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The unitary character group of abelian unipotent groups

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Abstract. We determine the Pontrjagin dual of all K-split abelian unipotent algebraic groups G over local fields K of positive characteristic. By giving a detailed study of the structure of such groups we show that there is always an isomorphism between the group G and its dual group $\widehat{G}$.

1. Introduction

In this paper we analyze the structure of abelian unipotent linear algebraic groups G over local fields of characteristic p. In particular we study their Pontrjagin dual, i.e., the group $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$ consisting of all continuous homomorphisms from G to the circle group $\mathbb{T}$, where G is equipped with the locally compact Hausdorff topology.

We consider these groups as abstract groups, equipped with the locally compact topology from the additive group of the underlying field K, which is commonly denoted by G_a. A large class of unipotent linear algebraic groups over local fields admit a finite composition series consisting of characteristic subgroups such that all consecutive quotient groups are isomorphic to the additive group G_a. Such groups are called K-split.

If the characteristic of the field K is equal to 0 then every unipotent linear algebraic group over K is K-split and every abelian K-split group over a local field of characteristic 0 is isomorphic to a finite product of copies of the group G_a. In particular, such groups are isomorphic to their dual groups. If K has positive characteristic, the situation becomes much more complicated. In this case there do exist abelian linear algebraic K-groups which are not K-split and the abelian K-split groups are usually not isomorphic to powers of K. A special class of K-split groups over such fields are given by the class of finite

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dimensional Witt groups (see Section 4 for the definition) and every K-split group is isogeneous to a finite product of finite-dimensional Witt groups.

The main result of this paper is a structure theorem (Theorem 4.16) which shows that each Witt group can be decomposed into a discrete and a compact part, each being isomorphic to the dual group of the other. This theorem is then used to show that every abelian K-split group G over a local field of characteristic p is isomorphic to its dual group $\widehat{G}$. We also give an explicit description of the characters of the first Witt group $W_1(K)$, where K is a local field of the form $\mathbb{F}_p((t))$ for some prime number p.

The paper is organized as follows. After a short preliminary section on local fields and duality for locally compact groups, we give in Section 3 a short discussion on the general structure of unipotent algebraic groups over local fields. The main results of this paper are given in Section 4, which is divided into four subsections: in the first subsection we recall the definition of $W_n(K)$, the nth Witt group of a local field K of characteristic p. In the second subsection we derive our structure theorem for Witt groups. In the third subsection we show that all Witt groups over local fields are self-dual and in the last subsection we give the explicit description of the characters of the first Witt groups. In the final Section 5 we use the fact that every abelian K-split group over a local field of characteristic p is isogeneous to a finite product of Witt group to prove that every abelian K-split group is topologically isomorphic to its dual group.

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2. Some preliminaries

A local field is a locally compact, non-discrete field. Up to isomorphism, the local fields of characteristic 0 are $\mathbb{R}$, $\mathbb{C}$, and finite algebraic extensions of the field $\mathbb{Q}_p$ of p-adic numbers for all prime numbers p ([9], §3 Theorem 5). Furthermore, every local field of characteristic $p > 0$ is isomorphic to a field of formal Laurent series in one variable, denoted by $\mathbb{F}_q((t))$, where q is some power of the prime p ([9], §4 Theorem 8). A field of this form can be constructed as follows.

Let $\mathbb{F}_q$ be the finite field with q elements, where q is a power of some prime $p > 0$, and let $\mathbb{F}_q[[t]]$ be the ring of formal power series in one variable over $\mathbb{F}_q$. The invertible elements of this ring are power series $x = \sum_{n=0}^{\infty} x_n t^n$, where $x_0 \neq 0$. Put

$$\mathbb{F}_q((t)) := \mathbb{F}_q[[t]](t^{-1}).$$

The elements of $\mathbb{F}_q((t))$ are formal Laurent series of the form $\sum_{n=n_0}^{\infty} a_n t^n$, where $n_0 \in \mathbb{Z}$ and $a_n \in \mathbb{F}_q$. Since every nonzero element $x \in \mathbb{F}_q((t))$ can be written uniquely as $x = t^l y$ for some $l \in \mathbb{Z}$ and some $y \in \mathbb{F}_q[[t]]^\times$, every nonzero element of $\mathbb{F}_q((t))$ has a multiplicative inverse. Thus $\mathbb{F}_q((t))$ is a field, it is the quotient field of the ring $\mathbb{F}_q[[t]]$.

One can define a norm on $\mathbb{F}_q((t))$ in the following way. We put $\|0\| = 0$ and for $0 \neq x \in \mathbb{F}_q((t))$ with $x = t^l y$ for some $l \in \mathbb{Z}$ and some $y \in \mathbb{F}_q[[t]]^\times$, we put $\|x\| = q^{-l}$. Equipped with this norm, the additive group of the field $\mathbb{F}_q((t))$ becomes a locally compact group.

Observe that the map

$$\phi : \mathbb{F}_q[[t]] \to \prod_{i=0}^{\infty} \mathbb{F}_q, \quad \sum_{n=0}^{\infty} a_n t^n \mapsto (a_n)_{n \in \mathbb{Z}_{\geq 0}}$$

defines an isomorphism of groups, which is bi-continuous with respect to the product topology on the compact group $\prod_{i=0}^{\infty} \mathbb{F}_q$. Furthermore, the map

$$\Phi : \mathbb{F}_q((t)) \to \bigoplus_{i=-1}^{-\infty} \mathbb{F}_q \times \prod_{i=0}^{\infty} \mathbb{F}_q, \quad \sum_{n=-m}^{\infty} a_n t^n \mapsto (a_n)_{n \in \mathbb{Z}_{\geq -m}},$$

is an isomorphism of additive groups and Φ is bi-continuous with respect to the discrete topology on the direct sum $\bigoplus_{i=-1}^{-\infty} \mathbb{F}_q$. By means of this map, we obtain a natural decomposition of the local field $\mathbb{F}_q((t))$ into a discrete and a compact part:

$$\mathbb{F}_q((t)) \cong \bigoplus_{i=1}^{\infty} \mathbb{F}_q \times \prod_{i=0}^{\infty} \mathbb{F}_q. \tag{1}$$

We now discuss duality of local fields. Recall that for any abelian locally compact group G, the Pontrjagin dual group $\widehat{G}$ is the group of all continuous group homomorphisms from G to the circle group $\mathbb{T}$ equipped with pointwise multiplication. The elements of $\widehat{G}$ are called characters of G. Given the topology of uniform convergence on compact sets of G, $\widehat{G}$ is again a locally compact group. We say that G is *self-dual* if there exists a topological isomorphism between G and $\widehat{G}$. For instance, if G is the additive group of a local field K then G is self-dual:

Proposition 2.1. *(*[9]*, Theorem 3) Let K be a non-discrete locally compact field and let χ be a non-trivial character of the additive group of K. Then the map $y \mapsto \chi_y$ from K to $\widehat{K}$, where $\chi_y(x) := \chi(xy)$, is an isomorphism of topological groups.*

The proposition implies in particular that the field $\mathbb{F}_q((t))$, viewed as an additive locally compact group, is selfdual. We want to observe that there is also a different way of exhibiting the selfduality of $\mathbb{F}_q((t))$, using the structure of its additive group. For this, we recall the following facts about the dual group of a locally compact abelian group G which can all be found in [3], Chapter 4.

(1) If $G_1, \ldots, G_n$ are locally compact abelian groups then

$$(G_1 \times \cdots \times G_n)\widehat{\ } \cong \widehat{G_1} \times \cdots \times \widehat{G_n}$$

and every finite abelian group is selfdual.

(2) If G is discrete then $\widehat{G}$ is compact and if G is compact then $\widehat{G}$ is discrete.

(3) If $G = \prod_{i\in I} G_i$, where each G_i is a compact abelian group then
$$\widehat{G} \cong \bigoplus_{i\in I} \widehat{G}_i.$$

(4) (Pontrjagin duality theorem) The map $\Phi : G \to \widehat{\widehat{G}}$ defined by $\Phi(x)(\chi) = \chi(x)$ is an isomorphism of topological groups.

Let G be the additive group of the local field $\mathbb{F}_q((t))$. Then we have by (1)
$$G \cong \bigoplus_{i=1}^{\infty} \mathbb{F}_q \times \prod_{i=0}^{\infty} \mathbb{F}_q$$
and using the facts listed above we obtain
$$\widehat{G} \cong \left(\bigoplus_{i=1}^{\infty} \mathbb{F}_q \times \prod_{i=0}^{\infty} \mathbb{F}_q\right)^{\widehat{}} \cong \left(\bigoplus_{i=1}^{\infty} \mathbb{F}_q\right)^{\widehat{}} \times \left(\prod_{i=0}^{\infty} \mathbb{F}_q\right)^{\widehat{}} \cong \prod_{i=0}^{\infty} \mathbb{F}_q \times \bigoplus_{i=1}^{\infty} \mathbb{F}_q \cong G.$$

3. The structure of unipotent linear algebraic groups

In this section we want to recall some results on the structure of unipotent linear algebraic groups which we shall use in this paper. By a linear algebraic group G over K we will always understand the K-rational points, $G(K)$, of such a group with the locally compact topology from the local field K.

The most basic example is the additive group of the underlying field K which we denote in the following by G_a. The group G_a is unipotent since it is isomorphic to the group $G = \left\{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right), x \in K\right\} \subseteq \mathrm{GL}_2(K)$. Recall the following definition.

Definition 3.1. Let V be a finite dimensional vector space over K and denote by $\mathrm{End}(V)$ the algebra of endomorphisms of V.

(a) An element $a \in \mathrm{End}(V)$ is called nilpotent if $a^m = 0$ for some $m \in \mathbb{N}$.
(b) An element $a \in \mathrm{End}(V)$ is called unipotent if $a - 1$ is nilpotent.
(c) A group $G \subseteq \mathrm{End}(V)$ is said to be unipotent if all its elements are unipotent.

Recall that a *linear* algebraic group G over K is a Zariski closed subgroup of $\mathrm{GL}_n(K)$ for some $n \in \mathbb{N}$. If K is a local field, then G is also closed with respect to the locally compact Hausdorff topology on $\mathrm{GL}_n(K)$ inherited from the locally compact topology on K.

Definition 3.2. Let $n \in \mathbb{N}$. We define
$$Tr_1(n,K) := \{A \in GL_n(K) \mid A_{ij} = 0 \text{ for } j < i \text{ and } A_{ii} = 1 \text{ for all } 1 \le i \le n\}$$
to be the algebra of upper triangular $n \times n$-matrices over K with each diagonal entry equal to 1.

Theorem 3.3. *([1], Theorem 4.8) A linear algebraic group defined over K is unipotent if and only if it is isomorphic to a closed algebraic subgroup of the upper triangular unipotent group $Tr_1(n, K)$ for some $n \in \mathbb{N}$. In particular, every unipotent linear algebraic group is nilpotent.*

An important class of unipotent linear algebraic groups is the class of K-split groups, which are defined as follows.

Definition 3.4. ([1], Definition 15.1) A unipotent linear algebraic group G defined over K is called K-split if it admits a composition series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_k = \{e\}$$

consisting of closed, normal linear algebraic subgroups of G such that G_i/G_{i+1} is isomorphic to the additive group G_a. In particular such a group is connected with respect to the Zariski topology ([7], V, 2.1).

Remark/Example 3.5. ([1], Corollary 15.5.) If the field K is perfect, in particular if the characteristic of K is equal to 0, then every unipotent linear algebraic group G over K is K-split. This is not true for local fields of positive characteristic.

Definition 3.6. ([7], V §3.1) Let G be a unipotent linear algebraic group over K. The group G is said to be K-wound if G does not admit a subgroup which is isomorphic to the additive group G_a.

Example 3.7. ([7], V §3.4) Let K be a local field of characteristic p and suppose that K is not perfect (i.e. $K^p \neq K$). Let t be an element of $K \setminus K^p$. One can show that the algebraic subgroup $H := \{(x, y) \mid x^p - ty^p = x\}$ of $G_a \times G_a$ is K-wound.

In [7], chapter V and VI, Oesterlé proves several interesting facts about the structure of K-wound groups.

Theorem 3.8. *([7], V §5) Let G be a unipotent linear algebraic group over a local field K of characteristic $p > 0$. Then the following are equivalent.*

(i) *G is K-wound.*
(ii) *The topological group G is compact.*

Theorem 3.9. *([7], VI §1) Every unipotent linear algebraic group G over K admits a largest algebraic normal subgroup N which is K-split. The quotient G/N is K-wound and it is the largest quotient of G with this property.*

It follows from Theorem 3.8 that every abelian K-wound group has a discrete dual group. So, unless they are finite, such groups can never be self-dual. From now on we will concentrate ourselves on K-split groups. Let G be K-split and let

$$G := Z_0(G) \trianglerighteq Z_1(G) \trianglerighteq \cdots \trianglerighteq Z_m(G) = \{e\} \tag{2}$$

be its descending central series. Notice that in this case $Z_{m-1}(G)$ is equal to the center of G. Since $(Z_i(G), Z_i(G)) \subseteq (Z_i(G), G)$ for all i, it follows that the canonical map

$$\varphi : Z_i(G)/(Z_i(G), Z_i(G)) \longrightarrow Z_i(G)/(Z_i(G), G)$$

is surjective. But we have $Z_i(G)/(Z_i(G), G) = Z_i(G)/Z_{i+1}(G)$ and since the quotient group $Z_i(G)/(Z_i(G), Z_i(G))$ is abelian it follows that all consecutive quotients $Z_i(G)/Z_{i+1}(G)$ appearing in the central series (2) are abelian unipotent K-split groups. Thus we can refine the central series (2) so that we obtain a new composition series in which all consecutive quotients are isomorphic to the additive group G_a.

As we have seen above, every unipotent K-split group is a multiple extension of groups of the type G_a. So in order to understand the structure of these groups more precisely, we first need to study the algebraic extensions of the additive group G_a with itself. For this we recall the following definitions.

An algebraic 2-cocycle f on G_a is a polynomial in two variables with coefficients in K satisfying the equations

$$f(x, 0) = f(0, x) = 0 \quad \text{for all } x \in G_a \qquad \text{and}$$
$$f(y, z) - f(x + y, z) + f(x, y + z) - f(x, y) = 0 \quad \text{for all } x, y, z \in G_a. \tag{3}$$

If $g : G_a \to G_a$ is any polynomial map, the function $h : G_a \times G_a \to G_a$ defined by

$$h(x, y) = g(x + y) - g(x) - g(y)$$

is an algebraic 2-cocycle and such a 2-cocycle is called trivial. The group of classes of algebraic 2-cocycles modulo the trivial 2-cocycles is denoted by $H^2(G_a, G_a)$. Every algebraic extension of the additive group G_a with itself is completely determined by an algebraic 2-cocycle $f : G_a \times G_a \to G_a$, and we will identify in the following such extensions with their 2-cocycles.

A 2-cocycle f is called symmetric if $f(x, y) = f(y, x)$ for all $x, y \in G_a$, and we denote the group of classes of symmetric, algebraic 2-cocycles modulo the trivial 2-cocycles by $\mathrm{Ext}(G_a, G_a)$. In characteristic $p > 0$, a non-trivial example of such a polynomial is

$$\omega(x, y) = \frac{1}{p}(x^p + y^p - (x + y)^p), \tag{4}$$

the 2-cocycle which defines the first Witt group over K. Finite-dimensional Witt groups will be introduced in Section 4 and we will see in Section 5 that there is in fact a close connection between abelian K-split groups and Witt groups.

But first we want to remark that the class of polynomial 2-cocycles of G_a is much greater then the class of symmetric polynomial 2-cocycles. It is easy to see that, for example, every bi-additive polynomial $f : G_a \times G_a \to G_a$ satisfies Equation (3). In characteristic p, the bi-additive polynomials of G_a are of the form

$$f(x,y) = \sum_{m,n} a_{m,n} x^{p^m} y^{p^n},$$

where all but finitely many coefficients $a_{m,n}$ are equal to zero. A simple example of an asymmetric bi-additive polynomial is given by

$$f(x,y) = x^p y.$$

The non-abelian group G corresponding to this asymmetric cocycle can be realized as

$$G = \left\{ \left(\begin{smallmatrix} 1 & x & y \\ 0 & 1 & x^p \\ 0 & 0 & 1 \end{smallmatrix} \right), x, y \in G_a \right\}.$$

In fact, we can show the following result.

Proposition 3.10. *Every algebraic* 2*-cocycle of* G_a *is the sum of a symmetric* 2*-cocycle and a bi-additive* 2*-cocycle.*

Proof. Notice first that if $f(x,y)$ is a 2-cocycle of G_a then the polynomials $\bar{f}(x,y) := f(y,x)$ and $g(x,y) := f(x,y) - f(y,x)$ are also 2-cocycles. Furthermore, we have

$$g(x, y+z) - g(x,y) - g(x,z) = 0,$$

for all $x, y, z \in G_a$, and since $g(x,y) = -g(y,x)$ it follows that the polynomial g is bi-additive. We distinguish between two cases.

If the characteristic of K is anything except 2, then we can write

$$f(x,y) = \frac{1}{2}\big(f(x,y) + f(y,x)\big) + \frac{1}{2} g(x,y)$$

and thus we can write f as a sum of a symmetric and a bi-additive 2-cocycle.

