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On semi-stable Fontaine theory in equal characteristic and good reduction of analytic Anderson motives

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On semi-stable Fontaine theory in equal characteristic and good reduction of analytic Anderson motives

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Introduction

p-adic Galois representations attached to elliptic curves

For a fixed prime number p, let K be a p-adic field with discrete valuation ring $(o_K, \mathfrak{m}_K)$ and perfect residue field $k = o_K/\mathfrak{m}_K$ of characteristic p; we write $G_K = \mathrm{Gal}(K^{\mathrm{alg}}/K)$ for the absolute Galois group of K where K^{alg}/K is a fixed algebraic closure; let $W = W(k)$ be the ring of Witt vectors over k, and $K_0 = \mathrm{Frac}(W)$. As is well-known from Fontaine theory ([3], [15], [26], [27], [39], [40], [75]), the crystalline period functor

$$D_{\mathrm{cris}} \colon \begin{pmatrix} p\text{-adic representations of } G_K \\ \text{in finite-dimensional } \mathbb{Q}_p\text{-vector spaces} \end{pmatrix} \to (\text{filtered isocrystals over } k)$$

induces an exact equivalence of categories between

— the category of those p-adic representations V of G_K which are *crystalline*, i.e., for which the canonical map

$$(V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{cris}})^{G_K} \otimes_{K_0} \mathbf{B}_{\mathrm{cris}} \to V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{cris}}$$

is an isomorphism, and

— the category of those filtered isocrystals over k which are *weakly admissible* in the sense of Fontaine.

For example, if E/K is an elliptic curve over K of good reduction then its special fiber E_0/k is an elliptic curve over k, and the p-adic Tate module

$$T_p(E) = \varprojlim_{(s)} E(K^{\mathrm{alg}})[p^s]$$

gives rise to the crystalline p-adic representation $V = V_p(E) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(E)$ of G_K; via D_{cris} the latter is mapped to $D_p(E_0/k) \otimes_W K_0$ where $D_p(E_0/k)$ denotes the Dieudonné module (defined over the Dieudonné ring $\mathcal{D}_k = W[F, V]$) associated to the special fiber E_0/k over k.

The finite W-module $D_p(E_0/k)$ is closely linked in a functorial way with the Barsotti-Tate group $E_0[p^\infty] = \varinjlim_{(s)} E_0[p^s]$ and, due to its (semi-)linear-algebra nature, is easier to understand than $E_0[p^\infty]$ itself; in fact, the object $E_0[p^\infty]$ can even be recovered from $D_p(E_0/k)$.

The structure of a filtered isocrystal on $D_{\text{cris}}(V)$ consists of two data: first of all, the action of the indeterminate $F \in \mathcal{D}_k$ induces a natural endomorphism of the abelian group $D_p(E_0/k)$ which is semi-linear with respect to the p-Frobenius lift $f\colon W \to W$ and becomes an isomorphism after inverting the uniformizer $p \in W$; the endomorphism F can also be obtained by letting crystalline cohomology intervene: denoting by $E_0^{(p)} = E_0 \otimes_{k, x \mapsto x^p} k$ the p-Frobenius pullback of E_0, the relative Frobenius-k-morphism $E_0 \to E_0^{(p)}$ induces a W-linear map

$$H^1_{\text{cris}}(E_0^{(p)}/W) = H^1_{\text{cris}}(E_0/W) \otimes_{W,f} W \to H^1_{\text{cris}}(E_0/W),$$

and there is an isomorphism of W-modules $D_p(E_0/k) \xrightarrow{\simeq} H^1_{\text{cris}}(E_0/W)$ which is compatible with F and the corresponding f-semi-linear endomorphism of $H^1_{\text{cris}}(E_0/W)$. Using this isomorphism, together with the comparison isomorphism

$$H^1_{\text{cris}}(E_0/W) \otimes_W o_K \simeq H^1_{dR}(\mathcal{E}/o_K)$$

where $\mathcal{E}/o_K$ is the (smooth) minimal Weierstraß model of E/K, the Hodge filtration on $H^1_{dR}(\mathcal{E}/o_K)$ induces on $(D_p(E_0/k) \otimes_W K_0) \otimes_{K_0} K$ an exhaustive and separated descending filtration by K-subspaces, which concludes our description of $D_{\text{cris}}(V_p(E))$; for a discussion of all this, see [15], [39].

If E/K is of split multiplicative (bad) reduction then the p-adic representation $V = V_p(E)$ is no longer crystalline, but rather *semi-stable*, and from Fontaine theory we know that via the semi-stable period functor

$$D_{\text{st}}\colon \begin{pmatrix} p\text{-adic representations of } G_K \\ \text{in finite-dimensional } \mathbb{Q}_p\text{-vector spaces} \end{pmatrix} \to (\text{filtered } (\varphi, N)\text{-modules over } k)$$

one associates to V a finite K_0-vector space $D_{\text{st}}(V)$ together with

— an automorphism φ of the abelian group $D_{\text{st}}(V)$ which is semi-linear with respect to the p-Frobenius lift $K_0 \to K_0$,

— a K_0-linear map $N\colon D_{\text{st}}(V) \to D_{\text{st}}(V)$ such that $N\varphi = p\varphi N$, and

— an exhaustive and separated descending filtration of $D_{\text{st}}(V) \otimes_{K_0} K$ by K-subspaces.

The structure of $D_{\mathrm{st}}(V)$ is more complicated to explain than in the good-reduction case, and we refer to [17] for a detailed discussion. Of course, the occuring phenomena arise from the fact that the special fiber E_0/k is not a smooth curve anymore, but rather has a nodal point. However, it should be emphasized that the "amount of complication" is very limited: by virtue of p-adic uniformization for semi-stable elliptic curves (or, more generally: semi-stable abelian varieties) it turns out that the step from good reduction to semi-stable reduction is encoded in a single additional datum on the associated filtered isocrystal of V: the monodromy operator N. More generally, it is well-known that an arbitrary $\mathbf{B}_{\mathrm{st}}$-admissible (that is: semi-stable) p-adic representation of G_K is crystalline if and only if its associated filtered (φ, N)-module has trivial monodromy, i.e., if and only if N acts as the zero map.

Local shtukas

In equal characteristic, we find a strongly contrary situation. To begin with, let us say a few words about the crystalline case: building upon work of R. Pink [63] on Hodge structures over function fields, A. Genestier and V. Lafforgue [34] have proposed an equal-characteristic analogue for the crystalline period functor D_{cris}. Here the notion of a crystalline Galois representation is replaced by that of a *local shtuka*: denoting by L an equal-characteristic complete discretely valued field containing a fixed finite field $\mathbb{F}$ of p-power order r, with valuation ring $(o_L, \mathfrak{m}_L)$ and perfect residue field $\ell = o_L/\mathfrak{m}_L$, a local shtuka over o_L is a finite free $o_L[\![z]\!]$-module $\hat{M}$ together with an isomorphism

$$F_{\hat{M}} \colon (\hat{M} \otimes_{o_L[\![z]\!],\sigma} o_L[\![z]\!])[\tfrac{1}{z-\zeta}] \to \hat{M}[\tfrac{1}{z-\zeta}]$$

of $o_L[\![z]\!][\frac{1}{z-\zeta}]$-modules, where $\zeta \in o_L - \{0\}$ is a fixed element, and where $\sigma \colon o_L[\![z]\!] \to o_L[\![z]\!]$ denotes the r-Frobenius lift defined by $z \mapsto z$ and $b \mapsto b^r$ for $b \in o_L$; a local shtuka $(\hat{M}, F_{\hat{M}})$ is said to be effective if $F_{\hat{M}}$ comes from an actual $o_L[\![z]\!]$-linear map $\hat{M} \otimes_{o_L[\![z]\!],\sigma} o_L[\![z]\!] \to \hat{M}$. The element $z - \zeta \in o_L[\![z]\!]$ appearing in the denominator stems from a distinguished Eisenstein polynomial employed in Breuil-Kisin's study of crystalline p-adic representations and finite flat group schemes ([14], [48]); in fact, the theory mentioned here lies at the very origin of the notion of a local shtuka.

When switching to equal characteristic, the ring of Witt vectors W from the p-adic world is replaced by the formal power series ring $\ell[\![z]\!]$ over the residue field $\ell = o_L/\mathfrak{m}_L$. If we suppose that $\zeta \in \mathfrak{m}_L$ then reduction of coefficients mod $\mathfrak{m}_L$ induces a canonical projection map $o_L[\![z]\!][\frac{1}{z-\zeta}] \to \ell(\!(z)\!)$, and the assignment $\hat{M} \mapsto \hat{M} \otimes_{o_L[\![z]\!]} \ell(\!(z)\!)$ associates to $(\hat{M}, F_{\hat{M}})$ the *z-isocrystal*

$$(D, F_D) = (\hat{M} \otimes_{o_L[\![z]\!]} \ell(\!(z)\!), F_{\hat{M}} \otimes \mathrm{id}).$$

This object naturally carries additional information, encoded in its *Hodge-Pink structure*, and one obtains a fully faithful functor from local shtukas over o_L to *z-isocrystals with Hodge-Pink structure* which is the analogue for Fontaine's crystalline period functor D_{cris} mentioned before; see [34], [40], [41].

Stressing the analogy between elliptic curves over K and Drinfeld modules over L, we may further illustrate the role of local shtukas: Let φ be a Drinfeld $\mathbb{F}[z]$-module over L; if φ is of *good reduction* in the sense of Drinfeld [21] then, in close analogy with the case of elliptic curves, by considering the collected z-power torsion $\bar{\varphi}[z^\infty]$ of the reduced Drinfeld $\mathbb{F}[z]$-module $\bar{\varphi}$ over ℓ one obtains a *z-divisible group* which, in fact, corresponds to a local shtuka over ℓ; see [22], [41], [52]. From this instance one can already tell that local shtukas play a double role: they not only take the place of crystalline p-adic representations, but also appear as analogues for Barsotti-Tate groups and, at the same time, their Dieudonné crystals. Furthermore, local shtukas are of a more general nature than crystalline p-adic representations. These circumstances already incorporate a moral reason for the fact that *bad* reduction seems to be less easy to capture in equal-characteristic arithmetic. The generality of local shtukas is also supported by the following instance: a very important feature about Drinfeld modules is that they can be mirrored by certain Drinfeld *shtukas* (also called *F-sheaves*), which are global objects of even more general nature; see [20], [22], [36]. Via formal completion a shtuka having coefficient scheme $\mathrm{Spec}(o_L)$ directly gives rise to a local shtuka over o_L, as is explained in [41].

Bad reduction

The original aim of research underlying the present thesis was to find a filler for the diagram of analogies

$$\begin{array}{ccc}
\left(\begin{array}{c}\text{semi-stable } p\text{-adic}\\ \text{representations of } G_K\end{array}\right) & \longleftrightarrow & (\ ?\ ) \\
| & & | \\
\left(\begin{array}{c}\text{crystalline } p\text{-adic}\\ \text{representations of } G_K\end{array}\right) & \longleftrightarrow & (\text{local shtukas over } o_L)
\end{array}$$

and to give an equal-characteristic analogue for Fontaine's semi-stable period functor D_{st}. In the above diagram, the missing objects would naturally be referred to as *semi-stable local shtukas*. However, we have to clarify from the outset that this aim lies beyond our capabilities. The situation in equal characteristic appears to be quite different from the p-adic case:

In order to describe bad reduction in equal characteristic, i.e., in order to say what a "semi-stable local shtuka" should be, one has to take several types of degeneration into account:

— First of all, arguing on the level of Drinfeld modules, Drinfeld's Tate uniformization theorem [21] can very well be compared with analytic uniformization for elliptic curves of split multiplicative reduction. The present work is mainly devoted to this instance.

— By work of F. Gardeyn [29], [30] it has been shown that Tate uniformization of Drinfeld modules may be carried out in terms of *(analytified) Anderson motives* or, more generally: of *analytic τ-sheaves*, and that Drinfeld modules of bad reduction give rise to objects called *(strongly) semi-stable τ-sheaves*. More generally, one of the merits of Gardeyn's work is to give a version of Tate uniformization for Anderson's *abelian t-modules*.

— Speaking in terms of the most general instance of Drinfeld shtukas, we encounter yet a different type of degeneration: in his work [49] on the proof of the Langlands conjecture for Gl_n over a global function field, L. Lafforgue has introduced objects which he called *chtoucas dégénérés*; these were first studied by Drinfeld [20] in the "rank 2"-case and were then generalized to higher rank by Lafforgue. Roughly speaking, one of the key insights for proving the Langlands conjecture was that the desired correspondence is realized by the cohomology of the moduli space of Lafforgue's chtoucas dégénérés, which in turn was first realized in the "rank 2"-case by Drinfeld.

Already from the first item one can derive phenomena which diverge from the p-adic case: we have already seen that to every good-reduction Drinfeld module over L (which amounts to a Drinfeld module over the *scheme* $\mathrm{Spec}(o_L)$) one can associate a local shtuka over o_L and therefore a z-isocrystal with Hodge-Pink structure; however, we will see that in contrast to the p-adic theory one cannot expect to obtain a

"*z-isocrystal with Hodge-Pink structure and monodromy operator*"

when starting with a bad-reduction Drinfeld module in the sense of [21].

Aim and contents of this thesis

In the present work we wish to give some evidence for the fact that, regarding bad reduction, the arithmetic over local function fields is quite different from p-adic arithmetic.

Let us give a description of the chapters of this thesis:

In the **first chapter** we study the connection between analytic Anderson motives and local shtukas at the residual characteristic place in a general fashion. We may illustrate this as follows: an *analytic Anderson motive* is a finite projective module M over the Tate algebra $L\langle z\rangle$ of strictly convergent power series ([9], [28]) in one indeterminate z over L, together with an injective $L\langle z\rangle$-linear map $F_M\colon M\otimes_{L\langle z\rangle,\sigma} L\langle z\rangle\to M$, where $\sigma\colon L\langle z\rangle\to L\langle z\rangle$ is the usual r-Frobenius lift, such that $\operatorname{coker}(F_M)$ is a finite-dimensional L-vector space and is annihilated by a power of the ideal $(z-\zeta)\subseteq L\langle z\rangle$. This means that analytic Anderson motives are an analytic variant of Anderson's t-motives living over the rigid-analytic closed unit ball $\mathbf{B}^1\subseteq\mathbb{A}^{1,\mathrm{an}}_L$. We show that an analytic Anderson motive (M,F_M) admits a good model (in the sense of Gardeyn [30]) over $o_L\langle z\rangle$ if and only if for a suitable effective local shtuka $(\hat{M},F_{\hat{M}})$ there is an $o_L[\![z]\!][1/\pi]$-linear isomorphism

$$M\otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi]\to\hat{M}[1/\pi]$$

which is compatible with the respective semi-linear data. The idea is that such a local shtuka arises via formal completion at the residual characteristic place from every good model. For a given analytic Anderson motive this also gives a precise characterization of its good models in terms of (effective) local shtukas. Adapting Gardeyn's theory [30] of good models for algebraic and analytic τ-sheaves to the aforementioned unit disc $\mathbf{B}^1$, we are able to further characterize good reduction of an algebraic Anderson t-motive in terms of its associated analytic Anderson motive; in particular, this gives a characterization of good models of algebraic Anderson t-motives in terms of local shtukas.

The **second chapter**, as well as the **third chapter**, is rather of a relative flavor: using the framework of Fontaine theory, in the second chapter we first explain that every p-adic Galois representation V which is an extension of $V_p(E)^\vee$ by $\mathbb{Q}_p$, where E/K is an elliptic curve of supersingular reduction, necessarily is crystalline. This is of course done by analyzing the associated filtered (φ,N)-module $D_{\mathrm{st}}(V)$ and showing that N has to act as the zero map. The considerations made here are certainly well-known to the experts. Turning to equal characteristic, in a next step we replace the elliptic curve E by a Drinfeld $\mathbb{F}[z]$-module φ over L which is of *good* supersingular reduction. We consider the associated analytic Anderson motive $M(\varphi)\otimes_{L[z]} L\langle z\rangle$ and, by choosing an $\mathbb{F}[z]$-lattice $\Lambda\subseteq\varphi(L^{\mathrm{sep}})$ of rank one and interpreting the Tate-uniformization morphism $\varphi\to_{\mathrm{an}}\varphi/\Lambda$ in terms of analytic Anderson motives, establish an extension structure of finite free $L\langle z\rangle$-modules with semi-linear data

$$0 \to (N, F_N) \to (M(\varphi/\Lambda) \otimes_{L[z]} L\langle z\rangle, \tau_{\varphi/\Lambda}) \to (M(\varphi) \otimes_{L[z]} L\langle z\rangle, \tau_{\varphi}) \to 0$$

where (N, F_N) is potentially trivial and of rank one. We now suppose that there is a hypothetical category of "semi-stable local shtukas" together with a hypothetical exact functor (corresponding to D_{st}) to a category of "z-isocrystals with Hodge-Pink structure and monodromy operator" which verify axioms being very close to those of $MF_K(\varphi, N)$ in the p-adic world, and we assume that the above sequence gives rise to a short exact sequence of "semi-stable local shtukas", and in particular to one of the hypothetical associated "$(\Phi, \mathcal{N})$-isocrystals"; we argue that the left-most and right-most term of the induced sequence have to be of trivial monodromy while the middle term has to be of properly bad reduction, due to its bad-reduction origin. However, by a similar argument as in the p-adic case, one can show that also the middle term has to be of trivial monodromy, which leads to a contradiction: according to our hypothesis, the monodromy operator of the isocrystal associated to a semi-stable local shtuka $\mathcal{M}$ is trivial if and only if $\mathcal{M}$ is actually of good reduction, i.e., a local shtuka over o_L; however, by the results from chapter 1, this cannot be the case.

Finally, in the third chapter, we are concerned with certain modules of Yoneda extension classes. Again, in the first part we consider the p-adic situation and study the Yoneda extension group $\mathrm{Ext}^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$ for the abelian categories of crystalline and semi-stable p-adic Galois representations, respectively; we explain that the group of crystalline extension classes lies as a $\mathbb{Q}_p$-hyperplane inside the group of semi-stable extension classes; this is done first via Galois cohomology and Kummer theory using the exact valuation sequence of the p-adic base field K, and via Fontaine theory. Again none of the considerations made on the p-adic side is expected to be original. Turning to equal characteristic, we again establish an analogous situation: motivated by Tate uniformization for an arbitrary bad-reduction Drinfeld $\mathbb{F}[z]$-module of rank 2, we study extensions of the form

$$0 \to (R, \sigma_R) \to (R^2, \left(\begin{smallmatrix} 1 & * \\ 0 & z-\zeta \end{smallmatrix}\right) \circ \sigma_R) \to (R, (z-\zeta) \circ \sigma_R) \to 0$$

where each of the (canonical) maps is compatible with the respective semi-linear data, and where (R, σ_R) is an $o_L[\![z]\!]$-algebra of a suitable type together with an extension $\sigma_R \colon R \to R$ of the r-Frobenius lift of $o_L[\![z]\!]$. The case $R = o_L[\![z]\!]$ corresponds to "crystalline" extensions of the Tate twist $R(1)$ by $R(0)$, and in case $R = o_L[\![z]\!][1/\pi]$ we speak of "semi-stable" extensions. Now, stressing the analogy between the multiplicative group scheme in the p-adic world and the Carlitz module in the function-field world, we discuss an equal-characteristic analogue for L of the p-adic valuation sequence for K and, using a result of B. Poonen [64], show that the quotient

$$\frac{\text{semi-stable Yoneda extensions of } R(1) \text{ by } R(0)}{\text{crystalline Yoneda extensions of } R(1) \text{ by } R(0)}$$

is free of countably infinite rank as $\mathbb{F}[\![z]\!]$-module if the residue field ℓ is finite.

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Contents

1 A local criterion for good reduction of analytic Anderson motives

Once and for all, we fix a finite field $\mathbb{F}$; its order $r = \#\mathbb{F}$ is a power of the prime number $p = \mathrm{char}(\mathbb{F})$.

Let $\mathcal{C}$ be a smooth and geometrically irreducible projective curve over $\mathbb{F}$ with function field $Q = \mathbb{F}(\mathcal{C})$. We fix a closed point $\infty \in \mathcal{C}$ and let $A = \Gamma(\mathcal{C} - \{\infty\}, \mathcal{O}_{\mathcal{C}})$ be the $\mathbb{F}$-algebra of those rational functions on $\mathcal{C}$ which are regular outside ∞, i.e.,

$$A = \{f \in Q, \quad x(f) \geq 0 \quad \text{for all closed points } x \in \mathcal{C} - \{\infty\}\},$$

where a closed point x is identified with the corresponding prime place of the global field Q.

Without proof we will use that the open part $\mathcal{C} - \{\infty\} \subseteq \mathcal{C}$ is *affine*, i.e., $\mathcal{C} - \{\infty\} = \mathrm{Spec}(A)$. In particular, A is a noetherian integral domain which moreover is immediately seen to be a Dedekind domain. The class number of A is finite; for a discussion, see [36], 4.1.

1.1 The characteristic place

Let o_L be an equi-characteristic complete discrete valuation ring containing the finite field $\mathbb{F}$, with quotient field $L = \mathrm{Frac}(o_L)$ and *perfect* residue field $\ell = o_L/\mathfrak{m}_L$, where $\mathfrak{m}_L \subseteq o_L$ is the sole maximal ideal of o_L; we fix a uniformizer $\pi = \pi_L$ of o_L, i.e., $\mathfrak{m}_L = (\pi)$. Let $|\cdot|$ denote the non-archimedean absolute value which, up to equivalence, corresponds to the discrete valuation $v = v_\pi = \mathrm{ord}_\pi(\cdot)$ on L normalized by $v(\pi) = 1$.

We assume that there is an o_L-valued point $c \in \mathcal{C}(o_L)$ such that the corresponding $\mathbb{F}$-morphism $c\colon \mathrm{Spec}(o_L) \to \mathcal{C}$ is dominant and factors via $\mathcal{C} - \{\infty\} \subseteq \mathcal{C}$; such a datum corresponds to a monomorphism of $\mathbb{F}$-algebras $c^*\colon A \to o_L$ which we call the *characteristic map.* We further assume that the closed point $V(\pi) \subseteq \mathrm{Spec}(o_L)$ is mapped to a closed point of $\mathrm{Spec}(A) \subseteq \mathcal{C}$; the latter corresponds to the kernel $\varepsilon \subseteq A$ of the composition $A \to o_L \to \ell$. So, in accordance with Drinfeld's terminology [21], we call ε the *(residue) characteristic.* In this spirit, we are encountering *mixed Drinfeld characteristic.*

Likewise, the prime place of the function field Q corresponding to the closed point ε of $\mathrm{Spec}(A) \subseteq \mathcal{C}$ is referred to as the *(residual) characteristic place of* Q. By continuity, the characteristic map $c^*\colon A \to o_L$ gives rise to an extension of complete discretely valued fields $Q_\varepsilon \subseteq L$ where $Q_\varepsilon = \mathrm{Frac}(\widehat{A}_\varepsilon)$ is the completion of Q at the characteristic place ε.

Lemma 1.1. *There is an $m \geq 1$ such that ε^m is a principal ideal of A.*

Proof. The closed point of $\mathrm{Spec}(A)$ corresponding to ε gives rise to a prime divisor D and hence to an element of the divisor class group $\mathrm{Cl}(\mathrm{Spec}(A)) = \mathrm{Div}(\mathrm{Spec}(A))/Q^\times$. This group equals the ideal class group of the Dedekind domain A and is therefore finite. This implies that the element $D \in \mathrm{Cl}(\mathrm{Spec}(A))$ is of finite order, which means that mD is a principal divisor for some $m \geq 1$, say $mD = \mathrm{div}(f)$ for some $f \in Q^\times$. Now if P varies among the closed points of $\mathrm{Spec}(A)$, we have $v_P(f) = m$ for $P = \varepsilon$ and $v_P(f) = 0$ otherwise, since D is a prime divisor, i.e., $v_P(f) \geq 0$ for all P. From this we may conclude $f \in A$ because the maximal ideals of A are precisely the prime ideals of height one. From $v_\varepsilon(f) = m$ it follows that $f \in A \cap \varepsilon^m A_\varepsilon = \varepsilon^m$. Conversely, let $g \in A \subseteq Q$ be a rational function such that $g \in \varepsilon^m$. If again P runs through the closed points of $\mathrm{Spec}(A)$, we have $v_P(g) \geq m$ for $P = \varepsilon$, and $v_P(g) \geq 0$ otherwise, i.e., $v_P(g/f) = v_P(g) - v_P(f) \geq 0$ for all P. Arguing as before, we get $g/f \in A_\mathfrak{p}$ for all maximal ideals $\mathfrak{p} \subseteq A$, and consequently $g/f \in A$, proving our claim. □

Example 1.2. Let $\mathcal{C} = \mathbb{P}^1_\mathbb{F}$ and let ∞ be the $\mathbb{F}$-rational point defined by $V(1/z) \subseteq \mathrm{Spec}(\mathbb{F}[1/z])$. Then A equals the polynomial ring $\mathbb{F}[z]$ in one indeterminate z over $\mathbb{F}$. In this situation we clearly have $\varepsilon = z\mathbb{F}[z]$ if and only if $\pi \mid z$ in o_L. □

Remark/Definition 1.3. According to Lemma 1.1, say we have $\varepsilon^m = (t)$; first of all we remark that ε cannot be nilpotent, i.e., $t \neq 0$; on the other hand, it is clear that t cannot be a unit in A. Now we know ([36], 4.1) that $A^\times = \mathbb{F}^\times$, and so we may conclude that $t \in A - \mathbb{F}$; the rational function t gives rise to a finite flat morphism $\mathcal{C} \to \mathbb{P}^1_\mathbb{F}$ ([54], 7.3.10, 4.3.10) and in particular induces a finite flat monomorphism of $\mathbb{F}$-algebras

$$\iota\colon \mathbb{F}[z] \to A$$

which identifies the indeterminate z with $t \in A$; clearly t is transcendent over the field $\mathbb{F}$, i.e., we may view t as an indeterminate over $\mathbb{F}$. –

The map $\iota\colon \mathbb{F}[z] \to A$ will be used frequently.

1.2 The base rings

In what follows, we will mainly be concerned with (semi-)linear algebra-objects which are defined over certain $A \otimes_{\mathbb{F}} o_L$-algebras. We abbreviate $A_{o_L} = A \otimes_{\mathbb{F}} o_L$ and furthermore write $A_L = A \otimes_{\mathbb{F}} L$ as well as $A_\ell = A \otimes_{\mathbb{F}} \ell$; i.e., $A_L \simeq A_{o_L}[1/\pi]$ and $A_\ell \simeq A_{o_L}/\pi A_{o_L}$.

The o_L-valued point $c \in \mathcal{C}(o_L)$ gives rise to a canonical morphism of $\mathbb{F}$-schemes $(\mathrm{id}, c)\colon \mathrm{Spec}(o_L) \to \mathcal{C} \otimes_{\mathbb{F}} o_L$, the associated *graph morphism*. In particular, as c factors via $\mathcal{C} - \{\infty\}$, there is a map of $\mathbb{F}$-algebras $\gamma = (\mathrm{id}, c^*)\colon A_{o_L} \to o_L$ which is surjective since it has a canonical section, naturally embedding o_L into A_{o_L}; at the same time, since c is dominant, γ yields that also A is naturally embedded into A_{o_L}.

We first gather together a couple of properties of the base rings defined so far, starting with the following Lemma. For the notion of excellence, see [EGA IV(2)], 7.8.

Lemma 1.4. *(i) The o_L-algebra A_{o_L} is excellent.*

(ii) A_ℓ and A_L are Dedekind domains.

Proof. As $\mathrm{Spec}(A)$ is noetherian, the inclusion $\mathrm{Spec}(A) \subseteq \mathcal{C}$ is quasi-compact, hence of finite type. Consequently, the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(\mathbb{F})$ is of finite type and, by base change, so is $\mathrm{Spec}(A_{o_L}) \to \mathrm{Spec}(o_L)$. By [EGA I(n)], I.6.3.5, the ring A_{o_L} therefore has to be noetherian. On the other hand, by [EGA IV(2)], 7.8.3, we conclude that A_{o_L} is excellent since the complete discrete valuation ring o_L is. In order to prove (ii), we just remark that $\mathcal{C} \otimes_{\mathbb{F}} K$ is smooth of relative dimension 1 over K and irreducible for every field extension $K/\mathbb{F}$. □

In particular, $A_{o_L} \subseteq A_L$ is a noetherian integral domain, and by virtue of the equality $A_\ell \simeq A_{o_L}/\pi A_{o_L}$ it follows that $\pi \in o_L$ gives rise to a prime element of A_{o_L}.

Definition 1.5. *Let $A_{o_L,\pi}$ (resp., $A_{o_L,(\varepsilon,\pi)}$) be the completion of the o_L-algebra A_{o_L} for the π-adic topology (resp., the (ε, π)-adic topology).*

By Krull's Theorem ([13], III.3.2), the ring A_{o_L} is separated for both the π-adic and the (ε, π)-adic topology.

Lemma 1.6. *The topological o_L-algebra $A_{o_L,\pi}$ is admissible in the sense of Raynaud, i.e., it is of topologically finite presentation and has no π-torsion. In particular, the L-algebra $A_{o_L,\pi}[1/\pi]$ is affinoid.*

Proof. First we remark that by construction $A_{o_L,\pi}$ is π-adically complete and separated. We have

$$A_{o_L,\pi}/\pi A_{o_L,\pi} \simeq A_{o_L}/\pi A_{o_L} \simeq A_\ell,$$

and the latter is a finitely generated ℓ-algebra. So, by [9], 2.3/10(a), it follows that $A_{o_L,\pi}$ is of topologically finite type over o_L, which means that it is isomorphic to a quotient of $o_L\langle \underline{x}\rangle$ for some finite system $\underline{x}$ of indeterminates; but $o_L\langle \underline{x}\rangle$ is noetherian by [9], 2.3/1, and so $A_{o_L,\pi}$ is even of topologically finite presentation. Since A_{o_L} is an integral domain, by [13], III.3.4.2, no power of π can give rise to a zero-divisor in $A_{o_L,\pi}$, i.e., $A_{o_L,\pi}$ does not have π-torsion and is therefore admissible. The last claim follows from [9], 2.4. □

In sections (1.5) and (1.6) the (geometric) role of these o_L-algebras will be further explained. For now, let us briefly explain the most important instance from which our base rings $A_{o_L,\pi}$ and $A_{o_L,(\varepsilon,\pi)}$ arise: If $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$ then we have $A_{o_L} = o_L[z]$ and correspondingly $A_L = L[z]$. Let us specify that $\varepsilon = z\mathbb{F}[z]$. Our choice of a uniformizer π gives rise to an identification $o_L = \ell[\![\pi]\!]$; see [69], II.4.2. Consequently $o_L[\![z]\!] = \ell[\![\pi]\!][\![z]\!] = \ell[\![\pi, z]\!]$, and the latter equals the (π, z)-adic completion of $\ell[\![\pi]\!][z]$. In this spirit we view $A_{o_L,(\varepsilon,\pi)}$ as a replacement, for general $\mathcal{C}$ and ε, of the o_L-algebra $o_L[\![z]\!]$.

On the other hand, the π-adic completion of $o_L[z]$ equals $o_L\langle z\rangle$, and since $L\langle z\rangle = o_L\langle z\rangle \otimes_{o_L} L$, we may view $A_{o_L,\pi}[1/\pi]$ as a replacement, for general $\mathcal{C}$, of the Tate algebra $L\langle z\rangle$ of strictly convergent power series in one indeterminate z over L, which serves as coordinate ring for the one-dimensional affinoid unit ball in classical rigid geometry.

The Tate algebra $L\langle z\rangle$ is obtained from the affine coordinate ring $L[z]$ via completion with respect to its Gauss norm defined by $||\sum_\nu^{<\infty} a_\nu z^\nu|| = \sup_\nu(|a_\nu|)$, where $|\cdot|$ is the π-adic absolute value of L; there is an obvious well-defined version of the Gauss norm for strictly convergent power series which makes $L\langle z\rangle$ into an L-Banach algebra, and one finds $o_L\langle z\rangle = \{f \in L\langle z\rangle, ||f|| \leq 1\}$; see [9], [28].

In the general case, this is mirrored as follows: There is a natural embedding $A_L \to A_{o_L,\pi}[1/\pi]$ which, for general $\mathcal{C}$, replaces the completion homomorphism $L[z] \to L\langle z\rangle$, and which itself can be regarded as a completion map with respect to the L-algebra norm-topology on the *reduced* affinoid L-algebra $A_{o_L,\pi}[1/\pi]$ and its restriction on A_L; see [9], 1.4/19. –

Note that the canonical homomorphism $A_{o_L} \to A_{o_L,(\varepsilon,\pi)}$ factors uniquely via $A_{o_L,\pi}$, where the induced map $A_{o_L,\pi} \to A_{o_L,(\varepsilon,\pi)}$ identifies $A_{o_L,(\varepsilon,\pi)}$ with the $(\varepsilon,\pi)A_{o_L,\pi}$-adic completion of $A_{o_L,\pi}$, which means that it has to be flat; moreover, there is a commutative diagram

$$\begin{array}{ccccc} A_{o_L} & \longrightarrow & A_{o_L,\pi} & \longrightarrow & A_{o_L,\pi}[1/\pi] \\ \| & & \downarrow & & \downarrow \\ A_{o_L} & \longrightarrow & A_{o_L,(\varepsilon,\pi)} & \longrightarrow & A_{o_L,(\varepsilon,\pi)}[1/\pi] \end{array}$$

where all arrows are injective and flat. In order to justify the injectivity for $A_{o_L,\pi} \to A_{o_L,(\varepsilon,\pi)}$, we claim that $A_{o_L,\pi}$ is $(\varepsilon,\pi)A_{o_L,\pi}$-adically separated. Again by Krull's Theorem it suffices to show that $A_{o_L,\pi}$ is an integral domain, which is accomplished by the following

Proposition 1.7. *$A_{o_L,\pi}$ is a regular integral domain.*

Proof. Let $X = \mathrm{Spec}(A_{o_L})$. Since A_{o_L} is excellent by 1.4, it follows from [EGA IV(2)], 7.8.3.1, that the regular locus $\mathrm{Reg}(X')$ of $X' = \mathrm{Spec}(A_{o_L,\pi})$ equals $f^{-1}(\mathrm{Reg}(X))$ where f is the canonical morphism $f\colon X' \to X$. Therefore, $X \to \mathrm{Spec}(o_L)$ being smooth, the scheme X' is regular, which implies that every local ring of $A_{o_L,\pi}$ at a prime ideal is an integral domain, i.e., X' is locally integral. It remains to show that X' is connected. Since $A_{o_L,\pi}/\pi A_{o_L,\pi} \simeq A_{o_L}/\pi A_{o_L}$ is an integral domain, the closed subset $V(\pi) \subseteq X'$ is connected. Suppose we have a nontrivial disjoint decomposition $X' = V(e) \cup V(1-e)$, where e, $1-e$ is a pair of orthogonal idempotents. From this we get $V(\pi) = V(\pi, e) \cup V(\pi, 1-e)$. Now the quotient $A_{o_L,\pi}/e$ is nontrivial, i.e., it contains a maximal ideal $\mathfrak{n}$. By [58], 8.1, $A_{o_L,\pi}/e$ is again π-adically complete and separated, so that $\pi A_{o_L,\pi}/e \subseteq \mathfrak{j}(A_{o_L,\pi}/e) \subseteq \mathfrak{n}$ (loc. cit., 8.2); this means that $V(\pi, e)$ cannot be empty; arguing similarly for the idempotent $1-e$ gives the desired contradiction, showing that X' has to be connected. Finally, by [EGA I(n)], I.4.5.6, we conclude that $A_{o_L,\pi}$ is an integral domain. □

Since $A_{o_L,\pi}$ is π-adically complete and separated, the following Lemma is immediately derived from [58], 8.2.

Lemma 1.8. *The element π lies in every maximal ideal of $A_{o_L,\pi}$.* □

Recall that there is a finite flat monomorphism of $\mathbb{F}$-algebras $\iota\colon \mathbb{F}[z] \to A$ which identifies the indeterminate z with the non-constant rational function $t \in A$ chosen in 1.3; the o_L-algebra homomorphism $\iota \otimes \mathrm{id} \colon o_L[z] \to A_{o_L}$, $\sum_\nu a_\nu z^\nu \mapsto \sum_\nu t^\nu \otimes a_\nu$, is finite flat, so that also the maps

$$o_L\langle z\rangle \to A_{o_L,\pi}, \quad L\langle z\rangle \to A_{o_L,\pi}[1/\pi], \quad o_L[\![z]\!] \to A_{o_L,(t,\pi)}, \quad \ell[z] \to A_\ell$$

are finite flat; here the (t,π)-adic completion $A_{o_L,(t,\pi)}$ of A_{o_L} equals $A_{o_L,(\varepsilon,\pi)}$ since $(\varepsilon,\pi)^m \subseteq (\varepsilon^m,\pi) = (t,\pi)$ in A_{o_L}; we have a commutative diagram

$$\begin{array}{ccccc} o_L[z] & \longrightarrow & o_L\langle z\rangle & \longrightarrow & o_L[\![z]\!] \\ \downarrow & & \downarrow & & \downarrow \\ A_{o_L} & \longrightarrow & A_{o_L,\pi} & \longrightarrow & A_{o_L,(\varepsilon,\pi)} \end{array}$$

where the horizontal arrows are completion maps and therefore flat, and where the vertical maps are finite flat.

1.3 Liftings of Frobenius

The r-Frobenius $\mathrm{Frob}_r \colon o_L \to o_L$, $x \mapsto x^r$, gives rise to an endomorphism

$$\sigma = \mathrm{id}_A \otimes \mathrm{Frob}_r \colon A_{o_L} \to A_{o_L}, \quad a \otimes x \mapsto a \otimes x^r,$$

which extends to give a map $\mathrm{id}_A \otimes \mathrm{Frob}_{r,L} \colon A_L \to A_L$ again denoted by σ. On the other hand, reducing mod π gives $\bar\sigma = \mathrm{id}_A \otimes \mathrm{Frob}_{r,\ell} \colon A_\ell \to A_\ell$; the latter is clearly an automorphism of the Dedekind domain A_ℓ.

The map $\sigma \colon A_{o_L} \to A_{o_L}$ is π-adically and (ε,π)-adically continuous and therefore extends to give endomorphisms $A_{o_L,\pi} \to A_{o_L,\pi}$ and $A_{o_L,(\varepsilon,\pi)} \to A_{o_L,(\varepsilon,\pi)}$, again denoted by σ.

Lemma 1.9. *In the commutative diagram*

$$\begin{array}{ccccc} A_{o_L} & \longrightarrow & A_{o_L,\pi} & \longrightarrow & A_{o_L,(\varepsilon,\pi)} \\ {\scriptstyle\sigma}\downarrow & & {\scriptstyle\sigma}\downarrow & & {\scriptstyle\sigma}\downarrow \\ A_{o_L} & \longrightarrow & A_{o_L,\pi} & \longrightarrow & A_{o_L,(\varepsilon,\pi)} \end{array}$$

both squares are cocartesian, and the vertical arrows are finite flat.

We let the proof be preceded by the following

Remark. As mentioned before (see (1.2)), our choice of a uniformizer π identifies o_L with $\ell[\![\pi]\!]$. Via this identification, the r-Frobenius $\mathrm{Frob}_{r,o_L} \colon o_L \to o_L$ is mirrored by the map $\ell[\![\pi]\!] \to \ell[\![\pi]\!]$, $\sum_{\nu=0}^\infty a_\nu \pi^\nu \mapsto \sum_{\nu=0}^\infty a_\nu^r \pi^{r\nu}$; using that $\mathrm{Frob}_{r,\ell}$ is an automorphism, this implies $(\mathrm{Frob}_{r,o_L})_* o_L = o_L\pi^0 \oplus ... \oplus o_L\pi^{r-1}$; furthermore, the map Frob_{r,o_L} is injective, i.e., $\mathrm{Spec}(\mathrm{Frob}_{r,o_L})$ is dominant, so that Frob_{r,o_L} has to be flat (for example, by [54], 4.3.9), i.e., we may summarize that the r-Frobenius $\mathrm{Frob}_{r,o_L}: o_L \to o_L$ is finite flat. –

Proof of Lemma 1.9. By [EGA IV(2)], 2.1.7, the product $\sigma = \mathrm{id}_A \otimes \mathrm{Frob}_{r,o_L}: A_{o_L} \to A_{o_L}$ is flat, and by [EGA II], 6.1.4/5, it is also finite. By base change, we conclude that $A_{o_L} \otimes_{\sigma, A_{o_L}} A_{o_L,\pi}$ is a flat $A_{o_L,\pi}$-module. Since $\sigma: A_{o_L} \to A_{o_L}$ is finite, this tensor product equals the π-adic completion of the A_{o_L}-module $\sigma_* A_{o_L}$. If we let $\mathfrak{a} = \sigma(\pi A_{o_L})A_{o_L} = \pi^r A_{o_L} = (\pi A_{o_L})^r$ and $\mathfrak{b} = \pi A_{o_L}$, we get $\mathfrak{b}^r = \mathfrak{a} \subseteq \mathfrak{b}$. Consequently, by [23], 7.14, the inverse systems $(A_{o_L}/\mathfrak{a}^n)_n$ and $(A_{o_L}/\mathfrak{b}^n)_n$ give the same limit, which shows at once that the square on the left is cocartesian, and that $\sigma: A_{o_L,\pi} \to A_{o_L,\pi}$ is flat; in particular, a base change argument now shows that the latter homomorphism is also finite. Similarly, we have $\sigma(\varepsilon,\pi)A_{o_L} = (\varepsilon, \pi^r) \subseteq (\varepsilon,\pi)$ as well as $(\varepsilon,\pi)^r \subseteq (\varepsilon,\pi^r)$, which proves that the displayed diagram qualifies $A_{o_L,(\varepsilon,\pi)}$ as tensor product $A_{o_L,(\varepsilon,\pi)} \otimes_{A_{o_L},\sigma} A_{o_L}$, and that $\sigma: A_{o_L,(\varepsilon,\pi)} \to A_{o_L,(\varepsilon,\pi)}$ is finite flat. But now it is merely a formal matter to show that also the square on the right has to be cocartesian. □

Finally, note that the embedding of o_L-algebras $\iota \otimes \mathrm{id}: o_L[z] \to A_{o_L}$ commutes with $\sigma: A_{o_L} \to A_{o_L}$ and the r-Frobenius lift of $o_L[z]$, given by

$$o_L[z] \to o_L[z], \quad \sum_\nu a_\nu z^\nu \mapsto \sum_\nu a_\nu^r z^\nu;$$

Consequently, also the embeddings

$$o_L\langle z\rangle \to A_{o_L,\pi}, \quad L\langle z\rangle \to A_{o_L,\pi}[1/\pi], \quad o_L[\![z]\!] \to A_{o_L,(t,\pi)}, \quad \ell[z] \to A_\ell$$

from the end of section (1.2) are Frobenius-equivariant.

1.4 Categories of Frobenius modules

Let $\underline{A}$ be an o_L-algebra together with a ring endomorphism $\sigma: \underline{A} \to \underline{A}$ such that σ and $\mathrm{Frob}_{r,o_L}: o_L \to o_L$ are compatible with the structure map $o_L \to \underline{A}$, i.e., such that σ extends Frob_{r,o_L}. For example, $\underline{A}$ could be any of the base rings considered in the previous sections.

Let $\underline{M}$ be any $\underline{A}$-module which comes equipped with an $\underline{A}$-linear map $F: \sigma^*\underline{M} \to \underline{M}$. Then F corresponds to a homomorphism of abelian groups $F^{\mathrm{sl}}: \underline{M} \to \underline{M}$ which is *semi*-linear with respect to $\sigma: \underline{A} \to \underline{A}$; namely, F^{sl} is obtained by composing F with the canonical σ-semi-linear map $\underline{M} \to \sigma^*\underline{M}$.

We define the category $\mathrm{FMod}(\underline{A})$ of *Frobenius $\underline{A}$-modules* (or simply *F-modules* over $\underline{A}$) as follows:

— An object of $\mathrm{FMod}(\underline{A})$ is a pair $(\underline{M}, F)$ consisting of an $\underline{A}$-module $\underline{M}$ which is finite projective (or, equivalently: locally free of finite rank), together with an *injective* $\underline{A}$-linear map $F = F_{\underline{M}}\colon \sigma^*\underline{M} \to \underline{M}$; the datum F (equivalently, F^{sl}) will usually be omitted from the notation, if no ambiguity can arise.

— As usual, a *morphism* of Frobenius $\underline{A}$-modules $(\underline{M}, F_{\underline{M}}) \to (\underline{N}, F_{\underline{N}})$ is an $\underline{A}$-linear map $\varphi\colon \underline{M} \to \underline{N}$ between the underlying $\underline{A}$-modules such that φ is *F-equivariant*, i.e., such that $\varphi \circ F_{\underline{M}} = F_{\underline{N}} \circ \sigma^*\varphi$ (or, equivalently: $\varphi \circ F^{\mathrm{sl}}_{\underline{M}} = F^{\mathrm{sl}}_{\underline{N}} \circ \varphi$); it is called an *isomorphism* if φ is an isomorphism of the underlying $\underline{A}$-modules.

It is an obvious conclusion that the forgetful functor from $\mathrm{FMod}(\underline{A})$ to the category of $\underline{A}$-modules is faithful. We further remark that for an isomorphism $\varphi\colon \underline{M} \to \underline{N}$ inside $\mathrm{FMod}(\underline{A})$ the inverse map of $\underline{A}$-modules φ^{-1} is automatically F-equivariant since the assignment $\underline{M} \mapsto \sigma^*\underline{M}$ is functorial on $\underline{A}$-modules.

Let $\underline{B}$ be a flat $\underline{A}$-algebra together with a ring endomorphism $\sigma\colon \underline{B} \to \underline{B}$ extending the Frobenius lift of $\underline{A}$, as explained before. Then the exact functor $\cdot \otimes_{\underline{A}} \underline{B}$ from $\underline{A}$-modules to $\underline{B}$-modules restricts to a functor $\mathrm{FMod}(\underline{A}) \to \mathrm{FMod}(\underline{B})$; if the structure map $\underline{A} \to \underline{B}$ is, in addition, injective then the induced functor on $\mathrm{FMod}(\underline{A})$ is faithful since, given a map $f\colon \underline{M} \to \underline{N}$ of finite projective $\underline{A}$-modules, restricting its image $f \otimes \mathrm{id}\colon \underline{M} \otimes_{\underline{A}} \underline{B} \to \underline{N} \otimes_{\underline{A}} \underline{B}$ to $\underline{M}$ gives back f. In particular, we obtain a natural commutative diagram of categories and faithful functors

$$\begin{array}{ccccc}
\mathrm{FMod}(A_{o_L}) & \longrightarrow & \mathrm{FMod}(A_{o_L,\pi}) & \longrightarrow & \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{FMod}(A_L) & \longrightarrow & \mathrm{FMod}(A_{o_L,\pi}[1/\pi]) & \longrightarrow & \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi])
\end{array}$$

1.5 Analytic Anderson motives

Since $(\mathcal{C} - \{\infty\}) \otimes_{\mathbb{F}} L = \mathrm{Spec}(A_L)$ is of finite type over L, one can consider its rigid analytification $\mathrm{Spec}(A_L)^{\mathrm{an}}$; see [9], [28]; in accordance with [6], we denote this rigid analytic L-space by $\mathfrak{A}(\infty)$.

On the other hand, the formal completion of the o_L-scheme $X = \mathrm{Spec}(A_{o_L})$ along its special fiber $V(\pi)$ leads to the formal o_L-scheme $\mathfrak{X} = \mathrm{Spf}(A_{o_L,\pi})$; see [EGA I(n)], I.10.8.3; its rigidification $\mathfrak{X}_{\mathrm{rig}}$ ([9], [28]) is given by the affinoid L-space $\mathfrak{A}(1)$ corresponding to the affinoid L-algebra $A_{o_L,\pi}[1/\pi]$ (see 1.6); this space can be regarded as the unit disc of the rigid analytic space $\mathfrak{A}(\infty)$; as opposed to its global counterpart $\mathfrak{A}(\infty)$, it corresponds to "radius of convergence 1", hence the notation.

Let $\mathfrak{J} \subseteq A_{o_L}$ be the ideal generated by the elements $a \otimes 1 - 1 \otimes c^*(a)$, where $a \in A$. For example, if $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$, i.e., $A = \mathbb{F}[z]$, then one easily computes that $\mathfrak{J} = (z - \zeta) \subseteq o_L[z]$ where $\zeta = c^*(z)$.

We intend to study the following instance of rigid analytic τ-sheaves over $A_{o_L,\pi}[1/\pi]$ on $\mathfrak{A}(1)$, in the sense of [6]. See also section (1.7).

Definition 1.10. *An* analytic Anderson $A(1)$-motive *is an object*

$$M_L \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$$

such that $\mathrm{coker}(F_{M_L})$ *is a finite-dimensional* L*-vector space and is annihilated by a power of* $\mathfrak{J}$*. A* morphism *of analytic Anderson* $A(1)$*-motives is defined as a morphism in the category* $\mathrm{FMod}(A_{o_L,\pi}[1/\pi])$.

Here the prefix "$A(1)$-" indicates that we are considering an analytic variant of Anderson A-motives over the rigid analytic unit disc associated to our chosen o_L-valued point $c \in \mathcal{C}(o_L)$; recall that an *Anderson A-motive* ([2], [36]) is an object $\underline{M} \in \mathrm{FMod}(A_L)$ such that $\mathrm{coker}(F_{\underline{M}})$ is a finite-dimensional L-vector space and is annihilated by a power of $\mathfrak{J}$; a *morphism* of Anderson A-motives is defined as a morphism inside $\mathrm{FMod}(A_L)$.

Proposition 1.11. *The natural functor* $\mathrm{FMod}(A_L) \to \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ *restricts to a functor*

$$(\textit{Anderson } A\textit{-motives}) \to (\textit{analytic Anderson } A(1)\textit{-motives}).$$

Proof. Let $\underline{M}$ be an Anderson A-motive. Then $\widehat{\underline{M}} = \underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ is flat over $A_{o_L,\pi}[1/\pi]$. Furthermore, any exact sequence of A_L-modules of the form $A_L^s \to \underline{M} \to 0$ yields an exact sequence $A_{o_L,\pi}[1/\pi]^s \to \widehat{\underline{M}} \to 0$, i.e., we may summarize that $\widehat{\underline{M}}$ is locally free of finite rank. Similarly one verifies that the map $F \otimes \mathrm{id}$ is again injective; let C be its cokernel; clearly C is finitely presented over the $L\langle z\rangle$-algebra $A_{o_L,\pi}[1/\pi]$, which in turn is finite over $L\langle z\rangle$; if $\mathrm{coker}(F)$ is annihilated by $\mathfrak{J}^d$, so is C (for details cf. the proof of 1.18); in particular, we have $(z-\zeta)^d C = 0$, where $\zeta \in L$ is defined in section (1.6); finally, by the Weierstraß Division Theorem for $L\langle z\rangle$ (see [9], 1.2/8), the quotient $L\langle z\rangle/(z-\zeta)^d$ is finite over L, and so C is a finite-dimensional L-vector space. □

For the following Lemma, recall the well-known fact ([9], [28]) that the Tate algebra $L\langle z\rangle$ is a factorial Dedekind domain, i.e., a principal ideal domain ([58], 20.7).

Lemma 1.12. *Let* M_L *be an analytic Anderson* $A(1)$*-motive. Then* M_L *is a finite free* $L\langle z\rangle$*-module via* $L\langle z\rangle \to A_{o_L,\pi}[1/\pi]$.

Proof. Being a composition of exact functors, the functor $\cdot \otimes_{L\langle z\rangle} M_L = (\cdot \otimes_{L\langle z\rangle} A_{o_L,\pi}[1/\pi]) \otimes_{A_{o_L,\pi}[1/\pi]} M_L$ is again exact. Furthermore, M_L is of finite presentation over $A_{o_L,\pi}[1/\pi]$, and the latter is finite over $L\langle z\rangle$, so M_L is also of finite presentation over $L\langle z\rangle$, and we may conclude that M_L is locally free of finite rank over $L\langle z\rangle$, which implies that it is torsion-free over $L\langle z\rangle$, hence free of finite rank, as $L\langle z\rangle$ is a principal ideal domain. □

Definition 1.13. *Let M_L be an analytic Anderson $A(1)$-motive. A* (formal) model *of M_L is an object $\mathcal{M} \in \mathrm{FMod}(A_{o_L,\pi})$ such that its image along the natural functor $\mathrm{FMod}(A_{o_L,\pi}) \to \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ is isomorphic to M_L inside $\mathrm{FMod}(A_{o_L,\pi}[1/\pi])$.*

For the moment, let $\mathcal{M}$ be any $A_{o_L,\pi}$-module, coming equipped with a σ-semi-linear map $\mathcal{M} \to \mathcal{M}$. To $\mathcal{M}$ we can associate its *reduction*

$$\mathcal{M}/\pi\mathcal{M} = \mathcal{M} \otimes_{o_L} \ell,$$

which is naturally a module over the Dedekind domain A_ℓ. The semi-linear datum $\mathcal{M} \to \mathcal{M}$ induces a canonical map of abelian groups $\mathcal{M}/\pi\mathcal{M} \to \mathcal{M}/\pi\mathcal{M}$ which is $\bar{\sigma}$-semi-linear; of course, the residue map $\mathcal{M} \to \mathcal{M}/\pi\mathcal{M}$ then automatically commutes with the respective semi-linear data on $\mathcal{M}$ and $\mathcal{M}/\pi\mathcal{M}$.

Note, however, that this does *not* induce a functor from $\mathrm{FMod}(A_{o_L,\pi})$ to $\mathrm{FMod}(A_\ell)$, since the induced F-map need not be injective. This circumstance lies at the origin of our study of good models:

Definition 1.14. *Let $\mathcal{M}$ be a model of an analytic Anderson $A(1)$-motive M_L. Then $\mathcal{M}$ is called a* good *model if*

1. *the induced A_ℓ-linear map*

$$\bar{\sigma}^*\mathcal{M}/\pi\mathcal{M} = \mathcal{M}/\pi\mathcal{M} \otimes_{A_\ell,\bar{\sigma}} A_\ell \to \mathcal{M}/\pi\mathcal{M}$$

is injective;

2. $\mathrm{coker}(F_{\mathcal{M}})$ *is a finite free o_L-module and is annihilated by $\mathfrak{J}^d$, for some $d \geq 0$.*

Example 1.15. Let $\underline{M}$ be an Anderson A-motive with *good reduction*, that is, there is a locally free A_{o_L}-module $\underline{\mathcal{M}}$ of finite rank together with an A_{o_L}-linear map $F^\circ\colon \underline{\mathcal{M}} \otimes_{A_{o_L},\sigma} A_{o_L} \to \underline{\mathcal{M}}$ such that there is an F-equivariant and A_L-linear isomorphism $\underline{\mathcal{M}}[1/\pi] \simeq \underline{M}$, in such a way that $\mathrm{coker}(F^\circ)$ is a finite free o_L-module and is annihilated by a power of $\mathfrak{J}$ (see also [41], 2.1.4); note that every such isomorphism gives rise to an F-equivariant embedding $\underline{\mathcal{M}} \hookrightarrow \underline{M}$ which shows that F° is automatically injective. Moreover the induced A_ℓ-linear map

$$\underline{\mathcal{M}}/\pi\underline{\mathcal{M}} \otimes_{A_\ell,\bar{\sigma}} A_\ell \to \underline{\mathcal{M}}/\pi\underline{\mathcal{M}}$$

is again injective by virtue of our requirements on $\operatorname{coker}(F^\circ)$, so that the π-adic completion $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi}$ of $\underline{\mathcal{M}}$ gives rise to a good model for the analytic Anderson $A(1)$-motive $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$; this is proven in 1.30 below. □

1.6 Local shtukas and the main theorem

As opposed to Drinfeld's shtukas (also called F-sheaves in [20], [22]), which are defined over base schemes involving the whole curve $\mathcal{C}$ and are therefore of global nature, local shtukas are associated to a fixed place of the curve $\mathcal{C}$; they are obtained via formal completion (along the fiber of this fixed place) from global objects like A-motives with good reduction, or Drinfeld shtukas; see [41], 2.1.4.

We intend to study (effective) local shtukas at the residual characteristic place ε and commence by giving some elementary remarks regarding the fiber $\varepsilon \times \operatorname{Spec}(o_L)$ of $\varepsilon \in \mathcal{C}$ along the projection $\mathcal{C} \otimes_{\mathbb{F}} o_L \to \mathcal{C}$. If $k(\varepsilon)$ denotes the residue field $A_\varepsilon/\varepsilon A_\varepsilon \simeq A/\varepsilon$ of ε, we have a canonical closed immersion

$$(\mathcal{C} \otimes_{\mathbb{F}} o_L) \times_{\mathcal{C}} \operatorname{Spec}(k(\varepsilon)) \to (\mathcal{C} \otimes_{\mathbb{F}} o_L) \times_{\mathcal{C}} U,$$

where $U \subseteq \mathcal{C}$ stands for the affine open neighborhood $\mathcal{C} - \{\infty\}$ of $\varepsilon \in \mathcal{C}$; this means that $\varepsilon \times \operatorname{Spec}(o_L)$ is contained in (and lies closed inside) the affine open subscheme $\operatorname{Spec}(A_{o_L}) = U \otimes_{\mathbb{F}} o_L \subseteq \mathcal{C} \otimes_{\mathbb{F}} o_L$, and we are led to considering the fiber $V(\varepsilon A_{o_L}) \subseteq \operatorname{Spec}(A_{o_L})$ of ε along $U \otimes_{\mathbb{F}} o_L \to U$; the formal completion of $U \otimes_{\mathbb{F}} o_L$ along this fiber is represented by the completion $A_{o_L,\varepsilon}$ of A_{o_L} for the εA_{o_L}-adic topology.

For example, if $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$ and $\varepsilon = z\mathbb{F}[z]$ then the ε-adic and the (ε, π)-adic completion of $A_{o_L} = o_L[z]$ coincide and are both equal to $o_L[\![z]\!]$. The following Lemma shows that, in fact, this is also true for general $\mathcal{C}$ and ε.

