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Cyclic and Hochschild homology of one-relator algebras via the X-complex of Cuntz and Quillen

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Cyclic and Hochschild homology of one-relator algebras via the X-complex of Cuntz and Quillen

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Zusammenfassung

In der vorliegenden Arbeit benutzen wir den Zugang von Cuntz und Quillen, um die zyklische Homologie von 1-R-Algebren zu untersuchen (eine 1-R-Algebra ist ein Quotient einer gemischten freien Algebra bzgl. einer einzigen definierenden Relation). So eine Algebra hat eine besondere quasi-freie Erweiterung, nämlich die gemischte freie Erweiterung. Wir zeigen für solche Algebren, dass die I-adische Filtrierung des X-Komplexes der dazugehörenden gemischten freien Erweiterung eine spezielle Form hat. Wir folgern daraus, dass in Dimensionen größer als 3 die zyklische Homologie solcher Algebren einfach periodisch ist. Für vier konkrete Beispiele (die irrationale Drehungsalgebra, die Weyl Algebra, ihre Modifikation mit einem invertierbaren Erzeuger und die Algebra der Laurent Polynome in zwei Variablen) bestimmen wir mit Hilfe des X-Komplexes vollständig die zyklische und Hochschildsche Homologie. Demnächst zeigen wir, wie man die Erzeuger der so berechneten zyklischen Homologie auch im Ω-Komplex finden kann und tun es für die oben erwähnten Beispiele. Wir zeigen auch, dass jede 1-R-Algebra eine freie Auflösung der Länge 2 besitzt, und schreiben solche Auflösungen für konkrete Beispiele auf. Schließlich beschreiben wir eine Methode, wie man mit Hilfe einer projektiven Auflösung der Länge n einer Algebra einen n-Zusammenhang auf dieser Algebra konstruiert und finden einen 2-Zusammenhang auf der irrationalen Drehungsalgebra.

Contents

Chapter 1

Introduction

Cyclic (co)homology was introduced independently about twenty years ago by Connes [9] and Tsygan [38]. They defined with different motivation two dual theories. In the work of Tsygan cyclic homology (called in his joint article with Feigin [20] additive K-functor) appears as an object that is isomorphic to the primitive part of the Lie homology of the matrix Lie algebra with coefficients in the trivial Lie module. Connes worked in the cohomological context and in [9] cyclic cohomology arises as a target of the Chern character from the K-homology. Cyclic homology (strictly speaking, its periodic version) can be regarded as a noncommutative variant of the de Rham cohomology. In the original definition the cyclic homology groups of an algebra are the homology groups of the quotient of the Hochschild complex by the action of finite cyclic groups. Next Loday and Quillen in [28] described cyclic homology as the homology of a certain bicomplex constructed from Hochschild complexes and bar complexes (as columns). In characteristic zero these definitions both give the same groups. To show this, Loday and Quillen constructed one more complex, the (b, B)-bicomplex, using the Hochschild boundary b and the Connes operator B. This very complex became later the most popular tool to define cyclic homology.

Another, quite different approach to cyclic homology, based on the X-complex (which is in fact a supercomplex) and quasi-free extensions, was introduced by Cuntz and Quillen in their joint work [13]. It can be considered as an analogue in the noncommutative setting of the approach of Hartshorne and Deligne to de Rham cohomology in algebraic geometry.

This new framework is a natural setting for the bivariant version of cyclic homology. It is also very convenient as a basis for various topological versions of cyclic (co)homology (e.g. [34], [35], [37]) as well as for equivariant ones (e.g. [3], [39]). It also turns out to be a very powerful and effective tool for establishing general homological properties, e.g. excision in bivariant periodic cyclic homology was proved within this context in [14] while in the classical context excision for cyclic homology was proved by Wodzicki [40] only for a certain class of nonunital algebras called H-unital algebras; Morita-invariance for certain nonunital algebras was treated in [11] also with the help of the X-complex (while the classical proof of Morita-invariance from [30] works only for H-unital algebras of Wodzicki [40]). Homological properties of topological and equivariant versions of cyclic homology are also treated successfully within this context ([10], [34], [35], [36], [37], [39]).

The question naturally arises whether one can compute the cyclic and Hochschild homology of certain algebras using the Cuntz-Quillen framework. The present work is an attempt to apply the Cuntz-Quillen theory to concrete computations.

Any algebra defined by generators and relations has a particularly nice quasi-free extension, namely a free one. The simplest case is that of one-relator associative algebras (i.e. quotients of free associative algebras by principal ideals), considered by Dicks in [17], where he obtains for them some homological results: an estimate on their global dimension, an exact sequence relating bifunctors Tor and Ext over a one-relator algebra $A = F/FwF$ itself, over its free extension F and over the so-called eigenring $E(w)$ of the element w, and some properties of the Poincaré series of such algebras. An important role in that work is played by the fact that the quotient $FwF/(FwF)^2$ is isomorphic to the tensor product $A \otimes_E A$ of the algebra A with itself over the eigenring $E = E(w)$ of the element w. This result can be considered as a version of the simple identity theorem of Lyndon, proved in [29] for one-relator groups (and used there for the computations of the group cohomology of such groups).

We consider in our work a more general case of a one-relator algebra, namely the case of a quotient of a *mixed* free algebra by a principal ideal. The identity theorem is also crucial for our work, but we use (and prove) a different version of it, which we discuss below.

We concentrate only on those one-relator algebras A for which the enveloping algebra $A^e \overset{def}{=} A \otimes A^{op}$ has no zero-divisors. The reason for this restriction will be clear in section 3.1, where the identity theorem is proved exactly for those algebras. This condition is not very restrictive though - we'll see that standard examples such as the algebra of Weyl or the (algebraic) irrational rotation algebra satisfy this additional condition. We show for such one-relator algebras that their Hochschild homology groups in dimensions greater than two are zero and compute the Hochschild, cyclic and periodic cyclic homology of three concrete algebras of this type in all dimensions.

This thesis is organized as follows (see also the diagram at the end of this introduction). In chapter 2 we describe the approach of Cuntz and Quillen to cyclic homology. We start with the definition of a (special) tower of supercomplexes and its cyclic, Hochschild and periodic cyclic homology and show how this new concept is related via the so-called Hodge tower to the classical notions of cyclic, Hochschild and periodic cyclic homology of an algebra A, which are defined with the help of the bicomplex $(\Omega A, B + b)$. Then we introduce the definition of the X-(super)complex of an arbitrary unital algebra R (which is the first level of the Hodge tower) and of its I-adic filtration, where I is a two-sided ideal of R; this filtration gives rise to a special tower $\mathcal{X}(R, I)$. The main statement of the second chapter, making computations possible, is proposition 2.3.6, which asserts that for any quasi-free extension R of an arbitrary algebra A ($A \cong R/I$) the I-adic tower $\mathcal{X}(R, I)$ is homotopy equivalent to the Hodge tower $\theta\Omega A$, hence its Hochschild, cyclic and periodic cyclic homology groups coincide with those of the algebra A. We also describe this homotopy equivalence explicitly, to be later able to find the generators of the cyclic and Hochschild homology of concrete algebras not only in the tower $\mathcal{X}(R, I)$, but also in the usual bicomplex $(\Omega A, B + b)$. This is done in two steps: in section 2.2 we consider the homotopy equivalence between the Hodge tower $\theta\Omega A$ and the tower $\mathcal{X}_A = \mathcal{X}(RA, IA)$, where RA is a particular quasi-free extension of an algebra A, the universal one, introduced in the same section, and then in section 2.3 we describe the way to construct a homotopy equivalence between $\mathcal{X}_A$ and $\mathcal{X}(R, I)$, where R is an arbitrary quasi-free extension of A. We also show that mixed free algebras are quasi-free. Then in section 2.4 we translate Connes' SBI-sequence into the Cuntz-Quillen setting. The last section of the second chapter is devoted to firs, which are by definition rings, where all left and right ideals are free as modules over the ring itself. We prove that any mixed free algebra is a fir and state Cohn's intersection property of firs that will be used in the proof of the identity theorem in the next chapter.

The goal of chapter 3 is to show that in dimensions higher than 2 the Hochschild homology of a one-relator algebra (with enveloping algebra without zero-divisors) is zero and that nothing new occurs in the cyclic homology - it is just periodic. The most important tool in this chapter is the identity theorem 3.1.4, which asserts that if I is a principal ideal of a mixed free algebra R such that the enveloping algebra $(R/I)^e$ has no zero-divisors, then the quotient I/I^2 is a free R/I-bimodule isomorphic to $R/I \otimes R/I$. The next step is a representation (for a one-relator algebra $A = R/I$ with an enveloping algebra without zero-divisors) of a quotient I^n/I^{n+1} as a tensor product, namely, $I^n/I^{n+1} \cong A^{\otimes(n+1)}$, due to which the complexes $gr^{2n}\mathcal{X}(R, I)$ for $n > 1$ (which delivers the Hochschild homology for even dimensions) and $gr^{2n+1}\mathcal{X}(R, I)$ for $n \geq 1$ (which delivers the

Hochschild homology for odd dimensions) can be rewritten as

$$A^{\otimes n} \overset{N}{\underset{(1-\lambda)}{\rightleftarrows}} A^{\otimes n},$$

where λ is the cyclic permutation and N is the norm operator, and

$$[A^{\otimes(n+1)}, A] \overset{0}{\underset{id}{\rightleftarrows}} [A^{\otimes(n+1)}, A],$$

respectively. It follows from this easily that the Hochschild homology of a one-relator algebra with an enveloping algebra without zero-divisors is zero in dimensions greater than 2.

Further we see that for such one-relator algebras the cyclic homology in all odd dimensions greater than 1 is isomorphic to HC_3 and that in all even dimensions greater than 0 it is isomorphic to HC_2, and the periodic one is given by $HP_+(A) \cong HC_2(A)$ and $HP_-(A) \cong HC_3(A)$.

Chapter 4 consists of computations of the Hochschild, cyclic and periodic cyclic homology for three concrete one-relator algebras (the algebraic counterpart A_θ^0 of the irrational rotation algebra, the Weyl algebra $A_{p,q}$ and the modified Weyl algebra $A_{p,p^{-1},q}$). They all have small mixed free extensions, and the cyclic and Hochschild homology of the first two examples is already known ([9], [2]), so one can compare our approach with the classical one. We take for each algebra its natural mixed free extension, write down the corresponding levels (and layers) of the I-adic tower $\mathcal{X}(R, I)$, and compute the cyclic homology in dimensions 0 and 1 and Hochschild homology in dimensions 0, 1, 2 directly. The second and the third cyclic homology of these three examples is computed using the SBI-sequence from chapter 2. For the higher dimensions and for the periodic cyclic homology we use the results of the previous chapter.

In chapter 5 we trace back what becomes of the generators of the cyclic and Hochschild homology of the algebras A_θ^0, $A_{p,q}$ and $A_{p,p^{-1},q}$, computed in chapter 4, under the homotopy equivalence between $\mathcal{X}(R, I)$ and the Hodge tower (which is described in chapter 2) and find this way the generators of the cyclic and Hochschild homology of those algebras in the usual bicomplex $(\Omega, B + b)$.

In chapter 6 we see another possibility to prove that a one-relator algebra $A = R/I$ with an enveloping algebra without zero-divisors has zero Hochschild homology groups in dimensions greater than two. Namely, we construct a free resolution of such an algebra of length 2. It can be obtained in two ways: either as a generalization of Dicks' result from [17] about one-relator associative algebras, or by splicing together two short exact sequences from the article [12] of Cuntz and Quillen. One of the terms of the free resolution obtained that way is the quotient I/I^2 from the identity theorem. For our examples $A_{p,q}$ and A^0_θ this resolution becomes the well-known Koszul resolution respectively the "ad hoc" resolution of Connes from [9]. In section 6.2 we show how a projective resolution of length n of an algebra A can be used to construct a connection on $\Omega^n A$. We prove that such a connection yields an explicit contractive homotopy on the n-th Hodge filtration of the completed Ω-complex $F^n(\hat{\Omega} A, b + B)$ and we construct as an example a 2-connection for the algebra A^0_θ using Connes' resolution.

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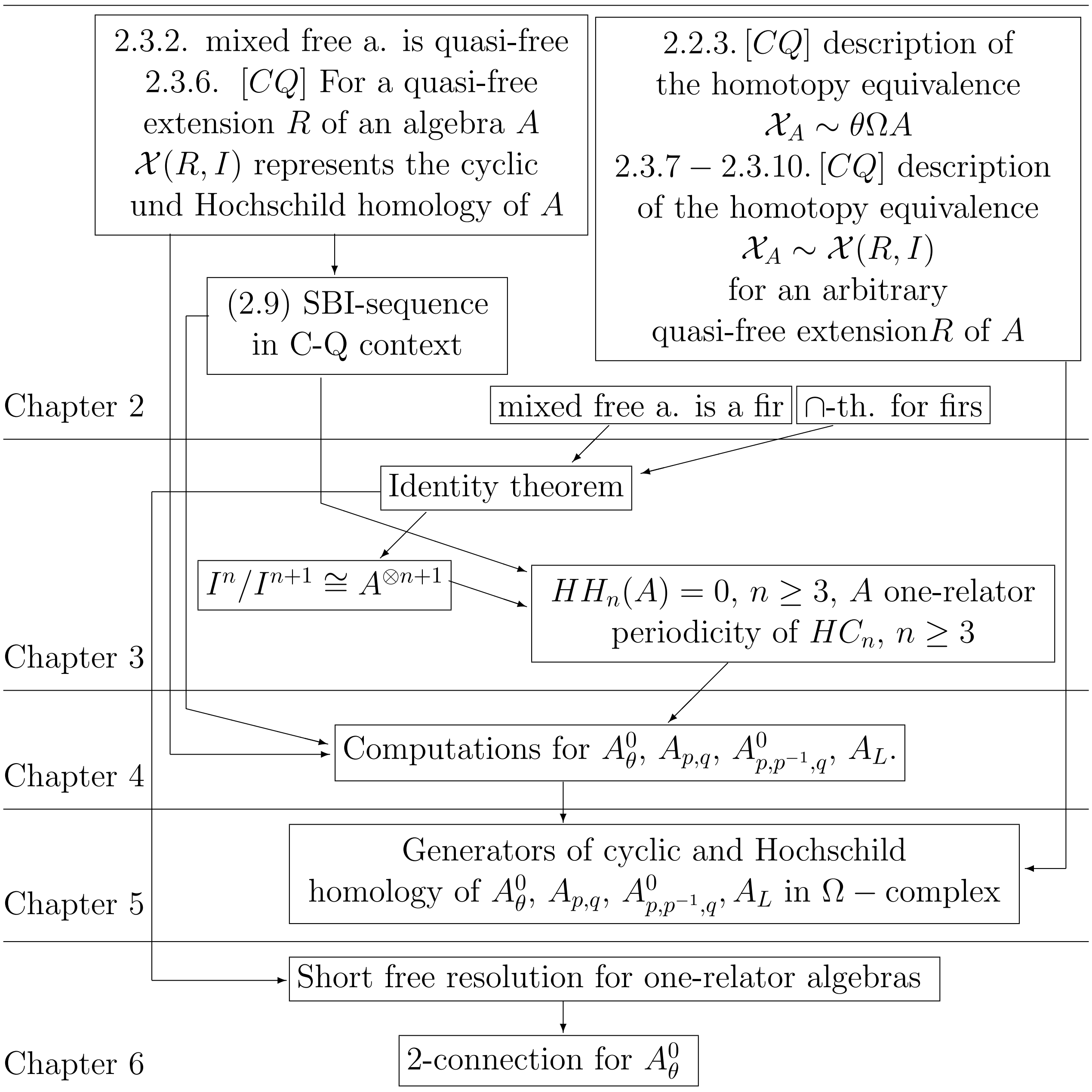
Interdependence of the results



Chapter 2

The Cuntz-Quillen framework and other preliminaries

2.1 Towers of supercomplexes, X-complex and its I-adic filtration

The detailed description of cyclic and Hochschild homology via towers of supercomplexes is given in [13]. We recall it here briefly.

Through the whole chapter "an algebra" means "a unital complex algebra", although the most constructions and results can be translated to the case of any characteristic zero field. Note also that this approach was extended onto the non-unital algebras in [14], but we do not need it since all examples considered in this work are unital.

A supercomplex is a $\mathbb{Z}/2$-graded vector space equipped with an odd operator of square zero.

A tower of supercomplexes is a bounded below inverse system $\mathcal{X} = (\mathcal{X}^n)_{n\in\mathbb{Z}}$ of supercomplexes such that the maps $\mathcal{X}^n \to \mathcal{X}^{n-1}$ are all surjective; $\mathcal{X}^n$ is called the n-th level of the tower $\mathcal{X}$.

Each tower defines a supercomplex $\widehat{\mathcal{X}} = \varprojlim \mathcal{X}^n$ together with the decreasing filtration $F^n\widehat{\mathcal{X}} = \operatorname{Ker}(\widehat{\mathcal{X}} \to \mathcal{X}^n)$ and, vice versa, each supercomplex K with a decreasing bounded below filtration $(F^nK)_{n\in\mathbb{Z}}$ that is complete in the topology induced by the filtration defines a tower $(\mathcal{X}^n)_{n\in\mathbb{Z}}$ via $\mathcal{X}^n = K/F^nK$, and this way one recovers from $\widehat{\mathcal{X}}$ the original tower.

The supercomplex

$$gr^n \mathcal{X} = \mathrm{Ker}\,(\mathcal{X}^n \to \mathcal{X}^{n-1}) = F^{n-1}\widehat{\mathcal{X}}/F^n\widehat{\mathcal{X}}$$

is called the n-th layer of the tower $\mathcal{X}$.

For so-called special towers, i.e. for such that the homology of the n-th layer is concentrated in degree $(n + 2\mathbb{Z})$, one defines the Hochschild, cyclic and periodic cyclic homology by

$$HH_n\mathcal{X} = H_{n+2\mathbb{Z}}(gr^n\mathcal{X}),$$

$$HC_n\mathcal{X} = H_{n+2\mathbb{Z}}(\mathcal{X}^n),$$

and

$$HP_\nu\mathcal{X} = H_\nu(\widehat{\mathcal{X}}),$$

respectively.

One defines naturally a morphism $f : \mathcal{X} \to \mathcal{X}'$ between two towers as a sequence $f = (f_n)_{n\in\mathbb{Z}}$ of homomorphisms of supercomplexes (that is of linear maps respecting the supercomplex structure) $f_n : \mathcal{X}^n \to \mathcal{X}'^n$, compatible with the surjections. Special towers and morphisms defined this way form a category that will be denoted by $\mathcal{T}$. For the computation of the cyclic and Hochschild homology of concrete algebras another category is more suitable, namely, the homotopy category of towers $Ho\mathcal{T}$, objects of which are also towers, but morphisms are classes of homotopic morphisms of towers; two morphisms f, g of towers are called homotopic to each other $(f \sim g)$ if there exists a sequence $h = (h_n)_{n\in\mathbb{Z}}$ of odd linear maps $h_n : \mathcal{X}^n \to \mathcal{X}'^n$ with the property $f - g = \partial h + h\partial$, where ∂ denotes the differential in $\mathcal{X}$. If two towers are homotopy equivalent (i.e., isomorphic in $Ho\mathcal{T}$), they have equal Hochschild, cyclic and periodic cyclic homology [13].

The definition of the Hochschild, cyclic and periodic cyclic homology of a tower is motivated by the corresponding definition for a mixed complex, which we now describe.

Definition 2.1.1 *([27], 2.5.13) A mixed complex* (C, b, B) *is a family of* $\mathbb{C}$*-vector spaces* $C_n, n \in \mathbb{N} \cup \{0\}$*, equipped with two maps,* *one of degree -1, $b : C_n \to C_{n-1}$, and the other one of degree $+1$, $B : C_n \to C_{n+1}$, satisfying*

$$b^2 = B^2 = bB + Bb = 0$$

Any mixed complex gives rise to a first-quadrant bicomplex, called (b, B)-bicomplex:

$$\begin{array}{ccccccc}
\dots & & \dots & & \dots & & \dots \\
\downarrow b & & \downarrow b & & \downarrow b & & \downarrow b \\
C_3 & \overset{B}{\leftarrow} & C_2 & \overset{B}{\leftarrow} & C_1 & \overset{B}{\leftarrow} & C_0 \\
\downarrow b & & \downarrow b & & \downarrow b & & \\
C_2 & \overset{B}{\leftarrow} & C_1 & \overset{B}{\leftarrow} & C_0 & & \\
\downarrow b & & \downarrow b & & & & \\
C_1 & \overset{B}{\leftarrow} & C_0 & & & & \\
\downarrow b & & & & & & \\
C_0 & & & & & &
\end{array} \qquad (2.1)$$

Definition 2.1.2 *The Hochschild homology $HH_*(C)$ of a mixed complex (C, b, B) is the homology of the first column of (b, B)-bicomplex, that is, of (C, b). The cyclic homology $HC_*(C)$ of a mixed complex (C, b, B) is the homology of the total complex of (b, B)-bicomplex (the total complex of a first-quadrant bicomplex consists of the direct sums of the finite diagonals of bicomplex and its differential is the sum of b and B). The periodic cyclic homology $HP_\nu(C)$ of a mixed complex (C, b, B) is the homology of the total complex (where one takes the direct products instead of the direct sums) of the periodic (b, B)-bicomplex, defined by*

$$
\begin{array}{ccccccccc}
 & & \dots & & \dots & & \dots & & \dots \\
 & & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b \\
\dots & \overset{B}{\leftarrow} & C_3 & \overset{B}{\leftarrow} & C_2 & \overset{B}{\leftarrow} & C_1 & \overset{B}{\leftarrow} & C_0 \\
 & & \downarrow b & & \downarrow b & & \downarrow b & & \\
\dots & \overset{B}{\leftarrow} & C_2 & \overset{B}{\leftarrow} & C_1 & \overset{B}{\leftarrow} & C_0 & & \\
 & & \downarrow b & & \downarrow b & & & & \\
\dots & \overset{B}{\leftarrow} & C_1 & \overset{B}{\leftarrow} & C_0 & & & & \\
 & & \downarrow b & & & & & & \\
\dots & \overset{B}{\leftarrow} & C_0 & & & & & &
\end{array}
$$

The definition of the Hochschild, cyclic and periodic cyclic homology of a special tower is a generalization of definition 2.1.2 in the following sense: if we consider (C, b, B) as a supercomplex $(C, B+b)$ with the even-odd grading and with the differential $B + b$, then the Hochschild, cyclic and periodic cyclic homology of the Hodge tower $\theta C = \theta(C, B + b)$ of $(C, B + b)$, which is defined by the filtration $F^n(C, B + b) = bC_{n+1} \oplus \bigoplus_{k>n} C_k$ on $(C, B + b)$, are exactly HH_nC, HC_nC and $HP_\nu C$.