If the characteristic of K is equal to 2, then $g(x,y) = f(x,y) + f(y,x)$ is bi-additive and symmetric. If f is of the form $f(x,y) = \sum a_{ij} x^i y^j$ then we have

$$g(x,y) = \sum (a_{ij} + a_{ji}) x^i y^j = \sum b_{ij} x^i y^j.$$

Furthermore we have $b_{ij} = b_{ji}$ and $b_{ii} = 2a_{ii} = 0$ for all i. Since $g(x,y)$ is bi-additive it follows that $b_{ij} = 0$ except for the case that i and j are both powers of 2. Now, let $h(x,y) = \sum_{i<j} b_{ij} x^i y^j$, then $h(x,y)$ is bi-additive and we have $h(x,y) + h(y,x) = g(x,y)$ for all $x, y \in G_a$. Since

$$f(x,y) + f(y,x) + h(x,y) + h(y,x) = 2g(x,y) = 0,$$

it follows that the sum $f(x,y) + h(x,y)$ is symmetric and hence $f(x,y)$ is the sum of a symmetric and a bi-additive 2-cocycle. □

4. WITT GROUPS

Finite-dimensional Witt groups over local fields K of characteristic p define an important class of abelian K-split groups. This section is divided into four subsections. In Subsection 4.1 we give, for every $n \in \mathbb{N}_0$, the definition of $W_n(K)$, the nth Witt group of a local field K of characteristic p. We prove in Subsection 4.2 that finite-dimensional Witt groups of any local field K of characteristic p can be decomposed into a discrete and a compact part, each being the dual of the other (Proposition 4.20). With this result we prove in Subsection 4.3 that, for every $n \in \mathbb{N}_0$ and every local field K of characteristic p, the topological group $W_n(K)$ is isomorphic to its dual group. Finally, we give in Subsection 4.4 an explicit description of the characters of the first Witt group of a local field K of the form $K = \mathbb{F}_p((t))$ for some prime number p.

4.1. **Definition of finite-dimensional Witt groups.** We first introduce the definition of the nth Witt ring $W_n(R)$ of a commutative ring R with unity. The approach we are following is given in [5].

Let $p \in \mathbb{N}$ be a fixed prime, let $n \in \mathbb{N}$, and consider the polynomial ring

$$A = \mathbb{Q}[X_0, Y_0, \ldots, X_n, Y_n].$$

We will define a new ring structure on the set A^{n+1} via the following procedure. Let $x = (x_0, x_1, \ldots, x_n) \in A^{n+1}$ and define a map $\phi : A^{n+1} \to A^{n+1}$ by

$$\phi((x_0, x_1, \ldots, x_n)) = (x^{(0)}, x^{(1)}, \ldots, x^{(n)}),$$

where

(5) $x^{(0)} := x_0$ and $x^{(i)} := x_0^{p^i} + p \cdot x_1^{p^{i-1}} + \cdots + p^{i-1} \cdot x_{i-1}^p + p^i \cdot x_i$ for $i \geq 1$.

Conversely, given an arbitrary vector $z = (x^{(0)}, x^{(1)}, \ldots, x^{(n)}) \in A^{n+1}$, define a map $\psi : A^{n+1} \to A^{n+1}$ by

$$\psi(z) = (x_0, x_1, \ldots, x_n),$$

where

(6) $x_0 := x^{(0)}$ and $x_i := \dfrac{1}{p^i}[x^{(i)} - x_0^{p^i} - p \cdot x_1^{p^{i-1}} - \cdots - p^{i-1} \cdot x_{i-1}^p]$ for $i \geq 1$.

The maps ϕ and ψ are inverse bijections. Using these maps we can introduce new binary operations of addition, denoted by $\oplus$, and multiplication, denoted by $\otimes$, on the set A^{n+1}. For this let $x, y \in A^{n+1}$ and define

$$x \oplus y := \psi(\phi(x) + \phi(y)) \text{ and } x \otimes y := \psi(\phi(x) \cdot \phi(y)). \tag{7}$$

That is, $(x \oplus y)^{(i)} = x^{(i)} + y^{(i)}$ and $(x \otimes y)^{(i)} = x^{(i)} \cdot y^{(i)}$ for $i \geq 0$. Note that, in general, $x \oplus y \neq x + y$ and $x \otimes y \neq x \cdot y$. We write $W_n(A)$ for the set A^{n+1} endowed with the operations $\oplus$ and $\otimes$ as given above. One can show that $W_n(A)$ is a commutative ring of characteristic 0, isomorphic to the ring A^{n+1} under $\phi : W_n(A) \to A^{n+1}$. Observe that by (5), (6) and (7), we obtain for example:

$$\begin{aligned}
(x \oplus y)_0 &= x_0 + y_0, \\
(x \oplus y)_1 &= x_1 + y_1 + \frac{1}{p}(x_0^p + y_0^p - (x_0 + y_0)^p), \\
(x \otimes y)_0 &= x_0 y_0, \\
(x \otimes y)_1 &= x_0^p y_1 + x_1 y_0^p + p \cdot x_1 y_1.
\end{aligned}$$

Theorem 4.1. *([4], Theorem 8.25) Let $n \in \mathbb{N}$ and let $x = (x_0, x_1, \dots, x_n)$ and $y = (y_0, y_1, \dots, y_n)$ be elements of $W_n(A)$. Let $x \circ y$ denote $x \oplus y$, $x \otimes y$, or $x \ominus y$. Then $(x \circ y)_i \in \mathbb{Z}[x_0, y_0, \dots x_i, y_i]$ for all $i \in \{0, \dots, n\}$.*

Using this result we can now define for all $0 \leq i \leq n$:

$$(x \oplus y)_i =: A_i(x_0, y_0, \dots, x_i, y_i) \in \mathbb{Z}[x_0, y_0, \dots, x_i, y_i],$$
$$(x \otimes y)_i =: M_i(x_0, y_0, \dots, x_i, y_i) \in \mathbb{Z}[x_0, y_0, \dots, x_i, y_i],$$
$$(x \ominus y)_i =: S_i(x_0, y_0, \dots, x_i, y_i) \in \mathbb{Z}[x_0, y_0, \dots, x_i, y_i].$$

With these properties we can pass from the ring $A = \mathbb{Q}[X_0, Y_0, \dots, X_n, Y_n]$ to any commutative ring R with identity. Let $\varphi : \mathbb{Z} \to R$ denote the natural homomorphism defined by $\varphi(c) = 1 \cdot c =: \bar{c}$ for $c \in \mathbb{Z}$. Let $\bar{A}_i(x_0, y_0, \dots, x_i, y_i)$ and $\bar{M}_i(x_0, y_0, \dots, x_i, y_i)$ be the polynomials in $R[x_0, y_0, \dots, x_i, y_i]$ obtained from $A_i(x_0, y_0, \dots, x_i, y_i)$ and $M_i[x_0, y_0, \dots, x_i, y_i]$, respectively, by replacing each coefficient $c \in \mathbb{Z}$ by $\bar{c} \in R$. We can now give the definition of the nth Witt ring of a commutative ring with unity.

Definition 4.2. Let R be any commutative ring with unity and let $n \geq 0$. The nth Witt ring $W_n(R)$ of R is defined to be the set of all $(n+1)$-tuples $x = (x_0, \dots, x_n)$, where $x_i \in R$ for every $i \in \{0, \dots, n\}$, with addition and multiplication defined as

$$(x \oplus x')_i =: \bar{A}_i(x_0, x'_0, \dots, x_i, x'_i) \quad \text{and}$$
$$(x \otimes x')_i =: \bar{M}_i(x_0, x'_0, \dots, x_i, x'_i).$$

By [4], Theorem 8.26, $W_n(R)$ is a commutative ring. The zero and identity elements of $W_n(R)$ are $0 = (0, 0, \dots, 0)$ and $(1, 0, \dots, 0)$, respectively.

Remark 4.3. Let R be any commutative ring with unity and let $n \geq 0$. Let $x = (x_0, \dots, x_n)$ and $x' = (x'_0, \dots, x'_n)$ be elements of $W_n(R)$, and consider $B := \mathbb{Z}[Y_0, Y'_0, \dots, Y_n, Y'_n]$. There exists a ring homomorphism θ from B to R, satisfying $\theta(c) = \bar{c}$ for $c \in \mathbb{Z}$, $\theta(Y_i) = x_i$, and $\theta(Y'_i) = x'_i$ for all $0 \leq i \leq n$. This map induces a ring homomorphism $\tilde{\theta} : W_n(B) \to W_n(R)$, where $\tilde{\theta}((a_0, \dots, a_n)) = (\theta(a_0), \dots, \theta(a_n))$, $a_i \in B$, satisfying $\tilde{\theta}((Y_0, \dots, Y_n)) = (x_0, \dots, x_n)$ and $\tilde{\theta}((Y'_0, \dots, Y'_n)) = (x'_0, \dots, x'_n)$. In this way we obtain a functor W_n from the category of commutative rings of characteristic p into the category of commutative rings. In particular, if S is a subring of R, then $W_n(S)$ is a subring of $W_n(R)$.

Remark 4.4. Let R be any commutative ring with unity and let $n \geq 0$. Let $x = (x_0, \dots, x_n)$ and $y = (y_0, \dots, y_n)$ be elements of $W_n(R)$. Then $(x \oplus y)_i = \bar{A}_i(x_0, y_0, \dots, x_i, y_i)$ for all $0 \leq i \leq n$ and it follows from Equation (5), (6), and (7) that $\bar{A}_i$ is of the form

$$\bar{A}_i(x_0, y_0, \dots, x_i, y_i) = x_i + y_i + \omega_i(x_0, y_0, \dots, x_{i-1}, y_{i-1}), \tag{8}$$

where ω_i, $i \in \{1, \dots, n\}$, is a polynomial in the variables $x_0, y_0, \dots, x_{i-1}, y_{i-1}$ with coefficients in R. Only ω_0 is equal to the zero polynomial. Note that the ω_i, $i \in \mathbb{N}$, can be regarded as 2-cocycles on the Witt-groups $W_{i-1}(R)$ with values in R and that the ith Witt-group $W_i(R)$ is realized as the central extension of $W_{i-1}(R)$ by the central subgroup R corresponding to this cocycle.

4.2. **The structure of finite-dimensional Witt groups.** In the following, let K be a local field of characteristic $p0$. The $(n+1)$-dimensional Witt ring $W_n(K)$, $n \in \mathbb{N}_{\geq 0}$, has the natural structure of an abelian, unipotent algebraic group. The elements of $W_n(K)$ are $(n+1)$-tuples $(x_0, \dots, x_n)$, where $x_i \in K$ and hence, as a set, $W_n(K) = K^{n+1}$. It follows from Definition 4.2 and the remarks following it that the maps $\pi : W_n \times W_n \to W_n$, where $\pi(x, y) = x \oplus y$ and $i : W_n \to W_n$, where $i(x) = -x$ are morphisms of affine varieties $K^{2(n+1)} \to K^{n+1}$ and $K^{n+1} \to K^{n+1}$, respectively. Therefore, $W_n(K)$ is a (n+1)-dimensional abelian (affine) algebraic group. In order to see that the group $W_n(K)$ is unipotent we introduce the following maps.

(1) The Shift map

$$S : W_n(K) \to W_n(K), (x_0, x_1, \dots, x_n) \mapsto (0, x_0, \dots, x_{n-1}) \text{ and}$$

(2) the Frobenius map

$$F : W_n(K) \to W_n(K), (x_0, x_1, \dots, x_n) \mapsto (x_0^p, x_1^p, \dots, x_n^p).$$

These maps have the following important properties.

Lemma 4.5. *([5], Theorem 13.6.)*

(i) *The Shift map S and the Frobenius map F are ring homomorphisms.*
(ii) *All elements $x = (x_0, \dots, x_n) \in W_n(K)$ satisfy the following equation*

$$\begin{aligned} &p^k x = S^k(F^k(x)) \quad \text{for all } 0 \leq k \leq n \text{ and hence} \\ &p^k(x_0, \dots, x_n) = (0, 0, \dots, 0, x_0^{p^k}, x_1^{p^k}, \dots, x_{n-k}^{p^k}). \end{aligned} \tag{9}$$

As a direct consequence of this lemma we obtain the following result.

Corollary 4.6. *If 1 denotes the vector $(1, 0 \dots, 0)$ in the nth Witt ring $W_n(K)$, then $p^{n+1} \cdot 1 = 0$ and $p^m \cdot 1 \neq 0$ whenever $m < n+1$. Thus each element of $W_n(K)$ has additive order a power of p, and hence $W_n(K)$ is a unipotent group. Moreover, p^{n+1} is the smallest power of p satisfying the condition $p^{n+1}x = 0$ for all $x \in W_n(K)$.*

For the rest of this section, we endow $W_n(K)$ with the locally compact topology of the field K and consider in this way $W_n(K)$ as an abelian locally compact group. In the following we will write W_n instead of $W_n(K)$.

As we have seen in Section 2, we can decompose the additive group of the field K into a product of a discrete and a compact part, each being the dual of the other. Our aim in this section is to derive a structure theorem for all finite-dimensional Witt groups over K, which shows that these groups consist, like the field K, of a discrete and a compact part, each being the dual of the other.

Recall that every non-discrete locally compact field of characteristic $p > 0$ is isomorphic to a field of formal Laurent series in one variable, $\mathbb{F}_q((t))$, where

q is some power of the prime p. We consider first the case that $q = p$, i.e., the case that the field K is isomorphic to $\mathbb{F}_p((t))$. We will use the following notation.

Notation 4.7. Let

- $k := \mathbb{Z}/p\mathbb{Z}$ be the finite field with p elements,
- $K := k((t))$ the field of Laurent-series over k,
- $K^+ := k[[t]] \subseteq K$ the power series ring over k, and
- $K^- := \{a_1t^{-1} + a_2t^{-2} + \ldots + a_nt^{-n} \mid n \in \mathbb{N}, a_i \in k\}$.

The set K^- is an additive subgroup of K which is also closed under multiplication. Clearly, every element $a \in K$ can be written uniquely as $a = a^+ + a^-$, where $a^+ \in K^+$ and $a^- \in K^-$.

Furthermore, if A is a finite abelian group with the discrete topology we define

- $A^\infty := \prod_{i=0}^\infty A = \{(a_0, a_1, \ldots) \mid a_i \in A\}$

to be the infinite direct product of A, which is a compact group with respect to the product topology. And we define

- $A^{(\infty)} := \bigoplus_{i=1}^\infty A = \{(a_1, a_2, \ldots) \mid a_i \in A, \;\; a_i = 0 \text{ for almost all } i\}$

to be the infinite direct sum, which is a discrete group with the usual direct-sum-topology. Moreover, let for every $n \in \mathbb{N}_0$

- $C_n := \mathbb{Z}/n\mathbb{Z}$ be the cyclic group with n elements,
- $W_n^+ := \{(x_0, \ldots, x_n) \in W_n \mid x_i \in K^+ \text{ for all } 0 \leq i \leq n\}$, and
- $W_n^- := \{(x_0, \ldots, x_n) \in W_n \mid x_i \in K^- \text{ for all } 0 \leq i \leq n\}$.

We recall that the field $K = \mathbb{F}_p((t))$ is isomorphic to the direct product of the discrete subgroup K^- and the compact subgroup K^+ and thus

$$K \cong K^- \times K^+ \cong \bigoplus_{i=1}^{\infty} k \times \prod_{i=0}^{\infty} k \cong C_p^{(\infty)} \times C_p^\infty.$$

The aim of this section is to show that the nth Witt group of K can be decomposed in a similar way. In fact we will prove

$$W_n \cong W_n^- \times W_n^+ \cong (C_{p^{n+1}}^{(\infty)} \times C_{p^n}^{(\infty)} \times \cdots \times C_p^{(\infty)}) \times (C_{p^{n+1}}^\infty \times C_{p^n}^\infty \times \cdots \times C_p^\infty).$$

The proof consists of several steps. At first we show that the nth Witt group W_n can be written as the direct product of its subgroups W_n^- and W_n^+.

Lemma 4.8.

(i) *The sets W_n^+ and W_n^-, defined as above, are subgroups of W_n and we have $W_n^+ \cap W_n^- = \{0\}$.*

(ii) *The map $\mu : W_n^+ \times W_n^- \to W_n, \;(x, y) \mapsto x \oplus y$ is a bi-continuous isomorphism.*

Proof. (i): In order to show that W_n^+ is a subgroup of W_n notice first that the neutral element $(0, \ldots, 0)$ is an element of W_n^+. Moreover, if $(x_0, \ldots, x_n) \in W_n^+$ then also $-(x_0, \ldots, x_n) \in W_n^+$, since it follows from (5), (6), and (7) that $-(x_0,\dots,x_n)=(-x_0,\dots,-x_n)$. Now, let $x=(x_0,\dots,x_n)$ and $x'=(x'_0,\dots,x'_n)$ be two arbitrary elements of W_n^+. Then we have $x_i,x'_i\in K^+$ for all $0\le i\le n$ and it follows from Definition 4.2 that $(x\oplus x')_i=\bar{A}_i(x_0,x'_0,\dots,x_i,x'_i)$, where $\bar{A}_i(x_0,x'_0,\dots,x_i,x'_i)$ denotes a polynomial in $x_0,\dots,x_i$ and $x'_0,\dots,x'_i$. Thus $\bar{A}_i(x_0,x'_0,\dots,x_i,x'_i)$ is itself an element of K^+ and we obtain $(x\oplus x')_i\in K^+$ for all $0\le i\le n$. This proves that W_n^+ is closed under addition. We can use the same arguments to show that W_n^- is a subgroup of W_n and obviously we have $W_n^+\cap W_n^-=\{0\}$.

