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Π^1_2-comprehension and the property of Ramsey

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0. INTRODUCTION

This thesis is a contribution to the reverse mathematics of Π^1_2-CA_0. We study this theory with respect to the property of Ramsey.

The axiom system which is mainly used in mathematics is ZFC. Nevertheless, to formalize big parts of ordinary (i.e. not set theoretic) mathematics, the full strength of ZFC is not needed. Most sets which are constructed in pure mathematics are already contained in a small initial segment of the constructible universe L, in particular they are of rank less than $\omega + \omega$. There are some exceptions, for example Martin's proof of Borel determinacy (see [Mar85]) which requires sets from L_{ω_1}, and these sets of higher type are really necessary as Friedman showed in [Fri71]. But even this proof is far from using the full strength of ZFC.

One axiom which contributes very much to the strength of ZFC is the power set axiom, and it turned out that it is possible to formalize wide parts of ordinary mathematics without using this axiom at all. That might be surprising because in a straightforward formalization of ordinary mathematics in ZFC, the power set axiom is heavily used. We need it for example to talk about the real numbers as a set and not only as a class, which makes it possible to talk about measures, i.e. sets of sets of real numbers. Nevertheless, it has turned out that one can develop even measure theory to a good extent in weak theories without power set axiom, see for example [Sim99]. This raises the question to find axiom systems which are strong enough to develop ordinary mathematics inside them such that the full strength of the axiom systems is really used.

First steps in that direction were done by Hilbert and Bernays in the chapter "Formalismen zur deduktiven Entwicklung der Analysis" in [HB70]. The formalism presented there was the predecessor of today's second order arithmetic (SOA). The language of SOA is a two-sorted language and contains variables for natural numbers and sets of natural numbers (i.e. reals). SOA is much weaker than ZFC, but it is still strong enough to develop broad parts of mathematics in it. The reason is that many objects which are relevant in ordinary mathematics can be coded into reals even if they are of higher type when they are developed in stronger axiom systems like ZFC.

For example a continuous function on the real numbers is at first sight a subset of $\mathbb{R} \times \mathbb{R}$, therefore essentially an element of the power set of $\mathbb{R}$, but it can be coded in a single real number by coding only the countably many values of rational points. Another example is measure theory. Although it is not possible to code each Lebesgue-nullset into a real (since there are more than $|\mathbb{R}|$ Lebesgue-nullsets), one can develop some measure theory in SOA. The identity outside of a Lebesgue-nullset is an equivalence relation on the measurable functions, and one encodes the equivalence classes of Lebesgue-measurable functions instead of the functions themselves. In that way it is possible to talk about mathematical objects which are encodable into real numbers.

If one has proved a theorem of ordinary mathematics in (a subsystem of) SOA, the question remains whether the main axioms which were used are really used in their full strength and can not be replaced by some weaker axioms. To answer this question one can take the theorem as an axiom and try to prove the main axioms back from the theorem. This is the main idea of the program of reverse mathematics which was originated by Harvey Friedman and pursued by Simpson and many others. It turned out that there are five main subsystems of SOA such that many theorems of ordinary mathematics are equivalent to one of these subsystems (see [Sim99]). For example, take the subsystem ACA_0 (which is an abbreviation for **a**rithmetical **c**omprehension **a**xiom) which is a conservative extension of Peano arithmetic. ACA_0 is equivalent over a weak base theory to the theorem of Bolzano-Weierstraß, i.e. every bounded sequence of real numbers has a convergent subsequence, and to the theorem that every countable vector space over a countable field has a basis. The strongest of these five subsystems is Π^1_1-CA_0 which is SOA with comprehension restricted to Π^1_1-formulas with parameters. It is for example equivalent over a weak base theory to the Cantor Bendixon theorem which says that every closed subset of $\mathbb{R}$ is the union of a countable set and a set which has no isolated points.

Recently first results about the reverse mathematics of Π^1_2-CA_0 were proved. Carl Mummert in [MS05] showed that a metrization theorem of topology is equivalent to Π^1_2-CA_0 over Π^1_1-CA_0. Michael Möllerfeld and I showed that Π^1_2-CA_0 shows the same Π^1_1-sentences as ACA_0 together with the assertion “all subsets of the Bairespace which are defined by Boolean combinations of $\mathbf{\Sigma^0_2}$-sets are determined”.

In this thesis we study the reverse mathematics of Π^1_2-comprehension with respect to the property of Ramsey. The property of Ramsey is a combinatorial property concerning sets of real numbers and is defined as follows. Let $\mathcal{X} \subset \mathcal{P}(\omega)$ be a set of real numbers. Then an infinite set $H \subset \omega$ is homogeneous for $\mathcal{X}$ if either each infinite subset of H is in $\mathcal{X}$ or each infinite subset of

H is not in $\mathcal{X}$. $\mathcal{X}$ has the property of Ramsey iff there exists a homogeneous set. In the first case we say that H accepts $\mathcal{X}$, in the second case H avoids $\mathcal{X}$.

In ZFC with the axiom of constructibility it is provable that there exists a Δ^1_2-set of reals which does not have the property of Ramsey. On the other side, Tanaka showed that $\mathbf{\Sigma^1_1}$-MI_0, a subsystem of SOA which asserts that each Σ^1_1-definable monotone operator has a smallest fixed point and which is much weaker than Π^1_2-CA_0, already proves that each Σ^1_1 set has the property of Ramsey. This shows there is no level of the analytic hierarchy which characterizes Π^1_2-CA_0 in terms of Ramseyness. Hence we have to find another way to characterize the Ramsey-strength of Π^1_2-CA_0. We will show that Π^1_2-comprehension shows the same Π^1_1-sentences as a theory of autonomous iterated Ramseyness, called R-calculus. The R-calculus guarantees the existence of homogeneous sets for all sets of reals which are first order definable and may use other homogeneous sets of already defined sets of reals. The homogeneous sets are uniformly in set parameters which occur in the defining formula of the set of reals.

In chapter 1, we give an overview of the reverse mathematics of the property of Ramsey for theories weaker than Π^1_2-CA_0 and introduce the R-calculus.

The proof of the main theorem heavily depends on the work of Michael Möllerfeld who characterized Π^1_2-CA_0 in terms of generalized recursion theory, see [Möl02]. In chapter 2, we present some of his results which are necessary for this work. He showed that Π^1_2-CA_0 shows the same Π^1_1-sentences as a system of iterated nonmonotone inductive definitions called σ-calculus. We especially need his game-quantifiers which are generalizations of the common first order quantifiers. Informally they can be defined by the following clauses.

- $\exists^0 x\varphi(x) \leftrightarrow \exists x\varphi(x)$
- $\exists^{n+1} x\varphi(x) \leftrightarrow (\forall^n x_0)(\forall^n x_1)(\forall^n x_2)\cdots \bigvee_{m\in\mathbb{N}} \varphi(\langle x_0,\ldots,x_m\rangle)$
- $\forall^n x\varphi(x) \leftrightarrow \neg\exists^n x\neg\varphi(x)$

Möllerfeld introduced the theory $\Game$ame which is a subsystem of SOA and has comprehension for all first order formulas which may contain game-quantifiers. The second clause of our informal definition is expressed by talking about least fixed points of inductive definitions (which eliminates the infinite formula in the second clause). He showed that $\Game$ame proves the same Π^1_1-sentences as Π^1_2-CA_0. At the end of chapter 2 we show that a modification of Möllerfeld's σ-calculus with some additional transfinite induction called σ^+-calculus shows the same Π^1_1-sentences as Π^1_2-CA_0.

In chapter 3, we embed the R-calculus into the σ^+-calculus which implies that each Π^1_1-sentence which is provable in the R-calculus is provable in Π^1_2-CA_0. Let $\varphi(X,\vec{Y})$ be a formula in the language of the σ^+-calculus with free set variables X and $\vec{Y}$ and without second order quantifiers. Then for each fixed parameters $\vec{Y}$, φ defines a set of real numbers (which is only a class in SOA). We have to build a set term $H(\vec{Y})$ uniformly in φ such that for all $\vec{Y}$, $H(\vec{Y})$ is homogeneous for the set of reals coded by $\varphi(X,\vec{Y})$. Since Möllerfeld showed that ⅁ame and the σ-calculus prove the same $\mathcal{L}_2$-sentences it suffices to construct in a uniform way homogeneous sets for all sets of reals which are first order definable with the use of game-quantifiers. In 3.1, we introduce codes in $\mathbb{R}$ for these sets of reals.

In 3.2 we give a proof in $\mathrm{ZFC}+\mathrm{CH}$ that all these sets of reals have homogeneous sets. For this we use that the sets of reals which locally have the property of Ramsey form a σ-algebra, and this σ-algebra contains a σ-ideal which is ccc. By "local Ramseyness" we mean the following. Let $F \subset \mathcal{P}(\omega)$ be an ultrafilter with some additional closure properties, called Ramsey ultrafilter (to prove that such an F exists one uses CH). We say that an infinite $H \subset \omega$ locally accepts the (code of a) set of reals X at a finite sequence of natural numbers s if all subsets of H which begin with s are in X. We denote this by $hom_+(s,H,X)$. $hom_-(s,H,X)$ means that H avoids X at s and is defined analogously. Finally $hom(s,H,X)$ means $hom_+(s,H,X)$ or $hom_-(s,H,X)$. We now define

$$C_F := \{B \subset \mathcal{P}(\omega) \mid (\forall s)(\forall S \in F)(\exists S' \subset S)[S' \in F \wedge hom(s,S',B)]\}$$

and

$$I_F := \{B \subset \mathcal{P}(\omega) \mid (\forall s)(\forall S \in F)(\exists S' \subset S)[S' \in F \wedge hom_-(s,S',B)]\}.$$

Then C_F is a σ-algebra containing all open sets and I_F is a σ-ideal and ccc (here antichain means that the intersection of each two sets is in I_F). This is due to Mathias, see [Mat77]. We then show that each σ-algebra on sets of reals which contains a ccc-σ-ideal is closed under $\exists^n$ and $\forall^n$; here we understand $\exists^n$ as operator on sets of reals via $\exists^n s X_s := \{x \mid \exists^n s(x \in X_s)\}$. To prove this we approximate the new set by sets of the σ-algebra in a transfinite recursion along a countable wellordering. The sets occurring in this approximation decrease monotonely, and they decrease fast with respect to the ideal, i.e. at each successor step the approximating set differs from its predecessor on a set not in the ideal. Since the ideal is ccc, these sets become eventually constant before ω_1.

Unfortunately, this proof does not directly transfer to SOA. We have to deal with two main problems.

- We do not have a wellordering of length ω_1 which we can use for the transfinite recursion mentioned above.
- We cannot talk about the ultrafilter F directly.

In chapter 3.3, we deal with the first problem. The idea is as follows. We already mentioned that the recursion comes to a halt after countably many steps since the ideal is ccc. This in turn is the case because we can index each $B \in C_F \backslash I_F$ by

$$i(B) := min\{s \mid (\exists S \in F) hom_+(s, S, B\}.$$

If B_1 and B_2 have the same index, then $(\exists S \in F) hom_+(i(B_1), S, B_1 \cap B_2)$ and $B_1 \cap B_2 \notin I_F$. Hence each two elements of an antichain have different indices, and because there are only countably many indices each antichain is countable. This helps us to find a suitable wellordering for our transfinite recursion, because as lined out above, at each successor step of the iteration we have a witness which is not in the ideal. Since two witnesses are always disjoint, each index occurs at most once. This gives us a canonical wellordering which can be constructed simultaneously with the iteration.

This wellordering makes it possible to wrap this transfinite recursion into a nonmonotone fixed point, where each stage of the fixed point corresponds to one step of the iteration. Here the main axioms of the σ^+-calculus are used. In chapter 3.3, we develop the technique of wrapping transfinite recursions with a simultaneous generated wellordering into a fixed point in general.

In chapter 3.4, we apply this to our situation and embed the R-calculus into the σ^+-calculus. Here we also deal with the second problem mentioned above. We do not need a whole Ramsey ultrafilter for our construction, but we can construct a filter simultaneously with our induction which contains sufficiently many sets.

In chapter 4, we prove the other direction by embedding the theory ⅁ame into the R-calculus. For this we introduce bounded versions of the game-quantifiers which are informally defined by the following clauses.

- $(\forall^1 x \leq B)\varphi(x) :\Leftrightarrow \exists k \exists f \forall n[f(n) \leq B^{\trianglerighteq k}(n) \wedge \varphi(\langle f(0), \ldots, f(n-1)\rangle)]$
- $(\exists^{n+1} x \leq B)\varphi(x) :\Leftrightarrow (\forall^n x_0 \leq B)(\forall^n x_1 \leq B) \cdots \bigvee_{m \in \omega} \varphi(\langle x_0, \ldots, x_m\rangle)$
- $(\forall^n x \leq B)\varphi(x) :\Leftrightarrow \neg(\exists^n x \leq B)\neg\varphi(x)$

The bound B is always an infinite set of natural numbers, and $B^{\trianglerighteq k}$ is B without its least k elements. $B^{\trianglerighteq k}(n)$ is the n'th element of $B^{\trianglerighteq k}$ if $B^{\trianglerighteq k}$ is ordered increasingly. It follows directly from König's lemma that the bounded

$\forall^1$-quantifier can be expressed arithmetically, because we only have to look for paths in the finitely branching tree which is left from B. This carries over to all game quantifiers, i.e. the bounded quantifier is of less complexity than the corresponding unbounded quantifier.

Let Q^n be $\exists^n$ if n is even and $\forall^n$ if n is odd. If $H_1(m) \leq H_2(m)$ for each m, we have a monotonicity property

$$(Q^n x \leq H_1)\varphi(x) \to (Q^n \leq H_2)\varphi(x) \to (Q^n x)\varphi(x),$$

hence the bounded quantifiers in some sense converge against the unbounded one if the bounds become thin enough.

To embed ∂ame into the R-calculus, we prove that

$$(Q^n x)\varphi(x, \vec{y}) \leftrightarrow (Q^n x \leq H)\varphi(x, \vec{y}),$$

where H is a homogeneous set for $R_{\vec{y}} := \{X \mid (Q^n x \leq X)\varphi(x, \vec{y})\}$ for each $\vec{y}$. This reduces the complexity of $(Q^n x)\varphi(x, \vec{y})$ and allows us to prove comprehension for this formula.

In chapter 5, we collect some consequences of the proof of our main theorem. In 5.1, we introduce and examine an analogon to recursive and hyperarithmatical encodibility (see [Sol78]) for the σ-calculus. In chapter 5.2, we use our technique from chapter 3.4 to prove that the σ^+-calculus shows that each set definable in the σ^+-calculus has the property of Baire. In chapter 5.3, we give reasons why Lebesgue-measurability can not be treated in this way. In chapter 5.4, we look at our results if the Bairespace is replaced by the space of monotone sequences of ordinals less than κ of length κ for an inaccessible cardinal κ.

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1. REVERSE MATHEMATICS OF THE PROPERTY OF RAMSEY

1.1 The property of Ramsey

The problem of Ramsey is the following question: Given a set X of reals (i.e. a set of sets of natural numbers), is there a infinite set H of natural numbers such that either all infinite subsets of H are in X or all infinite subsets of H are not in X? Such a set H is called homogeneous for X. We say that H accepts (avoids) X iff all infinite subsets of H are (not) in X.

It is easy to see that ZFC proves the existence of a set of reals which does not have the property of Ramsey. Define the equivalence relation

$$X \sim Y \Leftrightarrow X \triangle Y \text{ is finite}$$

and let X^* be a chosen representative of the equivalence class of X. Then

$$\{X \mid \text{the number of elements of } X \triangle X^* \text{ is even}\}$$

does not have the property of Ramsey.

Under $V = L$ we can compute the complexity of this set. Let $<_L$ be the Δ^1_2-wellordering of the reals. For $X \subset \omega$ and $y \in \omega$, let $X(y)$ be the set which arises from X by changing finitely many elements of X in a way coded by y. Then the set of representatives

$$M := \{X \subset \omega \mid (\forall y \in \omega) \neg [X(y) <_L X]\}$$

is also Δ^1_2. Define

$$A := \{X \subset \omega \mid (\exists y \in \omega) X(y) \in M \text{ and } |X \triangle X(y)| \text{ is even}\}.$$

Then A is a Δ^1_2-set which has not the property of Ramsey. We have proved the following well known theorem.

Theorem 1.1.1. ZFC $+V = L$ *proves the existence of a* Δ^1_2*-set which is not Ramsey. Hence* ZFC *does not prove that each* Δ^1_2*-set is Ramsey.*

Silver proved the following theorem.

Theorem 1.1.2. ZFC *proves that every* Σ^1_1*-set has the property of Ramsey. If there exists a measurable cardinal then each* Σ^1_2*-set has the property of Ramsey.*

For the proof see [Sil70]. Ellentuck gave a different proof for the first assertion in [Ell74].

1.2 The property of Ramsey in second order arithmetic

We will give a short overview of the reverse mathematics of the property of Ramsey for subsystems of second order arithmetic which are weaker than Π^1_2-CA_0. For subsystems of second order arithmetic, see [Sim99]. Some methods used for these weaker systems will reoccur in the proof of our main theorem 1.3.4. The following theorem is due to Mansfield [Man78]. Avigad [Avi98] gave the following simplified proof.

Theorem 1.2.1. ATR_0 *proves that every open set has the property of Ramsey.*

Proof. Let S be a code of the open set O, i.e. a real X is in O iff an initial sequence of X is in S. From now on we identify a set of natural numbers X with the sequence which orders the elements of X increasingly, and sequences are always monotone sequences. Assume that there is no homogeneous set which avoids O. Let

$$T := \{s \mid \text{ no subsequence of } s \text{ is in } S\}.$$

Then T is a tree and wellfounded by our assumption. By recursion on the Kleene-Brouwer ordering of T, we will define a set U_s for each monotone sequence s and label each s either as good or bad such that the following is true:

1. U_s is infinite for each s
2. $U_s \subset_{\mathrm{f}} U_t$ (i.e. $U_s \backslash U_t$ is finite) for $t \leq^{KB}_T s$
3. if s is good then for all $n \in U_s$, $s^\frown\langle n\rangle$ is good
4. if s is bad then for all $n \in U_s$, $s^\frown\langle n\rangle$ is bad

For $s \notin T$, we put $U_s := \omega$ and label s good if the shortest initial sequence of s which is not in T is in S, otherwise s is bad. To label the elements s of T by recursion on its Kleene-Brouwer ordering, we need the following lemma.

Lemma 1.2.2. *Suppose that for each $t \leq_T^{KB} s$ the set U_t satisfies the conditions 1. and 2. Then there is an infinite set Z such that $Z \subset_{\mathrm{f}} U_t$ for each $t \leq_T^{KB} s$.*

Proof. If s is the $\leq_T^{KB}$-minimal element let $U_s := \omega$, if s is the successor of t let $U_s := U_t$. In the limit case, we take a diagonal intersection. Let $(t_i)_{i\in\omega}$ be an enumeration of the $t \leq_T^{KB} s$. Let u_0 be the least element of U_{t_0} and u_{i+1} the least element of

$$\bigcap_{j\leq i}\{x \in U_{t_i} \mid x > u_i\}.$$

Then $Z := \{u_i \mid i \in \omega\}$ fulfills the claim. □

We now define the sets U_s and label each s as good or bad by recursion along $\leq_T^{KB}$. At stage s, by the lemma choose an infinite set Z such that $Z \subset_{\mathrm{f}} U_t$ for each $t \leq_T^{KB} s$. Let $W := \{n \in Z \mid s^\frown\langle n\rangle \text{ is good}\}$. If W is infinite, then s is good and $U_s := W$. Otherwise, s is bad and $U_s := Z\backslash W$.

We first prove that $\langle\rangle$ is good. If not, we could build an infinite increasing sequence $x_1, x_2, \ldots$ such that every subsequence is bad by taking $x_0 \in U_{\langle\rangle}$ and $x_{n+1} \in \bigcap_{t\subset\langle x_1,\ldots,x_n\rangle} U_t$ (this set is not empty since $U_{\langle\rangle} \subset_{\mathrm{f}} U_t$ for each t). By our assumption $X := \{x_i \mid i \in \omega\}$ has a subset which is an element of O, i.e. we have a finite sequence $s \in S$ with elements from X. Take s minimal with that property, i.e. all proper subsequences of s are not in S. Then s is not in T and any proper initial segment is in T, so by definition s is good, which is a contradiction because s is a subsequence of X. This finishes the proof that $\langle\rangle$ is good.

As above, we construct a set X every increasing subsequence of which is good. We claim that every infinite subset Y of X is in O. Since T is wellfounded, there is a shortest initial sequence s of Y which is not in T. Since s is good we have s in S, hence $Y \in O$. □

The notion "s is good (bad)" can be interpreted as "the set O is locally big (small) at s" in the following way: For a finite monotone sequence s and an infinite set X let

$$[s, X] := \{Y \mid Y \text{ infinite} \wedge Y \subset s \cup X \wedge s \text{ is an initial sequence of } Y\}.$$

Then "s is good (bad)" implies $[s, X] \subset O$ ($[s, X] \subset O^c$) for some suitable infinite set X. This is true because like in the above proof you can choose X in a way that each sequence which starts with s and continues in X is good (bad). We will often use this notion of locally big or small sets in the proof of our main theorem.

We could even prove a slightly stronger result than theorem 1.2.1: For each open set O and each $[s, X]$ there exists an infinite homogeneous set $H \subset X$ such that either $[s, H] \subset O$ or $[s, H] \subset O^c$. A set with this property we call completely Ramsey. The proof is analogous to 1.2.1. We define

$$T := \{t \subset X \mid \text{ for no subsequence } t' \text{ of } t,\, s^\frown t' \text{ is in } S\}$$

and start with $U_s := X$ instead of $U_s := \omega$ for the $\leq_T^{KB}$-minimal element s. Then we can assure that all sets U_t are subsets of X.

Notice that the lemma in the proof of 1.2.1 somehow replaces the use of an ultrafilter. If we would carry out the proof in a stronger axiom system (for example ZFC) where we can talk about a non principal ultrafilter on ω, the lemma becomes superfluous by labeling s good and $U_s := \{n \,|\, s^\frown\langle n\rangle \text{ is good}\}$ if $\{n \mid s^\frown\langle n\rangle \text{ is good}\}$ is in the ultrafilter; otherwise s is bad and $U_s := \{n \mid s^\frown\langle n\rangle \text{ is bad}\}$. In the rest of the proof we only need that the intersection of finitely many U_s is infinite which we now obtain from the ultrafilter property instead of the lemma.

When we prove our main theorem, the situation will be similar: We will have a proof which uses a non principal ultrafilter on ω and formalize it in a subsystem of second order arithmetic by exploiting that not the whole ultrafilter is used in the original proof. We construct the filter by recursion with properties like 1. and 2. in the above proof, and when we need the property of an ultrafilter for a special set A, by 1. and 2. we always can add either A or A^c to the filter we have constructed until this point. At limit stages we will use an argument similar to the lemma in the above proof.

[Sim99] gives a simple proof (due to Jockusch) of the reversal of theorem 1.2.1.

Theorem 1.2.3. RCA_0 *proves that the Ramsey theorem for clopen sets implies arithmetical transfinite recursion.*

For the proof of the next theorem see [Sim99], theorem VI.6.4.

Theorem 1.2.4. RCA_0 *proves that the following is equivalent:*

- Π^1_1-CA
- Σ^0_∞-*RT, i.e. "each arithmetical set is Ramsey"*
- Δ^0_2-*RT*

The following theorem is due to Tanaka (see [Tan89]).

Theorem 1.2.5. ACA_0 *proves that* Π^1_1-*TR is equivalent to* Δ^1_1-*RT.*

In [Tan89] Tanaka further considers a system of Σ^1_1 monotone inductive definitions called $\Sigma^1_1 - MI_0$ which is ACA_0 plus the assertion that any Σ^1_1-definable monotone operator Γ has a smallest fixed point. He proves (in [Tan89]):

Theorem 1.2.6. ACA_0 *proves that* $\Sigma^1_1 - MI_0$ *is equivalent to* Σ^1_1-*RT.*

The following tabular summarizes the previous results.

Theory	Ramsey strength	Reference
ZFC	$< \Delta^1_2$	[Jec03], corollary 25.28
$\Sigma^1_1 - MI_0$	Σ^1_1	[Tan89]
Π^1_1-TR	Δ^1_1	[Tan89]
Π^1_1-CA_0	Δ^2_0, Σ^0_∞	[Sim99], theorem VI.6.4.
ATR_0	Σ^0_1, Δ^0_1	[Sim99], Theorem V.9.7

Tab. 1.1: Ramsey strength

1.3 A system of autonomous iterated Ramseyness

We define an axiom system based on second order arithmetic which postulates the existence of all sets which are first order definable by iterated Ramseyness, called R-calculus. We will claim axiomatically the existence of homogeneous sets for first order definable sets of reals. It would be convenient to have the sharper property that for countably many first order definable sets of reals there always is an infinite subset of ω which is simultaneously homogeneous for all these sets. Unfortunately, such a system is inconsistent as the following example shows. Define

$$R_s := \{X \mid s \subset X \subset \omega \wedge X \text{ is infinite}\}$$

for each finite subset s of ω. Assume that H is homogeneous for all R_s. For $h \in H$, H can not avoid $R_{\{h\}}$, but since $H \backslash \{h\} \notin R_{\{h\}}$ it can not accept $R_{\{h\}}$ either.

To avoid this contradiction we only claim the existence of a set which becomes homogeneous after removing an initial segment.

Definition 1.3.1 (language of the R-calculus $\mathcal{L}_R$)**.** $\mathcal{L}_R$ is the language $\mathcal{L}_2$ of second order arithmetic extended by the following clause:

If $\varphi(\vec{x}, X)$ is a first order $\mathcal{L}_R$-formula then $R\vec{x}X\varphi(\vec{x}, X)$ is a set term of $\mathcal{L}_R$. The free variables of $R\vec{x}X\varphi(\vec{x}, X)$ are the free variables of $\varphi(\vec{x}, X)$ except for $\vec{x}$ and X.

The intended meaning of $R\vec{x}X\varphi(\vec{x}, X)$ is a set such that for all $\vec{x}$, we can remove an initial segment of $R\vec{x}X\varphi(\vec{x}, X)$ such that the remaining set is homogeneous for $\{X \mid \varphi(\vec{x}, X)\}$.

Definition 1.3.2 (R-calculus)**.** The R-calculus comprises the axioms of $\mathrm{ACA}_0(\mathcal{L}_R)$ (ACA_0 in the language of $\mathcal{L}_R$, that means we have comprehension for all first order $\mathcal{L}_R$-formulas) and the following schemes for all first order formulas $\varphi(\vec{x}, \vec{z}, X, \vec{Z})$:

- $\forall\vec{Z}\forall\vec{z}R\vec{x}X\varphi(\vec{x}, \vec{z}, X, \vec{Z})$ is infinite
- $\forall\vec{Z}\forall\vec{z}\forall\vec{y}\exists k\big[\forall^\infty Y\big(Y \subset (R\vec{x}X\varphi(\vec{x}, \vec{z}, X, \vec{Z}))^{\trianglerighteq k} \to \varphi(\vec{y}, \vec{z}, Y, \vec{Z})\big)$
  $\vee\ \forall^\infty Y\big(Y \subset (RX\vec{x}\varphi(\vec{x}, \vec{z}, X, \vec{Z}))^{\trianglerighteq k} \to \neg\varphi(\vec{y}, \vec{z}, Y, \vec{Z})\big)\big]$,

where $X^{\trianglerighteq n}$ is X without the least $n-1$ elements of X and $\forall^\infty Y(\dots)$ is an abbreviation for $\forall Y\big((\forall n)(\exists m > n)m \in Y \to \dots\big)$ and means "for all infinite sets Y".

We introduce subsystems of the R-calculus which only allow a certain number of R-nestings.

Definition 1.3.3 (subsystems of the R-calculus)**.** Let $\mathcal{L}_{R_0} := \mathcal{L}_2$. If $\varphi(\vec{x}, X)$ is an $\mathcal{L}_{R_n}$-formula, then $R\vec{x}X\varphi(\vec{x}, X)$ is an $\mathcal{L}_{R_{n+1}}$-set term. φ is an $\mathcal{L}_{R_n}$-formula iff it contains only $\mathcal{L}_{R_n}$-set terms. The R_n-calculus is a theory in the language $\mathcal{L}_{R_n}$ and contains the axioms of the R-calculus which are in $\mathcal{L}_{R_n}$.

Our goal is to prove the following theorem.

Theorem 1.3.4. *The R-calculus and Π^1_2-CA_0 prove the same Π^1_1-sentences of $\mathcal{L}_2$.*

We introduce a second system of autonomous iterated Ramseyness where the crucial axiom scheme has a more canonical form than in the R-calculus. This new calculus contains additionally some uniform recursion along the natural numbers.

Definition 1.3.5 (language of the RI-calculus $\mathcal{L}_{RI}$)**.** $\mathcal{L}_{RI}$ is the language $\mathcal{L}_2$ of second order arithmetic extended by the following clause:

If $\varphi(X)$ is a first order $\mathcal{L}_{RI}$-formula then $(RX\varphi(X))(U_\varphi)$ with a new free variable U_φ is a set term of $\mathcal{L}_{RI}$. The free variables of $(RX\varphi(X))(U_\varphi)$ are the free variables of $\varphi(X)$ except for X and a new free variable U_φ.

If $\psi(x,y,X)$ is a first order $\mathcal{L}_{RI}$-formula then $IxyX\psi(x,y,X)$ is a set term of $\mathcal{L}_{RI}$. The free variables of $IxyX\psi(x,y,X)$ are the free variables of $\psi(x,y,X)$ except for x, y and X.