Lemma 1.16. *The canonical map $A_{o_L,\varepsilon} \to A_{o_L,(\varepsilon,\pi)}$ is an isomorphism.*

Proof. First we remark that we have a canonical isomorphism

$$A_{o_L} \simeq A \otimes_{\mathbb{F}[z], z \mapsto z} o_L[z];$$

note that this is an isomorphism of $o_L[z]$-algebras. The composition $o_L[z] \to A_{o_L} \to A_{o_L,\varepsilon}$, mapping z to (the image of) t, is (z)-ε-adically continuous and hence induces a map $o_L[\![z]\!] \to A_{o_L,\varepsilon}$ which in turn gives rise to a canonical map $A \otimes_{\mathbb{F}[z]} o_L[\![z]\!] \to A_{o_L,\varepsilon}$. We claim that the latter is an isomorphism. Indeed, the tensor product $A \otimes_{\mathbb{F}[z]} o_L[\![z]\!]$ equals the z-adic completion of the (finite) $o_L[z]$-algebra A_{o_L} and is therefore z-adically complete; as $\varepsilon^m = (t)$ in A, the canonical map $A_{o_L} \to A \otimes_{\mathbb{F}[z]} o_L[\![z]\!]$ is

ε-(z)-adically continuous and thus extends to give a map $A_{o_L,\varepsilon} \to A \otimes_{\mathbb{F}[z]} o_L[\![z]\!]$ which is the desired inverse: it is trivial that $A_{o_L} \to A_{o_L,\varepsilon}$ factors via the identity of $A_{o_L,\varepsilon}$, and on the other hand, it factors via $A_{o_L} \to A \otimes_{\mathbb{F}[z]} o_L[\![z]\!]$, which in turn factors via $A_{o_L,\varepsilon}$; so by the universal property of the map $A_{o_L} \to A_{o_L,\varepsilon}$ we see that the composition $A_{o_L,\varepsilon} \to A \otimes_{\mathbb{F}[z]} o_L[\![z]\!] \to A_{o_L,\varepsilon}$ necessarily equals the identity. In the same way one argues in order to show that $A \otimes_{\mathbb{F}[z]} o_L[\![z]\!] \to A_{o_L,\varepsilon} \to A \otimes_{\mathbb{F}[z]} o_L[\![z]\!]$ also equals the identity. Finally, we just remark that $o_L[\![z]\!]$ also equals the (z,π)-adic completion of $o_L[z]$ and that $A_{o_L,(t,\pi)}$ equals $A_{o_L,(\varepsilon,\pi)}$; so, replacing $\varepsilon A_{o_L} \subseteq A_{o_L}$ by (ε,π) and thereby imitating the arguments given so far, we realize that the canonical map $A \otimes_{\mathbb{F}[z]} o_L[\![z]\!] \to A_{o_L,(\varepsilon,\pi)}$ is an isomorphism, proving our claim. □

Definition 1.17. *An* (effective) local shtuka *at ε over o_L is an object*

$$\hat{M} \in \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)})$$

such that $\mathrm{coker}(F_{\hat{M}})$ *is a finite free o_L-module and is annihilated by a power of $\mathfrak{J}$.*

Arguing like in the proof of 1.12, one easily verifies that via the embedding $o_L[\![z]\!] \to A_{o_L,(\varepsilon,\pi)}$ a local shtuka $\hat{M}$ gives rise to a finite free $o_L[\![z]\!]$-module. Furthermore, using the isomorphism $A \otimes_{\mathbb{F}[z]} o_L[\![z]\!] \to A_{o_L,(\varepsilon,\pi)}$ (see the proof of 1.16), one shows that there is a canonical isomorphism

$$\hat{M} \otimes_{(A_{o_L,(\varepsilon,\pi)}),\sigma} A_{o_L,(\varepsilon,\pi)} \simeq \hat{M} \otimes_{o_L[\![z]\!],\sigma} o_L[\![z]\!]$$

(for details cf. the corresponding argument for $A_{o_L,\pi}$ and $o_L\langle z\rangle$ on p. 18).

Let $\zeta \in o_L$ be the image of the rational function $t \in A$ under the characteristic map $c^*\colon A \to o_L$. By choice of t we obtain that $\pi \mid \zeta$ in o_L. If $\mathfrak{J} \subseteq A_{o_L}$ denotes the ideal generated by the elements $a \otimes 1 - 1 \otimes c^*(a)$, where $a \in A$, via the embedding $\iota \otimes \mathrm{id}\colon o_L[z] \to A_{o_L}$ we get $(z-\zeta)A_{o_L} \subseteq \mathfrak{J}$.

Remark. Let $\hat{M}$ be a local shtuka at ε in the sense of the above Definition, and let C be the cokernel of $F\colon \sigma^*\hat{M} \to \hat{M}$, say with $\mathfrak{J}^d C = 0$. In particular, this implies $(z-\zeta)^d C = 0$, so that applying the functor $\cdot \otimes_{o_L[\![z]\!]} o_L[\![z]\!][\frac{1}{z-\zeta}]$ to

$$0 \to \sigma^*\hat{M} \to \hat{M} \to C \to 0$$

yields an isomorphism $\sigma^*\hat{M}[\frac{1}{z-\zeta}] \to \hat{M}[\frac{1}{z-\zeta}]$ of $o_L[\![z]\!][\frac{1}{z-\zeta}]$-modules. In particular, $\hat{M}$ gives rise to a local shtuka in the sense of [41], 2.1.1, over the formal (one-point) o_L-scheme $\mathrm{Spf}(o_L)$. –

The following criterion for good reduction of analytic Anderson $A(1)$-motives becomes highly plausible when looking at the commutative square

$$\begin{array}{ccc} \mathrm{FMod}(A_{o_L,\pi}) & \longrightarrow & \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}) \\ \downarrow & & \downarrow \\ \mathrm{FMod}(A_{o_L,\pi}[1/\pi]) & \longrightarrow & \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi]) \end{array}$$

and can also be regarded as a *good-reduction Local-Global Principle at the characteristic place.* In the course of its proof we will make extensive use of the embedding $\iota\colon \mathbb{F}[z] \to A$, as defined in 1.3, and its various descendants over o_L.

Theorem 1.18. *Let M_L be an analytic Anderson $A(1)$-motive such that* $\mathrm{coker}(F_{M_L})$ *is annihilated by $\mathfrak{J}^d$ say. Then the following assertions are equivalent:*

(i) M_L admits a good model;

(ii) There is

- *a local shtuka $\hat{M}$ at ε such that* $\mathrm{coker}(F_{\hat{M}})$ *is a finite free o_L-module and is annihilated by $\mathfrak{J}^d$,*
- *an isomorphism*

$$M_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \simeq \hat{M}[1/\pi]$$

inside $\mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi])$.

Proof. In order to show that (ii) implies (i), let

$$f\colon M_L \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi] \to \hat{M}[1/\pi]$$

be an isomorphism of $A_{o_L,(\varepsilon,\pi)}[1/\pi]$-modules as displayed in the assertion of the Theorem. We have canonical F-equivariant $A_{o_L,\pi}$-linear maps

$$i\colon M_L \to M_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi], \qquad j\colon \hat{M} \to \hat{M}[1/\pi]$$

where i (resp., j) is injective since M_L (resp., $\hat{M}$) is flat. Let

$$\mathcal{M} = \mathrm{im}(i) \cap f^{-1}(\mathrm{im}(j)).$$

We claim that $\mathcal{M}$ gives rise to a good model of M_L. First we remark that, by virtue of the linearity of f, the $A_{o_L,\pi}$-module structure of $M_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi]$ restricts to an $A_{o_L,\pi}$-module structure of $\mathcal{M}$. Furthermore, by the F-equivariance of f and i, the semi-linear map $F^{\mathrm{sl}}_{M_L} \otimes \sigma$ restricts to a map of abelian groups $F^{\mathrm{sl}}_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{M}$ which of course is semi-linear with respect to $\sigma\colon A_{o_L,\pi} \to A_{o_L,\pi}$ and makes the $A_{o_L,\pi}$-linear inclusion $\mathcal{M} \hookrightarrow M_L$ F-equivariant (this embedding already shows that

$\mathcal{M}$ is a torsion-free $A_{o_L,\pi}$-module). The latter map gives rise to an $A_{o_L,\pi}[1/\pi]$-linear embedding $\mathcal{M}[1/\pi] \hookrightarrow M_L[1/\pi] \simeq M_L$, of which we claim that it is, in fact, an isomorphism. Indeed, let $m \in M_L$; there is an $s \geq 0$ such that $\pi^s f(m \otimes 1) \in \mathrm{im}(j)$, i.e., $\pi^s m \otimes 1 \in \mathcal{M}$, and it becomes clear that $(\pi^s m \otimes 1)/\pi^s$ is mapped to m.

– In the commutative diagram

$$\begin{array}{ccccc} \sigma^* M_L & \xrightarrow{\sigma^* i} & \sigma^*(M_L \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi]) & \xrightarrow{\sigma^* f} & \sigma^* \hat{M}[1/\pi] \\ \downarrow & & \downarrow & & \downarrow \\ M_L & \xrightarrow[i]{} & M_L \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi] & \xrightarrow[f]{} & \hat{M}[1/\pi] \end{array}$$

where σ stands for the Frobenius lift of $A_{o_L,\pi}$, we claim that $\sigma^*\mathcal{M} = \sigma^*\mathrm{im}(i) \cap (\sigma^* f)^{-1}(\sigma^*\mathrm{im}(j))$. In order to see this, we consider the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{im}(i) & \longrightarrow & M_L \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi] & \xrightarrow{\mathrm{pr}_1} & \mathrm{coker}(i) & \longrightarrow & 0 \\ & & & & \downarrow f & & & & \\ 0 & \longrightarrow & \mathrm{im}(j) & \longrightarrow & \hat{M}[1/\pi] & \xrightarrow{\mathrm{pr}_2} & \mathrm{coker}(j) & \longrightarrow & 0 \end{array}$$

and remark that $\mathcal{M}$ is characterized by the short exact sequence

$$0 \to \mathcal{M} \to M_L \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi] \xrightarrow{\binom{\mathrm{pr}_1}{\mathrm{pr}_2 \circ f}} \mathrm{coker}(i) \oplus \mathrm{coker}(j).$$

Now the functor $\cdot \otimes_{(A_{o_L,\pi}),\sigma} A_{o_L,\pi}$ respects kernels and finite direct sums, so that we obtain a short exact sequence

$$0 \to \sigma^*\mathcal{M} \to \sigma^*(M_L \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi]) \to \sigma^*\mathrm{coker}(i) \oplus \sigma^*\mathrm{coker}(j)$$

where the rightmost arrow is given by $(\sigma^*\mathrm{pr}_1, \sigma^*\mathrm{pr}_2 \circ \sigma^* f)$. So finally, by applying $\sigma^*(\cdot)$ to the above diagram, we get the desired equality.

– Applying the exact functor $\cdot \otimes_{(A_{o_L,\pi}),\sigma} A_{o_L,\pi}$ to the embedding $\mathcal{M} \hookrightarrow M_L$ gives a commutative diagram

$$\begin{array}{ccc} \mathcal{M} \otimes_{(A_{o_L,\pi}),\sigma} A_{o_L,\pi} & \longrightarrow & M_L \otimes_{(A_{o_L,\pi}),\sigma} A_{o_L,\pi} \\ {\scriptstyle F_{\mathcal{M}}} \downarrow & & \downarrow {\scriptstyle F_{M_L}} \\ \mathcal{M} & \longrightarrow & M_L \end{array}$$

where the left-hand vertical map $F_{\mathcal{M}} \colon \sigma^*\mathcal{M} \to \mathcal{M}$ has to be injective because the other three appearing maps are; here it just remains to remark that $M_L \otimes_{(A_{o_L,\pi}),\sigma} A_{o_L,\pi} \simeq M_L \otimes_{A_{o_L,\pi}[1/\pi],\sigma} A_{o_L,\pi}[1/\pi]$.

– Next we claim that $\mathfrak{J}^d\mathrm{coker}(F_{\mathcal{M}}) = 0$, where $\mathfrak{J} = (a\otimes 1 - 1\otimes c^*(a), a\in A) \subseteq A_{o_L}$, and where we are provided that both $\mathrm{coker}(F_{M_L})$ and $\mathrm{coker}(F_{\hat{M}})$ are annihilated by $\mathfrak{J}^d$. Let $x = \sum_\nu \alpha_\nu m_\nu \otimes 1 \in \mathfrak{J}^d\mathcal{M}$ where $\alpha_\nu \in \mathfrak{J}^d$. By assumption there is a (unique) $y \in \sigma^* M_L$ such that $\sum_\nu \alpha_\nu m_\nu = F_{M_L}(y)$; we have to show that, regarding y as an element of $\sigma^*\mathrm{im}(i) \simeq \mathrm{im}(\sigma^* i)$, we have $y \in \sigma^*\mathcal{M} = \sigma^*\mathrm{im}(i) \cap (\sigma^* f)^{-1}(\sigma^*\mathrm{im}(j))$. So it remains to see that $(\sigma^* f)(y) \in \mathrm{im}(\sigma^* j)$. Indeed, inside $\hat{M}[1/\pi]$ we have $f(x) = f(F(y)) = F((\sigma^* f)(y))$; on the other hand, the linearity of f and j gives that $f(x) = \sum_\nu \alpha_\nu f(m_\nu \otimes 1) = j(y')$ for some $y' \in \mathfrak{J}^d\hat{M} \subseteq \mathrm{im}(F_{\hat{M}})$, say $y' = F_{\hat{M}}(y'')$ for a $y'' \in \sigma^*\hat{M}$, i.e., $f(x) = F((\sigma^* j)(y''))$; so finally, since $F\colon \sigma^*\hat{M}[1/\pi] \to \hat{M}[1/\pi]$ is injective, we obtain that $(\sigma^* f)(y) = (\sigma^* j)(y'')$, as desired.

– The given $A_{o_L,(\varepsilon,\pi)}[1/\pi]$-linear isomorphism $f : M_L \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi] \to \hat{M}[1/\pi]$ gives rise to an $o_L[\![z]\!][1/\pi]$-linear isomorphism

$$\widetilde{f}\colon M_L \otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi] \to \hat{M} \otimes_{o_L[\![z]\!]} o_L[\![z]\!][1/\pi],$$

for we have

$$\begin{aligned} M_L \otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi] &\simeq M_L \otimes_{A_{o_L,\pi}[1/\pi]} (A_{o_L,\pi} \otimes_{o_L\langle z\rangle} o_L[\![z]\!])[1/\pi] \\ &\simeq M_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi], \\ \hat{M} \otimes_{o_L[\![z]\!]} o_L[\![z]\!][1/\pi] &\simeq \hat{M} \otimes_{A_{o_L,(\varepsilon,\pi)}} (A_{o_L,(\varepsilon,\pi)} \otimes_{o_L[\![z]\!]} o_L[\![z]\!][1/\pi]) \\ &\simeq \hat{M} \otimes_{A_{o_L,(\varepsilon,\pi)}} A_{o_L,(\varepsilon,\pi)}[1/\pi]. \end{aligned}$$

From the corresponding property of the isomorphism f and the F-equivariance of $o_L[z] \to A_{o_L}$ we derive that also $\widetilde{f}$ is F-equivariant. Furthermore, analogous to what we have seen before, we have natural maps

$$\widetilde{i}\colon M_L \to M_L \otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi], \qquad \widetilde{j}\colon \hat{M} \to \hat{M} \otimes_{o_L[\![z]\!]} o_L[\![z]\!][1/\pi]$$

where $\widetilde{i}$ (resp., $\widetilde{j}$) is $L\langle z\rangle$-linear (resp., $o_L[\![z]\!]$-linear) and injective. Let

$$\widetilde{\mathcal{M}} = \mathrm{im}(\widetilde{i}) \cap \widetilde{f}^{-1}(\mathrm{im}(\widetilde{j})).$$

Then the isomorphism displayed above restricts to an isomorphism

$$\widetilde{\mathcal{M}} \xrightarrow{\simeq} \mathcal{M}$$

which is $o_L\langle z\rangle$-linear; here $\mathcal{M}$ becomes an $o_L\langle z\rangle$-module via the embedding $o_L\langle z\rangle \to A_{o_L,\pi}$; in particular, we obtain an $\ell[z]$-linear isomorphism $\widetilde{\mathcal{M}}/\pi\widetilde{\mathcal{M}} \simeq \mathcal{M}/\pi\mathcal{M}$.

– In the following step we are going to show that $\widetilde{\mathcal{M}}$ is finitely presented over $o_L\langle z\rangle$, which implies that $\widetilde{\mathcal{M}}/\pi\widetilde{\mathcal{M}}$ will be finitely presented over $\ell[z]$; since we have an $o_L\langle z\rangle$-linear isomorphism $\mathcal{M} \simeq \widetilde{\mathcal{M}}$, it also follows that $\mathcal{M}$ is finitely presented over $o_L\langle z\rangle$; in particular $\mathcal{M}$ will be finitely presented over $A_{o_L,\pi}$, and finally we may conclude that the reduction $\mathcal{M}/\pi\mathcal{M}$ will be finitely presented over $\ell[z]$ and over A_ℓ.

Let $(e_1, ..., e_m)$ be an $L\langle z\rangle$-basis of M_L; see 1.12; furthermore, let $(d_1, ..., d_n)$ be a basis for $\hat{M}$ over $o_L[\![z]\!]$. Note that our choice of basis for M_L gives rise to an isomorphism $M_L \otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi] \simeq o_L[\![z]\!][1/\pi]^m$; for every $\nu = 1, ..., n$ we consider $\widetilde{f}^{-1}(d_\nu)$ and regard it as an element of the right-hand side of this isomorphism. We choose $N \geq 0$ big enough, such that $\widetilde{f}^{-1}(\pi^N d_\nu) \in o_L[\![z]\!]^m$ for all ν, say

$$\widetilde{f}^{-1}(\pi^N d_\nu) = (\rho_{\nu,1}, ..., \rho_{\nu,m})$$

where $\rho_{\nu,\mu} \in o_L[\![z]\!]$. Now let $x \in \widetilde{\mathcal{M}}$. Via $\widetilde{f}$ we obtain, say, $\widetilde{f}(x) = \sum_\nu \lambda_\nu d_\nu$ in $\hat{M}$, with suitable $\lambda_\nu \in o_L[\![z]\!]$; consequently $\widetilde{f}(\pi^N x) = \sum_\nu \lambda_\nu(\pi^N d_\nu)$, so that the image of $\pi^N x$ in $o_L[\![z]\!]^m$ corresponds to the family of scalars

$$(\sum_\nu \lambda_\nu \rho_{\nu,1}, ..., \sum_\nu \lambda_\nu \rho_{\nu,m}) \in o_L[\![z]\!]^m.$$

Now, since the embedding $\widetilde{i}$ is $L\langle z\rangle$-linear and since the images of the e_μ constitute an $o_L[\![z]\!][1/\pi]$-basis of $M_L \otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi]$, writing $\pi^N x \in M_L$ as a linear combination over $L\langle z\rangle$ has to yield $\pi^N x = \sum_\mu(\sum_\nu \lambda_\nu \rho_{\nu,\mu})e_\mu$, i.e., the appearing scalars $\alpha_\mu = \sum_\nu \lambda_\nu \rho_{\nu,\mu}$ have, in fact, to be elements of $o_L\langle z\rangle = L\langle z\rangle \cap o_L[\![z]\!]$. Inside M_L we may write $x = \pi^{-N}\pi^N x = \sum_\mu \alpha_\mu \pi^{-N} e_\mu$, so that we may conclude

$$\widetilde{\mathcal{M}} \subseteq \sum_\mu o_L\langle z\rangle \pi^{-N} e_\mu.$$

Finally, being a submodule of a finitely generated module over a noetherian ring, $\widetilde{\mathcal{M}}$ has to be of finite presentation.

– We claim that $\widetilde{\mathcal{M}}/\pi\widetilde{\mathcal{M}}$ is torsion-free over $\ell[z]$; this will imply that $\widetilde{\mathcal{M}}/\pi\widetilde{\mathcal{M}}$ is finite free over $\ell[z]$, since it is already of finite presentation. Furthermore, it follows that $\mathcal{M}/\pi\mathcal{M}$ is torsion-free and hence free over $\ell[z]$.

Let $\widetilde{x} \in \widetilde{\mathcal{M}}$, and let $\lambda = \sum_s \lambda_s z^s \in o_L\langle z\rangle$ be such that $\pi \nmid \lambda$ and $\lambda\widetilde{x} \in \pi\widetilde{\mathcal{M}}$, say $\lambda\widetilde{x} = \pi\widetilde{y}$ for some $\widetilde{y} \in \widetilde{\mathcal{M}}$. In order to prove that $\widetilde{\mathcal{M}}/\pi\widetilde{\mathcal{M}}$ is torsion-free we must show that $\widetilde{x} \in \pi\widetilde{\mathcal{M}}$. First suppose that $\lambda \in o_L\langle z\rangle \cap o_L[\![z]\!]^\times$. We consider $\pi^{-1}\widetilde{x} \in M_L$; in fact, this element lies in $\widetilde{\mathcal{M}}$, since we have $\widetilde{f}(\pi^{-1}\widetilde{x}) = \pi^{-1}\lambda^{-1}\widetilde{f}(\lambda\widetilde{x}) = \lambda^{-1}\widetilde{f}(\widetilde{y}) \in \hat{M}$; consequently $\widetilde{x} = \pi(\pi^{-1}\widetilde{x}) \in \pi\widetilde{\mathcal{M}}$. Now suppose that $\pi \mid \lambda_0$; this means we find $\lambda' \in o_L[z]$ and $\lambda'' \in o_L\langle z\rangle \cap o_L[\![z]\!]^\times$ such that $\lambda = \pi\lambda' + z^N\lambda''$ for some $N \geq 1$; we have $\pi\widetilde{y} = \lambda\widetilde{x} = \pi\lambda'\widetilde{x} + z^N\lambda''\widetilde{x}$; suppose we have already shown that $z^n\widetilde{x} \in \pi\widetilde{\mathcal{M}}$ implies $\widetilde{x} \in \pi\widetilde{\mathcal{M}}$ for any $n \geq 0$; we claim that $z^N\widetilde{x} \in \pi\widetilde{\mathcal{M}}$ which, by assumption, will imply that $\widetilde{x} \in \pi\widetilde{\mathcal{M}}$; indeed, we have

$$\widetilde{f}(\pi^{-1}z^N\widetilde{x}) = \lambda''^{-1}\pi^{-1}\widetilde{f}(\lambda''z^N\widetilde{x}) = \lambda''^{-1}\pi^{-1}\widetilde{f}(\pi\widetilde{y} - \pi\lambda'\widetilde{x}) = \lambda''^{-1}\widetilde{f}(\widetilde{y} - \lambda'\widetilde{x}) \in \hat{M},$$

which shows that $\pi^{-1}z^N\widetilde{x} \in \widetilde{\mathcal{M}}$, i.e., $z^N\widetilde{x} = \pi(\pi^{-1}z^N\widetilde{x}) \in \pi\widetilde{\mathcal{M}}$. So it remains to show that $z^n\widetilde{x} \in \pi\widetilde{\mathcal{M}}$ implies $\widetilde{x} \in \pi\widetilde{\mathcal{M}}$ for any $n \geq 0$. By induction, it suffices to consider the case $n = 1$. So suppose $z\widetilde{x} \in \pi\widetilde{\mathcal{M}}$, say $z\widetilde{x} = \pi\widetilde{y}$; let $\widetilde{f}(\widetilde{x}) = \sum_\nu \beta_\nu d_\nu$, where $(d_1, ..., d_n)$ is the finite $o_L[\![z]\!]$-basis of $\hat{M}$ fixed before. The relation $z\widetilde{x} = \pi\widetilde{y}$ implies that $\pi \mid z\beta_\nu$ for every index ν, so that $\pi \mid \beta_\nu$ for every ν. Arguing similarly as before, one now immediately shows that $\pi^{-1}\widetilde{x} \in M_L$ necessarily maps via $\widetilde{f}$ to an element of $\hat{M}$, i.e., $\widetilde{x} \in \pi\widetilde{\mathcal{M}}$.

- As an auxiliary step in order to show that $\mathcal{M}$ is locally free of finite rank over $A_{o_L,\pi}$, we claim that the reduction $\mathcal{M}/\pi\mathcal{M}$ is locally free of finite rank over A_ℓ. Indeed, now that we know that $\mathcal{M}/\pi\mathcal{M}$ is of finite presentation over A_ℓ, it suffices to prove flatness over A_ℓ. Since A_ℓ is a Dedekind domain, by [13], VII.10.22, we only need to show that $\mathcal{M}/\pi\mathcal{M}$ is torsion-free over A_ℓ (hence projective, hence flat).
  Since $\mathcal{M}/\pi\mathcal{M}$ is free over $\ell[z]$, it is flat and we get an embedding $\mathcal{M}/\pi\mathcal{M} \to \mathcal{M}/\pi\mathcal{M}\otimes_{\ell[z]}\ell(z)$; there is a canonical isomorphism $\mathcal{M}/\pi\mathcal{M}\otimes_{\ell[z]}\ell(z) \simeq \mathcal{M}/\pi\mathcal{M}\otimes_{A_\ell} (A_\ell \otimes_{\ell[z]} \ell(z))$, and we claim that $A_\ell \otimes_{\ell[z]} \ell(z) \simeq \mathrm{Frac}(A_\ell)$; indeed, $S = \ell[z] - \{0\}$ gives rise to a multiplicative subset of A_ℓ not containing zero, and $A_\ell \otimes_{\ell[z]} \ell(z) \simeq S^{-1}A_\ell$; furthermore, the embedding $\ell(z) \to A_\ell\otimes_{\ell[z]}\ell(z)$ is finite, and $\ell(z)$ is a field, so $A_\ell \otimes_{\ell[z]} \ell(z)$ also is a field; consequently the canonical map $A_\ell \to S^{-1}A_\ell$ factors via $\mathrm{Frac}(A_\ell)$, and it is directly seen that the induced map $\mathrm{Frac}(A_\ell) \to S^{-1}A_\ell$ is, in fact, an isomorphism.
  Let $\alpha \in A_\ell - \{0\}$ and $x \in \mathcal{M}/\pi\mathcal{M}$ be such that $\alpha x = 0$; by regarding αx as an element of $\mathcal{M}/\pi\mathcal{M} \otimes_{A_\ell} \mathrm{Frac}(A_\ell)$, we get $x = \alpha^{-1}\alpha x = 0$, as desired.

- Relying on the preceding step, we claim that $\mathcal{M}$ is locally free of finite rank over $A_{o_L,\pi}$ where again it only remains to show that $\mathcal{M}$ is flat over $A_{o_L,\pi}$.
  First we remark that, since $A_{o_L,\pi}$ is π-adically complete and separated, we have $\pi A_{o_L,\pi} \subseteq \mathfrak{j}(A_{o_L,\pi})$, and the $A_{o_L,\pi}$-module $\mathcal{M}$ is finitely generated, so that $\mathcal{M}$ is π-adically *ideally Hausdorff* in the sense of [13], III.5.1. In the preceding step we have shown that $\mathcal{M}/\pi\mathcal{M}$ is flat over $A_\ell \simeq A_{o_L,\pi}/\pi A_{o_L,\pi}$, and we know that $\mathcal{M}$ has no π-torsion, so that the canonical map $\pi A_{o_L,\pi} \otimes_{A_{o_L,\pi}} \mathcal{M} \to \pi\mathcal{M}$ is an isomorphism; therefore, by Bourbaki's Flatness Criterion [13], III.5.2.1(iii), we may conclude that $\mathcal{M}$ is indeed flat over $A_{o_L,\pi}$.

- Our next aim is to show that the kernel V of $\mathcal{M}/\pi\mathcal{M}\otimes_{A_\ell,\bar{\sigma}}A_\ell \to \mathcal{M}/\pi\mathcal{M}$ is trivial, i.e., that $\mathcal{M}$ is good in the sense of 1.14. Since we have a canonical isomorphism

$$\mathcal{M}/\pi\mathcal{M} \otimes_{\ell[z],\bar{\sigma}} \ell[z] \simeq \mathcal{M}/\pi\mathcal{M} \otimes_{A_\ell,\bar{\sigma}} A_\ell$$

and since the abelian subgroup V of the right-hand side corresponds to the kernel V' of $\mathcal{M}/\pi\mathcal{M} \otimes_{\ell[z],\bar{\sigma}} \ell[z] \to \mathcal{M}/\pi\mathcal{M}$, is suffices to show that V' is trivial.
We have already shown that $\mathfrak{J}^d\mathcal{M} \subseteq \mathrm{im}(F_\mathcal{M})$; in particular, we have the following chain of $o_L\langle z\rangle$-modules

$$(z-\zeta)^d\mathcal{M} \subseteq \mathrm{im}(F_\mathcal{M}) \subseteq \mathcal{M};$$

the element $\zeta \in o_L$ is zero mod π, and we obtain

$$z^d(\mathcal{M}/\pi\mathcal{M}) \subseteq \mathrm{im}(\mathcal{M}/\pi\mathcal{M} \otimes_{\ell[z],\bar{\sigma}} \ell[z] \to \mathcal{M}/\pi\mathcal{M}) \subseteq \mathcal{M}/\pi\mathcal{M};$$

we know that $\mathcal{M}/\pi\mathcal{M}$ is finite free over $\ell[z]$; so 1.19 below shows that the middle term W' in the latter chain has full rank inside $\mathcal{M}/\pi\mathcal{M}$. Finally, taking ranks in the (split) short exact sequence of finite free $\ell[z]$-modules

$$0 \to V' \to \mathcal{M}/\pi\mathcal{M} \otimes_{\ell[z],\bar{\sigma}} \ell[z] \to W' \to 0$$

accomplishes the proof that V' indeed is trivial.

– As we will now prove, the module $\mathcal{M}$ is finite free over $o_L\langle z\rangle$. To see this, let $(m_1, ..., m_s)$ be a lifting in $\mathcal{M}$ of a basis of $\mathcal{M}/\pi\mathcal{M}$. Let $\varphi\colon o_L\langle z\rangle^s \to \mathcal{M}$ be the $o_L\langle z\rangle$-linear map which sends the k-th vector of the canonical basis of $o_L\langle z\rangle^s$ to m_k. We claim that φ is an isomorphism. Indeed, by the choice of the m_k, the quotient $\mathrm{coker}(\varphi)/\pi\mathrm{coker}(\varphi)$ is trivial, and Lemma 1.8 shows that $\pi \in \mathfrak{j}(o_L\langle z\rangle)$; now Nakayama's Lemma ([58], 2.2) shows that the finitely generated $o_L\langle z\rangle$-module $\mathrm{coker}(\varphi)$ is trivial; finally, applying the Snake Lemma to the commutative diagram with short exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\varphi) & \longrightarrow & o_L\langle z\rangle^s & \longrightarrow & \mathcal{M} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & \ker(\varphi) & \longrightarrow & o_L\langle z\rangle^s & \longrightarrow & \mathcal{M} & \longrightarrow & 0 \end{array}$$

shows that $\ker(\varphi)/\pi\ker(\varphi) = 0$, so that, again by Nakayama's Lemma, also $\ker(\varphi)$ is trivial.

– It remains to prove that the cokernel C of $\mathcal{M} \otimes_{(A_{o_L,\pi}),\sigma} A_{o_L,\pi} \to \mathcal{M}$ is a finite free o_L-module.
In a first step we show that C is finitely presented over o_L. Since $\pi \in \mathfrak{j}(A_{o_L,\pi})$ and since C, being a quotient of $\mathcal{M}$, is finitely presented over the noetherian ring

$A_{o_L,\pi}$, we conclude by Krull's Theorem ([58], 8.10) that C is π-adically separated. By [58], 8.4, it now suffices to show that $C/\pi C$ is a finite-dimensional ℓ-vector space (because any lift of an ℓ-basis of $C/\pi C$ will then be a system of generators for C over o_L); indeed, from what we have seen so far, $C/\pi C$ is finitely presented over $A_{o_L,\pi}$ and hence over A_ℓ. As $\ell[z] \to A_\ell$ is finite, it follows that $C/\pi C$ is of finite presentation over $\ell[z]$. Moreover, from $(z-\zeta)^d C = 0$ it follows that $z^d(C/\pi C) = 0$, i.e., $C/\pi C$ is finitely presented over $\ell[z]/z^d$; but the latter is a finite-dimensional ℓ-vector space, and so we may conclude that $C/\pi C$ is indeed finite-dimensional over the residue field ℓ.

In a second step we show that C is a flat o_L-module, which will imply that C is finite free over the local ring o_L. Since we have just seen that $C/\pi C$ is free and hence flat over ℓ, we only need to prove that C has trivial π-torsion; then Bourbaki's Flatness Criterion [13], III.5.2.1(iii), will yield the desired result.

By imitating the argument given in the proof of 1.16 one shows that the canonical map $A \otimes_{\mathbb{F}[z]} o_L\langle z\rangle \to A_{o_L,\pi}$ is an isomorphism of $o_L\langle z\rangle$-algebras, and the canonical isomorphism $(\mathrm{id},\sigma)\colon o_L\langle z\rangle \otimes_{o_L\langle z\rangle,\sigma} o_L\langle z\rangle \to o_L\langle z\rangle$, $f\otimes g \mapsto \sigma(f)g$, gives rise to the composition

$$\begin{aligned} A_{o_L,\pi} \otimes_{o_L\langle z\rangle,\sigma} o_L\langle z\rangle &\simeq (A \otimes_{\mathbb{F}[z]} o_L\langle z\rangle) \otimes_{o_L\langle z\rangle,\sigma} o_L\langle z\rangle \\ &\simeq A \otimes_{\mathbb{F}[z]} (o_L\langle z\rangle \otimes_{o_L\langle z\rangle,\sigma} o_L\langle z\rangle) \\ &\simeq A \otimes_{\mathbb{F}[z]} o_L\langle z\rangle \\ &\simeq A_{o_L,\pi} \end{aligned}$$

which induces an isomorphism of $A_{o_L,\pi}$-modules $A_{o_L,\pi} \otimes_{o_L\langle z\rangle,\sigma} o_L\langle z\rangle \simeq \sigma_* A_{o_L,\pi}$, showing that $\mathcal{M} \otimes_{o_L\langle z\rangle,\sigma} o_L\langle z\rangle \simeq \mathcal{M} \otimes_{(A_{o_L,\pi}),\sigma} A_{o_L,\pi}$. Therefore it suffices to consider the cokernel C' of the map $\mathcal{M} \otimes_{o_L\langle z\rangle,\sigma} o_L\langle z\rangle \to \mathcal{M}$ and to show that C' has no π-torsion.

So let $\pi x \in \mathrm{im}(\mathcal{M} \otimes_{o_L\langle z\rangle,\sigma} o_L\langle z\rangle \to \mathcal{M})$, say there is an element $y \in \mathcal{M} \otimes_{o_L\langle z\rangle,\sigma} o_L\langle z\rangle$ which is mapped to πx; note that y is uniquely determined by πx. There is a canonical epimorphism $\ell[z] \otimes_{o_L\langle z\rangle,\sigma} o_L\langle z\rangle \to \ell[z]$ giving rise to a commutative diagram

$$\begin{CD} \mathcal{M} \otimes_{o_L\langle z\rangle,\sigma} o_L\langle z\rangle @>>> \mathcal{M} \\ @VVV @VVV \\ \mathcal{M}/\pi\mathcal{M} \otimes_{\ell[z],\bar\sigma} \ell[z] @>>> \mathcal{M}/\pi\mathcal{M} \end{CD}$$

where the horizontal maps are injective and the vertical maps are surjective, and where the left-hand projection is obtained via the composition of natural maps

$$\mathcal{M} \otimes_{o_L\langle z\rangle,\sigma} o_L\langle z\rangle \to \mathcal{M}/\pi\mathcal{M} \otimes_{o_L\langle z\rangle,\sigma} o_L\langle z\rangle \to \mathcal{M}/\pi\mathcal{M} \otimes_{\ell[z],\bar\sigma} \ell[z].$$

In the upper row of the above diagram both modules are free of the same rank over $o_L\langle z\rangle$, while in the bottom row both modules are free of the same rank over

$\ell[z]$.
Since πx goes to zero under the right-hand projection, it follows that y goes to zero under the left-hand projection. Let $(m_1, ..., m_s)$ be a lift in $\mathcal{M}$ of an $\ell[z]$-basis $(\bar{m}_1, ..., \bar{m}_s)$ of $\mathcal{M}/\pi\mathcal{M}$; as we have seen before, every such lift is an $o_L\langle z\rangle$-basis of $\mathcal{M}$; writing y in terms of the basis $(m_1\otimes 1, ..., m_s\otimes 1)$ yields $y \in \pi(\mathcal{M}\otimes_{o_L\langle z\rangle,\sigma} o_L\langle z\rangle)$, and since $\mathcal{M}$ is a torsion-free $o_L\langle z\rangle$-module, we are done. Hence we have shown that $\mathcal{M}$ gives rise to a good model for M_L.

Conversely, in order to show that (i) implies (ii), suppose that $\mathcal{M}$ is a good model of M_L. We define

$$\widehat{\mathcal{M}} = \mathcal{M} \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)},$$

i.e., $\widehat{\mathcal{M}}$ equals the completion of $\mathcal{M}$ for the $(\varepsilon, \pi)A_{o_L,\pi}$-adic topology. It is clear that every fixed F-equivariant isomorphism of $A_{o_L,\pi}[1/\pi]$-modules $\mathcal{M}[1/\pi] \simeq M_L$ gives rise to a natural F-equivariant $A_{o_L,(\varepsilon,\pi)}[1/\pi]$-linear isomorphism

$$M_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \simeq \widehat{\mathcal{M}}[1/\pi].$$

We claim that $\widehat{\mathcal{M}}$ is a local shtuka. Indeed, by standard base change arguments, $\widehat{\mathcal{M}}$ is again locally free of finite rank; furthermore, since the completion map $A_{o_L,\pi} \to A_{o_L,(\varepsilon,\pi)}$ is Frobenius-equivariant and flat, we indeed obtain an injective map

$$\widehat{\mathcal{M}} \otimes_{(A_{o_L,(\varepsilon,\pi)}),\sigma} A_{o_L,(\varepsilon,\pi)} \to \widehat{\mathcal{M}}.$$

Let C' be its cokernel, and let $C = \mathrm{coker}(F_{\mathcal{M}})$, i.e.,

$$C' \simeq C \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}.$$

First we claim that C' is annihilated by $\mathfrak{J}^d$, i.e., that $\mathfrak{J}^d\widehat{\mathcal{M}}$ lies in the image of the latter map, which is $F_{\mathcal{M}} \otimes \mathrm{id}$. So let $x = \sum_\nu \lambda_\nu x_\nu \in \mathfrak{J}^d\widehat{\mathcal{M}}$, where $\lambda_\nu \in \mathfrak{J}^d$ and $x_\nu = \sum_\mu y_{\mu\nu} \otimes a_{\mu\nu} \in \widehat{\mathcal{M}} = \mathcal{M} \otimes A_{o_L,(\varepsilon,\pi)}$; this gives

$$x = \sum_{\mu,\nu} \lambda_\nu y_{\mu\nu} \otimes a_{\mu\nu} = \sum_{\mu,\nu} F_{\mathcal{M}}(y'_{\mu\nu}) \otimes a_{\mu\nu} = (F_{\mathcal{M}} \otimes \mathrm{id})(\sum_{\mu,\nu} y'_{\mu\nu} \otimes a_{\mu\nu}).$$

In particular, note that C' is annihilated by $(z-\zeta)^d \subseteq o_L[z]$. It remains to show that C' is a finite free o_L-module. However, this is clear, for we have

$$\begin{aligned}
C &= C/(z-\zeta)^d C \\
&\simeq C \otimes_{A_{o_L,\pi}} A_{o_L,\pi}/(z-\zeta)^d \\
&\simeq C \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}/(z-\zeta)^d \\
&\simeq (C \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}) \otimes_{A_{o_L,(\varepsilon,\pi)}} A_{o_L,(\varepsilon,\pi)}/(z-\zeta)^d \\
&\simeq C' \otimes_{A_{o_L,(\varepsilon,\pi)}} A_{o_L,(\varepsilon,\pi)}/(z-\zeta)^d \\
&\simeq C'/(z-\zeta)^d C' \\
&= C'
\end{aligned}$$

by virtue of Lemma 1.20 below. In particular, this argument shows that the cokernel C is not affected by (ε, π)-adic completion, i.e., it is (ε, π)-adically complete. $\square$

Lemma 1.19. *Let R be a principal ideal domain, and let M be a finite free R-module; furthermore, let $a \in R - \{0\}$, and let $U \subseteq M$ be a submodule such that $aM \subseteq U$. Then the rank of U equals the rank of M.*

Proof. Let d be the rank of M. It suffices to consider the case $U = aM$: regarding the chain $aM \subseteq U \subseteq M$, U is finite free inside M while aM is finite free inside U, i.e., $\mathrm{rk}(aM) \leq \mathrm{rk}(U) \leq d$. But given a basis $(m_1, ..., m_d)$ of M, the system $(am_1, ..., am_d)$ is free and generates aM. $\square$

Lemma 1.20. *Let $e \geq 1$. There are natural isomorphisms*

$$A_{o_L}/(z-\zeta)^e \simeq A_{o_L,\pi}/(z-\zeta)^e \simeq A_{o_L,(\varepsilon,\pi)}/(z-\zeta)^e.$$

Proof. First of all, we claim that the canonical map $\alpha\colon o_L[z]/(z-\zeta)^e \to o_L\langle z\rangle/(z-\zeta)^e$ is an isomorphism; in order to see this, we consider some $f \in o_L\langle z\rangle$. By the Weierstraß Division Theorem for the Tate algebra $L\langle z\rangle$ ([9], 1.2/8) there is a unique $a \in L\langle z\rangle$ as well as a unique $b \in L[z]$ of degree $< e$ such that $f = a(z-\zeta)^e + b$; moreover we have $1 \geq ||f|| = \max(||a||, ||b||)$, where $||\cdot||$ denotes the Gauss norm of $L\langle z\rangle$, i.e., a and b have their coefficients in o_L. In particular, this shows that α is surjective. Now let $g \in o_L[z]$ be such that there is some $a \in o_L\langle z\rangle$ satisfying $g = a(z-\zeta)^e$ in $o_L\langle z\rangle$; note that necessarily a is uniquely determined by g. In order to show that α is injective, we have to prove that a lies in $o_L[z]$. Indeed, by the Division Theorem for $L[z]$ there is a uniquely determined $a' \in L[z]$ such that $g = a'(z-\zeta)^e$. Therefore, by uniqueness, a has to lie in $o_L[z]$. This accomplishes the proof that α is an isomorphism. Taking the Weierstraß Division Theorem for $o_L[\![z]\!]$ ([13], VII.3.8.5) into account, an analogous argument shows that also the canonical map $\beta\colon o_L\langle z\rangle/(z-\zeta)^e \to o_L[\![z]\!]/(z-\zeta)^e$ is an isomorphism, and at the same time one realizes that the three o_L-algebras involved in the composition

$$o_L[z]/(z-\zeta)^e \xrightarrow{\alpha} o_L\langle z\rangle/(z-\zeta)^e \xrightarrow{\beta} o_L[\![z]\!]/(z-\zeta)^e$$

are free over o_L of the same rank e. Finally, applying the functor $\cdot \otimes_{o_L[z]} A_{o_L}$ to this composition completes the proof. $\square$

Corollary 1.21. *Let M_L be an analytic Anderson $A(1)$-motive. Then there is a (1:1)-correspondence*

$$\{\textit{good models of } M_L\}/\!\sim \;\overset{(1:1)}{\longleftrightarrow} \left\{ \begin{array}{l} \textit{pairs } (\hat{M}, f) \textit{ consisting of} \\ \bullet \textit{ a local shtuka } \hat{M} \textit{ at } \varepsilon, \\ \bullet \textit{ an isomorphism in } \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi]) \\ \quad f\colon M_L \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi] \simeq \hat{M}[1/\pi] \end{array} \right\}/\!\sim$$

where $(\cdot)/\sim$ indicates taking isomorphism classes, and where on the right-hand side an isomorphism of pairs $(\hat{M}, f) \xrightarrow{\simeq} (\hat{N}, g)$ is defined to be an isomorphism of local shtukas $\hat{M} \to \hat{N}$ which in the obvious manner is compatible with f and g. In particular, if M_L admits a good model $\mathcal{M}$, one obtains equalities

$$\mathrm{rk}_{L\langle z\rangle}(M_L) = \mathrm{rk}_{o_L\langle z\rangle}(\mathcal{M}) = \mathrm{rk}_{o_L[\![z]\!]}(\hat{M})$$

where $\hat{M}$ is a corresponding local shtuka at ε.

Proof. Suppose that $\mathcal{M}$ is a good model of M_L. In the proof of 1.18 we have seen that its completion $\widehat{\mathcal{M}} = \mathcal{M} \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}$ is a local shtuka at ε; let $\mathcal{M}[1/\pi] \simeq M_L$ be an F-equivariant isomorphism of $A_{o_L,\pi}[1/\pi]$-modules; it induces a natural isomorphism

$$f\colon (\mathcal{M} \otimes_{A_{o_L,\pi}} A_{o_L,\pi}[1/\pi]) \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \xrightarrow{\simeq} \widehat{\mathcal{M}} \otimes_{A_{o_L,(\varepsilon,\pi)}} A_{o_L,(\varepsilon,\pi)}[1/\pi]$$

which is F-equivariant, and which is immediately seen to verify

$$\mathcal{M} = \mathcal{M}[1/\pi] \cap f^{-1}(\widehat{\mathcal{M}}).$$

Indeed, the isomorphism f clearly maps $\mathcal{M} \subseteq \mathcal{M}[1/\pi]$ to $\widehat{\mathcal{M}}$. Conversely, consider an element $m/\pi^s \in \mathcal{M}[1/\pi]$; the element $(m \otimes 1/\pi^s) \otimes 1$ of the domain of f is mapped to m/π^s, where m is viewed via the embedding $\mathcal{M} \hookrightarrow \widehat{\mathcal{M}}$ as an element of the completion $\widehat{\mathcal{M}}$; note that $\mathcal{M}$ is flat over $A_{o_L,\pi}$, so that it can be identified with its image inside $\widehat{\mathcal{M}}$; as $A_{o_L,(\varepsilon,\pi)}$ has no π-torsion, we see that, by hypothesis, we may indeed write $m = \pi^s m'$ for some $m' \in \mathcal{M}$. Therefore we may conclude that the construction given in 1.18 retrieves $\mathcal{M}$ from the local shtuka $\widehat{\mathcal{M}}$.
It remains to show that, given a local shtuka $\hat{M}$ together with an isomorphism $f\colon M_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \simeq \hat{M}[1/\pi]$, the $(\varepsilon,\pi)A_{o_L,\pi}$-adic completion of the good model $\mathcal{M} = M_L \cap f^{-1}(\hat{M})$ gained in the above construction gives back $\hat{M}$. By construction of $\mathcal{M}$, the map f restricts to an embedding $\mathcal{M} \hookrightarrow \hat{M}$, which in turn induces an F-equivariant and $A_{o_L,(\varepsilon,\pi)}$-linear map

$$\psi\colon \mathcal{M} \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)} \to \hat{M}.$$

Our aim is to show that the map ψ is, in fact, an isomorphism. We have a canonical isomorphism $\mathcal{M} \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)} \simeq \mathcal{M} \otimes_{o_L\langle z\rangle} o_L[\![z]\!]$, and we know that $\mathcal{M}$ is finite free over $o_L\langle z\rangle$. We claim that

$$\mathrm{rk}_{o_L[\![z]\!]}(\mathcal{M} \otimes_{o_L\langle z\rangle} o_L[\![z]\!]) = \mathrm{rk}_{o_L[\![z]\!]}(\hat{M}).$$

Indeed, in the proof of 1.18 we have seen that the given isomorphism f is mirrored by an $o_L[\![z]\!][1/\pi]$-linear isomorphism

$$M_L \otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi] \simeq \hat{M} \otimes_{o_L[\![z]\!]} o_L[\![z]\!][1/\pi],$$

showing that $\mathrm{rk}_{L\langle z\rangle}(M_L) = \mathrm{rk}_{o_L[\![z]\!]}(\hat{M})$; on the other hand, we have $\mathcal{M} \otimes_{o_L\langle z\rangle} L\langle z\rangle \simeq M_L$, i.e., $\mathrm{rk}_{o_L\langle z\rangle}(\mathcal{M}) = \mathrm{rk}_{L\langle z\rangle}(M_L)$. In particular, ψ is a map between finite-free $o_L[\![z]\!]$-modules of the same rank s. We fix an $o_L[\![z]\!]$-basis $\mathfrak{B}$ (resp., $\mathfrak{C}$) of $\mathcal{M} \otimes_{o_L\langle z\rangle} o_L[\![z]\!]$ (resp., of $\hat{M}$) and let $\mathbf{A} = {}_{\mathfrak{C}}[\psi]_{\mathfrak{B}} \in o_L[\![z]\!]^{s\times s}$ be the matrix which describes ψ with respect to $\mathfrak{B}$ and $\mathfrak{C}$; likewise, we let

$$\mathbf{T} = {}_{\mathfrak{B}}[F_{\mathcal{M}\otimes_{o_L\langle z\rangle} o_L[\![z]\!]}]_{\sigma^*\mathfrak{B}}, \qquad \mathbf{T}' = {}_{\mathfrak{C}}[F_{\hat{M}}]_{\sigma^*\mathfrak{C}},$$

so that $\mathbf{AT} = \mathbf{T}'\sigma(\mathbf{A})$ by virtue of the F-equivariance of ψ. In order to see that ψ is an isomorphism, we need to show that $\det(\mathbf{A})$ is a unit in $o_L[\![z]\!]$. To begin with, an elementary application of the Weierstraß Division Theorem for $o_L[\![z]\!]$ ([13], VII.3.8.5) shows that the kernel of the epimorphism $o_L[\![z]\!] \to o_L$, $z \mapsto \zeta$, is generated by $z-\zeta$, so that the latter is a prime element of $o_L[\![z]\!]$; furthermore, recall that $o_L[\![z]\!]$, being a regular local ring, is factorial ([58], 20.3). We know that $\mathcal{M} \otimes_{o_L\langle z\rangle} o_L[\![z]\!]$ is a local shtuka, so that $F_{\mathcal{M}\otimes_{o_L\langle z\rangle} o_L[\![z]\!]}$ becomes an isomorphism after inverting $z-\zeta$ which means that $\det(\mathbf{T})^{-1}$ is a unit of $o_L[\![z]\!][\frac{1}{z-\zeta}]$; say we have a relation $(z-\zeta)^e = \det(\mathbf{T})u$ in $o_L[\![z]\!]$, for some $e \geq 0$ and some $u \in o_L[\![z]\!]$; by a comparison of powers of $z-\zeta$, we may assume that u is not divided by $z-\zeta$; in this equation there is only one prime element of $o_L[\![z]\!]$ occuring on both sides, which, by factoriality, implies that u has to be a unit in $o_L[\![z]\!]$; let $(z-\zeta)^{e'} = \det(\mathbf{T}')u'$ be the corresponding relation for the local shtuka $\hat{M}$, with a unit $u' \in o_L[\![z]\!]^\times$ and some suitable $e' \geq 0$. Since $\mathcal{M} \otimes_{o_L\langle z\rangle} o_L[\![z]\!] \to \hat{M}$ becomes an isomorphism after inverting π, we see that $\det(\mathbf{A}) \in o_L[\![z]\!][1/\pi]^\times$; note that the natural reduction-mod-z map $o_L[\![z]\!] \to o_L$, $h \mapsto h(0)$, induces an epimorphism of abelian groups $o_L[\![z]\!][\frac{1}{\pi}]^\times \to L^\times$, so that (the absolute term of) $\det(\mathbf{A})$ gives rise to an element α of $L^\times$. By virtue of the relations derived above, the equation $\det(\mathbf{A})\det(\mathbf{T}) = \det(\mathbf{T}')\sigma(\det(\mathbf{A}))$ yields

$$\det(\mathbf{A})u^{-1}(z-\zeta)^e = u'^{-1}(z-\zeta)^{e'}\sigma(\det(\mathbf{A}))$$

which modulo z gives $\alpha^{q-1} = \frac{u'(0)}{u(0)}(-\zeta)^{e-e'}$ in $L^\times$. Suppose for a moment that $e = e'$; in this case it follows at once that α is a unit in o_L, so that $\det(\mathbf{A})$ is a unit in $o_L[\![z]\!]$. Therefore it remains to verify that our assumption $e = e'$ is justified. This can be seen as follows: The reduction-mod-π map $o_L[\![z]\!] \to \ell[\![z]\!]$ is an epimorphism with kernel $\pi o_L[\![z]\!]$, and via applying the functor $\cdot \otimes_{o_L[\![z]\!]} \ell[\![z]\!]$ to $F_{\hat{M}}\colon \sigma^*\hat{M} \to \hat{M}$ we obtain a commutative diagram

$$\begin{array}{ccc} \sigma^*\hat{M} = \hat{M} \otimes_{o_L[\![z]\!],\sigma} o_L[\![z]\!] & \longrightarrow & \hat{M} \\ \downarrow & & \downarrow \\ \bar{\sigma}^*\hat{M}/\pi\hat{M} = \hat{M}/\pi\hat{M} \otimes_{\ell[\![z]\!],\bar{\sigma}} \ell[\![z]\!] & \longrightarrow & \hat{M}/\pi\hat{M} \end{array}$$

where in the upper row (resp., the bottom row) both modules are finite free of the same rank over $o_L[\![z]\!]$ (resp., over $\ell[\![z]\!]$) and the arrow is given by $F_{\hat{M}}$ (resp., by

$\bar{F} = F_{\hat{M}} \otimes \mathrm{id}_{\ell[\![z]\!]}$). The reduced matrix $\overline{\mathbf{T}'} \in \ell[\![z]\!]^{s \times s}$ describes the map $\bar{F}$ with respect to the $\ell[\![z]\!]$-bases $\overline{\sigma^* \mathfrak{C}} = \bar{\sigma}^* \bar{\mathfrak{C}}$ of $\bar{\sigma}^* \hat{M}/\pi \hat{M}$ and $\bar{\mathfrak{C}}$ of $\hat{M}/\pi\hat{M}$ respectively, and from what we have seen before, we derive the relation $\det(\overline{\mathbf{T}'})\overline{u'} = z^{e'}$, i.e.,

$$e' = \mathrm{ord}_z(\det(\overline{\mathbf{T}'})),$$

the latter being true since $\overline{u'} \in \ell[\![z]\!]^{\times}$; in particular we have $\det(\overline{\mathbf{T}'}) \in \ell[\![z]\!] - \{0\}$. A similar observation for the local shtuka $\mathcal{M} \otimes_{o_L\langle z\rangle} o_L[\![z]\!]$ instead of $\hat{M}$ shows that $e = \mathrm{ord}_z(\det(\overline{\mathbf{T}}))$. Let

$$C = \mathrm{coker}(F_{\mathcal{M}\otimes_{o_L\langle z\rangle} o_L[\![z]\!]}), \qquad C' = \mathrm{coker}(F_{\hat{M}}).$$

Multiplication with the matrix $\overline{\mathbf{T}'}$ gives rise to a finite presentation

$$\ell[\![z]\!]^s \to \ell[\![z]\!]^s \to C'/\pi C' \to 0.$$

Taking determinants in an equation of the form $\mathbf{S}_1 \overline{\mathbf{T}'} \mathbf{S}_2 = \mathrm{Diag}(\alpha_1, ..., \alpha_d, 0, 0, ..., 0)$, where $\mathbf{S}_1, \mathbf{S}_2 \in \mathrm{Gl}_s(\ell[\![z]\!])$ are suitable matrices such that $\alpha_1, ..., \alpha_d \in \ell[\![z]\!] - \{0\}$ are the elementary divisors of $\overline{\mathbf{T}'}$ (see [12], VII.4.5.1), yields that necessarily $d = s$, so that $C'/\pi C'$ is a torsion $\ell[\![z]\!]$-module and

$$C'/\pi C' \simeq \ell[\![z]\!]/\alpha_1\ell[\![z]\!] \oplus ... \oplus \ell[\![z]\!]/\alpha_s\ell[\![z]\!] \simeq \ell^{n_1} \oplus ... \oplus \ell^{n_s}$$

where $n_j = \mathrm{ord}_z(\alpha_j)$ and $\sum_j n_j = e'$, i.e.,

$$e' = \mathrm{ord}_z(\det(\overline{\mathbf{T}'})) = \mathrm{rk}_\ell(C'/\pi C') = \mathrm{rk}_{o_L}(C'),$$

the latter equation being valid since $C'/\pi C' \simeq C' \otimes_{o_L[\![z]\!]} \ell[\![z]\!]$. Finally, imitating this argument for the local shtuka $\mathcal{M} \otimes_{o_L\langle z\rangle} o_L[\![z]\!]$ yields that

$$e = \mathrm{ord}_z(\det(\overline{\mathbf{T}})) = \mathrm{rk}_\ell(C/\pi C) = \mathrm{rk}_{o_L}(C).$$

So it remains to show that $\mathrm{rk}_{o_L}(C) = \mathrm{rk}_{o_L}(C')$. Indeed, we know that $\psi\colon \mathcal{M} \otimes_{o_L\langle z\rangle} o_L[\![z]\!] \to \hat{M}$ gives back f in the generic fiber, which means that ψ is an isomorphism after inverting π; therefore, inverting π in the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sigma^*(\mathcal{M} \otimes_{o_L\langle z\rangle} o_L[\![z]\!]) & \longrightarrow & \mathcal{M} \otimes_{o_L\langle z\rangle} o_L[\![z]\!] & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma^*\psi} & & \downarrow{\scriptstyle \psi} & & \downarrow & & \\ 0 & \longrightarrow & \sigma^*\hat{M} & \longrightarrow & \hat{M} & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

exhibits $(\sigma^*\psi)[1/\pi] = \sigma^*(\psi[1/\pi])$ and $\psi[1/\pi]$ as $o_L[\![z]\!][1/\pi]$-linear isomorphisms, so that the Snake Lemma yields $C'[1/\pi] \simeq C[1/\pi]$, and we obtain

$$\mathrm{rk}_{o_L}(C') = \dim_L(C'[1/\pi]) = \dim_L(C[1/\pi]) = \mathrm{rk}_{o_L}(C),$$

as desired. □

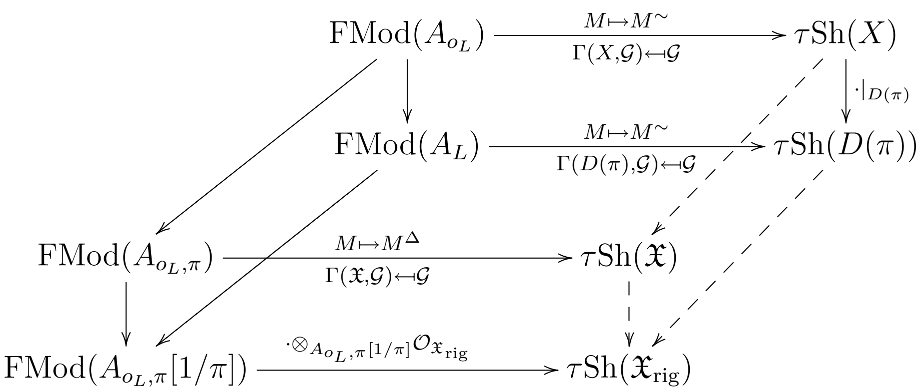
1.7 Algebraic, formal, and analytic τ-sheaves

Let X denote the o_L-scheme $(\mathcal{C}-\{\infty\})\otimes_{\mathbb{F}} o_L = \mathrm{Spec}(A_{o_L})$; its generic fiber $D(\pi) \subseteq X$ is well-known to be the open affine part corresponding to the L-algebra A_L. As mentioned before (see (1.5)), completing X along its special fiber $V(\pi)$ yields the affine formal o_L-scheme $\mathfrak{X} = \mathrm{Spf}(A_{o_L,\pi})$, which in turn gives rise to the affinoid L-space $\mathfrak{A}(1) = \mathfrak{X}_{\mathrm{rig}} = \mathrm{Sp}(A_{o_L,\pi}[1/\pi])$. The Frobenius lift on A_{o_L} (resp., A_L; resp., $A_{o_L,\pi}$; resp., $A_{o_L,\pi}[1/\pi]$) gives rise to an endomorphism of X (resp., $D(\pi)$; resp., $\mathfrak{X}$; resp., $\mathfrak{X}_{\mathrm{rig}}$) again denoted by σ.

