Münster J. of Math. **13** (2020), 519–540
DOI 10.17879/90169642993
urn:nbn:de:hbz:6-90169643343

Münster Journal of Mathematics

Deninger's conjectures and Weil–Arakelov cohomology

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(Communicated by Guido Kings)

Dedicated to Christopher Deninger on his 60th birthday

Abstract. We conjecture the existence of a long exact sequence relating Deninger's conjectural cohomology to Weil–Arakelov cohomology, the latter being unconditionally defined. We prove this conjecture for smooth projective varieties over finite fields whose Weil-étale motivic cohomology groups are finitely generated. Then we explain the consequences that such an exact sequence would have.

1. Introduction

Christopher Deninger has conjectured the existence of a certain cohomology theory for arithmetic schemes (i.e., separated schemes of finite type over the integers), which would explain many conjectural properties of the corresponding Zeta-functions. The shape that such a cohomology should take was described in a long series of papers (see [3, 5, 4, 6, 7, 8, 9, 10, 11]). The resulting conjectural framework generalizes Weil's conjectures to arbitrary—in particular, possibly flat—arithmetic schemes. More recently, the authors of this note have defined some cohomological complexes of $\mathbb{R}$-vector spaces attached to proper regular arithmetic schemes, which we call Weil–Arakelov cohomology, see [13]. The "secondary Euler characteristic" of the Weil–Arakelov cohomology groups of the arithmetic scheme $\mathcal{X}$ with coefficients in $\mathbb{R}(n)$ conjecturally gives the vanishing order of the Zeta-function $\zeta(\mathcal{X}, s)$ at $s = n \in \mathbb{Z}$. Under standard assumptions, Weil–Arakelov cohomology has an integral structure which conjecturally gives Zeta-values up to sign. In this note, we conjecture the existence of a long exact sequence relating Deninger's conjectural cohomology to Weil–Arakelov cohomology, the latter being unconditionally defined. We prove this conjecture for smooth projective varieties over finite fields whose Weil-étale

The first author was supported by a grant from the Simons foundation.
The second author was supported by ANR-15-CE40-0002.

motivic cohomology groups are finitely generated. Then we explain the consequences that such an exact sequence would have, using elementary linear algebra. For example, it would imply our vanishing order formula. It would also "explain" Beilinson's conjectures relating motivic cohomology to Deligne cohomology, as well as the perfectness of the Arakelov intersection pairing between the Arakelov–Chow groups of Gillet–Soulé. In the last section of this note, we recall from [13] the statement of our special value conjecture, and briefly mention how it fits into Deninger's formalism.

2. Deninger's conjectures on Zeta-functions

The Zeta-function of an arithmetic scheme $\mathcal{X}$ is defined by the product (see [34])

$$\zeta(\mathcal{X}, s) = \prod_{x \in \mathcal{X}_0} (1 - N(x)^{-s})^{-1},$$

which converges in the half-plane $\Re(s) > \dim(\mathcal{X})$ (see [34, Theorem 1]), where $\mathcal{X}_0$ is the set of closed points of $\mathcal{X}$, $N(x)$ is the cardinality of the residue field $\kappa(x)$, and $\dim(\mathcal{X})$ is the Krull dimension of $\mathcal{X}$. It is conjectured that $\zeta(\mathcal{X}, s)$ has a meromorphic continuation to the whole complex plane. We have

$$\zeta(\mathcal{X}, s) = \prod_{p < \infty} \zeta(\mathcal{X}_p, s),$$

where the product is indexed over the set of prime numbers and $\mathcal{X}_p := \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{F}_p$.

If $\mathcal{X} \to \operatorname{Spec}(\mathbb{Z})$ is projective, flat and regular, an Arakelov compactification $\overline{\mathcal{X}}$ of $\mathcal{X}$ is defined as follows. We view the fiber of $\overline{\mathcal{X}}$ over $\mathbb{R}$ as the complex analytic variety $\mathcal{X}(\mathbb{C})$ endowed with its obvious $G_{\mathbb{R}}$-action, where $G_{\mathbb{R}}$ is the Galois group of $\mathbb{C}/\mathbb{R}$. An "integral structure over ∞" is then given by the choice a Kähler metric ω on $\mathcal{X}(\mathbb{C})$ compatible with the $G_{\mathbb{R}}$-action, i.e., such that $F_\infty^*(\omega) = -\omega$, where $F_\infty \in G_{\mathbb{R}}$ is complex conjugation. We denote by $\mathcal{X}_\infty$ the pair $(\mathcal{X}(\mathbb{C}), \omega)$ endowed with its $G_{\mathbb{R}}$-action, and we set $\overline{\mathcal{X}} := (\mathcal{X}, \mathcal{X}_\infty)$. The dimension of $\overline{\mathcal{X}}$, which we denote by $\dim(\overline{\mathcal{X}})$, is defined to be the Krull dimension of the scheme $\mathcal{X}$.

The Zeta-function of $\overline{\mathcal{X}}$ is defined by

$$\zeta(\overline{\mathcal{X}}, s) = \zeta(\mathcal{X}, s) \cdot \zeta(\mathcal{X}_\infty, s) = \prod_{p \leq \infty} \zeta(\mathcal{X}_p, s),$$

where $\zeta(\mathcal{X}_\infty, s)$ is a product of Gamma factors, depending on the Hodge structure over $\mathbb{R}$ on Betti cohomology $H^*(\mathcal{X}(\mathbb{C}), \mathbb{C})$. More precisely, let

$$H^i(\mathcal{X}(\mathbb{C}), \mathbb{C}) \simeq \bigoplus_{p+q=i} H^q(\mathcal{X}(\mathbb{C}), \Omega^p) =: \bigoplus_{p+q=i} H^{p,q}$$

be the Hodge decomposition and let

$$h^{p,q} = \dim_{\mathbb{C}} H^{p,q}, \quad h^{p,\pm} = \dim_{\mathbb{C}} (H^{p,p})^{F_\infty = \pm(-1)^p}$$

be the Hodge numbers. One defines

$$\zeta(\mathcal{X}_\infty, s) := \prod_{i\in\mathbb{Z}} L_\infty(h^i(\mathcal{X}_\mathbb{Q}), s)^{(-1)^i},$$

where (see [35, Section 3])

$$(1)\quad L_\infty(h^i(\mathcal{X}_\mathbb{Q}), s) := \prod_{p<q;\, p+q=i} \Gamma_\mathbb{C}(s-p)^{h^{p,q}} \cdot \prod_{p=\frac{i}{2}} \Gamma_\mathbb{R}(s-p)^{h^{p,+}}\Gamma_\mathbb{R}(s-p+1)^{h^{p,-}}$$

and

$$\Gamma_\mathbb{C}(s) = (2\pi)^{-s}\Gamma(s), \quad \Gamma_\mathbb{R}(s) = 2^{-\frac{1}{2}}\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big).$$

The statement of Deninger's conjecture requires the notion of zeta-regularized determinant, which we now recall from [4, Section 1].

Definition 2.1. Let Θ be an endomorphism of a complex vector space V of countable dimension. We say that $\det_\infty(\Theta \mid V)$ *is defined* if the following conditions hold:

(i) V is the direct sum of finite-dimensional Θ-invariant subspaces. For any $\alpha \in \mathbb{C}$, there are at most finitely many of these subspaces on which α occurs as an eigenvalue.

(ii) Let $\mathrm{Sp}(\Theta \mid V)$ be the set of eigenvalues of Θ counted with their algebraic multiplicities. We consider

$$\zeta_{\Theta|V}(s) := \sum_{0\neq\alpha\in\mathrm{Sp}(\Theta|V)} \alpha^{-s}, \quad \text{with } -\pi < \arg(\alpha) \leq \pi.$$

We assume that $\zeta_{\Theta|V}(s)$ converges absolutely for $\Re(s) \gg 0$ and that $\zeta_{\Theta|V}(s)$ has an analytic continuation to $\Re(s) > -\epsilon$ for some $\epsilon > 0$, which is holomorphic at $s = 0$.

It follows from condition (i) above that $V^{\Theta\sim\alpha} := \operatorname{colim}\operatorname{Ker}(\Theta-\alpha)^m$, where the colimit is indexed over $m \in \mathbb{N}$, is finite-dimensional for any $\alpha \in \mathbb{C}$. The *algebraic multiplicity* of the eigenvalue α is defined as the dimension of $V^{\Theta\sim\alpha}$. Under these two conditions, we define

$$\det_\infty(\Theta \mid V) := \exp(-\zeta'_{\Theta|V}(0)) \quad \text{if } 0 \notin \mathrm{Sp}(\Theta \mid V)$$

and

$$\det_\infty(\Theta \mid V) := 0 \quad \text{if } 0 \in \mathrm{Sp}(\Theta \mid V).$$

Notation 2.2. We use the notation $\mathfrak{X}$ to denote either an arithmetic scheme $\mathcal{X} = (\mathcal{X}, \varnothing)$ or an Arakelov compactification $\overline{\mathcal{X}} = (\mathcal{X}, \mathcal{X}_\infty)$ of a projective regular flat arithmetic scheme $\mathcal{X}$. In the latter case, we say that $\mathfrak{X} \to \overline{\mathrm{Spec}(\mathbb{Z})}$ is projective, flat and regular. We say that $\mathfrak{X} \to \overline{\mathrm{Spec}(\mathbb{Z})}$ is projective and regular if either $\mathfrak{X} \to \overline{\mathrm{Spec}(\mathbb{Z})}$ is projective, flat and regular, or if $\mathfrak{X} = (\mathcal{X}, \varnothing)$, where $\mathcal{X}$ is a projective smooth scheme over a finite field.

The following conjecture is due to Deninger (see [3, 5, 4, 6, 7, 8, 9, 10, 11]).

Conjecture 2.3 (Deninger)**.** *On the category of arithmetic schemes and their Arakelov compactifications, there exists a cohomology theory given by (possibly infinite-dimensional) complex vector spaces $H^*_{\mathrm{dyn,c}}(\mathfrak{X})$ and $H^*_{\mathrm{dyn}}(\mathfrak{X})$ endowed with an $\mathbb{R}$-action φ^t, which satisfies the following properties:*

(i) *One has $H^i_{\mathrm{dyn,c}}(\mathfrak{X}) = 0$ for $i < 0$ and $i > 2 \cdot \dim(\mathfrak{X})$.*

(ii) *One has*

$$\zeta(\mathfrak{X}, s) = \prod_{i=0}^{2d} \det\nolimits_\infty\Big(\frac{s \cdot \mathrm{Id} - \Theta}{2\pi} \;\Big|\; H^i_{\mathrm{dyn,c}}(\mathfrak{X})\Big)^{(-1)^{i+1}},$$

where $\Theta := \lim_{t\to 0} \frac{1}{t}(\varphi^t - \mathrm{Id})$, and the right-hand side is defined (in the sense of Definition 2.1).

