_2k_7(-s_1-s_2-s_3)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)} + \frac{3e^{s_1+s_2+s_3}s_3k_7(-s_1-s_2-s_3)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)} \\
&+ \frac{3e^{s_3}G_1(s_1)k_7(-s_3)}{2s_2s_4} + \frac{3e^{s_3}G_2(s_1,s_2)k_7(-s_3)}{2s_4} + \frac{3e^{s_3}k_7(-s_3)}{2s_2(s_1+s_2)s_4} \\
&+ \frac{3e^{s_3+s_4}G_1(s_1)k_7(-s_3-s_4)}{2s_2(s_2+s_3)} + \frac{3e^{s_3+s_4}G_1(s_1)k_7(-s_3-s_4)}{2s_3(s_2+s_3)} \\
&+ \frac{3e^{s_3+s_4}G_2(s_1,s_2)k_7(-s_3-s_4)}{2s_3} + \frac{3e^{s_3+s_4}k_7(-s_3-s_4)}{(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} \\
&+ \frac{3e^{s_3+s_4}s_1k_7(-s_3-s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} + \frac{3e^{s_3+s_4}s_3k_7(-s_3-s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} \\
&+ \frac{3e^{s_3+s_4}s_1k_7(-s_3-s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)} + \frac{3e^{s_3+s_4}s_2k_7(-s_3-s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)} \\
&+ \frac{3e^{s_2+s_3+s_4}G_1(s_1)k_7(-s_2-s_3-s_4)}{2s_2s_4} + \frac{3e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{2s_1(s_1+s_2)s_4} \\
&+ \frac{3e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{2s_2(s_1+s_2)s_4} + \frac{3e^{s_1+s_2+s_3+s_4}s_2k_7(-s_1-s_2-s_3-s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} \\
&+ \frac{3e^{s_1+s_2+s_3+s_4}s_3k_7(-s_1-s_2-s_3-s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} + \frac{3e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{2s_1s_4(s_3+s_4)} \\
&- \frac{3}{2}e^{s_4}G_3(s_1,s_2,s_3)k_7(-s_4) + \frac{e^{s_1+s_2}k_{14}(-s_1-s_2,s_1)}{4s_3(s_3+s_4)} + \frac{e^{s_1+s_2+s_3}k_{14}(-s_1-s_2-s_3,s_1)}{4s_2s_4} \\
&+ \frac{e^{s_1+s_2+s_3}k_{14}(-s_1-s_2-s_3,s_1+s_2)}{4s_1s_4} + \frac{e^{s_3+s_4}G_1(s_1)k_{14}(-s_3-s_4,s_3)}{4s_2} \\
&+ \frac{1}{4}e^{s_3+s_4}G_2(s_1,s_2)k_{14}(-s_3-s_4,s_3) + \frac{e^{s_3+s_4}k_{14}(-s_3-s_4,s_3)}{4s_2(s_1+s_2)} \\
&+ \frac{e^{s_2+s_3+s_4}G_1(s_1)k_{14}(-s_2-s_3-s_4,s_2)}{4s_4} + \frac{e^{s_2+s_3+s_4}k_{14}(-s_2-s_3-s_4,s_2)}{4s_1s_4}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{14}(-s_2-s_3-s_4,s_2+s_3)}{4s_3}+\frac{e^{s_2+s_3+s_4}k_{14}(-s_2-s_3-s_4,s_2+s_3)}{4s_1s_3}\\
&+\frac{e^{s_1+s_2+s_3+s_4}k_{14}(-s_1-s_2-s_3-s_4,s_1)}{4s_2(s_2+s_3)}+\frac{e^{s_1+s_2+s_3+s_4}k_{14}(-s_1-s_2-s_3-s_4,s_1)}{4s_4(s_3+s_4)}\\
&+\frac{e^{s_1+s_2+s_3+s_4}k_{14}(-s_1-s_2-s_3-s_4,s_1+s_2)}{4s_1s_3}+\frac{e^{s_1+s_2+s_3+s_4}k_{14}(-s_1-s_2-s_3-s_4,s_1+s_2)}{4s_2s_4}\\
&+\frac{e^{s_1+s_2+s_3+s_4}k_{14}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{4s_1(s_1+s_2)}\\
&+\frac{e^{s_1+s_2+s_3+s_4}k_{14}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{4s_3(s_2+s_3)}\\
&+\frac{k_{15}(s_1,s_2)}{4s_3(s_3+s_4)}+\frac{k_{15}(s_1,s_2+s_3)}{4s_2s_4}+\frac{k_{15}(s_1,s_2+s_3+s_4)}{4s_2(s_2+s_3)}+\frac{k_{15}(s_1,s_2+s_3+s_4)}{4s_4(s_3+s_4)}\\
&+\frac{G_1(s_1)k_{15}(s_2,s_3+s_4)}{4s_4}+\frac{k_{15}(s_2,s_3+s_4)}{4s_1s_4}+\frac{k_{15}(s_1+s_2,s_3)}{4s_1s_4}+\frac{k_{15}(s_1+s_2,s_3+s_4)}{4s_1s_3}\\
&+\frac{k_{15}(s_1+s_2,s_3+s_4)}{4s_2s_4}+\frac{G_1(s_1)k_{15}(s_3,s_4)}{4s_2}+\frac{1}{4}G_2(s_1,s_2)k_{15}(s_3,s_4)+\frac{k_{15}(s_3,s_4)}{4s_2(s_1+s_2)}\\
&+\frac{G_1(s_1)k_{15}(s_2+s_3,s_4)}{4s_3}+\frac{k_{15}(s_2+s_3,s_4)}{4s_1s_3}+\frac{k_{15}(s_1+s_2+s_3,s_4)}{4s_1(s_1+s_2)}+\frac{k_{15}(s_1+s_2+s_3,s_4)}{4s_3(s_2+s_3)}\\
&+\frac{e^{s_1}k_{16}(s_2,-s_1-s_2)}{4s_3(s_3+s_4)}+\frac{e^{s_1+s_2}k_{16}(s_3,-s_1-s_2-s_3)}{4s_1s_4}+\frac{e^{s_1}k_{16}(s_2+s_3,-s_1-s_2-s_3)}{4s_2s_4}\\
&+\frac{e^{s_3}G_1(s_1)k_{16}(s_4,-s_3-s_4)}{4s_2}+\frac{1}{4}e^{s_3}G_2(s_1,s_2)k_{16}(s_4,-s_3-s_4)+\frac{e^{s_3}k_{16}(s_4,-s_3-s_4)}{4s_2(s_1+s_2)}\\
&+\frac{e^{s_2+s_3}G_1(s_1)k_{16}(s_4,-s_2-s_3-s_4)}{4s_3}+\frac{e^{s_2+s_3}k_{16}(s_4,-s_2-s_3-s_4)}{4s_1s_3}\\
&+\frac{e^{s_1+s_2+s_3}k_{16}(s_4,-s_1-s_2-s_3-s_4)}{4s_1(s_1+s_2)}+\frac{e^{s_1+s_2+s_3}k_{16}(s_4,-s_1-s_2-s_3-s_4)}{4s_3(s_2+s_3)}\\
&+\frac{e^{s_2}G_1(s_1)k_{16}(s_3+s_4,-s_2-s_3-s_4)}{4s_4}+\frac{e^{s_2}k_{16}(s_3+s_4,-s_2-s_3-s_4)}{4s_1s_4}\\
&+\frac{e^{s_1+s_2}k_{16}(s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_1s_3}+\frac{e^{s_1+s_2}k_{16}(s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_2s_4}\\
&+\frac{e^{s_1}k_{16}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_2(s_2+s_3)}+\frac{e^{s_1}k_{16}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_4(s_3+s_4)}\\
&+\frac{k_{20}(s_1,s_2,s_3)}{16s_4}+\frac{k_{20}(s_1,s_2,s_3+s_4)}{16s_3}+\frac{k_{20}(s_1,s_2+s_3,s_4)}{16s_2}+\frac{e^{s_1}k_{20}(s_2,s_3,-s_1-s_2-s_3)}{16s_4}\\
&-\frac{1}{16}G_1(s_1)k_{20}(s_2,s_3,s_4)+\frac{e^{s_1}k_{20}(s_2,s_3,s_4)}{16(s_1+s_2+s_3+s_4)}+\frac{e^{s_1}k_{20}(s_2,s_3+s_4,-s_1-s_2-s_3-s_4)}{16s_3}\\
&+\frac{k_{20}(s_1+s_2,s_3,s_4)}{16s_1}+\frac{e^{s_1+s_2+s_3}k_{20}(-s_1-s_2-s_3,s_1,s_2)}{16s_4}+\frac{e^{s_1+s_2}k_{20}(s_3,-s_1-s_2-s_3,s_1)}{16s_4}\\
&-\frac{1}{16}e^{s_2}G_1(s_1)k_{20}(s_3,s_4,-s_2-s_3-s_4)+\frac{e^{s_1+s_2}k_{20}(s_3,s_4,-s_2-s_3-s_4)}{16(s_1+s_2+s_3+s_4)}\\
&+\frac{e^{s_1+s_2}k_{20}(s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_1}+\frac{e^{s_1}k_{20}(s_2+s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_2}\\
&-\frac{1}{16}e^{s_2+s_3+s_4}G_1(s_1)k_{20}(-s_2-s_3-s_4,s_2,s_3)+\frac{e^{s_1+s_2+s_3+s_4}k_{20}(-s_2-s_3-s_4,s_2,s_3)}{16(s_1+s_2+s_3+s_4)}\\
&+\frac{e^{s_1+s_2+s_3+s_4}k_{20}(-s_1-s_2-s_3-s_4,s_1,s_2)}{16s_3}+\frac{e^{s_1+s_2+s_3+s_4}k_{20}(-s_1-s_2-s_3-s_4,s_1,s_2+s_3)}{16s_2}\\
&+\frac{e^{s_1+s_2+s_3+s_4}k_{20}(-s_1-s_2-s_3-s_4,s_1+s_2,s_3)}{16s_1}-\frac{1}{16}e^{s_2+s_3}G_1(s_1)k_{20}(s_4,-s_2-s_3-s_4,s_2)\\
&+\frac{e^{s_1+s_2+s_3}k_{20}(s_4,-s_2-s_3-s_4,s_2)}{16(s_1+s_2+s_3+s_4)}+\frac{e^{s_1+s_2+s_3}k_{20}(s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_2}\\
&+\frac{e^{s_1+s_2+s_3}k_{20}(s_4,-s_1-s_2-s_3-s_4,s_1+s_2)}{16s_1}+\frac{e^{s_1+s_2}k_{20}(s_3+s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_3}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{k_{20}(s_2,s_3,s_4)}{16s_1} - \frac{e^{s_2}k_{20}(s_3,s_4,-s_2-s_3-s_4)}{16s_1} - \frac{e^{s_2+s_3+s_4}k_{20}(-s_2-s_3-s_4,s_2,s_3)}{16s_1}\\
&- \frac{e^{s_2+s_3}k_{20}(s_4,-s_2-s_3-s_4,s_2)}{16s_1} - \frac{e^{s_2+s_3+s_4}G_1(s_1)k_{14}(-s_2-s_3-s_4,s_2+s_3)}{4s_2}\\
&- \frac{G_1(s_1)k_{15}(s_2+s_3,s_4)}{4s_2} - \frac{e^{s_2+s_3}G_1(s_1)k_{16}(s_4,-s_2-s_3-s_4)}{4s_2} - \frac{k_{20}(s_1+s_2,s_3,s_4)}{16s_2}\\
&- \frac{e^{s_1+s_2}k_{20}(s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_2} - \frac{e^{s_1+s_2+s_3+s_4}k_{20}(-s_1-s_2-s_3-s_4,s_1+s_2,s_3)}{16s_2}\\
&- \frac{e^{s_1+s_2+s_3}k_{20}(s_4,-s_1-s_2-s_3-s_4,s_1+s_2)}{16s_2} - \frac{e^{s_2+s_3+s_4}k_{14}(-s_2-s_3-s_4,s_2+s_3)}{4s_1(s_1+s_2)}\\
&- \frac{k_{15}(s_2+s_3,s_4)}{4s_1(s_1+s_2)} - \frac{e^{s_2+s_3}k_{16}(s_4,-s_2-s_3-s_4)}{4s_1(s_1+s_2)} - \frac{e^{s_2+s_3+s_4}k_{14}(-s_2-s_3-s_4,s_2+s_3)}{4s_2(s_1+s_2)}\\
&- \frac{k_{15}(s_2+s_3,s_4)}{4s_2(s_1+s_2)} - \frac{e^{s_2+s_3}k_{16}(s_4,-s_2-s_3-s_4)}{4s_2(s_1+s_2)} - \frac{3G_2(s_1,s_2)k_6(s_4)}{2s_3} - \frac{3e^{s_4}G_2(s_1,s_2)k_7(-s_4)}{2s_3}\\
&- \frac{e^{s_2+s_3+s_4}G_1(s_1)k_{14}(-s_2-s_3-s_4,s_2)}{4s_3} - \frac{G_1(s_1)k_{15}(s_2,s_3+s_4)}{4s_3}\\
&- \frac{e^{s_2}G_1(s_1)k_{16}(s_3+s_4,-s_2-s_3-s_4)}{4s_3} - \frac{k_{20}(s_1,s_2+s_3,s_4)}{16s_3}\\
&- \frac{e^{s_1}k_{20}(s_2+s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_3} - \frac{e^{s_1+s_2+s_3+s_4}k_{20}(-s_1-s_2-s_3-s_4,s_1,s_2+s_3)}{16s_3}\\
&- \frac{e^{s_1+s_2+s_3}k_{20}(s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_3} - \frac{e^{s_2+s_3+s_4}k_{14}(-s_2-s_3-s_4,s_2)}{4s_1s_3}\\
&- \frac{e^{s_1+s_2+s_3+s_4}k_{14}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{4s_1s_3} - \frac{k_{15}(s_2,s_3+s_4)}{4s_1s_3} - \frac{k_{15}(s_1+s_2+s_3,s_4)}{4s_1s_3}\\
&- \frac{e^{s_1+s_2+s_3}k_{16}(s_4,-s_1-s_2-s_3-s_4)}{4s_1s_3} - \frac{e^{s_2}k_{16}(s_3+s_4,-s_2-s_3-s_4)}{4s_1s_3}\\
&- \frac{3G_1(s_1)k_6(s_2+s_3+s_4)}{2s_2(s_2+s_3)} - \frac{3e^{s_2+s_3+s_4}G_1(s_1)k_7(-s_2-s_3-s_4)}{2s_2(s_2+s_3)}\\
&- \frac{e^{s_1+s_2+s_3+s_4}k_{14}(-s_1-s_2-s_3-s_4,s_1+s_2)}{4s_2(s_2+s_3)} - \frac{k_{15}(s_1+s_2,s_3+s_4)}{4s_2(s_2+s_3)}\\
&- \frac{e^{s_1+s_2}k_{16}(s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_2(s_2+s_3)} - \frac{3G_1(s_1)k_6(s_4)}{2s_3(s_2+s_3)} - \frac{3e^{s_4}G_1(s_1)k_7(-s_4)}{2s_3(s_2+s_3)}\\
&- \frac{e^{s_1+s_2+s_3+s_4}k_{14}(-s_1-s_2-s_3-s_4,s_1+s_2)}{4s_3(s_2+s_3)} - \frac{k_{15}(s_1+s_2,s_3+s_4)}{4s_3(s_2+s_3)}\\
&- \frac{e^{s_1+s_2}k_{16}(s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_3(s_2+s_3)} - \frac{3k_6(s_2+s_3+s_4)}{(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&- \frac{3e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} - \frac{3s_2k_6(s_2+s_3+s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&- \frac{3s_3k_6(s_2+s_3+s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} - \frac{3e^{s_2+s_3+s_4}s_2k_7(-s_2-s_3-s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&- \frac{3e^{s_2+s_3+s_4}s_3k_7(-s_2-s_3-s_4)}{2s_1(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} - \frac{3s_1k_6(s_2+s_3+s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&- \frac{3s_3k_6(s_2+s_3+s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} - \frac{3e^{s_2+s_3+s_4}s_1k_7(-s_2-s_3-s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)}\\
&- \frac{3e^{s_2+s_3+s_4}s_3k_7(-s_2-s_3-s_4)}{2s_2(s_1+s_2)(s_2+s_3)(s_1+s_2+s_3)} - \frac{3s_1k_6(s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}\\
&- \frac{3s_2k_6(s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)} - \frac{3e^{s_4}s_1k_7(-s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}\\
&- \frac{3e^{s_4}s_2k_7(-s_4)}{2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)} - \frac{3G_2(s_1,s_2)k_6(s_3+s_4)}{2s_4} - \frac{3e^{s_3+s_4}G_2(s_1,s_2)k_7(-s_3-s_4)}{2s_4}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{e^{s_2+s_3}G_1(s_1)k_{14}(-s_2-s_3,s_2)}{4s_4}-\frac{G_1(s_1)k_{15}(s_2,s_3)}{4s_4}-\frac{e^{s_2}G_1(s_1)k_{16}(s_3,-s_2-s_3)}{4s_4}\\
&-\frac{k_{20}(s_1,s_2,s_3+s_4)}{16s_4}-\frac{e^{s_1}k_{20}(s_2,s_3+s_4,-s_1-s_2-s_3-s_4)}{16s_4}\\
&-\frac{e^{s_1+s_2+s_3+s_4}k_{20}(-s_1-s_2-s_3-s_4,s_1,s_2)}{16s_4}-\frac{e^{s_1+s_2}k_{20}(s_3+s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_4}\\
&-\frac{e^{s_2+s_3}k_{14}(-s_2-s_3,s_2)}{4s_1s_4}-\frac{e^{s_1+s_2+s_3+s_4}k_{14}(-s_1-s_2-s_3-s_4,s_1+s_2)}{4s_1s_4}-\frac{k_{15}(s_2,s_3)}{4s_1s_4}\\
&-\frac{k_{15}(s_1+s_2,s_3+s_4)}{4s_1s_4}-\frac{e^{s_2}k_{16}(s_3,-s_2-s_3)}{4s_1s_4}-\frac{e^{s_1+s_2}k_{16}(s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_1s_4}\\
&-\frac{3G_1(s_1)k_6(s_2+s_3)}{2s_2s_4}-\frac{3G_1(s_1)k_6(s_3+s_4)}{2s_2s_4}-\frac{3e^{s_2+s_3}G_1(s_1)k_7(-s_2-s_3)}{2s_2s_4}\\
&-\frac{3e^{s_3+s_4}G_1(s_1)k_7(-s_3-s_4)}{2s_2s_4}-\frac{e^{s_1+s_2+s_3}k_{14}(-s_1-s_2-s_3,s_1+s_2)}{4s_2s_4}\\
&-\frac{e^{s_1+s_2+s_3+s_4}k_{14}(-s_1-s_2-s_3-s_4,s_1)}{4s_2s_4}-\frac{k_{15}(s_1,s_2+s_3+s_4)}{4s_2s_4}-\frac{k_{15}(s_1+s_2,s_3)}{4s_2s_4}\\
&-\frac{e^{s_1+s_2}k_{16}(s_3,-s_1-s_2-s_3)}{4s_2s_4}-\frac{e^{s_1}k_{16}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_2s_4}-\frac{3k_6(s_2+s_3)}{2s_1(s_1+s_2)s_4}\\
&-\frac{3k_6(s_1+s_2+s_3+s_4)}{2s_1(s_1+s_2)s_4}-\frac{3e^{s_2+s_3}k_7(-s_2-s_3)}{2s_1(s_1+s_2)s_4}-\frac{3e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{2s_1(s_1+s_2)s_4}\\
&-\frac{3k_6(s_2+s_3)}{2s_2(s_1+s_2)s_4}-\frac{3k_6(s_3+s_4)}{2s_2(s_1+s_2)s_4}-\frac{3e^{s_2+s_3}k_7(-s_2-s_3)}{2s_2(s_1+s_2)s_4}-\frac{3e^{s_3+s_4}k_7(-s_3-s_4)}{2s_2(s_1+s_2)s_4}\\
&-\frac{3G_1(s_1)k_6(s_2)}{2s_3(s_3+s_4)}-\frac{3e^{s_2}G_1(s_1)k_7(-s_2)}{2s_3(s_3+s_4)}-\frac{e^{s_1+s_2+s_3}k_{14}(-s_1-s_2-s_3,s_1)}{4s_3(s_3+s_4)}-\frac{k_{15}(s_1,s_2+s_3)}{4s_3(s_3+s_4)}\\
&-\frac{e^{s_1}k_{16}(s_2+s_3,-s_1-s_2-s_3)}{4s_3(s_3+s_4)}-\frac{3k_6(s_2)}{2s_1s_3(s_3+s_4)}-\frac{3k_6(s_1+s_2+s_3)}{2s_1s_3(s_3+s_4)}-\frac{3e^{s_2}k_7(-s_2)}{2s_1s_3(s_3+s_4)}\\
&-\frac{3e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{2s_1s_3(s_3+s_4)}-\frac{3G_1(s_1)k_6(s_2+s_3+s_4)}{2s_4(s_3+s_4)}-\frac{3e^{s_2+s_3+s_4}G_1(s_1)k_7(-s_2-s_3-s_4)}{2s_4(s_3+s_4)}\\
&-\frac{e^{s_1+s_2+s_3}k_{14}(-s_1-s_2-s_3,s_1)}{4s_4(s_3+s_4)}-\frac{k_{15}(s_1,s_2+s_3)}{4s_4(s_3+s_4)}-\frac{e^{s_1}k_{16}(s_2+s_3,-s_1-s_2-s_3)}{4s_4(s_3+s_4)}\\
&-\frac{3k_6(s_1+s_2+s_3)}{2s_1s_4(s_3+s_4)}-\frac{3k_6(s_2+s_3+s_4)}{2s_1s_4(s_3+s_4)}-\frac{3e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{2s_1s_4(s_3+s_4)}\\
&-\frac{3e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{2s_1s_4(s_3+s_4)}-\frac{3k_6(s_1+s_2)}{(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{3e^{s_1+s_2}k_7(-s_1-s_2)}{(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}-\frac{3s_3k_6(s_1+s_2)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{3s_4k_6(s_1+s_2)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}-\frac{3e^{s_1+s_2}s_3k_7(-s_1-s_2)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{3e^{s_1+s_2}s_4k_7(-s_1-s_2)}{2s_2(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}-\frac{3s_2k_6(s_1+s_2)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{3s_4k_6(s_1+s_2)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}-\frac{3e^{s_1+s_2}s_2k_7(-s_1-s_2)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{3e^{s_1+s_2}s_4k_7(-s_1-s_2)}{2s_3(s_2+s_3)(s_3+s_4)(s_2+s_3+s_4)}-\frac{3s_2k_6(s_1+s_2+s_3+s_4)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{3s_3k_6(s_1+s_2+s_3+s_4)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}-\frac{3e^{s_1+s_2+s_3+s_4}s_2k_7(-s_1-s_2-s_3-s_4)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}\\
&-\frac{3e^{s_1+s_2+s_3+s_4}s_3k_7(-s_1-s_2-s_3-s_4)}{2(s_2+s_3)s_4(s_3+s_4)(s_2+s_3+s_4)}-\frac{k_{20}(s_1,s_2,s_3)}{16(s_1+s_2+s_3+s_4)}-\frac{e^{s_1}k_{20}(s_2,s_3,-s_1-s_2-s_3)}{16(s_1+s_2+s_3+s_4)}\\
&-\frac{e^{s_1+s_2+s_3}k_{20}(-s_1-s_2-s_3,s_1,s_2)}{16(s_1+s_2+s_3+s_4)}-\frac{e^{s_1+s_2}k_{20}(s_3,-s_1-s_2-s_3,s_1)}{16(s_1+s_2+s_3+s_4)}.
\end{aligned}
$$

APPENDIX B. LENGTHY FUNCTIONAL RELATIONS AMONG $k_3, \ldots, k_{20}$

In Section 7 we found functional relations that involve only the functions $k_3, \ldots, k_{20}$. In this appendix we provide more functional relations of this type.

B.1. **Functional relation between $\boldsymbol{k_3, k_5, k_6, \ldots, k_{16}, k_{17}, k_{19}, k_{20}}$.** The explicit formulas given in Appendix C can be used to see that $2\widetilde{K}_{16} = 3\widetilde{K}_{10}$, which combined with the finite difference expressions (63) and (69), gives the following functional relation:

$$
\begin{aligned}
&-\frac{3e^{s_1+s_2}k_3(-s_1-s_2)}{4s_1s_3}+\frac{3e^{s_2}G_1(s_1)k_3(-s_2)}{4s_3}+\frac{3e^{s_2}k_3(-s_2)}{4s_1s_3}+\frac{3G_1(s_1)k_3(s_2)}{4s_3}+\frac{3k_3(s_2)}{4s_1s_3}\\
&+\frac{3e^{s_1+s_2+s_3}k_3(-s_1-s_2-s_3)}{4s_1s_3}+\frac{3k_3(s_1+s_2+s_3)}{4s_1s_3}+\frac{e^{s_1+s_2}k_5(-s_1-s_2)}{s_1s_3}+\frac{k_5(s_1+s_2)}{s_1s_3}\\
&+\frac{e^{s_2+s_3}G_1(s_1)k_5(-s_2-s_3)}{s_3}+\frac{e^{s_2+s_3}k_5(-s_2-s_3)}{s_1s_3}+\frac{G_1(s_1)k_5(s_2+s_3)}{s_3}+\frac{k_5(s_2+s_3)}{s_1s_3}+\frac{k_6(s_1)}{2s_2s_3}\\
&+\frac{k_6(s_3)}{4s_1(s_1+s_2)}+\frac{k_6(s_3)}{4s_2(s_1+s_2)}+\frac{k_6(s_1+s_2+s_3)}{4s_1s_2}+\frac{k_6(s_1+s_2+s_3)}{2s_2(s_2+s_3)}+\frac{k_6(s_1+s_2+s_3)}{2s_3(s_2+s_3)}\\
&+\frac{e^{s_1}k_7(-s_1)}{2s_2s_3}+\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{4s_1s_2}+\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{2s_2(s_2+s_3)}\\
&+\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{2s_3(s_2+s_3)}+\frac{e^{s_3}k_7(-s_3)}{4s_1(s_1+s_2)}+\frac{e^{s_3}k_7(-s_3)}{4s_2(s_1+s_2)}+\frac{3k_8(s_1,s_2)}{4s_3}-\frac{3}{4}G_1(s_1)k_8(s_2,s_3)\\
&+\frac{3k_8(s_1+s_2,s_3)}{4s_1}+\frac{3k_9(s_1,s_2)}{8s_3}+\frac{3k_9(s_1,s_2+s_3)}{8s_2}+\frac{3e^{s_1+s_2}k_9(-s_1-s_2,s_1)}{8(s_1+s_2+s_3)}-\frac{3}{8}G_1(s_1)k_9(s_2,s_3)\\
&+\frac{3k_9(s_1+s_2,s_3)}{8s_1}+\frac{3e^{s_1+s_2+s_3}k_9(-s_1-s_2-s_3,s_1)}{8s_3}+\frac{3}{8}e^{s_2}G_1(s_1)k_9(s_3,-s_2-s_3)\\
&+\frac{3e^{s_2}k_9(s_3,-s_2-s_3)}{8s_1}+\frac{3e^{s_1+s_2+s_3}k_{10}(s_1,s_2)}{8(-s_1-s_2-s_3)}+\frac{3e^{s_1+s_2+s_3}k_{10}(s_1,s_2)}{8(s_1+s_2+s_3)}+\frac{3k_{10}(s_1,s_2+s_3)}{8s_3}\\
&+\frac{3e^{s_1}k_{10}(s_2,-s_1-s_2)}{8s_3}+\frac{3}{8}e^{s_2+s_3}G_1(s_1)k_{10}(-s_2-s_3,s_2)+\frac{3e^{s_2+s_3}k_{10}(-s_2-s_3,s_2)}{8s_1}\\
&-\frac{3}{8}e^{s_2}G_1(s_1)k_{10}(s_3,-s_2-s_3)+\frac{3e^{s_1+s_2}k_{10}(s_3,-s_1-s_2-s_3)}{8s_1}+\frac{3e^{s_1}k_{10}(s_2+s_3,-s_1-s_2-s_3)}{8s_2}\\
&+\frac{3e^{s_1}k_{11}(s_2,-s_1-s_2)}{4s_3}-\frac{3}{4}e^{s_2}G_1(s_1)k_{11}(s_3,-s_2-s_3)+\frac{3e^{s_1+s_2}k_{11}(s_3,-s_1-s_2-s_3)}{4s_1}\\
&+\frac{3e^{s_1+s_2}k_{12}(-s_1-s_2,s_1)}{4s_3}-\frac{3}{4}e^{s_2+s_3}G_1(s_1)k_{12}(-s_2-s_3,s_2)\\
&+\frac{3e^{s_1+s_2+s_3}k_{12}(-s_1-s_2-s_3,s_1+s_2)}{4s_1}+\frac{3e^{s_1+s_2}k_{13}(-s_1-s_2,s_1)}{8s_3}+\frac{3e^{s_1}k_{13}(s_2,-s_1-s_2)}{8(s_1+s_2+s_3)}\\
&+\frac{3}{8}G_1(s_1)k_{13}(s_2,s_3)+\frac{3k_{13}(s_2,s_3)}{8s_1}-\frac{3}{8}e^{s_2+s_3}G_1(s_1)k_{13}(-s_2-s_3,s_2)\\
&+\frac{3e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1)}{8s_2}+\frac{3e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_1}\\
&+\frac{3e^{s_1}k_{13}(s_2+s_3,-s_1-s_2-s_3)}{8s_3}+\frac{e^{s_1}k_{14}(s_2,-s_1-s_2)}{4s_3}-\frac{1}{4}G_1(s_1)k_{14}(s_2,s_3)+\frac{e^{s_1}k_{14}(s_2,s_3)}{4(s_1+s_2+s_3)}\\
&+\frac{k_{14}(s_1+s_2,s_3)}{4s_1}+\frac{1}{2}e^{s_2+s_3}G_1(s_1)k_{14}(-s_2-s_3,s_2)+\frac{e^{s_2+s_3}k_{14}(-s_2-s_3,s_2)}{2s_1}\\
&+\frac{e^{s_1+s_2+s_3}k_{14}(-s_1-s_2-s_3,s_1)}{2s_3}+\frac{e^{s_1+s_2+s_3}k_{14}(-s_1-s_2-s_3,s_1+s_2)}{4s_2}+\frac{k_{15}(s_1,s_2+s_3)}{2s_3}\\
&+\frac{e^{s_1+s_2}k_{15}(-s_1-s_2,s_1)}{4s_3}+\frac{1}{2}G_1(s_1)k_{15}(s_2,s_3)+\frac{k_{15}(s_2,s_3)}{2s_1}+\frac{k_{15}(s_1+s_2,s_3)}{4s_2}\\
&-\frac{1}{4}e^{s_2}G_1(s_1)k_{15}(s_3,-s_2-s_3)+\frac{e^{s_1+s_2}k_{15}(s_3,-s_2-s_3)}{4(s_1+s_2+s_3)}+\frac{e^{s_1+s_2}k_{15}(s_3,-s_1-s_2-s_3)}{4s_1}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{k_{16}(s_1,s_2)}{4(-s_1-s_2-s_3)}+\frac{k_{16}(s_1,s_2)}{4s_3}-\frac{1}{4}e^{s_2+s_3}G_1(s_1)k_{16}(-s_2-s_3,s_2)+\frac{e^{s_1+s_2+s_3}k_{16}(-s_2-s_3,s_2)}{4(s_1+s_2+s_3)}\\
&+\frac{e^{s_1+s_2+s_3}k_{16}(-s_1-s_2-s_3,s_1+s_2)}{4s_1}+\frac{1}{2}e^{s_2}G_1(s_1)k_{16}(s_3,-s_2-s_3)+\frac{e^{s_2}k_{16}(s_3,-s_2-s_3)}{2s_1}\\
&+\frac{e^{s_1+s_2}k_{16}(s_3,-s_1-s_2-s_3)}{4s_2}+\frac{e^{s_1}k_{16}(s_2+s_3,-s_1-s_2-s_3)}{2s_3}+\frac{3}{16}k_{17}(s_1,s_2,s_3)\\
&+\frac{3}{16}e^{s_1}k_{17}(s_2,s_3,-s_1-s_2-s_3)+\frac{3}{16}e^{s_1+s_2+s_3}k_{17}(-s_1-s_2-s_3,s_1,s_2)\\
&+\frac{3}{16}e^{s_1+s_2}k_{17}(s_3,-s_1-s_2-s_3,s_1)+\frac{3}{16}k_{19}(s_1,s_2,s_3)+\frac{3}{16}e^{s_1}k_{19}(s_2,s_3,-s_1-s_2-s_3)\\
&+\frac{3}{16}e^{s_1+s_2+s_3}k_{19}(-s_1-s_2-s_3,s_1,s_2)+\frac{3}{16}e^{s_1+s_2}k_{19}(s_3,-s_1-s_2-s_3,s_1)-\frac{1}{8}k_{20}(s_1,s_2,s_3)\\
&-\frac{1}{8}e^{s_1}k_{20}(s_2,s_3,-s_1-s_2-s_3)-\frac{1}{8}e^{s_1+s_2+s_3}k_{20}(-s_1-s_2-s_3,s_1,s_2)\\
&-\frac{1}{8}e^{s_1+s_2}k_{20}(s_3,-s_1-s_2-s_3,s_1)-\frac{e^{s_1+s_2+s_3}k_{14}(-s_1-s_2-s_3,s_1+s_2)}{2s_1}-\frac{k_{15}(s_1+s_2,s_3)}{2s_1}\\
&-\frac{e^{s_1+s_2}k_{16}(s_3,-s_1-s_2-s_3)}{2s_1}-\frac{3k_8(s_2,s_3)}{4s_1}-\frac{3e^{s_2}k_{11}(s_3,-s_2-s_3)}{4s_1}-\frac{3e^{s_2+s_3}k_{12}(-s_2-s_3,s_2)}{4s_1}\\
&-\frac{k_{14}(s_2,s_3)}{4s_1}-\frac{e^{s_2}k_{15}(s_3,-s_2-s_3)}{4s_1}-\frac{e^{s_2+s_3}k_{16}(-s_2-s_3,s_2)}{4s_1}-\frac{3k_9(s_2,s_3)}{8s_1}\\
&-\frac{3e^{s_1+s_2}k_9(s_3,-s_1-s_2-s_3)}{8s_1}-\frac{3e^{s_1+s_2+s_3}k_{10}(-s_1-s_2-s_3,s_1+s_2)}{8s_1}-\frac{3e^{s_2}k_{10}(s_3,-s_2-s_3)}{8s_1}\\
&-\frac{3k_{13}(s_1+s_2,s_3)}{8s_1}-\frac{3e^{s_2+s_3}k_{13}(-s_2-s_3,s_2)}{8s_1}-\frac{e^{s_1+s_2+s_3}k_{14}(-s_1-s_2-s_3,s_1)}{4s_2}\\
&-\frac{k_{15}(s_1,s_2+s_3)}{4s_2}-\frac{e^{s_1}k_{16}(s_2+s_3,-s_1-s_2-s_3)}{4s_2}-\frac{3k_9(s_1+s_2,s_3)}{8s_2}\\
&-\frac{3e^{s_1+s_2}k_{10}(s_3,-s_1-s_2-s_3)}{8s_2}-\frac{3e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_2}-\frac{k_6(s_3)}{4s_1s_2}-\frac{e^{s_3}k_7(-s_3)}{4s_1s_2}\\
&-\frac{k_6(s_1+s_2+s_3)}{4s_1(s_1+s_2)}-\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{4s_1(s_1+s_2)}-\frac{k_6(s_1+s_2+s_3)}{4s_2(s_1+s_2)}\\
&-\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{4s_2(s_1+s_2)}-\frac{e^{s_1+s_2+s_3}k_{16}(s_1,s_2)}{4(-s_1-s_2-s_3)}-\frac{3k_{10}(s_1,s_2)}{8(-s_1-s_2-s_3)}\\
&-\frac{e^{s_2}G_1(s_1)k_5(-s_2)}{s_3}-\frac{G_1(s_1)k_5(s_2)}{s_3}-\frac{e^{s_1+s_2}k_{14}(-s_1-s_2,s_1)}{2s_3}-\frac{k_{15}(s_1,s_2)}{2s_3}\\
&-\frac{e^{s_1}k_{16}(s_2,-s_1-s_2)}{2s_3}-\frac{3e^{s_2+s_3}G_1(s_1)k_3(-s_2-s_3)}{4s_3}-\frac{3G_1(s_1)k_3(s_2+s_3)}{4s_3}-\frac{3k_8(s_1,s_2+s_3)}{4s_3}\\
&-\frac{3e^{s_1}k_{11}(s_2+s_3,-s_1-s_2-s_3)}{4s_3}-\frac{3e^{s_1+s_2+s_3}k_{12}(-s_1-s_2-s_3,s_1)}{4s_3}\\
&-\frac{e^{s_1}k_{14}(s_2+s_3,-s_1-s_2-s_3)}{4s_3}-\frac{e^{s_1+s_2+s_3}k_{15}(-s_1-s_2-s_3,s_1)}{4s_3}-\frac{k_{16}(s_1,s_2+s_3)}{4s_3}\\
&-\frac{3k_9(s_1,s_2+s_3)}{8s_3}-\frac{3e^{s_1+s_2}k_9(-s_1-s_2,s_1)}{8s_3}-\frac{3k_{10}(s_1,s_2)}{8s_3}-\frac{3e^{s_1}k_{10}(s_2+s_3,-s_1-s_2-s_3)}{8s_3}\\
&-\frac{3e^{s_1}k_{13}(s_2,-s_1-s_2)}{8s_3}-\frac{3e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1)}{8s_3}-\frac{e^{s_2}k_5(-s_2)}{s_1s_3}-\frac{k_5(s_2)}{s_1s_3}\\
&-\frac{e^{s_1+s_2+s_3}k_5(-s_1-s_2-s_3)}{s_1s_3}-\frac{k_5(s_1+s_2+s_3)}{s_1s_3}-\frac{3k_3(s_1+s_2)}{4s_1s_3}-\frac{3e^{s_2+s_3}k_3(-s_2-s_3)}{4s_1s_3}\\
&-\frac{3k_3(s_2+s_3)}{4s_1s_3}-\frac{k_6(s_1+s_2+s_3)}{2s_2s_3}-\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{2s_2s_3}-\frac{k_6(s_1)}{2s_2(s_2+s_3)}-\frac{e^{s_1}k_7(-s_1)}{2s_2(s_2+s_3)}\\
&-\frac{k_6(s_1)}{2s_3(s_2+s_3)}-\frac{e^{s_1}k_7(-s_1)}{2s_3(s_2+s_3)}-\frac{e^{s_1}k_{14}(s_2,-s_1-s_2)}{4(s_1+s_2+s_3)}-\frac{e^{s_1+s_2}k_{15}(-s_1-s_2,s_1)}{4(s_1+s_2+s_3)}
\end{aligned}
$$

$$- \frac{e^{s_1+s_2+s_3}k_{16}(s_1,s_2)}{4(s_1+s_2+s_3)} - \frac{3e^{s_1+s_2}k_9(s_3,-s_2-s_3)}{8(s_1+s_2+s_3)} - \frac{3e^{s_1+s_2+s_3}k_{10}(-s_2-s_3,s_2)}{8(s_1+s_2+s_3)} - \frac{3e^{s_1}k_{13}(s_2,s_3)}{8(s_1+s_2+s_3)} = 0.$$