(ii): We show first that the map μ is a homomorphism. For this, let (x,y) and (v,z) be two elements of $W_n^+\times W_n^-$. Since the group W_n is commutative we obtain

$$\begin{aligned}\mu((x,y)+(v,z))&=\mu((x\oplus v,y\oplus z))=(x\oplus v)\oplus(y\oplus z)\\&=(x\oplus y)\oplus(v\oplus z)=\mu((x,y))\oplus\mu((v,z)).\end{aligned}$$

In order to prove the injectivity of the map μ, let $x=(x_0,\dots,x_n)\in W_n^+$ and let $y=(y_0,\dots,y_n)\in W_n^-$ with $\mu((x,y))=(0,\dots,0)$. Then

$$(0,0,\dots,0)=(\bar{A}_0(x_0,y_0),\bar{A}_1(x_0,x_1,y_0,y_1),\dots,\bar{A}_n(x_0,\dots,x_n,y_0,\dots,y_n))$$

and by comparing the components of the vectors above we obtain for all $0\le i\le n$:

$$0=\bar{A}_i(x_0,y_0,\dots,x_i,y_i).$$

Rewriting the expression $\bar{A}_i(x_0,y_0,\dots,x_i,y_i)$ by means of (8) of Remark 4.4, we obtain for all $0\le i\le n$:

$$0=x_i+y_i+\omega_i(x_0,y_0,\dots,x_{i-1},y_{i-1}). \tag{10}$$

The proof proceeds by induction on $i\in\{0,\dots,n\}$.

If $i=0$, Equation (10) yields $0=x_0+y_0$ and since $x_0\in K^+$ and $y_0\in K^-$, it follows that $x_0=0$ and $y_0=0$, proving the base case.

So let $i\in\{0,\dots,n\}$ be fixed and suppose that $x_j=y_j=0$ for all $0\le j\le i$. Then we have $\omega_{i+1}(x_0,y_0,\dots,x_i,y_i)=0$ and it follows from (10) that $0=x_{i+1}+y_{i+1}$. But since $x_{i+1}\in K^+$ and $y_{i+1}\in K^-$, we obtain $x_{i+1}=y_{i+1}=0$.

In order to prove that the map μ is surjective let $x=(x_0,\dots,x_n)$ be an arbitrary element of W_n. We need to show that there exist two elements $y^-\in W_n^-$ and $y^+\in W_n^+$ with $\mu((y^-,y^+))=x$. We define these elements $y^-=(y_0^-,\dots,y_n^-)$ and $y^+=(y_0^+,\dots,y_n^+)$ via the following procedure. For each $a\in K$ let $a^-\in K^-$ and $a^+\in K^+$ be those elements, such that $a=a^-+a^+$ and define

$$\begin{aligned}&y_0^-:=x_0^-,\qquad y_0^+:=x_0^+,\\&y_k^-:=x_k^--\omega_k((y_0^-,\dots,y_{k-1}^-),(y_0^+,\dots,y_{k-1}^+))^-\qquad\text{and}\\&y_k^+:=x_k^+-\omega_k((y_0^-,\dots,y_{k-1}^-),(y_0^+,\dots,y_{k-1}^+))^+\qquad\text{for all }1\le k\le n.\end{aligned}$$

It follows directly from this definition that $y^-\in W_n^-$, $y^+\in W_n^+$ and clearly we have

$$\mu((y^-,y^+))=(y_0^-,\dots,y_n^-)\oplus(y_0^+,\dots,y_n^+)=(x_0,\dots,x_n).$$

It only remains to show that μ is bi-continuous. But μ is the addition map in a topological group and thus continuous by definition. Since every continuous, bijective homomorphism between σ-compact locally compact groups is open, it follows that μ is bi-continuous. $\square$

In a second step we construct an isomorphism Λ between the subgroup W_n^+ of W_n and the compact group $(C_{p^{n+1}}^\infty \times C_{p^n}^\infty \times \cdots \times C_p^\infty)$ and an isomorphism Ψ between the subgroup W_n^- of W_n and the discrete group $(C_{p^{n+1}}^{(\infty)} \times C_{p^n}^{(\infty)} \times \cdots \times C_p^{(\infty)})$. For this we define maps Λ_k and Ψ_k, $k = 0, \dots, n$, which will be the "componentwise building blocks" for the maps Λ and Ψ, respectively. Since the definition of these maps is not canonical we explain the idea of the construction by means of the following special case.

Example 4.9. Let $p = 2$ and consider the local field $K = \mathbb{F}_2((t))$. The first Witt group $W_1(K)$ of K consists of the set of pairs $\{(x_0, x_1) \mid x_0, x_1 \in K\}$, where addition is defined as

$$(x_0, x_1) \oplus (y_0, y_1) := (x_0 + y_0, x_1 + y_1 + x_0 y_0).$$

We can view $W_1(K)$ as the group $G := \left\{ \left(\begin{smallmatrix} 1 & x_0 & x_1 \\ 0 & 1 & x_0 \\ 0 & 0 & 1 \end{smallmatrix}\right), x_0, x_1 \in K \right\}$, since the map

$$\Phi : W_1(K) \to G,\ (x_0, x_1) \mapsto \left(\begin{smallmatrix} 1 & x_0 & x_1 \\ 0 & 1 & x_0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$$

defines an isomorphism of topological groups.

In order to define an isomorphism Λ between the subgroup $W_1^+(K)$ and the compact group $C_{p^2}^\infty \times C_p^\infty$, we introduce first two homomorphisms, $\Lambda_1 : C_{p^2}^\infty \to W_1^+(K)$ and $\Lambda_0 : C_p^\infty \to W_1^+(K)$. Notice that $C_{p^2} = C_4 = \mathbb{Z}/4\mathbb{Z}$ and $C_p = C_2 = \mathbb{Z}/2\mathbb{Z}$ and we identify elements of $\mathbb{Z}/4\mathbb{Z}$ and of $\mathbb{Z}/2\mathbb{Z}$ with numbers in $\{0,1,2,3\}$ and in $\{0,1\}$, respectively, in the canonical way. Thus we can multiply elements $a_m \in C_4$ with pairs $(x_0, x_1) \in W_1^+(K)$, where we understand the product as the a_m-fold sum of the pair (x_0, x_1) in $W_1^+(K)$. We can now define

$$\Lambda_1 : C_4^\infty \to W_1^+(K),\ (a_m)_{m \in \mathbb{N}_0} \mapsto \sum_{m \in \mathbb{N}_0} a_m (t^m, 0)$$

and we will prove in Lemma 4.12 that Λ_1 is a well-defined group homomorphism. Notice that

$$\begin{aligned}
0 \cdot (t^m, 0) &= (0, 0), \\
1 \cdot (t^m, 0) &= (t^m, 0), \\
2 \cdot (t^m, 0) &= (t^m, 0) \oplus (t^m, 0) \ = (0, t^{2m}), \qquad \text{and} \\
3 \cdot (t^m, 0) &= (0, t^{2m}) \oplus (t^m, 0) = (t^m, t^{2m}).
\end{aligned}$$

This computation shows that the image of Λ_1 is equal to the subgroup $K^+ \times (K^+)^2$ of $W_1^+(K)$ and since we want to obtain an isomorphism between

$C_4^\infty \times C_2^\infty$ and $W_1^+(K)$, we define a second map Λ_0 as follows:

$$\Lambda_0 : C_2^\infty \to W_1^+(K), \quad (a_m)_{m\in\mathbb{N}_0} \mapsto \sum_{\substack{m\in\mathbb{N}_0 \\ m\notin 2\,\mathbb{N}_0}} a_m(0, t^m).$$

But we do not want to use only every second term of the sequence $(a_m)_{m\in\mathbb{N}_0}$ (although we want to multiply a_m, $m \in \mathbb{N}_0$, only with the even powers of t), and thus we define the function f to be the unique monotone bijective function from $\mathbb{N}_0 \backslash 2\,\mathbb{N}_0$ to $\mathbb{N}_0$ and modify the definition of Λ_0 as follows:

$$\Lambda_0 : C_2^\infty \to W_1^+(K), \quad (a_m)_{m\in\mathbb{N}_0} \mapsto \sum_{\substack{m\in\mathbb{N}_0 \\ m\notin 2\,\mathbb{N}_0}} a_{f(m)}(0, t^m).$$

We will show in Lemma 4.12 that Λ_0 is a well-defined group homomorphism and we will prove in Proposition 4.14 that the map

$$\Lambda : C_4^\infty \times C_2^\infty \to W_1^+(K), \quad (a^{(1)}, a^{(0)}) \mapsto \Lambda_1(a^{(1)}) \oplus \Lambda_0(a^{(0)})$$

defines an isomorphism of topological groups.

In a similar way we will define an isomorphism Ψ between the subgroup $W_1^-(K)$ and the discrete group $C_{p^2}^{(\infty)} \times C_p^{(\infty)}$.

Recall that W_n denotes the nth Witt group of the field of Laurent series $\mathbb{F}_p((t))$. Generalizing the idea above, we introduce the following notation.

Definition 4.10. Let p be any prime number and $J := \mathbb{N}_0 \setminus p\,\mathbb{N}_0$. Define f to be the unique monotone bijective function from J to $\mathbb{N}_0$.

Definition 4.11. Let $n \in \mathbb{N}_0$ be fixed and define

$$\Lambda_n : C_{p^{n+1}}^\infty \to W_n^+, \quad (a_m^{(n)})_{m\in\mathbb{N}_0} \mapsto \sum_{m\in\mathbb{N}_0} a_m^{(n)}(t^m, 0, \ldots, 0). \tag{11}$$

We view $a_m^{(n)} \in C_{p^{n+1}} = \mathbb{Z}/p^{n+1}\,\mathbb{Z}$ as an integer between 0 and $p^{n+1}-1$ in the canonical way and understand the product $a_m^{(n)}(t^m, 0, \ldots, 0)$ as the $a_m^{(n)}$-fold sum of the $(n+1)$-tuple $(t^m, 0, \ldots, 0)$ in W_n^+. Furthermore, we define for every $0 \le k \le n-1$:

$$\Lambda_k : C_{p^{k+1}}^\infty \to W_n^+, \quad (a_m^{(k)})_{m\in\mathbb{N}_0} \mapsto \sum_{\substack{m\in\mathbb{N}_0 \\ m\notin p\,\mathbb{N}_0}} a_{f(m)}^{(k)}(0, \ldots, t^m, \ldots, 0), \tag{12}$$

where the term t^m is at the $(n+1-k)$th position in the $(n+1)$-tuple $(0, \ldots, t^m, \ldots, 0)$. Again, we view $a_{f(m)}^{(k)} \in C_{p^{k+1}} = \mathbb{Z}/p^{k+1}\,\mathbb{Z}$ as an integer between 0 and $p^{k+1}-1$ in the canonical way and understand the product $a_{f(m)}^{(k)}(0, \ldots, t^m, \ldots, 0)$ as the $a_{f(m)}^{(k)}$-fold sum in W_n^+ of the $(n+1)$-tuple $(0, \ldots, t^m, \ldots, 0)$.

Furthermore, we define

$$\Psi_n : C_{p^{n+1}}^{(\infty)} \to W_n^+, \quad (b_m^{(n)})_{m\in\mathbb{N}} \mapsto \sum_{m\in\mathbb{N}} b_m^{(n)}(t^{-m}, 0, \ldots, 0) \tag{13}$$

and for every $0 \leq k \leq n-1$:

$$\Psi_k : C^{(\infty)}_{p^{k+1}} \to W_n^-, \quad (b_m^{(k)})_{m\in\mathbb{N}} \mapsto \sum_{\substack{m\in\mathbb{N}\\ m\notin p\,\mathbb{N}}} b^{(k)}_{f(m)}(0,\ldots,t^{-m},\ldots,0), \tag{14}$$

where the term t^{-m} stands at the $(n+1-k)$th position in the $(n+1)$-tuple $(0,\ldots,t^{-m},\ldots,0)$. Notice that the sums, appearing in (13) and (14), are finite.

Lemma 4.12. *The map Λ_k is a continuous group homomorphism for every $k \in \{0,\ldots,n\}$.*

Proof. In order to prove that Λ_k is a well-defined map for every $k \in \{0,\ldots,n\}$, we observe that if $(G,+)$ is any abelian group and $g_0,\ldots,g_i \in G$, $i \in \mathbb{N}_0$, then the map $\varphi^i : \mathbb{Z}^{i+1} \to G, (z_0,\ldots,z_i) \mapsto \sum_{m=0}^i z_m g_m$ is a group homomorphism. So if $G = W_n^+$ and if $g_i = (0,\ldots,t^i,\ldots,0)$, $i \in \mathbb{N}_0$, is a vector in W_n^+, where the term t^i stands at the $(n+1-k)$th position in this $(n+1)$-tuple, we obtain for every $0 \leq k \leq n-1$ and every $i \in \mathbb{N}_0$ a group homomorphism

$$\varphi_k^i : \mathbb{Z}^{i+1} \to W_n^+, (z_0,\ldots,z_i) \mapsto \sum_{m=0}^{i} z_m(0,\ldots,t^m,\ldots,0).$$

By the same argument we obtain a group homomorphism

$$\varphi_n^i : \mathbb{Z}^{i+1} \to W_n^+, (z_0,\ldots,z_i) \mapsto \sum_{m=0}^{i} z_m(t^m,\ldots,0,\ldots,0).$$

Moreover, if we denote by $V : W_k \to W_{k+1}$, $(x_0,\ldots,x_k) \mapsto (0,x_0,\ldots,x_k)$ the Shift homomorphism, then one can show (see for example [5]) that for every $k \in \{0,\ldots,n-1\}$, the image $V^{n-k}(W_k) \subseteq W_n$ is isomorphic to W_k. Since $p^{k+1}(x_0,\ldots,x_k) = (0,\ldots,0)$ for every vector $(x_0,\ldots,x_k) \in W_{k+1}$ (Corollary 4.6), we obtain for every $i \in \mathbb{N}_0$ and every $0 \leq k \leq n-1$ a well-defined group homomorphism

$$\Lambda_k^i : (\mathbb{Z}/p^{k+1}\,\mathbb{Z})^{i+1} \to W_n^+,\ (a_0^{(k)},\ldots,a_i^{(k)}) \mapsto \sum_{\substack{m=0\\ m\notin p\,\mathbb{N}_0}}^{i} a^{(k)}_{f(m)}(0,\ldots,t^m,\ldots,0).$$

Furthermore we obtain a well-defined group homomorphism

$$\Lambda_n^i : (\mathbb{Z}/p^{n+1}\,\mathbb{Z})^{i+1} \to W_n^+,\ (a_0^{(n)},\ldots,a_i^{(n)}) \mapsto \sum_{m=0}^{i} a_m^{(n)}(t^m,0,\ldots,0).$$

Using the definition of addition in the nth Witt group W_n we can rewrite for every $k \in \{0,\ldots,n-1\}$, the $a^{(k)}_{f(m)}$-fold sum of the vector $(0,\ldots,t^m,\ldots,0)$ in the following way:

$$a^{(k)}_{f(m)}(0,\ldots,t^m,\ldots,0) = (0,\ldots,0,a^{(k)}_{f(m)}t^m, c^{(k)}_{n+1-k}(m),\ldots,c^{(k)}_n(m)), \tag{15}$$

where every term $c_j^{(k)}(m)$, $j \in \{n+1-k,\ldots,n\}$ is a polynomial in t, whose smallest exponent of t is greater than or equal to m. (If $k=0$, then the terms

$c_j^{(0)}(m)$ do not occur.) Consequently, the sequence $(\Lambda_k^i(a_0^{(k)}, \ldots, a_i^{(k)}))_{i\in\mathbb{N}_0}$ converges in W_n to the element

$$\sum_{\substack{m=0\\ m\notin p\,\mathbb{N}_0}}^{\infty} a_{f(m)}^{(k)}(0, \ldots, t^m, \ldots, 0) = \Lambda_k((a_m^{(k)})_{m\in\mathbb{N}_0})$$

for every $k = 0, \ldots, n-1$. Moreover, the sequence $\Lambda_n^i((a_0^{(n)}, \ldots a_i^{(n)})_{i\in\mathbb{N}_0}$ converges in W_n to the element

$$\sum_{m=0}^{\infty} a_m^{(n)}(t^m, 0, \ldots, 0) = \Lambda_n((a_m^{(n)})_{m\in\mathbb{N}_0}).$$

Now, let $k \in \{0, \ldots, n-1\}$ be fixed. Since the map Λ_k^i is an additive group homomorphism for every $i \in \mathbb{N}_0$, we obtain for all sequences $a^{(k)}$ and $b^{(k)}$ in $C_{p^{k+1}}^{\infty}$:

$$\begin{aligned}
\Lambda_k(a^{(k)} + b^{(k)}) &= \lim_{i\to\infty} \Lambda_k^i((a_0^{(k)}, \ldots, a_i^{(k)}) + (b_0^{(k)}, \ldots, b_i^{(k)})) \\
&= \lim_{i\to\infty} (\Lambda_k^i((a_0^{(k)}, \ldots, a_i^{(k)})) \oplus \Lambda_k^i((b_0^{(k)}, \ldots, b_i^{(k)}))) \\
&= \lim_{i\to\infty} \Lambda_k^i((a_0^{(k)}, \ldots, a_i^{(k)})) \oplus \lim_{i\to\infty} \Lambda_k^i((b_0^{(k)}, \ldots, b_i^{(k)})) \\
&= \Lambda_k(a^{(k)}) \oplus \Lambda_k(b^{(k)}).
\end{aligned}$$

Hence Λ_k is an additive group homomorphism and it follows by the same argument that Λ_n is an additive group homomorphism.