The notion of a mixed complex and its Hochschild, cyclic and periodic cyclic homology is in turn a generalization of the classical definition of the Hochschild, cyclic and periodic cyclic homology of an algebra. This definition is based on the (Ω, b, B)-complex,which we now describe.

For a unital algebra R the R-bimodule of non-commutative differential n-forms on R is defined (e.g. [27], §2.6) by $\Omega^0 R = R$ and

$$\Omega^n R = R \otimes \overline{R}^{\otimes n},$$

where $\overline{R} = R/\mathbb{C}$. An elementary tensor from $\Omega^n R$ is denoted by $r_0 dr_1 \dots dr_n$. The left R-module structure on ΩR is obvious; the right one is defined by the Leibniz rule $d(rt) = dr \cdot t + rdt$. With the differential map

$$
\begin{array}{cccc}
d: & \Omega^n R & \to & \Omega^{n+1} R \\
 & r_0 dr_1 \dots r_n & \mapsto & dr_0 dr_1 \dots r_n
\end{array}
$$

(d^2 is obviously zero) and with the product induced by the Leibniz rule, $\Omega R = \oplus_{n\geq 0}\Omega^n R$ becomes a differential graded algebra.

On ΩR there are some other important operators. The first one is the Hochschild boundary b, which is defined by the rule

$$b(\omega dr) = (-1)^{|\omega|}[\omega, r]$$

for homogeneous forms of positive degree $|\omega|$ and $b = 0$ on 0-forms. It's easy to check that $b^2 = 0$. Using b and d, one obtains the Karoubi operator κ of degree zero, defined by

$$\kappa = 1 - (db + bd).$$

It's clear that the operator κ defined this way commutes with b and d. Explicitly

$$\kappa(\omega dr) = (-1)|\omega|^d r \cdot \omega$$

for homogeneous forms of positive degree and in degree zero κ is the identity. Finally, the Connes' operator B is defined on $\Omega^n R$ by

$$B = \sum_{i=0}^{n} \kappa^i d,$$

and it obviously commutes with κ. Explicitly,

$$B(r^0 dr^1 ... dr^n) = \sum_{i=0}^{n} (-1)^{ni} dr^i ... dr^n dr^0 ... dr^{i-1}. \tag{2.2}$$

By the direct computation, one obtains the relations

$$Bb + bB = B^2 = 0.$$

Therefore, $\Omega = \Omega R$ with the operators b and B is a mixed complex (Ω, b, B) and its Hochschild, cyclic and periodic cyclic homology are by definition the Hochschild, cyclic and periodic cyclic homology of the algebra R.

The first level of the Hodge tower $\theta\Omega$ of the supercomplex $(\Omega R, B + b)$ plays a very important role in the whole theory, so it received a special notation:

$$X(R) = \Omega R / F^1 \Omega R.$$

It is a supercomplex

$$X(R) : R \underset{\bar{b}}{\overset{\natural\, d}{\rightleftarrows}} \Omega^1 R_\natural,$$

where $\Omega^1 R_\natural = \Omega^1 R/[\Omega^1 R, R]$, $\natural : \Omega^1 R \to \Omega^1 R_\natural$ is a quotient map, and $\bar{b}$ is defined by the rule $\bar{b}(\natural r_1 dr_2) = [r_1, r_2]$.

If I is an ideal in R, one defines the I-adic filtration on $X(R)$ by

$$F_I^{2n+1} X(R) : I^{n+1} \rightleftarrows \natural(I^{n+1} dR + I^n dI),$$

$$F_I^{2n} X(R) : I^{n+1} + [I^n, R] \rightleftarrows \natural(I^n dR),$$

for $n \geq 0$, and $F_I^p X(R) = X(R)$ for $p < 0$. One can also write

$$F_I^{2n+1} X(R) : I^{n+1} \rightleftarrows \natural(I^n dI),$$

since $\natural(I^{n+1} dR) \subset \natural(I^n dI)$. This filtration defines a (special) tower $\mathcal{X}(R, I) = (\mathcal{X}^p(R, I))_{p\in\mathbb{Z}}$ by $\mathcal{X}^p(R, I) = X(R)/F_I^p X(R)$.

2.2 The universal extension RA

An extension of a (unital) algebra A is a unital algebra R together with its proper two-sided ideal such that the sequence

$$0 \to I \to R \to A \to 0 \qquad (2.3)$$

is exact. Further in this paper if we want to say that an algebra R with its ideal I and sequence (2.3) is an extension of an algebra A, we simply write "an extension $A = R/I$".

Any unital algebra A possesses a special extension

$$0 \to IA \to RA \to A \to 0$$

such that its IA-adic tower $\mathcal{X}(RA, IA)$ is homotopically equivalent to the Hodge tower of $(\Omega A, B + b)$. We will describe it now in detail.

For an arbitrary unital algebra A, the algebra RA is defined in [12] as the following quotient of the unital tensor algebra $T_1(A) = \bigoplus_{i \geq 0} A^{\otimes i}$ of the vector space A:

$$RA = T_1(A)/T_1(A)(1_T - 1_A)T_1(A),$$

where 1_T and 1_A mean the units of $T_1(A)$ respectively of A. In other words, $RA = T_1(\overline{A}) = \bigoplus_{i \geq 0} \overline{A}^{\otimes i}$ is a unital tensor algebra of the reduced algebra $\overline{A}$.

The algebra RA is universal in the following sense ([12]):

Let

$$\begin{array}{rccl} \hat{\rho}: & A & \to & RA \\ & 1 \neq a & \mapsto & a \\ & 1_A & \mapsto & 1_{RA} \end{array}$$ be the following linear map :

Then there exists for each algebra R with 1 and for each linear map $\rho : A \to R$ with $\rho(1_A) = 1_R$ a unique homomorphism $\varphi : RA \to R$ such that $\varphi \circ \hat{\rho} = \rho$ holds. In particular, there exists a unique homomorphism $\psi : RA \to A$ such that $\psi \circ \hat{\rho} = \mathrm{id}_A$. The ideal $\mathrm{Ker}\,\psi$ plays an important role in the theory and is denoted by IA.

One defines on the DG algebra Ω the so-called Fedosov product by $\omega \circ \xi = \omega\xi - (-1)^{|\omega|} d\omega d\xi$ for ω homogeneous of degree $|\omega|$ and extends it by linearity to all forms.

Proposition 2.2.1 *[12] Let $\hat{\omega}$ be the curvature of $\hat{\rho}$ (that means $\hat{\omega}(a_1, a_2) = \hat{\rho}(a_1 a_2) - \hat{\rho}(a_1) \otimes \hat{\rho}(a_2)$). Then there exists a canonical isomorphism between RA and the algebra $\Omega^+ A$ of even differential forms with the Fedosov product, defined by*

$$\hat{\rho}(a_0) \otimes \hat{\omega}(a_1, a_2) \otimes \cdots \otimes \hat{\omega}(a_{2n-1}, a_{2n}) \leftrightarrow a_0 da_1 \cdots da_{2n}.$$

The canonical derivation from RA into $\Omega^1(RA)$ will be denoted by δ to avoid confusion with the derivation d on A. The differential from the odd degree to the even one in $X(RA)$ will be denoted by β in order to differ it from the operator b on ΩA. One identifies $\Omega^- A$ with $\Omega^1(RA)_\natural$ by $xda \leftrightarrow \natural(x\delta a)$ [12] .

In that way the complex $X(RA)$ is identified with the complex

$$X : \Omega^+ A \overset{\natural\delta}{\underset{\beta}{\rightleftarrows}} \Omega^- A,$$

which has the same underlying $\mathbb{Z}_2$-graded vector space as Ω, but other differentials:

$$\beta = b - (1+k)d \tag{2.4}$$

$$\natural\delta = -N_{\kappa^2} b + B, \tag{2.5}$$

where by definition $N_{\kappa^2} = \sum_{j=0}^{n-1} \kappa^{2j}$ on $\Omega^{2n}A$.

The IA-adic filtration on $X(RA)$ coincides (under this identification) with the Hodge filtration on ΩA.

The next theorem shows that the tower $\mathcal{X}_A = \mathcal{X}(RA, IA)$ represents the Hochschild, cyclic and periodic cyclic homology of the algebra A.

Theorem 2.2.2 *([13], theorem 6.2).* $\mathcal{X}_A \cong \theta\Omega A$ *in* $Ho\mathcal{T}$.

We give separately the detailed description of this homotopy equivalence. This is done by the following

Proposition 2.2.3 *Let P be the spectral projection on Ω as well as on X corresponding to the eigenvalue 1 of the Karoubi operator κ and let*

$$X = PX \oplus P^\perp X,$$

$$\Omega = P\Omega \oplus P^\perp\Omega$$

be the corresponding decomposition of supercomplexes. Then

1. *the supercomplexes $P^\perp X$ and $P^\perp\Omega$ are contractible;*

2. *the scaling operator*

$$\begin{array}{cccc} c: & PX & \rightarrow & P\Omega \\ & \omega_q & \mapsto & c_q\omega_q \end{array}$$

(where ω_q is a differential form from $\Omega^q A$) is a homotopy equivalence; the constants are $c_{2n} = c_{2n+1} = (-1)^n n!$.

The same is true for the towers (X/F^pX), $(\Omega/F^p\Omega)$, not only for complexes X and Ω.

Remark that, as was already mentioned, the Karoubi operator commutes with the differential $b + B$ of Ω as well as with the differentials β and $\natural\delta$ of X (it can be seen easily from (2.4) and (2.5)), hence the spectral projection P is compatible with the supercomplex structure.

The first claim of the proposition is proved in [13], §3, and the second one is theorem 6.2 of [13].

2.3 X-complex of a quasi-free extension

The extension RA of A is though very "large". In fact, it is possible to compute the Hochschild, cyclic and periodic cyclic homology of A with the help of an arbitrary quasi-free extension of A.

We start with the basic definition:

Definition 2.3.1 *([12], §3). An algebra R is called quasi-free if the following equivalent conditions are satisfied:*

- *the Hochschild cohomology $H^n(R, M)$ is zero for all $n > 1$ and for any R-bimodule M,*
- *$\Omega^1 R$ is a projective R-bimodule,*
- *for any square-zero extension $0 \to J \to S \xrightarrow{\varphi} R \to 0$ (square-zero means $J^2 = 0$) there exists a lifting homomorphism $R \xrightarrow{\psi} S$ (i.e. $\varphi \circ \psi = id_R$).*

Example 2.3.2 *Quasi-free algebras (cf.[12], proposition...).*

1. *Every free algebra is quasi-free.*

   Proof. *In each square-zero extension $0 \to J \to S \xrightarrow{\varphi} R \to 0$ one lifts arbitrary the generators of R and, since R is free, this lifting can be extended to a homomorphism $R \to S$.* □

2. *Every free group algebra $R = \mathbb{C}G$ (G is a free group) is quasi-free.*

   Proof. *Let $\{x_1, x_2, ...\}$ be the free generators of the group G, let $0 \to J \to S \xrightarrow{\varphi} R \to 0$ be a square-zero extension. One lifts arbitrary each generator x_i to y_i and each inverse x_i^{-1} to z_i. In general, $z_i y_i \neq 1$, but $(z_i y_i - 1)$ lies in J. Put $y_i\prime = 2y_i - y_i z_i y_i$. Then $\varphi(y_i\prime) = x_i$ and*

$$z_i y\prime - 1 = z_i(2y_i\prime - y_i\prime z_i y_i\prime) - 1 = -(1 - z_i y_i\prime)^2 = 0,$$

   since it lies in J^2. Similarly for $(z_i y_i - 1)$. Thus $y_i\prime$ lifts x_i, z_i lifts x_i^{-1} and they are inverse to each other; this can be then extended to the desired homomorphism $R \to S$. □

3. *Similarly, mixed free algebras, which we define below, are quasi-free.*

Definition 2.3.3 *([6]) Let k be a field, let X, Y be two disjoint sets. A mixed free algebra over k on X, Y is defined as a k-algebra $k\langle X, Y, Y^{-1}\rangle$ generated by X, Y and inverses to Y that is universal for Y-inverting maps of $X \sqcup Y$ into k-algebras.*

Now we are ready to define the main object of this work:

Definition 2.3.4 *An algebra is called a one-relator algebra if it is a quotient of a mixed free algebra by a principal ideal.*

Note that it follows from the definition that any one-relator algebra comes naturally equipped with a quasi-free extension, namely with a mixed free one.

The next two results make clear what is the goal of the consideration of quasi-free extensions:

Proposition 2.3.5 *([13], corollary 9.4): Let A be an algebra. For every extension $A = R/I$ there is a canonical morphism*

$$\mathcal{X}_A \xrightarrow{\psi} \mathcal{X}(R, I)$$

in the homotopy category of towers $Ho\mathcal{T}$. If R is quasi-free, then ψ is an isomorphism in $Ho\mathcal{T}$.

From theorem 2.2.2 and proposition 2.3.5 follows

Proposition 2.3.6 *For each quasi-free extension $A = R/I$ there exists a canonical homotopy equivalence*

$$\mathcal{X}(R, I) \sim \theta\Omega A$$

and, therefore, the tower $\mathcal{X}(R, I)$ represents the Hochschild, cyclic and periodic cyclic homology of the algebra A.

To describe explicitly how the morphism ψ from proposition 2.3.5 is constructed, we need to pass to towers of algebras. A tower of algebras (R_n) consists of an algebra R_n for each $n \in \mathbb{N} \cup \{0\}$ and of a surjective homomorphism $R_{n+1} \to R_n$ also for each $n \in \mathbb{N} \cup \{0\}$. A homomorphism φ of two such towers (R_n) and (S_n) is a system of algebra homomorphisms $\varphi_n : R_n \to S_n$, compatible with the surjections in (R_n) and (S_n).

Lemma 2.3.7 *([13], lemma 4.3) Let R be an algebra with an ideal I, let for $n \in \mathbb{N}$ $R_n = R/I^{n+1}$ and $I_n = I/I^{n+1}$. Then for any $k \leq 2n + 1$ the map of supercomplexes*

$$\mathcal{X}^k(R, I) \to \mathcal{X}^k(R_n, I_n),$$

which is induced by the canonical projection $R \to R/I^{n+1}$, is an isomorphism. It follows that for any algebra S with any ideal J each homomorphism of towers of algebras

$$\varphi : (R_n) \to (S_n)$$

induces a homomorphism of towers of supercomplexes

$$\varphi_* : \mathcal{X}(R, I) \to \mathcal{X}(S, J)$$

(where $S_n = S/J^{n+1}$).

Proposition 2.3.8 *(see [13], corollary 9.4) The morphism ψ from 2.3.5 is obtained by choosing any homomorphism of towers of algebras $(R_nA) \to (R_n)$ (where $R_n = R/I^{n+1}$ and $R_nA = RA/IA^{n+1}$) that lifts the identity map $A \to R/I$ and taking the induced map from $\mathcal{X}(RA, IA)$ into $\mathcal{X}(R, I)$.*

Now we state two other results that make precise how homomorphisms of towers of algebras can be obtained:

Theorem 2.3.9 *([13], theorem 9.3) Let $A = R/I$, $B = S/J$ be two extensions of algebras with R quasi-free. Then each homomorphism $v : A \to B$ can be lifted to a homomorphism of towers $u : (R_n) \to (S_n)$. Moreover, the homotopy class of the induced map $u_* : \mathcal{X}(R, I) \to \mathcal{X}(S, J)$ depends only on v.*

Lemma 2.3.10 *([13], lemma 8.6.) For any two pairs (R, I) and (S, J) of algebras with ideals the following data are equivalent:*

1. *a homomorphism $u : (R_n) \to (S_n)$ of towers of algebras;*
2. *a compatible system of maps $u^n : R \to S_n$ such that $u^0(I) = 0$. (Note that the map u^n becomes then a homomorphism of pairs $(R, I) \to (S_n, J_n)$ and that it induces therefore a homomorphism $u^n_* : \mathcal{X}(R, I) \to \mathcal{X}(S_n, J_n)$).*

2.4 SBI-sequence in the Cuntz-Quillen context

For any algebra there is an exact sequence called Connes' SBI-sequence, which connects Hochschild and cyclic homology of that algebra. It follows from the periodicity of the (b, B)-bicomplex (2.1):

Theorem 2.4.1 *([27], theorem 2.2.1). For any associative k-algebra A there is a natural long exact sequence*

$$\ldots \to HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} \ldots$$

Here I is a natural inclusion, B is defined by (2.2), and the periodicity map S is obtained by factoring out the first column of the (b, B)-bicomplex (2.1).

It is also possible to deduce the SBI-sequence in the Cuntz-Quillen setting of cyclic homology. To do that, we recall from [33], section 2.4, that if one has a short exact sequence of complexes

$$0 \to K \xrightarrow{\iota} L \xrightarrow{\pi} M \to 0, \qquad (2.6)$$

then there is a long exact sequence of homology groups

$$\ldots \to H_n(K) \xrightarrow{\iota_*} H_n(L) \xrightarrow{\pi_*} H_n(M) \xrightarrow{\beta} H_{n-1}(K) \to \ldots,$$

where ι_* and π_* are induced by ι and π respectively and the connecting homomorphism $\beta : H_n(M) \to H_{n-1}(K)$ is constructed in the following way (we denote differentials of all three complexes by ∂, since it will be always clear to which complex the differential belongs). Let m be a cycle in M_{n+1}. Since π is an epimorphism, there is an element l from L_n such that $\pi l = m$. Now since $\partial m = 0$, one has $\pi \partial l = 0$. Since the sequence (2.6) is exact, there is a unique cycle k from K_{n-1} such that $\iota k = \partial l$; this can be written as the following commutative diagram:

$$\begin{array}{ccccc} & & l & \to & m \\ & & \downarrow & & \downarrow \\ k & \to & \partial l & \to & 0 \end{array} \tag{2.7}$$

in

$$\begin{array}{ccccc} & & L_n & \xrightarrow{\pi} & M_n \\ & & \downarrow\partial & & \downarrow \\ K_{n-1} & \xrightarrow{\iota} & L_{n-1} & \to & M_{n-1}. \end{array}$$

The homology class of k depends only on the homology class of m (and does not depend on the choice of the representative m and on the choice of the pre-image l of m). Assigning to each homology class (represented by a cycle m from M_n) the homology class of the cycle k constructed in that way, one defines the desired homomorphism β.

Now we consider for an algebra A and its quasi-free extension R the following two exact sequences of supercomplexes:

$$0 \to gr^{n+1}\mathcal{X}(R,I) \to \mathcal{X}^{n+1}(R,I) \to \mathcal{X}^n(R,I) \to 0$$

and

$$0 \to gr^n\mathcal{X}(R,I) \to \mathcal{X}^n(R,I) \to \mathcal{X}^{n-1}(R,I) \to 0.$$

From these two sequences we obtain two exact sequences of homology groups; we write down only those segments of them that we need (we omit (R,I) by $\mathcal{X}$ for the sake of space):

$$H_{n+1}(gr^{n+1}\mathcal{X}) \xrightarrow{\widetilde{I}} H_{n+1}(\mathcal{X}^{n+1}) \to H_{n+1}(\mathcal{X}^{n}) \to H_{n}(gr^{n+1}\mathcal{X})$$

and

$$H_{n+1}(gr^{n}\mathcal{X}) \to H_{n+1}(\mathcal{X}^{n}) \to H_{n+1}(\mathcal{X}^{n-1}) \xrightarrow{\widetilde{B}} H_{n}(gr^{n}\mathcal{X}) \to H_{n}(\mathcal{X}^{n}).$$

With proposition 2.3.6 and with the fact that $\mathcal{X}(R,I)$ is a special tower, the first sequence becomes

$$HH_{n+1}(A) \xrightarrow{\widetilde{I}} HC_{n+1}(A) \to H_{n+1}(\mathcal{X}^{n}(R,I)) \to 0 \tag{2.8}$$

and the second one becomes

$$0 \to H_{n+1}(\mathcal{X}^{n}(R,I)) \to HC_{n-1}(A) \xrightarrow{\widetilde{B}} HH_{n}(A) \to HC_{n}(A). \tag{2.9}$$

Splicing them together, we obtain for each n a sequence

$$HH_{n+1}(A) \xrightarrow{\widetilde{I}} HC_{n+1}(A) \xrightarrow{\widetilde{S}} HC_{n-1}(A) \xrightarrow{\widetilde{B}} HH_{n}(A) \xrightarrow{\widetilde{I}} HC_{n}(A),$$

from which it follows that there is a long exact sequence

$$\ldots \to HH_{n+1}(A) \xrightarrow{\widetilde{I}} HC_{n+1}(A) \xrightarrow{\widetilde{S}} HC_{n-1}(A) \xrightarrow{\widetilde{B}} HH_{n}(A) \to \ldots \tag{2.10}$$

Here the map $\widetilde{I} : HH_{n+1}(A) \to HC_{n+1}(A)$ is just an inclusion, i.e. for $n = 2k$

$$\begin{array}{llcl} \widetilde{I}_{2k}: & HH_{2k}(A) & \to & HC_{2k}(A) \\ & h.cl.(i + (I^{k+1} + [I^k, R])) & \mapsto & h.cl.(i + (I^{k+1} + [I^k, R])), \end{array}$$

(h.cl.(.) denotes the homology class of an element) and for $n = 2k+1$

$$\begin{array}{llcl} \widetilde{I}_{2k+1}: & HH_{2k+1}(A) & \to & HC_{2k+1}(A) \\ & h.cl.(\natural\omega + \natural I^k dI) & \mapsto & h.cl.(\natural\omega + \natural I^k dI); \end{array}$$

the map $\widetilde{S} : HC_{n+1}(A) \to HC_{n-1}(A)$ is a quotient map, i.e. for $n = 2k$

$$\begin{array}{llcl} \widetilde{S}_{2k}: & HC_{2k}(A) & \to & HC_{2k-2}(A) \\ & h.cl.(i + (I^{k+1} + [I^k, R])) & \mapsto & h.cl.(i + (I^{k} + [I^{k-1}, R])) \end{array}$$

and for $n = 2k+1$

$$\begin{array}{rccc} \widetilde{S}_{2k+1}: & HC_{2k+1}(A) & \rightarrow & HC_{2k-1}(A) \\ & h.cl.(\natural\omega + \natural I^k dI) & \mapsto & h.cl.(\natural\omega + \natural I^{k-1} dI), \end{array}$$

and $\widetilde{B}$ is the connecting homomorphism. Explicitly, for $n = 2k$ the diagram (2.7) becomes

$$\begin{array}{ccccc} & & \natural\omega + \natural I^k dR & \rightarrow & \natural\omega + \natural I^{k-1} dI \\ & & \downarrow & & \downarrow \\ \bar{b}(\omega) + (I^{k+1} + [I^k, R]) & \rightarrow & \bar{b}(\omega) + (I^{k+1} + [I^k, R]) & \rightarrow & 0 \end{array}$$

and thus

$$\begin{array}{rccc} \widetilde{B}_{2k-1}: & HC_{2k-1}(A) & \rightarrow & HH_{2k}(A) \\ & h.cl.(\natural\omega + \natural I^{k-1} dI) & \mapsto & h.cl.(\bar{b}(\omega) + (I^{k+1} + [I^k, R])) \end{array}$$

For $n = 2k+1$ the diagram (2.7) becomes

$$\begin{array}{ccccc} & & r + I^{k+1} & \rightarrow & r + (I^{k+1} + [I^k, R]) \\ & & \downarrow & & \downarrow \\ \natural dr + \natural I^k dI & \rightarrow & \natural dr + \natural I^k dI & \rightarrow & 0 \end{array}$$

and thus

$$\begin{array}{rccc} \widetilde{B}_{2k}: & HC_{2k}(A) & \rightarrow & HH_{2k+1}(A) \\ & h.cl.(r + (I^{k+1} + [I^k, R])) & \mapsto & h.cl.(\natural dr + \natural I^k dI) \end{array}$$

2.5 Free ideal rings, a review

In this section we prove that any mixed algebra belongs to a special class of rings, called free ideal rings (firs). Firs have a nice intersection property for two-sided ideals which will be used in the proof of the identity theorem in the next chapter.