The intended meaning of $(RX\varphi(X))(U_\varphi)$ is a subset of U_φ which is homogeneous for $\{X \mid \varphi(X)\}$. $IxyX\psi(x,y,X)$ codes an ω-iteration of ψ-comprehension.

Definition 1.3.6 (RI-calculus)**.** The RI-calculus comprises the axioms of $\mathrm{ACA}_0(\mathcal{L}_{RI})$ and the following schemes for all first order formulas $\varphi(X)$ and $\psi(x,y,X)$:

$$\begin{aligned}&\forall U_\varphi\big[\forall^\infty Y\big(Y \subset (RX\varphi(X))(U_\varphi) \to \varphi(Y)\big)\\&\qquad \vee \forall^\infty Y\big(Y \subset (RX\varphi(X))(U_\varphi) \to \neg\varphi(Y)\big)\big],\end{aligned}$$

$$\forall U_\varphi[(RX\varphi(X))(U_\varphi) \subset U_\varphi]$$

$$\forall U_\varphi[(RX\varphi(X))(U_\varphi) \text{ is infinite}]$$

and

$$\forall z[\psi(z,0,\omega) \leftrightarrow z \in (IxyX\psi(x,y,X))_0] \wedge$$

$$\forall z\forall w[\psi(z,w+1,(IxyX\psi(x,y,X))_w) \leftrightarrow z \in (IxyX\psi(x,y,X))_{w+1}].$$

As in the definition of the R-calculus, φ and ψ may contain further parameters $\vec{Z}$ and $\vec{z}$. From now on, we will suppress them in most cases.

Lemma 1.3.7. *Each $\mathcal{L}_2$-sentence provable in the R-calculus is provable in the RI-calculus.*

Proof. We define an embedding $^* : \mathcal{L}_R \to \mathcal{L}_{RI}$ which is the identity on $\mathcal{L}_2$ and show that the translations of the axioms of the R-calculus are provable in the RI-calculus. A first idea could be as follows: We construct the homogeneous set by an iteration. We start with ω, and at the m-th step we take a subset that is homogeneous for $\varphi((m)_0, \dots (m)_n, X)$. Then for all $\vec{y}$ the intersection of these sets is homogeneous for $\varphi(\vec{y}, X)$. The problem is that this intersection could be finite or even empty. We therefore change our construction as follows: If X is the set at the m-th step of the iteration, we remove only elements which are bigger than the m-th element of X and leave the first m

elements unchanged. So we construct a kind of a kernel which grows with the induction and whose elements are never removed in the future. Then for each $\vec{y}$ we can remove an initial segment of this kernel such that the remaining set is homogeneous for $\varphi(\vec{y}, X)$.

For each $\mathcal{L}_R$-formula $\varphi(\vec{y}, Z)$ we define an $\mathcal{L}_{RI}$-formula

$$\widetilde{\varphi}(x, y, Z) :\equiv \big(y \in Seq \wedge x \in (RX\varphi^*((y)_0, \dots (y)_{n-1}, X))(Z^{\rhd y})\big) \vee x \in Z^{\unlhd y},$$

where n is the length of the sequence coded by y, $X^{\unlhd y}$ is the set of the y least elements of X and $X^{\rhd y} := X \backslash X^{\unlhd y}$. We abbreviate $I := IxyZ\widetilde{\varphi}(x, y, Z)$ and define

$$(R\vec{y}X\varphi(\vec{y}, X))^* := \bigcup_z ((I)_z)^{\unlhd z}.$$

Since

$$(I)_y = RX\varphi^*((y)_0, \dots, (y)_{n-1}, X)(((I)_{y-1})^{\rhd y}) \cup ((I)_{y-1})^{\unlhd y}$$

we obtain

$$((I)_y)^{\rhd y} \subset RX\varphi^*((y)_0, \dots, (y)_{n-1}, X)(((I)_{y-1})^{\rhd y}),$$

so $((I)_y)^{\rhd y}$ is homogeneous for $\varphi^*((y)_0, \dots (y)_{n-1}, X)$ and it suffices to show for all y

$$\bigcup_z ((I)_z)^{\unlhd z} \subset (I)_y.$$

If $z \geq y$ we have $((I)_z)^{\unlhd z} \subset (I)_z \subset (I)_y$, if $z < y$ we obtain $((I)_z)^{\unlhd z} = ((I)_y)^{\unlhd z} \subset (I)_y$ because of $((I)_x)^{\unlhd x} = ((I)_y)^{\unlhd x}$ for all $y > x$. □

2. SOME CHARACTERIZATIONS OF Π^1_2-CA$_0$

Möllerfeld showed in [Möl02] that Π^1_2-CA$_0$ is strongly connected to other subsystems of second order arithmetic which we will introduce in this chapter. The μ-calculus is a system of iterated monotone inductive definitions, the σ-calculus is its nonmonotone analogue. The theory ⅁ame can talk about generalizations of the quantifiers $\forall$ and $\exists$ which can be described by games. Möllerfeld showed that these three theories prove the same $\mathcal{L}_2$-sentences. Moreover, he showed that they prove the same Π^1_1-sentences of $\mathcal{L}_2$ as Π^1_2-CA$_0$. In the last part of this chapter, we introduce the σ^+-calculus, a variation of the σ-calculus which proves the same Π^1_1-sentences but has more transfinite induction.

2.1 *The μ-calculus*

A monotone inductive definition is given by an operator $\Gamma : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ which is monotone, i.e. $X \subset Y \to \Gamma(X) \subset \Gamma(Y)$. A fixed point of Γ is a set F such that $\Gamma(F) = F$. To see that each monotone operator has a fixed point we define the stages of the inductive definition given by Γ as follows:

- $I^0 := \emptyset$
- $I^{\alpha+1} := \Gamma(I^\alpha)$
- $I^\lambda := \cup_{\alpha<\lambda} I^\alpha$ for $\lambda \in Lim$

By monotonicity of Γ we obtain $I^\alpha \subset I^\beta$ for $\alpha \leq \beta$, therefore the stages I^α have to be eventually constant before ω_1 by cardinality reasons. This shows that a fixed point always exists. A least fixed point of Γ is a fixed point of Γ which is a subset of each fixed point of Γ. If $I_\alpha = \Gamma(I_\alpha)$ then I_α is the least fixed point of Γ.

We want to talk about monotone operators and its least fixed points in a theory which is based on second order arithmetic. An operator Γ is represented by a formula $\varphi(x, X)$ via $\Gamma(X) = \{x \mid \varphi(x, X)\}$. If $\varphi(x, X)$ is X-positive (i.e. X occurs in φ only inside an even number of negations), then the operator given by φ is monotone. A theory which formalizes the theory of iterated monotone inductive definitions is the μ-calculus.

Definition 2.1.1 (Language of the μ-calculus $\mathcal{L}_\mu$)**.** We start with the language of second order arithmetic and add a set-constructor μ: For each X-positive formula $\varphi(x, X)$ which contains no second order quantification we add a set term $\mu x X \varphi(x, X)$ which intends to denote the least fixed point of the monotone operator $\Gamma_\varphi(X) := \{x \mid \varphi(x, X)\}$. The free variables of $\mu x X \varphi(x, X)$ are the free variables of $\varphi(x, X)$ except for x and X. A free variable Y occurs positively in $t \in \mu x X \varphi(x, X)$ ($t \notin \mu x X \varphi(x, X)$) iff φ is Y-positive (Y-negative). φ may contain further μ-terms such that nestings of fixed points are possible.

Definition 2.1.2. Let $\varphi(x, X)$ be an X-positive formula. Then

- $\mathrm{Cl}(\varphi, Z) :\equiv \forall x(\varphi(x, Z) \to x \in Z)$
- $\mathrm{EFP}(z, \varphi) :\equiv \forall Z(\mathrm{Cl}(\varphi, Z) \to z \in Z)$
- $\mathrm{LFP}(Z, \varphi) :\equiv \mathrm{Cl}(Z, \varphi) \land \forall x(x \in Z \to \mathrm{EFP}(x, \varphi))$.

The formulas mean "Z is **cl**osed under φ", "z is in **e**ach **f**ixed **p**oint of φ" and "Z is the (with respect to set inclusion) **l**east **f**ixed **p**oint of φ". The first part of the conjunction in the definition of LFP we call the first fixed point axiom, the second one is the second fixed point axiom.

Lemma 2.1.3. ACA_0 *proves for each X-positive formula φ*

$$\forall Z\big([\forall z(z \in Z \leftrightarrow \mathrm{EFP}(z, \varphi))] \to \mathrm{LFP}(Z, \varphi)\big).$$

Proof. We only have to show $\mathrm{Cl}(Z, \varphi)$, hence assume $\varphi(x, Z)$. We have to show $x \in Z$, i.e. $\mathrm{EFP}(x, \varphi)$. Hence assume $\mathrm{Cl}(\varphi, X)$, and we have to show $x \in X$. $\mathrm{Cl}(\varphi, X)$ together with $\forall z(z \in Z \leftrightarrow \mathrm{EFP}(z, \varphi))$ implies $Z \subset X$, therefore $\varphi(x, X)$ since φ is X-positive. This implies $x \in X$ by $\mathrm{Cl}(\varphi, X)$. □

Definition 2.1.4 (μ-calculus)**.** The μ-calculus is formulated in $\mathcal{L}_\mu$ and contains the following axioms:

- the axioms of ACA_0 (see [Sim99]) with comprehension for all $\mathcal{L}_\mu$-formulas without second order quantifiers
- $\mathrm{LFP}(\mu x X \varphi(x, X), \varphi(x, X))$ for each X-positive φ without second order quantifiers

If we order the elements of a fixed point according to the stage I^α in which they first enter the fixed point we obtain a prewellordering. The general definition of a prewellordering is as follows.

Definition 2.1.5. $\preceq, \prec$ is a prewellordering on P iff

- $\forall x(x \in P \rightarrow x \preceq x)$
- $\forall x, y(x \preceq y \rightarrow x \in P)$
- $\forall x, y(x \preceq x \wedge y \not\preceq x \rightarrow x \prec y)$
- $\forall x, y(x \prec y \rightarrow x \preceq y)$
- $\forall x, y, z[x \prec y \preceq z \rightarrow x \prec z]$
- $\prec$ is wellfounded

We abbreviate this by $\mathrm{PWO}(P, \preceq, \prec)$.

Lemma 2.1.6. $\mathrm{PWO}(P, \preceq, \prec)$ *implies*

- $\forall x(x \not\prec x)$
- $\forall x(x \in P \leftrightarrow x \preceq x)$
- $\forall x, y(x \prec y \rightarrow y \not\preceq x)$
- $\forall x, y(x \in P \wedge y \notin P \rightarrow x \prec y)$
- $\forall x, y, z(x \preceq y \wedge y \preceq z \rightarrow x \preceq z)$
- $\forall x, y, z(x \preceq y \wedge y \prec z \rightarrow x \prec z)$
- $\forall x, y(x \in P \wedge y \not\prec x \rightarrow x \preceq y)$.

Lemma 2.1.7 (stage comparison)**.** *For each X-positive $\mathcal{L}_\mu$-formula $\varphi(x, X)$ there exist uniformly in φ relations $\preceq_\varphi$ and $\prec_\varphi$ such that the μ-calculus proves*

$$\mathrm{PWO}(\mu x X \varphi(x, X), \preceq_\varphi, \prec_\varphi) \wedge \forall x \forall y[x \preceq_\varphi y \leftrightarrow x \prec_\varphi y \vee \varphi(x, \{z \mid z \prec_\varphi x\}]).$$

Proof. In [Tap99] there is a proof of this theorem for the theory ID_1 (which can talk about fixed points of monotone operators but does not admit nestings of fixed points like the μ-calculus), and this proof transfers to the μ-calculus. □

2.2 The σ-calculus

For a nonmonotone operator Γ (i.e. an arbitrary $\Gamma : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$), a least fixed point in the above sense does not need to exist. For example, take an operator Γ such that $0 \in \Gamma(X) \Leftrightarrow 0 \notin X$. To obtain a suitable notion of a fixed point of a nonmonotone operator we alter the definition of the stages as follows:

- $J^\alpha := \Gamma(J^{<\alpha}) \cup J^{<\alpha}$
- $J^{<\alpha} := \cup_{\beta<\alpha} J^\beta$

We call $J^\infty := \cup_\alpha J^\alpha$ the fixed point of the nonmonotone inductive definition. By the definition of the stages, we again obtain $J_\alpha \subset J_\beta$ for $\alpha \leq \beta$, and the stages J^α become eventually constant before ω_1. We define a stage comparison relation by

- $x \preceq y \Leftrightarrow \exists\alpha[x \in J^\alpha \wedge y \notin J^{<\alpha}]$
- $x \prec y \Leftrightarrow \exists\alpha[x \in J^\alpha \wedge y \notin J^\alpha]$.

Then the stage comparison relation $\preceq, \prec$ of the fixed point of Γ is the unique PWO such that

$$\forall x \forall y[x \preceq y \leftrightarrow x \prec y \vee x \in \Gamma(\{z \mid z \prec y\})].$$

Möllerfeld's σ-calculus is a system of iterated nonmonotone inductive definitions. It has a set term $\sigma x X \varphi(x, X)$ for each formula φ which is intended to code the stage comparison relation $\preceq$ of the operator given by $\Gamma(X) := \{x \mid \varphi(x, X)\}$. Then the fixed point J^∞ is represented by $I(\sigma x X \varphi(x, X))$ with $I(X) := \{x \mid x \preceq_X x\}$, where $\preceq_X$ is the relation coded by X. We now give a formal definition of the σ-calculus.

Definition 2.2.1. For each (not necessarily X-positive) formula $\varphi(x, X)$ let IGF$(\varphi, \preceq_\varphi)$ be an abbreviation for

$$\text{PWO}(\preceq_\varphi, \prec_\varphi) \wedge \forall x \forall y[x \preceq_\varphi y \leftrightarrow x \prec_\varphi y \vee \varphi(x, \{z \mid z \prec_\varphi x\}]),$$

where $\prec_\varphi$ is the irreflexive part of $\preceq_\varphi$, i.e. $x \prec_\varphi y \Leftrightarrow x \preceq_\varphi y \wedge \neg y \preceq_\varphi x$.

Lemma 2.2.2. *If $\varphi(x, X)$ is a first order X-positive formula of $\mathcal{L}_2$ then* ACA$_0$ *proves*

$$\forall Z[\text{IGF}(\varphi, Z) \to \text{LFP}(\varphi, I(Z))].$$

Proof. see [Möl02], 3.17 □

Definition 2.2.3 (Language of the σ-calculus $\mathcal{L}_\sigma$)**.** We add a set constructor σ to the language of second order arithmetic: For each (not necessarily X-positive) formula $\varphi(x, X)$ which contains no second order quantifiers we add a set term $\sigma x X \varphi(x, X)$. The free variables of $\sigma x X \varphi(x, X)$ are the free variables of $\varphi(x, X)$ except for x and X. φ may contain further σ-terms, i.e. like in the μ-calculus nestings of fixed points are possible.

Definition 2.2.4 (σ-calculus)**.** The σ-calculus is formulated in $\mathcal{L}_\sigma$ and contains the following axioms:

- the axioms of ACA_0 with comprehension for all $\mathcal{L}_\sigma$-formulas without second order quantifiers
- $\mathrm{IGF}(\varphi(x, X), \sigma x X \varphi(x, X))$ for each $\mathcal{L}_\sigma$-formula φ without second order quantifiers

Theorem 2.2.5 (Möllerfeld)**.** *The μ-calculus and the σ-calculus prove the same sentences of second order arithmetic. Π^1_2-CA_0, the σ-calculus and the μ-calculus prove the same Π^1_1-sentences, so they are proof-theoretically equivalent.*

For the proof see [Möl02], theorems 3.21 and 10.6. We now introduce some subsystems of the σ-calculus.

Definition 2.2.6 (σ_n-formulas and terms)**.** The σ_0-formulas are the $\mathbf{\Sigma^0_2}$-formulas. If $\varphi(x, X)$ is a σ_n-formula then $\sigma x X \varphi(x, X)$ is a σ_n-term. If $\varphi(X)$ is an $\mathcal{L}_2$-formula where X occurs negatively and T is a σ_n-term, then $\varphi(I(T))$ is a σ_{n+1}-formula.

Definition 2.2.7 (σ_n-calculus)**.** The σ_n-calculus is the σ-calculus with fixed point axioms only for σ_n-terms.

In the notion of [Möl02], the σ_n-calculus is $Ind(\pi^n 2)$.

Lemma 2.2.8. *The σ-calculus proves the same $\mathcal{L}_\sigma$-sentences as the theory $\bigcup_{n\in\omega}(\sigma_n$-calculus$)$.*

Proof. We only have to show that each $\mathcal{L}_\sigma$-sentence provable in the σ-calculus is also provable in $\bigcup_{n\in\omega}(\sigma_n$-calculus$)$. It is sufficient to prove that for each σ_m-term T_m there is a σ_{m+1}-term T_{m+1} with $(T_m)^c = (T_{m+1})_1$ because then each formula which contains only σ_m-terms is equivalent to a σ_{m+1}-formula. The proof is by induction on m. If T_{m+1} is of the shape $\sigma x X \varphi(x, X)$ for a σ_{m+1}-formula φ, choose a σ_{m+2}-formula ψ such that

- $\psi(\langle 0, x\rangle, X) \leftrightarrow \varphi(x, (X)_0)$
- $\psi(\langle 1, x\rangle, X) \leftrightarrow \big([\forall y \varphi(y, (X)_0) \to y \in (X)_0] \wedge x \notin (X)_0\big)$.

Here we needed the induction hypothesis, because the σ_m-terms which occur negatively in φ occur positively and negatively in ψ and we replace the positive occurrences by a negative occurrence of a σ_{m+1}-term by induction hypothesis. Then $(\sigma x X \varphi(x, X))^c = (\sigma x X \psi(x, X))_1$. □

In the rest of this section we present some of Möllerfeld's results about the σ-calculus. He showed in [Möl02] that the σ_n-calculus proves the existence of β-model of the σ_m-calculus for $m < n$. A β-model is a model M such that for all Σ^1_1-sentences with parameters from M, φ is true iff φ holds in M. The central notion which is used in the proofs of these results is that of a Spector class which was first introduced by Moschovakis ([Mos74]). Möllerfeld formalized his results in subsystems of second order arithmetic. Roughly speaking, a Spector class is a set of sets of natural numbers which is closed under the formations of fixed points of inductive processes.

Definition 2.2.9 (Spector class)**.** Let s be a primitive recursive function and $e_{\preceq}$, $e_{\prec}$ be natural numbers. Working in ACA_0, let T be a set. Then $\langle T, e_{\preceq}, e_{\prec}, s\rangle$ is a Spector class iff

- $\mathrm{PWO}(T, (T)_{e_{\preceq}}, (T)_{e_{\prec}})$
- for each X-positive arithmetical formula $\varphi(x, y, X)$, we have

$$\forall y[(T)_{s(\ulcorner \varphi \urcorner, y)} = \{x \mid \varphi(x, y, T)\}].$$

If $\langle T, e_{\preceq}, e_{\prec}, s\rangle$ is a Spector class, we abbreviate $\exists x[X = (T)_x]$ by $X \in T$.

Spector classes are not closed under complements, i.e. $X \in T$ does not imply $X^c \in T$. By $\Delta(T)$ we denote the self dual part of a Spector class, i.e.

$$X \in \Delta(T) \Leftrightarrow \exists x, y[X = (T)_x \wedge X^c = (T)_y].$$

Lemma 2.2.10 (boundedness, [Möl02], 6.5)**.** *Let $\varphi(\vec{x}, X^+, Y^+)$ be a first order formula of $\mathcal{L}_2$ with no other free variables. Working in* ACA_0*, let T be a Spector class and $P \in T \backslash \Delta(T)$. Let $\prec \in T$ be a prewellordering on P. Then*

$$\forall \vec{x} \forall y[\varphi(\vec{x}, P, T) \to (\exists y \in P)\varphi(\vec{x}, P(\prec y), T)]$$

with $P(\prec y) := \{x \in P \mid x \prec y\}$.

Proof. Otherwise

$$\exists\vec{x}\forall y[y \notin P \leftrightarrow \varphi(\vec{x}, P(\prec y), T)],$$

which implies $P \in \Delta(T)$, contradiction. □

The next lemma provides an analogue of the hyperarithmetical hierarchy.

Lemma 2.2.11 (good parametrization of Δ, [Möl02], 6.7)**.** *For $e_{\preceq}$, $e_{\prec}$, s there are natural numbers $e_{\mathcal{I}}$, $e_{\mathcal{H}}$ and $e_{\hat{\mathcal{H}}}$ such that* ACA_0 *proves: Let $\langle T, e_{\preceq}, e_{\prec}, s\rangle$ be a Spector class and*

- $\mathcal{I}(T) := (T)_{e_{\mathcal{I}}}$
- $\mathcal{H}(T) := (T)_{e_{\mathcal{H}}}$
- $\hat{\mathcal{H}}(T) := (T)^c_{e_{\hat{\mathcal{H}}}}$.

Then

- $\forall Z \in \Delta(T)\exists x \in \mathcal{I}(T)[Z = (\mathcal{H}(T))_x]$
- $\forall x \in \mathcal{I}(T)[(\mathcal{H}(T))_x = (\hat{\mathcal{H}}(T))_x]$
- $\forall x \in \mathcal{I}(T)[(\mathcal{H}(T))_x \in \Delta(T)]$.

Proof. Let

- $\mathcal{I}(T) := \{\langle a, b\rangle \mid b \in T\}$
- $\mathcal{H}(T) := \{\langle\langle a, b\rangle, x\rangle \mid b \in T \wedge x \in T(\prec b)_a\}$
- $\hat{\mathcal{H}}(T) := \{\langle\langle a, b\rangle, x\rangle \mid b \not\preceq \langle a, x\rangle\}$.

For the first assertion, let $Z = (T)_x$ and $Z^c = (T)_y$ for some x, y. Applying the boundedness lemma 2.2.10 for $\varphi(x, y, X, Y) :\equiv \forall z[z \in (X)_x \vee z \in (Y)_y]$ and $P = T$, we obtain

$$\varphi(x, y, T, T) \rightarrow (\exists b \in T)\varphi(x, y, T(\prec b), T).$$

Since the premise follows from $(T)_x = (T)^c_y$ we obtain $Z = T(\prec b)_x$ from the conclusion. We put $u := \langle x, b\rangle$, hence $u \in \mathcal{I}(T)$ and $Z = \mathcal{H}(T)_u$.

For the second assertion, take $\langle a, b\rangle \in \mathcal{I}(T)$. Since $b \in T$ we obtain

$$\begin{aligned} x \in (\mathcal{H}(T))_{\langle a,b\rangle} &\leftrightarrow x \in T(\prec b)_a \leftrightarrow \langle a, x\rangle \prec b \\ &\leftrightarrow \neg[b \preceq \langle a, x\rangle] \leftrightarrow x \in \hat{\mathcal{H}}(T)_{\langle a,b\rangle}. \end{aligned}$$

The third claim follows directly from the second. □

The next lemma says how we can make the first assertion of the previous lemma effective, i.e. how to compute x primitive recursively from the T-indices of Z and Z^c.

Lemma 2.2.12 (index computation, [Möl02], 6.9)**.** *For $e_\preceq$, $e_\prec$, s there are primitive recursive functions s_Δ, s_1 and s_2 such that* ACA$_0$ *proves the following. If $\langle T, e_\preceq, e_\prec, s\rangle$ is a Spector class then*

$$\forall x_0, x_1\big(T_{x_0} = T^c_{x_1} \to [s_\Delta(x_0, x_1) \in \mathcal{I}(T) \wedge T_{x_0} = (\mathcal{H}(T))_{s_\Delta(x_0,x_1)}]\big).$$

If $a \in \mathcal{I}(T)$ then $(\mathcal{H}(T))_a = (T)_{s_1(a)} = ((T)_{s_2(a)})^c$.

Proof. To prove the first assertion, let

$$C := \{c \mid \exists y[y \notin T_{x_1} \wedge \langle x_0, y\rangle \not\prec \langle c, c\rangle]\}.$$

Then there is a c_0 such that $C = (T_{c_0})^c$. Towards a contradiction assume $c_0 \in C$. Then $y \notin T_{x_1}$, hence $y \in T_{x_0}$, and together with $\langle x_0, y\rangle \not\prec \langle c_0, c_0\rangle$ this implies $\langle c_0, c_0\rangle \in T$, hence $c_0 \in T_{c_0} = C^c$, contradiction. From $c_0 \notin C$ we obtain

$$\forall y[y \in T_{x_0} \leftrightarrow \langle x_0, y\rangle \prec \langle c_0, c_0\rangle],$$

and the result follows for $s_\Delta(x_0, x_1) := \langle x_0, \langle c_0, c_0\rangle\rangle$.

s_1 (s_2 resp.) can be directly defined using s and $e_\prec$ (s and $e_\preceq$ resp.). □

Lemma 2.2.13 (comprehension in Δ, [Möl02], 6.8)**.** *For each $e_\preceq$, $e_\prec$, s there is a primitive recursive function t such that for each first order $\mathcal{L}_2$-formula $\varphi(y, \vec{x}, X_1, \ldots X_n)$ with no other free variables* ACA$_0$ *proves: Let $\langle T, e_\preceq, e_\prec, s\rangle$ be a Spector class. Then we have for all $\vec{x}$ and all $a_1, \ldots, a_n \in \mathcal{I}(T)$*

- $t(\ulcorner\varphi\urcorner, \langle\vec{x}\rangle, \langle a_1, \ldots, a_n\rangle) \in \mathcal{I}(T)$
- $\mathcal{H}(T)_{t(\ulcorner\varphi\urcorner,\langle\vec{x}\rangle,\langle a_1,\ldots,a_n\rangle)} = \{y \mid \varphi(y, \vec{x}, \mathcal{H}(T)_{a_1}, \ldots, \mathcal{H}(T)_{a_n})\}$.

Proof. Without loss of generality φ is of the form $\varphi(y, x, X)$. $\widetilde{\varphi}(y, x, X^+, Y^-)$ emerges from $\varphi(y, x, X)$ by distinguishing the positive and negative occurrences of X, i.e. $\widetilde{\varphi}(y, x, X, X) \leftrightarrow \varphi(y, x, X)$. We compute T-indices of

$$\{y \mid \widetilde{\varphi}(y, x, T_{s_1(a)}, (T_{s_2(a)})^c)\} \text{ and } \{y \mid \neg\widetilde{\varphi}(y, x, (T_{s_2(a)})^c, T_{s_1(a)})\}$$

using s, and by lemma 2.2.12 we obtain the $\mathcal{I}(T)$-index. □

Lemma 2.2.14 ([Möl02],6.14)**.** *For each n there exists a σ_n-term $S^n(\vec{X})$, natural numbers $e_{\preceq n}$, $e_{\prec n}$ and a primitive recursive function s_n such that for each σ_n-formula $\psi(x, \vec{y}, X, \vec{X})$ and each first order formula $\varphi(x, \vec{y}, X^+, \vec{X})$ the σ_n-calculus proves*

- $I(S^n(\vec{X}))_{\langle\ulcorner\psi\urcorner,\langle\vec{y}\rangle\rangle} = I(\sigma x X \psi(x,\vec{y},X,\vec{X}))$
- $x \in I(S^n(\vec{X}))_{s_n(\ulcorner\varphi\urcorner,\langle\vec{y}\rangle)} \leftrightarrow \varphi(x,\vec{y},I(S^n(\vec{X})),\vec{X})$.

Let $T^n(\vec{X}) := I(S^n(\vec{X}))$. Then for each $\vec{X}$, $\langle T^n(\vec{X}), e_{\preceq_n}, e_{\prec_n}, s_n\rangle$ is a Spector class.

Proof. We prove the theorem by metainduction on n. Using the induction hypothesis for $n-1$, we have an universal σ_n-formula U_n, i.e. $U_n(\ulcorner\varphi\urcorner,\vec{x},\vec{X}) \leftrightarrow \varphi(\vec{x},\vec{X})$; if $n = 0$ we just fix an universal $\mathbf{\Sigma^0_2}$-formula, if $n > 0$ then U_n is essentially $x_0 \notin I(S^{n-1}(\vec{X}))_{s_{n-1}(\ulcorner\varphi\urcorner,\langle\vec{y}\rangle)}$. We define

$$\begin{aligned}\chi^n(z,X,\vec{X}) :\equiv &[z = \langle\langle\ulcorner\psi\urcorner,\vec{y}\rangle,x\rangle \wedge U_n(\ulcorner\psi\urcorner,x,\vec{y},(X)_{\langle\ulcorner\psi\urcorner,\vec{y}\rangle},\vec{X})]\\ &\vee[z = \langle\langle\langle\ulcorner\psi\urcorner,w\rangle,\vec{y}\rangle,x\rangle \wedge \langle w,x\rangle \in (X)_{\langle\ulcorner\psi\urcorner,\vec{y}\rangle}].\end{aligned}$$

Then χ^n is a σ_n-formula which satisfies

- $\chi^n(\langle\langle\ulcorner\psi\urcorner,\vec{y}\rangle,x\rangle,X,\vec{X}) \leftrightarrow \psi(x,\vec{y},(X)_{\langle\ulcorner\psi\urcorner,\vec{y}\rangle},\vec{X})$
- $\chi^n(\langle\langle\langle\ulcorner\psi\urcorner,w\rangle,\vec{y}\rangle,x\rangle,X,\vec{X}) \leftrightarrow \langle w,x\rangle \in (X)_{\langle\ulcorner\psi\urcorner,\vec{y}\rangle}$

for each σ_n-formula $\psi(x,\vec{y},X,\vec{X})$, hence

$$I(\sigma x X \chi^n(x,X,\vec{X}))_{\langle\ulcorner\psi\urcorner,\vec{y}\rangle} = I(\sigma x X \psi(x,\vec{y},X,\vec{X})) \tag{2.1}$$

and

$$\begin{aligned}I(\sigma x X \chi^n(x,X,\vec{X}))_{\langle\langle\ulcorner\psi\urcorner,w\rangle,\vec{y}\rangle} &= (I(\sigma x X \chi^n(x,X,\vec{X}))_{\langle\ulcorner\psi\urcorner,\vec{y}\rangle})_w\\ &= I(\sigma x X \psi(x,\vec{y},X,\vec{X}))_w.\end{aligned} \tag{2.2}$$

We define $S^n(\vec{X}) := \sigma x X \chi^n(x,X,\vec{X})$. Then the first assertion follows from (2.1).