For every $\bullet \in \{X, D(\pi), \mathfrak{X}, \mathfrak{X}_{\mathrm{rig}}\}$ we define the category $\tau\mathrm{Sh}(\bullet)$ of $\bullet$-τ-*sheaves* as follows:

The objects of $\tau\mathrm{Sh}(\bullet)$ are pairs $(\mathcal{G}, F)$ where $\mathcal{G}$ is a sheaf of $\mathcal{O}_\bullet$-modules which is locally free of finite rank (in the sense suitable for the choice of $\bullet$), together with a morphism of $\mathcal{O}_\bullet$-modules $F = F_{\mathcal{G}}\colon \sigma^*\mathcal{G} \to \mathcal{G}$ with trivial kernel (see also the remarks below); in $\tau\mathrm{Sh}(\bullet)$ a *morphism* of pairs $(\mathcal{G}, F_{\mathcal{G}}) \to (\mathcal{G}', F_{\mathcal{G}'})$ is defined to be a morphism of $\mathcal{O}_\bullet$-modules $\mathcal{G} \to \mathcal{G}'$ which is compatible with $F_{\mathcal{G}}$ and $F_{\mathcal{G}'}$.

There is a commutative diagram of categories and functors



on which we give the following remarks:

— In the upper, algebraic part of this diagram, it is well-known that the horizontal arrows are well-defined and moreover are equivalences of categories, and that the vertical arrows are faithful; see [EGA I(n)], I.1.3. Summarizing this part of the diagram we may say that algebraic τ-sheaves on X (resp., $D(\pi)$) are mirrored by Frobenius modules over A_{o_L} (resp., A_L) in the displayed manner; the involved objects were studied in [30].

— By [EGA I(n)], I.10.10.8 and 0.7.2.5, the assignment $M \mapsto M^\Delta$ sets up an (exact) equivalence between the category of finite projective $A_{o_L,\pi}$-modules and the category of locally free $\mathcal{O}_{\mathfrak{X}}$-modules of finite rank, and by [EGA I(n)], I.10.10.5, this equivalence restricts to $\mathrm{FMod}(A_{o_L,\pi}) \simeq \tau\mathrm{Sh}(\mathfrak{X})$.

— Let us briefly explain why the functor $\mathrm{FMod}(A_{o_L,\pi}[1/\pi]) \to \tau\mathrm{Sh}(\mathfrak{X}_{\mathrm{rig}})$ is well-defined: By Lemma 1.22(i) below, the assignment

$$M \mapsto M \otimes_{A_{o_L,\pi}[1/\pi]} \mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}}$$

maps finite projective $A_{o_L,\pi}[1/\pi]$-modules to locally free $\mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}}$-modules of finite rank, and it is well-known ([10], 9.4.2/2) that it is exact and fully faithful; finally, by 1.22(ii), it indeed restricts to a functor on Frobenius modules; as such, it is again fully faithful, and by Kiehl's Theorem ([10], 9.4.3/3) in combination with 1.22, it is essentially surjective, i.e., gives an equivalence of categories.

— The (dashed) functor $\tau\mathrm{Sh}(X) \to \tau\mathrm{Sh}(\mathfrak{X})$ is obtained via

$$\mathcal{G} \mapsto (\Gamma(X, \mathcal{G}) \otimes_{A_{o_L}} A_{o_L,\pi})^{\Delta},$$

and similarly for $\tau\mathrm{Sh}(D(\pi)) \to \tau\mathrm{Sh}(\mathfrak{X}_{\mathrm{rig}})$. By construction, these functors are faithful.

— The remaining (dashed) functor $\tau\mathrm{Sh}(\mathfrak{X}) \to \tau\mathrm{Sh}(\mathfrak{X}_{\mathrm{rig}})$ is obtained via the assignment

$$\mathcal{G} \mapsto \Gamma(\mathfrak{X}, \mathcal{G})[1/\pi] \otimes_{A_{o_L,\pi}[1/\pi]} \mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}}$$

and, by construction, is faithful. –

Lemma 1.22. *Let K be a complete non-archimedean valued field. Let $\underline{A}$ be an affinoid K-algebra, and let $\underline{X} = \mathrm{Sp}(\underline{A})$ be the associated affinoid K-space.*

(i) Suppose that $\underline{A}$ is integral, and let $\underline{M}$ be an $\underline{A}$-module; then $\underline{M}$ is locally free of finite rank d if and only if the associated $\mathcal{O}_{\underline{X}}$-module $\mathcal{F} = \underline{M} \otimes_{\underline{A}} \mathcal{O}_{\underline{X}}$ is locally free of finite rank d.

(ii) ([9]) Let $\underline{Y} = \mathrm{Sp}(\underline{B})$ be another affinoid K-space, and let $\mathrm{Sp}(\varphi)\colon \underline{Y} \to \underline{X}$ be a morphism of affinoid K-spaces, associated to a K-algebra homomorphism $\varphi\colon \underline{A} \to \underline{B}$. If $\underline{M}$ is an $\underline{A}$-module then

$$\mathrm{Sp}(\varphi)^*(\underline{M} \otimes_{\underline{A}} \mathcal{O}_{\underline{X}}) \simeq (\underline{M} \otimes_{\underline{A}} \underline{B}) \otimes_{\underline{B}} \mathcal{O}_{\underline{Y}}.$$

Proof. We begin with the proof of (i). For the "only if"-part, by [28], 4.5.1, it suffices to show that for every point $x \in \underline{X}$ the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{\underline{X},x}$-module of rank d; indeed, a fixed point $x \in \underline{X}$ corresponds to a maximal ideal $\mathfrak{m} \subseteq \underline{A}$, and we know that $\underline{M} \otimes_{\underline{A}} \underline{A}_{\mathfrak{m}}$ is a free $\underline{A}_{\mathfrak{m}}$-module of rank d; since $\underline{M}$ is of finite type,

by [10], 9.4.2/6, the canonical map $\underline{M} \otimes_{\underline{A}} \mathcal{O}_{\underline{X},x} \to \mathcal{F}_x$ is an isomorphism, and we have $\underline{M} \otimes_{\underline{A}} \mathcal{O}_{\underline{X},x} \simeq (\underline{M} \otimes_{\underline{A}} \underline{A}_{\mathfrak{m}}) \otimes_{\underline{A}_{\mathfrak{m}}} \mathcal{O}_{\underline{X},x}$ which gives the "only if"-part. Conversely, suppose that $\mathcal{F} = \underline{M} \otimes_{\underline{A}} \mathcal{O}_{\underline{X}}$ is locally free of rank d; then for every $x \in \underline{X}$ the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{\underline{X},x}$-module of rank d. By Kiehl's Theorem ([10], 9.4.3/3) there is a finite $\underline{A}$-module $\underline{N}$ such that $\mathcal{F} = \underline{N} \otimes_{\underline{A}} \mathcal{O}_{\underline{X}}$, and by the exactness properties of the functor $\cdot \otimes_{\underline{A}} \mathcal{O}_{\underline{X}}$ the module $\underline{N}$ has to be isomorphic to $\underline{M}$, which in particular means that $\underline{M}$ itself is finite. As mentioned before, using that $\underline{M}$ is finite, there is a canonical isomorphism $\mathcal{F}_x \simeq (\underline{M} \otimes_{\underline{A}} \underline{A}_{\mathfrak{m}}) \otimes_{\underline{A}_{\mathfrak{m}}} \mathcal{O}_{\underline{X},x}$ where $\mathfrak{m} \subseteq \underline{A}$ is the maximal ideal corresponding to a chosen point $x \in X$. By [28], 4.6.1, the natural map $\underline{A}_{\mathfrak{m}} \to \mathcal{O}_{\underline{X},x}$ is faithfully flat, which implies that, since $\mathcal{F}_x$ is free, the $\underline{A}_{\mathfrak{m}}$-module $\underline{M} \otimes_{\underline{A}} \underline{A}_{\mathfrak{m}}$ is a locally free $\underline{A}_{\mathfrak{m}}$-module and hence is free since $\underline{A}_{\mathfrak{m}}$ is a local ring. Going back and forth in the henceforth established equivalence, one sees that ranks are preserved.
In order to explain (ii), we briefly reproduce the remarks given at the end of section 1.13 in [9]: If $\underline{N}$ is a $\underline{B}$-module then $\mathrm{Sp}(\varphi)_*(\underline{N} \otimes_{\underline{B}} \mathcal{O}_{\underline{Y}}) \simeq (\varphi_* \underline{N}) \otimes_{\underline{A}} \mathcal{O}_{\underline{X}}$, so that the adjunction formula

$$\mathrm{Hom}_{\mathcal{O}_{\underline{Y}}}(\mathrm{Sp}(\varphi)^* \mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{O}_{\underline{X}}}(\mathcal{F}, \mathrm{Sp}(\varphi)_* \mathcal{G})$$

for $\mathcal{O}_{\underline{X}}$-modules $\mathcal{F}$ and $\mathcal{O}_{\underline{Y}}$-modules $\mathcal{G}$ completes the proof. □

In section (1.5) we have explained how to attach a canonical reduction (defined over the Dedekind domain A_ℓ) to every Frobenius module $\mathcal{M} \in \mathrm{FMod}(A_{o_L,\pi})$; the notion of good models for analytic Anderson $A(1)$-motives was based on the circumstance that the assignment $\mathcal{M} \mapsto \mathcal{M}/\pi\mathcal{M}$ does not induce a functor from $\mathrm{FMod}(A_{o_L,\pi})$ to $\mathrm{FMod}(A_\ell)$. For (algebraic) Frobenius modules over A_{o_L} we have an analogous situation: Given an object $\mathcal{M} \in \mathrm{FMod}(A_{o_L})$, the A_ℓ-module $\mathcal{M}/\pi\mathcal{M}$ is called the *reduction* of $\mathcal{M}$.

Proposition 1.23. *Let $\mathcal{M} \in \mathrm{FMod}(A_{o_L})$, and let $\widehat{\mathcal{M}}$ be its image under the natural functor $\mathrm{FMod}(A_{o_L}) \to \mathrm{FMod}(A_{o_L,\pi})$, i.e., the underlying $A_{o_L,\pi}$-module $\widehat{\mathcal{M}}$ equals the π-adic completion of $\mathcal{M}$. Then the reduction $\widehat{\mathcal{M}}/\pi\widehat{\mathcal{M}}$ of $\widehat{\mathcal{M}}$ is canonically isomorphic to the reduction $\mathcal{M}/\pi\mathcal{M}$ of $\mathcal{M}$.* □

Let $\mathcal{M} \in \mathrm{FMod}(A_{o_L})$. Following Gardeyn [30], we call $\mathcal{M}$ A_{o_L}-*maximal* if for every $\mathcal{N} \in \mathrm{FMod}(A_{o_L})$ the canonical map

$$\mathrm{Hom}_{\mathrm{FMod}(A_{o_L})}(\mathcal{N}, \mathcal{M}) \to \mathrm{Hom}_{\mathrm{FMod}(A_L)}(\mathcal{N}[1/\pi], \mathcal{M}[1/\pi])$$

is surjective (and hence bijective); correspondingly, an object $\mathcal{M}' \in \mathrm{FMod}(A_{o_L,\pi})$ is called $A_{o_L,\pi}$-*maximal* if for every $\mathcal{N}' \in \mathrm{FMod}(A_{o_L,\pi})$ the canonical map

$$\mathrm{Hom}_{\mathrm{FMod}(A_{o_L,\pi})}(\mathcal{N}', \mathcal{M}') \to \mathrm{Hom}_{\mathrm{FMod}(A_{o_L,\pi}[1/\pi])}(\mathcal{N}'[1/\pi], \mathcal{M}'[1/\pi])$$

is surjective (and hence bijective).

Let $M \in \mathrm{FMod}(A_L)$; an object $\mathcal{M} \in \mathrm{FMod}(A_{o_L})$ is called an A_{o_L}*-maximal model* for M if $\mathcal{M}[1/\pi] \simeq M$ inside $\mathrm{FMod}(A_L)$ (i.e., $\mathcal{M}$ is a *model* for M) and if $\mathcal{M}$ is an A_{o_L}-maximal object. Correspondingly, given $M' \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$, an object $\mathcal{M}' \in \mathrm{FMod}(A_{o_L,\pi})$ is called an $A_{o_L,\pi}$*-maximal model* for M' if $\mathcal{M}'[1/\pi] \simeq M'$ inside $\mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ and if $\mathcal{M}'$ is $A_{o_L,\pi}$-maximal.

Suppose that $\mathcal{M}, \mathcal{N} \in \mathrm{FMod}(A_{o_L})$ are both (A_{o_L}-)maximal models of some given $M \in \mathrm{FMod}(A_L)$; fixing isomorphisms $f\colon \mathcal{M}[1/\pi] \to M$ and $g\colon \mathcal{N}[1/\pi] \to M$ inside $\mathrm{FMod}(A_L)$, the composition $f^{-1}g\colon \mathcal{N}[1/\pi] \to \mathcal{M}[1/\pi]$ (resp., $g^{-1}f\colon \mathcal{M}[1/\pi] \to \mathcal{N}[1/\pi]$) corresponds to a unique morphism $\underline{f^{-1}g}\colon \mathcal{N} \to \mathcal{M}$ (resp., $\underline{g^{-1}f}\colon \mathcal{M} \to \mathcal{N}$) inside $\mathrm{FMod}(A_{o_L})$, and the commutative diagram with injective vertical arrows

$$\begin{array}{ccccc} & & \overset{\mathrm{id}}{\frown} & & \\ \mathcal{N}[1/\pi] & \xrightarrow{f^{-1}g} & \mathcal{M}[1/\pi] & \xrightarrow{g^{-1}f} & \mathcal{N}[1/\pi] \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{N} & \xrightarrow[\underline{f^{-1}g}]{} & \mathcal{M} & \xrightarrow[\underline{g^{-1}f}]{} & \mathcal{N} \end{array}$$

shows that $\underline{g^{-1}f} \circ \underline{f^{-1}g} = \mathrm{id}_{\mathcal{N}}$; similarly one shows that $\underline{f^{-1}g} \circ \underline{g^{-1}f} = \mathrm{id}_{\mathcal{M}}$. In this sense the A_{o_L}-maximal model of M, if it exists, is unique up to unique isomorphism inside $\mathrm{FMod}(A_{o_L})$. By an analogous argumentation, the same is true for $A_{o_L,\pi}$-maximal models of objects of $\mathrm{FMod}(A_{o_L,\pi}[1/\pi])$.

The existence of (A_{o_L}- and $A_{o_L,\pi}$-)maximal models has been established in [30]. To begin with, we recall the algebraic case in the following Lemma. Let $\varpi \in X = \mathrm{Spec}(A_{o_L})$ be the point corresponding to the ideal $\mathfrak{p} = \mathfrak{p}_\varpi = \pi A_{o_L}$. Let $R_\varpi = (A_{o_L})_{\mathfrak{p}}$; since A_{o_L} is a regular integral domain, it follows that R_ϖ is a discrete valuation ring with uniformizer π and residue field $\mathrm{Frac}(A_\ell)$, and its fraction field equals $F_\varpi = \mathrm{Frac}(A_{o_L})$. The Frobenius lift $\sigma\colon A_{o_L} \to A_{o_L}$ naturally extends to give an endomorphism $\sigma\colon R_\varpi \to R_\varpi$.

Lemma 1.24 ([30]). *Let $M \in \mathrm{FMod}(A_L)$. Then the following assertions hold:*

(i) M admits a model.

(ii) M admits an A_{o_L}-maximal model, which is unique up to unique isomorphism.

(iii) Let $\mathcal{M} \in \mathrm{FMod}(A_{o_L})$ be any model of M. Then $\mathcal{M}$ is a good *model, i.e., $\mathcal{M}/\pi\mathcal{M} \in \mathrm{FMod}(A_\ell)$, if and only if the induced R_ϖ-linear map*

$$\mathcal{M}_{\mathfrak{p}} \otimes_{R_\varpi,\sigma} R_\varpi \to \mathcal{M}_{\mathfrak{p}}$$

is an isomorphism.

(iv) If a model $\mathcal{M} \in \mathrm{FMod}(A_{o_L})$ of M is good, then it is A_{o_L}-maximal. □

Proof. For (i) (resp., (ii); resp., (iii); resp., (iv)), see [30], 2.2 (resp., 2.13(i); resp., 2.10(i); resp., 2.13(ii)). □

The key for the existence of maximal models lies in a result which F. Gardeyn [30] calls *Lafforgue's Lemma* (according to its appearance in [49]) and which is originally due to S. Langton [51]. In what follows, we give a brief account of the versions of this result which we will need for our purposes.

When completing the o_L-scheme $X = \mathrm{Spec}(A_{o_L})$ along its special fiber $V(\pi) \subseteq X$, the point $\varpi \in X$ is mirrored by the point $\underline{\varpi}$ of $\mathfrak{X} = \mathrm{Spf}(A_{o_L,\pi})$ which corresponds to the (π-adically open) prime ideal $\mathfrak{q} = \mathfrak{p}A_{o_L,\pi} = \pi A_{o_L,\pi}$. Since $A_{o_L,\pi}$ is a regular integral domain by 1.7 and moreover $\mathfrak{q}$ is a principal ideal, the local ring $S_{\underline{\varpi}} = (A_{o_L,\pi})_{\mathfrak{q}}$ is a regular local ring of dimension 1, i.e., $S_{\underline{\varpi}}$ is a discrete valuation ring with uniformizer π and residue field $\mathrm{Frac}(A_\ell)$, whose fraction field equals $F_{\underline{\varpi}} = \mathrm{Frac}(A_{o_L,\pi})$. As $\bar{\sigma}\colon A_\ell \to A_\ell$ is an automorphism, the Frobenius lift $\sigma\colon A_{o_L,\pi} \to A_{o_L,\pi}$ extends to give an endomorphism $\sigma\colon S_{\underline{\varpi}} \to S_{\underline{\varpi}}$; the completion map $A_{o_L} \to A_{o_L,\pi}$ induces an isomorphism $A_{o_L}/\pi A_{o_L} \simeq A_{o_L,\pi}/\pi A_{o_L,\pi}$ and hence an unramified embedding of discrete valuation rings $R_\varpi \to S_{\underline{\varpi}}$ which is of residue degree 1. Finally, note that, a priori, $S_{\underline{\varpi}}$ is *not* the local ring of the structure sheaf $\mathcal{O}_{\mathfrak{X}}$ at the point $\underline{\varpi} \in \mathfrak{X}$, even though the residue field of $S_{\underline{\varpi}}$ equals the residue field of $\mathcal{O}_{\mathfrak{X},\underline{\varpi}}$; see [EGA I(n)], I.10.1.6.

We define the category FL_X as follows:

— An *object* of FL_X is a triple (N, P, i) where $N \in \mathrm{FMod}(A_L)$, $P \in \mathrm{FMod}(R_\varpi)$, and $i\colon N \otimes_{A_L} F_\varpi \to P \otimes_{R_\varpi} F_\varpi$ is an F_ϖ-linear isomorphism.

— A *morphism* $(N, P, i) \to (N', P', i')$ is given by a couple

$$(N \to N', P \to P') \in \mathrm{Hom}_{\mathrm{FMod}(A_L)}(N, N') \times \mathrm{Hom}_{\mathrm{FMod}(R_\varpi)}(P, P')$$

such that the induced diagram

$$\begin{array}{ccc} N \otimes_{A_L} F_\varpi & \xrightarrow{i} & P \otimes_{R_\varpi} F_\varpi \\ \downarrow & & \downarrow \\ N' \otimes_{A_L} F_\varpi & \xrightarrow[i']{} & P' \otimes_{R_\varpi} F_\varpi \end{array}$$

commutes.

Analogously, we define a category $FL_{\mathfrak{X}}$ by the following data:

— An *object* of $FL_{\mathfrak{X}}$ is a triple (N, P, i) where $N \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$, $P \in \mathrm{FMod}(S_{\underline{\varpi}})$, and $i\colon N \otimes_{A_{o_L,\pi}[1/\pi]} F_{\underline{\varpi}} \to P \otimes_{S_{\underline{\varpi}}} F_{\underline{\varpi}}$ is an $F_{\underline{\varpi}}$-linear isomorphism.

— A *morphism* $(N, P, i) \to (N', P', i')$ is given by a couple

$$(N \to N', P \to P') \in \mathrm{Hom}_{\mathrm{FMod}(A_{o_L,\pi}[1/\pi])}(N, N') \times \mathrm{Hom}_{\mathrm{FMod}(S_{\underline{\varpi}})}(P, P')$$

such that the induced diagram

$$\begin{array}{ccc} N \otimes_{A_{o_L,\pi}[1/\pi]} F_{\underline{\varpi}} & \xrightarrow{i} & P \otimes_{S_{\underline{\varpi}}} F_{\underline{\varpi}} \\ \downarrow & & \downarrow \\ N' \otimes_{A_{o_L,\pi}[1/\pi]} F_{\underline{\varpi}} & \xrightarrow[i']{} & P' \otimes_{S_{\underline{\varpi}}} F_{\underline{\varpi}} \end{array}$$

commutes.

Now we state what Gardeyn calls "Lafforgue's Lemma".

Lemma 1.25. *(i) ([30]) There is an equivalence of categories*

$$\begin{array}{rcl} \mathrm{FMod}(A_{o_L}) & \to & FL_X, \\ M & \mapsto & (M[1/\pi],\ M_{\mathfrak{p}},\ M[1/\pi] \otimes_{A_L} F_{\varpi} \simeq M_{\mathfrak{p}} \otimes_{R_{\varpi}} F_{\varpi}); \end{array}$$

in particular, a τ-sheaf $M^{\sim} \in \tau\mathrm{Sh}(X)$ on X can be reconstructed from the data consisting of its restriction $M[1/\pi]^{\sim}$ to the generic fiber $D(\pi) \subseteq X$, together with the finite free R_{ϖ}-module $M_{\mathfrak{p}}$.

(ii) There is an equivalence of categories

$$\begin{array}{rcl} \mathrm{FMod}(A_{o_L,\pi}) & \to & FL_{\mathfrak{X}}, \\ M & \mapsto & (M[1/\pi],\ M_{\mathfrak{q}},\ M[1/\pi] \otimes_{A_{o_L,\pi}[1/\pi]} F_{\underline{\varpi}} \simeq M_{\mathfrak{q}} \otimes_{S_{\underline{\varpi}}} F_{\underline{\varpi}}); \end{array}$$

in particular, a τ-sheaf $M^{\Delta} \in \tau\mathrm{Sh}(\mathfrak{X})$ on $\mathfrak{X}$ can be reconstructed from the data consisting of its associated τ-sheaf $M[1/\pi] \otimes_{A_{o_L,\pi}[1/\pi]} \mathcal{O}_{\mathfrak{X}_{\mathrm{rig}}}$ on the Raynaud fiber $\mathfrak{X}_{\mathrm{rig}}$, together with the finite free $S_{\underline{\varpi}}$-module $M_{\mathfrak{q}}$.

The asserted equivalences give rise to an obvious commutative diagram of categories and functors

$$\begin{array}{ccc} \mathrm{FMod}(A_{o_L}) & \xrightarrow{\simeq} & FL_X \\ \downarrow & & \downarrow \\ \mathrm{FMod}(A_{o_L,\pi}) & \xrightarrow[\simeq]{} & FL_{\mathfrak{X}} \end{array}$$

where the functor $FL_X \to FL_{\mathfrak{X}}$ is given by

$$(N, P, f) \mapsto$$

$$(N \otimes_{A_L} A_{o_L,\pi}[1/\pi], \quad P \otimes_{R_{\varpi}} S_{\underline{\varpi}}, \quad (N \otimes_{A_L} F_{\varpi}) \otimes_{F_{\varpi}} F_{\underline{\varpi}} \xrightarrow{f \otimes \mathrm{id}_{F_{\underline{\varpi}}}} (P \otimes_{R_{\varpi}} F_{\varpi}) \otimes_{F_{\varpi}} F_{\underline{\varpi}}).$$

Proof of 1.25. For (i) we refer to [30], 2.9; the proof of (ii) is accomplished by the following Lemma 1.26. □

Lemma 1.26. *Let N be a locally free $A_{o_L,\pi}[1/\pi]$-module of rank d, and let P be a free $S_{\underline{\varpi}}$-submodule of rank d of $N \otimes_{A_{o_L,\pi}[1/\pi]} F_{\underline{\varpi}}$ such that the induced $F_{\underline{\varpi}}$-linear inclusion $P[1/\pi] \subseteq N \otimes_{A_{o_L,\pi}[1/\pi]} F_{\underline{\varpi}}$ is an isomorphism of $F_{\underline{\varpi}}$-vector spaces. Then there is a locally free $A_{o_L,\pi}$-module M of rank d such that $M[1/\pi] \simeq N$ and $M_{\mathfrak{q}} \simeq P$, and M is unique up to isomorphisms of $A_{o_L,\pi}$-modules.*

Proof of Lemma 1.26. We may identify N with the image of the canonical embedding $N \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,\pi}[1/\pi] \hookrightarrow N \otimes_{A_{o_L,\pi}[1/\pi]} F_{\underline{\varpi}}$. Let $M = N \cap P$; this is an $A_{o_L,\pi}$-submodule of $N \otimes_{A_{o_L,\pi}[1/\pi]} F_{\underline{\varpi}}$. First of all, adapting techniques given in [51], 3.6, we show that M is a finitely generated $A_{o_L,\pi}$-module. Indeed, we know that N is finitely generated over $A_{o_L,\pi}[1/\pi]$, say $N = \sum_{i=1}^{e} A_{o_L,\pi}[1/\pi]n_i$, and that P is finite free over $S_{\underline{\varpi}}$, say $P = \oplus_{j=1}^{d} S_{\underline{\varpi}} p_j$. Without loss of generality we may assume that $n_i \in M$ for every $i = 1, ..., e$. Indeed, viewing $n_1, ..., n_e$ as elements of $N \otimes_{A_{o_L,\pi}[1/\pi]} F_{\underline{\varpi}}$, there are integers $\nu_1, ..., \nu_e \geq 0$ such that $\pi^{\nu_i} n_i \in P$; as π is a unit in $A_{o_L,\pi}[1/\pi]$, the elements $\pi^{\nu_1} n_1, ..., \pi^{\nu_e} n_e$ still constitute a system of generators for N over $A_{o_L,\pi}[1/\pi]$. Next we note that the basis elements $p_1, ..., p_d$ of P give rise to an $F_{\underline{\varpi}}$-linearly independent family of $N \otimes_{A_{o_L,\pi}[1/\pi]} F_{\underline{\varpi}}$; furthermore, as $n_i \in P$ for all i, we may write $n_i = \sum_{j=1}^{d} \lambda_{ij} p_j$ for uniquely determined scalars $\lambda_{ij} \in S_{\underline{\varpi}}$; collecting the denominators of the λ_{ij}, we see that there is an element $c \in A_{o_L,\pi} - \mathfrak{q}$ such that $\lambda'_{ij} = c\lambda_{ij} \in A_{o_L,\pi}$ for all i, j; the element c gives rise to a unit of $S_{\underline{\varpi}}$, so that the elements $p'_1 = c^{-1}p_1, ..., p'_d = c^{-1}p_d$ still constitute a basis of P over $S_{\underline{\varpi}}$, and we get $n_i = \sum_{j=1}^{d} \lambda'_{ij} p'_j$ for every i. Let $m \in M$ be an arbitrary element, say

$$m = \sum_{i=1}^{e} \alpha_i n_i = \sum_{j=1}^{d} \beta_j p'_j$$

where $\alpha_i \in A_{o_L,\pi}[1/\pi]$ and $\beta_j \in S_{\underline{\varpi}}$. We obtain $m = \sum_{j=1}^{d}(\sum_{i=1}^{e} \alpha_i \lambda'_{ij}) p'_j$, so that $\beta_j = (\sum_{i=1}^{e} \alpha_i \lambda'_{ij})$ inside $F_{\underline{\varpi}}$ for every j, the latter equation being true since $p'_1, ..., p'_d$ are $F_{\underline{\varpi}}$-linearly independent; the same equation shows that $\beta_j \in A_{o_L,\pi}[1/\pi] \cap S_{\underline{\varpi}}$ for every j, and it is easy to see that the latter intersection inside $F_{\underline{\varpi}}$ does, in fact, equal $A_{o_L,\pi}$. Therefore $\beta_j \in A_{o_L,\pi}$ for every j, and we may conclude that $M \subseteq \sum_{j=1}^{d} A_{o_L,\pi} p'_j$; since $A_{o_L,\pi}$ is noetherian, M itself has to be finitely generated; in particular, M is of finite presentation. Next we remark that there is an $A_{o_L,\pi}[1/\pi]$-linear isomorphism $M[1/\pi] \simeq N$, for we have equalities

$$M[1/\pi] \simeq N[1/\pi] \cap P[1/\pi] \simeq N$$

of $A_{o_L,\pi}[1/\pi]$-submodules of $N \otimes_{A_{o_L,\pi}[1/\pi]} F_{\underline{\varpi}}$, which can be explained as follows: the canonical map $N \to N[1/\pi]$, $n \mapsto n/1$, is an $A_{o_L,\pi}[1/\pi]$-linear isomorphism, and the natural isomorphism $P[1/\pi] \simeq N \otimes_{A_{o_L,\pi}[1/\pi]} F_{\underline{\varpi}}$ is in particular $A_{o_L,\pi}[1/\pi]$-linear, so that it exhibits the displayed intersection as being isomorphic to N, as desired. Furthermore we have

$$M_{\mathfrak{q}} \simeq N_{\mathfrak{q}} \cap P_{\mathfrak{q}} \simeq P,$$

the latter relation being valid for the following reason: there are canonical $S_{\underline{\varpi}}$-linear isomorphisms

$$N_{\mathfrak{q}} \simeq N \otimes_{A_{o_L,\pi}} S_{\underline{\varpi}} \simeq N \otimes_{A_{o_L,\pi}[1/\pi]} (A_{o_L,\pi}[1/\pi] \otimes_{A_{o_L,\pi}} S_{\underline{\varpi}}) \simeq N \otimes_{A_{o_L,\pi}[1/\pi]} F_{\underline{\varpi}};$$

note that the canonical map of $A_{o_L,\pi}$-algebras $A_{o_L,\pi}[1/\pi] \otimes_{A_{o_L,\pi}} S_{\underline{\varpi}} \to F_{\underline{\varpi}}$ is an $S_{\underline{\varpi}}$-linear isomorphism; on the other hand, also the canonical map $P \to P_{\mathfrak{q}}$, $r \mapsto r/1$, is an $S_{\underline{\varpi}}$-linear isomorphism, so that the desired relation is established. Finally we claim that M is a flat $A_{o_L,\pi}$-module. Indeed, since $\pi M = \pi N \cap \pi P = N \cap \pi P$, the $A_{o_L,\pi}$-linear inclusion $M \subseteq P$ induces an embedding of A_ℓ-modules $M/\pi M \hookrightarrow P/\pi P$; here we note that

$$P/\pi P \simeq P \otimes_{S_{\underline{\varpi}}} \kappa(\underline{\varpi});$$

the residue field $\kappa(\underline{\varpi}) = S_{\underline{\varpi}}/\pi S_{\underline{\varpi}}$ is canonically isomorphic to $\mathrm{Frac}(A_\ell)$, so that $P/\pi P$ is a finite $\mathrm{Frac}(A_\ell)$-vector space; therefore, its A_ℓ-submodule $M/\pi M$ is torsion-free and hence projective, the latter being true since A_ℓ is a Dedekind domain; therefore $M/\pi M$ is flat over A_ℓ, which (for example, by [13], III.5.2(iii)) implies that M has to be flat over $A_{o_L,\pi}$ since $M \subseteq P$ has trivial π-torsion. □

We may draw the following

Conclusion 1.27. There are obvious equivalences of categories

$$L_{\mathfrak{X}} \simeq L'_{\mathfrak{X}} \simeq \mathrm{fPrj}(A_{o_L,\pi})$$

where

- $L_{\mathfrak{X}}$ is the category whose *objects* are given by triples (N, P, f) where N is a locally free $A_{o_L,\pi}[1/\pi]$-module of finite rank, P is a finite free $S_{\underline{\varpi}}$-module and $f\colon P[1/\pi] \xrightarrow{\simeq} N \otimes_{A_{o_L,\pi}[1/\pi]} F_{\underline{\varpi}}$ is an $F_{\underline{\varpi}}$-linear isomorphism, and where a *morphism* $(N, P, f) \to (N', P', f')$ is given by a tuple $(u, v) \in \mathrm{Hom}_{S_{\underline{\varpi}}}(P, P') \times \mathrm{Hom}_{A_{o_L,\pi}[1/\pi]}(N, N')$ such that $f' \circ (u \otimes \mathrm{id}) = (v \otimes \mathrm{id}) \circ f$;

- $L'_{\mathfrak{X}}$ is the category whose *objects* are given by pairs (N, P) where N is a locally free $A_{o_L,\pi}[1/\pi]$-module of finite rank, together with a finite free $S_{\underline{\varpi}}$-submodule P of $N \otimes_{A_{o_L,\pi}[1/\pi]} F_{\underline{\varpi}}$ such that the induced $F_{\underline{\varpi}}$-linear inclusion $P[1/\pi] \subseteq N \otimes_{A_{o_L,\pi}[1/\pi]} F_{\underline{\varpi}}$ is an $F_{\underline{\varpi}}$-linear isomorphism, and where a *morphism* $(N, P) \to (N', P')$ is given by an $A_{o_L,\pi}[1/\pi]$-linear map $w\colon N \to N'$ such that $(w \otimes \mathrm{id})(P) \subseteq P'$;

- $\mathrm{fPrj}(A_{o_L,\pi})$ is the full subcategory of $\mathrm{Mod}(A_{o_L,\pi})$ consisting of the locally free $A_{o_L,\pi}$-modules of finite rank.

See [30], 2.7, for a similar characterization of the category $\mathrm{fPrj}(A_{o_L})$. –

Proposition 1.28. *The following assertions hold:*

(i) Every $M \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ admits a maximal model, which is unique up to unique isomorphism.

(ii) If $M \in \mathrm{FMod}(A_L)$ is given and if $\mathcal{M} \in \mathrm{FMod}(A_{o_L})$ is an A_{o_L}-maximal model of M then $\mathcal{M} \otimes_{A_{o_L}} A_{o_L,\pi} \in \mathrm{FMod}(A_{o_L,\pi})$ is an $A_{o_L,\pi}$-maximal model of $M \otimes_{A_L} A_{o_L,\pi}[1/\pi] \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$.

(iii) Let $M \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$, and let $\mathcal{M} \in \mathrm{FMod}(A_{o_L,\pi})$ be any model of M. Then $\mathcal{M}$ is a good *model, i.e., $\mathcal{M}/\pi\mathcal{M} \in \mathrm{FMod}(A_\ell)$, if and only if the induced $S_{\underline{\varpi}}$-linear map*

$$\mathcal{M}_{\mathfrak{q}} \otimes_{S_{\underline{\varpi}},\sigma} S_{\underline{\varpi}} \to \mathcal{M}_{\mathfrak{q}}$$

is an isomorphism.

(iv) Let $M \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ and let $\mathcal{M} \in \mathrm{FMod}(A_{o_L,\pi})$ be a model of M; if $\mathcal{M}$ is a good model, i.e., $\mathcal{M}/\pi\mathcal{M} \in \mathrm{FMod}(A_\ell)$, then it is $A_{o_L,\pi}$-maximal.

Proof. For (i) (resp., (ii); resp., (iii); resp., (iv)), see [30], 3.3(i) (resp. 3.4(i); resp. 2.10(i); resp., 2.13(ii)). We remark that, by virtue of 1.25, 1.26, the proofs of the cited results carry over verbatim to the situation at hand. □

We may conclude:

Proposition 1.29. *A Frobenius A_L-module M admits a good model over A_{o_L} if and only if $M \otimes_{A_L} A_{o_L,\pi}[1/\pi] \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ admits a good model over $A_{o_L,\pi}$; then, up to isomorphism inside $\mathrm{FMod}(A_{o_L,\pi})$, a good model of $M \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ is given by $\mathcal{M} \otimes_{A_{o_L}} A_{o_L,\pi}$ where $\mathcal{M}$ is a good model of M.*

Proof. First suppose that M admits a good model $\mathcal{M} \in \mathrm{FMod}(A_{o_L})$. It follows that $\mathcal{M}$ is an A_{o_L}-maximal model of M; as such, the latter is unique up to unique isomorphism inside $\mathrm{FMod}(A_{o_L})$; furthermore, its image $\mathcal{M} \otimes_{A_{o_L}} A_{o_L,\pi}$ inside $\mathrm{FMod}(A_{o_L,\pi})$ is an $A_{o_L,\pi}$-maximal model of $M \otimes_{A_L} A_{o_L,\pi}[1/\pi]$, and as such it is unique up to unique isomorphism. Since the reduction of $\mathcal{M}$ is canonically isomorphic to the reduction of $\mathcal{M} \otimes_{A_{o_L}} A_{o_L,\pi}$, it follows that the latter is a good model. Conversely, suppose that $M \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ admits a good model $\mathcal{M}' \in \mathrm{FMod}(A_{o_L,\pi})$. Necessarily $\mathcal{M}'$ is a maximal model. We know that there is an A_{o_L}-maximal model $\mathcal{M} \in \mathrm{FMod}(A_{o_L})$ of M such that $\mathcal{M} \otimes_{A_{o_L}} A_{o_L,\pi} \simeq \mathcal{M}'$, and that the reduction of $\mathcal{M}'$ is canonically isomorphic to the reduction of $\mathcal{M}$. Since $\mathcal{M}'$ is a good model, so is $\mathcal{M}$, which completes the proof. □

Remark. In [30], especially the case of *bad reduction* has been studied, i.e., the case of those τ-sheaves which do not admit a good model; while, as we have seen, a model always exists and moreover can be chosen in a maximal possible manner, the induced τ-(or F-)map in the reduction of the maximal model can, for example, be nilpotent; in such cases the τ-sheaf at hand will not itself be of good reduction, but will rather *contain* a good-reduction τ-sheaf of a certain rank, at least after a suitable finite extension of the base field L. For example, if $\underline{M}$ is (the analytic τ-sheaf associated to) the A-motive of a Drinfeld A-module φ over L with stable but bad reduction ([21]) then $\underline{M}$ does not possess a good model but is rather a *semi-stable* analytic τ-sheaf in the sense of [30]; more precisely: Drinfeld's Tate Uniformization-Theorem ([21]) applied to φ can be carried out in terms of (necessarily non-algebraic) morphisms of A-motives ([29]), which clarifies the semi-stable structure of the τ-sheaf $\underline{M}$. We will come back to this in section (2.2). –

In 1.11 we have seen that a natural source for analytic Anderson $A(1)$-motives is incorporated by Anderson A-motives. So, in the case when a given analytic Anderson $A(1)$-motive comes from an A-motive, one is naturally led to asking for a characterization of the existence of a good model. For the following, also see Example 1.15.

Proposition 1.30. *Let $\underline{M}$ be an Anderson A-motive. Then the following assertions are equivalent:*

(i) There is a locally free A_{o_L}-module $\underline{\mathcal{M}}$ of finite rank, together with an A_{o_L}-linear map

$$F^\circ \colon \underline{\mathcal{M}} \otimes_{A_{o_L},\sigma} A_{o_L} \to \underline{\mathcal{M}}$$

such that

- *there is an A_L-linear and F-equivariant isomorphism $\underline{\mathcal{M}} \otimes_{o_L} L \simeq \underline{M}$,*
- $\operatorname{coker}(F^\circ)$ *is a finite free o_L-module and is annihilated by a power of $\mathfrak{J}$,*

(ii) The associated analytic Anderson $A(1)$-motive $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ admits a good model in the sense of 1.13 and 1.14.

Proof. First we show that (i) implies (ii). So let $(\underline{\mathcal{M}}, F^\circ)$ be given in accordance with (i). We claim that the π-adic completion

$$\widehat{\underline{\mathcal{M}}} = \underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi}$$

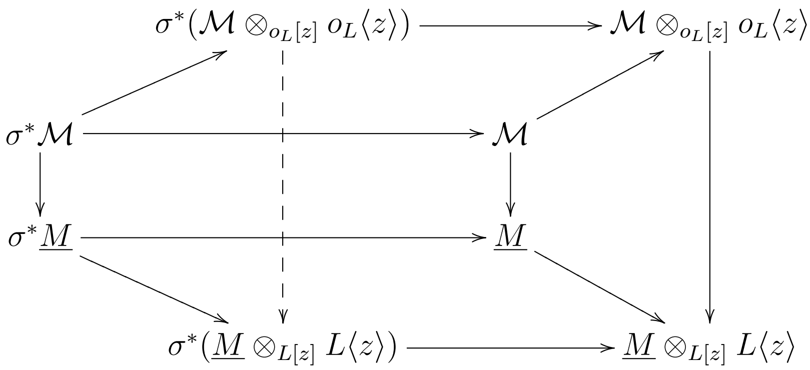
of $\underline{\mathcal{M}}$ is a good model for the analytic Anderson $A(1)$-motive $\underline{\mathcal{M}} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$. Imitating the arguments given in the last part of the proof of Theorem 1.18 shows that $\widehat{\underline{\mathcal{M}}}$ is locally free of finite rank over $A_{o_L,\pi}$, that $F^\circ \otimes \mathrm{id}$ is again injective and that $\operatorname{coker}(F^\circ \otimes \mathrm{id})$ is finite free over o_L and annihilated by a power of $\mathfrak{J}$; it is clear that $\widehat{\underline{\mathcal{M}}}[1/\pi]$ is indeed isomorphic to $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ as desired. It remains to see that $\widehat{\underline{\mathcal{M}}}$ is a good model. However, since the projection map $A_{o_L} \to A_\ell$ naturally factors via $A_{o_L,\pi}$, we have $\widehat{\underline{\mathcal{M}}}/\pi\widehat{\underline{\mathcal{M}}} \simeq \underline{\mathcal{M}}/\pi\underline{\mathcal{M}}$. Conversely, in order to show that (ii) implies (i), suppose that for a given Anderson A-motive $\underline{M}$, its analytification $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ admits a good model $\mathcal{M}'$ in the strong sense of 1.13. In particular, by 1.29, the F-module $\underline{M}$ over A_L admits a good model $\mathcal{M} \in \mathrm{FMod}(A_{o_L})$ in the sense of F-modules, and it remains to show that $\mathcal{M}$ is a good model of M in the strong sense, i.e., that $C = \operatorname{coker}(F_{\mathcal{M}})$ is a finite free o_L-module and is annihilated by a power of the ideal $\mathfrak{J} \subseteq A_{o_L}$. We start with the latter claim. Let $\mathfrak{J}^d \operatorname{coker}(F_{\underline{M}}) = 0$ say, and let $x \in \mathfrak{J}^d\mathcal{M}$. We need to show that $x \in \operatorname{im}(F_{\mathcal{M}})$. Since the good model of $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ as an F-module is uniquely determined up to unique isomorphism, by 1.29 we may assume that $\mathcal{M} \otimes_{A_{o_L}} A_{o_L,\pi}$ (which is necessarily isomorphic to $\mathcal{M}'$) is a good model of $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ in the strong sense. We remark that $\mathcal{M}$, being in particular a finite projective $o_L[z]$-module, is in fact finite free over $o_L[z]$ (see [70], p. 457), say with finite basis $\mathfrak{B}$; furthermore, recall that we have canonical isomorphisms

$$\mathcal{M} \otimes_{A_{o_L}} A_{o_L,\pi} \simeq \mathcal{M} \otimes_{o_L[z]} o_L\langle z\rangle, \qquad \mathcal{M} \otimes_{A_{o_L},\sigma} A_{o_L} \simeq \mathcal{M} \otimes_{o_L[z],\sigma} o_L[z];$$

in particular, we get

$$\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi] \simeq \underline{M} \otimes_{L[z]} L\langle z\rangle, \qquad \underline{M} \otimes_{L[z],\sigma} L[z] \simeq \underline{M} \otimes_{A_L,\sigma} A_L;$$

note that $A_{o_L,\pi}[1/\pi] \simeq (A_{o_L} \otimes_{o_L[z]} o_L\langle z\rangle)[1/\pi] \simeq A_L \otimes_{L[z]} L\langle z\rangle$. Fixing an isomorphism $\mathcal{M}[1/\pi] \simeq \underline{M}$ inside $\mathrm{FMod}(A_L)$, the $o_L[z]$-basis $\mathfrak{B}$ of $\mathcal{M}$ induces an $L[z]$-basis on $\mathcal{M}[1/\pi]$ and hence on $\underline{M}$, which in turn gives rise to a canonical induced basis on each remaining entry of the commutative diagram



where each arrow is injective. Our chosen element $x \in \mathfrak{J}^d\mathcal{M}$ in particular lies in $\mathfrak{J}^d\underline{M}$, so that there is a uniquely determined $y \in \sigma^*\underline{M}$ such that $x = F_{\underline{M}}(y)$. On the other hand, x gives rise to an element of $\mathcal{M} \otimes_{o_L[z]} o_L\langle z\rangle$; according to our assumption, we know that the cokernel of the map $\sigma^*(\mathcal{M} \otimes_{o_L[z]} o_L\langle z\rangle) \to \mathcal{M} \otimes_{o_L[z]} o_L\langle z\rangle$ is annihilated by a power of $\mathfrak{J}$, and since $\mathcal{M} \otimes_{o_L[z]} o_L\langle z\rangle$ is a good model in the strong sense of $\underline{M} \otimes_{L[z]} L\langle z\rangle$, we have seen in the proof of 1.18 that we may, in fact, take the power $\mathfrak{J}^d$; this implies that there is a uniquely determined element $y' \in \sigma^*(\mathcal{M} \otimes_{o_L[z]} o_L\langle z\rangle)$ which is mapped to (the image of) x in $\mathcal{M} \otimes_{o_L[z]} o_L\langle z\rangle$. Finally, since y' is necessarily mapped to (the image of) y via the dashed vertical arrow, writing y' in terms of the $o_L\langle z\rangle$-basis induced by $\mathfrak{B}$ and keeping track of linear combinations shows that the coefficients of y' have, in fact, to lie inside $o_L\langle z\rangle \cap L[z] = o_L[z]$, which proves that $\mathfrak{J}^d C = 0$. In particular, this means $(z-\zeta)^d C = 0$, i.e., C is finitely generated over $o_L[z]/(z-\zeta)^d$ and hence over o_L; it only remains to show that C is flat over o_L; indeed, consider the short exact sequence

$$0 \to \sigma^*\mathcal{M} \to \mathcal{M} \to C \to 0;$$

we see that applying the functor $\cdot \otimes_{o_L} \ell$ to this sequence exhibits $C/\pi C$ as a (necessarily flat) ℓ-vector space. On the other hand, by virtue of our hypothesis upon $\underline{M}$, applying the functor $\cdot \otimes_{o_L} L$ to the same sequence shows that $C[1/\pi]$ is a finite-dimensional L-vector space and therefore flat; it remains to see that C does not have π-torsion; in order to prove this, we need to see that $\pi x \in \mathrm{im}(F_{\mathcal{M}})$ for a given $x \in \mathcal{M}$ implies $x \in \mathrm{im}(F_{\mathcal{M}})$; we again use that $\mathcal{M}$ is finite free over $o_L[z]$ and remark that, since $\mathcal{M}$ is a good model of $\underline{M}$ as an F-module, the bottom horizontal arrow in the commutative diagram

$$\begin{array}{ccc} \sigma^*\mathcal{M} & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ \bar{\sigma}^*(\mathcal{M} \otimes_{o_L[z]} \ell[z]) & \longrightarrow & \mathcal{M} \otimes_{o_L[z]} \ell[z] \end{array}$$

is injective; furthermore, we remark that the vertical maps are surjective and that in the upper (resp., bottom) row both modules are finite free over $o_L[z]$ (resp., over $\ell[z]$) of the same rank. From $\pi x \in \mathrm{im}(F_{\mathcal{M}})$ it follows that there is a uniquely determined $y \in \sigma^*\mathcal{M}$ such that $\pi x = F_{\mathcal{M}}(y)$; since πx goes to zero under the right-hand projection, necessarily y has to go to zero via the left-hand projection; a chosen $o_L[z]$-basis of $\mathcal{M}$ induces bases of each of the other entries of the above diagram; keeping track of coefficients in linear combinations one verifies that $y \in \pi\sigma^*\mathcal{M}$; finally, since $\mathcal{M}$ is torsion-free, we obtain $x = F_{\mathcal{M}}(y)$, as desired; so, for example, by [9], 2.6/1, we may conclude that C is flat over o_L. □

Using that the canonical map $A_{o_L,\varepsilon} \to A_{o_L,(\varepsilon,\pi)}$ is an isomorphism, we obtain

Corollary 1.31. *Let $\underline{M}$ be an Anderson A-motive such that* $\mathrm{coker}(F_{\underline{M}})$ *is annihilated by $\mathfrak{J}^d$ say. Then the following assertions are equivalent:*

(i) $\underline{M}$ admits a good model $\underline{\mathcal{M}}$, i.e., there is an object $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ such that $\mathrm{coker}(F_{\underline{\mathcal{M}}})$ *is a finite free o_L-module and is annihilated by a power of $\mathfrak{J}^d$, together with an isomorphism $\underline{\mathcal{M}}[1/\pi] \simeq \underline{M}$ inside* $\mathrm{FMod}(A_L)$*;*

(ii) There is

- *a local shtuka* $\hat{M}$ *at* ε *such that* $\operatorname{coker}(F_{\hat{M}})$ *is a finite free* o_L*-module and is annihilated by* $\mathfrak{J}^d$*,*
- *an isomorphism*

$$\underline{M} \otimes_{A_L} A_{o_L,\varepsilon}[1/\pi] \simeq \hat{M}[1/\pi]$$

inside $\mathrm{FMod}(A_{o_L,\varepsilon}[1/\pi])$.

In particular, we obtain a one-to-one correspondence between (isomorphism classes of) good models of $\underline{M}$ *and (isomorphism classes of) pairs* $(\hat{M}, f)$ *consisting of a local shtuka* $\hat{M}$ *at* ε *and an isomorphism* $f\colon \underline{M} \otimes_{A_L} A_{o_L,\varepsilon}[1/\pi] \xrightarrow{\simeq} \hat{M}[1/\pi]$ *inside* $\mathrm{FMod}(A_{o_L,\varepsilon}[1/\pi])$. □

2 The monodromy of certain extension structures

2.1 Crystalline extensions attached to elliptic curves of supersingular reduction

Let K be a mixed-characteristic complete discretely valued field and let $o_K \subseteq K$ be its valuation ring; let $\mathfrak{m}_K = (\pi)$ be the sole maximal ideal of o_K where $\pi = \pi_K \in o_K$ is a fixed uniformizer of K. The characteristic of the residue field $k = o_K/\mathfrak{m}_K$ is given by a prime number p. We assume k to be a *perfect* (not necessarily finite) extension of the prime field $\mathbb{F}_p$. We fix an algebraic closure K^{alg}/K and let $G_K = \text{Gal}(K^{\text{alg}}/K)$.

Let $F = W(k)[1/p]$ where $W(k)$ is the ring of Witt vectors over the perfect field k; let $\sigma\colon F \to F$ be the p-Frobenius lift. The field F is a complete discretely valued field with uniformizer p which naturally embeds into K; the extension K/F is finite, so that K^{alg} gives rise to an algebraic closure of F for which we write F^{alg}; we denote by F^{ur} the compositum of all finite unramified subextensions of F^{alg}/F; the valuation of F extends uniquely to give a valuation on F^{ur}, and one can show that the residue field of the completion $\widehat{F^{\text{ur}}}$ is an algebraic closure of k which we denote by k^{alg}. We have $\widehat{F^{\text{ur}}} = W(k^{\text{alg}})[1/p]$, i.e., the extension k^{alg}/k on the level of residue fields is mirrored by the extension $\widehat{F^{\text{ur}}}/F$.

2.1.1 Elliptic curves and p-adic Galois representations

Let E be an elliptic curve over K, i.e., E is a smooth projective curve over K which is isomorphic over K to $\text{Proj}(K[u,v,w]/f) \subseteq \mathbb{P}^2_K$ where the homogeneous polynomial $f \in K[u,v,w]$ is given by

$$f = v^2w + a_1uvw + a_3vw^2 - u^3 - a_2u^2w - a_4uw^2 - a_6w^3$$

for suitable $a_1, a_2, a_3, a_4, a_6 \in K$. It is well-known that, fixing a rational point $e \in E(K)$, the pair (E, e) has the structure of a geometrically integral commutative K-group scheme with unit section given by e, in such a way that for every pair of points $x, y \in E(K^{\mathrm{alg}})$ their sum $z = x + y$ in the abelian group $E(K^{\mathrm{alg}})$ is characterized by the linear equivalence relation of divisors $(z) + (e) \sim (x) + (y)$ on $E_{K^{\mathrm{alg}}}$; see [54], 9.4.

For every $n \geq 1$ the kernel $E[p^n]$ of the K-morphism $[p^n]\colon E \to E$ (multiplication by p^n) is a finite K-group scheme of order p^{2n}, and the kernel $E[p^n](K^{\mathrm{alg}})$ of the map of abstract abelian groups $p^n\colon E(K^{\mathrm{alg}}) \to E(K^{\mathrm{alg}})$ is a finite abelian group which is isomorphic to $(\mathbb{Z}/p^n)^2$ as a $\mathbb{Z}/p^n$-module (see [60], p. 64); there is a natural continuous action of the Galois group G_K on $E[p^n](K^{\mathrm{alg}})$ which is given coordinatewise. The inverse limit of these groups,

$$T_p(E) = \varprojlim_n E[p^n](K^{\mathrm{alg}}),$$

called the *p-adic Tate module* of E, is a free $\mathbb{Z}_p$-module of rank 2 and carries a natural induced continuous action of the group G_K; see [71], III.7. The 2-dimensional $\mathbb{Q}_p$-vector space

$$V_p(E) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(E)$$

together with its induced continuous G_K-action is a *p-adic representation* of G_K.

By virtue of a suitable change of variables we may assume that the coefficients $a_i \in K$ of the Weierstraß equation for E given by the polynomial $f \in K[u, v, w]$ lie inside o_K; consequently the discriminant $\Delta \in \mathbb{Z}[a_1, ..., a_6]$ associated to f will also have non-negative valuation; adjusting the a_i further, we may assume that the resulting 2-dimensional regular projective o_K-scheme $\mathscr{E} = \mathrm{Proj}(o_K[u, v, w]/f) \subseteq \mathbb{P}^2_{o_K}$ is *minimal* in the sense that the valuation $v_K(\Delta) \geq 0$ of $\Delta \in o_K - \{0\}/o_K^\times$ becomes minimal; note that mod $o_K^\times$ the discriminant Δ only depends on $\mathscr{E}$; see [54], 10.2.

We assume E to be of *good reduction*, which is to say that the minimal discriminant Δ attached to E is a unit in o_K. The minimal Weierstraß model $\mathscr{E} \subseteq \mathbb{P}^2_{o_K}$ discussed above is then smooth over o_K; in fact, $\mathscr{E}$ is a Néron model for E (see [72], IV.6.3), and the special fiber $\mathscr{E}_0 = \mathscr{E} \otimes_{o_K} k$ is an elliptic curve over k which corresponds to the Weierstraß equation given by the reduced polynomial $\bar{f} \in k[u, v, w]$.

We further assume that the reduced elliptic curve $\mathscr{E}_0$ is *supersingular*, which can be characterized by saying that the abelian group $\mathscr{E}_0[p](k^{\mathrm{alg}})$ is trivial, i.e., the finite k-group scheme $\mathscr{E}_0[p]$ has no geometric points of order p (see [53]).

2.1.2 Crystalline and semi-stable Fontaine theory

Recall ([3], [15], [27]) that inside the category $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ of p-adic representations of the Galois group G_K there are several arithmetically significant full subcategories.

We are particularly interested in two of them, namely the *crystalline* and the *semi-stable* p-adic representations. To begin with, we recall that there is a functor

$$D_{\mathrm{st}} \colon \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to MF_K(\varphi, N), \qquad V \mapsto (V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{st}})^{G_K},$$

into the additive category of *filtered* (φ, N)*-modules* over the field F; here $\mathbf{B}_{\mathrm{st}}$ denotes the semi-stable period ring from Fontaine theory ([3]).

In order to explain the category $MF_K(\varphi, N)$, let $F[\varphi, N]$ be the skew polynomial ring over F with the commutation rules $\varphi f = \sigma(f)\varphi$, $Nf = fN$ for all $f \in F$, and $N\varphi = p\varphi N$; the first two relations can be rephrased by saying that φ, the *Frobenius*, acts σ-semi-linearly and N, the *monodromy operator*, acts F-linearly. A *filtered* (φ, N)*-module* is a pair $(D, (\mathrm{Fil}^i D_K)_{i\in\mathbb{Z}})$ consisting of a left $F[\varphi, N]$-module D which is finite-dimensional as an F-vector space and on which φ acts bijectively, together with an exhaustive and separated descending filtration $(\mathrm{Fil}^i D_K)_{i\in\mathbb{Z}}$ of $D_K = D \otimes_F K$ by K-subspaces; a *morphism* of filtered (φ, N)-modules is a map $D \to D'$ of left $F[\varphi, N]$-modules such that the induced map $D_K \to D'_K$ is compatible with the filtrations.

Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. One calls V a $\mathbf{B}_{\mathrm{st}}$*-admissible* or, equivalently: a *semi-stable* p-adic representation if the natural map $(V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{st}})^{G_K} \otimes_F \mathbf{B}_{\mathrm{st}} \to V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{st}}$ is an isomorphism; the category $\mathrm{Rep}_{\mathrm{st}}(G_K)$ of semi-stable p-adic representations of G_K is an abelian full subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, and the restriction of D_{st} to this category is additive and fully faithful. Inside $\mathrm{Rep}_{\mathrm{st}}(G_K)$ there is an abelian full subcategory $\mathrm{Rep}_{\mathrm{cris}}(G_K)$ whose objects are called *crystalline* p-adic representations of G_K, and which can be characterized as follows: a semi-stable p-adic representation V is *crystalline* if and only if N acts trivially on $D_{\mathrm{st}}(V)$. There is a functor

$$D_{\mathrm{cris}} \colon \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to MF_K(\varphi), \qquad V \mapsto (V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{cris}})^{G_K},$$

where $MF_K(\varphi)$ denotes the full subcategory of $MF_K(\varphi, N)$ consisting of those filtered (φ, N)-modules on which N acts trivially, and where $\mathbf{B}_{\mathrm{cris}}$ denotes the crystalline period ring from Fontaine theory ([3]). Similarly as described above, a p-adic representation V is crystalline if and only if V is $\mathbf{B}_{\mathrm{cris}}$-admissible, which is to say that the natural map $(V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{cris}})^{G_K} \otimes_F \mathbf{B}_{\mathrm{cris}} \to V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{cris}}$ is an isomorphism. If $V \in \mathrm{Rep}_{\mathrm{st}}(G_K)$ is crystalline then $D_{\mathrm{cris}}(V) = D_{\mathrm{st}}(V)$.