The definition of a free ideal ring generalizes the concept of a principal ideal domain:

Definition 2.5.1 *([6], section 1.2) A free left (right) ideal ring, or a left (right) fir, is a ring, in which all left (right) ideals are free of unique rank as left (right) modules over that ring. A fir or a two-sided fir is both a left and a right fir.*

Over a right fir R every submodule of a free right R-module is again free (it follows from theorem 5.3 of [4]). A (left) fir R has invariant basis number (i.e. any free R-module is of unique rank); a fir is always an integral domain ([6], section 1.2).

In the commutative ring R any two elements are R-linearly dependent (since $a \cdot b = b \cdot a$). This implies that no ideal can have a basis of more than one element. It follows

Proposition 2.5.2 *[5] A commutative ring is a fir if and only if it is a principal ideal domain.*

This can be interpreted like this: one can consider a fir as a "non-commutativisation" of a principal ideal domain.

The fact that any mixed free algebra is a fir follows from the general result of Cohn about free products of firs. In order to state it, we start with the definition of a free product of algebras:

Definition 2.5.3 *A free product of a family of unital algebras $(A_i)_{i \in I}$ over a field k is their coproduct in the category of unital associative k-algebras; a free product of two unital k-algebras A and B is denoted by $A * B$ (cf. [27], E.2.6.2).*

That means that a free product of a family $(A_i)_{i \in I}$ of unital k-algebras is a unital k-algebra A together with a family of algebra homomorphisms $(j_i : A_i \to A)_{i \in I}$ such that, given a unital k-algebra B and a family of homomorphisms $(\varphi_i : A_i \to B)_{i \in I}$, there is a unique homomorphism $\varphi : A \to B$ such that $\varphi \circ j_i = \varphi_i$ for any $i \in I$ (cf. [24], chapter I, §11). This is called the universal property of a free product.

It follows immediately from the universal property that the free product of a family $(A_i)_{i \in I}$ is unique up to isomorphism. To show that it exists, we consider the free algebra on all elements of all A_i as generators and its quotient by the ideal generated by all elements of the form $(s \cdot t - st)$, where elements s and t belong to the same A_i, $s \cdot t$ is their product in the free algebra, and st is their product in A_i. This quotient is a free product of $(A_i)_{i \in I}$.

We remark here that Cohn in his articles [8], [5] uses a different definition of a free product, given in [8] not only for algebras over a field, but for Λ-rings for an arbitrary ring Λ (a Λ-ring is a ring with Λ as a subring), namely a Λ-ring R is a free product of Λ-rings R_i, $i \in I$ if $(R_i)_{i\in I}$ is a family of subrings of R such that

1. $R_i \cap R_k = \Lambda$ for all $i \neq k$ from I;
2. if for each $i \in I$ X_i is a set of generators of R_i, then $\cup X_i$ is a set of generators of R;
3. if C_i is a set of defining relations of R_i, $i \in I$ (in terms of the generating set X_i), then $\cup C_i$ is a set of defining relations of R.

The free product of Λ-rings in the sense of Cohn does not always exist, but for the algebras over some field k it exists (it follows from theorem 4.7 of [8], since a field is a regular ring) and clearly has the universal property. Thus for the algebras over a field the free product in the sense of Cohn and the free product in the sense of universal algebra coincide.

Proposition 2.5.4 *([5], corollary 1 of theorem 4.3) If k is a field, (R_λ) is a family of right firs that are augmented k-algebras such that the augmentation module in R_λ is a right ideal, then the free product of (R_λ) is a right fir.*

(Recall from [5] that a k-ring is called augmented if there is a k-linear map $\epsilon : R \to k$ such that $j \circ \epsilon = id_k$, where j is the canonical embedding of k into R. The augmentation module is by definition $\operatorname{Ker} \epsilon$; it is a k-subspace of R.)

The proof of the main theorem of this section relies on that proposition:

Theorem 2.5.5 *Any mixed free algebra over a field k is a two-sided fir.*

Proof. A mixed free algebra $k\langle X, Y, Y^{-1}\rangle$ is a free product of the polynomial algebras $k[x]$ $(x \in X)$ and of the group algebras $k[y, y^{-1}]$

($y \in Y$) of the infinite cyclic groups. The first ones are firs by 2.5.2, since $k[x]$ is a principal ideal domain ([24], theorem IV.1.2). The group ring $k[y, y^{-1}]$ is a ring of quotients of $k[y]$ with respect to the regular multiplicative system $\{y^n | n \in \mathbb{N}\}$ (cf. [41], section 4.9). Applying theorem 4.15(c) of [41] (which claims that any ideal J of the ring of quotients R_M of the commutative ring R with respect to the multiplicative system M is an extended ideal, which means that there exists an ideal I of R such that $J = R_M\gamma(I)$, where γ is the canonical embedding of R into R_M), we see that $k[y, y^{-1}]$ is also a principal ideal domain.

An augmentation for both $k[x]$ and $k[y, y^{-1}]$ can be defined by $\epsilon(\sum \lambda_i x^i) = \sum \lambda_i$. Clearly, the augmentation module is then an ideal in both $k[x]$ and $k[y, y^{-1}]$. Therefore, the mixed free algebra $k\langle X, Y, Y^{-1}\rangle$ is a two-sided fir by theorem 2.5.4 and its left version. $\square$

We also want to mention here (although we don't need it in the sequel) that another approach to the proof of this theorem can be found in [26], where it is not only shown that any submodule of a (right) free module over a free associative algebra or over a free group algebra is free, but also its generators are given and the formula for its rank is obtained, analogous to the Schreier's generators and Schreier's formula for a subgroup of a finite group. Namely, if G is a free monoid or a free group on r generators (or a free product of a free monoid and a free group with r generators on the whole), $F = kG$ and M is a submodule of codimension n in a free F-module N of rank l, then

$$\text{rank} M = n(r-1) + l.$$

Now we state the property of free ideal rings which is the reason why we consider them in this work:

Theorem 2.5.6 *(intersection theorem, [7], corollary of theorem 3.3)*

Let R be a two-sided fir. Then for any proper ideal I in R we have

$$\bigcap_{n\in\mathbb{N}} I^n = \{0\}.$$

Chapter 3

Hochschild and cyclic homology of one-relator algebras, higher dimensions

In this chapter we show that for a one-relator algebra the levels higher than 2 do not contribute to the cyclic homology. In order to do this, we prove that for a one-relator algebra $A = R/I$ (where R is a mixed free algebra, I is a principal ideal) such that the enveloping algebra A^e has no zero-divisors there is an isomorphism $I^n/I^{n+1} \cong A^{\otimes(n+1)}$.

3.1 Identity theorem

If R is an algebra over a field k and I is an ideal generated by some elements $\{z_j | j \in \Lambda\}$, then an arbitrary element of the ideal I can be written in the form

$$i = \sum_{k=1}^{m} s_k z_{j_k} r_k. \tag{3.1}$$

The identity problem is the question of "how unique" expressions of that form for an element i of the ideal are. Or, equivalently, the problem of finding all "identities" of the form

$$\sum_{k=1}^{m} s_k z_{j_k} r_k = 0. \tag{3.2}$$

For us the case of a principal ideal I (generated by an element w) is of interest. The first obvious source of non-uniqueness in (3.1) is the fact that for all s, r and t from F the element $rwswt$ can be written in two ways: as $rw(swt)$ and as $(rws)wt$. Thus for all s, r and t from F the difference $rw(swt) - (rws)wt$ is of the form (3.1) and zero.

In some cases all non-uniqueness arises in some sense in that way, as was proved by Dicks in [18]. Namely, let $F = k\langle x_0, \ldots, x_n\rangle$ be a free associative algebra freely generated by $x_0, \ldots, x_n$, let J denote the ideal in F generated by $x_0, \ldots, x_n$ and let M denote the free monoid freely generated by $x_0, \ldots, x_n$ (note that F is then a monoid algebra of M). A non-zero element r of J is called aperiodic if $rf \in JrF$ implies $f \in FrF$ for all $f \in F$.

Proposition 3.1.1 *([18]) Suppose that there are a commutative monoid $(T, +, 0, >)$, well-ordered as a monoid, and a monoid morphism $\phi : M \to T$ such that $\phi(x_i) > 0$ for $i = 0, \ldots n$. If for an element $w = \sum_{a \in M} \lambda_a a$ ($\lambda_a \in k$) from F the element $w_\phi = \sum_{a \in S} \lambda_a a$, where $S = \{a | \phi(a) = max\{\phi(b) | b \in M, \lambda_b \neq 0\}\}$, is aperiodic, then the annihilator of w in the enveloping algebra F^e is generated by*

$$\{(wa \otimes 1^o - 1 \otimes (aw)^o | a \in M\}$$

(by definition the enveloping algebra of an algebra B is the algebra $B^e = B \otimes B^{op}$).

That means exactly that the sum $\sum_k s_k w r_k$ is zero if and only if in F^e

$$\sum_k s_k \otimes r_k^o = \sum_i (a_i w b_i \otimes c_i^o - a_i \otimes (b_i w c_i)^o)$$

for some $a_i, b_i, c_i \in F$, $i = 1, \ldots l$.

From this proposition it follows easily that, under the conditions of proposition 3.1.1, for the principal ideal $I = FwF$ the F/I-bimodule I/I^2 is freely generated by $w + I^2$ ([18], theorem 1(iii)), i.e. passing to the quotient I/I^2 "kills" all non-trivial identities (3.2). For our goals this weaker result would be sufficient, but we need to extend it somehow onto the *mixed* free algebras. We do it in the following way.

In his paper [25] Lewin constructs for a mixed free k-algebra R with two ideals U and V an embedding of the k-algebra R/UV into the k-algebra $\begin{pmatrix} R/V & 0 \\ T & R/U \end{pmatrix}$, where $T = T(R; R/U, R/V)$ is the R/U-R/V-bimodule of the universal derivation δ, which is defined by the rule $r + UV \mapsto \begin{pmatrix} r+V & 0 \\ \delta(r) & r+U \end{pmatrix}$. (For k-algebras R, R', R'' and for homomorphisms

$$\begin{array}{rcl} R & \to & R', \\ r & \mapsto & r' \end{array} \qquad\qquad \begin{array}{rcl} R & \to & R'', \\ r & \mapsto & r'' \end{array}$$

a k-linear map $\delta : R \to M$ into an R'-R''-bimodule M is called a derivation if for all a, b from R

$$\delta(ab) = a'\delta(b) + \delta(a)b''.$$

A R'-R''-bimodule $T(R; R', R'')$ is said to be the module of the universal derivation $\delta : R \to T(R; R', R'')$ if for any R'-R''-bimodule M and any derivation $\tilde{\delta} : R \to M$ there is a unique homomorphism of R'-R''-bimodules $\sigma : T \to M$ such that $\tilde{\delta} = \sigma \circ \delta$.)

That embedding gives rise to an embedding of $U \cap V/UV$ into $T(R; R/U, R/V)$ defined by $z + UV \mapsto \delta(z)$, which is a free R/U-R/V-bimodule by the following proposition:

Proposition 3.1.2 *([25], corollary 5) If $R = k\langle X \rangle = k\langle Y, Z, Z^{-1} \rangle$ is a mixed free algebra, R' and R'' are k-algebras and $R \to R'$ and $R \to R''$ are k-homomorphisms, then the module of the universal derivation $T(R; R', R'')$ is a free R'-R''-bimodule. In particular, the R-bimodule $T(R)$ of its universal R-derivation δ is a free R-bimodule, freely generated by on $\{\delta(x) | x \in X\}$, and for any R-bimodule M every setting of $\gamma(x) \in M$ on R defines a derivation $\gamma : R \to M$.*

That proves the following

Theorem 3.1.3 *([25], theorem 7) Let k be a field, let $X = Y \sqcup Z$ be a set, let $R = k\langle X\rangle = k\langle Y, Z, Z^{-1}\rangle$ be a mixed free algebra. Let U, V be two-sided ideals in $k\langle X\rangle$. Then $U \cap V/UV$ can be embedded into a free R/U-R/V-bimodule.*

The next result is a generalization of theorem 8 of [25].

Theorem 3.1.4 *(Simple identity theorem) Let k be a field, let $X = Y \sqcup Z$ be a set, let $R = k\langle X\rangle = k\langle Y, Z, Z^{-1}\rangle$ be a mixed free algebra and let w be an element of R such that the enveloping algebra $(R/I)^e$ (where $I = RwR$) has no zero-divisors. Then I/I^2 is a free R/I-bimodule generated by $w + I^2$ and, therefore,*

$$\begin{array}{rcl} I/I^2 & \cong & R/I \otimes R/I \\ w + I^2 & \mapsto & 1 \otimes 1. \end{array}$$

Proof runs exactly as in [25], theorem 8, using the fact that a mixed free algebra is a fir. Namely, by theorem 3.1.3, I/I^2 can be embedded into a free R/I-R/I-bimodule. Since the algebra $(R/I)^e$ has no zero-divisors, each non-zero single-generated submodule of a free R/I-bimodule is again free. I/I^2 is generated by a single element (namely, by $w + I^2$) and due to theorem 2.5.6 the intersection $\bigcap I^n = \{0\}$, hence $I/I^2 \neq 0$ and is thus free. $\square$

There is a whole class of one-relator algebras satisfying the conditions of the identity theorem, for which the simple identity theorem of Lewin ([25], theorem 8) was proved: if F is free and w is a Lie element of F, then F/I is the universal enveloping algebra of a Lie algebra L, generated by X and the single relation $w = 0$. $F/I \otimes (F/I)^{op}$ is then the universal enveloping algebra of $L \oplus L^{op}$ and has therefore no zero divisors ([22], theorem V.6). In the next chapter we consider three more examples of one-relator algebras, for which the identity theorem holds.

3.2 Quotient I^n/I^{n+1}.

In this section we prove the following fact:

$$I^n/I^{n+1} \cong A^{\otimes(n+1)} \tag{3.3}$$

as A-bimodules.

To do that we state first the following propositions from [12]:

Proposition 3.2.1 *([12], proposition 5.2) For any hereditary algebra R (that means that any submodule of a projective left module over R is projective) and its two-sided ideal I the associated graded algebra $gr_I R = \oplus I^n/I^{n+1}$ is the tensor algebra on the R/I-bimodule I/I^2. The isomorphism*

$$\mu_n : I/I^2 \otimes_{R/I} I^n/I^{n+1} \longrightarrow I^{n+1}/I^{n+2}$$

is induced by the multiplication map.

Proposition 3.2.2 *([12], proposition 5.2) Any quasi-free algebra is hereditary.*

From these two propositions it follows

Proposition 3.2.3 *For a one-relator algebra $A = R/I$ $(I = RwR)$ such that the enveloping algebra A^e has no zero-divisors the A-bimodule homomorphism*

$$\begin{array}{cccc} \varphi_n : & I^n/I^{n+1} & \longrightarrow & A^{\otimes(n+1)} \\ & x_0wx_1w\dots wx_n + I^{n+1} & \mapsto & \overline{x_0}\otimes\dots\otimes\overline{x_n} \end{array}$$

(where $\overline{x}$ is the image of an element $x \in R$ under the canonical epimorphism $R \to A$) is an isomorphism.

Proof. Since any mixed free algebra is quasi-free, it follows from proposition 3.2.1 and proposition 3.2.2 that for each n there is an isomorphism

$$I^n/I^{n+1} \cong (I/I^2)^{\otimes_A n},$$

and this isomorphism is induced by the multiplication map. It's well known that the map of A-bimodules

$$\begin{array}{rcl} A \otimes_A A & \to & A \\ a \otimes_A b & \mapsto & ab \end{array}$$

is an isomorphism (e.g. (3.7) of chapter V of [33]), and that its inverse takes an arbitrary element a from A into $a \otimes_A 1 \in A \otimes_A A$. By the identity theorem 3.1.4, $I/I^2 \cong A \otimes A$. One can now easily see that the composition of the following sequence of maps

$$A^{\otimes(n+1)} \to (A \otimes A)^{\otimes_A n} \to (I/I^2)^{\otimes_A n} \to I^n/I^{n+1},$$

taking

$$\overline{x_0} \otimes \ldots \otimes \overline{x_n} \mapsto \overline{x_0} \otimes (\overline{x_1} \otimes_A 1) \otimes (\overline{x_2} \otimes_A 1) \ldots \otimes (\overline{x_{n-1}} \otimes_A 1) \otimes \overline{x_n} \mapsto$$
$$(\overline{x_0} w \overline{x_1} + I^2) \otimes_A (1 w \overline{x_1} + I^2) \otimes_A \ldots (1 w \overline{x_n} + I^2) \mapsto x_0 w x_1 w \ldots w x_n + I^{n+1},$$

is an isomorphism. □

Note that another way to see that the n-th quotient I^n/I^{n+1} is an n-fold tensor product of the first one I/I^2 follows from 2.5.4 with the help of the following lemma of Cohn:

Lemma 3.2.4 *([7], proposition 2.1) Let R be a ring and let J be a two-sided ideal in R. If J is free as a right R-module, then the associated graded algebra $gr_J R$ is the tensor algebra on the R/J-bimodule J/J^2.*

Proof. Denote J^n/J^{n+1} by M_n. One shows first that the multiplication map $J^n \otimes_F J \to J^{n+1}$ is an isomorphism, using the fact that J is free as a right R-module. Let $E = \{e_i\}$ be a basis of J as a right R-module. Consider the set of n-products $E^n = \{e_{i_1} \ldots e_{i_n} | \, e_{i_k} \in E\}$. Then E^n is again right linearly independent over R and is a basis of J^n. The mapping

$$(e_{i_1} \ldots e_{i_n}, e_i) \mapsto e_{i_1} \ldots e_{i_n} e_i$$

is R-balanced. It induces the multiplication map $J^n \otimes_R J \to J^{n+1}$, under which

$$\sum e_{i_1} \dots e_{i_n} a_{i_1 \dots i_n} \otimes e_i b_i \mapsto \sum e_{i_1} \dots e_{i_n} a_{i_1 \dots i_n} e_i b_i.$$

This map is obviously surjective. To see that it is also injective, take $a = \sum e_{i_1} \dots e_{i_n} \otimes e_i a_{i_1,\dots i_n,i}$ in the kernel. Then $\sum e_{i_1} \dots e_{i_n} e_i a_{i_1,\dots i_n,i}$, hence all $a_{i_1,\dots i_n,i} = 0$ and $a = 0$.

Then one takes for M_n the short exact sequence

$$0 \to J^{n+1} \to J^n \to M_n \to 0,$$

tensors it up (over R) with the corresponding short exact sequence for M_1, and gets the following commutative diagram:

$$\begin{array}{ccccccc}
J^{n+3} & \to & J^{n+2} & \to & M_n \otimes_R J^2 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
J^{n+2} & \to & J^{n+1} & \to & M_n \otimes_R J & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
J^{n+1} \otimes_R M_1 & \to & J^n \otimes_R M_1 & \to & M_n \otimes_R M_1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

The middle row yields an isomorphism $M_n \otimes_R J \cong M_{n+1}$. The image of the map $M_n \otimes_R J^2 \to M_n \otimes_R J$ is zero. One gets now from the right column that $M_n \otimes_R J \cong M_n \otimes_R M_1$ and, therefore, $M_n \otimes_R M_1 \cong M_{n+1}$.

One checks at last that $M_n \otimes_R M_1 \cong M_n \otimes_{R/J} M_1$. □

Since in a fir each right ideal is free, one can apply this lemma to the case of the mixed free algebra R and its ideal I.

3.3 Higher Hochschild homology, even case

Further in this chapter A denotes a one-relator algebra, $A = R/I$, such that $I = RwR$ and R, w satisfy the conditions of the identity theorem.

Let us compute $HH_{2n}(A)$, $n > 1$. By proposition 2.3.6, $HH_{2n}(A) \cong H_+(gr^{2n}\mathcal{X}(R,I))$. The left-hand side of the complex

$$gr^{2n}\mathcal{X}(R,I) : I^n/I^{n+1} + [I^n,R] \rightleftarrows \natural I^{n-1}dI/\natural I^n dR$$

can be identified with $A^{\otimes n}$ by the map

$$\overline{x_0 w \dots w x_n} \mapsto \overline{x_n x_0} \otimes \dots \otimes \overline{x_{n-1}}.$$

This map is really an isomorphism. To prove this, we start with the fact that for all $x_0, \dots x_n$ from R the equality

$$\overline{x_0 w \dots w x_n} = \overline{x_n x_0 w \dots w}$$

holds (since the difference $x_0 w \dots w x_n - x_n x_0 w \dots w$ lies in $[I^n, R]$), from which it follows that with the new notation

$$S_n = \operatorname{span}\{x_0 w \dots w x_{n-1} w | x_0, \dots x_{n-1} \in R\}$$

we have

$$I^n/[I^n,R] + I^{n+1} \cong S_n/(I^{n+1} \cap S_n).$$

The homomorphism φ_n from proposition 3.2.3 induces in the obvious way on $S_n/(I^{n+1} \cap S_n)$ a $\mathbb{C}$-linear (injective, since φ_n is injective) map $\tilde{\varphi}_n$ into $A^{\otimes(n+1)}$, the image of which is $A^{\otimes n} \otimes 1 \cong A^{\otimes n}$.