The function s_n is defined by recursion on the build up of φ as follows.
If φ is $t(x,\vec{y}) \in X_k$, define $\psi(x,\vec{y},X,\vec{X}) :\equiv x \in X_k$; then

$$t(x,\vec{y}) \in X_k \leftrightarrow x \in I(S^n(\vec{X}))_{\langle\ulcorner\psi\urcorner,\langle\vec{y}\rangle\rangle}$$

by (2.1) and we define $s_n(\ulcorner\varphi\urcorner,\langle\vec{y}\rangle) := \langle\ulcorner\psi\urcorner,\vec{y}\rangle$.
If φ is of the shape $t(x,\vec{y}) \in X$, we obtain a σ_n-formula $\widetilde{\varphi}$ such that

- $\widetilde{\varphi}(\langle 0,x\rangle,\vec{y},X,\vec{X}) \leftrightarrow \chi^n(x,(X)_0,\vec{X})$
- $\widetilde{\varphi}(\langle 1,x\rangle,\vec{y},X,\vec{X}) \leftrightarrow t(x,\vec{y}) \in (X)_0$

holds true. Hence

$$I(\sigma x X \widetilde{\varphi}(x, \vec{y}, X, \vec{X}))_0 = I(S^n(\vec{X}))$$

and

$$t(x, \vec{y}) \in I(S^n(\vec{X})) \leftrightarrow x \in I(\sigma x X \widetilde{\varphi}(x, \vec{y}, X, \vec{X}))_1 \leftrightarrow x \in I(S^n(\vec{X}))_{\langle\langle \ulcorner \widetilde{\varphi} \urcorner, 1\rangle, \vec{y}\rangle},$$

using (2.2) for the last equivalence. We therefore define $s_n(\ulcorner \varphi \urcorner, \langle \vec{y} \rangle) := \langle\langle \ulcorner \widetilde{\varphi} \urcorner, 1\rangle, \vec{y}\rangle$.

If φ is of the form $\forall y \varphi'(x, y, \vec{y}, X, \vec{X})$ we choose $\widetilde{\varphi}$ such that

- $\widetilde{\varphi}(\langle 0, x\rangle, \vec{y}, X, \vec{X}) \leftrightarrow \chi^n(x, (X)_0, \vec{X})$
- $\widetilde{\varphi}(\langle 1, x\rangle, \vec{y}, X, \vec{X}) \leftrightarrow (\forall y) x \in ((X)_0)_{s_n(\ulcorner \varphi' \urcorner, \langle y, \vec{y}\rangle)}$,

therefore

$$I(\sigma x X \widetilde{\varphi}(x, \vec{y}, X, \vec{X}))_0 = I(S^n(\vec{X}))$$

and

$$\begin{aligned} & (\forall y) \varphi'(x, y, \vec{y}, X, \vec{X}) \\ \leftrightarrow & (\forall y)[x \in I(S^n(\vec{X}))_{s_n(\ulcorner \varphi' \urcorner, \langle y, \vec{y}\rangle)}] \\ \leftrightarrow & x \in I(\sigma x X \widetilde{\varphi}(x, \vec{y}, X, \vec{X}))_1 \\ \leftrightarrow & x \in I(S^n(\vec{X}))_{\langle\langle \ulcorner \widetilde{\varphi} \urcorner, 1\rangle, \vec{y}\rangle} \end{aligned}$$

as in the previous case, using the induction hypothesis for the first equivalence. Hence define $s_n(\ulcorner \varphi \urcorner, \langle \vec{y} \rangle) := \langle\langle \ulcorner \widetilde{\varphi} \urcorner, 1\rangle, \vec{y}\rangle$.

Let $T^n(\vec{X}) := I(S^n(\vec{X}))$. It remains to find $e_{\preceq_n}$ and $e_{\prec_n}$ such that $\langle T^n(\vec{X}), e_{\preceq_n}, e_{\prec_n}, s_n\rangle$ becomes a Spector class. Choose $\widetilde{\chi^n}$ such that

- $\widetilde{\chi^n}(\langle 0, x\rangle, X, \vec{X}) \leftrightarrow \chi^n(x, (X)_0, \vec{X})$
- $\widetilde{\chi^n}(\langle 1, \langle x, y\rangle\rangle, X, \vec{X}) \leftrightarrow \chi^n(x, (X)_0, \vec{X}) \wedge y \notin (X)_0$
- $\widetilde{\chi^n}(\langle 2, \langle x, y\rangle\rangle, X, \vec{X}) \leftrightarrow x \in (X)_0 \wedge y \notin (X)_0$.

$\widetilde{\chi^n}$ is a σ_n-formula, and $I(\sigma x X \widetilde{\chi^n}(x, X, \vec{X}))_1$ and $I(\sigma x X \widetilde{\chi^n}(x, X, \vec{X}))_2$ code a PWO of $T^n(\vec{X})$. Now we can compute $e_{\preceq_n}$ and $e_{\prec_n}$ with the first part of the lemma. □

Lemma 2.2.15 ([Möl02],9.8). *For each n, there exists a primitive recursive function r^{n+1} such that for each a σ_n-formula $\varphi(x, X, \vec{y}, Y_1, \dots, Y_m)$ the σ_{n+1}-calculus proves*

$$\begin{aligned}\forall \vec{X} \forall \vec{y} (\forall a_1, \dots, a_m \in & \mathcal{I}(T^{n+1}(\vec{X}))) \\ & [r^n(\ulcorner \varphi \urcorner, \langle \vec{y} \rangle, \langle a_1, \dots, a_m \rangle) \in \mathcal{I}(T^{n+1}(\vec{X})) \\ & \wedge \mathrm{IGF}(\varphi(x, X, \vec{y}, \mathcal{H}(T^{n+1}(\vec{X}))_{a_1}, \dots, \mathcal{H}(T^{n+1}(\vec{X}))_{a_m}), \\ & \qquad \mathcal{H}(T^{n+1}(\vec{X}))_{r^n(\ulcorner \varphi \urcorner, \langle \vec{y} \rangle, \langle a_1, \dots, a_m \rangle)})]\end{aligned}$$

with $\mathcal{I}$ and $\mathcal{H}$ from lemma 2.2.11.

Proof. To simplify notations let $m = 1$ and $a := a_1$. Choose ψ such that

- $\psi(\langle 0, x \rangle, \vec{y}, X, \vec{X}) :\equiv \chi^{n+1}(x, (X)_0, \vec{X})$
- $\psi(\langle 1, x \rangle, \vec{y}, X, \vec{X}) :\equiv \forall z[z \in ((X)_0)_{s_1(a)} \vee z \in ((X)_0)_{s_2(a)}] \wedge x \in ((X)_0)_{s_1(a)}$
- $\psi(\langle 2, x \rangle, \vec{y}, X, \vec{X}) :\equiv \forall z[z \in ((X)_0)_{s_1(a)} \vee z \in ((X)_0)_{s_2(a)}] \wedge \varphi(x, (X)_2, \vec{y}, (X)_1)$
- $\psi(\langle 3, x \rangle, \vec{y}, X, \vec{X}) :\equiv \forall z[z \in ((X)_0)_{s_1(a)} \vee z \in ((X)_0)_{s_2(a)}] \wedge \forall z[\varphi(z, (X)_2, \vec{y}, (X)_1) \to z \in (X)_2] \wedge x \notin (X)_2$.

Then ψ is a σ_{n+1}-formula with

- $I(\sigma x X \psi(x, \vec{y}, X, \vec{X}))_0 = T^{n+1}(\vec{X})$
- $I(\sigma x X \psi(x, \vec{y}, X, \vec{X}))_1 = \mathcal{H}(T^{n+1}(\vec{X}))_a$
- $\mathrm{IGF}(\varphi(x, X, \vec{y}, \mathcal{H}(T^{n+1}(\vec{X}))_a), I(\sigma x X \psi(x, \vec{y}, X, \vec{X}))_2)$
- $I(\sigma x X \psi(x, \vec{y}, X, \vec{X}))_2 = (I(\sigma x X \psi(x, \vec{y}, X, \vec{X}))_3)^c$.

Therefore we can compute an $T^{n+1}(\vec{X})$-index of $I(\sigma x X \psi(x, \vec{y}, X, \vec{X}))$ using the first part of lemma 2.2.14. Then we obtain $T^{n+1}(\vec{X})$-indices of the fixed point of φ and its compliment by s_n (again from 2.2.14). Now we can compute $r^{n+1}(\ulcorner \varphi \urcorner, \langle \vec{y} \rangle, \langle a \rangle)$ using lemma 2.2.12. □

We will need these results in section 2.5 when we prove that the σ-calculus enriched by some transfinite recursion still proves the same Π^1_1-sentences as the σ-calculus.

2.3 *The theory* ⅁ame

Möllerfeld introduces in [Möl02] a theory which can talk about generalizations of the quantifiers $\forall$ and $\exists$. A generalized quantifier Q on ω is a subset of $\mathcal{P}(\omega)$ such that

- $\emptyset \notin \mathsf{Q}$
- $\mathsf{Q} \neq \emptyset$
- $X \subset Y \wedge X \in \mathsf{Q} \Rightarrow Y \in \mathsf{Q}$.

Let $(\mathsf{Q}x)\varphi(x)$ be an abbreviation for $\{x \mid \varphi(x)\} \in \mathsf{Q}$. We can interpret the $\forall$- and the $\exists$-quantifier as generalized quantifiers by $\forall = \{\mathbb{N}\}$ and $\exists = \{X \subset \mathbb{N} \mid X \neq \emptyset\}$.

For each generalized quantifier Q the inverse quantifier $\overline{\mathsf{Q}} := \{X^c | X \notin \mathsf{Q}\}$ is again a generalized quantifier. It holds $\overline{\overline{\mathsf{Q}}} = \mathsf{Q}$, $(\overline{\mathsf{Q}}x)\varphi(x) \Leftrightarrow \neg(\mathsf{Q}x)\neg\varphi(x)$ and $\overline{\exists} = \forall$.

For a generalized quantifier Q we define the next quantifier $\mathsf{Q}^\vee$ by

$$(\mathsf{Q}^\vee x)\varphi(x) :\equiv (\overline{\mathsf{Q}}x_0)(\overline{\mathsf{Q}}x_1)(\overline{\mathsf{Q}}x_2)\cdots \bigvee_{n\in\mathbb{N}} \varphi(\langle x_0, \ldots, x_n\rangle).$$

For example, $(\exists^\vee x)\varphi(x)$ holds if and only if for each function $f : \omega \to \omega$ there is an $n \in \omega$ such that $\varphi(\langle f(1), \ldots, f(n)\rangle)$. For the inverse quantifier we have

$$(\overline{\exists^\vee}x)\varphi(x) \Leftrightarrow \exists f \forall n \varphi(\langle f(1), \ldots, f(n)\rangle).$$

This quantifier is known as the Souslin-quantifier. We next introduce a hierarchy of generalized quantifiers which arises from $\exists$ by taking inverse and next quantifiers.

Definition 2.3.1 (game-quantifiers)**.** Let

- $\exists^0 := \exists$
- $\exists^{n+1} := (\exists^n)^\vee$
- $\forall^n := \overline{\exists^n}$.

These quantifiers are called game-quantifiers because they can be described by games. Take for example the formula $\exists^1 x\varphi(x)$. Imagine the following two player game: Player II plays natural numbers $x_1, x_2, \ldots$, and after finitely many (possibly zero) natural number player I says "stop". If $x_1, \ldots, x_n$ are the natural number played so far, then Player I wins if

$\varphi(\langle x_1, \ldots, x_n \rangle)$ is true, otherwise player II wins. Player I has a winning strategy in this game iff $\exists^1 x \varphi(x)$ holds. For the games describing $\exists^n$ for $n \geq 2$, cf. [Hei03].

We now introduce a theory based on second order arithmetic in which we can talk about the game-quantifiers. We first introduce the language of this theory.

Definition 2.3.2 (the languages $\mathcal{L}_{\supset}$ and $\mathcal{L}_{\supset_n}$)**.** Let $\mathcal{L}_{\supset}$ be the language $\mathcal{L}_2$ extended by new quantifiers $\exists^n$ and $\forall^n$ for each $n \in \omega$. In the inductive definition of the formulas the quantifiers $\exists^n$ and $\forall^n$ are treated like the quantifiers $\exists$ and $\forall$. By $\mathcal{L}_{\supset_n}$ we denote the language which contains only quantifiers $\exists^m$ and $\forall^m$ for $m \leq n$. The new quantifiers are among the first order quantifiers.

Definition 2.3.3 (the theories Ɔame and Ɔame_n)**.** The theory Ɔame is formulated in $\mathcal{L}_{\supset}$ and contains the following axioms and axiom schemes:

- the axioms from ACA_0, with comprehension for all $\mathcal{L}_{\supset}$-formulas without second order quantifiers (which allows game-quantifiers)
- $(\exists^0 x)\varphi(x) \leftrightarrow (\exists x)\varphi(x)$
- $(\exists^{n+1} x)\varphi(x, \vec{y}, \vec{Y}) \leftrightarrow \text{EFP}(\langle\rangle, \varphi(x, \vec{y}, \vec{Y}) \vee (\forall^n z) x^\frown\langle z \rangle \in X)$
- $(\forall^n x)\varphi(x) \leftrightarrow \neg(\exists^n x)\neg\varphi(x)$

for an $\mathcal{L}_{\supset}$-formula φ without second order quantifiers.

Ɔame_n is Ɔame restricted to $\mathcal{L}_{\supset_n}$.

Since $\varphi(x, \vec{y}, \vec{Y}) \vee (\forall^n z) x^\frown\langle z \rangle \in X$ is X-positive it makes sense to consider

$$\text{EFP}(\langle\rangle, \varphi(x, \vec{y}, \vec{Y}) \vee (\forall^n z) x^\frown\langle z \rangle \in X). \tag{2.3}$$

This second order formula replaces the infinite first order formula from the definition of $\mathsf{Q}^\vee$, and we can see as follows that both definitions have the same meaning. Assume

$$(\forall^n x_0)(\forall^n x_1)(\forall^n x_2) \cdots \bigvee_{m \in \mathbb{N}} \varphi(\langle x_0, \ldots, x_m \rangle).$$

Then there exists a wellfounded tree T such that φ holds at all its leafs and each inner node s is $\forall^n$-branching, i.e. $\forall^n x(s^\frown\langle x \rangle \in T)$. But all elements of each such T enter consecutively each fixed point of the operator given by $\varphi(x, \vec{y}, \vec{Y}) \vee (\forall^n z) x^\frown\langle z \rangle \in X)$, therefore $\langle\rangle$ enters, too. This implies (2.3). Conversely, (2.3) implies that each fixed point contains such a tree T, hence $(\forall^n x_0)(\forall^n x_1)(\forall^n x_2) \cdots \bigvee_{m \in \mathbb{N}} \varphi(\langle x_0, \ldots, x_m \rangle)$.

To prove some basic facts about these quantifiers in Ɔame, we need the following technical lemma.

Lemma 2.3.4. *Let* $\varphi(x, X)$ *be an* X*-positive* $\mathcal{L}_2$*-formula and* f *a primitive recursive function. Let*

$$\psi(x, X) :\equiv \varphi(f(x), f(X)) \text{ with } f(X) := \{f(x) \mid x \in X\}.$$

Then ACA_0 *proves: For all sets* I_φ *and* I_ψ *such that* $\mathrm{LFP}(I_\varphi, \varphi)$, $\mathrm{LFP}(I_\psi, \psi)$ *and*

$$\varphi(f(x), X \cup (im(f))^c) \to \varphi(f(x), X)$$

it holds

$$f(I_\psi) = I_\varphi \cap im(f).$$

Proof. Since

$$\forall x[\varphi(x, I_\varphi) \to x \in I_\varphi]$$

we obtain for $f^{-1}(X) := \{x \mid f(x) \in X\}$ since φ is X-positive

$$\forall x[\varphi(f(x), f(f^{-1}(I_\varphi))) \to f(x) \in I_\varphi],$$

which implies

$$\forall x[\psi(x, f^{-1}(I_\varphi)) \to x \in f^{-1}(I_\varphi)].$$

Hence $I_\psi \subset f^{-1}(I_\varphi)$ and $f(I_\psi) \subset I_\varphi$. For the other direction,

$$\varphi(x, f(I_\psi) \cup (im(f))^c)$$

implies by the additional assumption

$$x \in (im(f))^c \vee \big(\exists y f(y) = x \wedge \psi(y, I_\psi)\big),$$

hence

$$x \in (im(f))^c \cup f(I_\psi)$$

by the first fixed point axiom. This shows $I_\varphi \subset (im(f))^c \cup f(I_\psi)$ which finishes the proof. □

Lemma 2.3.5. $\partial\mathrm{ame}_{\mathrm{n+1}}$ *proves*

- $(\exists^{n+1} x)\varphi(x) \leftrightarrow (\forall^n k)(\exists^{n+1} x)\varphi(\langle k\rangle^\frown x) \vee \varphi(\langle\rangle)$
- $(\forall^{n+1} x)\varphi(x) \leftrightarrow (\exists^n k)(\forall^{n+1} x)\varphi(\langle k\rangle^\frown x) \wedge \varphi(\langle\rangle)$.

Proof. Let

$$I_{\varphi(x)\vee(\forall^n y)x^\frown\langle y\rangle\in X} := \{s \mid \mathrm{EFP}(s, \varphi(x) \vee (\forall^n y)x^\frown\langle y\rangle \in X)\};$$

$\eth\mathrm{ame}_{\mathrm{n+1}}$ proves the existence of this set since

$$\begin{aligned} &\mathrm{EFP}(s, \varphi(x) \vee (\forall^n y)x^\frown\langle y\rangle \in X) \\ \leftrightarrow\, &\mathrm{EFP}(\langle\rangle, \varphi(s^\frown x) \vee (\forall^n y)x^\frown\langle y\rangle \in X) \leftrightarrow (\exists^{n+1} x)\varphi(s^\frown x). \end{aligned}$$

By lemma 2.1.3 we obtain

$$\mathrm{LFP}(I_{\varphi(x)\vee(\forall^n y)x^\frown\langle y\rangle\in X}, \varphi(x) \vee (\forall^n y)x^\frown\langle y\rangle \in X). \tag{2.4}$$

We now compute

$$\begin{aligned} &(\exists^{n+1} x)\varphi(x) \\ \leftrightarrow &\langle\rangle \in I_{\varphi(x)\vee(\forall^n y)x^\frown\langle y\rangle\in X} \\ \leftrightarrow &\varphi(\langle\rangle) \vee (\forall^n k)\langle k\rangle \in I_{\varphi(x)\vee(\forall^n y)x^\frown\langle y\rangle\in X} \text{ (by 2.4)} \\ \leftrightarrow &\varphi(\langle\rangle) \vee (\forall^n k)\langle\rangle \in I_{\varphi(\langle k\rangle^\frown x)\vee(\forall^n y)x^\frown\langle y\rangle\in X} \text{ (by lemma 2.3.4 for} \\ &\hspace{18em} f(x) = \langle k\rangle^\frown x) \\ \leftrightarrow &(\forall^n k)(\exists^{n+1} x)\varphi(\langle k\rangle^\frown x) \vee \varphi(\langle\rangle). \end{aligned}$$

The second equivalence follows directly from the first. □

Lemma 2.3.6. *For each first order $\mathcal{L}_\eth$-formula $\varphi(x)$ we have uniformly in φ a first order $\mathcal{L}_\eth$-formula $\widetilde{\varphi}(x)$ such that the* ∂ame *proves*

- $(\forall^{n+1} x)\varphi(x) \rightarrow (\forall^{n+1} x)\widetilde{\varphi}(x)$
- $\forall x[\widetilde{\varphi}(x) \rightarrow \varphi(x)]$
- $\forall x[\widetilde{\varphi}(x) \rightarrow \exists^n y\widetilde{\varphi}(x^\frown\langle y\rangle)]$.

Proof. Choose $\widetilde{\varphi}(x) := (\forall^{n+1} y)\varphi(x^\frown y)$. By the definition of the generalized quantifiers we have

$$(\forall^{n+1} x)\varphi(x) \leftrightarrow \exists X[\langle\rangle \in X \wedge \forall x \in X(\varphi(x) \wedge (\exists^n z)x^\frown\langle z\rangle \in X)]. \tag{2.5}$$

Assume $(\forall^{n+1} x)\varphi(x)$ and let X be a witness for this according to (2.5). If $x \in X$ then $(\forall^{n+1} y)\varphi(x^\frown y)$ with witness $\{z \mid x^\frown z \in X\}$, hence $(\forall^{n+1} x)\widetilde{\varphi}(x)$ with witness X.

For the third claim assume $(\forall^{n+1} y)\varphi(x^\frown y)$. By the second assertion of lemma 2.3.5 this implies $(\exists^n z)(\forall^{n+1} y)\varphi(x^\frown\langle z\rangle^\frown y)$, hence $(\exists^n z)\widetilde{\varphi}(x^\frown\langle z\rangle)$. The proof of the second claim is analogous using also lemma 2.3.5. □

Lemma 2.3.7. $\Game$ame *proves for each first order* $\mathcal{L}_\Game$*-formula* $\varphi(x)$

$$\big(\varphi(\langle\rangle) \wedge \forall s[\varphi(s) \to \exists^n x\varphi(s^\frown\langle x\rangle)]\big) \to \forall^{n+1}x\varphi(x).$$

Proof. This follows from

$$(\forall^{n+1}x)\varphi(x) \leftrightarrow \exists X[\langle\rangle \in X \wedge \forall x \in X(\varphi(x) \wedge (\exists^n z)x^\frown\langle z\rangle \in X)]$$

for $X := \{x \mid \varphi(x)\}$. □

In the rest of this section, we will prove the following

Theorem 2.3.8 ([Möl02], 2.11)**.** *The* μ*-calculus and* $\Game$ame *prove the same* $\mathcal{L}_2$*-sentences.*

The proof of this theorem given in [Möl02] is for a slightly different definition of the $\exists^{n+1}$-quantifier which is

$$\exists^{n+1}x\varphi(x,\vec{y},\vec{Y}) \leftrightarrow$$
$$\mathrm{EFP}(\langle\rangle, \varphi(x,\vec{y},\vec{Y}) \vee (\exists^n z_1)(\forall^n z_2)(\forall z_3)(\exists z_4)x^\frown\langle z_1,z_2,z_3,z_4\rangle \in X).$$

Since the $\exists^{n+1}$-quantifier of definition 2.3.3 is (at first glance) weaker then this $\exists^{n+1}$-quantifier, the proof becomes a bit longer. The key is the following lemma which is essentially proposition 2.9 of [Möl02].

Lemma 2.3.9. *Let* $\varphi(x,X,\vec{y},\vec{Y})$ *be a first order* $\mathcal{L}_{\Game_n}$*-formula. Then there exists a first order* $\mathcal{L}_{\Game_{n+1}}$*-formula* $\widetilde{\varphi}(z,\vec{y},\vec{Y})$ *such that* $\Game$ame$_{\mathrm{n+1}}$ *proves:*

$$\forall z,\vec{y},\vec{Y}[\widetilde{\varphi}(z,\vec{y},\vec{Y}) \leftrightarrow \mathrm{EFP}(z,\varphi)],$$

i.e. $LFP(\varphi(x,X,\vec{y},\vec{Y}), \{z \mid \widetilde{\varphi}(z,\vec{y},\vec{Y})\})$.

Proof of theorem 2.3.8. We only have to show that each $\mathcal{L}_2$-sentences provable in the μ-calculus is already provable in $\Game$ame. For this it is sufficient to give an embedding $^* : \mathcal{L}_\mu \to \mathcal{L}_\Game$ which is the identity on $\mathcal{L}_2$ and show the translations of the axioms of the μ-calculus in $\Game$ame. We define

$$(\mu x X\varphi(x,X,\vec{y},\vec{Y}))^* := \{z \mid \widetilde{\varphi^*}(z,\vec{y},\vec{Y})\}$$

with the notion $\tilde{}$ from lemma 2.3.9. We define * to commute with quantifiers, boolean connectives and negation and to be the identity on $\mathcal{L}_2$. Then the translation of the defining axioms of the μ-terms are provable in $\Game$ame by lemma 2.3.9. All other axioms are directly translated into axioms of $\Game$ame. □

It remains to prove lemma 2.3.9. We first define for fixed n, $\vec{x}$, $\vec{Y}$ and $\vec{Z}$ a semi-formal calculus for formulas which may contain the quantifier $\forall^n$, number parameters $\vec{x}$ and set parameters $\vec{Y}$ ($\vec{Z}$ resp.) which occur positively (negatively resp.).

Definition 2.3.10 (the language of $\vdash^{\vec{x},\vec{Y},\vec{Z}}_{\forall^n}$)**.** The terms are built up from number parameters $\vec{x}$ and symbols for primitive recursive functions. The atomic formulas are $s < t$, $s \not< t$, $s = t$, $s \neq t$, $s \in Y_i$, $s \notin Z_i$, $s \in P$ and $s \notin P$ for terms s and t and a new set variable P. Formulas are built up from atomic formulas by $\wedge$, $\vee$ and $\forall^n$. Since we have no negation symbol, the quantifier $\forall^n$ only occurs positively, and we have no $\exists^n$-quantifiers.

Definition 2.3.11 (the semi-formal calculus $\vdash^{\vec{x},\vec{Y},\vec{Z}}_{\forall^n}$)**.** Let Γ be a finite set of formulas in the language defined above. This finite set should be read as disjunction. Then $\vdash^{\vec{x},\vec{Y},\vec{Z}}_{\forall^n} \Gamma$ holds if and only if one of the following holds:

- Γ contains a true atomic sentence
- $s \in P$ and $s \notin P$ are both formulas of Γ
- $A_0 \wedge A_1 \in \Gamma$ and $\vdash^{\vec{x},\vec{Y},\vec{Z}}_{\forall^n} \Gamma, A_i$ for $i = 0$ and $i = 1$
- $A_0 \vee A_1 \in \Gamma$ and $\vdash^{\vec{x},\vec{Y},\vec{Z}}_{\forall^n} \Gamma, A_i$ for $i = 0$ or $i = 1$
- $\forall^n x\varphi(x) \in \Gamma$ and $\forall^n m \vdash^{\vec{x},\vec{Y},\vec{Z}}_{\forall^n} \Gamma, \varphi(m)$

Our next aim is to formalize the calculus in $\Game\mathrm{ame}_{\mathrm{n+1}}$ by expressing $\vdash^{\vec{x},\vec{Y},\vec{Z}}_{\forall^n} \Gamma$ as a $\exists^{n+1}$-formula. Let p be the number of $\forall^n$-quantifiers which occur in Γ. For each p, we fix a partial truth predicate $U(m, \vec{x}, \vec{Y}, \vec{Z}, P)$ with the intended meaning "m codes a formula in the language of $\vdash^{\vec{x},\vec{Y},\vec{Z}}_{\forall^n}$ which contains at most p different $\forall^n$-quantifiers and the formula coded by m is true". For each p, we can express U inside $\mathcal{L}_{\Game_n}$. We further have a function S such that $\forall x[\varphi(x) \leftrightarrow U(S(\ulcorner\varphi\urcorner, x))]$. We will use U not only for formulas but also for finite sets of formulas, and the code of a finite set of formulas is the code of the disjunction of its elements. We will define a search tree ST for each Γ such that each subtree T with the properties

- the leaves of T are also leaves of ST
- T is $\forall^n$-branching at each node which is not a leaf

will witness a derivation of $\overset{\vec{x},\vec{Y},\vec{Z}}{\underset{\forall^n}{|\!\!-\!\!-}}\ \Gamma$. This search tree is defined by the arithmetical predicate $ST(\langle s, f\rangle, g, \vec{x}, \vec{Y}, \vec{Z})$ with the intended meaning "s is an element of the search tree for the set Γ coded by g in the calculus with parameters $\vec{x}, \vec{Y}, \vec{Z}$, and the set of formulas which belongs to the node s is coded by f." ST is defined by the following clauses:

- $ST(\langle\langle\rangle, f\rangle, g, \vec{x}, \vec{Y}, \vec{Z}) \leftrightarrow f = g$
- If $ST(\langle s, \ulcorner t(\vec{x}) \in Y_i, \Delta\urcorner\rangle, g, \vec{x}, \vec{Y}, \vec{Z})$
  then $t(\vec{x}) \in Y_i \to \forall m, x \neg ST(\langle s^\frown\langle m\rangle, x\rangle, g, \vec{x}, \vec{Y}, \vec{Z})$
  and $t(\vec{x}) \notin Y_i \to \forall m, x[ST(\langle s^\frown\langle m\rangle, x\rangle, g, \vec{x}, \vec{Y}, \vec{Z}) \leftrightarrow x = \ulcorner\Delta\urcorner]$
- If $ST(\langle s, \ulcorner t(\vec{x}) \notin Z_i, \Delta\urcorner\rangle, g, \vec{x}, \vec{Y}, \vec{Z})$
  then $t(\vec{x}) \notin Z_i \to \forall m, x \neg ST(\langle s^\frown\langle m\rangle, x\rangle, g, \vec{x}, \vec{Y}, \vec{Z})$
  and $t(\vec{x}) \in Z_i \to \forall m, x[ST(\langle s^\frown\langle m\rangle, x\rangle, g, \vec{x}, \vec{Y}, \vec{Z}) \leftrightarrow x = \ulcorner\Delta\urcorner]$
- If $ST(\langle s, \ulcorner t(\vec{x}) \in P, \Delta\urcorner\rangle, g, \vec{x}, \vec{Y}, \vec{Z})$
  then $[t(\vec{x}) \notin P] \in \Delta \to \forall m, x \neg ST(\langle s^\frown\langle m\rangle, x\rangle, g, \vec{x}, \vec{Y}, \vec{Z})$
  and $[t(\vec{x}) \notin P] \notin \Delta \to \forall m, x[ST(\langle s^\frown\langle m\rangle, x\rangle, g, \vec{x}, \vec{Y}, \vec{Z})$
  $\leftrightarrow x = \ulcorner\Delta, t(\vec{x}) \in P\urcorner]$
- If $ST(\langle s, \ulcorner t(\vec{x}) \notin P, \Delta\urcorner\rangle, g, \vec{x}, \vec{Y}, \vec{Z})$
  then $[t(\vec{x}) \in P] \in \Delta \to \forall m, x \neg ST(\langle s^\frown\langle m\rangle, x\rangle, g, \vec{x}, \vec{Y}, \vec{Z})$
  and $[t(\vec{x}) \in P] \notin \Delta \to \forall m, x[ST(\langle s^\frown\langle m\rangle, x\rangle, g, \vec{x}, \vec{Y}, \vec{Z})$
  $\leftrightarrow x = \ulcorner\Delta, t(\vec{x}) \notin P\urcorner]$
- If $ST(\langle s, \ulcorner\varphi \vee \psi, \Delta\urcorner\rangle, g, \vec{x}, \vec{Y}, \vec{Z})$
  then $\forall m, x[ST(\langle s^\frown\langle m\rangle, x\rangle, g, \vec{x}, \vec{Y}, \vec{Z}) \leftrightarrow x = \ulcorner\Delta, \varphi, \psi\urcorner]$
- If $ST(\langle s, \ulcorner\varphi \wedge \psi, \Delta\urcorner\rangle, g, \vec{x}, \vec{Y}, \vec{Z})$
  then $\forall m, x[ST(\langle s^\frown\langle m\rangle, x\rangle, g, \vec{x}, \vec{Y}, \vec{Z}) \leftrightarrow [(m = \langle\rangle \wedge x = \ulcorner\Delta, \varphi\urcorner) \vee$
  $(m \neq \langle\rangle \wedge x = \ulcorner\Delta, \psi\urcorner)]]$

- If $ST(\langle s, \ulcorner\forall^n x\varphi(x), \Delta\urcorner\rangle, g, \vec{x}, \vec{Y}, \vec{Z})$
  then $\forall m, y[ST(\langle s^\frown\langle m\rangle, y\rangle, g, \vec{x}, \vec{Y}, \vec{Z}) \leftrightarrow y = \ulcorner\Delta\urcorner{}^\frown S(\ulcorner\varphi\urcorner, m)]$

It follows directly from this definition that each node is either a leaf or is $\forall$-branching, i.e. all its successors are in the tree. The search tree has the following crucial property: For each node s, the set of formulas belonging to s is derivable in the calculus iff s is a leaf or if there are $\forall^n$-many successors of s such that all sets of formulas which belong to these successors are derivable. In the case of the conjunction, for example, this is true since $(\forall^n x)\varphi(x)$ implies $\varphi(\langle\rangle) \wedge (\exists n \neq \langle\rangle)\varphi(n)$, therefore a $\forall^n$-branching of a $\wedge$-node contains the φ-successor and a ψ-successor. Therefore if $\forall^n$-many successors of a $\wedge$-node are derivable, a φ- and a ψ-successor are derivable.