(iii) *If $\mathfrak{X}$ is regular of pure dimension d, one has a trace map*

$$\mathrm{Tr}\colon H^{2d}_{\mathrm{dyn,c}}(\mathfrak{X}) \to \mathbb{C}(-d)$$

and an $\mathbb{R}$-equivariant pairing

$$H^i_{\mathrm{dyn,c}}(\mathfrak{X}) \times H^{2d-i}_{\mathrm{dyn}}(\mathfrak{X}) \xrightarrow{\cup} H^{2d}_{\mathrm{dyn,c}}(\mathfrak{X}) \xrightarrow{\mathrm{Tr}} \mathbb{C}(-d)$$

such that the induced pairing between the ρ-eigenspace of Θ on $H^i_{\mathrm{dyn,c}}(\mathfrak{X})$ and the $(d-\rho)$-eigenspace of Θ on $H^{2d-i}_{\mathrm{dyn}}(\mathfrak{X})$ is perfect, for all $\rho \in \mathbb{C}$. Here $\mathbb{C}(-d)$ denotes the vector space $\mathbb{C}$ with $\mathbb{R}$-action $e^{d\cdot t}$.

(iv) *If $\mathfrak{X} \to \overline{\mathrm{Spec}(\mathbb{Z})}$ is projective and regular, there exists an $\mathbb{R}$-equivariant and $\mathbb{C}$-antilinear Hodge $*$-operator*

$$* : H^i_{\mathrm{dyn}}(\mathfrak{X}) \to H^{2d-i}_{\mathrm{dyn}}(\mathfrak{X})(d-i)$$

such that

$$H^i_{\mathrm{dyn}}(\mathfrak{X}) \times H^i_{\mathrm{dyn}}(\mathfrak{X}) \to \mathbb{C}, \quad (x, y) \mapsto \mathrm{Tr}_{\mathcal{X}}(x \cup *y),$$

is an hermitian scalar product on $H^i_{\mathrm{dyn}}(\mathfrak{X})$. Here, $H^*_{\mathrm{dyn}}(\mathfrak{X})(n)$ stands for $H^*_{\mathrm{dyn}}(\mathfrak{X})$ with $\mathbb{R}$-action $e^{-n\cdot t}\varphi^t$.

(v) *If $\mathfrak{X} \to \overline{\mathrm{Spec}(\mathbb{Z})}$ is projective and regular, then the function*

$$L(h^i(\mathfrak{X}), s) := \det\nolimits_\infty\Big(\frac{s \cdot \mathrm{Id} - \Theta}{2\pi} \;\Big|\; H^i_{\mathrm{dyn,c}}(\mathfrak{X})\Big)$$

defines a holomorphic function on the entire complex plane, whose zeroes lie on the line $\Re(s) = i/2$.

3. Weil–Arakelov cohomology with real coefficients

In this section, we recall from [13, Sections 2, 4] the definition of Weil–Arakelov cohomology with $\mathbb{R}$-coefficients. For a regular arithmetic scheme $\mathcal{X}$ and an integer $n \geq 0$, we denote by $\mathbb{Z}(n)(\mathcal{X}) := z^n(\mathcal{X}, 2n - *)$ Bloch's cycle complex (see [1, 16, 26, 27]), which we consider as a complex of Zariski sheaves on $\mathcal{X}$. For any abelian group A, we set $A(n) := \mathbb{Z}(n) \otimes_{\mathbb{Z}} A$ and we consider the Zariski hypercohomology $R\Gamma(\mathcal{X}, A(n))$ of the complex of sheaves $A(n)$.

3.1. **Cohomology with compact support.** Let $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ be proper and regular. We define motivic cohomology with compact support $R\Gamma_c(\mathcal{X}, \mathbb{R}(n))$ as the mapping fiber of the regulator map $R\Gamma(\mathcal{X}, \mathbb{R}(n)) \xrightarrow{\mathrm{reg}} R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}}, \mathbb{R}(n))$, so that we have an exact triangle

$$R\Gamma_c(\mathcal{X}, \mathbb{R}(n)) \to R\Gamma(\mathcal{X}, \mathbb{R}(n)) \xrightarrow{\mathrm{reg}} R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}}, \mathbb{R}(n)) \to ,$$

where

$$R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}}, \mathbb{R}(n)) := R\Gamma(G_{\mathbb{R}}, \mathcal{X}(\mathbb{C}), \mathbb{R}(n)_{\mathcal{D}})$$

denotes Deligne cohomology. A similar construction was done by Goncharov in [20] and later by Holmstrom and Scholbach in [21]. Recall that $\mathbb{R}(n)_{\mathcal{D}}$ is the following complex of $G_{\mathbb{R}}$-equivariant sheaves on the manifold $\mathcal{X}(\mathbb{C})$:

$$\mathbb{R}(n)_{\mathcal{D}} := \big[(2i\pi)^n\mathbb{R} \to \Omega^0_{\mathcal{X}(\mathbb{C})} \to \cdots \to \Omega^{n-1}_{\mathcal{X}(\mathbb{C})}\big].$$

Note that if $\mathcal{X}$ is proper regular over a finite field, then

$$R\Gamma_c(\mathcal{X}, \mathbb{R}(n)) = R\Gamma(\mathcal{X}, \mathbb{R}(n)) \quad \text{for any } n \in \mathbb{Z}.$$

For $n < 0$, we have by definition $R\Gamma(\mathcal{X}, \mathbb{R}(n)) = 0$ and $\mathbb{R}(n)_{\mathcal{D}} = (2i\pi)^n\mathbb{R}$, hence

$$R\Gamma_c(\mathcal{X}, \mathbb{R}(n)) = R\Gamma(G_{\mathbb{R}}, \mathcal{X}(\mathbb{C}), (2i\pi)^n\mathbb{R})[-1].$$

Finally, we define

$$R\Gamma_c(\mathcal{X}, \mathbb{C}(n)) := R\Gamma_c(\mathcal{X}, \mathbb{R}(n)) \otimes_{\mathbb{R}} \mathbb{C}$$

for any $\mathcal{X}$ proper regular and any $n \in \mathbb{Z}$.

3.2. **Cohomology of Arakelov compactifications.** Let $\mathcal{X}$ be a projective flat regular arithmetic scheme and let $\overline{\mathcal{X}} := (\mathcal{X}, \mathcal{X}_\infty)$ be an Arakelov compactification, where $\mathcal{X}_\infty$ is the pair $(\mathcal{X}(\mathbb{C}), \omega)$ endowed with its $G_{\mathbb{R}}$-action. The Kähler metric ω provides a canonical morphism of complexes (see [13, Remark 2.13])

$$R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}}, \mathbb{R}(n)) \to \tau^{<2n} R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}}, \mathbb{R}(n)),$$

which is a retract of the adjunction map

$$\tau^{<2n} R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}}, \mathbb{R}(n)) \to R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}}, \mathbb{R}(n)).$$

We define $R\Gamma(\overline{\mathcal{X}}, \mathbb{R}(n))$ as the mapping fiber of the composite map

$$R\Gamma(\mathcal{X}, \mathbb{R}(n)) \to R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}}, \mathbb{R}(n)) \to \tau^{<2n} R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}}, \mathbb{R}(n))$$

so that there is an exact triangle

$$R\Gamma(\overline{\mathcal{X}}, \mathbb{R}(n)) \to R\Gamma(\mathcal{X}, \mathbb{R}(n)) \to \tau^{<2n} R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}}, \mathbb{R}(n)) \to .$$

Note that if $\mathcal{X}$ is proper regular over a finite field, then we define $\overline{\mathcal{X}} := \mathcal{X}$, and therefore we have

$$R\Gamma(\overline{\mathcal{X}}, \mathbb{R}(n)) = R\Gamma(\mathcal{X}, \mathbb{R}(n)) \quad \text{for any } n \in \mathbb{Z}.$$

In both cases, we set

$$R\Gamma(\overline{\mathcal{X}}, \mathbb{C}(n)) := R\Gamma(\overline{\mathcal{X}}, \mathbb{R}(n)) \otimes \mathbb{C}.$$

3.3. **Weil–Arakelov cohomology.** Let $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ be proper and regular. The Weil–Arakelov cohomology with real coefficients is defined as follows:

$$
\begin{aligned}
R\Gamma_{\mathrm{ar}}(\mathcal{X}, \tilde{\mathbb{R}}(n)) &:= R\Gamma(\mathcal{X}, \mathbb{R}(n)) \oplus R\Gamma(\mathcal{X}, \mathbb{R}(n))[-1], \\
R\Gamma_{\mathrm{ar}}(\overline{\mathcal{X}}, \tilde{\mathbb{R}}(n)) &:= R\Gamma(\overline{\mathcal{X}}, \mathbb{R}(n)) \oplus R\Gamma(\overline{\mathcal{X}}, \mathbb{R}(n))[-1], \\
R\Gamma_{\mathrm{ar,c}}(\mathcal{X}, \tilde{\mathbb{R}}(n)) &:= R\Gamma_c(\mathcal{X}, \mathbb{R}(n)) \oplus R\Gamma_c(\mathcal{X}, \mathbb{R}(n))[-1].
\end{aligned} \tag{2}
$$

If $\mathcal{X}$ lies over a finite field $\mathbb{F}_q$, then one has

$$R\Gamma_{\mathrm{ar,c}}(\mathcal{X}, \tilde{\mathbb{R}}(n)) = R\Gamma_{\mathrm{ar}}(\mathcal{X}, \tilde{\mathbb{R}}(n)) = R\Gamma_{\mathrm{ar}}(\overline{\mathcal{X}}, \tilde{\mathbb{R}}(n)).$$

Finally, we set

$$
\begin{aligned}
R\Gamma_{\mathrm{ar}}(\mathcal{X}, \tilde{\mathbb{C}}(n)) &:= R\Gamma_{\mathrm{ar}}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \otimes_{\mathbb{R}} \mathbb{C}, \\
R\Gamma_{\mathrm{ar}}(\overline{\mathcal{X}}, \tilde{\mathbb{C}}(n)) &:= R\Gamma_{\mathrm{ar}}(\overline{\mathcal{X}}, \tilde{\mathbb{R}}(n)) \otimes_{\mathbb{R}} \mathbb{C}, \\
R\Gamma_{\mathrm{ar,c}}(\mathcal{X}, \tilde{\mathbb{C}}(n)) &:= R\Gamma_{\mathrm{ar,c}}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \otimes_{\mathbb{R}} \mathbb{C}.
\end{aligned}
$$