It remains to show that each of the maps Λ_k, $k = 0, \ldots, n$, is continuous. For this, we observe that the infinite direct product $C_{p^{k+1}}^{\infty}$ is clearly a compact group with respect to the product topology. Furthermore, the group W_n^+ is, as a topological space, isomorphic to $(K^+)^n$, the n-fold direct product of compact groups and thus itself compact. So in order to show that the map Λ_k is continuous, it suffices to show that Λ_k is componentwise continuous. We will show that $\pi_j \circ \Lambda_k$ is continuous for every $j \in \{1, \ldots, n+1\}$, where $\pi_j : W_n^+ \to K^+, (x_0, \ldots, x_n) \mapsto x_{j+1}$ denotes the projection onto the jth component.

Let $k \in \{0, \ldots, n\}$. We have

$$\Lambda_k((a_m^{(k)})_{m\in\mathbb{N}_0}) = \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} a_{f(m)}^{(k)}(0, \ldots, t^m, \ldots, 0), \tag{16}$$

where the term t^m is at the $(n+1-k)$th position in the $(n+1)$-tuple $(0, \ldots, t^m, \ldots, 0)$. Hence $\pi_j \circ \Lambda_k((a_m)_{m\in\mathbb{N}_0}) = 0$ for every $j \in \{1, \ldots, n-k\}$ and in particular $\pi_j \circ \Lambda_k$ is continuous for every $j \in \{1, \ldots, n-k\}$. To see that $\pi_j \circ \Lambda_k$ is also continuous for every $j \in \{n+1-k, \ldots, n+1\}$ we observe that the locally compact topology on K is constructed in a way that open sets of the form $U_r :=< t^r >$ with $r \in \mathbb{N}_0$ form a neighborhoodbasis of $0 \in K$. Let U_r, $r \in \mathbb{N}_0$, be such a neighborhood in K. We show that there exists a neighborhood V_{s_r} of $0 \in C^{\infty}_{p^{k+1}}$ such that $(\pi_j \circ \Lambda_k)(V_{s_r}) \subseteq U_r$ for all $j \in \{n+1-k, \dots, n+1\}$. For this, put $s_r := f^{-1}(r)$ and define

$$V_{s_r} := \left\{ (a^{(k)}_m)_{m \in \mathbb{N}_0} \in C^{\infty}_{p^{k+1}} \mid a^{(k)}_i = 0 \text{ for all } i = 0, \dots, s_r - 1 \right\}.$$

This is a neighborhood of 0 in the product topology of $C^{\infty}_{p^{k+1}}$.
Let $(0, \dots, 0, a^{(k)}_{s_r}, a^{(k)}_{s_r+1}, \dots)$ be an arbitrary element of V_{s_r}, then

$$(\pi_j \circ \Lambda_k)(0, \dots, 0, a^{(k)}_{s_r}, a^{(k)}_{s_r+1}, \dots) \in U_r$$

for all $j \in \{n+1-k, \dots, n+1\}$, since we have seen in (15) that the smallest exponent of t appearing in each nonzero entry of

$$\sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\,\mathbb{N}_0}} a^{(k)}_{f(m)}(0, \dots, t^m, \dots, 0)$$

is at least $f(s_r) = r$. Hence the map $\pi_j \circ \Lambda_k$ is continuous for every $j \in \{n+1-k, \dots, n+1\}$, which proves that Λ_k is continuous. The continuity of Λ_n can be obtained in the same way. □

Next, we obtain the same result for the maps $\Psi_k, k = 0, \dots, n$.

Lemma 4.13. *The map Ψ_k, as defined in (14), is a continuous group homomorphism for every $k \in \{0, \dots n-1\}$. Furthermore, the map Ψ_n, as defined in (13), is a continuous group homomorphism.*

Proof. We can apply the same arguments as in the proof of Lemma 4.12. In fact, the proof is even simpler since all the sums appearing in the definition of the maps $\Psi_k, k \in \{0, \dots n-1\}$ and Ψ_n are finite. □

We will now state and prove the key result of this section, namely that the sum of all the continuous group homomorphisms Λ_k, $k \in \{0, \dots, n\}$, yields an isomorphism Λ between the group W_n^+ and the compact group $C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_p$. Furthermore, we prove that the sum of all the continuous group homomorphisms Ψ_k, $k \in \{0, \dots, n\}$, yields an isomorphism Ψ between the group W_n^- and the discrete group $C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_p$.

Proposition 4.14.

(i) *The map*

$$\Lambda : C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_p \to W_n^+,$$
$$(a^{(n)}, a^{(n-1)}, \dots, a^{(0)}) \mapsto \Lambda_n(a^{(n)}) \oplus \Lambda_{n-1}(a^{(n-1)}) \oplus \cdots \oplus \Lambda_0(a^{(0)})$$

is an isomorphism of topological groups.

(ii) *The map*

$$\Psi : C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_p \to W_n^-,$$
$$(b^{(n)}, b^{(n-1)}, \dots, b^{(0)}) \mapsto \Psi_n(b^{(n)}) \oplus \Psi_{n-1}(b^{(n-1)}) \oplus \cdots \oplus \Psi_0(b^{(0)})$$

is an isomorphism of topological groups.

Before we give a proof of this proposition we rewrite the sum appearing in the definition of the map Λ in the following way. Recall that we have for all vectors $(a^{(n)}, a^{(n-1)}, \dots, a^{(0)}) \in C^\infty_{p^{n+1}} \times C^\infty_{p^n} \times \cdots \times C^\infty_p$,

$$\Lambda((a^{(n)}, a^{(n-1)}, \dots, a^{(0)})) = \Lambda_n(a^{(n)}) \oplus \Lambda_{n-1}(a^{(n-1)}) \oplus \cdots \oplus \Lambda_0(a^{(0)}) = \sum_{m\in\mathbb{N}_0} a^{(n)}_m (t^m, 0, \dots, 0) \oplus \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} a^{(n-1)}_{f(m)} (0, t^m, 0, \dots, 0) \oplus \cdots \oplus \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} a^{(0)}_{f(m)} (0, \dots, 0, t^m).$$

We may write every coefficient $a^{(k)}_{f(m)} \in C_{p^{k+1}} = \mathbb{Z}/p^{k+1}\mathbb{Z}$, $0 \le k < n$, $m \in \mathbb{N}_0$, with respect to its p-adic expansion, i.e., we can find uniquely determined numbers $0 \le a^{(k)}_{f(m)_j} \le p-1$, $j = 0, \dots, k$, such that

$$a^{(k)}_{f(m)} = a^{(k)}_{f(m)_0} + a^{(k)}_{f(m)_1} p + a^{(k)}_{f(m)_2} p^2 + \cdots + a^{(k)}_{f(m)_k} p^k. \tag{17}$$

In the same way we can write every coefficient $a^{(n)}_m \in C_{p^{n+1}} = \mathbb{Z}/p^{n+1}\mathbb{Z}$, $m \in \mathbb{N}_0$, with respect to its p-adic expansion, i.e., we can find uniquely determined numbers $0 \le a^{(n)}_{m_j} \le p-1$, $j = 0, \dots, n$, such that

$$a^{(n)}_{(m)} = a^{(n)}_{m_0} + a^{(n)}_{m_1} p + a^{(n)}_{m_2} p^2 + \cdots + a^{(n)}_{m_n} p^n. \tag{18}$$

Using (17) and (18) we obtain for all $0 \le k \le n-1$:

$$\begin{aligned} a^{(k)}_{f(m)}(0, \dots, t^m, \dots, 0) &= (a^{(k)}_{f(m)_0} + a^{(k)}_{f(m)_1} p + \cdots + a^{(k)}_{f(m)_k} p^k)(0, \dots, t^m, \dots, 0) = \\ &a^{(k)}_{f(m)_0}(0, \dots, t^m, \dots, 0) \oplus a^{(k)}_{f(m)_1} p\,(0, \dots, t^m, \dots, 0) \oplus \\ &\cdots \oplus a^{(k)}_{f(m)_k} p^k\,(0, \dots, t^m, \dots, 0). \end{aligned}$$

If we apply part (ii) of Lemma 4.5 to the last expression we obtain

$$\begin{aligned} a^{(k)}_{f(m)}(0, \dots, t^m, \dots, 0) &= \\ &a^{(k)}_{f(m)_0}(0, \dots, t^m, \dots, 0) \oplus a^{(k)}_{f(m)_1}(0, \dots, t^{mp}, \dots, 0) \oplus \\ &\cdots \oplus a^{(k)}_{f(m)_k}(0, \dots, 0, t^{mp^k}). \end{aligned}$$

Using the definition of addition in the nth Witt group we obtain, as in (15), for every $0 \le k \le n-1$ and $0 \le j \le k$:

$$a^{(k)}_{f(m)_j}(0, \dots, t^{mp^j}, \dots, 0) = (0, \dots, 0, a^{(k)}_{f(m)_j} t^{mp^j}, c^{(j,k)}_{n+1-k+j}(m), \dots, c^{(j,k)}_n(m)),$$

where all the terms $c^{(j,k)}_{n-k+1+j}(m), \dots, c^{(j,k)}_n(m)$ are polynomials in t, whose smallest exponent of t is greater than or equal to m and whose coefficients are uniquely determined by the coefficients $a^{(k)}_{f(m)_j}$. Notice that if $k=0$, then the terms $c^{(j,0)}$ do not occur. Furthermore, we have for all $0 \le j \le n$:

$$a^{(n)}_{m_j}(0,\dots,t^{mp^j},\dots,0) = (0,\dots,0,a^{(n)}_{m_j}t^{mp^j},c^{(j,n)}_{j+1}(m),\dots,c^{(j,n)}_n(m)),$$

where all the terms $c^{(j,n)}_{j+1}(m),\dots,c^{(j,n)}_n(m)$ are polynomials in t, whose smallest exponent of t is at least m and whose coefficients are uniquely determined by the coefficients $a^{(n)}_{f(m)_j}$.

With this notation we obtain, for every $0 \le k \le n-1$,

$$\begin{aligned}
(19)\quad & a^{(k)}_{f(m)}(0,\dots,t^m,\dots,0) = \\
&\quad (0,\dots,0,a^{(k)}_{f(m)_0}t^m,c^{(0,k)}_{n+1-k},\dots,c^{(0,k)}_n) \\
&\quad \oplus (0,\dots,0,a^{(k)}_{f(m)_1}t^{mp},c^{(1,k)}_{n+1-k+1},\dots,c^{(1,k)}_n)\oplus \\
&\quad \cdots \oplus (0,\dots,0,a^{(k)}_{f(m)_k}t^{mp^k}).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
(20)\quad & a^{(k)}_m(t^m,0,\dots,0) = \\
&\quad (a^{(n)}_{m_0}t^m,c^{(0,n)}_1,\dots,c^{(0,n)}_n)\oplus(0,a^{(n)}_{m_1}t^{mp},c^{(1,n)}_2,\dots,c^{(1,n)}_n)\oplus \\
&\quad \cdots \oplus (0,\dots,0,a^{(n)}_{m_n}t^{mp^n}).
\end{aligned}$$

If we use (19) for every $k \in \{0,\dots,n-1\}$ and (20) for $k=n$, then we obtain

$$\begin{aligned}
&\Lambda((a^{(n)},a^{(n-1)},\dots,a^{(0)})) \\
&= \sum_{m\in\mathbb{N}_0} a^{(n)}_m(t^m,0,\dots,0) \oplus \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} a^{(n-1)}_{f(m)}(0,t^m,0,\dots,0) \\
&\qquad \oplus\cdots\oplus \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} a^{(0)}_{f(m)}(0,\dots,0,t^m) \\
&= \sum_{m\in\mathbb{N}_0} [(a^{(n)}_{m_0}t^m,c^{(0,n)}_1(m),\dots,c^{(0,n)}_n(m)) \\
&\quad \oplus(0,a^{(n)}_{m_1}t^{mp},c^{(1,n)}_2(m),\dots,c^{(1,n)}_n(m))\oplus\cdots\oplus(0,\dots,0,a^{(n)}_{m_n}t^{mp^n})] \\
&\quad \oplus \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} [(0,a^{(n-1)}_{f(m)_0}t^m,c^{(0,n-1)}_2(m),\dots,c^{(0,n-1)}_n(m)) \\
&\quad \oplus(0,0,a^{(n-1)}_{f(m)_1}t^{mp},c^{(1,n-1)}_3(m),\dots,c^{(1,n-1)}_n(m)) \\
&\qquad\qquad \oplus\cdots\oplus(0,\dots,0,a^{(n-1)}_{f(m)_{n-1}}t^{mp^{n-1}})]\oplus \\
&\quad \cdots\oplus \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} (0,\dots,0,a^{(0)}_{f(m)_0}t^m).
\end{aligned}$$

Changing the order of summation in a suitable way yields

$$
\begin{aligned}
(21)\qquad &\Lambda((a^{(n)}, a^{(n-1)}, \dots, a^{(0)}))\\
&= \sum_{m\in\mathbb{N}_0} (a^{(n)}_{m_0} t^m, c^{(0,n)}_1(m), \dots, c^{(0,n)}_n(m))\\
&\oplus \sum_{m\in\mathbb{N}_0} (0, a^{(n)}_{m_1} t^{mp}, c^{(1,n)}_2(m), \dots, c^{(1,n)}_n(m))\\
&\oplus \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} (0, a^{(n-1)}_{f(m)_0} t^m, c^{(0,n-1)}_2(m), \dots, c^{(0,n-1)}_n(m))\\
&\oplus \sum_{m\in\mathbb{N}_0} (0, 0, a^{(n)}_{m_2} t^{mp^2}, c^{(2,n)}_3(m), \dots, c^{(2,n)}_n(m))\\
&\oplus \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} [(0, 0, a^{(n-1)}_{f(m)_1} t^{mp}, c^{(1,n-1)}_3(m), \dots, c^{(1,n-1)}_n(m))\\
&\oplus (0, 0, a^{(n-2)}_{f(m)_0} t^m, c^{(0,n-2)}_3(m), \dots, c^{(0,n-2)}_n(m))]\oplus\\
&\cdots \oplus \sum_{m\in\mathbb{N}_0} (0, \dots, 0, a^{(n)}_{m_n} t^{mp^n})\\
&\oplus \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} [(0, \dots, 0, a^{(n-1)}_{f(m)_{n-1}} t^{mp^{n-1}})\\
&\qquad \oplus (0, \dots, 0, a^{(n-2)}_{f(m)_{n-2}} t^{mp^{n-2}}) \oplus \dots \oplus (0, \dots, 0, a^{(0)}_{f(m)_0} t^m)].
\end{aligned}
$$

To avoid lengthy descriptions we will use the following notation.

Notation 4.15. Let $t, s \in \mathbb{Z}$ with $t \mid s$, and let $a, b \in \mathbb{Z}/s\,\mathbb{Z} = C_s$. We say that $a = b \bmod t$ if $a + t\cdot C_s = b + t\cdot C_s$.

We now prove Proposition 4.14.

Proof. To (i): Observe first that, as a direct consequence of Lemma 4.12, the map Λ is a well-defined group homomorphism. So it remains to show that Λ is onto, one-to-one, and bi-continuous.

In order to prove the surjectivity of Λ, let $(x^{(n)}, x^{(n-1)}, \dots, x^{(0)})$ be an arbitrary vector of W_n^+. Each entry $x^{(k)}$, $k = 0, \dots, n$, of this vector is of the form

$$x^{(k)} = \sum_{i=0}^{\infty} x^{(k)}_i t^i \quad \text{ with } x^{(k)}_i \in C_p = \mathbb{Z}/p\,\mathbb{Z}.$$

We need to show that there exists a vector

$$(a^{(n)}, a^{(n-1)}, \dots, a^{(0)}) \in C^\infty_{p^{n+1}} \times C^\infty_{p^n} \times \dots \times C^\infty_p$$

with the property that

$$\Lambda((a^{(n)}, a^{(n-1)}, \dots, a^{(0)})) = (x^{(n)}, x^{(n-1)}, \dots, x^{(0)}). \tag{22}$$

We prove by induction on k, $k \in \{0, \ldots, n\}$, the statement
$I(k)$: We can find

(1.) a series $a^{(n)} \in C^{\infty}_{p^{n+1}}$, which is uniquely determined mod p^{k+1}, i.e., each element $a^{(n)}_m \in C_{p^{n+1}}$, $m \in \mathbb{N}_0$, of the series $a^{(n)}$ is uniquely determined mod p^{k+1}, and

(2.) for every $1 \leq j \leq k$, a series $a^{(n-j)} \in C^{\infty}_{p^{n-j+1}}$, which is uniquely determined mod p^{k+1-j}, i.e., each element $a^{(n-j)}_{f(m)} \in C_{p^{n-j+1}}$, $m \in \mathbb{N}_0$, of the series $a^{(n-j)}$ is uniquely determined mod p^{k+1-j},

such that the vector $(a^{(n)}, a^{(n-1)}, \ldots, a^{(0)})$ satisfies Equation (22).