The right-hand side of the same complex is identified under proposition 3.2.3 with $A^{\otimes n}$ by the map

$$\overline{\natural x_0 w \dots w x_{n-1} d(xwy)} \mapsto \overline{y x_0} \otimes \dots \otimes \overline{x_{n-1} x}.$$

This map is really an isomorphism. To show that, we note first that by (4.1)

$$\natural x_0 w \dots w x_{n-1} d(xwy) = \natural w y x_0 w \dots w x_{n-1} dx + \\ \natural y x_0 w \dots w x_{n-1} x dw + \natural x_0 w \dots w x_{n-1} x w dy,$$

whence in $I^{n-1}dI/\natural I^n dR$ one has

$$\overline{\natural x_0 w \dots w x_{n-1} d(xwy)} = \overline{\natural y x_0 w \dots w x_{n-1} x dw}.$$

Further, we obtain with the new notation

$$T_n = \operatorname{span}\{\overline{\natural x_0 w \dots w x_{n-1} dw} | x_0, \dots x_{n-1} \in R\}$$

the following relation:

$$\natural I^{n-1}dI/\natural I^n dR \cong T_n/(\natural I^n dR \cap T_n).$$

The map φ_{n-1} from proposition 3.2.3 induces on $T_n/(\natural I^n dR \cap T_n)$ in the obvious way an injective $\mathbb{C}$-linear map $\hat{\varphi}_{n-1}$ into $A^{\otimes n}$; it is easy to see that its image is the whole of $A^{\otimes n}$.

Now we compute the differentials. By (4.1),

$$\natural d(\overline{x_0 w \ldots w x_{n-1} w}) = \overline{\natural x_0 w \ldots w x_{n-1} dw} + \overline{\natural w d(x_0 w \ldots w x_{n-1})} =$$
$$\overline{\natural x_0 w \ldots w x_{n-1} dw} + \overline{\natural w x_0 w \ldots w dx_{n-1}} + \overline{\natural x_{n-1} w d(x_0 w \ldots w x_{n-1})},$$

where the second summand is zero, since $\natural w x_0 w \ldots w dx_{n-1}$ lies in $I^n dR$; with the third summand we proceed as with the element $\natural d(\overline{x_0 w \ldots w x_{n-1} w})$ itself and so on; at the end we obtain that

$$\natural d(\overline{x_0 w \ldots w x_{n-1} w}) = \overline{\natural x_0 w \ldots w x_{n-1} dw} +$$
$$\overline{\natural x_{n-1} w x_0 w \ldots w x_{n-2} dw} + \ldots + \overline{\natural x_1 w \ldots w x_{n-1} w x_0 dw}.$$

The map $\overline{b}$ on an arbitrary element from $I^{n-1}dI/\natural I^n dR$ is

$$\overline{b}(\overline{\natural x_0 w \ldots w x_{n-1} dw}) = \overline{[x_0 w \ldots w x_{n-1} w, w]} =$$
$$\overline{x_0 w \ldots w x_{n-1} w} - \overline{x_{n-1} w x_0 w \ldots x_{n-2} w}.$$

Finally, the complex $F_I^{2n-1}X(R)/F_I^{2n}X(R)$ becomes

$$A^{\otimes n} \underset{(1-\lambda)}{\overset{N}{\rightleftarrows}} A^{\otimes n}$$

where $\lambda(a_0 \otimes \ldots \otimes a_{n-1}) = a_{n-1} \otimes a_0 \otimes \ldots \otimes a_{n-2}$ is the cyclic permutation and $N = 1 + \lambda + \lambda^2 + \ldots \lambda^{n-1}$ is the norm operator. It is well known (e.g.[27], appendix C.4) that $Ker N = Im(1-\lambda)$, hence $HH_{2n}(A) = 0$ for $n > 1$.

3.4 Higher Hochschild homology, odd case

Let us now compute $HH_{2n+1}(A)$, $n \geq 1$.

By proposition 2.3.6, $HH_{2n}(A) \cong H_-(gr^{2n+1}\mathcal{X}(R,I))$. The left-hand side of the complex

$$gr^{2n+1}\mathcal{X}(R,I) : I^{n+1} + [I^n, R]/I^{n+1} \rightleftarrows \natural I^n dR/\natural(I^{n+1}dR + I^n dI)$$

is isomorphic to $[I^n, R]/(I^{n+1} \cap [I^n, R])$, which is identified with $[A^{\otimes(n+1)}, A]$ with the help of the map

$$\overline{[x_0 w \dots w x_n, y]} \mapsto [\overline{x_0} \otimes \dots \otimes \overline{x_n}, \overline{y}].$$

This map is really an isomorphism because the homomorphism φ_n from proposition 3.2.3 obviously induces on $[I^n, R]/(I^{n+1} \cap [I^n, R])$ an injective $\mathbb{C}$-linear map $\tilde{\varphi}_n$ into $A^{\otimes(n+1)}$, the image of which is $[A^{\otimes n+1}, A]$, since

$$\tilde{\varphi}(x_0 w \dots w x_n y - y x_0 w \dots w x_n) = \\ \overline{x_0} \otimes \dots \otimes \overline{x_n y} - \overline{y x_0} \otimes \dots \otimes \overline{x_n} = [\overline{x_0} \otimes \dots \otimes \overline{x_n}, \overline{y}].$$

The right-hand side of the same complex is identified with $[A^{\otimes(n+1)}, A]$ by the map

$$\begin{array}{rlcl} \psi_n : & \natural I^n dR/\natural(I^{n+1}dR + I^n dI) & \to & [A^{\otimes(n+1)}, A] \\ & \overline{\natural x_0 w \dots w x_n dr} & \mapsto & [\overline{x_0} \otimes \dots \otimes \overline{x_n}, \overline{r}]. \end{array}$$

To prove that it is really an isomorphism, we start with showing that it is well-defined: first, it respects the equality (4.1), since

$$\psi_n(\overline{\natural r_0 w \dots w r_n d(st)}) = [\overline{r_0} \otimes \dots \otimes \overline{r_n}, \overline{st}] = \\ [\overline{r_0} \otimes \dots \otimes \overline{r_n s}, \overline{t}] + [\overline{t r_0} \otimes \dots \otimes \overline{r_n}, \overline{s}] = \\ \psi_n(\overline{\natural r_0 w \dots w r_n s dt}) + \psi_n(\overline{\natural t r_0 w \dots w r_n ds});$$

second, it vanishes on $\natural(I^{n+1}dR + I^n dI)$. It is obviously surjective. To show that it is injective, suppose that $\psi_n(z + \natural(I^{n+1}dR + I^n dI)) = 0$ for some element $z = \sum_{j=1}^{l} \sum_{i=1}^{k_j} \natural x_0^i w \dots w x_n^i dr_j$ from $\natural I^n dR$. The equality $\psi_n(z + \natural(I^{n+1}dR + I^n dI)) = 0$ implies that

$$\sum_{j=1}^{l} \sum_{i=1}^{k_j} [\overline{x_0^i} \otimes \dots \otimes \overline{x_n^i}, \overline{r_j}] = 0$$

in $A^{\otimes(n+1)}$. It follows that in I^n/I^{n+1}

$$\sum_{j=1}^{l}\sum_{i=1}^{k_j}(x_0^i w\dots w x_n^i r_j - r_j x_0^i w\dots w x_n^i) + I^{n+1} = 0.$$

Our element can be represented like

$$z = \sum_{j=1}^{l}\sum_{i=1}^{k_j}(\natural x_0^i w\dots w d(x_n^i r_j) - \natural r_j x_0^i w\dots w dx_n^i),$$

which lies in $I^{n+1}dR + I^n dI$, since

$$\sum_{j=1}^{l}\sum_{i=1}^{k_j}(x_0^i w\dots w x_n^i r_j - r_j x_0^i w\dots w x_n^i) \in I^{n+1},$$

Therefore, $z+\natural(I^{n+1}dR+I^n dI) = 0$ in $\natural I^n dR/\natural(I^{n+1}dR+I^n dI)$ and ψ_n is thus injective.

Finally, the complex $F_I^{2n}X(R)/F_I^{2n+1}X(R)$ becomes

$$[A^{\otimes(n+1)}, A] \underset{(id)}{\overset{0}{\rightleftarrows}} [A^{\otimes(n+1)}, A]$$

and, therefore, $HH_{2n+1}(A) = 0$ for $n \geq 1$.

3.5 Higher cyclic homology and periodic cyclic homology

For $n > 3$, for any one-relator algebra A satisfying the conditions of the identity theorem 3.1.4 the corresponding segment of the SBI-sequence (2.10) becomes

$$0 \xrightarrow{\widetilde{I}} HC_n(A) \xrightarrow{\widetilde{S}} HC_{n-1}(A) \xrightarrow{\widetilde{B}} 0$$

and it follows inductively that

$$HC_{2k+1}(A) \cong HC_3(A),$$

$$HC_{2k}(A) \cong HC_2(A)$$

for any $k \geq 1$.

To deal with the periodic cyclic homology, we recall from section 5.1.10 of [27] that if for an algebra A for all n greater than some N_0 the periodicity maps $S : HC_n(A) \to HC_{n-2}(A)$ are surjective, then

$$HP_{n+2\mathbb{Z}} \cong \lim_{\leftarrow r} HC_{n+2r}(A).$$

It is the case for any one-relator algebra A such that its enveloping algebra has no zero-divisors and it follows that

$$HP_0(A) \cong HC_2(A)$$

and

$$HP_1(A) \cong HC_3(A).$$

Chapter 4

Examples of computations

In this chapter we compute the Hochschild, cyclic and periodic cyclic homology of three concrete one-relator algebras: of the algebraic counterpart A_θ^0 of the irrational rotation algebra, of the Weyl algebra $A_{p,q}$ and of the modified Weyl algebra $A_{p,p^{-1},q}$ with one invertible generator. Each of these algebras has a two-generated mixed free extension and satisfies the conditions of the identity theorem. Hochschild and cyclic homology of the first two algebras is already known (e.g.[9], [2]), so it is possible to compare our approach with the standard one.

4.1 Cyclic and Hochschild homology of the algebra A_θ^0

Definition 4.1.1 *The algebra A_θ^0 consists of complex Laurent polynomials in two variables u and v with the commutation relation $vu = \lambda uv$, where λ is a complex number, $\lambda = e^{2\pi i\theta}$ with θ irrational from the interval $(0, 1)$.*

A_θ^0 is an algebraic counterpart of the locally convex irrational rotation algebra A_θ (or the so-called noncommutative torus), the cyclic homology groups of which are described in [9].

A_θ^0 is a one-relator algebra. To see that, we take the following extension of A_θ^0: $R_\theta = \mathbb{C}(F_2)$ is the group algebra over $\mathbb{C}$ of the free group $F_2 = F(u, v)$ and I is the ideal generated by $w_\theta = vu - \lambda uv$. Further in this section we usually omit the subscript θ and write R

for R_θ and w for w_θ.

Then I is a kernel of the surjective homomorphism

$$\begin{array}{rcl} R & \to & A_\theta^0 \\ u & \mapsto & u, \\ v & \mapsto & v, \\ 1_R & \mapsto & 1_{A_\theta^0} \end{array}$$

and R is a mixed free (in particular, quasi-free) extension of A_θ^0.

The extension R_θ and the element w_θ satisfy the conditions of the identity theorem 3.1.4. To see that, we remark that the enveloping algebra $A_\theta^0 \otimes (A_\theta^0)^{op}$ can be considered as the algebra of formal finite sums of the form $\sum a_{i,j,k,l} u^i v^j (u')^k (v')^l$, with the multiplication induced by the following relations: $vu = \lambda uv$, $v'u' = \lambda^{-1} u'v'$, and u, v commute with u', v'. If we now order $A_\theta^0 \otimes (A_\theta^0)^{op}$ lexicographically, then any element of it has a maximal monomial and this order is compatible with the multiplication. The maximal monomial of the product of two arbitrary non-zero elements is the product of the maximal monomials of these elements, whence this maximal monomial is not zero and, therefore, the product itself is not zero. Thus $A_\theta^0 \otimes (A_\theta^0)^{op}$ has no zero-divisors.

4.1.1 Zero cyclic and Hochschild homology of A_θ^0

By proposition 2.3.6, one has

$$HC_0(A_\theta^0) = HH_0(A_\theta^0) \cong H_+(\mathcal{X}^0(R, I))$$

and the complex $\mathcal{X}^0(R, I)$ is of the form

$$\mathcal{X}^0(R, I) : (A_\theta^0)_\natural \rightleftarrows 0$$

(not depending on R, I). An arbitrary commutator in $[A_\theta^0, A_\theta^0]$ is of the following form:

$$[u^{n_1} v^{n_2}, u^{m_1} v^{m_2}] = (\lambda^{n_2 m_1} - \lambda^{n_1 m_2}) u^{n_1+m_1} v^{n_2+m_2}.$$

For all n_1, n_2, m_1, m_2 such that $n_1 + m_1 = n_2 + m_2 = 0$ we have $(\lambda^{n_2 m_1} - \lambda^{n_1 m_2}) = 0$ and, therefore, $1 \notin [A_\theta^0, A_\theta^0]$. For all $u^s v^t$ with

$(s,t) \neq (0,0)$ one can find n_1, n_2, m_1, m_2 such that $s = n_1 + m_1$, $t = n_2 + m_2$ and $(\lambda^{n_2 m_1} - \lambda^{n_1 m_2}) \neq 0$; thus $u^s v^t$ lies in $[A^0_\theta, A^0_\theta]$. It follows that $HC_0(A^0_\theta) = HH_0(A^0_\theta) \cong \mathbb{C}$ and that the homology class of $\overline{1}$ forms the basis of $HC_0(A^0_\theta)$ and $HH_0(A^0_\theta)$ in $\mathcal{X}^0(R,I)$.

4.1.2 First cyclic homology of A^0_θ

The first cyclic homology group of A^0_θ is by proposition 2.3.6

$$HC_1(A^0_\theta) \cong H_-(\mathcal{X}^1(R,I))$$

and the complex $\mathcal{X}^1(R,I)$ is of the form

$$\mathcal{X}^1(R,I) : R/I \underset{\overline{b}}{\overset{\natural\, d}{\rightleftarrows}} \Omega^1 R_\natural / \natural(IdR + RdI).$$

Due to the equality $[xdy, z] = xd(yz) - xydz - zxdy$, we have in $\Omega^1 R_\natural$ the relation

$$\natural xd(yz) = \natural xydz + \natural zxdy. \tag{4.1}$$

(note that this relation holds in $\Omega T_\natural$ for an arbitrary algebra T). In particular, each element from $(\Omega^1 R)_\natural$ can be (inductively) rewritten as a linear combination of the following elements:

$$\{\natural rdu,\ \natural rdv |\ r \text{ is a monomial in}\, u, v\}.$$

In the quotient $\Omega^1 R_\natural / \natural(IdR + RdI)$, we can always take r in $\overline{\natural rdu}$ and $\overline{\natural rdv}$ of the form $u^{n_1} v^{n_2}$ (since we factorize by $\natural(IdR)$).
Thus the quotient $\Omega^1 R_\natural / \natural(IdR + RdI)$ is generated by

$$\{\overline{\natural u^{n_1} v^{n_2} du}, \overline{\natural u^{n_1} v^{n_2} dv} | n_1, n_2 \in \mathbb{Z}\}.$$

Since

$$\overline{\natural u^{n_1} v^{n_2} dw} = \overline{\natural u^{n_1} v^{n_2} d(vu - \lambda uv)} =$$
$$\overline{\natural u^{n_1} v^{(n_2+1)} du} + \overline{\natural u^{(n_1+1)} v^{n_2} dv}) - \lambda \overline{\natural u^{n_1} v^{n_2} udv} - \lambda \overline{\natural vu^{n_1} v^{n_2} du} =$$
$$(1 - \lambda^{n_2+1}) \overline{\natural u^{(n_1+1)} v^{n_2} dv} + (1 - \lambda^{n_1+1}) \overline{\natural u^{n_1} v^{(n_2+1)} du},$$

we have with the new notation

$$\mathcal{M} := \operatorname{span}\{\overline{\natural u^{n_1} v^{n_2+1} du}, \overline{\natural u^{n_1+1} v^{n_2} dv} | n_1, n_2 \in \mathbb{Z}\} \text{ and}$$

$$\mathcal{N} := \operatorname{span}\{(1-\lambda^{n_1+1})\overline{\natural u^{n_1} v^{n_2+1} du} + (1-\lambda^{n_2+1})\overline{\natural u^{n_1+1} v^{n_2} dv} | n_1, n_2 \in \mathbb{Z}\}$$

the following relation: $\Omega^1 R_\natural / \natural(IdR + RdI) = \mathcal{M}/\mathcal{N}$.

Let now

$$\left(\sum_{(n_1,n_2)\in\mathbb{Z}^2} \left(a_{n_1,n_2} \overline{\natural u^{n_1+1} v^{n_2} dv} + c_{n_1,n_2} \overline{\natural u^{n_1} v^{n_2+1} du} \right) \right) \in \operatorname{Ker} \overline{b},$$

(where almost all a_{n_1,n_2}, c_{n_1,n_2} are zero). This means that

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} \left(a_{n_1,n_2}(1-\lambda^{n_1+1}) + c_{n_1,n_2}(\lambda^{n_2+1}-1) \right) \overline{u^{n_1+1} v^{n_2+1}} = \overline{0}$$

in $A_\theta^0 \cong R/I$, where $\overline{r}$ denotes the image of the element $r \in R$ under the quotient map $R \to R/I$. This implies that

$$a_{n_1,n_2}(1-\lambda^{n_1+1}) + c_{n_1,n_2}(\lambda^{n_2+1}-1) = 0 \tag{4.2}$$

for all n_1, n_2 from $\mathbb{Z}$.

It follows from (4.2) that
for $n_1 \neq -1$, $n_2 \neq -1$ we have $c_{n_1,n_2} = a_{n_1,n_2}(1-\lambda^{n_1+1})/(\lambda^{n_2+1}-1)$;
for $n_2 \neq -1$ the coefficient a_{-1,n_2} is arbitrary and $c_{-1,n_2} = 0$;
for $n_1 \neq -1$ the coefficient $c_{n_1,-1}$ is arbitrary and $a_{n_1,-1} = 0$
and for $n_1 = n_2 = -1$ both $a_{-1,-1}$, $c_{-1,-1}$ are arbitrary.

With the notation

$$\mathcal{M}_0 := \operatorname{span}\{\overline{\natural u^n du}, \overline{\natural v^n dv},\ n \in \mathbb{Z},\ (1-\lambda^{n_1+1})\overline{\natural u^{n_1} v^{(n_2+1)} du} + (1-\lambda^{n_2+1})\overline{\natural u^{(n_1+1)} v^{n_2} dv} |\ n_1, n_2 \in \mathbb{Z} \setminus \{-1\}\}$$

we have the relation

$$\operatorname{Ker} \overline{b} = \mathcal{M}_0/\mathcal{N} = \operatorname{span}\{\overline{\natural u^{-1} du}, \overline{\natural v^{-1} dv}\}.$$

To compute $\operatorname{Im} \natural d$, we note that $R/I \cong \mathbb{C} \cdot 1 \oplus [R/I, R/I]$ (as we have seen in subsection 4.1.1); $d(1) = 0$ and

$$\natural d|_{[-,-]} = 0, \tag{4.3}$$

since by (4.1)

$$\natural d([x, y]) = \natural d(xy - yx) = \natural x dy + \natural y dx - \natural y dx - \natural x dy = 0.$$

Thus $\natural d(R/I) = 0$ and, therefore, $HC_1(A^0_\theta) \cong \mathbb{C}^2$; the basis of $HC_1(A^0_\theta)$ in $\mathcal{X}^1(R, I)$ consists of the homology classes of $\overline{\natural u^{-1} du}$ and $\overline{\natural v^{-1} dv}$.

4.1.3 First Hochschild homology of A^0_θ

The first Hochschild homology group of A^0_θ is by proposition 2.3.6

$$HH_1(A^0_\theta) \cong H_-(gr^1 \mathcal{X}(R, I))$$

and the complex $gr^1 \mathcal{X}(R, I)$ is of the form

$$gr^1 \mathcal{X}(R, I) : [R, R] + I/I \underset{\overline{b}}{\overset{\natural\, d}{\rightleftarrows}} \Omega^1 R_\natural / \natural(IdR + RdI).$$

As in the previous subsection, $\operatorname{Ker} \overline{b} = \operatorname{span}\{\overline{\natural u^{-1} du}, \overline{\natural v^{-1} dv}\}$. On the other hand, $\operatorname{Im} \natural d = 0$, since $[R, R] + I \cong [R, R]/[R, R] \cap I$ and since $\natural d|_{[R,R]} = 0$ by (4.3). Thus $HH_1(A^0_\theta) \cong \mathbb{C}^2$. The basis of $HH_1(A^0_\theta)$ in $gr^1 \mathcal{X}(R, I)$ consists of the homology classes of $\overline{\natural u^{-1} du}$ and $\overline{\natural v^{-1} dv}$.

4.1.4 Second Hochschild homology of A^0_θ

The second Hochschild homology group of A^0_θ is by proposition 2.3.6

$$HH_2(A^0_\theta) \cong H_+(gr^2 \mathcal{X}(R, I))$$

and the complex $gr^2\mathcal{X}(R,I)$ is of the form:

$$gr^2\mathcal{X}(R,I) : I/(I^2+[I,R]) \underset{\overline{b}}{\overset{\natural\, d}{\rightleftarrows}} \natural(IdR+RdI)/\natural IdR.$$

Note that the basis of $\Omega^1 R_\natural/\natural IdR$ is given by

$$\{\overline{\natural u^s v^t du}, \overline{\natural u^s v^t dv}, \mid s,t\in\mathbb{Z}\}.$$

We use in what follows the notation $\nu = (n_1, n_2) \in \mathbb{Z}^2$.

Since $I/I^2 \cong A^0_\theta \otimes A^0_\theta$ (recall that $w = vu - \lambda uv$ is the generator of the principal ideal I; then the mentioned isomorphism is given by $bwc + I^2 \mapsto \overline{b}\otimes\overline{c}$; $\overline{x}$ denotes here the image of an element $x\in R$ under the quotient map $R\to R/I(\cong A^0_\theta)$) and since in $I/(I^2+[I,R])$ we have

$$bwc + (I^2+[I,R]) = cbw + (I^2+[I,R]),$$

we can identify $I/(I^2+[I,R])$ with A^0_θ by

$$bwc + (I^2+[I,R]) \mapsto \overline{cb}.$$

Now we compute the differential $\natural d$. For an element a form R of the form $a = \sum_{\nu\in\mathbb{Z}^2} a_\nu u^{n_1}v^{n_2}$, $a_\nu\in\mathbb{C}$, almost all a_ν are zero, we have

$$\natural d(\overline{aw}) = \overline{\natural(av-\lambda va)du} + \overline{\natural(ua-\lambda au)dv} = \sum_{\nu\in\mathbb{Z}^2}(1-\lambda^{n_1})a_{n_1-1,n_2-1}\overline{\natural u^{n_1-1}v^{n_2}du} + \sum_{\nu\in\mathbb{Z}^2}(1-\lambda^{n_2})a_{n_1-1,n_2-1}\overline{\natural u^{n_1}v^{n_2-1}dv},$$

and $\natural d(\overline{aw}) = 0$ implies that

$$\begin{cases}(1-\lambda^{n_1})a_{n_1-1,n_2-1} = 0,\\ (1-\lambda^{n_2})a_{n_1-1,n_2-1} = 0.\end{cases}$$

It follows that $a_{n_1,n_2} = 0$ if $n_1 \neq -1$ or $n_2 \neq -1$ and that $a_{-1,-1}$ is arbitrary. Therefore, $\operatorname{Ker}\natural d = \operatorname{span}\{u^{-1}v^{-1}w\}$. On the other hand, $\bar{b}(\natural(IdR + RdI)) \subset [I, R]$ whence $\mathrm{Im}\bar{b} = 0$ in $I/(I^2 + [I, R])$. Therefore, $HH_2(A^0_\theta) \cong \mathbb{C}$ and its basis in $gr^2\mathcal{X}(R, I)$ is the homology class of $\overline{u^{-1}v^{-1}w}$.