B.2. **Functional relation between $k_6, k_7, \ldots, k_{13}, k_{17}, k_{18}, k_{19}$.** Finally, one can use the equality $\widetilde{K}_{18} = \widetilde{K}_{19}$, which can be observed by using the final explicit formulas written in Appendix C, and the different finite difference expressions (70) and (71) for these functions to obtain the following functional relation:

$$\frac{k_6(s_1)}{2s_2(s_2+s_3)s_4} - \frac{k_6(s_1)}{4s_2s_3(s_2+s_3+s_4)} - \frac{k_6(s_1)}{4s_2(s_2+s_3)(s_2+s_3+s_4)} + \frac{k_6(s_1)}{4s_3(s_2+s_3)(s_2+s_3+s_4)}$$
$$- \frac{k_6(s_1)}{2s_2s_4(s_2+s_3+s_4)} + \frac{G_1(s_1)k_6(s_2)}{2s_3(s_3+s_4)} + \frac{G_1(s_1)k_6(s_2)}{2s_4(s_3+s_4)} + \frac{k_6(s_2)}{2s_1s_3(s_3+s_4)} + \frac{k_6(s_2)}{2s_1s_4(s_3+s_4)}$$
$$+ \frac{k_6(s_1+s_2)}{2s_1s_3s_4} + \frac{k_6(s_1+s_2)}{2s_2s_3(s_3+s_4)} + \frac{k_6(s_1+s_2)}{2s_2s_4(s_3+s_4)} + \frac{k_6(s_1+s_2+s_3)}{4s_2s_3s_4} + \frac{G_1(s_1)k_6(s_4)}{4s_2(s_2+s_3)}$$
$$+ \frac{G_1(s_1)k_6(s_4)}{4s_3(s_2+s_3)} + \frac{k_6(s_4)}{4s_1s_2(s_2+s_3)} + \frac{k_6(s_4)}{4s_2(s_1+s_2)(s_1+s_2+s_3)} + \frac{k_6(s_4)}{4s_3(s_2+s_3)(s_1+s_2+s_3)}$$
$$+ \frac{G_1(s_1)k_6(s_2+s_3+s_4)}{4s_2s_3} + \frac{G_1(s_1)k_6(s_2+s_3+s_4)}{2s_3s_4} + \frac{k_6(s_2+s_3+s_4)}{4s_1s_2s_3} + \frac{k_6(s_2+s_3+s_4)}{2s_1s_3s_4}$$
$$+ \frac{k_6(s_1+s_2+s_3+s_4)}{4s_2(s_1+s_2)s_3} + \frac{k_6(s_1+s_2+s_3+s_4)}{4s_1s_3(s_2+s_3)} + \frac{k_6(s_1+s_2+s_3+s_4)}{4s_1s_2(s_1+s_2+s_3)} + \frac{k_6(s_1+s_2+s_3+s_4)}{4s_2s_3s_4}$$
$$+ \frac{k_6(s_1+s_2+s_3+s_4)}{4s_3(s_2+s_3)s_4} + \frac{k_6(s_1+s_2+s_3+s_4)}{2s_1s_3(s_3+s_4)} + \frac{k_6(s_1+s_2+s_3+s_4)}{2s_1s_4(s_3+s_4)} + \frac{k_6(s_1+s_2+s_3+s_4)}{4s_2s_3(s_2+s_3+s_4)}$$
$$+ \frac{k_6(s_1+s_2+s_3+s_4)}{4s_2(s_2+s_3)(s_2+s_3+s_4)} + \frac{k_6(s_1+s_2+s_3+s_4)}{2s_2s_4(s_2+s_3+s_4)} + \frac{e^{s_1}k_7(-s_1)}{2s_2(s_2+s_3)s_4} + \frac{e^{s_1}k_7(-s_1)}{4s_3(s_2+s_3)(s_2+s_3+s_4)}$$
$$+ \frac{e^{s_1+s_2}k_7(-s_1-s_2)}{2s_1s_3s_4} + \frac{e^{s_1+s_2}k_7(-s_1-s_2)}{2s_2s_3(s_3+s_4)} + \frac{e^{s_1+s_2}k_7(-s_1-s_2)}{2s_2s_4(s_3+s_4)} + \frac{e^{s_2}G_1(s_1)k_7(-s_2)}{2s_3(s_3+s_4)}$$
$$+ \frac{e^{s_2}G_1(s_1)k_7(-s_2)}{2s_4(s_3+s_4)} + \frac{e^{s_2}k_7(-s_2)}{2s_1s_3(s_3+s_4)} + \frac{e^{s_2}k_7(-s_2)}{2s_1s_4(s_3+s_4)} + \frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{4s_2s_3s_4}$$
$$+ \frac{e^{s_2+s_3+s_4}G_1(s_1)k_7(-s_2-s_3-s_4)}{4s_2s_3} + \frac{e^{s_2+s_3+s_4}G_1(s_1)k_7(-s_2-s_3-s_4)}{2s_3s_4}$$
$$+ \frac{e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{4s_1s_2s_3} + \frac{e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{2s_1s_3s_4}$$
$$+ \frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{4s_2(s_1+s_2)s_3} + \frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{4s_1s_3(s_2+s_3)}$$
$$+ \frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{4s_1s_2(s_1+s_2+s_3)} + \frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{4s_2s_3s_4}$$
$$+ \frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{4s_3(s_2+s_3)s_4} + \frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{2s_1s_3(s_3+s_4)}$$
$$+ \frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{2s_1s_4(s_3+s_4)} + \frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{4s_2s_3(s_2+s_3+s_4)}$$
$$+ \frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{4s_2(s_2+s_3)(s_2+s_3+s_4)} + \frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{2s_2s_4(s_2+s_3+s_4)}$$
$$+ \frac{e^{s_4}G_1(s_1)k_7(-s_4)}{4s_2(s_2+s_3)} + \frac{e^{s_4}G_1(s_1)k_7(-s_4)}{4s_3(s_2+s_3)} + \frac{e^{s_4}k_7(-s_4)}{4s_1s_2(s_2+s_3)} + \frac{e^{s_4}k_7(-s_4)}{4s_2(s_1+s_2)(s_1+s_2+s_3)}$$
$$+ \frac{e^{s_4}k_7(-s_4)}{4s_3(s_2+s_3)(s_1+s_2+s_3)} + \frac{k_8(s_1,s_2+s_3)}{4s_2s_4} + \frac{k_8(s_1,s_2+s_3+s_4)}{8s_3(s_2+s_3)} + \frac{G_1(s_1)k_8(s_2,s_3)}{4s_4}$$
$$+ \frac{k_8(s_2,s_3)}{4s_1s_4} + \frac{k_8(s_1+s_2,s_3+s_4)}{4s_1s_3} + \frac{k_8(s_1+s_2,s_3+s_4)}{4s_2s_3} + \frac{k_8(s_1+s_2,s_3+s_4)}{4s_1s_4}$$
$$+ \frac{k_8(s_1+s_2,s_3+s_4)}{4s_2s_4} + \frac{G_1(s_1)k_8(s_2+s_3,s_4)}{4s_3} + \frac{k_8(s_2+s_3,s_4)}{4s_1s_3} + \frac{k_8(s_1+s_2+s_3,s_4)}{8s_2(s_2+s_3)} + \frac{k_9(s_1,s_2)}{8s_3s_4}$$

$$+\frac{k_9(s_1,s_2+s_3+s_4)}{8s_2(s_2+s_3)}+\frac{k_9(s_1,s_2+s_3+s_4)}{8s_2s_4}+\frac{k_9(s_1,s_2+s_3+s_4)}{8s_3(s_3+s_4)}+\frac{k_9(s_1,s_2+s_3+s_4)}{8s_4(s_3+s_4)}$$
$$+\frac{G_1(s_1)k_9(s_2,s_3+s_4)}{8s_3}+\frac{G_1(s_1)k_9(s_2,s_3+s_4)}{8s_4}+\frac{k_9(s_2,s_3+s_4)}{8s_1s_3}+\frac{k_9(s_2,s_3+s_4)}{8s_1s_4}$$
$$+\frac{k_9(s_1+s_2,s_3)}{8s_1s_4}+\frac{k_9(s_1+s_2,s_3)}{8s_2s_4}+\frac{k_9(s_1+s_2+s_3,s_4)}{8s_1s_3}+\frac{k_9(s_1+s_2+s_3,s_4)}{8s_2s_3}$$
$$+\frac{e^{s_1}k_{10}(s_2,-s_1-s_2)}{8s_3s_4}+\frac{e^{s_1+s_2}k_{10}(s_3,-s_1-s_2-s_3)}{8s_1s_4}+\frac{e^{s_1+s_2}k_{10}(s_3,-s_1-s_2-s_3)}{8s_2s_4}$$
$$+\frac{e^{s_1+s_2+s_3}k_{10}(s_4,-s_1-s_2-s_3-s_4)}{8s_1s_3}+\frac{e^{s_1+s_2+s_3}k_{10}(s_4,-s_1-s_2-s_3-s_4)}{8s_2s_3}$$
$$+\frac{e^{s_2}G_1(s_1)k_{10}(s_3+s_4,-s_2-s_3-s_4)}{8s_3}+\frac{e^{s_2}G_1(s_1)k_{10}(s_3+s_4,-s_2-s_3-s_4)}{8s_4}$$
$$+\frac{e^{s_2}k_{10}(s_3+s_4,-s_2-s_3-s_4)}{8s_1s_3}+\frac{e^{s_2}k_{10}(s_3+s_4,-s_2-s_3-s_4)}{8s_1s_4}$$
$$+\frac{e^{s_1}k_{10}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_2(s_2+s_3)}+\frac{e^{s_1}k_{10}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_2s_4}$$
$$+\frac{e^{s_1}k_{10}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_3(s_3+s_4)}+\frac{e^{s_1}k_{10}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_4(s_3+s_4)}$$
$$+\frac{e^{s_2}G_1(s_1)k_{11}(s_3,-s_2-s_3)}{4s_4}+\frac{e^{s_2}k_{11}(s_3,-s_2-s_3)}{4s_1s_4}+\frac{e^{s_1}k_{11}(s_2+s_3,-s_1-s_2-s_3)}{4s_2s_4}$$
$$+\frac{e^{s_2+s_3}G_1(s_1)k_{11}(s_4,-s_2-s_3-s_4)}{4s_3}+\frac{e^{s_2+s_3}k_{11}(s_4,-s_2-s_3-s_4)}{4s_1s_3}$$
$$+\frac{e^{s_1+s_2+s_3}k_{11}(s_4,-s_1-s_2-s_3-s_4)}{8s_2(s_2+s_3)}+\frac{e^{s_1+s_2}k_{11}(s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_1s_3}$$
$$+\frac{e^{s_1+s_2}k_{11}(s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_2s_3}+\frac{e^{s_1+s_2}k_{11}(s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_1s_4}$$
$$+\frac{e^{s_1+s_2}k_{11}(s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_2s_4}+\frac{e^{s_1}k_{11}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_3(s_2+s_3)}$$
$$+\frac{e^{s_2+s_3}G_1(s_1)k_{12}(-s_2-s_3,s_2)}{4s_4}+\frac{e^{s_2+s_3}k_{12}(-s_2-s_3,s_2)}{4s_1s_4}+\frac{e^{s_1+s_2+s_3}k_{12}(-s_1-s_2-s_3,s_1)}{4s_2s_4}$$
$$+\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{12}(-s_2-s_3-s_4,s_2+s_3)}{4s_3}+\frac{e^{s_2+s_3+s_4}k_{12}(-s_2-s_3-s_4,s_2+s_3)}{4s_1s_3}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{12}(-s_1-s_2-s_3-s_4,s_1)}{8s_3(s_2+s_3)}+\frac{e^{s_1+s_2+s_3+s_4}k_{12}(-s_1-s_2-s_3-s_4,s_1+s_2)}{4s_1s_3}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{12}(-s_1-s_2-s_3-s_4,s_1+s_2)}{4s_2s_3}+\frac{e^{s_1+s_2+s_3+s_4}k_{12}(-s_1-s_2-s_3-s_4,s_1+s_2)}{4s_1s_4}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{12}(-s_1-s_2-s_3-s_4,s_1+s_2)}{4s_2s_4}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{12}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_2(s_2+s_3)}+\frac{e^{s_1+s_2}k_{13}(-s_1-s_2,s_1)}{8s_3s_4}$$
$$+\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_1s_4}+\frac{e^{s_1+s_2+s_3}k_{13}(-s_1-s_2-s_3,s_1+s_2)}{8s_2s_4}$$
$$+\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2)}{8s_3}+\frac{e^{s_2+s_3+s_4}G_1(s_1)k_{13}(-s_2-s_3-s_4,s_2)}{8s_4}$$
$$+\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2)}{8s_1s_3}+\frac{e^{s_2+s_3+s_4}k_{13}(-s_2-s_3-s_4,s_2)}{8s_1s_4}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8s_2(s_2+s_3)}+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8s_2s_4}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8s_3(s_3+s_4)}+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8s_4(s_3+s_4)}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_1s_3}$$

$$+\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_2s_3}+\frac{k_{17}(s_1,s_2+s_3,s_4)}{16s_2}$$
$$+\frac{e^{s_1}k_{17}(s_2,s_3,-s_1-s_2-s_3)}{16(s_1+s_2+s_3+s_4)}+\frac{1}{16}G_1(s_1)k_{17}(s_2,s_3,s_4)+\frac{k_{17}(s_2,s_3,s_4)}{16s_1}$$
$$+\frac{e^{s_1}k_{17}(s_2,s_3+s_4,-s_1-s_2-s_3-s_4)}{16s_3}+\frac{e^{s_1}k_{17}(s_2,s_3+s_4,-s_1-s_2-s_3-s_4)}{16s_4}$$
$$+\frac{e^{s_1+s_2+s_3}k_{17}(-s_1-s_2-s_3,s_1,s_2)}{16(s_1+s_2+s_3+s_4)}+\frac{e^{s_1+s_2+s_3+s_4}k_{17}(-s_1-s_2-s_3-s_4,s_1,s_2)}{16s_3}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{17}(-s_1-s_2-s_3-s_4,s_1,s_2)}{16s_4}+\frac{1}{16}e^{s_2+s_3}G_1(s_1)k_{17}(s_4,-s_2-s_3-s_4,s_2)$$
$$+\frac{e^{s_2+s_3}k_{17}(s_4,-s_2-s_3-s_4,s_2)}{16s_1}+\frac{e^{s_1+s_2+s_3}k_{17}(s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_2}$$
$$+\frac{k_{18}(s_1,s_2,s_3)}{16(-s_1-s_2-s_3-s_4)}+\frac{k_{18}(s_1,s_2,s_3)}{16s_4}+\frac{k_{18}(s_1,s_2+s_3,s_4)}{16s_3}+\frac{e^{s_1}k_{18}(s_2,s_3,-s_1-s_2-s_3)}{16s_4}$$
$$-\frac{1}{16}G_1(s_1)k_{18}(s_2,s_3,s_4)+\frac{e^{s_1}k_{18}(s_2,s_3,s_4)}{16(s_1+s_2+s_3+s_4)}+\frac{k_{18}(s_1+s_2,s_3,s_4)}{16s_1}+\frac{k_{18}(s_1+s_2,s_3,s_4)}{16s_2}$$
$$+\frac{e^{s_1+s_2+s_3}k_{18}(-s_1-s_2-s_3,s_1,s_2)}{16s_4}+\frac{e^{s_1+s_2}k_{18}(s_3,-s_1-s_2-s_3,s_1)}{16s_4}$$
$$-\frac{1}{16}e^{s_2}G_1(s_1)k_{18}(s_3,s_4,-s_2-s_3-s_4)+\frac{e^{s_1+s_2}k_{18}(s_3,s_4,-s_2-s_3-s_4)}{16(s_1+s_2+s_3+s_4)}$$
$$+\frac{e^{s_1+s_2}k_{18}(s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_1}+\frac{e^{s_1+s_2}k_{18}(s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_2}$$
$$+\frac{e^{s_1}k_{18}(s_2+s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_3}-\frac{1}{16}e^{s_2+s_3+s_4}G_1(s_1)k_{18}(-s_2-s_3-s_4,s_2,s_3)$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{18}(-s_2-s_3-s_4,s_2,s_3)}{16(s_1+s_2+s_3+s_4)}+\frac{e^{s_1+s_2+s_3+s_4}k_{18}(-s_1-s_2-s_3-s_4,s_1,s_2+s_3)}{16s_3}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{18}(-s_1-s_2-s_3-s_4,s_1+s_2,s_3)}{16s_1}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{18}(-s_1-s_2-s_3-s_4,s_1+s_2,s_3)}{16s_2}-\frac{1}{16}e^{s_2+s_3}G_1(s_1)k_{18}(s_4,-s_2-s_3-s_4,s_2)$$
$$+\frac{e^{s_1+s_2+s_3}k_{18}(s_4,-s_2-s_3-s_4,s_2)}{16(s_1+s_2+s_3+s_4)}+\frac{e^{s_1+s_2+s_3}k_{18}(s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_3}$$
$$+\frac{e^{s_1+s_2+s_3}k_{18}(s_4,-s_1-s_2-s_3-s_4,s_1+s_2)}{16s_1}+\frac{e^{s_1+s_2+s_3}k_{18}(s_4,-s_1-s_2-s_3-s_4,s_1+s_2)}{16s_2}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{19}(s_1,s_2,s_3)}{16(-s_1-s_2-s_3-s_4)}+\frac{e^{s_1+s_2+s_3+s_4}k_{19}(s_1,s_2,s_3)}{16(s_1+s_2+s_3+s_4)}+\frac{k_{19}(s_1,s_2,s_3+s_4)}{16s_3}$$
$$+\frac{k_{19}(s_1,s_2,s_3+s_4)}{16s_4}+\frac{e^{s_1+s_2}k_{19}(s_3,-s_1-s_2-s_3,s_1)}{16(s_1+s_2+s_3+s_4)}+\frac{1}{16}e^{s_2}G_1(s_1)k_{19}(s_3,s_4,-s_2-s_3-s_4)$$
$$+\frac{e^{s_2}k_{19}(s_3,s_4,-s_2-s_3-s_4)}{16s_1}+\frac{e^{s_1}k_{19}(s_2+s_3,s_4,-s_1-s_2-s_3-s_4)}{16s_2}$$
$$+\frac{1}{16}e^{s_2+s_3+s_4}G_1(s_1)k_{19}(-s_2-s_3-s_4,s_2,s_3)+\frac{e^{s_2+s_3+s_4}k_{19}(-s_2-s_3-s_4,s_2,s_3)}{16s_1}$$
$$+\frac{e^{s_1+s_2+s_3+s_4}k_{19}(-s_1-s_2-s_3-s_4,s_1,s_2+s_3)}{16s_2}+\frac{e^{s_1+s_2}k_{19}(s_3+s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_3}$$
$$+\frac{e^{s_1+s_2}k_{19}(s_3+s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_4}-\frac{k_{17}(s_1+s_2,s_3,s_4)}{16s_1}$$
$$-\frac{e^{s_1+s_2+s_3}k_{17}(s_4,-s_1-s_2-s_3-s_4,s_1+s_2)}{16s_1}-\frac{k_{18}(s_2,s_3,s_4)}{16s_1}-\frac{e^{s_2}k_{18}(s_3,s_4,-s_2-s_3-s_4)}{16s_1}$$
$$-\frac{e^{s_2+s_3+s_4}k_{18}(-s_2-s_3-s_4,s_2,s_3)}{16s_1}-\frac{e^{s_2+s_3}k_{18}(s_4,-s_2-s_3-s_4,s_2)}{16s_1}$$

$$- \frac{e^{s_1+s_2} k_{19}(s_3, s_4, -s_1 - s_2 - s_3 - s_4)}{16 s_1} - \frac{e^{s_1+s_2+s_3+s_4} k_{19}(-s_1 - s_2 - s_3 - s_4, s_1 + s_2, s_3)}{16 s_1}$$
$$- \frac{k_{17}(s_1 + s_2, s_3, s_4)}{16 s_2} - \frac{e^{s_1+s_2+s_3} k_{17}(s_4, -s_1 - s_2 - s_3 - s_4, s_1 + s_2)}{16 s_2} - \frac{k_{18}(s_1, s_2 + s_3, s_4)}{16 s_2}$$
$$- \frac{e^{s_1} k_{18}(s_2 + s_3, s_4, -s_1 - s_2 - s_3 - s_4)}{16 s_2} - \frac{e^{s_1+s_2+s_3+s_4} k_{18}(-s_1 - s_2 - s_3 - s_4, s_1, s_2 + s_3)}{16 s_2}$$
$$- \frac{e^{s_1+s_2+s_3} k_{18}(s_4, -s_1 - s_2 - s_3 - s_4, s_1)}{16 s_2} - \frac{e^{s_1+s_2} k_{19}(s_3, s_4, -s_1 - s_2 - s_3 - s_4)}{16 s_2}$$
$$- \frac{e^{s_1+s_2+s_3+s_4} k_{19}(-s_1 - s_2 - s_3 - s_4, s_1 + s_2, s_3)}{16 s_2} - \frac{G_1(s_1) k_8(s_2, s_3 + s_4)}{4 s_3}$$
$$- \frac{e^{s_2} G_1(s_1) k_{11}(s_3 + s_4, -s_2 - s_3 - s_4)}{4 s_3} - \frac{e^{s_2+s_3+s_4} G_1(s_1) k_{12}(-s_2 - s_3 - s_4, s_2)}{4 s_3}$$
$$- \frac{G_1(s_1) k_9(s_2 + s_3, s_4)}{8 s_3} - \frac{e^{s_2+s_3} G_1(s_1) k_{10}(s_4, -s_2 - s_3 - s_4)}{8 s_3}$$
$$- \frac{e^{s_2+s_3+s_4} G_1(s_1) k_{13}(-s_2 - s_3 - s_4, s_2 + s_3)}{8 s_3} - \frac{e^{s_1} k_{17}(s_2 + s_3, s_4, -s_1 - s_2 - s_3 - s_4)}{16 s_3}$$
$$- \frac{e^{s_1+s_2+s_3+s_4} k_{17}(-s_1 - s_2 - s_3 - s_4, s_1, s_2 + s_3)}{16 s_3} - \frac{k_{18}(s_1, s_2, s_3 + s_4)}{16 s_3}$$
$$- \frac{e^{s_1} k_{18}(s_2, s_3 + s_4, -s_1 - s_2 - s_3 - s_4)}{16 s_3} - \frac{e^{s_1+s_2+s_3+s_4} k_{18}(-s_1 - s_2 - s_3 - s_4, s_1, s_2)}{16 s_3}$$
$$- \frac{e^{s_1+s_2} k_{18}(s_3 + s_4, -s_1 - s_2 - s_3 - s_4, s_1)}{16 s_3} - \frac{k_{19}(s_1, s_2 + s_3, s_4)}{16 s_3}$$
$$- \frac{e^{s_1+s_2+s_3} k_{19}(s_4, -s_1 - s_2 - s_3 - s_4, s_1)}{16 s_3} - \frac{k_8(s_2, s_3 + s_4)}{4 s_1 s_3} - \frac{k_8(s_1 + s_2 + s_3, s_4)}{4 s_1 s_3}$$
$$- \frac{e^{s_1+s_2+s_3} k_{11}(s_4, -s_1 - s_2 - s_3 - s_4)}{4 s_1 s_3} - \frac{e^{s_2} k_{11}(s_3 + s_4, -s_2 - s_3 - s_4)}{4 s_1 s_3}$$
$$- \frac{e^{s_2+s_3+s_4} k_{12}(-s_2 - s_3 - s_4, s_2)}{4 s_1 s_3} - \frac{e^{s_1+s_2+s_3+s_4} k_{12}(-s_1 - s_2 - s_3 - s_4, s_1 + s_2 + s_3)}{4 s_1 s_3}$$
$$- \frac{k_9(s_1 + s_2, s_3 + s_4)}{8 s_1 s_3} - \frac{k_9(s_2 + s_3, s_4)}{8 s_1 s_3} - \frac{e^{s_2+s_3} k_{10}(s_4, -s_2 - s_3 - s_4)}{8 s_1 s_3}$$
$$- \frac{e^{s_1+s_2} k_{10}(s_3 + s_4, -s_1 - s_2 - s_3 - s_4)}{8 s_1 s_3} - \frac{e^{s_2+s_3+s_4} k_{13}(-s_2 - s_3 - s_4, s_2 + s_3)}{8 s_1 s_3}$$
$$- \frac{e^{s_1+s_2+s_3+s_4} k_{13}(-s_1 - s_2 - s_3 - s_4, s_1 + s_2)}{8 s_1 s_3} - \frac{G_1(s_1) k_6(s_4)}{4 s_2 s_3} - \frac{e^{s_4} G_1(s_1) k_7(-s_4)}{4 s_2 s_3}$$
$$- \frac{k_8(s_1, s_2 + s_3 + s_4)}{8 s_2 s_3} - \frac{k_8(s_1 + s_2 + s_3, s_4)}{8 s_2 s_3} - \frac{k_9(s_1 + s_2, s_3 + s_4)}{8 s_2 s_3}$$
$$- \frac{e^{s_1+s_2} k_{10}(s_3 + s_4, -s_1 - s_2 - s_3 - s_4)}{8 s_2 s_3} - \frac{e^{s_1+s_2+s_3} k_{11}(s_4, -s_1 - s_2 - s_3 - s_4)}{8 s_2 s_3}$$
$$- \frac{e^{s_1} k_{11}(s_2 + s_3 + s_4, -s_1 - s_2 - s_3 - s_4)}{8 s_2 s_3} - \frac{e^{s_1+s_2+s_3+s_4} k_{12}(-s_1 - s_2 - s_3 - s_4, s_1)}{8 s_2 s_3}$$
$$- \frac{e^{s_1+s_2+s_3+s_4} k_{12}(-s_1 - s_2 - s_3 - s_4, s_1 + s_2 + s_3)}{8 s_2 s_3}$$
$$- \frac{e^{s_1+s_2+s_3+s_4} k_{13}(-s_1 - s_2 - s_3 - s_4, s_1 + s_2)}{8 s_2 s_3} - \frac{k_6(s_1 + s_2 + s_3 + s_4)}{4 s_1 s_2 s_3}$$
$$- \frac{e^{s_1+s_2+s_3+s_4} k_7(-s_1 - s_2 - s_3 - s_4)}{4 s_1 s_2 s_3} - \frac{k_6(s_4)}{4 s_2 (s_1 + s_2) s_3} - \frac{e^{s_4} k_7(-s_4)}{4 s_2 (s_1 + s_2) s_3}$$
$$- \frac{G_1(s_1) k_6(s_2 + s_3 + s_4)}{4 s_2 (s_2 + s_3)} - \frac{e^{s_2+s_3+s_4} G_1(s_1) k_7(-s_2 - s_3 - s_4)}{4 s_2 (s_2 + s_3)} - \frac{k_8(s_1, s_2 + s_3 + s_4)}{8 s_2 (s_2 + s_3)}$$
$$- \frac{k_9(s_1 + s_2 + s_3, s_4)}{8 s_2 (s_2 + s_3)} - \frac{e^{s_1+s_2+s_3} k_{10}(s_4, -s_1 - s_2 - s_3 - s_4)}{8 s_2 (s_2 + s_3)}$$

$$
\begin{aligned}
&- \frac{e^{s_1} k_{11}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_2(s_2+s_3)} - \frac{e^{s_1+s_2+s_3+s_4} k_{12}(-s_1-s_2-s_3-s_4,s_1)}{8s_2(s_2+s_3)} \\
&- \frac{e^{s_1+s_2+s_3+s_4} k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_2(s_2+s_3)} - \frac{k_6(s_2+s_3+s_4)}{4s_1s_2(s_2+s_3)} \\
&- \frac{e^{s_2+s_3+s_4} k_7(-s_2-s_3-s_4)}{4s_1s_2(s_2+s_3)} - \frac{G_1(s_1)k_6(s_2+s_3+s_4)}{4s_3(s_2+s_3)} - \frac{e^{s_2+s_3+s_4} G_1(s_1)k_7(-s_2-s_3-s_4)}{4s_3(s_2+s_3)} \\
&- \frac{k_8(s_1+s_2+s_3,s_4)}{8s_3(s_2+s_3)} - \frac{e^{s_1+s_2+s_3} k_{11}(s_4,-s_1-s_2-s_3-s_4)}{8s_3(s_2+s_3)} \\
&- \frac{e^{s_1+s_2+s_3+s_4} k_{12}(-s_1-s_2-s_3-s_4,s_1+s_2+s_3)}{8s_3(s_2+s_3)} - \frac{k_6(s_2+s_3+s_4)}{4s_1s_3(s_2+s_3)} \\
&- \frac{e^{s_2+s_3+s_4} k_7(-s_2-s_3-s_4)}{4s_1s_3(s_2+s_3)} - \frac{k_6(s_4)}{4s_1s_2(s_1+s_2+s_3)} - \frac{e^{s_4} k_7(-s_4)}{4s_1s_2(s_1+s_2+s_3)} \\
&- \frac{k_6(s_1+s_2+s_3+s_4)}{4s_2(s_1+s_2)(s_1+s_2+s_3)} - \frac{e^{s_1+s_2+s_3+s_4} k_7(-s_1-s_2-s_3-s_4)}{4s_2(s_1+s_2)(s_1+s_2+s_3)} \\
&- \frac{k_6(s_1+s_2+s_3+s_4)}{4s_3(s_2+s_3)(s_1+s_2+s_3)} - \frac{e^{s_1+s_2+s_3+s_4} k_7(-s_1-s_2-s_3-s_4)}{4s_3(s_2+s_3)(s_1+s_2+s_3)} \\
&- \frac{e^{s_1+s_2+s_3+s_4} k_{18}(s_1,s_2,s_3)}{16(-s_1-s_2-s_3-s_4)} - \frac{k_{19}(s_1,s_2,s_3)}{16(-s_1-s_2-s_3-s_4)} - \frac{G_1(s_1)k_8(s_2,s_3+s_4)}{4s_4} \\
&- \frac{e^{s_2} G_1(s_1)k_{11}(s_3+s_4,-s_2-s_3-s_4)}{4s_4} - \frac{e^{s_2+s_3+s_4} G_1(s_1)k_{12}(-s_2-s_3-s_4,s_2)}{4s_4} \\
&- \frac{G_1(s_1)k_9(s_2,s_3)}{8s_4} - \frac{e^{s_2} G_1(s_1)k_{10}(s_3,-s_2-s_3)}{8s_4} - \frac{e^{s_2+s_3} G_1(s_1)k_{13}(-s_2-s_3,s_2)}{8s_4} \\
&- \frac{e^{s_1} k_{17}(s_2,s_3,-s_1-s_2-s_3)}{16s_4} - \frac{e^{s_1+s_2+s_3} k_{17}(-s_1-s_2-s_3,s_1,s_2)}{16s_4} - \frac{k_{18}(s_1,s_2,s_3+s_4)}{16s_4} \\
&- \frac{e^{s_1} k_{18}(s_2,s_3+s_4,-s_1-s_2-s_3-s_4)}{16s_4} - \frac{e^{s_1+s_2+s_3+s_4} k_{18}(-s_1-s_2-s_3-s_4,s_1,s_2)}{16s_4} \\
&- \frac{e^{s_1+s_2} k_{18}(s_3+s_4,-s_1-s_2-s_3-s_4,s_1)}{16s_4} - \frac{k_{19}(s_1,s_2,s_3)}{16s_4} - \frac{e^{s_1+s_2} k_{19}(s_3,-s_1-s_2-s_3,s_1)}{16s_4} \\
&- \frac{k_8(s_2,s_3+s_4)}{4s_1s_4} - \frac{k_8(s_1+s_2,s_3)}{4s_1s_4} - \frac{e^{s_1+s_2} k_{11}(s_3,-s_1-s_2-s_3)}{4s_1s_4} \\
&- \frac{e^{s_2} k_{11}(s_3+s_4,-s_2-s_3-s_4)}{4s_1s_4} - \frac{e^{s_1+s_2+s_3} k_{12}(-s_1-s_2-s_3,s_1+s_2)}{4s_1s_4} \\
&- \frac{e^{s_2+s_3+s_4} k_{12}(-s_2-s_3-s_4,s_2)}{4s_1s_4} - \frac{k_9(s_2,s_3)}{8s_1s_4} - \frac{k_9(s_1+s_2,s_3+s_4)}{8s_1s_4} - \frac{e^{s_2} k_{10}(s_3,-s_2-s_3)}{8s_1s_4} \\
&- \frac{e^{s_1+s_2} k_{10}(s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_1s_4} - \frac{e^{s_2+s_3} k_{13}(-s_2-s_3,s_2)}{8s_1s_4} \\
&- \frac{e^{s_1+s_2+s_3+s_4} k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_1s_4} - \frac{k_8(s_1,s_2+s_3+s_4)}{4s_2s_4} - \frac{k_8(s_1+s_2,s_3)}{4s_2s_4} \\
&- \frac{e^{s_1+s_2} k_{11}(s_3,-s_1-s_2-s_3)}{4s_2s_4} - \frac{e^{s_1} k_{11}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{4s_2s_4} \\
&- \frac{e^{s_1+s_2+s_3} k_{12}(-s_1-s_2-s_3,s_1+s_2)}{4s_2s_4} - \frac{e^{s_1+s_2+s_3+s_4} k_{12}(-s_1-s_2-s_3-s_4,s_1)}{4s_2s_4} \\
&- \frac{k_9(s_1,s_2+s_3)}{8s_2s_4} - \frac{k_9(s_1+s_2,s_3+s_4)}{8s_2s_4} - \frac{e^{s_1} k_{10}(s_2+s_3,-s_1-s_2-s_3)}{8s_2s_4} \\
&- \frac{e^{s_1+s_2} k_{10}(s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_2s_4} - \frac{e^{s_1+s_2+s_3} k_{13}(-s_1-s_2-s_3,s_1)}{8s_2s_4} \\
&- \frac{e^{s_1+s_2+s_3+s_4} k_{13}(-s_1-s_2-s_3-s_4,s_1+s_2)}{8s_2s_4} - \frac{G_1(s_1)k_6(s_2)}{2s_3s_4} - \frac{e^{s_2} G_1(s_1)k_7(-s_2)}{2s_3s_4} \\
&- \frac{k_9(s_1,s_2+s_3+s_4)}{8s_3s_4} - \frac{e^{s_1} k_{10}(s_2+s_3+s_4,-s_1-s_2-s_3-s_4)}{8s_3s_4}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{e^{s_1+s_2+s_3+s_4}k_{13}(-s_1-s_2-s_3-s_4,s_1)}{8s_3s_4}-\frac{k_6(s_2)}{2s_1s_3s_4}-\frac{k_6(s_1+s_2+s_3+s_4)}{2s_1s_3s_4}-\frac{e^{s_2}k_7(-s_2)}{2s_1s_3s_4}\\
&-\frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{2s_1s_3s_4}-\frac{k_6(s_1+s_2)}{2s_2s_3s_4}-\frac{e^{s_1+s_2}k_7(-s_1-s_2)}{2s_2s_3s_4}-\frac{k_6(s_1+s_2+s_3)}{4s_2(s_2+s_3)s_4}\\
&-\frac{k_6(s_1+s_2+s_3+s_4)}{4s_2(s_2+s_3)s_4}-\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{4s_2(s_2+s_3)s_4}-\frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{4s_2(s_2+s_3)s_4}\\
&-\frac{k_6(s_1+s_2+s_3)}{4s_3(s_2+s_3)s_4}-\frac{e^{s_1+s_2+s_3}k_7(-s_1-s_2-s_3)}{4s_3(s_2+s_3)s_4}-\frac{G_1(s_1)k_6(s_2+s_3+s_4)}{2s_3(s_3+s_4)}\\
&-\frac{e^{s_2+s_3+s_4}G_1(s_1)k_7(-s_2-s_3-s_4)}{2s_3(s_3+s_4)}-\frac{k_9(s_1,s_2)}{8s_3(s_3+s_4)}-\frac{e^{s_1}k_{10}(s_2,-s_1-s_2)}{8s_3(s_3+s_4)}\\
&-\frac{e^{s_1+s_2}k_{13}(-s_1-s_2,s_1)}{8s_3(s_3+s_4)}-\frac{k_6(s_1+s_2)}{2s_1s_3(s_3+s_4)}-\frac{k_6(s_2+s_3+s_4)}{2s_1s_3(s_3+s_4)}-\frac{e^{s_1+s_2}k_7(-s_1-s_2)}{2s_1s_3(s_3+s_4)}\\
&-\frac{e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{2s_1s_3(s_3+s_4)}-\frac{k_6(s_1+s_2+s_3+s_4)}{2s_2s_3(s_3+s_4)}-\frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{2s_2s_3(s_3+s_4)}\\
&-\frac{G_1(s_1)k_6(s_2+s_3+s_4)}{2s_4(s_3+s_4)}-\frac{e^{s_2+s_3+s_4}G_1(s_1)k_7(-s_2-s_3-s_4)}{2s_4(s_3+s_4)}-\frac{k_9(s_1,s_2)}{8s_4(s_3+s_4)}\\
&-\frac{e^{s_1}k_{10}(s_2,-s_1-s_2)}{8s_4(s_3+s_4)}-\frac{e^{s_1+s_2}k_{13}(-s_1-s_2,s_1)}{8s_4(s_3+s_4)}-\frac{k_6(s_1+s_2)}{2s_1s_4(s_3+s_4)}-\frac{k_6(s_2+s_3+s_4)}{2s_1s_4(s_3+s_4)}\\
&-\frac{e^{s_1+s_2}k_7(-s_1-s_2)}{2s_1s_4(s_3+s_4)}-\frac{e^{s_2+s_3+s_4}k_7(-s_2-s_3-s_4)}{2s_1s_4(s_3+s_4)}-\frac{k_6(s_1+s_2+s_3+s_4)}{2s_2s_4(s_3+s_4)}\\
&-\frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{2s_2s_4(s_3+s_4)}-\frac{e^{s_1}k_7(-s_1)}{4s_2s_3(s_2+s_3+s_4)}-\frac{e^{s_1}k_7(-s_1)}{4s_2(s_2+s_3)(s_2+s_3+s_4)}\\
&-\frac{k_6(s_1+s_2+s_3+s_4)}{4s_3(s_2+s_3)(s_2+s_3+s_4)}-\frac{e^{s_1+s_2+s_3+s_4}k_7(-s_1-s_2-s_3-s_4)}{4s_3(s_2+s_3)(s_2+s_3+s_4)}-\frac{e^{s_1}k_7(-s_1)}{2s_2s_4(s_2+s_3+s_4)}\\
&-\frac{e^{s_1}k_{17}(s_2,s_3,s_4)}{16(s_1+s_2+s_3+s_4)}-\frac{e^{s_1+s_2+s_3}k_{17}(s_4,-s_2-s_3-s_4,s_2)}{16(s_1+s_2+s_3+s_4)}-\frac{e^{s_1+s_2+s_3+s_4}k_{18}(s_1,s_2,s_3)}{16(s_1+s_2+s_3+s_4)}\\
&-\frac{e^{s_1}k_{18}(s_2,s_3,-s_1-s_2-s_3)}{16(s_1+s_2+s_3+s_4)}-\frac{e^{s_1+s_2+s_3}k_{18}(-s_1-s_2-s_3,s_1,s_2)}{16(s_1+s_2+s_3+s_4)}\\
&-\frac{e^{s_1+s_2}k_{18}(s_3,-s_1-s_2-s_3,s_1)}{16(s_1+s_2+s_3+s_4)}-\frac{e^{s_1+s_2}k_{19}(s_3,s_4,-s_2-s_3-s_4)}{16(s_1+s_2+s_3+s_4)}\\
&-\frac{e^{s_1+s_2+s_3+s_4}k_{19}(-s_2-s_3-s_4,s_2,s_3)}{16(s_1+s_2+s_3+s_4)}=0.
\end{aligned}
$$

Appendix C. Lengthy explicit formulas

In Section 9 we presented the explicit formula for the one and two variable functions $K_1,\dots,K_7$ that appear in formula (6) for the term $a_4\in C^\infty(\mathbb{T}^2_\theta)$. Since the expressions associated with the three and four variable functions $K_8,\dots,K_{20}$ are quite lengthy, they are provided in this appendix.

C.1. The three variable functions $K_8,\dots,K_{16}$.

First we cover the explicit formulas for the three variable functions $K_8,K_9,\dots,K_{16}$.