3.4. **The map $\cup\boldsymbol{\theta}$.** For any proper regular arithmetic scheme $\mathcal{X}$, we define

$$\cup\theta\colon R\Gamma_{\mathrm{ar,c}}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \to R\Gamma_{\mathrm{ar,c}}(\mathcal{X}, \tilde{\mathbb{R}}(n))[1]$$

as the map

$$R\Gamma_c(\mathcal{X}, \mathbb{R}(n)) \oplus R\Gamma_c(\mathcal{X}, \mathbb{R}(n))[-1] \xrightarrow{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]} R\Gamma_c(\mathcal{X}, \mathbb{R}(n))[1] \oplus R\Gamma_c(\mathcal{X}, \mathbb{R}(n)),$$

and similarly over $\overline{\mathcal{X}}$ and without compact support. In other words, $\cup\theta$ is the projection on the first direct summand followed by the inclusion of the second direct summand. The map $\cup\theta$ induces an acyclic complex

$$\cdots \xrightarrow{\cup\theta} H^i_{\mathrm{ar,c}}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \xrightarrow{\cup\theta} H^{i+1}_{\mathrm{ar,c}}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \xrightarrow{\cup\theta} \cdots. \tag{3}$$

Of course, $\cup\theta$ induces a $\mathbb{C}$-linear map

$$\cup\theta\colon H^i_{\mathrm{ar,c}}(\mathcal{X}, \tilde{\mathbb{C}}(n)) \to H^{i+1}_{\mathrm{ar,c}}(\mathcal{X}, \tilde{\mathbb{C}}(n)),$$

and similarly over $\overline{\mathcal{X}}$ and without compact support. Note that we have

$$\mathrm{Ker}\big(\cup\theta\colon H^i_{\mathrm{ar,c}}(\mathcal{X}, \tilde{\mathbb{C}}(n)) \to H^{i+1}_{\mathrm{ar,c}}(\mathcal{X}, \tilde{\mathbb{C}}(n))\big) \simeq H^{i-1}_c(\mathcal{X}, \mathbb{C}(n)) \tag{4}$$

and

$$\mathrm{Coker}\big(\cup\theta\colon H^i_{\mathrm{ar,c}}(\mathcal{X}, \tilde{\mathbb{C}}(n)) \to H^{i+1}_{\mathrm{ar,c}}(\mathcal{X}, \tilde{\mathbb{C}}(n))\big) \simeq H^{i+1}_c(\mathcal{X}, \mathbb{C}(n)), \tag{5}$$

and similarly over $\overline{\mathcal{X}}$ and without compact support.

3.5. **Weil–Arakelov cohomology and cohomology of the Weil-étale topos.** Let $\mathcal{X}$ be a proper regular scheme over $\mathrm{Spec}(\mathbb{F}_q)$. We denote by $\mathcal{X}_W$ the Weil-étale topos and by

$$R\Gamma(\mathcal{X}_W, \mathbb{Z}(n)) \simeq R\Gamma(W_k, R\Gamma(\mathcal{X}_{\overline{\mathbb{F}}_q,et}, \mathbb{Z}(n)))$$

the Weil-étale motivic cohomology in the sense of [28] and [17]. By [17, Theorem 7.1 (c)], we have an isomorphism in the derived category

$$R\Gamma(\mathcal{X}_W, \mathbb{Z}(n)) \otimes \mathbb{R} \xrightarrow{\iota_{\mathbb{F}_q}} R\Gamma(\mathcal{X}, \mathbb{R}(n)) \oplus R\Gamma(\mathcal{X}, \mathbb{R}(n))[-1],$$

which depends on the base field $\mathbb{F}_q$. We define $\iota_{\mathbb{F}_1}$ as the map $\iota_{\mathbb{F}_q}$ followed by the map

$$R\Gamma(\mathcal{X}, \mathbb{R}(n)) \oplus R\Gamma(\mathcal{X}, \mathbb{R}(n))[-1] \xrightarrow{(1,\log(q)^{-1})} R\Gamma(\mathcal{X}, \mathbb{R}(n)) \oplus R\Gamma(\mathcal{X}, \mathbb{R}(n))[-1].$$

Then the isomorphism

$$R\Gamma(\mathcal{X}_W, \mathbb{Z}(n)) \otimes \mathbb{R} \xrightarrow{\iota_{\mathbb{F}_1}} R\Gamma(\mathcal{X}, \mathbb{R}(n)) \oplus R\Gamma(\mathcal{X}, \mathbb{R}(n))[-1]$$

does not depend on the base field.

Suppose now that $\mathcal{X}$ has pure dimension d. Under the assumptions $\mathbf{L}(\mathcal{X}_W, n)$ and $\mathbf{L}(\mathcal{X}_W, d-n)$ of [13, Section 3.6], the square

$$\begin{array}{ccc} R\Gamma(\mathcal{X}_W, \mathbb{Z}(n)) & \xrightarrow{\mathrm{Id}\otimes 1} & R\Gamma(\mathcal{X}_W, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{R} \\ \downarrow \simeq & & \downarrow \iota_{\mathbb{F}_1} \\ R\Gamma_W(\mathcal{X}, \mathbb{Z}(n)) & \xrightarrow{\beta_{\mathcal{X}}} & R\Gamma(\mathcal{X}, \mathbb{R}(n)) \oplus R\Gamma(\mathcal{X}, \mathbb{R}(n))[-1] \end{array}$$

commutes, where $R\Gamma_W(\mathcal{X}, \mathbb{Z}(n))$ is the Weil-étale complex defined in [13, Section 3], the left vertical isomorphism is [13, Theorem 3.20], and $\beta_{\mathcal{X}}$ is the map defined in [13, Proposition 4.4].

Remark 3.6. Let $\mathcal{X}$ be a proper regular arithmetic scheme of pure dimension d. The map $\beta_{\mathcal{X}}$ of [13, Proposition 4.4] involves the map

$$B\colon R\Gamma_c(\mathcal{X}, \mathbb{R}(n)) \to R\Gamma(\mathcal{X}, \mathbb{R}(d-n))^*[-2d]$$

induced by the pairing

$$R\Gamma_c(\mathcal{X}, \mathbb{R}(n)) \otimes R\Gamma(\mathcal{X}, \mathbb{R}(d-n)) \to R\Gamma_c(\mathcal{X}, \mathbb{R}(d)) \xrightarrow{\mathrm{tr}} \mathbb{R}[-2d]. \tag{6}$$

Here the trace map

$$R\Gamma_c(\mathcal{X}, \mathbb{R}(d)) \xrightarrow{\mathrm{tr}} \mathbb{R}[-2d]$$

is tacitly defined (see the proof of [13, Lemma 2.3 (a)]) as

$$R\Gamma_c(\mathcal{X}, \mathbb{R}(d)) \to R\Gamma_c(\mathrm{Spec}(\mathbb{Z}), \mathbb{R}(1))[-2d+2] \xrightarrow{\sim} \mathbb{R}[-2d].$$

If $\mathcal{X}$ lies over a finite field, then (6) is the pairing

$$\begin{aligned} R\Gamma(\mathcal{X}, \mathbb{R}(n)) \otimes R\Gamma(\mathcal{X}, \mathbb{R}(d-n)) &\to CH^d(\mathcal{X})_{\mathbb{R}}[-2d] \\ &\to CH^1(\overline{\mathrm{Spec}(\mathbb{Z})})_{\mathbb{R}}[-2d] \xrightarrow{\sim} \mathbb{R}[-2d], \end{aligned}$$

where $CH^1(\overline{\mathrm{Spec}(\mathbb{Z})})$ is the Arakelov–Chow group of [18].

3.7. **The map $\cup\theta$ versus $\cup e_{\mathbb{F}_q}$.** Let $\mathcal{X}$ be a regular scheme which is proper over the finite field $\mathbb{F}_q$. Recall that we denote by $R\Gamma(\mathcal{X}_W, \mathbb{Z}(n))$ the cohomology of the Weil-étale topos with coefficients in the motivic complex $\mathbb{Z}(n)$. The fundamental class

$$e_{\mathbb{F}_q} \in H^1(\mathrm{Spec}(\mathbb{F}_q)_W, \mathbb{Z}) \simeq \mathrm{Hom}(W_{\mathbb{F}_q}, \mathbb{Z})$$

is the class mapping the arithmetic Frobenius $F \in W_{\mathbb{F}_q}$ to $1 \in \mathbb{Z}$. Cup-product with $e_{\mathbb{F}_q}$ yields a map

$$R\Gamma(\mathcal{X}_W, \mathbb{Z}(n)) \otimes \mathbb{R} \xrightarrow{\cup e_{\mathbb{F}_q}} R\Gamma(\mathcal{X}_W, \mathbb{Z}(n)) \otimes \mathbb{R}[1].$$

Proposition 3.8. *The square*

$$\begin{array}{ccc} R\Gamma(\mathcal{X}_W, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{R} & \xrightarrow{\log(q)\cdot\cup e_{\mathbb{F}_q}} & R\Gamma(\mathcal{X}_W, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{R}[1] \\ \downarrow{\scriptstyle \iota_{\mathbb{F}_1}} & & \downarrow{\scriptstyle \iota_{\mathbb{F}_1}[1]} \\ R\Gamma_{\mathrm{ar}}(\mathcal{X}, \mathbb{R}(n)) & \xrightarrow{\cup\theta} & R\Gamma_{\mathrm{ar}}(\mathcal{X}, \mathbb{R}(n)) \end{array}$$

commutes, where $\iota_{\mathbb{F}_1}$ is the isomorphism defined in Section 3.5.