4.1.5 Second cyclic homology of A^0_θ

The second cyclic homology group of A^0_θ is by proposition 2.3.6

$$HC_2(A^0_\theta) \cong H_+(\mathcal{X}^2(R, I))$$

and the complex $\mathcal{X}^2(R, I)$ is of the following form:

$$\mathcal{X}^2(R, I) : R/(I^2 + [I, R]) \underset{\bar{b}}{\overset{\natural d}{\rightleftarrows}} \Omega^1 R_\natural / \natural IdR.$$

It is possible to compute $H_+(\mathcal{X}^2(R, I))$ directly, but such a computation is extremely technical and long. The simpler way is to use the SBI-sequence (2.10). We consider the following segment of the SBI-sequence:

$$HC_1(A^0_\theta) \overset{\widetilde{B}_1}{\to} HH_2(A^0_\theta) \overset{\widetilde{I}_2}{\to} HC_2(A^0_\theta) \overset{\widetilde{S}_2}{\to} HC_0(A^0_\theta) \overset{\widetilde{B}_0}{\to} HH_1(A^0_\theta). \tag{4.4}$$

Note that $\widetilde{B}_1(HC_1(A^0_\theta)) = 0$, since the generators of $HC_1(A^0_\theta)$ in $\mathcal{X}^1(R, I)$ computed in subsection 4.1.2 are the homology classes of $\overline{\natural u^{-1}du}$ and $\overline{\natural v^{-1}dv}$ and since

$$\widetilde{B}_1(h.cl.(\natural u^{-1}du + \natural RdI)) = h.cl.(\bar{b}(u^{-1}du) + (I^2 + [I, R])) = 0$$

and, similarly,

$$\widetilde{B}_1(h.cl.((\natural v^{-1}dv + \natural RdI)) = 0.$$

Note also that $\widetilde{B}_0(HC_0(A^0_\theta)) = 0$, since the generator of $HC_0(A^0_\theta)$ computed in subsection 4.1.1 is the homology class of $\overline{1}$ and since

$$\widetilde{B}_0(h.cl.(1 + (I + [R, R]))) = h.cl.(\natural d(1) + RdI) = 0.$$

Therefore, from exact sequence (4.4) we obtain the exact sequence

$$0 \to HH_2(A_\theta^0) \xrightarrow{\widetilde{I}_2} HC_2(A_\theta^0) \xrightarrow{\widetilde{S}_2} HC_0(A_\theta^0) \to 0$$

and the second cyclic homology $HC_2(A_\theta^0)$ is a direct sum of $HH_2(A_\theta^0)$ and $HC_0(A_\theta^0)$. The generators of $HC_2(A_\theta^0)$ are the homology classes of $\widetilde{I}_2(u^{-1}v^{-1}w + (I^2 + [I,R]))$ (since the generator of $HH_2(A_\theta^0)$ in $gr^2\mathcal{X}(R,I)$ is the homology class of $\overline{u^{-1}v^{-1}w}$) and of the pre-image of $1 + (I + [R,R])$ under $\widetilde{S}_2$. So we conclude that $HC_2(A_\theta^0) \cong \mathbb{C}^2$ and its basis in $\mathcal{X}^2(R,I)$ consists of the homology classes of $\overline{u^{-1}v^{-1}w}$ and $\overline{1}$.

4.1.6 Higher cyclic homology and periodic cyclic homology of A_θ^0

To compute $HC_3(A_\theta^0)$, we consider the following segment of sequence (2.10):

$$0 = HH_3(A_\theta^0) \xrightarrow{\widetilde{I}_3} HC_3(A_\theta^0) \xrightarrow{\widetilde{S}_3} HC_1(A_\theta^0) \xrightarrow{\widetilde{B}_1} HH_2(A_\theta^0) \qquad (4.5)$$

As we have seen in subsection 4.1.5, $B(HC_1(A_\theta^0)) = 0$ in $HH_2(A_\theta^0)$. Now sequence (4.5) becomes

$$0 \to HC_3(A_\theta^0) \to HC_1(A_\theta^0) \to 0$$

and it follows that $HC_3(A_\theta^0) \cong HC_1(A_\theta^0) \cong \mathbb{C}^2$.

Now the results of section 3.5 yield that

$$HH_n(A_\theta^0) = 0$$

for all $n > 2$, that

$$HC_n(A_\theta^0) \cong \mathbb{C}^2$$

for any $n \geq 2$ and that

$$HP_*(A_\theta^0) \cong \mathbb{C}^2$$

for $* = 0, 1$.

4.2 Cyclic and Hochschild homology of the Weyl algebra $A_{p,q}$

Definition 4.2.1 *The Weyl algebra $A_{p,q}$ consists of complex polynomials in two variables p and q with the commutation relation $[p,q]=1$.*

Take the following unital extension of $A_{p,q}$: $R_{p,q}=\mathbb{C}\langle p,q\rangle$ is the free associative $\mathbb{C}$-algebra with generators p and q. Let I be the ideal generated by $w_{p,q}=[p,q]-1$. Further in this section we usually simply write R for $R_{p,q}$ and w for $w_{p,q}$.

Then I is the kernel of the surjective homomorphism

$$\begin{array}{rcl} R & \to & A_{p,q} \\ p & \mapsto & p, \\ q & \mapsto & q, \\ 1_R & \mapsto & 1_{A_{p,q}} \end{array}$$

and R is a quasi-free extension of $A_{p,q}$.

We note here that the algebra $R_{p,q}$ and the element $w_{p,q}$ satisfy the conditions of the identity theorem 3.1.4. To see that, we remark that $A_{p,q}\otimes A_{p,q}^{op}$ is again the algebra of Weyl, in two pairs of generators, and has therefore no zero-divisors (e.g. [19]).

One proves inductively the following

Lemma 4.2.2 *([19], 2.1) In $A_{p,q}$ the following commutation relation holds:*

$$[p^s,q^t]=\sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{l!}\prod_{j=0}^{l-1}\left((s-j)(t-j)\right)p^{s-l}q^{t-l}. \qquad (4.6)$$

Note that in fact only a finite number of summands in (4.6) is not zero, namely for $l=1,\dots,min\ \{s,t\}$ (compare also with formula (2.4) from [21], which holds in a more general situation.)

Sometimes it is useful to rewrite (4.6) in the following way:

$$q^tp^s=\sum_{l=0}^{\infty}(-1)^l l!\binom{s}{l}\binom{t}{l}p^{s-l}q^{t-l}. \qquad (4.7)$$

We use here the standard conventions that for $k > n$ from $\mathbb{N}$ the binomial coefficient $\binom{n}{k}$ is equal to zero, that $\binom{n}{0} = 1$ and that $0! = 1$.

We will in fact use formula (4.7) only in the simple particular case where either s or t is equal to 1. Note also that from (4.6) it follows that for any polynomial $g \in \mathbb{C}[x]$ one has $[g(p), q] = g'(p)$ and $[p, g(q)] = g'(q)$, where g' is the derivation of g.

4.2.1 Zero cyclic and Hochschild homology of $A_{p,q}$

As in subsection 4.1.1, $HC_0(A_{p,q}) = HH_0(A_{p,q}) \cong (A_{p,q})_\natural$. Since for all n, k one has $[p, p^k q^{n+1}] = (n+1)p^k q^n$, each monomial $p^k q^n$ lies in $[A_{p,q}, A_{p,q}]$. It follows that $(A_{p,q})_\natural = 0$ and thus $HC_0(A_{p,q}) = HH_0(A_{p,q}) = 0$.

4.2.2 First cyclic and Hochschild homology of $A_{p,q}$

The first cyclic homology group of $A_{p,q}$ is by proposition 2.3.6

$$HC_1(A_{p,q}) \cong H_-(\mathcal{X}^1(R, I))$$

and the complex $\mathcal{X}^1(R, I)$ is of the form

$$\mathcal{X}^1(R, I) : R/I \underset{\overline{b}}{\overset{\natural\, d}{\rightleftarrows}} \Omega^1 R_\natural / \natural(IdR + RdI).$$

Similarly to subsection 4.1.2, one has with the notation

$$\mathcal{S} := \operatorname{span}\{\overline{\natural p^{s-1} q^t dp}, \overline{\natural p^t q^{s-1} dq} | s \in \mathbb{N}, t \in \mathbb{N} \cup \{0\}\} \text{ and}$$

$$\mathcal{T} := \operatorname{span}\{n_1 \overline{\natural p^{n_1 - 1} q^{n_2} dp} + n_2 \overline{\natural p^{n_1} q^{n_2 - 1} dq} | n_1, n_2 \in \mathbb{N} \cup \{0\}\}$$

(by convention, the second summand in the definition of $\mathcal{T}$ is zero for $n_2 = 0$) the relation $\Omega^1 R_\natural / \natural(IdR + RdI) = \mathcal{S}/\mathcal{T}$.

Let now

$$\left(\sum_{n_1, n_2} \left(a_{n_1, n_2} \overline{\natural p^{n_1 - 1} q^{n_2} dp} + c_{n_1, n_2} \overline{\natural p^{n_1} q^{n_2 - 1} dq} \right) \right) \in \operatorname{Ker} \overline{b},$$

where by definition $a_{0,n_2} = c_{n_1,0} = 0$ for all n_1, n_2 and almost all a_{n_1,n_2}, c_{n_1,n_2} are zero. This implies

$$\sum_{n_1,n_2} (-n_2 a_{n_1,n_2} + n_1 c_{n_1,n_2}) \overline{p^{n_1-1} q^{n_2-1}} = \overline{0}$$

in R/I and it follows that

$$-n_1 a_{n_1,n_2} + n_2 c_{n_1,n_2} = 0 \tag{4.8}$$

for all n_1, n_2 from $\mathbb{N} \cup \{0\}$.

It follows from (4.8) that $\operatorname{Ker} \overline{b}$ lies in $\mathcal{T}$ and, therefore, $HC_1(A_{p,q}) = 0$.

Similarly, $HH_1(A_{p,q}) = 0$.

4.2.3 Second Hochschild homology of $A_{p,q}$

The second Hochschild homology group of $A_{p,q}$ is by proposition 2.3.6

$$HH_2(A_{p,q}) \cong H_+(gr^2 \mathcal{X}(R,I))$$

and the complex $gr^2\mathcal{X}(R,I)$ is of the form

$$gr^2\mathcal{X}(R,I) : I/(I^2 + [I,R]) \underset{\overline{b}}{\overset{\natural d}{\rightleftarrows}} \natural(IdR + RdI)/\natural IdR.$$

As in subsection 4.1.4, each element $\overline{j}$ from $I/(I^2 + [I,R])$ can be represented in the form $\overline{j} = \overline{aw}$, where $a = \sum\limits_{n_1,n_2=0}^{\infty} a_{n_1,n_2} p^{n_1} q^{n_2}$, $a_{n_1,n_2} \in \mathbb{C}$, almost all a_{n_1,n_2} are zero, and $w = [p,q] - 1$ is the generator of the principal ideal I.

In $\natural(IdR + RdI)$ one has

$$\natural d(\overline{aw}) = \overline{\natural adw} + \overline{\natural wda} = \overline{\natural ad(pq - qp)} = \overline{\natural[a,p]dq} - \overline{\natural[a,q]dp} =$$
$$\sum_{n_1,n_2=0}^{\infty} (-n_2 a_{n_1,n_2} \overline{\natural p^{n_1} q^{n_2-1} dq} + n_1 a_{n_1,n_2} \overline{\natural p^{n_1-1} q^{n_2} dp}).$$

Now $\natural d(\overline{aw}) = 0$ implies that for all n_1, n_2

$$\begin{cases} n_1 a_{n_1,n_2} = 0, \\ n_2 a_{n_1,n_2} = 0, \end{cases}$$

which yields that $a_{n_1,n_2} = 0$ for $(n_1, n_2) \neq (0,0)$ and that $a_{0,0}$ is arbitrary.

Therefore, Ker $\natural d$ is generated by $\overline{w}$. We have, on the other hand, that $\overline{b}(\natural(IdR + RdI)) \subset [I, R]$ and thus Im $\overline{b} = 0$ in $I/(I^2 + [I, R])$. We get $HH_2(A_{p,q}) \cong \mathbb{C}$ and its basis in $gr^2\mathcal{X}(R, I)$ is the homology class of $\overline{w}$.

4.2.4 Second cyclic homology of $A_{p,q}$

The second cyclic homology group of $A_{p,q}$ is by proposition 2.3.6

$$HC_2(A^0_\theta) \cong H_+(\mathcal{X}^2(R, I))$$

and the complex $\mathcal{X}^2(R, I)$ is of the form

$$\mathcal{X}^2(R, I) : R/(I^2 + [I, R]) \underset{\overline{b}}{\overset{\natural\, d}{\rightleftarrows}} \Omega^1 R_\natural / \natural IdR.$$

As well as for the algebra A^0_θ, one could compute $H_+(\mathcal{X}^2(R, I))$ directly, but such a computation involves the commutation formula (4.7) in the general form and some nasty technical combinatorial results. It is easier to use the SBI-sequence. We consider the following segment of sequence (2.10):

$$0 = HC_1(A_{p,q}) \overset{\widetilde{B}_1}{\to} HH_2(A_{p,q}) \overset{\widetilde{I}_2}{\to} HC_2(A_{p,q}) \overset{\widetilde{S}_2}{\to} HC_0(A_{p,q}) = 0.$$

It follows that the second cyclic homology $HC_2(A_{p,q})$ is isomorphic to $HH_2(A_{p,q})$. The generator of $HC_2(A_{p,q})$ is the homology class of $\widetilde{I}_2(w + (I^2 + [I, R]))$. Thus $HC_2(A_{p,q}) \cong \mathbb{C}$ and its basis in $\mathcal{X}^2(R, I)$ consists of the homology class of $\overline{w}$.

4.2.5 Higher cyclic homology and periodic cyclic homology of $A_{p,q}$

To compute $HC_3(A_{p,q})$, we consider the following segment of sequence (2.10):

$$0 = HH_3(A_{p,q}) \to HC_3(A_{p,q}) \to HC_1(A_{p,q}) = 0,$$

hence $HC_3(A_{p,q}) = 0$

From the results of section 3.5 it follows that

$$HH_n(A_{p,q}) = 0$$

for all $n > 2$, that

$$HC_{2k+1}(A_{p,q}) = 0,$$
$$HC_{2k}(A_{p,q}) \cong \mathbb{C}$$

for any $k \geq 1$, and that

$$HP_0(A_{p,q}) \cong \mathbb{C}$$

$$HP_1(A_{p,q}) = 0.$$

4.3 Cyclic and Hochschild homology of the Weyl-type algebra with one invertible generator

In this section we consider the following modification of the Weyl algebra:

Definition 4.3.1 *The algebra $A_{p,p^{-1},q}$ is defined as the quotient of the mixed free algebra $R_{p,p^{-1},q} = \mathbb{C}\langle p, p^{-1}, q\rangle$ by the same relation as the Weyl algebra, i.e. by the principal ideal I generated by the element $w_{p,q} = [p, q] - 1$.*

The extension $R_{p,p^{-1},q}$ and the element $w_{p,q}$ satisfy the conditions of the identity theorem 3.1.4, the argument is similar to the one used for the algebra A^0_θ.

Further in this section we omit the subscripts and simply write R for $R_{p,p^{-1},q}$ and w for $w_{p,q}$.

For the computations we need to know how to write an arbitrary element of $A_{p,p^{-1},q}$ in the canonical form

$$\sum_{n_1\in\mathbb{Z}}\sum_{n_2=0}^{\infty} a_{n_1,n_2}p^{n_1}q^{n_2},$$

where almost all a_{n_1,n_2} are zero. The question is how to commute the powers of p and q. For positive powers of p the answer is given by lemma 4.2.2. The rest is done with the following

Lemma 4.3.2 *For negative powers of p the same commutator formula (4.6) as for positive holds; formula (4.7) is also true if one generalizes the binomial coefficients $\binom{n}{k}$ to negative values of n by $\binom{n}{k} = \frac{1}{k!}n(n-1)\dots(n-l+1)$ for $n \in \mathbb{Z}, k \in \mathbb{N}$.*

Proof. 1. We prove by induction on t that

$$q^t p^{-1} = \sum_{l=0}^{\infty}(-1)^l l! \binom{-1}{l}\binom{t}{l} p^{-1-l}q^{t-l}.$$

The base of induction is $t = 1$: multiplying the equality $[p, q] = 1$ on the left and on the right by p^{-1}, one obtains $qp^{-1} = p^{-1}q + p^{-2}$.

Now multiplying the equality $[p, q^t] = tq^{t-1}$ on the left and on the right by p^{-1}, we obtain

$$q^t p^{-1} = p^{-1}q^t + tp^{-1}q^{t-1}p^{-1},$$

the right-hand side of which is by the inductive assumption (and with the remark that $\binom{-1}{l} = (-1)^{-1}$) equal to

$$p^{-1}q^t + tp^{-1}\sum_{l=0}^{\infty}(-1)^l l!\binom{-1}{l}\binom{t-1}{l}p^{-1-l}q^{t-1-l} =$$

$$p^{-1}q^t + \sum_{l=0}^{\infty} l!t\binom{t-1}{l}p^{-2-l}q^{t-1-l} =$$

$$p^{-1}q^t + \sum_{l=0}^{\infty} t(t-1)\dots(t-1-l+1)p^{-2-l}q^{t-1-l} =$$

$$p^{-1}q^t + \sum_{k=1}^{\infty} t(t-1)\dots(t-k+1)p^{-1-k}q^{t-k} =$$

$$\sum_{k=0}^{\infty}(-1)^k k!\binom{-1}{k}\binom{t}{k}p^{-1-k}q^{t-k},$$

which completes the induction step.

2. Now we prove by induction on $s \in \mathbb{N}$ that

$$q^t p^{-s} = \sum_{l=0}^{\infty}(-1)^l l!\binom{-s}{l}\binom{t}{l}p^{-s-l}q^{t-l}.$$

The base of induction is the case $s = 1$, which we just have proved. Now by the inductive assumption

$$q^t p^{-s} = q^t p^{-s+1}\cdot p^{-1} = \sum_{l=0}^{\infty}(-1)^l l!\binom{-s+1}{l}\binom{t}{l}p^{-s+1-l}q^{t-l}p^{-1} =$$

$$\sum_{l=0}^{\infty}(-1)^l l!\binom{-s+1}{l}\binom{t}{l}\sum_{k=0}^{\infty}(-1)^k k!\binom{-1}{k}\binom{t-l}{k}p^{-s-l-k}q^{t-l-k} =$$

$$\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{l+k} l!k!\binom{-s+1}{l}\binom{t}{l}\binom{-1}{k}\binom{t-l}{k}p^{-s-l-k}q^{t-l-k}.$$

Observe that

$$l!k!\binom{t}{l}\binom{t-l}{k} = \frac{t!}{(t-l-k)!} = (l+k)!\binom{t}{l+k}.$$

Now the substitution $m = l + k$ yields

$$q^t p^{-s} = \sum_{l=0}^{\infty}\sum_{m=l}^{\infty}(-1)^m m!\binom{-s+1}{l}\binom{t}{m}\binom{-1}{m-l}p^{-s-m}q^{t-m} =$$

$$\sum_{m=0}^{\infty}(-1)^m m!\binom{t}{m}\left(\sum_{l=0}^{m}\binom{-s+1}{l}(-1)^{m-l}\right)p^{-s-m}q^{t-m}.$$

One can easily see by induction on m that

$$\sum_{l=0}^{m}\binom{-s+1}{l}(-1)^{m-l} = \binom{-s}{m},$$

which completes the proof. $\square$

Note also that the equality $[g(p), q] = g'(p)$ holds for any Laurent polynomial $g \in \mathbb{C}[x, x^{-1}]$ as well.

As well as for the Weyl algebra we rally use (4.6) and (4.7) only in the simple case, where $s = 1$ or $t = 1$.

4.3.1 Zero cyclic and Hochschild homology of $A_{p,p^{-1},q}$

As in subsection 4.1.1, $HC_0(A_{p,p^{-1},q}) = HH_0(A_{p,p^{-1},q}) \cong (A_{p,p^{-1},q})_\natural$. Exactly as for the Weyl algebra, $[p, p^k q^{n+1}] = (n+1)p^k q^n$ for all $k \in \mathbb{Z}, n \in \mathbb{N}$, hence $[A_{p,p^{-1},q}, A_{p,p^{-1},q}] = A_{p,p^{-1},q}$ and, therefore, $HC_0(A_{p,p^{-1},q}) = HH_0(A_{p,p^{-1},q}) = 0$.

4.3.2 First cyclic and Hochschild homology of $A_{p,p^{-1},q}$.

The first cyclic homology group of $A_{p,p^{-1},q}$ is by proposition 2.3.6

$$HC_1(A_{p,p^{-1},q}) \cong H_-(\mathcal{X}^1(R, I))$$

and the complex $\mathcal{X}^1(R, I)$ is of the form

$$\mathcal{X}^1(R, I) : R/I \underset{\bar{b}}{\overset{\natural\, d}{\rightleftarrows}} \Omega^1 R_\natural / \natural(IdR + RdI).$$

Similarly to subsection 4.1.2, one has with the notation

$$\widetilde{\mathcal{S}} := \mathrm{span}(\{\overline{\natural p^{s-1}q^t dp} | s \in \mathbb{Z}, t \in \mathbb{N} \cup \{0\}\} \cup \{\overline{\natural p^s q^{t-1} dq} | s \in \mathbb{Z}, t \in \mathbb{N}\})$$

and

$$\widetilde{\mathcal{T}} := \mathrm{span}\{n_1 \overline{\natural p^{n_1-1} q^{n_2} dp} + n_2 \overline{\natural p^{n_1} q^{n_2-1} dq} | n_1 \in Z, n_2 \in \mathbb{N} \cup \{0\}\}$$

(by convention, the second summand in the definition of $\widetilde{\mathcal{T}}$ is zero for $n_2 = 0$) the equality $\Omega^1 R_\natural / \natural(IdR + RdI) = \widetilde{\mathcal{S}}/\widetilde{\mathcal{T}}$.