Notice that this proves then the surjectivity of Λ since by $I(n)$ we can find series

$$a^{(n)} \in C^{\infty}_{p^{n+1}}, a^{(n-1)} \in C^{\infty}_{p^n}, \ldots, \text{ and } a^{(0)} \in C^{\infty}_{p}$$

such that the vector $(a^{(n)}, a^{(n-1)}, \ldots, a^{(0)})$ satisfies (22).

If $k = 0$, we use the summation formula (21) to compare the first component of the vector $\Lambda((a^{(n)}, a^{(n-1)}, \ldots, a^{(0)}))$ with the first component of the vector $(x^{(n)}, x^{(n-1)}, \ldots, x^{(0)})$. This yields the following conditions for the series $a^{(n)} \in C^{\infty}_{p^{n+1}}$:

$$\sum_{m\in\mathbb{N}_0} a^{(n)}_{m_0} t^m = \sum_{m\in\mathbb{N}_0} x^{(n)}_m t^m.$$

By comparing the coefficients of these sums we obtain the defining equation:

$$a^{(n)}_{m_0} := x^{(n)}_m \quad \forall m \in \mathbb{N}_0 \,. \tag{23}$$

But this means that the coefficients $a^{(n)}_m$, $m \in \mathbb{N}_0$, of the series $a^{(n)} \in C^{\infty}_{p^{n+1}}$ are determined mod p and we have proven the base case $I(0)$.

If $k = 1$, we use again formula (21) to compare the second component of the vector $\Lambda((a^{(n)}, a^{(n-1)}, \ldots, a^{(0)}))$ with the second component of the vector $(x^{(n)}, x^{(n-1)}, \ldots, x^{(0)})$. This leads to the following conditions for the series $a^{(n)} \in C^{\infty}_{p^{n+1}}$ and $a^{(n-1)} \in C^{\infty}_{p^n}$:

$$\sum_{m\in\mathbb{N}_0} a^{(n)}_{m_1} t^{mp} + \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} (a^{(n-1)}_{f(m)_0} t^m + c^{(0,n)}_1(m)) = \sum_{m\in\mathbb{N}_0} x^{(n-1)}_m t^m. \tag{24}$$

Notice that there do not appear the same exponents of t in the expressions

$$\sum_{m\in\mathbb{N}_0} a^{(n)}_{m_1} t^{mp} \quad \text{and} \quad \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} a^{(n-1)}_{f(m)_0} t^m.$$

Furthermore, all coefficients appearing in the polynomials $c^{(0,n)}_1(m)$, $m \in \mathbb{N}_0$, depend only on the numbers $a^{(n)}_{m_0}$, which are already uniquely defined by Equation (23). Thus if we compare coefficients in (24) we obtain the following defining equations:

$$
(25)\qquad \begin{aligned} a_{m_1}^{(n)} &:= x_{mp}^{(n-1)} - F(mp) &&\text{for all } m \in \mathbb{N}_0 \text{ and} \\ a_{f(m)_0}^{(n-1)} &:= x_m^{(n-1)} - F(m) &&\text{for all } m \in \mathbb{N}_0 \setminus p\,\mathbb{N}_0, \end{aligned}
$$

where $F(m)$ and $F(mp)$ denote some numbers, which depend only on the coefficients of the term $c_1^{(0,n)}(m)$ and hence on the numbers $a_{m_0}^{(n)}$, $m \in \mathbb{N}_0$. Thus the series, $a^{(n)}$ and $a^{(n-1)}$, are determined mod p^2 and mod p, respectively, and we have proven the statement $I(1)$.

Let $k \in \{0, \ldots, n-1\}$ be fixed and assume that $I(j)$ holds for every $0 \le j \le k$. In order to prove the statement $I(k+1)$ we use again formula (21) and compare the $(k+2)$nd component of the vector $\Lambda((a^{(n)}, a^{(n-1)}, \ldots, a^{(0)}))$ with the $(k+2)$nd component of the vector $(x^{(n)}, x^{(n-1)}, \ldots, x^{(0)})$. This yields the following condition for the series $a^{(n)} \in C_{p^{n+1}}^{\infty}, a^{(n-1)} \in C_{p^n}^{\infty}, \ldots$, and $a^{n-(k+1)} \in C_{p^{n-k}}^{\infty}$:

$$
(26)\qquad \sum_{m \in \mathbb{N}_0} a_{m_{k+1}}^{(n)} t^{mp^{k+1}} + \sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\,\mathbb{N}_0}} [a_{f(m)_k}^{(n-1)} t^{mp^k} + \cdots + a_{f(m)_0}^{(n-(k+1))} t^m] + X(m) = \sum_{m \in \mathbb{N}_0} x_m^{(n-(k+1))} t^m,
$$

where $X(m)$ is a polynomial in t, whose coefficients consist of linear combinations in the numbers

$$
a_{m_0}^{(n)}, a_{m_1}^{(n)}, \ldots, a_{m_k}^{(n)}; a_{f(m)_0}^{(n-1)}, \ldots, a_{f(m)_{k-1}}^{(n-1)}; a_{f(m)_0}^{(n-2)}, \ldots, a_{f(m)_{k-2}}^{(n-2)}; \ldots; a_{f(m)_0}^{(n-k)}.
$$

But it follows from the induction hypothesis that these numbers are already uniquely determined by the elements of the series $x^{(n)}, x^{(n-1)}, \ldots$, and $x^{(n-k)}$. Furthermore, there do not appear the same exponents of t in the sums

$$
\sum_{m \in \mathbb{N}_0} a_{m_{k+1}}^{(n)} t^{mp^{k+1}}, \quad \sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\,\mathbb{N}_0}} a_{f(m)_k}^{(n-1)} t^{mp^k}, \cdots, \text{ and } \sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\,\mathbb{N}_0}} a_{f(m)_0}^{(n-(k+1))} t^m,
$$

so that comparing coefficients in (26) leads to the following defining equations for elements of the series $a^{(n)}, a^{(n-1)}, a^{(n-2)}, \ldots$, and $a^{(n-(k+1))}$:

$$
\begin{aligned}
a_{m_{k+1}}^{(n)} &:= x_{mp^{k+1}}^{(n-(k+1))} - F(mp^{k+1}) &&\text{for all } m \in \mathbb{N}_0 \\
a_{f(m)_k}^{(n-1)} &:= x_{mp^k}^{(n-(k+1))} - F(mp^k) &&\text{for all } m \in \mathbb{N}_0 \setminus p\,\mathbb{N}_0 \\
a_{f(m)_{k-1}}^{(n-2)} &:= x_{mp^{k-1}}^{(n-(k+1))} - F(mp^{k-1}) &&\text{for all } m \in \mathbb{N}_0 \setminus p\,\mathbb{N}_0 \\
&\ \vdots \qquad\qquad \vdots &&\qquad \vdots \\
a_{f(m)_0}^{(n-(k+1))} &:= x_m^{(n-(k+1))} - F(m) &&\text{for all } m \in \mathbb{N}_0 \setminus p\,\mathbb{N}_0\,.
\end{aligned}
$$

Therefore, the series $a^{(n)}$, $a^{(n-1)}$, $a^{(n-2)}$, …, $a^{(n-(k+1))}$ are determined modulo p^{k+2}, p^{k+1}, p^k …, p, respectively, and so we have proven $I(k+1)$.

To prove the injectivity of the map Λ, let

$$(a^{(n)}, a^{(n-1)}, \dots, a^{(0)}) \in C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p}$$

with

$$\Lambda((a^{(n)}, a^{(n-1)}, \dots, a^{(0)})) = (0, \dots, 0) \in W_n^+.$$

We prove by induction on k, $k \in \{0, \dots, n\}$, the statement
$I(k)$: The series $a^{(n)}, a^{(n-1)}, \dots, a^{(n-k)}$ satisfy

(1.) $a^{(n)} = 0 \mod p^{k+1}$, i.e., $a^{(n)}_m = 0 \mod p^{k+1}$ for all $m \in \mathbb{N}_0$, and
(2.) $a^{(n-j)} = 0 \mod p^{k+1-j}$ for all $j \in \{1, \dots, k\}$, i.e., $a^{(n-j)}_{f(m)} = 0 \mod p^{k+1-j}$ for all $m \in \mathbb{N}_0$.

If $k = 0$, we obtain with formula (21)

$$\sum_{m \in \mathbb{N}_0} a^{(n)}_{m_0} t^m = 0, \tag{27}$$

and hence

$$a^{(n)}_{m_0} = 0 \text{ for all } m \in \mathbb{N}_0,$$

which proves the base case $I(0)$.

So let $k \in \{0, \dots, n-1\}$ be fixed and assume that $I(j)$ holds for every $j \in \{0, \dots, k\}$. Then we have, for every $m \in \mathbb{N}_0$,

(1.) $a^{(n)}_m = 0 \mod p^{k+1}$ and hence $a^{(n)}_{m_0} = a^{(n)}_{m_1} = \cdots = a^{(n)}_{m_k} = 0$, and
(2.) $a^{(n-j)}_{f(m)} = 0 \mod p^{k+1-j}$ for all $j \in \{1, \dots, k\}$ and hence

$$a^{(n-j)}_{f(m)_0} = a^{(n-j)}_{f(m)_1} = \cdots = a^{(n-j)}_{f(m)_{k-j}} = 0.$$

Thus, if we set the $(k+2)$nd component of the vector $\Lambda((a^{(n)}, a^{(n-1)}, \dots, a^{(0)}))$ equal to zero, we obtain, from formula (21), the following equation

$$\sum_{m \in \mathbb{N}_0} a^{(n)}_{m_{k+1}} t^{mp^{k+1}} + \sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\,\mathbb{N}_0}} [a^{(n-1)}_{f(m)_k} t^{mp^k} + \cdots + a^{(n-(k+1))}_{f(m)_0} t^m] + X(m) = 0, \tag{28}$$

where $X(m)$ is a polynomial in t, whose coefficients consist of linear combinations in the numbers

$$a^{(n)}_{m_0}, a^{(n)}_{m_1}, \dots, a^{(n)}_{m_k}, a^{(n-1)}_{f(m)_0}, \dots, a^{(n-1)}_{f(m)_{k-1}}, a^{(n-2)}_{f(m)_0}, \dots, a^{(n-2)}_{f(m)_{k-2}}, \dots, a^{(n-k)}_{f(m)_0}.$$

But it follows from the induction hypothesis that these numbers are all equal to zero and hence $X(m) = 0$. Therefore, (28) turns into

$$\sum_{m \in \mathbb{N}_0} a^{(n)}_{m_{k+1}} t^{mp^{k+1}} + \sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\,\mathbb{N}_0}} [a^{(n-1)}_{f(m)_k} t^{mp^k} + \cdots + a^{(n-(k+1))}_{f(m)_0} t^m] = 0. \tag{29}$$

Since there do not appear the same exponents of t in the sums

$$\sum_{m \in \mathbb{N}_0} a^{(n)}_{m_{k+1}} t^{mp^{k+1}}, \sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\,\mathbb{N}_0}} a^{(n-1)}_{f(m)_k} t^{mp^k}, \cdots, \text{ and } \sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\,\mathbb{N}_0}} a^{(n-(k+1))}_{f(m)_0} t^m,$$

we may easily compare coefficients in (29) and obtain

$$\begin{aligned} a^{(n)}_{m_{k+1}} &= 0 \quad \text{for all } m \in \mathbb{N}_0, \\ a^{(n-1)}_{f(m)_k} &= 0 \quad \text{for all } m \in \mathbb{N}_0 \setminus p\,\mathbb{N}_0, \\ a^{(n-2)}_{f(m)_{k-1}} &= 0 \quad \text{for all } m \in \mathbb{N}_0 \setminus p\,\mathbb{N}_0, \\ \vdots \quad &\vdots \qquad\qquad \vdots \\ a^{(n-(k+1))}_{f(m)_0} &= 0 \quad \text{for all } m \in \mathbb{N}_0 \setminus p\,\mathbb{N}_0\,. \end{aligned}$$

This proves the statement $I(k+1)$ and hence the injectivity of the map Λ.

It remains to prove that the map Λ is bi-continuous. But we have seen already in Lemma 4.12 that each map $\Lambda_k : C^{\infty}_{p^{k+1}} \to W_n^+$, $k \in \{0, \dots, n\}$ is bi-continuous. Hence Λ is, as the sum of bi-continuous maps, itself bi-continuous.

The proof of part (ii) is similar. □

We now establish the main theorem of this section. It summarizes the results obtained so far and gives us precise information about the structure of the nth Witt group, $W_n(K)$, of the field $K = \mathbb{F}_p((t))$.

Theorem 4.16. *Let $K = \mathbb{F}_p((t))$ for some prime p, let $n \in \mathbb{N}_0$, and let W_n be the nth Witt group of K. The map*

$$\begin{aligned} \Theta : (C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p}) \times (C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_{p}) \longrightarrow W_n, \\ ((a^{(n)}, a^{(n-1)}, \dots, a^{(0)}), (b^{(n)}, b^{(n-1)}, \dots, b^{(0)})) \longmapsto \\ \Lambda((a^{(n)}, a^{(n-1)}, \dots, a^{(0)})) \oplus \Psi((b^{(n)}, b^{(n-1)}, \dots, b^{(0)})), \end{aligned}$$

where Λ and Ψ are defined as in Proposition 4.14, is an isomorphism of topological groups.

Proof. The map

$$\mu : W_n^+ \times W_n^- \to W_n, (x, y) \mapsto x \oplus y$$

is an isomorphism of topological groups (Lemma 4.8) and we have

$$\Theta((a, b)) = \mu(\Lambda(a), \Psi(b))$$

for all $a \in C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p}$ and $b \in C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_{p}$. Since, by Proposition 4.14, both maps Λ and Ψ are isomorphisms of topological groups, it follows that Θ is, as the composition of isomorphisms, itself an isomorphism of topological groups. □

We may also obtain a more general version of Theorem 4.16, i.e., a similar decomposition of the nth Witt group of every local field of characteristic p. For this, let p be a prime, let $k = \mathbb{F}_{p^r}$ for some fixed $r \in \mathbb{N}$, and let $K := \mathbb{F}_{p^r}((t))$ be the field of formal Laurent series over k. As for the field $\mathbb{F}_p((t))$ (see Notation 4.7), we introduce the following notations. We define

- $K^+ := k[[t]] \subseteq K$ to be the power series ring over k,
- $K^- := \{a_1 t^{-1} + a_2 t^{-2} + \ldots + a_n t^{-n} \mid n \in \mathbb{N},\ a_i \in k\}$,

- $W_n^+(K) := \{(x_0, \dots, x_n) \in W_n \mid x_i \in K^+ \text{ for all } 0 \le i \le n\}$, and
- $W_n^-(K) := \{(x_0, \dots, x_n) \in W_n \mid x_i \in K^- \text{ for all } 0 \le i \le n\}$.

The set K^- is an additive subgroup of K which is also closed under multiplication. Clearly, every element $a \in K$ can be written uniquely as $a = a^+ + a^-$, where $a^+ \in K^+$ and $a^- \in K^-$. Furthermore, we can decompose the nth Witt group $W_n(K)$ into a direct product of its subgroups $W_n^+(K)$ and $W_n^-(K)$.

Lemma 4.17.

(i) *The sets $W_n^+(K)$ and $W_n^-(K)$, defined as above, are subgroups of $W_n(K)$ and we have $W_n^+(K) \cap W_n^-(K) = \{0\}$.*

(ii) *The map $\mu : W_n^+(K) \times W_n^-(K) \to W_n(K)$, $(x, y) \mapsto x \oplus y$ is a bicontinuous isomorphism.*

Proof. The proof of Lemma 4.8 goes through without any modifications. □

We will show in the remaining part of this section that, for every $n \in \mathbb{N}_0$, the nth Witt group $W_n(K)$ can be decomposed as follows.

$$\begin{aligned} W_n(K) &\cong W_n^-(K) \times W_n^+(K) \\ &\cong (C_{p^{n+1}}^{(\infty)})^r \times (C_{p^n}^{(\infty)})^r \times \cdots \times (C_p^{(\infty)})^r \\ &\qquad \times (C_{p^{n+1}}^{\infty})^r \times (C_{p^n}^{\infty})^r \times \cdots \times (C_p^{\infty})^r. \end{aligned}$$

We observe that the finite field $k = \mathbb{F}_{p^r}$ is a vector space over the finite field $\mathbb{F}_p$, and we can choose elements $\omega_0, \omega_1, \dots, \omega_{r-1} \in \mathbb{F}_{p^r}$ such that the set $\{\omega_0, \omega_1, \dots, \omega_{r-1}\}$ is a basis of $\mathbb{F}_{p^r}$ over $\mathbb{F}_p$, i.e., for every element $x \in \mathbb{F}_{p^r}$ there exist unique elements ${}^0a, {}^1a, \dots, {}^{r-1}a \in \mathbb{F}_p$ such that $x = \sum_{i=0}^{r-1} {}^i a \omega_i$.