For example, since our elliptic curve E is of good reduction, the associated p-adic representation $V_p(E)$ is crystalline; see [3], II.3.2.

Our aim is to show the following

Proposition 2.1. *Let*

$$0 \to \mathbb{Q}_p \to V \to V_p(E)^\vee \to 0$$

be an extension inside the category $\mathrm{Rep}_{\mathrm{st}}(G_K)$*. Then* V *is crystalline.*

We refer to section (3.2) for some general remarks on extensions of G_K-representations.

2.1.3 Isocrystals

Via forgetting about the filtration, any filtered φ-module $(D, \mathrm{Fil}^\bullet D_K) \in MF_K(\varphi)$ gives rise to a φ-*isocrystal* over k which means that D is a module over the skew polynomial ring $F[\varphi]$ over $F = W(k)[1/p]$ with the commutation rule $\varphi f = \sigma(f)\varphi$ for all $f \in F$, that D is finite-dimensional as an F-vector space, and that φ acts bijectively on D.

Example 2.2. Fixing integers $m, n \in \mathbb{Z}$, $n > 0$, the F-vector space

$$D_{m,n} = F[\varphi]/F[\varphi](\varphi^n - p^m)$$

gives rise to a φ-isocrystal over k in the following manner: the polynomial $\varphi^n - p^m \in F[\varphi]$ is clearly of degree n, and via the classical argument one verifies that the elements $\varphi^0 = 1$, φ, ..., φ^{n-1} constitute an F-basis of $D_{m,n}$, so that $\dim_F D_{m,n} = n$. Multiplication from the left with (the image of) φ gives rise to a map of abelian groups $D_{m,n} \to D_{m,n}$ which, according to the commutation rule $\varphi f = \sigma(f)\varphi$ for $f \in F$, is semi-linear with respect to the p-Frobenius lift $\sigma\colon F \to F$; furthermore, by virtue of

$$\varphi \sum_{i=0}^{n-1} \alpha_i \varphi^i = \sum_{i=0}^{n-1} \sigma(\alpha_i)\varphi^{i+1} = p^m \sigma(\alpha_{n-1}) + \sigma(\alpha_1)\varphi + ... + \sigma(\alpha_{n-2})\varphi^{n-1},$$

the map $\varphi(\cdot)\colon D_{m,n} \to D_{m,n}$ is, with respect to the F-basis $(\varphi^0, \varphi^1, ..., \varphi^{n-1})$, described ($\sigma$-semi-linearly!) by the matrix

$$\mathbf{A}_{m,n} = \left(\begin{smallmatrix} 0 & p^m \\ \mathrm{Id}_{n-1} & 0 \end{smallmatrix} \right) \in F^{n \times n};$$

in fact, $\mathbf{A}_{m,n}$ is invertible and one has $\mathbf{A}_{m,n}^n = p^m \mathrm{Id}_n$; we may summarize that the obtained $F[\varphi]$-module $D_{m,n}$ is a φ-isocrystal over k, and that

$$p^m \mathbf{A}_{m,n}^{-1} = \mathbf{A}_{m,n}^{n-1} = \left(\begin{smallmatrix} 0 & p^m \mathrm{Id}_{n-1} \\ 1 & 0 \end{smallmatrix} \right).$$

□

For the following Lemma, recall that given a φ-isocrystal D over k, an F-subspace $D' \subseteq D$ is called a *sub-isocrystal* of D if D' is also an $F[\varphi]$-submodule of D and if φ acts bijectively on D'.

Lemma 2.3. *For* $(m,n)=1$ *the* φ*-isocrystal* $\widehat{F^{\mathrm{ur}}}[\varphi]/\widehat{F^{\mathrm{ur}}}[\varphi](\varphi^n - p^m)$ *over* k^{alg} *is* simple*, i.e., it admits no proper sub-isocrystals* $\neq 0$.

Proof. See [76], 6.27. □

We are particularly interested in the filtered φ-module

$$\underline{D} = D_{\mathrm{st}}(V_p(E)) \in MF_K(\varphi)$$

which, forgetting about the filtration, gives rise to a φ-isocrystal over k. Since our elliptic curve E has supersingular reduction, it is well-known ([3], [53]) that the Newton-polygon of the isocrystal $\underline{D}$ consists of two consecutive segments of horizontal distance 1 and slope 1/2. Let us briefly discuss what this means. Given a φ-isocrystal D over k, let $\varphi_D\colon D \to D$ be the σ-semi-linear automorphism by which φ acts on D; then the $\widehat{F^{\mathrm{ur}}}$-vector space $D \otimes_F \widehat{F^{\mathrm{ur}}}$ acquires a natural $\widehat{F^{\mathrm{ur}}}[\varphi]$-module structure such that φ acts via $\varphi_D \otimes \sigma$, making $D \otimes_F \widehat{F^{\mathrm{ur}}}$ into a φ-isocrystal over k^{alg}. Recall the

Theorem 2.4 (Dieudonné-Manin – [57]). *Let* D *be a nontrivial* φ*-isocrystal over* k*, and let* $\widehat{D} = D \otimes_F \widehat{F^{\mathrm{ur}}}$ *be the associated* φ*-isocrystal over* k^{alg}*. Then there is a uniquely determined finite ascending sequence of rational numbers* $m_1/n_1 < \ldots < m_N/n_N$ *such that* $n_\nu > 0$*,* $(m_\nu, n_\nu) = 1$ *for all* ν *and*

$$\widehat{D} \simeq (\widehat{D}_{m_1,n_1} \oplus \ldots \oplus \widehat{D}_{m_1,n_1}) \oplus \ldots \oplus (\widehat{D}_{m_N,n_N} \oplus \ldots \oplus \widehat{D}_{m_N,n_N})$$

where $\widehat{D}_{m_\nu,n_\nu} = \widehat{F^{\mathrm{ur}}}[\varphi]/\widehat{F^{\mathrm{ur}}}[\varphi](\varphi^{n_\nu} - p^{m_\nu})$*, with the structure of* φ*-isocrystal discussed in 2.2, and where* $\widehat{D}_{m_\nu,n_\nu}$ *occurs* e_{m_ν,n_ν} *times on the right-hand side.*

By what we have seen in 2.2 it is clear that $\dim_{\widehat{F^{\mathrm{ur}}}} \widehat{D}_{m_\nu,n_\nu}^{\oplus e_{m_\nu,n_\nu}} = n_\nu e_{m_\nu,n_\nu}$.

Given a nontrivial φ-isocrystal D over k, the *Newton polygon* of D is the convex polygon with leftmost endpoint $(0,0)$ and consisting of $m_\nu e_{m_\nu,n_\nu}$ consecutive segments of horizontal distance 1 and slope m_ν/n_ν; for all this, see [15], [47], [76].

In particular, returning to the φ-isocrystal $\underline{D} = D_{\mathrm{st}}(V_p(E))$ over k attached to our elliptic curve E, we may conclude that

$$\underline{D} \otimes_F \widehat{F^{\mathrm{ur}}} \simeq \widehat{F^{\mathrm{ur}}}[\varphi]/\widehat{F^{\mathrm{ur}}}[\varphi](\varphi^2 - p)$$

as φ-isocrystals over k^{alg}, i.e., fixing the obvious basis of the right-hand side discussed in 2.2, the action of φ is given σ-semi-linearly by the matrix $\left(\begin{smallmatrix} 0 & p \\ 1 & 0 \end{smallmatrix}\right)$; by virtue of 2.3 we see that $\underline{D} \otimes_F \widehat{F^{\mathrm{ur}}}$ is a simple φ-isocrystal over k^{alg}.

Recall ([15], [27]) that the functor D_{st} respects duals, which in particular means that

$$D_{\mathrm{st}}(V_p(E)^\vee) \simeq \underline{D}^\vee;$$

here we remark that for a φ-isocrystal D over k its F-linear dual $D^\vee = \mathrm{Hom}_F(D, F)$ is made into a φ-isocrystal over k by letting φ act via $D^\vee \to D^\vee$, $\alpha \mapsto \sigma \circ \alpha \circ \varphi_D^{-1}$ where $\varphi_D \colon D \to D$ is the σ-semi-linear automorphism by which φ acts on D. For example, if $D = D_{m,n}$ for $m, n \in \mathbb{Z}$, $n > 0$, then with respect to the dual basis $(1^\vee, \varphi^\vee, ..., (\varphi^{n-1})^\vee)$, the φ-action $D_{m,n}^\vee \to D_{m,n}^\vee$ is described σ-semi-linearly by the matrix $\mathbf{A}_{-m,n}$ from 2.2; this follows from the relation $p^m \mathbf{A}_{m,n}^{-1} = \mathbf{A}_{m,n}^{n-1}$ which is to say that $\varphi_{D_{m,n}}^{-1}(\varphi^0) = p^{-m}\varphi^{n-1}$ and $\varphi_{D_{m,n}}^{-1}(\varphi^j) = \varphi^{j-1}$ for $j \geq 1$. We may conclude that $D_{m,n}^\vee \simeq D_{-m,n}$; the same is of course true for φ-isocrystals over k^{alg}.

Since $\underline{D}$ is finite-dimensional over F, there is a natural isomorphism $\underline{D}^\vee \otimes_F \widehat{F^{\mathrm{ur}}} \simeq (\underline{D} \otimes_F \widehat{F^{\mathrm{ur}}})^\vee$ of $\widehat{F^{\mathrm{ur}}}$-vector spaces which is, in fact, an isomorphism of φ-isocrystals over k^{alg}. In particular, since we have

$$(\underline{D} \otimes_F \widehat{F^{\mathrm{ur}}})^\vee \simeq \widehat{D}_{-1,2} = \widehat{F^{\mathrm{ur}}}[\varphi]/\widehat{F^{\mathrm{ur}}}[\varphi](\varphi^2 - p^{-1}),$$

it follows that $\underline{D}^\vee \otimes_F \widehat{F^{\mathrm{ur}}}$ is again a simple φ-isocrystal over k^{alg}.

In a next step, we aim at showing that, in fact, also $\underline{D}^\vee$ is simple as a φ-isocrystal over k. Indeed, the functor $\cdot \otimes_F \widehat{F^{\mathrm{ur}}}$ from F-vector spaces to $\widehat{F^{\mathrm{ur}}}$-vector spaces restricts to a left-exact functor

$$\cdot \otimes_F \widehat{F^{\mathrm{ur}}} \colon (\varphi\text{-isocrystals over } k) \to (\varphi\text{-isocrystals over } k^{\mathrm{alg}}),$$

so that for any sub-isocrystal $D' \subseteq \underline{D}^\vee$ over k we obtain a sub-isocrystal $D' \otimes_F \widehat{F^{\mathrm{ur}}} \subseteq \underline{D}^\vee \otimes_F \widehat{F^{\mathrm{ur}}}$ over k^{alg}. Since $\widehat{F^{\mathrm{ur}}}$ is faithfully flat over F, $\underline{D}^\vee \otimes_F \widehat{F^{\mathrm{ur}}}$ being simple indeed implies that $\underline{D}^\vee$ is simple.

Proof of Proposition 2.1. Suppose we are given a short exact sequence $0 \to \mathbb{Q}_p \to V \to V_p(E)^\vee \to 0$ inside the abelian category $\mathrm{Rep}_{\mathrm{st}}(G_K)$. Applying the exact functor D_{st} to this sequence yields a short exact sequence of filtered (φ, N)-modules

$$0 \to F \to D_{\mathrm{st}}(V) \to \underline{D}^\vee \to 0$$

where N acts trivially on F and $\underline{D}^\vee$. Since the maps in this sequence are in particular compatible with the action of N, we obtain a commutative diagram of F-vector spaces with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \xrightarrow{\ i\ } & D_{\mathrm{st}}(V) & \xrightarrow{\ \mathrm{pr}\ } & \underline{D}^\vee & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & F & \xrightarrow{\ i\ } & D_{\mathrm{st}}(V) & \xrightarrow{\ \mathrm{pr}\ } & \underline{D}^\vee & \longrightarrow & 0
\end{array}$$

which by virtue of the Snake Lemma gives rise to an F-linear map $d\colon \underline{D}^\vee \to F$ satisfying $N = i \circ d \circ \mathrm{pr}$; for this, recall that d is defined by the following diagram chase: for a given $x \in \underline{D}^\vee$, choose any lift $x' \in D_{\mathrm{st}}(V)$ along pr; then $d(x)$ is defined to be the uniquely determined $y \in F$ such that $i(y) = N(x')$. From the relation $N\varphi = p\varphi N$ it follows that, by construction of d, we have $d \circ \varphi_{\underline{D}^\vee} = p\varphi_F \circ d$, which means that $d\colon \underline{D}^\vee \to F$ respects the φ-actions only up to the factor p. This in turn immediately implies that $d(x) = 0$ if and only if $d(\varphi(x)) = 0$, i.e., the φ-action on $\underline{D}$ restricts to an automorphism of the abelian group $\ker(d)$, which, in fact, is σ-semi-linear; we may conclude that $\ker(d)$ becomes a sub-isocrystal of $\underline{D}^\vee$ in this way. An obvious comparison of F-dimensions shows that $\ker(d)$ has to be nontrivial. However, the φ-isocrystal $\underline{D}^\vee$ is simple, i.e., $\ker(d) = \underline{D}^\vee$, and we may summarize that $N = 0$ on $D_{\mathrm{st}}(V)$. □

2.2 A non-crystalline Dieudonné module in equal characteristic

In this section we discuss a situation analogous to that in section (2.1) and construct an example which will show that, stressing the analogy between Drinfeld modules and elliptic curves ([21], [31], [36]), a natural analogue of 2.1 turns out to be false. More specifically, we will show that there is a short exact sequence of Dieudonné modules ([52]) exhibiting a non-crystalline extension structure where one would expect a local shtuka.

2.2.1 Mixed Drinfeld characteristic

Retaining the notation from (1.1), let L be a complete discretely valued field containing the finite field $\mathbb{F}$ fixed in the beginning, and let $o_L \subseteq L$ be its valuation ring, with sole maximal ideal $\mathfrak{m}_L = (\pi)$ where $\pi \in o_L$ is a fixed uniformizer of L; we denote by $v = v_\pi = \mathrm{ord}_\pi(\cdot)$ the discrete valuation on L normalized by $v(\pi) = 1$; the residue field $\ell = o_L/\mathfrak{m}_L$ is always supposed to be a perfect extension of $\mathbb{F}$; we recall that the choice of π identifies o_L with $\ell[\![\pi]\!]$ and L with $\ell(\!(\pi)\!)$. Let $G_L = \mathrm{Gal}(L^{\mathrm{sep}}/L)$ be the absolute Galois group of L where L^{sep}/L is a fixed separable closure; furthermore, let ℓ^{alg}/ℓ be a fixed algebraic closure of the perfect field ℓ.

We will work exclusively with Drinfeld $\mathbb{F}[z]$-modules, i.e., in the notation of (1.1) we specify $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$ and let $A = \Gamma(\mathbb{P}^1_{\mathbb{F}} - \{\infty\}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}}}) = \mathbb{F}[z]$ where the point ∞ is defined by $V(1/z) \subseteq \mathrm{Spec}(\mathbb{F}[1/z])$.

The characteristic monomorphism of $\mathbb{F}$-algebras $c^*\colon \mathbb{F}[z] \to o_L$ is clearly determined by the image $\zeta \in o_L$ of z; let us specify that $\pi \mid \zeta$; consequently we have $0 < \mathrm{ord}_\pi(\zeta) < \infty$, and the image of z in the residue field ℓ will be zero; we may conclude that the residual characteristic place is given by $\varepsilon = z\mathbb{F}[z]$, and that c^* induces an extension of complete discretely valued fields $L/\mathbb{F}(\!(\zeta)\!)$.

So far, we have the following analogies between the world of mixed-characteristic complete valuation rings and our scenery of *mixed Drinfeld characteristic*:

$o_K/\mathbb{Z}_p$ p-adically complete extension	$o_L/\mathbb{F}[\![\zeta]\!]$ ζ-adically complete extension
$\mathbb{Z} \hookrightarrow o_K$ natural map	$c^*\colon A \hookrightarrow o_L$ choice of characteristic map
$p \in p\mathbb{Z}_p \subseteq \mathfrak{m}_K$	assumption $c^*(z) \in \mathfrak{m}_L$
$(p) \subseteq \mathbb{Z}$ residue characteristic	$(z) \subseteq A$ kernel of $A \hookrightarrow o_L \to \ell$
$\mathbb{Z} \hookrightarrow \mathbb{Z}_{(p)} \hookrightarrow o_K$	$A \hookrightarrow A_{(z)} \hookrightarrow o_L$
$\widehat{\mathbb{Z}_{(p)}} \cong \mathbb{Z}_p$	$\widehat{A_{(z)}} \cong \mathbb{F}[\![z]\!]$

2.2.2 Tate uniformization of Drinfeld modules, and supersingular reduction

Recall that the skew polynomial ring $L[\tau]$ with the commutation rule $\tau\alpha = \alpha^r\tau$ for $\alpha \in L$ corresponds to those L-endomorphisms of the additive group scheme $\mathbb{G}_{a,L} = \mathrm{Spec}(L[x])$ which are $\mathbb{F}$-linear; especially, the distinguished element τ corresponds to the r-Frobenius $\mathbb{G}_{a,L} \to \mathbb{G}_{a,L}$ defined by $x \mapsto x^r$; see [59], 1.3.

Definition 2.5. *A* Drinfeld ($\mathbb{F}[z]$-)module *over L ([21], [31], [36]) is a ring homomorphism $\varphi\colon \mathbb{F}[z] \to L[\tau]$ such that* $\mathrm{im}(\varphi) \nsubseteq L$ *and such that the triangle*

$$\begin{array}{ccc} \mathbb{F}[z] & \xrightarrow{\quad c^* \quad} & L \\ & {\scriptstyle\varphi}\searrow \qquad \nearrow{\scriptstyle\partial_L} & \\ & L[\tau] & \end{array}$$

is commutative.

Here the ring homomorphism $\partial_L\colon L[\tau] \to L$ is given by $\sum_\nu \alpha_\nu\tau^\nu \mapsto \alpha_0$. In [21] Drinfeld modules are called *elliptic* ($\mathbb{F}[z]$-)*modules.* In the future we shall suppress the prefix "$\mathbb{F}[z]$-", for the reason that the underlying curve and characteristic map are fixed.

A *homomorphism* $\varphi \to \varphi'$ of Drinfeld modules $\varphi\colon \mathbb{F}[z] \to L[\tau]$, $\varphi'\colon \mathbb{F}[z] \to L[\tau]$ over L is defined to be a skew polynomial $\lambda \in L[\tau]$ such that $\lambda\varphi_f = \varphi'_f\lambda$ for all $f \in \mathbb{F}[z]$. A homomorphism $\lambda\colon \varphi \to \varphi'$ is called an *isogeny* (resp., an *isomorphism*) if $\lambda \neq 0$ (resp., if $\deg_\tau(\lambda) = 0$), and then φ and φ' are called *isogenous* (resp., *isomorphic*); from the defining relations it follows directly that there is an isogeny $\varphi \to \varphi'$ only if $\mathrm{rk}(\varphi) = \mathrm{rk}(\varphi')$; if there is an isomorphism $\varphi \to \varphi'$ then one writes $\varphi \simeq \varphi'$. Being isomorphic is clearly an equivalence relation of Drinfeld modules over L, and by virtue of the existence of dual isogenies ([59], 3.2) the same is true for the relation of being isogenous; furthermore, the isogeny relation clearly preserves the rank of representatives; in particular, so does the isomorphy relation.

It is clear that a Drinfeld module $\varphi\colon \mathbb{F}[z] \to L[\tau]$ is already determined by the image

$$\varphi_z = \zeta + \alpha_1\tau + ... + \alpha_n\tau^n \in L[\tau]$$

of z; in fact, the number $n = \deg_\tau(\varphi_z)$ is always positive and equals the *rank* of φ; the latter is characterized by the relations $\deg_\tau(\varphi_f) = -n \cdot \mathrm{ord}_{1/z}(f)$ for all $f \in \mathbb{F}[z]$; here $\mathrm{ord}_{1/z}$ denotes the valuation at ∞ of the function field $\mathbb{F}(z)$ of the underlying curve $\mathbb{P}^1_{\mathbb{F}}$. Note that, since $c^*\colon \mathbb{F}[z] \to o_L$ is supposed to be injective, the *characteristic* of a Drinfeld module $\varphi\colon \mathbb{F}[z] \to L[\tau]$ over L, i.e., the point of $\mathbb{P}^1_{\mathbb{F}}$ associated to the ideal $\ker(\partial_L \circ \varphi) \subseteq \mathbb{F}[z]$, always equals the generic point in the situation at hand; in this sense, every Drinfeld module over L considered here is of *generic characteristic*; see [21], 2A.

Given a Drinfeld module $\varphi\colon \mathbb{F}[z] \to L[\tau]$, the abelian group L acquires an additional $\mathbb{F}[z]$-module structure by letting $f \in \mathbb{F}[z]$ act on $x \in L$ via $\varphi_f x = \zeta x + \sum_{\mu=1}^m \alpha_\mu x^{r^\mu}$ where $\varphi_f = \zeta + \sum_{\mu=1}^m \alpha_\mu\tau^\mu$, and the condition $\mathrm{im}(\varphi) \nsubseteq L$ assures that this module structure differs from the structure of an $\mathbb{F}[z]$-algebra on L induced by $c^*\colon \mathbb{F}[z] \to L$. In the sequel we will denote by $\varphi(L)$ the $\mathbb{F}[z]$-module with underlying abelian group L and $\mathbb{F}[z]$-action induced by φ; note that in the described way we get an $\mathbb{F}[z]$-module structure via φ on every L-algebra R; we denote this $\mathbb{F}[z]$-module by $\varphi(R)$.

We have already distinguished the place $\varepsilon = \{z = 0\}$ of our underlying curve $\mathbb{P}^1_{\mathbb{F}}$. We will be particularly interested in the behavior of a Drinfeld module $\varphi\colon \mathbb{F}[z] \to L[\tau]$ over L at ε. Let L^{alg}/L be an algebraic closure of L. For every $n \geq 1$ the equation $z^n x = 0$ $(x \in \varphi(L^{\mathrm{alg}}))$ is a separable polynomial equation over L. Therefore, let

$$\varphi[z^n](L^{\mathrm{sep}}) = \{x \in \varphi(L^{\mathrm{sep}}), \quad z^n x = 0\}$$

where L^{sep}/L is our fixed separable closure. It is obvious that the group G_L acts naturally on $\varphi[z^n](L^{\mathrm{sep}})$, and by [59], 2.5(a), we have

$$\varphi[z^n](L^{\mathrm{sep}}) \simeq (\mathbb{F}[z]/z^n)^{\mathrm{rk}(\varphi)}.$$

There are natural transition maps $\varphi[z^{n+1}](L^{\mathrm{sep}}) \to \varphi[z^n](L^{\mathrm{sep}})$, given by scalar multiplication with z, which are G_L-equivariant, and the projective limit

$$T_z\varphi = \varprojlim_n \varphi[z^n](L^{\mathrm{sep}})$$

is a free $\mathbb{F}[\![z]\!]$-module of rank $\mathrm{rk}(\varphi)$ which carries a natural action of G_L. We call $T_z\varphi$ the *z-adic Tate module* of φ. We further let $V_z\varphi = T_z\varphi \otimes_{\mathbb{F}[\![z]\!]} \mathbb{F}(\!(z)\!)$; see [59], 3.3.

Similarly as in 2.5, one defines a *Drinfeld* ($\mathbb{F}[z]$-)*module* over ℓ to be a ring homomorphism $\psi\colon \mathbb{F}[z] \to \ell[\tau]$ such that $\mathrm{im}(\psi) \nsubseteq \ell$ and such that $\partial_\ell \circ \psi\colon \mathbb{F}[z] \to \ell$ equals the composition $\mathbb{F}[z] \to o_L \to \ell$ induced by c^*.

Note that every Drinfeld module $\psi\colon \mathbb{F}[z] \to \ell[\tau]$ over ℓ considered here is of *finite characteristic* $\ker(\partial_\ell \circ \psi) = \varepsilon = z\mathbb{F}[z]$.

For example, let $\varphi\colon \mathbb{F}[z] \to L[\tau]$ be a Drinfeld module over L such that $\mathrm{im}(\varphi) \subseteq o_L[\tau] \subseteq L[\tau]$. Then, denoting by $\bar{f}$ the reduction in $\ell[\tau]$ of a skew polynomial $f \in o_L[\tau]$, one can ask whether the assignment $z \mapsto \bar{\varphi}_z$ defines a Drinfeld module over ℓ:

Definition 2.6. *A Drinfeld module $\varphi\colon \mathbb{F}[z] \to L[\tau]$ is called* stable *if there is a Drinfeld module $\psi\colon \mathbb{F}[z] \to L[\tau]$, called an* integral model *for φ, such that*

- *$\varphi \simeq \psi$,*
- *$\psi_f \in o_L[\tau]$ for every $f \in \mathbb{F}[z]$,*
- *the ring homomorphism $\mathbb{F}[z] \to \ell[\tau]$, $z \mapsto \bar{\psi}_z$, defines a Drinfeld module over ℓ.*

If $\varphi\colon \mathbb{F}[z] \to L[\tau]$ is stable, with a suitable integral model ψ as in the previous Definition, then $\mathrm{rk}(\bar{\psi}) \leq \mathrm{rk}(\varphi)$; if moreover ψ can be chosen in such a way that equality holds then φ is said to be of *good reduction.* For the following Theorem, recall that given a Drinfeld module $\varphi\colon \mathbb{F}[z] \to L[\tau]$, a *lattice* inside φ is defined to be a finite projective $\mathbb{F}[z]$-submodule $\Lambda \subseteq \varphi(L^{\mathrm{sep}})$ such that every ball of finite radius inside L^{sep} contains at most finitely many points of Λ and such that $\rho(\Lambda) \subseteq \Lambda$ for every $\rho \in G_L$.

Theorem 2.7 (Tate Uniformization – [21], 7.2). *There is a bijection between*

- *the set of isomorphism classes of stable Drinfeld modules over L of rank d, and*

— *the set of isomorphism classes of pairs* (ψ, Λ) *where* ψ *is a good-reduction Drinfeld module over* L *of rank* $d' \le d$*, and where* $\Lambda \subseteq \psi(L^{\mathrm{sep}})$ *is a lattice inside* ψ *such that* $(\mathrm{rk}_{\mathbb{F}[z]}\Lambda =) \dim_{\mathbb{F}(z)} \Lambda \otimes_{\mathbb{F}[z]} \mathbb{F}(z) = d - d'$.

In this Theorem, the asserted bijection can be described as follows: For a stable Drinfeld module $\varphi\colon \mathbb{F}[z] \to L[\tau]$ of rank d with a choice of an integral model $\varphi'\colon \mathbb{F}[z] \to o_L[\tau]$ there is

- a Drinfeld module $\psi\colon \mathbb{F}[z] \to L[\tau]$ of rank $d' = \mathrm{rk}(\overline{\varphi'})$ such that $\mathrm{im}(\psi) \subseteq o_L[\tau]$ and such that $z \mapsto \overline{\psi_z}$ defines a Drinfeld module $\mathbb{F}[z] \to \ell[\tau]$ of rank d' (in particular, ψ is of good reduction),
- a skew formal power series $u = 1 + \sum_{j=1}^{\infty} a_j \tau^j \in o_L\{\{\tau\}\}$ such that $\mathrm{ord}_\pi(a_j) \ge 1$ for all j and such that
$$u\psi_f = \varphi'_f u$$
for all $f \in \mathbb{F}[z]$; here $o_L\{\{\tau\}\}$ denotes the ring of skew formal power series $\sum_{j=0}^{\infty} b_j \tau^j$ having coefficients in o_L, with the commutation rule $\tau\alpha = \alpha^r \tau$ for $\alpha \in o_L$; in [21], 7.2, Drinfeld goes on to show that moreover u verifies the convergence condition $\mathrm{ord}_\pi(a_j)/r^j \to \infty$ $(j \to \infty)$ and that $a_j \in \mathfrak{m}_L$ for all $j \ge 1$ (in particular, reducing the relations $u\psi_f = \varphi'_f u \bmod \mathfrak{m}_L$ gives $\bar{\psi} \simeq \bar{\varphi}$);

the pair (u, ψ) is uniquely determined by φ'; note that u can be interpreted as an *analytic* homomorphism of Drinfeld modules $\psi \to_{\mathrm{an}} \varphi'$ which, due to rank reasons, can *not* in general represent a nontrivial homomorphism of Drinfeld modules in the (algebraic) sense defined before. Finally, the isomorphism class of φ corresponds to the isomorphism class of the pair (ψ, Λ), where the lattice Λ inside ψ is given by

$$\Lambda = \ker(u) = \{x \in L^{\mathrm{sep}},\ x + \sum_{j=1}^{\infty} a_j x^{r^j} = 0\};$$

the latter is a free $\mathbb{F}[z]$-module of rank $d - d'$; note that for every formal series $\sum_{j=0}^{\infty} b_j \tau^j \in o_L\{\{\tau\}\}$ additionally verifying the convergence condition $\mathrm{ord}_\pi(b_j)/r^j \to \infty$ and every $x \in L^{\mathrm{alg}}$ (resp., $x \in L^{\mathrm{sep}}$) the series $\sum_{j=0}^{\infty} b_j x^{r^j}$ converges in L^{alg} (resp., in L^{sep}) since the field extension $L(x)$ is finite (resp., finite separable) and therefore complete. Conversely, the Drinfeld module over L obtained from the exponential function associated to Λ (as constructed in analytic uniformization theory [21], [36] for Drinfeld modules over the completion $\mathbb{C}_\infty$ of an algebraic closure of $\mathbb{F}((1/z))$) is isomorphic to φ, which concludes the description of the asserted bijection.

For every Drinfeld module $\varphi\colon \mathbb{F}[z] \to L[\tau]$ over L such that $\mathrm{im}(\varphi) \subseteq o_L[\tau]$ and such that $z \mapsto \overline{\varphi_z}$ defines a Drinfeld module over ℓ, the latter is a Drinfeld module over the *perfect* field ℓ, with Drinfeld characteristic given by the place ε associated to

$(z) \subseteq \mathbb{F}[z]$. This parallels the situation of elliptic curves of semi-stable reduction over p-adic fields, which are fields of characteristic zero, such that the associated reduced curve is defined over a perfect field of positive characteristic; therefore we see that, switching from elliptic curves to Drinfeld modules, the scenario of mixed characteristic in the sense of *rings* is replaced by the scenario of mixed *Drinfeld* characteristic.

Recall that for a given Drinfeld module $\psi\colon \mathbb{F}[z] \to \ell[\tau]$ over ℓ and every $n \geq 1$ we have

$$\psi(\ell^{\mathrm{alg}})[z^n] = \{x \in \psi(\ell^{\mathrm{alg}}),\ z^n x = 0\} \simeq (\mathbb{F}[z]/z^n)^{\mathrm{rk}(\psi)-\mathrm{ht}(\psi)},$$

where the positive integer $\mathrm{rk}(\psi)$ (resp., $\mathrm{ht}(\psi)$) denotes the *rank* (resp., the *height*) of ψ, and where $\mathrm{ht}(\psi) \leq \mathrm{rk}(\psi)$; see [59], 2.3, 2.5.

Definition 2.8. *Let $\psi\colon \mathbb{F}[z] \to \ell[\tau]$ be a Drinfeld module over ℓ. Then ψ is called* supersingular *if $\psi(\ell^{\mathrm{alg}})[z] = 0$ (i.e., $\mathrm{ht}(\psi) = \mathrm{rk}(\psi)$).*

By virtue of the isomorphisms displayed above, the condition $\psi[z](\ell^{\mathrm{alg}}) = 0$ is equivalent to saying that for all $n \geq 1$ one has $\psi(\ell^{\mathrm{alg}})[z^n] = 0$. For a couple of different characterizations and a deeper study of supersingularity for Drinfeld modules over *finite* fields, see [32], 5.1. There is a tight analogy with the situation for elliptic curves, as we have encountered in section (2.1); see [71], V.3.

Example 2.9. The simplest example of a Drinfeld module is incorporated by the *Carlitz module* which is

$$C\colon \mathbb{F}[z] \to L[\tau], \qquad C_z = \zeta + \tau$$

(see [36], 3.3). The reduction of C is given by

$$\bar{C}\colon \mathbb{F}[z] \to \ell[\tau], \quad z \mapsto \tau.$$

This is a Drinfeld module over ℓ (in fact, the Carlitz module over ℓ) which is supersingular: indeed, for $x \in \bar{C}(\ell^{\mathrm{alg}})$ we have $zx = 0$ if and only if $x^r = 0$, i.e., $x = 0$.

2.2.3 Analytic Anderson motives

Let $\varphi\colon \mathbb{F}[z] \to L[\tau]$ be a Drinfeld module over L of rank $d = \mathrm{rk}(\varphi)$. Recall ([2], [36]) that the L-vector space $M(\varphi) = L[\tau]$ becomes an $L[z]$-module by letting z act on $f \in M(\varphi)$ via $zf = f\varphi_z$; the $L[z]$-module $M(\varphi)$ is free of rank d, with basis given by $1, \tau, ..., \tau^{d-1}$; see [36], 5.4.1. Furthermore, the map

$$M(\varphi) \to M(\varphi),\ f \mapsto \tau f,$$

is an endomorphism of the abelian group $M(\varphi)$, which, according to the commutation rule $\tau\alpha = \alpha^r\tau$ in $L[\tau]$ for $\alpha \in L$, is semi-linear with respect to the r-Frobenius lift $\sigma\colon L[z] \to L[z]$, $\sum_{\nu=0}^n a_\nu z^\nu \mapsto \sum_{\nu=0}^n a_\nu^r z^\nu$; furthermore, the pair $(M(\varphi), \tau(\cdot))$ gives rise to an object of the category $\mathrm{FMod}(\mathbb{F}[z] \otimes_{\mathbb{F}} L)$ from (1.4), i.e., the associated $L[z]$-linear map $\sigma^* M(\varphi) \to M(\varphi)$, $m \otimes a \mapsto a\tau m$, is injective.

For a given homomorphism of Drinfeld modules $\lambda\colon \varphi \to \varphi'$, from $(f\varphi'_z)\lambda = (f\lambda)\varphi_z$ for $f \in L[\tau]$ it follows that the L-linear map $M(\lambda)\colon M(\varphi') \to M(\varphi)$, $f \mapsto f\lambda$, verifies $M(\lambda)(zf) = zM(\lambda)(f)$, i.e., $M(\lambda)$ is $L[z]$-linear, and the assignments

$$\varphi \mapsto M(\varphi), \qquad (\lambda\colon \varphi \to \varphi') \mapsto (M(\lambda)\colon M(\varphi') \to M(\varphi))$$

define a contravariant fully faithful functor from the category of Drinfeld modules over L to the category of *Anderson* $\mathbb{F}[z]$*-motives*; see [36], 5.4.11.

Now suppose that $\varphi\colon \mathbb{F}[z] \to L[\tau]$ verifies $\mathrm{im}(\varphi) \subseteq o_L[\tau]$; note that this is the case if and only if $\varphi_z \in o_L[\tau]$. Further, suppose that $z \mapsto \overline{\varphi_z}$ defines a Drinfeld module over ℓ, of rank $d' \leq d = \mathrm{rk}(\varphi)$. Again by [21], 7.2, there is a unique good-reduction Drinfeld module $\psi\colon \mathbb{F}[z] \to L[\tau]$ of rank d' such that $\mathrm{im}(\psi) \subseteq o_L[\tau]$, together with a formal power series $u = 1 + \sum_{j=1}^\infty a_j x^{r^j} \in o_L[\![x]\!]$ such that $\mathrm{ord}_\pi(a_j) \geq 1$ for all j, verifying the convergence condition $\mathrm{ord}_\pi(a_j)/r^j \to \infty$ for $j \to \infty$, as well as the relations $u\psi_f = \varphi_f u$ for all $f \in \mathbb{F}[z]$; like in the context of 2.7 we interpret the power series u as an analytic homomorphism of Drinfeld modules $u\colon \psi \to_{\mathrm{an}} \varphi$. The following Theorem relies crucially on work of Gardeyn, [29], and shows how, by virtue of u, Tate uniformization can be carried out in terms of (analytic) Anderson motives.

Theorem 2.10. *The analytic homomorphism* $u\colon \psi \to_{\mathrm{an}} \varphi$ *of Drinfeld modules gives rise to a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M(\varphi) \otimes_{L[z]} L\langle z\rangle & \longrightarrow & M(\psi) \otimes_{L[z]} L\langle z\rangle & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F_N} & & \downarrow{\scriptstyle \tau\otimes\sigma} & & \downarrow{\scriptstyle \tau\otimes\sigma} & & \\ 0 & \longrightarrow & N & \longrightarrow & M(\varphi) \otimes_{L[z]} L\langle z\rangle & \longrightarrow & M(\psi) \otimes_{L[z]} L\langle z\rangle & \longrightarrow & 0 \end{array}$$

with a finite free $L\langle z\rangle$*-module* N *of rank* $s = d - d'$*, where the horizontal maps are* $L\langle z\rangle$*-linear and the vertical maps are semi-linear with respect to the* r*-Frobenius lift* $\sigma\colon L\langle z\rangle \to L\langle z\rangle$, $\sum_{j=0}^\infty a_j z^j \mapsto \sum_{j=0}^\infty a_j^r z^j$*; moreover, there is a finite field extension* L'/L *such that one has a commutative diagram*

$$\begin{array}{ccc} N \otimes_{L\langle z\rangle} L'\langle z\rangle & \xrightarrow[\iota]{\sim} & L'\langle z\rangle^s \\ \downarrow{\scriptstyle F_N\otimes\sigma} & & \downarrow{\scriptstyle \sigma^{\oplus s}} \\ N \otimes_{L\langle z\rangle} L'\langle z\rangle & \xrightarrow[\iota]{\sim} & L'\langle z\rangle^s \end{array}$$

where the horizontal map $\iota\colon N \otimes_{L\langle z\rangle} L'\langle z\rangle \to L'\langle z\rangle^s$ *is an* $L'\langle z\rangle$*-linear isomorphism and the vertical maps are semi-linear with respect to the* r*-Frobenius lift of* $L'\langle z\rangle$*; in this sense, the couple* (N, F_N) *is* potentially trivial*.*

Note that with respect to the canonical basis $\mathfrak{E} = (e_1, ..., e_s)$ of $L'\langle z\rangle^s$ the map $\sigma^{\oplus s}\colon L'\langle z\rangle^s \to L'\langle z\rangle^s$ is described σ-semi-linearly by the unit matrix $\mathrm{Id}_s \in L'\langle z\rangle^{s\times s}$; let $\mathfrak{B} = (b_1, ..., b_s)$ be the $L'\langle z\rangle$-basis of $N \otimes_{L\langle z\rangle} L'\langle z\rangle$ defined by $e_i = \iota(b_i)$; the condition $\iota \circ (\tau_N \otimes \sigma) = \sigma^{\oplus s} \circ \iota$ asserted in the Theorem amounts to saying that $(F_N \otimes \sigma)(b_i) = b_i$ for every index i.

Proof of Theorem 2.10. By [29], 1.2, the analytic morphism $\psi \to_{\mathrm{an}} \varphi$ induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{N} & \longrightarrow & M(\varphi) \otimes_{L[z]} L\langle\langle z\rangle\rangle & \longrightarrow & M(\psi) \otimes_{L[z]} L\langle\langle z\rangle\rangle & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F_N} & & \downarrow{\scriptstyle \tau\otimes\sigma} & & \downarrow{\scriptstyle \tau\otimes\sigma} & & \\ 0 & \longrightarrow & \widetilde{N} & \longrightarrow & M(\varphi) \otimes_{L[z]} L\langle\langle z\rangle\rangle & \longrightarrow & M(\psi) \otimes_{L[z]} L\langle\langle z\rangle\rangle & \longrightarrow & 0 \end{array}$$

with a finite (and necessarily free) $L\langle\langle z\rangle\rangle$-module $\widetilde{N}$ of rank $s \geq 0$, where the horizontal maps are $L\langle\langle z\rangle\rangle$-linear and the vertical maps are semi-linear with respect to the r-Frobenius lift $\sigma\colon L\langle\langle z\rangle\rangle \to L\langle\langle z\rangle\rangle$, $\sum_{j=0}^{\infty} a_j z^j \mapsto \sum_{j=0}^{\infty} a_j^r z^j$; here $L\langle\langle z\rangle\rangle$ denotes the subring of $L[\![z]\!]$ consisting of those formal power series $\sum_{j=0}^{\infty} b_j z^j$ satisfying $\mathrm{ord}_\pi(b_j)/r^j \to \infty$ as $j \to \infty$; furthermore, for some finite field extension L'/L there is a commutative diagram

$$\begin{array}{ccc} \widetilde{N} \otimes_{L\langle\langle z\rangle\rangle} L'\langle\langle z\rangle\rangle & \xrightarrow[\iota]{\sim} & L'\langle\langle z\rangle\rangle^s \\ {\scriptstyle \tau_N\otimes\sigma}\downarrow & & \downarrow{\scriptstyle \sigma^{\oplus s}} \\ \widetilde{N} \otimes_{L\langle\langle z\rangle\rangle} L'\langle\langle z\rangle\rangle & \xrightarrow[\iota]{\sim} & L'\langle\langle z\rangle\rangle^s \end{array}$$

where the horizontal map $\iota\colon \widetilde{N} \otimes_{L\langle\langle z\rangle\rangle} L'\langle\langle z\rangle\rangle \to L'\langle\langle z\rangle\rangle^s$ is an $L'\langle\langle z\rangle\rangle$-linear isomorphism and the vertical maps are semi-linear with respect to the r-Frobenius lift of $L'\langle\langle z\rangle\rangle$; we briefly write M (resp., M') for $M(\varphi) \otimes_{L[z]} L\langle\langle z\rangle\rangle$ (resp., for $M(\psi) \otimes_{L[z]} L\langle\langle z\rangle\rangle$) and observe that the underlying exact sequence of $L\langle\langle z\rangle\rangle$-modules $0 \to \widetilde{N} \to M \to M' \to 0$ has to be split and therefore exhibits M as the $L\langle\langle z\rangle\rangle$-linear direct sum of $\widetilde{N}$ and M'; as the tensor product is compatible with direct sums, applying the functor $\cdot \otimes_{L\langle\langle z\rangle\rangle} L\langle z\rangle$ yields an exact sequence of free $L\langle z\rangle$-modules

$$0 \to \widetilde{N} \otimes_{L\langle\langle z\rangle\rangle} L\langle z\rangle \to M \otimes_{L\langle\langle z\rangle\rangle} L\langle z\rangle \to M' \otimes_{L\langle\langle z\rangle\rangle} L\langle z\rangle \to 0$$

which of course is again split and where each map is again compatible with the semi-linear data. Since functors preserve isomorphisms, applying $\cdot \otimes_{L'\langle\langle z\rangle\rangle} L'\langle z\rangle$ to the above commutative square yields an $L'\langle z\rangle$-linear isomorphism

$$(\widetilde{N} \otimes_{L\langle\langle z\rangle\rangle} L\langle z\rangle) \otimes_{L\langle z\rangle} L'\langle z\rangle \simeq (\widetilde{N} \otimes_{L\langle\langle z\rangle\rangle} L'\langle\langle z\rangle\rangle) \otimes_{L'\langle\langle z\rangle\rangle} L'\langle z\rangle \xrightarrow{\sim} L'\langle z\rangle$$

which is compatible with the semi-linear data. Setting $N = \widetilde{N} \otimes_{L\langle\langle z\rangle\rangle} L\langle z\rangle$ and observing that $M \otimes_{L\langle\langle z\rangle\rangle} L\langle z\rangle \simeq M(\varphi) \otimes_{L[z]} L\langle z\rangle$ (likewise for M'), we see that we are done. □

2.2.4 Genestier-Lafforgue's analogue for the crystalline period functor $D_{\mathbf{cris}}$

As mentioned before, given a p-adic field K there is a functor $D_{\text{cris}}\colon \text{Rep}_{\mathbb{Q}_p}(G_K) \to MF_K(\varphi)$, defined by J.M. Fontaine, which induces an equivalence between

- the full subcategory $\text{Rep}_{\text{cris}}(G_K)$ of $\text{Rep}_{\mathbb{Q}_p}(G_K)$ consisting of those p-adic representations of the absolute Galois group G_K of K which are *crystalline*, and
- the full subcategory $MF_K(\varphi)^{\text{wa}}$ of $MF_K(\varphi)$ consisting of those filtered φ-modules over the field $F = W(k)[1/p]$ (k being the residue field of the valuation ring of K) which are *weakly admissible*;

see [27] for a discussion of this equivalence, and cf. (2.1.2) for a brief discussion of the categories $\text{Rep}_{\text{cris}}(G_K)$ and $MF_K(\varphi)$.

Turning to equal characteristic, and retaining our complete discretely valued field L from before, we now briefly describe an analogue for the functor D_{cris} which was first defined by A. Genestier and V. Lafforgue in [34] and thorougly studied in [34], [41].

Definition 2.11. *A* local shtuka *(over o_L) is a pair $(\hat{M}, F_{\hat{M}})$ consisting of a finite free $o_L[\![z]\!]$-module $\hat{M}$, together with an $o_L[\![z]\!][\frac{1}{z-\zeta}]$-linear isomorphism*

$$F_{\hat{M}}\colon \sigma^*\hat{M}[\tfrac{1}{z-\zeta}] \to \hat{M}[\tfrac{1}{z-\zeta}]$$

where $\sigma^*\hat{M} = \hat{M} \otimes_{o_L[\![z]\!],\sigma} o_L[\![z]\!]$ and where $\sigma\colon o_L[\![z]\!] \to o_L[\![z]\!]$ is the r-Frobenius lift of $o_L[\![z]\!]$ defined by $\sum_{j=0}^{\infty} a_j z^j \mapsto \sum_{j=0}^{\infty} a_j^r z^j$. A morphism *of local shtukas $(\hat{M}, F_{\hat{M}}) \to (\hat{N}, F_{\hat{N}})$ is an $o_L[\![z]\!]$-linear map $f\colon \hat{M} \to \hat{N}$ such that $f \circ F_{\hat{M}} = F_{\hat{N}} \circ \sigma^* f$. An* isogeny *of local shtukas is a morphism $f\colon (\hat{M}, F_{\hat{M}}) \to (\hat{N}, F_{\hat{N}})$ such that there is a morphism $g\colon (\hat{N}, F_{\hat{N}}) \to (\hat{M}, F_{\hat{M}})$ and an integer $e \geq 0$ such that $g \circ f = z^e$ and $f \circ g = z^e$.*

Remark. Let us briefly indicate that the element $z - \zeta \in o_L[\![z]\!]$ appearing in the denominator stems from a distinguished Eisenstein polynomial employed in Breuil-Kisin's study ([14], [48]) of crystalline p-adic representations of the Galois group of a p-adic field; more precisely, in the notation of [48], the category of local shtukas provides an analogue for the category $\text{BT}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$; see [40], [41] for a discussion of this analogy.

Note that the isomorphism $F_{\hat{M}}$ need not be induced by an actual $o_L[\![z]\!]$-linear map $\sigma^*\hat{M} \to \hat{M}$; if, however, such a map exists then the local shtuka $(\hat{M}, F_{\hat{M}})$ is called *effective*. For example, as we have studied in chapter 1, the local shtuka associated to (a good model of) an *Anderson motive* of *good* reduction via ε-adic formal completion is always effective; in particular, an effective local shtuka can be associated to every good-reduction Drinfeld module over L; besides these examples, see [41], 2.1.4, for an account of the most important sources from which local shtukas arise.

Already here it should be stressed that local shtukas are limited to a scenario of *good reduction*, a circumstance which will be further discussed below.

Definition 2.12. *A* z-isocrystal (*or* local isoshtuka) *over* ℓ *is a pair* (D, F_D) *consisting of a finite* $\ell(\!(z)\!)$*-vector space* D*, together with an* $\ell(\!(z)\!)$*-linear isomorphism* $\sigma^*D \to D$ *where* $\sigma^*D = D \otimes_{\ell(\!(z)\!),\sigma} \ell(\!(z)\!)$ *and where* $\sigma\colon \ell(\!(z)\!) \to \ell(\!(z)\!)$ *is induced by the* r*-Frobenius lift* $\ell[\![z]\!] \to \ell[\![z]\!]$ *defined by* $\sum_{j=0}^{\infty} a_j z^j \mapsto \sum_{j=0}^{\infty} a_j^r z^j$*. A* morphism *of* z*-isocrystals* $(D, F_D) \to (D', F_{D'})$ *is an* $\ell(\!(z)\!)$*-linear map* $f\colon D \to D'$ *such that* $f \circ F_D = F_{D'} \circ \sigma^* f$*.*

Given a morphism $f\colon (D, F_D) \to (D', F_{D'})$ of z-isocrystals, the semi-linear map $F_D^{\mathrm{sl}}\colon D \to D$ restricts to a semi-linear map $F_D^{\mathrm{sl}}\colon \ker(f) \to \ker(f)$ which in turn induces an $\ell(\!(z)\!)$-linear map $\sigma^* \ker(f) \to \ker(f)$; the latter is clearly injective and, looking at dimensions, therefore has to be an isomorphism again, so that one obtains a canonical structure of a z-isocrystal on $\ker(f)$; similarly one obtains a canonical structure of a z-isocrystal on $\mathrm{im}(f) \subseteq D'$.

Let us next discuss the analogue for the category $MF_K(\varphi)$ of *filtered* isocrystals from Fontaine theory as proposed in [34]. To begin with, we remark that, according to [69], II.4.8, there is a unique ring homomorphism $\ell \to o_L$ which is a section of the residue map $o_L \to \ell$. The section $\ell \to o_L$ induces a canonical homomorphism

$$\ell(\!(z)\!) \to L[\![z-\zeta]\!], \quad z \mapsto \zeta + (z-\zeta),$$

where $L[\![z-\zeta]\!]$ denotes "the" equal-characteristic complete discrete valuation ring with uniformizer $z-\zeta$ and residue field L (see [40], 2.9); let $L(\!(z-\zeta)\!) = L[\![z-\zeta]\!][\frac{1}{z-\zeta}]$.

Definition 2.13. *A* z-isocrystal with Hodge-Pink structure *(over* L*) is a triple* $(D, F_D, \mathfrak{q}_D)$ *where* (D, F_D) *is a* z*-isocrystal over* ℓ *and where*

$$\mathfrak{q}_D \subseteq \sigma^* D \otimes_{\ell(\!(z)\!)} L(\!(z-\zeta)\!)$$

is an $L[\![z-\zeta]\!]$*-lattice of full rank. A* morphism *of* z*-isocrystals with Hodge-Pink structure* $(D, F_D, \mathfrak{q}_D) \to (D', F_{D'}, \mathfrak{q}_{D'})$ *is a morphism of* z*-isocrystals* $f\colon (D, F_D) \to$

$(D', F_{D'})$ *such that*

$$\sigma^* f \otimes \mathrm{id}\colon \sigma^* D \otimes_{\ell((z))} L((z-\zeta)) \to \sigma^* D' \otimes_{\ell((z))} L((z-\zeta))$$

verifies $(\sigma^* f \otimes \mathrm{id})(\mathfrak{q}_D) \subseteq \mathfrak{q}_{D'}$. *We denote the category of* z*-isocrystals with Hodge-Pink structure by* $M_{\ell((z))}(F, \mathfrak{q})$.

See [41], 2.2.3, for a comparison between the concept of *filtered Frobenius-isocrystals* from Fontaine theory on the one hand, and the concept of z*-isocrystals with Hodge-Pink structure* on the other hand.

Now let $(\hat{M}, F_{\hat{M}})$ be a local shtuka over o_L; it gives rise to a z-isocrystal (D, F_D) with Hodge-Pink structure $\mathfrak{q}_D$ as follows: the underlying $\ell((z))$-vector space is given by $D = \hat{M} \otimes_{o_L[\![z]\!]} \ell((z))$; accordingly one defines $F_D = F_{\hat{M}} \otimes \mathrm{id}$. In order to associate a Hodge-Pink structure to the pair (D, F_D), one employs the following

Lemma 2.14 ([34], [41]). *There is a unique functorial isomorphism*

$$\delta_{\hat{M}}\colon \hat{M} \otimes_{o_L[\![z]\!]} o_L[\![z, z^{-1}\}[1/t_-] \xrightarrow{\simeq} D \otimes_{\ell((z))} o_L[\![z, z^{-1}\}[1/t_-]$$

which satisfies $\delta_{\hat{M}} \circ F_{\hat{M}} = F_D \circ \sigma^* \delta_{\hat{M}}$ *and which mod* π *reduces to the identity.*

Here the o_L-algebra $o_L[\![z, z^{-1}\}$ consists of those (infinite-tail) formal Laurent series $\sum_{j=-\infty}^{\infty} b_j z^j$ such that $b_j \in o_L$ and $|b_j| \cdot |\zeta|^{rj} \to 0$ ($j \to -\infty$) for all $r > 0$, and the element $t_- \in o_L[\![z, z^{-1}\}$ is defined as the limit of the sequence

$$\Big(\prod_{j=0}^{n} \frac{1}{z}(z - \zeta^{r^j})\Big)_{n \geq 0} = \Big(\sum_{j=0}^{n+1} \big(\frac{1}{z}\big)^{n+1-j} (-1)^{n+1-j} \sum_{0 \leq \nu_1 < \ldots < \nu_{n+j-1} \leq n} \zeta^{r^{\nu_1} + \ldots + r^{\nu_{n+1-j}}}\Big)_{n \geq 0}$$

inside $o_L[\![z, z^{-1}\}$. There is a canonical map $o_L[\![z, z^{-1}\} \to L[\![z-\zeta]\!]$ which is given by the inclusion $o_L \hookrightarrow L$ and $z \mapsto \zeta + (z - \zeta)$ and which extends to a map $o_L[\![z, z^{-1}\}[1/t_-] \to L[\![z-\zeta]\!]$; by applying the functor $\cdot \otimes_{o_L[\![z,z^{-1}\}[1/t_-]} L[\![z-\zeta]\!]$ to the isomorphism

$$\sigma^* \delta_{\hat{M}}\colon \sigma^* \hat{M} \otimes_{o_L[\![z]\!]} o_L[\![z, z^{-1}\}[1/\sigma(t_-)] \xrightarrow{\simeq} \sigma^* D \otimes_{\ell((z))} o_L[\![z, z^{-1}\}[1/\sigma(t_-)],$$

we obtain an isomorphism $\sigma^* \hat{M} \otimes_{o_L[\![z]\!]} L[\![z-\zeta]\!] \xrightarrow{\simeq} \sigma^* D \otimes_{\ell((z))} L[\![z-\zeta]\!]$ which is again denoted by $\sigma^* \delta_{\hat{M}}$; note that $\sigma^* D \otimes_{\ell((z))} L[\![z-\zeta]\!]$ is an $L[\![z-\zeta]\!]$-lattice of full rank inside $\sigma^* D \otimes_{\ell((z))} L((z-\zeta))$; it is called the *tautological lattice*. Similarly, by composing the reduction map $o_L[\![z]\!][\frac{1}{z-\zeta}] \to \ell((z))$ with the ring homomorphism $\ell((z)) \to L((z-\zeta))$, $z \mapsto \zeta + (z-\zeta)$, the isomorphism $F_{\hat{M}}\colon \sigma^* \hat{M} \otimes_{o_L[\![z]\!]} o_L[\![z]\!][\frac{1}{z-\zeta}] \xrightarrow{\simeq} \hat{M} \otimes_{o_L[\![z]\!]} o_L[\![z]\!][\frac{1}{z-\zeta}]$ induces an isomorphism

$$\sigma^* \hat{M} \otimes_{o_L[\![z]\!]} L((z-\zeta)) \xrightarrow{\simeq} \hat{M} \otimes_{o_L[\![z]\!]} L((z-\zeta))$$

which is again denoted by $F_{\hat{M}}$. Finally, in this notation, the Hodge-Pink structure associated to the z-isocrystal (D, F_D) is given by

$$\mathfrak{q}_D = \sigma^*\delta_{\hat{M}} \circ (\sigma^* F_{\hat{M}})^{-1}(\hat{M} \otimes_{o_L[\![z]\!]} L[\![z-\zeta]\!]) \subseteq \sigma^* D \otimes_{\ell(\!(z)\!)} L(\!(z-\zeta)\!).$$

The assignment $(\hat{M}, F_{\hat{M}}) \mapsto (D, F_D, \mathfrak{q}_D)$ defines a functor

$$\mathbb{H}\colon (\text{local shtukas over } o_L) \to M_{\ell(\!(z)\!)}(F, \mathfrak{q}).$$

Localizing the category of local shtukas over o_L by the class of isogenies yields, by definition, the category of *local shtukas over* o_L *up to isogeny.* The functor $\mathbb{H}$ sends isogenies of local shukas to isomorphisms of the associated z-isocrystals with Hodge-Pink structure and therefore, by the universal property of localization, factors uniquely up to equivalence of functors via

$$\mathbb{H}_{\text{iso}}\colon (\text{local shtukas over } o_L \text{ up to isogeny}) \to M_{\ell(\!(z)\!)}(F, \mathfrak{q}).$$

For the following Theorem we remark that, given two local shtukas $(\hat{M}, F_{\hat{M}})$, $(\hat{N}, F_{\hat{N}})$, their *tensor product* is given by the local shtuka $(\hat{M} \otimes_{o_L[\![z]\!]} \hat{N}, F_{\hat{M}} \otimes F_{\hat{N}})$, and the *dual* of $(\hat{M}, F_{\hat{M}})$ is given by the $o_L[\![z]\!]$-module $\hat{M}^\vee = \operatorname{Hom}_{o_L[\![z]\!]}(\hat{M}, o_L[\![z]\!])$ together with $F_{\hat{M}^\vee}\colon \sigma^*\hat{M}^\vee[\frac{1}{z-\zeta}] \to \hat{M}^\vee[\frac{1}{z-\zeta}]$ defined by the commutative diagram

$$\begin{array}{ccc}
\sigma^*\hat{M}^\vee[\frac{1}{z-\zeta}] & \xrightarrow{F_{\hat{M}^\vee}} & \hat{M}^\vee[\frac{1}{z-\zeta}] \\
\downarrow \simeq & & \downarrow \simeq \\
\operatorname{Hom}_{o_L[\![z]\!][\frac{1}{z-\zeta}]}(\sigma^*\hat{M}[\frac{1}{z-\zeta}], o_L[\![z]\!][\frac{1}{z-\zeta}]) & \xrightarrow[\cdot\circ F_{\hat{M}}^{-1}]{} & \operatorname{Hom}_{o_L[\![z]\!][\frac{1}{z-\zeta}]}(\hat{M}[\frac{1}{z-\zeta}], o_L[\![z]\!][\frac{1}{z-\zeta}])
\end{array}$$

using that the Frobenius lift $\sigma\colon o_L[\![z]\!] \to o_L[\![z]\!]$ is flat and hence $\sigma^*(\hat{M}^\vee) \simeq (\sigma^*\hat{M})^\vee$; in this spirit, one may write $F_{\hat{M}^\vee} = (F_{\hat{M}}^{-1})^\vee$. Similarly, given two z-isocrystals with Hodge-Pink structure $(D, F_D, \mathfrak{q}_D)$, $(D', F_{D'}, \mathfrak{q}_{D'})$, their *tensor product* is given by the triple $(D \otimes_{\ell(\!(z)\!)} D', F_D \otimes F_{D'}, \mathfrak{q}_D \otimes_{L[\![z-\zeta]\!]} \mathfrak{q}_{D'})$, and the dual of $(D, F_D, \mathfrak{q}_D)$ is given by the triple $(D^\vee, (F_D^{-1})^\vee, \operatorname{Hom}_{L[\![z-\zeta]\!]}(\mathfrak{q}_D, L[\![z-\zeta]\!]))$.