Proof. We consider the following diagram:

$$\begin{array}{ccc} R\Gamma(\mathcal{X}_W, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{R} & \xrightarrow{\log(q)\cdot\cup e_{\mathbb{F}_q}} & R\Gamma(\mathcal{X}_W, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{R}[1] \\ \downarrow{\scriptstyle \iota_{\mathbb{F}_q}} & & \downarrow{\scriptstyle \iota_{\mathbb{F}_q}[1]} \\ R\Gamma(\mathcal{X}, \mathbb{R}(n)) \oplus R\Gamma(\mathcal{X}, \mathbb{R}(n))[-1] & \xrightarrow{\left[\begin{smallmatrix} 0 & \log(q) \\ 0 & 0 \end{smallmatrix}\right]} & R\Gamma(\mathcal{X}, \mathbb{R}(n))[1] \oplus R\Gamma(\mathcal{X}, \mathbb{R}(n)) \\ \downarrow{\scriptstyle (1,\log(q)^{-1})} & & \downarrow{\scriptstyle (1,\log(q)^{-1})} \\ R\Gamma(\mathcal{X}, \mathbb{R}(n)) \oplus R\Gamma(\mathcal{X}, \mathbb{R}(n))[-1] & \xrightarrow{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]} & R\Gamma(\mathcal{X}, \mathbb{R}(n))[1] \oplus R\Gamma(\mathcal{X}, \mathbb{R}(n))[-1]. \end{array}$$

The lower square obviously commutes and the upper square commutes by [17, Theorem 7.1 (c)]. The result then follows from the definitions of $\iota_{\mathbb{F}_1}$ and $\cup\theta$. □

In particular, $\log(q) \cdot \cup e_{\mathbb{F}_q}$ does not depend on base field $\mathbb{F}_q$ whereas $\cup e_{\mathbb{F}_q}$ does. For example, if $q = p^f$, then one has $e_{\mathbb{F}_p} = f \cdot e_{\mathbb{F}_q}$ in $H^1(\mathrm{Spec}(\mathbb{F}_q)_W, \mathbb{Z})$, and therefore

$$\log(q) \cdot \cup e_{\mathbb{F}_q} = \log(p) \cdot f \cdot \cup e_{\mathbb{F}_q} = \log(p) \cdot \cup e_{\mathbb{F}_p}.$$

We refer to [29] and [12] for a more geometric definition of $\cup\theta$ in the case $n = 0$, which in particular explains the factor $\log(q)$ appearing in the commutative square of Proposition 3.8, see also [32, p. 47].

4. THE LONG EXACT SEQUENCE

Throughout this section we assume Conjecture 2.3, and we use Notation 2.2. In particular, we use the notation $\mathfrak{X}$ to denote either an arithmetic scheme or an Arakelov compactification of a projective regular flat arithmetic scheme.

Conjecture 4.1. *For any $\mathfrak{X}$ and any $n \in \mathbb{Z}$, we have a long exact sequence*

$$\cdots \to H^i_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) \to H^i_{\mathrm{dyn,c}}(\mathfrak{X}) \xrightarrow{\Theta - n} H^i_{\mathrm{dyn,c}}(\mathfrak{X}) \to H^{i+1}_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) \to \cdots$$

such that the composite map

$$H^i_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) \to H^i_{\mathrm{dyn,c}}(\mathfrak{X}) \to H^{i+1}_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n))$$

coincides with $\cup\theta$, and similarly without compact support.

4.2. **Varieties over finite fields.** For $\mathfrak{X}$ a smooth projective scheme over $\mathbb{F}_q$, Deninger defined in [5] vector spaces

$$H^i_{\mathrm{dyn}}(\mathfrak{X}) := \mathbb{D}(H^i(\mathfrak{X}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l, \sigma} \mathbb{C}) \tag{7}$$

endowed with an endomorphism Θ satisfying Conjecture 2.3 (except for statement (iv)), at least if the Frobenius acts semi-simply on the l-adic cohomology groups $H^i(\mathfrak{X}_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$. We prove below that our Conjecture 4.1 holds true if one defines $H^i_{\mathrm{dyn}}(\mathfrak{X})$ by (7) and if the Weil-étale cohomology groups $H^i(\mathfrak{X}_W, \mathbb{Z}(n))$ defined in [28] and [17] are finitely generated for all $i \in \mathbb{Z}$.

Theorem 4.3. *Let $\mathfrak{X}$ be a smooth projective connected variety over a finite field $\mathbb{F}_q$. Assume that the Weil-étale cohomology groups $H^i(\mathfrak{X}_W, \mathbb{Z}(n))$ are finitely generated for all $i \in \mathbb{Z}$. Then Conjecture 4.1 holds for $\mathfrak{X}$ and n.*

Proof. We recall some notation from [5]. We set $k = \mathbb{F}_q$ and we denote by $W_k \subset G_k$ the Weil group of k, i.e., the discrete subgroup of G_k generated by the Frobenius. Let $\phi \in W_k$ be the geometric Frobenius. We have the isomorphism

$$W_k \xrightarrow{\|-\|} q^{\mathbb{Z}}$$

such that $\|\phi\| = q^{-1}$, where $q^{\mathbb{Z}}$ is the subgroup of $\mathbb{R}^\times_{>0}$ generated by q. Applying the logarithm isomorphism $\mathbb{R}^\times_{>0} \xrightarrow{\sim} \mathbb{R}$, we get

$$W_k \xrightarrow{\sim} \mathbb{Z} \cdot \log(q) \subset \mathbb{R}.$$

We denote by $B := \mathbb{C}[\mathbb{C}]$ the group ring over $\mathbb{C}$ with coefficients in $\mathbb{C}$. An element of B is written in the form $\sum_{\alpha \in \mathbb{C}} r_\alpha e^\alpha$, with $\alpha, r_\alpha \in \mathbb{C}$. Then W_k acts on $B := \mathbb{C}[\mathbb{C}]$ by $w \cdot (e^\alpha) := \|w\|^\alpha \cdot e^\alpha$ for any $w \in W_k$. Moreover, B has a $\mathbb{C}$-linear derivation Θ defined by $\Theta(e^\alpha) = \alpha \cdot e^\alpha$. For any $\lambda \in \mathbb{C}$, one defines

$$\mathbb{L}_\lambda := B^{\phi = -\lambda} = \mathbb{C}[\mathrm{Log}_q(\lambda)],$$

where $\mathrm{Log}_q(\lambda) \subset \mathbb{C}$ denotes the set of complex numbers α satisfying $q^\alpha = \lambda$. Hence an element of $\mathbb{L}_\lambda$ is of the form $\sum_{\alpha \in \mathrm{Log}_p(\lambda)} r_\alpha e^\alpha$. Finally, we consider the ring $\mathbb{L} := \mathbb{L}_1 := \mathbb{C}[\mathrm{Log}_q(1)]$ endowed with its derivation Θ. Note that $\mathbb{L}_\lambda$ is a free $\mathbb{L}$-module of rank one for any $\lambda \in \mathbb{C}$.

If V is a finite-dimensional complex representation of the Weil group W_k, one defines

$$\mathbb{D}(V) := (V \otimes_{\mathbb{C}} B)^{W_k} \simeq \bigoplus_{\lambda \in \mathbb{C}} V^{\phi = \lambda} \otimes_{\mathbb{C}} \mathbb{L}_\lambda.$$

Then $\mathbb{D}(V)$ is a finitely generated free $\mathbb{L}$-module endowed with the Θ-action induced by the one defined on B. Note the following fact: If $V^{\phi=\lambda}$ is the eigenspace of ϕ for the eigenvalue λ, then for any $\alpha \in \mathbb{C}$ such that $q^\alpha = \lambda$, the subspace $(V^{\phi=\lambda} \otimes \mathbb{C} \cdot e^\alpha) \subset \mathbb{D}(V)$ is the eigenspace of Θ for the eigenvalue α. Note also that if D is a $\mathbb{L}[\Theta]$-module and $n \in \mathbb{Z}$, we may define the twist $D(n)$ (see [5, 1.6]) in such a way that the functor $\mathbb{D}$ commutes with twists.

Finally, if V is a finite-dimensional complex representation of W_k, then $\mathbb{D}(V)$ is a countable direct sum of 1-dimensional complex vector spaces. One may therefore endow $\mathbb{D}(V)$ with a topology which makes it a locally convex, Hausdorff and quasi-complete topological vector space. Then the topological group $\mathbb{R}$ acts continuously on $\mathbb{D}(V)$ as follows: $t \in \mathbb{R}$ acts as the automorphism $\exp(t \cdot \Theta)$ of $\mathbb{D}(V)$. This $\mathbb{R}$-action is in fact $\mathbb{C}$-linear and differentiable.

Lemma 4.4. *Let V be a finite-dimensional complex representation of W_k. We set $\phi_n := q^{-n}\phi$. There is a diagram with exact rows*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H^0(W_k, V(n)) & \xrightarrow{\ i\ } & V & \xrightarrow{\phi_n - 1} & V & \xrightarrow{\ s\ } & H^1(W_k, V(n)) & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow{\scriptstyle \simeq} & & & & & & {\scriptstyle \simeq}\uparrow{\scriptstyle l} & & \\
0 & \longrightarrow & H^0(\mathbb{R}, \mathbb{D}(V)(n)) & \xrightarrow{\ i'\ } & \mathbb{D}(V) & \xrightarrow{\Theta - n} & \mathbb{D}(V) & \xrightarrow{\ s'\ } & H^1(\mathbb{R}, \mathbb{D}(V)(n)) & \longrightarrow & 0,
\end{array}$$

where the vertical maps are isomorphisms, and such that the square

$$\begin{array}{ccccc}
H^0(W_k, V(n)) & \xrightarrow{\ i\ } & V & \xrightarrow{\ s\ } & H^1(W_k, V(n)) \\
{\scriptstyle f}\downarrow & & & & \uparrow{\scriptstyle -\log(q)^{-1} \cdot l} \\
H^0(\mathbb{R}, \mathbb{D}(V)(n)) & \xrightarrow{\ i'\ } & \mathbb{D}(V) & \xrightarrow{\ s'\ } & H^1(\mathbb{R}, \mathbb{D}(V)(n))
\end{array}$$

commutes. Here $H^i(\mathbb{R}, \mathbb{D}(V))$ is the cohomology of the topological group $\mathbb{R}$ with coefficients in the $\mathbb{R}$-equivariant topological vector space $\mathbb{D}(V)$. In particular, we have an exact sequence

$$0 \to H^0(W_k, V(n)) \xrightarrow{i' \circ f} \mathbb{D}(V) \xrightarrow{\Theta - n} \mathbb{D}(V) \xrightarrow{l \circ s'} H^1(W_k, V(n)) \to 0 \tag{8}$$

such that the map

$$H^0(W_k, V(n)) \xrightarrow{i' \circ f} \mathbb{D}(V) \xrightarrow{l \circ s'} H^1(W_k, V(n))$$

coincides with $-\log(q) \cdot s \circ i$.

Proof. In view of the identifications $\mathbb{D}(V(n)) = \mathbb{D}(V)(n)$, $(\mathbb{D}(V)(n), \Theta) = (\mathbb{D}(V), \Theta - n)$ and $(V(n), \phi) = (V, \phi_n)$, one may suppose $n = 0$.

The fact that the top row of the first diagram is exact is clear since we have $R\Gamma(W_k, V) \simeq [V \xrightarrow{\phi - 1} V]$.