In the same way as in Definition 4.11, we can now define maps Λ_k and Ψ_k, $k \in \{0, 1, \dots, n\}$, between the r-fold direct product of $C_{p^{k+1}}^{\infty}$ and $W_n^+(K)$, and between the r-fold direct product of $C_{p^{k+1}}^{(\infty)}$ and $W_n^-(K)$. Recall that $f : \mathbb{N}_0 \setminus p\,\mathbb{N}_0 \to \mathbb{N}_0$ denotes the unique monotone bijective function from $\mathbb{N}_0 \setminus p\,\mathbb{N}_0$ to $\mathbb{N}_0$.

Definition 4.18. Let $n \in \mathbb{N}_0$ be fixed and define

$$\begin{aligned} &\Lambda_n : (C_{p^{n+1}}^{\infty})^r \longrightarrow W_n^+(K), \\ &\quad \big(({}^0a_m^{(n)})_{m\in\mathbb{N}_0}, ({}^1a_m^{(n)})_{m\in\mathbb{N}_0}, \dots, ({}^{r-1}a_m^{(n)})_{m\in\mathbb{N}_0}\big) \longmapsto \\ &\qquad\qquad \sum_{m\in\mathbb{N}_0} \big({}^0a_m^{(n)}(\omega_0 t^m, 0, \dots, 0) \oplus {}^1a_m^{(n)}(\omega_1 t^m, 0, \dots, 0) \oplus \\ &\qquad\qquad\qquad\qquad \cdots \oplus {}^{r-1}a_m^{(n)}(\omega_{r-1} t^m, 0, \dots, 0)\big). \end{aligned}$$

For every $i = 0, \dots, r-1$, we view ${}^i a_m^{(n)} \in C_{p^{n+1}} = \mathbb{Z}/p^{n+1}\mathbb{Z}$ as an integer between 0 and $p^{n+1} - 1$ in the canonical way and understand the product ${}^i a_m^{(n)}(\omega_i t^m, 0, \dots, 0)$ as the ${}^i a_m^{(n)}$-fold sum of the $(n+1)$-tuple $(\omega_i t^m, 0, \dots, 0)$ in $W_n^+(K)$.

Furthermore, we define for every $0 \leq k \leq n-1$:

$$\begin{aligned}
\Lambda_k : (C^{\infty}_{p^{k+1}})^r &\longrightarrow W_n^+(K), \\
\big(({}^0a_m^{(k)})_{m\in\mathbb{N}_0}, ({}^1a_m^{(k)})_{m\in\mathbb{N}_0}, \ldots, ({}^{r-1}a_m^{(k)})_{m\in\mathbb{N}_0}\big) &\longmapsto \\
\sum_{\substack{m\in\mathbb{N}_0 \\ m\notin p\,\mathbb{N}_0}} \big({}^0a_{f(m)}^{(k)}(0,\ldots,0,\omega_0^{p^{n-k}}t^m,\ldots,0) \oplus& \\
{}^1a_{f(m)}^{(k)}(0,\ldots,0,\omega_1^{p^{n-k}}t^m,0,\ldots,0) \oplus& \\
\cdots \oplus {}^{r-1}a_{f(m)}^{(k)}(0,\ldots,0,\omega_{r-1}^{p^{n-k}}t^m,0,\ldots,0)\big),&
\end{aligned}$$

where, for every $i = 0,\ldots,r-1$, the term $\omega_i^{p^{n-k}}t^m$ is at the $(n+1-k)$th position in the $(n+1)$-tuple $(0,\ldots,\omega_i^{p^{n-k}}t^m,\ldots,0)$. Again, we view $a_{f(m)}^{(k)} \in C_{p^{k+1}} = \mathbb{Z}/p^{k+1}\mathbb{Z}$ as an integer between 0 and $p^{k+1}-1$ in the canonical way and understand the product ${}^ia_{f(m)}^{(k)}(0,\ldots,\omega_i^{p^{n-k}}t^m,\ldots,0)$ as the ${}^ia_{f(m)}^{(k)}$-fold sum in $W_n^+(K)$ of the $(n+1)$-tuple $(0,\ldots,\omega_i^{p^{n-k}}t^m,\ldots,0)$.

Similarly, we define

$$\begin{aligned}
\Psi_n : (C^{(\infty)}_{p^{n+1}})^r &\longrightarrow W_n^+(K), \\
\big(({}^0b_m^{(n)})_{m\in\mathbb{N}_0}, ({}^1b_m^{(n)})_{m\in\mathbb{N}_0}, \ldots, ({}^{r-1}b_m^{(n)})_{m\in\mathbb{N}_0}\big) &\longmapsto \\
\sum_{m\in\mathbb{N}_0} \big({}^0b_m^{(n)}(\omega_0t^{-m},0,\ldots,0) \oplus {}^1b_m^{(n)}(\omega_1t^{-m},0,\ldots,0) \oplus& \\
\cdots \oplus {}^{r-1}b_m^{(n)}(\omega_{r-1}t^{-m},0,\ldots,0)\big)&
\end{aligned}$$

and for every $0 \leq k \leq n-1$:

$$\begin{aligned}
\Psi_k : (C^{(\infty)}_{p^{k+1}})^r &\to W_n^-(K), \\
\big(({}^0b_m^{(k)})_{m\in\mathbb{N}_0}, ({}^1b_m^{(k)})_{m\in\mathbb{N}_0}, \ldots, ({}^{r-1}b_m^{(k)})_{m\in\mathbb{N}_0}\big) &\longmapsto \\
\sum_{\substack{m\in\mathbb{N}_0 \\ m\notin p\,\mathbb{N}_0}} \big({}^0b_{f(m)}^{(k)}(0,\ldots,0,\omega_0^{p^{n-k}}t^{-m},\ldots,0) \oplus& \\
{}^1b_{f(m)}^{(k)}(0,\ldots,0,\omega_1^{p^{n-k}}t^{-m},0,\ldots,0) \oplus& \\
\cdots \oplus {}^{r-1}b_{f(m)}^{(k)}(0,\ldots,0,\omega_{r-1}^{p^{n-k}}t^{-m},0,\ldots,0)\big),&
\end{aligned}$$

where, for every $i = 0,\ldots,r-1$, the term $\omega_i^{p^{n-k}}t^{-m}$ is at the $(n+1-k)$th position in the $(n+1)$-tuple $(0,\ldots,\omega_i^{p^{n-k}}t^{-m},\ldots,0)$. Again, we view $b_{f(m)}^{(k)} \in C_{p^{k+1}} = \mathbb{Z}/p^{k+1}\mathbb{Z}$ as an integer between 0 and $p^{k+1}-1$ in the canonical way and understand the product ${}^ib_{f(m)}^{(k)}(0,\ldots,\omega_i^{p^{n-k}}t^{-m},\ldots,0)$ as the ${}^ib_{f(m)}^{(k)}$-fold sum in $W_n^-(K)$ of the $(n+1)$-tuple $(0,\ldots,\omega_i^{p^{n-k}}t^{-m},\ldots,0)$.

Lemma 4.19. *The maps Λ_k and Ψ_k, as defined in Definition 4.18, are continuous group homomorphisms for every $k \in \{0, \dots, n\}$.*

Proof. Let $k \in \{0, \dots, n\}$. For every $i \in \{0, \dots, r-1\}$, we obtain by the same arguments as in the proof of Lemma 4.12 a continuous group homomorphism

$$\begin{aligned} \Lambda_k^i : C^{\infty}_{p^{k+1}} &\longrightarrow W_n^+(K), \\ ({}^i a_m^{(k)})_{m \in \mathbb{N}_0} &\longmapsto \sum_{m \in \mathbb{N}_0\, m \notin p\,\mathbb{N}_0} {}^i a_{f(m)}^{(k)}(0, \dots, 0, \omega_i^{p^{n-k}} t^m, \dots, 0). \end{aligned}$$

Since

$$\Lambda_k\big(({}^0a^{(k)}, {}^1a^{(k)}, \dots, {}^{r-1}a^{(k)})\big) = \Lambda_k^0({}^0a^{(k)}) \oplus \Lambda_k^1({}^1a^{(k)}) \oplus \dots \oplus \Lambda_k^{r-1}({}^{r-1}a^{(k)}),$$

it follows that the map Λ_k is, as the sum of the continuous group homomorphisms Λ_k^i, $i = 0, \dots, r-1$, itself a well-defined, continuous group homomorphism. $\square$

Proposition 4.20. *Let $K = \mathbb{F}_q((t))$, where $q = p^r$ for some prime $p > 0$ and some $r \in \mathbb{N}$, let $n \in \mathbb{N}_0$, and let $W_n(K)$ be the nth Witt group of K.*

(i) *The map*

$$\begin{aligned} \Lambda : (C^{\infty}_{p^{n+1}})^r \times (C^{\infty}_{p^n})^r \times \dots \times (C^{\infty}_{p})^r &\longrightarrow W_n^+(K), \\ \big(({}^0a^{(n)}, {}^1a^{(n)}, \dots, {}^{r-1}a^{(n)}), ({}^0a^{(n-1)}, {}^1a^{(n-1)}, \dots, {}^{r-1}a^{(n-1)}), &\\ \dots, ({}^0a^{(0)}, {}^1a^{(0)}, \dots, {}^{r-1}a^{(0)})\big) &\longmapsto \\ \Lambda_n\big(({}^0a^{(n)}, {}^1a^{(n)}, \dots, {}^{r-1}a^{(n)})\big) \oplus \Lambda_{n-1}\big(({}^0a^{(n-1)}, {}^1a^{(n-1)}, \dots, {}^{r-1}a^{(n-1)})\big) \oplus &\\ \dots \oplus \Lambda_0\big(({}^0a^{(0)}, {}^1a^{(0)}, \dots, {}^{r-1}a^{(0)})\big) & \end{aligned}$$

is an isomorphism of topological groups.

(ii) *The map*

$$\begin{aligned} \Psi : (C^{(\infty)}_{p^{n+1}})^r \times (C^{(\infty)}_{p^n})^r \times \dots \times (C^{(\infty)}_{p})^r &\longrightarrow W_n^-(K), \\ \big(({}^0b^{(n)}, {}^1b^{(n)}, \dots, {}^{r-1}b^{(n)}), ({}^0b^{(n-1)}, {}^1b^{(n-1)}, \dots, {}^{r-1}b^{(n-1)}), &\\ \dots, ({}^0b^{(0)}, {}^1b^{(0)}, \dots, {}^{r-1}b^{(0)})\big) &\longmapsto \\ \Psi_n\big(({}^0b^{(n)}, {}^1b^{(n)}, \dots, {}^{r-1}b^{(n)})\big) \oplus \Psi_{n-1}\big(({}^0b^{(n-1)}, {}^1b^{(n-1)}, \dots, {}^{r-1}b^{(n-1)})\big) \oplus &\\ \dots \oplus \Psi_0\big(({}^0b^{(0)}, {}^1b^{(0)}, \dots, {}^{r-1}b^{(0)})\big) & \end{aligned}$$

is an isomorphism of topological groups.

Proof. We observe first that if the set $\{\omega_0, \omega_1, \dots, \omega_{r-1}\}$ is a basis of the finite field $\mathbb{F}_{p^r}$ over the finite field $\mathbb{F}_p$, then also the set $\{\omega_0^{p^i}, \omega_1^{p^i}, \dots, \omega_{r-1}^{p^i}\}$ is, for every $i \in \mathbb{N}$. With this fact, the proof of the bijectivity of the maps Λ and Ψ is a straightforward application of the proof of Proposition 4.14. $\square$

In the same way as in Theorem 4.16 we may now obtain the following decomposition of the nth Witt group of any local field K into a discrete and a compact part.

Proposition 4.21. *Let $K = \mathbb{F}_q((t))$, where $q = p^r$ for some prime $p > 0$ and some $r \in \mathbb{N}$, let $n \in \mathbb{N}_0$, and let $W_n(K)$ be the nth Witt group of K. Then*

$$W_n(K) \cong (C^{\infty}_{p^{n+1}})^r \times \cdots \times (C^{\infty}_{p})^r \times (C^{(\infty)}_{p^{n+1}})^r \times \cdots \times (C^{(\infty)}_{p})^r. \tag{30}$$

Proof. The map

$$\mu : W_n^+(K) \times W_n^-(K) \to W_n(K),\ (x, y) \mapsto x \oplus y$$

is an isomorphism of topological groups (Lemma 4.17) and we have

$$\Theta((a, b)) = \mu(\Lambda(a), \Psi(b))$$

for all $a \in (C^{\infty}_{p^{n+1}})^r \times (C^{\infty}_{p^n})^r \times \cdots \times (C^{\infty}_{p})^r$ and $b \in (C^{(\infty)}_{p^{n+1}})^r \times (C^{(\infty)}_{p^n})^r \times \cdots \times (C^{(\infty)}_{p})^r$. Since, by Proposition 4.20, both maps Λ and Ψ are isomorphisms of topological groups, it follows that Θ is, as the composition of isomorphisms, itself an isomorphism of topological groups. □

4.3. **Duality of Witt groups.** With the detailed information about the structure of finite-dimensional Witt groups over local fields of characteristic $p > 0$, it is now easy to see that such groups are topologically isomorphic to their dual groups.

Proposition 4.22. *Let $K = \mathbb{F}_p((t))$ for some prime p. The nth Witt group of K is, as a topological group, selfdual for every $n \in \mathbb{N}_0$, i.e., $\widehat{W_n(K)} \cong W_n(K)$.*

Proof. The proof of this proposition follows directly from Theorem 4.16 and the facts (1)-(4) about the dual group of locally compact abelian groups listed in Section 2. Theorem 4.16 yields

$$W_n(K) \cong (C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p}) \times (C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_{p}).$$

Since C_{p^k} is a finite cyclic group we have $(C_{p^k})\widehat{\ } \cong C_{p^k}$ for every $k \in \{1, \ldots, n+1\}$. Additionally, we have for every $k \in \{1, \ldots, n+1\}$,

$$\left(C^{\infty}_{p^k}\right)\widehat{\ } \cong \left(\prod_{i=0}^{\infty} C_{p^k}\right)\widehat{\ } \cong \bigoplus_{i=0}^{\infty} (C_{p^k})\widehat{\ } \cong \bigoplus_{i=0}^{\infty} C_{p^k} = C^{(\infty)}_{p^k}$$

and

$$\left(C^{(\infty)}_{p^k}\right)\widehat{\ } \cong \left(\bigoplus_{i=1}^{\infty} C_{p^k}\right)\widehat{\ } \cong \prod_{i=1}^{\infty} (C_{p^k})\widehat{\ } \cong \prod_{i=1}^{\infty} C_{p^k} = C^{\infty}_{p^k}.$$

With these results we obtain

$$\begin{aligned}\widehat{W_n(K)} &\cong (C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p} \times C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_{p})\widehat{\ } \\ &\cong C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_{p} \times C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p} \\ &\cong W_n(K).\end{aligned}$$

□

Corollary 4.23. *For any local field K of characteristic p and every $n \in \mathbb{N}_0$, the nth Witt group $W_n(K)$ is isomorphic to its dual group, as a topological group.*

Proof. Let K be any local field of characteristic p. Then K is isomorphic to a field of formal Laurent series in one indeterminate with coefficients in a finite field of characteristic p, i.e., $K \cong \mathbb{F}_q((t))$, where $q = p^r$ for some $r \in \mathbb{N}$. But by Proposition 4.21 we have

$$\begin{aligned} W_n(K) &\cong \big((C^{\infty}_{p^{n+1}})^r \times \cdots \times (C^{\infty}_{p})^r\big) \times \big((C^{(\infty)}_{p^{n+1}})^r \times \cdots \times (C^{(\infty)}_{p})^r\big) \\ &\cong \big((C^{\infty}_{p^{n+1}} \times \cdots \times C^{\infty}_{p}) \times (C^{(\infty)}_{p^{n+1}} \times \cdots \times C^{(\infty)}_{p})\big)^r, \end{aligned}$$

and thus, we obtain by the same arguments as in the proof of Proposition 4.22

$$\begin{aligned} \widehat{W_n(K)} &\cong \big((C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p} \times C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_{p})^r\big)\widehat{} \\ &\cong \big((C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p} \times C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_{p})\widehat{}\,\big)^r \\ &\cong \big(C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_{p} \times C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p}\big)^r \\ &\cong W_n(K). \end{aligned}$$

□

4.4. **Characters of the first Witt group.** In this subsection, we give an explicit description of the characters of the first Witt group $W_1(K)$, where $K = \mathbb{F}_p((t))$ for some prime p.