Theorem 2.15 ([34], [41]). *The functor* $\mathbb{H}_{\text{iso}}$ *is exact, fully faithful, and it respects tensor products and duals.*

The study of the categories introduced here and of the functor $\mathbb{H}$ is referred to as *Hodge-Pink theory*; see [34], [41], [63]. In [34] the functor $\mathbb{H}_{\text{iso}}$ is proposed as an equal-characteristic analogue for the functor

$$D_{\text{cris}}\colon \operatorname{Rep}_{\text{cris}}(G_K) \to MF_K(\varphi)$$

where G_K is the absolute Galois group of a given p-adic field K. See [40], and [41], 2.3.6, for a discussion of the analogy between *local shtukas* on the one hand, and *crystalline p-adic representations* in the sense of Fontaine on the other hand.

2.2.5 Semi-stable local shtukas and z-isocrystals with Hodge-Pink structure and monodromy

After having discussed the "crystalline level" in the previous section, namely the functor $\mathbb{H}_{\text{iso}}$ which plays the role of Fontaine's functor $D_{\text{cris}}\colon \text{Rep}_{\text{cris}}(G_K) \to MF_K(\varphi)$ in equal characteristic, we now turn to the "semi-stable level", and for this we first recall the commutative diagram of categories and fully faithful functors

$$\begin{array}{ccc} \text{Rep}_{\text{st}}(G_K) & \longrightarrow & MF_K(\varphi, N) \\ \uparrow & & \uparrow \\ \text{Rep}_{\text{cris}}(G_K) & \longrightarrow & MF_K(\varphi) \end{array}$$

which was already discussed in section (2.1.2). Viewing the (isogeny) category of local shtukas over o_L as an analogue for crystalline p-adic representations ([40], 5.3, 5.4), we now intend to study a hypothetical analogue for the category $\text{Rep}_{\text{st}}(G_K)$ of *semi-stable* p-adic representations à la Fontaine.

Digression 2.16. We have mentioned earlier that local shtukas and, correspondingly, crystalline p-adic representations have to be seen as *good-reduction* objects. Let us look at the p-adic world and let us say a few words about the geometric picture standing behind Fontaine's theory. In [38], Grothendieck posed the question of whether there is a functorial relation between the p-adic étale cohomology $H^*_{\text{ét}}(X \otimes_K K^{\text{alg}}, \mathbb{Q}_p)$ of a smooth proper scheme X of *good* reduction over a p-adic field K on the one hand, and the crystalline cohomology $H^*_{\text{crys}}(X/K)$ of X on the other hand. Such a functorial relation is, in fact, provided by Fontaine's crystalline period functor D_{cris}, i.e., the $\mathbb{Q}_p$-vector space $H^*_{\text{ét}}(X \otimes_K K^{\text{alg}}, \mathbb{Q}_p)$ indeed is a crystalline p-adic representation of the absolute Galois group G_K of the field K (as was shown by G. Faltings), the abelian group $H^*_{\text{crys}}(X/K)$ gives rise to an object of $MF_K(\varphi)$ (shown by P. Deligne and L. Illusie), and $D_{\text{cris}}(H^*_{\text{ét}}(X \otimes_K K^{\text{alg}}, \mathbb{Q}_p))$ is indeed isomorphic to $H^*_{\text{crys}}(X/K)$ as objects of $MF_K(\varphi)$; see [3], [15], [26], [27], [39]. There is a very similar geometric picture which is related to Fontaine's functor D_{st} – namely, given a proper and smooth scheme X of *semi-stable* reduction over a p-adic field K, a conjecture of J. M. Fontaine and U. Jannsen states that there is a functorial isomorphism

$$D_{\text{st}}(H^*_{\text{ét}}(X \otimes_K K^{\text{alg}}, \mathbb{Q}_p)) \simeq H^*_{\text{log-crys}}(X)$$

where $H^*_{\text{log-crys}}(X)$ denotes the *log-crystalline cohomology* of the scheme X. The proof of this conjecture has been accomplished by T. Tsuji; see [75] for a survey. –

Turning again to equal characteristic, we have already seen that local shtukas are functorially associated to global objects such as Drinfeld modules or Anderson motives, where one has to restrict to those objects which are of *good* reduction or rather, in terms of the most general instance of *Drinfeld shtukas*: those which do not possess *degenerators* ([20], [49]).

Taking the case of *bad* reduction into account, a given local shtuka $(\hat{M}, F_{\hat{M}})$ should be seen as a canonical good model for the associated pair

$$(\hat{M} \otimes_{o_L[\![z]\!]} (o_L[\![z]\!] \otimes_{o_L} L), F_{\hat{M}} \otimes \mathrm{id})$$

which is then said to be of good reduction. This point of view ties in with the general "philosophy" of reduction, such as in the case of elliptic curves/abelian varieties or Drinfeld modules/Anderson motives.

In order to discuss a hypothetical analogue for Fontaine's functor $D_{\mathrm{st}} \colon \mathrm{Rep}_{\mathrm{st}}(G_K) \to MF_K(\varphi, N)$ we commence by studying the following analogue for the category $MF_K(\varphi, N)$. We define the category $M_{\ell(\!(z)\!)}(F, \mathfrak{q}, \mathcal{N})$ of *z-isocrystals with Hodge-Pink structure and monodromy operator* as follows:

- An *object* of $M_{\ell(\!(z)\!)}(F, \mathfrak{q}, \mathcal{N})$ is given by a pair $((D, F_D, \mathfrak{q}_D), \mathcal{N}_D)$ where $(D, F_D, \mathfrak{q}_D)$ is a z-isocrystal with Hodge-Pink structure and where $\mathcal{N}_D \colon D \to D$ is an $\ell(\!(z)\!)$-linear map, called the *monodromy operator*, such that $\mathcal{N}_D \circ F_D = \lambda_D F_D \circ \sigma^* \mathcal{N}_D$ for a suitable $\lambda_D \in \ell[\![z]\!] - \ell[\![z]\!]^\times$.
- A *morphism* $((D, F_D, \mathfrak{q}_D), \mathcal{N}_D) \to ((D', F_{D'}, \mathfrak{q}_{D'}), \mathcal{N}_{D'})$ inside $M_{\ell(\!(z)\!)}(F, \mathfrak{q}, \mathcal{N})$ is given by a morphism $f \colon (D, F_D, \mathfrak{q}_D) \to (D', F_{D'}, \mathfrak{q}_{D'})$ of z-isocrystals with Hodge-Pink structure such that $\mathcal{N}_{D'} \circ f = f \circ \mathcal{N}_D$.

A sequence of morphisms

$$0 \to ((D', F_{D'}, \mathfrak{q}_{D'}), \mathcal{N}_{D'}) \xrightarrow{f} ((D, F_D, \mathfrak{q}_D), \mathcal{N}_D) \xrightarrow{g} ((D'', F_{D''}, \mathfrak{q}_{D''}), \mathcal{N}_{D''}) \to 0$$

inside $M_{\ell(\!(z)\!)}(F, \mathfrak{q}, \mathcal{N})$ is said to be *exact* if the underlying sequence of z-isocrystals with Hodge-Pink structure $0 \to (D', F_{D'}, \mathfrak{q}_{D'}) \to (D, F_D, \mathfrak{q}_D) \to (D'', F_{D''}, \mathfrak{q}_{D''}) \to 0$ is an exact sequence of z-isocrystals such that $(\sigma^* g \otimes \mathrm{id})(\mathfrak{q}_D) = \mathfrak{q}_{D''}$ and such that $\sigma^* f \otimes \mathrm{id}$ identifies $\mathfrak{q}_{D'}$ with $\mathfrak{q}_D \cap \sigma^* D \otimes_{\ell(\!(z)\!)} L(\!(z - \zeta)\!)$.

Remark. Note that for an object $((D, F_D, \mathfrak{q}_D), \mathcal{N}_D)$ of $M_{\ell(\!(z)\!)}(F, \mathfrak{q}, \mathcal{N})$ we do not impose a relation between the monodromy operator $\mathcal{N}_D$ and the Hodge-Pink structure $\mathfrak{q}_D$ of the underlying z-isocrystal. This parallels the situation in Fontaine theory where for an object $((D, \varphi_D, \mathrm{Fil}^\bullet D_K), N_D)$ of $MF_K(\varphi, N)$ the filtration $\mathrm{Fil}^\bullet D_K$ of the underlying filtered isocrystal is not related to the monodromy operator $N_D \colon D \to D$; cf. [14], [15], [27]. Also note that the relation $\mathcal{N}_D \circ F_D = \lambda_D F_D \circ \sigma^* \mathcal{N}_D$ is equivalent to $\mathcal{N} \circ F_D^{\mathrm{sl}} = \lambda_D F_D^{\mathrm{sl}} \circ \mathcal{N}_D$ where $F_D^{\mathrm{sl}} \colon D \to D$ is the semi-linear map corresponding to the isomorphism $F_D \colon \sigma^* D \to D$.

There is an obvious fully faithful and exact "inclusion" functor

$$M_{\ell((z))}(F, \mathfrak{q}) \to M_{\ell((z))}(F, \mathfrak{q}, \mathcal{N}), \quad (D, F_D, \mathfrak{q}_D) \mapsto ((D, F_D, \mathfrak{q}_D), \mathcal{N}_D = 0),$$

which admits a faithful and exact "retraction" given by

$$M_{\ell((z))}(F, \mathfrak{q}, \mathcal{N}) \to M_{\ell((z))}(F, \mathfrak{q}), \quad ((D, F_D, \mathfrak{q}_D), \mathcal{N}_D) \mapsto (D, F_D, \mathfrak{q}_D).$$

Of course both categories appearing here admit an obvious faithful and exact forgetful functor into the category $\mathrm{Mod}(\ell((z)))$ of $\ell((z))$-vector spaces.

Lemma 2.17. *Let $((D, F_D, \mathfrak{q}_D), \mathcal{N}_D)$ be an object of $M_{\ell((z))}(F, \mathfrak{q}, \mathcal{N})$ such that*

$$\dim_{\ell((z))} D = 1.$$

Then $\mathcal{N}_D = 0$.

Proof. For a fixed basis element $d \in D$ the map $F_D \colon \sigma^* D \to D$ corresponds to the map $\ell((z)) \to \ell((z))$, $x \mapsto f_D x$, where $f_D = {}_d[F_D]_{\sigma^* d} \in \ell((z))^\times$, and $\mathcal{N}_D \colon D \to D$ corresponds to the map $\ell((z)) \to \ell((z))$, $x \mapsto n_D x$, where $n_D = {}_d[\mathcal{N}_D]_d \in \ell((z))$. Now, as $n_D = {}_d[\mathcal{N}_D]_d = {}_{\sigma^* d}[\sigma^* \mathcal{N}_D]_{\sigma^* d}$, we obtain a relation $\lambda_D f_D \sigma(n_D) = f_D n_D$ inside $\ell((z))$; applying $\mathrm{ord}_z(\cdot)$ on both sides we realize that, by virtue of $\mathrm{ord}_z(n_D) = \mathrm{ord}_z(\sigma(n_D))$, this relation cannot be valid unless $n_D = 0$. $\square$

Hypothesis 2.18. *Suppose there is*

- *a category $\underline{\mathcal{S}}$ whose objects are called* semi-stable local shtukas over L, *and which admits a notion of exact sequence, together with an exact functor*

$$\underline{i} \colon (\text{local shtukas over } o_L) \to \underline{\mathcal{S}};$$

- *an exact functor*

$$\mathbb{H}_{\mathrm{st}} \colon \underline{\mathcal{S}} \to M_{\ell((z))}(F, \mathfrak{q}, \mathcal{N})$$

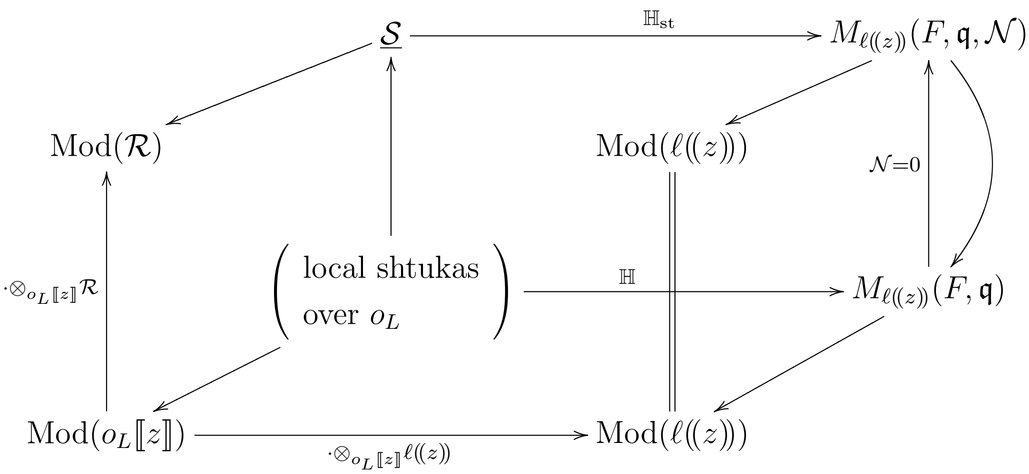
  which, up to equivalence of functors, restricts to the functor $\mathbb{H}$ on local shtukas over o_L;

- *for $\underline{M} \in \underline{\mathcal{S}}$ one has $\mathcal{N}_{\mathbb{H}_{\mathrm{st}}(\underline{M})} = 0$ if and only if $\underline{M}$ comes from a local shtuka over o_L, i.e., lies in the essential image of $\underline{i}$.*

Remark. The hypothetical category $\underline{\mathcal{S}}$ could be expected to admit a (universal) functor into the category $\mathrm{FMod}(\mathcal{R})$ where $\mathcal{R}$ is a suitable $o_L[\![z]\!] \otimes_{o_L} L$-algebra; the image of a semi-stable local shtuka $\underline{M} \in \underline{\mathcal{S}}$ over L via such a "forgetful" functor would then be interpreted as the underlying module of $\underline{M}$; such a situation would truly generalize the case of the category of local shtukas over o_L which indeed admits an exact forgetful functor into the category of (finite free) $o_L[\![z]\!]$-modules. However, it would not be obvious how to define the hypothetical analogue

$$\mathbb{H}_{\mathrm{st}} \colon \underline{\mathcal{S}} \to M_{\ell(\!(z)\!)}(F, \mathfrak{q}, \mathcal{N})$$

for Fontaine's functor $D_{\mathrm{st}} \colon \mathrm{Rep}_{\mathrm{st}}(G_K) \to MF_K(\varphi, N)$ on underlying modules, requiring that $\mathbb{H}_{\mathrm{st}}$ be an extension of $\mathbb{H}$. To begin with, supposing additionally that for a given local shtuka $(\hat{M}, F_{\hat{M}})$ the underlying $\mathcal{R}$-module of the associated object of $\underline{\mathcal{S}}$ is (functorially) isomorphic $\hat{M} \otimes_{o_L[\![z]\!]} \mathcal{R}$ would lead to a commutative diagram



However, there cannot be an o_L-algebra homomorphism $\mathcal{R} \to \ell(\!(z)\!)$ which replaces the reduction map $o_L[\![z]\!] \to \ell(\!(z)\!)$ appearing in the good-reduction case, for the image of π in $\mathcal{R}$ is a unit, whereas it is zero in $\ell(\!(z)\!)$. Given a local shtuka $(\hat{M}, F_{\hat{M}})$ over o_L, a hypothetical natural isomorphism of $\ell(\!(z)\!)$-vector spaces

$$(\hat{M} \otimes_{o_L[\![z]\!]} \mathcal{R}) \otimes_{\mathcal{R}} \ell(\!(z)\!) \simeq \hat{M} \otimes_{o_L[\![z]\!]} \ell(\!(z)\!)$$

expressing $\mathbb{H}_{\mathrm{st}}(\hat{M}) \simeq \mathbb{H}(\hat{M})$ is therefore not available, so that it becomes impossible for $\mathbb{H}_{\mathrm{st}}$ to act as $\cdot \otimes_{\mathcal{R}} \ell(\!(z)\!)$ on underlying modules for any $o_L[\![z]\!] \otimes_{o_L} L$-algebra $\mathcal{R}$. –

2.2.6 A non-crystalline Dieudonné module in equal characteristic

We consider the Drinfeld module over L given by

$$\varphi \colon \mathbb{F}[z] \to L[\tau], \quad z \mapsto \zeta + \tau^2.$$

The Drinfeld module φ clearly has integral coefficients, i.e., $\mathrm{im}(\varphi) \subseteq o_L[\tau]$, and φ is of good reduction: the reduced Drinfeld module over ℓ is given by

$$\bar{\varphi}\colon \mathbb{F}[z] \to \ell[\tau], \quad z \mapsto \tau^2;$$

the latter is a supersingular Drinfeld module, i.e., φ is of supersingular reduction, for we have

$$\bar{\varphi}[z](\ell^{\mathrm{alg}}) = \{x \in \ell^{\mathrm{alg}}, x^{q^2} = 0\} = 0.$$

Inside the $\mathbb{F}[z]$-module $\varphi(L^{\mathrm{sep}})$ we consider the $\mathbb{F}[z]$-lattice of rank 1 given by

$$\Lambda = \mathbb{F}[z]\zeta^{-1} = \{\varphi_\lambda(\zeta^{-1}), \lambda \in \mathbb{F}[z]\}.$$

Indeed $\Lambda \subseteq \varphi(L^{\mathrm{sep}})$ is a free $\mathbb{F}[z]$-submodule of rank 1 which is ρ-stable for every $\rho \in G_L$ and, by virtue of $|\zeta^{-1}| > 1$, is discrete in the sense that every bounded ball inside L^{sep} contains at most finitely many elements of Λ.

By the Tate uniformization Theorem the pair (φ, Λ) gives rise to a *bad*-reduction Drinfeld module φ' over L of rank $3 = \mathrm{rk}(\varphi) + \mathrm{rk}_{\mathbb{F}[z]}(\Lambda)$ whose isomorphism class corresponds to the isomorphism class of (φ, Λ) via the bijection described in (2.2.2). Moreover, by 2.10, the Tate uniformization map $\varphi \to_{\mathrm{an}} \varphi'$ may be carried out in terms of analytic Anderson motives: we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & M(\varphi') \otimes_{L[z]} L\langle z\rangle & \longrightarrow & M(\varphi) \otimes_{L[z]} L\langle z\rangle & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau_N} & & \downarrow{\scriptstyle \tau\otimes\sigma} & & \downarrow{\scriptstyle \tau\otimes\sigma} & & \\
0 & \longrightarrow & N & \longrightarrow & M(\varphi') \otimes_{L[z]} L\langle z\rangle & \longrightarrow & M(\varphi) \otimes_{L[z]} L\langle z\rangle & \longrightarrow & 0
\end{array}$$

where the vertical maps are semi-linear with respect to the Frobenius lift σ of $L\langle z\rangle$. Tensoring this diagram over $L\langle z\rangle$ with $o_L[\![z]\!][1/\pi] \simeq o_L[\![z]\!] \otimes_{o_L} L$ yields a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N \otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi] & \longrightarrow & M(\varphi') \otimes_{L[z]} o_L[\![z]\!][1/\pi] & \longrightarrow & M(\varphi) \otimes_{L[z]} o_L[\![z]\!][1/\pi] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau_N\otimes\sigma} & & \downarrow{\scriptstyle \tau\otimes\sigma} & & \downarrow{\scriptstyle \tau\otimes\sigma} & & \\
0 & \longrightarrow & N \otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi] & \longrightarrow & M(\varphi') \otimes_{L[z]} o_L[\![z]\!][1/\pi] & \longrightarrow & M(\varphi) \otimes_{L[z]} o_L[\![z]\!][1/\pi] & \longrightarrow & 0
\end{array}$$

where now σ denotes the Frobenius lift of $o_L[\![z]\!][1/\pi]$. Taking up the notation from (2.2.5) we want to explain how this diagram can give rise to a short exact sequence inside the hypothetical category $\underline{\mathcal{S}}$ of *semi-stable local shtukas*. For this purpose it is, in the first place, desirable to establish the following Hypothesis; before we state it, note that there is a canonical faithful, exact functor

$$\begin{array}{rcl}
(\textit{effective}\text{ local shtukas over } o_L) & \to & \mathrm{FMod}(o_L[\![z]\!][1/\pi]), \\
(\hat{M}, F_{\hat{M}}) & \mapsto & (\hat{M}[1/\pi], F_{\hat{M}}[1/\pi]),
\end{array}$$

which to every effective local shtuka $\underline{\hat{M}} = (\hat{M}, F_{\hat{M}})$ associates the F-module over $o_L[\![z]\!][1/\pi]$ of which $\underline{\hat{M}}$ is a "canonical good model".

Hypothesis 2.19. *There is an exact functor*

$$s\colon \mathrm{FMod}(o_L[\![z]\!][1/\pi]) \to \underline{\mathcal{S}}$$

such that $\dim_{\ell(\!(z)\!)}(\mathbb{H}_{\mathrm{st}} \circ s)(\underline{M}) = \mathrm{rk}_{o_L[\![z]\!][1/\pi]}(\underline{M})$ *for* $\underline{M} \in \mathrm{FMod}(o_L[\![z]\!][1/\pi])$, *and such that the diagram of categories and functors*

$$\begin{array}{ccc} & \mathrm{FMod}(o_L[\![z]\!][1/\pi]) & \\ \nearrow & & \searrow^{s} \\ \left(\begin{array}{c}\text{effective local}\\ \text{shtukas over } o_L\end{array}\right) & \xrightarrow{\quad \underline{i} \quad} & \underline{\mathcal{S}} \end{array}$$

is commutative (up to equivalence of functors).

Here the category $\mathrm{FMod}(o_L[\![z]\!][1/\pi])$ was defined in (1.4). The hypothetical functor s generalizes the functor which assigns a local shtuka over o_L to (good models of) good-reduction Anderson motives (see [41], 2.1.4).

Regarding the exact sequence $0 \to N \otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi] \to M(\varphi') \otimes_{L[z]} o_L[\![z]\!][1/\pi] \to M(\varphi) \otimes_{L[z]} o_L[\![z]\!][1/\pi] \to 0$ which we want to transfer via $\mathbb{H}_{\mathrm{st}}$ into the category $M_{\ell(\!(z)\!)}(F, \mathfrak{q}, \mathcal{N})$, we make the following

Remark 2.20.

– According to the above Hypothesis 2.19, applying $\mathbb{H}_{\mathrm{st}} \circ s$ to $N \otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi]$ will in particular give a z-isocrystal with Hodge-Pink structure whose underlying $\ell(\!(z)\!)$-vector space is of dimension 1. Therefore, by 2.17, the object $(\mathbb{H}_{\mathrm{st}} \circ s)(N \otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi])$ of $M_{\ell(\!(z)\!)}(F, \mathfrak{q}, \mathcal{N})$ has trivial monodromy operator (which means that, by 2.18, the semi-stable local shtuka $s(N \otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi])$ would have to come from a local shtuka over o_L).

– Since φ is of good reduction as a Drinfeld module it follows that the Anderson motive $M(\varphi)$ admits a good model $\mathcal{M}(\varphi)$ as an algebraic τ-sheaf à la Gardeyn, and even in a stronger sense. In order to explain this, we study the additional semi-linear structure of $M(\varphi)$ which is given by

$$\tau\colon M(\varphi) \to M(\varphi), \quad m \mapsto \tau m.$$

First of all, as was mentioned before, the L-vector space $M(\varphi) = L[\tau]$ becomes an $L[z]$-module via $zf = f\varphi_z$ for $f \in L[z]$. As such, $M(\varphi)$ is finite free of rank 2. The map $\tau\colon M(\varphi) \to M(\varphi)$ induces an $L[z]$-linear map

$$\tau^{\mathrm{lin}}\colon M(\varphi) \otimes_{L[z],\sigma} L[z] \to M(\varphi), \quad m \otimes f \mapsto f\tau m,$$

which, fixing the basis $(1,\tau)$ of $M(\varphi)$, is described by the matrix $\left(\begin{smallmatrix} 0 & z-\zeta \\ 1 & 0 \end{smallmatrix}\right)$. Let $C = \mathrm{coker}(\tau^{\mathrm{lin}})$ be the cokernel of τ^{lin}. We claim that C is a one-dimensional L-vector space and that $(z-\zeta)C = 0$. Indeed, clearly there is an isomorphism

$$C \simeq L[z]^2/\left(\begin{smallmatrix} 0 & z-\zeta \\ 1 & 0 \end{smallmatrix}\right)L[z]^2$$

which is induced by our choice of basis, and the projection $L[z]^2 \to L[z]$, $(a,b) \mapsto a$, induces an isomorphism $L[z]^2/\left(\begin{smallmatrix} 0 & z-\zeta \\ 1 & 0 \end{smallmatrix}\right)L[z]^2 \simeq L[z]/(z-\zeta)$; the latter is isomorphic to L via $L[z] \to L$, $z \mapsto \zeta$, so our claim follows.

Next we claim that the $o_L[z]$-module $\mathcal{M}(\varphi) = o_L[z]^2$ together with the injective $o_L[z]$-linear map

$$F_{\mathcal{M}(\varphi)} \colon o_L[z]^2 \otimes_{o_L[z],\sigma} o_L[z] \to o_L[z]^2, \quad \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \otimes 1 \mapsto \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \quad \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \otimes 1 \mapsto (z-\zeta)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right),$$

is a good model for $M(\varphi)$ in the sense that

- there is an isomorphism of $L[z]$-modules $\mathcal{M}(\varphi) \otimes_{o_L} L \simeq M(\varphi)$ which is compatible with $F_{\mathcal{M}(\varphi)} \otimes \mathrm{id}$ on $\mathcal{M}(\varphi) \otimes_{o_L} L$ and τ^{lin} on $M(\varphi)$,
- $\mathrm{coker}(F_{\mathcal{M}(\varphi)})$ is an o_L-module of rank 1 and is annihilated by $z-\zeta$.

The first item is clear, and for the second item it remains to verify that we may imitate the above argument with $L[z]$ replaced by $o_L[z]$. Indeed, we have an isomorphism $\mathrm{coker}(F_{\mathcal{M}(\varphi)}) \simeq o_L[z]^2/\left(\begin{smallmatrix} 0 & z-\zeta \\ 1 & 0 \end{smallmatrix}\right)o_L[z]^2$ which is induced by the canonical basis, and by virtue of the isomorphism $o_L[z]^2/\left(\begin{smallmatrix} 0 & z-\zeta \\ 1 & 0 \end{smallmatrix}\right)o_L[z]^2 \xrightarrow{\simeq} o_L[z]/(z-\zeta)$ induced by the projection $(a,b) \mapsto a$ it remains to show that $o_L[z]/(z-\zeta)$ is isomorphic to o_L via the map $o_L[z] \to o_L$, $z \mapsto \zeta$. So let $f \in o_L[z]$ be such that $f(\zeta) = 0$; here we may interpret f as an element of $L[z]$, so that we find a unique $g \in L[z]$ verifying $f = (z-\zeta)g$; we may further interpret this equation as being valid inside $L\langle z\rangle$, so that $||g|| \leq 1$ (for example, by [9], 1.2/8) where $||\cdot||$ denotes the Gauss-Norm of $L\langle z\rangle$, i.e., $g \in o_L[z]$. We may conclude (see [41], 2.1.4) that the (z)-adic completion $(\mathcal{M}(\varphi) \otimes_{o_L[z]} o_L[\![z]\!], F_{\mathcal{M}(\varphi)} \otimes \mathrm{id})$ of the model $(\mathcal{M}(\varphi), F_{\mathcal{M}(\varphi)})$ is an effective local shtuka over o_L, and we see that this local shtuka verifies

$$(\mathcal{M}(\varphi) \otimes_{o_L[z]} o_L[\![z]\!]) \otimes_{o_L[\![z]\!]} o_L[\![z]\!][1/\pi] \simeq (M(\varphi) \otimes_{L[z]} L\langle z\rangle) \otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi];$$

this isomorphism is compatible with $(F^{\mathrm{sl}}_{\mathcal{M}(\varphi)} \otimes \sigma_{o_L[\![z]\!]}) \otimes \sigma_{o_L[\![z]\!][1/\pi]}$ and $(\tau \otimes \sigma_{L\langle z\rangle}) \otimes \sigma_{o_L[\![z]\!]}[1/\pi]$. Finally, applying the functor $\mathbb{H}$ to the local shtuka $(\mathcal{M}(\varphi) \otimes_{o_L[z]} o_L[\![z]\!], F_{\mathcal{M}(\varphi)} \otimes \mathrm{id})$ gives the z-isocrystal (D, F_D) where $D = \ell(\!(z)\!)^2$ and where $F_D \colon \ell(\!(z)\!)^2 \otimes_{\ell(\!(z)\!),\sigma} \ell(\!(z)\!) \to \ell(\!(z)\!)^2$ is, with respect to the canonical basis of $\ell(\!(z)\!)^2$, described by the matrix $\left(\begin{smallmatrix} 0 & z \\ 1 & 0 \end{smallmatrix}\right)$. In particular, the monodromy operator of the corresponding object of $M_{\ell(\!(z)\!)}(F, \mathfrak{q}, \mathcal{N})$ is trivial.

- The middle term $M(\varphi') \otimes_{L[z]} o_L[\![z]\!][1/\pi]$ is of bad-reduction origin and therefore should certainly give rise to an object of $\underline{\mathcal{S}}$ which is *properly semi-stable*, i.e., it should not eventually turn out to come from a local shtuka over o_L like the right-hand term $M(\varphi)\otimes_{L[z]} o_L[\![z]\!][1/\pi]$ does (see the previous item). However, below we will see that, in fact, the z-isocrystal $(\mathbb{H}_{\mathrm{st}}\circ s)(M(\varphi')\otimes_{L[z]} o_L[\![z]\!][1/\pi])$ has trivial monodromy. –

Now we finally study the situation on the level of the associated z-isocrystals. To begin with, applying the functor s to our exact sequence

$$0\to N\otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi]\to M(\varphi')\otimes_{L[z]} o_L[\![z]\!][1/\pi]\to M(\varphi)\otimes_{L[z]} o_L[\![z]\!][1/\pi]\to 0$$

inside $\mathrm{FMod}(o_L[\![z]\!][1/\pi])$ yields a short exact sequence $0\to s_N\to s_{M(\varphi')}\to s_{M(\varphi)}\to 0$ inside the hypothetical category $\underline{\mathcal{S}}$ of semi-stable local shtukas. Retaining our hypotheses 2.18, 2.19, and recalling what we have seen in section (2.2.5), the associated exact sequence

$$0\to\mathbb{H}_{\mathrm{st}}(s_N)\to\mathbb{H}_{\mathrm{st}}(s_{M(\varphi')})\to\mathbb{H}_{\mathrm{st}}(s_{M(\varphi)})\to 0$$

inside $M_{\ell(\!(z)\!)}(F,\mathfrak{q},\mathcal{N})$ is, in particular, a short exact sequence of z-isocrystals with Hodge-Pink structure, and we obtain the following commutative diagram of $\ell(\!(z)\!)$-vector spaces with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{H}_{\mathrm{st}}(s_N) & \xrightarrow{i} & \mathbb{H}_{\mathrm{st}}(s_{M(\varphi')}) & \xrightarrow{\mathrm{pr}} & \mathbb{H}_{\mathrm{st}}(s_{M(\varphi)}) & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle \mathcal{N}_{\mathbb{H}_{\mathrm{st}}(s_N)}=0} & & \Big\downarrow{\scriptstyle \mathcal{N}_{\mathbb{H}_{\mathrm{st}}(s_{M(\varphi')})}} & & \Big\downarrow{\scriptstyle \mathcal{N}_{\mathbb{H}_{\mathrm{st}}(s_{M(\varphi)})}=0} & & \\
0 & \longrightarrow & \mathbb{H}_{\mathrm{st}}(s_N) & \xrightarrow{i} & \mathbb{H}_{\mathrm{st}}(s_{M(\varphi')}) & \xrightarrow{\mathrm{pr}} & \mathbb{H}_{\mathrm{st}}(s_{M(\varphi)}) & \longrightarrow & 0
\end{array}$$

By virtue of the Snake Lemma there is an $\ell(\!(z)\!)$-linear map

$$d\colon \mathbb{H}_{\mathrm{st}}(s_{M(\varphi)})\to\mathbb{H}_{\mathrm{st}}(s_N)$$

such that $\mathcal{N}_{\mathbb{H}_{\mathrm{st}}(s_{M(\varphi')})}=i\circ d\circ\mathrm{pr}$. From this, by

$$\begin{aligned}
i(d(F^{\mathrm{sl}}_{\mathbb{H}_{\mathrm{st}}(s_{M(\varphi)})}(\mathrm{pr}(y)))) &= i(d(\mathrm{pr}(F^{\mathrm{sl}}_{\mathbb{H}_{\mathrm{st}}(s_{M(\varphi')})}(y))))\\
&= \mathcal{N}_{\mathbb{H}_{\mathrm{st}}(s_{M(\varphi')})}(F^{\mathrm{sl}}_{\mathbb{H}_{\mathrm{st}}(s_{M(\varphi')})}(y))\\
&= \lambda_{\mathbb{H}_{\mathrm{st}}(s_{M(\varphi')})}F^{\mathrm{sl}}_{\mathbb{H}_{\mathrm{st}}(s_{M(\varphi')})}(\mathcal{N}_{\mathbb{H}_{\mathrm{st}}(s_{M(\varphi')})}(y))\\
&= i(\lambda_{\mathbb{H}_{\mathrm{st}}(s_{M(\varphi')})}F^{\mathrm{sl}}_{\mathbb{H}_{\mathrm{st}}(s_N)}(d(\mathrm{pr}(y))))
\end{aligned}$$

for every $y\in\mathbb{H}_{\mathrm{st}}(s_{M(\varphi')})$, it follows that

$$d\circ F^{\mathrm{sl}}_{\mathbb{H}_{\mathrm{st}}(s_{M(\varphi)})}=\lambda_{\mathbb{H}_{\mathrm{st}}(s_{M(\varphi')})}F^{\mathrm{sl}}_{\mathbb{H}_{\mathrm{st}}(s_N)}\circ d.$$

In particular, the map $F^{\mathrm{sl}}_{\mathbb{H}_{\mathrm{st}}(s_{M(\varphi)})}\colon\mathbb{H}_{\mathrm{st}}(s_{M(\varphi)})\to\mathbb{H}_{\mathrm{st}}(s_{M(\varphi)})$ restricts to a map

$$F^{\mathrm{sl}}_{\mathbb{H}_{\mathrm{st}}(s_{M(\varphi)})}\colon\ \ker(d)\to\ker(d)$$

which of course is again semi-linear; it follows at once that the corresponding $\ell(\!(z)\!)$-linear map $\sigma^* \ker(d) \to \ker(d)$ is a monomorphism of $\ell(\!(z)\!)$-vector spaces, hence an isomorphism. We may conclude that the $\ell(\!(z)\!)$-linear subspace $\ker(d) \subseteq \mathbb{H}_{\mathrm{st}}(s_{M(\varphi)})$ gives rise to a sub-z-isocrystal of $\mathbb{H}_{\mathrm{st}}(s_{M(\varphi)})$; note that we ignore Hodge-Pink structures in this place.

Our aim is to show that the subspace $\ker(d)$ equals $\mathbb{H}_{\mathrm{st}}(s_{M(\varphi)})$. In order to achieve this we recall that, according to our Hypothesis 2.18, 2.19 as well as the above remarks on the structure of $M(\varphi)$, the underlying z-isocrystal of $\mathbb{H}_{\mathrm{st}}(s_{M(\varphi)})$ is isomorphic to the z-isocrystal

$$\underline{D} = \left(\ell(\!(z)\!)^2, \quad \left(\begin{smallmatrix} 0 & z \\ 1 & 0 \end{smallmatrix}\right) \cdot \sigma\right)$$

where $\left(\begin{smallmatrix} 0 & z \\ 1 & 0 \end{smallmatrix}\right) \cdot \sigma$ denotes the map $\ell(\!(z)\!)^2 \to \ell(\!(z)\!)^2$ which is, with respect to the canonical basis of $\ell(\!(z)\!)^2$, semi-linearly described by the matrix $\left(\begin{smallmatrix} 0 & z \\ 1 & 0 \end{smallmatrix}\right)$. However, the latter z-isocrystal is simple, i.e., it admits no nonzero proper subobjects; indeed, by [52], 2.4.5, the associated Dieudonné-$\ell^{\mathrm{alg}}(\!(z)\!)$-module

$$\underline{D} \otimes_{\ell(\!(z)\!)} \ell^{\mathrm{alg}}(\!(z)\!) = \left(\ell^{\mathrm{alg}}(\!(z)\!)^2, \quad \left(\begin{smallmatrix} 0 & z \\ 1 & 0 \end{smallmatrix}\right) \cdot \sigma\right)$$

has to be simple, and we may conclude that, consequently, $\underline{D}$ is simple, the latter being true since the field extension $\ell^{\mathrm{alg}}(\!(z)\!)/\ell(\!(z)\!)$ is faithfully flat; note that, according to loc. cit., the structure theory of z-isocrystals over an algebraically closed residue field very much parallels the corresponding theory over (residue fields of) p-adic fields as indicated in (2.1.3); see also [40], 3.6. Finally, we may conclude that the inclusion $\ker(d) \subseteq \mathbb{H}_{\mathrm{st}}(s_{M(\varphi')})$ has, in fact, to be an equality since, looking at dimensions, $\ker(d)$ has to be a nontrivial subobject of the simple z-isocrystal $\mathbb{H}_{\mathrm{st}}(s_{M(\varphi)})$, i.e., $\mathcal{N}_{\mathbb{H}_{\mathrm{st}}(s_{M(\varphi')})}$ is trivial, so that the z-isocrystal $\mathbb{H}_{\mathrm{st}}(s_{M(\varphi')})$ comes from a local shtuka over o_L, more precisely: there is an effective local shtuka $\underline{\hat{M}} = (\hat{M}, F_{\hat{M}})$ such that

$$M(\varphi') \otimes_{L[z]} o_L[\![z]\!][1/\pi] \simeq \hat{M} \otimes_{o_L[\![z]\!]} o_L[\![z]\!][1/\pi]$$

inside $\mathrm{FMod}(o_L[\![z]\!][1/\pi])$. However, according to 1.21, this is a contradiction since $M(\varphi') \otimes_{L[z]} L\langle z\rangle$ does not admit a good model.

3 Crystalline and semi-stable extension classes in mixed and equal characteristic

Let K be a p-adic field. In a first step we briefly discuss the 2-dimensional p-adic representation of $G_K = \mathrm{Gal}(K^{\mathrm{alg}}/K)$ given by the p-adic Tate module $T_p(E)$ of an elliptic curve E over K of split multiplicative reduction; see [3], [15].

3.1 Tate elliptic curves

Retaining the notation from (2.1), let K be a mixed-characteristic complete discretely valued field of prime residue characteristic $p > 0$, with perfect residue field $k = o_K/\mathfrak{m}_K$ where $o_K \subseteq K$ denotes the valuation ring of K and $\mathfrak{m}_K \subseteq o_K$ its sole maximal ideal; let $\pi \in K$ be a fixed uniformizer. To begin with, we cite two Theorems due to J. Tate.

Theorem 3.1 (Tate Elliptic Curves). *Let $q \in K^\times$ be such that $|q| < 1$, and let*

$$s_k(q) = \sum_{n=1}^{\infty} \frac{n^k q^n}{1-q^n}, \quad a_4(q) = -5s_3(q), \quad a_6(q) = \frac{5s_3(q) + 7s_5(q)}{12}.$$

The series $a_4(q)$ and $a_6(q)$ converge in K. Define the projective curve $E_q \subseteq \mathbb{P}^2_{K^{\mathrm{alg}}}$ by the Weierstraß equation

$$E_q\colon \qquad Y^2Z + XYZ = X^3 + a_4(q)XZ^2 + a_6(q)Z^3.$$

(i) E_q is an elliptic curve defined over K with discriminant $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$, and with j-invariant $j(E_q)$ whose q-expansion is given by

$$j(E_q) = \tfrac{1}{q} + 744 + 196884q + 21493760q^2 + \ldots \in \tfrac{1}{q} + \mathbb{Z}[\![q]\!]$$

(cf. [72], I.7.4).

(ii) The series

$$\begin{aligned} X(u,q) &= \sum_{n=-\infty}^{\infty} \frac{q^n u}{(1-q^n u)^2} - 2s_1(q), \\ Y(u,q) &= \sum_{n=-\infty}^{\infty} \frac{(q^n u)^2}{(1-q^n u)^3} + s_1(q) \end{aligned}$$

converge for every $u \in K^{\mathrm{alg}} - q^{\mathbb{Z}}$. They define a surjective and G_K-equivariant homomorphism of abelian groups

$$(K^{\mathrm{alg}})^\times \to E_q(K^{\mathrm{alg}}), \quad u \mapsto \begin{cases} (X(u,q), Y(u,q)) & \text{if } u \notin q^{\mathbb{Z}}, \\ O & \text{if } u \in q^{\mathbb{Z}}, \end{cases}$$

whose kernel equals $q^{\mathbb{Z}} \subseteq (K^{\mathrm{alg}})^\times$; in particular, for every algebraic field extension K'/K, it induces an isomorphism of abelian groups

$$(K')^\times / q^{\mathbb{Z}} \xrightarrow{\simeq} E_q(K').$$

Proof. See [72], Theorem V.3.1 and Remark V.3.2.1. □

Theorem 3.2 (p-adic Uniformization). *Suppose that k is finite. Let E/K be an elliptic curve with $|j(E)| > 1$.*

(i) There is a unique $q \in (K^{\mathrm{alg}})^\times$ with $|q| < 1$ such that E is isomorphic over K^{alg} to the Tate elliptic curve E_q. Further, this value of q lies in K.

(ii) Let q be chosen as in (i). Then E is isomorphic to E_q over K if and only if E has split multiplicative reduction.

Proof. See [72], V.5.3. □

Relying on Theorem 3.1, we next discuss the structure of $V_p(E_q) = T_p(E_q) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for a Tate elliptic curve E_q/K. To begin with, we recall that the p-adic Tate module

$$T_p(\mathbb{G}_{m,K}) = \varprojlim_n \mathbb{G}_m(K^{\mathrm{alg}})[p^n]$$

of the multiplicative group scheme $\mathbb{G}_{m,K}$ is a free $\mathbb{Z}_p$-module of rank 1. Let $e = (\varepsilon^{(n)})_n \in T_p(\mathbb{G}_{m,K})$ be a $\mathbb{Z}_p$-basis, i.e., let $\varepsilon^{(n)} \in (K^{\mathrm{alg}})^\times$ be a primitive p^n-th root of unity for every n, subject to the relations $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$; in particular $\varepsilon^{(0)} = 1$, $\varepsilon^{(1)} \neq 1$; the $\mathbb{Z}_p$-module $T_p(\mathbb{G}_{m,K})$ carries an action of G_K which is given by the cyclotomic character $\chi\colon G_K \to \mathbb{Z}_p^\times$, and the resulting G_K-module is denoted $\mathbb{Z}_p(1)$; namely, composing the natural action

$$G_K \to \mathrm{Aut}_{\mathbb{Z}_p}(\mathbb{Z}_p(1)), \quad \rho \mapsto ((\zeta_{p^n})_n \mapsto (\rho.\zeta_{p^n})_n),$$

with the isomorphism $\mathrm{Aut}_{\mathbb{Z}_p}(\mathbb{Z}_p(1)) \simeq \mathbb{Z}_p^\times$ belonging to the chosen $\mathbb{Z}_p$-basis $e = (\varepsilon^{(n)})_n$ yields the cyclotomic character $\chi\colon G_K \to \mathbb{Z}_p^\times$, i.e., $\rho.e = \chi(\rho)e = (\rho.\varepsilon^{(n)})_n$; we obtain the G_K-representation on $\mathbb{Z}_p$-linear maps

$$G_K \to \mathrm{Aut}_{\mathbb{Z}_p}(\mathbb{Z}_p(1)), \quad \rho \mapsto \begin{pmatrix} \mathbb{Z}_p(1) & \to & \mathbb{Z}_p(1) \\ x & \mapsto & \chi(\rho)x \end{pmatrix}.$$

Note that for every $\rho \in G_K$ the associated automorphism $\rho\colon \mathbb{Z}_p(1) \to \mathbb{Z}_p(1)$ is independent of the chosen $\mathbb{Z}_p$-basis e, since every coordinate-change relation takes place inside the commutative ring $\mathbb{Z}_p$.

Let $\mathbb{Q}_p(1) = \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, endowed with the induced $\mathbb{Q}_p$-linear action of G_K.

Proposition 3.3. *Let $q \in K^\times$ be such that $|q| < 1$, and let E_q/K be the associated elliptic curve from 3.1. Then there is a short exact sequence of G_K-equivariant $\mathbb{Z}_p$-linear maps*

$$0 \to \mathbb{Z}_p(1) \xrightarrow{i} T_p(E_q) \xrightarrow{\mathrm{pr}} \mathbb{Z}_p \to 0;$$

in particular, there is an extension $0 \to \mathbb{Q}_p(1) \to V_p(E_q) \to \mathbb{Q}_p \to 0$ inside the abelian category $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$.

It follows immediately that fixing the $\mathbb{Z}_p$-basis $(i(1), \mathrm{pr}(1))$ of $T_p(E_q)$ the action of $\rho \in G_K$ is given by a matrix of the form $\left(\begin{smallmatrix} \chi(\rho) & * \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Gl}_2(\mathbb{Z}_p)$; below we will further analyze this.

Proof. By Tate's Theorem 3.1 there is an isomorphism of G_K-modules $(K^{\mathrm{alg}})^\times/q^{\mathbb{Z}} \xrightarrow{\simeq} E_q(K^{\mathrm{alg}})$ which corresponds to a G_K-equivariant short exact sequence of abelian groups

$$0 \to q^{\mathbb{Z}} \to (K^{\mathrm{alg}})^\times \to E_q(K^{\mathrm{alg}}) \to 0;$$

here we note that, as q lies inside the base field K, the G_K-action on $q^{\mathbb{Z}}$ has to be trivial. Let $n \geq 1$; applying the Snake Lemma to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & q^{\mathbb{Z}} & \longrightarrow & (K^{\mathrm{alg}})^\times & \longrightarrow & E_q(K^{\mathrm{alg}}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p^n} & & \downarrow{\scriptstyle p^n} & & \downarrow{\scriptstyle p^n} & & \\ 0 & \longrightarrow & q^{\mathbb{Z}} & \longrightarrow & (K^{\mathrm{alg}})^\times & \longrightarrow & E_q(K^{\mathrm{alg}}) & \longrightarrow & 0 \end{array}$$

yields an exact sequence of $\mathbb{Z}/p^n$-modules

$$0 \to q^{\mathbb{Z}}[p^n] \to (K^{\mathrm{alg}})^\times[p^n] \to E_q(K^{\mathrm{alg}})[p^n] \to$$

$$\to q^{\mathbb{Z}}/(q^{\mathbb{Z}})^{p^n} \to (K^{\mathrm{alg}})^\times/((K^{\mathrm{alg}})^\times)^{p^n} \to E_q(K^{\mathrm{alg}})/p^n E_q(K^{\mathrm{alg}}) \to 0;$$

it is clear that $q^{\mathbb{Z}}[p^n]$ and $(K^{\mathrm{alg}})^\times/((K^{\mathrm{alg}})^\times)^{p^n}$ have to be trivial, the latter since for every given nonzero $x \in K^{\mathrm{alg}}$ the polynomial $u^{p^n} - x \in K^{\mathrm{alg}}[u]$ splits up into linear factors. We obtain a short exact sequence of $\mathbb{Z}/p^n$-modules

$$1 \to (K^{\mathrm{alg}})^\times[p^n] \to E_q(K^{\mathrm{alg}})[p^n] \to q^{\mathbb{Z}}/(q^{\mathbb{Z}})^{p^n} \to 1$$

which, in fact, is G_K-equivariant. Letting n vary, this gives a projective system of short exact sequences, the transition maps being induced by multiplication with p;

we observe that these are G_K-equivariant, and that the Mittag-Leffler condition is met, the latter since $p\colon (K^{\mathrm{alg}})^\times[p^{n+1}] \to (K^{\mathrm{alg}})^\times[p^n]$ is surjective for every n; we may summarize that taking the limit yields a short exact sequence of $\mathbb{Z}_p$-modules

$$0 \to \mathbb{Z}_p(1) \to T_p(E_q) \to q^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \to 0;$$

the abelian group $q^{\mathbb{Z}}$ is canonically isomorphic to $\mathbb{Z}$, so that $q^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ equals $\mathbb{Z}_p$, having trivial G_K-action. The proof is complete. □

We will see below that the exact sequence $0 \to \mathbb{Q}_p(1) \to V_p(E_q) \to \mathbb{Q}_p \to 0$ does, in fact, give rise to a Yoneda extension class of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$ for the abelian category $\mathrm{Rep}_{\mathrm{st}}(G_K)$ of semi-stable p-adic representations of G_K.

3.2 Yoneda extension classes of p-adic representations, and Galois cohomology

3.2.1 Kummer theory

Let M be a fixed topological G_K-module, i.e., a topological abelian group which is equipped with a continuous action of the pro-finite group G_K. We recall (cf. Appendix B of [65]) that a *1-cocycle* (resp., a *1-coboundary*) is a map of sets $\gamma\colon G_K \to M$ such that $\gamma_{\rho'\rho} = \gamma_{\rho'} + \rho'.\gamma_\rho$ for all $\rho, \rho' \in G_K$ (resp., such that $\gamma_\rho = \rho.m - m$ for a suitable $m \in M$ and all $\rho \in G_K$); it is well-known that the 1-cocycles constitute an abelian group under pointwise operation, of which the 1-coboundaries are a subgroup; by definition, the abelian group $C^1(G_K, \mathbb{Q}_p(1))$ (resp., $B^1(G_K, \mathbb{Q}_p(1))$) consists of all *continuous* 1-cocyles (resp., of all those *continuous* 1-cocycles which are 1-coboundaries), and one defines the group $H^1(G_K, M)$, called the *1st cohomology group*, to be the quotient $C^1(G_K, M)/B^1(G_K, M)$; note that in the present context the group $H^1(G_K, M)$ is abelian. We commence by stating the well-known

Lemma 3.4 (Kummer theory). *(i) ([69]) For every $n \geq 0$ there is a natural map of abelian groups $\delta_n\colon K^\times \to H^1(G_K, \mu_{p^n}(K^{\mathrm{alg}}))$ which is defined as follows: for a given $q \in K^\times$ choose a p^n-th root $q^{1/p^n} \in (K^{\mathrm{alg}})^\times$ of q; then $\delta_n(q)$ is defined to be the class of the* 1*-cocycle $G_K \to \mu_{p^n}(K^{\mathrm{alg}})$, $\rho \mapsto \rho.q^{1/p^n}/q^{1/p^n}$; the map δ_n induces an isomorphism of $\mathbb{Z}/p^n$-modules*

$$K^\times/(K^\times)^{p^n} \to H^1(G_K, \mu_{p^n}(K^{\mathrm{alg}})).$$

(ii) ([3]) There is a natural isomorphism of $\mathbb{Z}_p$-modules

$$\begin{aligned} \widehat{K^\times} = \varprojlim_{(n)} K^\times/(K^\times)^{p^n} &\to H^1(G_K, \mathbb{Z}_p(1)), \\ (\overline{q_n})_n &\mapsto \text{class of } (\rho \mapsto (\rho.q_n^{1/p^n}/q_n^{1/p^n})_n); \end{aligned}$$

in particular, this induces an isomorphism of $\mathbb{Q}_p$-vector spaces

$$\widehat{K^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\cong} H^1(G_K, \mathbb{Q}_p(1))$$

where the right-hand side becomes a $\mathbb{Q}_p$-vector space by pointwise operations on 1-cocycles.

Proof. Let $n \geq 0$ be fixed; the finite group $\mu_{p^n}(K^{\text{alg}})$ carries the discrete topology, and (for example, by [71], B.2.2.) from the short exact sequence of abelian groups

$$1 \to \mu_{p^n}(K^{\text{alg}}) \to (K^{\text{alg}})^\times \xrightarrow{p^n} (K^{\text{alg}})^\times \to 1$$

we obtain the long exact cohomology sequence

$$\mu_{p^n}(K^{\text{alg}})^{G_K} \to ((K^{\text{alg}})^\times)^{G_K} \xrightarrow{p^n} ((K^{\text{alg}})^\times)^{G_K} \xrightarrow{\delta}$$

$$\xrightarrow{\delta} H^1(G_K, \mu_{p^n}(K^{\text{alg}})) \to H^1(G_K, (K^{\text{alg}})^\times) \to \ldots$$

where $H^1(G_K, (K^{\text{alg}})^\times)$ is trivial by "Hilbert 90" (see [69], X.1.2); we obtain a short exact sequence

$$1 \to K^\times \xrightarrow{p^n} K^\times \xrightarrow{\delta} H^1(G_K, \mu_{p^n}(K^{\text{alg}})) \to 1$$

which proves (i). In order to explain (ii), we first remark that by [65], B.2.3, the unique map

$$H^1(G_K, \mathbb{Z}_p(1)) \to \varprojlim_{(n)} H^1(G_K, \mu_{p^n}(K^{\text{alg}}))$$

making the diagram

$$\begin{array}{ccc} C^1(G_K, \mathbb{Z}_p(1)) & \longrightarrow & \varprojlim_{(n)} C^1(G_K, \mu_{p^n}(K^{\text{alg}})) \\ \downarrow & & \downarrow \\ H^1(G_K, \mathbb{Z}_p(1)) & \longrightarrow & \varprojlim_{(n)} H^1(G_K, \mu_{p^n}(K^{\text{alg}})) \end{array}$$

commutative is an isomorphism of abelian groups; in particular, by (i), this induces a natural isomorphism

$$H^1(G_K, \mathbb{Z}_p(1)) \xrightarrow{\cong} \widehat{K^\times};$$

the induced map

$$\widehat{K^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^1(G_K, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is clearly again an isomorphism; finally, by [65], B.2.4, we realize that the $\mathbb{Q}_p$-vector space $H^1(G_K, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is naturally isomorphic to $H^1(G_K, \mathbb{Q}_p(1))$. □

3.2.2 The Baer sum

Next we discuss the $\mathbb{Q}_p$-vector space $\operatorname{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p, \mathbb{Q}_p(1))$ whose underlying abelian group consists of the Yoneda extension classes of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$ inside the abelian category $\operatorname{Rep}_{\mathbb{Q}_p}(G_K)$; our discussion will follow closely [62]. Recall that the category $\operatorname{Rep}_{\mathbb{Q}_p}(G_K)$, whose objects are finite-dimensional $\mathbb{Q}_p$-vector spaces endowed with a continuous G_K-action (morphisms being G_K-equivariant $\mathbb{Q}_p$-linear maps), becomes an abelian category in a natural way since kernels, cokernels, images, and coimages of G_K-equivariant $\mathbb{Q}_p$-linear maps naturally acquire a continuous G_K-action by $\mathbb{Q}_p$-linear automorphisms.

Let $A, B \in \operatorname{Rep}_{\mathbb{Q}_p}(G_K)$ be two fixed p-adic representations. Two extensions $0 \to B \to V \to A \to 0$ and $0 \to B \to V' \to A \to 0$ inside $\operatorname{Rep}_{\mathbb{Q}_p}(G_K)$ are said to be *Yoneda equivalent* if there is a G_K-equivariant isomorphism of $\mathbb{Q}_p$-vector spaces $f\colon V \to V'$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & V & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & {\scriptstyle f}\downarrow{\scriptstyle \simeq} & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & V' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

commute; this clearly defines an equivalence relation on the set of extensions of the type $0 \to B \to \cdot \to A \to 0$, and the set of equivalence classes is denoted by $\operatorname{Ext}^1_{\mathbb{Q}_p[G_K]}(A, B)$. This set is made into an abelian group via the *Baer sum*; in order to describe this group structure, we recall that the direct sum $A \oplus B$ of A and B is given in an obvious way by the direct sum of underlying vector spaces and, like in any abelian category, at the same time gives rise to both a categorial product and a categorial coproduct; let $C \in \operatorname{Rep}_{\mathbb{Q}_p}(G_K)$ be a third object, and suppose that there are morphisms $a\colon A \to C$ and $b\colon B \to C$ inside $\operatorname{Rep}_{\mathbb{Q}_p}(G_K)$; the *pullback* $A \times_C B$ with respect to a, b is characterized by the exact sequence

$$0 \to A \times_C B \to A \oplus B \overset{(a,-b)}{\to} C$$

inside $\operatorname{Rep}_{\mathbb{Q}_p}(G_K)$; the direct-sum G_K-action on $A \oplus B$ restricts to a G_K-action on $A \times_C B$, i.e., the action of $\rho \in G_K$ on $(x, y) \in A \times_C B$ is given by

$$\rho.(x, y) = (\rho.x, \rho.y);$$

furthermore, there is a natural isomorphism of abelian groups

$$\operatorname{Hom}(T, A \times_C B) \simeq \operatorname{Hom}(T, A) \times_{\operatorname{Hom}(T,C)} \operatorname{Hom}(T, B)$$

for every $T \in \operatorname{Rep}_{\mathbb{Q}_p}(G_K)$, which is induced by the projections of $A \oplus B$ (seen as a product); dually, suppose that there are arrows $a'\colon D \to A$, $b'\colon D \to B$ for some object $D \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$; then the *pushout* $A \amalg^D B$ with respect to a', b' is characterized by the exact sequence

$$D \overset{(a',-b')}{\to} A \oplus B \to A \amalg^D B \to 0$$

inside $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$; the direct-sum G_K-action of $A \oplus B$ restricts to a G_K-action of the $\mathbb{Q}_p$-linear subspace $\mathrm{im}(a', -b') \subseteq A \oplus B$, so that one obtains an induced G_K-action on $A \amalg^D B$; more precisely, a given $\rho \in G_K$ acts on $\overline{(x,y)} \in A \amalg^D B$ by

$$\rho.\overline{(x,y)} = \overline{\rho.(x,y)} = \overline{(\rho.x, \rho.y)};$$

furthermore, there is a natural isomorphism of abelian groups

$$\mathrm{Hom}(A \amalg^D B, T) \simeq \mathrm{Hom}(A,T) \times_{\mathrm{Hom}(D,T)} \mathrm{Hom}(B,T)$$

for every $T \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, which is induced by the coprojections of $A \oplus B$ (seen as a coproduct). Let $\xi \in \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(A,B)$ be the class of

$$0 \to B \overset{i}{\to} V \overset{\mathrm{pr}}{\to} A \to 0,$$

and let $f\colon C \to A$, $g\colon B \to D$ be morphisms inside $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$; one defines $f^*(\xi) = \xi \cdot f$ to be the class of

$$0 \to B \overset{(0,i)}{\to} V \times_A C \to C \to 0$$

in $\mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(C,B)$, and $g_*(\xi) = g \cdot \xi$ to be the class of

$$0 \to D \to D \amalg^B V \overset{(0,\mathrm{pr})}{\to} A \to 0$$

in $\mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(A,D)$; one can show that

$$g \cdot (\xi \cdot f) = (g \cdot \xi) \cdot f$$

(cf. [62], Lemma 2, p. 230), i.e., the expression $g \cdot \xi \cdot f$ is well-defined. Now we can describe the Baer sum of two classes ξ and η in $\mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(A,B)$ where ξ (resp., η) is induced by the extension $0 \to B \to V \to A \to 0$ (resp., by $0 \to B \to V' \to A \to 0$) say; let $\xi \oplus \eta$ be the class in $\mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(A \oplus A, B \oplus B)$ of the induced direct-sum sequence $0 \to B \oplus B \to V \oplus V' \to A \oplus A \to 0$; furthermore, let $d = (\mathrm{id}, \mathrm{id})\colon A \to A \oplus A$ be the diagonal, and $s = (\mathrm{id}, \mathrm{id})\colon B \oplus B \to B$ the sum; now the *Baer sum* of ξ and η is defined to be

$$\xi + \eta = s \cdot (\xi \oplus \eta) \cdot d;$$

the resulting map

$$+\colon \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(A,B) \times \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(A,B) \to \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(A,B)$$

makes the set $\mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(A,B)$ into an abelian group whose zero element is given by the class of the canonical split extension $0 \to B \to B \oplus A \to A \to 0$ where $B \oplus A$ carries the direct-sum G_K-action; furthermore, the additive inverse of ξ is given by $(-\mathrm{id}_B)\cdot\xi = \xi\cdot(-\mathrm{id}_A)$; for all this, see [62], section 2. The $\mathbb{Q}_p$-vector space structure of the abelian group $\mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p, \mathbb{Q}_p(1))$ will be discussed below.