Since $\mathbb{D}(V)$ is a locally convex, Hausdorff and quasi-complete topological vector space, we have (see [2, IX, Proposition 5.6])

$$H^i(\mathbb{R}, \mathbb{D}(V)) \simeq H^i(\mathrm{Lie}(\mathbb{R}), \mathbb{D}(V)),$$

where the right-hand side is the cohomology of the Lie algebra $\mathbb{R} \simeq \mathrm{Lie}(\mathbb{R})$ which acts on $\mathbb{D}(V)$ by $t \mapsto t \cdot \Theta$. The exactness of the second row of the first diagram of the lemma then follows from the canonical isomorphisms

$$H^0(\mathrm{Lie}(\mathbb{R}), \mathbb{D}(V)) \simeq \mathbb{D}(V)^{\Theta=0} \quad \text{and} \quad H^1(\mathrm{Lie}(\mathbb{R}), \mathbb{D}(V)) \simeq \mathbb{D}(V)_{\Theta=0}.$$

Here $\mathbb{D}(V)^{\Theta=0}$ (respectively, $\mathbb{D}(V)_{\Theta=0}$) denotes the kernel (respectively, the cokernel) of $\Theta\colon \mathbb{D}(V) \to \mathbb{D}(V)$.

The vertical isomorphism f in the first diagram of the lemma is the following map

$$\begin{aligned} H^0(W_k, V) = V^{\phi=1} &\to V^{\phi=1} \otimes_{\mathbb{C}} \mathbb{C} \cdot e^0 \\ &= V^{\phi=1} \otimes_{\mathbb{C}} \mathbb{L}^{\Theta=0} = \mathbb{D}(V)^{\Theta=0} = H^0(\mathbb{R}, \mathbb{D}(V)), \end{aligned}$$

where $V^{\phi=1} \to V^{\phi=1} \otimes_{\mathbb{C}} \mathbb{C} \cdot e^0$ sends v to $v \otimes e^0$.

The vertical isomorphism l is obtained as follows. We have

$$\begin{aligned} H^1(\mathbb{R}, \mathbb{D}(V)) &\simeq H^1(\mathbb{R}, V^{\phi=1} \otimes_{\mathbb{C}} \mathbb{L}) \oplus H^1(\mathbb{R}, \oplus_{\lambda \neq 1} V^{\phi=\lambda} \otimes_{\mathbb{C}} \mathbb{L}_\lambda) \\ &\simeq H^1(\mathbb{R}, V^{\phi=1} \otimes_{\mathbb{C}} \mathbb{L}) \\ &\simeq H^1(\mathbb{R}, V^{\phi=1} \otimes_{\mathbb{C}} \mathbb{C} \cdot e^0) \oplus \big(\oplus_{\alpha \neq 0} H^1(\mathbb{R}, V^{\phi=1} \otimes_{\mathbb{C}} \mathbb{C} \cdot e^\alpha)\big) \\ &\simeq H^1(\mathbb{R}, V^{\phi=1} \otimes_{\mathbb{C}} \mathbb{C} \cdot e^0) \\ &\simeq \mathrm{Hom}(\mathbb{R}, V^{\phi=1}). \end{aligned}$$

Hence the map $H^1(\mathbb{R}, \mathbb{D}(V)) \to H^1(W_k, V^{\phi=1})$ may be identified with

$$l\colon \mathrm{Hom}(\mathbb{R}, V^{\phi=1}) \to \mathrm{Hom}(W_k, V^{\phi=1}),$$

which is in turn induced by the morphism

$$W_k \to \mathbb{R}, \quad \phi \mapsto -\log(q).$$

It follows that the second diagram of the lemma can be identified with the following one:

$$\begin{array}{ccccccc} V^{\phi=1} & \xrightarrow{i} & V & \xrightarrow{s} & \mathrm{Hom}(W_k, V^{\phi=1}) & \xrightarrow{f \mapsto f(\phi)} & V^{\phi=1} \\ \downarrow{\scriptstyle \mathrm{Id}} & & & & \uparrow{\scriptstyle -\log(q)^{-1} \cdot l} & & \uparrow{\scriptstyle \mathrm{Id}} \\ V^{\phi=1} & \xrightarrow{i'} & \mathbb{D}(V) & \xrightarrow{s'} & \mathrm{Hom}(\mathbb{R}, V^{\phi=1}) & \xrightarrow{f \mapsto f(1)} & V^{\phi=1}. \end{array}$$

Here the right square commutes. Moreover, the composition of the maps of the top row is the identity of $V^{\phi=1}$, and similarly the composition of the maps of the bottom row is the identity of $V^{\phi=1}$. Hence the left square commutes. The result follows. □

Let $\mathfrak{X}$ be a smooth projective variety over $k = \mathbb{F}_q$. We set

$$V^i := H^i(\mathfrak{X}_{\bar{k}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l, \sigma} \mathbb{C},$$

where $\bar{k}$ is an algebraic closure, $l \neq p$ is a prime number away from the characteristic, $H^i(\mathfrak{X}_{\bar{k}}, \mathbb{Q}_l)$ is étale l-adic cohomology and $\sigma\colon \mathbb{Q}_l \to \mathbb{C}$ is a fixed embedding. We consider V^i as a finite-dimensional complex representation of W_k, and we still denote by ϕ the geometric Frobenius. The quasi-isomorphism

$$R\Gamma(\mathfrak{X}, \mathbb{Q}_l(n)) \simeq \mathrm{holim}\big(R\Gamma(\mathfrak{X}_{\bar{k}}, \mathbb{Q}_l(n)) \xrightarrow{\phi-1} R\Gamma(\mathfrak{X}_{\bar{k}}, \mathbb{Q}_l(n))\big)$$

yields the long exact sequence

$$\cdots \to H^i(\mathfrak{X}, \mathbb{Q}_l(n)) \to H^i(\mathfrak{X}_{\bar{k}}, \mathbb{Q}_l) \xrightarrow{\phi_n-1} H^i(\mathfrak{X}_{\bar{k}}, \mathbb{Q}_l) \to H^{i+1}(\mathfrak{X}, \mathbb{Q}_l(n)) \to \cdots,$$

where $\phi_n := q^{-n}\phi$. Applying $(-) \otimes_{\mathbb{Q}_l,\sigma} \mathbb{C}$, we obtain

$$\cdots \to H^i(\mathfrak{X}, \mathbb{Q}_l(n)) \otimes_{\mathbb{Q}_l,\sigma} \mathbb{C} \to V^i \xrightarrow{\phi_n-1} V^i \to H^{i+1}(\mathfrak{X}, \mathbb{Q}_l(n)) \otimes_{\mathbb{Q}_l,\sigma} \mathbb{C} \to \cdots.$$

In view of (8), we obtain the long exact sequence

$$\begin{aligned} \cdots \to H^i(\mathfrak{X}, \mathbb{Q}_l(n)) \otimes_{\mathbb{Q}_l,\sigma} \mathbb{C} \to \mathbb{D}(V^i) \xrightarrow{\Theta-n} \mathbb{D}(V^i) \\ \to H^{i+1}(\mathfrak{X}, \mathbb{Q}_l(n)) \otimes_{\mathbb{Q}_l,\sigma} \mathbb{C} \to \cdots. \end{aligned}$$

Consider the Weil-étale motivic cohomology

$$R\Gamma(\mathfrak{X}_W, \mathbb{Z}(n)) := R\Gamma(W_k, R\Gamma(\mathfrak{X}_{et}, \mathbb{Z}(n)))$$

in the sense of [17]. Since the groups $H^i(\mathfrak{X}_W, \mathbb{Z}(n)) := H^i(R\Gamma(\mathfrak{X}_W, \mathbb{Z}(n)))$ are finitely generated by assumption, we have by [17, Theorem 8.4] an isomorphism

$$H^i(\mathfrak{X}_W, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q}_l \simeq H^i(\mathfrak{X}, \mathbb{Q}_l(n)),$$

hence

$$H^i(\mathfrak{X}_W, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^i(\mathfrak{X}, \mathbb{Q}_l(n)) \otimes_{\mathbb{Q}_l,\sigma} \mathbb{C} \tag{9}$$

for any $i \in \mathbb{Z}$. Hence Lemma 4.4 gives the long exact sequence

$$\cdots \to H^i(\mathfrak{X}_W, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathbb{D}(V^i) \xrightarrow{\Theta-n} \mathbb{D}(V^i) \to H^{i+1}(\mathfrak{X}_W, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{C} \to \cdots$$

such that the composite map

$$H^i(\mathfrak{X}_W, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{C} \to \mathbb{D}(V^i) \to H^{i+1}(\mathfrak{X}_W, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{C} \tag{10}$$

coincides, under isomorphism (9), with

$$H^i(\mathfrak{X}, \mathbb{Q}_l(n)) \otimes_{\mathbb{Q}_l,\sigma} \mathbb{C} \to V^i \to H^{i+1}(\mathfrak{X}, \mathbb{Q}_l(n)) \otimes_{\mathbb{Q}_l,\sigma} \mathbb{C} \tag{11}$$

multiplied by the factor $-\log(q)$. But (11) coincides with $\cup(-e_k)$, where $-e_k \in H^1(W_k, \mathbb{Z}) = \mathrm{Hom}(W_k, \mathbb{Z})$ is the class mapping ϕ to 1, and $\cup e_k$ is the map $H^i(\mathfrak{X}, \mathbb{Q}_l(n)) \to H^{i+1}(\mathfrak{X}, \mathbb{Q}_l(n))$ defined by cup-product with e_k (see the proof of [13, Lemma 5.20]). We obtain

$$(10) = \log(q) \cdot \cup e_k.$$

Hence the result follows from Proposition 3.8. □

4.5. **The vanishing order conjecture.** Recall that we denote the kernel (resp. the cokernel) of the map

$$\Theta - n\colon H^i_{\mathrm{dyn,c}}(\mathfrak{X}) \to H^i_{\mathrm{dyn,c}}(\mathfrak{X})$$

by $H^i_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta=n}$ (resp. by $H^i_{\mathrm{dyn,c}}(\mathfrak{X})_{\Theta=n}$). We say that Θ is semisimple at n if the composite map

$$H^i_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta=n} \to H^i_{\mathrm{dyn,c}}(\mathfrak{X}) \to H^i_{\mathrm{dyn,c}}(\mathfrak{X})_{\Theta=n} \tag{12}$$

is an isomorphism for all $i \in \mathbb{Z}$.