If Γ is a sequence of formulas belonging to a node s let $k(m, \ulcorner\Gamma\urcorner)$ be the code of the sequence of formulas belonging to $s^\frown\langle m\rangle$. Then k is arithmetically definable. One shows by induction on the length of t that

$$ST(\langle s_1, f\rangle, g_1, \vec{x}, \vec{Y}, \vec{Z}) \wedge ST(\langle s_2, f\rangle, g_2, \vec{x}, \vec{Y}, \vec{Z})$$

implies

$$ST(\langle s_1{}^\frown t, h\rangle, g_1, \vec{x}, \vec{Y}, \vec{Z}) \leftrightarrow ST(\langle s_2{}^\frown t, h\rangle, g_2, \vec{x}, \vec{Y}, \vec{Z})$$

for all h. Therefore we obtain

$$ST(\langle\langle m\rangle^\frown s, f\rangle, g, \vec{x}, \vec{Y}, \vec{Z}) \leftrightarrow ST(\langle s, f\rangle, k(m, g), \vec{x}, \vec{Y}, \vec{Z}). \tag{2.6}$$

We are now ready to talk about $\left|\frac{\vec{x}, \vec{Y}, \vec{Z}}{\forall^n}\right. \Gamma$ inside $\partial\mathrm{ame}_{n+1}$. By the property of the tree mentioned above a set of formulas Γ is derivable iff there exists a subtree of the search tree such that all nodes of the subtree are either leafs or $\forall^n$-branching, hence if the tree is "$\forall^n$-wellfounded". But this is exactly the property checked by an $\exists^{n+1}$-quantifier.

Definition 2.3.12. Let $\left|\frac{\vec{x}, \vec{Y}, \vec{Z}}{\forall^n}\right. \Gamma$ be an abbreviation for the $\mathcal{L}_{\partial_{n+1}}$-formula $(\exists^{n+1} s)\rho(s, \ulcorner\Gamma\urcorner, \vec{x}, \vec{Y}, \vec{Z})$, where $\rho(s, \ulcorner\Gamma\urcorner, \vec{x}, \vec{Y}, \vec{Z})$ means that s is a leaf of the search tree which belongs to Γ, i.e.

$$\rho(s, \ulcorner\Gamma\urcorner, \vec{x}, \vec{Y}, \vec{Z}) :\Leftrightarrow$$
$$\exists t[ST(\langle s, t\rangle, \ulcorner\Gamma\urcorner, \vec{x}, \vec{Y}, \vec{Z}) \wedge \forall n \forall t' \neg ST(\langle s^\frown\langle n\rangle, t'\rangle, \ulcorner\Gamma\urcorner, \vec{x}, \vec{Y}, \vec{Z})].$$

Lemma 2.3.13. $\partial\mathrm{ame}_{n+1}$ *proves all rules for* $\left|\frac{\vec{x}, \vec{Y}, \vec{Z}}{\forall^n}\right. \Gamma$ *from definition 2.3.11.*

Proof. We only consider the case $\Gamma = \forall^n x\varphi(x), \Delta$. By lemma 2.3.5, we obtain

$$(\exists^{n+1}s)\rho(s, \ulcorner\Gamma\urcorner, \vec{x}, \vec{Y}, \vec{Z}) \leftrightarrow \rho(\langle\rangle, \ulcorner\Gamma\urcorner, \vec{x}, \vec{Y}, \vec{Z}) \vee (\forall^n m)(\exists^{n+1}s)\rho(\langle m\rangle^\frown s, \ulcorner\Gamma\urcorner, \vec{x}, \vec{Y}, \vec{Z}).$$

Since $\rho(\langle\rangle, \ulcorner\forall^n x\varphi(x), \Delta\urcorner, \vec{x}, \vec{Y}, \vec{Z})$ is always false, we obtain with (2.6)

$$\left|\frac{\vec{x}, \vec{Y}, \vec{Z}}{\forall^n}\right. \forall^n x\varphi(x), \Delta \leftrightarrow (\forall^n m)(\exists^{n+1}s)\rho(s, k(m, \ulcorner\forall^n x\varphi(x), \Delta\urcorner), \vec{x}, \vec{Y}, \vec{Z}).$$

which implies

$$\left|\frac{\vec{x}, \vec{Y}, \vec{Z}}{\forall^n}\right. \forall^n x\varphi(x), \Delta \leftrightarrow (\forall^n m)\left|\frac{\vec{x}, \vec{Y}, \vec{Z}}{\forall^n}\right. \Delta, \varphi(m)$$

by the definition of k. □

Theorem 2.3.14 (correctness theorem)**.** *Let $\varphi(X, \vec{x}, \vec{Y}^+, \vec{Z}^-)$ be a $\forall^n$- formula. Then $\partial\mathrm{ame}_{n+1}$ proves*

$$\forall\vec{x}\forall\vec{Y}\forall\vec{Z}\big(\left|\frac{\vec{x}, \vec{Y}, \vec{Z}}{\forall^n}\right. \varphi(P, \vec{x}, \vec{Y}, \vec{Z}) \to \forall X\varphi(X, \vec{x}, \vec{Y}, \vec{Z})\big).$$

Proof. Let

$$\psi(s, X, \ulcorner\Gamma\urcorner, \vec{x}, \vec{Y}, \vec{Z}) :\equiv \rho(s, \ulcorner\Gamma\urcorner, \vec{x}, \vec{Y}, \vec{Z}) \vee (\forall^n x)[s^\frown\langle x\rangle \in X].$$

Let M be the set of nodes with true sets of formulas, i.e.

$$M := \{s \mid \exists t[U(t, \vec{x}, \vec{Y}, \vec{Z}, P) \wedge ST(\langle s, t\rangle, \ulcorner\Gamma\urcorner, \vec{x}, \vec{Y}, \vec{Z})]\}.$$

We first show

$$(\forall P)I_\psi \subset M. \tag{2.7}$$

Assume $\psi(s, M)$. We have to show $s \in M$. Let us consider the crucial case that $\psi(s, M)$ holds because of $(\forall^n x)s^\frown\langle x\rangle \in M$ and that the leftmost formula of the sequence belonging to s is of the form $\forall^n x\varphi(x)$, i.e.

$$ST(\langle s, \ulcorner\forall^n x\varphi(x), \Delta\urcorner\rangle, \ulcorner\Gamma\urcorner, \vec{x}, \vec{Y}, \vec{Z}).$$

By the definition of the search tree this implies

$$(\forall m)ST(\langle s^\frown\langle m\rangle, \ulcorner\Delta\urcorner{}^\frown S(\ulcorner\varphi\urcorner, m)\rangle, \ulcorner\Gamma\urcorner, \vec{x}, \vec{Y}, \vec{Z})$$

and together with $(\forall^n x)s^\frown\langle x\rangle \in M$ we obtain

$$(\forall^n m)U(\ulcorner\Delta\urcorner{}^\frown S(\ulcorner\varphi\urcorner, m)).$$

But this implies $U(\ulcorner\Delta, \forall^n x\varphi(x)\urcorner)$ by the properties of the truth predicate U, and hence $s \in M$. This finishes the proof of (2.7). Applying (2.7) for $\psi(s, X, \ulcorner\varphi\urcorner, \vec{x}, \vec{Y}, \vec{Z})$ delivers the claim. □

To show a partial completeness theorem we need the following lemma.

Lemma 2.3.15. *For all* $\forall^n$*-formulas* $\psi(X^+,\vec{x},\vec{Y},\vec{Z})$ *and* $\varphi(z,X^+,\vec{x},\vec{Y}^+,\vec{Z}^-)$ *the theory* $\Game\mathrm{ame}_{\mathrm{n+1}}$ *proves*

$$\forall\vec{x}\forall\vec{Y}\forall\vec{Z}\big(\psi(M,\vec{x},\vec{Y},\vec{Z}) \to \big|\!\frac{\vec{x},\vec{Y},\vec{Z}}{\forall^n}\ \neg\,\mathrm{Cl}(\varphi,P),\psi(P,\vec{x},\vec{Y},\vec{Z})\big)$$

with

$$M := \{z \mid \big|\!\frac{\vec{x},\vec{Y},\vec{Z}}{\forall^n}\ \neg\,\mathrm{Cl}(\varphi,P), z\in P\}.$$

Proof. We induct on ψ. If ψ is of the form $t(\vec{x})\in X$ then $t(\vec{x})\in M$ and the claim follows from the definition of M.

Assume $\psi(X^+,\vec{x},\vec{Y},\vec{Z})$ is $\forall^n z\psi_0(X^+,z,\vec{x},\vec{Y},\vec{Z})$. From $\psi(M,\vec{x},\vec{Y},\vec{Z})$ we obtain

$$(\forall^n z)\psi_0(M,z,\vec{x},\vec{Y},\vec{Z}),$$

and the induction hypothesis implies

$$\forall^n z\big|\!\frac{\vec{x},\vec{Y},\vec{Z}}{\forall^n}\ \neg\,\mathrm{Cl}(\varphi,P),\psi_0(P,z,\vec{x},\vec{Y},\vec{Z}).$$

Then lemma 2.3.13 yields

$$\big|\!\frac{\vec{x},\vec{Y},\vec{Z}}{\forall^n}\ \neg\,\mathrm{Cl}(\varphi,P),(\forall^n z)\psi_0(P,z,\vec{x},\vec{Y},\vec{Z}).$$

The remaining cases are similar. □

Theorem 2.3.16 (partial completeness)**.** *Let* $\varphi(z,X^+,\vec{x},\vec{Y}^+,\vec{Z}^-)$ *be a* $\forall^n$*-formula. Then* $\Game\mathrm{ame}_{\mathrm{n+1}}$ *proves*

$$\forall\vec{Y}\forall\vec{Z}\forall\vec{x}\forall z[\mathrm{EFP}(z,\varphi) \leftrightarrow \big|\!\frac{\vec{x},\vec{Y},\vec{Z}}{\forall^n}\ \neg\,\mathrm{Cl}(\varphi,P), z\in P].$$

Proof. The direction from right to left follows from the correctness theorem 2.3.14. For the other direction, put

$$M(\vec{x},\vec{Y},\vec{Z}) := \{z \mid \big|\!\frac{\vec{x},\vec{Y},\vec{Z}}{\forall^n}\ \neg\,\mathrm{Cl}(\varphi,P), z\in P\}.$$

We have to show

$$\forall z(\mathrm{EFP}(z,\varphi)\to z\in M)$$

which follows directly from $\mathrm{Cl}(\varphi, M)$. So assume $\varphi(z, M)$, which by lemma 2.3.15 implies

$$\Big|\frac{\vec{x}, \vec{Y}, \vec{Z}}{\forall^n} \neg\, \mathrm{Cl}(\varphi, P), \varphi(z, P)$$

Since $z \in P$ follows from $\varphi(z, P)$ by $\mathrm{Cl}(\varphi, P)$ we obtain $z \in M$ which finishes the proof. □

Using this theorem we can describe least fixed points of $\forall^n$-formulas with first order $\mathcal{L}_{\mathfrak{D}_{n+1}}$-formulas. To apply this in the proof of lemma 2.3.9 we have to justify why we can assume that the formula φ of lemma 2.3.9 contains no quantifiers but $\forall^n$ (after pulling the negations inwards to the atomic formulas). We will at first remove the quantifiers $\exists^n$ by coding the inductive process which produces the $\exists^n$-quantifier and the inductive process which produces the fixed point of φ together into one inductive process. For this we have to prove a formalized version of the transitivity theorem, which says that if a monotone inductive definition occurs positively in another monotone inductive definition, then they can be coded together into one inductive definition (lemma 2.3.18). To prove this theorem without using ordinals we need the following lemma which allows us to choose an element of minimal stage (with a certain property) in a fixed point.

Lemma 2.3.17. *For all formulas $\varphi(x, X^+)$ ACA_0 proves: For all sets I_φ, Y such that $\mathrm{LFP}(I_\varphi, \varphi)$ and $Y \cap I_\varphi \neq \emptyset$ there exists $z \in Y$ such that $\varphi(z, Y^c \cap I_\varphi)$.*

Proof. Assume $(\forall z \in Y)\neg\varphi(z, Y^c \cap I_\varphi)$. We first show that $Y^c \cap I_\varphi$ is a fixed point of φ. Therefore assume $\varphi(x, Y^c \cap I_\varphi)$. By our first assumption this implies $x \notin Y$. $\varphi(x, Y^c \cap I_\varphi)$ also implies $x \in I_\varphi$ by monotonicity and the first fixed point axiom for φ, therefore we have $x \in Y^c \cap I_\varphi$, and we have proved that $Y^c \cap I_\varphi$ is a fixed point of φ. Since I_φ is the least fixed point we have $I_\varphi \subset Y^c$ which is a contradiction to $Y \cap I_\varphi \neq \emptyset$. □

Lemma 2.3.18. *For all first order formulas $\varphi(x, X^+, Y^+)$ and $\psi(y, X^+, Y^+)$ and each set term $T(X)$ there exists uniformly in φ and ψ a formula $\chi(z, Z^+)$ such that ACA_0 proves: If $\forall X\, \mathrm{LFP}(T(X), \psi(y, X, Y))$ then*

$$\forall Z[\mathrm{LFP}(Z, \chi) \to \mathrm{LFP}((Z)_0, \varphi(x, X, T(X)))].$$

Proof. Let

$$\begin{aligned} \chi(z, Z) :\equiv \exists x[&(z = \langle 0, x\rangle \wedge \varphi(x, (Z)_0, (Z)_1)) \\ &\vee (z = \langle 1, x\rangle \wedge \psi(x, (Z)_0, (Z)_1))]. \end{aligned} \tag{2.8}$$

To show the first fixed point axiom assume $\varphi(x,(Z)_0,T((Z)_0))$ for a Z with $\mathrm{LFP}(Z,\chi)$. We first claim $T((Z)_0)\subset(Z)_1$. It suffices to show

$$\forall y[\psi(y,(Z)_0,(Z)_1)\to y\in(Z)_1]$$

because of $\mathrm{LFP}(T((Z)_0),\psi(y,(Z)_0,Y))$. Therefore assume $\psi(y,(Z)_0,(Z)_1)$ which implies $\chi(\langle 1,y\rangle,Z)$, hence $\langle 1,y\rangle\in Z$ by $\mathrm{LFP}(Z,\chi)$. This finishes the proof of the claim.

As $T((Z)_0)$ occurs positively in $\varphi(x,(Z)_0,T((Z)_0))$ the claim implies $\varphi(x,(Z)_0,(Z)_1)$, hence $\chi(\langle 0,x\rangle,Z)$ and $\langle 0,x\rangle\in Z$ by the first fixed point axiom for χ. Therefore we have $x\in(Z)_0$ and the first fixed point axiom is proved.

It remains to show

$$\forall X\big[\forall x\big(\varphi(x,X,T(X))\to x\in X\big)\to(Z)_0\subset X\big].$$

Assume $x\in(Z)_0\backslash X$. Let

$$M:=\langle 0,X\rangle\cup\langle 1,T(X)\rangle \text{ with } \langle n,X\rangle:=\{\langle n,x\rangle\mid x\in X\}.$$

Then $\langle 0,x\rangle\in M^c\cap Z\neq\emptyset$. With lemma 2.3.17 we find a $z\in M^c$ such that $\chi(z,M\cap Z)$. Now z is either of the form $\langle 0,y\rangle$ or $\langle 1,y\rangle$ for some y. In the first case we obtain

$$\varphi(y,(M\cap Z)_0,(M\cap Z)_1)$$

which implies

$$\varphi(y,X\cap(Z)_0,T(X)\cap(Z)_1).$$

By positivity we obtain $\varphi(y,X,T(X))$, therefore $y\in X$ by hypothesis, which implies $z\in M$, contradiction. If z is of the form $\langle 1,y\rangle$ we analogously obtain $\psi(y,X,T(X))$ which implies $y\in T(X)$ by $\mathrm{LFP}(T(X),\psi(y,X,Y))$, hence again $z\in M$, contradiction. □

The next lemma allows us to assume without loss of generality that the formula φ of lemma 2.3.9 contains no $\exists^{n-1}$-quantifiers.

Lemma 2.3.19. *If $\varphi(x,X^+,\vec{y},\vec{Y})$ is an $\mathcal{L}_{\partial_n}$-formula without negation symbols (but predicate symbols $\in$ and $\notin$) and if φ contains at most k quantifiers $\exists^n$, then there exists an $\mathcal{L}_{\partial_n}$-formula $\chi(x,X,\vec{y},\vec{Y})$ which contains no $\exists^n$-quantifiers such that* $\partial\mathrm{ame_n}$ *proves*

$$\forall\vec{y},\vec{Y},Z\big(\mathrm{LFP}(Z,\chi(x,X,\vec{y},\vec{Y}))\to\mathrm{LFP}((Z)_{\langle\rangle},\varphi(x,X,\vec{y},\vec{Y}))\big)$$

if $n = 0$ *and*

$$\forall \vec{y}, \vec{Y}, Z\big(\mathrm{LFP}(Z, \chi(x, X, \vec{y}, \vec{Y})) \to \mathrm{LFP}((\underbrace{(Z)_0 \ldots)_0}_{k-times}, \varphi(x, X, \vec{y}, \vec{Y}))\big).$$

if $n > 0$.

Proof. If $n = 0$ then

$$\mathrm{EFP}\left(z, \varphi(x, X, \vec{y}, \vec{Y})\right) \tag{2.9}$$

is a Π^1_1-formula, which by Kleenes normal form theorem (see [Sim99], lemma V.1.4) is equivalent to the assertion that certain tree which is describable without unbounded quantifiers is wellfounded. We therefore can find a formula $\widetilde{\varphi}$ without unbounded quantifiers such that (2.9) is equivalent to

$$\mathrm{EFP}\left(\langle\langle\rangle, z\rangle, \widetilde{\varphi}(x, \vec{y}, \vec{Y}) \vee \forall y \langle (x)_0{}^\frown\langle y\rangle, (x)_1\rangle \in X\right).$$

Then the claim follows for

$$\chi(x, X, \vec{y}, \vec{Y}) :\equiv \widetilde{\varphi}(x, \vec{y}, \vec{Y}) \vee \forall y \langle (x)_0{}^\frown\langle y\rangle, (x)_1\rangle \in X.$$

If $n > 0$ we induct on k. Assume φ contains a subformula $(\exists^n z)\rho(z, X)$. Let

$$T(X) := \{s \mid (\exists^n z)\rho(s^\frown z, X)\}$$

and

$$\psi(z, X^+, Z^+) :\equiv \rho(z, X) \vee (\forall^{n-1} y) z^\frown\langle y\rangle \in Z.$$

As in the proof of lemma 2.3.5 we obtain $\mathrm{LFP}(T(X), \psi(z, X, Z))$. We obtain a $\mathcal{L}_{\ni_n}$-formula $\widetilde{\varphi}(x, X^+, W^+)$ such that

$$\varphi(x, X) \leftrightarrow \widetilde{\varphi}(x, X, T(X))$$

by replacing $(\exists^n z)\rho(z, X)$ by $\langle\rangle \in T(X)$. $\widetilde{\varphi}$ and ρ contain together at most $k-1$ quantifiers $\exists^n$ and W occurs positively in $\widetilde{\varphi}$ since φ contains no negation symbols. By lemma 2.3.18 there exists a formula χ such that

$$\forall Z\big(\mathrm{LFP}(Z, \chi) \to \mathrm{LFP}((Z)_0, \varphi(x, X, T(X)))\big).$$

Since $\widetilde{\varphi}$ and ψ contain at most $k - 1$ quantifiers $\exists^n$ we see from (2.8) in the proof of 2.3.18 that χ contains at most $k - 1$ quantifiers $\exists^n$. Hence we can apply the induction hypothesis to χ and obtain the claim. □

The next lemma allows us to code each quantifier $\forall^m$ and $\exists^m$ for $m < n$ into a $\forall^n$-quantifier.

Lemma 2.3.20. *For each $m \leq n$ the theory ⅁ame_n proves*

- $\exists^m x\varphi((x)_0) \leftrightarrow \forall^{m-1} x\varphi(x)$
- $\exists^m x\varphi(f_m(x)) \leftrightarrow \exists^{m-1} x\varphi(x)$

with $f_1(x) := lh(x)$ and $f_{i+1}(\langle x_0, \dots, x_j\rangle) := \langle f_i(x_0), \dots, f_i(x_j)\rangle$.

Proof. The first equivalence follows directly from lemma 2.3.5, the second is proved by induction on m. For $m = 1$ the claim follows from

$$(\exists^1 x)\varphi(lh(x)) \leftrightarrow (\forall f)(\exists n)\varphi(lh(\langle f(1), \dots, f(n)\rangle)) \leftrightarrow (\exists n)\varphi(n).$$

For $m > 1$ we obtain

$$\begin{aligned}
&(\exists^m x)\varphi(f_m(x))\\
\leftrightarrow&\forall X\big[\forall x\big(\big(\varphi(f_m(x)) \vee (\forall^{m-1} y)x^\frown\langle y\rangle \in X\big) \to x \in X\big) \to \langle\rangle \in X\big]\\
\leftrightarrow&\forall X\big[\forall x\big(\big(\varphi(f_m(x)) \vee (\forall^{m-1} y)f_m(x^\frown\langle y\rangle) \in f_m(X)\big) \to x \in X\big)\\
&\qquad \to \langle\rangle \in f_m(X)\big]\\
\leftrightarrow&\forall X\big[\forall x\big(\big(\varphi(f_m(x)) \vee (\forall^{m-1} y)f_m(x)^\frown\langle f_{m-1}(y)\rangle \in f_m(X)\big) \to x \in X\big)\\
&\qquad \to \langle\rangle \in f_m(X)\big]\\
\leftrightarrow&\forall X\big[\forall x\big(\big(\varphi(x) \vee (\forall^{m-1} y)x^\frown\langle f_{m-1}(y)\rangle \in X\big) \to x \in X\big)\\
&\qquad \to \langle\rangle \in X\big] \text{ (by lemma 2.3.4)}\\
\leftrightarrow&\forall X\big[\forall x\big(\big(\varphi(x) \vee (\forall^{m-2} y)x^\frown\langle y\rangle \in X\big) \to x \in X\big)\\
&\qquad \to \langle\rangle \in X\big] \text{ (by induction hypothesis)}\\
\leftrightarrow&(\exists^{m-1} x)\varphi(x).
\end{aligned}$$

□

Proof of lemma 2.3.9. We may assume that φ contains no negation symbols (but predicate symbols $\in$ and $\notin$). We further may assume that φ contains no $\exists^n$-quantifiers by lemma 2.3.19. By lemma 2.3.20, we remove all other quantifiers $\forall^m, \exists^m$ with $m < n$ by replacing them by $\forall^n$. Therefore we may assume that φ contains no quantifiers but $\forall^n$. Now the first claim follows from the partial completeness theorem 2.3.16 (the bounded quantifiers which came in in the first case of the proof of 2.3.19 are no problem because in the search tree proof of 2.3.16, they can be treated as $\wedge$ and $\vee$). The second claim follows with lemma 2.1.3. □

We will need this lemma in chapter 4.

2.4 The σ^+-calculus

Definition 2.4.1 (σ^+-calculus)**.** Let

$$TI(X,\varphi) :\equiv LO(X) \wedge \big(\big(\forall x(\forall y(y <_X x \to \varphi(x)) \to \varphi(x))\big) \to \forall x \varphi(x)\big)$$

be the formula of transfinite induction along X with respect to φ; $LO(X)$ means that X codes a linear ordering denoted by $<_X$.

$\Pi^1_1(\mathcal{L}_\sigma)$-TI is the scheme

$$\forall X(WO(X) \to TI(X,\varphi)) \text{ for all } \Pi^1_1\text{-formulas } \varphi \text{ of } \mathcal{L}_\sigma$$

of $\Pi^1_1(\mathcal{L}_\sigma)$-transfinite induction, where $WO(X)$ means that X codes a well-ordering. The scheme of transfinite induction is also known as bar induction.

The σ^+-calculus is the σ-calculus extended by $\Pi^1_1(\mathcal{L}_\sigma)$-TI.

Lemma 2.4.2. *Each $\mathcal{L}_2$-sentence provable in the σ^+-calculus is provable in Π^1_2-CA$_0$.*

For the proof we need the following lemma.