This discussion carries over verbatim to the abelian category $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ of $\mathbb{Z}_p$-linear p-adic representations of G_K whose objects are *finitely generated* (not necessarily free) $\mathbb{Z}_p$-modules which are endowed with a continuous action of G_K; morphisms in the category $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ are G_K-equivariant $\mathbb{Z}_p$-linear maps. We denote by $\mathrm{Ext}^1_{\mathbb{Z}_p[G_K]}(\mathbb{Z}_p, \mathbb{Z}_p(1))$ the abelian group of Yoneda extension classes of $\mathbb{Z}_p$ by $\mathbb{Z}_p(1)$, the group law being given by the Baer sum.

Remark. Even though the underlying $\mathbb{Z}_p$-module of an object of $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ is not in general free, one observes that given an extension $0 \to \mathbb{Z}_p(1) \to M \to \mathbb{Z}_p \to 0$ of $\mathbb{Z}_p$-linear p-adic representations the $\mathbb{Z}_p$-module M is, in fact, always free of rank 2; indeed, by the Snake Lemma, the functor on $\mathbb{Z}_p$-modules defined by $P \mapsto T(P) = \ker(P \to P \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ is left-exact, so that M is torsion-free and therefore free. –

3.2.3 Yoneda extensions and Galois cohomology

Following [75], 2.3.2 (see also [18], 5.1), we define a map

$$C^1(G_K, \mathbb{Z}_p(1)) \to \mathrm{Ext}^1_{\mathbb{Z}_p[G_K]}(\mathbb{Z}_p, \mathbb{Z}_p(1))$$

as follows: the image of a 1-cocycle $c\colon G_K \to \mathbb{Z}_p(1)$ is defined to be the class of the extension

$$e_c\colon \quad 0 \to \mathbb{Z}_p(1) \xrightarrow{i_c} \mathbb{Z}_p(1) \oplus \mathbb{Z}_p \xrightarrow{\mathrm{pr}_c} \mathbb{Z}_p \to 0$$

where $\rho \in G_K$ acts on $\mathbb{Z}_p(1) \oplus \mathbb{Z}_p$ via

$$\rho\colon \quad (y,x) \mapsto (\rho.y + xc_\rho, x)$$

for $x \in \mathbb{Z}_p$, $y \in \mathbb{Z}_p(1)$; by virtue of the cocycle condition on c, one immediately obtains $\rho'.(\rho.(y,x)) = (\rho'\rho).(y,x)$ for $\rho, \rho' \in G_K$, and $c_{\mathrm{id}} = c_{\mathrm{id}\cdot\mathrm{id}}$ implies $\mathrm{id}.(y,x) = (y,x)$; we further remark that by the continuity of c the induced $\mathbb{Z}_p$-linear action

$$G_K \times (\mathbb{Z}_p(1) \oplus \mathbb{Z}_p) \to \mathbb{Z}_p(1) \oplus \mathbb{Z}_p$$

becomes continuous. Finally, observe that with respect to the basis $((e,0),(0,1))$ of $\mathbb{Z}_p(1) \oplus \mathbb{Z}_p$ (where the $\mathbb{Z}_p$-basis $e = (\varepsilon^{(n)})_n$ of $\mathbb{Z}_p(1)$ was chosen in section (3.1)) the action of $\rho \in G_K$ is described by the matrix $\left(\begin{smallmatrix} \chi(\rho) & c_\rho \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Gl}_2(\mathbb{Z}_p)$.

For later use we want to give a detailed explanation of the following well-known

Proposition 3.5 ([18]). *The map* $C^1(G_K, \mathbb{Z}_p(1)) \to \mathrm{Ext}^1_{\mathbb{Z}_p[G_K]}(\mathbb{Z}_p, \mathbb{Z}_p(1))$*,* $c \mapsto [e_c]$*, induces an isomorphism of abelian groups*

$$H^1(G_K, \mathbb{Z}_p(1)) \xrightarrow{\simeq} \mathrm{Ext}^1_{\mathbb{Z}_p[G_K]}(\mathbb{Z}_p, \mathbb{Z}_p(1)).$$

Proof. We have to show that $c \mapsto [e_c]$ defines a surjective group homomorphism with kernel $B^1(G_K, \mathbb{Z}_p(1))$. First of all, it is clear that the trivial cocycle is mapped to the class of the canonical split extension $0 \to \mathbb{Z}_p(1) \to \mathbb{Z}_p(1) \oplus \mathbb{Z}_p \to \mathbb{Z}_p \to 0$; in order to see that the (pointwise) sum of two cocycles $c, c' \colon G_K \to \mathbb{Z}_p(1)$ is mapped to the Baer sum $[e_c] + [e_{c'}]$, we study the latter, proceeding as follows: let $s \colon \mathbb{Z}_p(1) \oplus \mathbb{Z}_p(1) \to \mathbb{Z}_p(1)$ be the sum and $d \colon \mathbb{Z}_p \to \mathbb{Z}_p \oplus \mathbb{Z}_p$ the diagonal; from the commutative diagram inside $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ with exact bottom row

$$\begin{array}{ccccccccc} & & & \underline{i}=(0,i_c\oplus i_{c'}) \nearrow & X & \xrightarrow{\underline{\mathrm{pr}}} & \mathbb{Z}_p & & \\ & & & & \downarrow & & \downarrow d & & \\ e_c \oplus e_{c'}\colon & 0 \longrightarrow & \mathbb{Z}_p(1)^{\oplus 2} & \xrightarrow[i_c\oplus i_{c'}]{} & (\mathbb{Z}_p(1)\oplus\mathbb{Z}_p)^{\oplus 2} & \xrightarrow[\mathrm{pr}_c\oplus\mathrm{pr}_{c'}]{} & \mathbb{Z}_p^{\oplus 2} & \longrightarrow 0 & \end{array}$$

where

$$\begin{aligned} X &= (\mathbb{Z}_p(1) \oplus \mathbb{Z}_p)^{\oplus 2} \times_{\mathrm{pr}_c\oplus\mathrm{pr}_{c'}, \mathbb{Z}_p^{\oplus 2}, d} \mathbb{Z}_p \\ &= \{ \left(\begin{smallmatrix} ((y,x),(y',x')) \\ x'' \end{smallmatrix} \right) \in (\mathbb{Z}_p(1) \oplus \mathbb{Z}_p)^{\oplus 2} \oplus \mathbb{Z}_p,\ d(x'') = (\mathrm{pr}_c \oplus \mathrm{pr}_{c'})((y,x),(y',x')) \} \\ &= \{ \left(\begin{smallmatrix} ((y,x),(y',x')) \\ x'' \end{smallmatrix} \right) \in (\mathbb{Z}_p(1) \oplus \mathbb{Z}_p)^{\oplus 2} \oplus \mathbb{Z}_p,\ (x'', x'') = (x, x') \text{ in } \mathbb{Z}_p^{\oplus 2} \} \end{aligned}$$

one obtains a G_K-equivariant extension

$$0 \to \mathbb{Z}_p(1)^{\oplus 2} \xrightarrow{\underline{i}} X \xrightarrow{\underline{\mathrm{pr}}} \mathbb{Z}_p \to 0;$$

here $\underline{i} = (0, i_c \oplus i_{c'}) \colon \mathbb{Z}_p(1)^{\oplus 2} \to X$ is given by $(y, y') \mapsto \left(\begin{smallmatrix} ((y,0),(y',0)) \\ 0 \end{smallmatrix} \right)$, and $\underline{\mathrm{pr}} \colon X \to \mathbb{Z}_p$ is the projection onto the second component; the latter extension gives rise to a pullback diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathbb{Z}_p(1)^{\oplus 2} & \xrightarrow{\underline{i}} & X & \xrightarrow{\underline{\mathrm{pr}}} & \mathbb{Z}_p & \longrightarrow 0 \\ & s \downarrow & & \downarrow & \nearrow (0,\underline{\mathrm{pr}}) & & \\ & \mathbb{Z}_p(1) & \xrightarrow[\underline{j}]{} & Y & & & \end{array}$$

where

$$Y = \mathbb{Z}_p(1) \amalg^{s, \mathbb{Z}_p(1)^{\oplus 2}, \underline{i}} X = (\mathbb{Z}_p(1) \oplus X)/\mathrm{im}(s, -\underline{i}),$$

with coprojections induced by those of $\mathbb{Z}_p(1) \oplus X$; we obtain a G_K-equivariant extension

$$0 \to \mathbb{Z}_p(1) \xrightarrow{\underline{j}} Y \xrightarrow{(0,\underline{\mathrm{pr}})} \mathbb{Z}_p \to 0$$

where $\underline{j} \colon \mathbb{Z}_p(1) \to Y$ is given by $y \mapsto \overline{(y, 0)}$, and where $(0, \underline{\mathrm{pr}}) \colon Y \to \mathbb{Z}_p$ maps the residue class of $(y'', \left(\begin{smallmatrix} ((y,x),(y',x')) \\ x'' \end{smallmatrix} \right))$ to x''. It is well-known that every short exact sequence of finite free modules is split, i.e., we obtain a $\mathbb{Z}_p$-linear section $\underline{w}\colon \mathbb{Z}_p \to Y$ of the projection $(0, \underline{\mathrm{pr}})\colon Y \to \mathbb{Z}_p$ which in the present situation maps $x \in \mathbb{Z}_p$ to the residue class of $(0, \left(\begin{smallmatrix}((0,x),(0,x))\\ x\end{smallmatrix}\right))$; note that $\underline{w}$ is *not* in general G_K-equivariant; we further recall that $\underline{w}$ induces a direct-sum decomposition of Y being $Y = \mathrm{im}(\underline{j}) \oplus \mathrm{im}(\underline{w})$. We observe that the residue class inside Y of a given element $(y'', \left(\begin{smallmatrix}((y,x),(y',x'))\\ x''\end{smallmatrix}\right)) \in \mathbb{Z}_p(1) \oplus X$ admits the couple

$$(y + y' + y'', \left(\begin{smallmatrix}((0,x),(0,x'))\\ x''\end{smallmatrix}\right)) \in \mathbb{Z}_p(1) \oplus X$$

as a representative; the latter may be decomposed as $(y+y'+y'', 0)+(0, \left(\begin{smallmatrix}((0,x),(0,x'))\\ x''\end{smallmatrix}\right))$ and, in view of the above characterization of X, its residue class therefore is mapped to $(y + y' + y'', x)$ via the G_K-equivariant isomorphism of $\mathbb{Z}_p$-modules

$$Y = \mathrm{im}(\underline{j}) \oplus \mathrm{im}(\underline{w}) \xrightarrow{\simeq} \mathbb{Z}_p(1) \oplus \mathbb{Z}_p,$$

where $\mathbb{Z}_p(1) \oplus \mathbb{Z}_p$ is endowed with the Galois action from the definition of $e_{c+c'}$; the latter isomorphism fits into a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_p(1) & \xrightarrow{\underline{j}} & Y & \xrightarrow{(0,\underline{\mathrm{pr}})} & \mathbb{Z}_p & \longrightarrow & 0 \\
 & & \| & & \downarrow{\simeq} & & \| & & \\
0 & \longrightarrow & \mathbb{Z}_p(1) & \xrightarrow{i_{c+c'}} & \mathbb{Z}_p(1) \oplus \mathbb{Z}_p & \xrightarrow{\mathrm{pr}_{c+c'}} & \mathbb{Z}_p & \longrightarrow & 0
\end{array}$$

inside the category $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$, and we may summarize that the extension class $[e_{c+c'}]$ is Yoneda-equivalent with the Baer sum $[e_c] + [e_{c'}]$, as desired. Next we show that the map $c \mapsto [e_c]$ is surjective. Let $\xi \in \mathrm{Ext}^1_{\mathbb{Z}_p[G_K]}(\mathbb{Z}_p, \mathbb{Z}_p(1))$ be the class of the extension

$$0 \to \mathbb{Z}_p(1) \xrightarrow{i} V \xrightarrow{\mathrm{pr}} \mathbb{Z}_p \to 0;$$

since the underlying extension of $\mathbb{Z}_p$-modules is split, there is a $\mathbb{Z}_p$-linear section $w\colon \mathbb{Z}_p \to V$ of the projection $\mathrm{pr}\colon V \to \mathbb{Z}_p$ which induces a $\mathbb{Z}_p$-linear isomorphism

$$\begin{array}{rcl}
V = \mathrm{im}(i) \oplus \mathrm{im}(w) & \xrightarrow{\simeq} & \mathbb{Z}_p(1) \oplus \mathbb{Z}_p, \\
v = (v - (w\mathrm{pr})(v)) + (w\mathrm{pr})(v) & \mapsto & (y_v, \mathrm{pr}(v)),
\end{array}$$

where $y_v \in \mathbb{Z}_p(1)$ is uniquely determined by the condition $i(y_v) = v - (w\mathrm{pr})(v)$; we observe that $w(1) \in V$ is mapped to $(0, 1) \in \mathbb{Z}_p(1) \oplus \mathbb{Z}_p$ via the latter isomorphism; on the other hand, for a given $\rho \in G_K$ we have

$$\rho.w(1) = \rho.w(1) - (w\mathrm{pr})(\rho.w(1)) + (w\mathrm{pr})(\rho.w(1)) = \rho.w(1) - w(\rho.1) + w(\rho.1),$$

so that the element $\rho.w(1)$ is mapped to the couple $(\gamma_\rho, 1) \in \mathbb{Z}_p(1) \oplus \mathbb{Z}_p$, where $\gamma_\rho \in \mathbb{Z}_p(1)$ is uniquely determined by the condition $i(\gamma_\rho) = \rho.w(1) - w(1)$; from

$$i(\gamma_{\rho'} + \rho'.\gamma_\rho) = (\rho'\rho).w(1) - w(1)$$

for $\rho, \rho' \in G_K$ it immediately follows that the assignment $\rho \mapsto \gamma_\rho$ defines a 1-cocycle $G_K \to \mathbb{Z}_p(1)$ which, by the continuity of the G_K-action on V, is continuous, and which we denote by γ; in order to see that the G_K-action of $\mathbb{Z}_p(1) \oplus \mathbb{Z}_p$ is indeed given according to e_γ, we just need to remark that for every $y \in \mathbb{Z}_p(1)$ and $x \in \mathbb{Z}_p$ a given $\rho \in G_K$ does act on $i(y)+w(x)$ as $i(\rho.y+x\gamma_\rho)+w(x)$. We may summarize that $\xi = [e_\gamma]$. It remains to verify that $[e_c]$ is trivial if and only if c is a 1-coboundary; indeed, suppose that given $c \in C^1(G_K, \mathbb{Z}_p(1))$ there is some $\alpha \in \mathbb{Z}_p(1)$ such that $c_\rho = \rho.\alpha - \alpha$ for all $\rho \in G_K$, i.e., that c is a 1-coboundary; then for every $\rho \in G_K$ the element $(-\alpha, 1)$ is fixed by the $\mathbb{Z}_p$-linear automorphism

$$\rho\colon \mathbb{Z}_p(1) \oplus \mathbb{Z}_p \to \mathbb{Z}_p(1) \oplus \mathbb{Z}_p, \quad (y, x) \mapsto (\rho.y + x(\rho.\alpha - \alpha), x),$$

associated to c, and the $\mathbb{Z}_p$-linear map $\mathbb{Z}_p \to \mathbb{Z}_p(1) \oplus \mathbb{Z}_p$ defined by $1 \mapsto (-\alpha, 1)$ is a G_K-equivariant section of the projection $\mathbb{Z}_p(1) \oplus \mathbb{Z}_p \to \mathbb{Z}_p$, i.e., $[e_c] = 0$; conversely, suppose that the extension

$$e_c\colon 0 \to \mathbb{Z}_p(1) \xrightarrow{i} \mathbb{Z}_p(1) \oplus \mathbb{Z}_p \xrightarrow{\mathrm{pr}} \mathbb{Z}_p \to 0$$

admits a G_K-equivariant $\mathbb{Z}_p$-linear section $w\colon \mathbb{Z}_p \to \mathbb{Z}_p(1) \oplus \mathbb{Z}_p$ of pr; from this we may conclude that $\rho.w(1) = w(1)$ for every $\rho \in G_K$, and $w(1) = (y, 1)$ for some $y \in \mathbb{Z}_p$; therefore

$$0 = \rho.w(1) - w(1) = (\rho.y + c_\rho, 1) - (y, 1) = (\rho.y - y + c_\rho, 0)$$

for every $\rho \in G_K$, i.e., c has to be a 1-coboundary. □

The proof in particular shows that the map

$$H^1(G_K, \mathbb{Z}_p(1)) \to \mathrm{Ext}^1_{\mathbb{Z}_p[G_K]}(\mathbb{Z}_p, \mathbb{Z}_p(1)), \quad \bar{c} \mapsto [e_c],$$

does, in fact, give rise to an isomorphism of $\mathbb{Z}_p$-modules: since $C^1(G_K, \mathbb{Z}_p(1))$ is a $\mathbb{Z}_p$-module by pointwise operation and admits $B^1(G_K, \mathbb{Z}_p(1))$ as a $\mathbb{Z}_p$-submodule, we may argue on the level of 1-cocycles, and on the other hand we may restrict our attention to extensions of the type e_c for varying 1-cocycles c; now there can be only one $\mathbb{Z}_p$-module structure on $\mathrm{Ext}^1_{\mathbb{Z}_p[G_K]}(\mathbb{Z}_p, \mathbb{Z}_p(1))$ such that the 1-cocycle λc is mapped to $\lambda[e_c]$ for every $\lambda \in \mathbb{Z}_p$, and the above arguments show that indeed all needed axioms are met by the obvious candidate; more generally, this $\mathbb{Z}_p$-vector space structure may be described in terms of pullbacks (or, equivalently, pushouts): for a given $\lambda \in \mathbb{Z}_p$ and the class $\xi \in \mathrm{Ext}^1_{\mathbb{Z}_p[G_K]}(\mathbb{Z}_p, \mathbb{Z}_p(1))$ of $0 \to \mathbb{Z}_p(1) \to V \to \mathbb{Z}_p \to 0$ say, define $\lambda\xi$ to be $\lambda^*\xi = \xi \cdot \lambda$, i.e., the class of

$$0 \to \mathbb{Z}_p(1) \to V \times_{\mathbb{Z}_p, \lambda} \mathbb{Z}_p \to \mathbb{Z}_p \to 0;$$

see [23], A3.26(e); indeed, from $\xi = [e_c]$ it follows that $\lambda\xi = [e_{\lambda c}]$.

Replacing $\mathbb{Z}_p$ by $\mathbb{Q}_p$ everywhere, we obtain

Corollary 3.6. *The map* $C^1(G_K, \mathbb{Q}_p(1)) \to \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p, \mathbb{Q}_p(1))$, $c \mapsto [e_c]$, *induces an isomorphism of* $\mathbb{Q}_p$*-vector spaces*

$$H^1(K, \mathbb{Q}_p(1)) \simeq \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p, \mathbb{Q}_p(1)).$$

□

By virtue of the natural isomorphism of $\mathbb{Q}_p$-vector spaces

$$H^1(K, \mathbb{Q}_p(1)) \simeq H^1(K, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

the preceding corollary implies

Corollary 3.7. *There is a natural isomorphism of* $\mathbb{Q}_p$*-vector spaces*

$$\mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p, \mathbb{Q}_p(1)) \simeq \mathrm{Ext}^1_{\mathbb{Z}_p[G_K]}(\mathbb{Z}_p, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

□

The connection with Galois Cohomology also yields

Corollary 3.8. *There is a natural isomorphism of* $\mathbb{Z}_p$*-modules*

$$\widehat{K^\times} \simeq \mathrm{Ext}^1_{\mathbb{Z}_p[G_K]}(\mathbb{Z}_p, \mathbb{Z}_p(1));$$

which in particular induces an isomorphism of $\mathbb{Q}_p$*-vector spaces*

$$\widehat{K^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p, \mathbb{Q}_p(1)).$$

□

Finally, in the notation of Proposition 3.3, we may draw the following conclusion:

Corollary 3.9. *Let* $q \in K^\times$ *be such that* $|q| < 1$, *and let* E_q/K *be the corresponding Tate elliptic curve; fixing a basis of the* p*-adic Tate module* $T_p(E_q)$, *for every* $\rho \in G_K$ *let* $c_\rho \in \mathbb{Z}_p$ *be such that the action of* ρ *on* $T_p(E_q)$ *is given by the matrix* $\left(\begin{smallmatrix} \chi(\rho) & c_\rho \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Gl}_2(\mathbb{Z}_p)$. *Then the map* $\rho \mapsto c_\rho$ *is a continuous* 1*-cocycle* $G_K \to \mathbb{Z}_p(1)$. □

3.3 Crystalline and semi-stable extensions of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$

Our aim in the present section is to explain the following Proposition. First we remark that the abelian group $\mathrm{Ext}^1_{\mathrm{cris}}(\mathbb{Q}_p, \mathbb{Q}_p(1))$ (resp., $\mathrm{Ext}^1_{\mathrm{st}}(\mathbb{Q}_p, \mathbb{Q}_p(1))$) of Yoneda extension classes of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$ inside the abelian category $\mathrm{Rep}_{\mathrm{cris}}(G_K)$ (resp.,

$\mathrm{Rep}_{\mathrm{st}}(G_K)$) may be viewed as a subgroup of $\mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p, \mathbb{Q}_p(1))$; namely, an extension class $\xi \in \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p, \mathbb{Q}_p(1))$ belongs to $\mathrm{Ext}^1_{\mathrm{cris}}(\mathbb{Q}_p, \mathbb{Q}_p(1))$ (resp., to $\mathrm{Ext}^1_{\mathrm{st}}$ $(\mathbb{Q}_p, \mathbb{Q}_p(1))$) if and only if for one (and hence for every) representative $0 \to \mathbb{Q}_p(1) \to V \to \mathbb{Q}_p \to 0$ of ξ the p-adic representation V is crystalline (resp., semi-stable).

Here we already use the well-known fact that the trivial p-adic representation $\mathbb{Q}_p$ is crystalline and, in particular, semi-stable; the same is, of course, also true for $\mathbb{Q}_p(1)$; for an argument, we refer to the remarks after 3.19.

Proposition 3.10. *There is an exact sequence of $\mathbb{Q}_p$-vector spaces*

$$0 \to \mathrm{Ext}^1_{\mathrm{cris}}(\mathbb{Q}_p, \mathbb{Q}_p(1)) \to \mathrm{Ext}^1_{\mathrm{st}}(\mathbb{Q}_p, \mathbb{Q}_p(1)) \to \mathbb{Q}_p \to 0.$$

3.3.1 The p-adic valuation sequence

Below we will give a proof of this result using Fontaine's characterization of crystalline and semi-stable p-adic representations of G_K in terms of weakly admissible filtered (φ, N)-modules (see [27]). First we give an argument letting Galois cohomology intervene, using the results of the previous section. We commence by proving the following

Lemma 3.11 ([15]). *The choice of a uniformizer $\pi \in o_K$ for the complete discretely valued field K gives rise to a split exact sequence of $\mathbb{Z}_p$-modules*

$$0 \to 1 + \mathfrak{m}_K \to \widehat{K^\times} \to \mathbb{Z}_p \to 0;$$

in particular, this induces an exact sequence of $\mathbb{Q}_p$-vector spaces

$$0 \to (1 + \mathfrak{m}_K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \widehat{K^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathbb{Q}_p \to 0.$$

Proof. The normalized discrete valuation on the field K gives rise to an exact sequence of abelian groups

$$0 \to o_K^\times \to K^\times \to \mathbb{Z} \to 0,$$

and our choice of a uniformizer $\pi \in o_K$ defines a section $\mathbb{Z} \to K^\times$ of the valuation map $K^\times \to \mathbb{Z}$ which is given by $1 \mapsto \pi$, i.e., the above sequence is split, and it therefore induces a direct-sum decomposition of the abelian group $K^\times$, being $K^\times = o_K^\times \oplus \mathbb{Z}$. Furthermore, there is a canonical exact sequence of abelian groups

$$1 \to 1 + \mathfrak{m}_K \to o_K^\times \to k^\times \to 1$$

which is split as well: using that k is perfect, and according to [69], II.4.8, let $\lambda\colon k \to o_K$ be the unique section of the residue map $o_K \to k$ being compatible with p-th powers; by loc. cit. the map λ is multiplicative, and it restricts to a map of abelian groups $k^\times \to o_K^\times$ which canonically renders the above sequence split; furthermore, the subgroup $\lambda(k^\times) \subseteq o_K^\times$ is p-divisible. We obtain a direct-sum decomposition of the abelian group $o_K^\times$, being $o_K^\times = \lambda(k^\times) \oplus (1 + \mathfrak{m}_K)$, and we may summarize that there is a (non-canonical) identification

$$K^\times = \lambda(k^\times) \oplus (1 + \mathfrak{m}_K) \oplus \mathbb{Z}.$$

Now fix an integer $n \geq 1$. Applying the functor $\cdot \otimes_\mathbb{Z} \mathbb{Z}/p^n$ to the latter identity, we get

$$K^\times/(K^\times)^{p^n} = \lambda(k^\times)/\lambda(k^\times)^{p^n} \oplus (1 + \mathfrak{m}_K)/(1 + \mathfrak{m}_K)^{p^n} \oplus \mathbb{Z}/p^n.$$

As the group $\lambda(k^\times)$ is p-divisible, the first summand on the right-hand side is trivial; we observe that for every $s \geq 1$ and $x \in K^\times$ we have $x^{p^{s+1}} = (x^p)^{p^s}$, i.e., the obvious transition map of abelian groups $(1 + \mathfrak{m}_K)/(1 + \mathfrak{m}_K)^{p^{s+1}} \to (1 + \mathfrak{m}_K)/(1 + \mathfrak{m}_K)^{p^s}$ is well-defined and surjective, i.e., the Mittag-Leffler condition is met; on the other hand, the abelian group $U = U^{(1)} = 1 + \mathfrak{m}_K$ is p-adically complete: the canonical map $U \to \varprojlim_{(s)} U/U^{p^s}$ has to be injective since $o_K^\times$ is the direct sum of U and the p-divisible group $\lambda(k^\times)$; the surjectivity of the completion map is seen as follows: given an element $((1 + \pi x_n)U^{p^n})_n$ of the projective limit, choosing a representative $1 + \pi x_n \in U$ of the n-th component for every $n \geq 1$ amounts to giving a Cauchy sequence $(1 + \pi x_n)_n$ with respect to the topology given by the p-adic filtration $(U^{p^n})_n$ of U; from $\pi \mid p$ it follows that $U^{p^n} \subseteq 1 + \mathfrak{m}_K^{n+1}$, and we obtain a relation $1 + \pi x_{n+1} = (1 + \pi x_n)(1 + \pi^{n+1} y_n)$ for every $n \geq 1$; by induction it follows that $1 + \pi x_{n+1} = \prod_{j=0}^n (1 + \pi^{j+1} y_j)$ where we set $y_0 = x_1$; it is now instantly verified that the infinite product $x' = \prod_{j=0}^\infty (1 + \pi^{j+1} y_j)$ is π-adically convergent and lies inside U; by a multiplicative version of the argument given in 3.23(ii) below, one now shows that p-adically $1 + \pi x_n \to x' = 1 + \pi x$ as $n \to \infty$. We may summarize that taking the projective limit over the above mod-p^n identifications gives a (necessarily split) exact sequence of $\mathbb{Z}_p$-modules

$$0 \to 1 + \mathfrak{m}_K \to \widehat{K^\times} \to \mathbb{Z}_p \to 0$$

which implies the desired result. □

Proposition 3.12 ([3], [75]). *(i) The isomorphism of $\mathbb{Q}_p$-vector spaces $\widehat{K^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p, \mathbb{Q}_p(1))$ from 3.8 restricts to an isomorphism of $\mathbb{Q}_p$-vector spaces*

$$\widehat{K^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\simeq} \mathrm{Ext}^1_{\mathrm{st}}(\mathbb{Q}_p, \mathbb{Q}_p(1)),$$

i.e., for every extension $0 \to \mathbb{Q}_p(1) \to V \to \mathbb{Q}_p \to 0$ inside $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ the p-adic representation V is $\mathbf{B}_{\mathrm{st}}$-admissible.

(ii) The isomorphism of $\mathbb{Q}_p$-vector spaces $\widehat{K^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{Ext}^1_{\mathrm{st}}(\mathbb{Q}_p, \mathbb{Q}_p(1))$ from (i) restricts to an isomorphism of $\mathbb{Q}_p$-vector spaces

$$(1 + \mathfrak{m}_K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\simeq} \mathrm{Ext}^1_{\mathrm{cris}}(\mathbb{Q}_p, \mathbb{Q}_p(1)).$$

Proof. See [3], II.4.4, and [75], 2.3.2. □

Corollary 3.13 ([3], [75]). *Let $q \in K^\times$ be such that $|q| < 1$, and let E_q/K be the corresponding Tate elliptic curve; let $T_p(E_q) = \varprojlim_{(n)} E_q[p^n](K^{\mathrm{alg}})$ be the p-adic Tate module of E_q, and let $V_p(E_q) = T_p(E_q) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ be the associated p-adic representation of G_K. Then $V_p(E_q)$ is $\mathbf{B}_{\mathrm{st}}$-admissible but not $\mathbf{B}_{\mathrm{cris}}$-admissible.*

Proof. By virtue of 3.3, it follows directly from 3.12(i) that $V_p(E_q)$ is semi-stable. The argumentation in [75], 2.3.2(2), shows that $V_p(E_q)$ cannot be crystalline. □

We may summarize that there is a commutative diagram of $\mathbb{Q}_p$-vector spaces with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (1+\mathfrak{m}_K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \longrightarrow & \widehat{K^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0 \\
 & & \downarrow \simeq & & \downarrow \simeq & & \| & & \\
0 & \longrightarrow & \mathrm{Ext}^1_{\mathrm{cris}}(\mathbb{Q}_p, \mathbb{Q}_p(1)) & \longrightarrow & \mathrm{Ext}^1_{\mathrm{st}}(\mathbb{Q}_p, \mathbb{Q}_p(1)) & \longrightarrow & \mathbb{Q}_p & \longrightarrow & 0
\end{array}$$

where the vertical arrows are isomorphisms. This proves Proposition 3.10. In what follows we view this result in a different angle.

3.3.2 The semi-stable period functor D_{st}

We recall that there is an exact equivalence

$$D_{\mathrm{st}} \colon \mathrm{Rep}_{\mathrm{st}}(G_K) \to MF_K(\varphi, N)^{\mathrm{wa}}$$

between the abelian category of semi-stable p-adic representations of the group G_K on the one hand, and the abelian category of weakly admissible filtered (φ, N)-modules over $F = \mathrm{Frac}(W(k))$ on the other hand. The additive category of filtered (φ, N)-modules over F has already been discussed in (2.1.2). We commence by explaining the notion of weak admissibility for filtered (φ, N)-modules.

Note that given an F-vector space V of dimension $d < \infty$, together with a map of abelian groups $\varphi \colon V \to V$ which is semi-linear (always with respect to the Frobenius lift $\sigma \colon F \xrightarrow{\simeq} F$), by $\wedge^d \varphi \colon \wedge^d V \to \wedge^d V$ we mean the semi-linear map of abelian groups corresponding to the F-linear map

$$\wedge^d \varphi^{\mathrm{lin}} \colon \sigma^*(\wedge^d V) \simeq \wedge^d(\sigma^* V) \to \wedge^d V.$$

The map $\wedge^d\varphi$ is again bijective if φ is: indeed, if $\mathfrak{B}$ is a fixed F-basis of V and if $A = {}_{\mathfrak{B}}[\varphi]^{(\sigma)}_{\mathfrak{B}} \in F^{d\times d}$ denotes the matrix describing φ (σ-semi-linearly) with respect to $\mathfrak{B}$ then

$$\det(A) = \det({}_{\mathfrak{B}}[\varphi^{\mathrm{lin}}]_{\sigma^*\mathfrak{B}}) = {}_{\wedge^d\mathfrak{B}}[\wedge^d\varphi^{\mathrm{lin}}]_{\wedge^d\sigma^*\mathfrak{B}} = {}_{\wedge^d\mathfrak{B}}[\wedge^d\varphi]^{(\sigma)}_{\wedge^d\mathfrak{B}};$$

now some calculations in σ-semi-linear algebra ([67]) show that $\det(A) \neq 0$ if and only if V is F-linearly generated by the image of φ, and the latter condition already implies that φ is injective; moreover, since σ is an automorphism of the field F, the image $\mathrm{im}(\varphi) \subseteq V$ is, in fact, an F-linear subspace of V, and so we may conclude that $\det(A) \neq 0$ if and only if φ is surjective if and only if φ is injective.

Definition 3.14. *Let $\underline{D} = (D, \varphi_D, N_D, (\mathrm{Fil}^i D_K)_i)$ be a filtered (φ, N)-module of dimension $d = \dim_F D$.*

(i) For any fixed basis $d \in D$ of the 1-dimensional F-vector space $\wedge^d D$ let $\lambda \in F^\times$ be the describing matrix of the semi-linear automorphism $\wedge^d\varphi_D \colon \wedge^d D \to \wedge^d D$ of the abelian group $\wedge^d D$ with respect to d. The Newton slope *of $\underline{D}$ is defined to be*

$$t_N(\underline{D}) = \mathrm{ord}_p(\lambda)$$

where $\mathrm{ord}_p(\lambda) \in \mathbb{Z}$ is the p-adic valuation of λ.

(ii) Let $(\mathrm{Fil}^i(\wedge^d D_K))_i$ be the induced filtration of $\wedge^d D_K \subseteq D_K \otimes_K \ldots \otimes_K D_K$ (d factors) where $\mathrm{Fil}^i(D_K \otimes_K \ldots \otimes_K D_K)$ is for every $i \in \mathbb{Z}$ given by

$$\sum_{i_1+\ldots+i_d=i} \mathrm{Fil}^{i_1}(D_K) \otimes_K \ldots \otimes_K \mathrm{Fil}^{i_d}(D_K) \subseteq D_K \otimes_K \ldots \otimes_K D_K.$$

The Hodge slope *$t_H(\underline{D})$ of $\underline{D}$ is defined to be the integer $i \in \mathbb{Z}$ such that $\mathrm{Fil}^i(\wedge^d D_K) = \wedge^d D_K$ and $\mathrm{Fil}^{i+1}(\wedge^d D_K) = 0$.*

The integers $t_N(\underline{D})$ and $t_H(\underline{D})$ associated to $\underline{D}$ are indeed well-defined; see [27], (6.4.2).

Lemma 3.15 ([27]). *Let $\underline{D} = (D, \varphi_D, N_D, (\mathrm{Fil}^i(D_K))_i)$ be a filtered (φ, N)-module. For every $i \in \mathbb{Z}$ let $\mathrm{gr}^i(D_K) = \mathrm{Fil}^i(D_K)/\mathrm{Fil}^{i+1}(D_K)$ be the associated i-*th graded object*. Then*

$$t_H(\underline{D}) = \sum_{i\in\mathbb{Z}} i \dim_K(\mathrm{gr}^i(D_K)).$$

Proof. See [27], 6.45. □

For the following Lemma, note that a sequence of finite-dimensional filtered K-vector spaces $0 \to V' \to V \to V'' \to 0$ is, by definition, exact if and only if the underlying sequence of K-vector spaces is exact and

$$\mathrm{Fil}^i(V') = \mathrm{Fil}^i(V) \cap V', \quad \mathrm{Fil}^i(V'') = (\mathrm{Fil}^i(V) + V')/V'$$

for every $i \in \mathbb{Z}$.

Lemma 3.16 ([27]). *Let* $0 \to \underline{D}' \to \underline{D} \to \underline{D}'' \to 0$ *be a short exact sequence of filtered* (φ, N)*-modules, i.e., a short exact sequence of left* $F[\varphi, N]$*-modules such that the induced* K*-linear sequence* $0 \to D'_K \to D_K \to D''_K \to 0$ *is an exact sequence of finite-dimensional filtered* K*-vector spaces. Then*

$$t_N(\underline{D}) = t_N(\underline{D}') + t_N(\underline{D}''), \qquad t_H(\underline{D}) = t_H(\underline{D}') + t_H(\underline{D}'').$$

Proof. See [27], 6.42, 6.46. □

For a given filtered (φ, N)-module $\underline{D} = (D, \varphi_D, N_D, (\mathrm{Fil}^i(D_K))_i)$ over F, a *subobject* of $\underline{D}$ consists of a (necessarily finite-dimensional) F-subspace $D' \subseteq D$ which is a left $F[\varphi, N]$-submodule of D, i.e., which is stable under φ_D and N_D, and such that $\mathrm{Fil}^i(D'_K) = \mathrm{Fil}^i(D_K) \cap D'_K$ for every $i \in \mathbb{Z}$; this may be rephrased by saying that a subobject of $\underline{D}$ corresponds to an exact sequence $0 \to \underline{D}' \to \underline{D}$ of filtered (φ, N)-modules. Note that for any subobject $\underline{D}'$ with underlying F-vector space D' the restriction $\varphi_D|_{D'} \colon D' \to D'$ is still injective and therefore, by the above characterization of σ-semi-linear bijections, is a semi-linear automorphism of D'.

Definition 3.17. *A filtered* (φ, N)*-module* $\underline{D} = (D, \varphi_D, N_D, (\mathrm{Fil}^i D_K)_i)$ *over* F *is called* weakly admissible *if*

— $t_H(\underline{D}) = t_N(\underline{D})$;

— *for any subobject* $\underline{D}' = (D', \varphi_D|_{D'}, N_D|_{D'}, (\mathrm{Fil}^i(D'_K))_i)$ *one has*

$$t_H(\underline{D}') \leq t_N(\underline{D}').$$

One denotes by $MF_K(\varphi, N)^{\mathrm{wa}}$ the full subcategory of $MF_K(\varphi, N)$ consisting of those filtered (φ, N)-modules which are weakly admissible. For the sake of completeness we state the well-known

Theorem 3.18 (Fontaine, Colmez-Fontaine). *(i) ([27])* $MF_K(\varphi, N)^{\mathrm{wa}}$ *is an abelian category.*

(ii) ([19], [26]) The functor

$$D_{\mathrm{st}} \colon \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to MF_K(\varphi, N), \quad V \mapsto (V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{st}})^{G_K},$$

induces an additive, exact equivalence between the category $\mathrm{Rep}_{\mathrm{st}}(G_K)$ *of* $\mathbf{B}_{\mathrm{st}}$*-admissible* p*-adic representations of* G_K *and the abelian category* $MF_K(\varphi, N)^{\mathrm{wa}}$ *of weakly admissible filtered* (φ, N)*-modules over* F. □

For example, the base field $F = \mathrm{Frac}(W(k))$ becomes a left $F[\varphi, N]$-module by setting

$$\varphi_F = \sigma\colon F \to F, \quad N_F = 0\colon F \to F;$$

if the filtration of $F_K \simeq K$ is given by

$$\mathrm{Fil}^i(F_K) = \begin{cases} F_K & \text{if } i \leq 0, \\ 0 & \text{if } i > 0 \end{cases}$$

for $i \in \mathbb{Z}$ then the collection $K\langle 0\rangle = (F, \varphi_F, N_F, (\mathrm{Fil}^i(K))_i)$ becomes a filtered (φ, N)-module which clearly is weakly admissible since $t_H(K\langle 0\rangle) = 0 = t_N(K\langle 0\rangle)$. The structure of a filtered (φ, N)-module on F can also be "twisted" – one defines a filtered (φ, N)-module $K\langle 1\rangle$ by equipping the abelian group $K\langle 1\rangle = F$ with the left $F[\varphi, N]$-action given by $\varphi_{K\langle 1\rangle} = \frac{1}{p}\sigma$ and $N_{K\langle 1\rangle} = N_F = 0$; the filtration of $K\langle 1\rangle_K \simeq K$ is given by

$$\mathrm{Fil}^i(K\langle 1\rangle_K) = \begin{cases} K\langle 1\rangle_K & \text{if } i \leq -1, \\ 0 & \text{if } i > -1 \end{cases}$$

for $i \in \mathbb{Z}$. Also $K\langle 1\rangle$ is weakly admissible, for we have $t_H(K\langle 1\rangle) = -1 = t_N(K\langle 1\rangle)$. It is well-known that $K\langle 0\rangle = D_{\mathrm{st}}(\mathbb{Q}_p)$; moreover we have

Lemma 3.19 ([27]). *The choice of a basis of $\mathbb{Z}_p(1)$ over $\mathbb{Z}_p$ induces an isomorphism*

$$D_{\mathrm{st}}(\mathbb{Q}_p(1)) \xrightarrow{\simeq} K\langle 1\rangle$$

inside $MF_K(\varphi, N)^{\mathrm{wa}}$.

Proof. See [27], (7.1.3). ☐

In particular, the p-adic representation $\mathbb{Q}_p(1)$ is crystalline. Similarly as in Theorem 3.18, the functor

$$D_{\mathrm{cris}}\colon \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to MF_K(\varphi), \quad V \mapsto (V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{cris}})^{G_K},$$

induces an additive, exact equivalence between the abelian category $\mathrm{Rep}_{\mathrm{cris}}(G_K)$ of crystalline p-adic representations of G_K and the abelian category $MF_K(\varphi)^{\mathrm{wa}}$ of weakly admissible filtered φ-modules over F, where weak admissibility is defined as in the case of filtered (φ, N)-modules; see [15], [26], [27].

We denote by $\mathrm{Ext}^1_{MF_K(\varphi,N)^{\mathrm{wa}}}(\cdot,\cdot)$ (resp., by $\mathrm{Ext}^1_{MF_K(\varphi)^{\mathrm{wa}}}(\cdot,\cdot)$) the Yoneda Ext^1-group with respect to the abelian category $MF_K(\varphi, N)^{\mathrm{wa}}$ (resp., $MF_K(\varphi)^{\mathrm{wa}}$), the group law being given by the Baer sum; we have explained this for the case of the abelian category $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ in section (3.2); for a general discussion, see [62].

Proposition 3.20. *The functor D_{st} induces isomorphisms of $\mathbb{Q}_p$-vector spaces*

$$\begin{aligned}\mathrm{Ext}^1_{\mathrm{st}}(\mathbb{Q}_p, \mathbb{Q}_p(1)) &\simeq \mathrm{Ext}^1_{MF_K(\varphi,N)^{\mathrm{wa}}}(K\langle 0\rangle, K\langle 1\rangle),\\ \mathrm{Ext}^1_{\mathrm{cris}}(\mathbb{Q}_p, \mathbb{Q}_p(1)) &\simeq \mathrm{Ext}^1_{MF_K(\varphi)^{\mathrm{wa}}}(K\langle 0\rangle, K\langle 1\rangle).\end{aligned}$$

Proof. We need merely remark that the functor D_{st} is additive, exact, and fully faithful; furthermore, it restricts to the functor D_{cris} on crystalline representations. In particular, Yoneda equivalence classes of extensions $0 \to \mathbb{Q}_p(1) \to \cdot \to \mathbb{Q}_p \to 0$ inside $\mathrm{Rep}_{\mathrm{st}}(G_K)$ correspond to Yoneda equivalence classes of the associated extensions of filtered (φ, N)-modules; the same is true in case $N = 0$, i.e., on the crystalline level. If we endow each of the abelian groups on the right-hand side with the usual F-vector space structure, using that $F^{\sigma=\mathrm{id}} = \mathbb{Q}_p$ we instantly see that the asserted isomorphisms are $\mathbb{Q}_p$-linear. □

Proposition 3.21. *There is a canonical exact sequence of $\mathbb{Q}_p$-vector spaces*

$$0 \to \mathrm{Ext}^1_{MF_K(\varphi)^{\mathrm{wa}}}(K\langle 0\rangle, K\langle 1\rangle) \to \mathrm{Ext}^1_{MF_K(\varphi,N)^{\mathrm{wa}}}(K\langle 0\rangle, K\langle 1\rangle) \to \mathbb{Q}_p \to 0.$$

Proof. Suppose we are given an extension

$$0 \to K\langle 1\rangle \xrightarrow{i} \underline{D} \xrightarrow{\mathrm{pr}} K\langle 0\rangle \to 0$$

of weakly admissible filtered (φ, N)-modules where D is the F-vector space underlying $\underline{D}$; let ξ be its Yoneda equivalence class with respect to the category $MF_K(\varphi, N)^{\mathrm{wa}}$. To begin with, we conclude from 3.16 that

$$t_N(\underline{D}) = t_N(K\langle 0\rangle) + t_N(K\langle 1\rangle) = -1 = t_H(K\langle 0\rangle) + t_H(K\langle 1\rangle) = t_H(\underline{D}).$$

Since the Frobenius lift $\sigma\colon W(k) \to W(k)$ is an automorphism of the ring of Witt vectors over k, the $W(k)$-submodule $W(k)$ of F is an $W(k)$-lattice of $K\langle 0\rangle$ verifying $\sigma W(k) = p^0 W(k)$; so, by [76], 6.18, the φ-isocrystal $K\langle 0\rangle$ is isoclinic. By the same argument, the φ-isocrystal $K\langle 1\rangle$ is isoclinic, for we have a relation $(\frac{1}{p}\sigma)W(k) = p^{-1}W(k)$. By [76], 6.21, our given extension of filtered (φ, N)-modules gives rise to a split extension of φ-isocrystals (forgetting about N_D and the filtration) via a unique φ-equivariant section $w\colon F \to D$ of the projection $\mathrm{pr}\colon D \to F$, i.e., there is a direct-sum decomposition of φ-isocrystals

$$\underline{D} = K\langle 1\rangle \oplus K\langle 0\rangle$$

showing that with respect to the canonical basis $\mathfrak{B} = (i(1), w(1))$ of D, the map φ_D is (σ-semi-linearly) described by the matrix

$${}_{\mathfrak{B}}[\varphi_D]^{(\sigma)}_{\mathfrak{B}} = \left(\begin{smallmatrix} 1/p & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Gl}_2(F).$$

From the N-equivariance of the map $i\colon K\langle 1\rangle \to D$ we immediately derive that $N_D(i(1)) = 0$; on the other hand, the identity of σ-semi-linear maps $N_D\varphi_D = p\varphi_D N_D$ yields that

$$\varphi_D(N_D(w(1))) = \tfrac{1}{p} N_D(w(1));$$

writing $N_D(w(1)) = \alpha i(1) + \beta w(1)$ we find that $\sigma(\alpha) = \alpha$, i.e., $\alpha \in F^{\sigma=\mathrm{id}} = \mathbb{Q}_p$, as well as $\beta = p\sigma(\beta)$; however, the latter relation implies that $\beta = 0$, for if we uniquely write, say, $\beta = \beta' p^n$, where $\beta' \in W(k)^\times$ and $n \in \mathbb{Z}$, we obtain $\sigma(\beta) = \sigma(\beta')p^n$ which leads to a contradiction since $\sigma(\beta')$ is a unit again; next we remark that given a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K\langle 1\rangle & \xrightarrow{i} & \underline{D} & \xrightarrow{\mathrm{pr}} & K\langle 0\rangle & \longrightarrow & 0 \\ & & \| & & \simeq\downarrow\iota & & \| & & \\ 0 & \longrightarrow & K\langle 1\rangle & \xrightarrow{i''} & \underline{D}'' & \xrightarrow{\mathrm{pr}''} & K\langle 0\rangle & \longrightarrow & 0 \end{array}$$

inside $MF_K(\varphi, N)^{\mathrm{wa}}$ where $\iota\colon \underline{D} \to \underline{D}''$ is an isomorphism of filtered (φ, N)-modules, together with a φ-equivariant section $w\colon K\langle 0\rangle \to D$ of $\mathrm{pr}\colon D \to K\langle 0\rangle$, the composition ιw is a φ-equivariant section of $\mathrm{pr}''\colon D'' \to K\langle 0\rangle$, and the couple $(i''(1), (\iota w)(1))$ is an F-basis of D''; the relation $N_{D''}((\iota w)(1)) = \alpha(\iota w)(1)$ now shows that the map

$$\mathrm{Ext}^1_{MF_K(\varphi,N)^{\mathrm{wa}}}(K\langle 0\rangle, K\langle 1\rangle) \to \mathbb{Q}_p, \quad \xi \mapsto \alpha,$$

is, in fact, well-defined; it obviously remains to be shown that this map is $\mathbb{Q}_p$-linear and surjective: by construction its kernel is already as desired; in order to prove the $\mathbb{Q}_p$-linearity, we study the Baer sum $\xi + \xi'$ where ξ' is the Yoneda equivalence class of an extension

$$0 \to K\langle 1\rangle \xrightarrow{i'} \underline{D}' \xrightarrow{\mathrm{pr}'} K\langle 0\rangle \to 0$$

inside $MF_K(\varphi, N)^{\mathrm{wa}}$; let $w'\colon F \to D'$ be the unique φ-equivariant section of the projection $\mathrm{pr}'\colon D' \to K\langle 0\rangle$; let $s\colon K\langle 1\rangle \oplus K\langle 1\rangle \to K\langle 1\rangle$ be the sum and $d\colon \to K\langle 0\rangle \oplus K\langle 0\rangle$ the diagonal; proceeding similarly as in the proof of 3.5, we derive an extension

$$0 \to K\langle 1\rangle \oplus K\langle 1\rangle \overset{\underline{i}=(i\oplus i',0)}{\to} X \to K\langle 0\rangle \to 0$$

inside $MF_K(\varphi, N)^{\mathrm{wa}}$, where

$$\begin{aligned} X &= (D \oplus D') \times_{\mathrm{pr}\oplus\mathrm{pr}', K\langle 0\rangle\oplus K\langle 0\rangle, d} K\langle 0\rangle \\ &= \{((a,b),z) \in (D \oplus D') \oplus K\langle 0\rangle, \quad \mathrm{pr}(a) = z = \mathrm{pr}'(b)\}; \end{aligned}$$

here $\underline{i}\colon K\langle 1\rangle \oplus K\langle 1\rangle \to X$ is given by $(x, y) \mapsto ((i(x), i'(y)), 0)$, and the map $X \to K\langle 0\rangle$ is given by the projection onto the second component; in a second step we obtain an extension

$$0 \to K\langle 1\rangle \xrightarrow{i_+} Y \xrightarrow{\mathrm{pr}_+} K\langle 0\rangle \to 0,$$

inside $MF_K(\varphi, N)^{\mathrm{wa}}$ where

$$\begin{aligned} Y &= K\langle 1\rangle \amalg^{s, K\langle 1\rangle \oplus K\langle 1\rangle, \underline{i}} X \\ &= (K\langle 1\rangle \oplus X)/\mathrm{im}(s, -\underline{i}), \end{aligned}$$

the maps being given as follows: via $i_+ \colon K\langle 1\rangle \to Y$ an element $x \in K\langle 1\rangle$ is sent to the class of $(x, 0) \in K\langle 1\rangle \oplus X$, and the class of a couple $(x, ((a, b), z)) \in K\langle 1\rangle \oplus X$ is via $\mathrm{pr}_+ \colon Y \to K\langle 0\rangle$ sent to $z \in K\langle 0\rangle$. Let $\underline{w} \colon K\langle 0\rangle \to Y$ be the F-linear map defined by

$$1 \mapsto \text{class of } (0, ((w(1), w'(1)), 1));$$

this map is evidently a section of the projection $\mathrm{pr}_+ \colon Y \to K\langle 0\rangle$ just described, and we claim that $\underline{w}$ is φ-equivariant: indeed, first of all, it is instantly seen that X is an $F[\varphi, N]$-submodule of $(D \oplus D') \oplus K\langle 0\rangle$ and that the canonical projection $K\langle 1\rangle \oplus X \to Y$ naturally becomes $F[\varphi, N]$-linear as well; but this said, the φ-equivariance of $\underline{w}$ is immediate. Similarly as above, the N-equivariance of $i_+ \colon K\langle 1\rangle \to Y$ shows that $N_Y(i_+(1)) = 0$; let us compute $N_Y(\underline{w}(1))$: using the equivalence relation defining Y we see that $N_Y(\underline{w}(1))$ equals the class of $(0, ((\alpha i(1), \alpha' i'(1)), 0))$, provided that $N_D(w(1)) = \alpha i(1)$ and $N_{D'}(w'(1)) = \alpha' i'(1)$; however, the latter equivalence class admits the element $(\alpha + \alpha') i_+(1)$ as a representative, which proves that the Baer sum $\xi + \xi'$ is, in fact, mapped to $\alpha + \alpha'$. Let $\lambda \in \mathbb{Q}_p$ be a scalar; in order to accomplish the proof of the desired $\mathbb{Q}_p$-linearity, we study the Yoneda equivalence class $\lambda^* \xi$ of the extension

$$0 \to K\langle 1\rangle \xrightarrow{i_\lambda} D \times_{\mathrm{pr}, K\langle 0\rangle, \lambda} K\langle 0\rangle \xrightarrow{\mathrm{pr}_\lambda} K\langle 0\rangle \to 0$$

where $i_\lambda \colon K\langle 1\rangle \to Z = D \times_{\mathrm{pr}, K\langle 0\rangle, \lambda} K\langle 0\rangle$ maps 1 to $(i(1), 0)$; we define an F-linear and φ-equivariant section $w_\lambda \colon K\langle 0\rangle \to Z$ of $\mathrm{pr}_\lambda \colon Z \to K\langle 0\rangle$ by $1 \mapsto (\lambda w(1), 1)$; note that here we make use of our requirement $\lambda \in F^{\sigma = \mathrm{id}}$; we finally remark that the resulting F-basis $\mathfrak{B}_\lambda = (i_\lambda(1), w_\lambda(1))$ of Z verifies $N_Z(i_\lambda(1)) = 0$ and $N_Z(w_\lambda(1)) = (\lambda\alpha) i_\lambda(1)$, provided that $N_D(w(1)) = \alpha i(1)$. Let us now show the desired surjectivity. Let $\lambda \in \mathbb{Q}_p$ be given. We construct an extension

$$0 \to K\langle 1\rangle \xrightarrow{i} \underline{D} \xrightarrow{\mathrm{pr}} K\langle 0\rangle \to 0,$$

inside $MF_K(\varphi, N)^{\mathrm{wa}}$ as follows: the F-vector space underlying $\underline{D}$ is $D = K\langle 1\rangle \oplus K\langle 0\rangle$, and i, pr are canonically given by $i \colon 1 \mapsto (1, 0)$, $\mathrm{pr} \colon (0, 1) \mapsto 1$; the structure of a left $F[\varphi, N]$-module on D is given by $\varphi_{\underline{D}} = \varphi_{K\langle 1\rangle} \oplus \varphi_F = (\frac{1}{p}\sigma) \oplus \sigma$, $N_{\underline{D}}((1, 0)) = 0$, $N_{\underline{D}}((0, 1)) = \lambda(1, 0)$; there is an obvious φ-equivariant F-linear section $w \colon K\langle 0\rangle \to D$ of $\mathrm{pr} \colon D \to K\langle 0\rangle$ which sends $1 \in F$ to $(0, 1) \in D$. Using the requirement of N-equivariance, the left $F[\varphi, N]$-module D admits only a single proper nontrivial left $F[\varphi, N]$-submodule which is given by $D' = \mathrm{im}(K\langle 1\rangle \hookrightarrow D) = F(1, 0)$. We have to define a filtration of $D_K = (K\langle 1\rangle \oplus K\langle 0\rangle)_K = K\langle 1\rangle_K \oplus K\langle 0\rangle_K$ such that $\underline{D}$

becomes weakly admissible; necessarily, by 3.15, such a filtration has to be of the type

$$\mathrm{Fil}^i(D_K) = \begin{cases} D_K & \text{for } i \leq -1, \\ \mathcal{L} & \text{for } i = 0, \\ 0 & \text{for } i \geq 1, \end{cases}$$

where $\mathcal{L} \subseteq D_K$ is a suitable 1-dimensional K-linear subspace of D_K. Setting

$$\mathcal{L} = \mathrm{im}(K\langle 0\rangle_K \hookrightarrow D_K) = K((0,1) \otimes 1),$$

the induced K-linear sequence

$$0 \to K\langle 1\rangle_K \to D_K \to K\langle 0\rangle_K \to 0$$

becomes an exact sequence of filtered K-vector spaces, by construction of $\mathrm{Fil}^{\cdot}(K\langle 1\rangle_K)$, $\mathrm{Fil}^{\cdot}(K\langle 0\rangle_K)$; finally, since $t_H(\underline{D}') = -1 = t_N(\underline{D}')$ and $t_H(\underline{D}) = -1 = t_N(\underline{D})$, we see that $\underline{D}$ is weakly admissibe, which concludes the proof. □

3.4 Yoneda extension classes and bad reduction in equal characteristic

Retaining the notation from chapter 2, let L be an equal-characteristic complete discretely valued field extension of $\mathbb{F}$, with valuation ring o_L and residue field $\ell = o_L/(\pi)$ where $\pi = \pi_L \in \mathfrak{m}_L$ is a fixed uniformizer; we denote by $v = v_\pi = \mathrm{ord}_\pi(\cdot)$ the discrete valuation on L normalized by $v(\pi) = 1$. We recall that the r-Frobenius lift of the o_L-algebra $o_L[\![z]\!]$ is given by the map

$$\sigma \colon o_L[\![z]\!] \to o_L[\![z]\!], \quad \sum_{j=0}^{\infty} a_j z^j \mapsto \sum_{j=0}^{\infty} a_j^r z^j.$$

We also take up the $\mathbb{F}$-algebra homomorphism $c^* \colon \mathbb{F}[t] \to o_L$, recalling that the image $\zeta \in o_L$ of the indeterminate t is supposed to be divided by π_L and therefore is zero in the residue field ℓ of L.