By Theorem 4.16 we have

$$\begin{aligned} W_1(K) &\cong \left(\bigoplus_{i=1}^{\infty} C_{p^2} \times \prod_{i=0}^{\infty} C_{p^2}\right) \times \left(\bigoplus_{i=1}^{\infty} C_{p} \times \prod_{i=0}^{\infty} C_{p}\right) \\ &\cong \bigoplus_{i=1}^{\infty}(C_{p^2} \times C_p) \times \prod_{i=0}^{\infty}(C_{p^2} \times C_p). \end{aligned}$$

So, in order to describe the characters of $W_1(K)$ we can use the isomorphism of Theorem 4.16 and describe instead the characters of the group

$$H := \bigoplus_{i=1}^{\infty}(C_{p^2} \times C_p) \times \prod_{i=0}^{\infty}(C_{p^2} \times C_p).$$

Since C_{p^j}, $j = 1, 2$, is a finite cyclic group, every character $\chi \in \widehat{C_{p^j}}$ is of the form

$$\chi = \chi_{v_j} : C_{p^j} \to \mathbb{T}, \quad s_j \mapsto \exp\left(\frac{2\pi i s_j v_j}{p^j}\right),$$

for some $v_j \in C_{p^j}$. Thus every character $\chi \in \widehat{C_{p^2} \times C_p}$ is of the form $\chi = \chi_v$, where $v = (v_1, v_2) \in C_{p^2} \times C_p$ and we have

$$\chi_v : C_{p^2} \times C_p \to \mathbb{T}, \quad \chi_v(s_1, s_2) = \chi_{v_1}(s_1) \cdot \chi_{v_2}(s_2),$$

where χ_{v_1} is a character of C_{p^2} and χ_{v_2} is a character of C_p, as above.

We define a "duality bracket" in the following way:

$$\langle v, s\rangle_{C^{p^2}\times C^p} := \langle v_1, s_1\rangle_{C^{p^2}} \cdot \langle v_2, s_2\rangle_{C^p} := \chi_{(v_1,v_2)}(s_1, s_2) = \chi_v(s). \tag{31}$$

Observe that

$$\left(\bigoplus_{i=1}^{\infty}(C_{p^2}\times C_p)\right)^{\widehat{}} \cong \prod_{i=0}^{\infty}(C_{p^2}\times C_p) \text{ and } \left(\prod_{i=0}^{\infty}(C_{p^2}\times C_p)\right)^{\widehat{}} \cong \bigoplus_{i=1}^{\infty}(C_{p^2}\times C_p).$$

Thus we can define a character $\chi_x \in \widehat{H}$, $x = (x_m)_{m\in\mathbb{Z}} \in H$ by defining it first on every component of the sequence $s = (s_m)_{m\in\mathbb{Z}} \in H$:

$$\chi_x(s_m) := \langle x_{-m}, s_m\rangle_{C^{p^2}\times C^p}.$$

The character $\chi_x \in \widehat{H}$, $x = (x_m)_{m\in\mathbb{Z}} \in H$, is then of the form

$$\chi_x(s) = \prod_{m\in\mathbb{Z}} \langle x_{-m}, s_m\rangle_{C^{p^2}\times C^p} \tag{32}$$

and it is clear that every character of H is of such a form. Notice that since only finitely many components with negative subscript of x and s are nonzero, the product in (32) is well-defined.

5. The structure of abelian K-split groups

In the following, let K be a local field. Recall that we denote by G_a the additive group of the field K. In this section we give a complete characterization of abelian K-split groups. As we have seen in Section 3, the basic building-blocks for these groups are the abelian, algebraic extensions of the additive group G_a with itself. Recall that we denote by $\mathrm{Ext}(G_a, G_a)$ the set of all group extensions given by symmetric algebraic 2-cocycles $f : G_a\times G_a \to G_a$ and we will identify such group extensions with the corresponding 2-cocycle. During this section we will follow an approach of Serre [8], chapter VII to the structure of commutative unipotent groups, state the most important results, and prove some additional facts, which will be needed in the next section.

Remark 5.1. A general assumption made in [8], chapter VII, is that the base field K is algebraically closed. But studying the relevant proofs in that chapter, one can show that this assumption can be removed. In fact, all the results cited in this section hold for any local field K.

Proposition 5.2. *([8], Proposition 8) In characteristic* 0, $\mathrm{Ext}(G_a, G_a) = 0$. *In characteristic* $p > 0$, *the* K*-vector space* $\mathrm{Ext}(G_a, G_a)$ *admits for a basis the* p^n*th powers* ($n \in \mathbb{N}_0$) *of the 2-cocycle* f *which defines the first Witt group* $W_1(K)$*:*

$$f(x, y) = \frac{1}{p}(x^p + y^p - (x + y)^p).$$

Note that the right hand side of the equation above should be considered as a formal sum.

We sketch briefly the idea of the proof. One writes the polynomial $g(x,y)$, which determines the group extension, in the form $\sum a_{ij}x^iy^j$. Then formula (3) translates into identities for the coefficients a_{ij} which allow one to determine explicitly all symmetric 2-cocycles. For the details of the computation see [6], §III.

Corollary 5.3. *(*[8]*, Corollary of Proposition 8) In characteristic* 0*, every commutative connected unipotent group is isomorphic to a product of copies of the additive group G_a.*

Proposition 5.2 indicates the relevance of finite-dimensional Witt groups in the field of abelian K-split groups. In the following we recall and state some facts concerning these groups, see also [8], chapter VII. The definition of the nth Witt group $W_n(K) =: W_n$ of a field K is given in Section 4.1. There exist two maps which are very useful in this context:

(1) the Shift homomorphism $S : W_n \longrightarrow W_{n+1}, (x_0, \ldots, x_n) \mapsto (0, x_0, \ldots, x_n)$ and
(2) the Restriction homomorphism $R : W_{n+1} \longrightarrow W_n, (x_0, \ldots, x_{n+1}) \mapsto (x_0, \ldots, x_n)$.

We should notice that this shift homomorphism does not coincide with the shift homomorphism $S : W_n \longrightarrow W_n$ as introduced earlier. The above homomorphisms commute with each other and we obtain, for all $m, n \in \mathbb{N}_0$, an exact sequence:

$$0 \longrightarrow W_m \xrightarrow{S^{n+1}} W_{n+m+1} \xrightarrow{R^{m+1}} W_n \longrightarrow 0. \tag{33}$$

We denote the corresponding element of $\mathrm{Ext}(W_n, W_m)$ by V_n^m. The following commutative diagram shows the effect of the restriction homomorphism R on these extensions

$$\begin{tikzcd}
0 \arrow[r] & W_m \arrow[r] \arrow[d, "R"] & W_{n+m+1} \arrow[r] \arrow[d, "R"] & W_n \arrow[r] \arrow[d, equal, "id"] & 0 \\
0 \arrow[r] & W_{m-1} \arrow[r] & W_{n+m} \arrow[r] & W_n \arrow[r] & 0.
\end{tikzcd}$$

Thus we obtain the formula

$$R_*(V_n^m) = V_n^{m-1},$$

where $R_*(V_n^m)$ denotes the pushout of V_n^m by the map R as indicated in the above diagram. Analogously, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & W_m \arrow[r] & W_{n+m+1} \arrow[r] & W_n \arrow[r] & 0 \\
0 \arrow[r] & W_m \arrow[r] \arrow[u, equal, "id"] & W_{n+m} \arrow[r] \arrow[u, "S"] & W_{n-1} \arrow[r] \arrow[u, "S"] & 0.
\end{tikzcd}$$

And thus we obtain the formula

$$S^*(V_n^m) = V_{n-1}^m, \tag{34}$$

where $S^*(V_n^m)$ denotes the pullback of V_n^m by the map S. In the same way one can show

$$S_*(V_n^m) = R^*(V_{n-1}^{m+1}).$$

We denote by $\mathcal{E}_n$ the ring of endomorphisms of the algebraic group W_n, $n \in \mathbb{N}_0$. The pushout operation $\varphi_*(V_n^m)$ and the pullback operation $\varphi^*(V_n^m)$ give the group $\operatorname{Ext}(W_n, W_m)$ the structure of a left module over $\mathcal{E}_m$ and a right module over $\mathcal{E}_n$, respectively, and these two structures are compatible in the above sense.

Remark 5.4. The group W_0 is just the additive group G_a and the exact sequence (33) shows that the nth Witt group W_n, $n \in \mathbb{N}_0$, is an iterated extension of the additive group G_a. For $m \leq n$, we can identify W_m with a subgroup of W_n by means of S^{n-m} and we have $W_m = p^{n-m} W_n$ (see also Lemma 4.5 in Section 4.2). Furthermore, the mth Witt groups W_m, $m \leq n$, are the only connected subgroups of W_n ([8], VII, Section 8).

The following definition is a useful instrument in algebraic geometry.

Definition 5.5.

(i) A homomorphism between two algebraic groups is called an isogeny if it is surjective with finite kernel.
(ii) We say that two algebraic groups G and H are isogeneous if there exist isogenies $f : G \to H$ and $g : H \to G$.

Remark 5.6. ([8], chapter VII) Let $n \in \mathbb{N}_0$ and let G be an abelian unipotent linear algebraic group. The following are equivalent:

(i) There exists an isogeny $f : G \to W_n$.
(ii) There exists an isogeny $g : W_n \to G$.

Lemma 5.7. *([8], VII, §2, Lemma 3) Every element $H \in \operatorname{Ext}(G_a, G_a)$ can be written uniquely as $H = \varphi^*(V_0^0)$ (or $\psi_*(V_0^0)$), where φ and ψ are elements of $\mathcal{E}_0$. Furthermore $\varphi^*(V_0^0)$ is the trivial extension if and only if φ is not an isogeny.*

Proof. The existence and uniqueness of φ works as follows. The element $V_0^0 \in \operatorname{Ext}(G_a, G_a)$ corresponds to a symmetric 2-cocycle $\omega : G_a \times G_a \to G_a$ which determines the first Witt group:

$$V_0^0 : \ 0 \longrightarrow G_a \longrightarrow W_1 \longrightarrow G_a \longrightarrow 0.$$

Let $H \in \operatorname{Ext}(G_a, G_a)$ be an abelian algebraic group extension of G_a. According to Proposition 5.2, the element H corresponds to a symmetric 2-cocycle of the form

$$f(x, y) = \sum_i a_i \, \omega(x, y)^{p^i} \text{ with } a_i \in K.$$

On the other hand, every endomorphisms φ of G_a can be written uniquely as

$$\varphi(x) = \sum_i b_i \, x^{p^i}.$$

Hence we have $H = \varphi^*(V_0^0)$ if and only if $b_i = a_i$ for all i, which proves the existence and uniqueness of φ. The other parts are similar. $\square$

With Lemma 5.7 we can obtain a useful characterization of the elements of $\mathrm{Ext}(G_a, G_a)$.

Corollary 5.8. *Let H be an element of* $\mathrm{Ext}(G_a, G_a)$. *Then H is either isomorphic (as an algebraic group) to $G_a \times G_a$ or isogeneous to the 2-dimensional Witt group $W_1(K)$.*

Proof. By Lemma 5.7 we can find a map $\varphi \in \mathrm{End}(G_a, G_a)$ such that $H = \varphi^*(V_0^0)$. If $H = \varphi^*(V_0^0) = 0$ then H splits, which means that H is isomorphic to $G_a \times G_a$. Otherwise the map φ is an isogeny, and since the corresponding pullback diagram

$$\begin{tikzcd} 0 \arrow[r] & G_a \arrow[r] & W_1 \arrow[r] & G_a \arrow[r] & 0 \\ 0 \arrow[r] & G_a \arrow[r] \arrow[u, equal, "id"'] & H \arrow[r] \arrow[u, "\phi"'] & G_a \arrow[r] \arrow[u, "\varphi"'] & 0 \end{tikzcd}$$

is commutative, it follows as an application of the Snake-Lemma that the map $\phi : H \to W_1$ is an isogeny. $\square$

There are also similar results for higher dimensional Witt groups.

Lemma 5.9. *([8], VII, §2, Lemma 6) Every element H of* $\mathrm{Ext}(W_n, G_a)$ *can be written as $H = \varphi^*(V_n^0)$ for some $\varphi \in \mathcal{E}_n$. One has $\varphi^*(V_n^0) = 0$ if and only if φ is not an isogeny.*

One can also reverse the roles of W_n and G_a.

Lemma 5.10. *([8], VII, §2, Lemma 6') Every element H of* $\mathrm{Ext}(G_a, W_n)$ *can be written as $H = \varphi_*(V_0^n)$ for some $\varphi \in \mathcal{E}_n$. One has $\varphi_*(V_0^n) = 0$ if and only if φ is not an isogeny.*

As in the case $n = 0$, we obtain a characterization of the elements of $\mathrm{Ext}(G_a, W_n)$ and $\mathrm{Ext}(W_n, G_a)$.

Corollary 5.11. *Let H be an element of either* $\mathrm{Ext}(G_a, W_n)$ *or* $\mathrm{Ext}(W_n, G_a)$. *Then H (i.e., the linear algebraic group defined by the exact sequence H) is either isomorphic to $W_n \times G_a$ or isogeneous to W_{n+1}.*

Proof. We will prove the corollary for $\mathrm{Ext}(G_a, W_n)$, the case of $\mathrm{Ext}(W_n, G_a)$ is similar. As in the two-dimensional case we have either $H = (\varphi)^* V_0^n = 0$ for some $\varphi \in \mathcal{E}_n$ and thus H splits and is isomorphic to $W_n \times G_a$, or there exists an isogeny φ from G_a to G_a such that H is the pullback of W_{n+1} and G_a under φ. Since the diagram

$$\begin{tikzcd} 0 \arrow[r] & W_n \arrow[r] & W_{n+1} \arrow[r] & G_a \arrow[r] & 0 \\ 0 \arrow[r] & W_n \arrow[r] \arrow[u, equal, "id"'] & H \arrow[r] \arrow[u, "\phi"'] & G_a \arrow[r] \arrow[u, "\varphi"'] & 0 \end{tikzcd}$$

is commutative, it follows as an application of the Snake-Lemma that the map $\phi : H \to W_n$ is an isogeny. $\square$

Lemma 5.12. *([8], VII, §2, Lemma 7) If $m \geq n$, every element $H \in \mathrm{Ext}(W_n, G_a)$ can be written as $H = f_*(V_m^0)$ with $f \in \mathrm{Hom}(W_n, W_m)$.*

The next theorem demonstrates the exact connection between abelian unipotent K-split groups and Witt groups.

Theorem 5.13. *([8], VII, §2, Theorem 1) Every commutative unipotent K-split group is isogeneous to a finite product of Witt groups.*

In order to get a better understanding of this theorem, we give a sketch of the proof.

Proof. Let G be a commutative unipotent K-split group of dimension $n \in \mathbb{N}$. We argue by induction on n. If $n = 1$ then $G = G_a = W_0(K)$ and there is nothing to prove.

So let $n \in \mathbb{N}$ and suppose that the theorem is shown for all abelian K-split groups of dimension less than n. The group G is an extension of a group H of dimension $n - 1$ by the group G_a. Applying the induction hypothesis to the group H yields an isogeny

$$f : \prod_{i=1}^{k} W_{n_i} \to H.$$

Put $W := \prod_{i=1}^{k} W_{n_i}$. The pullback $f^*(G)$ is an extension of W by G_a and this pullback is isogeneous to G:

$$\begin{tikzcd}
0 \arrow[r] & G_a \arrow[r] & G \arrow[r] & H \arrow[r] & 0 \\
0 \arrow[r] & G_a \arrow[r] \arrow[u, equal, "id"'] & f^*(G) \arrow[r] \arrow[u, "F"'] & W \arrow[r] \arrow[u, "f"'] & 0.
\end{tikzcd}$$

Thus it suffices to show that $f^*(G)$ is isogeneous to a product of Witt groups. In other words we are reduced to the case where $H = W$. Replacing $f^*(G)$ by G, let us denote the extension in question by $\gamma \in \mathrm{Ext}(W, G_a)$.

The extension γ is defined by a family of elements $\gamma_i \in \mathrm{Ext}(W_{n_i}, G_a)$. Suppose that $n_1 \geq n_i$ for all i and let $V = \prod_{i=2}^{k} W_{n_i}$. We are going to distinguish two cases.

1.) $\gamma_1 = 0$. The group G is then the product of W_{n_1} and the extension of V by G_a, defined by the system $(\gamma_i)_{i\geq 2}$. By the induction hypothesis, this extension of V by G_a is isogeneous to a product of Witt groups and hence G is isogeneous to a product of Witt groups.

2.) $\gamma_1 \neq 0$. Let $\beta = (\beta_i) \in \mathrm{Ext}(W, G_a)$ be the element defined by $\beta_1 = V_{n_1}^0$ and $\beta_i = 0$ for $i \geq 2$. The extension G' corresponding to β is the product $W_{n_1+1} \times V$. We are going to show the existence of an isogeny $\varphi : W \to W$ such that $\varphi^*(G')$ is isomorphic to G:

$$\begin{tikzcd} 0 \arrow[r] & G_a \arrow[r] & G' \arrow[r] & W \arrow[r] & 0 \\ 0 \arrow[r] & G_a \arrow[r] \arrow[u, equal, "id"'] & \varphi^*(G') \arrow[r] \arrow[u, "\phi"'] & W \arrow[r] \arrow[u, "\varphi"'] & 0. \end{tikzcd}$$

It will follow from this that G is isogeneous to G', which is a product of Witt groups.