C.1.1. *The function K_8.* We have

$$K_8(s_1,s_2,s_3)=\sum_{i=1}^{18}K_{8,i}(s_1,s_2,s_3),$$

where

$$K_{8,1}(s_1,s_2,s_3)=-\frac{4\pi e^{\frac{3s_1}{2}+\frac{5s_2}{2}}((e^{s_1}(2e^{s_2}-1)-1)s_1+(e^{s_1}-1)s_2)}{(e^{s_2}-1)(e^{s_1+s_2}-1)^2(e^{\frac{s_3}{2}}-1)^2s_1s_2(s_1+s_2)},$$

$$K_{8,2}(s_1,s_2,s_3)=\frac{4\pi e^{\frac{3s_1}{2}+\frac{5s_2}{2}}((e^{s_1}(2e^{s_2}-1)-1)s_1+(e^{s_1}-1)s_2)}{(e^{s_2}-1)(e^{s_1+s_2}-1)^2(e^{\frac{s_3}{2}}+1)^2s_1s_2(s_1+s_2)},$$

$$K_{8,3}(s_1,s_2,s_3)=-\frac{2\pi e^{\frac{3s_1}{2}}(s_2-2s_3)}{(e^{s_1}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)^3s_1s_2s_3},$$

$$K_{8,4}(s_1,s_2,s_3)=-\frac{2\pi e^{\frac{3s_1}{2}}(s_2-2s_3)}{(e^{s_1}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)^3s_1s_2s_3},$$

$$K_{8,5}(s_1,s_2,s_3)=-\frac{3\pi(s_1^2-s_3s_1-s_2^2+2s_3^2+s_2s_3)}{(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^4s_1(s_1+s_2)s_3(s_2+s_3)},$$

$$K_{8,6}(s_1,s_2,s_3)=\frac{3\pi(s_1^2-s_3s_1-s_2^2+2s_3^2+s_2s_3)}{(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^4s_1(s_1+s_2)s_3(s_2+s_3)},$$

$$K_{8,7}(s_1,s_2,s_3)=\frac{4\pi e^{\frac{3s_1}{2}}(2(e^{s_2}(s_3-3)-s_3+2)s_3+s_2(-2s_3+e^{s_2}(2s_3-1)-1))}{(e^{s_1}-1)(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)s_1s_2s_3(s_2+s_3)},$$

$$K_{8,8}(s_1,s_2,s_3)=\frac{4\pi e^{\frac{3s_1}{2}}(2(e^{s_2}(s_3-3)-s_3+2)s_3+s_2(-2s_3+e^{s_2}(2s_3-1)-1))}{(e^{s_1}-1)(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)s_1s_2s_3(s_2+s_3)},$$

$$K_{8,9}(s_1,s_2,s_3)=\frac{\pi e^{\frac{3s_1}{2}}((e^{s_2}+3)s_2^2+(-7s_3+e^{s_2}(15s_3+4)-4)s_2+2s_3(-5s_3+e^{s_2}(7s_3-4)+4))}{(e^{s_1}-1)(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)^2s_1s_2s_3(s_2+s_3)}$$

$$K_{8,10}(s_1,s_2,s_3)$$
$$=\frac{\pi e^{\frac{3s_1}{2}}(-(e^{s_2}+3)s_2^2+(7s_3-e^{s_2}(15s_3+4)+4)s_2-2s_3(-5s_3+e^{s_2}(7s_3-4)+4))}{(e^{s_1}-1)(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)^2s_1s_2s_3(s_2+s_3)},$$

$$K_{8,11}(s_1,s_2,s_3)=\frac{8\pi e^{\frac{3}{2}(s_1+s_2)}(e^{s_2}(e^{s_1}-1)s_2+s_1(-e^{s_1+2s_2}(s_3-2)+e^{s_2}(e^{s_1}+1)(s_3-1)-s_3))}{(e^{s_2}-1)(e^{s_1+s_2}-1)^2(e^{\frac{s_3}{2}}-1)s_1s_2(s_1+s_2)s_3},$$

$$K_{8,12}(s_1,s_2,s_3)=\frac{8\pi e^{\frac{3}{2}(s_1+s_2)}(e^{s_2}(e^{s_1}-1)s_2+s_1(-e^{s_1+2s_2}(s_3-2)+e^{s_2}(e^{s_1}+1)(s_3-1)-s_3))}{(e^{s_2}-1)(e^{s_1+s_2}-1)^2(e^{\frac{s_3}{2}}+1)s_1s_2(s_1+s_2)s_3},$$

$$\begin{aligned}
&K_{8,13}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^3s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)\\
&=\pi(((e^{s_1}(e^{s_2}(7-5e^{s_1})+9)-11)s_2-4e^{s_1}(e^{s_2}-1)s_3)s_1^3+((e^{s_1}(e^{s_2}(11-5e^{s_1})+9)-15)s_2^2\\
&\quad+2(3(e^{s_1}-1)(e^{s_1+s_2}-1)-2(e^{s_1}(2e^{s_2}-3)+1)s_3)s_2-8e^{s_1}(e^{s_2}-1)s_3^2)s_1^2\\
&\quad+((e^{s_1}(e^{s_2}(5e^{s_1}+1)-9)+3)s_2^3-4(e^{s_1}(e^{s_2}(4e^{s_1}-3)-3)+2)s_3s_2^2\\
&\quad+s_3((e^{s_1}(25-7e^{s_2}(3e^{s_1}-1))-11)s_3-6(e^{s_1}-1)(e^{s_1+s_2}-1))s_2-4e^{s_1}(e^{s_2}-1)s_3^3)s_1\\
&\quad+s_2(s_2+s_3)(7s_2^2-11s_3s_2-6s_2-18s_3^2+12s_3+e^{2s_1+s_2}(5s_2^2-3(7s_3+2)s_2+2s_3(6-13s_3))\\
&\quad+e^{s_1}(-3(e^{s_2}+3)s_2^2+(13s_3+e^{s_2}(19s_3+6)+6)s_2+2(e^{s_2}+1)s_3(11s_3-6)))),
\end{aligned}$$

$$K_{8,14}(s_1,s_2,s_3)$$
$$=\frac{K_{8,13}^{\mathrm{num}}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^3s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},$$

$$\begin{aligned}
&K_{8,15}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)\\
&=4\pi(-2(e^{s_1+s_2}-1)(e^{s_1}(e^{s_2}-1)s_3-(e^{s_1}-1)s_2)s_1^3+(4(e^{s_1}-1)(e^{s_1+s_2}-1)s_2^2\\
&\quad+((-8e^{s_1}+4e^{s_1+s_2}-6e^{2(s_1+s_2)}+8e^{2s_1+s_2}+2)s_3+4e^{s_1}+6e^{s_1+s_2}+e^{2(s_1+s_2)}-6e^{2s_1+s_2}\\
&\quad-5)s_2-2e^{s_1}(e^{s_2}-1)s_3(2(e^{s_1+s_2}-1)s_3-3e^{s_1+s_2}+2))s_1^2+(2(e^{s_1}-1)(e^{s_1+s_2}-1)s_2^3\\
&\quad-2(e^{s_1+s_2}-1)((-5e^{s_1}+e^{s_1+s_2}+2e^{2s_1+s_2}+2)s_3+e^{s_1}-e^{2s_1+s_2}-2)s_2^2\\
&\quad-s_3(2(5e^{s_1}-e^{s_1+s_2}+2e^{2(s_1+s_2)}-7e^{2s_1+s_2}+2e^{3s_1+2s_2}-1)s_3-6e^{s_1}+10e^{s_1+s_2}\\
&\quad-11e^{2(s_1+s_2)}+10e^{2s_1+s_2}-2e^{3s_1+2s_2}-1)s_2-2e^{s_1}(e^{s_2}-1)s_3^2((e^{s_1+s_2}-1)s_3\\
&\quad-3e^{s_1+s_2}+2))s_1-s_2(s_2+s_3)(2s_3((2e^{s_1}-1)s_3(e^{s_1+s_2}-1)^2-5e^{s_1}-6e^{s_1+s_2}+4e^{2(s_1+s_2)}
\end{aligned}$$

$$+ 11e^{2s_1+s_2} - 7e^{3s_1+2s_2} + 3) + s_2(2(2e^{s_1} - 1)s_3(e^{s_1+s_2} - 1)^2 + 2e^{s_1} + 2e^{s_1+s_2} + e^{2(s_1+s_2)} - 2e^{2s_1+s_2} - 2e^{3s_1+2s_2} - 1))),$$

$$K_{8,16}(s_1, s_2, s_3) = \frac{K^{\rm num}_{8,15}(s_1, s_2, s_3)}{(e^{s_1} - 1)(e^{s_1+s_2} - 1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)} + 1)s_1 s_2(s_1 + s_2)s_3(s_2 + s_3)(s_1 + s_2 + s_3)},$$

$$K_{8,17}(s_1, s_2, s_3)2(e^{s_1} - 1)(e^{s_1+s_2} - 1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)} - 1)^2 s_1 s_2(s_1 + s_2)s_3(s_2 + s_3)(s_1 + s_2 + s_3)$$
$$= -\pi(((-29 + 23e^{s_1} + 38e^{s_1+s_2} - e^{2(s_1+s_2)} - 34e^{2s_1+s_2} + 3e^{3s_1+2s_2})s_2$$
$$+ 4e^{s_1}(-1 + e^{s_2})(-5 + 7e^{s_1+s_2})s_3)s_1^3 + ((-49 + 31e^{s_1} + 62e^{s_1+s_2} - 5e^{2(s_1+s_2)} - 50e^{2s_1+s_2}$$
$$+ 11e^{3s_1+2s_2})s_2^2 + 2(2(-5 + 17e^{s_1} - 9e^{s_1+s_2} + 20e^{2(s_1+s_2)} - 25e^{2s_1+s_2} + 2e^{3s_1+2s_2})s_3$$
$$- 15e^{s_1} - 30e^{s_1+s_2} + 11e^{2(s_1+s_2)} + 22e^{2s_1+s_2} - 7e^{3s_1+2s_2} + 19)s_2$$
$$+ 8e^{s_1}(-1 + e^{s_2})s_3((-5 + 7e^{s_1+s_2})s_3 - 2e^{s_1+s_2} + 2))s_1^2 + ((-11 - 7e^{s_1} + 10e^{s_1+s_2}$$
$$- 7e^{2(s_1+s_2)} + 2e^{2s_1+s_2} + 13e^{3s_1+2s_2})s_2^3 + 4(4(-1 + e^{s_1+s_2})^2 + (-10 + 19e^{s_1} + 9e^{s_1+s_2}$$
$$- e^{2(s_1+s_2)} - 41e^{2s_1+s_2} + 24e^{3s_1+2s_2})s_3)s_2^2 + s_3(2(-11 - e^{s_1} + 30e^{s_1+s_2} - 19e^{2(s_1+s_2)}$$
$$- 6e^{2s_1+s_2} + 7e^{3s_1+2s_2}) + (-29 + 103e^{s_1} + 6e^{s_1+s_2} + 31e^{2(s_1+s_2)} - 194e^{2s_1+s_2}$$
$$+ 83e^{3s_1+2s_2})s_3)s_2 + 4e^{s_1}(-1 + e^{s_2})s_3^2((-5 + 7e^{s_1+s_2})s_3 - 4e^{s_1+s_2} + 4))s_1$$
$$+ s_2(s_2 + s_3)((9 - 15e^{s_1} - 14e^{s_1+s_2} - 3e^{2(s_1+s_2)} + 18e^{2s_1+s_2} + 5e^{3s_1+2s_2})s_2^2$$
$$+ ((-29 + 43e^{s_1} + 66e^{s_1+s_2} - 53e^{2(s_1+s_2)} - 110e^{2s_1+s_2} + 83e^{3s_1+2s_2})s_3 + 30e^{s_1} + 28e^{s_1+s_2}$$
$$- 6e^{2(s_1+s_2)} - 44e^{2s_1+s_2} + 14e^{3s_1+2s_2} - 22)s_2 + 2s_3((-19 + 29e^{s_1} + 40e^{s_1+s_2} - 25e^{2(s_1+s_2)}$$
$$- 64e^{2s_1+s_2} + 39e^{3s_1+2s_2})s_3 - 38e^{s_1} - 68e^{s_1+s_2} + 38e^{2(s_1+s_2)} + 84e^{2s_1+s_2}$$
$$- 46e^{3s_1+2s_2} + 30))),$$

$$K_{8,18}(s_1, s_2, s_3) = \frac{-K^{\rm num}_{8,17}(s_1, s_2, s_3)}{2(e^{s_1} - 1)(e^{s_1+s_2} - 1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)} + 1)^2 s_1 s_2(s_1 + s_2)s_3(s_2 + s_3)(s_1 + s_2 + s_3)}.$$

Therefore, we can write the function K_8 as a quotient

$$K_8(s_1, s_2, s_3) = \frac{K^{\rm num}_8(s_1, s_2, s_3)}{K^{\rm den}_8(s_1, s_2, s_3)},$$

where

$$(73)\qquad K^{\rm den}_8(s_1, s_2, s_3) = (e^{s_1} - 1)(e^{s_2} - 1)(e^{s_1+s_2} - 1)^2(e^{s_3} - 1)^2(e^{s_2+s_3} - 1)^3 \times (e^{s_1+s_2+s_3} - 1)^4 s_1 s_2(s_1 + s_2)s_3(s_2 + s_3)(s_1 + s_2 + s_3)$$

and $K^{\rm num}_8(s_1, s_2, s_3)$ is a polynomial in s_1, s_2, s_3, $e^{s_1/2}$, $e^{s_2/2}$ and $e^{s_3/2}$. The points (i, j, m) and (n, p, q) such that $s_1^i s_2^j s_3^m e^{ns_1/2} e^{ps_2/2} e^{qs_3/2}$ appears in $K^{\rm num}_8(s_1, s_2, s_3)$ are plotted in Figure 14 and Figure 15.

C.1.2. *The function K_9.* We have

$$K_9(s_1, s_2, s_3) = \sum_{i=1}^{18} K_{9,i}(s_1, s_2, s_3),$$

where

$$K_{9,1}(s_1, s_2, s_3) = -\frac{8\pi e^{\frac{3}{2}(s_1+s_2)}(e^{s_2}(s_1 - s_2 + e^{s_1}(s_1 + s_2)) - 2s_1)}{(e^{s_2} - 1)(e^{s_1+s_2} - 1)^2(e^{\frac{s_3}{2}} - 1)^2 s_1 s_2(s_1 + s_2)},$$

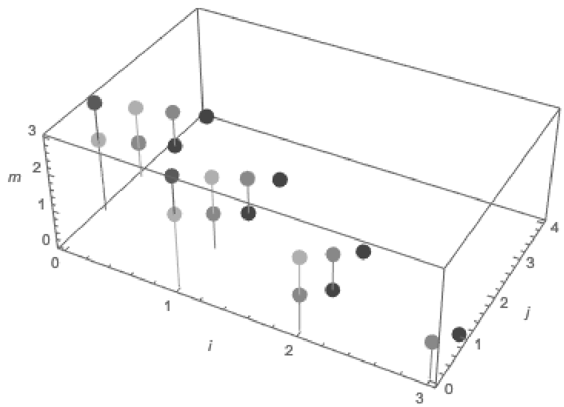

FIGURE 14. The points (i,j,m) such that $s_1^i\, s_2^j\, s_3^m\, e^{ns_1/2}\, e^{ps_2/2}\, e^{qs_3/2}$ appears in the expressions for $K_8^{\mathrm{num}}, K_9^{\mathrm{num}}, \ldots, K_{16}^{\mathrm{num}}$.

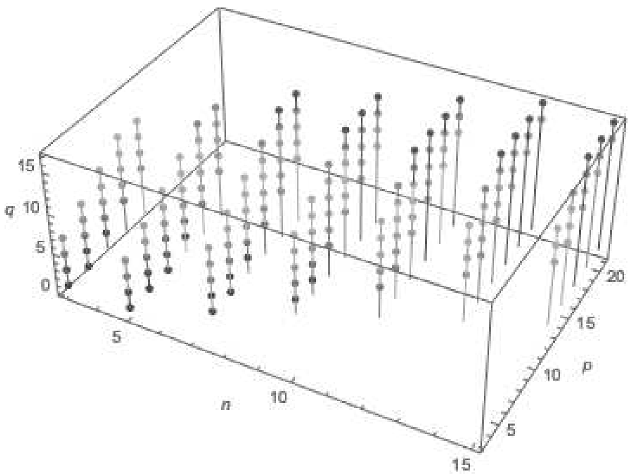

FIGURE 15. The points (n,p,q) such that $s_1^i\, s_2^j\, s_3^m\, e^{ns_1/2}\, e^{ps_2/2}\, e^{qs_3/2}$ appears in the expressions for K_8^{num} and K_{15}^{num}.

$$K_{9,2}(s_1,s_2,s_3) = \frac{8\pi e^{\frac{3}{2}(s_1+s_2)}(e^{s_2}(s_1-s_2+e^{s_1}(s_1+s_2))-2s_1)}{(e^{s_2}-1)(e^{s_1+s_2}-1)^2(e^{\frac{s_3}{2}}+1)^2 s_1 s_2 (s_1+s_2)},$$

$$K_{9,3}(s_1,s_2,s_3) = -\frac{4\pi e^{\frac{3s_1}{2}}(s_2-2s_3)}{(e^{s_1}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)^3 s_1 s_2 s_3},$$

$$K_{9,4}(s_1,s_2,s_3) = -\frac{4\pi e^{\frac{3s_1}{2}}(s_2-2s_3)}{(e^{s_1}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)^3 s_1 s_2 s_3},$$

$$K_{9,5}(s_1,s_2,s_3) = -\frac{16\pi e^{\frac{3}{2}(s_1+s_2)}(e^{s_2}(s_1-s_2+e^{s_1}(s_1+s_2))-2s_1)(s_3-1)}{(e^{s_2}-1)(e^{s_1+s_2}-1)^2(e^{\frac{s_3}{2}}-1)s_1s_2(s_1+s_2)s_3},$$

$$K_{9,6}(s_1,s_2,s_3) = -\frac{16\pi e^{\frac{3}{2}(s_1+s_2)}(e^{s_2}(s_1-s_2+e^{s_1}(s_1+s_2))-2s_1)(s_3-1)}{(e^{s_2}-1)(e^{s_1+s_2}-1)^2(e^{\frac{s_3}{2}}+1)s_1s_2(s_1+s_2)s_3},$$

$$K_{9,7}(s_1,s_2,s_3) = -\frac{6\pi(s_1^2-s_3s_1-(s_2-2s_3)(s_2+s_3))}{(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^4s_1(s_1+s_2)s_3(s_2+s_3)},$$

$$K_{9,8}(s_1,s_2,s_3) = \frac{6\pi(s_1^2-s_3s_1-(s_2-2s_3)(s_2+s_3))}{(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^4s_1(s_1+s_2)s_3(s_2+s_3)},$$

$$K_{9,9}(s_1,s_2,s_3) = \frac{8\pi e^{\frac{3s_1}{2}}(2s_2^2+(2(e^{s_2}+1)s_3+e^{s_2}-3)s_2+2(e^{s_2}(s_3-2)+1)s_3)}{(e^{s_1}-1)(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)s_1s_2s_3(s_2+s_3)},$$

$$K_{9,10}(s_1,s_2,s_3) = \frac{8\pi e^{\frac{3s_1}{2}}(2s_2^2+(2(e^{s_2}+1)s_3+e^{s_2}-3)s_2+2(e^{s_2}(s_3-2)+1)s_3)}{(e^{s_1}-1)(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)s_1s_2s_3(s_2+s_3)},$$

$$K_{9,11}(s_1,s_2,s_3) = \frac{2\pi e^{\frac{3s_1}{2}}((7-3e^{s_2})s_2^2+(s_3+e^{s_2}(7s_3+4)-4)s_2+2s_3(-3s_3+e^{s_2}(5s_3-4)+4))}{(e^{s_1}-1)(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)^2s_1s_2s_3(s_2+s_3)},$$

$$K_{9,12}(s_1,s_2,s_3) = \frac{2\pi e^{\frac{3s_1}{2}}((3e^{s_2}-7)s_2^2-(s_3+e^{s_2}(7s_3+4)-4)s_2-2s_3(-3s_3+e^{s_2}(5s_3-4)+4))}{(e^{s_1}-1)(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)^2s_1s_2s_3(s_2+s_3)},$$

$$\begin{aligned}
&K_{9,13}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^3s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)\\
&\quad = -2\pi(((-9e^{s_1}-7e^{s_1+s_2}+5e^{2s_1+s_2}+11)s_2+4e^{s_1}(e^{s_2}-1)s_3)s_1^3\\
&\qquad +((-5e^{s_1}-7e^{s_1+s_2}+e^{2s_1+s_2}+11)s_2^2-2(3(e^{s_1}-1)(e^{s_1+s_2}-1)\\
&\qquad +2e^{s_1}(-3e^{s_2}+e^{s_1+s_2}+2)s_3)s_2+8e^{s_1}(e^{s_2}-1)s_3^2)s_1^2+((17e^{s_1}+7e^{s_1+s_2}-13e^{2s_1+s_2}-11)s_2^3\\
&\qquad +4(e^{s_1}+e^{s_1+s_2}-2)s_3s_2^2+s_3(6(e^{s_1}-1)(e^{s_1+s_2}-1)+(-17e^{s_1}+e^{s_1+s_2}+13e^{2s_1+s_2}\\
&\qquad +3)s_3)s_2+4e^{s_1}(e^{s_2}-1)s_3^3)s_1-s_2(s_2+s_3)((-13e^{s_1}-7e^{s_1+s_2}+9e^{2s_1+s_2}+11)s_2^2\\
&\qquad +((5e^{s_1}+11e^{s_1+s_2}-13e^{2s_1+s_2}-3)s_3-6(e^{s_1}-1)(e^{s_1+s_2}-1))s_2\\
&\qquad -2s_3((-9e^{s_1}-9e^{s_1+s_2}+11e^{2s_1+s_2}+7)s_3-6(e^{s_1}-1)(e^{s_1+s_2}-1)))),
\end{aligned}$$

$$K_{9,14}(s_1,s_2,s_3) = \frac{K_{9,13}^{\mathrm{num}}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^3s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},$$

$$\begin{aligned}
&K_{9,15}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)\\
&\quad = 8\pi(2((1-2e^{s_1+s_2}+e^{2s_1+s_2})s_2-e^{2s_1+s_2}(-1+e^{s_2})s_3)s_1^3+(-2(-1-2e^{s_1}+2e^{s_1+s_2}\\
&\qquad +e^{2s_1+s_2})s_2^2-(2e^{s_1}(-2-e^{s_1+s_2}+3e^{s_1+2s_2})s_3-2e^{s_1}-8e^{s_1+s_2}+e^{2(s_1+s_2)}+4e^{2s_1+s_2}\\
&\qquad +5)s_2-2e^{s_1}(-1+e^{s_2})s_3(2e^{s_1+s_2}s_3-2e^{s_1+s_2}+1))s_1^2-(2(1-4e^{s_1}-2e^{s_1+s_2}+5e^{2s_1+s_2})s_2^3\\
&\qquad +2(e^{s_1}(3-4e^{s_1+s_2}+e^{2(s_1+s_2)})+(2-6e^{s_1}-4e^{s_1+s_2}+e^{2(s_1+s_2)}+5e^{2s_1+s_2}\\
&\qquad +2e^{3s_1+2s_2})s_3)s_2^2+s_3((2-4e^{s_1}-4e^{s_1+s_2}+4e^{2(s_1+s_2)}-2e^{2s_1+s_2}+4e^{3s_1+2s_2})s_3+4e^{s_1}\\
&\qquad +12e^{s_1+s_2}-9e^{2(s_1+s_2)}-4e^{2s_1+s_2}+2e^{3s_1+2s_2}-5)s_2+2e^{s_1}(-1+e^{s_2})s_3^2(e^{s_1+s_2}s_3\\
&\qquad -2e^{s_1+s_2}+1))s_1-s_2(s_2+s_3)((2-4e^{s_1}-4e^{s_1+s_2}+6e^{2s_1+s_2})s_2^2+((2-4e^{s_1}-4e^{s_1+s_2}\\
&\qquad -2e^{2(s_1+s_2)}+4e^{2s_1+s_2}+4e^{3s_1+2s_2})s_3+8e^{s_1}+8e^{s_1+s_2}-e^{2(s_1+s_2)}-12e^{2s_1+s_2}\\
&\qquad +2e^{3s_1+2s_2}-5)s_2+2s_3(e^{2s_1+s_2}(-1-e^{s_2}+2e^{s_1+s_2})s_3-2e^{s_1}-3e^{s_1+s_2}+3e^{2(s_1+s_2)}\\
&\qquad +6e^{2s_1+s_2}-5e^{3s_1+2s_2}+1))),
\end{aligned}$$

$$K_{9,16}(s_1,s_2,s_3) = \frac{K_{9,15}^{\mathrm{num}}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},$$

$$\begin{aligned}
&K_{9,17}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^2s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)\\
&\quad = -\pi(((-29+15e^{s_1}+46e^{s_1+s_2}-9e^{2(s_1+s_2)}-26e^{2s_1+s_2}+3e^{3s_1+2s_2})s_2
\end{aligned}$$

$$+ 4e^{s_1}(-1+e^{s_2})(-3+5e^{s_1+s_2})s_3)s_1^3 - ((29+13e^{s_1} - 46e^{s_1+s_2} + 9e^{2(s_1+s_2)} - 14e^{2s_1+s_2} + 9e^{3s_1+2s_2})s_2^2 + 2(2e^{s_1}(-2+9e^{s_2} + 5e^{s_1+s_2} + 3e^{2(s_1+s_2)} - 15e^{s_1+2s_2})s_3 + 15e^{s_1} + 30e^{s_1+s_2} - 11e^{2(s_1+s_2)} - 22e^{2s_1+s_2} + 7e^{3s_1+2s_2} - 19)s_2 - 8e^{s_1}(-1+e^{s_2})s_3((-3+5e^{s_1+s_2})s_3 - 2e^{s_1+s_2} + 2))s_1^2 + ((29 - 71e^{s_1} - 46e^{s_1+s_2} + 9e^{2(s_1+s_2)} + 106e^{2s_1+s_2} - 27e^{3s_1+2s_2})s_2^3 + 4(4e^{s_1}(-1+e^{s_1+s_2})^2 + (10 - 15e^{s_1} - 17e^{s_1+s_2} + 5e^{2(s_1+s_2)} + 13e^{2s_1+s_2} + 4e^{3s_1+2s_2})s_3)s_2^2 + s_3((11+23e^{s_1} - 34e^{s_1+s_2} + 31e^{2(s_1+s_2)} - 74e^{2s_1+s_2} + 43e^{3s_1+2s_2})s_3 + 14e^{s_1} + 92e^{s_1+s_2} - 54e^{2(s_1+s_2)} - 44e^{2s_1+s_2} + 30e^{3s_1+2s_2} - 38)s_2 + 4e^{s_1}(-1+e^{s_2})s_3^2((-3+5e^{s_1+s_2})s_3 - 4e^{s_1+s_2} + 4))s_1 - s_2(s_2+s_3)((-29+43e^{s_1} + 46e^{s_1+s_2} - 9e^{2(s_1+s_2)} - 66e^{2s_1+s_2} + 15e^{3s_1+2s_2})s_2^2 + ((-11+13e^{s_1} - 2e^{s_1+s_2} + 29e^{2(s_1+s_2)} + 14e^{2s_1+s_2} - 43e^{3s_1+2s_2})s_3 - 46e^{s_1} - 60e^{s_1+s_2} + 22e^{2(s_1+s_2)} + 76e^{2s_1+s_2} - 30e^{3s_1+2s_2} + 38)s_2 - 2s_3((-9+15e^{s_1} + 24e^{s_1+s_2} - 19e^{2(s_1+s_2)} - 40e^{2s_1+s_2} + 29e^{3s_1+2s_2})s_3 - 30e^{s_1} - 52e^{s_1+s_2} + 30e^{2(s_1+s_2)} + 68e^{2s_1+s_2} - 38e^{3s_1+2s_2} + 22))),$$

$$K_{9,18}(s_1,s_2,s_3) = \frac{-K_{9,17}^{\mathrm{num}}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^2 s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}.$$

By putting together the above expressions, we have

$$K_9(s_1,s_2,s_3) = \frac{K_9^{\mathrm{num}}(s_1,s_2,s_3)}{K_9^{\mathrm{den}}(s_1,s_2,s_3)},$$

where

$$K_9^{\mathrm{den}}(s_1,s_2,s_3) = K_8^{\mathrm{den}}(s_1,s_2,s_3),$$

which is given by (73), and K_9^{num} is a polynomial in $s_1, s_2, s_3, e^{s_1/2}, e^{s_2/2}, e^{s_3/2}$. The points (i,j,m) and (n,p,q) such that $s_1^i s_2^j s_3^m e^{ns_1/2} e^{ps_2/2} e^{qs_3/2}$ appears in $K_9^{\mathrm{num}}(s_1,s_2,s_3)$ are plotted in Figure 14 and Figure 16.

C.1.3. *The functions K_{10}, K_{11}, K_{16}.* We have

$$K_{10}(s_1,s_2,s_3) = \sum_{i=1}^{16} K_{10,i}(s_1,s_2,s_3),$$

where

$$K_{10,1}(s_1,s_2,s_3) = \frac{16\pi e^{\frac{3}{2}(s_1+s_2)}(-(e^{s_1+2s_2}(e^{s_2}(e^{s_1}+1)-3)+1)s_1 - e^{s_2}(e^{s_1}-1)(e^{s_2}+e^{s_1+2s_2}-2)s_2)}{(e^{s_2}-1)^2(e^{s_1+s_2}-1)^3(e^{\frac{s_3}{2}}-1)s_1 s_2(s_1+s_2)},$$

$$K_{10,2}(s_1,s_2,s_3) = \frac{16\pi e^{\frac{3}{2}(s_1+s_2)}(-(e^{s_1+2s_2}(e^{s_2}(e^{s_1}+1)-3)+1)s_1 - e^{s_2}(e^{s_1}-1)(e^{s_2}+e^{s_1+2s_2}-2)s_2)}{(e^{s_2}-1)^2(e^{s_1+s_2}-1)^3(e^{\frac{s_3}{2}}+1)s_1 s_2(s_1+s_2)},$$

$$K_{10,3}(s_1,s_2,s_3) = -\frac{4\pi e^{\frac{3s_1}{2}}(2s_2-s_3)}{(e^{s_1}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)^3 s_1 s_2 s_3},$$

$$K_{10,4}(s_1,s_2,s_3) = -\frac{4\pi e^{\frac{3s_1}{2}}(2s_2-s_3)}{(e^{s_1}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)^3 s_1 s_2 s_3},$$

$$K_{10,5}(s_1,s_2,s_3) = \frac{6\pi(-s_1^2 + (s_2+2s_3)s_1 + (2s_2-s_3)(s_2+s_3))}{(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^4 s_1(s_1+s_2)s_3(s_2+s_3)},$$

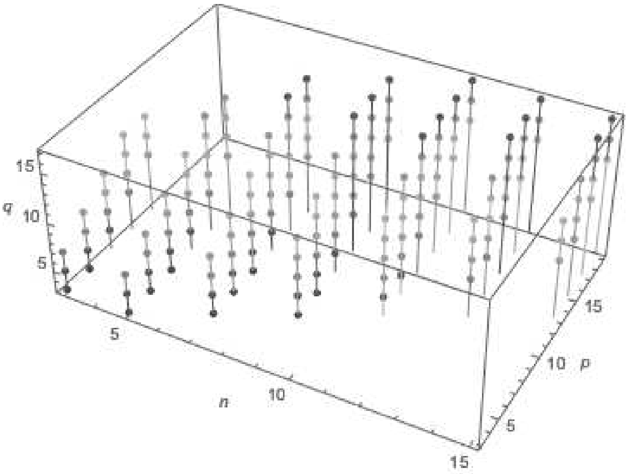

FIGURE 16. The points (n,p,q) such that $s_1^i\, s_2^j\, s_3^m\, e^{ns_1/2}\, e^{ps_2/2}\, e^{qs_3/2}$ appears in the expression for K_9^{num}.

$$K_{10,6}(s_1,s_2,s_3)=\frac{6\pi(s_1^2-(s_2+2s_3)s_1-(2s_2-s_3)(s_2+s_3))}{(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^4 s_1(s_1+s_2)s_3(s_2+s_3)},$$

$$K_{10,7}(s_1,s_2,s_3)=\frac{2\pi e^{\frac{3s_1}{2}}((10-6e^{s_2})s_2^2+(7s_3+e^{s_2}(s_3+8)-8)s_2+s_3(-3s_3+e^{s_2}(7s_3-4)+4))}{(e^{s_1}-1)(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)^2 s_1s_2s_3(s_2+s_3)},$$

$$K_{10,8}(s_1,s_2,s_3)=\frac{2\pi e^{\frac{3s_1}{2}}(2(3e^{s_2}-5)s_2^2-(7s_3+e^{s_2}(s_3+8)-8)s_2-s_3(-3s_3+e^{s_2}(7s_3-4)+4))}{(e^{s_1}-1)(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)^2 s_1s_2s_3(s_2+s_3)},$$

$$K_{10,9}(s_1,s_2,s_3)$$
$$=\frac{8\pi e^{\frac{3s_1}{2}}((4e^{s_2}-2)s_2^2+2((2e^{s_2}+e^{2s_2}-1)s_3-3e^{s_2}+e^{2s_2}+2)s_2+s_3(2e^{2s_2}s_3+4e^{s_2}-3e^{2s_2}-1))}{(e^{s_1}-1)(e^{s_2}-1)^2(e^{\frac{1}{2}(s_2+s_3)}-1)s_1s_2s_3(s_2+s_3)},$$

$$K_{10,10}(s_1,s_2,s_3)$$
$$=\frac{8\pi e^{\frac{3s_1}{2}}((4e^{s_2}-2)s_2^2+2((2e^{s_2}+e^{2s_2}-1)s_3-3e^{s_2}+e^{2s_2}+2)s_2+s_3(2e^{2s_2}s_3+4e^{s_2}-3e^{2s_2}-1))}{(e^{s_1}-1)(e^{s_2}-1)^2(e^{\frac{1}{2}(s_2+s_3)}+1)s_1s_2s_3(s_2+s_3)},$$

$$K_{10,11}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^3 s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)$$
$$\begin{aligned}
&=-2\pi(((-7e^{s_1}-9e^{s_1+s_2}+5e^{2s_1+s_2}+11)s_2+2e^{s_1}(e^{s_2}-1)s_3)s_1^3+(-4(2e^{s_1}+1)(e^{s_1+s_2}-1)s_2^2\\
&\quad+((9e^{s_1}+11e^{s_1+s_2}-13e^{2s_1+s_2}-7)s_3-6(e^{s_1}-1)(e^{s_1+s_2}-1))s_2+4e^{s_1}(e^{s_2}-1)s_3^2)s_1^2\\
&\quad+((37e^{s_1}+19e^{s_1+s_2}-31e^{2s_1+s_2}-25)s_2^3+(6(e^{s_1}-1)(e^{s_1+s_2}-1)+(46e^{s_1}+26e^{s_1+s_2}\\
&\quad-36e^{2s_1+s_2}-36)s_3)s_2^2-s_3((-7e^{s_1}-9e^{s_1+s_2}+5e^{2s_1+s_2}+11)s_3-12(e^{s_1}-1)(e^{s_1+s_2}-1))s_2\\
&\quad+2e^{s_1}(e^{s_2}-1)s_3^3)s_1-s_2(s_2+s_3)(2(-11e^{s_1}-7e^{s_1+s_2}+9e^{2s_1+s_2}+9)s_2^2+((-13e^{s_1}-3e^{s_1+s_2}\\
&\quad+5e^{2s_1+s_2}+11)s_3-12(e^{s_1}-1)(e^{s_1+s_2}-1))s_2-s_3((-9e^{s_1}-11e^{s_1+s_2}+13e^{2s_1+s_2}+7)s_3\\
&\quad-6(e^{s_1}-1)(e^{s_1+s_2}-1)))),
\end{aligned}$$

$$K_{10,12}(s_1,s_2,s_3)$$
$$=\frac{K_{10,11}^{\mathrm{num}}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^3 s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},$$

$$
\begin{aligned}
&K_{10,13}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^2 s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)\\
&\quad=\pi(((-29+9e^{s_1}+52e^{s_1+s_2}-15e^{2(s_1+s_2)}-20e^{2s_1+s_2}+3e^{3s_1+2s_2})s_2\\
&\qquad+2e^{s_1}(-1+e^{s_2})(-3+7e^{s_1+s_2})s_3)s_1^3-(4(1+2e^{s_1})(5-8e^{s_1+s_2}+3e^{2(s_1+s_2)})s_2^2\\
&\qquad+((-9+31e^{s_1}+38e^{s_1+s_2}-45e^{2(s_1+s_2)}-42e^{2s_1+s_2}+27e^{3s_1+2s_2})s_3+22e^{s_1}+68e^{s_1+s_2}\\
&\qquad-30e^{2(s_1+s_2)}-36e^{2s_1+s_2}+14e^{3s_1+2s_2}-38)s_2-4e^{s_1}(-1+e^{s_2})s_3((-3+7e^{s_1+s_2})s_3\\
&\qquad-2e^{s_1+s_2}+2))s_1^2+((47-107e^{s_1}-92e^{s_1+s_2}+21e^{2(s_1+s_2)}+188e^{2s_1+s_2}-57e^{3s_1+2s_2})s_2^3\\
&\qquad-2((-38+69e^{s_1}+77e^{s_1+s_2}-19e^{2(s_1+s_2)}-111e^{2s_1+s_2}+22e^{3s_1+2s_2})s_3-27e^{s_1}-18e^{s_1+s_2}\\
&\qquad+7e^{2(s_1+s_2)}+50e^{2s_1+s_2}-23e^{3s_1+2s_2}+11)s_2^2+s_3(60(-1+e^{s_1})(-1+e^{s_1+s_2})^2+(29-25e^{s_1}\\
&\qquad-68e^{s_1+s_2}+31e^{2(s_1+s_2)}+20e^{2s_1+s_2}+13e^{3s_1+2s_2})s_3)s_2+2e^{s_1}(-1+e^{s_2})s_3^2((-3\\
&\qquad+7e^{s_1+s_2})s_3-4e^{s_1+s_2}+4))s_1-s_2(s_2+s_3)(2(-19+29e^{s_1}+36e^{s_1+s_2}-9e^{2(s_1+s_2)}\\
&\qquad-52e^{2s_1+s_2}+15e^{3s_1+2s_2})s_2^2+((-29+43e^{s_1}+50e^{s_1+s_2}+11e^{2(s_1+s_2)}-62e^{2s_1+s_2}\\
&\qquad-13e^{3s_1+2s_2})s_3-76e^{s_1}-104e^{s_1+s_2}+44e^{2(s_1+s_2)}+136e^{2s_1+s_2}-60e^{3s_1+2s_2}+60)s_2\\
&\qquad-s_3((-9+15e^{s_1}+22e^{s_1+s_2}-29e^{2(s_1+s_2)}-42e^{2s_1+s_2}+43e^{3s_1+2s_2})s_3-30e^{s_1}-60e^{s_1+s_2}\\
&\qquad+38e^{2(s_1+s_2)}+76e^{2s_1+s_2}-46e^{3s_1+2s_2}+22))),\\
&K_{10,14}(s_1,s_2,s_3)\\
&\quad=\frac{-K_{10,13}^{\mathrm{num}}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^2 s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},\\
&K_{10,15}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)^3(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)\\
&\quad=8\pi(2((-1+3e^{s_1+s_2}-3e^{2(s_1+s_2)}+e^{3s_1+2s_2})s_2-e^{3s_1+2s_2}(-1+e^{s_2})s_3)s_1^3\\
&\qquad-(2(1+2e^{s_1})(1-3e^{s_1+s_2}+2e^{2(s_1+s_2)})s_2^2+(2e^{s_1}(2-6e^{s_1+s_2}+2e^{2(s_1+s_2)}-e^{s_1+2s_2}\\
&\qquad+3e^{2s_1+3s_2})s_3+e^{s_1}+14e^{s_1+s_2}-11e^{2(s_1+s_2)}+2e^{3(s_1+s_2)}-4e^{2s_1+s_2}+3e^{3s_1+2s_2}-5)s_2\\
&\qquad+e^{s_1}(-1+e^{s_2})s_3(4e^{2(s_1+s_2)}s_3+4e^{s_1+s_2}-3e^{2(s_1+s_2)}-1))s_1^2-(2(-1+4e^{s_1}+3e^{s_1+s_2}\\
&\qquad-5e^{2(s_1+s_2)}-12e^{2s_1+s_2}+11e^{3s_1+2s_2})s_2^3+(2(-2+6e^{s_1}+6e^{s_1+s_2}-10e^{2(s_1+s_2)}+e^{3(s_1+s_2)}\\
&\qquad-18e^{2s_1+s_2}+15e^{3s_1+2s_2}+2e^{4s_1+3s_2})s_3-9e^{s_1}-4e^{s_1+s_2}+3e^{2(s_1+s_2)}+24e^{2s_1+s_2}\\
&\qquad-19e^{3s_1+2s_2}+4e^{4s_1+3s_2}+1)s_2^2+2s_3((3-4e^{s_1}-4e^{s_1+s_2}+2e^{2s_1+s_2})(-1+e^{s_1+s_2})^2\\
&\qquad+(-1+2e^{s_1}+3e^{s_1+s_2}-5e^{2(s_1+s_2)}+2e^{3(s_1+s_2)}-6e^{2s_1+s_2}+3e^{3s_1+2s_2}+2e^{4s_1+3s_2})s_3)s_2\\
&\qquad+e^{s_1}(-1+e^{s_2})s_3^2(2e^{2(s_1+s_2)}s_3+4e^{s_1+s_2}-3e^{2(s_1+s_2)}-1))s_1-s_2(s_2+s_3)(2(-1+2e^{s_1}\\
&\qquad+3e^{s_1+s_2}-4e^{2(s_1+s_2)}-6e^{2s_1+s_2}+6e^{3s_1+2s_2})s_2^2+2((-1+2e^{s_1}+3e^{s_1+s_2}-5e^{2(s_1+s_2)}\\
&\qquad-e^{3(s_1+s_2)}-6e^{2s_1+s_2}+6e^{3s_1+2s_2}+2e^{4s_1+3s_2})s_3-5e^{s_1}-9e^{s_1+s_2}+7e^{2(s_1+s_2)}-e^{3(s_1+s_2)}\\
&\qquad+14e^{2s_1+s_2}-11e^{3s_1+2s_2}+2e^{4s_1+3s_2}+3)s_2+s_3(2e^{2(s_1+s_2)}(-1-e^{s_1+s_2}+2e^{2s_1+s_2})s_3\\
&\qquad+2e^{s_1}+3e^{s_1+s_2}-7e^{2(s_1+s_2)}+5e^{3(s_1+s_2)}-8e^{2s_1+s_2}+14e^{3s_1+2s_2}-8e^{4s_1+3s_2}-1))),\\
&K_{10,16}(s_1,s_2,s_3)\\
&\quad=\frac{K_{10,15}^{\mathrm{num}}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)^3(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}.
\end{aligned}
$$

In fact, by putting together the above expressions, we can write

$$K_{10}(s_1,s_2,s_3)=\frac{K_{10}^{\mathrm{num}}(s_1,s_2,s_3)}{K_{10}^{\mathrm{den}}(s_1,s_2,s_3)},$$

where

$$
\begin{aligned}
K_{10}^{\mathrm{den}}(s_1,s_2,s_3)&=(e^{s_1}-1)(e^{s_2}-1)^2(e^{s_1+s_2}-1)^3(e^{s_3}-1)(e^{s_2+s_3}-1)^3\\
&\quad\times(e^{s_1+s_2+s_3}-1)^4 s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)
\end{aligned}
$$

and K_{10}^{num} is a polynomial in $s_1, s_2, s_3, e^{s_1/2}, e^{s_2/2}, e^{s_3/2}$. The points (i,j,m) and (n,p,q) such that $s_1^i s_2^j s_3^m e^{ns_1/2} e^{ps_2/2} e^{qs_3/2}$ appears in $K_{10}^{\mathrm{num}}(s_1,s_2,s_3)$ are plotted in Figure 14 and Figure 17.

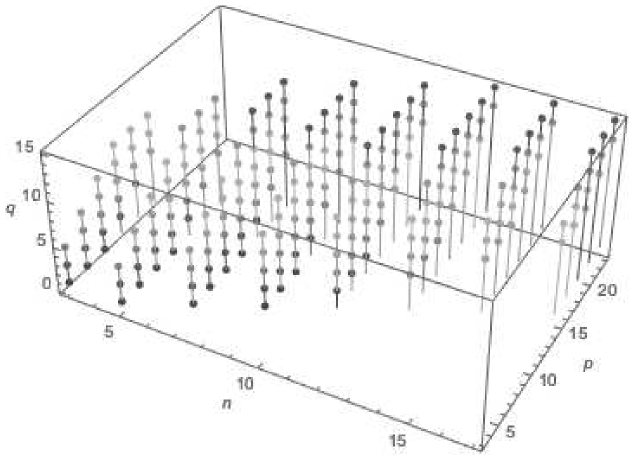

FIGURE 17. The points (n,p,q) such that $s_1^i\, s_2^j\, s_3^m\, e^{ns_1/2}\, e^{ps_2/2}\, e^{qs_3/2}$ appears in the expression for K_{10}^{num}.