Lemma 2.4.3. ∂ame *shows that each first order $\mathcal{L}_{\partial}$-formula φ is equivalent to both a Π^1_2- and a Σ^1_2-formula of $\mathcal{L}_2$. An analogous assertion holds for the μ- and the σ-calculus.*

Proof. We show by induction on n that for each Δ^1_2-formula $\varphi(x)$ of $\mathcal{L}_2$, $\exists^n x\varphi(x)$ is also Δ^1_2. It holds

$$\exists^n x\varphi(x) \leftrightarrow \exists T\big[T \text{ wellfounded tree} \wedge \\ \forall s \in T[(s \text{ is a leaf } \wedge \varphi(s)) \vee (\forall^{n-1}y)s^\frown\langle y\rangle \in T]\big].$$

For the direction from right to left one shows $(\forall s \in T)\exists^n x\varphi(s^\frown x)$ by induction on the Kleene-Brouwer-ordering of T. For the other direction, we define the set

$$T := \{t \mid (\forall s \subset t)(\exists^n x)\varphi(s^\frown x) \wedge (\forall s \subsetneq t)\neg\varphi(s) \wedge (\forall s_1 \subsetneq s_2 \subset t)s_2 \prec s_1\},$$

where $\prec$ is the stage comparison relation according to lemma 2.1.7 on the least fixed point of $\varphi(x) \vee (\forall^{n-1}y)x^\frown\langle y\rangle \in X$ which is $\{s \mid (\exists^n x)\varphi(s^\frown x)\}$. T is wellfounded by the wellfoundedness of $\prec$ and not empty by $\exists^n x\varphi(x)$, hence the right side holds for this T. Since

$$\exists^n x\varphi(x) \leftrightarrow \mathrm{EFP}(\langle\rangle, \varphi(x) \vee (\forall^{n-1}y)x^\frown\langle y\rangle \in X)$$

$\exists^n x\varphi(x)$ is also equivalent to a Π^1_2-formula. Using the embedding from the μ-calculus to ∂ame which exists by the proof of theorem 2.3.8 and an embedding from the σ-calculus to the μ-calculus which exists by theorem 3.20 of [Möl02] the result transfers to the μ- and the σ-calculus. □

Proof of lemma 2.4.2. By the previous lemma, the σ-calculus plus Π^1_2-CA proves comprehension for all Π^1_1-formulas of the language $\mathcal{L}_\sigma$, hence

$$\sigma\text{-calculus} + \Pi^1_2\text{-CA} \vdash \sigma^+\text{-calculus}.$$

Since the σ-calculus plus Π^1_2-CA proves the same $\mathcal{L}_2$-sentences as Π^1_2-CA$_0$ (see [Möl02], chapter 10a) the claim follows. □

Definition 2.4.4 (countable coded ω-models)**.** A countable coded ω-model is a set W which is intended to code the structure

$$M = (\omega, \mathcal{S}_M, +, \cdot, 0, 1, <)$$

with

$$\mathcal{S}_M = \{(W)_n \mid n \in \omega\}.$$

Definition 2.4.5 (evaluation of $\mathcal{L}_{\sigma_n}$-formulas)**.** Let M be a countable coded ω-model. An n-evaluation for M is a pair of functions (f, g), where f is a function from the (codes of) $\mathcal{L}_{\sigma_n}$-sentences with parameters in $\omega \cup \mathcal{S}_M$ to $\{0, 1\}$ and g is a function from the set-terms of $\mathcal{L}_{\sigma_n}$ without free variables and with parameters from $\omega \cup \mathcal{S}_M$ to ω which obey the canonical clauses, for example:

$$f(\forall X \varphi(X)) = \begin{cases} 1 & \text{if } \forall n f((\varphi(X)_n)) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

$$f(t \in \sigma x X \varphi(x, X)) = \begin{cases} 1 & \text{if } t \in g(\sigma x X \varphi(x, X)) \\ 0 & \text{otherwise} \end{cases} \tag{2.11}$$

$$f(IGF(\varphi, (W)_{g(\sigma x X \varphi(x,X))})) = 1 \tag{2.12}$$

Definition 2.4.6. A countable coded ω-model of the σ_n-calculus is a countable coded ω-model together with an n-evaluation function which maps each axiom of the σ_n-calculus to 1. We say that φ holds in this model if $f(\varphi) = 1$.

Definition 2.4.7 (β-models)**.** A β-model M is an ω-model such that for each Π^1_1-sentences with parameters from M, φ is true if and only if φ holds in M.

Lemma 2.4.8 ([Möl02], theorem 9.10.)**.** *For each n the σ_{n+1}-calculus proves: There is a countable coded β-model of the σ_n-calculus.*

Proof. Let $\mathcal{S}_M = \mathcal{H}(T^{n+1})$ with the notation from section 2.2. Then g can be defined directly from the function r^{n+1} (see lemma 2.2.15) such that we have for each σ_n-formula $\varphi(x, X, \vec{y}, \vec{Y})$

$$(\forall \vec{Y} \in \mathcal{S}_M)(\forall \vec{y})(\mathcal{S}_M)_{g(\sigma x X \varphi(x,X,\vec{y},\vec{Y}))} = \sigma x X \varphi(x, X, \vec{y}, \vec{Y}). \tag{2.13}$$

To define f, remember that $T^{n+1} = I(\sigma x X \chi(x, X))$ for a σ_{n+1}-formula χ (see proof of lemma 2.2.14). We define a formula $\varphi(x, X)$ whose fixed point is intended to code the graph of f. Choose $\varphi(x, X)$ such that

- $\varphi(\langle 0, x\rangle, X) \leftrightarrow \chi(x, (X)_0)$
- $\varphi(\langle 1, \langle \ulcorner y \in (\mathcal{S}_M)_x \urcorner, 0\rangle\rangle, X) \leftrightarrow \langle x, y\rangle \in ((X)_0)_{e_{\hat{\mathcal{H}}}}$
- $\varphi(\langle 1, \langle \ulcorner y \in (\mathcal{S}_M)_x \urcorner, 1\rangle\rangle, X) \leftrightarrow \langle x, y\rangle \in ((X)_0)_{e_{\mathcal{H}}}$
- $\varphi(\langle 1, \langle \ulcorner y \notin (\mathcal{S}_M)_x \urcorner, 0\rangle\rangle, X) \leftrightarrow \langle x, y\rangle \in ((X)_0)_{e_{\mathcal{H}}}$
- $\varphi(\langle 1, \langle \ulcorner y \notin (\mathcal{S}_M)_x \urcorner, 1\rangle\rangle, X) \leftrightarrow \langle x, y\rangle \in ((X)_0)_{e_{\hat{\mathcal{H}}}}$
- $\varphi(\langle 1, \langle \ulcorner y \in \sigma x X \varphi(x, X) \urcorner, 0\rangle\rangle, X) \leftrightarrow y \in (((X)_0)_{e_{\hat{\mathcal{H}}}})_{g(\ulcorner \sigma x X \varphi(x,X) \urcorner)}$
- $\varphi(\langle 1, \langle \ulcorner \exists y \psi(y) \urcorner, 0\rangle\rangle, X) \leftrightarrow \forall y \langle \ulcorner \psi(y) \urcorner, 0\rangle \in X_1$
- $\varphi(\langle 1, \langle \ulcorner \forall Y \psi(Y) \urcorner, 1\rangle\rangle, X) \leftrightarrow \forall y \langle \ulcorner \psi((\mathcal{S}_M)_y) \urcorner, 1\rangle \in X_1$

and so on. Choose f such that $(\sigma x X \varphi(x, X))_1$ is the graph of f. Then for each first order $\mathcal{L}_{\sigma_n}$-formula $\varphi(\vec{y}, \vec{Y})$ we can show

$$(\forall \vec{Y} \in \mathcal{S}_M)(\forall \vec{y}) f(\ulcorner \varphi(\vec{y}, \vec{Y}) \urcorner) = 1 \leftrightarrow \varphi(\vec{y}, \vec{Y}). \tag{2.14}$$

If φ is a Π^1_1-formula of $\mathcal{L}_{\sigma_n}$ the implication from right to left is still true. The proof is by induction on φ using (2.13) for the σ_n-set terms. Since all fixed point axioms are Π^1_1 in $\mathcal{L}_{\sigma_n}$, (2.12) follows.

We show next that f maps each instance of the comprehension scheme to 1. By 2.13, each first order $\mathcal{L}_{\sigma_n}$-formula $\varphi(x)$ is equivalent to an $\mathcal{L}_2$-formula with parameters from $\mathcal{S}_M$, hence by lemma 2.2.13 about comprehension in Δ, $\{x \mid \varphi(x)\}$ is in $\mathcal{S}_M$. Now the claim follows from (2.14).

It remains to show that this model is a β-model. By Kleenes basis theorem (see [Sim99], lemma VII.1.7) it is sufficient to show that comprehension for all Π^1_1-formulas of $\mathcal{L}_2$ holds in $\mathcal{S}_M$. Fix a Π^1_1-formula of $\mathcal{L}_2$. By Kleenes normal form theorem (see [Sim99], lemma V.1.4) we can assume that it is of the form $\forall X \exists m \varphi(X[m], \vec{y}, \vec{Y})$ with a quantifier free formula φ and $X[m] := \langle \xi_0, \ldots, \xi_{m-1}\rangle$ where $\xi_i = 1$ if $i \in X$ and 0 otherwise. Let

$$\psi(x, X, \vec{y}, \vec{Y}) :\equiv \varphi(x, \vec{y}, \vec{Y}) \vee (\forall y) x^\frown\langle y\rangle \in X.$$

Then

$$\forall X \exists m \varphi(X[m], \vec{y}, \vec{Y}) \leftrightarrow \langle\rangle \in I(\sigma x X \psi(x, X, \vec{y}, \vec{Y})),$$

hence it is equivalent to a first order formula for which we have comprehension. □

Lemma 2.4.9. ATR_0 *proves that all countable coded β-models satisfy* $\Pi^1_\infty - TI$.

Proof. see [Sim99], lemma VII.2.15. □

Theorem 2.4.10. *The σ-calculus and the σ^+-calculus prove the same Π^1_1-sentences of $\mathcal{L}_2$.*

Proof. Let φ be a Π^1_1-sentence provable in the σ^+-calculus. Then there is an n such that φ is provable in the σ^+_n-calculus. By lemma 2.4.8 the σ-calculus proves that there is a countable coded β-model of the σ_n-calculus, which is by lemma 2.4.9 a model of the σ^+_n-calculus. Therefore φ holds in this β-model, which implies that φ is a theorem of the σ-calculus. □

3. EMBEDDING THE R-CALCULUS

3.1 Sets of reals in the σ-calculus

The goal of this section is to give codes for sets of reals which are definable in the σ-calculus.

Definition 3.1.1 (code of a real)**.** A real is an infinite strictly monotone sequence of natural numbers which is coded by the set of the members of the sequence.

This definition is slightly different from the usual one by claiming that the sequences should be monotone. This has the advantage that each set of natural numbers codes a real which is not the case in the usual definition, and this is appropriate for our goals because the property of Ramsey talks about sets of natural numbers and not about sequences. This alteration of the definition makes no real difference because we equip this space with a topology which is isomorphic to the Baire space with the usual topology.

Definition 3.1.2 (topology on ${}^{\omega}\omega^{mon}$)**.** Let ${}^{\omega}\omega^{mon}$ be the set of all infinite strictly monotone sequences. Let

$$\mathcal{N}_n = \begin{cases} \{X \mid m \in X\} & \text{if } n = 2m \\ \{X \mid m \notin X\} & \text{if } n = 2m+1 \end{cases}.$$

The sets $\mathcal{N}_n$ form a basis of a topology.

This topology is canonically isomorphic to the Baire space with the usual topology (by $f : {}^{\omega}\omega^{mon} \rightarrow {}^{\omega}\omega, (x_n)_{n\in\omega} \mapsto (y_n)_{n\in\omega}$ with $y_0 := x_0$ and $y_{n+1} := x_{n+1} - x_n - 1$).

Definition 3.1.3 ($\forall^n$ and $\exists^n$ on sets of reals)**.** Let A_m be a set of reals for each $m \in \omega$. Then

$$\forall^n m A_m := \{x \mid \forall^n m(x \in A_m)\} \text{ and } \exists^n m A_m := \{x \mid \exists^n m(x \in A_m)\}.$$

We want to code the sets of reals which can be obtained from the basis sets by applications of $\forall^m$ and $\exists^m$ with $m \leq n$ for a fixed natural number n.

Definition 3.1.4 (n-codes, simple n-codes)**.** An n-code C is a wellfounded tree T_C together with a function $f_C : T \to \omega$ such that

- for all interior nodes s it is $f_C(s) \leq 2n+1$
- $\forall s \in T_C[\exists m(s^\frown\langle m\rangle \in T_C) \to \forall m(s^\frown\langle m\rangle \in T_C)]$.

With each n-code, we associate a set of reals as follows. If $T_C = \{\langle\rangle\}$, then (T_C, f_C) codes the basis set $\mathcal{N}_{f_C(\langle\rangle)}$. Otherwise, $\langle m\rangle$ is in the tree for each m. Let $T^m_C := \{s \mid \langle m\rangle^\frown s\}$, $f^m_C(s) = f_C(\langle m\rangle^\frown s)$ and A_m be the set of reals coded by (T^m_C, f^m_C). If $f_C(\langle\rangle) = 2k+1$ then (T_C, f_C) codes $\exists^k x A_x$, if $f_C(\langle\rangle) = 2k$ then (T_C, f_C) codes $\forall^k x A_x$.

If C_i is an n-code of A_i for each $i \in \omega$ let $\exists^m i C_i$ ($\forall^m i C_i$ resp.) be the canonically code of $\exists^m i A_i$ ($\forall^m i A_i$ resp.) for $m \leq n$. $\exists^m i C_i$ ($\forall^m i C_i$ resp.) is computable by arithmetical comprehension from a set coding all C_i.

A simple n-code is an n-code where only $f(\langle\rangle)$ may equal $2n$ or $2n+1$ and all other inner nodes are labeled with numbers less than $2n$.

Definition 3.1.5. For an n-code (T, f) and $s \in T$ let

$$(T, f) \upharpoonright s := (\{t \mid s^\frown t \in T\}, g)$$

with $g(t) = f(s^\frown t)$. Then $(T, f) \upharpoonright s$ is again an n-code.

Lemma 3.1.6. *For each $n \in \omega$ there is an $\mathcal{L}_\sigma$-formula $\varphi_{\in^n(X,C)}$ (we will write $X \in^n C$ instead) such that the σ-calculus proves for all X and for all n-codes C*

- $X \in^n C \leftrightarrow m \in X$ *if* $f_C(\langle\rangle) = 2m$ *and* $T_C = \{\langle\rangle\}$
- $X \in^n C \leftrightarrow m \notin X$ *if* $f_C(\langle\rangle) = 2m+1$ *and* $T_C = \{\langle\rangle\}$
- $X \in^n C \leftrightarrow \forall^m i(X \in^n C \upharpoonright \langle i\rangle)$ *if* $f_C(\langle\rangle) = 2m$ *and* $T_C \neq \{\langle\rangle\}$
- $X \in^n C \leftrightarrow \exists^m i(X \in^n C \upharpoonright \langle i\rangle)$ *if* $f_C(\langle\rangle) = 2m+1$ *and* $T_C \neq \{\langle\rangle\}$.

Proof. Let $X \in^n C :\equiv \langle\rangle \in I(\sigma y Y \varphi(y, Y, X, C))$ with

$$\begin{aligned}
&\varphi(y, Y, X, C) :\equiv \\
&\qquad [y \text{ is a leaf of } T_C \wedge \exists m[(f_C(y) = 2m \wedge m \in X)\vee \\
&\qquad\qquad (f_C(y) = 2m+1 \wedge m \notin X)]] \\
&\qquad \vee[y \text{ is not a leaf of } T_C \wedge \bigvee_{m\leq n} [(f_C(y) = 2m \wedge (\forall^m x) y^\frown\langle x\rangle \in Y)\vee \\
&\qquad\qquad (f_C(y) = 2m+1 \wedge (\exists^m x) y^\frown\langle x\rangle \in Y)]].
\end{aligned}$$

Since φ is Y-positive we have $\mathrm{LFP}(I(\sigma y Y \varphi(y, Y, X, C)), \varphi)$ by lemma 2.2.2. We first show

$$X \in^n C \restriction s \leftrightarrow s \in I(\sigma y Y \varphi(y, Y, X, C)) \tag{3.1}$$

for each $s \in T_C$. Let $f_s(x) = s^\frown x$. Then

$$\varphi(f_s(y), f_s(Y), X, C) \leftrightarrow \varphi(y, Y, X, C \restriction s),$$

and lemma 2.3.4 implies

$$f_s(I_{\varphi(y,Y,X,C\restriction s)}) = I_{\varphi(y,Y,X,C)} \cap im(f_s),$$

where I_φ is a set with $\mathrm{LFP}(I_\varphi, \varphi)$. Therefore we have shown (3.1). Now we prove the third equivalence of the lemma as follows.

$$\begin{aligned} X \in^n C \leftrightarrow & \langle\rangle \in I(\sigma y Y \varphi(y, Y, X, C)) \\ \leftrightarrow & \varphi(\langle\rangle, I(\sigma y Y \varphi(y, Y, X, C)), X, C) \\ \leftrightarrow & (\forall^m i)\langle i \rangle \in I(\sigma y Y \varphi(y, Y, X, C)) \\ \leftrightarrow & (\forall^m i) X \in^n C \restriction \langle i \rangle \text{ by } (3.1) \end{aligned}$$

The other cases are similar. □

Definition 3.1.7. Let

$$X =^n_U Y :\equiv X, Y \text{ are } n\text{-codes and } \forall Z \subset U (Z \in^n X \leftrightarrow Z \in^n Y).$$

3.2 *Proving Ramseyness in* ZFC + CH

We want to prove in ZFC + CH that every set of reals that has an n-code has the property of Ramsey. We first fix some notations.

Definition 3.2.1. Let Seq^{mon} denote the set of all finite, strictly monotone sequences. We will not distinguish between a sequence and the natural number coding the sequence. Let $\max s$ be the maximal element of the sequence s with $\max\langle\rangle := -1$. Further let $X \backslash s := \{x \in X \mid x > \max s\}$. $s \subset_b U$ is an abbreviation for "$s = \langle s_1, \ldots, s_n \rangle$ is a monotone sequence, U is an infinite set of natural numbers and U begins with s, i.e. the n least elements of U are $s_1, \ldots, s_n$".

Definition 3.2.2. X diagonalizes $\{X_s \mid s \in Seq^{mon}\}$ if $X \subset X_{\langle\rangle}$ and if $\max s \in X$ implies $X \backslash s \subset X_s$.

Definition 3.2.3. F is a Ramsey ultrafilter iff

- $F \subset \mathcal{P}(\omega)$ is an ultrafilter
- F contains no finite set
- F is closed under diagonalization, i.e. if $X_s \in F$ for each $s \in Seq^{mon}$ then there is an $X \in F$ which diagonalizes $\{X_s \mid s \in Seq^{mon}\}$.

Mathias proves in [Mat77]:

Theorem 3.2.4. ZFC + CH *proves that there is a Ramsey ultrafilter F.*

Definition 3.2.5. Let $s \in Seq^{mon}$, U an infinite subset of ω and $X \subset \mathcal{P}(\omega)$. We define a notion of locally homogeneousness as follows.

- $hom_+(s, U, X) :\equiv \forall^\infty Y(s \subset_b Y \subset U \rightarrow Y \in X)$
- $hom_-(s, U, X) :\equiv \forall^\infty Y(s \subset_b Y \subset U \rightarrow Y \notin X)$
- $hom(s, U, X) :\equiv hom_+(s, U, X) \vee hom_-(s, U, X)$.

Let

$$C_F := \{B \subset \mathcal{P}(\omega) \mid (\forall s \in Seq^{mon})(\forall S \in F)(\exists S' \subset S)\\ [S' \in F \wedge hom(s, S', B)]\}$$

and

$$I_F := \{B \subset \mathcal{P}(\omega) \mid (\forall s \in Seq^{mon})(\forall S \in F)(\exists S' \subset S)\\ [S' \in F \wedge hom_-(s, S', B)]\}.$$

Definition 3.2.6. C is a σ-algebra if it contains the empty set, is closed under complements and countable unions. $I \subset C$ is a σ-ideal if it is closed under subsets and countable unions. I is ccc if there are no uncountable antichains, i.e. there is no uncountable set $\{B_i \mid i \in J\}$ such that $B_i \in C \backslash I$ for all $i \in J$ and $B_i \cap B_j \in I_F$ for all $i \neq j$.

In [Mat77] it is shown that if F is a Ramsey ultrafilter, C_F is a σ-algebra containing all open sets and I_F is a ccc σ-ideal in C_F.

We now show that each σ-algebra containing a ccc σ-ideal is closed under the operations $\forall^m$ and $\exists^m$. This is a generalization of theorem 5.14 in [And01] which states the closedness under the Souslin operator $\mathcal{A}$ which is the $\forall^1$-quantifier in our terminology.

Theorem 3.2.7. ZFC *proves: Assume that $(X, \mathcal{T})$ is a topological space and $\mathcal{S} \subset \mathcal{P}(X)$ is a σ-algebra which contains all open sets. Let $\mathcal{I}$ be a ccc σ-ideal in $\mathcal{S}$. Then $\mathcal{S}$ contains all sets which have an n-code.*

Proof. We induct on n. It is enough to show that $\mathcal{S}$ is closed under $\forall^n$ because of $\exists^n sB_s = (\forall^n sB_s^c)^c$ and $\mathcal{S}$ is closed under complements.

Assume $A = \forall^{n+1} sA_s$ with $A_s \in \mathcal{S}$ for all $s \in Seq$. For $\alpha < \omega_1$ and $s \in Seq$ let

$$A_s^0 = A_s$$

$$A_s^{\alpha+1} = \begin{cases} A_s^\alpha \cap \exists^n m A^\alpha_{s^\frown\langle m\rangle} & \text{if } A_s^\alpha \backslash \exists^n m A^\alpha_{s^\frown\langle m\rangle} \notin \mathcal{I} \\ A_s^\alpha & \text{otherwise} \end{cases}$$

$$A_s^\lambda = \bigcap_{a<\lambda} A_s^\alpha \text{ if } \lambda \in Lim.$$

By induction hypothesis $\mathcal{S}$ is closed under $\exists^n$, hence

$$(\forall\alpha < \omega_1)(\forall s \in Seq) A_s^\alpha \in \mathcal{S}$$

by induction on α. Furthermore

$$C_s := \{\alpha < \omega_1 \mid A_s^\alpha \backslash A_s^{\alpha+1} \notin \mathcal{I}\}$$

is countable, because otherwise $\{A_s^\alpha \backslash A_s^{\alpha+1} \mid \alpha \in C_s\}$ would witness that $\mathcal{I}$ is not ccc. By regularity of ω_1 we have

$$(\forall s \in Seq)(\exists \gamma_s < \omega_1)(\forall \beta \geq \gamma_s) A_s^{\gamma_s} = A_s^\beta$$

and again with regularity of ω_1

$$(\exists \gamma < \omega_1)(\forall s \in Seq)(\forall \beta \geq \gamma) A_s^\gamma = A_s^\beta.$$

Together with the definition of A_s^α we obtain from $A_s^\gamma = A_s^{\gamma+1}$

$$(\forall s \in Seq) A_s^\gamma \backslash \exists^n m A^\gamma_{s^\frown\langle m\rangle} \in \mathcal{I},$$

and since $\mathcal{I}$ is closed under countable unions we get

$$M := \bigcup_{s\in Seq} A_s^\gamma \backslash \exists^n m A^\gamma_{s^\frown\langle m\rangle} \in \mathcal{I}.$$

By the definition of M we have

$$(\forall s \in Seq) M^c \cap A_s^\gamma \subset \exists^n m A^\gamma_{s^\frown\langle m\rangle},$$

and together with the definition of the $\forall^{n+1}$-quantifier we get

$$M^c \cap A^\gamma_{\langle\rangle} \subset \forall^{n+1} m A^\gamma_m,$$

hence

$$A^\gamma_{\langle\rangle} \backslash \forall^{n+1} m A^\gamma_m \subset M \in \mathcal{I}. \tag{3.2}$$

By induction on α we show for all u

$$A^\alpha_u \supset \forall^{n+1} m A_{u^\frown m}. \tag{3.3}$$

Assume $A^{\alpha+1}_u = A^\alpha_u \cap \exists^n k A^\alpha_{u^\frown\langle k\rangle}$ in the successor step. By induction hypothesis and lemma 2.3.5 we obtain

$$\exists^n k A^\alpha_{u^\frown\langle k\rangle} \supset \exists^n k \forall^{n+1} m A_{u^\frown\langle k\rangle^\frown m} = \forall^{n+1} m A_{u^\frown m}$$

which proves the claim.

From (3.3) we obtain

$$A^\gamma_{\langle\rangle} \supset \forall^{n+1} m A_m \supset \forall^{n+1} m A^\gamma_m,$$

and together with (3.2) and $A^\gamma_{\langle\rangle} \in \mathcal{S}$ this implies $\forall^{n+1} m A_m \in \mathcal{S}$. □

By applying the theorem to C_F and I_F we get the

Corollary 3.2.8. ZFC + CH *proves that every set of reals which has an n-code is Ramsey.*

If we want to carry out this proof in subsystems of second order arithmetic like the σ-calculus we encounter two problems: How can we get a wellordering which is long enough to iterate the A^α_s's and how can we replace the ultrafilter? The next section is a preparation to meet the first problem.

3.3 Iterations along wellorderings

The aim of this section is to prepare the construction of a wellordering which is long enough for the transfinite recursion of the A^α_s in the proof of 3.2.7. We have seen there that I being ccc is the reason why this iteration comes to a halt before ω_1, and therefore it is not amazing that the key to the construction of our wellordering is in the proof that I_F is ccc.

Lemma 3.3.1 ([Mat77],proposition 1.11)**.** *I_F is ccc.*

Proof. Let $B \in C_F \backslash I_F$. $B \notin I_F$ implies

$$(\exists s \in Seq^{mon})(\exists S \in F)(\forall S' \subset S)[S' \in F \rightarrow \neg hom_-(s, S', B)],$$

which together with $B \in C_F$ entails

$$(\exists s \in Seq^{mon})(\exists S \in F)(\exists S' \subset S)[S' \in F \wedge hom(s, S', B) \wedge \neg hom_-(s, S', B)],$$

i.e.

$$(\exists s \in Seq^{mon})(\exists S \in F) hom_+(s, S, B).$$

For each $B \in C_F \backslash I_F$, we define its index

$$i(B) := \min\{s \mid (\exists S \in F) hom_+(s, S, B)\}.$$

If $B_1 \cap B_2 \in I_F$ then $i(B_1) \neq i(B_2)$ because $i(B_1) = i(B_2) =: i$ implies

$$hom_+(i, S_j, B_j) \text{ for } j \in \{1, 2\},$$

therefore $hom_+(i, S_1 \cap S_2, B_j)$ since $hom_+(i, S, B)$ stays true if S becomes smaller. But this implies $hom_+(i, S_1 \cap S_2, B_1 \cap B_2)$ which is a contradiction to $B_1 \cap B_2 \in I_F$.

Therefore in each antichain there are no two elements with the same index. Because we only have countable many indices, there are no uncountable antichains. □

We will construct the wellordering for the transfinite recursion of the A_s^α simultaneously with the sets A_s^α themselves. Assume that we have already done the recursion along a suitable wellordering of order type α. If

$$(\forall s) A_s^\alpha \backslash \exists^n m A_{s^\frown\langle m\rangle}^\alpha \in I_F,$$

the A_s^β are constant for $\beta \geq \alpha$ and the recursion is already finished. If $A_s^\alpha \backslash \exists^n m A_{s^\frown\langle m\rangle}$ is not in I_F for some s, we have to extend our wellordering by one further element; for this element we choose $\langle s, i\rangle$ where i is the index of the set $A_s^\alpha \backslash \exists^n m A_{s^\frown\langle m\rangle}^\alpha$ (see the proof of 3.3.1). If at some later point of the iteration a set $A_s^\beta \backslash \exists^n m A_{s^\frown\langle m\rangle}^\beta$ claims that the wellordering has to be extended by one further element, then the index of $A_s^\beta \backslash \exists^n m A_{s^\frown\langle m\rangle}^\beta$ has to be different from all indices which already occured in the wellordering. But this is the case since

$$A_s^\beta \backslash \exists^n m A_{s^\frown\langle m\rangle} \subset A_s^\beta \subset A_s^{\alpha+1} = A_s^\alpha \cap \exists^n m A_{s^\frown\langle m\rangle}^\alpha$$

implies

$$(A^\alpha_s \backslash \exists^n m A^\alpha_{s^\frown\langle m\rangle}) \cap (A^\beta_s \backslash \exists^n m A^\beta_{s^\frown\langle m\rangle}) = \emptyset \in I_F,$$

hence the indices of $A^\alpha_s \backslash \exists^n m A^\alpha_{s^\frown\langle m\rangle}$ and $A^\beta_s \backslash \exists^n m A^\beta_{s^\frown\langle m\rangle}$ are different by the argument in the proof of 3.3.1. Therefore it is not possible that a index occurs twice, and we really obtain a wellordering. In this section, we will make this argument more explicit.

Let us consider a non-monotone inductive process given by a formula φ in the following way.

Definition 3.3.2. Let $\varphi(x, X, Y)$ be a first order $\mathcal{L}_\sigma$-formula and W a wellordering. I is an iteration of φ along W if

- $\forall x \in I[x \in Seq \wedge (lh(x) = 1 \vee lh(x) = 2)]$
- $(I)^0 = field(W)$
- $\forall i \in field(W)[(I)_i = \{x \mid \varphi(x, I^{<i}, W^{<i})\}]$
  where $I^{<i} := \{\langle x, y\rangle \in I \mid \langle x, i\rangle \in W\}$

with $(X)^n := \{x \mid \exists y \in Seq \cap X[lh(y) \geq n + 1 \wedge x = (y)_n]\}$. We allow sequences of length one in I because the empty set may occur during the iteration.

Given an inductive process by a formula φ we want to iterate it along a wellordering which is constructed simultaneously with the iteration of φ. The wellordering will be given by a formula $\psi(i, I, W)$ with the intended meaning: if I is an iteration of φ along W then we extend W by the new maximal element i,which adds one iteration step of φ.

Definition 3.3.3. Let $\varphi(x, X, Y)$ and $\psi(i, X, Y)$ be first order $\mathcal{L}_\sigma$-formulas. (I, W) is a ψ-iteration of φ if

- I is an iteration of φ along W
- $\forall i \in field(W)\psi(i, I^{<i}, W^{<i})$.

Definition 3.3.4. Let $\varphi(x, X, Y)$ and $\psi(i, X, Y)$ be first order $\mathcal{L}_\sigma$-formulas. (I, W) is a maximal ψ-iteration of φ if

- (I, W) is a ψ-iteration of φ
- $\neg\exists i\psi(i, I, W)$.

To guarantee the existence of such an iteration we have to ensure that the new element of the wellordering W suggested by ψ is not yet an element of W.

Definition 3.3.5. Let $\varphi(x, X, Y)$ and $\psi(i, X, Y)$ be first order $\mathcal{L}_\sigma$-formulas. ψ is called suitable for φ if

- $\forall X, Y \forall x, y[(\psi(x, X, Y) \wedge \psi(y, X, Y)) \rightarrow x = y]$
- $\forall I, W \forall i[((I, W)$ is ψ-iteration of $\varphi \wedge \psi(i, I, W)) \rightarrow i \notin \mathit{field}(W)]$.