Let us now compute $\mathrm{Ker}\, \overline{b}$. Let an element

$$z = \sum_{n_1 \in \mathbb{Z}} \sum_{n_2 \in \mathbb{N} \cup \{0\}} \left(a_{n_1,n_2} \overline{\natural p^{n_1-1} q^{n_2} dp} + c_{n_1,n_2} \overline{\natural p^{n_1} q^{n_2-1} dq} \right)$$

lie in $\mathrm{Ker}\, \overline{b}$ (where by definition $c_{n_1,0} = 0$ for all n_2 and almost all a_{n_1,n_2}, c_{n_1,n_2} are zero). This implies that

$$\sum_{n_1 \in \mathbb{Z}} \sum_{n_2 \in \mathbb{N} \cup \{0\}} (-n_2 a_{n_1,n_2} + n_1 c_{n_1,n_2}) \overline{p^{n_1-1} q^{n_2-1}} = \overline{0}$$

in R/I and it follows that

$$-n_1 a_{n_1,n_2} + n_2 c_{n_1,n_2} = 0 \tag{4.9}$$

for all n_1 from $\mathbb{Z}$, , n_2 from $\mathbb{N} \cup \{0\}$. This yields

$$z = \sum_{n_1 \neq 0} \sum_{n_2} \frac{a_{n_1,n_2}}{n_1} \left(n_1 \overline{\natural p^{n_1-1} q^{n_2} dp} + n_2 \overline{\natural p^{n_1} q^{n_2-1} dq} \right) + a_{0,0} \overline{\natural p^{-1} dp}$$

The double sum lies in $\widetilde{\mathcal{T}}$. It follows that the basis of $\mathrm{Ker}\, \overline{b}$ consists of the single element $\overline{\natural p^{-1} dp}$.

The image $\natural d(R/I)$ is zero due to (4.3), since $R/I = [R/I, R/I]$ and, therefore, $HC_1(A_{p,p^{-1},q}) \cong \mathbb{C}$; the basis of $HC_1(A_{p,p^{-1},q})$ in $\mathcal{X}^1(R, I)$ consists of the homology class of $\overline{\natural p^{-1} dp}$.

Similarly, $HH_1(A_{p,p^{-1},q}) \cong \mathbb{C}$ and the basis of $HH_1(A_{p,p^{-1},q})$ consists of the homology class of $\overline{\natural p^{-1} dp}$.

4.3.3 Second Hochschild homology of $A_{p,p^{-1},q}$

The second Hochschild homology group of $A_{p,p^{-1},q}$ is computed exactly as for the Weyl algebra; $HH_2(A_{p,p^{-1},q}) \cong \mathbb{C}$ with the basis in $gr^2\mathcal{X}(R,I)$ given by the homology class of $\overline{w}$.

4.3.4 Second cyclic homology of $A_{p,p^{-1},q}$

The second cyclic homology group of A^0_θ is by proposition 2.3.6

$$HC_2(A_{p,p^{-1},q}) \cong H_+(\mathcal{X}^2(R,I))$$

and the complex $\mathcal{X}^2(R,I)$ is of the following form:

$$\mathcal{X}^2(R,I) : R/(I^2 + [I,R]) \underset{\overline{b}}{\overset{\natural\, d}{\rightleftarrows}} \Omega^1 R_\natural / \natural I dR.$$

Again, instead of the direct computation of $H_+(\mathcal{X}^2(R,I))$ (involving (4.7) in the general form and very technical), we use the SBI-sequence (2.10). We consider the following segment of it:

$$HC_1(A_{p,p^{-1},q}) \xrightarrow{\widetilde{B}_1} HH_2(A_{p,p^{-1},q}) \xrightarrow{\widetilde{I}_2} HC_2(A_{p,p^{-1},q}) \xrightarrow{\widetilde{S}_2} HC_0(A_{p,p^{-1},q}) = 0. \tag{4.10}$$

Note that $\widetilde{B}_1(HC_1(A_{p,p^{-1},q})) = 0$, since the generator of $HC_1(A_{p,p^{-1},q})$ in $\mathcal{X}^1(R,I)$ computed in subsection 4.3.2 is the homology class of $\overline{\natural p^{-1} dp}$ and since

$$\widetilde{B}_1(h.cl.(\natural p^{-1}dp + \natural R dI)) = h.cl.(\overline{b}(p^{-1}dp) + (I^2 + [I,R])) = 0.$$

Therefore, from exact sequence (4.10) we obtain the exact sequence

$$0 \to HH_2(A_{p,p^{-1},q}) \xrightarrow{\widetilde{I}_2} HC_2(A^0_\theta) \to 0$$

It follows that the second cyclic homology $HC_2(A_{p,p^{-1},q})$ is isomorphic to $HH_2(A_{p,p^{-1},q})$ and that its generator is the homology class of $\widetilde{I}_2(w + (I^2 + [I,R]))$. Thus $HC_2(A_{p,p^{-1},q}) \cong \mathbb{C}$ and its basis in $\mathcal{X}^2(R,I)$ consists of the homology class of $\overline{w}$.

4.3.5 Higher cyclic homology and periodic cyclic homology of $A_{p,p^{-1},q}$

To compute $HC_3(A_{p,p^{-1},q})$, we consider the following segment of sequence (2.10):

$$0 = HH_3(A^0_\theta) \stackrel{\widetilde{I}_3}{\to} HC_3(A_{p,p^{-1},q}) \stackrel{\widetilde{S}_3}{\to} HC_1(A_{p,p^{-1},q}) \stackrel{\widetilde{B}_1}{\to} HH_2(A_{p,p^{-1},q}) \tag{4.11}$$

As we have seen in subsection 4.3.4, $B(HC_1(A^0_\theta)) = 0$ in $HH_2(A^0_\theta)$, hence sequence (4.11) becomes

$$0 \to HC_3(A_{p,p^{-1},q}) \to HC_1(A_{p,p^{-1},q}) \to 0$$

and it follows that $HC_3(A_{p,p^{-1},q}) \cong HC_1(A_{p,p^{-1},q}) \cong \mathbb{C}$.

Now the results of section 3.5 yield that

$$HH_n(A_{p,p^{-1},q}) = 0$$

for all $n > 2$,

$$HC_n(A_{p,p^{-1},q}) \cong \mathbb{C}$$

for any $n \geq 2$ and

$$HP_*(A_{p,p^{-1},q}) \cong \mathbb{C}$$

for $* = 0, 1$.

One could get the impression that for a one-relator algebra A $HC_3(A) \cong HC_1(A)$ is always true. In fact, it is not so. To see this, we consider now one more example.

4.4 Cyclic and Hochschild homology of the algebra of Laurent polynomials in two variables

Definition 4.4.1 *The algebra A_L consists of commutative complex Laurent polynomials in two variables u and v.*

A_L is a one-relator algebra. To see that, we take the following extension of A_L: $R_L = \mathbb{C}(F_2)$ is the complex group algebra of the free group $F_2 = F(u, v)$ and I is the ideal generated by $w_L = [v, u]$.

Then $A_L \cong R/I$ and R_L is a mixed free (in particular, quasi-free) extension of A_L.

The extension R_L and the element w_L satisfy the conditions of the identity theorem 3.1.4. It can be seen with the help of an argument similar to the one used for the algebra A_θ^0.

Further in this section we usually omit the subscript L and write R for R_L and w for w_L.

4.4.1 Zero cyclic and Hochschild homology of A_L

As in subsection 4.1.1, $HC_0(A_L) = HH_0(A_L) \cong (A_L)_\natural$. Since A_L is commutative, $[A_L, A_L] = 0$ and $(A_L)_\natural = A_L$, and, therefore, $HC_0(A_L) = HH_0(A_L) = \mathbb{C}^\infty$ and its basis in $\mathcal{X}^0(R, I)$ consists of the homology classes of all monomials $\overline{u^{n_1}v^{n_2}}$ $(n_1, n_2 \in \mathbb{Z})$.

4.4.2 First cyclic homology of A_L

The first cyclic homology group of A_L is by proposition 2.3.6

$$HC_1(A_L) \cong H_-(\mathcal{X}^1(R, I))$$

and the complex $\mathcal{X}^1(R, I)$ is of the form

$$\mathcal{X}^1(R, I) : R/I \underset{\overline{b}}{\overset{\natural\, d}{\rightleftarrows}} \Omega^1 R_\natural / \natural(IdR + RdI).$$

As well as for the algebra A_θ^0, with the help of equality (4.1) one can rewrite each element from $(\Omega^1 R)_\natural$ as a linear combination of the following elements:

$$\{\natural r du,\ \natural r dv |\ r \text{ is a monomial in } u, v\}.$$

In the quotient $\Omega^1 R_\natural / \natural(IdR + RdI)$, we can always take r in $\overline{\natural r du}$ and $\overline{\natural r dv}$ of the form $u^{n_1}v^{n_2}$ (since we factorize by $\natural(IdR)$). Now note that in fact in our case $\natural RdI$ lies in $\natural IdR$ (and is thus equal to it), since

$$\natural ad(bwc) = \natural abwdc + \natural cabdw + \natural wcadb = \natural abwdc + \natural wcadb + \natural[cab,u]dv + \natural[v,cab]du$$

and since the commutator of an arbitrary element of R with u or v obviously lies in I. Thus the left-hand side of $\mathcal{X}^1(R,I)$ is equal to $\Omega^1 R_\natural/\natural(IdR)$ and its basis is given by

$$\{\overline{\natural u^{n_1-1}v^{n_2}du}, \overline{\natural u^{n_1}v^{n_2-1}dv} | n_1, n_2 \in \mathbb{Z}\}.$$

The map $\overline{b}$ is zero on the whole $\Omega^1 R_\natural/\natural(IdR)$, since R/I is commutative.

We introduce now a new notation:

$$\sum_{i\in[n_1,n_2]} = \begin{cases} \sum\limits_{i=n_1}^{n_2-1}, & n_1 < n_2; \\ 0, & n_1 = n_2; \\ -\sum\limits_{i=n_2}^{n_1-1}, & n_1 > n_2. \end{cases} \tag{4.12}$$

To compute $\operatorname{Im}\natural d$, we prove first the following

Lemma 4.4.2 *In $\Omega^1 R_\natural/\natural(IdR)$ the equality*

$$\overline{\natural d(u^{n_1}v^{n_2})} = n_1\overline{\natural u^{n_1-1}v^{n_2}du} + n_2\overline{\natural u^{n_1}v^{n_2-1}dv}$$

holds for all $n_1, n_2 \in \mathbb{Z}$.

Proof. We compute two following sums of commutators (which, as all commutators, represent zero in $\Omega^1 R_\natural/\natural(IdR)$):

$$\overline{0} = \sum_{k_2\in[0,n_2]} \overline{\natural[v^{k_2}d(u^{n_1}v^{n_2-1-k_2}), v]} = \sum_{k_2\in[0,n_2]} \Big(\overline{\natural v^{k_2}d(u^{n_1}v^{n_2-k_2})} - \overline{\natural u^{n_1}v^{n_2-1}dv} - \overline{\natural v^{(k_2+1)}d(u^{n_1}v^{n_2-1-k_2})}\Big) =$$

$$- \sum_{k_2 \in [0,n_2]} \big(u^{n_1} v^{n_2-1} dv + v^{k_2} d(u^{n_1} v^{n_2-k_2})\big) -$$

$$\sum_{l_2 \in [1,n_2+1]} v^{l_2} d(u^{n_1} v^{n_2-l_2}) =$$

$$- \sum_{k_2 \in [0,n_2]} \overline{\natural u^{n_1} v^{n_2-1} dv} + \overline{\natural d(u^{n_1} v^{n_2})} - \overline{\natural v^{n_2} du^{n_1}} =$$

$$- n_2 \overline{\natural u^{n_1} v^{n_2-1} dv} + \overline{\natural d(u^{n_1} v^{n_2})} - \overline{\natural v^{n_2} du^{n_1}};$$

similarly,

$$\overline{0} = \sum_{k_1 \in [0,n_1]} \overline{\natural [u^{k_1} v^{n_2} du^{n_1-1-k_1}, u]} =$$

$$- n_1 \overline{\natural u^{n_1-1} v^{n_2} du} + \overline{\natural v^{n_2} du^{n_1}}.$$

By summing these two representations for $\overline{0}$ we obtain the desired equality. $\square$

It follows from the lemma that $\operatorname{Im} \natural d$ consists of the vectors

$$\{\overline{\natural u^{n_1-1} du}, \overline{\natural v^{n_2-1} dv}, (n_1 \overline{\natural u^{n_1-1} v^{n_2} du} + n_2 \overline{\natural u^{n_1} v^{n_2-1} dv}) | n_1, n_2 \neq 0\}.$$

Therefore, $HC_1(A_L) = \operatorname{Ker} \overline{b} / \operatorname{Im} \natural d \cong \mathbb{C}^\infty$; the basis of $HC_1(A_L)$ in $\mathcal{X}^1(R,I)$ consists of the homology classes of $\overline{\natural u^{-1} du}$, $\overline{\natural v^{-1} dv}$ and $\overline{\natural u^{n_1-1} v^{n_2} du}$ $(n_1, n_2 \neq 0)$.

4.4.3 First Hochschild homology of A_L

The first Hochschild homology group of A_L is by proposition 2.3.6

$$HH_1(A_L) \cong H_-(gr^1 \mathcal{X}(R,I))$$

and the complex $gr^1 \mathcal{X}(R,I)$ is of the form

$$gr^1 \mathcal{X}(R,I) : [R,R] + I/I \underset{\overline{b}}{\overset{\natural\, d}{\rightleftarrows}} \Omega^1 R_\natural / \natural(IdR + RdI).$$

As well as in the previous subsection,

$$\operatorname{Ker}\overline{b} = \operatorname{span}\{\overline{\natural u^{n_1-1}v^{n_2}du}, \overline{\natural u^{n_1}v^{n_2-1}dv} | n_1, n_2 \in \mathbb{Z}\}.$$

On the other hand, $\operatorname{Im}\natural d = 0$, since $[R,R] + I \cong [R,R]/[R,R] \cap I$ and since $\natural d|_{[R,R]} = 0$ by (4.3). Thus $HH_1(A_L) \cong \mathbb{C}^\infty$. The basis of $HH_1(A_L)$ in $gr^1\mathcal{X}(R,I)$ consists of the homology classes of all $\overline{\natural u^{n_1-1}v^{n_2}du}$, $\overline{\natural u^{n_1}v^{n_2-1}dv}$, $n_1, n_2 \in \mathbb{Z}$.

Another possibility to compute $HC_1(A_L)$ were to compute first $HH_1(A_L)$ and to use then the SBI-sequence (2.10), but it makes in fact not much difference, since the operator $\widetilde{B}_0$ is essentially $\natural d$.

4.4.4 Second Hochschild homology of A_L

The second Hochschild homology group of A_L is by proposition 2.3.6

$$HH_2(A_L) \cong H_+(gr^2\mathcal{X}(R,I))$$

and the complex $gr^2\mathcal{X}(R,I)$ is of the form:

$$gr^2\mathcal{X}(R,I) : I/(I^2 + [I,R]) \underset{\overline{b}}{\overset{\natural\, d}{\rightleftarrows}} \natural(IdR + RdI)/\natural IdR.$$

As in subsection 4.1.4, we can identify $I/(I^2 + [I,R])$ with A_L by

$$bwc + (I^2 + [I,R]) \mapsto \overline{cb}$$

(where $\overline{x}$ denotes the image of an element $x \in R$ under the quotient map $R \to R/I (\cong A_L)$ and $w = w_L$ is the generator of the principal ideal I). The left-hand side of the complex $gr^2\mathcal{X}(R,I)$ is zero, since we have seen in subsection 4.4.2 that $\natural(RdI) = \natural(IdR)$. Therefore, $HH_2(A_L) \cong \mathbb{C}^\infty$ and its basis in $gr^2\mathcal{X}(R,I)$ is given by the homology classes of all elements $\overline{u^{n_1}v^{n_2}w}$, $n_1, n_2 \in \mathbb{Z}$.

4.4.5 Second cyclic homology of A_L

The second cyclic homology group of A_L is by proposition 2.3.6

$$HC_2(A_L) \cong H_+(\mathcal{X}^2(R,I))$$

and the complex $\mathcal{X}^2(R,I)$ is of the following form:

$$\mathcal{X}^2(R,I) : R/(I^2 + [I,R]) \underset{\bar{b}}{\overset{\natural\, d}{\rightleftarrows}} \Omega^1 R_\natural / \natural I dR.$$

We will not compute $H_+(\mathcal{X}^2(R,I))$ directly, but use the SBI-sequence (2.10). We consider the following segment of the SBI-sequence:

$$HC_1(A_L) \overset{\widetilde{B}_1}{\to} HH_2(A_L) \overset{\widetilde{I}_2}{\to} HC_2(A_L) \overset{\widetilde{S}_2}{\to} HC_0(A_L) \overset{\widetilde{B}_0}{\to} HH_1(A_L). \tag{4.13}$$

From it follows the exact sequence

$$0 \to HH_2(A_L)/\widetilde{B}_1(HC_1(A_L)) \to HC_2(A_L) \to \mathrm{Ker}\, \widetilde{B}_0 \to 0$$

and the second cyclic homology $HC_2(A_L)$ is thus a direct sum of $HH_2(A_L)/\widetilde{B}_1(HC_1(A_L))$ and $\mathrm{Ker}\, \widetilde{B}_0$.

For the computation of $\widetilde{B}_1(HC_1(A_L))$ we need the following technical

Lemma 4.4.3 *In R the equality*

$$v^n u - u v^n = \sum_{l \in [0,n]} v^{n-1-l} w v^l$$

(where $w = [v,u]$ is the generator of the principal ideal I) holds for all $n \in \mathbb{Z}$.

Proof. We prove the claim first for $n > 0$ by induction on n. For $n = 1$ we have $vu - uv = w$; for $n > 1$ we obtain by the inductive assumption

$$v^n u - uv^n = v(v^{n-1}u - uv^{n-1}) + vuv^{n-1} - uv^n = \\ v \sum_{l\in[0,n-1]} v^{n-2-l}wv^l + (vu-uv)v^{n-1} = \sum_{l\in[0,n]} v^{n-1-l}wv^l.$$

Now we prove the claim for $n=-m$, $m>0$ by induction on m. For $m=1$ one has

$$v^{-1}u - uv^{-1} = v^{-1}(-vu+uv)v^{-1} = \\ -v^{-1}wv^{-1} = \sum_{l\in[0,-1]} v^{-1-1-l}wv^l;$$

for $m>1$ by the inductive assumption one has

$$v^{-m}u - uv^{-m} = \\ v^{-1}(v^{-m+1}u - uv^{-m+1}) + v^{-1}uv^{-m+1} - uv^{-m} = \\ v^{-1} \sum_{l\in[0,-m+1]} v^{-m-l}wv^l + (v^{-1}u - uv^{-1})v^{-m+1} = \\ \sum_{l\in[0,-m+1]} v^{-m-1-l}wv^l - v^{-1}wv^{-m} = \sum_{l\in[0,-m]} v^{-m-1-l}wv^l. \qquad \square$$

We evaluate now $\widetilde{B}_1$ on the generators of $HC_1(A_L)$ computed in subsection 4.4.2:

$$\widetilde{B}_1(\natural u^{n_1-1}v^{n_2}du + \natural(IdR+RdI)) = \\ \overline{b}(\natural u^{n_1-1}v^{n_2}du) + (I^2+[I,R]) = u^{n_1-1}[v^{n_2},u] + (I^2+[I,R]) = \\ u^{n_1-1} \sum_{l\in[0,n_2]} v^{n_2-1-l}wv^l + (I^2+[I,R]) = \\ \sum_{l\in[0,n_2]} v^l u^{n_1-1}v^{n_2-1-l}w + (I^2+[I,R]) = \\ \sum_{l\in[0,n_2]} u^{n_1-1}v^{n_2-1}w + (I^2+[I,R]) = n_2 u^{n_1-1}v^{n_2-1}w + (I^2+[I,R]),$$

$$\widetilde{B}_1(\natural u^{-1}du + \natural(IdR + RdI) = 0,$$
$$\widetilde{B}_1(\natural v^{-1}dv + \natural(IdR + RdI) = 0.$$

Therefore, $\widetilde{B}_1(HC_1(A_L) = \text{span}\{\overline{u^{n_1}v^{n_2}w} | n_1, n_2 \neq -1\}$, whence $HH_1(A_L)/\widetilde{B}_1(HC_1(A_L) \cong \mathbb{C}$ and its basis consists of the homology class of $u^{-1}v^{-1}w + (I^2 + [I, R])$.

Compute now Ker $\widetilde{B}_0$. Let

$$0 = \widetilde{B}_0(\sum_{n_1,n_2\in\mathbb{Z}} a_{n_1,n_2} u^{n_1} v^{n_2} + (I + [R, R])) =$$
$$\sum_{n_1,n_2\in\mathbb{Z}} a_{n_1,n_2} \natural d(u^{n_1} v^{n_2}) + \natural(RdI) =$$
$$\sum_{n_1,n_2\in\mathbb{Z}} a_{n_1,n_2}(n_1 \natural u^{n_1-1} v^{n_2} du + n_2 \natural u^{n_1} v^{n_2-1} dv) + \natural(RdI).$$

It follows that $a_{n_1,n_2} = 0$ for $(n_1, n_2) \neq (0, 0)$ and that $a_{0,0}$ is arbitrary, hence Ker $\widetilde{B}_0 \cong \mathbb{C}$ and its basis consists of the homology class of $1 + (I + [R, R])$.

Now the generators of $HC_2(A_L)$ are the homology classes of $\widetilde{I}_2(u^{-1}v^{-1}w + (I^2 + [I, R]))$ and of the pre-image of $1 + (I + [R, R])$ under $\widetilde{S}_2$. So we conclude that $HC_2(A_L) \cong \mathbb{C}^2$ and its basis in $\mathcal{X}^2(R, I)$ consists of the homology classes of $\overline{u^{-1}v^{-1}w}$ and $\overline{1}$.

4.4.6 Higher cyclic homology and periodic cyclic homology of A_L

To compute $HC_3(A_L)$, we consider the following segment of sequence (2.10):

$$0 = HH_3(A_L) \xrightarrow{\widetilde{I}_3} HC_3(A_L) \xrightarrow{\widetilde{S}_3} HC_1(A_L) \xrightarrow{\widetilde{B}_1} HH_2(A_L) \qquad (4.14)$$

From this follows that $HC_3(A_L) \cong \text{Ker}\,\widetilde{B}_1$.