The functions K_{11} and K_{16} are scalar multiples of K_{10}:

$$K_{11}(s_1,s_2,s_3) = \frac{1}{2} K_{10}(s_1,s_2,s_3),$$
$$K_{16}(s_1,s_2,s_3) = \frac{3}{2} K_{10}(s_1,s_2,s_3).$$

C.1.4. *The function K_{12}.* We have

$$K_{12}(s_1,s_2,s_3) = \sum_{i=1}^{14} K_{12,i}(s_1,s_2,s_3),$$

where

$$K_{12,1}(s_1,s_2,s_3) = \frac{8\pi e^{\frac{3}{2}(s_1+s_2)}\left(e^{s_2}(2e^{s_1}-1)s_1 + s_1 - e^{s_1+2s_2}(s_1 - s_2 + e^{s_1}(s_1+s_2))\right)}{(e^{s_2}-1)(e^{s_1+s_2}-1)^3(e^{\frac{s_3}{2}}-1)s_1 s_2(s_1+s_2)},$$

$$K_{12,2}(s_1,s_2,s_3) = \frac{8\pi e^{\frac{3}{2}(s_1+s_2)}\left(e^{s_2}(2e^{s_1}-1)s_1 + s_1 - e^{s_1+2s_2}(s_1 - s_2 + e^{s_1}(s_1+s_2))\right)}{(e^{s_2}-1)(e^{s_1+s_2}-1)^3(e^{\frac{s_3}{2}}+1)s_1 s_2(s_1+s_2)},$$

$$K_{12,3}(s_1,s_2,s_3) = -\frac{4\pi e^{\frac{5s_1}{2}}(s_2-s_3)}{(e^{s_1}-1)^2(e^{\frac{1}{2}(s_2+s_3)}-1)^2 s_1 s_2 s_3},$$

$$K_{12,4}(s_1,s_2,s_3) = \frac{4\pi e^{\frac{5s_1}{2}}(s_2-s_3)}{(e^{s_1}-1)^2(e^{\frac{1}{2}(s_2+s_3)}+1)^2 s_1 s_2 s_3},$$

$$K_{12,5}(s_1,s_2,s_3) = -\frac{3\pi(2s_1^2 + (s_2-s_3)s_1 - s_2^2 + s_3^2)}{(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^4 s_1(s_1+s_2)s_3(s_2+s_3)},$$

$$K_{12,6}(s_1,s_2,s_3)=\frac{3\pi(2s_1^2+(s_2-s_3)s_1-s_2^2+s_3^2)}{(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^4 s_1(s_1+s_2)s_3(s_2+s_3)},$$

$$\begin{aligned}
&K_{12,7}(s_1,s_2,s_3)\\
&\quad=\frac{8\pi e^{\frac{3s_1}{2}}(e^{s_1}s_2^2+(e^{s_1}(s_3+e^{s_2}(s_3+1)-1)-(e^{s_2}-1)s_3)s_2+s_3(e^{s_2}(e^{s_1}(s_3-1)-s_3)+e^{s_1}+s_3))}{(e^{s_1}-1)^2(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)s_1s_2s_3(s_2+s_3)},\\
&K_{12,8}(s_1,s_2,s_3)\\
&\quad=\frac{8\pi e^{\frac{3s_1}{2}}(e^{s_1}s_2^2+(e^{s_1}(s_3+e^{s_2}(s_3+1)-1)-(e^{s_2}-1)s_3)s_2+s_3(e^{s_2}(e^{s_1}(s_3-1)-s_3)+e^{s_1}+s_3))}{(e^{s_1}-1)^2(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)s_1s_2s_3(s_2+s_3)},
\end{aligned}$$

$$\begin{aligned}
&K_{12,9}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^3 s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)\\
&\quad=\pi(-2((-7e^{s_1}-7e^{s_1+s_2}+5e^{2s_1+s_2}+9)s_2+2e^{s_1}(e^{s_2}-1)s_3)s_1^3+((13e^{s_1}+19e^{s_1+s_2}\\
&\qquad-7e^{2s_1+s_2}-25)s_2^2+(12(e^{s_1}-1)(e^{s_1+s_2}-1)+(11e^{s_1}-7e^{s_1+s_2}+3e^{2s_1+s_2}-7)s_3)s_2\\
&\qquad-8e^{s_1}(e^{s_2}-1)s_3^2)s_1^2+2(2(4e^{s_1}-1)(e^{s_1+s_2}-1)s_2^3+(3(e^{s_1}-1)(e^{s_1+s_2}-1)+(-6e^{s_1}\\
&\qquad-6e^{s_1+s_2}+8e^{2s_1+s_2}+4)s_3)s_2^2-s_3(3(e^{s_1}-1)(e^{s_1+s_2}-1)+(-4e^{s_1}+6e^{s_1+s_2}-2)s_3)s_2\\
&\qquad-2e^{s_1}(e^{s_2}-1)s_3^3)s_1+s_2(s_2+s_3)((-15e^{s_1}-9e^{s_1+s_2}+13e^{2s_1+s_2}+11)s_2^2-2(e^{s_1}-1)(2s_3\\
&\qquad+3e^{s_1+s_2}-3)s_2-s_3((-11e^{s_1}-9e^{s_1+s_2}+13e^{2s_1+s_2}+7)s_3-6(e^{s_1}-1)(e^{s_1+s_2}-1)))),\\
&K_{12,10}(s_1,s_2,s_3)\\
&\quad=\frac{K_{12,9}^{\mathrm{num}}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^3 s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},\\
&K_{12,11}(s_1,s_2,s_3)2(e^{s_1}-1)^2(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^2 s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)\\
&\quad=\pi(2((-19+28e^{s_1}-5e^{2s_1}+36e^{s_1+s_2}-9e^{2(s_1+s_2)}-56e^{2s_1+s_2}+12e^{3s_1+s_2}+12e^{3s_1+2s_2}\\
&\qquad+e^{4s_1+2s_2})s_2-2e^{s_1}(-1+e^{s_2})(1-3e^{s_1}-5e^{s_1+s_2}+7e^{2s_1+s_2})s_3)s_1^3+((-47+34e^{s_1}+37e^{2s_1}\\
&\qquad+92e^{s_1+s_2}-21e^{2(s_1+s_2)}-88e^{2s_1+s_2}-52e^{3s_1+s_2}+6e^{3s_1+2s_2}+39e^{4s_1+2s_2})s_2^2\\
&\qquad+(4(-1+e^{s_1})(-15+11e^{s_1}+26e^{s_1+s_2}-11e^{2(s_1+s_2)}-18e^{2s_1+s_2}+7e^{3s_1+2s_2})+(-9-10e^{s_1}\\
&\qquad+11e^{2s_1}+8e^{s_1+s_2}+57e^{2(s_1+s_2)}+8e^{3s_1+s_2}-102e^{3s_1+2s_2}+37e^{4s_1+2s_2})s_3)s_2\\
&\qquad-8e^{s_1}(-1+e^{s_2})s_3((1-3e^{s_1}-5e^{s_1+s_2}+7e^{2s_1+s_2})s_3-2(-1+e^{s_1})(-1+e^{s_1+s_2})))s_1^2\\
&\qquad+2(2(5-25e^{s_1}+26e^{2s_1}-8e^{s_1+s_2}+3e^{2(s_1+s_2)}+40e^{2s_1+s_2}-44e^{3s_1+s_2}-15e^{3s_1+2s_2}\\
&\qquad+18e^{4s_1+2s_2})s_2^3+(2(10-37e^{s_1}+29e^{2s_1}-17e^{s_1+s_2}+11e^{2(s_1+s_2)}+54e^{2s_1+s_2}-41e^{3s_1+s_2}\\
&\qquad-25e^{3s_1+2s_2}+16e^{4s_1+2s_2})s_3-(-1+e^{s_1})(11+5e^{s_1}-18e^{s_1+s_2}+7e^{2(s_1+s_2)}-14e^{2s_1+s_2}\\
&\qquad+9e^{3s_1+2s_2}))s_2^2-s_3((-1+e^{s_1})(-19+11e^{s_1}+50e^{s_1+s_2}-31e^{2(s_1+s_2)}-34e^{2s_1+s_2}\\
&\qquad+23e^{3s_1+2s_2})+2(-5+11e^{s_1}+10e^{s_1+s_2}-13e^{2(s_1+s_2)}-12e^{2s_1+s_2}-10e^{3s_1+s_2}+17e^{3s_1+2s_2}\\
&\qquad+2e^{4s_1+2s_2})s_3)s_2-2e^{s_1}(-1+e^{s_2})s_3^2((1-3e^{s_1}-5e^{s_1+s_2}+7e^{2s_1+s_2})s_3\\
&\qquad-4(-1+e^{s_1})(-1+e^{s_1+s_2})))s_1+s_2(s_2+s_3)((29-78e^{s_1}+57e^{2s_1}-52e^{s_1+s_2}+15e^{2(s_1+s_2)}\\
&\qquad+136e^{2s_1+s_2}-100e^{3s_1+s_2}-42e^{3s_1+2s_2}+35e^{4s_1+2s_2})s_2^2-2(-1+e^{s_1})(2(5-9e^{s_1}-7e^{s_1+s_2}\\
&\qquad-2e^{2(s_1+s_2)}+11e^{2s_1+s_2}+2e^{3s_1+2s_2})s_3+27e^{s_1}+34e^{s_1+s_2}-15e^{2(s_1+s_2)}-50e^{2s_1+s_2}\\
&\qquad+23e^{3s_1+2s_2}-19)s_2-s_3((9-22e^{s_1}+21e^{2s_1}-24e^{s_1+s_2}+23e^{2(s_1+s_2)}+64e^{2s_1+s_2}\\
&\qquad-56e^{3s_1+s_2}-58e^{3s_1+2s_2}+43e^{4s_1+2s_2})s_3-2(-1+e^{s_1})(-11+19e^{s_1}+26e^{s_1+s_2}\\
&\qquad-15e^{2(s_1+s_2)}-42e^{2s_1+s_2}+23e^{3s_1+2s_2})))),\\
&K_{12,12}(s_1,s_2,s_3)\\
&\quad=\frac{-K_{12,11}^{\mathrm{num}}(s_1,s_2,s_3)}{2(e^{s_1}-1)^2(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^2 s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},\\
&K_{12,13}(s_1,s_2,s_3)(e^{s_1}-1)^2(e^{s_1+s_2}-1)^3(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)
\end{aligned}$$

$$
\begin{aligned}
&= 4\pi(2((1 - e^{s_1} - e^{2s_1} - 3e^{s_1+s_2} + 4e^{2(s_1+s_2)} + 3e^{2s_1+s_2} + 2e^{3s_1+s_2} - 6e^{3s_1+2s_2} + e^{4s_1+2s_2})s_2 \\
&\quad - e^{2s_1}(-1 + e^{s_2})(-1 - e^{s_2} + 2e^{s_1+s_2} + e^{2(s_1+s_2)} - e^{s_1+2s_2})s_3)s_1^3 - 2((-1 - e^{s_1} + 5e^{2s_1} \\
&\quad + 3e^{s_1+s_2} - 5e^{2(s_1+s_2)} + 3e^{2s_1+s_2} - 12e^{3s_1+s_2} + 4e^{3s_1+2s_2} + 4e^{4s_1+2s_2})s_2^2 + (e^{s_1}(-2 + 7e^{s_1} \\
&\quad + 6e^{s_1+s_2} + 7e^{2(s_1+s_2)} + 3e^{3(s_1+s_2)} - 16e^{2s_1+s_2} - 4e^{s_1+2s_2} + 2e^{3s_1+2s_2} - 3e^{2s_1+3s_2})s_3 \\
&\quad - 4e^{s_1} - 9e^{s_1+s_2} + 7e^{2(s_1+s_2)} - e^{3(s_1+s_2)} + 13e^{2s_1+s_2} - e^{3s_1+s_2} - 10e^{3s_1+2s_2} + e^{4s_1+3s_2} \\
&\quad + e^{5s_1+3s_2} + 3)s_2 + e^{2s_1}(-1 + e^{s_2})s_3(2(-1 - e^{s_2} + 2e^{s_1+s_2} + e^{2(s_1+s_2)} - e^{s_1+2s_2})s_3 - 2e^{s_2} \\
&\quad + 4e^{s_1+s_2} - 3e^{2(s_1+s_2)} + 2e^{s_1+2s_2} - 1))s_1^2 - (2(1 - 5e^{s_1} + 7e^{2s_1} - 3e^{s_1+s_2} + 2e^{2(s_1+s_2)} \\
&\quad + 15e^{2s_1+s_2} - 18e^{3s_1+s_2} - 10e^{3s_1+2s_2} + 11e^{4s_1+2s_2})s_2^3 + (2(2 - 8e^{s_1} + 13e^{2s_1} - 6e^{s_1+s_2} \\
&\quad + 3e^{2(s_1+s_2)} - e^{3(s_1+s_2)} + 24e^{2s_1+s_2} - 32e^{3s_1+s_2} - 11e^{3s_1+2s_2} + 15e^{4s_1+2s_2} - e^{4s_1+3s_2} \\
&\quad + 2e^{5s_1+3s_2})s_3 + 6e^{s_1} - 11e^{2s_1} - 4e^{s_1+s_2} + 3e^{2(s_1+s_2)} - 12e^{2s_1+s_2} + 28e^{3s_1+s_2} + 10e^{3s_1+2s_2} \\
&\quad - 25e^{4s_1+2s_2} - 4e^{4s_1+3s_2} + 8e^{5s_1+3s_2} + 1)s_2^2 + s_3(2(1 - 3e^{s_1} + 7e^{2s_1} - 3e^{s_1+s_2} - 2e^{3(s_1+s_2)} \\
&\quad + 9e^{2s_1+s_2} - 16e^{3s_1+s_2} + 2e^{3s_1+2s_2} + 3e^{4s_1+2s_2} + 2e^{5s_1+3s_2})s_3 + 14e^{s_1} - 7e^{2s_1} + 14e^{s_1+s_2} \\
&\quad - 19e^{2(s_1+s_2)} + 10e^{3(s_1+s_2)} - 34e^{2s_1+s_2} + 14e^{3s_1+s_2} + 38e^{3s_1+2s_2} - 13e^{4s_1+2s_2} - 18e^{4s_1+3s_2} \\
&\quad + 6e^{5s_1+3s_2} - 5)s_2 + 2e^{2s_1}(-1 + e^{s_2})s_3^2((-1 - e^{s_2} + 2e^{s_1+s_2} + e^{2(s_1+s_2)} - e^{s_1+2s_2})s_3 - 2e^{s_2} \\
&\quad + 4e^{s_1+s_2} - 3e^{2(s_1+s_2)} + 2e^{s_1+2s_2} - 1))s_1 - s_2(s_2 + s_3)(2(1 - 3e^{s_1} + 3e^{2s_1} - 3e^{s_1+s_2} \\
&\quad + 3e^{2(s_1+s_2)} + 9e^{2s_1+s_2} - 8e^{3s_1+s_2} - 8e^{3s_1+2s_2} + 6e^{4s_1+2s_2})s_2^2 + (2(1 - 3e^{s_1} + 4e^{2s_1} \\
&\quad - 3e^{s_1+s_2} + 3e^{2(s_1+s_2)} + e^{3(s_1+s_2)} + 9e^{2s_1+s_2} - 10e^{3s_1+s_2} - 7e^{3s_1+2s_2} + 6e^{4s_1+2s_2} \\
&\quad - 3e^{4s_1+3s_2} + 2e^{5s_1+3s_2})s_3 + 14e^{s_1} - 11e^{2s_1} + 14e^{s_1+s_2} - 11e^{2(s_1+s_2)} + 2e^{3(s_1+s_2)} \\
&\quad - 38e^{2s_1+s_2} + 30e^{3s_1+s_2} + 30e^{3s_1+2s_2} - 25e^{4s_1+2s_2} - 6e^{4s_1+3s_2} + 6e^{5s_1+3s_2} - 5)s_2 \\
&\quad + s_3(2e^{2s_1}(1 - 2e^{s_1+s_2} + 2e^{3(s_1+s_2)} + e^{s_1+2s_2} + e^{s_1+3s_2} - 3e^{2s_1+3s_2})s_3 - 2e^{s_1} + 3e^{2s_1} \\
&\quad - 4e^{s_1+s_2} + 7e^{2(s_1+s_2)} - 4e^{3(s_1+s_2)} + 10e^{2s_1+s_2} - 12e^{3s_1+s_2} - 18e^{3s_1+2s_2} + 17e^{4s_1+2s_2} \\
&\quad + 10e^{4s_1+3s_2} - 8e^{5s_1+3s_2} + 1))),
\end{aligned}
$$

$$
\begin{aligned}
&K_{12,14}(s_1, s_2, s_3) \\
&= \frac{K_{12,13}^{\mathrm{num}}(s_1, s_2, s_3)}{(e^{s_1} - 1)^2(e^{s_1+s_2} - 1)^3(e^{\frac{1}{2}(s_1+s_2+s_3)} + 1)s_1 s_2(s_1 + s_2)s_3(s_2 + s_3)(s_1 + s_2 + s_3)}.
\end{aligned}
$$

Using the above expressions, we have

$$K_{12}(s_1, s_2, s_3) = \frac{K_{12}^{\mathrm{num}}(s_1, s_2, s_3)}{K_{12}^{\mathrm{den}}(s_1, s_2, s_3)},$$

where

(74)

$$
\begin{aligned}
K_{12}^{\mathrm{den}}(s_1, s_2, s_3) &= (e^{s_1} - 1)^2(e^{s_2} - 1)(e^{s_1+s_2} - 1)^3(e^{s_3} - 1)(e^{s_2+s_3} - 1)^2 \\
&\quad \times (e^{s_1+s_2+s_3} - 1)^4 s_1 s_2(s_1 + s_2)s_3(s_2 + s_3)(s_1 + s_2 + s_3)
\end{aligned}
$$

and K_{12}^{num} is a polynomial in $s_1, s_2, s_3, e^{s_1/2}, e^{s_2/2}, e^{s_3/2}$. The points (i, j, m) and (n, p, q) such that $s_1^i s_2^j s_3^m e^{ns_1/2} e^{ps_2/2} e^{qs_3/2}$ appears in $K_{12}^{\mathrm{num}}(s_1, s_2, s_3)$ are plotted in Figure 14 and Figure 18.

C.1.5. *The function K_{13}.* We have

$$K_{13}(s_1, s_2, s_3) = \sum_{i=1}^{14} K_{13,i}(s_1, s_2, s_3),$$

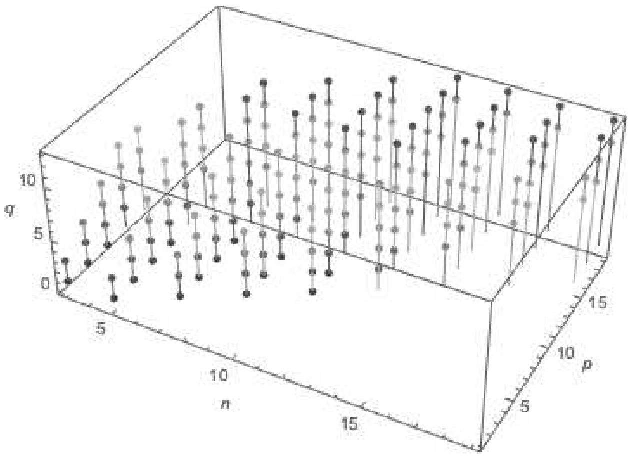

FIGURE 18. The points (n, p, q) such that $s_1^i\, s_2^j\, s_3^m\, e^{ns_1/2}\, e^{ps_2/2}\, e^{qs_3/2}$ appears in the expressions for K_{12}^{num} and K_{14}^{num}.

where

$$K_{13,1}(s_1,s_2,s_3) = -\frac{16\pi e^{\frac{3}{2}(s_1+s_2)}((e^{s_1+s_2}(2e^{s_2}-1)-1)s_1 + e^{s_2}(e^{s_1}-1)s_2)}{(e^{s_2}-1)(e^{s_1+s_2}-1)^3(e^{\frac{s_3}{2}}-1)s_1 s_2(s_1+s_2)},$$

$$K_{13,2}(s_1,s_2,s_3) = -\frac{16\pi e^{\frac{3}{2}(s_1+s_2)}((e^{s_1+s_2}(2e^{s_2}-1)-1)s_1 + e^{s_2}(e^{s_1}-1)s_2)}{(e^{s_2}-1)(e^{s_1+s_2}-1)^3(e^{\frac{s_3}{2}}+1)s_1 s_2(s_1+s_2)},$$

$$K_{13,3}(s_1,s_2,s_3) = -\frac{8\pi e^{\frac{3s_1}{2}}(s_2-s_3)}{(e^{s_1}-1)^2(e^{\frac{1}{2}(s_2+s_3)}-1)^2 s_1 s_2 s_3},$$

$$K_{13,4}(s_1,s_2,s_3) = \frac{8\pi e^{\frac{3s_1}{2}}(s_2-s_3)}{(e^{s_1}-1)^2(e^{\frac{1}{2}(s_2+s_3)}+1)^2 s_1 s_2 s_3},$$

$$K_{13,5}(s_1,s_2,s_3) = -\frac{6\pi(2s_1^2+(s_2-s_3)s_1-s_2^2+s_3^2)}{(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^4 s_1(s_1+s_2)s_3(s_2+s_3)},$$

$$K_{13,6}(s_1,s_2,s_3) = \frac{6\pi(2s_1^2+(s_2-s_3)s_1-s_2^2+s_3^2)}{(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^4 s_1(s_1+s_2)s_3(s_2+s_3)},$$

$$K_{13,7}(s_1,s_2,s_3) = \frac{16\pi e^{\frac{3s_1}{2}}(s_2^2+(e^{s_2}(2s_3+1)-1)s_2+s_3(-s_3+e^{s_2}(2s_3-1)+1))}{(e^{s_1}-1)^2(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)s_1 s_2 s_3(s_2+s_3)},$$

$$K_{13,8}(s_1,s_2,s_3) = \frac{16\pi e^{\frac{3s_1}{2}}(s_2^2+(e^{s_2}(2s_3+1)-1)s_2+s_3(-s_3+e^{s_2}(2s_3-1)+1))}{(e^{s_1}-1)^2(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)s_1 s_2 s_3(s_2+s_3)},$$

$$\begin{aligned}
&K_{13,9}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^3 s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)\\
&\quad = -2\pi(2((-9e^{s_1}-9e^{s_1+s_2}+7e^{2s_1+s_2}+11)s_2+2e^{s_1}(e^{s_2}-1)s_3)s_1^3+((-25e^{s_1}-31e^{s_1+s_2}\\
&\qquad +19e^{2s_1+s_2}+37)s_2^2+((-19e^{s_1}-e^{s_1+s_2}+5e^{2s_1+s_2}+15)s_3-12(e^{s_1}-1)(e^{s_1+s_2}-1))s_2\\
&\qquad +8e^{s_1}(e^{s_2}-1)s_3^2)s_1^2+(-4(e^{s_1}+2)(e^{s_1+s_2}-1)s_2^3+(-6(e^{s_1}-1)(e^{s_1+s_2}-1)-4(e^{s_1}+e^{s_1+s_2}\\
&\qquad -2)s_3)s_2^2+2s_3(3(e^{s_1}-1)(e^{s_1+s_2}-1)+2e^{s_1}(2e^{s_2}+e^{s_1+s_2}-3)s_3)s_2+4e^{s_1}(e^{s_2}-1)s_3^3)s_1
\end{aligned}$$

$$- s_2(s_2+s_3)((-11e^{s_1}-5e^{s_1+s_2}+9e^{2s_1+s_2}+7)s_2^2-2(e^{s_1}-1)(2e^{s_1+s_2}s_3+3e^{s_1+s_2}-3)s_2$$
$$- s_3((-11e^{s_1}-9e^{s_1+s_2}+13e^{2s_1+s_2}+7)s_3-6(e^{s_1}-1)(e^{s_1+s_2}-1)))),$$

$$K_{13,10}(s_1,s_2,s_3)$$
$$=\frac{K^{\rm num}_{13,9}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^3s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},$$

$$K_{13,11}(s_1,s_2,s_3)(e^{s_1}-1)^2(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^2s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)$$
$$=-\pi(2((29-52e^{s_1}+19e^{2s_1}-52e^{s_1+s_2}+15e^{2(s_1+s_2)}+96e^{2s_1+s_2}-36e^{3s_1+s_2}-28e^{3s_1+2s_2}$$
$$+9e^{4s_1+2s_2})s_2+2e^{s_1}(-1+e^{s_2})(3-5e^{s_1}-7e^{s_1+s_2}+9e^{2s_1+s_2})s_3)s_1^3+((107-178e^{s_1}$$
$$+47e^{2s_1}-188e^{s_1+s_2}+57e^{2(s_1+s_2)}+328e^{2s_1+s_2}-92e^{3s_1+s_2}-102e^{3s_1+2s_2}+21e^{4s_1+2s_2})s_2^2$$
$$+((49-110e^{s_1}+69e^{2s_1}-48e^{s_1+s_2}-57e^{2(s_1+s_2)}+160e^{2s_1+s_2}-128e^{3s_1+s_2}+62e^{3s_1+2s_2}$$
$$+3e^{4s_1+2s_2})s_3-4(-1+e^{s_1})(-19+15e^{s_1}+34e^{s_1+s_2}-15e^{2(s_1+s_2)}-26e^{2s_1+s_2}$$
$$+11e^{3s_1+2s_2}))s_2+8e^{s_1}(-1+e^{s_2})s_3((3-5e^{s_1}-7e^{s_1+s_2}+9e^{2s_1+s_2})s_3-2(-1+e^{s_1})(-1$$
$$+e^{s_1+s_2})))s_1^2-2(2(-10+11e^{s_1}+5e^{2s_1}+16e^{s_1+s_2}-6e^{2(s_1+s_2)}-20e^{2s_1+s_2}-8e^{3s_1+s_2}$$
$$+9e^{3s_1+2s_2}+3e^{4s_1+2s_2})s_2^3+((-1+e^{s_1})(-27+11e^{s_1}+50e^{s_1+s_2}-23e^{2(s_1+s_2)}-18e^{2s_1+s_2}$$
$$+7e^{3s_1+2s_2})-2(10-17e^{s_1}+5e^{2s_1}-9e^{s_1+s_2}-5e^{2(s_1+s_2)}+26e^{2s_1+s_2}-13e^{3s_1+s_2}$$
$$-e^{3s_1+2s_2}+4e^{4s_1+2s_2})s_3)s_2^2-s_3((-1+e^{s_1})(-11+3e^{s_1}+34e^{s_1+s_2}-23e^{2(s_1+s_2)}$$
$$-18e^{2s_1+s_2}+15e^{3s_1+2s_2})+2e^{s_1}(-9+15e^{s_1}+10e^{s_2}+8e^{s_1+s_2}+17e^{2(s_1+s_2)}-30e^{2s_1+s_2}$$
$$-18e^{s_1+2s_2}+7e^{3s_1+2s_2})s_3)s_2-2e^{s_1}(-1+e^{s_2})s_3^2((3-5e^{s_1}-7e^{s_1+s_2}+9e^{2s_1+s_2})s_3$$
$$-4(-1+e^{s_1})(-1+e^{s_1+s_2})))s_1-s_2(s_2+s_3)((9-30e^{s_1}+29e^{2s_1}-20e^{s_1+s_2}+3e^{2(s_1+s_2)}$$
$$+56e^{2s_1+s_2}-52e^{3s_1+s_2}-10e^{3s_1+2s_2}+15e^{4s_1+2s_2})s_2^2-2(-1+e^{s_1})(2e^{s_1+s_2}(-1-3e^{s_1}$$
$$-3e^{s_1+s_2}+7e^{2s_1+s_2})s_3+19e^{s_1}+18e^{s_1+s_2}-7e^{2(s_1+s_2)}-34e^{2s_1+s_2}+15e^{3s_1+2s_2}-11)s_2$$
$$-s_3((9-30e^{s_1}+29e^{2s_1}-16e^{s_1+s_2}+15e^{2(s_1+s_2)}+64e^{2s_1+s_2}-64e^{3s_1+s_2}-50e^{3s_1+2s_2}$$
$$+43e^{4s_1+2s_2})s_3-2(-1+e^{s_1})(-11+19e^{s_1}+26e^{s_1+s_2}-15e^{2(s_1+s_2)}-42e^{2s_1+s_2}$$
$$+23e^{3s_1+2s_2})))),$$

$$K_{13,12}(s_1,s_2,s_3)$$
$$=\frac{-K^{\rm num}_{13,11}(s_1,s_2,s_3)}{(e^{s_1}-1)^2(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^2s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},$$

$$K_{13,13}(s_1,s_2,s_3)(e^{s_1}-1)^2(e^{s_1+s_2}-1)^3(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)$$
$$=8\pi(2((2-4e^{s_1}+e^{2s_1}-6e^{s_1+s_2}+6e^{2(s_1+s_2)}+11e^{2s_1+s_2}-3e^{3s_1+s_2}-11e^{3s_1+2s_2}$$
$$+4e^{4s_1+2s_2})s_2-e^{2s_1}(-1+e^{s_2})(1-3e^{s_1+s_2}+4e^{2(s_1+s_2)}-2e^{s_1+2s_2})s_3)s_1^3+2((4-8e^{s_1}$$
$$+e^{2s_1}-12e^{s_1+s_2}+11e^{2(s_1+s_2)}+21e^{2s_1+s_2}-3e^{3s_1+s_2}-19e^{3s_1+2s_2}+5e^{4s_1+2s_2})s_2^2$$
$$+((2-4e^{s_1}+3e^{2s_1}-6e^{s_1+s_2}+5e^{2(s_1+s_2)}+6e^{3(s_1+s_2)}+7e^{2s_1+s_2}-9e^{3s_1+s_2}-5e^{3s_1+2s_2}$$
$$+13e^{4s_1+2s_2}-12e^{4s_1+3s_2})s_3+9e^{s_1}-3e^{2s_1}+14e^{s_1+s_2}-11e^{2(s_1+s_2)}+2e^{3(s_1+s_2)}$$
$$-26e^{2s_1+s_2}+9e^{3s_1+s_2}+21e^{3s_1+2s_2}-7e^{4s_1+2s_2}-4e^{4s_1+3s_2}+e^{5s_1+3s_2}-5)s_2$$
$$-e^{s_1}(-1+e^{s_2})s_3(2e^{s_1}(1-3e^{s_1+s_2}+4e^{2(s_1+s_2)}-2e^{s_1+2s_2})s_3-2e^{s_1}-4e^{s_1+s_2}+3e^{2(s_1+s_2)}$$
$$+6e^{2s_1+s_2}-4e^{3s_1+2s_2}+1))s_1^2-(2(-2+4e^{s_1}+e^{2s_1}+6e^{s_1+s_2}-4e^{2(s_1+s_2)}-9e^{2s_1+s_2}$$
$$-3e^{3s_1+s_2}+5e^{3s_1+2s_2}+2e^{4s_1+2s_2})s_2^3+(2(-2+4e^{s_1}-e^{2s_1}+6e^{s_1+s_2}-2e^{2(s_1+s_2)}$$
$$-4e^{3(s_1+s_2)}-5e^{2s_1+s_2}+3e^{3s_1+s_2}-3e^{3s_1+2s_2}-6e^{4s_1+2s_2}+8e^{4s_1+3s_2}+2e^{5s_1+3s_2})s_3$$
$$-14e^{s_1}+e^{2s_1}-24e^{s_1+s_2}+19e^{2(s_1+s_2)}-4e^{3(s_1+s_2)}+40e^{2s_1+s_2}-4e^{3s_1+s_2}-34e^{3s_1+2s_2}$$
$$+3e^{4s_1+2s_2}+8e^{4s_1+3s_2}+9)s_2^2+s_3(2e^{2s_1}(-3+5e^{s_2}+2e^{2s_2}+9e^{s_1+s_2}-12e^{2(s_1+s_2)}$$
$$+2e^{3(s_1+s_2)}-9e^{s_1+2s_2}-6e^{s_1+3s_2}+12e^{2s_1+3s_2})s_3+3e^{2s_1}+8e^{s_1+s_2}-19e^{2(s_1+s_2)}$$

$$
\begin{aligned}
&+ 12e^{3(s_1+s_2)} - 4e^{2s_1+s_2} - 10e^{3s_1+s_2} + 20e^{3s_1+2s_2} + 5e^{4s_1+2s_2} - 16e^{4s_1+3s_2} \\
&+ 2e^{5s_1+3s_2} - 1)s_2 + 2e^{s_1}(-1 + e^{s_2})s_3^2(e^{s_1}(1 - 3e^{s_1+s_2} + 4e^{2(s_1+s_2)} - 2e^{s_1+2s_2})s_3 - 2e^{s_1} \\
&- 4e^{s_1+s_2} + 3e^{2(s_1+s_2)} + 6e^{2s_1+s_2} - 4e^{3s_1+2s_2} + 1))s_1 - s_2(s_2 + s_3)(2e^{2s_1}(1 + e^{s_2} + e^{2s_2} \\
&- 3e^{s_1+s_2} + 3e^{2(s_1+s_2)} - 3e^{s_1+2s_2})s_2^2 + (4e^{2s_1+s_2}(1 + e^{s_2} - 3e^{s_1+s_2} + e^{3s_1+2s_2})s_3 + 4e^{s_1} \\
&- 5e^{2s_1} + 4e^{s_1+s_2} - 3e^{2(s_1+s_2)} - 12e^{2s_1+s_2} + 14e^{3s_1+s_2} + 8e^{3s_1+2s_2} - 11e^{4s_1+2s_2} \\
&+ 2e^{5s_1+3s_2} - 1)s_2 + s_3(2e^{2s_1}(-1 + e^{s_2} + e^{2s_2} + 3e^{s_1+s_2} - 3e^{2(s_1+s_2)} + 2e^{3(s_1+s_2)} \\
&- 3e^{s_1+2s_2})s_3 - 4e^{s_1} + 5e^{2s_1} - 2e^{s_1+s_2} + 3e^{2(s_1+s_2)} - 2e^{3(s_1+s_2)} + 12e^{2s_1+s_2} - 16e^{3s_1+s_2} \\
&- 16e^{3s_1+2s_2} + 19e^{4s_1+2s_2} + 8e^{4s_1+3s_2} - 8e^{5s_1+3s_2} + 1))),
\end{aligned}
$$

$$
\begin{aligned}
&K_{13,14}(s_1, s_2, s_3) \\
&= \frac{K_{13,13}^{\mathrm{num}}(s_1, s_2, s_3)}{(e^{s_1} - 1)^2(e^{s_1+s_2} - 1)^3(e^{\frac{1}{2}(s_1+s_2+s_3)} + 1)s_1 s_2(s_1 + s_2)s_3(s_2 + s_3)(s_1 + s_2 + s_3)}.
\end{aligned}
$$

Using the above expressions, we have

$$K_{13}(s_1, s_2, s_3) = \frac{K_{13}^{\mathrm{num}}(s_1, s_2, s_3)}{K_{13}^{\mathrm{den}}(s_1, s_2, s_3)},$$

where

$$K_{13}^{\mathrm{den}}(s_1, s_2, s_3) = K_{12}^{\mathrm{den}}(s_1, s_2, s_3),$$

which is given by (74), and K_{13}^{num} is a polynomial in $s_1, s_2, s_3, e^{s_1/2}, e^{s_2/2}, e^{s_3/2}$. The points (i, j, m) and (n, p, q) such that $s_1^i s_2^j s_3^m e^{ns_1/2} e^{ps_2/2} e^{qs_3/2}$ appears in $K_{13}^{\mathrm{num}}(s_1, s_2, s_3)$ are plotted in Figure 14 and Figure 19.

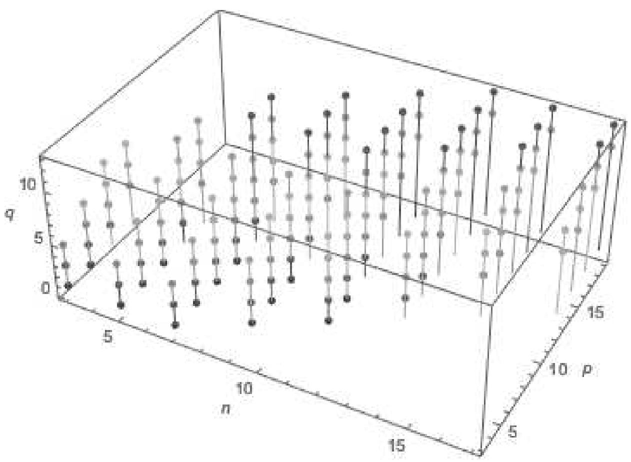

FIGURE 19. The points (n, p, q) such that $s_1^i\, s_2^j\, s_3^m\, e^{ns_1/2}\, e^{ps_2/2}\, e^{qs_3/2}$ appears in the expression for K_{13}^{num}.

C.1.6. *The function K_{14}.* We have

$$K_{14}(s_1,s_2,s_3)=\sum_{i=1}^{14}K_{14,i}(s_1,s_2,s_3),$$

where

$$K_{14,1}(s_1,s_2,s_3)=\frac{8\pi e^{\frac{3}{2}(s_1+s_2)}(-(e^{s_2}(e^{s_1}(e^{s_2}(e^{s_1}+5)-4)+1)-3)s_1-e^{s_2}(e^{s_1}-1)(e^{s_1+s_2}+2)s_2)}{(e^{s_2}-1)(e^{s_1+s_2}-1)^3(e^{\frac{s_3}{2}}-1)s_1s_2(s_1+s_2)},$$

$$K_{14,2}(s_1,s_2,s_3)=\frac{8\pi e^{\frac{3}{2}(s_1+s_2)}(-(e^{s_2}(e^{s_1}(e^{s_2}(e^{s_1}+5)-4)+1)-3)s_1-e^{s_2}(e^{s_1}-1)(e^{s_1+s_2}+2)s_2)}{(e^{s_2}-1)(e^{s_1+s_2}-1)^3(e^{\frac{s_3}{2}}+1)s_1s_2(s_1+s_2)},$$

$$K_{14,3}(s_1,s_2,s_3)=-\frac{4\pi e^{\frac{3s_1}{2}}(e^{s_1}+2)(s_2-s_3)}{(e^{s_1}-1)^2(e^{\frac{1}{2}(s_2+s_3)}-1)^2s_1s_2s_3},$$

$$K_{14,4}(s_1,s_2,s_3)=\frac{4\pi e^{\frac{3s_1}{2}}(e^{s_1}+2)(s_2-s_3)}{(e^{s_1}-1)^2(e^{\frac{1}{2}(s_2+s_3)}+1)^2s_1s_2s_3},$$

$$K_{14,5}(s_1,s_2,s_3)=-\frac{9\pi(2s_1^2+(s_2-s_3)s_1-s_2^2+s_3^2)}{(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^4s_1(s_1+s_2)s_3(s_2+s_3)},$$

$$K_{14,6}(s_1,s_2,s_3)=\frac{9\pi(2s_1^2+(s_2-s_3)s_1-s_2^2+s_3^2)}{(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^4s_1(s_1+s_2)s_3(s_2+s_3)},$$

$$\begin{aligned}
&K_{14,7}(s_1,s_2,s_3)(e^{s_1}-1)^2(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)s_1s_2s_3(s_2+s_3)\\
&\quad=8\pi e^{\frac{3s_1}{2}}((e^{s_1}+2)s_2^2+((e^{s_1}+2)(e^{s_2}-1)+(e^{s_1}+3e^{s_2}+e^{s_1+s_2}+1)s_3)s_2\\
&\qquad+s_3((3e^{s_2}+e^{s_1+s_2}-1)s_3-(e^{s_1}+2)(e^{s_2}-1))),
\end{aligned}$$

$$K_{14,8}(s_1,s_2,s_3)=\frac{K_{14,7}^{\mathrm{num}}(s_1,s_2,s_3)}{(e^{s_1}-1)^2(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)s_1s_2s_3(s_2+s_3)},$$

$$\begin{aligned}
&K_{14,9}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^3s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)\\
&\quad=-\pi(2((-25e^{s_1}-25e^{s_1+s_2}+19e^{2s_1+s_2}+31)s_2+6e^{s_1}(e^{s_2}-1)s_3)s_1^3+(9(-7e^{s_1}-9e^{s_1+s_2}\\
&\qquad+5e^{2s_1+s_2}+11)s_2^2+((-49e^{s_1}+5e^{s_1+s_2}+7e^{2s_1+s_2}+37)s_3-36(e^{s_1}-1)(e^{s_1+s_2}-1))s_2\\
&\qquad+24e^{s_1}(e^{s_2}-1)s_3^2)s_1^2-2(6(2e^{s_1}+1)(e^{s_1+s_2}-1)s_2^3+(9(e^{s_1}-1)(e^{s_1+s_2}-1)+(-2e^{s_1}\\
&\qquad-2e^{s_1+s_2}+8e^{2s_1+s_2}-4)s_3)s_2^2-s_3(9(e^{s_1}-1)(e^{s_1+s_2}-1)+2(-8e^{s_1}+7e^{s_1+s_2}\\
&\qquad+2e^{2s_1+s_2}-1)s_3)s_2-6e^{s_1}(e^{s_2}-1)s_3^3)s_1-s_2(s_2+s_3)((-37e^{s_1}-19e^{s_1+s_2}\\
&\qquad+31e^{2s_1+s_2}+25)s_2^2-2(e^{s_1}-1)(9(e^{s_1+s_2}-1)+(4e^{s_1+s_2}+2)s_3)s_2-3s_3((-11e^{s_1}-9e^{s_1+s_2}\\
&\qquad+13e^{2s_1+s_2}+7)s_3-6(e^{s_1}-1)(e^{s_1+s_2}-1)))),
\end{aligned}$$

$$K_{14,10}(s_1,s_2,s_3)=\frac{K_{14,9}^{\mathrm{num}}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^3s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},$$

$$\begin{aligned}
&K_{14,11}(s_1,s_2,s_3)2(e^{s_1}-1)^2(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^2s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)\\
&\quad=-\pi(2((77-132e^{s_1}+43e^{2s_1}-140e^{s_1+s_2}+39e^{2(s_1+s_2)}+248e^{2s_1+s_2}-84e^{3s_1+s_2}-68e^{3s_1+2s_2}\\
&\qquad+17e^{4s_1+2s_2})s_2+2e^{s_1}(-1+e^{s_2})(7-13e^{s_1}-19e^{s_1+s_2}+25e^{2s_1+s_2})s_3)s_1^3+(3(87-130e^{s_1}\\
&\qquad+19e^{2s_1}-156e^{s_1+s_2}+45e^{2(s_1+s_2)}+248e^{2s_1+s_2}-44e^{3s_1+s_2}-70e^{3s_1+2s_2}+e^{4s_1+2s_2})s_2^2\\
&\qquad-(4(-1+e^{s_1})(-53+41e^{s_1}+94e^{s_1+s_2}-41e^{2(s_1+s_2)}-70e^{2s_1+s_2}+29e^{3s_1+2s_2})\\
&\qquad+(-107+210e^{s_1}-127e^{2s_1}+104e^{s_1+s_2}+171e^{2(s_1+s_2)}-320e^{2s_1+s_2}+264e^{3s_1+s_2}\\
&\qquad-226e^{3s_1+2s_2}+31e^{4s_1+2s_2})s_3)s_2+8e^{s_1}(-1+e^{s_2})s_3((7-13e^{s_1}-19e^{s_1+s_2}+25e^{2s_1+s_2})s_3\\
&\qquad-6(-1+e^{s_1})(-1+e^{s_1+s_2})))s_1^2-2(6(-5-e^{s_1}+12e^{2s_1}+8e^{s_1+s_2}-3e^{2(s_1+s_2)}-20e^{3s_1+s_2}\\
&\qquad+e^{3s_1+2s_2}+8e^{4s_1+2s_2})s_2^3+((-1+e^{s_1})(-65+17e^{s_1}+118e^{s_1+s_2}-53e^{2(s_1+s_2)}-22e^{2s_1+s_2}
\end{aligned}$$