3.4.1 Motivation: Semi-stable Drinfeld modules

We have seen earlier how the p-adic Tate module of a Tate elliptic curve naturally becomes an extension of $\mathbb{Z}_p$ by $\mathbb{Z}_p(1)$ as Galois modules.

In this section we want to study an analogous situation in equal characteristic; here the most basic objects to study are Drinfeld modules of bad reduction; their behavior will lead to our principal object of interest. Let $\alpha \in o_L$ be a non-unit, i.e., such that $\pi \mid \alpha$. We consider the Drinfeld $\mathbb{F}[z]$-module $\varphi\colon \mathbb{F}[z] \to L[\tau]$ given by $z \mapsto \zeta + \tau + \alpha\tau^2$. Clearly φ is of bad reduction: whereas φ is of rank 2, its reduced Drinfeld module over ℓ is of rank 1. By Drinfeld's Tate uniformization theorem there is a Drinfeld $\mathbb{F}[z]$-module $\psi\colon \mathbb{F}[z] \to L[\tau]$ of good reduction and rank 1, together with an analytic morphism $\psi \to_{\mathrm{an}} \varphi$; the latter is given by a formal power series $u \in o_L[[x]]$ of the form $u = x + \sum_{\nu \geq 1} u_\nu x^{r^\nu}$ verifying $v(u_\nu)/r^\nu \to \infty$ as $\nu \to \infty$, and $u\psi_a = \varphi_a u$ for all $a \in \mathbb{F}[z]$. By Theorem 2.10 there is an exact sequence

$$0 \to N \to M(\varphi) \otimes_{L[z]} L\langle z\rangle \to M(\psi) \otimes_{L[z]} L\langle z\rangle \to 0$$

compatible with the respective semi-linear data, together with a finite field extension L'/L such that the pair $(N \otimes_{L\langle z\rangle} L'\langle z\rangle, \tau_N \otimes \sigma)$ is isomorphic to $(L'\langle z\rangle, \sigma)$. Note that the underlying sequence of $L\langle z\rangle$-modules is split, but in general the splitting will not be F-equivariant. We know that, up to a unit $c \in o_L^\times$, the τ-action on $M(\psi) \otimes_{L[z]} L\langle z\rangle$ is with respect to the canonical basis $1 \in M(\psi)$ given by $z - \zeta$, and since the F_N-action is trivial over L', we see that *after replacing* L *by* $L'(c^{\frac{1}{r-1}})$ the τ-action of $M(\varphi) \otimes_{L[z]} L\langle z\rangle$ with respect to its composed $L\langle z\rangle$-basis is given by a matrix of the form

$$\left(\begin{smallmatrix} 1 & * \\ 0 & z-\zeta \end{smallmatrix}\right) \in L\langle z\rangle^{2\times 2}.$$

Since φ is of bad reduction as a Drinfeld module, the resulting object $M(\varphi) \otimes_{L[z]} o_L[\![z]\!][1/\pi]$ should give rise to a proper "semi-stable local shtuka"; note that a priori we do not have a chance to remedy negative powers of π in the coefficients. We may summarize that, indicating by $\mathcal{O}(n)$ for $n \geq 0$ the object $(o_L[\![z]\!][1/\pi], F = (z-\zeta)^n \circ \sigma)$ we obtain an extension structure

$$0 \to \mathcal{O}(0) \to M(\varphi) \otimes_{L[z]} o_L[\![z]\!][1/\pi] \to \mathcal{O}(1) \to 0.$$

By permitting finite base field extensions in the p-adic case, here and in the following discussion we may ignore the circumstance that we have to extend the base field L in order to obtain the described extension structure of $M(\varphi) \otimes_{L[z]} L\langle z\rangle$ (rather than being able to obtain this structure in a *rational* way).

3.4.2 The Carlitz module

We want to exhibit the circumstance ([74]) that, based on the analogy between $\mathbb{Z}$ and $\mathbb{F}[t]$, the Carlitz module $C\colon \mathbb{F}[t] \to L[\tau]$ defined by $C_t = \zeta + \tau$ provides a function-field analogue for the multiplicative K-group scheme

$$\mathbb{G}_{m,K} = \mathrm{Spec}(K[u, u^{-1}])$$

in the following manner: recall that the group scheme $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[u, u^{-1}])$ represents the functor

$$(\text{schemes}) \to (\text{abelian groups}), \qquad S \mapsto \Gamma(S, \mathcal{O}_S)^\times.$$

Stressing that $\mathrm{Spec}(\mathbb{Z})$ is the final object in the category of schemes and that abelian groups correspond to $\mathbb{Z}$-modules, one observes that $\mathbb{G}_m$ parallels the functor

$$(\mathbb{F}[t]\text{-schemes}) \to (\mathbb{F}[z]\text{-modules}), \quad S \mapsto \Gamma(S, \mathcal{O}_S),$$

where z acts on the $\mathbb{F}$-vector space $\Gamma(S, \mathcal{O}_S)$ via the $\mathbb{F}$-linear endomorphism $x \mapsto tx + x^r$; note that via this functor, for every L-algebra R the $\mathbb{F}[t]$-scheme $\mathrm{Spec}(R)$ goes to the $\mathbb{F}[z]$-module $C(R)$; here we use that every L-algebra becomes an $\mathbb{F}[t]$-algebra via our fixed characteristic map of $\mathbb{F}$-algebras $\mathbb{F}[t] \to L$, $t \mapsto \zeta$.

In this spirit we may regard the z-adic Tate module

$$\mathbb{F}[\![z]\!](1) = T_z(C) = \varprojlim_{n \geq 1} C(L^{\mathrm{sep}})[z^n]$$

as a z-adic analogue for $\mathbb{Z}_p(1)$; for every $n \geq 1$ the abelian group $C(L^{\mathrm{alg}})[z^n]$ is naturally an $\mathbb{F}[z]/z^n$-module which is free of rank 1 (for example, by [59], 2.5(a)); by virtue of the following elementary Lemma, the abelian group $C(L^{\mathrm{alg}})[z^n]$ consists of the roots of a separable polynomial over L, so that, in fact, the abelian group $C(L^{\mathrm{sep}})[z^n]$ is free of rank 1 as a module over $\mathbb{F}[z]/z^n$, and therefore we may summarize that $T_z(C)$ is a free $\mathbb{F}[\![z]\!]$-module of rank 1.

Lemma 3.22. *Via the isomorphism*

$$\begin{aligned} L[\tau] &\xrightarrow{\simeq} \mathrm{End}_{(\mathrm{GrSch}/L), \mathbb{F}-\mathrm{lin}}(\mathbb{G}_{a,L}), \\ \sum_{\nu=0}^{s} \alpha_\nu \tau^\nu &\mapsto \mathrm{Spec}(L[x] \to L[x],\ x \mapsto \sum_{\nu=0}^{s} \alpha_\nu x^{r^\nu}), \end{aligned}$$

where $L[\tau]$ is the skew polynomial ring over L with the commutation rule $\tau\alpha = \alpha^r\tau$ for $\alpha \in L$, every power of $\zeta+\tau \in L[\tau]$ corresponds to (an endomorphism $\mathbb{G}_{a,L} \to \mathbb{G}_{a,L}$ given by) a separable polynomial over L.

Proof. The skew polynomial $\zeta + \tau \in L[\tau]$ corresponds to $f_0 = \zeta x + x^r \in L[x]$ whose formal derivative is

$$(d/dx)(\zeta x + x^r) = \zeta \in L^\times.$$

Therefore $\gcd(f_0, (d/dx)f_0) = 1$. To accomplish the proof, it suffices to show that for $f \in L[x]$ the condition $\deg((d/dx)f) = 0$ implies $\deg((d/dx)(\zeta f + f^r)) = 0$. However, this is immediate, for we have $(d/dx)(\zeta f + f^r) = \zeta(d/dx)f$. □

Let us briefly discuss the z-adic analogue for the p-adic cyclotomic character $\chi_K : G_K \to \mathbb{Z}_p^\times$. Let $(t_n)_{n\geq 0} \in T_z(C)$ be a coherent sequence where $t_n \in C(L^{\mathrm{sep}})[z^{n+1}]$ is an $\mathbb{F}[z]/z^{n+1}$-basis, in particular

$$t_0^{r-1} = -\zeta, \qquad \zeta t_n + t_n^r = t_{n-1} \ (n \geq 1).$$

This sequence gives rise to an isomorphism of $\mathbb{F}[\![z]\!]$-modules

$$\mathrm{Aut}_{\mathbb{F}[\![z]\!]}(T_z(C)) \simeq \mathbb{F}[\![z]\!]^\times;$$

we consider the element $\mathbf{t}_+ = \sum_n t_n z^n \in L_\infty[\![z]\!]^\times$ where $L_\infty = L((t_n)_{n\geq 0})$; note that $t_0 \in L_\infty^\times$. Let $\sigma = \sigma_{L_\infty} \colon L_\infty[\![z]\!] \to L_\infty[\![z]\!]$ be the r-Frobenius lift of $L_\infty[\![z]\!]$. By construction $\sigma(\mathbf{t}_+)$ equals $(z-\zeta)\mathbf{t}_+$. Let $\gamma \in G_L$. Since $\mathbf{t}_+$ is a unit in $L_\infty[\![z]\!]$, there is a well-defined element $\chi_L(\gamma) \in L_\infty[\![z]\!]$ such that

$$\sum_{n=0}^{\infty} \gamma(t_n) z^n = \chi_L(\gamma)\mathbf{t}_+.$$

Since ζ lies in the ground field L, the element $\chi_L(\gamma)$ is σ-invariant: we have

$$\sigma(\chi_L(\gamma)) = \sigma(\sum_{n\geq 0}^{\infty} \gamma(t_n)z^n)\sigma(\mathbf{t}_+)^{-1} = (z-\zeta)\chi_L(\gamma)\mathbf{t}_+\sigma(\mathbf{t}_+)^{-1} = \chi_L(\gamma),$$

i.e., the coefficients of $\chi_L(\gamma)$ lie in the splitting field of the polynomial $x^r - x \in L[x]$ inside L^{sep}, i.e., $\chi_L(\gamma) \in \mathbb{F}[\![z]\!]^\times$; because of the defining relation of $\chi_L(\gamma)$, the absolute term of $\chi_L(\gamma)$ has to be nontrivial. We obtain a character $\chi_L \colon G_L \to \mathbb{F}[\![z]\!]^\times$ which is our analogue of χ_K, and which induces a canonical embedding $\mathrm{Gal}(L_\infty/L) \hookrightarrow \mathbb{F}[\![z]\!]^\times$; see [40], 1.3.

3.4.3 The valuation sequence

In (3.1) we have recalled that the abelian group $E(K)$ of K-rational points of a Tate elliptic curve E/K over a complete discretely valued field extension $K/\mathbb{Q}_p$ is naturally isomorphic to the unit group $K^\times$ modulo a $\mathbb{Z}$-lattice of the form $q^{\mathbb{Z}}$ for a uniquely determined parameter $q \in K^\times$. By virtue of the period $q \in K^\times$, the p-adic representation $V_p(E)$ associated to E/K acquires a natural structure of a semi-stable, non-crystalline extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$. As we have seen in 3.10, the $\mathbb{Q}_p$-vector space $\mathrm{Ext}^1_{\mathrm{cris}}(\mathbb{Q}_p, \mathbb{Q}_p(1))$ of Yoneda-equivalence classes of crystalline extensions of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$ sits as a $\mathbb{Q}_p$-hyperplane inside the corresponding $\mathbb{Q}_p$-vector space $\mathrm{Ext}^1_{\mathrm{st}}(\mathbb{Q}_p, \mathbb{Q}_p(1))$ for the semi-stable category; this in turn is mirrored by the $\mathbb{Q}_p$-linear short exact sequence

$$0 \to (1+\mathfrak{m}_K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \widehat{K^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathbb{Q}_p \to 0$$

which has its origin in a unique splitting of the canonical exact sequence of abelian groups

$$1 \to 1+\mathfrak{m}_K \to o_K^\times \to k^\times \to 1$$

combined with a (non-canonical) splitting of the canonical valuation sequence

$$1 \to o_K^\times \to K^\times \xrightarrow{v_K} \mathbb{Z} \to 1$$

for the p-adic field K. Every splitting of the latter sequence corresponds to the choice of a uniformizer for K. Stressing the analogy between the multiplicative group scheme $\mathbb{G}_m$ and the Carlitz module C, we may regard the valuation sequence as being an analogue for the $\mathbb{F}[z]$-linear exact sequence

$$0 \to C(o_L) \to C(L) \to C(L)/C(o_L) \to 0$$

where L is our equal-characteristic complete discretely valued base field; note that the latter sequence does not admit a canonical $\mathbb{F}[z]$-linear splitting. Furthermore, looking at the above kernel sequence for the reduction map $o_K^\times \to k^\times$, the exact sequence of $\mathbb{F}[z]$-modules

$$0 \to C(\mathfrak{m}_L) \to C(o_L) \to C(\ell) \to 0$$

indicates that the kernel $C(\mathfrak{m}_L)$ may be viewed as a function-field analogue for the principal-unit group $1 + \mathfrak{m}_K \subseteq o_K^\times$ of the p-adic field K; note that the $\mathbb{F}$-linear subspace $\mathfrak{m}_L$ of o_L is indeed an $\mathbb{F}[z]$-submodule of $C(o_L)$, so that writing $C(\mathfrak{m}_L)$ actually makes sense.

Proposition 3.23. *(i) For every $n \geq 1$ the $\mathbb{F}[z]$-linear inclusion $C(\mathfrak{m}_L) \subseteq C(o_L)$ restricts to the equality*

$$C(\mathfrak{m}_L)[z^n] = C(o_L)[z^n]$$

of $\mathbb{F}[z]/z^n$-modules.

(ii) The $\mathbb{F}[z]$-module $C(\mathfrak{m}_L)$ is z-adically complete.

(iii) For every $n \geq 1$ the canonical map

$$C(\mathfrak{m}_L)/z^n C(\mathfrak{m}_L) \to C(o_L)/z^n C(o_L)$$

of $\mathbb{F}[z]/z^n$-modules is an isomorphism; in particular, there is a canonical isomorphism of $\mathbb{F}[\![z]\!]$-modules

$$C(\mathfrak{m}_L) \xrightarrow{\simeq} \widehat{C(o_L)}^{(z)}.$$

Proof. For every $n \geq 1$, apply the Snake Lemma to the commutative diagram of $\mathbb{F}[z]$-linear maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(\mathfrak{m}_L) & \longrightarrow & C(o_L) & \longrightarrow & C(\ell) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle z^n} & & \downarrow{\scriptstyle z^n} & & \downarrow{\scriptstyle z^n} & & \\ 0 & \longrightarrow & C(\mathfrak{m}_L) & \longrightarrow & C(o_L) & \longrightarrow & C(\ell) & \longrightarrow & 0 \end{array}$$

with exact rows. We have already seen earlier that the Carlitz module is of supersingular reduction, and moreover

$$C(\ell)[z^n] = \{x \in \ell, x^{r^n} = 0\} = 0.$$

This implies at once part (i) as well as that for every $n \geq 1$ there is a canonical exact sequence of $\mathbb{F}[z]/z^n$-linear maps

$$0 \to C(\mathfrak{m}_L)/z^n C(\mathfrak{m}_L) \to C(o_L)/z^n C(o_L) \to C(\ell)/z^n C(\ell) \to 0;$$

in particular, there is a canonical exact sequence of $\mathbb{F}[\![z]\!]$-linear maps

$$0 \to \widehat{C(\mathfrak{m}_L)}^{(z)} \to \widehat{C(o_L)}^{(z)} \to \widehat{C(\ell)}^{(z)} \to 0;$$

the latter being true since the Mittag-Leffler condition is clearly met. Finally, however, since ℓ is perfect, the map $z^n \colon C(\ell) \to C(\ell)$ is surjective, so that $C(\ell)/z^n C(\ell)$ is trivial for every $n \geq 1$. It remains to show that the canonical map

$$C(\mathfrak{m}_L) \to \varprojlim_{(s)} C(\mathfrak{m}_L)/z^s C(\mathfrak{m}_L)$$

is an isomorphism. First of all, we note that for every $s \geq 1$ we have an inclusion $zC(\mathfrak{m}_L^s) \subseteq \mathfrak{m}_L^{s+1}$, and using this we show by induction on s that $z^s C(\mathfrak{m}_L) \subseteq \mathfrak{m}_L^{s+1}$ for all $s \geq 1$: our claim holds true if $s = 1$ since $r \geq 2$, and for any fixed s we have

$$z^s C(\mathfrak{m}_L) = z(z^{s-1} C(\mathfrak{m}_L)) \subseteq zC(\mathfrak{m}_L^s) \subseteq \mathfrak{m}_L^{s+1}$$

provided that $z^{s-1} C(\mathfrak{m}_L) \subseteq \mathfrak{m}_L^s$; we may conclude that

$$\cap_{s \geq 1} z^s C(\mathfrak{m}_L) \subseteq \cap_{s \geq 1} \mathfrak{m}_L^{s+1} = 0,$$

i.e., the canonical map $C(\mathfrak{m}_L) \to \widehat{C(\mathfrak{m}_L)}^{(z)}$ is injective or, in other words: $C(\mathfrak{m}_L)$ is z-adically separated. In order to show that the displayed map is also surjective, we fix a coherent sequence

$$(x_s[z^s])_s \in \widehat{C(\mathfrak{m}_L)}^{(z)}$$

of residue classes $x_s[z^s] \in C(\mathfrak{m}_L)/z^s C(\mathfrak{m}_L)$; in particular, we are provided that $x_{s+1} - x_s \in z^s C(\mathfrak{m}_L)$ for every $s \geq 1$, so we find elements $y_s, w_s \in \mathfrak{m}_L$ such that

$$x_{s+1} - x_s = z^s y_s = \pi^{s+1} w_s,$$

where the latter equality follows from what we have seen above; the series $x_1 + \sum_{s=1}^{\infty} x_{s+1} - x_s$ converges inside o_L and gives an element $x \in \mathfrak{m}_L$ for we have

$$v(x) = v(x_1 + \sum_{s=1}^{\infty} x_{s+1} - x_s) \geq \min(v(x_1), v(\sum_{s=1}^{\infty} \pi^{s+1} w_s)) \geq 1;$$

here v denotes as usual the discrete valuation of L normalized by $v(\pi) = 1$; we need yet to verify that $x - x_s \in z^s C(\mathfrak{m}_L)$ for every s; indeed, we have

$$\begin{aligned} x - x_s &= x_1 - x_s + \sum_{\nu=1}^{\infty} x_{\nu+1} - x_\nu \\ &= \sum_{\nu=1}^{\infty} x_{\nu+1} - x_\nu - \sum_{\nu=1}^{s-1} x_{\nu+1} - x_\nu \\ &= \sum_{\nu=s}^{\infty} x_{\nu+1} - x_\nu \\ &= \sum_{\nu=s}^{\infty} z^\nu y_\nu = z^s w \end{aligned}$$

where $w = \sum_{\nu=s}^{\infty} z^{\nu-s} y_\nu \in C(\mathfrak{m}_L)$, see Lemma 3.24 below; note that $z^{\nu-s} y_\nu \in C(\mathfrak{m}_L^{\nu-s+1})$ for every $\nu \geq s$, as we have seen above; therefore the defining series for w converges in o_L, and indeed $v(w) \geq 1$. □

Lemma 3.24. *Let $\alpha = \sum_{n=1}^{\infty} \alpha_n$ be a convergent series inside o_L; then $z^s \alpha = \sum_{n=1}^{\infty} z^s \alpha_n$ inside $C(o_L)$ for every $s \geq 1$.*

Proof. We have

$$z\alpha = \zeta\alpha + \alpha^r = \sum_{n=1}^{\infty} (\zeta\alpha_n + \alpha_n^r) = \sum_{n=1}^{\infty} z\alpha_n.$$

So the claim follows by induction. □

3.4.4 Analytic uniformization

Having fixed a separable closure L^{sep}/L, we denote by $G_L = \text{Gal}(L^{\text{sep}}/L)$ the absolute Galois group of our complete discretely valued field L. Let $\Lambda \subseteq C(L^{\text{sep}})$ be a *lattice* of rank d, i.e., a finite projective (hence free) $\mathbb{F}[z]$-submodule of $C(L^{\text{sep}})$ of rank d such that $\rho(\Lambda) \subseteq \Lambda$ for every $\rho \in G_L$. According to Drinfeld's Tate uniformization theorem, which we have already discussed in section (2.2.2), the couple (C, Λ) corresponds up to isomorphism to a bad-reduction Drinfeld $\mathbb{F}[z]$-module of rank $d+1$, which we denote by C/Λ; note that C/Λ plays the role of a Tate elliptic curve $E_q = \mathbb{G}_{m,K}/q^{\mathbb{Z}}$ in the p-adic world; furthermore, note that Tate uniformization for Drinfeld modules is actually broader than that of elliptic curves: in order to uniformize all stable Drinfeld modules over L, one would have to allow lattices inside Drinfeld modules of higher rank (not only inside C).

According to what we have recorded in (2.2.2), one may write down the uniformization of C/Λ in terms of an exact sequence

$$0 \to \Lambda \to C \to_{\text{an}} C/\Lambda \to 0$$

where one has to note that, as indicated by the notation, the arrow $C \to_{\mathrm{an}} C/\Lambda$ is merely an analytic morphism of Drinfeld modules, not an algebraic one; this morphism is given by a formal power series $u = x + \sum_{\nu=1}^{\infty} u_\nu x^{r^\nu} \in o_L[\![x]\!]$ verifying $v(u_\nu)/r^\nu \to \infty$ as $\nu \to \infty$ and $uC_a = (C/\Lambda)_a u$ for all $a \in \mathbb{F}[z]$; the power series u induces an $\mathbb{F}[z]$-linear map $C(L^{\mathrm{sep}}) \to (C/\Lambda)(L^{\mathrm{sep}})$ by $\xi \mapsto u(\xi)$; namely, for every $\xi \in L^{\mathrm{sep}}$ the field extension $L(\xi)$ is finite separable and therefore complete, i.e., $u(\xi) \in L(\xi) \subseteq L^{\mathrm{sep}}$; the resulting map is clearly $\mathbb{F}$-linear, and we have

$$u(z\xi) = u(C_z(\xi)) = (C/\Lambda)_z(u(\xi)) = zu(\xi),$$

i.e., the map defined by u is indeed $\mathbb{F}[z]$-linear. We obtain an exact sequence of $\mathbb{F}[z]$-modules $0 \to \Lambda \to C(L^{\mathrm{sep}}) \to C/\Lambda(L^{\mathrm{sep}}) \to 0$. Now let $n \geq 1$; applying the Snake Lemma to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & C(L^{\mathrm{sep}}) & \longrightarrow & C/\Lambda(L^{\mathrm{sep}}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle z^n} & & \downarrow{\scriptstyle z^n} & & \downarrow{\scriptstyle z^n} & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & C(L^{\mathrm{sep}}) & \longrightarrow & C/\Lambda(L^{\mathrm{sep}}) & \longrightarrow & 0 \end{array}$$

we get an exact sequence

$$0 \to \Lambda[z^n] \to C(L^{\mathrm{sep}})[z^n] \to C/\Lambda(L^{\mathrm{sep}})[z^n] \to$$

$$\to \Lambda/z^n\Lambda \to C(L^{\mathrm{sep}})/z^nC(L^{\mathrm{sep}}) \to C/\Lambda(L^{\mathrm{sep}})/z^n(C/\Lambda(L^{\mathrm{sep}})) \to 0$$

Here $\Lambda[z^n] = 0$ since Λ is free over $\mathbb{F}[z]$; we claim that $C(L^{\mathrm{sep}})/z^nC(L^{\mathrm{sep}})$ is trivial as well: indeed, arguing as in 3.22 we see that for every $\beta \in L^{\mathrm{sep}}$ the relation $z^n\alpha = \beta$ inside $C(L^{\mathrm{alg}})$ corresponds to a separable polynomial equation over L and therefore does, in fact, admit a solution inside $C(L^{\mathrm{sep}})$. Therefore we obtain an exact sequence of $\mathbb{F}[z]/z^n$-modules

$$0 \to C(L^{\mathrm{sep}})[z^n] \to C/\Lambda(L^{\mathrm{sep}})[z^n] \to \Lambda/z^n\Lambda \to 0$$

for every n. Since the $\mathbb{F}[z]$-linear map $z\colon C(L^{\mathrm{sep}})[z^{n+1}] \to C(L^{\mathrm{sep}})[z^n]$ is surjective for every n, we see that for the resulting projective system of exact sequences the Mittag-Leffler condition is met, so that in the projective limit we get an exact sequence

$$0 \to T_z(C) \to T_z(C/\Lambda) \to \Lambda \otimes_{\mathbb{F}[z]} \mathbb{F}[\![z]\!] \to 0.$$

As a G_L-module $T_z(C)$ equals $\mathbb{F}[\![z]\!](1)$, as we have discussed in (3.4.2); similarly we have a natural G_L-action on the Tate module $T_z(C/\Lambda)$, leading to a G_L-representation which should be regarded as *non-crystalline*, i.e., it cannot correspond to a local shtuka over o_L, due to its bad-reduction origin.

In the case of a Tate elliptic curve E_q/K we have seen that the exact uniformization sequence $1 \to q^{\mathbb{Z}} \to \mathbb{G}_m(K^{\mathrm{alg}}) \to E_q(K^{\mathrm{alg}}) \to 0$ induces an extension of $\mathbb{Z}_p$-modules $0 \to \mathbb{Z}_p(1) \to T_p(E_q) \to \mathbb{Z}_p(0) \to 0$ compatible with the G_K-actions. The uniformizing parameter q of a Tate elliptic curve is *always* an element of K and therefore is fixed under the action of G_K. Consequently we may write $q^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathbb{Z}_p(0)$.

In equal characteristic, however, the situation is slightly different: We consider the fixed lattice $\Lambda \subseteq C(L^{\mathrm{sep}})$ from above. Let $\rho \in G_L$ be any L-automorphism of L^{sep}; in particular, ρ fixes $\mathbb{F}$; moreover, being a ring homomorphism, ρ is compatible with $z\colon C(L^{\mathrm{sep}}) \to C(L^{\mathrm{sep}})$, i.e., ρ restricts to an $\mathbb{F}[z]$-linear automorphism

$$\rho\colon \Lambda \to \Lambda,$$

and fixing any $\mathbb{F}[z]$-basis yields a homomorphism of groups $G_L \to \mathrm{Gl}_d(\mathbb{F}[z])$. Assume that $d = \mathrm{rk}_{\mathbb{F}[z]}(\Lambda) = 1$, say with $\mathbb{F}[z]$-basis $\lambda \in \Lambda$. This setting is supposed to be in tightest analogy with the uniformization of Tate elliptic curves. Thus we wish to relate the z-adic completion $\Lambda \otimes_{\mathbb{F}[z]} \mathbb{F}[\![z]\!]$ to the trivial Tate twist $\mathbb{F}[\![z]\!](0)$: Let $\alpha(\rho) \in \mathbb{F}[z]^\times = \mathbb{F}^\times$ be such that $\rho.\lambda = \alpha(\rho)\lambda$. This scalar is clearly independent of the choice of λ, and we may summarize that the induced representation

$$G_L \to \mathrm{Aut}_{\mathbb{F}[\![z]\!]}(\Lambda \otimes_{\mathbb{F}[z]} \mathbb{F}[\![z]\!]) = \mathbb{F}[\![z]\!]^\times$$

factors via $\mathbb{F}^\times$; one further observes that $\alpha(\rho) = 1$ holds for all ρ if and only if $\Lambda \subseteq C(L)$; in particular, without imposing any restriction upon Λ, our desired relation between $\Lambda \otimes_{\mathbb{F}[z]} \mathbb{F}[\![z]\!]$ and the 0-th Tate twist of $\mathbb{F}[\![z]\!]$ fails to be true, which means that these two are in general not isomorphic: the scalar $\alpha(\rho)$ may vary inside the finite group $\mathbb{F}^\times$.

3.4.5 Yoneda extensions of Tate-twist quasi-crystals

Let R be a noetherian integral domain which is a flat $o_L[\![z]\!]$-algebra, together with a ring endomorphism $\sigma_R\colon R \to R$ which is an extension of the r-Frobenius lift $\sigma\colon o_L[\![z]\!] \to o_L[\![z]\!]$. We define a category $\sigma\mathrm{Mod}(R)$ as follows: an *object* of $\sigma\mathrm{Mod}(R)$ is a couple (M, φ_M) where M is a finitely generated R-module together with a σ_R-semi-linear map $\varphi_M\colon M \to M$; a *morphism* $(M, \varphi_M) \to (N, \varphi_N)$ is defined to be a φ-equivariant R-linear map $M \to N$. Given an object (M, φ_M), the datum φ_M will usually be omitted from the notation.

For example, for every $n \in \mathbb{Z}$, $n \geq 0$, the n-th *Tate object* of $\sigma\mathrm{Mod}(R)$ is given by

$$R(n) = (R, (z - \zeta)^n \circ \sigma_R).$$

We may write R instead of $R(0)$. If $R = o_L[\![z]\!]$ then every effective local shtuka gives rise to an object of $\sigma\mathrm{Mod}(R)$; we take this instance as a motivation for calling an object of $\sigma\mathrm{Mod}(\cdot)$ a *quasi-crystal over* R, for (effective) local shtukas over o_L correspond to Dieudonné crystals of p-adic Barsotti-Tate groups ([40], [41]); accordingly, for every effective local shtuka $\hat{M}$ its mod-π_L reduction $\hat{M} \otimes_{o_L[\![z]\!]} \ell[\![z]\!]$ gives rise to a z-isocrystal (with Hodge-Pink structure) by inverting the scalar z; note that $\ell[\![z]\!]$ corresponds to the ring of Witt vectors over the residue field in the p-adic world; see [40].

Lemma 3.25. *The category* $\sigma\mathrm{Mod}(R)$ *is abelian.*

Proof. It is clear how to define finite bi-products and, since the base ring R is noetherian, also kernels and cokernels of morphisms. □

In particular, for any two objects $M, M' \in \sigma\mathrm{Mod}(R)$ the set $\mathrm{Ext}^1_{\sigma,R}(M, M')$ of Yoneda extension classes of M by M' with respect to $\sigma\mathrm{Mod}(R)$ is an abelian group under the Baer sum; see [62].

Let us consider the following special case: suppose that a given extension $0 \to R \xrightarrow{i} M \xrightarrow{\mathrm{pr}} R(1) \to 0$ inside $\sigma\mathrm{Mod}(R)$ admits an R-linear section $w\colon R(1) \to M$ of $\mathrm{pr}\colon M \to R(1)$ so that, fixing this section, M is canonically isomorphic to $R \oplus R(1)$ as an R-module; necessarily M is free with basis $\mathcal{B}_w = (i(1), w(1))$; fixing this basis, there is a unique $b \in R$ such that the given extension amounts to a commutative diagram of R-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & R \oplus R(1) & \longrightarrow & R(1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma_R} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}1 & b\\ 0 & z-\zeta\end{smallmatrix}\right)\circ\sigma_R} & & \downarrow{\scriptstyle (z-\zeta)\circ\sigma_R} & & \\ 0 & \longrightarrow & R & \longrightarrow & R \oplus R(1) & \longrightarrow & R(1) & \longrightarrow & 0 \end{array}$$

where the rows are exact sequences of R-linear maps and where the vertical maps are σ_R-semi-linear; let us exhibit that two extensions of this type are Yoneda equivalent if and only if there is some $u \in R$ such that

$$\left(\begin{smallmatrix}1 & b\\ 0 & z-\zeta\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & \sigma_R(u)\\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1 & u\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & b'\\ 0 & z-\zeta\end{smallmatrix}\right),$$

which is to say that $\sigma_R(u) + b = b' + u(z - \zeta)$; here $b' \in R$ corresponds (in the manner just described) to an extension $0 \to R \to M' \to R(1) \to 0$ with a fixed R-linear splitting $w'\colon R(1) \to M'$.

We intend to follow this line of thought and commence by stating the obvious

Proposition 3.26. *For every extension class* $\xi \in \mathrm{Ext}^1_{\sigma,R}(R(1), R)$ *every representative of* ξ *admits an* R*-linear section.*

Proof. We need merely remark that the R-module underlying $R(1)$ is free of rank one. □

It is clear that the binary relation $\sim$ on R defined by

$$b \sim b' :\Longleftrightarrow \text{There is some } u \in R \text{ such that } \sigma_R(u) + b = b' + u(z - \zeta)$$

is an equivalence relation. The set $R/\sim$ of equivalence classes for $\sim$ naturally becomes an abelian group via

$$+\colon (R/\sim) \times (R/\sim) \to (R/\sim), \quad ([b], [b']) \mapsto [b + b'].$$

Example. $z \sim \zeta + 1$, in particular $[z - \zeta] = [1]$. –

Let $b, b' \in R$. If $\lambda \in R^{\sigma_R = \mathrm{id}}$ then $b \sim b'$ implies $\lambda b \sim \lambda b'$, i.e., the map

$$R^{\sigma_R = \mathrm{id}} \times (R/\sim) \to (R/\sim), \quad (\lambda, [b]) \mapsto [\lambda b],$$

is well-defined; it clearly makes $R/\sim$ into an $R^{\sigma_R = \mathrm{id}}$-module; in particular, $R/\sim$ is an $\mathbb{F}[\![z]\!]$-module. Furthermore, given $\lambda \in R^{\sigma_R = \mathrm{id}}$, the map

$$\lambda\colon R(n) \to R(n)$$

becomes φ-equivariant for every $n \geq 0$; recall that the usual $R^{\sigma_R = \mathrm{id}}$-module structure of the abelian group $\mathrm{Ext}^1_{\sigma,R}(R(1), R)$ is given by

$$\lambda\xi = \text{class of} \quad 0 \to R \to M \times_{\mathrm{pr}, R(1), \lambda} R(1) \to R(1) \to 0,$$

where the class ξ is represented by $0 \to R \to M \xrightarrow{\mathrm{pr}} R(1) \to 0$.

Proposition 3.27. *There is a canonical isomorphism of abelian groups*

$$\mathrm{Ext}^1_{\sigma,R}(R(1), R) \xrightarrow{\simeq} R/\sim$$

which is $R^{\sigma_R = \mathrm{id}}$-linear, and which is natural in the sense that if R' is a noetherian domain being a flat R-algebra, together with an extension $\sigma_{R'}\colon R' \to R'$ of σ_R, then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_{\sigma,R}(R(1), R) & \xrightarrow{\simeq} & R/\sim_R \\ \downarrow & & \downarrow \\ \mathrm{Ext}^1_{\sigma,R'}(R'(1), R') & \xrightarrow{\simeq} & R'/\sim_{R'} \end{array}$$

where the vertical maps are defined in the obvious manner.

Proof. Let $\xi \in \operatorname{Ext}^1_{\sigma,R}(R(1), R)$ be the class of

$$0 \to R \xrightarrow{i} M \xrightarrow{\mathrm{pr}} R(1) \to 0$$

say, and let $w\colon R(1) \to M$ be an R-linear section of the projection $\mathrm{pr}\colon M \to R(1)$; let $\mathcal{B}_w = (i(1), w(1))$ be the resulting R-basis of M. If $w'\colon R(1) \to M$ is another section of pr with resulting basis $\mathcal{B}_{w'} = (i(1), w'(1))$ then we have

$${}_{\mathcal{B}_{w'}}[\mathrm{id}_M]_{\mathcal{B}_w} = \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Gl}_2(R)$$

where $r \in R$ is uniquely determined by the relation $i(r) = w(1) - w'(1)$. From

$$\varphi_M(w(1)) = \varphi_M(w(1)) - (z-\zeta)w(1) + (z-\zeta)w(1)$$

we may conclude that

$${}_{\mathcal{B}_w}[\varphi_M]^{(\sigma_R)}_{\mathcal{B}_w} = \left(\begin{smallmatrix} 1 & \rho_w \\ 0 & z-\zeta \end{smallmatrix}\right) \in R^{2\times 2}$$

where $\rho_w \in R$ is uniquely determined by the relation $i(\rho_w) = \varphi_M(w(1)) - (z-\zeta)w(1)$, i.e., ρ_w is trivial if and only if the section w is φ-equivariant. Proceeding analogously with the section $w'\colon R(1) \to M$, the coordinate change

$${}_{\mathcal{B}_{w'}}[\varphi_M]^{(\sigma_R)}_{\mathcal{B}_{w'}} = {}_{\mathcal{B}_{w'}}[\mathrm{id}_M]_{\mathcal{B}_w} \cdot {}_{\mathcal{B}_w}[\varphi_M]^{(\sigma_R)}_{\mathcal{B}_w} \cdot \sigma_R({}_{\mathcal{B}_{w'}}[\mathrm{id}_M]_{\mathcal{B}_w})^{-1}$$

shows that $\rho_w \sim \rho_{w'}$ in R. Next we have to show that the assignment

$$\operatorname{Ext}^1_{\sigma,R}(R(1), R) \to R/{\sim}, \quad \xi \mapsto [\rho_w],$$

is well-defined. Let $0 \to R \xrightarrow{i'} M' \xrightarrow{\mathrm{pr}'} R(1) \to 0$ be another representative of ξ, and let $\iota\colon M \to M'$ be a corresponding φ-equivariant isomorphism of R-modules. We already know that ιw is an R-linear section of pr' and that the couple $\mathcal{B}'_{\iota w} = (i'(1), (\iota w)(1))$ constitutes an R-basis of M'. It just remains to remark that writing $\varphi_M(w(1)) = \rho_w i(1) + (z-\zeta)w(1)$ we obtain $\varphi_{M'}((\iota w)(1)) = \rho_w i'(1) + (z-\zeta)(\iota w)(1)$, i.e., we get

$${}_{\mathcal{B}_w}[\varphi_M]^{(\sigma_R)}_{\mathcal{B}_w} = \left(\begin{smallmatrix} 1 & \rho_w \\ 0 & z-\zeta \end{smallmatrix}\right) = {}_{\mathcal{B}_{\iota w}}[\varphi_{M'}]^{(\sigma_R)}_{\mathcal{B}_{\iota w}};$$

so, by the above considerations, we may conclude that the assignment $\xi \mapsto [\rho_w]$ is indeed well-defined. By analyzing the Baer sum $\xi + \xi'$ of two given extension classes $\xi, \xi' \in \operatorname{Ext}^1_{\sigma,R}(R(1), R)$ similarly as in 3.21, one easily verifies that the map $\operatorname{Ext}^1_{\sigma,R}(R(1), R) \to R/{\sim}$ just defined is additive, and obviously it sends the trivial extension class to $0 = [0] \in R/{\sim}$. The asserted naturality and $R^{\sigma_R=\mathrm{id}}$-linearity are clear. From the considerations made so far, it also becomes immediately clear that $\xi \in \operatorname{Ext}^1_{\sigma,R}(R(1), R)$ is mapped to 0 if and only if $\xi = 0$, i.e., the map $\operatorname{Ext}^1_{\sigma,R}(R(1), R) \to R/{\sim}$ is injective. In order to prove surjectivity, we need merely remark that given an equivalence class $[\rho]$ for some $\rho \in R$, one can consider the trivial extension of R-modules $0 \to R \to R \oplus R(1) \to R(1) \to 0$ where the R-linear maps $R \to R \oplus R(1)$ and $R \oplus R(1) \to R(1)$ are canonically given by $1 \mapsto (1,0)$ and $(0,1) \mapsto 1$ respectively; in particular, there is a canonical R-linear section $R(1) \to R \oplus R(1)$ of the projection map which is given by $1 \mapsto (0,1)$; if we define $\varphi_{R\oplus R(1)}\colon R\oplus R(1) \to R\oplus R(1)$ with respect to the canonical basis of $R\oplus R(1)$ (σ_R-semi-linearly) by the matrix $\left(\begin{smallmatrix} 1 & \rho \\ 0 & z-\zeta \end{smallmatrix}\right)$ then it becomes clear that the class of the gained extension inside $\mathrm{Ext}^1_{\sigma,R}(R(1),R)$ is mapped to $[\rho] \in R/\!\sim$. $\square$

Let us give another description of the $R^{\sigma=\mathrm{id}}$-module $R/\!\sim$, relying on the fact that the canonical map $R \to R/\!\sim$ is $R^{\sigma=\mathrm{id}}$-linear. We observe that this map is surjective and that its kernel coincides with the image of the $R^{\sigma=\mathrm{id}}$-linear map

$$\eta_R\colon R \to R, \qquad u \mapsto \sigma(u) - u(z-\zeta),$$

i.e., there is an exact sequence of $R^{\sigma=\mathrm{id}}$-modules

$$R \overset{\eta_R}{\to} R \to (R/\!\sim) \to 0,$$

which means that $(R/\!\sim) = \mathrm{coker}(\eta_R)$.

3.4.6 "Crystalline" and "semi-stable" Yoneda extensions

Let

$$\begin{aligned} \mathit{YE}^1_{\mathrm{cris}} &= \mathrm{Ext}^1_{\sigma,o_L[\![z]\!]}(o_L[\![z]\!](1),\ o_L[\![z]\!]), \\ \mathit{YE}^1_{\mathrm{st}} &= \mathrm{Ext}^1_{\sigma,o_L[\![z]\!][1/\pi]}(o_L[\![z]\!][1/\pi](1),\ o_L[\![z]\!][1/\pi]). \end{aligned}$$

Note that, since $\pi \in o_L[\![z]\!]$ is not σ-invariant, it does not make sense to write down expressions like "$\mathit{YE}^1_{\mathrm{cris}}[1/\pi]$"; in particular, it should be noted that $\mathit{YE}^1_{\mathrm{st}}$ cannot arise from $\mathit{YE}^1_{\mathrm{cris}}$ by "inverting π".

Proposition 3.28. *The obvious $\mathbb{F}[\![z]\!]$-linear map $\mathit{YE}^1_{\mathrm{cris}} \to \mathit{YE}^1_{\mathrm{st}}$ is injective.*

Proof. Let $x \in o_L[\![z]\!]$ be given such that we have an equation $x = \frac{\sigma(u)}{\pi^{rn}} - \frac{u}{\pi^n}(z-\zeta)$ inside $o_L[\![z]\!][1/\pi]$ for a suitable $u \in o_L[\![z]\!]$ and $n \geq 0$; in particular, this implies $\pi^{rn}x = \sigma(u) - \pi^{(r-1)n}u(z-\zeta)$. Suppose that $n > 0$. We may assume that $u \in o_L[\![z]\!] - \pi o_L[\![z]\!]$, i.e., that the reduction mod π of u does not vanish. However, reducing the latter equation mod π yields $\bar{\sigma}(\bar{u}) = 0$, where $\bar{\sigma}\colon \ell[\![z]\!] \to \ell[\![z]\!]$ denotes the r-Frobenius lift, i.e., we obtain $u \in \pi o_L[\![z]\!]$, a contradiction. $\square$

The $\mathbb{F}[\![z]\!]$-module of Yoneda extension classes introduced in the last section always carries the z-adic topology. As an example, using the identity

$$\mathit{YE}^1_{\mathrm{cris}} = \mathrm{coker}(\eta_{o_L[\![z]\!]}\colon o_L[\![z]\!] \to o_L[\![z]\!])$$

let us study this topology on the $\mathbb{F}[\![z]\!]$-module $\mathit{YE}^1_{\mathrm{cris}}$. We will restrict ourselves to the module of crystalline extension classes, i.e., to the case of good reduction. Writing $\eta = \eta_{o_L[\![z]\!]}$ we prove the

Proposition 3.29. *The canonical map*

$$\operatorname{coker}(\eta) \to \varprojlim_s \operatorname{coker}(\eta)/z^s \operatorname{coker}(\eta)$$

is an isomorphism of $\mathbb{F}[\![z]\!]$*-modules, i.e., the* $\mathbb{F}[\![z]\!]$*-module* YE^1_{cris} *is* z*-adically complete.*

Proof. Let us first show z-adic separatedness, i.e., we first claim that the displayed map is injective. Let $f \in o_L[\![z]\!]$, $f = \sum_\nu f_\nu z^\nu$, be such that $\bar{f} = f[\mathrm{im}(\eta)]$ lies in the kernel, which is to say that for every $s \geq 1$ there exists an element $u_s = \sum_\nu u_{s\nu} z^\nu \in o_L[\![z]\!]$ such that $z^s \mid (f - \eta(u_s))$ for every s; so for every $s \geq 1$ we get relations

$$f_0 - u_{s0}^r - \zeta u_{s0} = 0, \quad f_\nu - u_{s\nu}^r + u_{s,\nu-1} - \zeta u_{s\nu} = 0 \quad (1 \leq \nu \leq s-1);$$

we claim that the sequence $(u_s)_{s\geq 1}$ admits a sub-sequence $(u_{s(k)})_{k\geq 1}$ which converges to an element $u \in o_L[\![z]\!]$, in the sense that for every integer $\varepsilon \geq 1$ we have $u - u_{s(k)} \in z^\varepsilon o_L[\![z]\!]$ for all $k \geq N(\varepsilon)$ say; here we may assume without loss of generality that $N(\varepsilon) \geq \varepsilon$; suppose for a moment that there is such a sequence $(u_{s(k)})_k$, with limit $u \in o_L[\![z]\!]$; let $\varepsilon \geq 1$, and let $k \geq 1$ be sufficiently large such that $u - u_{s(k)} \in z^\varepsilon o_L[\![z]\!]$ and $s(k) \geq \varepsilon$; then we get

$$f - \eta(u) = f - \eta(u_{s(k)}) + \eta(u_{s(k)}) - \eta(u) = f - \eta(u_{s(k)}) + \eta(u_{s(k)} - u)$$

which lies in $z^\varepsilon o_L[\![z]\!]$ since $f - \eta(u_{s(k)})$ is divided by $z^{s(k)}$ and therefore also by z^ε, and since $\eta(u_{s(k)} - u)$ also is divided by z^ε; this implies that $f = \eta(u)$, i.e., that $f[\mathrm{im}(\eta)] = 0$. So it remains to find a convergent subsequence $(u_{s(k)})_k$. Using the relations $f_0 - u_{s0}^r - \zeta u_{s0} = 0$ for $s \geq 1$ we find that $u_{s0} - u_{s'0} \in C(o_L)[z]$ for all $s, s' \geq 1$; consequently the set

$$D = \{u_{s0} - u_{s'0}, \quad s, s' \geq 1\}$$

has to be finite. For every $d \in D$ let

$$A_d = \{s \geq 1, \quad u_{10} - u_{s0} = d\};$$

since the partition $\{A_d\}_{d\in D}$ of the set $\mathbb{N}^{\geq 1}$ is finite, there has to be some $d \in D$ such that A_d is infinite; we fix such a d; now, given any $s, s' \in A_d$, we obtain $u_{s0} = u_{s'0}$, i.e., for all $s \in A_d$ the u_{s0} have the same value; this yields a constant subsequence $(u_{s(k),0})_{k\geq 1}$ of $(u_{s0})_{s\geq 1}$ given by $\{s(k), k \geq 1\} = A_d$. Next we consider the relations

$$f_1 = u_{s1}^r - u_{s0} + \zeta u_{s1}, \quad s \geq 2;$$

in particular, these relations are valid for $s \in \{s(k)\}_{k\geq 1} \cap \mathbb{N}^{\geq 2}$; using that the sequence $(u_{s(k),0})_k$ is constant, they yield that $u_{s(k),1} - u_{s(k'),1} \in C(o_L)[z]$ for all

$k, k' \geq 1$ such that $s(k), s(k') \geq 2$; as above we show that the sequence $(u_{s(k),1})_{k\geq 1}$ admits a constant subsequence $(u_{s(k)(k_1),1})_{k_1\geq 1}$; inductively, fixing $\nu \geq 2$ and using the relations

$$f_\nu = u_{s\nu}^r - u_{s,\nu-1} + \zeta u_{s\nu}, \quad s \geq \nu + 1,$$

we can now prove that the sequence $(u_{s(k)(k_1)\dots(k_{\nu-1}),\nu})_{k_{\nu-1}\geq 1}$ admits a constant subsequence $(u_{s(k)(k_1)\dots(k_{\nu-1})(k_\nu),\nu})_{k_\nu\geq 1}$; now we define a sequence $(v_j)_{j\geq 1}$ by $v_1 = u_{s(1)}$, $v_2 = u_{s(k)(2)}$, $v_3 = u_{s(k)(k_1)(3)}$, $v_4 = u_{s(k)(k_1)(k_2)(4)}$, etc.; we write $v_j = \sum_\mu v_{j\mu} z^\mu$ for every j; by construction, for every $\mu \geq 0$ the sequence $(v_{j\mu})_j$ becomes stationary, i.e., $(v_j)_j$ is z-adically a Cauchy sequence and therefore converges to some element $u \in o_L[\![z]\!]$, as desired.

In order to show surjectivity, we consider a sequence of elements $f_s = \sum_\nu f_{s\nu} z^\nu \in o_L[\![z]\!]$ such that for every $s \geq 1$ there exists some $u_s \in o_L[\![z]\!]$ verifying $f_{s+1} - f_s - \eta(u_s) \in z^s o_L[\![z]\!]$; we define a sequence $(f'_s)_{s\geq 1}$ in $o_L[\![z]\!]$ by

$$f'_s = f_s - \sum_{\mu=1}^{s-1} \eta(u_\mu) \quad (s \geq 1);$$

by construction, for every $s \geq 1$ we obtain $\overline{f_s} = \overline{f'_s}$ in $\operatorname{coker}(\eta)$, and

$$f'_{s+1} - f'_s = f_{s+1} - f_s - \eta(u_s);$$

we claim that the sequence $(f'_s)_s$ converges z-adically to some $f \in o_L[\![z]\!]$; indeed, given any integer $\varepsilon \geq 1$, the latter relations yield immediately that $f'_{s+1} - f'_s \in z^s o_L[\![z]\!] \subseteq z^\varepsilon o_L[\![z]\!]$ for all $s \geq N(\varepsilon) = \varepsilon$; the residue class $\bar{f} = f[\operatorname{im}(\eta)]$ is now the desired preimage: we have to show that $\bar{f}[z^s] = \overline{f_s}[z^s]$ for every $s \geq 1$; since the element $(\overline{f_s}[z^s])_s$ of the projective limit is a coherent sequence of residue classes, it suffices to show that for fixed $s \geq 1$ there is some $n \geq s$ such that $\bar{f}[z^s] = \overline{f_n}[z^s]$; so let $n \geq 1$ be large enough such that $n \geq s$ and such that $f - f'_n \in z^s o_L[\![z]\!]$; this is possible by construction of f, and it means that we find some $g \in o_L[\![z]\!]$ such that $f = f'_n + z^s g$; in particular, $\bar{f} = \overline{f'_n + z^s g}$ in $\operatorname{coker}(\eta)$; we claim that $\overline{f'_n + z^s g}[z^s] = \overline{f_n}[z^s]$, which then implies $\bar{f}[z^s] = \overline{f_n}[z^s]$, as desired; indeed, our claim is equivalent to saying that $\overline{f'_n + z^s g} - \overline{f_n} \in z^s \operatorname{coker}(\eta)$, and the latter actually holds true since $\overline{f'_n} = \overline{f_n}$ in $\operatorname{coker}(\eta)$. □

3.4.7 Study of the Carlitz action over an equal-characteristic local field

In the present section we will mainly be concerned with the following results.

Theorem 3.30. *The $\mathbb{F}[\![z]\!]$-linear map*

$$\Psi\colon C(L)/C(o_L) \otimes_{\mathbb{F}[z]} \mathbb{F}[\![z]\!] \to YE^1_{\mathrm{st}}/YE^1_{\mathrm{cris}}, \qquad \bar{x} \otimes f \mapsto f\overline{[x]},$$

is injective. If the residue field $\ell = o_L/\mathfrak{m}_L$ is finite then Ψ is an isomorphism.

Corollary 3.31. *Suppose that the residue field ℓ of L is finite. Then the $\mathbb{F}(\!(z)\!)$-vector space $YE^1_{\mathrm{st}}/YE^1_{\mathrm{cris}} \otimes_{\mathbb{F}[\![z]\!]} \mathbb{F}(\!(z)\!)$ is of countably infinite dimension.*

In order to prove this corollary, we will apply a result due to B. Poonen [64] showing that the $\mathbb{F}[z]$-module $C(L)/C(o_L)$ is free of countably infinite rank, provided that ℓ is finite. The proof of 3.31 will be given after 3.37 below.

Proof of Theorem 3.30. Let us first convince ourselves of well-definedness: let $x, x' \in C(L)$ be such that $x = \xi + x'$ for some $\xi \in C(o_L)$; we claim that $[x] - [x'] \in YE^1_{\mathrm{cris}}$; however, this is clear since $w(z-\zeta) + (x - x') = g + \sigma(w)$ is met by $g = \xi$ and $w = 0$. The map

$$C(L)/C(o_L) \times \mathbb{F}[\![z]\!] \to YE^1_{\mathrm{st}}/YE^1_{\mathrm{cris}}, \qquad (\bar{x}, f) \mapsto f\overline{[x]},$$

is $\mathbb{F}[z]$-bilinear and therefore induces the displayed $\mathbb{F}[\![z]\!]$-linear map Ψ. In order to show injectivity, let $c = \sum_{\nu=1}^m \overline{\xi_\nu} \otimes f_\nu \in C(L)/C(o_L) \otimes_{\mathbb{F}[z]} \mathbb{F}[\![z]\!]$ be any finite sum of elementary tensors $\overline{\xi_\nu} \otimes f_\nu$ such that $\Psi(c) = \sum_{\nu=1}^m f_\nu \overline{[\xi_\nu]} = 0$. It suffices to show that $f_\nu = 0$ for every ν. By erasing elementary tensors being zero and renumbering we may assume

$$v(\xi_1) \le v(\xi_2) \le ... \le v(\xi_m) < 0,$$

where v is the discrete valuation of L normalized by $v(\pi) = 1$; for every ν write $\xi_\nu = \pi^{v(\xi_\nu)} u_\nu$ where $u_\nu \in o_L^\times$. Furthermore, we may assume that

(i) If $s < 0$ is an integer such that $s = v(\xi_\nu)$ for some ν then $r \nmid s$.

(ii) If $s < 0$ is an integer such that $r \nmid s$ then the system $(\xi_\nu \colon v(\xi_\nu) = s)$ is linearly independent inside the $\mathbb{F}$-vector space $\pi^s o_L / \pi^{s+1} o_L$.