Compute Ker $\widetilde{B}_1$. Let

$$0 = \widetilde{B}_1\Big(\sum_{n_1,n_2\neq 0} a_{n_1,n_2}\natural u^{n_1-1}v^{n_2}du + a\natural u^{-1}du + c\natural v^{-1}dv + \natural(RdI)\Big) = \sum_{n_1,n_2\neq 0} a_{n_1,n_2} n_2 u^{n_1}v^{n_2}w + (I^2+[I,R]),$$

it follows that $a_{n_1,n_2} = 0$ for all $n_1, n_2 \neq 0$ and that a and c can be arbitrary. Therefore, $\operatorname{Ker}\widetilde{B}_1 \cong \mathbb{C}^2$ and its basis consists of the homology classes of $\overline{\natural u^{-1}du}$ and $\overline{\natural v^{-1}dv}$, from which we conclude that $HC_3(A_L) \cong \mathbb{C}^2$ and that its basis consists of the homology classes of $\natural u^{-1}du + \natural(I^2dR + IdI)$ and $\natural v^{-1}dv + \natural(I^2dR + IdI)$ (which are the pre-images of the homology classes of $\overline{\natural u^{-1}du}$ and $\overline{\natural v^{-1}dv} \in HC_1(A_L)$ under $\widetilde{S}_3$).

Now the results of section 3.5 yield that

$$HH_n(A_L) = 0$$

for all $n > 2$, that

$$HC_n(A_L) \cong \mathbb{C}^2$$

for any $n \geq 2$ and that

$$HP_*(A_L) \cong \mathbb{C}^2$$

for $* = 0, 1$.

We remark here also that in the case of the *rational* (algebraic) rotation algebra A^0_θ (for $\theta = \frac{k}{m}$ rational) one has the similar situation: $HC_1(A^0_{\frac{k}{m}})$ is infinitely-dimensional and $HC_3(A^0_{\frac{k}{m}})$ is two-dimensional; the computations are analogue.

Chapter 5

Generators in the complex $(\Omega, b + B)$

We computed for $A = A^0_\theta, A_{p,q}, A_{p,p^{-1},q}$ and $i = 0, 1, 2$ the basis of $HC_i(A)$ in the tower $\mathcal{X}(R, I)$. To compute the basis of those homology groups in the tower $\theta\Omega A$, we follow the detailed description of the homotopy equivalence $\mathcal{X}(R, I) \sim \mathcal{X}(RA, IA)$ given in section 2.3 and of the homotopy equivalence $\mathcal{X}(RA, IA) \sim \theta\Omega A$ given in section 2.2.

5.1 Generators of $HC_i(A_{p,q})$ in the complex $(\Omega, b + B)$

In this case both extensions $RA_{p,q}$ and $R = R_{p,q}$ of $A_{p,q}$ are free, hence there exist the following homomorphisms:

$$\begin{array}{llcl} f: & R & \longrightarrow & RA_{p,q} \\ & v_1 \dots v_k & \mapsto & v_1 \otimes \dots \otimes v_k, \end{array}$$

where $v_i = p$ or q, and

$$\begin{array}{llcl} g: & RA^0_\theta & \longrightarrow & R \\ & p^{n_1}q^{m_1} \otimes \dots \otimes p^{n_k}q^{m_k} & \mapsto & p^{n_1}v^{m_1} \dots p^{n_k}q^{m_k}. \end{array}$$

We have $f(I) \subseteq IA_{p,q}$, $g(IA_{p,q}) \subseteq I$. It is true that $gf = id_R$ and, therefore, $g_*f_* = id_{\mathcal{X}(R,I)}$. The map $fg : ((RA_{p,q})_n) \to ((RA_{p,q})_n)$ is a lifting of the identity map $A^0_\theta \to A^0_\theta$, thus by theorem 2.3.9 we have $f_*g_* \sim id_{\mathcal{X}(RA^0_\theta, IA^0_\theta)}$. Therefore, $f_* : \mathcal{X}(R, I) \to$

$\mathcal{X}(RA_{p,q}, IA_{p,q})$ is a homotopy equivalence. It implies that the homology class of $\overline{p \otimes q - q \otimes p - 1} = f_*(\overline{w_{p,q}})$ forms the basis of $HC_2(A_{p,q})$ in $\mathcal{X}^2(RA_{p,q}, IA_{p,q})$. Now under the identification $\mathcal{X}(RA_{p,q}, IA_{p,q}) \leftrightarrow X = (\Omega, \beta \oplus \delta)$ described in the section 2.2, the element $\overline{p \otimes q - q \otimes p - 1}$ of $\mathcal{X}^2(RA_{p,q}, IA_{p,q})$ corresponds to the element $\overline{dqdp - dpdq}$ of $X/F^2(X)$, since

$$p \otimes q - q \otimes p - 1 = -\hat{\omega}(p,q) + \hat{\rho}(pq) + \hat{\omega}(q,p) - \hat{\rho}(qp) - 1 = \\ \hat{\omega}(q,p) - \hat{\omega}(p,q) - \hat{\rho}(pq - qp - 1) = \hat{\omega}(q,p) - \hat{\omega}(p,q),$$

since $\hat{\omega}(q,p) - \hat{\omega}(p,q)$ corresponds to $dqdp - dpdq$, and since the IA-adic filtration corresponds to the Hodge filtration under the mentioned identification.

Thus the homology class of $\overline{dqdp - dpdq}$ forms the basis of $HC_2(A_{p,q})$ in $X/F^2(X)$.

Let now $x = dqdp - dpdq$. We represent x in the form $x = Px + y$, $y = P^{\perp}x$, where P is the spectral projection corresponding to the eigenvalue 1 of the Karoubi operator. The homology class of y in (X/F^2X) as well as in $\Omega A^0_\theta/F^2(\Omega A^0_\theta)$ is zero because the complexes $P^{\perp}X/F^2P^{\perp}X$ and $P^{\perp}\Omega A_{p,q}/F^2(P^{\perp}\Omega A_{p,q})$ are contractible by proposition 2.2.3. The homology class of $c_2\overline{Px}$ forms the basis von $HC_2(A^0_\theta)$ in $\Omega A_{p,q}/F^2(\Omega A_{p,q})$, also by proposition 2.2.3. But if we add to $c_2\overline{Px}$ the element $c_2\overline{y}$, we do not change its homology class. It follows that the homology class of $c_2\overline{x}$ forms the basis of $HC_2(A_{p,q})$ in $\Omega A_{p,q}/F^2(\Omega A_{p,q})$. The constant c_2 can be omitted, therefore, the generator of $HC_2(A_{p,q}) \cong \mathbb{C}$ (as well as of $HH_2(A_{p,q}) \cong \mathbb{C}$) in $(\Omega A_{p,q}, B + b)$ is given by the homology class of $dqdp - dpdq$.

5.2 Generators of $HC_i(A^0_\theta)$ in the complex $(\Omega, b + B)$

Now we recover the generators of $HC_i(A^0_\theta)$ $(i = 0, 1, 2)$ in the usual Ω-complex. The procedure here is more complicated than for $A_{p,q}$, since in this case $R = R_\theta$ is not free and, therefore, there is no homomorphism from R into RA_θ^0. So we have to work with a homomorphism of towers of algebras $(R_n) \to ((R_n A_\theta^0))$ (where $R_n = R/I^{n+1}$ and $R_n A = RA/(IA)^{n+1}$).

Define

$$\varphi^n : R \to RA_\theta^0/(IA_\theta^0)^{n+1}$$

by setting it on the generators as follows:

$$\varphi^n(u) = u + (IA_\theta^0)^{n+1},$$

$$\varphi^n(u^{-1}) = \sum_{k=1}^{n+1} \binom{n+1}{k} (-1)^{k-1} u^{-1} \otimes (u \otimes u^{-1})^{\otimes(k-1)} + (IA_\theta^0)^{n+1},$$

$$\varphi^n(v) = v + (IA_\theta^0)^{n+1},$$

$$\varphi^n(v^{-1}) = \sum_{k=1}^{n+1} \binom{n+1}{k} (-1)^{k-1} v^{-1} \otimes (v \otimes v^{-1})^{\otimes(k-1)} + (IA_\theta^0)^{n+1}.$$

This really defines a homomorphism, since

$$\varphi^n(u)\varphi^n(u^{-1}) - 1 = \sum_{k=1}^{n+1} \binom{n+1}{k} (-1)^{k-1} (u \otimes u^{-1})^{\otimes k} - 1 + (IA_\theta^0)^{n+1} =$$

$$- \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{k} (u \otimes u^{-1})^{\otimes k} + (IA_\theta^0)^{n+1} =$$

$$- (1 - u \otimes u^{-1})^{\otimes(n+1)} + (IA_\theta^0)^{n+1} = 0 + (IA_\theta^0)^{n+1},$$

since $(1 - u \otimes u^{-1})^{\otimes(n+1)} \in (IA_\theta^0)^{n+1}$ and, similarly, $\varphi^n(u^{-1})\varphi^n(u) = \varphi^n(v)\varphi^n(v^{-1}) = \varphi^n(v^{-1})\varphi^n(v) = 1 + (IA_\theta^0)^{n+1}$.

Check that the system (φ^n) is compatible, i.e. that $\pi^n \varphi^n = \varphi^{n-1}$, where $\pi^n : RA_\theta^0/(IA_\theta^0)^{n+1} \to RA_\theta^0/(IA_\theta^0)^n$ is the canonical surjection.

$\pi^n\varphi^n(u) = \varphi^{n-1}(u)$ is obvious.

$$(\pi^n\varphi^n - \varphi^{n-1})(u^{-1}) = \sum_{k=1}^{n+1}\binom{n+1}{k}(-1)^{k-1}u^{-1}\otimes(u\otimes u^{-1})^{\otimes(k-1)} -$$
$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}u^{-1}\otimes(u\otimes u^{-1})^{\otimes(k-1)} + (IA_\theta^0)^n =$$
$$(-1)^n u^{-1}\otimes(u\otimes u^{-1})^{\otimes n} +$$
$$\sum_{k=1}^{n}(-1)^{k-1}\left(\binom{n+1}{k} - \binom{n}{k}\right)u^{-1}\otimes(u\otimes u^{-1})^{\otimes(k-1)} + (IA_\theta^0)^n =$$
$$u^{-1}\otimes\sum_{l=0}^{n}(-1)^l\binom{n}{l}(u\otimes u^{-1})^{\otimes l} + (IA_\theta^0)^n =$$
$$u^{-1}\otimes(1-u\otimes u^{-1})^{\otimes n} + (IA_\theta^0)^n = 0,$$

since $(1-u\otimes u^{-1})^{\otimes n}\in(IA_\theta^0)^n$. Similarly for v, hence the system (φ^n) is really compatible. $\varphi^0(I) = 0$ is obvious.

The system of homomorphisms (φ^n) defines a homomorphism of towers $\varphi : (R_n) \to (R_nA_\theta^0)$ by lemma 2.3.10, which lifts the identity map $R_0 = A_\theta^0 \xrightarrow{=} A_\theta^0 = R_0A_\theta^0$.

The homomorphism of algebras

$$\begin{array}{rccc} \psi' : & RA_\theta^0 & \to & R \\ & u^{n_1}v^{m_1}\otimes\ldots\otimes u^{n_l}v^{m_l} & \mapsto & u^{n_1}v^{m_1}\ldots u^{n_l}v^{m_l} \end{array}$$

induces also a homomorphism of towers $\psi : (R_nA_\theta^0) \to (R_n)$ and the compositions $\psi\circ\varphi$ and $\varphi\circ\psi$ both lift the identity map $A_\theta^0 \xrightarrow{=} A_\theta^0$. We conclude by theorem 2.3.9 that for the induced morphisms of towers of supercomplexes one has in the category $Ho\mathcal{T}$ the equalities $\varphi_*\circ\psi_* = id_{\mathcal{X}(RA_\theta^0, IA_\theta^0)}$ and $\psi_*\circ\varphi_* = id_{\mathcal{X}(R,I)}$, hence the induced map $\varphi_* : \mathcal{X}(R,I) \to \mathcal{X}(RA_\theta^0, IA_\theta^0)$ is an isomorphism in $Ho\mathcal{T}$.

For any $k \le 2n+1$ consider the sequence of maps

$$\mathcal{X}^k(R,I) \xrightarrow{\varphi_*^n} \mathcal{X}^k(R_nA_\theta^0, I_nA_\theta^0) \xrightarrow{\cong} \mathcal{X}^k(RA_\theta^0, IA_\theta^0), \qquad (5.1)$$

where the second one is the inverse of the canonical projection

$\mathcal{X}^k(RA^0_\theta, IA^0_\theta) \to \mathcal{X}^k(R_nA^0_\theta, I_nA^0_\theta)$, which is an isomorphism by lemma 2.3.7.

Their composition is exactly the k-th level of the isomorphism φ_*. Now we see that, in order to find the generators of $HC_k(A^0_\theta)$ in $\mathcal{X}^k(RA^0_\theta, IA^0_\theta)$, one needs only to find a suitable number n, to write down sequence (5.1) for it and to compute what becomes of the generators of $HC_k(A^0_\theta)$ in $\mathcal{X}^k(R, I)$ under this sequence of maps.

One has for $k = n = 0$

$$\begin{array}{ccccc} \mathcal{X}^0(R,I) & \xrightarrow{\varphi^0_*} & \mathcal{X}^0(R_0A^0_\theta, I_0A^0_\theta) & \xrightarrow{\cong} & \mathcal{X}^0(RA^0_\theta, IA^0_\theta) \\ \overline{1} & \mapsto & \overline{1 + IA^0_\theta} & \mapsto & \overline{1}, \end{array}$$

whence the generator of $HC_0(A^0_\theta)$ in $\mathcal{X}^0(RA^0_\theta, IA^0_\theta)$ is the homology class of $\overline{1}$. Under the identification $\mathcal{X}(RA^0_\theta) \leftrightarrow X = (\Omega, \beta \oplus \delta)$ described in section 2.2, we see that the homology class of $\overline{1}$ forms the basis of $HC_0(A^0_\theta)$ in $X/F^0(X)$. By proposition 2.2.3, since the addition (or subtraction) of $P^\perp x$ to (from) an arbitrary element x and scaling do not change its homology class, the homology class of 1 forms the basis of $HC_0(A^0_\theta)$ in $(\Omega A^0_\theta, B + b)$.

One has for $k = 1$, $n = 0$

$$\begin{array}{ccccc} \mathcal{X}^1(R,I) & \xrightarrow{\varphi^0_*} & \mathcal{X}^1(R_0A^0_\theta, I_0A^0_\theta) & \xrightarrow{\cong} & \mathcal{X}^1(RA^0_\theta, IA^0_\theta) \\ \overline{\natural u^{-1}du} & \mapsto & \overline{\natural(u^{-1} + IA^0_\theta)\delta(u + IA^0_\theta)} & \mapsto & \overline{\natural u^{-1}\delta u} \\ \overline{\natural v^{-1}dv} & \mapsto & \overline{\natural(v^{-1} + IA^0_\theta)\delta(v + IA^0_\theta)} & \mapsto & \overline{\natural v^{-1}\delta v}, \end{array}$$

hence the basis of $HC_1(A^0_\theta)$ in $\mathcal{X}^1(RA^0_\theta, IA^0_\theta)$ consists of the homology classes of $\overline{\natural u^{-1}\delta u}$ and $\overline{\natural v^{-1}\delta v}$. It follows that the basis of $HC_1(A^0_\theta)$ in $X/F^1(X)$ is given by the homology classes of $\overline{u^{-1}du}$ and $\overline{v^{-1}dv}$. By proposition 2.2.3, since the addition (or subtraction) of $P^\perp x$ to (from) an arbitrary element x does not change its homology class and since the scaling constant does not change it either, the homology classes of $u^{-1}du$ and $v^{-1}dv$ form the basis of $HC_1(A^0_\theta)$ in $(\Omega A^0_\theta, B + b)$.

One has for $k = 2$, $n = 1$

$$\begin{array}{ccccc} \mathcal{X}^2(R,I) & \xrightarrow{\varphi^1_*} & \mathcal{X}^2(R_1A^0_\theta, I_1A^0_\theta) & \xrightarrow{\cong} & \mathcal{X}^2(RA^0_\theta, IA^0_\theta) \\ \overline{1} & \mapsto & \overline{1} + (IA^0_\theta)^2 & \mapsto & \overline{1} \end{array},$$

$$\varphi^1_*(\overline{u^{-1}v^{-1}w}) =$$
$$\overline{(2u^{-1} - u^{-1} \otimes u \otimes u^{-1}) \otimes (2v^{-1} - v^{-1} \otimes v \otimes v^{-1}) \otimes w' + (IA^0_\theta)^2}$$

(where $w' = v \otimes u - \lambda u \otimes v$) and the homology class of it is taken by the second map into the homology class of $\overline{(2u^{-1} - u^{-1} \otimes u \otimes u^{-1}) \otimes (2v^{-1} - v^{-1} \otimes v \otimes v^{-1}) \otimes w'}$, which is, therefore, the generator of $HC_2(A^0_\theta)$ in $\mathcal{X}^2(RA^0_\theta, IA^0_\theta)$. To find the image of it in $X/F^2(X)$, we need to rewrite it in the form with $\hat{\rho}$ and $\hat{\omega}$. First, in $\mathcal{X}^2(RA^0_\theta, IA^0_\theta)$

$$\overline{w'} = \overline{v \otimes u - \lambda u \otimes v} = \overline{\hat{\rho}(vu) - \hat{\omega}(v,u) - \lambda\hat{\rho}(uv) + \lambda\hat{\omega}(u,v)} =$$
$$\overline{\lambda\hat{\omega}(u,v) - \hat{\omega}(v,u)} \in IA^0_\theta.$$

Further, in the same complex

$$\overline{u^{-1} \otimes v^{-1} \otimes w'} = \overline{(\hat{\rho}(u^{-1}v^{-1}) - \hat{\omega}(u^{-1}, v^{-1})) \otimes w'} =$$
$$\overline{\hat{\rho}(u^{-1}v^{-1}) \otimes (\lambda\hat{\omega}(u,v) - \hat{\omega}(v,u))},$$

since $\hat{\omega}(u^{-1}, v^{-1}) \otimes w'$ lies in $(IA^0_\theta)^2$. Similarly,

$$\overline{u^{-1} \otimes u \otimes u^{-1} \otimes v^{-1} \otimes w'} =$$
$$\overline{(\hat{\rho}(1) - \hat{\omega}(u^{-1}, u)) \otimes (\hat{\rho}(u^{-1}v^{-1}) - \hat{\omega}(u^{-1}, v^{-1})) \otimes w'} =$$
$$\overline{\hat{\rho}(u^{-1}v^{-1}) \otimes (\lambda\hat{\omega}(u,v) - \hat{\omega}(v,u))},$$

also similarly,

$$\overline{u^{-1} \otimes v^{-1} \otimes v \otimes v^{-1} \otimes w'} \quad = \quad \overline{\hat{\rho}(u^{-1}v^{-1}) \otimes (\lambda\hat{\omega}(u,v) - \hat{\omega}(v,u))},$$

and also similarly,

$$\overline{u^{-1} \otimes u \otimes u^{-1} \otimes v^{-1} \otimes v \otimes v^{-1} \otimes w'} =$$
$$\overline{\hat{\rho}(u^{-1}v^{-1}) \otimes (\lambda\hat{\omega}(u,v) - \hat{\omega}(v,u))}.$$

Therefore,

$$\overline{(2u^{-1} - u^{-1} \otimes u \otimes u^{-1}) \otimes (2v^{-1} - v^{-1} \otimes v \otimes v^{-1}) \otimes w'} =$$
$$\overline{4u^{-1} \otimes v \otimes w'} - 2\overline{u^{-1} \otimes u \otimes u^{-1} \otimes v \otimes w'} -$$
$$2\overline{u^{-1} \otimes v^{-1} \otimes v \otimes v^{-1} \otimes w'} +$$
$$\overline{u^{-1} \otimes u \otimes u^{-1} \otimes v^{-1} \otimes v \otimes v^{-1} \otimes w'} =$$
$$\overline{\hat{\rho}(u^{-1}v^{-1}) \otimes (\lambda\hat{\omega}(u,v) - \hat{\omega}(v,u))}.$$

It follows that the generators of $HC_2(A^0_\theta)$ in $X/F^2(X)$ are the homology classes of $\overline{1}$ and of $\overline{\lambda u^{-1}v^{-1}dudv - u^{-1}v^{-1}dvdu}$. By proposition 2.2.3, since the addition (or subtraction) of $P^\perp x$ to (from) an arbitrary element x does not change its homology class and since the scaling constant can be omitted, the basis of $HC_2(A^0_\theta)$ in $(\Omega A^0_\theta, B{+}b)$ is given by the classes of 1 and of $\lambda u^{-1}v^{-1}dudv - u^{-1}v^{-1}dvdu$.

5.3 Generators of $HC_i(A_{p,p^{-1},q})$ in the complex $(\Omega, b+B)$

To recover the generators of $HC_i(A_{p,p^{-1},q})$ $(i = 0,1,2)$ in the Ω-complex, we proceed similarly as for the algebra A^0_θ.

Define

$$\varphi^n : R \to RA_{p,p^{-1},q}/(IA_{p,p^{-1},q})^{n+1}$$

by setting it on the generators as follows:

$$\varphi^n(p) = p + (IA_{p,p^{-1},q})^{n+1},$$

$$\varphi^n(p^{-1}) = \sum_{k=1}^{n+1} \binom{n+1}{k} (-1)^{k-1} p^{-1} \otimes (p \otimes p^{-1})^{\otimes(k-1)} + (IA_{p,p^{-1},q})^{n+1},$$

$$\varphi^n(q) = q + (IA_{p,p^{-1},q})^{n+1}.$$

The same argument as for the algebra A^0_θ shows that each φ^n defined this way is really a homomorphism, that the system (φ^n) is compatible with $\varphi^0(I) = 0$, and that the induced map of the towers of supercomplexes $\varphi_* : \mathcal{X}(R,I) \to \mathcal{X}(RA_{p,p^{-1},q}, IA_{p,p^{-1},q})$ is an isomorphism in $Ho\mathcal{T}$.

The procedure used for the algebra A_θ^0 works in the case of $A_{p,p^{-1},q}$ as well. To find the generator of $HC_k(A_{p,p^{-1},q})$ in $\mathcal{X}^k(RA_{p,p^{-1},q}, IA_{p,p^{-1},q})$, one takes the least n such that $k \leq 2n+1$, considers the sequence of maps

$$\mathcal{X}^k(R,I) \xrightarrow{\varphi_*^n} \mathcal{X}^k(R_nA_{p,p^{-1},q}, I_nA_{p,p^{-1},q}) \xrightarrow{\cong} \mathcal{X}^k(RA_{p,p^{-1},q}, IA_{p,p^{-1},q}) \tag{5.2}$$

for it, and computes the image of the generators of $HC_k(A_{p,p^{-1},q})$ in $\mathcal{X}^k(R,I)$ under this sequence of maps.