$$+ 5e^{3s_1+2s_2}) + 2(-10 - 3e^{s_1} + 19e^{2s_1} + e^{s_1+s_2} + 21e^{2(s_1+s_2)} + 2e^{2s_1+s_2} - 15e^{3s_1+s_2}$$
$$- 23e^{3s_1+2s_2} + 8e^{4s_1+2s_2})s_3)s_2^2 - s_3((-1 + e^{s_1})(-41 + 17e^{s_1} + 118e^{s_1+s_2} - 77e^{2(s_1+s_2)}$$
$$- 70e^{2s_1+s_2} + 53e^{3s_1+2s_2}) + 2(-5 - 7e^{s_1} + 30e^{2s_1} + 30e^{s_1+s_2} - 49e^{2(s_1+s_2)} + 4e^{2s_1+s_2}$$
$$- 70e^{3s_1+s_2} + 51e^{3s_1+2s_2} + 16e^{4s_1+2s_2})s_3)s_2 - 2e^{s_1}(-1 + e^{s_2})s_3^2((7 - 13e^{s_1} - 19e^{s_1+s_2}$$
$$+ 25e^{2s_1+s_2})s_3 - 12(-1 + e^{s_1})(-1 + e^{s_1+s_2})))s_1 - s_2(s_2 + s_3)((47 - 138e^{s_1} + 115e^{2s_1}$$
$$- 92e^{s_1+s_2} + 21e^{2(s_1+s_2)} + 248e^{2s_1+s_2} - 204e^{3s_1+s_2} - 62e^{3s_1+2s_2} + 65e^{4s_1+2s_2})s_2^2$$
$$- 2(-1 + e^{s_1})(2(5 - 9e^{s_1} - 9e^{s_1+s_2} - 8e^{2(s_1+s_2)} + 5e^{2s_1+s_2} + 16e^{3s_1+2s_2})s_3 + 65e^{s_1}$$
$$+ 70e^{s_1+s_2} - 29e^{2(s_1+s_2)} - 118e^{2s_1+s_2} + 53e^{3s_1+2s_2} - 41)s_2 - s_3((27 - 82e^{s_1} + 79e^{2s_1}$$
$$- 56e^{s_1+s_2} + 53e^{2(s_1+s_2)} + 192e^{2s_1+s_2} - 184e^{3s_1+s_2} - 158e^{3s_1+2s_2} + 129e^{4s_1+2s_2})s_3$$
$$- 6(-1 + e^{s_1})(-11 + 19e^{s_1} + 26e^{s_1+s_2} - 15e^{2(s_1+s_2)} - 42e^{2s_1+s_2} + 23e^{3s_1+2s_2})))),$$

$$K_{14,12}(s_1, s_2, s_3) = \frac{-K_{14,11}^{\mathrm{num}}(s_1,s_2,s_3)}{2(e^{s_1}-1)^2(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^2 s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},$$

$$K_{14,13}(s_1, s_2, s_3)(e^{s_1} - 1)^2(e^{s_1+s_2} - 1)^3(e^{\frac{1}{2}(s_1+s_2+s_3)} - 1)s_1 s_2(s_1 + s_2)s_3(s_2 + s_3)(s_1 + s_2 + s_3)$$
$$= 4\pi(2((5 - 9e^{s_1} + e^{2s_1} - 15e^{s_1+s_2} + 16e^{2(s_1+s_2)} + 25e^{2s_1+s_2} - 4e^{3s_1+s_2} - 28e^{3s_1+2s_2}$$
$$+ 9e^{4s_1+2s_2})s_2 - e^{2s_1}(-1 + e^{s_2})(1 - e^{s_2} - 4e^{s_1+s_2} + 9e^{2(s_1+s_2)} - 5e^{s_1+2s_2})s_3)s_1^3$$
$$+ 2(3(3 - 5e^{s_1} - e^{2s_1} - 9e^{s_1+s_2} + 9e^{2(s_1+s_2)} + 13e^{2s_1+s_2} + 2e^{3s_1+s_2} - 14e^{3s_1+2s_2}$$
$$+ 2e^{4s_1+2s_2})s_2^2 + (-(-4 + 6e^{s_1} + e^{2s_1} + 12e^{s_1+s_2} - 14e^{2(s_1+s_2)} - 15e^{3(s_1+s_2)} - 8e^{2s_1+s_2}$$
$$+ 2e^{3s_1+s_2} + 17e^{3s_1+2s_2} - 24e^{4s_1+2s_2} + 27e^{4s_1+3s_2})s_3 + 22e^{s_1} - 6e^{2s_1} + 37e^{s_1+s_2}$$
$$- 29e^{2(s_1+s_2)} + 5e^{3(s_1+s_2)} - 65e^{2s_1+s_2} + 19e^{3s_1+s_2} + 52e^{3s_1+2s_2} - 14e^{4s_1+2s_2} - 9e^{4s_1+3s_2}$$
$$+ e^{5s_1+3s_2} - 13)s_2 - e^{s_1}(-1 + e^{s_2})s_3(2e^{s_1}(1 - e^{s_2} - 4e^{s_1+s_2} + 9e^{2(s_1+s_2)} - 5e^{s_1+2s_2})s_3$$
$$- 5e^{s_1} - 10e^{s_1+s_2} + 8e^{2(s_1+s_2)} + 16e^{2s_1+s_2} - 11e^{3s_1+2s_2} + 2))s_1^2 - (6(-1 + e^{s_1} + 3e^{2s_1}$$
$$+ 3e^{s_1+s_2} - 2e^{2(s_1+s_2)} - e^{2s_1+s_2} - 8e^{3s_1+s_2} + 5e^{4s_1+2s_2})s_2^3 + (2(-2 + 11e^{2s_1} + 6e^{s_1+s_2}$$
$$- e^{2(s_1+s_2)} - 9e^{3(s_1+s_2)} + 14e^{2s_1+s_2} - 26e^{3s_1+s_2} - 17e^{3s_1+2s_2} + 3e^{4s_1+2s_2} + 15e^{4s_1+3s_2}$$
$$+ 6e^{5s_1+3s_2})s_3 - 22e^{s_1} - 9e^{2s_1} - 52e^{s_1+s_2} + 41e^{2(s_1+s_2)} - 8e^{3(s_1+s_2)} + 68e^{2s_1+s_2}$$
$$+ 20e^{3s_1+s_2} - 58e^{3s_1+2s_2} - 19e^{4s_1+2s_2} + 12e^{4s_1+3s_2} + 8e^{5s_1+3s_2} + 19)s_2^2 + s_3(2(1 - 3e^{s_1}$$
$$+ e^{2s_1} - 3e^{s_1+s_2} + 4e^{2(s_1+s_2)} - 14e^{3(s_1+s_2)} + 19e^{2s_1+s_2} + 2e^{3s_1+s_2} - 16e^{3s_1+2s_2}$$
$$- 21e^{4s_1+2s_2} + 24e^{4s_1+3s_2} + 6e^{5s_1+3s_2})s_3 + 14e^{s_1} - e^{2s_1} + 30e^{s_1+s_2} - 57e^{2(s_1+s_2)}$$
$$+ 34e^{3(s_1+s_2)} - 42e^{2s_1+s_2} - 6e^{3s_1+s_2} + 78e^{3s_1+2s_2} - 3e^{4s_1+2s_2} - 50e^{4s_1+3s_2}$$
$$+ 10e^{5s_1+3s_2} - 7)s_2 + 2e^{s_1}(-1 + e^{s_2})s_3^2(e^{s_1}(1 - e^{s_2} - 4e^{s_1+s_2} + 9e^{2(s_1+s_2)} - 5e^{s_1+2s_2})s_3$$
$$- 5e^{s_1} - 10e^{s_1+s_2} + 8e^{2(s_1+s_2)} + 16e^{2s_1+s_2} - 11e^{3s_1+2s_2} + 2))s_1 - s_2(s_2 + s_3)(2(1 - 3e^{s_1}$$
$$+ 5e^{2s_1} - 3e^{s_1+s_2} + 5e^{2(s_1+s_2)} + 11e^{2s_1+s_2} - 14e^{3s_1+s_2} - 14e^{3s_1+2s_2} + 12e^{4s_1+2s_2})s_2^2$$
$$+ (2(1 - 3e^{s_1} + 4e^{2s_1} - 3e^{s_1+s_2} + 7e^{2(s_1+s_2)} + e^{3(s_1+s_2)} + 13e^{2s_1+s_2} - 10e^{3s_1+s_2}$$
$$- 19e^{3s_1+2s_2} + 6e^{4s_1+2s_2} - 3e^{4s_1+3s_2} + 6e^{5s_1+3s_2})s_3 + 22e^{s_1} - 21e^{2s_1} + 22e^{s_1+s_2}$$
$$- 17e^{2(s_1+s_2)} + 2e^{3(s_1+s_2)} - 62e^{2s_1+s_2} + 58e^{3s_1+s_2} + 46e^{3s_1+2s_2} - 47e^{4s_1+2s_2} - 6e^{4s_1+3s_2}$$
$$+ 10e^{5s_1+3s_2} - 7)s_2 + s_3(2e^{2s_1}(-1 + 2e^{s_2} + 2e^{2s_2} + 4e^{s_1+s_2} - 6e^{2(s_1+s_2)} + 6e^{3(s_1+s_2)}$$
$$- 5e^{s_1+2s_2} + e^{s_1+3s_2} - 3e^{2s_1+3s_2})s_3 - 10e^{s_1} + 13e^{2s_1} - 8e^{s_1+s_2} + 13e^{2(s_1+s_2)} - 8e^{3(s_1+s_2)}$$
$$+ 34e^{2s_1+s_2} - 44e^{3s_1+s_2} - 50e^{3s_1+2s_2} + 55e^{4s_1+2s_2} + 26e^{4s_1+3s_2} - 24e^{5s_1+3s_2} + 3))),$$

$$K_{14,14}(s_1, s_2, s_3) = \frac{K_{14,13}^{\mathrm{num}}(s_1,s_2,s_3)}{(e^{s_1}-1)^2(e^{s_1+s_2}-1)^3(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}.$$

By putting together the above expressions, we have

$$K_{14}(s_1,s_2,s_3)=\frac{K_{14}^{\mathrm{num}}(s_1,s_2,s_3)}{K_{14}^{\mathrm{den}}(s_1,s_2,s_3)},$$

where

$$K_{14}^{\mathrm{den}}(s_1,s_2,s_3)=K_{12}^{\mathrm{den}}(s_1,s_2,s_3),$$

which is given by (74), and K_{14}^{num} is a polynomial in $s_1,s_2,s_3,e^{s_1/2},e^{s_2/2},e^{s_3/2}$. The points (i,j,m) and (n,p,q) such that $s_1^i s_2^j s_3^m e^{ns_1/2}e^{ps_2/2}e^{qs_3/2}$ appears in the expression of $K_{14}^{\mathrm{num}}(s_1,s_2,s_3)$ are plotted in Figure 14 and Figure 18.

C.1.7. *The function K_{15}.* We have

$$K_{15}(s_1,s_2,s_3)=\sum_{i=1}^{18}K_{15,i}(s_1,s_2,s_3),$$

where

$$K_{15,1}(s_1,s_2,s_3)=-\frac{4\pi e^{\frac{3}{2}(s_1+s_2)}((e^{s_2}(e^{s_1}(2e^{s_2}+1)+1)-4)s_1+3e^{s_2}(e^{s_1}-1)s_2)}{(e^{s_2}-1)(e^{s_1+s_2}-1)^2(e^{\frac{s_3}{2}}-1)^2s_1s_2(s_1+s_2)},$$

$$K_{15,2}(s_1,s_2,s_3)=\frac{4\pi e^{\frac{3}{2}(s_1+s_2)}((e^{s_2}(e^{s_1}(2e^{s_2}+1)+1)-4)s_1+3e^{s_2}(e^{s_1}-1)s_2)}{(e^{s_2}-1)(e^{s_1+s_2}-1)^2(e^{\frac{s_3}{2}}+1)^2s_1s_2(s_1+s_2)},$$

$$K_{15,3}(s_1,s_2,s_3)=-\frac{6\pi e^{\frac{3s_1}{2}}(s_2-2s_3)}{(e^{s_1}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)^3s_1s_2s_3},$$

$$K_{15,4}(s_1,s_2,s_3)=-\frac{6\pi e^{\frac{3s_1}{2}}(s_2-2s_3)}{(e^{s_1}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)^3s_1s_2s_3},$$

$$K_{15,5}(s_1,s_2,s_3)=-\frac{9\pi(s_1^2-s_3s_1-(s_2-2s_3)(s_2+s_3))}{(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^4s_1(s_1+s_2)s_3(s_2+s_3)},$$

$$K_{15,6}(s_1,s_2,s_3)=\frac{9\pi(s_1^2-s_3s_1-(s_2-2s_3)(s_2+s_3))}{(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^4s_1(s_1+s_2)s_3(s_2+s_3)},$$

$$K_{15,7}(s_1,s_2,s_3)=\frac{4\pi e^{\frac{3s_1}{2}}(4s_2^2+((6e^{s_2}+2)s_3+e^{s_2}-7)s_2+2s_3(-s_3+e^{s_2}(3s_3-7)+4))}{(e^{s_1}-1)(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)s_1s_2s_3(s_2+s_3)},$$

$$K_{15,8}(s_1,s_2,s_3)=\frac{4\pi e^{\frac{3s_1}{2}}(4s_2^2+((6e^{s_2}+2)s_3+e^{s_2}-7)s_2+2s_3(-s_3+e^{s_2}(3s_3-7)+4))}{(e^{s_1}-1)(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)s_1s_2s_3(s_2+s_3)},$$

$$K_{15,9}(s_1,s_2,s_3)=\frac{\pi e^{\frac{3s_1}{2}}((17-5e^{s_2})s_2^2+(-5s_3+e^{s_2}(29s_3+12)-12)s_2+2s_3(-11s_3+e^{s_2}(17s_3-12)+12))}{(e^{s_1}-1)(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)^2s_1s_2s_3(s_2+s_3)},$$

$$K_{15,10}(s_1,s_2,s_3)=\frac{\pi e^{\frac{3s_1}{2}}((5e^{s_2}-17)s_2^2+(5s_3-e^{s_2}(29s_3+12)+12)s_2-2s_3(-11s_3+e^{s_2}(17s_3-12)+12))}{(e^{s_1}-1)(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)^2s_1s_2s_3(s_2+s_3)},$$

$$K_{15,11}(s_1,s_2,s_3)$$
$$=\frac{8\pi e^{\frac{3}{2}(s_1+s_2)}(s_1(-e^{s_1+2s_2}(s_3-2)-e^{s_2}(e^{s_1}+1)(s_3-1)+3s_3-4)-e^{s_2}(e^{s_1}-1)s_2(2s_3-3))}{(e^{s_2}-1)(e^{s_1+s_2}-1)^2(e^{\frac{s_3}{2}}-1)s_1s_2(s_1+s_2)s_3},$$

$$K_{15,12}(s_1,s_2,s_3)$$
$$=\frac{8\pi e^{\frac{3}{2}(s_1+s_2)}(s_1(-e^{s_1+2s_2}(s_3-2)-e^{s_2}(e^{s_1}+1)(s_3-1)+3s_3-4)-e^{s_2}(e^{s_1}-1)s_2(2s_3-3))}{(e^{s_2}-1)(e^{s_1+s_2}-1)^2(e^{\frac{s_3}{2}}+1)s_1s_2(s_1+s_2)s_3},$$

$$
\begin{aligned}
&K_{15,13}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^3 s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)\\
&\quad=\pi(-3((-9e^{s_1}-7e^{s_1+s_2}+5e^{2s_1+s_2}+11)s_2+4e^{s_1}(e^{s_2}-1)s_3)s_1^3+((19e^{s_1}+25e^{s_1+s_2}\\
&\qquad-7e^{2s_1+s_2}-37)s_2^2+2(9(e^{s_1}-1)(e^{s_1+s_2}-1)+2(7e^{s_1}-8e^{s_1+s_2}+2e^{2s_1+s_2}-1)s_3)s_2\\
&\qquad-24e^{s_1}(e^{s_2}-1)s_3^2)s_1^2+((-43e^{s_1}-13e^{s_1+s_2}+31e^{2s_1+s_2}+25)s_2^3-4(-e^{s_1}-e^{s_1+s_2}\\
&\qquad+4e^{2s_1+s_2}-2)s_3 s_2^2-s_3(18(e^{s_1}-1)(e^{s_1+s_2}-1)+(-59e^{s_1}-5e^{s_1+s_2}+47e^{2s_1+s_2}+17)s_3)s_2\\
&\qquad-12e^{s_1}(e^{s_2}-1)s_3^3)s_1+s_2(s_2+s_3)((-35e^{s_1}-17e^{s_1+s_2}+23e^{2s_1+s_2}+29)s_2^2+((23e^{s_1}\\
&\qquad+41e^{s_1+s_2}-47e^{2s_1+s_2}-17)s_3-18(e^{s_1}-1)(e^{s_1+s_2}-1))s_2-2s_3((-29e^{s_1}-29e^{s_1+s_2}\\
&\qquad+35e^{2s_1+s_2}+23)s_3-18(e^{s_1}-1)(e^{s_1+s_2}-1)))),
\end{aligned}
$$

$$
K_{15,14}(s_1,s_2,s_3)=\frac{K_{15,13}^{\mathrm{num}}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^3 s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},
$$

$$
\begin{aligned}
&K_{15,15}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)\\
&\quad=4\pi(2((3-e^{s_1}-5e^{s_1+s_2}+3e^{2s_1+s_2})s_2-e^{s_1}(-1+e^{s_2})(-1+3e^{s_1+s_2})s_3)s_1^3+(4(2+e^{s_1}\\
&\qquad-3e^{s_1+s_2})s_2^2+((2+4e^{s_1+s_2}-18e^{2(s_1+s_2)}+12e^{2s_1+s_2})s_3+8e^{s_1}+22e^{s_1+s_2}-e^{2(s_1+s_2)}\\
&\qquad-14e^{2s_1+s_2}-15)s_2-2e^{s_1}(-1+e^{s_2})s_3((-2+6e^{s_1+s_2})s_3-7e^{s_1+s_2}+4))s_1^2-(2(1-7e^{s_1}\\
&\qquad-3e^{s_1+s_2}+9e^{2s_1+s_2})s_2^3+2((2-7e^{s_1}-7e^{s_1+s_2}+3e^{2(s_1+s_2)}+3e^{2s_1+s_2}+6e^{3s_1+2s_2})s_3\\
&\qquad+5e^{s_1}-2e^{s_1+s_2}-6e^{2s_1+s_2}+e^{3s_1+2s_2}+2)s_2^2+s_3(2(1+e^{s_1}-5e^{s_1+s_2}+6e^{2(s_1+s_2)}\\
&\qquad-9e^{2s_1+s_2}+6e^{3s_1+2s_2})s_3+2e^{s_1}+34e^{s_1+s_2}-29e^{2(s_1+s_2)}+2e^{2s_1+s_2}+2e^{3s_1+2s_2}-11)s_2\\
&\qquad+2e^{s_1}(-1+e^{s_2})s_3^2((-1+3e^{s_1+s_2})s_3-7e^{s_1+s_2}+4))s_1-s_2(s_2+s_3)(4(1-2e^{s_1}-2e^{s_1+s_2}\\
&\qquad+3e^{2s_1+s_2})s_2^2+(2(1-2e^{s_1}-2e^{s_1+s_2}-3e^{2(s_1+s_2)}+6e^{3s_1+2s_2})s_3+18e^{s_1}+18e^{s_1+s_2}\\
&\qquad-e^{2(s_1+s_2)}-26e^{2s_1+s_2}+2e^{3s_1+2s_2}-11)s_2+2s_3((-1+2e^{s_1}+2e^{s_1+s_2}-3e^{2(s_1+s_2)}\\
&\qquad-6e^{2s_1+s_2}+6e^{3s_1+2s_2})s_3-9e^{s_1}-12e^{s_1+s_2}+10e^{2(s_1+s_2)}+23e^{2s_1+s_2}-17e^{3s_1+2s_2}+5))),
\end{aligned}
$$

$$
K_{15,16}(s_1,s_2,s_3)=\frac{K_{15,15}^{\mathrm{num}}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},
$$

$$
\begin{aligned}
&K_{15,17}(s_1,s_2,s_3)2(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^2 s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)\\
&\quad=\pi(((87-53e^{s_1}-130e^{s_1+s_2}+19e^{2(s_1+s_2)}+86e^{2s_1+s_2}-9e^{3s_1+2s_2})s_2-4e^{s_1}(-1+e^{s_2})(-11\\
&\qquad+17e^{s_1+s_2})s_3)s_1^3+((107-5e^{s_1}-154e^{s_1+s_2}+23e^{2(s_1+s_2)}+22e^{2s_1+s_2}+7e^{3s_1+2s_2})s_2^2\\
&\qquad+2(2(5-21e^{s_1}+27e^{s_1+s_2}-50e^{2(s_1+s_2)}+35e^{2s_1+s_2}+4e^{3s_1+2s_2})s_3+45e^{s_1}+90e^{s_1+s_2}\\
&\qquad-33e^{2(s_1+s_2)}-66e^{2s_1+s_2}+21e^{3s_1+2s_2}-57)s_2-8e^{s_1}(-1+e^{s_2})s_3((-11+17e^{s_1+s_2})s_3\\
&\qquad-6e^{s_1+s_2}+6))s_1^2+((-47+149e^{s_1}+82e^{s_1+s_2}-11e^{2(s_1+s_2)}-214e^{2s_1+s_2}+41e^{3s_1+2s_2})s_2^3\\
&\qquad-4(4(1+2e^{s_1})(-1+e^{s_1+s_2})^2+(10-11e^{s_1}-25e^{s_1+s_2}+9e^{2(s_1+s_2)}-15e^{2s_1+s_2}\\
&\qquad+32e^{3s_1+2s_2})s_3)s_2^2-s_3(2(-49+13e^{s_1}+122e^{s_1+s_2}-73e^{2(s_1+s_2)}-50e^{2s_1+s_2}+37e^{3s_1+2s_2})\\
&\qquad+(-7+149e^{s_1}-62e^{s_1+s_2}+93e^{2(s_1+s_2)}-342e^{2s_1+s_2}+169e^{3s_1+2s_2})s_3)s_2\\
&\qquad-4e^{s_1}(-1+e^{s_2})s_3^2((-11+17e^{s_1+s_2})s_3-12(-1+e^{s_1+s_2})))s_1+s_2(s_2+s_3)((-67+101e^{s_1}\\
&\qquad+106e^{s_1+s_2}-15e^{2(s_1+s_2)}-150e^{2s_1+s_2}+25e^{3s_1+2s_2})s_2^2+(2(49-61e^{s_1}-74e^{s_1+s_2}\\
&\qquad+25e^{2(s_1+s_2)}+98e^{2s_1+s_2}-37e^{3s_1+2s_2})+(7-17e^{s_1}-70e^{s_1+s_2}+111e^{2(s_1+s_2)}+138e^{2s_1+s_2}\\
&\qquad-169e^{3s_1+2s_2})s_3)s_2-2s_3((-37+59e^{s_1}+88e^{s_1+s_2}-63e^{2(s_1+s_2)}-144e^{2s_1+s_2}\\
&\qquad+97e^{3s_1+2s_2})s_3-98e^{s_1}-172e^{s_1+s_2}+98e^{2(s_1+s_2)}+220e^{2s_1+s_2}-122e^{3s_1+2s_2}+74))),
\end{aligned}
$$

$$
K_{15,18}(s_1,s_2,s_3)=\frac{-K_{15,17}^{\mathrm{num}}(s_1,s_2,s_3)}{2(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^2 s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)}.
$$

Using the above expressions, we have

$$K_{15}(s_1,s_2,s_3)=\frac{K_{15}^{\mathrm{num}}(s_1,s_2,s_3)}{K_{15}^{\mathrm{den}}(s_1,s_2,s_3)},$$

where

$$K_{15}^{\mathrm{den}}(s_1,s_2,s_3)=K_{8}^{\mathrm{den}}(s_1,s_2,s_3),$$

which is given by (73), and K_{15}^{num} is a polynomial in $s_1,s_2,s_3,e^{s_1/2},e^{s_2/2},e^{s_3/2}$. The points (i,j,m) and (n,p,q) such that $s_1^i s_2^j s_3^m e^{ns_1/2}e^{ps_2/2}e^{qs_3/2}$ appears in the expression of $K_{15}^{\mathrm{num}}(s_1,s_2,s_3)$ are plotted in Figure 14 and Figure 15.

C.2. **The four variable functions $K_{17},\dots,K_{20}$.** The functions of four variables $K_{17},K_{18},K_{19},K_{20}$ appearing in (6) have very lengthy expressions. These functions are of the form

$$K_j(s_1,s_2,s_3,s_4)=\frac{K_j^{\mathrm{num}}(s_1,s_2,s_3,s_4)}{K_j^{\mathrm{den}}(s_1,s_2,s_3,s_4)},\quad j=17,18,19,20,$$

where each $K_j^{\mathrm{num}}(s_1,s_2,s_3,s_4)$ is a polynomial in s_1, s_2, s_3, s_4, $e^{s_1/2}$, $e^{s_1/2}$, $e^{s_3/2}$, $e^{s_4/2}$, and

$$\begin{aligned}(75)\quad & K_{17}^{\mathrm{den}}(s_1,s_2,s_3,s_4)\\ &=K_{18}^{\mathrm{den}}(s_1,s_2,s_3,s_4)=K_{19}^{\mathrm{den}}(s_1,s_2,s_3,s_4)=K_{20}^{\mathrm{den}}(s_1,s_2,s_3,s_4)\\ &=(e^{s_1}-1)(e^{s_2}-1)(e^{s_1+s_2}-1)^2(e^{s_3}-1)(e^{s_2+s_3}-1)^2\\ &\quad\times(e^{s_1+s_2+s_3}-1)^3(e^{s_4}-1)(e^{s_3+s_4}-1)^2(e^{s_2+s_3+s_4}-1)^3\\ &\quad\times(e^{s_1+s_2+s_3+s_4}-1)^4 s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)s_4(s_3+s_4)\\ &\quad\times(s_2+s_3+s_4)(s_1+s_2+s_3+s_4).\end{aligned}$$

In fact, it turns out that

$$K_{18}(s_1,s_2,s_3,s_4)=K_{19}(s_1,s_2,s_3,s_4).$$

It is made clear in Section 3, using the functional relations stated in Theorem 3.1, that the functions $K_{17},K_{18},K_{19},K_{20}$ can be constructed from the one, two and three variable functions $K_1,\dots,K_{16}$ presented in Section 9 and earlier in this Appendix, the functions $G_1,\dots,G_4$ given by (9), and the following three variable functions:

$$k_j(s_1,s_2,s_3)=K_j(s_1,s_2,s_3,-s_1-s_2-s_3),\quad j=17,18,19,20,$$

which were in fact introduced already by (8). Therefore, in the rest of this appendix we present the latter functions explicitly.

C.2.1. *The function k_{17}.* We have

$$k_{17}(s_1,s_2,s_3)=K_{17}(s_1,s_2,s_3,-s_1-s_2-s_3)=\sum_{i=1}^{30}k_{17,i}(s_1,s_2,s_3),$$

where

$$k_{17,1}(s_1,s_2,s_3)(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4 s_1^2(s_1+s_2)^2(s_1+s_2+s_3)^2$$
$$= 16\pi e^{2s_1}(2e^{3s_2}-8e^{3(s_1+s_2)}-39e^{2s_1+s_2}-40e^{s_1+2s_2}+10e^{4s_1+2s_2}+5e^{2s_1+4s_2}+16e^{5s_1+4s_2}$$
$$+12e^{4s_1+5s_2}-12)s_2(s_2+s_3),$$

$$k_{17,2}(s_1,s_2,s_3) = \frac{64\pi e^{2s_1}(2e^{3s_2}-2e^{4s_1+2s_2}+7e^{4s_1+5s_2}-7)s_2(s_2+s_3)}{5(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4 s_1(s_1+s_2)^2(s_1+s_2+s_3)^2},$$

$$k_{17,3}(s_1,s_2,s_3)\frac{3}{16}(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4 s_1^2(s_1+s_2)^2(s_1+s_2+s_3)^2$$
$$= \pi(-(-4e^{s_1}-28e^{3s_1}-5e^{4s_1}-e^{s_2}-10e^{2(s_1+s_2)}+20e^{3(s_1+s_2)}+4e^{5(s_1+s_2)}-20e^{2s_1+s_2}$$
$$-116e^{3s_1+s_2}+20e^{5s_1+s_2}+4e^{s_1+2s_2}+104e^{4s_1+2s_2}-188e^{5s_1+2s_2}-200e^{4s_1+3s_2}$$
$$+142e^{6s_1+3s_2}+20e^{7s_1+3s_2}+4e^{3s_1+4s_2}+140e^{5s_1+4s_2}-76e^{6s_1+4s_2}-5e^{8s_1+4s_2}-e^{4s_1+5s_2}$$
$$+28e^{7s_1+5s_2}+5e^{8s_1+5s_2}+1))s_2(s_2+s_3),$$

$$k_{17,4}(s_1,s_2,s_3)\frac{3}{16}(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4 s_1(s_1+s_2)^2 s_3(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= \pi(-11e^{s_1}-29e^{3s_1}+11e^{4s_1}-2e^{s_2}+2e^{s_1+s_2}+82e^{3(s_1+s_2)}-53e^{4(s_1+s_2)}-88e^{3s_1+s_2}$$
$$+106e^{4s_1+s_2}-136e^{4s_1+3s_2}+26e^{6s_1+3s_2}-26e^{7s_1+3s_2}+11e^{3s_1+4s_2}-83e^{6s_1+4s_2}$$
$$+26e^{7s_1+4s_2}+2)s_2^3,$$

$$k_{17,5}(s_1,s_2,s_3)\frac{1}{16}(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4 s_1(s_1+s_2)^2 s_3(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= \pi e^{s_1}s_2^3(2e^{2s_1+4s_2}s_2(e^{s_1}-1)^5+9e^{s_1}+3e^{2s_2}+8e^{s_1+s_2}+8e^{2(s_1+s_2)}+33e^{4(s_1+s_2)}$$
$$-14e^{4s_1+s_2}-11e^{s_1+2s_2}+24e^{3s_1+2s_2}-43e^{4s_1+2s_2}+19e^{5s_1+2s_2}-6e^{s_1+3s_2}+24e^{4s_1+3s_2}),$$

$$k_{17,6}(s_1,s_2,s_3)\frac{3}{32}(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4(s_1+s_2)^2 s_3(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= -\frac{1}{2}\pi((19e^{s_1}+37e^{3s_1}-13e^{4s_1}+4e^{s_2}-10e^{s_1+s_2}-50e^{3(s_1+s_2)}+73e^{4(s_1+s_2)}+80e^{3s_1+s_2}$$
$$-104e^{4s_1+s_2}+68e^{4s_1+3s_2}-64e^{6s_1+3s_2}+34e^{7s_1+3s_2}-19e^{3s_1+4s_2}+43e^{6s_1+4s_2}$$
$$+14e^{7s_1+4s_2}-4)s_1^2+(49e^{s_1}+103e^{3s_1}-37e^{4s_1}+10e^{s_2}-22e^{s_1+s_2}-182e^{3(s_1+s_2)}$$
$$+199e^{4(s_1+s_2)}+248e^{3s_1+s_2}-314e^{4s_1+s_2}+272e^{4s_1+3s_2}-154e^{6s_1+3s_2}+94e^{7s_1+3s_2}$$
$$-49e^{3s_1+4s_2}+169e^{6s_1+4s_2}+2e^{7s_1+4s_2}-10)s_2s_1-(-41e^{s_1}-95e^{3s_1}+35e^{4s_1}-8e^{s_2}$$
$$+14e^{s_1+s_2}+214e^{3(s_1+s_2)}-179e^{4(s_1+s_2)}-256e^{3s_1+s_2}+316e^{4s_1+s_2}-340e^{4s_1+3s_2}$$
$$+116e^{6s_1+3s_2}-86e^{7s_1+3s_2}+41e^{3s_1+4s_2}-209e^{6s_1+4s_2}+38e^{7s_1+4s_2}+8)s_2^2),$$

$$k_{17,7}(s_1,s_2,s_3)\frac{1}{32}(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4(s_1+s_2)^2 s_3(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= -\frac{1}{2}e^{s_1}\pi(2e^{s_1}(-1+e^{s_1})(-1+e^{s_2})(-e^{s_1}-2e^{2s_2}-e^{s_1+s_2}-2e^{2(s_1+s_2)}+6e^{3(s_1+s_2)}$$
$$+4e^{2s_1+s_2}+5e^{s_1+2s_2}-4e^{3s_1+2s_2}+3e^{s_1+3s_2}-8e^{2s_1+3s_2})s_1^3+(-2e^{s_1}(-1+e^{s_1})(-3e^{s_1}$$
$$-6e^{2s_2}+6e^{3s_2}-18e^{2(s_1+s_2)}+30e^{3(s_1+s_2)}-4e^{4(s_1+s_2)}+12e^{2s_1+s_2}+18e^{s_1+2s_2}$$
$$-12e^{3s_1+2s_2}-6e^{s_1+3s_2}-18e^{2s_1+3s_2}-8e^{s_1+4s_2}+20e^{2s_1+4s_2}-12e^{3s_1+4s_2}+e^{5s_1+4s_2})s_2$$
$$-13e^{s_1}-3e^{2s_2}-4e^{s_1+s_2}-16e^{2(s_1+s_2)}-33e^{4(s_1+s_2)}+24e^{5(s_1+s_2)}+14e^{4s_1+s_2}+13e^{s_1+2s_2}$$
$$-12e^{3s_1+2s_2}+35e^{4s_1+2s_2}-17e^{5s_1+2s_2}+4e^{s_1+3s_2}-4e^{7s_1+4s_2}+4e^{3s_1+5s_2}-16e^{4s_1+5s_2}$$
$$-16e^{6s_1+5s_2}+4e^{7s_1+5s_2})s_1^2-s_2(6e^{s_1}(-1+e^{s_1})(-e^{s_1}-2e^{2s_2}+2e^{3s_2}-6e^{2(s_1+s_2)}$$
$$+10e^{3(s_1+s_2)}-4e^{4(s_1+s_2)}+4e^{2s_1+s_2}+6e^{s_1+2s_2}-4e^{3s_1+2s_2}-2e^{s_1+3s_2}-6e^{2s_1+3s_2}$$
$$-2e^{s_1+4s_2}+4e^{2s_1+4s_2}+e^{5s_1+4s_2})s_2+35e^{s_1}+9e^{2s_2}+16e^{s_1+s_2}+40e^{2(s_1+s_2)}+99e^{4(s_1+s_2)}$$
$$-48e^{5(s_1+s_2)}-42e^{4s_1+s_2}-37e^{s_1+2s_2}+48e^{3s_1+2s_2}-113e^{4s_1+2s_2}+53e^{5s_1+2s_2}-14e^{s_1+3s_2}$$
$$+24e^{4s_1+3s_2}+8e^{7s_1+4s_2}-8e^{3s_1+5s_2}+32e^{4s_1+5s_2}+32e^{6s_1+5s_2}-8e^{7s_1+5s_2})s_1$$
$$-s_2^2(2e^{s_1}(-1+e^{s_1})(-e^{s_1}-2e^{2s_2}+2e^{3s_2}-6e^{2(s_1+s_2)}+10e^{3(s_1+s_2)}-12e^{4(s_1+s_2)}$$

$$+ 4e^{2s_1+s_2} + 6e^{s_1+2s_2} - 4e^{3s_1+2s_2} - 2e^{s_1+3s_2} - 6e^{2s_1+3s_2} - 4e^{2s_1+4s_2} + 12e^{3s_1+4s_2}$$
$$+ 3e^{5s_1+4s_2})s_2 + 31e^{s_1} + 9e^{2s_2} + 20e^{s_1+s_2} + 32e^{2(s_1+s_2)} + 99e^{4(s_1+s_2)} - 24e^{5(s_1+s_2)}$$
$$- 42e^{4s_1+s_2} - 35e^{s_1+2s_2} + 60e^{3s_1+2s_2} - 121e^{4s_1+2s_2} + 55e^{5s_1+2s_2} - 16e^{s_1+3s_2} + 48e^{4s_1+3s_2}$$
$$+ 4e^{7s_1+4s_2} - 4e^{3s_1+5s_2} + 16e^{4s_1+5s_2} + 16e^{6s_1+5s_2} - 4e^{7s_1+5s_2})),$$

$$k_{17,8}(s_1,s_2,s_3)\frac{3}{32}(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= \frac{1}{2}\pi(4e^{2s_1+s_2}(8 + 74e^{s_1} + 13e^{s_2} + 98e^{2(s_1+s_2)} - 131e^{3(s_1+s_2)} - 98e^{2s_1+s_2} - 88e^{3s_1+s_2}$$
$$+ 10e^{s_1+2s_2} + 89e^{4s_1+2s_2} + 25e^{4s_1+3s_2})s_2^2 - e^{s_1}(55 - 55e^{3s_1} - 13e^{2s_2} - 176e^{s_1+s_2}$$
$$- 256e^{3(s_1+s_2)} - 79e^{4(s_1+s_2)} + 182e^{4s_1+s_2} + 149e^{s_1+2s_2} - 128e^{3s_1+2s_2} - 155e^{4s_1+2s_2}$$
$$+ 94e^{6s_1+3s_2} - 8e^{7s_1+4s_2} + 40e^{3s_1+5s_2} - 164e^{4s_1+5s_2} + 20e^{6s_1+5s_2} + 8e^{7s_1+5s_2})s_2$$
$$- 2s_3(2e^{2s_1+s_2}(-4 - 40e^{s_1} - 5e^{s_2} - 46e^{2(s_1+s_2)} + 40e^{3(s_1+s_2)} + 70e^{2s_1+s_2} + 50e^{3s_1+s_2}$$
$$- 2e^{s_1+2s_2} - 37e^{4s_1+2s_2} + 34e^{4s_1+3s_2})s_3 + 14e^{s_1} - 26e^{3s_1} - 14e^{4s_1} + e^{s_2} - 14e^{s_1+s_2}$$
$$+ 31e^{2(s_1+s_2)} - 104e^{3(s_1+s_2)} + 206e^{4(s_1+s_2)} - 118e^{5(s_1+s_2)} - 79e^{2s_1+s_2} - 94e^{3s_1+s_2}$$
$$+ 146e^{4s_1+s_2} + 40e^{5s_1+s_2} - 244e^{5s_1+2s_2} - 344e^{4s_1+3s_2} + 131e^{6s_1+3s_2} + 20e^{7s_1+3s_2}$$
$$- 52e^{3s_1+4s_2} + 46e^{5s_1+4s_2} - 179e^{6s_1+4s_2} - 14e^{7s_1+4s_2} - 7e^{8s_1+4s_2} + 7e^{8s_1+5s_2} - 1)$$
$$+ 2s_1(-2e^{2s_1+s_2}(-16 - 94e^{s_1} - 23e^{s_2} - 154e^{2(s_1+s_2)} + 25e^{3(s_1+s_2)} + 58e^{2s_1+s_2} + 83e^{3s_1+s_2}$$
$$- 38e^{s_1+2s_2} - 127e^{4s_1+2s_2} + 67e^{4s_1+3s_2})s_2 - 17e^{s_1} - 19e^{3s_1} + 17e^{4s_1} - e^{s_2} + 44e^{3(s_1+s_2)}$$
$$+ 4e^{3s_1+s_2} - 7e^{s_1+2s_2} + 4e^{4s_1+2s_2} + 140e^{4s_1+3s_2} + e^{6s_1+3s_2} - 32e^{7s_1+3s_2} + 31e^{3s_1+4s_2}$$
$$+ 74e^{6s_1+4s_2} - 26e^{4s_1+5s_2} + 56e^{7s_1+5s_2} - 2e^{2s_1+s_2}(-16 - 100e^{s_1} - 17e^{s_2} - 142e^{2(s_1+s_2)}$$
$$+ 25e^{3(s_1+s_2)} + 118e^{2s_1+s_2} + 92e^{3s_1+s_2} - 26e^{s_1+2s_2} - 97e^{4s_1+2s_2} + 121e^{4s_1+3s_2})s_3 + 1)),$$

$$k_{17,9}(s_1,s_2,s_3)\frac{3}{32}(e^{s_1}-1)^4(e^{s_1+s_2}-1)^4 s_2(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= \frac{1}{2}\pi e^{s_1}((-7e^{3s_1} - 7e^{s_2} - 37e^{3(s_1+s_2)} - 5e^{3s_1+s_2} + 27e^{3s_1+2s_2} + 10e^{6s_1+3s_2} - 8e^{3s_1+4s_2}$$
$$+ 20e^{6s_1+4s_2} + 7)s_1^2 + 2(-e^{3s_1} - e^{s_2} - 85e^{3(s_1+s_2)} + 43e^{3s_1+s_2} + 27e^{3s_1+2s_2} + 4e^{6s_1+3s_2}$$
$$- 14e^{3s_1+4s_2} + 26e^{6s_1+4s_2} + 1)s_3^2),$$

$$k_{17,10}(s_1,s_2,s_3)\frac{1}{32}(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= \frac{1}{2}\pi(2e^{2s_1}(-8 + 2e^{s_1} - 4e^{2s_1} + 8e^{s_2} + 10e^{2s_2} - 10e^{3s_2} + 36e^{s_1+s_2} - 4e^{2(s_1+s_2)} - 24e^{3(s_1+s_2)}$$
$$- 46e^{4(s_1+s_2)} - 4e^{5(s_1+s_2)} - 10e^{2s_1+s_2} + 16e^{3s_1+s_2} - 72e^{s_1+2s_2} - 20e^{3s_1+2s_2} - 14e^{4s_1+2s_2}$$
$$+ 20e^{s_1+3s_2} + 68e^{2s_1+3s_2} + 50e^{4s_1+3s_2} - 4e^{5s_1+3s_2} + 13e^{s_1+4s_2} - 47e^{2s_1+4s_2} + 26e^{3s_1+4s_2}$$
$$+ 3e^{5s_1+4s_2} + e^{6s_1+4s_2} + 2e^{2s_1+5s_2} - 8e^{3s_1+5s_2} + 20e^{4s_1+5s_2})s_1^2 - 2e^{s_1}(e^{2s_1+s_2}(-6e^{s_1}$$
$$+ 104e^{s_2} - 8e^{2(s_1+s_2)} + 39e^{s_1+3s_2} + 18e^{4s_1+3s_2})s_2 - 6e^{s_1} - 8e^{s_2} - 5e^{s_1+s_2} + 20e^{2(s_1+s_2)}$$
$$- 15e^{4(s_1+s_2)} + 41e^{5(s_1+s_2)} - 2e^{3s_1+s_2} + 16e^{4s_1+s_2} + 12e^{s_1+2s_2} - 17e^{4s_1+2s_2} - 16e^{5s_1+2s_2}$$
$$- e^{s_1+3s_2} + 52e^{4s_1+3s_2} + 47e^{3s_1+4s_2} + 8e^{6s_1+4s_2} - 5e^{7s_1+4s_2} - 36e^{4s_1+5s_2} + 5e^{7s_1+5s_2}$$
$$- e^{2s_1+s_2}(24e^{s_1} - 88e^{s_2} + 64e^{2(s_1+s_2)} - 29e^{s_1+3s_2} + 5e^{4s_1+3s_2})s_3)s_1 - 2e^{3s_1+s_2}(-26e^{s_1}$$
$$+ 56e^{s_2} - 56e^{2(s_1+s_2)} - 19e^{s_1+3s_2} + 45e^{4s_1+3s_2})s_2^2 - 2e^{2s_1}s_3(2e^{s_1+s_2}(-10e^{s_1} + 12e^{s_2}$$
$$- 24e^{2(s_1+s_2)} - 2e^{s_1+3s_2} + 3e^{4s_1+3s_2})s_3 + 7e^{3s_2} - 8e^{2(s_1+s_2)} + 92e^{3(s_1+s_2)} - 2e^{5(s_1+s_2)}$$
$$+ 92e^{s_1+2s_2} - 13e^{4s_1+2s_2} + 10e^{2s_1+5s_2} + 29e^{4s_1+5s_2} + 9) + s_2(-2e^{2s_1+s_2}(-8 - 76e^{s_1}$$
$$- 50e^{2s_1} - 10e^{s_2} + 72e^{s_1+s_2} - 92e^{2(s_1+s_2)} + 98e^{3(s_1+s_2)} + 124e^{2s_1+s_2} + 90e^{3s_1+s_2}$$
$$- 4e^{s_1+2s_2} - 120e^{3s_1+2s_2} - 74e^{4s_1+2s_2} - 19e^{2s_1+3s_2} + 30e^{4s_1+3s_2} + 36e^{5s_1+3s_2})s_3 + 9e^{2s_1}$$
$$- 11e^{3s_1} - 2e^{s_2} + 14e^{s_1+s_2} + 114e^{3(s_1+s_2)} - 135e^{4(s_1+s_2)} - 12e^{3s_1+s_2} + 2e^{4s_1+s_2}$$