Proposition 4.6. *Assume Conjecture 4.1. Then Θ is semisimple at n.*

Proof. Recall from (3) that

$$\cdots \xrightarrow{\cup\theta} H^i_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) \xrightarrow{\cup\theta} H^{i+1}_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) \xrightarrow{\cup\theta} \cdots \tag{13}$$

is an acyclic complex. Conjecture 4.1 gives the following diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{i-2}_{\mathrm{dyn,c}}(\mathfrak{X})_{\Theta=n} & \longrightarrow & H^{i-1}_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) & \longrightarrow & H^{i-1}_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta=n} & \longrightarrow & 0 \\
 & & & \swarrow & \alpha^{i-1} & \swarrow & & & \\
0 & \longrightarrow & H^{i-1}_{\mathrm{dyn,c}}(\mathfrak{X})_{\Theta=n} & \longrightarrow & H^{i}_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) & \xrightarrow{s} & H^{i}_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta=n} & \longrightarrow & 0 \\
 & & & \swarrow & \alpha^{i} & \swarrow & & & \\
0 & \longrightarrow & H^{i}_{\mathrm{dyn,c}}(\mathfrak{X})_{\Theta=n} & \xrightarrow[\iota]{} & H^{i+1}_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) & \longrightarrow & H^{i+1}_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta=n} & \longrightarrow & 0,
\end{array}$$

where s is induced by the map $H^i_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) \to H^i_{\mathrm{dyn,c}}(\mathfrak{X})$ appearing in the long exact sequence of Conjecture 4.1, ι is induced by the map $H^i_{\mathrm{dyn,c}}(\mathfrak{X}) \to H^{i+1}_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n))$, and α^i is the map (12) defined above. We denote the map $H^i_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) \xrightarrow{\cup\theta} H^{i+1}_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n))$ by θ^i. By Conjecture 4.1, we have

$$\theta^i = \iota \circ \alpha^i \circ s$$

for all i. Identifying $H^{i-1}_{\mathrm{dyn,c}}(\mathfrak{X})_{\Theta=n}$ with its image in $H^i_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n))$, we obtain

$$\mathrm{Im}(\theta^{i-1}) = \mathrm{Im}(\alpha^{i-1}) \subseteq H^{i-1}_{\mathrm{dyn,c}}(\mathfrak{X})_{\Theta=n} \subseteq s^{-1}(\mathrm{Ker}(\alpha^i)) = \mathrm{Ker}(\theta^i).$$

Since (13) is acyclic, we have $\mathrm{Im}(\theta^{i-1}) = \mathrm{Ker}(\theta^i)$, hence

$$\mathrm{Im}(\alpha^{i-1}) = H^{i-1}_{\mathrm{dyn,c}}(\mathfrak{X})_{\Theta=n} = s^{-1}(\mathrm{Ker}(\alpha^i)). \tag{14}$$

Therefore, α^{i-1} is surjective and we have

$$s^{-1}(\mathrm{Ker}(\alpha^i)) = H^{i-1}_{\mathrm{dyn,c}}(\mathfrak{X})_{\Theta=n} = s^{-1}(0). \tag{15}$$

Applying s to (15), we obtain $\mathrm{Ker}(\alpha^i) = 0$, since s is surjective. Hence α^i is injective. □

Proposition 4.7. *Assume Conjecture 4.1, and set* $d := \dim(\mathfrak{X})$. *Then* $H^i_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n))$ *is finite-dimensional for all* i, *zero for* $i \notin [0, 2d+1]$, *and one has*

$$\operatorname{ord}_{s=n}\zeta(\mathfrak{X}, s) = \sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \dim_{\mathbb{C}} H^i_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) \tag{16}$$

for any $n \in \mathbb{Z}$.

Proof. By Conjecture 2.3 (ii), the multiplicity of the eigenvalue n of Θ acting on $H^i_{\mathrm{dyn,c}}(\mathfrak{X})$ is finite (see Definition 2.1), and therefore the eigenspace $H^i_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta=n}$ is finite-dimensional for all $i \in \mathbb{Z}$. Since $H^i_{\mathrm{dyn,c}}(\mathfrak{X})_{\Theta=n}$ is isomorphic to $H^i_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta=n}$, it is finite-dimensional as well. The short exact sequence

$$0 \to H^{i-1}_{\mathrm{dyn,c}}(\mathfrak{X})_{\Theta=n} \to H^i_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) \to H^i_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta=n} \to 0 \tag{17}$$

then shows that $H^i_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n))$ is finite-dimensional for all i and zero for $i \notin [0, 2d+1]$. We obtain

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \dim_{\mathbb{C}} H^i_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) \\
&\quad = \sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \dim_{\mathbb{C}} H^{i-1}_{\mathrm{dyn,c}}(\mathfrak{X})_{\Theta=n} + \sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \dim_{\mathbb{C}} H^i_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta=n} \\
&\quad = \sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \dim_{\mathbb{C}} H^{i-1}_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta=n} + \sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \dim_{\mathbb{C}} H^i_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta=n} \\
&\quad = -\sum_{i\in\mathbb{Z}}(-1)^i \cdot \dim_{\mathbb{C}} H^i_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta=n} \\
&\quad = -\sum_{i\in\mathbb{Z}}(-1)^i \cdot \dim_{\mathbb{C}} H^i_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta\sim n},
\end{aligned}$$

where the second and the last equalities follow from the fact that Θ is semisimple at n by Proposition 4.6. Here we denote

$$H^i_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta\sim n} := \operatorname{colim} \operatorname{Ker}(\Theta - n)^m,$$

as in Definition 2.1. It remains to show the identity

$$\operatorname{ord}_{s=n}\zeta(\mathfrak{X}, s) = -\sum_{i\in\mathbb{Z}}(-1)^i \cdot \dim_{\mathbb{C}} H^i_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta\sim n},$$

which, by Conjecture 2.3 (ii), follows from

$$\operatorname{ord}_{s=n}\det{}_\infty\Big(\frac{s\cdot \mathrm{Id} - \Theta}{2\pi} \;\Big|\; H^i_{\mathrm{dyn,c}}(\mathfrak{X})\Big) = \dim_{\mathbb{C}} H^i_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta\sim n}. \tag{18}$$

For the sake of completeness, we show (18). To ease notations, we set $\Theta_s := \frac{s\cdot\mathrm{Id}-\Theta}{2\pi}$ and $V := H^i_{\mathrm{dyn,c}}(\mathfrak{X})$. We have

$$\det\nolimits_\infty(\Theta_s \mid V) = \det\nolimits_\infty(\Theta_s \mid V^{\Theta\sim n}) \cdot \det\nolimits_\infty(\Theta_s \mid V/V^{\Theta\sim n}) \tag{19}$$

$$= \det(\Theta_s \mid V^{\Theta\sim n}) \cdot \det\nolimits_\infty(\Theta_s \mid V/V^{\Theta\sim n}) \tag{20}$$

$$= \Big(\frac{s-n}{2\pi}\Big)^{\dim(V^{\Theta\sim n})} \cdot \det\nolimits_\infty(\Theta_s \mid V/V^{\Theta\sim n}), \tag{21}$$

where the symbol det in (20) denotes the usual determinant. Indeed, the identity (20) is valid since $V^{\Theta\sim n}$ is finite-dimensional, and (19) is (an easy case of) [4, Lemma 1.2] applied to the Θ_s-equivariant short exact sequence

$$0 \to V^{\Theta\sim n} \to V \to V/V^{\Theta\sim n} \to 0.$$

Since 0 does not occur as an eigenvalue of Θ_n acting on $V/V^{\Theta\sim n}$, we have by definition

$$\det\nolimits_\infty(\Theta_n \mid V/V^{\Theta\sim n}) := \exp(-\zeta'_{\Theta_n|V/V^{\Theta\sim n}}(0)) \neq 0,$$

hence $\det_\infty(\Theta_s \mid V/V^{\Theta\sim n})$ does not vanish at $s = n$. Therefore (18) follows from (21). □

Remark 4.8. Assume that either $\mathfrak{X} \to \mathrm{Spec}(\mathbb{Z})$ is flat, proper and regular, or that $\mathfrak{X} \to \overline{\mathrm{Spec}(\mathbb{Z})}$ is flat, projective and regular. Suppose, moreover, that the motivic L-function $L(h^i(\mathfrak{X}_\mathbb{Q}), s)$ satisfy the expected meromorphic continuation and functional equation for all $0 \le i \le 2d-2$. Then, by [12, Theorem 9.1] and [13, Proposition 5.13], the conclusion of Proposition 4.7 holds for $n = 0$.

4.9. The eigenspace $H^i_{\mathrm{dyn},c}(\mathfrak{X})^{\Theta=n}$ and duality.

Proposition 4.10. *Assume Conjecture 4.1. For any $i, n \in \mathbb{Z}$, we have isomorphisms*

$$H^i_c(\mathfrak{X}, \mathbb{C}(n)) \simeq H^i_{\mathrm{dyn},c}(\mathfrak{X})^{\Theta=n} \simeq H^i_{\mathrm{dyn},c}(\mathfrak{X})_{\Theta=n}$$

and

$$H^i(\mathfrak{X}, \mathbb{C}(n)) \simeq H^i_{\mathrm{dyn}}(\mathfrak{X})^{\Theta=n} \simeq H^i_{\mathrm{dyn}}(\mathfrak{X})_{\Theta=n}.$$

Proof. The kernel of $\cup\theta\colon H^{i+1}_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) \to H^{i+2}_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n))$ can be identified with $H^i_{\mathrm{dyn,c}}(\mathfrak{X})_{\Theta=n}$ by (14) on the one hand, and with $H^i_c(\mathfrak{X}, \mathbb{C}(n))$ by (4) on the other. So we have isomorphisms $H^i_c(\mathfrak{X}, \mathbb{C}(n)) \simeq H^i_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta=n} \simeq H^i_{\mathrm{dyn,c}}(\mathfrak{X})_{\Theta=n}$, and similarly without compact support. □

Proposition 4.11. *Assume Conjecture 4.1 and that $\mathfrak{X}$ has pure dimension d. There is a trace map*

$$H^{2d}_c(\mathfrak{X}, \mathbb{C}(d)) \xrightarrow{\sim} H^{2d}_{\mathrm{dyn},c}(\mathfrak{X})^{\Theta=d} \xrightarrow{\mathrm{Tr}} \mathbb{C}$$

and a perfect pairing

$$H^i_c(\mathfrak{X}, \mathbb{C}(n)) \times H^{2d-i}(\mathfrak{X}, \mathbb{C}(d-n)) \to H^{2d}_c(\mathfrak{X}, \mathbb{C}(d)) \to \mathbb{C} \tag{22}$$

of finite-dimensional vector spaces, for any $i, n \in \mathbb{Z}$.