Theorem 3.3.6. *For first order $\mathcal{L}_\sigma$-formulas $\varphi(x, X, Y)$ and $\psi(i, X, Y)$ there exist set terms I and W uniformly in φ and ψ such that the σ-calculus proves: If ψ is suitable for φ then (I, W) is a maximal ψ-iteration of φ.*

Proof. Let

$$\begin{aligned}\tau(\langle a, z\rangle, X) :\equiv &[a = 0 \wedge \psi(z, (X)_1, (X)_2)] \\ &\vee[a = 1 \wedge z = \langle i, x\rangle \wedge \psi(i, (X)_1, (X)_2) \wedge \varphi(x, (X)_1, (X)_2)] \\ &\vee[a = 2 \wedge z = \langle x, y\rangle \wedge \psi(y, (X)_1, (X)_2) \wedge x \in (X)_0 \wedge y \notin (X)_0]\end{aligned}$$

Here we used the abbreviation $z = \langle x, i\rangle \wedge \varphi(x, i)$ for $z \in \mathit{Seq} \wedge lh(z) = 2 \wedge \varphi((z)_0, (z)_1)$. We will show that $(I(\sigma x X \tau(x, X)))_2$ is the strict part of a prewellordering with field $(I(\sigma x X \tau(x, X)))_0$ and that $(I(\sigma x X \tau(x, X)))_1$ is a maximal ψ-iteration of φ along this prewellordering. Let $\preceq, \prec$ be the prewellordering on $I(\sigma x X \tau(x, X))$. We first show

$$\langle x, y\rangle \in (I(\sigma x X \tau(x, X)))_2 \rightarrow \langle 0, x\rangle \prec \langle 0, y\rangle. \tag{3.4}$$

$\langle x, y\rangle \in (I(\sigma x X \tau(x, X)))_2$ implies $\langle 2, \langle x, y\rangle\rangle \preceq \langle 2, \langle x, y\rangle\rangle$, hence

$$\tau(\langle 2, \langle x, y\rangle\rangle, \{z \mid z \prec \langle 2, \langle x, y\rangle\rangle\})$$

by the fixed point axioms. This implies

$$\langle 0, x\rangle \prec \langle 2, \langle x, y\rangle\rangle \text{ and } \langle 0, y\rangle \not\prec \langle 2, \langle x, y\rangle\rangle$$

by the definition of τ, therefore

$$\langle 2, \langle x, y\rangle\rangle \preceq \langle 0, y\rangle$$

since $\langle 2, \langle x, y\rangle\rangle \in I(\sigma x X \tau(x, X))$. This finishes the proof of (3.4).

From the wellfoundedness of $\prec$ together with (3.4) we obtain that the relation coded by $(I(\sigma x X \tau(x, X)))_2$ is wellfounded. Now we can show by induction along $\prec$ that for each z, $(\{x \mid x \prec z\})_1$ is a ψ-iteration of φ along the prewellordering with strict part $(\{x \mid x \prec z\})_2$ and field $(\{x \mid x \prec z\})_0$. Notice that we can express this without second order quantifiers since we have already proved the wellfoundedness.

It remains to show that the iteration is maximal. Assume

$$\exists i \psi\big(i, (I(\sigma x X \tau(x,X)))_1, (I(\sigma x X \tau(x,X)))_2\big).$$

Then we obtain $\tau(\langle 0,i\rangle, I(\sigma x X \tau(x,X)))$, therefore $\langle 0,i\rangle \in I(\sigma x X \tau(x,X))$ by the fixed point axioms. But then i is in the field of the wellordering which is a contradiction since ψ is suitable for φ. □

3.4 The embedding

We now formalize the proof in section 3.2 in the σ^+-calculus. We have already described the idea at the beginning of section 3.3, but there is still the question left how to replace the ultrafilter F which is needed for the definition of the σ-algebra C_F and the σ-ideal I_F. We construct a filter which is fine enough simultaneously with the sets A^α_s. At step α of the recursion, we construct an infinite set of natural numbers F_α which codes the filter $\{X \subset \omega \mid F_\alpha \backslash X \text{ is finite}\}$. We make sure that $F_\beta \backslash F_\alpha$ is finite for $\beta > \alpha$, therefore the filter comprises more sets with increasing α. Thus C_{F_α} and I_{F_α} also become bigger with increasing α.[1]

Let us assume that we have done the iteration up to α. Using the induction hypothesis we choose F_α such that for each s and i we can remove an initial segment from the code F_α and obtain ${F_\alpha}^{>k}$ for some $k \in \omega$ such that

$$hom(i, {F_\alpha}^{>k}, A^\alpha_s \backslash \exists^n m A^\alpha_{s^\frown\langle m\rangle}).$$

This guarantees $A^\alpha_s \backslash \exists^n m A^\alpha_{s^\frown\langle m\rangle} \in C_{F_\alpha}$ for all s. Let us assume that the iteration is not yet maximal, therefore we have $\langle s,i\rangle$ which is minimal such that $A^\alpha_s \backslash \exists^n m A^\alpha_{s^\frown\langle m\rangle} \notin I_{F_\alpha}$ and i is the index of $A^\alpha_s \backslash \exists^n m A^\alpha_{s^\frown\langle m\rangle}$, i.e.

$$i := \min\{s \mid (\exists S \in F_\alpha) hom_+(s, S, A^\alpha_s \backslash \exists^n m A^\alpha_{s^\frown\langle m\rangle})\};$$

i exists since $A^\alpha_s \backslash \exists^n m A^\alpha_{s^\frown\langle m\rangle} \notin I_{F_\alpha}$ implies

$$(\exists i)(\exists k) \neg hom_-(i, {F_\alpha}^{>k}, A^\alpha_s \backslash \exists^n m A^\alpha_{s^\frown\langle m\rangle}),$$

[1] Let $\alpha < \beta$ and $B \in C_{F_\alpha}$, therefore for each s there exists $S_s \in F_\alpha$ such that $hom(s, S_s, B)$. But then

$$\forall s \forall S \in F_\beta [S \cap S_s \in F_\beta \wedge hom(s, S \cap S_s, B)],$$

i.e. $B \in C_{F_\beta}$.

hence

$$(\exists i)(\exists k)hom_+(i, F_\alpha{}^{>k}, A_s^\alpha \backslash \exists^n m A^\alpha_{s^\frown\langle m\rangle}).$$

We choose $\langle s, i\rangle$ as new element of our wellordering. Notice that for all future enlargements F of F_α, $A_s^\alpha \backslash \exists^n m A^\alpha_{s^\frown\langle m\rangle}$ will always stay out of I_F. If $\langle s, i\rangle$ would be chosen a second time at stage $\beta > \alpha$, we would have

$$(\exists l)hom_+(i, F_\beta{}^{>l}, A_s^\beta \backslash \exists^n m A^\beta_{s^\frown\langle m\rangle}).$$

But this is a contradiction since $A_s^\alpha \backslash \exists^n m A^\alpha_{s^\frown\langle m\rangle}$ and $A_s^\beta \backslash \exists^n m A^\beta_{s^\frown\langle m\rangle}$ are disjoint because of $A_s^\beta \subset A_s^\alpha \cap \exists^n m A^\alpha_{s^\frown\langle m\rangle}$. Hence the index $\langle s, i\rangle$ can not be chosen a second time. At limit points λ of our iteration, we define F_λ as the diagonal intersection of the $(F_\alpha)_{\alpha<\lambda}$. This works because we defined that all X such that $F\backslash X$ is finite are in the filter coded by F and not all X such that $F \subset X$. Let us now spell out the details.

Definition 3.4.1 (code of a filter)**.** For sets X and Y let

$$X^{>n} := \{x \in X \mid x > n\}$$

and

$$X \subset_\infty Y :\equiv \exists n(X^{>n} \subset Y).$$

A code of a filter is an infinite set of natural numbers F. The filter coded by F is $\{X \mid F \subset_\infty X\}$.

Such filters are not closed under diagonalization, but for a given family $\{X_s \mid s \in Seq^{mon}\}$ where all X_s are elements of a filter we can find a finer filter which contains a diagonalizing set. As the next lemma shows, this is already provable in ACA_0.

Lemma 3.4.2. ACA_0 *proves: If F is a code of a filter and X is a set such that $F \subset_\infty X_s$ for all $s \in Seq^{mon}$ then there is a set G (uniform in F and X) such that $G \subset_\infty F$ (i.e. the filter coded by F is a subfilter of the filter coded by G) and G diagonalizes the X_s.*

Proof. Let $S_i := \bigcap\{X_s \mid max(s) = i\}$, then $F \subset_\infty S_i$ for each i since it is a cut of finitely many sets since s is a strictly monotone sequence. Let $g_0 := 0$ and

$$g_{i+1} = \text{ the least element of } \bigcap_{j\leq i} S_{g_j}^{>g_i},$$

then $\{g_i \mid j < i\} \subset S_{g_j}$ for each j. Let $G := \{g_i \mid i \in \omega\}$. Then we have for each s with $\max(s) = g_j$

$$G \backslash s = \{g_i \mid g_j < g_i\} = \{g_i \mid j < i\} \subset S_{g_j} \subset X_s,$$

hence G diagonalizes the X_s. □

If F_α is the filter at stage α of our iteration we will ensure that F_α becomes finer with increasing α, i.e. $F_\alpha \subset_\infty F_\beta$ for $\alpha > \beta$. We prepare the limit steps of the iteration by the following lemma:

Lemma 3.4.3. *There exists an arithmetical formula $\varphi_{Diag}(x, X)$ (we will write $x \in Diag(X)$ instead) such that* ACA_0 *proves:*

$$\forall n\big(\infty(X)_n \wedge (X)_{n+1} \subset_\infty (X)_n\big) \to \big(\infty(Diag(X)) \wedge (\forall n) Diag(X) \subset_\infty (X)_n\big)$$

with $\infty(X) :\equiv \forall x(\exists y > x)(y \in X)$.

Proof. Let $x_0 := 0$ and x_{i+1} be the least element of $\bigcap_{j \leq i+1}(X)_j$ which is greater than x_i. Choose $Diag(X) := \{x_i \mid i \in \omega\}$. □

Lemma 3.4.4. *There is an arithmetical formula $\varphi_{cof}(x, X)$ (we will write $x \in cof(X)$ instead) such that* ACA_0 *proves: If X codes an ordering with no maximal element then $cof(X)$ codes a cofinal sequence.*

Proof. Choose $x_0 \in field(X)$ and x_{n+1} such that $x_n <_X x_{n+1}$ and $i <_X x_{n+1}$ for all $i < n$ with $i \in field(X)$. Let $cof(X) := \{x_i \mid i \in \omega\}$. □

Definition 3.4.5. In analogy to 3.2.5 we define

$$hom^n_+(s, U, X) :\equiv \forall^\infty Y(s \subset_b Y \subset s \cup U \to Y \in^n X),$$

$$hom^n_-(s, U, X) :\equiv \forall^\infty Y(s \subset_b Y \subset s \cup U \to Y \notin^n X),$$

$$hom^n(s, U, X) :\equiv hom^n_+(s, U, X) \vee hom^n_-(s, U, X).$$

Here we identify the sequence s with the set of its elements.

The following lemma follows directly from this definition.

Lemma 3.4.6. *For all n,* ACA_0 *proves*

- $(hom^n_+(s, U, X) \wedge \widetilde{U} \subset U \wedge X \subset^n \widetilde{X}) \to hom^n_+(s, \widetilde{U}, \widetilde{X})$
- $(hom^n_-(s, U, X) \wedge \widetilde{U} \subset U \wedge \widetilde{X} \subset^n X) \to hom^n_-(s, \widetilde{U}, \widetilde{X})$
- $(hom^n(s, U, X) \wedge \widetilde{U} \subset U) \to hom^n(s, \widetilde{U}, X)$.

Theorem 3.4.7. *For all $n \in \omega$ there exists an $\mathcal{L}_\sigma$-term $H^n(s, U, X)$ such that the σ^+-calculus proves: for all $s \in Seq^{mon}$, all infinite U and for each X which is a n-code of reals it holds*

- $s \subset_b H^n(s, U, X) \subset s \cup U$
- $hom^n(s, H^n(s, U, X), X)$.

Proof. The proof is by metainduction on n. In each step of the induction, we first prove the theorem for simple n-codes and then for arbitrary n-codes. For simple 0-codes, the claim follows from theorem 1.2.1 which states the property of Ramsey for open sets. The step from simple 0-codes to arbitrary 0-codes is as in the induction step.

From now on assume that the theorem is proved for n. All lemmata until lemma 3.4.21 are part of the induction step. We first prove the theorem for simple $n+1$-codes C whose root codes a $\forall^{n+1}$-branching by formalizing the proof of theorem 3.2.4 in the σ^+-calculus.

We would like to generalize our induction hypothesis in the following way. Given ω-many sets of reals $(A_i)_{i\in\omega}$ with an n-code, we would like to have one set of natural numbers which is homogeneous for all A_i´s simultaneously. But as the example at the beginning of section 1.3 shows, this is impossible. However, we can find an infinite subset H of ω such that for each $i \in \omega$ we can remove an initial segment of H such that the remaining set is homogeneous for A_i. This is done in the following lemma.

Lemma 3.4.8. *There exists an $\mathcal{L}_\sigma$-term $H^n_\omega(U, X)$ such that the σ^+-calculus proves: If $(X)_i$ is an n-code for each $i \in \omega$ then*

$$(\forall U)(\forall s \in Seq^{mon})(\forall i)[hom^n(s, H^n_\omega(U, X)^{\triangleright\langle s,i\rangle}, (X)_i) \wedge H^n_\omega(U, X) \subset_\infty U],$$

where $X^{\triangleright n}$ is X without the n least elements of X.

Proof. By recursion on p we define a sequence of infinite sets $(S_p)_{p\in\omega}$ such that $S_p \subset S_q \subset U$ for $p > q$ and $hom^n(s, S_{\langle s,i\rangle}, (X)_i)$ for each s, i. We start with $S_0 := U$. If $p = \langle s, i\rangle$ let

$$S_p := H^n(s, S_{p-1}, (X)_i) \backslash \{i \mid i \text{ is an element of } s\},$$

otherwise $S_p := S_{p-1}$. Then the properties of the S_p follow from the main induction hypothesis.

Let k_0 be the least element of S_0 and

$$k_{i+1} := \text{ the least element of } \bigcap_{j\leq i+1} S_j \text{ which is bigger than } k_i.$$

Then $H^n_\omega(U, X) := \{k_i \mid i \in \omega\}$ satisfies the claim since $H^n_\omega(U, X)^{\triangleright\langle s,i\rangle} \subset S_{\langle s,i\rangle}$. □

Definition 3.4.9. Let

$$big^n(s,U,X) :\equiv H^n(s,U,X) \in^n X,$$

i.e. X is "big" in s,U.

The following lemma we obtain immediately from the induction hypothesis of the induction on n.

Lemma 3.4.10. *The σ^+-calculus proves for all n-codes X*

$$(\forall U)(\forall s \in Seq^{mon})[big^n(s,U,X) \to hom^n_+(s,H^n(s,U,X),X)]$$

and

$$(\forall U)(\forall s \in Seq^{mon})[\neg big^n(s,U,X) \to hom^n_-(s,H^n(s,U,X),X)].$$

For the construction of $H^{n+1}(s,V,C)$ fix a simple $\forall^{n+1}$-code C and an infinite set V.

Definition 3.4.11. Let $max(i,X)$ denote that i is the maximal element of the ordering X and $lim(X)$ that X has no maximal element. We define sets $T(X,Y)$ and $F(X,Y)$ by arithmetical comprehension.

$$\begin{aligned}\langle s,x\rangle \in T(X,Y) :\equiv &(Y=\emptyset \wedge x \in C\restriction\langle s\rangle)\\ &\vee\exists i[max(i,Y)\wedge\langle i,\langle 0,\langle s,x\rangle\rangle\rangle \in X]\\ &\vee[lim(Y)\wedge x\in\forall^0 n\{\langle n,z\rangle \mid \exists y[\langle n,y\rangle\in cof(Y)\wedge\\ &\qquad\qquad \langle y,\langle 0,\langle s,z\rangle\rangle\rangle\in X]\}]\end{aligned}$$

$$\begin{aligned}x \in \widetilde{F}(X,Y) :\equiv &(Y=\emptyset\wedge x\in V)\\ &\vee\exists i(max(i,Y)\wedge\langle i,\langle 1,x\rangle\rangle\in X)\\ &\vee[lim(Y)\wedge x\in Diag(Cof(X,Y))]\end{aligned}$$

$$x\in F(X,Y) :\equiv H^n_\omega(\widetilde{F}(X,Y),Z) \text{ with } (Z)_s = T(X,Y)_s\backslash\exists^n k T(X,Y)_{s^\frown\langle k\rangle}$$

To understand this definition suppose that X codes an iteration along the wellordering Y such that $(((X)_i)_0)_s$ codes A^α_s if i is an element of $field(Y)$ with ordertype α and $((X)_i)_1$ is a code of the stage α of the filter. Then $T(X,Y)_s$ is the n-code of the the "actual" set A_s which is A^α_s if α is the ordertype of the maximal element of Y, and in case that the ordertype of Y is a limit ordinal it is $\bigcap_{n\in\omega} A^{\alpha_n}_s$, where $(\alpha_n)_{n\in\omega}$ is a cofinal sequence in α. To express $\bigcap_{n\in\omega} A_s^{\alpha_n}$ we use the notation of definition 3.1.4. $F(X,Y)$ codes the "actual" filter which is the diagonal intersection in the limit case.[2] Let

$$\psi(j,X,Y) :\equiv j \text{ is the least } \langle s,i\rangle \text{ such that } \\ big^n(i,F(X,Y),T(X,Y)_s\backslash\exists^n kT(X,Y)_{s^\frown\langle k\rangle})$$

and

$$\begin{aligned}\varphi(x,X,Y) :\equiv \\ \exists\langle s,i\rangle\big(\psi(\langle s,i\rangle,X,Y)\wedge \\ \exists t,y[x=\langle 0,\langle t,y\rangle\rangle\wedge[(s\neq t\wedge y\in T(X,Y)_s)\vee \\ (s=t\wedge y\in T(X,Y)_t\cap\exists^n iT(X,Y)_{t^\frown\langle i\rangle})]] \\ \vee\exists y[x=\langle 1,y\rangle \\ \wedge y\in H^n(i,F(X,Y),T(X,Y)_s\backslash\exists^n kT(X,Y)_{s^\frown\langle k\rangle})]\big).\end{aligned}$$

Lemma 3.4.12. *The σ^+-calculus proves: If (I,W) is a ψ-iteration of φ then*

$$(\forall s\in Seq^{mon})(\forall i)hom^n(i,F(I,W)^{\triangleright\langle s,i\rangle},T(I,W)_s\backslash\exists^n kT(I,W)_{s^\frown\langle k\rangle}).$$

Proof. This follows directly from the definition of $F(I,W)$ and lemma 3.4.8. □

Lemma 3.4.13. *The σ^+-calculus proves: If (I,W) is a ψ-iteration of φ and $w=\langle s,i\rangle\in field(W)$ then*

$$(\exists m)hom^n_-(i,F(I;W)^{\triangleright m},T(I,W)_s).$$

Proof. Since $w\in field(W)$ we have $\psi(w,I^{<w},W^{<w})$ by the definition of a ψ-iteration 3.3.3, hence by the definition of ψ

$$big^n(i,F(I^{<w},W^{<w}),T(I^{<w},W^{<w})_s\backslash\exists^n kT(I^{<w},W^{<w})_{s^\frown\langle k\rangle}). \tag{3.5}$$

By the definition 3.3.2 of an iteration together with the definition of φ we obtain

$$((I)_w)_1=H^n(i,F(I^{<w},W^{<w}),T(I^{<w},W^{<w})_s\backslash\exists^n kT(I^{<w},W^{<w})_{s^\frown\langle k\rangle}). \tag{3.6}$$

With lemma 3.4.10 we get from (3.5) and (3.6)

$$hom^n_+(i,((I)_w)_1,T(I^{<w},W^{<w})_s\backslash\exists^n kT(I^{<w},W^{<w})_{s^\frown\langle k\rangle})$$

[2] Since we code an ordering $\prec$ by $\{\langle x,y\rangle\mid x\prec y\}$ the code makes no difference between orderings with zero and one element because both are coded by the empty set. Hence $max(i,Y)$ does also depend on X from which we can decode the field of the ordering, but we suppress that.

which entails

$$hom^n_-(i, ((I)_w)_1, T(I^{<w}, W^{<w})_s \cap \exists^n k T(I^{<w}, W^{<w})_{s^\frown\langle k\rangle}).$$

Since

$$T(I,W)_s \subset^n (((I)_w)_0)_s = T(I^{<w}, W^{<w})_s \cap \exists^n k T(I^{<w}, W^{<w})_{s^\frown\langle k\rangle}$$

and $F(I;W) \subset_\infty ((I)_w)_1$ we get by monotonicity lemma 3.4.6

$$(\exists m) hom^n_-(i, F(I;W)^{\triangleright m}, T(I,W)_s).$$

□

Lemma 3.4.14. *The σ^+-calculus proves: ψ is suitable for φ.*

Proof. Assume that (I,W) is a ψ-iteration of φ and $w = \langle s, i\rangle \in field(W)$ with $\psi(w, I, W)$. Then we get

$$(\exists m) hom^n_-(i, F(I;W)^{\triangleright m}, T(I,W)_s)$$

by lemma 3.4.13, which yields

$$(\exists m) hom^n_-(i, F(I;W)^{>m}, T(I,W)_s \backslash \exists^n k T(I,W)_{s^\frown\langle k\rangle})$$

by monotonicity of hom^n_- (lemma 3.4.6). But this is a contradiction to $\psi(w, I, W)$. □

Lemma 3.4.15. *There exists a $\mathcal{L}_\sigma$-term $K(F,X)$ such that the σ^+-calculus proves: If $(X)_i$ is an n-code for each i and if F is infinite then*

$$(\forall s \in Seq^{mon}) K(F,X)_s \text{ is infinite,}$$

$$(\forall s \in Seq^{mon}) K(F,X)_s \subset F$$

and

$$(\forall s \in Seq^{mon})(\forall i) hom^n_-(s, F^{\triangleright\langle s,i\rangle}, (X)_i) \to (\forall s) hom^n_-(s, K(F,X)_s, \bigcup_{i\in\omega}(X)_i). \tag{3.7}$$

Proof. Since $\langle s^\frown t, t\rangle \geq \langle s^\frown t, i\rangle$ for all $i \leq t$ we obtain for all t such that $s^\frown t$ is strictly monotone

$$(\forall i \leq t) hom^n_-(s^\frown t, F^{\triangleright\langle s^\frown t, t\rangle}, X_i)$$

by the premise of (3.7). Define

$$S_t := \begin{cases} F^{\triangleright\langle s^\frown t,t\rangle} & \text{if } s^\frown t \text{ is strictly monotone} \\ S_t := F & \text{otherwise.} \end{cases}$$

Let K_s be a diagonalization of the S_t according to lemma 3.4.2. Towards a contradiction assume $s \subset_b Z \subset s \cup K_s$ and $Z \in^n X_j$. If t is the sequence of the first j elements of Z which are greater than the elements of s we have $\langle s^\frown t, t\rangle \geq \langle s^\frown t, j\rangle$, hence $hom^n_-(s^\frown t, F^{\triangleright\langle s^\frown t,t\rangle}, X_j)$. This implies $hom^n_-(s^\frown t, K_s, X_j)$, hence $Z \notin^n X_j$, contradiction. □

Lemma 3.4.16. *The σ^+-calculus proves: There is a maximal ψ-iteration (I,W) of φ uniformly in φ and ψ. Furthermore there is a set K uniformly in φ and ψ such that $K_s \subset V$ and*

$$(\forall s \in Seq^{mon}) hom^n_-(s, K_s, \bigcup_t T(I,W)_t \backslash \exists^n k T(I,W)_{t^\frown\langle k\rangle}).$$

Proof. The first claim follows directly from theorem 3.3.6 since ψ is suitable for φ by lemma 3.4.14. Since the iteration is maximal we have $\forall i \neg\psi(i,I,W)$, i.e.

$$(\forall s \in Seq^{mon})(\forall t)\neg big^n(s, F(I,W), T(I,W)_t \backslash \exists^n k T(I,W)_{t^\frown\langle k\rangle}). \quad (3.8)$$

We also have

$$(\forall s \in Seq^{mon})(\forall t) hom^n(s, F(I,W)^{\triangleright\langle s,t\rangle}, T(I,W)_t \backslash \exists^n k T(I,W)_{t^\frown\langle k\rangle})$$

by lemma 3.4.12, and since (3.8) excludes hom^n_+ we obtain

$$(\forall s \in Seq^{mon})(\forall t) hom^n_-(s, F(I,W)^{\triangleright\langle s,t\rangle}, T(I,W)_t \backslash \exists^n k T(I,W)_{t^\frown\langle k\rangle}),$$

which yields the claim by lemma 3.4.15. □

Lemma 3.4.17. *The σ^+-calculus proves: If (I,W) is a ψ-iteration of φ and 0_W is the least element of the wellordering W, then it holds for all s and X*

$$(\forall^{n+1} x) X \in^n (((I)_{0_W})_0)_{s^\frown x} \leftrightarrow (\forall^{n+1} x) X \in^n T(I,W)_{s^\frown x}.$$

Proof. The direction from right to left holds because of

$$X \in^n T(I,W)_{s^\frown x} \to X \in^n (((I)_{0_W})_0)_{s^\frown x}$$

which is provable for each X by transfinite induction along W. For the other direction we apply lemma 2.3.6 for $\varphi(s,x,X) :\equiv X \in^n (((I)_{0_W})_0)_{s^\frown x}$ and prove

$$\forall w \in field(W) \forall s,x,X[\widetilde{\varphi}(s,x,X) \to X \in^n (((I)_w)_0)_{s^\frown x}] \tag{3.9}$$

by induction on w. For $w = 0_W$ this follows from the second claim of lemma 2.3.6. For the successor case, we first notice that

$$(((I)_w)_0)_s = T(I^{\leq w}, W^{\leq w})_s \tag{3.10}$$

by the definition of T. If w' is the W-successor of w then

$$(((I)_{w'})_0)_{s^\frown x} = ((\{y \mid \varphi(y, I^{\leq w}, W^{\leq w})\})_0)_{s^\frown x}$$

by the definition of an iteration (3.3.2). Therefore $(((I)_{w'})_0)_{s^\frown x}$ is either $(((I)_w)_0)_{s^\frown x}$ or $(((I)_w)_0)_{s^\frown x} \cap \exists^n y(((I)_w)_0)_{s^\frown x^\frown \langle y \rangle}$ by the definition of φ and (3.10). In the first case the claim follows directly from the induction hypothesis, in the second case from the third claim of 2.3.6 together with the induction hypothesis. If w is a limit then

$$(\forall s,x)\widetilde{\varphi}(s,x,X) \to X \in^n T(I^{<w}, W^{<w})_{s^\frown x}$$

by induction hypothesis together with the definition of T, therefore we obtain

$$(\forall s,x)\widetilde{\varphi}(s,x,X) \to (\exists^n y) X \in^n T(I^{<w}, W^{<w})_{s^\frown x^\frown \langle y \rangle}$$

with the third claim of 2.3.6. Hence

$$X \in^n ((\{x \mid \varphi(x, I^{<w}, W^{<w})\})_0)_{s^\frown x} = (((I)_w)_0)_{s^\frown x}$$

which finishes the proof of (3.9).

To prove the direction from left to right assume $\forall^{n+1} x \varphi(x)$. Lemma 2.3.6 implies $\forall^{n+1} x \widetilde{\varphi}(x)$, and from (3.9) we obtain

$$(\forall^{n+1} x)(\forall w \in field(W)) X \in^n (((I)_w)_0)_{s^\frown x}$$

which yields the claim. □

Lemma 3.4.18. *The σ^+-calculus proves: If (I,W) is a maximal φ-iteration of ψ and K is as in lemma 3.4.16 then*

$$(\forall s \in Seq^{mon})(\forall X)\big(s \subset_b X \subset K_s \\ \to [X \in^n T(I,W)_{\langle\rangle} \leftrightarrow X \in^{n+1} (\forall^{n+1} x) T(I,W)_x]\big).$$

Proof. The direction from right to left follows directly from the definition of $\forall^{n+1}$. For the other direction it suffices to show

$$(\forall s \in Seq^{mon}) hom_{-}^{n+1}(s, K_s, T(I,W)_{\langle\rangle} \backslash (\forall^{n+1} x) T(I,W)_x). \tag{3.11}$$

Lemma 2.3.7 implies

$$(\varphi(\langle\rangle) \wedge \neg \forall^{n+1} x \varphi(x)) \rightarrow \exists t(\varphi(t) \wedge \neg \exists^n x \varphi(t^\frown \langle x \rangle)),$$

for each first order φ, and if we set $\varphi(s, X) :\equiv X \in^n T(I,W)_s$ we obtain

$$T(I,W)_{\langle\rangle} \backslash (\forall^{n+1} x) T(I,W)_x \subset^n \bigcup_t T(I,W)_t \backslash \exists^n k T(I,W)_{t^\frown \langle k \rangle}.$$

Together with 3.4.16 this proves 3.11. □

We are now able to prove theorem 3.4.7 for simple $n+1$-codes whose root codes a $\forall^{n+1}$-branching. Let

$$H^{n+1}(s, V, C) := H^n(s, K_s, T(I,W)_{\langle\rangle}).$$

Then we have

$$s \subset_b H^{n+1}(s, V, C) \subset s \cup K_s \subset s \cup F(I,W) \subset s \cup V$$

and

$$hom^n(s, H^{n+1}(s, V, C), T(I,W)_{\langle\rangle}).$$

By lemma 3.4.18 this implies

$$hom^{n+1}(s, H^{n+1}(s, V, C), (\forall^{n+1} x) T(I,W)_x),$$

and lemma 3.4.17 yields

$$hom^{n+1}(s, H^{n+1}(s, V, C), C).$$

This finishes the proof of theorem 3.4.7 for simple $n+1$-codes with $\forall^{n+1}$-branching root. Since a set which is homogeneous for X is also homogeneous for X^c this extends to simple $n+1$-codes with $\exists^{n+1}$-branching root. We now generalize the result for arbitrary $n+1$-codes.