$$
\begin{aligned}
&- 120e^{3s_1+2s_2} + 71e^{6s_1+2s_2} - 18e^{2s_1+3s_2} - 72e^{5s_1+3s_2} - 78e^{6s_1+3s_2} + 29e^{3s_1+4s_2}\\
&+ 39e^{6s_1+4s_2} + 38e^{7s_1+4s_2} - 32e^{6s_1+5s_2} + 2)),
\end{aligned}
$$

$$
\begin{aligned}
&k_{17,11}(s_1,s_2,s_3)\frac{5}{32}(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2\\
&= \pi(2e^{2s_1}(-11e^{3s_2} - 19e^{4s_1+2s_2} + 54e^{4s_1+5s_2} - 24)s_2^2 + (8e^{s_1} - 72e^{2s_1} - 67e^{3s_1} - 7e^{4s_1} + 2e^{s_2}\\
&- 88e^{5(s_1+s_2)} + 28e^{5s_1+s_2} - 8e^{s_1+2s_2} - 12e^{6s_1+2s_2} - 18e^{2s_1+3s_2} - 32e^{7s_1+3s_2} - 13e^{3s_1+4s_2}\\
&+ 38e^{8s_1+4s_2} + 22e^{4s_1+5s_2} + 192e^{6s_1+5s_2} + 12e^{7s_1+5s_2} + 2e^{8s_1+5s_2} - 2)s_3s_2 + 6e^{2s_1}(-e^{3s_2}\\
&+ e^{4s_1+2s_2} + 14e^{4s_1+5s_2} - 4)s_3^2 + 2e^{2s_1}s_1((-33e^{3s_2} - 42e^{4s_1+2s_2} + 92e^{4s_1+5s_2} - 32)s_2\\
&+ (-23e^{3s_2} - 7e^{4s_1+2s_2} + 122e^{4s_1+5s_2} - 32)s_3)),
\end{aligned}
$$

$$
\begin{aligned}
&k_{17,12}(s_1,s_2,s_3)\frac{15}{32}(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2\\
&= \pi((16e^{s_1} - 104e^{3s_1} - 34e^{4s_1} + 4e^{s_2} - 136e^{5(s_1+s_2)} + 136e^{5s_1+s_2} - 16e^{s_1+2s_2} - 44e^{7s_1+3s_2}\\
&- 31e^{3s_1+4s_2} + 131e^{8s_1+4s_2} + 34e^{4s_1+5s_2} + 44e^{7s_1+5s_2} + 4e^{8s_1+5s_2} - 4)s_2^2 + 2(4e^{s_1} - 41e^{3s_1}\\
&- e^{4s_1} + e^{s_2} - 64e^{5(s_1+s_2)} + 4e^{5s_1+s_2} - 4e^{s_1+2s_2} - 26e^{7s_1+3s_2} - 4e^{3s_1+4s_2} + 14e^{8s_1+4s_2}\\
&+ 16e^{4s_1+5s_2} - 4e^{7s_1+5s_2} + e^{8s_1+5s_2} - 1)s_3^2 + s_1((8e^{s_1} - 37e^{3s_1} - 77e^{4s_1} + 2e^{s_2}\\
&- 248e^{5(s_1+s_2)} + 308e^{5s_1+s_2} - 8e^{s_1+2s_2} - 112e^{7s_1+3s_2} + 187e^{3s_1+4s_2} + 88e^{8s_1+4s_2}\\
&+ 62e^{4s_1+5s_2} - 68e^{7s_1+5s_2} + 2e^{8s_1+5s_2} - 2)s_2 + (8e^{s_1} - 112e^{3s_1} - 32e^{4s_1} + 2e^{s_2}\\
&- 368e^{5(s_1+s_2)} + 128e^{5s_1+s_2} - 8e^{s_1+2s_2} - 172e^{7s_1+3s_2} + 127e^{3s_1+4s_2} + 43e^{8s_1+4s_2}\\
&+ 92e^{4s_1+5s_2} - 128e^{7s_1+5s_2} + 2e^{8s_1+5s_2} - 2)s_3)),
\end{aligned}
$$

$$
\begin{aligned}
&k_{17,13}(s_1,s_2,s_3)\frac{1}{32}(e^{s_1}-1)^4(e^{s_1+s_2}-1)^4s_2(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2\\
&= -\frac{1}{2}e^{s_1}\pi(-2e^{s_1}(-1+e^{s_2})(-2+e^{s_1} - e^{2s_1} - 2e^{s_2} + 10e^{s_1+s_2} - 11e^{2(s_1+s_2)} - 6e^{2s_1+s_2}\\
&+ 4e^{3s_1+s_2} + 3e^{s_1+2s_2} + 6e^{3s_1+2s_2} - 4e^{4s_1+2s_2} + 2e^{4s_1+3s_2})s_1^3 + e^{s_1}(2(-4 - e^{s_1} - e^{2s_1}\\
&+ 4e^{2s_2} + 19e^{s_1+s_2} - 15e^{2(s_1+s_2)} - 2e^{3(s_1+s_2)} - 16e^{4(s_1+s_2)} + e^{2s_1+s_2} + 4e^{3s_1+s_2}\\
&- 13e^{s_1+2s_2} - 10e^{3s_1+2s_2} - 2e^{4s_1+2s_2} - 5e^{s_1+3s_2} + 17e^{2s_1+3s_2} + 18e^{4s_1+3s_2} - 4e^{5s_1+3s_2}\\
&- 2e^{2s_1+4s_2} + 8e^{3s_1+4s_2} + 4e^{5s_1+4s_2})s_3 - 3e^{s_1} + e^{s_2} - 4e^{2s_2} + e^{s_1+s_2} - 9e^{3(s_1+s_2)}\\
&+ 14e^{4(s_1+s_2)} - 6e^{3s_1+s_2} + e^{s_1+2s_2} + 7e^{3s_1+2s_2} + 5e^{4s_1+2s_2} - e^{s_1+3s_2} + e^{4s_1+3s_2}\\
&- 12e^{3s_1+4s_2} + 2e^{6s_1+4s_2} + 3)s_1^2 + s_3(3(-1+e^{s_1})(1 + 2e^{s_1} + e^{2s_1} - e^{s_2} - 2e^{s_1+s_2}\\
&+ 7e^{2(s_1+s_2)} - 18e^{3(s_1+s_2)} + 12e^{4(s_1+s_2)} - 11e^{2s_1+s_2} - 2e^{3s_1+s_2} + 16e^{3s_1+2s_2} + e^{4s_1+2s_2}\\
&+ 5e^{2s_1+3s_2} - e^{4s_1+3s_2} - 2e^{5s_1+3s_2} - 4e^{3s_1+4s_2} - 6e^{5s_1+4s_2} + 2e^{6s_1+4s_2}) + 4e^{s_1}(-1 - e^{s_1}\\
&+ e^{2s_2} + 5e^{s_1+s_2} - 5e^{2(s_1+s_2)} + 2e^{3(s_1+s_2)} - 7e^{4(s_1+s_2)} + 3e^{2s_1+s_2} - 3e^{s_1+2s_2} - 6e^{3s_1+2s_2}\\
&+ e^{4s_1+2s_2} - e^{s_1+3s_2} + 3e^{2s_1+3s_2} + 6e^{4s_1+3s_2} - 2e^{5s_1+3s_2} - e^{2s_1+4s_2} + 4e^{3s_1+4s_2}\\
&+ 2e^{5s_1+4s_2})s_3)s_1 + 2e^{s_1}(-3 + 3e^{s_1} + e^{s_2} + 2e^{2s_2} + 13e^{s_1+s_2} - 21e^{3(s_1+s_2)} + 20e^{4(s_1+s_2)}\\
&- 11e^{s_1+2s_2} + 19e^{3s_1+2s_2} - e^{4s_1+2s_2} - 7e^{s_1+3s_2} + e^{4s_1+3s_2} - 18e^{3s_1+4s_2} + 2e^{6s_1+4s_2})s_3^2),
\end{aligned}
$$

$$
\begin{aligned}
&k_{17,14}(s_1,s_2,s_3)\frac{3}{32}(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4s_1(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2\\
&= \frac{1}{2}\pi(4e^{2s_1+s_2}(1 + 22e^{s_1} + 2e^{s_2} + 22e^{2(s_1+s_2)} - 76e^{3(s_1+s_2)} - 46e^{2s_1+s_2} - 32e^{3s_1+s_2}\\
&- 4e^{s_1+2s_2} + 22e^{4s_1+2s_2} + 35e^{4s_1+3s_2})s_2^3 + e^{s_1+s_2}(4e^{s_1}(2 + 44e^{s_1} + 4e^{s_2} + 44e^{2(s_1+s_2)}\\
&- 107e^{3(s_1+s_2)} - 92e^{2s_1+s_2} - 64e^{3s_1+s_2} - 8e^{s_1+2s_2} + 44e^{4s_1+2s_2} + 25e^{4s_1+3s_2})s_3 + 160e^{s_1}\\
&- 125e^{3s_1} - 124e^{4s_1} + 16e^{s_2} - 82e^{s_1+s_2} - 137e^{3(s_1+s_2)} - 20e^{4(s_1+s_2)} - 16e^{3s_1+s_2}\\
&+ 280e^{4s_1+s_2} - 160e^{4s_1+3s_2} + 200e^{6s_1+3s_2} - 11e^{7s_1+3s_2} + 5e^{3s_1+4s_2} - 140e^{6s_1+4s_2}\\
&+ 11e^{7s_1+4s_2} + 8)s_2^2 + e^{s_1}s_3(4e^{s_1+s_2}(1 + 22e^{s_1} + 2e^{s_2} + 22e^{2(s_1+s_2)} - 46e^{3(s_1+s_2)}\\
&- 46e^{2s_1+s_2} - 32e^{3s_1+s_2} - 4e^{s_1+2s_2} + 22e^{4s_1+2s_2} + 5e^{4s_1+3s_2})s_3 + 29e^{3s_1} + 5e^{2s_2}
\end{aligned}
$$

$$+ 196e^{s_1+s_2} + 656e^{3(s_1+s_2)} - 469e^{4(s_1+s_2)} - 106e^{4s_1+s_2} - 49e^{s_1+2s_2} - 176e^{3s_1+2s_2}$$
$$+ 715e^{4s_1+2s_2} - 26e^{6s_1+3s_2} - 26e^{7s_1+4s_2} + 10e^{3s_1+5s_2} - 44e^{4s_1+5s_2} - 244e^{6s_1+5s_2}$$
$$+ 26e^{7s_1+5s_2} - 11)s_2 - (1 - 2e^{s_1} - 46e^{3s_1} - 7e^{4s_1} - e^{s_2} - 40e^{3(s_1+s_2)} - 152e^{3s_1+s_2}$$
$$+ 2e^{s_1+2s_2} + 88e^{4s_1+2s_2} - 280e^{4s_1+3s_2} + 274e^{6s_1+3s_2} - 8e^{7s_1+3s_2} - 2e^{3s_1+4s_2}$$
$$- 142e^{6s_1+4s_2} - 5e^{4s_1+5s_2} + 104e^{7s_1+5s_2})s_3^2),$$

$$k_{17,15}(s_1,s_2,s_3)\frac{1}{32}(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4 s_1(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= \frac{1}{2}\pi(-4e^{3s_1+s_2}(-7e^{s_1} + 4e^{s_2} - 16e^{2(s_1+s_2)} - 11e^{s_1+3s_2} + 12e^{4s_1+3s_2})s_2^3 + (-2e^{3s_1+s_2}(-28e^{s_1}$$
$$+ 16e^{s_2} - 64e^{2(s_1+s_2)} - 29e^{s_1+3s_2} + 33e^{4s_1+3s_2})s_3 - 8e^{s_1} - 12e^{2s_1} + 8e^{3s_1} + 11e^{4s_1} - e^{s_2}$$
$$+ 76e^{3(s_1+s_2)} + 28e^{3s_1+s_2} - 120e^{3s_1+2s_2} + 54e^{6s_1+2s_2} - 14e^{2s_1+3s_2} + 80e^{4s_1+3s_2}$$
$$+ 8e^{5s_1+3s_2} - 130e^{6s_1+3s_2} - 20e^{7s_1+3s_2} + 8e^{3s_1+4s_2} + 28e^{6s_1+4s_2} + 48e^{6s_1+5s_2} + 1)s_2^2$$
$$+ e^{s_1}s_3(-4e^{2s_1+s_2}(-7e^{s_1} + 4e^{s_2} - 16e^{2(s_1+s_2)} - 6e^{s_1+3s_2} + 7e^{4s_1+3s_2})s_3 - 39e^{s_1} + 33e^{2s_1}$$
$$+ 2e^{s_2} - 208e^{2(s_1+s_2)} + 84e^{5(s_1+s_2)} + 108e^{2s_1+s_2} - 140e^{3s_1+s_2} + 43e^{5s_1+2s_2} - 10e^{s_1+3s_2}$$
$$+ 62e^{2s_1+3s_2} - 32e^{4s_1+3s_2} - 230e^{5s_1+3s_2} + 5e^{2s_1+4s_2} - 33e^{3s_1+4s_2} + 103e^{5s_1+4s_2}$$
$$+ 90e^{6s_1+4s_2})s_2 + e^{2s_1}(-18 + 20e^{s_2} - 2e^{3s_2} - 16e^{3(s_1+s_2)} - 63e^{2s_1+s_2} - 8e^{3s_1+s_2}$$
$$- 80e^{s_1+2s_2} + 102e^{3s_1+2s_2} + 8e^{4s_1+2s_2} - 5e^{2s_1+4s_2} - 70e^{3s_1+4s_2} + 32e^{5s_1+4s_2} - 5e^{6s_1+4s_2}$$
$$- 8e^{3s_1+5s_2} + 36e^{4s_1+5s_2} + 5e^{6s_1+5s_2})s_3^2),$$

$$k_{17,16}(s_1,s_2,s_3)\frac{15}{32}(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4 s_1(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= \pi s_2(2(4e^{s_1} - 26e^{3s_1} - e^{4s_1} + e^{s_2} - 4e^{5(s_1+s_2)} + 4e^{5s_1+s_2} - 4e^{s_1+2s_2} + 4e^{7s_1+3s_2}$$
$$- 34e^{3s_1+4s_2} + 29e^{8s_1+4s_2} + e^{4s_1+5s_2} + 26e^{7s_1+5s_2} + e^{8s_1+5s_2} - 1)s_2^2 + (16e^{s_1} - 104e^{3s_1}$$
$$- 4e^{4s_1} + 4e^{s_2} - 16e^{5(s_1+s_2)} + 16e^{5s_1+s_2} - 16e^{s_1+2s_2} + 16e^{7s_1+3s_2} - 91e^{3s_1+4s_2}$$
$$+ 71e^{8s_1+4s_2} + 4e^{4s_1+5s_2} + 104e^{7s_1+5s_2} + 4e^{8s_1+5s_2} - 4)s_3s_2 + 2(4e^{s_1} - 26e^{3s_1} - e^{4s_1} + e^{s_2}$$
$$- 4e^{5(s_1+s_2)} + 4e^{5s_1+s_2} - 4e^{s_1+2s_2} + 4e^{7s_1+3s_2} - 19e^{3s_1+4s_2} + 14e^{8s_1+4s_2} + e^{4s_1+5s_2}$$
$$+ 26e^{7s_1+5s_2} + e^{8s_1+5s_2} - 1)s_3^2),$$

$$k_{17,17}(s_1,s_2,s_3) = \frac{4\pi(s_1^2 - s_3s_1 - s_2^2 + s_3^2)}{(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^4 s_1(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},$$

$$k_{17,18}(s_1,s_2,s_3) = \frac{4\pi(s_1^2 - s_3s_1 - s_2^2 + s_3^2)}{(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^4 s_1(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},$$

$$k_{17,19}(s_1,s_2,s_3) = \frac{8\pi e^{2s_1}(s_2-s_3)}{(e^{s_1}-1)^2(e^{\frac{1}{2}(s_2+s_3)}-1)^2 s_1s_2s_3(s_1+s_2+s_3)},$$

$$k_{17,20}(s_1,s_2,s_3) = \frac{8\pi e^{2s_1}(s_2-s_3)}{(e^{s_1}-1)^2(e^{\frac{1}{2}(s_2+s_3)}+1)^2 s_1s_2s_3(s_1+s_2+s_3)},$$

$$k_{17,21}(s_1,s_2,s_3)(e^{s_1}-1)^3(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)s_1s_2s_3(s_2+s_3)(s_1+s_2+s_3)$$
$$= -8\pi e^{2s_1}((-e^{s_1} - e^{s_2} + 3e^{s_1+s_2} - 1)s_2^2 + 2(e^{s_1}-1)((4e^{s_2}-2)s_3 + e^{s_2} - 1)s_2$$
$$+ s_3((-3e^{s_1} - 7e^{s_2} + 5e^{s_1+s_2} + 5)s_3 - 2(e^{s_1}-1)(e^{s_2}-1))),$$

$$k_{17,22}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^3 s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= -4\pi(((-3e^{s_1} - 3e^{s_1+s_2} + 2e^{2s_1+s_2} + 4)s_2 + e^{s_1}(e^{s_2}-1)s_3)s_1^3 + ((-3e^{s_1} - 5e^{s_1+s_2}$$
$$+ 2e^{2s_1+s_2} + 6)s_2^2 + ((-3e^{s_1} + e^{s_1+s_2} + 2)s_3 - 2(e^{s_1}-1)(e^{s_1+s_2}-1))s_2 + 2e^{s_1}(e^{s_2}-1)s_3^2)s_1^2$$
$$- (e^{s_1}(e^{s_2} + 2e^{s_1+s_2} - 3)s_2^3 - (-e^{s_1} - 3e^{s_1+s_2} + 2e^{2s_1+s_2} + 2)s_3s_2^2$$
$$- s_3(2(e^{s_1}-1)(e^{s_1+s_2}-1) + (-5e^{s_1} - e^{s_1+s_2} + 4e^{2s_1+s_2} + 2)s_3)s_2 - e^{s_1}(e^{s_2}-1)s_3^3)s_1$$

$$- s_2(s_2+s_3)((-3e^{s_1}-e^{s_1+s_2}+2e^{2s_1+s_2}+2)s_2^2-2(e^{s_1}-1)((2e^{s_1+s_2}-1)s_3+e^{s_1+s_2}-1)s_2 - s_3((-5e^{s_1}-5e^{s_1+s_2}+6e^{2s_1+s_2}+4)s_3-2(e^{s_1}-1)(e^{s_1+s_2}-1)))),$$

$$k_{17,23}(s_1,s_2,s_3)(e^{s_1}-1)^3(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)s_1s_2s_3(s_2+s_3)(s_1+s_2+s_3) = 8\pi e^{2s_1}((-e^{s_1}-e^{s_2}+3e^{s_1+s_2}-1)s_2^2+2(e^{s_1}-1)((4e^{s_2}-2)s_3+e^{s_2}-1)s_2+s_3((-3e^{s_1}-7e^{s_2} + 5e^{s_1+s_2}+5)s_3-2(e^{s_1}-1)(e^{s_2}-1))),$$

$$k_{17,24}(s_1,s_2,s_3) = \frac{-k_{17,22}^{\mathrm{num}}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^3s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2},$$

$$k_{17,25}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^2s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= 2\pi(((-11+10e^{s_1}+13e^{s_1+s_2}+2e^{2(s_1+s_2)}-15e^{2s_1+s_2}+e^{3s_1+2s_2})s_2 + e^{s_1}(-1+e^{s_2})(-9+13e^{s_1+s_2})s_3)s_1^3+((-21+18e^{s_1}+25e^{s_1+s_2}-31e^{2s_1+s_2}+9e^{3s_1+2s_2})s_2^2 + ((-10+35e^{s_1}-15e^{s_1+s_2}+37e^{2(s_1+s_2)}-55e^{2s_1+s_2}+8e^{3s_1+2s_2})s_3-2(-7+5e^{s_1} + 12e^{s_1+s_2}-5e^{2(s_1+s_2)}-8e^{2s_1+s_2}+3e^{3s_1+2s_2}))s_2+2e^{s_1}(-1+e^{s_2})s_3((-9+13e^{s_1+s_2})s_3 - 2e^{s_1+s_2}+2))s_1^2+((-9+6e^{s_1}+11e^{s_1+s_2}-6e^{2(s_1+s_2)}-17e^{2s_1+s_2}+15e^{3s_1+2s_2})s_2^3 + (8(-1+e^{s_1+s_2})^2+(-19+42e^{s_1}+15e^{s_1+s_2}-89e^{2s_1+s_2}+51e^{3s_1+2s_2})s_3)s_2^2+s_3(2(-3+e^{s_1} + 8e^{s_1+s_2}-5e^{2(s_1+s_2)}-4e^{2s_1+s_2}+3e^{3s_1+2s_2})+(-10+45e^{s_1}-5e^{s_1+s_2}+19e^{2(s_1+s_2)} - 85e^{2s_1+s_2}+36e^{3s_1+2s_2})s_3)s_2+e^{s_1}(-1+e^{s_2})s_3^2((-9+13e^{s_1+s_2})s_3-4e^{s_1+s_2}+4))s_1 + s_2(s_2+s_3)((1-2e^{s_1}-e^{s_1+s_2}-4e^{2(s_1+s_2)}-e^{2s_1+s_2}+7e^{3s_1+2s_2})s_2^2+2((-5+9e^{s_1} + 11e^{s_1+s_2}-10e^{2(s_1+s_2)}-23e^{2s_1+s_2}+18e^{3s_1+2s_2})s_3+5e^{s_1}+4e^{s_1+s_2}-e^{2(s_1+s_2)} - 8e^{2s_1+s_2}+3e^{3s_1+2s_2}-3)s_2+s_3(2(7-9e^{s_1}-16e^{s_1+s_2}+9e^{2(s_1+s_2)}+20e^{2s_1+s_2} - 11e^{3s_1+2s_2})+(-11+20e^{s_1}+23e^{s_1+s_2}-16e^{2(s_1+s_2)}-45e^{2s_1+s_2}+29e^{3s_1+2s_2})s_3))),$$

$$k_{17,26}(s_1,s_2,s_3) = \frac{k_{17,25}^{\mathrm{num}}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^2s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2},$$

$$k_{17,27}(s_1,s_2,s_3)3(e^{s_1}-1)^3(e^{s_1+s_2}-1)^3(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= 2\pi(3((-5+16e^{s_1}-17e^{2s_1}+14e^{3s_1}+8e^{s_1+s_2}+11e^{2(s_1+s_2)}-22e^{3(s_1+s_2)}-23e^{2s_1+s_2} + 30e^{3s_1+s_2}-39e^{4s_1+s_2}-30e^{3s_1+2s_2}+11e^{4s_1+2s_2}+32e^{5s_1+2s_2}+61e^{4s_1+3s_2}-48e^{5s_1+3s_2} + e^{6s_1+3s_2})s_2-e^{s_1}(-7+30e^{s_1}-15e^{2s_1}+7e^{s_2}-4e^{s_1+s_2}+57e^{2(s_1+s_2)}-70e^{3(s_1+s_2)} - 69e^{2s_1+s_2}+42e^{3s_1+s_2}-26e^{s_1+2s_2}+28e^{3s_1+2s_2}-35e^{4s_1+2s_2}+27e^{2s_1+3s_2} + 35e^{4s_1+3s_2})s_3)s_1^3-(3(11-36e^{s_1}+39e^{2s_1}-38e^{3s_1}-18e^{s_1+s_2}-25e^{2(s_1+s_2)}+40e^{3(s_1+s_2)} + 51e^{2s_1+s_2}-72e^{3s_1+s_2}+111e^{4s_1+s_2}+66e^{3s_1+2s_2}-9e^{4s_1+2s_2}-104e^{5s_1+2s_2}-105e^{4s_1+3s_2} + 66e^{5s_1+3s_2}+23e^{6s_1+3s_2})s_2^2+(2(-23+23e^{s_1}+45e^{s_1+s_2}-9e^{2(s_1+s_2)}-13e^{3(s_1+s_2)} - 57e^{2s_1+s_2}+33e^{3s_1+2s_2}+e^{4s_1+3s_2})(-1+e^{s_1})^2+3(6-41e^{s_1}+112e^{2s_1}-69e^{3s_1}+11e^{s_1+s_2} - 92e^{2(s_1+s_2)}+99e^{3(s_1+s_2)}+16e^{2s_1+s_2}-249e^{3s_1+s_2}+198e^{4s_1+s_2}+207e^{3s_1+2s_2} + 86e^{4s_1+2s_2}-177e^{5s_1+2s_2}-254e^{4s_1+3s_2}+123e^{5s_1+3s_2}+24e^{6s_1+3s_2})s_3)s_2 + 6e^{s_1}(-1+e^{s_2})s_3((7-30e^{s_1}+15e^{2s_1}-26e^{s_1+s_2}+27e^{2(s_1+s_2)}+84e^{2s_1+s_2}-42e^{3s_1+s_2} - 70e^{3s_1+2s_2}+35e^{4s_1+2s_2})s_3-4(-1+e^{s_1})^2(2-5e^{s_1+s_2}+3e^{2(s_1+s_2)})))s_1^2+(-3(7-24e^{s_1} + 27e^{2s_1}-34e^{3s_1}-12e^{s_1+s_2}-17e^{2(s_1+s_2)}+14e^{3(s_1+s_2)}+33e^{2s_1+s_2}-54e^{3s_1+s_2} + 105e^{4s_1+s_2}+42e^{3s_1+2s_2}+15e^{4s_1+2s_2}-112e^{5s_1+2s_2}-27e^{4s_1+3s_2}-12e^{5s_1+3s_2} + 49e^{6s_1+3s_2})s_2^3-3((13-64e^{s_1}+137e^{2s_1}-94e^{3s_1}-14e^{s_1+s_2}-63e^{2(s_1+s_2)}+56e^{3(s_1+s_2)} + 89e^{2s_1+s_2}-336e^{3s_1+s_2}+285e^{4s_1+s_2}+114e^{3s_1+2s_2}+217e^{4s_1+2s_2}-292e^{5s_1+2s_2} - 115e^{4s_1+3s_2}-42e^{5s_1+3s_2}+109e^{6s_1+3s_2})s_3-16(-1+e^{s_1})^2(-1+e^{s_1+s_2})^2(1-e^{s_1} + e^{2s_1+s_2}))s_2^2+s_3(2(-1+e^{s_1})^2(1-49e^{s_1}+45e^{s_1+s_2}-105e^{2(s_1+s_2)}+59e^{3(s_1+s_2)}$$

$$+ 135e^{2s_1+s_2} - 111e^{3s_1+2s_2} + 25e^{4s_1+3s_2}) - 3(6 - 47e^{s_1} + 140e^{2s_1} - 75e^{3s_1} + 5e^{s_1+s_2}$$
$$- 72e^{2(s_1+s_2)} + 69e^{3(s_1+s_2)} + 52e^{2s_1+s_2} - 351e^{3s_1+s_2} + 222e^{4s_1+s_2} + 129e^{3s_1+2s_2}$$
$$+ 230e^{4s_1+2s_2} - 215e^{5s_1+2s_2} - 158e^{4s_1+3s_2} + 5e^{5s_1+3s_2} + 60e^{6s_1+3s_2})s_3)s_2$$
$$- 3e^{s_1}(-1+e^{s_2})s_3^2((7 - 30e^{s_1} + 15e^{2s_1} - 26e^{s_1+s_2} + 27e^{2(s_1+s_2)} + 84e^{2s_1+s_2} - 42e^{3s_1+s_2}$$
$$- 70e^{3s_1+2s_2} + 35e^{4s_1+2s_2})s_3 - 8(-1+e^{s_1})^2(2 - 5e^{s_1+s_2} + 3e^{2(s_1+s_2)})))s_1$$
$$- s_2(s_2+s_3)(3(1 - 4e^{s_1} + 5e^{2s_1} - 10e^{3s_1} - 2e^{s_1+s_2} - 3e^{2(s_1+s_2)} - 4e^{3(s_1+s_2)} + 5e^{2s_1+s_2}$$
$$- 12e^{3s_1+s_2} + 33e^{4s_1+s_2} + 6e^{3s_1+2s_2} + 13e^{4s_1+2s_2} - 40e^{5s_1+2s_2} + 17e^{4s_1+3s_2} - 30e^{5s_1+3s_2}$$
$$+ 25e^{6s_1+3s_2})s_2^2 + 2(-1+e^{s_1})(3(-3 + 10e^{s_1} - 15e^{2s_1} + 8e^{s_1+s_2} - 3e^{2(s_1+s_2)} + 6e^{3(s_1+s_2)}$$
$$- 24e^{2s_1+s_2} + 48e^{3s_1+s_2} + 18e^{3s_1+2s_2} - 55e^{4s_1+2s_2} - 20e^{4s_1+3s_2} + 30e^{5s_1+3s_2})s_3$$
$$- (-1+e^{s_1})(1 - e^{s_1} - 3e^{s_1+s_2} + 15e^{2(s_1+s_2)} - 13e^{3(s_1+s_2)} + 15e^{2s_1+s_2} - 39e^{3s_1+2s_2}$$
$$+ 25e^{4s_1+3s_2}))s_2 + s_3(3(5 - 22e^{s_1} + 45e^{2s_1} - 20e^{3s_1} - 14e^{s_1+s_2} + 9e^{2(s_1+s_2)} - 8e^{3(s_1+s_2)}$$
$$+ 59e^{2s_1+s_2} - 132e^{3s_1+s_2} + 63e^{4s_1+s_2} - 48e^{3s_1+2s_2} + 133e^{4s_1+2s_2} - 70e^{5s_1+2s_2} + 35e^{4s_1+3s_2}$$
$$- 70e^{5s_1+3s_2} + 35e^{6s_1+3s_2})s_3 - 2(-1+e^{s_1})^2(23 - 47e^{s_1} - 69e^{s_1+s_2} + 81e^{2(s_1+s_2)}$$
$$- 35e^{3(s_1+s_2)} + 153e^{2s_1+s_2} - 177e^{3s_1+2s_2} + 71e^{4s_1+3s_2})))),$$

$$k_{17,28}(s_1,s_2,s_3) = \frac{k_{17,27}^{\text{num}}(s_1,s_2,s_3)}{3(e^{s_1}-1)^3(e^{s_1+s_2}-1)^3(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2},$$

$$k_{17,29}(s_1,s_2,s_3) = -\frac{16\pi e^{2s_1+3s_2}((e^{s_1}(2e^{s_2}-1)-1)s_1 + (e^{s_1}-1)s_2)}{(e^{s_2}-1)(e^{s_1+s_2}-1)^3(e^{\frac{s_3}{2}}+1)s_1s_2(s_1+s_2)(s_1+s_2+s_3)},$$

$$k_{17,30}(s_1,s_2,s_3) = \frac{16\pi e^{2s_1+3s_2}((e^{s_1}(2e^{s_2}-1)-1)s_1 + (e^{s_1}-1)s_2)}{(e^{s_2}-1)(e^{s_1+s_2}-1)^3(e^{\frac{s_3}{2}}-1)s_1s_2(s_1+s_2)(s_1+s_2+s_3)}.$$

C.2.2. *The functions* k_{18} *and* k_{19}. As we mentioned earlier, the functions K_{18} and K_{19} appearing in (6) are identical. Therefore, the derived three variable functions k_{18} and k_{19} match precisely. For the function k_{18} we have

$$k_{18}(s_1,s_2,s_3) = K_{18}(s_1,s_2,s_3,-s_1-s_2-s_3) = \sum_{i=1}^{30} k_{18,i}(s_1,s_2,s_3),$$

where

$$k_{18,1}(s_1,s_2,s_3)(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4 s_1^2(s_1+s_2)^2(s_1+s_2+s_3)^2$$
$$= 16\pi e^{2s_1}(2e^{3s_2} - 39e^{2s_1+s_2} - 40e^{s_1+2s_2} + 10e^{4s_1+2s_2} - 8e^{3s_1+3s_2} + 5e^{2s_1+4s_2} + 16e^{5s_1+4s_2}$$
$$+ 12e^{4s_1+5s_2} - 12)s_2(s_2+s_3),$$

$$k_{18,2}(s_1,s_2,s_3)3(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4 s_1^2(s_1+s_2)^2(s_1+s_2+s_3)^2$$
$$= -16\pi(-4e^{s_1} - 28e^{3s_1} - 5e^{4s_1} - e^{s_2} - 20e^{2s_1+s_2} - 116e^{3s_1+s_2} + 20e^{5s_1+s_2} + 4e^{s_1+2s_2}$$
$$- 10e^{2s_1+2s_2} + 104e^{4s_1+2s_2} - 188e^{5s_1+2s_2} + 20e^{3s_1+3s_2} - 200e^{4s_1+3s_2} + 142e^{6s_1+3s_2}$$
$$+ 20e^{7s_1+3s_2} + 4e^{3s_1+4s_2} + 140e^{5s_1+4s_2} - 76e^{6s_1+4s_2} - 5e^{8s_1+4s_2} - e^{4s_1+5s_2} + 4e^{5s_1+5s_2}$$
$$+ 28e^{7s_1+5s_2} + 5e^{8s_1+5s_2} + 1)s_2(s_2+s_3),$$

$$k_{18,3}(s_1,s_2,s_3)3(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4 s_1(s_1+s_2)^2 s_3(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= -16\pi(-25e^{s_1} - 31e^{3s_1} + 7e^{4s_1} - 4e^{s_2} + 4e^{s_1+s_2} - 140e^{3s_1+s_2} + 104e^{4s_1+s_2} + 104e^{3s_1+3s_2}$$
$$- 140e^{4s_1+3s_2} + 4e^{6s_1+3s_2} - 4e^{7s_1+3s_2} + 7e^{3s_1+4s_2} - 31e^{4s_1+4s_2} - 25e^{6s_1+4s_2}$$
$$+ 4e^{7s_1+4s_2} + 4)s_2^3,$$

$$k_{18,4}(s_1,s_2,s_3)(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4 s_1(s_1+s_2)^2 s_3(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= -16\pi e^{s_1}s_2^3(2e^{s_1+3s_2}(e^{s_1+s_2}-2)s_2(e^{s_1}-1)^5 + 15e^{s_1} + 7e^{2s_2} + 20e^{s_1+s_2} - 8e^{4s_1+s_2}$$

$$- 27e^{s_1+2s_2} + 20e^{2s_1+2s_2} + 20e^{3s_1+2s_2} - 27e^{4s_1+2s_2} + 7e^{5s_1+2s_2} - 8e^{s_1+3s_2} + 20e^{4s_1+3s_2} + 15e^{4s_1+4s_2}),$$

$$k_{18,5}(s_1,s_2,s_3)3(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4(s_1+s_2)^2s_3(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= 16\pi((-7e^{s_1} - e^{3s_1} + e^{4s_1} - 4e^{s_2} - 14e^{s_1+s_2} - 80e^{3s_1+s_2} + 56e^{4s_1+s_2} + 74e^{3s_1+3s_2} - 92e^{4s_1+3s_2} + 4e^{6s_1+3s_2} - 10e^{7s_1+3s_2} + 7e^{3s_1+4s_2} - 25e^{4s_1+4s_2} - 31e^{6s_1+4s_2} + 10e^{7s_1+4s_2} + 4)s_1^2 + (11e^{s_1} + 29e^{3s_1} - 5e^{4s_1} - 4e^{s_2} - 32e^{s_1+s_2} - 20e^{3s_1+s_2} + 8e^{4s_1+s_2} + 44e^{3s_1+3s_2} - 44e^{4s_1+3s_2} + 4e^{6s_1+3s_2} - 16e^{7s_1+3s_2} + 7e^{3s_1+4s_2} - 19e^{4s_1+4s_2} - 37e^{6s_1+4s_2} + 16e^{7s_1+4s_2} + 4)s_2s_1 + (43e^{s_1} + 61e^{3s_1} - 13e^{4s_1} + 4e^{s_2} - 22e^{s_1+s_2} + 200e^{3s_1+s_2} - 152e^{4s_1+s_2} - 134e^{3s_1+3s_2} + 188e^{4s_1+3s_2} - 4e^{6s_1+3s_2} - 2e^{7s_1+3s_2} - 7e^{3s_1+4s_2} + 37e^{4s_1+4s_2} + 19e^{6s_1+4s_2} + 2e^{7s_1+4s_2} - 4)s_2^2),$$

$$k_{18,6}(s_1,s_2,s_3)(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4(s_1+s_2)^2s_3(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= -16e^{s_1}\pi(2e^{s_1}(-1+e^{s_1})(-1+e^{s_2})(-2+e^{s_1} - 2e^{s_2} + 2e^{2s_2} + 9e^{s_1+s_2} - 4e^{2s_1+s_2} - 3e^{s_1+2s_2} - 4e^{2s_1+2s_2} + 2e^{3s_1+2s_2} - e^{s_1+3s_2} + 2e^{2s_1+3s_2})s_1^3 + (2e^{s_1}(-1+e^{s_1})(6 - 3e^{s_1} - 12e^{2s_2} + 4e^{3s_2} - 24e^{s_1+s_2} + 12e^{2s_1+s_2} + 36e^{s_1+2s_2} - 6e^{3s_1+2s_2} + 2e^{s_1+3s_2} - 30e^{2s_1+3s_2} + 14e^{3s_1+3s_2} - 2e^{4s_1+3s_2} - 2e^{s_1+4s_2} + 2e^{2s_1+4s_2} + 6e^{3s_1+4s_2} - 4e^{4s_1+4s_2} + e^{5s_1+4s_2})s_2 - e^{s_1} - 7e^{2s_2} - 20e^{s_1+s_2} + 6e^{4s_1+s_2} + 13e^{s_1+2s_2} - 20e^{3s_1+2s_2} + 23e^{4s_1+2s_2} - 9e^{5s_1+2s_2} + 8e^{s_1+3s_2} - 16e^{4s_1+3s_2} - 13e^{4s_1+4s_2})s_1^2 + s_2(6e^{s_1}(-1+e^{s_1})(2 - e^{s_1} - 4e^{2s_2} - 8e^{s_1+s_2} + 4e^{2s_1+s_2} + 12e^{s_1+2s_2} - 2e^{3s_1+2s_2} + 6e^{s_1+3s_2} - 18e^{2s_1+3s_2} + 10e^{3s_1+3s_2} - 2e^{4s_1+3s_2} - 2e^{2s_1+4s_2} + 6e^{3s_1+4s_2} - 4e^{4s_1+4s_2} + e^{5s_1+4s_2})s_2 + 13e^{s_1} - 7e^{2s_2} - 20e^{s_1+s_2} + 4e^{4s_1+s_2} - e^{s_1+2s_2} + 20e^{2s_1+2s_2} - 20e^{3s_1+2s_2} + 19e^{4s_1+2s_2} - 11e^{5s_1+2s_2} + 8e^{s_1+3s_2} - 12e^{4s_1+3s_2} - 11e^{4s_1+4s_2})s_1 + s_2^2(2e^{s_1}(-1+e^{s_1})(2 - e^{s_1} - 4e^{2s_2} - 4e^{3s_2} - 8e^{s_1+s_2} + 4e^{2s_1+s_2} + 12e^{s_1+2s_2} - 2e^{3s_1+2s_2} + 22e^{s_1+3s_2} - 42e^{2s_1+3s_2} + 26e^{3s_1+3s_2} - 6e^{4s_1+3s_2} + 2e^{s_1+4s_2} - 10e^{2s_1+4s_2} + 18e^{3s_1+4s_2} - 12e^{4s_1+4s_2} + 3e^{5s_1+4s_2})s_2 + 29e^{s_1} + 7e^{2s_2} + 20e^{s_1+s_2} - 10e^{4s_1+s_2} - 41e^{s_1+2s_2} + 40e^{2s_1+2s_2} + 20e^{3s_1+2s_2} - 31e^{4s_1+2s_2} + 5e^{5s_1+2s_2} - 8e^{s_1+3s_2} + 24e^{4s_1+3s_2} + 17e^{4s_1+4s_2})),$$

$$k_{18,7}(s_1,s_2,s_3)5(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4s_1(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= 64\pi e^{2s_1}s_2((-13e^{3s_2} - 2e^{4s_1+2s_2} + 7e^{4s_1+5s_2} - 7)s_2^2 + (-11e^{3s_2} - 4e^{4s_1+2s_2} + 14e^{4s_1+5s_2} - 14)s_3s_2 + (-3e^{3s_2} - 2e^{4s_1+2s_2} + 7e^{4s_1+5s_2} - 7)s_3^2),$$