(Recall that $r = \#\mathbb{F}$.) This will be justified below. Considering our assumption $\Psi(c) = 0$, one first observes that $0 = \sum_{\nu=1}^m f_\nu \overline{[\xi_\nu]}$ equals the residue class of $[\sum_{\nu=1}^m \xi_\nu f_\nu]$ in $YE^1_{\mathrm{st}}/YE^1_{\mathrm{cris}}$, i.e., we find some $w \in o_L[\![z]\!][1/\pi]$ and $c' = \sum_{i=0}^\infty c'_i z^i \in o_L[\![z]\!]$ such that

$$w(z-\zeta) + \sum_{\nu=1}^m \xi_\nu f_\nu = c' + \sigma(w);$$

let us write $w = \sum_{i=0}^\infty w_i z^i$ where $w_i \in L$ and $v(w_i) \ge -N$ for some integer $N \ge 0$ and all i; note that we view the power series $f_\nu = \sum_{i=0}^\infty f_{\nu i} z^i \in \mathbb{F}[\![z]\!]$ as elements of $o_L[\![z]\!]$, so that $\xi_\nu f_\nu \in o_L[\![z]\!][1/\pi]$ for every ν; with these conventions the above equation yields

$$w_{i-1} - \zeta w_i + \sum_{\nu=1}^m \xi_\nu f_{\nu i} = c'_i + w_i^r \qquad (i \ge 0),$$

where we let $w_{-1} = 0$. Now suppose that there is some ν such that $f_\nu \ne 0$. We define

$$i_0 = \min\{i,\ f_{\nu i} \ne 0 \text{ for some } \nu\}, \quad \nu_1 = \min\{\nu,\ f_{\nu i_0} \ne 0\};$$

let $n = v(\xi_{\nu_1}) < 0$ be the valuation of ξ_{ν_1}, and

$$\nu_2 = \max\{\nu,\ v(\xi_\nu) = n\}.$$

We may note that by virtue of (i) we have $r \nmid n$. We claim that $v(\sum_{\nu=1}^m \xi_\nu f_{\nu i_0}) = n < 0$; indeed, by choice of ν_1 and ν_2 we may write

$$\begin{aligned}\sum_{\nu=1}^m \xi_\nu f_{\nu i_0} &= \sum_{\nu_1 \le \nu \le \nu_2} \xi_\nu f_{\nu i_0} + \sum_{\nu_2 < \nu \le m} \xi_\nu f_{\nu i_0} \\ &= \pi^n \sum_{\nu_1 \le \nu \le \nu_2} u_\nu f_{\nu i_0} + \pi^{n+1} \sum_{\nu_2 < \nu \le m} \pi^{v(\xi_\nu)-(n+1)} u_\nu f_{\nu i_0};\end{aligned}$$

therefore the first summand here is of valuation $\geq n$ whereas the second one is of valuation $\geq n+1$; by virtue of (ii) we realize that $v(\sum_{\nu_1 \le \nu \le \nu_2} \xi_\nu f_{\nu i_0}) = n$, i.e., the first summand does, in fact, have valuation n, and so by the triangle inequality for v our claim follows. Using this, in a next step we show that

$$rv(w_{i_0}) = n,$$

i.e., $r \mid n$, a contradiction. Let us first consider the case $i_0 > 0$: supposing for a moment that $v(w_{i_0-1}) \geq 0$ we get

$$0 > v(\sum_{\nu=1}^m \xi_\nu f_{\nu i_0}) = v(w_{i_0}^r + \zeta w_{i_0} - w_{i_0-1} + c'_{i_0}) = v(w_{i_0}^r + \zeta w_{i_0});$$

this implies $v(w_{i_0}) < 0$, and consequently $v(w_{i_0}^r) < v(\zeta w_{i_0})$, so we may conclude that $n = v(\sum_{\nu=1}^m \xi_\nu f_{\nu i_0}) = v(w_{i_0}^r)$, as desired; we have yet to justify our assumption $v(w_{i_0-1}) \geq 0$, and in order to achieve this we show inductively that $v(w_i) \geq 0$ for $0 \leq i < i_0$, starting with $i = 0$: let us assume that $v(w_0) < 0$; noting that $\sum_{\nu=1}^m \xi_\nu f_{\nu,0} = 0$, from the equation $0 = w_0^r + c'_0 + \zeta w_0$ we conclude that $0 > v(w_0^r) \geq \min(v(c'_0), v(\zeta w_0))$; if $v(c'_0) > v(\zeta w_0)$ then we get $0 > v(w_0^r) = v(\zeta w_0)$ which leads to a contradiction, and $v(c'_0) \leq v(\zeta w_0)$ cannot happen either, i.e., our assumption is false and therefore $v(w_0) \geq 0$; proceeding inductively, we now suppose that for some $1 \leq j \leq i_0 - 1$ we have $v(w_{j-1}) \geq 0$; we show that assuming $v(w_j) < 0$ leads to a contradiction: indeed, from

$$0 > v(w_j^r) = v(w_{j-1} - c'_j - \zeta w_j) \geq \min(v(w_{j-1} - c'_j), v(\zeta w_j))$$

it follows that $0 > v(w_j^r) \geq v(\zeta w_j)$ since $v(w_{j-1} - c'_j) \geq 0$. This contradiction concludes our proof for $rv(w_{i_0}) = n$ in the case $i_0 > 0$. It remains to consider the case $i_0 = 0$; here we have

$$0 > v(\sum_{\nu=1}^m \xi_\nu f_{\nu,0}) = v(w_0^r + \zeta w_0 + c'_0) \geq \min(v(w_0^r + \zeta w_0), v(c'_0)) = v(w_0^r + \zeta w_0)$$

which implies $v(w_0) < 0$ and therefore $v(w_0^r) < v(\zeta w_0)$, i.e., we get

$$n = v(\sum_{\nu=1}^{m} \xi_\nu f_{\nu,0}) = v(w_0^r + \zeta w_0 + c_0') = v(w_0^r),$$

as desired. Next we have to justify our assumptions (i), (ii). Let us start with (i). We need to either show that (i) is already met by our fixed representation $c = \sum_{\nu=1}^{m} \xi_\nu \otimes f_\nu$ of c by elementary tensors or, if this is not the case, how to produce another representation of c by elementary tensors, i.e.,

$$c = \sum_{\nu=1}^{m} \xi_\nu \otimes f_\nu = \sum_{\nu'=1}^{m'} \xi'_{\nu'} \otimes f'_{\nu'},$$

such that all the $\xi'_{\nu'}$ meet (i); for furnishing such a new representation we would use our assumption $v(\xi_1) \leq ... \leq v(\xi_m) < 0$ on the ξ_ν; we further remark that such a new representation does not affect $\Psi(c)$, for we have

$$0 = \Psi(c) = \sum_{\nu=1}^{m} f_\nu \overline{[\xi_\nu]} = \Psi(\sum_{\nu=1}^{m} \xi_\nu \otimes f_\nu) = \Psi(\sum_{\nu'=1}^{m'} \xi'_{\nu'} \otimes f'_{\nu'}) = \sum_{\nu'=1}^{m'} f'_{\nu'} \overline{[\xi'_{\nu'}]}.$$

Clearly (i) is true for all $s < v(\xi_1)$ since in this case there is no ν such that $s = v(\xi_\nu)$. Let $s_0 < 0$ be the smallest s such that $r \mid s$ and such that $v(\xi_\nu) = s$ for some ν, say $v(\xi_{\nu_0}) = s_0$; clearly we have $v(\xi_1) \leq s_0 < 0$ and $r \mid s_0$; using the identification $L = \ell(\!(\pi)\!)$, let us write $\xi_{\nu_0} = \sum_{j \geq s_0} \gamma_j \pi^j$ where $\gamma_{s_0} \in \ell^\times$; let us write γ_{ν_0} instead of γ_{s_0}; we get $\xi_{\nu_0} = \gamma_{\nu_0} \pi^{s_0} + \widetilde{\xi_{\nu_0}}$ where $\widetilde{\xi_{\nu_0}} \in \ell(\!(\pi)\!)$ is an element of π-adic valuation $v(\widetilde{\xi_{\nu_0}}) = \mathrm{ord}_\pi(\widetilde{\xi_{\nu_0}}) > s_0$; since ℓ is perfect, we have a unique r-th root $\gamma_{\nu_0}^{1/r}$ of $\gamma_{\nu_0} \in \ell^\times$; the $\mathbb{F}[z]$-action of $L = \ell(\!(\pi)\!)$ being understood to come from the Carlitz module $C(L) = C(\ell(\!(\pi)\!))$, we write

$$\begin{aligned}
\gamma_{s_0}^{1/r} \pi^{s_0/r} \otimes z &= z(\gamma_{\nu_0}^{1/r} \pi^{s_0/r}) \otimes 1 \\
&= (\gamma_{\nu_0} \pi^{s_0} + \zeta \gamma_{\nu_0}^{1/r} \pi^{s_0/r}) \otimes 1 \\
&= (\xi_{\nu_0} - \widetilde{\xi_{\nu_0}} + \zeta \gamma_{\nu_0}^{1/r} \pi^{s_0/r}) \otimes 1 \\
&= \xi_{\nu_0} \otimes 1 + (\zeta \gamma_{\nu_0}^{1/r} \pi^{s_0/r} - \widetilde{\xi_{\nu_0}}) \otimes 1,
\end{aligned}$$

which in turn gives a new representation of c by elementary tensors, namely

$$\begin{aligned}
c = \sum_{\nu=1}^{m} \xi_\nu \otimes f_\nu &= \sum_{\nu \neq \nu_0} \xi_\nu \otimes f_\nu + \xi_{\nu_0} \otimes f_{\nu_0} \\
&= \sum_{\nu \neq \nu_0} \xi_\nu \otimes f_\nu + \gamma_{\nu_0}^{1/r} \pi^{s_0/r} \otimes z f_{\nu_0} - (\zeta \gamma_{\nu_0}^{1/r} \pi^{s_0/r} - \widetilde{\xi_{\nu_0}}) \otimes f_{\nu_0}
\end{aligned}$$

Here we have $v(\gamma_{\nu_0}^{1/r} \pi^{s_0/r}) = s_0/r > s_0$ and

$$v(\zeta \gamma_{\nu_0}^{1/r} \pi^{s_0/r} - \widetilde{\xi_{\nu_0}}) \geq \min(s_0/r, v(\widetilde{\xi_{\nu_0}})) > s_0;$$

we proceed like this for every remaining ν such that $v(\xi_\nu) = s_0$, obtaining

$$c = \sum_{\nu : v(\xi_\nu) \neq s_0} \xi_\nu \otimes f_\nu + \sum_{\nu : v(\xi_\nu) = s_0} \gamma_\nu^{1/r} \pi^{s_0/r} \otimes z f_\nu - (\zeta \gamma_\nu^{1/r} \pi^{s_0/r} - \widetilde{\xi_\nu}) \otimes f_\nu;$$

now if $s < s_0$ is any integer such that $r \mid s$ then we claim that in the henceforth gained family

$$(\xi_\nu)_{\{\nu \colon v(\xi_\nu) \neq s_0\}} \cup (\gamma_\nu^{1/r} \pi^{s_0/r})_{\{\nu \colon v(\xi_\nu) = s_0\}} \cup (\widetilde{\xi_\nu} - \zeta \gamma_\nu^{1/r} \pi^{s_0/r})_{\{\nu \colon v(\xi_\nu) = s_0\}}$$

there is none of valuation s: indeed, by choice of s_0 there is no ν such that $v(\xi_\nu) = s$, and the remaining elements are of valuation $> s_0$. Furthermore, there is no element in this new family which is of valuation s_0. We may conclude that our assumption (i) was justified. Let us turn to (ii); here we proceed analogously as in (i). First of all, we remark that for every integer $s < v(\xi_1)$ the system $(\xi_\nu \colon v(\xi_\nu) = s)$ is the empty system and is therefore linearly independent inside the $\mathbb{F}$-vector space $V_s = \pi^s o_L / \pi^{s+1} o_L$. Let $s_0 < 0$ be the smallest s such that $r \nmid s$ and such that the system $(\xi_\nu \colon v(\xi_\nu) = s)$ is linearly dependent inside the $\mathbb{F}$-vector space V_s; in particular, we get that $r \nmid s_0$ and that $(\xi_\nu \colon v(\xi_\nu) = s_0)$ in linearly dependent inside the $\mathbb{F}$-vector space V_{s_0}. From our assumption $v(\xi_1) \leq ... \leq v(\xi_m)$ we obtain that there are indices $\nu_1, \nu_2 \in \{1, ..., m\}$ such that

$$v(\xi_1) \leq ... \leq v(\xi_{\nu_1 - 1}) < v(\xi_{\nu_1}) = ... = v(\xi_{\nu_2}) < v(\xi_{\nu_2 + 1}) \leq ... \leq v(\xi_m)$$

where $v(\xi_\nu) = s_0$ for all $\nu \in \{\nu_1, ..., \nu_2\}$; therefore we may say that $v(\xi_\nu) = s_0$ if and only if $\nu_1 \leq \nu \leq \nu_2$; note that, since a single non-zero element of any vector space is always linearly independent, actually $\nu_1 < \nu_2$; since the elements $\xi_{\nu_1}, ..., \xi_{\nu_2}$ are of the same valuation, we may re-arrange them and write down a linear-combination

$$\overline{\xi_{\nu_2}} = \sum_{\nu = \nu_1}^{\nu_2 - 1} \alpha_\nu \overline{\xi_\nu}$$

inside V_{s_0} where not all of the scalars $\alpha_\nu \in \mathbb{F}$ are zero; we may rephrase this by writing $\xi_{\nu_2} = \sum_{\nu=\nu_1}^{\nu_2 - 1} \alpha_\nu \xi_\nu + \widetilde{\xi_{\nu_2}}$ with a suitable $\widetilde{\xi_{\nu_2}} \in \pi^{s_0 + 1} o_L$; setting $\alpha_\nu = 0$ for $\nu \notin \{\nu_1, ..., \nu_2\}$ we may also write

$$\xi_{\nu_2} = \sum_{\nu \neq \nu_2} \alpha_\nu \xi_\nu + \widetilde{\xi_{\nu_2}}.$$

Consequently

$$\begin{aligned} c = \sum_\nu \xi_\nu \otimes f_\nu &= \sum_{\nu \neq \nu_2} \xi_\nu \otimes f_\nu + \xi_{\nu_2} \otimes f_{\nu_2} \\ &= \sum_{\nu \neq \nu_2} \xi_\nu \otimes (f_\nu + \alpha_\nu f_{\nu_2}) + \widetilde{\xi_{\nu_2}} \otimes f_{\nu_2} \end{aligned}$$

using that $\alpha_\nu \in \mathbb{F} \subseteq \mathbb{F}[z]$; we may conclude that via replacing ξ_{ν_2} by $\widetilde{\xi_{\nu_2}}$ in the system $(\xi_\nu : v(\xi_\nu) = s_0)$ we obtain a system where the "new" ξ_{ν_2} is no longer of valuation s_0, but of valuation $\geq s_0 + 1$; we proceed like this until the system $(\xi_\nu : v(\xi_\nu) = s_0)$ is linearly independent inside V_{s_0}, a process which has to terminate at the system of one single element, if not earlier. Note that in this construction we did not use that $r \nmid s_0$. Now let $s < s_0$ be any integer such that $r \nmid s$; by choice of s_0 the system $(\xi_\nu : v(\xi_\nu) = s)$ is linearly independent inside V_s; moreover, we have found a representation by elementary tensors $c = \sum_{\nu'} \xi'_{\nu'} \otimes f'_{\nu'}$ for c such that the system $(\xi'_{\nu'} : v(\xi'_{\nu'}) = s_0)$ is linearly independent inside V_{s_0}. Finally, we may summarize that also (ii) is justified. Let us show that Ψ is surjective provided that ℓ is finite. Let $\overline{[f]} \in \mathit{YE}^1_{\mathrm{st}}/\mathit{YE}^1_{\mathrm{cris}}$, say with $f = \sum_{i=0}^\infty f_i z^i \in L[\![z]\!]$ such that $v(f_i) \geq -N$ for some $N \geq 0$ and all i. Using the identification $L = \ell(\!(\pi)\!)$ we may write $o_L[\![z]\!] = \ell[\![\pi]\!][\![z]\!] = \ell[\![\pi, z]\!]$; let, say, $f_i = \sum_{j \geq -N} f_{ij}\pi^j \in \ell(\!(\pi)\!)$ for every i; then

$$f = \sum_{i\geq 0} f_i z^i = \sum_{i \geq 0}(\sum_{j\geq -N} f_{ij}\pi^j)z^i = \sum_{j\geq -N}(\sum_{i\geq 0} f_{ij}z^i)\pi^j = \sum_{j \geq -N}\sum_{a\in\ell} a f_{a,j}\pi^j$$

as elements of $o_L[\![z]\!][1/\pi] = \ell[\![z]\!](\!(\pi)\!)$, where $f_{a,j} = \sum_{i:\, f_{ij}=a} z^i \in \mathbb{F}[\![z]\!]$ for every $j \geq -N$. We claim that the element

$$\sum_{j=-N}^{-1}\sum_{a\in\ell} \overline{a\pi^j} \otimes f_{a,j}$$

is via Ψ mapped to the residue class of $[f]$; indeed, we need to find $w \in o_L[\![z]\!][1/\pi]$ and $g \in o_L[\![z]\!]$ such that

$$w(z-\zeta) + (\sum_{j\in\{-N,\ldots,-1\},\, a\in\ell} (af_{a,j})\pi^j - f) = g + \sigma(w),$$

which is met by $w = 0$ and $g = \sum_{j\geq 0,\, a\in\ell}(af_{a,j})\pi^j$. □

We remark that there is a commutative diagram of $\mathbb{F}[z]$-linear maps with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(o_L)\otimes_{\mathbb{F}[z]}\mathbb{F}[\![z]\!] & \longrightarrow & C(L)\otimes_{\mathbb{F}[z]}\mathbb{F}[\![z]\!] & \longrightarrow & C(L)/C(o_L)\otimes_{\mathbb{F}[z]}\mathbb{F}[\![z]\!] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathit{YE}^1_{\mathrm{cris}} & \longrightarrow & \mathit{YE}^1_{\mathrm{st}} & \longrightarrow & \mathit{YE}^1_{\mathrm{st}}/\mathit{YE}^1_{\mathrm{cris}} & \longrightarrow & 0 \end{array}$$

which is induced by the $\mathbb{F}[z]$-linear map

$$C(L) \to o_L[\![z]\!][1/\pi]/\sim, \qquad x \mapsto [x],$$

using the natural isomorphism $\mathit{YE}^1_{\mathrm{st}} \simeq o_L[\![z]\!][1/\pi]/\sim$ discussed before. In 3.30 we have already realized that the right-hand vertical map is always injective, and is even an isomorphism if the residue field ℓ is finite. Let us give some further remarks regarding this diagram: we consider the $\mathbb{F}[\![z]\!]$-linear map

$$\eta\colon o_L[\![z]\!] \to o_L[\![z]\!], \quad u \mapsto \sigma(u) - u(z-\zeta).$$

The kernel of $C(o_L) \to YE^1_{\mathrm{cris}}$ clearly equals $C(o_L) \cap \mathrm{im}(\eta)$. Moreover, we have

Proposition 3.32.

$$\begin{aligned} \ker(C(o_L) \to YE^1_{\mathrm{cris}}) &= \cap_{s\geq 1} z^s C(o_L) \subseteq C(o_L) \\ \ker(C(L) \to YE^1_{\mathrm{st}}) &= \cup_{n\geq 0} \cap_{s\geq 1} z^s(\pi^{-n} o_L) \subseteq C(L). \end{aligned}$$

We will see below that, in fact, the two kernels described here do coincide. Note that for $n > 0$ the subgroup $\pi^{-n} o_L \subseteq L$ is not an $\mathbb{F}$-linear subspace of L and that $z\colon C(L) \to C(L)$ does not restrict to a map $\pi^{-n} o_L \to \pi^{-n} o_L$, so that the inclusion $\pi^{-n} o_L \subseteq C(L)$ cannot become $\mathbb{F}[z]$-linear.

Proof. Let $x \in C(L)$. Suppose there is an integer $n \geq 0$ together with an element $u = \sum_{\nu=0}^{\infty} u_\nu z^\nu \in o_L[\![z]\!]$ such that we have an equation

$$x = \frac{\sigma(u)}{\pi^{rn}} - \frac{u}{\pi^n}(z-\zeta) = \sum_{\nu=0}^{\infty} (\pi^{-rn} u_\nu^r - \pi^{-n} u_{\nu-1} + \pi^{-n} u_\nu \zeta) z^\nu$$

inside $o_L[\![z]\!][1/\pi]$, where we let $u_{-1} = 0$; a comparison of coefficients yields

$$x = \pi^{-rn} u_0^r + \pi^{-n} u_0 \zeta, \qquad \pi^{-n} u_{\nu-1} = \pi^{-rn} u_\nu^r + \pi^{-n} u_\nu \zeta \quad (\nu \geq 1),$$

i.e., $x = z(\pi^{-n} u_0) \in C(L)$, and $\pi^{-n} u_{\nu-1} = z(\pi^{-n} u_\nu) \in C(L)$ for every $\nu \geq 1$; this proves the second asserted equation; if we let $n = 0$ and $x \in C(o_L)$ then the same argument shows that also the first equation is true. □

The map $\eta\colon o_L[\![z]\!] \to o_L[\![z]\!]$ induces an $\mathbb{F}[\![z]\!]$-linear map $\ell[\![z]\!] \to \ell[\![z]\!]$ defined by $u \mapsto \bar{\sigma}(u) - zu$, where $\bar{\sigma}\colon \ell[\![z]\!] \to \ell[\![z]\!]$ denotes the r-Frobenius lift.

Proposition 3.33. *The map η is injective, and the commutative diagram of $\mathbb{F}[\![z]\!]$-linear maps with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi o_L[\![z]\!] & \longrightarrow & o_L[\![z]\!] & \longrightarrow & \ell[\![z]\!] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\eta} & & \downarrow & & \\ 0 & \longrightarrow & \pi o_L[\![z]\!] & \longrightarrow & o_L[\![z]\!] & \longrightarrow & \ell[\![z]\!] & \longrightarrow & 0 \end{array}$$

induces a short exact sequence of $\mathbb{F}[\![z]\!]$-linear maps

$$0 \to \mathrm{coker}(\pi o_L[\![z]\!] \to \pi o_L[\![z]\!]) \to \mathrm{coker}(o_L[\![z]\!] \xrightarrow{\eta} o_L[\![z]\!]) \to \mathrm{coker}(\ell[\![z]\!] \to \ell[\![z]\!]) \to 0.$$

Proof. Let $u = \sum_{\nu=0}^{\infty} u_\nu z^\nu \in o_L[\![z]\!]$. Provided we have a relation $\sigma(u) = u(z-\zeta)$ it follows that $u_\nu^r + \zeta u_\nu = u_{\nu-1}$ for every $\nu \geq 0$, where we let $u_{-1} = 0$. Suppose that $u_0 \neq 0$; this implies $u_0^{r-1} = -\zeta$, and we obtain a relation

$$(*) \qquad (u_\nu/u_0)^r - u_\nu/u_0 = (u_{\nu-1}/u_0)(-1/\zeta)$$

inside L for every $\nu \geq 0$. Let v be the discrete valuation of L normalized by $v(\pi) = 1$. We get $\min(rv(u_1/u_0), v(u_1/u_0)) \leq v((u_1/u_0)^r - u_1/u_0) = -v(\zeta) < 0$ and necessarily $v(u_1/u_0) < 0$. From $v((u_1/u_0)^r - u_1/u_0) = -v(\zeta)$ it now follows that $rv(u_1/u_0) = -v(\zeta)$, and by induction, using the above relations $(*)$, one verifies that

$$r^\nu v(u_\nu/u_0) = -v(\zeta)$$

for every $\nu \geq 0$. In particular, this gives

$$v(u_\nu) = -v(\zeta)/r^\nu + v(u_0)$$

for all $\nu \geq 0$, which is a contradiction since $0 < v(\zeta) < \infty$. It follows that $u_0 = 0$, and by induction, using the relations $u_\nu^r + \zeta u_\nu = u_{\nu-1}$ for $\nu \geq 1$, our argument shows that $u_\nu = 0$ for all $\nu \geq 0$. In particular, we get

$$\ker(\pi o_L[\![z]\!] \xrightarrow{\eta} \pi o_L[\![z]\!]) \subseteq \ker(o_L[\![z]\!] \xrightarrow{\eta} o_L[\![z]\!]) = 0,$$

and it only remains to remark that

$$\ker(\ell[\![z]\!] \to \ell[\![z]\!]) = \{\sum_{\nu=0}^{\infty} u_\nu z^\nu, u_0^r = 0, u_\nu^r = u_{\nu-1} \text{ for all } \nu \geq 1\} = 0;$$

so by the Snake Lemma our claim follows. □

Remark. The equalities $C(\mathfrak{m}_L)[z^s] = C(o_L)[z^s]$ $(s \geq 1)$ from 3.23 do as well imply that the kernel of $\eta\colon o_L[\![z]\!] \to o_L[\![z]\!]$ coincides with the kernel of the restriction $\pi o_L[\![z]\!] \to \pi o_L[\![z]\!]$, for we have

$$\begin{aligned}
\ker(o_L[\![z]\!] \xrightarrow{\eta} o_L[\![z]\!]) &= \{u = \sum_{\nu=0}^{\infty} u_\nu z^\nu \in o_L[\![z]\!], (u_\nu)_{\nu\geq 0} \in \varprojlim_{s\geq 1} C(o_L)[z^s]\} \\
&= \{u = \sum_{\nu=0}^{\infty} u_\nu z^\nu \in o_L[\![z]\!], (u_\nu)_{\nu\geq 0} \in \varprojlim_{s\geq 1} C(\mathfrak{m}_L)[z^s]\} \\
&= \ker(\mathfrak{m}_L[\![z]\!] \xrightarrow{\eta} \mathfrak{m}_L[\![z]\!]). -
\end{aligned}$$

Lemma 3.34. *Suppose that the residue field ℓ of L is finite, i.e., that L is an equal-characteristic local field. Then there exist*

— *a finite field extension $L'/\mathbb{F}(\zeta)$ and*

— *a prime place $\mathfrak{p}$ of L' lying over the place of $\mathbb{F}(\zeta)$ given by $(\zeta) \subseteq \mathbb{F}[\zeta]$*

such that $L'_{\mathfrak{p}} = L$.

Proof. Our assumption amounts to saying that $L/\mathbb{F}(\!(\zeta)\!)$ is a finite field extension. Let $\mathbb{F}(\!(\zeta)\!)_{\mathrm{sep}}$ be the separable closure of $\mathbb{F}(\!(\zeta)\!)$ inside L; this is evidently again a local field. Let $\pi_s \in \mathbb{F}(\!(\zeta)\!)_{\mathrm{sep}}$ be a fixed uniformizer, and let ℓ_s be the (finite) residue field of $\mathbb{F}(\!(\zeta)\!)_{\mathrm{sep}}$; we obtain an identification $\mathbb{F}(\!(\zeta)\!)_{\mathrm{sep}} = \ell_s(\!(\pi_s)\!)$ where, ℓ_s being perfect, the right-hand side may be regarded as the field of (finite-tail) formal Laurent series over ℓ_s; we claim that, in fact, we have $\ell_s = \ell$: indeed, the finite field extension $L/\mathbb{F}(\!(\zeta)\!)_{\mathrm{sep}}$ is purely inseparable, and some elementary considerations in field theory show that the degree of L over $\mathbb{F}(\!(\zeta)\!)_{\mathrm{sep}}$ is a power of p, say $[L : \mathbb{F}(\!(\zeta)\!)_{\mathrm{sep}}] = p^m$; by Lemma 3.35 below, using the identification $\mathbb{F}(\!(\zeta)\!)_{\mathrm{sep}} = \ell_s(\!(\pi_s)\!)$, the field L is isomorphic to $\ell_s(\!(\pi_s^{1/p^m})\!)$, i.e., the finite field extension $L/\mathbb{F}(\!(\zeta)\!)_{\mathrm{sep}}$ is totally ramified. Let us now consider the separable finite field extension $\mathbb{F}(\!(\zeta)\!)_{\mathrm{sep}}/\mathbb{F}(\!(\zeta)\!)$; the element $\zeta \in \mathbb{F}(\!(\zeta)\!)^\times$ of the base field gives rise to the subring $\ell[\zeta] \subseteq \ell[\![\pi_s]\!]$ of polynomials in ζ over ℓ; let $\zeta = e\pi_s^n$ for some $e = \sum_{\nu=0}^{\infty} e_\nu \pi_s^\nu \in \ell[\![\pi_s]\!]^\times$ and $n > 0$, say; we claim that ζ is transcendent over ℓ: indeed, for any polynomial expression $f = \sum_{\nu=0}^{d} a_\nu \zeta^\nu \in \ell[\![\pi_s]\!]$ such that $a_\nu \neq 0$ for at least one index ν, we may assume without loss of generality that $a_0 \neq 0$, so that

$$f = \sum_{\nu=0}^{d} a_\nu \zeta^\nu = \sum_{\nu=0}^{d} a_\nu (e_0 \pi_s^n + e_1 \pi_s^{n+1} + \ldots)^\nu = a_0 + (a_1 e_0)\pi_s^n + \ldots,$$

i.e., $f \in \ell[\![\pi_s]\!]^\times$, and in particular $f \neq 0$; the inclusion $\ell[\zeta] \subseteq \ell[\![\pi_s]\!]$ induces monomorphisms of rings $\ell[\zeta]/(\zeta^\nu) \hookrightarrow \ell[\![\pi_s]\!]/(\pi_s^{n\nu})$ for every $\nu \geq 1$, which in the projective limit give a finite embedding of complete discrete valuation rings $\ell[\![\zeta]\!] \hookrightarrow \ell[\![\pi_s]\!]$. We consider the corresponding totally ramified finite extension of local fields $\ell(\!(\pi_s)\!)/\ell(\!(\zeta)\!)$; it is well-known that $\ell(\!(\pi_s)\!) = \ell(\!(\zeta)\!)(\pi_s)$ (for example, by [25], Corollary (2.9)/2); let $f = \sum_\nu f_\nu x^\nu \in \ell(\!(\zeta)\!)[u]$ be the minimal polynomial of π_s over $\ell(\!(\zeta)\!)$; since the global field $\ell(\zeta)$ lies (ζ)-adically dense inside the local field $\ell(\!(\zeta)\!)$, every coefficient f_ν may be approximated by some element of $\ell(\zeta)$, i.e., for any given range $N > 0$ we can find some polynomial $g = \sum_\nu g_\nu u^\nu \in \ell(\zeta)[u]$ such that $\deg(f) = \deg(g)$ and

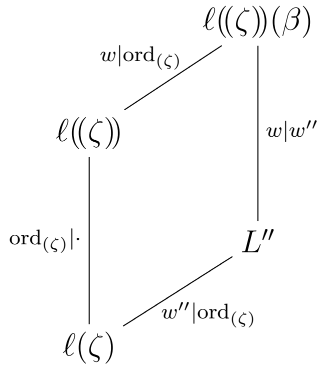
$$\min_\nu \operatorname{ord}_\zeta(f_\nu - g_\nu) > N;$$

therefore, by Krasner's Lemma [10], 3.4.2/3, there is an element $\beta \in \ell(\!(\zeta)\!)^{\mathrm{alg}}$ such that $g(\beta) = 0$ and

$$\mathbb{F}(\!(\zeta)\!)_{\mathrm{sep}} = \ell(\!(\zeta)\!)(\pi_s) = \ell(\!(\zeta)\!)(\beta);$$

in particular, the element β is algebraic over $\ell(\zeta)$, so that $L'' = \ell(\zeta)(\beta)$ is finite over $\ell(\zeta)$; the (ζ)-adic valuation $\operatorname{ord}_{(\zeta)}$ on $\ell(\zeta)$ canonically extends to the natural valuation $\operatorname{ord}_{(\zeta)}$ on $\ell(\!(\zeta)\!)$ which in turn uniquely extends to a discrete valuation w on the finite extension $\ell(\!(\zeta)\!)(\beta)$, in the sense that $w|_{\ell(\!(\zeta)\!)}$ is equivalent with $\operatorname{ord}_{(\zeta)}$ on

$\ell((\zeta))$; we remark that the field $\ell((\zeta))(\beta)$ is complete with respect to w; let $w'' = w|_{L''}$; by [25], Theorem (2.6), the field L'' lies dense inside $\ell((\zeta))(\beta)$ with respect to the topology induced by w, and moreover $w''|_{\ell(\zeta)}$ induces the (ζ)-adic topology on $\ell(\zeta)$; we obtain the diagram



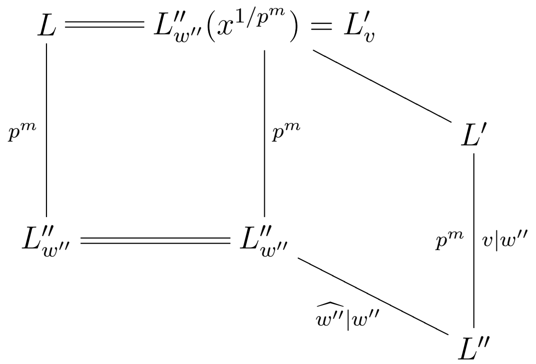
of extensions of discretely valued fields; by the universal property of the (ζ)-adic completion $(L''_{w''}, \widehat{w''})$ of L'' (see, for example, [50], XII.2.1) there is a unique isomorphism of valued fields

$$(L''_{w''}, \widehat{w''}) \simeq (\ell((\zeta))(\beta), w)$$

being compatible with the respective canonical embeddings of L''; we may summarize that $\mathbb{F}((\zeta))_{\text{sep}} = L''_{w''}$. Returning to the purely inseparable extension $L/\mathbb{F}((\zeta))_{\text{sep}}$, we consider the valuation w'' of the field L'' and choose an element $x \in L''$ such that $1 = w''(x) = \widehat{w''}(x)$; necessarily $x \in L''_{w''}$ is a uniformizer of the local field $L''_{w''}$. We adjoin a p^m-th root x^{1/p^m} of x to L''. The polynomial $f = u^{p^m} - x \in L''[u]$ is Eisenstein and therefore irreducible, since x is a prime element of the valuation ring $o_{w''}$ of L'' for w'', i.e., f is the minimal polynomial of x^{1/p^m} over L''. Let $L' = L''(x^{1/p^m})$. Since the extension L'/L'' is purely inseparable, there is a unique discrete valuation v on L' such that $v|_{L''}$ is equivalent with w'' (for details see the proof of 3.35(i) below, or [25], (2.6)); similarly as before, from [25], Theorem (2.6), it follows that $L''_{w''}(x^{1/p^m})$ is the v-adic completion of L'. Furthermore, since the residue field of the local field $L''_{w''}$ is perfect, we learn from 3.35(iii) below that $L^{p^m} = L''_{w''}$; in particular, L contains a p^m-th root of the uniformizer x, and we may conclude that $L''_{w''}(x^{1/p^m}) \subseteq L$; for reasons of degree, from

$$[L''_{w''}(x^{1/p^m}) : L''_{w''}] = p^m = \deg(f)$$

it follows that the latter inclusion of fields has to be an equality. We summarize our findings in the diagram



of extensions of discretely valued fields. The proof is complete. □

Lemma 3.35. *Let κ be a field of characteristic $p > 0$ and let x be an indeterminate over κ; let $m \geq 1$ be an integer.*

(i) The (x)-adic valuation $\mathrm{ord}_{(x)}$ on $\kappa(x)$ extends uniquely to a discrete valuation on the purely inseparable finite field extension $\kappa(x)(x^{1/p^m})$, and the field $\kappa(\!(x)\!)(x^{1/p^m})$ is the (x)-adic completion of $\kappa(x)(x^{1/p^m})$.

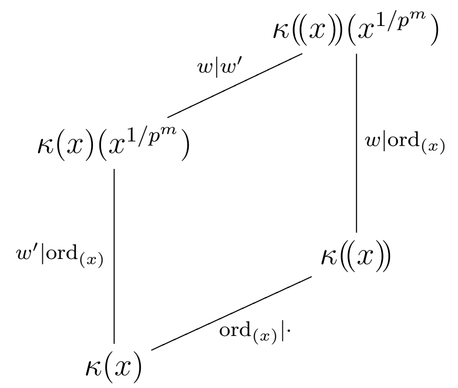
(ii) The purely inseparable finite field extension $\kappa(\!(x)\!)(x^{1/p^m})/\kappa(\!(x)\!)$ is totally ramified of degree p^m.

(iii) Let κ be perfect, and let $E/\kappa(\!(x)\!)$ be a purely inseparable finite field extension of degree p^m. Then $\kappa(\!(x)\!) = E^{p^m} = \mathrm{im}(\mathrm{Frob}_E^m \colon E \to E)$. In particular,

$$E = \kappa(\!(x)\!)^{1/p^m} = \{\alpha^{1/p^m}, \alpha \in \kappa(\!(x)\!)\}.$$

(iv) Let κ be perfect. Then every purely inseparable finite field extension of $\kappa(\!(x)\!)$ is of the form $\kappa(\!(x^{1/p^s})\!) = \kappa(\!(x)\!)(x^{1/p^s})$ for some $s \geq 0$.

Proof of 3.35. The polynomial $f = u^{p^m} - x \in \kappa(x)[u]$ is Eisenstein over $\kappa[x]$ with respect to the prime element $x \in \kappa[x]$ and therefore irreducible over $\kappa(x)$; we consider the finite field extension $\kappa(x)(x^{1/p^m})$ of $\kappa(x)$ of degree p^m and remark that the root x^{1/p^m} has to be transcendent over κ since x is; furthermore, f being the minimal polynomial of x^{1/p^m} over $\kappa(x)$, the element x^{1/p^m} is purely inseparable over $\kappa(x)$. One can also view f as an element of $\kappa(\!(x)\!)[u]$; as such it is Eisenstein over $\kappa[\![x]\!]$ and therefore irreducible over $\kappa(\!(x)\!)$. Let us show part (i). Let w be the unique discrete valuation on $\kappa(\!(x)\!)(x^{1/p^m})$ such that $w|_{\kappa(\!(x)\!)}$ is equivalent with $\mathrm{ord}_{(x)}$ on $\kappa(\!(x)\!)$; by [25], Theorem (2.6), the field $\kappa(x)(x^{1/p^m})$ lies dense inside $\kappa(\!(x)\!)(x^{1/p^m})$ with respect to the topology induced by w, and the restriction of $w' = w|_{\kappa(x)(x^{1/p^m})}$ to $\kappa(x)$ is equivalent with $\mathrm{ord}_{(x)}$. We obtain the diagram



of extensions of valued fields; in order to show the uniqueness of w', suppose that $w'' \neq w'$ is another discrete valuation on $\kappa(x)(x^{1/p^m})$ such that $w''|_{\kappa(x)}$ is equivalent with $\mathrm{ord}_{(x)}$ on $\kappa(x)$; then there is some $\alpha \in \kappa(x)(x^{1/p^m})$ such that $w'(\alpha) \neq w''(\alpha)$; now consider the purely inseparable field extension $\kappa(x)(\alpha)/\kappa(x)$; the minimal polynomial of α over $\kappa(x)$ is purely inseparable and therefore admits only a single linear factor over a fixed algebraic closure of $\kappa(x)$, so that by [25], Theorem (2.6), the (x)-adic valuation $\mathrm{ord}_{(x)}$ of $\kappa(x)$ can only admit a single extension to $\kappa(x)(\alpha)$, which is a contradiction; we may summarize that the valuation w' is the unique extension of $\mathrm{ord}_{(x)}$ on $\kappa(x)$ to $\kappa(x)(x^{1/p^m})$ (this argument can be carried out more generally, see [25], Corollary (2.6)); finally, by the universal property of (x)-adic completion ([50], VII.2.1) we find that there is a unique isomorphism of valued fields

$$(\kappa((x))(x^{1/p^m}), w) \simeq (\kappa(x)(x^{1/p^m})_{w'}, \widehat{w'})$$

over $\kappa(x)(x^{1/p^m})$, as desired. Let us turn to part (ii). It is well-known that we have

$$w(\alpha) = \tfrac{1}{f}\mathrm{ord}_{(x)}(N_{\kappa((x))(x^{1/p^m})/\kappa((x))}(\alpha))$$

for every $\alpha \in \kappa((x))(x^{1/p^m})$, where $f = f(w|\mathrm{ord}_{(x)})$ is the residue degree of the field extension $\kappa((x))(x^{1/p^m})/\kappa((x))$; letting $e = e(w|\mathrm{ord}_{(x)})$ denote the ramification index of this extension, one calculates

$$w(x) = \tfrac{1}{f}\mathrm{ord}_{(x)}(x^{p^m}) = e,$$

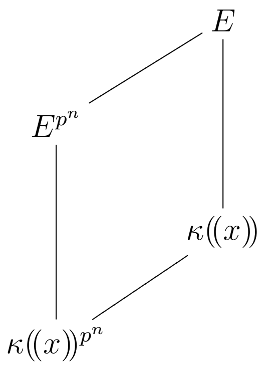
where we use that $p^m = [\kappa((x))(x^{1/p^m}) : \kappa((x))] = ef$; on the other hand, we find $w(x) = p^m w(x^{1/p^m})$, i.e., $e = p^m w(x^{1/p^m})$; in combination with the equation $p^m = ef$ this yields

$$e = p^m, \qquad f = 1, \qquad w(x^{1/p^m}) = 1,$$

i.e., the field extension $\kappa((x))(x^{1/p^m})/\kappa((x))$ is totally ramified of degree p^m; in particular, the residue field of $\kappa((x))(x^{1/p^m})$ equals κ, and x^{1/p^m} is a uniformizer of $\kappa((x))(x^{1/p^m})$. This accomplishes the proof of part (ii). In order to prove (iii), we imitate an argument given in [55], X.1.3, for the case of a global function field: let, say, $E = \kappa(\!(x)\!)(\alpha_1, ..., \alpha_s)$, where each of the elements $\alpha_1, ..., \alpha_s$ is algebraic and purely inseparable over $\kappa(\!(x)\!)$; for every $j = 1, ..., s$ let $f_j \in \kappa(\!(x)\!)[u]$ be the minimal polynomial of α_j; then f_j is of the form $u^{p^{m_j}} - \alpha'_j$ for some $m_j \geq 1$, and α'_j is the p^{m_j}-th power of α_j for every j; in particular, $\deg(f_j) = p^{m_j}$ for every j; let $n = \max_j(m_j)$; necessarily $E^{p^n} \subseteq \kappa(\!(x)\!)$, and from $[E : \kappa(\!(x)\!)] = p^m$ it follows that $n \leq m$. We claim that $[E : E^{p^n}] = p^n$; indeed, we have isomorphisms of fields

$$\mathrm{Frob}_E^n \colon E \xrightarrow{\simeq} E^{p^n}, \qquad \mathrm{Frob}^n_{\kappa(\!(x)\!)} \colon \kappa(\!(x)\!) \xrightarrow{\simeq} \kappa(\!(x)\!)^{p^n}$$

which show that $[E^{p^n} : \kappa(\!(x)\!)^{p^n}] = [E : \kappa(\!(x)\!)]$; from the diagram of field extensions



we therefore get that $[E : E^{p^n}] = [\kappa(\!(x)\!) : \kappa(\!(x)\!)^{p^n}]$. Since κ is perfect, it follows that $\kappa(\!(x)\!)^{p^n} = \kappa(\!(x^{p^n})\!)$, and with the aid of part (i) we realize that $\kappa(\!(x)\!) = \kappa(\!(x^{p^n})\!)(x)$, therefore

$$p^n = [\kappa(\!(x)\!) : \kappa(\!(x^{p^n})\!)] = [E : E^{p^n}],$$

and our claim follows. From $E^{p^n} \subseteq \kappa(\!(x)\!) \subseteq E$ we obtain that $[E : \kappa(\!(x)\!)] \leq [E : E^{p^n}]$ and hence $m \leq n$; we may summarize that $m = n$, and so, for reasons of degree, the inclusion $E^{p^n} \subseteq \kappa(\!(x)\!)$ has to be an equality. Finally, let us discuss (iv). Let $E/\kappa(\!(x)\!)$ be a purely inseparable finite field extension, say of degree p^s for some $s \geq 1$, the case $s = 0$ being trivial. From (iii) it follows that $E = \kappa(\!(x)\!)^{1/p^s}$, in particular, the field E contains a uniquely determined p^s-th root x^{1/p^s} of x, i.e., $\kappa(\!(x)\!)(x^{1/p^s}) \subseteq E$. In part (i) we have realized that the field $\kappa(\!(x)\!)(x^{1/p^s})$ is purely inseparable of degree p^s over $\kappa(\!(x)\!)$, and we may conclude that the inclusion $\kappa(\!(x)\!)(x^{1/p^s}) \subseteq E$ has to be an equality. ☐

Proposition 3.36. *Suppose that the residue field ℓ is finite.*

(i) The $\mathbb{F}[z]$-module $C(L)/C(o_L)$ is torsion-free, i.e., the kernel of the canonical $\mathbb{F}[z]$-linear map $C(L)/C(o_L) \to C(L)/C(o_L) \otimes_{\mathbb{F}[z]} \mathbb{F}(z)$ is trivial.

(ii) The $\mathbb{F}[z]$-module $C(L)/C(o_L)$ is free of countably infinite rank.

In particular, the $\mathbb{F}(z)$-vector space $C(L)/C(o_L) \otimes_{\mathbb{F}[z]} \mathbb{F}(z)$ is of countably infinite dimension.

Proof. From 3.34 we know that the local field L arises as the completion of a global function field $L'/\mathbb{F}(\zeta)$ with respect to a prime place $\mathfrak{p}|(\zeta)$. The Carlitz module $C : \mathbb{F}[z] \to L'[\tau]$ over L', defined by $z \mapsto \zeta + \tau$, where $L'[\tau] = \mathrm{End}_{(\mathrm{GrSch}/L'),\mathbb{F}-\mathrm{lin}}(\mathbb{G}_{a,L'})$, is certainly defined over $o_\mathfrak{p}$, and in particular over $o_L = \widehat{o_\mathfrak{p}}$. In order to show (i), for a given $x \in C(L)$ let $\bar{x} \in C(L)/C(o_L)$ be such that $\alpha\bar{x} = 0$ for some $\alpha \in \mathbb{F}[z] - \{0\}$, i.e., $\alpha x \in C(o_L)$; by 3.37(ii) below, this implies $V(x) = 0$, i.e., $v(x) \geq 0$ by 3.37(iii), which in turn means that $\bar{x} = 0$ in $C(L)/C(o_L)$. Now Theorem 2 in [64] immediately implies (ii); note that in order to apply the cited result in loc. cit. one can take S to be any nonempty finite set of prime places of L' such that $\mathfrak{p} \notin S$. □

Let v be the discrete valuation of L normalized by $v(\pi) = 1$, and suppose that the residue field ℓ is finite. By [64], Proposition 1.(1), for every $x \in C(L)$ the limit

$$V(x) = \lim_{n\to\infty} \frac{\min(0, v(z^n x))}{r^n}$$

exists; this gives a function $V\colon C(L) \to \mathbb{R}$ which plays the role of a local height function à la Néron-Tate associated to the Carlitz module

$$C\colon \mathbb{F}[z] \to L[\tau] = \mathrm{End}_{(\mathrm{GrSch}/L),\mathbb{F}-\mathrm{lin}}(\mathbb{G}_{a,L}), \qquad z \mapsto \zeta + \tau,$$

over L; see [64], §3, for details.

Lemma 3.37 ([64]). *Suppose that the residue field ℓ is finite. Let $x \in C(L)$.*

(i) $V(x) = 0$ if and only if $\alpha x \in C(o_L)$ for some $\alpha \in \mathbb{F}[z] - \{0\}$.

(ii) $V(x) = \min(0, v(x))$.

Proof. For (i) (resp., (ii)) see Proposition 4.(3) (resp., Proposition 4.(4)) in [64]. □

Finally, we are able to prove the main result of the present section.

Proof of Corollary 3.31. We need merely remark that by 3.30 there is a canonical $\mathbb{F}[\![z]\!]$-linear isomorphism $C(L)/C(o_L)\otimes_{\mathbb{F}[z]}\mathbb{F}[\![z]\!] \to YE^1_{\mathrm{st}}/YE^1_{\mathrm{cris}}$, and that $C(L)/C(o_L)$ is free of infinite rank over $\mathbb{F}[z]$. □

3.4.8 Inverting isogenies

We have seen earlier that $YE^1_{\mathrm{cris}} = o_L[\![z]\!]/\!\sim$ naturally is an $\mathbb{F}[\![z]\!]$-module. Setting $F = \mathbb{F}(\!(z)\!)$ and $o = o_F = \mathbb{F}[\![z]\!]$ we now consider the F-vector space

$$YE^1_{\mathrm{cris}}[\tfrac{1}{z}] \simeq YE^1_{\mathrm{cris}} \otimes_o F.$$

We let

$$\mathrm{Iso}YE^1_{\mathrm{cris}} = \mathrm{Ext}^1_{\sigma, o_L(\!(z)\!)}(o_L(\!(z)\!)(1), o_L(\!(z)\!)),$$

i.e., the o-module $\mathrm{Iso}YE^1_{\mathrm{cris}}$ is naturally isomorphic to $o_L(\!(z)\!)/\sim$ where $\sim$ denotes the usual equivalence relation on $o_L(\!(z)\!)$ as discussed in 3.27. In fact, since the action of $z \in o$ on $\mathrm{Iso}YE^1_{\mathrm{cris}}$ is an automorphism $\rho_z\colon \mathrm{Iso}YE^1_{\mathrm{cris}} \to \mathrm{Iso}YE^1_{\mathrm{cris}}$ of the underlying abelian group, the o-module structure on $\mathrm{Iso}YE^1_{\mathrm{cris}}$ naturally extends to an action of $F = o[\frac{1}{z}]$ in such a way that z^{-1} acts by ρ_z^{-1}.

Remark. The prefix "Iso-" is motivated by the characterization of the isogeny category of (good-reduction) local shtukas over o_L; see [34], §§2, 7, and [41], §2. –

Proposition 3.38. *The map*

$$YE^1_{\mathrm{cris}}[\tfrac{1}{z}] \to \mathrm{Iso}YE^1_{\mathrm{cris}} = o_L(\!(z)\!)/\sim_{o_L(\!(z)\!)}, \qquad [f]/z^n \mapsto [z^{-n}f],$$

is an isomorphism of F-vector spaces.

Proof. Let us first check that the displayed map is well-defined. Before doing so, we remark that if $g, h \in o_L[\![z]\!]$ then $[g] = [h]$ with respect to $\sim_{o_L[\![z]\!]}$ clearly implies that $[g] = [h]$ with respect to $\sim_{o_L(\!(z)\!)}$. Now let $[f]/z^n = [g]/z^m$ inside $YE^1_{\mathrm{cris}}[\frac{1}{z}]$, i.e., there is some $s \geq 0$ such that $[z^{s+m}f] = [z^{s+n}g]$ with respect to $\sim_{o_L(\!(z)\!)}$, and therefore, via multiplying with z^{-s-m-n}, we obtain that $[z^{-n}f] = [z^{-m}g]$, as desired. It is a straightforward matter to show F-linearity and surjectivity; let us briefly explain injectivity: suppose that there is some $u \in o_L(\!(z)\!)$ such that $\sigma(u) + z^{-n}f = u(z-\zeta)$, i.e., that $[z^{-n}f] = [0]$; for $N \gg 0$ we get $z^{n+N}u \in o_L[\![z]\!]$ and therefore $z^N[f] = [0]$ over $o_L[\![z]\!]$, but this means that $[f]/z^n = 0$ inside $YE^1_{\mathrm{cris}}[\frac{1}{z}]$. □

Replacing YE^1_{cris} by YE^1_{st}, we arrive at the following situation: let

$$\begin{aligned}\mathrm{Iso}YE^1_{\mathrm{st}} &= \mathrm{Ext}^1_{\sigma, o_L(\!(z)\!)[1/\pi]}(o_L(\!(z)\!)[1/\pi](1),\ o_L(\!(z)\!)[1/\pi]) \\ &= o_L(\!(z)\!)[1/\pi]/\sim_{o_L(\!(z)\!)[1/\pi]}\ .\end{aligned}$$

Again, the scalar $z \in o$ acts by an automorphism on this abelian group, i.e., the multiplication-by-z map $\rho'_z\colon \mathrm{Iso}YE^1_{\mathrm{st}} \to \mathrm{Iso}YE^1_{\mathrm{st}}$ is bijective; as before, we may conclude that $\mathrm{Iso}YE^1_{\mathrm{st}}$ naturally becomes an F-vector space, and that z^{-1} acts via $(\rho'_z)^{-1}$.

Remark. Similarly as mentioned earlier, since $\pi \in o_L[\![z]\!]$ is not σ-invariant, the abelian group $\mathrm{Iso}YE^1_{\mathrm{st}}$, being merely an F-vector space, cannot arise from $\mathrm{Iso}YE^1_{\mathrm{cris}}$ by "inverting π". –

Proposition 3.39. *The map*

$$YE^1_{\mathrm{st}}[\tfrac{1}{z}] \to \mathrm{Iso}YE^1_{\mathrm{st}}, \qquad [\pi^{-n}f]/z^m \mapsto [z^{-m}f/\pi^n],$$

is an isomorphism of F-vector spaces.

Proof. We remark that $\pi^{-n}g \sim \pi^{-m}h$ with respect to $\sim_{o_L[\![z]\!][1/\pi]}$ implies $\pi^{-n}g \sim \pi^{-m}h$ with respect to $\sim_{o_L(\!(z)\!)[1/\pi]}$. This being said, the proof is entirely analogous to that of 3.38. □

Appendix: A brief dictionary

Mixed Characteristic/Number Fields	Equal Characteristic/Function Fields
	Let $\mathbb{F}$ be a finite field, $\#\mathbb{F} = r < \infty$, $\mathrm{char}(\mathbb{F}) = p$
$\mathbb{Z}$	$\Gamma(\mathbb{P}^1_{\mathbb{F}} - \{z = \infty\}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{F}}}) = \mathbb{F}[z]$
$\mathbb{Q}$	$\mathbb{F}(\mathbb{P}^1_{\mathbb{F}}) = \mathbb{F}(z)$
$\lvert\cdot\rvert_\infty$ the archimedean absolute value on $\mathbb{Q}$	$\lvert\cdot\rvert_\infty$ on $\mathbb{F}(z)^\times$ defined by $\lvert\frac{f}{g}\rvert_\infty = r^{\deg(f)-\deg(g)}$
$\mathbb{R} = \widehat{\mathbb{Q}}^{\lvert\cdot\rvert_\infty}$	$\mathbb{R}_\infty = \mathbb{F}(\!(\frac{1}{z})\!) = \widehat{\mathbb{F}(z)}^{\lvert\cdot\rvert_\infty}$ completion at ∞
$\mathbb{C}$	$\mathbb{C}_\infty = \widehat{\mathbb{R}_\infty^{\mathrm{alg}}}$
o_K complete mixed-char. DVR	o_L complete discretely valued $\mathbb{F}$-algebra
$K = \mathrm{Frac}(o_K)$	$L = \mathrm{Frac}(o_L)$
$G_K = \mathrm{Gal}(K^{\mathrm{alg}}/K)$	$G_L = \mathrm{Gal}(L^{\mathrm{sep}}/L)$
$p = \mathrm{char}(k)$, k the residue field of o_K	$p = \mathrm{char}(\ell)$, ℓ the residue field of o_L
$\mathbb{Z} \hookrightarrow o_K$ canonical map	$\mathbb{F}[z] \hookrightarrow o_L$ an embedding of $\mathbb{F}$-alg., $z \mapsto: \zeta \in o_L$
$p = 0$ in k	Assumption $\zeta = 0$ in ℓ
$(p) \subseteq \mathbb{Z}$ residue characteristic	$(z) \subseteq \mathbb{F}[z]$ kernel of $\mathbb{F}[z] \hookrightarrow o_L \to \ell$
$k \to k$, $x \mapsto x^p$, p-Frobenius	$\ell \to \ell$, $x \mapsto x^r$, r-Frobenius
$W(k) \to W(k)$, $[x] \mapsto [x]^p$, p-Frobenius lift	$\ell[\![z]\!] \to \ell[\![z]\!]$, $\sum_i b_i z^i \mapsto \sum_i b_i^r z^i$, r-Frobenius lift
$\mathbb{Z}_p$ (p)-adic completion of $\mathbb{Z}$	$\mathbb{F}[\![z]\!]$ (z)-adic completion of $\mathbb{F}[z]$
$\mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}]$	$\mathbb{F}(\!(z)\!) = \mathbb{F}[\![z]\!][\frac{1}{z}]$
$o_K/\mathbb{Z}_p$ complete ring extension	$o_L/\mathbb{F}[\![\zeta]\!]$ complete ring extension
E Elliptic curve over K	E Drinfeld $\mathbb{F}[z]$-module over L, $E_z = \zeta + \sum_{i=1}^{<\infty} a_i\tau^i$, where $\tau\colon \mathbb{G}_{a,L} \to \mathbb{G}_{a,L}$ r-Frobenius
$E(K)$ group of K-rational points of E	$E(L)$ $\mathbb{F}[z]$-module: $zx = \zeta x + \sum_{i=1}^{<\infty} a_i x^{r^i}$
$\mathbb{G}_m$ multiplicative group scheme	C Carlitz $\mathbb{F}[z]$-module
$\mathbb{G}_m(K) = K^\times$ unit group of K	$C(L)$ $\mathbb{F}[z]$-module: $zx = \zeta x + x^r$
$\mathbb{Z}_p(1) = T_p(\mathbb{G}_{m,K}) = \varprojlim_{n\geq 1}\mathbb{G}_m(K^{\mathrm{alg}})[p^n]$	$\mathbb{F}[\![z]\!](1) = T_z(C) = \varprojlim_{n\geq 1} C(L^{\mathrm{sep}})[z^n]$

$\mathrm{Aut}_{\mathbb{Z}_p}(\mathbb{Z}_p(1)) \simeq \mathbb{Z}_p^\times$ via $(\varepsilon_n)_{n\geq 1}$, where ε_n is a generator of the cyclic group $\mathbb{G}_m(K^{\mathrm{alg}})[p^n]$ s.t. $(\varepsilon_n)_{n\geq 1} \in T_p(\mathbb{G}_{m,K})$ $K_\infty = K(\varepsilon_1, \varepsilon_2, ...) \subseteq K^{\mathrm{alg}}$ $\gamma \in G_K$: $\mathbb{Z}_p(1) \to \mathbb{Z}_p(1)$, $x \mapsto \chi_K(\gamma)x$, natural G_K-action via the cyclotomic character $\chi_K\colon G_K \to \mathbb{Z}_p^\times$	$\mathrm{Aut}_{\mathbb{F}[\![z]\!]}(\mathbb{F}[\![z]\!](1)) \simeq \mathbb{F}[\![z]\!]^\times$ via $(t_n)_{n\geq 0}$, where t_n is an $\mathbb{F}[z]/(z^{n+1})$-basis of $C(L^{\mathrm{sep}})[z^{n+1}]$ s.t. $(t_n)_{n\geq 0} \in T_z(C)$ $L_\infty = L(t_0, t_1, ...) \subseteq L^{\mathrm{sep}}$ $\gamma \in G_L$: $\mathbb{F}[\![z]\!](1) \to \mathbb{F}[\![z]\!](1)$, $x \mapsto \chi_L(\gamma)x$, G_L-action via the cyclotomic character $\chi_L\colon G_L \to \mathbb{F}[\![z]\!]^\times$ defined by $\sum_n \gamma(t_n)z^n = \chi_L(\gamma)\mathbf{t}_+$ where $\mathbf{t}_+ = \sum_n t_n z^n \in L_\infty[\![z]\!]^\times$
$1 \to o_K^\times \to K^\times \stackrel{v_K}{\to} \mathbb{Z} \to 0$ valuation sequence $1 \to 1 + \mathfrak{m}_K \to o_K^\times \to k^\times \to 1$ principal units	$0 \to C(o_L) \to C(L) \to C(L)/C(o_L) \to 0$ $0 \to C(\mathfrak{m}_L) \to C(o_L) \to C(\ell) \to 0$
$1 \to q^{\mathbb{Z}} \to (K^{\mathrm{alg}})^\times \to E(K^{\mathrm{alg}}) \to 0$, $\lvert q\rvert_K < 1$, p-adic uniformization of Tate elliptic curves E	$0 \to \Lambda \to C \to_{\mathrm{an}} E' \to 0$ Drinfeld's analytic uniformization of quotients E' of C by G_L-invariant finite free $\mathbb{F}[z]$-submodules $\Lambda \subseteq C(L^{\mathrm{sep}})$
The action of G_K on the uniformization lattice $q^{\mathbb{Z}} \subseteq (K^{\mathrm{alg}})^\times$ is always trivial, i.e., $q^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathbb{Z}_p(0)$	The uniformization lattice $\Lambda \subseteq C(L^{\mathrm{sep}})$ does not in general lie inside $C(L)$, and if $\mathrm{rk}(\Lambda) = 1$ then $G_L \to \mathrm{Aut}_{\mathbb{F}[\![z]\!]}(\Lambda \otimes_{\mathbb{F}[z]} \mathbb{F}[\![z]\!])$ factors via $\mathbb{F}^\times$
crystalline Yoneda-extension classes $(1 + \mathfrak{m}_K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathrm{Ext}^1_{\mathrm{cris}}(\mathbb{Q}_p, \mathbb{Q}_p(1))$	 $C(\mathfrak{m}_L) \otimes_{\mathbb{F}[\![z]\!]} \mathbb{F}(\!(z)\!) \to \mathit{YE}^1_{\mathrm{cris}} \otimes_{\mathbb{F}[\![z]\!]} \mathbb{F}(\!(z)\!)$
semi-stable Yoneda-extension classes $\widehat{K^\times}^{(p)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathrm{Ext}^1_{\mathrm{st}}(\mathbb{Q}_p, \mathbb{Q}_p(1))$	 $\widehat{C(L)}^{(z)} \otimes_{\mathbb{F}[\![z]\!]} \mathbb{F}(\!(z)\!) \to \widehat{\mathit{YE}^1_{\mathrm{st}}}^{(z)} \otimes_{\mathbb{F}[\![z]\!]} \mathbb{F}(\!(z)\!)$
Quotients of Ext^1-modules $\mathrm{Ext}^1_{\mathrm{st}}(\mathbb{Q}_p, \mathbb{Q}_p(1))/\mathrm{Ext}^1_{\mathrm{cris}}(\mathbb{Q}_p, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p$	 $\mathit{YE}^1_{\mathrm{st}}/\mathit{YE}^1_{\mathrm{cris}} \otimes_{\mathbb{F}[\![z]\!]} \mathbb{F}(\!(z)\!) \simeq \oplus_{\mathbb{N}}\mathbb{F}(\!(z)\!)$ if $\#\ell < \infty$

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