The idea is as follows: Assume that we have a code C for a set of reals and a set-term H such that $(\forall s)(\forall^\infty W) hom(s, H(s, W, C), C)$. Then for each infinite set W and each $s \subset_b W$ we can find an infinite set U ($:= s \cup H(s, W, C)$) such that $s \subset_b U \subset W$ and the set coded by C is open in U, i.e. $C =_U P$ for a code of an open set P with $=_U$ as in definition 3.1.7. For a given $n+1$-code we will define sets U_s by transfinite recursion along the Kleene-Brouwer ordering on the code C such that the set coded by the part of the tree above s is open in U_s. Then we obtain a homogeneous set for C using the property of Ramsey for open sets.

Lemma 3.4.19. *There are $\mathcal{L}_\sigma$-terms $U(m,i,W,O)$ and $P(m,i,W,O)$ such that the σ^+-calculus proves: If $m \leq n+1$ and O_j is a code of an open set for each j and W is an infinite set then $P(m,i,W,O)$ is a code of an open set, $U(m,i,W,O)$ is infinite and*

$$W^{\trianglelefteq i} \subset_b U(m,i,W,O) \subset W$$

and

$$\forall^m j O_j =_{U(m,i,W,O)} P(m,i,W,O).$$

(Remember that $X^{\trianglelefteq n}$ is the set of the n least elements of X.) A similar assertion holds for $\exists^m$.

Proof. We first construct an infinite set $V \subset W$ such that

$$(\forall s \subset W^{\trianglelefteq i}) hom^{n+1}(s, V, \forall^m j O_j).$$

Since $W^{\trianglelefteq i}$ is finite, we can do this by a recursion of finite length using theorem 3.4.7 for simple m-codes in each recursion step. Now we choose $U := V \cup W^{\trianglelefteq i}$ and P coding $\bigcup\{\mathcal{N}_s \mid s \subset W^{\trianglelefteq i} \wedge big^{n+1}(s, V, \forall^m j O_j)\}$ which satisfy the claim. □

We will now iterate lemma 3.4.19 along the Kleene-Brouwer ordering $<^C_{KB}$ of the tree T_C which belongs to the code C. We have to make sure that at limit stages λ we find an infinite set U_λ which is contained in all U_α for $\alpha < \lambda$. For this we have to define the U_α's such that they agree on long enough initial segments. To make this more precise we need the following notation.

Definition 3.4.20. For a sequence $s = \langle s_0, \ldots, s_n \rangle$ let $\Sigma s := \Sigma_{i \leq n} s_i$.

If s has order type α and t has order type β in $<^C_{KB}$ with $\alpha < \beta$, then U_α and U_β will agree on the first $min\{\Sigma s \mid t_1 \leq^X_{KB} s \leq^X_{KB} t_2\}$ elements. In the proof of the next lemma we will see that this is sufficient to handle the limit steps.

Lemma 3.4.21. *There are $\mathcal{L}_\sigma$-terms $U(C,W)$ and $O(C,W)$ such that the σ^+-calculus proves for each $n+1$-code C and for each infinite set W*

1. *$(U(C))_s$ is an infinite subset of W for all s*
2. *$\forall s \in C(C \restriction s =_{U(C)_s} O(C)_s)$*
3. *$\forall t_2 <^C_{KB} t_1 (U(C)_{t_1} \subset U(C)_{t_2})$*
4. *$\forall t_1 <^C_{KB} t_2 (U(C)_{t_1}^{\trianglelefteq \min\{\Sigma s \mid t_1 \leq^C_{KB} s \leq^C_{KB} t_2\}} = U(C)_{t_2}^{\trianglelefteq \min\{\Sigma s \mid t_1 \leq^C_{KB} s \leq^C_{KB} t_2\}})$.*

Proof. We define $U(C)$ and $O(C)$ by simultaneous recursion along $<^C_{KB}$. Let $s \in C$ and assume $U(C)_t$ and $O(C)_t$ are already defined for $t <^C_{KB} s$. Let

$$V := \begin{cases} W & \text{if } s \text{ is } <^C_{KB}\text{-minimal} \\ U(C)_{\widetilde{s}} & \text{if } s \text{ is the successor of } \widetilde{s} \\ \bigcup_{m\in\omega} U(C)^{\trianglelefteq \Sigma s^\frown\langle m\rangle}_{s^\frown\langle m\rangle} & \text{if } s \text{ is limit element} \end{cases} \tag{3.12}$$

We claim that in the limit case

$$V \subset U(C)_{s^\frown\langle i\rangle} \text{ for all } i \in \omega. \tag{3.13}$$

Assume $x \in U(C)^{\trianglelefteq \Sigma s^\frown\langle m\rangle}_{s^\frown\langle m\rangle}$. Then $x \in U(C)_{s^\frown\langle i\rangle}$ for all $i \leq m$ by 3. of the induction hypothesis. For $i > m$ this follows from 4. since

$$\min\{\Sigma t \mid s^\frown\langle m\rangle \leq^C_{KB} t \leq^C_{KB} s^\frown\langle i\rangle\} = \Sigma s^\frown\langle m\rangle$$

which implies

$$U(C)^{\trianglelefteq \Sigma s^\frown\langle m\rangle}_{s^\frown\langle m\rangle} = U(C)^{\trianglelefteq \Sigma s^\frown\langle m\rangle}_{s^\frown\langle i\rangle}.$$

This proves (3.13).

By 2. of the induction hypothesis we get

$$C \restriction s^\frown\langle m\rangle =_{U(C)_{s^\frown\langle m\rangle}} O(C)_{s^\frown\langle m\rangle},$$

therefore by lemma 3.4.19 we obtain an infinite set $U(C)_s$ and $O(C)_s$ such that

$$V^{\trianglelefteq \Sigma s} \subset_b U(C)_s \subset V \text{ and } C \restriction s =_{U(C)_s} O(C)_s.$$

This shows 1. and 2., and 3. follows directly from (3.12) and $U(C)_s \subset V$ in the successor case and from (3.13) and $U(C)_s \subset V$ in the limit case.

It remains to show 4. If s is the successor of $\widetilde{s}$ we have by induction hypothesis for each $t \leq^C_{KB} \widetilde{s}$

$$U(C)_t^{\trianglelefteq \min\{\Sigma u \mid t \leq^C_{KB} u \leq^C_{KB} \widetilde{s}\}} = V^{\trianglelefteq \min\{\Sigma u \mid t \leq^C_{KB} u \leq^C_{KB} \widetilde{s}\}}.$$

Because $\widetilde{s} <^C_{KB} s$ implies

$$\min\{\Sigma u \mid t \leq^C_{KB} u \leq^C_{KB} s\} \leq \min\{\Sigma u \mid t \leq^C_{KB} u \leq^C_{KB} \widetilde{s}\}$$

we obtain with $V^{\trianglelefteq \Sigma s} \subset_b U(C)_s$

$$\begin{aligned} U(C)_t^{\trianglelefteq \min\{\Sigma u \mid t \leq^C_{KB} u \leq^C_{KB} s\}} &= V^{\trianglelefteq \min\{\Sigma u \mid t \leq^C_{KB} u \leq^C_{KB} s\}} \\ &= U(C)_s^{\trianglelefteq \min\{\Sigma u \mid t \leq^C_{KB} u \leq^C_{KB} s\}}. \end{aligned}$$

In the limit case we first observe that $U(C)^{\trianglelefteq\Sigma s}_{s^\frown\langle m\rangle}$ does not depend on m by induction hypothesis 4., hence we get by definition of V

$$V^{\trianglelefteq\Sigma s} = U(C)^{\trianglelefteq\Sigma s}_{s^\frown\langle m\rangle} \text{ for all } m. \tag{3.14}$$

Now assume $t <^C_{KB} s$. Since s is limit there is an m such that $t <^C_{KB} s^\frown\langle m\rangle$. The induction hypothesis yields

$$U(C)_t^{\trianglelefteq\min\{\Sigma u \mid t\leq^C_{KB}u\leq^C_{KB}s^\frown\langle m\rangle\}} = U(C)_{s^\frown\langle m\rangle}^{\trianglelefteq\min\{\Sigma u \mid t\leq^C_{KB}u\leq^C_{KB}s^\frown\langle m\rangle\}}, \tag{3.15}$$

hence

$$\begin{aligned}
&U(C)_t^{\trianglelefteq\min\{\Sigma u \mid t\leq^C_{KB}u\leq^C_{KB}s\}} \\
=&U(C)_{s^\frown\langle m\rangle}^{\trianglelefteq\min\{\Sigma u \mid t\leq^C_{KB}u\leq^C_{KB}s\}} && \text{(by (3.15) and } \min\{\cdots s\} \leq \min\{\cdots s^\frown\langle m\rangle\}) \\
=&V^{\trianglelefteq\min\{\Sigma u \mid t\leq^C_{KB}u\leq^C_{KB}s\}} && \text{(by (3.14) and } \min\{\cdots\} \leq \Sigma s) \\
=&U(C)_s^{\trianglelefteq\min\{\Sigma u \mid t\leq^C_{KB}u\leq^C_{KB}s\}} && \text{(by } V^{\trianglelefteq\Sigma s} \subset_b U(C)_s).
\end{aligned}$$

This together with induction hypothesis for 4. implies 4.

Because

$$(\forall s \in C) C \restriction s =_{U(C)_s} O(C)_s$$

is a $\Pi^1_1(\mathcal{L}_\sigma)$-formula we need for its proof by transfinite induction on $<^C_{KB}$ the transfinite induction scheme of the σ^+-calculus. □

Lemma 3.4.21 together with the proof of Ramseyness of open sets (theorem 1.2.1) yield directly theorem 3.4.7. □

Lemma 3.4.22. *For each first order $\mathcal{L}_\sigma$-formula $\varphi(X, \vec{y}, \vec{Y})$ there is a first order $\mathcal{L}_2$-formula $\varphi^*(x, \vec{y}, \vec{Y})$ and an $n_\varphi \in \omega$ such that the σ-calculus proves*

- $\forall\vec{y}\forall\vec{Y}\{x \mid \varphi^*(x, \vec{y}, \vec{Y})\}$ *is an n_φ-code*
- $\forall\vec{y}\forall\vec{Y}\forall X(\varphi(X, \vec{y}, \vec{Y}) \leftrightarrow X \in^{n_\varphi} \{x \mid \varphi^*(x, \vec{y}, \vec{Y})\})$.

Proof. By proposition 3.7 in [Möl02] there is an n_φ such that φ is equivalent to an $\mathcal{L}_{\eth_{n_\varphi}}$-formula $\tilde{\varphi}$. By induction on $\tilde{\varphi}$ we can find an $\mathcal{L}_2$-formula φ^* which describes the n_φ-code belonging to $\tilde{\varphi}$. □

Definition 3.4.23. Let $(\cdot)^\sigma : \mathcal{L}_{RI} \to \mathcal{L}_\sigma$ such that

$$(RX\varphi(X)(U_\varphi))^\sigma := H^{n_\varphi}(\langle\rangle, U_\varphi, \{x \mid (\varphi^\sigma)^*(x)\})$$

and $(\cdot)^\sigma$ commutes with all logical connectives and quantifiers. The set terms $IxyX\psi(x, y, X)$ (which code an ω-iteration of ψ-comprehension) are translated into some appropriate fixed point.

Lemma 3.4.24. $(\cdot)^\sigma$ *is an embedding from the RI-calculus into the* σ^+*-calculus, i.e.* $(\varphi)^\sigma$ *is provable in the* σ^+*-calculus for each axiom* φ *of the RI-calculus.*

Proof. In the case of a crucial axiom we have to show

$$\forall^\infty U_\varphi[H^{n_\varphi}(\langle\rangle, U_\varphi, \{x \mid (\varphi^\sigma)^*(x)\}) \subset U_\varphi] \tag{3.16}$$

and

$$\begin{aligned}\forall^\infty U_\varphi[&\forall^\infty Y(Y \subset H^{n_\varphi}(\langle\rangle, U_\varphi, \{x \mid (\varphi^\sigma)^*(x)\}) \to \varphi^\sigma(Y))\\ &\vee\forall^\infty Y(Y \subset H^{n_\varphi}(\langle\rangle, U_\varphi, \{x \mid (\varphi^\sigma)^*(x)\}) \to \neg\varphi^\sigma(Y))]\end{aligned} \tag{3.17}$$

for each $\mathcal{L}_\sigma$-formula φ. Theorem 3.4.7 yields (3.16) and

$$\begin{aligned}&\forall^\infty Y(Y \subset H^{n_\varphi}(\langle\rangle, U_\varphi, \{x \mid (\varphi^\sigma)^*(x)\}) \to Y \in^n \{x \mid (\varphi^\sigma)^*(x)\})\\ \vee&\forall^\infty Y(Y \subset H^{n_\varphi}(\langle\rangle, U_\varphi, \{x \mid (\varphi^\sigma)^*(x)\}) \to Y \notin^n \{x \mid (\varphi^\sigma)^*(x)\}),\end{aligned}$$

and together with 3.4.22 we obtain (3.17). □

Theorem 3.4.25.

$$R\text{-}calculus \leq_{\mathcal{L}_2} RI\text{-}calculus \leq_{\mathcal{L}_2} \sigma^+\text{-}calculus =_{\Pi^1_1} \sigma\text{-}calculus$$

and

$$R\text{-}calculus \leq_{\mathcal{L}_2} \Pi^1_2\text{-CA}_0$$

Proof. The first claim holds by lemma 1.3.7, the second by lemma 3.4.24 and the third by theorem 2.4.10. The last claim follows with lemma 2.4.2. □

4. THE REVERSAL

Definition 4.1. For a set B of natural numbers let $B(n)$ be the n-th element of B if B is ordered increasingly. For a first order formula φ let

$$(\forall^1 x \leq B)\varphi(x) :\equiv \exists k \exists f \forall n[f(n) \leq B^{\trianglerighteq k}(n) \wedge \varphi(\langle f(0), \ldots, f(n-1)\rangle)].$$

In the presence of König´s lemma (which is provable in ACA_0, see [Sim99], theorem III.7.2) $(\forall^1 x \leq B)\varphi(x)$ is equivalent to the first order formula

$$(\exists k)(\exists^\infty s \in Seq)(\forall n < lh(s))[(s)_n \leq B^{\trianglerighteq k}(n) \wedge \varphi(\langle\rangle) \wedge \varphi(\langle s_0, \ldots, s_n\rangle)].$$

Let $(\exists^1 x \leq B)\varphi(x) :\equiv \neg(\forall^1 x \leq B)\neg\varphi(x)$. We iterate the bounded quantifiers in the same way as the unbounded quantifiers.

Definition 4.2 (bounded generalized quantifiers)**.** For $n \geq 2$ let

$$(\exists^n x \leq B)\varphi(x) :\equiv$$
$$\forall X[\forall x[(\varphi(x) \vee (\forall^{n-1} y \leq B)x^\frown\langle y\rangle \in X) \to x \in X] \to \langle\rangle \in X]$$

and

$$(\forall^n x \leq B)\varphi(x) :\equiv$$
$$\exists X[\langle\rangle \in X \wedge \forall x(x \in X \to (\varphi(x) \wedge (\exists^{n-1} y \leq B)x^\frown\langle y\rangle \in X))].$$

By applying $\forall X\psi(X) \leftrightarrow \forall X\psi(X^c)$ to the definition of $(\exists^n x \leq B)\varphi(x)$ we immediately obtain by induction on n

$$(\forall^n x \leq B)\varphi(x) \leftrightarrow \neg(\exists^n x \leq B)\neg\varphi(x).$$

The quantifiers $\exists^n x \leq B$ and $\forall^n x \leq B$ are can be expressed without second order quantifiers in $\mathcal{L}_{\Game_{n-1}}$. This follows by induction on n using lemma 2.3.9 in the induction step.

Theorem 4.3. *There exists an embedding* $^* : \mathcal{L}_\Game \to \mathcal{L}_R$ *which is the identity on* $\mathcal{L}_2$ *such that we have for all* $\mathcal{L}_\Game$*-formulas* φ*: If* φ *is provable in* $\Game$ame *then* φ^* *is provable in the R-calculus.*

Proof. The idea of the proof is as follows. We show by metainduction on n that the R-calculus proves the translations of all defining axioms of $\forall^n$ and $\exists^n$. Because $\forall^n$ has different monotonicity properties for odd and even n (see lemma 4.5 and lemma 4.7) we have to distinguish two cases. If n is odd, we first show

$$\exists H\big(\forall\vec{y}[\forall^n x\varphi(x,\vec{y}) \leftrightarrow (\forall^n x \leq H)\varphi(x,\vec{y})]\big). \tag{4.1}$$

The direction from right to left is true for each H by some monotonicity property, see lemma 4.7. For the other direction, notice that $(\forall^n x \leq H)\varphi(x,\vec{y})$ becomes "more true" if H is replaced by a "thinner" set (see again lemma 4.7). This implies that if there exists a set H such that $(\forall^n x \leq H)\varphi(x,\vec{y})$ is true, then there is no rejecting homogeneous set for $(\forall^n x \leq X)\varphi(x,\vec{y})$, because it would have a subset which is thinner than H. Therefore if there exists an H with $(\forall^n x \leq H)\varphi(x,\vec{y})$ then each set which is homogeneous for $(\forall^n x \leq X)\varphi(x,\vec{y})$ has to be an accepting homogeneous set. Since

$$\forall\vec{y}\big(\forall^n x\varphi(x,\vec{y}) \rightarrow \exists H(\forall^n x \leq H)\varphi(x,\vec{y})\big) \tag{4.2}$$

(see lemma 4.4) we can show that for each set H which is homogeneous for $(\forall^n x \leq X)\varphi(x,\vec{y})$ for all $\vec{y}$ it holds

$$\forall^n x\varphi(x,\vec{y}) \rightarrow (\forall^n x \leq H)\varphi(x,\vec{y}).$$

Because a change of some initial segment of H does not alter the truth of $(\forall^n x \leq H)\varphi(x,\vec{y})$ it even suffices to find a set H such that for each $\vec{y}$, we can remove an initial segment of H such that the remaining set is homogeneous for $(\forall^n x \leq X)\varphi(x,\vec{y})$, and this set exists in the R-calculus. Hence we have shown 4.1. Since $(\forall^n x \leq H)\varphi(x,\vec{y})$ is a formula of less complexity than $\forall^n x\varphi(x,\vec{y})$ and because at this stage of our induction we have already proved comprehension for $(\forall^n x \leq H)\varphi(x,\vec{y})$, we have shown comprehension for $\forall^n x\varphi(x,\vec{y})$.

If n is even, we have the inverse monotonicity properties as in the odd case, i.e. $(\exists^n x \leq H)\varphi(x)$ becomes "more true" if H is replaced by a subset of H. It suffices to give a set term H such that

$$\exists^n x\varphi(x) \leftrightarrow (\exists^n x \leq H)\varphi(x);$$

as in the previous case H has to be independent of the free number variables in φ which we suppress here. The direction from right to left holds independently from the choice of H by a monotonicity property, see lemma 4.7. For the other direction, it is sufficient to choose H such that

$$\{m \mid (\exists^n y \leq H)\varphi(m^\frown y)\}$$

is closed under the operator

$$\varphi(z) \vee (\forall^{n-1}x) z^\frown \langle x \rangle \in Z$$

which generates the fixed point belonging to $\exists^n y \varphi(y)$. This is sufficient because then $(\exists^n y)\varphi(y)$ implies that $\langle\rangle$ is in this fixed point which is contained in $\{m \mid (\exists^n y \leq H)\varphi(m^\frown y)\}$, hence $(\exists^n y \leq H)\varphi(y)$. Define

$$H := RmX[(\exists^n x \leq X)\varphi(m^\frown x)].$$

We assume

$$\varphi(m) \vee (\forall^{n-1}x)(\exists^n y \leq H)\varphi(m^\frown \langle x \rangle^\frown y)$$

and have to show

$$(\exists^n y \leq H)\varphi(m^\frown y).$$

In the case that $\varphi(m)$ holds this follows from the definition of $\exists^n x \leq B$. In the case of

$$(\forall^{n-1}x)(\exists^n y \leq H)\varphi(m^\frown \langle x \rangle^\frown y)$$

by (4.2) we find a bound Z (which depends on m) such that

$$(\forall^{n-1}x \leq Z)(\exists^n y \leq H)\varphi(m^\frown \langle x \rangle^\frown y).$$

For a $\widetilde{Z}$ with $Z \leq \widetilde{Z}$ and $H \leq \widetilde{Z}$ we obtain by monotonicity

$$(\forall^{n-1}x \leq \widetilde{Z})(\exists^n y \leq \widetilde{Z})\varphi(m^\frown \langle x \rangle^\frown y),$$

hence

$$(\exists^n y \leq \widetilde{Z})\varphi(m^\frown y).$$

This implies that for no m, H is rejecting because we could always find a subset majorizing $\widetilde{Z}$. Therefore H has to be accetping for all m, hence $(\exists^n y \leq H)\varphi(m^\frown y)$.

Now let us carry out the proof in detail. Let * commute with all logical connectives such that

$$(\forall^n x \varphi(x, \vec{y}, \vec{Y}))^* :\equiv \forall^n x \leq R^n_\varphi(\vec{Y})\varphi^*(x, \vec{y}, \vec{Y})$$
$$\text{with } R^n_\varphi(\vec{Y}) := R\vec{y}X[(\forall^n x \leq X)\varphi^*(x, \vec{y}, \vec{Y})]$$

if n is odd and

$$(\exists^n x\varphi(x,\vec{y},\vec{Y}))^* :\equiv \exists^n x \le R^n_\varphi(\vec{Y})\varphi^*(x,\vec{y},\vec{Y})$$
$$\text{with } R^n_\varphi(\vec{Y}) := R\vec{y}X[\exists^n x \le X\varphi^*(x,\vec{y},\vec{Y})]$$

if n is even.

We prove by metainduction on n that the translations of the defining axioms for $\forall^n$ and $\exists^n$ are provable in the R-calculus. Let us assume that this is proved for $n-1$ with n odd. To see that the definitions of $(\forall^n x\varphi(x,\vec{y},\vec{Y}))^*$ and $(\exists^n x\varphi(x,\vec{y},\vec{Y}))^*$ make sense we have to check that the formulas occuring inside the R-terms are first order. Therefore we have to prove that $\mathcal{L}_R$-formulas of the form $(\exists^n x \le H)\varphi(x)$, which are formulas with second order quantifiers by definition 4.2, are equivalent to formulas without second order quantifiers. For $n=1$ this is the case since $(\forall^1 x \le H)\varphi(x)$ is equivalent to

$$(\exists^\infty s \in Seq)(\forall n < lh(s))[(s)_n \le H(n) \wedge \varphi(\langle\rangle) \wedge \varphi(\langle (s)_0,\dots,(s)_n\rangle)]$$

and $(\exists^1 x \le H)\varphi(x)$ is $\neg(\forall^1 x \le H)\neg\varphi(x)$. Let

$$\psi^{n-1}(x,X,Y,H) :\equiv x \in Y \vee (\forall^{n-1} z \le H)x^\frown\langle z\rangle \in X.$$

Since ψ^{n-1} is an $\mathcal{L}_{\Game_{n-2}}$-formula there exists by lemma 2.3.9 an $\mathcal{L}_{\Game_{n-1}}$-formula $\widetilde{\psi^{n-1}}$ such that $\Game\mathrm{ame}_{\text{n-1}}$ proves

$$(\forall Y,H)\,\mathrm{LFP}(\{x \mid \widetilde{\psi^{n-1}}(x,Y,H)\},\psi^{n-1}). \tag{4.3}$$

By induction hypothesis the *-translation

$$(\forall Y,H)\,\mathrm{LFP}(\{x \mid \widetilde{\psi^{n-1}}^*(x,Y,H)\},\psi^{n-1*}) \tag{4.4}$$

is provable in the R-calculus. (4.3) implies that

$$\forall Y[(\exists^n x \le H)(x \in Y) \leftrightarrow \widetilde{\psi^{n-1}}(\langle\rangle,Y,H)] \tag{4.5}$$

is provable in $\Game\mathrm{ame}_{\text{n-1}}$ and (4.4) implies that

$$(\exists^n x \le H)\varphi(x) \leftrightarrow \widetilde{\psi^{n-1}}^*(\langle\rangle,\{x \mid \varphi(x)\},H) \tag{4.6}$$

is provable in the R-calculus for each $\mathcal{L}_R$-formula φ. Therefore the R-calculus can talk about the quantifiers $\forall^n x \le X$ and $\exists^n x \le X$ without using second order quantifiers. (4.5) and (4.6) also imply that the R-calculus proves

$$(\exists^n x \le H)\varphi^*(x) \leftrightarrow (\exists^n x \le H\varphi(x))^* \tag{4.7}$$

for each $\mathcal{L}_\Game$-formula φ.

Let $\forall^n\varphi(x)$ be an abbreviation for the formula

$$\exists X[\langle\rangle \in X \wedge \forall x(x \in X \to (\varphi(x) \wedge (\exists^{n-1} y)x^\frown\langle y\rangle \in X))]. \tag{4.8}$$

Lemma 4.4. *Let n be an odd natural number. For each $\mathcal{L}_{\supset}$-formula φ the R-calculus proves*

$$\forall^n x\varphi^*(x,\vec{y},\vec{Y}) \to \exists^\infty H(\forall^n x \le H)\varphi^*(x,\vec{y},\vec{Y}),$$

where $\exists^\infty H$ means "there exists an infinite set H".

Proof. For $n = 1$ take

$$H := \{y \mid (\exists x \in X \cap Seq)(\exists n) \\ y = (x)_n \wedge (\forall z \in Seq)((z <_{KB} x \wedge lh(z) = n) \to y \notin X)\},$$

where X is the set which exists by (4.8) for φ^* instead of φ. Then H is the leftmost path through X, and $f(n) := H(n)$ satisfies the definition of $\forall^1 x \le H$. For $n > 1$ we need the following induction hypothesis (which is shown in the case "n even" in lemma 4.10): For each $\mathcal{L}_{\supset}$-formula $\psi(m,x,X)$ we have

$$\forall X\exists^\infty H\forall m[\exists^{n-1}x\psi^*(m,x,X) \leftrightarrow \exists^{n-1}x \le H\psi^*(m,x,X)]. \tag{4.9}$$

Let $\psi(m,x,X) := m^\frown\langle x\rangle \in X$. By definition of $\forall^n x\varphi^*(x)$ there is an X such that

$$\langle\rangle \in X \wedge \forall x(x \in X \to (\varphi^*(x) \wedge (\exists^{n-1}y)x^\frown\langle y\rangle \in X)).$$

For this X there exists an infinite H according to (4.9), and we obtain

$$\langle\rangle \in X \wedge \forall x(x \in X \to (\varphi^*(x) \wedge (\exists^{n-1}y \le H)x^\frown\langle y\rangle \in X)),$$

which is the definition of $(\forall^n x \le H)\varphi^*(x)$. □

Lemma 4.5 (monotonicity for odd n)**.** *For each $\mathcal{L}_R$-formula φ the R-calculus proves*

$$[(\forall^n x \le H)\varphi(x) \wedge H' \ge H] \to (\forall^n x \le H')\varphi(x)$$

and

$$(\forall^n x \le H)\varphi(x) \to \forall^n x\varphi(x).$$

Furthermore, we have

$$\forall H\forall H'\forall i\forall j[H^{\trianglerighteq i} = H'^{\trianglerighteq j} \to ((\forall^n x \le H)\varphi(x) \leftrightarrow (\forall^n x \le H')\varphi(x))].$$

Proof. This follows directly from the definitions in the case $n = 1$, and in the case $n > 1$ from the monotonicity lemma in the case $n - 1$ (lemma 4.7). □

Lemma 4.6. *Let n be odd. The R-calculus proves all *-translations of the defining axioms for $\forall^n$, i.e. for each $\mathcal{L}_{\supset}$-formula φ the R-calculus proves*

$$\forall\vec{Y}\forall\vec{y}[\forall^n x\varphi^*(x,\vec{y},\vec{Y}) \leftrightarrow (\forall^n x\varphi(x,\vec{y},\vec{Y}))^*].$$

Proof. Because of the second claim of the monotonicity lemma we only have to show the direction from left to right. Assume $\forall^n x\varphi^*(x,\vec{y},\vec{Y})$. By lemma 4.4 this implies

$$(\exists^\infty H)(\forall^n x \leq H)\varphi^*(x,\vec{y},\vec{Y}).$$

Since for each H and each k there is a $X \subset R^n_\varphi(\vec{Y})^{\trianglerighteq k}$ such that $H \leq X$ we obtain by the monotonicity lemma

$$(\forall k)(\exists^\infty X \subset R^n_\varphi(\vec{Y})^{\trianglerighteq k})(\forall^n x \leq X)\varphi^*(x,\vec{y},\vec{Y}).$$

By the definition of $R^n_\varphi(\vec{Y})$ and the main axiom of the R-calculus we obtain

$$(\exists k)(\forall^\infty X \subset R^n_\varphi(\vec{Y})^{\trianglerighteq k})(\forall^n x \leq X)\varphi^*(x,\vec{y},\vec{Y}),$$

which implies $(\exists k)(\forall^n x \leq R^n_\varphi(\vec{Y})^{\trianglerighteq k})\varphi^*(x,\vec{y},\vec{Y})$. Since the truth of any formula $(\forall^n x \leq B)\psi(x)$ does not change if we change an initial segment of B (see 4.5) we obtain $(\forall^n x \leq R^n_\varphi(\vec{Y}))\varphi^*(x,\vec{y},\vec{Y})$ which is $(\forall^n x\varphi(x,\vec{y},\vec{Y}))^*$. □

This finishes the induction step from $n-1$ to n if n is odd. Let us now consider the case that n is even.