$$k_{18,8}(s_1,s_2,s_3)3(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= -16\pi(4e^{2s_1+s_2}(-8 - 35e^{s_1} + 14e^{s_2} + 194e^{2s_1+s_2} + 82e^{3s_1+s_2} - 76e^{s_1+2s_2} + 136e^{2s_1+2s_2} + 52e^{4s_1+2s_2} - 13e^{3s_1+3s_2} + 140e^{4s_1+3s_2})s_2^2 + (14 - 97e^{s_1} - 215e^{3s_1} + 13e^{4s_1} - 14e^{s_2} - 676e^{3s_1+s_2} + 61e^{s_1+2s_2} + 380e^{4s_1+2s_2} + 256e^{3s_1+3s_2} - 908e^{4s_1+3s_2} + 512e^{6s_1+3s_2} + 20e^{7s_1+3s_2} + 47e^{3s_1+4s_2} - 329e^{6s_1+4s_2} - 10e^{4s_1+5s_2} + 136e^{7s_1+5s_2})s_2 + 2e^{s_1}s_3((-1+e^{s_2})(23 + 7e^{3s_1} + 10e^{s_2} - 190e^{3s_1+s_2} - 360e^{3s_1+2s_2} - 38e^{3s_1+3s_2} - 28e^{6s_1+3s_2} - 4e^{3s_1+4s_2} + 67e^{6s_1+4s_2}) + 2e^{s_1+s_2}(-4 - 25e^{s_1} + 4e^{s_2} + 94e^{2s_1+s_2} + 44e^{3s_1+s_2} - 2e^{s_1+2s_2} - 16e^{2s_1+2s_2} - 22e^{4s_1+2s_2} + 28e^{3s_1+3s_2} + 37e^{4s_1+3s_2})s_3) + 2s_1(2e^{2s_1+s_2}(-16 - 4e^{s_1} + 40e^{s_2} + 286e^{2s_1+s_2} + 71e^{3s_1+s_2} - 23e^{s_1+2s_2} + 56e^{2s_1+2s_2} + 38e^{4s_1+2s_2} + 31e^{3s_1+3s_2} + 118e^{4s_1+3s_2})s_2 - 22e^{s_1} - 62e^{3s_1} + 7e^{4s_1} - 2e^{s_2} - 97e^{3s_1+s_2} + e^{s_1+2s_2} + 86e^{4s_1+2s_2} + 22e^{3s_1+3s_2} - 158e^{4s_1+3s_2} + 173e^{6s_1+3s_2} - 16e^{7s_1+3s_2} + 23e^{3s_1+4s_2} - 113e^{6s_1+4s_2} - 7e^{4s_1+5s_2} + 55e^{7s_1+5s_2} + 2e^{2s_1+s_2}(-16 - 31e^{s_1} + 28e^{s_2} + 262e^{2s_1+s_2} + 74e^{3s_1+s_2} + 13e^{s_1+2s_2} - 58e^{2s_1+2s_2} - 31e^{4s_1+2s_2} + 67e^{3s_1+3s_2} + 82e^{4s_1+3s_2})s_3 + 2)),$$

$$k_{18,9}(s_1,s_2,s_3)3(e^{s_1}-1)^4(e^{s_1+s_2}-1)^4 s_2(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= -16\pi e^{s_1}((-5e^{3s_1}-5e^{s_2}-55e^{3s_1+s_2}-63e^{3s_1+2s_2}-23e^{3s_1+3s_2}+14e^{6s_1+3s_2}-4e^{3s_1+4s_2}$$
$$+28e^{6s_1+4s_2}+5)s_1^2+2(4e^{3s_1}-2e^{s_2}-e^{3s_1+s_2}-45e^{3s_1+2s_2}-17e^{3s_1+3s_2}-7e^{6s_1+3s_2}$$
$$-e^{3s_1+4s_2}+13e^{6s_1+4s_2}+2)s_3^2),$$

$$k_{18,10}(s_1,s_2,s_3)(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= 16\pi(-2e^{2s_1}(-2+16e^{s_1}-4e^{2s_1}-8e^{s_2}+20e^{2s_2}-8e^{3s_2}+6e^{s_1+s_2}-64e^{2s_1+s_2}+16e^{3s_1+s_2}$$
$$-48e^{s_1+2s_2}+116e^{2s_1+2s_2}+16e^{3s_1+2s_2}-4e^{4s_1+2s_2}+12e^{s_1+3s_2}-16e^{2s_1+3s_2}-92e^{3s_1+3s_2}$$
$$+8e^{4s_1+3s_2}-4e^{5s_1+3s_2}+3e^{s_1+4s_2}-7e^{2s_1+4s_2}+30e^{3s_1+4s_2}+24e^{4s_1+4s_2}-e^{5s_1+4s_2}$$
$$+e^{6s_1+4s_2}-8e^{4s_1+5s_2}-2e^{5s_1+5s_2})s_1^2+2e^{s_1}(e^{2s_1+s_2}(106e^{s_1}+64e^{s_2}+228e^{2s_1+2s_2}$$
$$-3e^{s_1+3s_2}+8e^{4s_1+3s_2})s_2-25e^{s_1}-7e^{s_2}+18e^{s_1+s_2}-44e^{3s_1+s_2}+2e^{s_1+2s_2}-38e^{2s_1+2s_2}$$
$$+65e^{4s_1+2s_2}+5e^{s_1+3s_2}+2e^{4s_1+3s_2}+20e^{3s_1+4s_2}-60e^{4s_1+4s_2}+13e^{6s_1+4s_2}-3e^{7s_1+4s_2}$$
$$-7e^{4s_1+5s_2}+20e^{5s_1+5s_2}+3e^{7s_1+5s_2}-e^{2s_1+s_2}(-86e^{s_1}-32e^{s_2}-152e^{2s_1+2s_2}-3e^{s_1+3s_2}$$
$$+11e^{4s_1+3s_2})s_3)s_1+2e^{3s_1+s_2}(68e^{s_1}+16e^{s_2}+244e^{2s_1+2s_2}-15e^{s_1+3s_2}+13e^{4s_1+3s_2})s_2^2$$
$$-2s_3(2e^{4s_1+s_2}(-15-e^{3s_2}-34e^{s_1+2s_2}+4e^{3s_1+3s_2})s_3+32e^{2s_1}-23e^{3s_1}-e^{s_2}-18e^{2s_1+s_2}$$
$$-67e^{3s_1+s_2}+16e^{5s_1+s_2}-7e^{2s_1+2s_2}+74e^{3s_1+2s_2}-104e^{5s_1+2s_2}-23e^{6s_1+2s_2}-7e^{2s_1+3s_2}$$
$$+10e^{3s_1+3s_2}+2e^{5s_1+3s_2}+93e^{6s_1+3s_2}+6e^{3s_1+4s_2}+83e^{5s_1+4s_2}-46e^{6s_1+4s_2}+3e^{5s_1+5s_2}$$
$$-24e^{6s_1+5s_2}+1)+e^{s_1}s_2(-2e^{s_1+s_2}(-8-48e^{s_1}-86e^{2s_1}+8e^{s_2}+180e^{2s_1+s_2}+82e^{3s_1+s_2}$$
$$-26e^{s_1+2s_2}+16e^{2s_1+2s_2}-244e^{3s_1+2s_2}-12e^{4s_1+2s_2}+e^{2s_1+3s_2}+38e^{3s_1+3s_2}+88e^{4s_1+3s_2}$$
$$+10e^{5s_1+3s_2})s_3-95e^{s_1}-12e^{s_2}-4e^{s_1+s_2}-214e^{3s_1+s_2}+71e^{s_1+2s_2}-196e^{2s_1+2s_2}$$
$$+259e^{4s_1+2s_2}+13e^{5s_1+2s_2}+28e^{s_1+3s_2}-68e^{4s_1+3s_2}+39e^{3s_1+4s_2}-183e^{4s_1+4s_2}$$
$$+52e^{6s_1+4s_2}-2e^{7s_1+4s_2}-8e^{4s_1+5s_2}+48e^{5s_1+5s_2}+2e^{7s_1+5s_2})),$$

$$k_{18,11}(s_1,s_2,s_3)5(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= 32\pi(2e^{2s_1}(-16e^{3s_2}-4e^{4s_1+2s_2}+24e^{4s_1+5s_2}-9)s_2^2+(8e^{s_1}-42e^{2s_1}-107e^{3s_1}+3e^{4s_1}+2e^{s_2}$$
$$-12e^{5s_1+s_2}-8e^{s_1+2s_2}-42e^{6s_1+2s_2}-8e^{2s_1+3s_2}+78e^{7s_1+3s_2}-3e^{3s_1+4s_2}-12e^{8s_1+4s_2}$$
$$+2e^{4s_1+5s_2}-8e^{5s_1+5s_2}+72e^{6s_1+5s_2}+72e^{7s_1+5s_2}+2e^{8s_1+5s_2}-2)s_3s_2+2e^{2s_1}(2e^{3s_2}$$
$$-12e^{4s_1+2s_2}+12e^{4s_1+5s_2}-7)s_3^2+2e^{2s_1}s_1((12e^{3s_2}+3e^{4s_1+2s_2}+32e^{4s_1+5s_2}+3)s_2$$
$$+(17e^{3s_2}-22e^{4s_1+2s_2}+32e^{4s_1+5s_2}-7)s_3)),$$

$$k_{18,12}(s_1,s_2,s_3)15(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= 32\pi((16e^{s_1}-254e^{3s_1}+26e^{4s_1}+4e^{s_2}-104e^{5s_1+s_2}-16e^{s_1+2s_2}+256e^{7s_1+3s_2}+29e^{3s_1+4s_2}$$
$$-79e^{8s_1+4s_2}+4e^{4s_1+5s_2}-16e^{5s_1+5s_2}+134e^{7s_1+5s_2}+4e^{8s_1+5s_2}-4)s_2^2+2(4e^{s_1}-56e^{3s_1}$$
$$-e^{4s_1}+e^{s_2}+4e^{5s_1+s_2}-4e^{s_1+2s_2}+34e^{7s_1+3s_2}-4e^{3s_1+4s_2}-e^{8s_1+4s_2}+e^{4s_1+5s_2}$$
$$-4e^{5s_1+5s_2}+41e^{7s_1+5s_2}+e^{8s_1+5s_2}-1)s_3^2+s_1((8e^{s_1}-397e^{3s_1}+73e^{4s_1}+2e^{s_2}$$
$$-292e^{5s_1+s_2}-8e^{s_1+2s_2}+218e^{7s_1+3s_2}-23e^{3s_1+4s_2}-62e^{8s_1+4s_2}+2e^{4s_1+5s_2}-8e^{5s_1+5s_2}$$
$$+112e^{7s_1+5s_2}+2e^{8s_1+5s_2}-2)s_2+(8e^{s_1}-322e^{3s_1}+28e^{4s_1}+2e^{s_2}-112e^{5s_1+s_2}-8e^{s_1+2s_2}$$
$$+158e^{7s_1+3s_2}-23e^{3s_1+4s_2}-17e^{8s_1+4s_2}+2e^{4s_1+5s_2}-8e^{5s_1+5s_2}+142e^{7s_1+5s_2}$$
$$+2e^{8s_1+5s_2}-2)s_3)),$$

$$k_{18,13}(s_1,s_2,s_3)(e^{s_1}-1)^4(e^{s_1+s_2}-1)^4 s_2(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= 16e^{s_1}\pi(2e^{s_1}(-1+e^{s_2})(-3e^{s_1}+e^{2s_1}+2e^{s_2}-6e^{s_1+s_2}+14e^{2s_1+s_2}-4e^{3s_1+s_2}-e^{s_1+2s_2}$$
$$+3e^{2s_1+2s_2}-10e^{3s_1+2s_2}+2e^{4s_1+2s_2}+2e^{4s_1+3s_2})s_1^3+e^{s_1}(2(7e^{s_1}-e^{2s_1}-4e^{s_2}$$
$$+4e^{2s_2}+5e^{s_1+s_2}-29e^{2s_1+s_2}+4e^{3s_1+s_2}-11e^{s_1+2s_2}+27e^{2s_1+2s_2}+14e^{3s_1+2s_2}$$
$$+2e^{4s_1+2s_2}-e^{s_1+3s_2}+3e^{2s_1+3s_2}-18e^{3s_1+3s_2}-6e^{4s_1+3s_2}-2e^{5s_1+3s_2}+4e^{4s_1+4s_2}$$

$$+ 2e^{5s_1+4s_2})s_3 - 9e^{s_1} - 5e^{s_2} - 13e^{s_1+s_2} - 2e^{3s_1+s_2} + 3e^{s_1+2s_2} - 27e^{3s_1+2s_2} + 3e^{4s_1+2s_2}$$
$$- 3e^{s_1+3s_2} - 27e^{3s_1+3s_2} + 27e^{4s_1+3s_2} - 4e^{3s_1+4s_2} + 10e^{4s_1+4s_2} + 2e^{6s_1+4s_2} + 9)s_1^2$$
$$+ s_3((-1 + e^{s_1})(3 - 14e^{s_1} - e^{2s_1} - 3e^{s_2} + 8e^{s_1+s_2} + 31e^{2s_1+s_2} + 12e^{3s_1+s_2} - 2e^{s_1+2s_2}$$
$$- 3e^{2s_1+2s_2} - 54e^{3s_1+2s_2} - 13e^{4s_1+2s_2} + 7e^{2s_1+3s_2} - 12e^{3s_1+3s_2} + 53e^{4s_1+3s_2} - 2e^{3s_1+4s_2}$$
$$+ 4e^{4s_1+4s_2} - 16e^{5s_1+4s_2} + 2e^{6s_1+4s_2}) + 4e^{s_1}(2e^{s_1} - e^{s_2} + e^{2s_2} + e^{s_1+s_2} - 8e^{2s_1+s_2}$$
$$- 3e^{s_1+2s_2} + 8e^{2s_1+2s_2} + 4e^{3s_1+2s_2} + 2e^{4s_1+2s_2} - 4e^{3s_1+3s_2} - 3e^{4s_1+3s_2} - e^{5s_1+3s_2}$$
$$+ e^{4s_1+4s_2} + e^{5s_1+4s_2})s_3)s_1 + 2e^{s_1}(4 - 2e^{s_1} - 3e^{s_2} + e^{2s_2} - 5e^{s_1+s_2} + 7e^{3s_1+s_2} - e^{s_1+2s_2}$$
$$- 7e^{3s_1+2s_2} - 8e^{4s_1+2s_2} - 2e^{s_1+3s_2} - 19e^{3s_1+3s_2} + 13e^{4s_1+3s_2} - e^{3s_1+4s_2} + 5e^{4s_1+4s_2})s_3^2),$$

$$k_{18,14}(s_1, s_2, s_3)3(e^{s_1} - 1)^4(e^{s_2} - 1)(e^{s_1+s_2} - 1)^4 s_1(s_1 + s_2)^2(s_2 + s_3)(s_1 + s_2 + s_3)^2$$
$$= -16\pi(4e^{2s_1+s_2}(-1 - 22e^{s_1} - 2e^{s_2} + 46e^{2s_1+s_2} + 32e^{3s_1+s_2} - 41e^{s_1+2s_2} + 68e^{2s_1+2s_2}$$
$$+ 23e^{4s_1+2s_2} - 14e^{3s_1+3s_2} + 55e^{4s_1+3s_2})s_2^3 + (4e^{2s_1+s_2}(-2 - 44e^{s_1} - 4e^{s_2} + 92e^{2s_1+s_2}$$
$$+ 64e^{3s_1+s_2} - 37e^{s_1+2s_2} + 46e^{2s_1+2s_2} + e^{4s_1+2s_2} + 17e^{3s_1+3s_2} + 65e^{4s_1+3s_2})s_3 - 62e^{s_1}$$
$$- 146e^{3s_1} - e^{4s_1} - 11e^{s_2} + 4e^{s_1+s_2} + 58e^{2s_1+s_2} - 610e^{3s_1+s_2} + 545e^{4s_1+s_2} + 14e^{5s_1+s_2}$$
$$+ 58e^{s_1+2s_2} - 196e^{2s_1+2s_2} + 320e^{4s_1+2s_2} - 650e^{5s_1+2s_2} + 214e^{3s_1+3s_2} - 832e^{4s_1+3s_2}$$
$$+ 380e^{6s_1+3s_2} + 58e^{7s_1+3s_2} + 14e^{3s_1+4s_2} - 31e^{4s_1+4s_2} + 398e^{5s_1+4s_2} - 230e^{6s_1+4s_2} -$$
$$140e^{7s_1+4s_2} - 11e^{8s_1+4s_2} - e^{4s_1+5s_2} - 2e^{5s_1+5s_2} + 82e^{7s_1+5s_2} + 11e^{8s_1+5s_2} + 11)s_2^2$$
$$+ s_3(4e^{2s_1+s_2}(-1 - 22e^{s_1} - 2e^{s_2} + 46e^{2s_1+s_2} + 32e^{3s_1+s_2} - 11e^{s_1+2s_2} + 8e^{2s_1+2s_2}$$
$$- 7e^{4s_1+2s_2} + 16e^{3s_1+3s_2} + 25e^{4s_1+3s_2})s_3 - 53e^{s_1} - 163e^{3s_1} - 19e^{4s_1} - 10e^{s_2} - 692e^{3s_1+s_2}$$
$$+ 53e^{s_1+2s_2} + 388e^{4s_1+2s_2} + 140e^{3s_1+3s_2} - 1012e^{4s_1+3s_2} + 580e^{6s_1+3s_2} + 112e^{7s_1+3s_2}$$
$$+ 7e^{3s_1+4s_2} - 277e^{6s_1+4s_2} - 2e^{4s_1+5s_2} + 128e^{7s_1+5s_2} + 10)s_2 + e^{s_1}(-16 - 11e^{3s_1} + 16e^{2s_2}$$
$$- 10e^{s_1+s_2} + 46e^{4s_1+s_2} - 50e^{s_1+2s_2} + 128e^{3s_1+2s_2} - 256e^{4s_1+2s_2} - 320e^{3s_1+3s_2} + 50e^{6s_1+3s_2}$$
$$+ 160e^{4s_1+4s_2} - 7e^{7s_1+4s_2} - e^{3s_1+5s_2} + 2e^{4s_1+5s_2} + 46e^{6s_1+5s_2} + 7e^{7s_1+5s_2})s_3^2),$$

$$k_{18,15}(s_1, s_2, s_3)(e^{s_1} - 1)^4(e^{s_2} - 1)(e^{s_1+s_2} - 1)^4 s_1(s_1 + s_2)^2(s_2 + s_3)(s_1 + s_2 + s_3)^2$$
$$= -16\pi(-4e^{3s_1+s_2}(7e^{s_1} - 4e^{s_2} + 46e^{2s_1+2s_2} - 4e^{s_1+3s_2} + 3e^{4s_1+3s_2})s_2^3 + 2e^{2s_1}(e^{s_1+s_2}(-28e^{s_1}$$
$$+ 16e^{s_2} - 124e^{2s_1+2s_2} + e^{s_1+3s_2} + 3e^{4s_1+3s_2})s_3 - 10e^{3s_2} + 88e^{s_1+2s_2} - 10e^{4s_1+2s_2}$$
$$+ 40e^{3s_1+3s_2} - 15e^{4s_1+5s_2} + 33)s_2^2 + e^{2s_1}s_3(4e^{s_1+s_2}(-7e^{s_1} + 4e^{s_2} - 26e^{2s_1+2s_2} - e^{s_1+3s_2}$$
$$+ 2e^{4s_1+3s_2})s_3 - 4e^{s_2} - 55e^{2s_2} - 16e^{3s_2} + 210e^{2s_1+s_2} + 28e^{3s_1+s_2} + 236e^{s_1+2s_2}$$
$$- 275e^{3s_1+2s_2} - 53e^{4s_1+2s_2} + 76e^{3s_1+3s_2} + 5e^{2s_1+4s_2} + 171e^{3s_1+4s_2} - 80e^{5s_1+4s_2}$$
$$- 6e^{6s_1+4s_2} - 48e^{4s_1+5s_2} + 6e^{6s_1+5s_2} + 75)s_2 - (-24e^{2s_1} + 16e^{3s_1} + e^{s_2} + 74e^{3s_1+s_2}$$
$$- 63e^{4s_1+s_2} - 80e^{3s_1+2s_2} + 26e^{6s_1+2s_2} + 4e^{2s_1+3s_2} - 10e^{3s_1+3s_2} - 16e^{5s_1+3s_2} - 68e^{6s_1+3s_2}$$
$$- 5e^{4s_1+4s_2} + 24e^{6s_1+4s_2} + 32e^{7s_1+4s_2} + 18e^{6s_1+5s_2} - 1)s_3^2),$$

$$k_{18,16}(s_1, s_2, s_3)15(e^{s_1} - 1)^4(e^{s_2} - 1)(e^{s_1+s_2} - 1)^4 s_1(s_1 + s_2)^2(s_2 + s_3)(s_1 + s_2 + s_3)^2$$
$$= 32\pi s_2(2(4e^{s_1} - 26e^{3s_1} - e^{4s_1} + e^{s_2} + 4e^{5s_1+s_2} - 4e^{s_1+2s_2} + 49e^{7s_1+3s_2} + 11e^{3s_1+4s_2}$$
$$- 16e^{8s_1+4s_2} + e^{4s_1+5s_2} - 4e^{5s_1+5s_2} + 26e^{7s_1+5s_2} + e^{8s_1+5s_2} - 1)s_2^2 + (16e^{s_1} - 104e^{3s_1}$$
$$- 4e^{4s_1} + 4e^{s_2} + 16e^{5s_1+s_2} - 16e^{s_1+2s_2} + 106e^{7s_1+3s_2} - e^{3s_1+4s_2} - 19e^{8s_1+4s_2} + 4e^{4s_1+5s_2}$$
$$- 16e^{5s_1+5s_2} + 104e^{7s_1+5s_2} + 4e^{8s_1+5s_2} - 4)s_3 s_2 + 2(4e^{s_1} - 26e^{3s_1} - e^{4s_1} + e^{s_2} + 4e^{5s_1+s_2}$$
$$- 4e^{s_1+2s_2} + 19e^{7s_1+3s_2} - 4e^{3s_1+4s_2} - e^{8s_1+4s_2} + e^{4s_1+5s_2} - 4e^{5s_1+5s_2} + 26e^{7s_1+5s_2}$$
$$+ e^{8s_1+5s_2} - 1)s_3^2),$$

$$k_{18,17}(s_1, s_2, s_3) = \frac{4\pi(s_1^2 - s_3 s_1 - s_2^2 + s_3^2)}{(e^{\frac{1}{2}(s_1+s_2+s_3)} - 1)^4 s_1(s_1 + s_2)s_3(s_2 + s_3)(s_1 + s_2 + s_3)},$$

$$k_{18,18}(s_1, s_2, s_3) = \frac{4\pi(s_1^2 - s_3 s_1 - s_2^2 + s_3^2)}{(e^{\frac{1}{2}(s_1+s_2+s_3)} + 1)^4 s_1(s_1 + s_2)s_3(s_2 + s_3)(s_1 + s_2 + s_3)},$$

$$k_{18,19}(s_1,s_2,s_3) = \frac{8\pi e^{2s_1}(s_2-s_3)}{(e^{s_1}-1)^2(e^{\frac{1}{2}(s_2+s_3)}-1)^2 s_1 s_2 s_3(s_1+s_2+s_3)},$$

$$k_{18,20}(s_1,s_2,s_3) = \frac{8\pi e^{2s_1}(s_2-s_3)}{(e^{s_1}-1)^2(e^{\frac{1}{2}(s_2+s_3)}+1)^2 s_1 s_2 s_3(s_1+s_2+s_3)},$$

$$\begin{aligned}
&k_{18,21}(s_1,s_2,s_3)(e^{s_1}-1)^3(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)s_1 s_2 s_3(s_2+s_3)(s_1+s_2+s_3)\\
&\quad = 8\pi e^{2s_1}((-3e^{s_1}-3e^{s_2}+e^{s_1+s_2}+5)s_2^2-2(e^{s_1}-1)(2s_3+e^{s_2}-1)s_2-s_3((e^{s_1}-3e^{s_2}\\
&\qquad +e^{s_1+s_2}+1)s_3-2(e^{s_1}-1)(e^{s_2}-1))),
\end{aligned}$$

$$\begin{aligned}
&k_{18,22}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^3 s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2\\
&\quad = -4\pi(((-3e^{s_1}-3e^{s_1+s_2}+2e^{2s_1+s_2}+4)s_2+e^{s_1}(e^{s_2}-1)s_3)s_1^3-((e^{s_1}+3e^{s_1+s_2}-4)s_2^2\\
&\qquad +(2(e^{s_1}-1)(e^{s_1+s_2}-1)+e^{s_1}(-3e^{s_2}+2e^{s_1+s_2}+1)s_3)s_2-2e^{s_1}(e^{s_2}-1)s_3^2)s_1^2-((-7e^{s_1}\\
&\qquad -3e^{s_1+s_2}+6e^{2s_1+s_2}+4)s_2^3+(-7e^{s_1}-5e^{s_1+s_2}+6e^{2s_1+s_2}+6)s_3 s_2^2\\
&\qquad -s_3(2(e^{s_1}-1)(e^{s_1+s_2}-1)+(-e^{s_1}+3e^{s_1+s_2}-2)s_3)s_2-e^{s_1}(e^{s_2}-1)s_3^3)s_1\\
&\qquad -s_2(s_2+s_3)((-5e^{s_1}-3e^{s_1+s_2}+4e^{2s_1+s_2}+4)s_2^2-2(e^{s_1}-1)(s_3+e^{s_1+s_2}-1)s_2\\
&\qquad -s_3((-3e^{s_1}-3e^{s_1+s_2}+4e^{2s_1+s_2}+2)s_3-2(e^{s_1}-1)(e^{s_1+s_2}-1)))),
\end{aligned}$$

$$k_{18,23}(s_1,s_2,s_3) = \frac{-k^{\mathrm{num}}_{18,21}(s_1,s_2,s_3)}{(e^{s_1}-1)^3(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)s_1 s_2 s_3(s_2+s_3)(s_1+s_2+s_3)},$$

$$k_{18,24}(s_1,s_2,s_3) = \frac{-k^{\mathrm{num}}_{18,22}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^3 s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2},$$

$$\begin{aligned}
&k_{18,25}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^2 s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2\\
&\quad = 2\pi(((-11+2e^{s_1}+21e^{s_1+s_2}-7e^{2s_1+s_2}-6e^{2s_1+2s_2}+e^{3s_1+2s_2})s_2+e^{s_1}(-1+e^{s_2})\\
&\qquad \times(-1+5e^{s_1+s_2})s_3)s_1^3-((11+16e^{s_1}-21e^{s_1+s_2}-21e^{2s_1+s_2}+6e^{2s_1+2s_2}+9e^{3s_1+2s_2})s_2^2\\
&\qquad +(2(-7+5e^{s_1}+12e^{s_1+s_2}-8e^{2s_1+s_2}-5e^{2s_1+2s_2}+3e^{3s_1+2s_2})+e^{s_1}(15+3e^{s_2}-13e^{s_1+s_2}\\
&\qquad -15e^{s_1+2s_2}+10e^{2s_1+2s_2})s_3)s_2-2e^{s_1}(-1+e^{s_2})s_3((-1+5e^{s_1+s_2})s_3-2e^{s_1+s_2}+2))s_1^2\\
&\qquad +((11-38e^{s_1}-21e^{s_1+s_2}+63e^{2s_1+s_2}+6e^{2s_1+2s_2}-21e^{3s_1+2s_2})s_2^3+(8e^{s_1}(-1+e^{s_1+s_2})^2\\
&\qquad +(21-54e^{s_1}-41e^{s_1+s_2}+79e^{2s_1+s_2}+16e^{2s_1+2s_2}-21e^{3s_1+2s_2})s_3)s_2^2+s_3(2(-7+5e^{s_1}\\
&\qquad +16e^{s_1+s_2}-12e^{2s_1+s_2}-9e^{2s_1+2s_2}+7e^{3s_1+2s_2})+(10-15e^{s_1}-21e^{s_1+s_2}+11e^{2s_1+s_2}\\
&\qquad +15e^{2s_1+2s_2})s_3)s_2+e^{s_1}(-1+e^{s_2})s_3^2((-1+5e^{s_1+s_2})s_3-4e^{s_1+s_2}+4))s_1-s_2(s_2+s_3)\\
&\qquad \times((-11+20e^{s_1}+21e^{s_1+s_2}-35e^{2s_1+s_2}-6e^{2s_1+2s_2}+11e^{3s_1+2s_2})s_2^2-2((5-9e^{s_1}-9e^{s_1+s_2}\\
&\qquad +13e^{2s_1+s_2})s_3+9e^{s_1}+12e^{s_1+s_2}-16e^{2s_1+s_2}-5e^{2s_1+2s_2}+7e^{3s_1+2s_2}-7)s_2-s_3((-1+2e^{s_1}\\
&\qquad +3e^{s_1+s_2}-9e^{2s_1+s_2}-6e^{2s_1+2s_2}+11e^{3s_1+2s_2})s_3-2(-3+5e^{s_1}+8e^{s_1+s_2}-12e^{2s_1+s_2}\\
&\qquad -5e^{2s_1+2s_2}+7e^{3s_1+2s_2})))),
\end{aligned}$$

$$k_{18,26}(s_1,s_2,s_3) = \frac{k^{\mathrm{num}}_{18,25}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^2 s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2}$$

$$\begin{aligned}
&k_{18,27}(s_1,s_2,s_3)3(e^{s_1}-1)^3(e^{s_1+s_2}-1)^3(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)s_1 s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2\\
&\quad = 2\pi(3((-5+8e^{s_1}+15e^{2s_1}-10e^{3s_1}+16e^{s_1+s_2}-23e^{2s_1+s_2}-42e^{3s_1+s_2}+25e^{4s_1+s_2}\\
&\qquad -21e^{2s_1+2s_2}+42e^{3s_1+2s_2}+11e^{4s_1+2s_2}-8e^{5s_1+2s_2}+2e^{3s_1+3s_2}-3e^{4s_1+3s_2}-8e^{5s_1+3s_2}\\
&\qquad +e^{6s_1+3s_2})s_2+e^{s_1}(-1+2e^{s_1}-9e^{2s_1}+e^{s_2}+4e^{s_1+s_2}-3e^{2s_1+s_2}+22e^{3s_1+s_2}-6e^{s_1+2s_2}\\
&\qquad +15e^{2s_1+2s_2}-28e^{3s_1+2s_2}-5e^{4s_1+2s_2}-3e^{2s_1+3s_2}+6e^{3s_1+3s_2}+5e^{4s_1+3s_2})s_3)s_1^3\\
&\qquad +(3(-5-6e^{s_1}+75e^{2s_1}-40e^{3s_1}+16e^{s_1+s_2}+27e^{2s_1+s_2}-222e^{3s_1+s_2}+107e^{4s_1+s_2}\\
&\qquad -21e^{2s_1+2s_2}+159e^{4s_1+2s_2}-66e^{5s_1+2s_2}+2e^{3s_1+3s_2}+3e^{4s_1+3s_2}-36e^{5s_1+3s_2}\\
&\qquad +7e^{6s_1+3s_2})s_2^2+(3e^{s_1}(-17+66e^{s_1}-57e^{2s_1}+3e^{s_2}+62e^{s_1+s_2}-189e^{2s_1+s_2}+148e^{3s_1+s_2}\\
&\qquad -18e^{s_1+2s_2}+3e^{2s_1+2s_2}+64e^{3s_1+2s_2}-73e^{4s_1+2s_2}-9e^{2s_1+3s_2}+24e^{3s_1+3s_2}-13e^{4s_1+3s_2}
\end{aligned}$$

$$
\begin{aligned}
&+ 6e^{5s_1+3s_2})s_3 - 2(-1+e^{s_1})^2(-23 - e^{s_1} + 69e^{s_1+s_2} - 9e^{2s_1+s_2} - 57e^{2s_1+2s_2} + 9e^{3s_1+2s_2}\\
&+ 11e^{3s_1+3s_2} + e^{4s_1+3s_2}))s_2 + 6e^{s_1}(-1+e^{s_2})s_3(4e^{s_1+s_2}(-1+e^{s_1+s_2})(-1+e^{s_1})^2\\
&+ (1 - 2e^{s_1} + 9e^{2s_1} - 6e^{s_1+s_2} + 12e^{2s_1+s_2} - 22e^{3s_1+s_2} - 3e^{2s_1+2s_2} + 6e^{3s_1+2s_2}\\
&+ 5e^{4s_1+2s_2})s_3))s_1^2 + (3(5 - 36e^{s_1} + 105e^{2s_1} - 50e^{3s_1} - 16e^{s_1+s_2} + 123e^{2s_1+s_2} - 318e^{3s_1+s_2}\\
&+ 139e^{4s_1+s_2} + 21e^{2s_1+2s_2} - 126e^{3s_1+2s_2} + 285e^{4s_1+2s_2} - 108e^{5s_1+2s_2} - 2e^{3s_1+3s_2}\\
&+ 15e^{4s_1+3s_2} - 48e^{5s_1+3s_2} + 11e^{6s_1+3s_2})s_2^3 + 3((11 - 64e^{s_1} + 159e^{2s_1} - 98e^{3s_1} - 34e^{s_1+s_2}\\
&+ 223e^{2s_1+s_2} - 480e^{3s_1+s_2} + 267e^{4s_1+s_2} + 39e^{2s_1+2s_2} - 210e^{3s_1+2s_2} + 383e^{4s_1+2s_2}\\
&- 188e^{5s_1+2s_2} - 8e^{3s_1+3s_2} + 27e^{4s_1+3s_2} - 38e^{5s_1+3s_2} + 11e^{6s_1+3s_2})s_3 - 16e^{s_1}(-1+e^{s_1})^2\\
&\times(-2+e^{s_1+s_2})(-1+e^{s_1+s_2})^2)s_2^2 + s_3(3(6 - 29e^{s_1} + 56e^{2s_1} - 57e^{3s_1} - 17e^{s_1+s_2} + 104e^{2s_1+s_2}\\
&- 165e^{3s_1+s_2} + 150e^{4s_1+s_2} + 12e^{2s_1+2s_2} - 69e^{3s_1+2s_2} + 70e^{4s_1+2s_2} - 85e^{5s_1+2s_2} - 9e^{3s_1+3s_2}\\
&+ 18e^{4s_1+3s_2} + 15e^{5s_1+3s_2})s_3 - 2(-1+e^{s_1})^2(23 - 47e^{s_1} - 69e^{s_1+s_2} + 105e^{2s_1+s_2}\\
&+ 81e^{2s_1+2s_2} - 81e^{3s_1+2s_2} - 35e^{3s_1+3s_2} + 23e^{4s_1+3s_2}))s_2 + 3e^{s_1}(-1+e^{s_2})s_3^2\\
&\times(8e^{s_1+s_2}(-1+e^{s_1+s_2})(-1+e^{s_1})^2 + (1 - 2e^{s_1} + 9e^{2s_1} - 6e^{s_1+s_2} + 12e^{2s_1+s_2} - 22e^{3s_1+s_2}\\
&- 3e^{2s_1+2s_2} + 6e^{3s_1+2s_2} + 5e^{4s_1+2s_2})s_3))s_1 + s_2(s_2+s_3)(3(5 - 22e^{s_1} + 45e^{2s_1} - 20e^{3s_1}\\
&- 16e^{s_1+s_2} + 73e^{2s_1+s_2} - 138e^{3s_1+s_2} + 57e^{4s_1+s_2} + 21e^{2s_1+2s_2} - 84e^{3s_1+2s_2} + 137e^{4s_1+2s_2}\\
&- 50e^{5s_1+2s_2} - 2e^{3s_1+3s_2} + 9e^{4s_1+3s_2} - 20e^{5s_1+3s_2} + 5e^{6s_1+3s_2})s_2^2\\
&- 2(-1+e^{s_1})((-1+e^{s_1})(23 - 47e^{s_1} - 69e^{s_1+s_2} + 129e^{2s_1+s_2} + 57e^{2s_1+2s_2} - 105e^{3s_1+2s_2}\\
&- 11e^{3s_1+3s_2} + 23e^{4s_1+3s_2}) + 3(3 - 10e^{s_1} + 15e^{2s_1} - 10e^{s_1+s_2} + 36e^{2s_1+s_2} - 42e^{3s_1+s_2}\\
&+ 15e^{2s_1+2s_2} - 42e^{3s_1+2s_2} + 35e^{4s_1+2s_2})s_3)s_2 + s_3(2(-1+e^{s_1})^2(-1 + e^{s_1} + 3e^{s_1+s_2}\\
&+ 9e^{2s_1+s_2} + 9e^{2s_1+2s_2} - 33e^{3s_1+2s_2} - 11e^{3s_1+3s_2} + 23e^{4s_1+3s_2}) - 3(-1 + 4e^{s_1} - 5e^{2s_1}\\
&+ 10e^{3s_1} + 4e^{s_1+s_2} - 19e^{2s_1+s_2} + 18e^{3s_1+s_2} - 27e^{4s_1+s_2} - 9e^{2s_1+2s_2} + 30e^{3s_1+2s_2}\\
&- 17e^{4s_1+2s_2} + 20e^{5s_1+2s_2} - 2e^{3s_1+3s_2} + 9e^{4s_1+3s_2} - 20e^{5s_1+3s_2} + 5e^{6s_1+3s_2})s_3))),
\end{aligned}
$$

$$k_{18,28}(s_1,s_2,s_3) = \frac{-k^{\text{num}}_{18,27}(s_1,s_2,s_3)}{3(e^{s_1}-1)^3(e^{s_1+s_2}-1)^3(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2},$$

$$k_{18,29}(s_1,s_2,s_3) = \frac{16\pi e^{2(s_1+s_2)}(2s_1 - e^{s_2}(s_1 - s_2 + e^{s_1}(s_1+s_2)))}{(e^{s_2}-1)(e^{s_1+s_2}-1)^3(e^{\frac{s_3}{2}}+1)s_1s_2(s_1+s_2)(s_1+s_2+s_3)},$$

$$k_{18,30}(s_1,s_2,s_3) = \frac{16\pi e^{2(s_1+s_2)}(e^{s_2}(s_1 - s_2 + e^{s_1}(s_1+s_2)) - 2s_1)}{(e^{s_2}-1)(e^{s_1+s_2}-1)^3(e^{\frac{s_3}{2}}-1)s_1s_2(s_1+s_2)(s_1+s_2+s_3)}.$$