Proof. This follows from Proposition 4.10, since Poincaré duality for Deninger's cohomology induces a perfect pairing

$$H^i_{\mathrm{dyn},c}(\mathfrak{X})^{\Theta=n} \times H^{2d-i}_{\mathrm{dyn}}(\mathfrak{X})^{\Theta=d-n} \xrightarrow{\cup} H^{2d}_{\mathrm{dyn,c}}(\mathfrak{X})^{\Theta=d} \xrightarrow{\mathrm{Tr}} \mathbb{C}. \qquad \square$$

For $\mathcal{X}$ proper regular over $\mathrm{Spec}(\mathbb{Z})$, a trace map

$$H^{2d}_c(\mathcal{X}, \mathbb{C}(d)) \xrightarrow{\mathrm{tr}} \mathbb{C}$$

and a pairing

$$H^i_c(\mathcal{X}, \mathbb{C}(n)) \times H^{2d-i}(\mathcal{X}, \mathbb{C}(d-n)) \to H^{2d}_c(\mathcal{X}, \mathbb{C}(d)) \xrightarrow{\mathrm{tr}} \mathbb{C} \tag{23}$$

are defined in [13, Section 2.2], where it is also shown that the perfectness of (23) is essentially equivalent to the classical conjecture of Beilinson relating motivic cohomology to Deligne cohomology (see [13, Conjecture 2.5] and also [33, Section 4]). In particular, Proposition 4.11 holds for $\mathfrak{X} = (\mathrm{Spec}(\mathcal{O}_F), \varnothing)$, and more generally for $\dim(\mathfrak{X}) \leq 1$.

Remark 4.12. The pairings (22) and (23) should coincide. Assuming this and Conjecture 4.1, the discussion above points out that Conjecture 2.3 (iii) implies Beilinson's conjectures relating motivic cohomology to Deligne cohomology.

Notation 4.13. Let $\mathfrak{X} \to \overline{\mathrm{Spec}(\mathbb{Z})}$ be projective, regular and connected.

If $\mathfrak{X}$ is moreover flat, then $\mathfrak{X}$ is an Arakelov compactification of a flat regular projective arithmetic scheme $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$. Then we denote by $CH^n(\mathfrak{X})_{\mathbb{R}}$ the Arakelov–Chow group with real coefficients (see [18, 5.1.1] and [19, 3.3.3]) and we consider

$$CH^n(\mathfrak{X})_{\mathbb{C}} := CH^n(\mathfrak{X})_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}.$$

If $\mathfrak{X}$ is not flat, then $\mathfrak{X}$ is a smooth projective connected variety over a finite field, and we denote by

$$CH^n(\mathfrak{X})_{\mathbb{C}} := CH^n(\mathfrak{X}) \otimes_{\mathbb{Z}} \mathbb{C}$$

the usual Chow group with $\mathbb{C}$-coefficients.

Proposition 4.14. *Assume Conjecture 4.1. Let $\mathfrak{X} \to \overline{\mathrm{Spec}(\mathbb{Z})}$ be projective regular of pure dimension d. Then we have*

(i) $H^i_{\mathrm{ar}}(\mathfrak{X}, \tilde{\mathbb{C}}(n))$ *is finite-dimensional for all i and* $H^i_{\mathrm{ar}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) = 0$ *for* $i \neq 2n, 2n+1$*;*
(ii) $CH^n(\mathfrak{X})_{\mathbb{C}} \simeq H^{2n}_{\mathrm{dyn}}(\mathfrak{X})^{\Theta=n}$;
(iii) $\mathrm{ord}_{s=n}\zeta(\mathfrak{X}, s) = -\dim_{\mathbb{C}} CH^n(\mathfrak{X})_{\mathbb{C}}$;
(iv) *a trace map* $CH^d(\mathfrak{X})_{\mathbb{C}} \to \mathbb{C}$ *and a perfect pairing*

$$CH^n(\mathfrak{X})_{\mathbb{C}} \times CH^{d-n}(\mathfrak{X})_{\mathbb{C}} \to CH^d(\mathfrak{X})_{\mathbb{C}} \to \mathbb{C};$$

(v) *the map* $(x, y) \mapsto \mathrm{Tr}(x \cup *y)$ *induces a positive definite hermitian form on* $CH^n(\mathfrak{X})_{\mathbb{C}}$.

Proof. The eigenvalues of Θ on $H^i_{\mathrm{dyn}}(\mathfrak{X})$ lie on the line $\mathfrak{R}(s) = i/2$, hence

$$H^i_{\mathrm{dyn}}(\mathfrak{X})_{\Theta=n} = H^i_{\mathrm{dyn}}(\mathfrak{X})^{\Theta=n} = 0$$

for $i \neq 2n$, so that (i) follows from the short exact sequence (17). By [13, Proposition 2.11] and Proposition 4.10, we have isomorphisms

$$CH^n(\mathfrak{X})_{\mathbb{C}} \simeq H^{2n}(\mathfrak{X}, \mathbb{C}(n)) \simeq H^{2n}_{\mathrm{dyn}}(\mathfrak{X})^{\Theta=n}.$$

We obtain

$$H^{2n}_{\mathrm{ar}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) \simeq H^{2n+1}_{\mathrm{ar}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) \simeq CH^n(\mathfrak{X})_{\mathbb{C}},$$

hence (iii) follows from (i) and Proposition 4.7. As in the previous proof, the perfect pairing (iv) is induced by Poincaré duality for Deninger's cohomology, and similarly for (v). □

Assume that $\mathfrak{X} = \overline{\mathcal{X}} \to \overline{\mathrm{Spec}(\mathbb{Z})}$ is projective, regular, of pure dimension d and moreover flat. By [13, Proposition 2.10], [13, Conjecture 2.5] (i.e., the Beilinson conjectures) implies Proposition 4.14 (i) as well as the existence of a perfect pairing

$$CH^n(\overline{\mathcal{X}})_{\mathbb{C}} \times CH^{d-n}(\overline{\mathcal{X}})_{\mathbb{C}} \to CH^d(\overline{\mathcal{X}})_{\mathbb{C}} \to \mathbb{C}. \tag{24}$$

Presumably, the pairing of Proposition 4.14 (iv) induced by duality for Deninger's cohomology, the pairing (24) and the Arakelov intersection pairing [18, 5.1.4] should coincide. By [13, Proposition 2.10], Proposition 4.14 (iii) follows from the vanishing order formula (16) for the incomplete zeta function $\zeta(\mathcal{X}, s)$. Finally, assertion (v) appears in [25, Proposition 3.1], see also [19, Conjecture 2].

Remark 4.15. One may reformulate Conjecture 4.1 as follows. First assume that Deninger's cohomology is the cohomology of a complex $R\Gamma_{\mathrm{dyn,c}}(\mathfrak{X})$ of $\mathbb{C}$-vector spaces endowed with an $\mathbb{R}$-action inducing a map $\Theta\colon R\Gamma_{\mathrm{dyn,c}}(\mathfrak{X}) \to R\Gamma_{\mathrm{dyn,c}}(\mathfrak{X})$. Then we expect an equivalence

$$R\Gamma_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) \to [R\Gamma_{\mathrm{dyn,c}}(\mathfrak{X}) \xrightarrow{\Theta-n} R\Gamma_{\mathrm{dyn,c}}(\mathfrak{X})],$$

where the right-hand side denotes the homotopy fiber of $\Theta - n$, such that the map $\cup\theta$ coincides with the composite morphism

$$R\Gamma_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n)) \to R\Gamma_{\mathrm{dyn,c}}(\mathfrak{X}) \to R\Gamma_{\mathrm{ar,c}}(\mathfrak{X}, \tilde{\mathbb{C}}(n))[1],$$

and similarly without compact support. Poincaré duality for Deninger's cohomology would take the form of a Θ-equivariant pairing

$$R\Gamma_{\mathrm{dyn,c}}(\mathfrak{X}) \otimes R\Gamma_{\mathrm{dyn}}(\mathfrak{X}) \to R\Gamma_{\mathrm{dyn,c}}(\mathfrak{X}) \to \mathbb{C}(-d)[-2d],$$

which is, in some sense, perfect. Then

$$[R\Gamma_{\mathrm{dyn,c}}(\mathfrak{X}) \xrightarrow{\Theta-n} R\Gamma_{\mathrm{dyn,c}}(\mathfrak{X})]$$

would be dual to

$$[R\Gamma_{\mathrm{dyn}}(\mathfrak{X}) \xleftarrow{\Theta-(d-n)} R\Gamma_{\mathrm{dyn}}(\mathfrak{X})],$$

which would give an equivalence

$$R\Gamma_{\mathrm{ar,c}}(\mathfrak{X},\tilde{\mathbb{C}}(n)) \xrightarrow{\sim} R\mathrm{Hom}(R\Gamma_{\mathrm{ar}}(\mathfrak{X},\tilde{\mathbb{C}}(d-n)),\mathbb{C}[-2d-1]),$$

as in [13, Proposition 4.2].

Remark 4.16. If one assumes that all Θ-eigenvalues on all $H^i_{\mathrm{dyn},c}(\mathfrak{X})$ occur semi-simply, the equation $\zeta(\mathfrak{X},\bar{s}) = \overline{\zeta(\mathfrak{X},s)}$ then implies that $H^i_{\mathrm{dyn},c}(\mathfrak{X})$ has a Θ-invariant real structure. Similarly, Weil–Arakelov cohomology has a real structure. The speculations mentioned in Section 4 should then also hold with $\mathbb{R}$-coefficients. The standard example of a supersingular elliptic curve over a finite field shows that this real structure might not be preserved under all endomorphisms of $\mathfrak{X}$.