Applying Lemma 5.12 to every $\gamma_i \in \mathrm{Ext}(W_{n_i}, G_a)$ yields homomorphism $f_i \in \mathrm{Hom}(W_{n_i}, W_{n_1})$ such that $\gamma_i = f_{i_*} V^0_{n_1}$. Define the map $\varphi : W \to W$ by

$$\varphi(w_1, w_2, \ldots, w_k) = (f_1(w_1) + f_2(w_2) + \cdots + f_k(w_k), w_2, \ldots, w_k).$$

Then $\varphi^*(\beta) = \gamma$. Since f_1 is surjective (see Lemma 5.9), it follows immediately that φ is surjective and every surjective homomorphism between two groups of the same dimension has a finite kernel. Thus the map φ defines the desired isogeny.

□

From now on we assume that K is a local field of characteristic $p > 0$. We now show that every abelian K-split group is self-dual. Indeed, this follows from our previous study of the Witt groups and the following result.

Proposition 5.14. *Let H be a unipotent linear algebraic group and suppose H is isogeneous to $G = W_n(K)$, the nth Witt group of the field $K = \mathbb{F}_p((t))$. Then H is topologically isomorphic to G.*

Proof. Since H is isogeneous to G, we can find a finite subgroup F of G such that $H \cong G/F$. So in order to show that H is isomorphic to G, it suffices to prove that $G \cong G/F$, where F is an arbitrary finite subgroup of G. But every finite subgroup F is of the form $F = \langle x_1, \ldots, x_k \rangle$ for some $x_1, \ldots, x_k \in G$ and we will prove that $G/\langle x \rangle \cong G$ for every $x \in G$, where $\langle x \rangle$ denotes the additive subgroup in G generated by x. (Notice that by Corollary 4.6 of Section 4.2 the group $G = W_n(K)$ is of exponent p^{n+1}, so in particular every element $x \in G$ has finite order and thus $\langle x \rangle$ is finite for every $x \in G$.) It then follows by an induction argument that $G/F \cong (G/\langle x_1 \rangle)/\langle x_2, \ldots, x_k \rangle \cong G/\langle x_2, \ldots, x_k \rangle \cong G$.

Recall that we denote by $C_n := \mathbb{Z}/n\mathbb{Z}$ the cyclic group with n elements. Furthermore we write A^∞ for the infinite direct product $\prod_{i=0}^\infty A$ of a finite abelian group A and $A^{(\infty)}$ for the infinite direct sum $\bigoplus_{i=0}^\infty A$.

By Theorem 4.16 of Section 4.2 we know that the topological group $G = W_n(K)$ is of the form

$$G \cong C_p^\infty \times C_{p^2}^\infty \times \cdots \times C_{p^{n+1}}^\infty \times C_p^{(\infty)} \times C_{p^2}^{(\infty)} \times \cdots \times C_{p^{n+1}}^{(\infty)}.$$

So if we define $H_{p^i} := C_{p^i}^\infty \times C_{p^i}^{(\infty)}$, then

$$G \cong H_p \times H_{p^2} \times \cdots \times H_{p^{n+1}}$$

and every $x \in G$ is of the form $x = (x^1, \dots, x^{n+1})$ with $x^i \in H_{p^i}$ for every $i \in \{1, \dots, n+1\}$.

Let $x \in G$ and suppose $x \neq 0$. The finite group $\langle x \rangle$ is a subgroup of $C_{p^{n+1}}$, and thus $\langle x \rangle \cong C_{p^m}$ for some $m \in \{1, \dots, n+1\}$. But every cyclic group C_{p^m}, $m \geq 1$, has a subgroup which is isomorphic to C_p. Thus, by replacing x by x^{p^k} for a suitable power k, we can assume without loss of generality that $\langle x \rangle \cong C_p$.

We consider two different cases:

1.) The intersection of the element x and the group H_p is not trivial, i.e., $x^1 \neq 0$. Then $\langle x^1 \rangle \cong C_p$ and $x^1 \in H_p$ is a Laurent series of the form $x^1 = (x^1_m)_{m \in \mathbb{Z}}$, where $x^1_m \in C_p$ for every $m \in \mathbb{Z}$. But the series x^1 generates the cyclic group C_p, and thus we can find an integer k such that $\langle x^1_k \rangle \cong C_p$. Observe that for every $y = (y^1, \dots, y^{n+1}) \in G$ we can find a unique element of the span $\langle x^1_k \rangle$ which is equal to y^1_k. We denote by $\bar{x}^1$ the element in H_p defined by $\bar{x}^1_k = x^1_k$ and $\bar{x}^1_m = 0$ for all $m \in \mathbb{Z} \setminus \{k\}$. We will show

(a) $G/\langle x \rangle \cong H_p/\langle \bar{x}^1 \rangle \times H_{p^2} \times \cdots \times H_{p^{n+1}}$ and

(b) $H_p/\langle \bar{x}^1 \rangle \cong H_p$.

It follows directly from (a) and (b) that $G/\langle x \rangle \cong G$.

In order to show part (a) we define the map

$$\Phi : G \longrightarrow G,\ y \mapsto y - \varphi(y),$$

where $\varphi(y) = x' \in \langle x \rangle$ with ${x'}^1_k = y^1_k$. We conclude from the above observation that Φ is well-defined and clearly, Φ is a group homomorphism. Furthermore, we have $y - \varphi(y) = 0$ if and only if $y = \varphi(y)$ if and only if $y \in \langle x \rangle$, which shows that $\ker(\Phi) = \langle x \rangle$. Hence $G/\langle x \rangle$ is isomorphic to the image of Φ, which is isomorphic to the direct product $H_p/\langle \bar{x}^1 \rangle \times H_{p^2} \times \cdots \times H_{p^{n+1}}$.

In order to prove part (b), we recall that

$$H_p \cong \bigoplus_{i=1}^{\infty} C_p \times \prod_{i=0}^{\infty} C_p \quad \text{and} \quad \langle \bar{x}^1 \rangle \cong \langle \bar{x}^1_k \rangle \cong C_p.$$

Without loss of generality we assume $k = 0$. Notice that if $y^1, z^1 \in [y^1] \in H_p/\langle \bar{x}^1 \rangle$ are two elements of the same coset, then $y^1 - z^1 \in \langle \bar{x}^1 \rangle$ which means that there exists a number $\lambda \in C_p$ such that $y^1_l - z^1_l = \lambda \bar{x}^1_l$ for all $l \in \mathbb{Z}$. In particular, if $y^1, z^1 \in [y^1]$ with $y^1_0 = z^1_0 = 0$ then we obtain $y^1_l = z^1_l = 0$ for all $l \in \mathbb{Z}$ and thus $y^1 = z^1$, since $\langle \bar{x}^1_0 \rangle \neq 0$. This means that in every coset $[y^1] \in H_p/\langle \bar{x}^1 \rangle$ there exists a unique element z^1 with $z^1_0 = 0$. We now define the map

$$\Psi : \bigoplus_{i=1}^{\infty} C_p \times \{0\} \times \prod_{i=1}^{\infty} C_p \longrightarrow H_p/\langle \bar{x}^1 \rangle, \quad y^1 \mapsto [y^1].$$

It follows directly from the above that Ψ is well-defined and it is not hard to see that Ψ is a group isomorphism. But the group $\bigoplus_{i=1}^{\infty} C_p \times \{0\} \times \prod_{i=1}^{\infty} C_p$ is obviously isomorphic to H_p, which completes the proof of part (b).

2.) The intersection of the element x and the group H_p is trivial, i.e., $x^1 = 0$. Let $i \in \{2, \dots, n+1\}$ be minimal with respect to the property that

$x^i \neq 0$. Since $\langle x\rangle \cong C_p$, we have $\langle x^i\rangle \cong C_p$. As in the case $i = 1$, we know that x^i is a Laurent series of the form $x^i = (x^i_m)_{m\in\mathbb{Z}}$ with $x^i_m \in C_{p^i}$ for every $m \in \mathbb{Z}$. Since $\langle x^i\rangle \cong C_p$, there exists $k \in \mathbb{Z}$ such that $\langle x^i_k\rangle \cong C_p$. So for every $y = (y^i, \ldots, y^{n+1}) \in H_{p^i} \times \cdots \times H_{p^{n+1}}$ we can find a unique element $z \in \langle x^i_k\rangle$ with $y^i_k + \big(C_{p^i}/C_p\big) = z + \big(C_{p^i}/C_p\big)$. (Or, if we view x^i_k as an element of $\{0, \ldots, p^i - 1\} \cong C_{p^i}$, then $z \equiv y^i_k \mod p$.) Denote by $\bar{x}^i$ the element of H_{p^i} defined by $\bar{x}^i_k = x^i_k$ and $\bar{x}^i_m = 0$ for all $m \in \mathbb{Z}\setminus\{k\}$. We have

$$\begin{aligned}&\big(H_p \times H_{p^2} \times \cdots \times H_{p^{n+1}}\big)/\langle x\rangle\\ &\qquad\qquad \cong H_p \times \cdots \times H_{p^{i-1}} \times \big(H_{p^i} \times \cdots \times H_{p^{n+1}}/\langle(x^i, \ldots, x^{n+1})\rangle\big)\end{aligned}$$

and claim that it suffices to prove the statements

(a) $H_{p^i} \times \cdots \times H_{p^{n+1}}/\langle(x^i, \ldots, x^{n+1})\rangle \cong H_{p^i}/\langle\bar{x}^i\rangle \times H_{p^{i+1}} \times \cdots \times H_{p^{n+1}}$ and

(b) $H_{p^i}/\langle\bar{x}^i\rangle \cong \bigoplus_{i=1}^\infty C_{p^i} \times (C_{p^i}/C_p) \times \prod_{i=1}^\infty C_{p^i}$.

Indeed, using (a) and (b) and the fact that $(\prod_{i=0}^\infty C_{p^{i-1}}) \times C_{p^{i-1}} \cong \prod_{i=0}^\infty C_{p^{i-1}}$ and $\prod_{i=1}^\infty C_{p^i} \cong \prod_{i=0}^\infty C_{p^i}$, we obtain

$$\begin{aligned}G/\langle x\rangle &\cong H_p \times \cdots \times H_{p^{i-1}} \times (H_{p^i}/\langle x^i\rangle) \times H_{p^{i+1}} \times \cdots \times H_{p^{n+1}}\\
&\cong \bigoplus_{i=1}^\infty C_p \times \prod_{i=0}^\infty C_p \times \cdots \times \bigoplus_{i=1}^\infty C_{p^{i-1}} \times \prod_{i=0}^\infty C_{p^{i-1}}\\
&\quad \times \big(\bigoplus_{i=1}^\infty C_{p^i} \times C_{p^{i-1}} \times \prod_{i=1}^\infty C_{p^i}\big) \times \cdots \times \bigoplus_{i=1}^\infty C_{p^{n+1}} \times \prod_{i=0}^\infty C_{p^{n+1}}\\
&\cong \bigoplus_{i=1}^\infty C_p \times \prod_{i=0}^\infty C_p \times \cdots \times \bigoplus_{i=1}^\infty C_{p^{i-1}} \times \prod_{i=0}^\infty C_{p^{i-1}}\\
&\quad \times \bigoplus_{i=1}^\infty C_{p^i} \times \prod_{i=0}^\infty C_{p^i} \times \cdots \times \bigoplus_{i=1}^\infty C_{p^{n+1}} \times \prod_{i=0}^\infty C_{p^{n+1}}\\
&\cong G.\end{aligned}$$

In order to prove the statement (a), we may use exactly the same idea as in the first case. We define a map

$$\Phi : H_{p^i} \times \cdots \times H_{p^{n+1}} \longrightarrow H_{p^i} \times \cdots \times H_{p^{n+1}},\ y \mapsto y - \varphi(y),$$

where $\varphi(y) = x' \in \langle(x^i, \ldots, x^{n+1})\rangle$ is defined so that $y^i_k + \big(C_{p^i}/C_p\big) = {x'}^i_k + \big(C_{p^i}/C_p\big)$. By the above remarks we know that Φ is a well-defined group homomorphism. The kernel of Φ is equal to $\langle(x^i, \ldots, x^{n+1})\rangle$ and hence the quotient group $H_{p^i} \times \cdots \times H_{p^{n+1}}/\langle(x^i, \ldots, x^{n+1})\rangle$ is isomorphic to the image of Φ, which is isomorphic to $H_{p^i}/\langle\bar{x}^i\rangle \times H_{p^{i+1}} \times \cdots \times H_{p^{n+1}}$.

For the proof of part (b), we assume without loss of generality that $k = 0$ and apply the same argument as above to the map

$$\Psi : \bigoplus_{i=1}^\infty C_{p^i} \times \prod_{i=0}^\infty C_{p^i} \longrightarrow \bigoplus_{i=1}^\infty C_{p^i} \times \prod_{i=0}^\infty C_{p^i}, \quad y \mapsto y - \psi(y),$$

where $\psi(y) = x' \in \langle x^i \rangle$ with $y_0^i + (C_{p^i}/C_p) = {x'}_0^i + (C_{p^i}/C_p)$. Since $\langle x_0^i \rangle \cong C_p$, it follows that the image of Ψ is isomorphic to $\bigoplus_{i=1}^{\infty} C_{p^i} \times C_{p^i}/C_p \times \prod_{i=1}^{\infty} C_{p^i}$, which finishes the proof. $\square$

Lemma 5.15. *Let G be a finite product of Witt groups of the field $K = \mathbb{F}_q((t))$, where $q = p^r$ for some prime p and some $r \in \mathbb{N}$, i.e., $G = \prod_{i=1}^{k} W_{n_i}(K)$ for some $k \in \mathbb{N}$ and some $n_i \in \mathbb{N}_0$, $i = 1, \dots, k$. Let n_j be the maximum of the set $\{n_i \mid i = 1, \dots, k\}$. Then G is, as a topological group, isomorphic to $W_{n_j}\big(\mathbb{F}_p((t))\big)$.*

Proof. Using Proposition 4.21, the topological group G is of the form

$$\begin{aligned} G \cong \prod_{i=1}^{k} &(C_p^{\infty})^r \times (C_{p^2}^{\infty})^r \times \cdots \\ &\times (C_{p^{n_i+1}}^{\infty})^r \times (C_p^{(\infty)})^r \times (C_{p^2}^{(\infty)})^r \times \cdots \times (C_{p^{n_i+1}}^{(\infty)})^r \\ \cong \prod_{i=1}^{k} &(C_p^{(\infty)} \times C_p^{\infty})^r \times (C_{p^2}^{(\infty)} \times C_{p^2}^{\infty})^r \times \cdots \times (C_{p^{n_i+1}}^{(\infty)} \times C_{p^{n_i+1}}^{\infty})^r. \end{aligned}$$

But for all $j = 1, \dots, n_i + 1$, we have

$$(C_{p^j}^{(\infty)} \times C_{p^j}^{\infty})^r \cong (C_{p^j}^{(\infty)} \times C_{p^j}^{\infty})$$

(as additive topological groups) and since the finite product $\prod_{i=1}^{k}(C_{p^j}^{(\infty)} \times C_{p^j}^{\infty})$ is topologically isomorphic to the group $C_{p^j}^{(\infty)} \times C_{p^j}^{\infty}$ for all $j = 1, \dots, n_i + 1$, it follows that

$$G \cong (C_p^{(\infty)} \times C_p^{\infty}) \times (C_{p^2}^{(\infty)} \times C_{p^2}^{\infty}) \times \cdots \times (C_{p^{n_j+1}}^{(\infty)} \times C_{p^{n_j+1}}^{\infty}) \cong W_{n_j}\big(\mathbb{F}_p((t))\big).$$

$\square$

Corollary 5.16. *If K is any local field of characteristic p and G a commutative K-split group then G is, as a topological group, isomorphic to its dual group.*

Proof. Let K be any local field of characteristic p. Then K is isomorphic to a field of formal Laurent series in one indeterminate with coefficients in a finite field of characteristic p, i.e., $K \cong \mathbb{F}_q((t))$, where $q = p^r$ for some $r \in \mathbb{N}$. Let G be a commutative K-split group. Then G is isogeneous to a finite product of Witt groups (Theorem 5.13), i.e., there exists $k \in \mathbb{N}$ and there exist $n_i \in \mathbb{N}$, $i = 1, \dots, k$, such that G is isogeneous to H, where

$$H = \prod_{i=1}^{k}(C_p^{(\infty)} \times C_p^{\infty})^r \times (C_{p^2}^{(\infty)} \times C_{p^2}^{\infty})^r \times \cdots \times (C_{p^{n_i+1}}^{(\infty)} \times C_{p^{n_i+1}}^{\infty})^r.$$

By Lemma 5.15, the group H is topologically isomorphic to $W_{n_j}\big(\mathbb{F}_p((t))\big)$ for some $n_j \in \{n_i \mid i = 1, \dots, k\}$ and thus G is isogeneous to the Witt group $W_{n_j}\big(\mathbb{F}_p((t))\big)$. It follows then from Proposition 5.14 that the topological group G is isomorphic to $W_{n_j}\big(\mathbb{F}_p((t))\big)$. Since every such finite dimensional Witt group is, as a topological group, self-dual (Proposition 4.22), it follows that G is self-dual. $\square$

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