C.2.3. *The function k_{20}.* We have

$$k_{20}(s_1,s_2,s_3) = K_{20}(s_1,s_2,s_3,-s_1-s_2-s_3) = \sum_{i=1}^{24} k_{20,i}(s_1,s_2,s_3),$$

where

$$k_{20,1}(s_1,s_2,s_3) = -\frac{16\pi(-3e^{s_1} + 33e^{2s_1} + 5e^{3s_1} + 1)s_2(s_2+s_3)}{(e^{s_1}-1)^3 s_1^2(s_1+s_2)^2(s_1+s_2+s_3)^2},$$

$$
\begin{aligned}
&k_{20,2}(s_1,s_2,s_3)(e^{s_1}-1)^2(e^{s_2}-1)(e^{s_1+s_2}-1)^4 s_1(s_1+s_2)^2 s_3(s_2+s_3)(s_1+s_2+s_3)^2\\
&\quad = -16\pi s_2^3(2e^{2s_1+3s_2}(e^{s_1+s_2}-4)s_2(e^{s_1}-1)^3 - 9e^{s_1} + e^{2s_1} - 2e^{s_2} - 2e^{s_1+s_2} + 30e^{2s_1+s_2}\\
&\qquad - 2e^{3s_1+s_2} + 11e^{s_1+2s_2} - 21e^{2s_1+2s_2} - 21e^{3s_1+2s_2} - 5e^{4s_1+2s_2} - 10e^{2s_1+3s_2} + 22e^{3s_1+3s_2}\\
&\qquad + 6e^{4s_1+3s_2} + 6e^{5s_1+3s_2} + e^{3s_1+4s_2} - e^{4s_1+4s_2} - 6e^{5s_1+4s_2} + 2),
\end{aligned}
$$

$$
\begin{aligned}
&k_{20,3}(s_1,s_2,s_3)(e^{s_1}-1)^3(e^{s_2}-1)(e^{s_1+s_2}-1)^4(s_1+s_2)^2 s_3(s_2+s_3)(s_1+s_2+s_3)^2\\
&\quad = -16\pi(2e^{2s_1}(-1+e^{s_2})(-4 + e^{s_1} - 4e^{s_2} + 2e^{2s_2} + 17e^{s_1+s_2} - 4e^{2s_1+s_2} - e^{s_1+2s_2} - 10e^{2s_1+2s_2}\\
&\qquad + e^{s_1+3s_2} - 4e^{2s_1+3s_2} + 6e^{3s_1+3s_2})s_1^3 + ((-1+e^{s_1})(-4 + 3e^{s_1} - 5e^{2s_1} + 4e^{s_2} + 14e^{s_1+s_2}
\end{aligned}
$$

$$- 20e^{2s_1+s_2} + 26e^{3s_1+s_2} - 17e^{s_1+2s_2} + 5e^{2s_1+2s_2} + 11e^{3s_1+2s_2} - 35e^{4s_1+2s_2} + 20e^{2s_1+3s_2}$$
$$- 26e^{3s_1+3s_2} + 12e^{4s_1+3s_2} + 18e^{5s_1+3s_2} - 11e^{3s_1+4s_2} + 19e^{4s_1+4s_2} - 10e^{5s_1+4s_2} - 4e^{6s_1+4s_2}$$
$$+ 4e^{4s_1+5s_2} - 8e^{5s_1+5s_2} + 4e^{6s_1+5s_2}) + 2e^{2s_1}(12 - 3e^{s_1} - 18e^{2s_2} + 2e^{3s_2} - 48e^{s_1+s_2}$$
$$+ 12e^{2s_1+s_2} + 54e^{s_1+2s_2} + 18e^{2s_1+2s_2} + 10e^{s_1+3s_2} - 42e^{2s_1+3s_2} - 2e^{3s_1+3s_2} - 4e^{4s_1+3s_2}$$
$$+ 4e^{s_1+4s_2} - 16e^{2s_1+4s_2} + 24e^{3s_1+4s_2} - 4e^{4s_1+4s_2} + e^{5s_1+4s_2})s_2)s_1^2 + (-1 + e^{s_1+s_2})$$
$$\times s_2((-1 + e^{s_1})(6 + 3e^{s_1} + 9e^{2s_1} - 6e^{s_2} - 20e^{s_1+s_2} + 13e^{2s_1+s_2} - 41e^{3s_1+s_2} + 17e^{s_1+2s_2}$$
$$- 9e^{2s_1+2s_2} + 12e^{3s_1+2s_2} + 34e^{4s_1+2s_2} - 13e^{2s_1+3s_2} + 21e^{3s_1+3s_2} - 18e^{4s_1+3s_2} - 8e^{5s_1+3s_2}$$
$$+ 8e^{3s_1+4s_2} - 16e^{4s_1+4s_2} + 8e^{5s_1+4s_2}) + 6e^{2s_1}(-4 + e^{s_1} + 6e^{2s_2} + 2e^{3s_2} + 12e^{s_1+s_2}$$
$$- 3e^{2s_1+s_2} - 18e^{s_1+2s_2} + 6e^{2s_1+2s_2} - 3e^{3s_1+2s_2} - 8e^{s_1+3s_2} + 12e^{2s_1+3s_2} - 4e^{3s_1+3s_2}$$
$$+ e^{4s_1+3s_2})s_2)s_1 + e^{s_1}s_2^2((-1 + e^{s_1})(-1 + e^{s_2})(15 + 3e^{s_1} + 5e^{s_2} - 37e^{s_1+s_2} - 22e^{2s_1+s_2}$$
$$+ 9e^{2s_1+2s_2} + 45e^{3s_1+2s_2} - 9e^{2s_1+3s_2} + 21e^{3s_1+3s_2} - 30e^{4s_1+3s_2} + 4e^{3s_1+4s_2} - 8e^{4s_1+4s_2}$$
$$+ 4e^{5s_1+4s_2}) + 2e^{s_1}(4 - e^{s_1} - 6e^{2s_2} - 10e^{3s_2} - 16e^{s_1+s_2} + 4e^{2s_1+s_2} + 18e^{s_1+2s_2} + 6e^{2s_1+2s_2}$$
$$+ 46e^{s_1+3s_2} - 78e^{2s_1+3s_2} + 42e^{3s_1+3s_2} - 12e^{4s_1+3s_2} + 4e^{s_1+4s_2} - 16e^{2s_1+4s_2} + 24e^{3s_1+4s_2}$$
$$- 12e^{4s_1+4s_2} + 3e^{5s_1+4s_2})s_2)),$$

$$k_{20,4}(s_1,s_2,s_3)3(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= 16\pi((139e^{s_1} + 397e^{3s_1} + 29e^{4s_1} + 22e^{s_2} + 152e^{2s_1+s_2} + 1316e^{3s_1+s_2} - 182e^{5s_1+s_2}$$
$$- 109e^{s_1+2s_2} + 277e^{2s_1+2s_2} - 632e^{4s_1+2s_2} + 1709e^{5s_1+2s_2} - 170e^{3s_1+3s_2} + 2072e^{4s_1+3s_2}$$
$$- 1258e^{6s_1+3s_2} - 134e^{7s_1+3s_2} - 7e^{3s_1+4s_2} - 1019e^{5s_1+4s_2} + 775e^{6s_1+4s_2} - 4e^{8s_1+4s_2}$$
$$- 20e^{4s_1+5s_2} + 116e^{5s_1+5s_2} - 292e^{7s_1+5s_2} + 4e^{8s_1+5s_2} - 22)s_2 - 2(-32e^{s_1} - 164e^{3s_1} - 28e^{4s_1}$$
$$- 5e^{s_2} - 187e^{2s_1+s_2} - 496e^{3s_1+s_2} + 136e^{5s_1+s_2} + 20e^{s_1+2s_2} - 11e^{2s_1+2s_2} + 316e^{4s_1+2s_2}$$
$$- 868e^{5s_1+2s_2} - 44e^{3s_1+3s_2} - 988e^{4s_1+3s_2} + 689e^{6s_1+3s_2} + 76e^{7s_1+3s_2} - 16e^{3s_1+4s_2}$$
$$+ 544e^{5s_1+4s_2} - 455e^{6s_1+4s_2} - 7e^{8s_1+4s_2} + 22e^{4s_1+5s_2} - 100e^{5s_1+5s_2} + 128e^{7s_1+5s_2}$$
$$+ 7e^{8s_1+5s_2} + 5)s_3),$$

$$k_{20,5}(s_1,s_2,s_3)(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= -16\pi(2e^{2s_1}(4 + 30e^{s_1} - 4e^{2s_1} - 24e^{s_2} + 30e^{2s_2} - 6e^{3s_2} - 24e^{s_1+s_2} - 118e^{2s_1+s_2} + 16e^{3s_1+s_2}$$
$$- 24e^{s_1+2s_2} + 236e^{2s_1+2s_2} + 52e^{3s_1+2s_2} + 6e^{4s_1+2s_2} + 4e^{s_1+3s_2} - 100e^{2s_1+3s_2} - 160e^{3s_1+3s_2}$$
$$- 34e^{4s_1+3s_2} - 4e^{5s_1+3s_2} - 7e^{s_1+4s_2} + 33e^{2s_1+4s_2} + 34e^{3s_1+4s_2} + 94e^{4s_1+4s_2} - 5e^{5s_1+4s_2}$$
$$+ e^{6s_1+4s_2} - 2e^{2s_1+5s_2} + 8e^{3s_1+5s_2} - 36e^{4s_1+5s_2})s_1^2 - 2((-1 + e^{s_1})(1 - 8e^{s_1} + 36e^{2s_1} + e^{3s_1}$$
$$- e^{s_2} + 5e^{s_1+s_2} - 36e^{2s_1+s_2} - 102e^{3s_1+s_2} - 16e^{4s_1+s_2} + 3e^{s_1+2s_2} + 11e^{2s_1+2s_2}$$
$$+ 107e^{3s_1+2s_2} + 163e^{4s_1+2s_2} + 16e^{5s_1+2s_2} - 11e^{2s_1+3s_2} - 11e^{3s_1+3s_2} - 163e^{4s_1+3s_2}$$
$$- 115e^{5s_1+3s_2} + 5e^{3s_1+4s_2} + 12e^{4s_1+4s_2} + 117e^{5s_1+4s_2} + 17e^{6s_1+4s_2} - e^{7s_1+4s_2} + 4e^{4s_1+5s_2}$$
$$- 18e^{5s_1+5s_2} - 17e^{6s_1+5s_2} + e^{7s_1+5s_2}) - e^{2s_1+s_2}(-32 - 68e^{s_1} - 218e^{2s_1} + 38e^{s_2} - 24e^{s_1+s_2}$$
$$+ 420e^{2s_1+s_2} + 150e^{3s_1+s_2} - 56e^{s_1+2s_2} - 28e^{2s_1+2s_2} - 464e^{3s_1+2s_2} - 34e^{4s_1+2s_2}$$
$$+ 45e^{2s_1+3s_2} + 58e^{3s_1+3s_2} + 202e^{4s_1+3s_2} + 2e^{5s_1+3s_2})s_2 - e^{2s_1+s_2}(-32 - 108e^{s_1}$$
$$- 196e^{2s_1} + 26e^{s_2} + 24e^{s_1+s_2} + 428e^{2s_1+s_2} + 160e^{3s_1+s_2} - 172e^{2s_1+2s_2} - 368e^{3s_1+2s_2}$$
$$- 106e^{4s_1+2s_2} + 23e^{2s_1+3s_2} + 106e^{3s_1+3s_2} + 190e^{4s_1+3s_2} + 17e^{5s_1+3s_2})s_3)s_1$$
$$+ e^{s_1}(2e^{s_1+s_2}(-16 - 96e^{s_1} - 162e^{2s_1} + 10e^{s_2} + 24e^{s_1+s_2} + 324e^{2s_1+s_2} + 168e^{3s_1+s_2}$$
$$- 108e^{s_1+2s_2} + 116e^{2s_1+2s_2} - 544e^{3s_1+2s_2} + 10e^{4s_1+2s_2} + 11e^{2s_1+3s_2} + 70e^{3s_1+3s_2}$$
$$+ 170e^{4s_1+3s_2} + 19e^{5s_1+3s_2})s_2^2 + (2e^{s_1+s_2}(-24 - 172e^{s_1} - 222e^{2s_1} + 6e^{s_2} + 72e^{s_1+s_2}$$
$$+ 484e^{2s_1+s_2} + 254e^{3s_1+s_2} - 56e^{s_1+2s_2} - 60e^{2s_1+2s_2} - 608e^{3s_1+2s_2} - 98e^{4s_1+2s_2}$$
$$- 17e^{2s_1+3s_2} + 174e^{3s_1+3s_2} + 206e^{4s_1+3s_2} + 56e^{5s_1+3s_2})s_3 + 181e^{s_1} + 10e^{s_2} + 426e^{3s_1+s_2}$$

$$+ 512e^{2s_1+2s_2} - 97e^{5s_1+2s_2} - 38e^{s_1+3s_2} + 208e^{4s_1+3s_2} + 57e^{3s_1+4s_2} - 142e^{6s_1+4s_2}$$
$$- 64e^{5s_1+5s_2})s_2 + 2s_3(2e^{s_1+s_2}(-4 - 30e^{s_1} - 40e^{2s_1} + e^{s_2} + 12e^{s_1+s_2} + 86e^{2s_1+s_2}$$
$$+ 46e^{3s_1+s_2} - 2e^{s_1+2s_2} - 26e^{2s_1+2s_2} - 92e^{3s_1+2s_2} - 27e^{4s_1+2s_2} - 4e^{2s_1+3s_2} + 32e^{3s_1+3s_2}$$
$$+ 36e^{4s_1+3s_2} + 11e^{5s_1+3s_2})s_3 + 73e^{s_1} + 4e^{s_2} + 180e^{3s_1+s_2} + 240e^{2s_1+2s_2} - 59e^{5s_1+2s_2}$$
$$- 7e^{s_1+3s_2} + 96e^{4s_1+3s_2} + 46e^{3s_1+4s_2} - 68e^{6s_1+4s_2} - 19e^{5s_1+5s_2}))),$$

$$k_{20,6}(s_1, s_2, s_3)3(e^{s_1} - 1)^4(e^{s_2} - 1)(e^{s_1+s_2} - 1)^4 s_1(s_1 + s_2)^2(s_2 + s_3)(s_1 + s_2 + s_3)^2$$
$$= -16\pi((19 - 100e^{s_1} - 316e^{3s_1} - 35e^{4s_1} - 19e^{s_2} - 44e^{2s_1+s_2} - 1304e^{3s_1+s_2} + 152e^{5s_1+s_2}$$
$$+ 100e^{s_1+2s_2} - 310e^{2s_1+2s_2} + 656e^{4s_1+2s_2} - 1580e^{5s_1+2s_2} + 200e^{3s_1+3s_2} - 1904e^{4s_1+3s_2}$$
$$+ 1150e^{6s_1+3s_2} + 176e^{7s_1+3s_2} + 4e^{3s_1+4s_2} + 956e^{5s_1+4s_2} - 544e^{6s_1+4s_2} - 11e^{8s_1+4s_2}$$
$$- 7e^{4s_1+5s_2} + 16e^{5s_1+5s_2} + 304e^{7s_1+5s_2} + 11e^{8s_1+5s_2})s_2^2 + (20 - 95e^{s_1} - 425e^{3s_1} - 67e^{4s_1}$$
$$- 20e^{s_2} - 220e^{2s_1+s_2} - 1708e^{3s_1+s_2} + 274e^{5s_1+s_2} + 101e^{s_1+2s_2} - 281e^{2s_1+2s_2} + 952e^{4s_1+2s_2}$$
$$- 2365e^{5s_1+2s_2} + 94e^{3s_1+3s_2} - 2680e^{4s_1+3s_2} + 1850e^{6s_1+3s_2} + 250e^{7s_1+3s_2} - e^{3s_1+4s_2}$$
$$+ 1495e^{5s_1+4s_2} - 863e^{6s_1+4s_2} - 10e^{8s_1+4s_2} - 14e^{4s_1+5s_2} + 44e^{5s_1+5s_2} + 500e^{7s_1+5s_2}$$
$$+ 10e^{8s_1+5s_2})s_3s_2 - (-7 + 34e^{s_1} + 142e^{3s_1} + 29e^{4s_1} + 7e^{s_2} + 80e^{2s_1+s_2} + 596e^{3s_1+s_2}$$
$$- 116e^{5s_1+s_2} - 34e^{s_1+2s_2} + 100e^{2s_1+2s_2} - 344e^{4s_1+2s_2} + 818e^{5s_1+2s_2} - 20e^{3s_1+3s_2}$$
$$+ 920e^{4s_1+3s_2} - 682e^{6s_1+3s_2} - 92e^{7s_1+3s_2} + 2e^{3s_1+4s_2} - 530e^{5s_1+4s_2} + 286e^{6s_1+4s_2}$$
$$- e^{8s_1+4s_2} + 7e^{4s_1+5s_2} - 28e^{5s_1+5s_2} - 196e^{7s_1+5s_2} + e^{8s_1+5s_2})s_3^2),$$

$$k_{20,7}(s_1, s_2, s_3)5(e^{s_1} - 1)^4(e^{s_2} - 1)(e^{s_1+s_2} - 1)^4(s_1 + s_2)^2(s_2 + s_3)(s_1 + s_2 + s_3)^2$$
$$= 32\pi((-4 + 16e^{s_1} - 84e^{2s_1} - 204e^{3s_1} + 6e^{4s_1} + 4e^{s_2} - 24e^{5s_1+s_2} - 16e^{s_1+2s_2} - 54e^{6s_1+2s_2}$$
$$- 86e^{2s_1+3s_2} + 156e^{7s_1+3s_2} + 9e^{3s_1+4s_2} - 9e^{8s_1+4s_2} + 14e^{4s_1+5s_2} - 56e^{5s_1+5s_2}$$
$$+ 204e^{6s_1+5s_2} + 104e^{7s_1+5s_2} + 4e^{8s_1+5s_2})s_2^2 + (-6 + 24e^{s_1} - 156e^{2s_1} - 281e^{3s_1} - e^{4s_1}$$
$$+ 6e^{s_2} + 4e^{5s_1+s_2} - 24e^{s_1+2s_2} - 96e^{6s_1+2s_2} - 34e^{2s_1+3s_2} + 124e^{7s_1+3s_2} - 19e^{3s_1+4s_2}$$
$$+ 14e^{8s_1+4s_2} + 26e^{4s_1+5s_2} - 104e^{5s_1+5s_2} + 336e^{6s_1+5s_2} + 156e^{7s_1+5s_2} + 6e^{8s_1+5s_2})s_3s_2$$
$$+ 2(-1 + 4e^{s_1} - 26e^{2s_1} - 51e^{3s_1} - e^{4s_1} + e^{s_2} + 4e^{5s_1+s_2} - 4e^{s_1+2s_2} - 21e^{6s_1+2s_2} + e^{2s_1+3s_2}$$
$$+ 14e^{7s_1+3s_2} - 4e^{3s_1+4s_2} + 4e^{8s_1+4s_2} + 6e^{4s_1+5s_2} - 24e^{5s_1+5s_2} + 66e^{6s_1+5s_2} + 26e^{7s_1+5s_2}$$
$$+ e^{8s_1+5s_2})s_3^2 + s_1((-2 + 8e^{s_1} - 52e^{2s_1} - 277e^{3s_1} + 23e^{4s_1} + 2e^{s_2} - 92e^{5s_1+s_2} - 8e^{s_1+2s_2}$$
$$- 72e^{6s_1+2s_2} - 18e^{2s_1+3s_2} + 108e^{7s_1+3s_2} + 47e^{3s_1+4s_2} - 12e^{8s_1+4s_2} + 22e^{4s_1+5s_2}$$
$$- 88e^{5s_1+5s_2} + 312e^{6s_1+5s_2} + 52e^{7s_1+5s_2} + 2e^{8s_1+5s_2})s_2 + (-2 + 8e^{s_1} - 92e^{2s_1} - 252e^{3s_1}$$
$$+ 8e^{4s_1} + 2e^{s_2} - 32e^{5s_1+s_2} - 8e^{s_1+2s_2} - 102e^{6s_1+2s_2} + 22e^{2s_1+3s_2} + 48e^{7s_1+3s_2} + 27e^{3s_1+4s_2}$$
$$+ 3e^{8s_1+4s_2} + 32e^{4s_1+5s_2} - 128e^{5s_1+5s_2} + 372e^{6s_1+5s_2} + 52e^{7s_1+5s_2} + 2e^{8s_1+5s_2})s_3)),$$

$$k_{20,8}(s_1, s_2, s_3)(e^{s_1} - 1)^4(e^{s_1+s_2} - 1)^4 s_2(s_1 + s_2)^2(s_2 + s_3)(s_1 + s_2 + s_3)^2$$
$$= 16e^{s_1}\pi(2e^{s_1}(-1 + e^{s_2})(-2 - 5e^{s_1} + e^{2s_1} + 2e^{s_2} - 2e^{s_1+s_2} + 22e^{2s_1+s_2} - 4e^{3s_1+s_2} + e^{s_1+2s_2}$$
$$- 5e^{2s_1+2s_2} - 14e^{3s_1+2s_2} + 6e^{4s_1+3s_2})s_1^3 + ((-1 + e^{s_1})(1 - 14e^{s_1} + e^{2s_1} - e^{s_2} + 10e^{s_1+s_2}$$
$$+ 37e^{2s_1+s_2} + 2e^{3s_1+s_2} - 4e^{s_1+2s_2} - 9e^{2s_1+2s_2} - 60e^{3s_1+2s_2} + e^{4s_1+2s_2} + 5e^{2s_1+3s_2}$$
$$+ 2e^{3s_1+3s_2} + 47e^{4s_1+3s_2} - 6e^{5s_1+3s_2} - 4e^{4s_1+4s_2} - 10e^{5s_1+4s_2} + 2e^{6s_1+4s_2})$$
$$+ 2e^{s_1}(-1 + e^{s_2})(-4 - 15e^{s_1} + e^{2s_1} + 4e^{s_2} - 6e^{s_1+s_2} + 60e^{2s_1+s_2} - 4e^{3s_1+s_2} + 3e^{s_1+2s_2}$$
$$- 9e^{2s_1+2s_2} - 42e^{3s_1+2s_2} - 6e^{4s_1+2s_2} + 2e^{2s_1+3s_2} - 8e^{3s_1+3s_2} + 24e^{4s_1+3s_2})s_3)s_1^2$$
$$+ s_3(4e^{s_1}(-1 + e^{s_2})(-1 - 5e^{s_1} + e^{s_2} - 2e^{s_1+s_2} + 19e^{2s_1+s_2} + e^{s_1+2s_2} - 2e^{2s_1+2s_2}$$
$$- 14e^{3s_1+2s_2} - 3e^{4s_1+2s_2} + e^{2s_1+3s_2} - 4e^{3s_1+3s_2} + 9e^{4s_1+3s_2})s_3 - (-1 + e^{s_1})(-3 + 34e^{s_1}$$
$$+ 5e^{2s_1} + 3e^{s_2} - 22e^{s_1+s_2} - 95e^{2s_1+s_2} - 30e^{3s_1+s_2} + 4e^{s_1+2s_2} + 27e^{2s_1+2s_2} + 156e^{3s_1+2s_2}$$
$$+ 29e^{4s_1+2s_2} + e^{2s_1+3s_2} - 30e^{3s_1+3s_2} - 109e^{4s_1+3s_2} - 6e^{5s_1+3s_2} - 8e^{3s_1+4s_2} + 28e^{4s_1+4s_2}$$

$$+ 14e^{5s_1+4s_2} + 2e^{6s_1+4s_2}))s_1 - 2(-1+e^{s_1})(-1+10e^{s_1} + 3e^{2s_1} + e^{s_2} - 6e^{s_1+s_2} - 29e^{2s_1+s_2} - 14e^{3s_1+s_2} + 9e^{2s_1+2s_2} + 48e^{3s_1+2s_2} + 15e^{4s_1+2s_2} + 3e^{2s_1+3s_2} - 14e^{3s_1+3s_2} - 31e^{4s_1+3s_2} - 6e^{5s_1+3s_2} - 4e^{3s_1+4s_2} + 12e^{4s_1+4s_2} + 2e^{5s_1+4s_2} + 2e^{6s_1+4s_2})s_3^2),$$

$$k_{20,9}(s_1,s_2,s_3)(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4 s_1(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= -16e^{s_1}\pi(4e^{s_1+s_2}(-1-22e^{s_1}-21e^{2s_1}-2e^{s_2}+12e^{s_1+s_2}+46e^{2s_1+s_2}+32e^{3s_1+s_2} - 26e^{s_1+2s_2}+38e^{2s_1+2s_2}-108e^{3s_1+2s_2}+8e^{4s_1+2s_2}-3e^{2s_1+3s_2}+16e^{3s_1+3s_2}+25e^{4s_1+3s_2} + 6e^{5s_1+3s_2})s_2^3 + e^{s_1}(2e^{s_2}(-4-88e^{s_1}-84e^{2s_1}-8e^{s_2}+48e^{s_1+s_2}+184e^{2s_1+s_2} + 128e^{3s_1+s_2}-44e^{s_1+2s_2}+32e^{2s_1+2s_2}-312e^{3s_1+2s_2}-28e^{4s_1+2s_2}-27e^{2s_1+3s_2} + 94e^{3s_1+3s_2}+70e^{4s_1+3s_2}+39e^{5s_1+3s_2})s_3 - 26e^{3s_2}+405e^{2s_1+s_2}+472e^{s_1+2s_2} - 94e^{4s_1+2s_2}+152e^{3s_1+3s_2}+25e^{2s_1+4s_2}-160e^{5s_1+4s_2}-108e^{4s_1+5s_2}+144)s_2^2 + s_3(4e^{s_1+s_2}(-1-22e^{s_1}-21e^{2s_1}-2e^{s_2}+12e^{s_1+s_2}+46e^{2s_1+s_2}+32e^{3s_1+s_2}-6e^{s_1+2s_2} - 2e^{2s_1+2s_2}-68e^{3s_1+2s_2}-12e^{4s_1+2s_2}-8e^{2s_1+3s_2}+26e^{3s_1+3s_2}+15e^{4s_1+3s_2} + 11e^{5s_1+3s_2})s_3 + 189e^{s_1}-2e^{s_2}+560e^{3s_1+s_2}+680e^{2s_1+2s_2}-149e^{5s_1+2s_2}-22e^{s_1+3s_2} + 184e^{4s_1+3s_2}+43e^{3s_1+4s_2}-250e^{6s_1+4s_2}-180e^{5s_1+5s_2})s_2 - 3e^{s_1}(-22+2e^{3s_2}-63e^{2s_1+s_2} - 80e^{s_1+2s_2}+20e^{4s_1+2s_2}-16e^{3s_1+3s_2}-5e^{2s_1+4s_2}+32e^{5s_1+4s_2}+24e^{4s_1+5s_2})s_3^2),$$

$$k_{20,10}(s_1,s_2,s_3)5(e^{s_1}-1)^4(e^{s_2}-1)(e^{s_1+s_2}-1)^4 s_1(s_1+s_2)^2(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= 32\pi s_2(2(4e^{s_1}-21e^{2s_1}-26e^{3s_1}-e^{4s_1}+e^{s_2}+4e^{5s_1+s_2}-4e^{s_1+2s_2}-6e^{6s_1+2s_2}-24e^{2s_1+3s_2} + 34e^{7s_1+3s_2}-4e^{3s_1+4s_2}-e^{8s_1+4s_2}+e^{4s_1+5s_2}-4e^{5s_1+5s_2}+21e^{6s_1+5s_2}+26e^{7s_1+5s_2} + e^{8s_1+5s_2}-1)s_2^2 + (16e^{s_1}-84e^{2s_1}-104e^{3s_1}-4e^{4s_1}+4e^{s_2}+16e^{5s_1+s_2}-16e^{s_1+2s_2} - 24e^{6s_1+2s_2}-36e^{2s_1+3s_2}+76e^{7s_1+3s_2}-31e^{3s_1+4s_2}+11e^{8s_1+4s_2}+4e^{4s_1+5s_2}-16e^{5s_1+5s_2} + 84e^{6s_1+5s_2}+104e^{7s_1+5s_2}+4e^{8s_1+5s_2}-4)s_3s_2 + 2(4e^{s_1}-21e^{2s_1}-26e^{3s_1}-e^{4s_1}+e^{s_2} + 4e^{5s_1+s_2}-4e^{s_1+2s_2}-6e^{6s_1+2s_2}-4e^{2s_1+3s_2}+14e^{7s_1+3s_2}-9e^{3s_1+4s_2}+4e^{8s_1+4s_2} + e^{4s_1+5s_2}-4e^{5s_1+5s_2}+21e^{6s_1+5s_2}+26e^{7s_1+5s_2}+e^{8s_1+5s_2}-1)s_3^2),$$

$$k_{20,11}(s_1,s_2,s_3) = \frac{12\pi(s_1^2-s_3s_1-s_2^2+s_3^2)}{(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^4 s_1(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},$$

$$k_{20,12}(s_1,s_2,s_3) = \frac{12\pi(s_1^2-s_3s_1-s_2^2+s_3^2)}{(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^4 s_1(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)},$$

$$k_{20,13}(s_1,s_2,s_3) = \frac{24\pi e^{2s_1}(s_2-s_3)}{(e^{s_1}-1)^2(e^{\frac{1}{2}(s_2+s_3)}-1)^2 s_1s_2s_3(s_1+s_2+s_3)},$$

$$k_{20,14}(s_1,s_2,s_3) = \frac{24\pi e^{2s_1}(s_2-s_3)}{(e^{s_1}-1)^2(e^{\frac{1}{2}(s_2+s_3)}+1)^2 s_1s_2s_3(s_1+s_2+s_3)},$$

$$k_{20,15}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^3 s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2$$
$$= -4\pi(3((-3e^{s_1}-3e^{s_1+s_2}+2e^{2s_1+s_2}+4)s_2+e^{s_1}(e^{s_2}-1)s_3)s_1^3 + ((-5e^{s_1}-11e^{s_1+s_2} + 2e^{2s_1+s_2}+14)s_2^2+((-5e^{s_1}+7e^{s_1+s_2}-4e^{2s_1+s_2}+2)s_3-6(e^{s_1}-1)(e^{s_1+s_2}-1))s_2 + 6e^{s_1}(e^{s_2}-1)s_3^2)s_1^2 + ((17e^{s_1}+5e^{s_1+s_2}-14e^{2s_1+s_2}-8)s_2^3 - (-13e^{s_1}-7e^{s_1+s_2} + 10e^{2s_1+s_2}+10)s_3s_2^2 + s_3(6(e^{s_1}-1)(e^{s_1+s_2}-1)+(-7e^{s_1}+5e^{s_1+s_2}+4e^{2s_1+s_2}-2)s_3)s_2 + 3e^{s_1}(e^{s_2}-1)s_3^3)s_1 - s_2(s_2+s_3)((-13e^{s_1}-7e^{s_1+s_2}+10e^{2s_1+s_2}+10)s_2^2 - 2(e^{s_1}-1)(2e^{s_1+s_2}s_3+s_3+3e^{s_1+s_2}-3)s_2 - s_3((-11e^{s_1}-11e^{s_1+s_2}+14e^{2s_1+s_2}+8)s_3 - 6(e^{s_1}-1)(e^{s_1+s_2}-1)))),$$

$$k_{20,16}(s_1,s_2,s_3)(e^{s_1}-1)^3(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}-1)s_1s_2s_3(s_2+s_3)(s_1+s_2+s_3)$$
$$= -8\pi e^{2s_1}((5e^{s_1}+5e^{s_2}+e^{s_1+s_2}-11)s_2^2+2(e^{s_1}-1)(3(e^{s_2}-1)+(4e^{s_2}+2)s_3)s_2 + s_3((-e^{s_1}-13e^{s_2}+7e^{s_1+s_2}+7)s_3-6(e^{s_1}-1)(e^{s_2}-1))),$$

$$k_{20,17}(s_1,s_2,s_3) = \frac{-k^{\mathrm{num}}_{20,16}(s_1,s_2,s_3)}{(e^{s_1}-1)^3(e^{s_2}-1)(e^{\frac{1}{2}(s_2+s_3)}+1)s_1s_2s_3(s_2+s_3)(s_1+s_2+s_3)},$$

$$\begin{aligned}
&k_{20,18}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^3s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2\\
&\quad= 4\pi(3((-3e^{s_1}-3e^{s_1+s_2}+2e^{2s_1+s_2}+4)s_2+e^{s_1}(e^{s_2}-1)s_3)s_1^3+((-5e^{s_1}-11e^{s_1+s_2}\\
&\qquad+2e^{2s_1+s_2}+14)s_2^2+((-5e^{s_1}+7e^{s_1+s_2}-4e^{2s_1+s_2}+2)s_3-6(e^{s_1}-1)(e^{s_1+s_2}-1))s_2\\
&\qquad+6e^{s_1}(e^{s_2}-1)s_3^2)s_1^2+((17e^{s_1}+5e^{s_1+s_2}-14e^{2s_1+s_2}-8)s_2^3-(-13e^{s_1}-7e^{s_1+s_2}\\
&\qquad+10e^{2s_1+s_2}+10)s_3s_2^2+s_3(6(e^{s_1}-1)(e^{s_1+s_2}-1)+(-7e^{s_1}+5e^{s_1+s_2}+4e^{2s_1+s_2}-2)s_3)s_2\\
&\qquad+3e^{s_1}(e^{s_2}-1)s_3^3)s_1-s_2(s_2+s_3)((-13e^{s_1}-7e^{s_1+s_2}+10e^{2s_1+s_2}+10)s_2^2\\
&\qquad-2(e^{s_1}-1)(2e^{s_1+s_2}s_3+s_3+3e^{s_1+s_2}-3)s_2-s_3((-11e^{s_1}-11e^{s_1+s_2}+14e^{2s_1+s_2}+8)s_3\\
&\qquad-6(e^{s_1}-1)(e^{s_1+s_2}-1)))),
\end{aligned}$$

$$\begin{aligned}
&k_{20,19}(s_1,s_2,s_3)(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)^2s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2\\
&\quad= 2\pi(((-33+14e^{s_1}+55e^{s_1+s_2}-29e^{2s_1+s_2}-10e^{2s_1+2s_2}+3e^{3s_1+2s_2})s_2+e^{s_1}(-1+e^{s_2})\\
&\qquad\times(-11+23e^{s_1+s_2})s_3)s_1^3-((43+14e^{s_1}-67e^{s_1+s_2}-11e^{2s_1+s_2}+12e^{2s_1+2s_2}+9e^{3s_1+2s_2})s_2^2\\
&\qquad+(6(-7+5e^{s_1}+12e^{s_1+s_2}-8e^{2s_1+s_2}-5e^{2s_1+2s_2}+3e^{3s_1+2s_2})+(10-5e^{s_1}+21e^{s_1+s_2}\\
&\qquad+29e^{2s_1+s_2}-67e^{2s_1+2s_2}+12e^{3s_1+2s_2})s_3)s_2-2e^{s_1}(-1+e^{s_2})s_3((-11+23e^{s_1+s_2})s_3\\
&\qquad-6e^{s_1+s_2}+6))s_1^2+((13-70e^{s_1}-31e^{s_1+s_2}+109e^{2s_1+s_2}+6e^{2s_1+2s_2}-27e^{3s_1+2s_2})s_2^3\\
&\qquad+(8(1+2e^{s_1})(-1+e^{s_1+s_2})^2+(23-66e^{s_1}-67e^{s_1+s_2}+69e^{2s_1+s_2}+32e^{2s_1+2s_2}\\
&\qquad+9e^{3s_1+2s_2})s_3)s_2^2+s_3((10+15e^{s_1}-47e^{s_1+s_2}-63e^{2s_1+s_2}+49e^{2s_1+2s_2}+36e^{3s_1+2s_2})s_3\\
&\qquad+22e^{s_1}+80e^{s_1+s_2}-56e^{2s_1+s_2}-46e^{2s_1+2s_2}+34e^{3s_1+2s_2}-34)s_2+e^{s_1}(-1+e^{s_2})s_3^2\\
&\qquad\times((-11+23e^{s_1+s_2})s_3-12(-1+e^{s_1+s_2})))s_1-s_2(s_2+s_3)((-23+42e^{s_1}+43e^{s_1+s_2}\\
&\qquad-69e^{2s_1+s_2}-8e^{2s_1+2s_2}+15e^{3s_1+2s_2})s_2^2-2((5-9e^{s_1}-7e^{s_1+s_2}+3e^{2s_1+s_2}-10e^{2s_1+2s_2}\\
&\qquad+18e^{3s_1+2s_2})s_3+23e^{s_1}+28e^{s_1+s_2}-40e^{2s_1+s_2}-11e^{2s_1+2s_2}+17e^{3s_1+2s_2}-17)s_2\\
&\qquad-s_3((-13+24e^{s_1}+29e^{s_1+s_2}-63e^{2s_1+s_2}-28e^{2s_1+2s_2}+51e^{3s_1+2s_2})s_3-38e^{s_1}-64e^{s_1+s_2}\\
&\qquad+88e^{2s_1+s_2}+38e^{2s_1+2s_2}-50e^{3s_1+2s_2}+26))),
\end{aligned}$$

$$k_{20,20}(s_1,s_2,s_3) = \frac{k^{\mathrm{num}}_{20,19}(s_1,s_2,s_3)}{(e^{s_1}-1)(e^{s_1+s_2}-1)^2(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)^2s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2},$$

$$\begin{aligned}
&k_{20,21}(s_1,s_2,s_3)(e^{s_1}-1)^3(e^{s_1+s_2}-1)^3(e^{\frac{1}{2}(s_1+s_2+s_3)}+1)s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2\\
&\quad= 2\pi(((-15+32e^{s_1}+13e^{2s_1}-6e^{3s_1}+40e^{s_1+s_2}-69e^{2s_1+s_2}-54e^{3s_1+s_2}+11e^{4s_1+s_2}\\
&\qquad-31e^{2s_1+2s_2}+54e^{3s_1+2s_2}+33e^{4s_1+2s_2}+16e^{5s_1+2s_2}-18e^{3s_1+3s_2}+55e^{4s_1+3s_2}\\
&\qquad-64e^{5s_1+3s_2}+3e^{6s_1+3s_2})s_2-e^{s_1}(-5+26e^{s_1}+3e^{2s_1}+5e^{s_2}-12e^{s_1+s_2}-63e^{2s_1+s_2}\\
&\qquad-2e^{3s_1+s_2}-14e^{s_1+2s_2}+27e^{2s_1+2s_2}+84e^{3s_1+2s_2}-25e^{4s_1+2s_2}+33e^{2s_1+3s_2}-82e^{3s_1+3s_2}\\
&\qquad+25e^{4s_1+3s_2})s_3)s_1^3-((21-24e^{s_1}-111e^{2s_1}+42e^{3s_1}-50e^{s_1+s_2}-3e^{2s_1+s_2}+372e^{3s_1+s_2}\\
&\qquad-103e^{4s_1+s_2}+17e^{2s_1+2s_2}+66e^{3s_1+2s_2}-327e^{4s_1+2s_2}+28e^{5s_1+2s_2}+36e^{3s_1+3s_2}\\
&\qquad-111e^{4s_1+3s_2}+138e^{5s_1+3s_2}+9e^{6s_1+3s_2})s_2^2+(2(-23+7e^{s_1}+61e^{s_1+s_2}-25e^{2s_1+s_2}\\
&\qquad-41e^{2s_1+2s_2}+17e^{3s_1+2s_2}+3e^{3s_1+3s_2}+e^{4s_1+3s_2})(-1+e^{s_1})^2+(6-7e^{s_1}-20e^{2s_1}+45e^{3s_1}\\
&\qquad+5e^{s_1+s_2}-108e^{2s_1+s_2}+129e^{3s_1+s_2}-98e^{4s_1+s_2}-56e^{2s_1+2s_2}+201e^{3s_1+2s_2}-42e^{4s_1+2s_2}\\
&\qquad-31e^{5s_1+2s_2}+117e^{3s_1+3s_2}-302e^{4s_1+3s_2}+149e^{5s_1+3s_2}+12e^{6s_1+3s_2})s_3)s_2\\
&\qquad+2e^{s_1}(-1+e^{s_2})s_3((5-26e^{s_1}-3e^{2s_1}-14e^{s_1+s_2}+60e^{2s_1+s_2}+2e^{3s_1+s_2}+33e^{2s_1+2s_2}\\
&\qquad-82e^{3s_1+2s_2}+25e^{4s_1+2s_2})s_3-4(-1+e^{s_1})^2(2-7e^{s_1+s_2}+5e^{2s_1+2s_2})))s_1^2+(-(-3+48e^{s_1}\\
&\qquad-183e^{2s_1}+66e^{3s_1}+20e^{s_1+s_2}-213e^{2s_1+s_2}+582e^{3s_1+s_2}-173e^{4s_1+s_2}-59e^{2s_1+2s_2}\\
&\qquad+294e^{3s_1+2s_2}-555e^{4s_1+2s_2}+104e^{5s_1+2s_2}+18e^{3s_1+3s_2}-57e^{4s_1+3s_2}+84e^{5s_1+3s_2}\\
&\qquad+27e^{6s_1+3s_2})s_2^3+(-16(-1+e^{s_1})^2(-1-3e^{s_1}+e^{2s_1+s_2})(-1+e^{s_1+s_2})^2-(-9+64e^{s_1}
\end{aligned}$$

$$
\begin{aligned}
&- 181e^{2s_1} + 102e^{3s_1} + 54e^{s_1+s_2} - 357e^{2s_1+s_2} + 624e^{3s_1+s_2} - 249e^{4s_1+s_2} - 141e^{2s_1+2s_2} \\
&+ 534e^{3s_1+2s_2} - 549e^{4s_1+2s_2} + 84e^{5s_1+2s_2} + 72e^{3s_1+3s_2} - 169e^{4s_1+3s_2} + 34e^{5s_1+3s_2} \\
&+ 87e^{6s_1+3s_2})s_3)s_2^2 + s_3((6 - 11e^{s_1} - 28e^{2s_1} - 39e^{3s_1} - 39e^{s_1+s_2} + 156e^{2s_1+s_2} + 21e^{3s_1+s_2} \\
&+ 78e^{4s_1+s_2} + 96e^{2s_1+2s_2} - 267e^{3s_1+2s_2} - 90e^{4s_1+2s_2} + 45e^{5s_1+2s_2} - 87e^{3s_1+3s_2} \\
&+ 194e^{4s_1+3s_2} + 25e^{5s_1+3s_2} - 60e^{6s_1+3s_2})s_3 - 2(-1 + e^{s_1})^2(15 - 15e^{s_1} - 61e^{s_1+s_2} \\
&+ 25e^{2s_1+s_2} + 89e^{2s_1+2s_2} - 17e^{3s_1+2s_2} - 43e^{3s_1+3s_2} + 7e^{4s_1+3s_2}))s_2 - e^{s_1}(-1 + e^{s_2})s_3^2 \\
&\times ((5 - 26e^{s_1} - 3e^{2s_1} - 14e^{s_1+s_2} + 60e^{2s_1+s_2} + 2e^{3s_1+s_2} + 33e^{2s_1+2s_2} - 82e^{3s_1+2s_2} \\
&+ 25e^{4s_1+2s_2})s_3 - 8(-1 + e^{s_1})^2(2 - 7e^{s_1+s_2} + 5e^{2s_1+2s_2})))s_1 - s_2(s_2 + s_3)((-9 + 40e^{s_1} \\
&- 85e^{2s_1} + 30e^{3s_1} + 30e^{s_1+s_2} - 141e^{2s_1+s_2} + 264e^{3s_1+s_2} - 81e^{4s_1+s_2} - 45e^{2s_1+2s_2} \\
&+ 174e^{3s_1+2s_2} - 261e^{4s_1+2s_2} + 60e^{5s_1+2s_2} - e^{4s_1+3s_2} + 10e^{5s_1+3s_2} + 15e^{6s_1+3s_2})s_2^2 \\
&+ 2(-1 + e^{s_1})((-1 + e^{s_1})(15 - 31e^{s_1} - 45e^{s_1+s_2} + 81e^{2s_1+s_2} + 33e^{2s_1+2s_2} - 57e^{3s_1+2s_2} \\
&- 3e^{3s_1+3s_2} + 7e^{4s_1+3s_2}) + (3 - 10e^{s_1} + 15e^{2s_1} - 12e^{s_1+s_2} + 48e^{2s_1+s_2} - 36e^{3s_1+s_2} \\
&+ 27e^{2s_1+2s_2} - 66e^{3s_1+2s_2} + 15e^{4s_1+2s_2} + 6e^{3s_1+3s_2} - 20e^{4s_1+3s_2} + 30e^{5s_1+3s_2})s_3)s_2 \\
&+ s_3((3 - 14e^{s_1} + 35e^{2s_1} - 6e^{s_1+s_2} + 21e^{2s_1+s_2} - 96e^{3s_1+s_2} + 9e^{4s_1+s_2} - 9e^{2s_1+2s_2} \\
&+ 12e^{3s_1+2s_2} + 99e^{4s_1+2s_2} - 30e^{5s_1+2s_2} - 12e^{3s_1+3s_2} + 53e^{4s_1+3s_2} - 110e^{5s_1+3s_2} \\
&+ 45e^{6s_1+3s_2})s_3 - 2(-1 + e^{s_1})^2(7 - 15e^{s_1} - 21e^{s_1+s_2} + 57e^{2s_1+s_2} + 33e^{2s_1+2s_2} - 81e^{3s_1+2s_2} \\
&- 19e^{3s_1+3s_2} + 39e^{4s_1+3s_2})))),
\end{aligned}
$$

$$k_{20,22}(s_1, s_2, s_3) = \frac{-k^{\text{num}}_{20,21}(s_1,s_2,s_3)}{(e^{s_1}-1)^3(e^{s_1+s_2}-1)^3(e^{\frac{1}{2}(s_1+s_2+s_3)}-1)s_1s_2(s_1+s_2)s_3(s_2+s_3)(s_1+s_2+s_3)^2},$$

$$k_{20,23}(s_1, s_2, s_3) = -\frac{16\pi e^{2(s_1+s_2)}((e^{s_2}(e^{s_1}(2e^{s_2} + 1) + 1) - 4)s_1 + 3e^{s_2}(e^{s_1} - 1)s_2)}{(e^{s_2} - 1)(e^{s_1+s_2} - 1)^3(e^{\frac{s_3}{2}} + 1)s_1s_2(s_1 + s_2)(s_1 + s_2 + s_3)},$$

$$k_{20,24}(s_1, s_2, s_3) = \frac{16\pi e^{2(s_1+s_2)}((e^{s_2}(e^{s_1}(2e^{s_2} + 1) + 1) - 4)s_1 + 3e^{s_2}(e^{s_1} - 1)s_2)}{(e^{s_2} - 1)(e^{s_1+s_2} - 1)^3(e^{\frac{s_3}{2}} - 1)s_1s_2(s_1 + s_2)(s_1 + s_2 + s_3)}.$$

Acknowledgments. The second author is indebted to Eric Schost for very helpful and generous conversations on computer algebra programming, which played a crucial role for handling the heavy calculations of the present paper. He also thanks the Institut des Hautes Études Scientifiques (I.H.E.S.), in particular Francois Bachelier in the I.T. department of the institute, for the excellent environment and facilities that were used for carrying out this work partially during stays in the Fall of 2013 and the Summer of 2015.

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Received November 9, 2018; accepted February 11, 2019

Alain Connes
Collége de France
3 rue d'Ulm, Paris F-75005 France;
I.H.E.S. and Ohio State University
E-mail: alain@connes.org
URL: http://alainconnes.org/en/

Farzad Fathizadeh
Department of Mathematics, Computational Foundry
Swansea University Bay Campus
SA1 8EN Swansea, United Kingdom;
Caltech and Max Planck Institute for Biological Cybernetics
E-mail: farzad.fathizadeh@swansea.ac.uk
URL: http://math.swansea.ac.uk/staff/farzad_fathizadeh/