5. Special values conjecture

In this section we recall the main conjecture of [13]. We consider proper regular arithmetic schemes over $\mathrm{Spec}(\mathbb{Z})$, and we use the notation $\mathcal{X}$ to denote such an arithmetic scheme. The complex $R\Gamma_{\mathrm{ar,c}}(\mathcal{X},\tilde{\mathbb{C}}(n))$ has a real structure $R\Gamma_{\mathrm{ar,c}}(\mathcal{X},\tilde{\mathbb{R}}(n))$, and in some sense a $\mathbb{Z}$-structure. Indeed, under standard assumptions (see [13, Section 4]), there is an object $R\Gamma_{\mathrm{ar,c}}(\mathcal{X},\mathbb{Z}(n)) \in \mathbf{D}^b(\mathrm{FLCA})$, where FLCA is the quasi-abelian category of locally compact abelian groups of finite ranks in the sense of [22], and an isomorphism

$$R\Gamma_{\mathrm{ar,c}}(\mathcal{X},\tilde{\mathbb{R}}(n)) \simeq R\Gamma_{\mathrm{ar,c}}(\mathcal{X},\mathbb{Z}(n)) \otimes^L \mathbb{R},$$

where the derived tensor product is defined as in [22]. More precisely, there is an exact triangle in $\mathbf{D}^b(\mathrm{FLCA})$

$$R\Gamma_{\mathrm{ar,c}}(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_{\mathrm{ar,c}}(\mathcal{X},\tilde{\mathbb{R}}(n)) \to R\Gamma_{\mathrm{ar,c}}(\mathcal{X},\tilde{\mathbb{R}}/\mathbb{Z}(n)) \tag{25}$$

such that the cohomology of $R\Gamma_{\mathrm{ar,c}}(\mathcal{X},\tilde{\mathbb{R}}/\mathbb{Z}(n))$ consists of compact groups (see [13, Definition 4.15]). Consider the tangent space functor

$$T_\infty\colon \mathbf{D}^b(\mathrm{FLCA}) \to \mathbf{D}^b(\mathrm{FLCA}), \quad K \mapsto R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(K,\mathbb{R}/\mathbb{Z}),\mathbb{R}),$$

where $R\underline{\mathrm{Hom}}$ is the internal Hom in $\mathbf{D}^b(\mathrm{FLCA})$.

5.1. **Statement of the conjecture.** Applying the triangulated functor T_∞ to (25), we obtain by [13, Remark 4.16] the exact triangle

$$R\Gamma_{dR}(\mathcal{X}_{\mathbb{R}}/\mathbb{R})/F^n[-2] \to R\Gamma_{\mathrm{ar,c}}(\mathcal{X},\tilde{\mathbb{R}}(n)) \to R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n))_{\mathbb{R}}, \tag{26}$$

where $R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n))$ is Weil-étale cohomology with compact support in the sense of [13, Section 3.8], which is a perfect complex of abelian groups, and $R\Gamma_{dR}(\mathcal{X}_{\mathbb{R}}/\mathbb{R})$ is algebraic de Rham cohomology. We have a canonical isomorphism

$$R\Gamma_{dR}(\mathcal{X}_{\mathbb{R}}/\mathbb{R})/F^n \simeq R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n \otimes \mathbb{R}, \tag{27}$$

where $R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n := R\Gamma(\mathcal{X}, L\Omega^{<n}_{\mathcal{X}/\mathbb{Z}})$ denotes derived de Rham cohomology modulo the n-th step of the Hodge filtration as defined in [24] (see also Remark 5.4), which is a perfect complex of abelian groups (see [13, Section 5.1]). This gives isomorphisms

$$\mathbb{R} \xrightarrow{\sim} \det_{\mathbb{R}} R\Gamma_{\mathrm{ar,c}}(\mathcal{X}, \tilde{\mathbb{R}}(n)) \tag{28}$$

$$\xrightarrow{\sim} \big(\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n)\big) \otimes \mathbb{R} \tag{29}$$

$$=: \Delta(\mathcal{X}/\mathbb{Z}, n) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{30}$$

where (28) is induced by the acyclic complex (3), and (29) is induced by (26) and (27). We denote the composite isomorphism by

$$\lambda_\infty\colon \mathbb{R} \xrightarrow{\sim} \Delta(\mathcal{X}/\mathbb{Z}, n) \otimes \mathbb{R}.$$

Finally, we define a rational number $C(\mathcal{X}, n) \in \mathbb{Q}^\times$ using p-adic Hodge theory (see [13, Section 5.4]). One shows that $C(\mathcal{X}, n) = 1$ for any $n \in \mathbb{Z}$ if $\mathcal{X}$ lies over a finite field, and that $C(\mathcal{X}, n) = 1$ for any $\mathcal{X}$ if $n \leq 1$. The conjecture below is [13, Conjecture 5.12].

Conjecture 5.2. *We have an identity*

$$\lambda_\infty(\zeta^*(\mathcal{X}, n)^{-1} \cdot C(\mathcal{X}, n)) \cdot \mathbb{Z} = \Delta(\mathcal{X}/\mathbb{Z}, n).$$

By [13, Theorem 5.27], if $\mathcal{X}$ is projective smooth over a number ring, then Conjecture 5.2 is equivalent to the Bloch–Kato conjecture in the formulation of Fontaine–Perrin-Riou (see [14] and [15]). On the other hand, if $\mathcal{X}$ has characteristic p, then Conjecture 5.2 is equivalent to the conjecture of Geisser–Lichtenbaum (see [28] and [17]), which holds, e.g., for curves. The formulation of Conjecture 5.2 in the case $n = 0$ was envisioned by Lichtenbaum [29], see also [31].

5.3. **A reformulation.** We now give a slight reformulation of Conjecture 5.2 using Conjecture 4.1. If $V^0 \to V^1$ is a map of complex vector spaces with finite-dimensional kernel and cokernel, one considers the complex $[V^0 \to V^1]$ concentrated in degrees $0, 1$ and one defines

$$\det_{\mathbb{C}}[V^0 \xrightarrow{f} V^1] := \det_{\mathbb{C}}\mathrm{Ker}(f) \otimes \det_{\mathbb{C}}^{-1}\mathrm{Coker}(f).$$

The isomorphism (29) and the long exact sequence of Conjecture 4.1 provide us with the isomorphism

$$\iota\colon \Delta(\mathcal{X}/\mathbb{Z}, n)_{\mathbb{C}} \xrightarrow{\sim} \det_{\mathbb{C}} R\Gamma_{\mathrm{ar,c}}(\mathcal{X}, \tilde{\mathbb{C}}(n)) \xrightarrow{\sim} \bigotimes_{i \in \mathbb{Z}} \det_{\mathbb{C}}^{(-1)^i}[H^i_{\mathrm{dyn,c}}(\mathcal{X}) \xrightarrow{\frac{n-\Theta}{2\pi}} H^i_{\mathrm{dyn,c}}(\mathcal{X})].$$

Moreover, one has a trivialization

$$\mathrm{ss}\colon \bigotimes_{i \in \mathbb{Z}} \det_{\mathbb{C}}^{(-1)^i}[H^i_{\mathrm{dyn,c}}(\mathcal{X}) \xrightarrow{\frac{n-\Theta}{2\pi}} H^i_{\mathrm{dyn,c}}(\mathcal{X})] \xrightarrow{\sim} \mathbb{C},$$

induced by the isomorphisms (12), and one may show that the diagram of isomorphisms

$$\begin{array}{ccc}
\Delta(\mathcal{X}/\mathbb{Z},n)\otimes\mathbb{C} & \xrightarrow{\ \iota\ } & \bigotimes_{i\in\mathbb{Z}}\det_{\mathbb{C}}^{(-1)^i}[H^i_{\mathrm{dyn,c}}(\mathcal{X}) \xrightarrow{\frac{n-\Theta}{2\pi}} H^i_{\mathrm{dyn,c}}(\mathcal{X})] \\
 & \searrow^{\lambda_{\infty,\mathbb{C}}^{-1}} & \big\downarrow \mathrm{ss} \\
 & & \mathbb{C}
\end{array}$$

commutes. Hence Conjecture 5.2 reads as follows: The complex line

$$\bigotimes_{i\in\mathbb{Z}}\det_{\mathbb{C}}^{(-1)^i}[H^i_{\mathrm{dyn,c}}(\mathcal{X}) \xrightarrow{\frac{n-\Theta}{2\pi}} H^i_{\mathrm{dyn,c}}(\mathcal{X})]$$

has a canonical lattice $\iota(\Delta(\mathcal{X}/\mathbb{Z},n))$ whose generator maps to

$$\pm\zeta^*(\mathcal{X},n)^{-1}\cdot C(\mathcal{X},n)$$

under the isomorphism ss induced by the semi-simplicity isomorphisms (12).

Remark 5.4. We expect

$$C(\mathcal{X},n)^{-1} = \prod_{i\le n-1;\, j}(n-1-i)!^{(-1)^{i+j}\dim_{\mathbb{Q}} H^j(\mathcal{X}_{\mathbb{Q}},\Omega^i)} \tag{31}$$

for any $n\in\mathbb{Z}$, but among schemes flat over $\mathbb{Z}$, this is currently only known for $\mathcal{X}=\mathrm{Spec}(\mathcal{O}_F)$, where F is a number field all of whose completions are absolutely abelian (see [13, Prop. 5.34]). In any case, (31) suggests the following modification of the derived de Rham complex. For a free $\mathbb{Z}$-algebra P, consider the subcomplex

$$\tilde{\Omega}^{<n}_{P/\mathbb{Z}} := \big[(n-1)!\Omega^0_{P/\mathbb{Z}} \to (n-2)!\Omega^1_{P/\mathbb{Z}} \to \cdots \to 0!\Omega^{n-1}_{P/\mathbb{Z}}\big]$$

of the truncated de Rham complex

$$\Omega^{<n}_{P/\mathbb{Z}} := \big[\Omega^0_{P/\mathbb{Z}} \to \Omega^1_{P/\mathbb{Z}} \to \cdots \to \Omega^{n-1}_{P/\mathbb{Z}}\big].$$

The complex $\tilde{\Omega}^{<n}_{P/\mathbb{Z}}$ is functorial in the free algebra P. For an arbitrary $\mathbb{Z}$-algebra A, let $P_\bullet(A)\to A$ be the standard simplicial resolution of A by free $\mathbb{Z}$-algebras [23]. Consider the simplicial complex $\tilde{\Omega}^{<n}_{P_\bullet(A)/\mathbb{Z}}$ and its total complex

$$L\tilde{\Omega}^{<n}_{A/\mathbb{Z}} := \mathrm{Tot}(\tilde{\Omega}^{<n}_{P_\bullet(A)/\mathbb{Z}}),$$

which is a subcomplex of the derived de Rham complex modulo the Hodge filtration

$$L\Omega^{<n}_{A/\mathbb{Z}} := \mathrm{Tot}(\Omega^{<n}_{P_\bullet(A)/\mathbb{Z}}).$$

One then defines $R\Gamma_{dR,c}(\mathcal{X}/\mathbb{Z})/F^n$ as the Zariski hypercohomology of the complex of sheaves $L\tilde{\Omega}^{<n}_{-/\mathbb{Z}}$ on $\mathcal{X}$. If one assumes (31) and if one replaces the complex $R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n$ by $R\Gamma_{dR,c}(\mathcal{X}/\mathbb{Z})/F^n$ in the formulation of Conjecture 5.2, then the correction factor $C(\mathcal{X},n)$ disappears.

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Received April 28, 2019; accepted October 9, 2019

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