Lemma 4.7 (monotonicity for even n)**.** *For each $\mathcal{L}_R$-formula φ the R-calculus proves*

$$[(\exists^n x \leq H)\varphi(x) \wedge H' \geq H] \rightarrow (\exists^n x \leq H')\varphi(x)$$

and

$$(\exists^n x \leq H)\varphi(x) \rightarrow \exists^n x\varphi(x).$$

Furthermore, we have

$$\forall H\forall H'\forall i\forall j[H^{\trianglerighteq i} = H'^{\trianglerighteq j} \rightarrow ((\exists^n x \leq H)\varphi(x) \leftrightarrow (\exists^n x \leq H')\varphi(x))].$$

Proof. This follows from monotonicity lemma 4.5 in the case $n-1$. □

Lemma 4.8. *For each $\mathcal{L}_R$-formula φ the R-calculus proves*

$$\varphi(\langle\rangle) \vee (\forall^{n-1} z \leq Z)(\exists^n x \leq Z)\varphi(\langle z\rangle^\frown x) \rightarrow (\exists^n x \leq Z)\varphi(x).$$

Proof. $\varphi(\langle\rangle) \vee (\forall^{n-1} z \leq Z)(\exists^n x \leq Z)\varphi(\langle z\rangle^\frown x)$ implies

$$\varphi(\langle\rangle) \vee (\forall^{n-1} z \leq Z)\widetilde{\psi^{n-1}}^*(\langle\rangle, \{x \mid \varphi(\langle z\rangle^\frown x)\}, Z) \tag{4.10}$$

by (4.6). We want to apply lemma 2.3.4 for

- $f(x) := \langle z\rangle^\frown x$
- $\varphi(x, X) := \psi^{n-1*}(x, X, \{y \mid \varphi(y)\}, Z)$
- $\psi(x, X) := \psi^{n-1*}(x, X, \{y \mid \varphi(\langle z\rangle^\frown y)\}, Z)$
- $I_\varphi := \{x \mid \widetilde{\psi^{n-1}}^*(x, \{y \mid \varphi(y)\}, Z)\}$
- $I_\psi := \{x \mid \widetilde{\psi^{n-1}}^*(x, \{y \mid \varphi(\langle z\rangle^\frown y)\}, Z)\}$.

The premises $\mathrm{LFP}(I_\varphi, \varphi)$ and $\mathrm{LFP}(I_\psi, \psi)$ hold by (4.4), hence the lemma delivers

$$\langle\rangle \in I_\psi \to f(\langle\rangle) \in I_\varphi.$$

Together with (4.10) we obtain

$$\varphi(\langle\rangle) \vee (\forall^{n-1} z \leq Z)\widetilde{\psi^{n-1}}^*(\langle z\rangle, \{x \mid \varphi(x)\}, Z).$$

This implies

$$\psi^{n-1*}\big(\langle\rangle, \{s \mid \widetilde{\psi^{n-1}}^*(s, \{x \mid \varphi(x)\}, Z)\}, \{x \mid \varphi(x)\}, Z\big)$$

by the *-translation of the definition of ψ^{n-1} together with (4.7). The first fixed point axiom for ψ^{n-1*} (which we have by lemma (4.4)) implies

$$\langle\rangle \in \{s \mid \widetilde{\psi^{n-1}}^*(s, \{x \mid \varphi(x)\}, Z)\}$$

which is $(\exists^n x \leq Z)\varphi(x)$ by (4.6). □

Lemma 4.9. *Let n be even. The R-calculus proves*

$$\mathrm{LFP}(\{s \mid (\exists^n x \varphi(s^\frown x, \vec{y}, \vec{Y}))^*\}, \chi^*)$$

for $\chi(x, \vec{y}, X, \vec{Y}) :\equiv \varphi(x, \vec{y}, \vec{Y}) \vee (\forall^{n-1} z) x^\frown\langle z\rangle \in X$.

Proof. We start with the first fixed point axiom, so assume

$$\chi^*(m, \{s \mid (\exists^n x \varphi(s^\frown x, \vec{y}, \vec{Y}))^*\}),$$

i.e.

$$\varphi^*(m, \vec{y}, \vec{Y}) \vee \big((\forall^{n-1} z)(\exists^n x)\varphi(m^\frown \langle z \rangle^\frown x, \vec{y}, \vec{Y})\big)^*,$$

hence by induction hypothesis (see lemma 4.6)

$$\varphi^*(m, \vec{y}, \vec{Y}) \vee (\forall^{n-1} z)((\exists^n x)\varphi(m^\frown \langle z \rangle^\frown x, \vec{y}, \vec{Y}))^*.$$

By lemma 4.4 this implies

$$\varphi^*(m, \vec{y}, \vec{Y}) \vee (\exists^\infty X)(\forall^{n-1} z \leq X)(\exists^n x \varphi(m^\frown \langle z \rangle^\frown x, \vec{y}, \vec{Y}))^*.$$

With the definition of * we get

$$\varphi^*(m, \vec{y}, \vec{Y}) \vee (\exists^\infty X)(\forall^{n-1} z \leq X)(\exists^n x \leq R(\vec{Y}))\varphi^*(m^\frown \langle z \rangle^\frown x, \vec{y}, \vec{Y})$$

for an appropriate $\mathcal{L}_R$-term R and by monotonicity we obtain

$$\varphi^*(m, \vec{y}, \vec{Y}) \vee (\exists^\infty Z)(\forall^{n-1} z \leq Z)(\exists^n x \leq Z)\varphi^*(m^\frown \langle z \rangle^\frown x, \vec{y}, \vec{Y})$$

which implies

$$(\exists^\infty Z)(\exists^n x \leq Z)\varphi^*(m^\frown x, \vec{y}, \vec{Y}) \tag{4.11}$$

by lemma 4.8. R^n_φ is homogeneous for $(\exists^n x \leq Z)\varphi^*(m^\frown x, \vec{y}, \vec{Y})$ and can not avoid it by (4.11), therefore we obtain

$$(\exists i)(\exists^n x \leq R^n_\varphi(\vec{Y})^{\trianglerighteq i})\varphi^*(m^\frown x, \vec{y}, \vec{Y})$$

which implies

$$(\exists^n x \leq R^n_\varphi(\vec{Y}))\varphi^*(m^\frown x, \vec{y}, \vec{Y})$$

because $\exists^n x \leq B$ is independent from initial segments of B (see 4.7). Hence we obtain

$$(\exists^n x \varphi(m^\frown x, \vec{y}, \vec{Y}))^*.$$

For the second fixed point axiom assume that I is a fixed point of χ^*, i.e.

$$\forall x[\chi^*(x, I) \to x \in I]. \tag{4.12}$$

For all x we have

$$\begin{aligned}
&\varphi^*(x,\vec{y},\vec{Y}) \vee (\forall^{n-1} y \leq R^n_\varphi(\vec{Y})) x^\frown\langle y\rangle \in I\\
\to&\varphi^*(x,\vec{y},\vec{Y}) \vee \forall^{n-1} y (x^\frown\langle y\rangle \in I)^* \text{ by monotonicity lemma 4.5}\\
\to&\varphi^*(x,\vec{y},\vec{Y}) \vee (\forall^{n-1} y (x^\frown\langle y\rangle \in I))^* \text{ by induction hypothesis (lemma 4.6)}\\
\to&\chi^*(x,I) \text{ by definition of } \chi\\
\to&x \in I \text{ by (4.12).}
\end{aligned} \tag{4.13}$$

Therefore I is a fixed point of $\psi^{n-1*}(x, X, \{x \mid \varphi^*(x,\vec{y},\vec{Y})\}, R^n_\varphi(\vec{Y}))$. This implies

$$\begin{aligned}
&\{m \mid \exists^n x \leq R^n_\varphi(\vec{Y})\varphi^*(m^\frown x,\vec{y},\vec{Y})\}\\
=&\{m \mid \widetilde{\psi^{n-1}}^*(\langle\rangle, \{x \mid \varphi^*(m^\frown x,\vec{y},\vec{Y})\}, R^n_\varphi(\vec{Y}))\} \text{ (by 4.6)}\\
\subset&\{m \mid \widetilde{\psi^{n-1}}^*(m, \{x \mid \varphi^*(x,\vec{y},\vec{Y})\}, R^n_\varphi(\vec{Y}))\} \text{ (with lemma 2.3.4 like}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in the proof of lemma 4.8)}\\
\subset&I \text{ (since } I \text{ is a fixed point of } \psi^{n-1*} \text{ and (4.4)).}
\end{aligned}$$

□

Lemma 4.10. *Let n be even. The R-calculus proves all *-translations of the defining axioms for $\exists^n$, i.e. for each $\mathcal{L}_\partial$-formula φ the R-calculus proves*

$$\forall\vec{Y}\forall\vec{y}\big(\exists^n x\varphi^*(x,\vec{y},\vec{Y}) \leftrightarrow (\exists^n x\varphi(x,\vec{y},\vec{Y}))^*\big).$$

Proof. This follows directly from the last lemma. □

Lemma 4.10 and lemma 4.6 finish the induction step in the proof of theorem 4.3. □

Theorem 4.11.

- ∂ame $\leq_{\mathcal{L}_2}$ *R-calculus*
- *σ-calculus* $\leq_{\mathcal{L}_2}$ *R-calculus*

Proof. The first claim follows directly from theorem 4.3, the second follows with

$$\sigma\text{-calculus} =_{\mathcal{L}_2} \partial\text{ame}$$

which is proved in [Möl02], theorem 10.6. □

Proof of theorem 1.3.4. Corollary 4.11 and corollary 3.4.25 imply

$$\sigma\text{-calculus} =_{\Pi^1_1} R\text{-calculus.}$$

Together with Möllerfeld´s result Π^1_2-$CA_0 =_{\Pi^1_1} \sigma$-calculus (see [Möl02], theorem 10.6) the proof is finished. □

5. CONSEQUENCES

5.1 *A consequence for encodeability of sets definable in the σ-calculus*

Definition 5.1.1. A set M of natural numbers is called recursively (hyperarithmetically) encodable iff for every infinite set $Z \subset \omega$ there is an infinite $X \subset Z$ such that M is recursive (hyperarithmetical) in X.

Jockusch showed in [Joc68] that every hyperarithmetic set is recursively encodable and Solovay in [Sol78] showed that each recursive encodable set is hyperarithmetic. If σ is the least Σ^1_1-reflecting ordinal then every set in $L_\sigma \cap \mathcal{P}(\omega)$ is hyperarithmetically encodable and every hyperarithmetical encodable set lies in L_σ. This is also due to Solovay in [Sol78].

Definition 5.1.2. A set M of natural numbers is ∂_n-encodable iff there is a $\mathcal{L}_{\partial_n}$-formula $\varphi(x, X)$ such that for each infinite $Z \subset \omega$ there is an infinite $X \subset Z$ such that $M = \{x \mid \varphi(x, X)\}$.

Corollary 5.1.3. *The σ^+-calculus proves that each first order $\mathcal{L}_{\partial_n}$-definable set of natural numbers M is ∂_{n-1}-encodable; moreover, for each first order $\mathcal{L}_{\partial_n}$-formula $\varphi(\vec{y}, \vec{Y})$ there exists uniformly in φ an $\mathcal{L}_\sigma$-term $T(\vec{Y})$ and an $\mathcal{L}_{\partial_{n-1}}$-formula $\psi(\vec{y}, Z, \vec{Y})$ such that the σ^+-calculus proves*

$$\forall \vec{Y} \forall \vec{y} (\forall Z \geq T(\vec{Y}))[\varphi(\vec{y}, \vec{Y}) \leftrightarrow \psi(\vec{y}, Z, \vec{Y})];$$

here $Z \geq T(\vec{Y})$ means that if we order Z and $T(\vec{Y})$ increasingly, then the n-th element of Z is greater or equal the n-th element of $T(\vec{Y})$ for all n.

Proof. We fix an enumeration ψ_i of all subformulas of φ which are of the form $Q_i z \chi_i(z, \vec{y}, \vec{Y})$ with $Q_i \in \{\forall^n, \exists^n\}$. By metainduction on j we prove that for each j there exists an $\mathcal{L}_\sigma$-term $T_j(\vec{Y})$ such that for each $i < j$ the σ^+-claculus proves

$$\forall \vec{Y} \forall \vec{y} (\forall Z \geq T_j(\vec{Y}))[Q_i z \chi_i(z, \vec{y}, \vec{Y}) \leftrightarrow (Q_i z \leq Z) \chi_i(z, \vec{y}, \vec{Y})]; \tag{5.1}$$

here $Q_i z \leq Z$ is the bounded quantifier from definition 4.2. We start with $T_0(\vec{Y}) := \omega$. Assume that $T_j(\vec{Y})$ is already constructed. By theorem 3.4.7

there is an $\mathcal{L}_\sigma$-term $\widetilde{T}(\vec{y},\vec{Y})$ which is a subset of $T_j(\vec{Y})$ and which is homogeneous for

$$\psi(\vec{y},Z,\vec{Y}) :\equiv (Q_j z \leq Z)\chi_j(z,\vec{y},\vec{Y}).$$

With an argument as in the proof of lemma 3.4.8 we obtain a term $T_{j+1}(\vec{Y})$ such that

$$\forall\vec{Y}\forall\vec{y}\exists m T_{j+1}(\vec{Y})^{>m} \geq \widetilde{T}(\vec{y},\vec{Y}).$$

Now it follows

$$\forall\vec{Y}\forall\vec{y}(\forall Z \subset T_{j+1}(\vec{Y}))[(Q_j z)\chi_j(z,\vec{y},\vec{Y}) \leftrightarrow (Q_j z \leq Z)\chi_j(z,\vec{y},\vec{Y})]$$

similar to the proof of lemma 4.6 in the case that $Q_i = \forall^n$ and n odd or $Q_i = \exists^n$ and n even and from the proof of lemma 4.10 in the other cases. Now 5.1 follows with the monotonicity lemmata 4.5 and 4.7.

We now obtain ψ from φ by replacing all generalized quantifiers Q_i by $Q_i \leq Z$. Let $T(\vec{Y}) := T_i(\vec{Y})$ if i is the number of generalized quantifiers occuring in φ. Now the claim follow directly from (5.1). □

5.2 *A consequence for the Baire property of sets definable in the σ-calculus*

Theorem 3.2.7 can also be applied to the property of Baire.

Corollary 5.2.1. ZFC *proves that each set with an n-code has the property of Baire.*

Proof. The sets of reals which have the property of Baire form a σ-algebra C which contains all open sets and the meager sets form an σ-ideal I. We have to show that I is ccc. For each $B \in C\backslash I$ let $i(B)$ let s be the least natural number such that $\mathcal{N}_s$ is comeager in B. If B_1 and B_2 are different elements of an antichain they can not have the same index i because otherwise $B_1 \cap B_2$ would be comeager in $\mathcal{N}_i$ and hence not in I. Since there are only countably many indices there is no uncountable antichain. Now the claim follows from 3.2.4. □

We will formalize this result in the σ^+-calculus with the methods of chapter 3.

Definition 5.2.2. Let X be an n-code of a set of real numbers. Y is called a Baire witness for X if

- $(Y)_0$ is a code of an open set
- $(Y)_1$ is code of a countable union of nowhere dense closed sets
- the symmetric difference of the sets coded by X and $(Y)_0$ is contained in the set coded by $(Y)_1$.

Lemma 5.2.3. *There exists an $\mathcal{L}_\sigma$-term $B_{bor}(X)$ such that the σ-calculus proves*

$$\forall X[X \textit{ Borel code } \rightarrow B_{bor}(X) \textit{ is a Baire witness for } X].$$

Proof. The usual prove that all Borel sets have the property of Baire is formalizable in the σ-calculus without any problems. □

Theorem 5.2.4. *For each n there exists an $\mathcal{L}_\sigma$-term $B^n(X)$ such that the σ^+-calculus shows: If X is n-code then $B^n(X)$ is a Baire witness for X.*

Proof. The proof is similar to the proof of theorem 3.4.7 by metainduction on n. We assume that the theorem is proved for n and first show the claim for a simple $n+1$-code X. In analogy to definition 3.4.11 we define

$$\begin{aligned}\langle s,x\rangle \in T(X,Y) :\equiv &(Y=\emptyset \wedge x \in C \restriction \langle s\rangle)\\ &\vee \exists i(max(i,Y) \wedge \langle i,\langle s,x\rangle\rangle \in X))\\ &\vee[lim(Y) \wedge x \in \forall^0 n\{\langle n,z\rangle \mid \exists y[\langle n,y\rangle \in cof(Y)\\ &\qquad\qquad \wedge \langle y,\langle s,z\rangle\rangle \in X]\}]\end{aligned}$$

As we have seen in the proof that the σ-ideal of the meager sets is ccc we can index a nonmeager set B by the least i such that B is comeager in $\mathcal{N}_i$, hence we define

$$\begin{aligned}\psi(j,X,Y) :\equiv\ & j \text{ is the least } \langle s,i\rangle \text{ such that}\\ &\langle 0,i\rangle \in B^n(T(X,Y)_s \backslash \exists^n k T(X,Y)_{s^\frown\langle k\rangle})\end{aligned}$$

and

$$\begin{aligned}\varphi(x,X,Y) :\equiv\ & \exists\langle s,i\rangle[\psi(\langle s,i\rangle,X,Y) \wedge \exists t,y[x=\langle t,y\rangle \wedge\\ &((s\neq t \wedge y \in T(X,Y)_s) \vee (s=t \wedge y \in T(X,Y)_t \cap \exists^n i T(X,Y)_{t^\frown\langle i\rangle}))]].\end{aligned}$$

Lemma 5.2.5. *Let (I,W) be a ψ-iteration of φ and $w=\langle s,i\rangle \in field(W)$. Then $\langle 0,i\rangle \notin B^n(T(I,W)_s)$.*

Proof. Since $\langle s,i\rangle \in field(W)$ we obtain from definition 3.3.3 of a ψ-iteration φ

$$\psi(\langle s,i\rangle, I^{<w}, W^{<w}).$$

By definition of φ this implies

$$\langle 0,i\rangle \in B^n(T(I^{<w},W^{<w})_s \backslash \exists^n k T(I^{<w},W^{<w})_{s^\frown\langle k\rangle}).$$

Because of

$$T(I^{\leq w},W^{\leq w})_s = T(I^{<w},W^{<w})_s \cap \exists^n k T(I^{<w},W^{<w})_{s^\frown\langle k\rangle})$$

the sets $T(I^{<w},W^{<w})_s \backslash \exists^n k T(I^{<w},W^{<w})_{s^\frown\langle k\rangle}$ and $T(I^{\leq w},W^{\leq w})_s$ are disjoint and we obtain

$$\langle 0,i\rangle \notin B^n(T(I^{\leq w},W^{\leq w})_s)$$

which yields the claim since $T(I^{\leq w},W^{\leq w})_s \supset T(I,W)_s$. □

Lemma 5.2.6. *ψ is suitable for φ.*

Proof. Assume that (I,W) is a ψ-iteration of φ such that $\psi(w,I,W)$ for a $w = \langle s,i\rangle \in field(W)$. By lemma 5.2.5 we obtain $\langle 0,i\rangle \notin B^n(T(I,W)_s)$ which entails

$$\langle 0,i\rangle \notin B^n(T(I,W)_s \backslash \exists^n k T(I,W)_{s^\frown\langle k\rangle}).$$

But this implies $\neg\psi(\langle s,i\rangle, I, W)$, contradiction. □

Let (I,W) be the maximal ψ-iteration of φ which exists by theorem 3.3.6.

Lemma 5.2.7. *There exists uniformly in I and W a code of a countable union of closed nowhere dense sets M such that*

$$\bigcup_s T(I,W)_s \backslash \exists^n k T(I,W)_{s^\frown\langle k\rangle} \subset M.$$

Proof. Since (I,W) is a maximal φ-iteration of ψ it holds

$$\forall s,i \langle 0,i\rangle \notin B^n(T(I,W)_s \cap \exists^n k T(I,W)_{s^\frown\langle k\rangle}),$$

i.e. the open set that belongs to $T(I,W)_s \backslash \exists^n k T(I,W)_{s^\frown\langle k\rangle}$ is the empty set for each n and we obtain

$$\bigcup_s T(I,W)_s \backslash \exists^n k T(I,W)_{s^\frown\langle k\rangle} \subset B^n(T(I,W)_s \backslash \exists^n k T(I,W)_{s^\frown\langle k\rangle})_1 =: M.$$

□

Lemma 5.2.8. *Let (I, W) be a φ-iteration of ψ and 0_W be the least element of the wellordering W. Then it holds for all X*

$$(\forall^{n+1}x)X \in^n ((I)_{0_W})_x \leftrightarrow (\forall^{n+1}x)X \in^n T(I, W)_x.$$

Proof. The proof is similar to that of lemma 3.4.17. □

Lemma 5.2.9. *It holds*

$$\forall X \notin^n M[X \in^n T(I, W)_{\langle\rangle} \leftrightarrow X \in^{n+1} (\forall^{n+1}x)T(I, W)_x].$$

Proof. The proof is similar to that of lemma 3.4.18. The direction from right to left follows again directly from the definition of $\forall^{n+1}$. For the other direction we obtain similarly to 3.4.18

$$T(I, W)_{\langle\rangle} \backslash (\forall^{n+1}x)T(I, W)_x \subset \bigcup_t T(I, W)_t \backslash \exists^n k T(I, W)_{t^\frown\langle k\rangle}.$$

Together with lemma 5.2.7 this yields the second direction. □

For a simple $n + 1$-code C we define $B^{n+1}(C)$ by

- $(B^{n+1}(C))_0 := T(I(C), W(C))_{\langle\rangle}$
- $(B^{n+1}(C))_1 := M \cup B^n(T(I(C), W(C))_{\langle\rangle})_1$.

Then we have for all X with $X \notin^n (B^{n+1}(C))_1$

$$\begin{aligned} & X \in^n (B^{n+1}(C))_0 \\ \leftrightarrow & X \in^n T(I, W)_{\langle\rangle} \\ \leftrightarrow & X \in^{n+1} (\forall^{n+1}x)T(I, W)_x \text{ by lemma 5.2.9} \\ \leftrightarrow & (\forall^{n+1}x)X \in^n ((I)_{0_W})_x \text{ by lemma 5.2.8} \\ \leftrightarrow & (\forall^{n+1}x)X \in^n C \restriction \langle x\rangle \\ \leftrightarrow & X \in^{n+1} C. \end{aligned}$$

This finishes the proof for simple $n+1$-codes. To prove the claim for arbitrary $n + 1$-codes C we define by transfinite recursion along the Kleene-Brower-ordering of C a set A uniform in C such that $(A)_s$ is a Baire witness for $C \restriction s$ (see definition 3.1.5) for each $s \in Seq$. Here we use the Baire witness for simple $n + 1$-codes at each $\forall^{n+1}$ and $\exists^{n+1}$-branch. Then we show by transfinite induction that A has this desired property. Here we need the transfinite induction which is available in the σ^+-calculus. This finishes the proof of 5.2.4. □

5.3 Lebesgue-measurability and sets definable in the σ-calculus

One might ask whether our technique is also applicable to Lebesgue - measurability, i.e. if the σ^+-calculus shows that each set with an n-code is Lebesgue-measurable. Unfortunately, we can not prove that, and the reason is as follows: Let C be the σ-algebra of all Lebesgue-measurable sets and I be the ccc σ-ideal of all sets of measure null. Let $(I)_{k\in\omega}$ be an enumeration of the open intervals with rational end points. Then the canonical candidate for the index of a set B would be the least k such that $\mu(I_k) = \mu(I_k \cap B)$. Then we need to prove that each $B \in C\backslash I$ has an index, but this is not true, as the following counterexample shows. We build a set B by altering the construction of the Cantor's discontinuum as follows. Start with $[0,1]$ and remove an interval of length $1/4$ from the midth of it. From the to remaining intervals, remove two intervals from their midth which together have length $1/8$, and so on. Taking the intersection after ω many steps, it remains a set B of measure $1/2$ such that in each open interval, there is another open interval which is disjoint from B. This B has no index. Lebesgue-measurability does not have the property "each set which is not small is locally big", and this property is crucial for our argument.

5.4 Generalization for an inaccessible cardinal

In this section we work in ZFC. Let κ be a fixed cardinal. We want to generalize the set constructors $\exists^n$ to sets which are indexed by ordinals less than κ. Let $\kappa^{<\omega}$ be the set of finite subsets of κ. We fix an injection from $\kappa^{<\omega}$ to κ, called $\langle\cdot\rangle$.

Definition 5.4.1. Let $\varphi(x,\vec{y})$ be an $\mathcal{L}_\in$-formula. Let

- $(\exists^0_\kappa x)\varphi(x,\vec{y}) :\equiv (\exists x \in \kappa)\varphi(x,\vec{y})$
- $(\exists^{n+1}_\kappa x)\varphi(x,\vec{y}) :\equiv (\forall X \subset \kappa^{<\omega})\big[(\forall x \in \kappa^{<\omega})[(\varphi(x,\vec{y}) \vee \forall^n_\kappa z(x^\frown\langle z\rangle \in X)) \rightarrow x \in X] \rightarrow \langle\rangle \in X\big]$
- $(\forall^n_\kappa x)\varphi(x,\vec{y}) :\equiv \neg(\exists^n_\kappa x)\neg\varphi(x,\vec{y})$.

If $A_x \subset {}^\kappa\kappa^{mon}$ for each $x \in \kappa$ let

$$\exists^n x A_x := \{y \mid \exists^n x(y \in A_x)\}.$$

Definition 5.4.2 (topology on ${}^{\kappa}\kappa^{mon}$)**.** Let ${}^{\kappa}\kappa^{mon}$ be the set of all strictly monotone sequences of length κ of ordinals less than κ and ${}^{<\kappa}\kappa^{mon}$ be the set of all such sequences of length less than κ. For $s \in {}^{<\kappa}\kappa^{mon}$ let

$$\mathcal{N}_s := \{x \in {}^{\kappa}\kappa^{mon} \mid s \text{ is an initial segment of } x\}.$$

We consider the topology which is given by the open basis sets $\mathcal{N}_s$.

Analogous to definition 3.1.4 we say that a subset of ${}^{\kappa}\kappa^{mon}$ has an n-κ-code if it can be generated from the open sets with the quantifiers $\exists^m_\kappa$ and $\forall^m_\kappa$ for $m \leq n$.

Theorem 5.4.3. ZFC *proves: Assume that* $(X, \mathcal{T})$ *is a topological space and* $\mathcal{S} \subset \mathcal{P}(X)$ *is a* κ*-algebra that contains all open sets. Let* $\mathcal{I}$ *be a* κ*-ideal in* $\mathcal{S}$*. Assume that* $\mathcal{I}$ *is* κ*-cc. Then* $\mathcal{S}$ *contains all sets which have an* n*-*κ*-code.*

Proof. The proof is analogous to the proof of theorem 3.2.7. We induct on n. Since the algebra is closed under κ-many intersections it is closed under $\exists^0_\kappa$. In the successor case, the iteration of the sets A_α (see proof of theorem 3.2.7) is defined for $\alpha < \kappa^+$, and since $\mathcal{I}$ is κ-cc the iteration comes to a halt before κ^+, and we can go on as in 3.2.7. □

To give an application of this theorem we define an analogon to the property of Baire suitable for ${}^{<\kappa}\kappa^{mon}$.

Definition 5.4.4. A subset of a topological space is κ-meager if it is contained in the union of κ-many closed nowhere dense sets. A set is κ-Baire if the symmetric difference with an open set is κ-meager.

Corollary 5.4.5. *Let* κ *be an inaccessible cardinal. Then all subsets of* ${}^{\kappa}\kappa^{mon}$ *which have an* n*-*κ*-code of some* n *are* κ*-Baire.*

Proof. Obviously the κ-Baire subsets of ${}^{\kappa}\kappa^{mon}$ form a κ-algebra C which contains all open sets and the meager sets form a κ-ideal I. We claim that I is κ-cc. Since κ is inaccessible we can fix an injection f from the subsets of κ of cardinality less than κ to κ. For $B \in C \backslash I$ let $i(B)$ be the least ordinal such that $\mathcal{N}_{f(i(B))}$ is comeager in B. Furthermore, for $B_1, B_2 \in C \backslash I$ with $i(B_1) = i(B_2)$ then $B_1 \cap B_2 \in C \backslash I$ because $B_1 \cap B_2$ is comeager in $\mathcal{N}_{f(i(B_1))}$. Hence each two elements of an antichain have different indices, and since we only have κ-many indices there are no antichains of size greater than κ. Hence I is κ-cc, and theorem 5.4.3 delivers the claim. □

One might ask if a similar result is true for the property of Ramsey. Unfortunately, this is not the case as the following lemma shows.

Lemma 5.4.6. *For each $\kappa > \omega$ there is an open subset U of ${}^{\kappa}\kappa^{mon}$ which has no homogeneous set, i.e. there is no $H \subset \kappa$ with $|H| = \kappa$ such that either each $V \subset H$ with $|V| = \kappa$ is in U or each such V is not in U.*

Proof. The proof is similar to the proof that there is a set of reals which has not the property of Ramsey from the beginning of chapter 1. For $V \in {}^{\kappa}\kappa^{mon}$ let $\widetilde{V}$ be the set of the first ω elements of V. Let $V \sim W$ iff $\widetilde{V} \triangle \widetilde{W}$ is finite. Then $\sim$ is an equivalence relation, and we can choose a representative from each equivalence class. If V^* is the representative in the equivalence class of V let

$$U := \{V \mid |\widetilde{V} \triangle \widetilde{V^*}| \text{ is even } \}.$$

U has no homogeneous set and since $V \in U$ depends only on the first ω elements of V and since $\kappa > \omega$ U is open. $\square$

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