For $k=1$, $n=0$ the sequence of maps

$$\mathcal{X}^1(R,I) \xrightarrow{\varphi_*^0} \mathcal{X}^1(R_0A_{p,p^{-1},q}, I_0A_{p,p^{-1},q}) \xrightarrow{\cong} \mathcal{X}^1(RA_{p,p^{-1},q}, IA_{p,p^{-1},q})$$

takes the element $\overline{\natural p^{-1}dp}$ into $\overline{\natural p^{-1}\delta p}$, whence the basis of the first cyclic homology $HC_1(A_{p,p^{-1},q})$ in $\mathcal{X}^1(RA_{p,p^{-1},q}, IA_{p,p^{-1},q})$ consists of the homology class of $\overline{\natural p^{-1}\delta p}$. It follows that the basis of $HC_1(A_{p,p^{-1},q})$ in $X/F^1(X)$ is given by the homology class of $\overline{p^{-1}dp}$. By proposition 2.2.3, since the addition (or subtraction) of $P^\perp x$ to (from) an arbitrary element x does not change its homology class, the homology class of $p^{-1}dp$ forms the basis of $HC_1(A_{p,p^{-1},q})$ in $(\Omega A_{p,p^{-1},q}, B+b)$.

For $k=2$, $n=1$ the sequence of maps

$$\mathcal{X}^2(R,I) \xrightarrow{\varphi_*^1} \mathcal{X}^2(R_1A_{p,p^{-1},q}, I_1A_{p,p^{-1},q}) \xrightarrow{\cong} \mathcal{X}^2(RA_{p,p^{-1},q}, IA_{p,p^{-1},q})$$

takes the element $\overline{w}$ of $\mathcal{X}^2(R,I)$ into the element $\overline{p\otimes q - q\otimes p - 1}$ of $\mathcal{X}^2(RA_{p,p^{-1},q}, IA_{p,p^{-1},q})$, and it follows (exactly as for the Weyl algebra) that the basis of $HC_2(A_{p,p^{-1},q})$ in $X/F^2(X)$ consists of the homology class of $\overline{dqdp - dpdq}$. By proposition 2.2.3, since the addition (or subtraction) of $P^\perp x$ to (from) an arbitrary element x does not change its homology class, the homology class of $dqdp - dpdq$ forms the basis of $HC_2(A_{p,p^{-1},q})$ in $(\Omega A_{p,p^{-1},q}, B+b)$.

5.4 Generators of $HC_i(A_L)$ in the complex $(\Omega, b+B)$

The generators of $HC_i(A_L)$ are recovered analogue to how it is done for the algebra A^0_θ. For each $n \in \mathbb{N} \cup \{0\}$ the map

$$\varphi^n : R_L \to RA_L/(IA_L)^{n+1}$$

is defined by exactly the same formula as

$$\varphi^n : R_\theta \to RA^0_\theta/(IA^0_\theta)^{n+1}$$

in section 5.2 and the computations are very similar. At the end one gets that the basis of $HC_0(A_L)$ in $(\Omega A_L, b+B)$ is $\{u^{n_1}v^{n_2} | n_1, n_2 \in \mathbb{Z}\}$, the basis of $HC_1(A_L)$ in $(\Omega A_L, b+B)$ is $\{u^{-1}du, v^{-1}dv, u^{n_1-1}v^{n_2}du | n_1, n_2 \neq 0\}$, the basis of $HC_2(A_L)$ in $(\Omega A_L, b+B)$ is $\{1, u^{-1}v^{-1}dudv - u^{-1}v^{-1}dvdu\}$ and the basis of $HC_3(A_L)$ in $(\Omega A_L, b+B)$ is $\{u^{-1}du, v^{-1}dv\}$.

Chapter 6

Short free resolutions and connections

The classical way of computing Hochschild homology is based on the fact that for any $\mathbb{C}$-algebra A and for any A-bimodule M there is an isomorphism

$$H_n(A, M) \cong Tor_n^{A^e}(M, A),$$

([27], proposition 1.1.13). So if one has a "nice" projective resolution of the algebra A as of an A-bimodule, then, in order to compute the Hochschild homology $HH_*(A)$, one tensors the resolution up with A (over A^e) and computes the homology of the obtained complex ([33], theorem V.8.1). In particular, if one has a finite projective resolution of A (that means that all its terms in dimensions greater than some n are zero), then one can conclude that $HH_k(A) = 0$ for $k > n$.

6.1 Short free resolution of one-relator algebras

Dicks proved in his article [17] for associative one-relator algebras (i.e. quotients of free associative algebras) that I/I^2 fits into a certain exact sequence of A-bimodules. We generalize his result to all one-relator algebras and note that in the situation of the identity theorem 3.1.4 we obtain a free resolution of A of the length 2.

Consider $(A \otimes A)^{(X)}$, which is the direct sum of X copies of $A \otimes A$ (note that it is an R-bimodule), and the following R-derivation

$\partial : R \to (A \otimes A)^{(X)}$: put $\partial x = [x] := 1 \otimes 1 \in (A \otimes A)_x$ on the generators of R; this defines a derivation by proposition 3.1.2. Remark that ∂ vanishes on I^2 (since $\partial(i_1 i_2) = i_1 \partial(i_2) + \partial(i_1) i_2$, which is zero in $(A \otimes A)^{(X)}$).

Theorem 6.1.1 *Let A be a one-relator algebra (that is $A = R/I$, $R = k\langle X \rangle = k\langle Z, Y, Y^{-1} \rangle$, $I = RwR$) with an integral enveloping algebra. Then the following sequence is a free resolution of A as of an A-bimodule:*

$$0 \to I/I^2 \xrightarrow{\alpha} (A \otimes A)^{(X)} \xrightarrow{\beta} A \otimes A (\xrightarrow{\mu} A \to 0), \qquad (6.1)$$

where $\alpha(i + I^2) = \partial i$, $\beta[x] = 1 \otimes \overline{x} - \overline{x} \otimes 1$ and μ is the multiplication map $\mu(a \otimes b) = ab$. It follows that $HH_n(A) = 0$ for all $n > 2$ (since a free resolution is in particular projective).

Proof. The fact that this sequence is exact is proved by Dicks in [17] for associative one-relator algebras (i.e. in the case of R free). We outline his proof, which works also in our case. First, α is well-defined, since ∂ vanishes on I^2. $\beta\alpha = 0$, since $\beta\partial : R \to A \otimes A$ is the derivation sending generators x into $1 \otimes \overline{x} - \overline{x} \otimes 1$ and, being unique by proposition 3.1.2, it is equal to the derivation that takes each element r from R into $1 \otimes \overline{r} - \overline{r} \otimes 1$, which vanishes on I. The fact that $\mu\beta = 0$ is obvious.
To show that the sequence is exact, one constructs the left A-linear contracting homotopy as follows: let $\{a_i | i \in J\}$ be a basis of the algebra A over k containing 1 and let $\{s_i | i \in J\}$ be its pre-image in R (i.e. $\overline{s_1} = a_i$), let for any $i \in J$, $x \in X$ the product $\overline{x} a_i = \sum_{j \in J} \lambda_{ij}^x a_j$ (where λ_{ij}^x are elements of k, only finite number of which are not zero). Then we define the left A-linear maps

$$\mu' : A \to A \otimes A,$$

$$\beta' : A \otimes A \to (A \otimes A)^{(X)},$$

and

$$\alpha' (A \otimes A)^{(X)} \to I/I^2$$

by the rule

$$\mu'(1) = 1 \otimes 1,$$
$$\beta'(1 \otimes a_j) = \partial s_j,$$
$$\alpha'([x]a_i) = (xs_i - \sum_{j \in J} \lambda_{ij}^x s_j) + I^2.$$

One checks then directly that $\mu\mu' = \mathrm{id}$, $\beta\beta' + \mu'\mu = \mathrm{id}$, $\alpha\alpha' + \beta'\beta = \mathrm{id}$ and $\alpha'\alpha = \mathrm{id}$, for details see [17], theorem 4.1. The A-bimodule I/I^2 is free by the identity theorem 3.1.4, $(A \otimes A)^{(X)}$ and $A \otimes A$ are obviously free. □

Observe that the bimodule of 1-forms of an arbitrary algebra R is the module of universal derivation, i.e. $T(R) = \Omega^1 F$ and the map $d : R \to \Omega^1 R$ is the universal derivation ([27], 2.6.1), thus for a mixed free algebra $R = \mathbb{C}\langle X \rangle$ the R-bimodule $\Omega^1 R$ is freely generated by $\{dx | x \in X\}$ by proposition 3.1.2.

We observe now that another way to obtain the resolution (6.1) is splicing together two exact sequences from [12]:

Proposition 6.1.2 *([12], proposition 2.5) One has an exact sequence of A-bimodules:*

$$0 \to \Omega^1 A \xrightarrow{j} A \otimes A \xrightarrow{\mu} A \to 0$$

where $j(a_0 da_1) = a_0 a_1 \otimes 1 - a_0 \otimes a_1$ and μ is the multiplication map.

Proposition 6.1.3 *([12], corollary 2.11) If $A = R/I$, where I is an ideal in R, then one has a short exact sequence of A-bimodules:*

$$0 \to I/I^2 \xrightarrow{\overline{d}} A \otimes_R \Omega^1 R \otimes_R A \xrightarrow{\pi} \Omega^1 A \to 0,$$

where the surjection π is induced by the canonical projection $R \to A$ and the map $\overline{d}$ is induced by the restriction of the canonical derivation $d : R \to \Omega^1 R$ to I.

Note that $A \otimes_R \Omega^1 R \otimes_R A$ is isomorphic to $(A \otimes A)^{(X)}$ and can be identified with it, since $\Omega_1 R$ is free on the generators $\{dx | x \in X\}$, and that the composition $j \circ \pi$ becomes β under this identification.

For $A_{p,q}$ one obtains this way the following resolution:

$$0 \to A_{p,q} \otimes A_{p,q} \xrightarrow{\alpha} (A_{p,q} \otimes A_{p,q}) \oplus (A_{p,q} \otimes A_{p,q}) \xrightarrow{\beta} A_{p,q} \otimes A_{p,q} (\xrightarrow{\mu} A_{p,q} \to 0),$$

where

$$\beta(1 \otimes 1, 0) = 1 \otimes p - p \otimes 1,$$

$$\beta(0, 1 \otimes 1) = 1 \otimes q - q \otimes 1,$$

$$\alpha(1 \otimes 1) = (q \otimes 1 - 1 \otimes q, -(p \otimes 1 - 1 \otimes p)),$$

which turns out to be the classical Koszul-type resolution, being rewritten like

$$0 \to \Lambda^2 L \xrightarrow{\partial} \Lambda^1 L \xrightarrow{\partial} \Lambda^0 L (\xrightarrow{\mu} A_{p,q} \to 0),$$

where L is a free $A_{p,q}$-bimodule (or $A_{p,q}^e$-module) with free generators e_1 and e_2, $\Lambda L = \bigoplus_i \Lambda^i L$ is the external algebra of L, l is a $A_{p,q}^e$-linear map

$$\begin{array}{rccc} l: & L & \to & A_{p,q}^e = A_{p,q} \otimes A_{p,q}^{op}, \\ & e_1 & \mapsto & 1 \otimes p - p \otimes 1, \\ & e_2 & \mapsto & 1 \otimes q - q \otimes 1, \end{array}$$

and the differential is a particular case of the usual Koszul differential $\partial(e_1 \wedge \ldots \wedge e_n) = \sum_{i=1}^{n} (-1)^i l(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n$ (e.g. [1], section 9.1). Similarly for the algebra $A_{p,p^{-1},q}$.

For A_θ^0 we become

$$0 \to A_\theta^0 \otimes A_\theta^0 \xrightarrow{\alpha} (A_\theta^0 \otimes A_\theta^0) \oplus (A_\theta^0 \otimes A_\theta^0) \xrightarrow{\beta} A_\theta^0 \otimes A_\theta^0 (\xrightarrow{\mu} A_\theta^0 \to 0),$$

where

$$\beta(1 \otimes 1, 0) = 1 \otimes u - u \otimes 1,$$

$$\beta(0, 1 \otimes 1) = 1 \otimes v - v \otimes 1,$$

$$\alpha(1 \otimes 1) = (v \otimes 1 - \lambda \otimes v, -(\lambda u \otimes 1 - 1 \otimes u)),$$

which is exactly the "ad hoc" resolution of Connes' ([9], lemma 48):

$$0 \to M_2 \xrightarrow{\delta_2} M_1 \xrightarrow{\delta_1} M_0 (\xrightarrow{\mu} A_\theta^0 \to 0),$$

where $M_i = A^0_\theta \otimes \Lambda_i \otimes A^0_\theta$, $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2$ is the exterior tensor algebra over a two-dimensional $\mathbb{C}$-vector space V with the basis e_1, e_2 and where

$$\delta_1(1 \otimes e_1 \otimes 1) = 1 \otimes u - u \otimes 1,$$
$$\delta_1(1 \otimes e_2 \otimes 1) = 1 \otimes v - v \otimes 1,$$
$$\delta_2(1 \otimes (e_1 \wedge e_2) \otimes 1) = (v \otimes 1 - \lambda \otimes v) \otimes e_1 - (\lambda u \otimes 1 - 1 \otimes u) \otimes e_2.$$

6.2 2-connection on A^0_θ

A projective resolution of length n of an algebra can be used to construct an n-connection on that algebra, as it was done in [36] for a tensor product of two free algebras. An n-connection on an algebra A in turn allows one to write down explicitly a partial contracting homotopy for the complex $(\hat{\Omega} A, B+b)$; this was done by Khalkhali in [23] for the cyclic cohomology (i.e. for the dual complex $((\Omega A)^*, B+b)$), we do this now for the cyclic homology.

Definition 6.2.1 *[23] Let $n \geq 0$. An n-connection on A is a linear map*

$$\nabla_n : \Omega^n A \to \Omega^{n+1} A$$

such that for all $a \in A$ and for all $\omega \in \Omega^n A$ the following equalities are fulfilled:

$$\nabla_n a\omega = a \nabla_n \omega$$

and

$$\nabla_n \omega a = \nabla_n \omega \cdot a + \omega da.$$

One extends ∇_n as follows: for $k \geq n$ one defines

$$\nabla_k : \Omega^k A \to \Omega^{k+1} A$$

by the rule

$$\nabla_k(a^0 da^1 \dots da^k) = \nabla_n(a^0 da^1 \dots da^n) a^{k+1} \dots a^k.$$

Then one has for all $k \geq n+1$ the equality

$$b\nabla_k + \nabla_{k-1} b = (-1)^n \mathrm{id} \tag{6.2}$$

([23], proposition 4.2), and it follows that the n-th Hodge filtration $F^n(\hat{\Omega}A, b) = (b\Omega^{n+1}A \oplus \prod_{k>n} \Omega^k A, b)$ is contractible, by (6.2) together with the fact that for $b\omega \in b\Omega^{n+1}A$

$$b\nabla_n(b\omega) = b(\mathrm{id} - b\nabla_{n+1})(\omega) = b\omega.$$

Lemma 6.2.2 *(perturbation lemma, [23]) Let the supercomplex (L, b) be a deformation retract of a supercomplex (M, b). This means that there exist two homomorphisms of complexes*

$$L \xrightarrow{i} M \xrightarrow{r} L$$

and a homotopy $h : M \to M$ such that

$$\begin{aligned} ri &= id_L, \\ ir &= id_M + bh + hb, \\ hi &= 0. \end{aligned}$$

Let B be a perturbation of the differential b such that $Bi = iB$ and the operator

$$K = \sum_{k=0}^{\infty} (Bh)^k$$

is well defined. Then the complex $(L, b+B)$ is a deformation retract of $(M, b+B)$. The corresponding homomorphisms

$$(L, b+B) \xrightarrow{I} (M, b+B) \xrightarrow{R} (L, b+B)$$

and the homotopy $H : (M, b+B) \to (M, b+B)$ are defined as follows: $R = rK$, $H = hK$ and $I = i$.

Applying lemma 6.2.2 to the case $M = F^n(\hat{\Omega}A, b)$, $L = 0$ and $h = \nabla$, we obtain a contracting homotopy $H = \nabla \sum_{k=0}^{\infty} (B\nabla)^k$ for the supercomplex $F^n(\hat{\Omega}A, b + B)$. It follows that if there is an n-connection on A then the periodic cyclic homology of A is the homology of the supercomplex $(\hat{\Omega}A, B + b)/F^n(\hat{\Omega}A, B + b)$.

Now we explain how an n-connection can be obtained from a projective resolution of an algebra, generalizing the construction introduced by Puschnigg in [36] for a tensor product of two free algebras. Then we apply this procedure to the algebra A_θ^0 and its Connes' resolution.

Let A be an algebra with a unit. One has for A ([13], §3) the standard free resolution

$$C_\bullet(A) \ldots \xrightarrow{\partial} \Omega^n A \otimes A \xrightarrow{\partial} \Omega^{n-1} A \otimes A \xrightarrow{\partial} \ldots \xrightarrow{\partial} \Omega^0 A \otimes A (\xrightarrow{\mu} A \to 0),$$

where

$$\partial(\omega da \otimes a') = (-1)^{|\omega|}(\omega a \otimes a' - \omega \otimes aa'),$$
$$\mu(a \otimes b) = ab.$$

If

$$P_\bullet : \ldots \to 0 \to 0 \to P_n \xrightarrow{\partial} P_{n-1} \xrightarrow{\partial} \ldots \xrightarrow{\partial} P_0 (\xrightarrow{\varepsilon} A \to 0)$$

is a (finite) projective resolution of A, then one can compare it with the standard resolution, that means find mutually inverse homotopy equivalences $\beta : C_\bullet(A) \to P_\bullet$ and $\gamma : P_\bullet \to C_\bullet(A)$ ([33], theorem III.6.1). Set $\alpha = \gamma\beta$. Then α is homotopic to $\mathrm{id}_{C_\bullet(A)}$. Let h be the corresponding homotopy, i.e. the sequence $(h_i)_{i \in \mathbb{N} \cup \{0\}}$ of maps $h_i : \Omega^i A \otimes A \to \Omega^{i+1} A \otimes A$ such that

$$\partial_i h_i + h_{i-1}\partial_{i-1} = id - \alpha_i. \tag{6.3}$$

Then the map $(\mathrm{id} - \partial_n h_n) \in End(C_n(A))$ is an idempotent, since

$$(\mathrm{id} - \partial_n h_n)^2 = \mathrm{id} - 2\partial_n h_n + \partial_n h_n \partial_n h_n = \\ \mathrm{id} - 2\partial_n h_n + \partial_n(\mathrm{id} - \partial_{n+1} h_{n+1}) h_n = \mathrm{id} - \partial_n h_n.$$

It follows that $(\mathrm{id} - \partial_n h_n) C_n(A)$ is a projective A-bimodule. Restricted to this A-bimodule, the map ∂_{n-1} becomes injective, whence it is an isomorphism between $(\mathrm{id} - \partial_n h_n) C_n(A)$ and $\partial_{n-1} C_n(A)$. The map

$$\begin{array}{rccc} \eta : & \Omega^n A & \to & \partial_{n-1}(\Omega^n A \otimes A) \\ & \omega & \mapsto & \partial_{n-1}(\omega \otimes 1) \end{array}$$

is an isomorphism of A-bimodules. The inverse isomorphism is given by $\partial_{n-1}(\omega \otimes a) \mapsto \omega \cdot a$. It follows that the map

$$\begin{array}{rccc} \varphi : & \Omega^n A & \to & (\mathrm{id} - \partial_n h_n)(\Omega^n A \otimes A) \\ & \omega & \mapsto & (\mathrm{id} - \partial_n h_n)(\omega \otimes 1) \end{array}$$

is an A-A-isomorphism. Considered as a map $\Omega^n A \to \Omega^n A \otimes A$, φ splits the multiplication map

$$\begin{array}{rccc} m : & \Omega^n A \otimes A & \to & A \\ & \omega \otimes a & \mapsto & \omega \cdot a, \end{array}$$

since

$$m\varphi(\omega) = m(\mathrm{id} - \partial_n h_n)(\omega \otimes 1) = \omega - m\partial_n h_n(\omega \otimes 1)$$

and since for an arbitrary element $\tilde{\omega} da \otimes a'$ from $\Omega^{n+1} A \otimes A$ one has

$$m\partial_n(\tilde{\omega} da \otimes a') = (-1)^{|\tilde{\omega}|} m(\tilde{\omega} a \otimes a' - \tilde{\omega} \otimes aa') = (-1)^{|\tilde{\omega}|}(\tilde{\omega} a \cdot a' - \tilde{\omega} \cdot aa') = 0.$$

Now we recall the following result of Cuntz an Quillen, relating the sections of the multiplication map and connections:

Proposition 6.2.3 *([12], proposition 8.1) For a right A-module E associating to every map $s : E \to E \otimes A$ that splits the multiplication map $\mu : E \otimes A \to A$ the Grassmanian connection $\nabla = \mu(1 \otimes d)s$ is a one-to-one correspondence between sections of μ and connections ∇ on E.*

By this proposition we get the connection $\nabla = m(1 \otimes d)\varphi$ on $\Omega^n A$. We remark here that the homotopy h can be constructed inductively (see the proof of theorem III.6.1 of [33]); that means that if one has $h_0, \ldots, h_k$ satisfying (6.3) then one can find h_{k+1} and so on. If we proceed this way, then it is unnecessary for our purposes to compute h_n, since $\varphi = \mathrm{id} - \partial_n h_n = \alpha_n + h_{n-1}\partial_{n-1}$ holds.

We consider now the free resolution of the algebra A^0_θ described in the previous section (written in the Connes' form):

$$M_\bullet : \ldots \to 0 \to M_2 \xrightarrow{\delta_2} M_1 \xrightarrow{\delta_1} M_0 (\xrightarrow{\mu} A^0_\theta \to 0),$$

$M_i = A^0_\theta \otimes \Lambda_i \otimes A^0_\theta$, $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2$ is the exterior tensor algebra over $V = \mathbb{C}(e_1, e_2)$,

$$\delta_1(1 \otimes e_1 \otimes 1) = 1 \otimes u - u \otimes 1,$$

$$\delta_1(1 \otimes e_2 \otimes 1) = 1 \otimes v - v \otimes 1,$$

$$\delta_2(1 \otimes (e_1 \wedge e_2) \otimes 1) = (v \otimes 1 - \lambda \otimes v) \otimes e_1 - (\lambda u \otimes 1 - 1 \otimes u) \otimes e_2.$$

The homotopy equivalences of the resolutions $\gamma : M_\bullet \to C_\bullet(A^0_\theta)$ and $\beta : C_\bullet(A^0_\theta) \to M_\bullet$ are constructed in the proof of lemma 51 of [9] (where they are denoted with h and k respectively). We rewrite the formulas for γ and β in the following way (taking into account that $M_0 \cong A^0_\theta \otimes A^0_\theta$ and $\Omega^0 A^0_\theta = A^0_\theta \otimes A^0_\theta$):

$$\gamma_0 = id_{A^0_\theta \otimes A^0_\theta},$$

$$\gamma_1(1 \otimes e_1 \otimes 1) = -du \otimes 1,$$

$$\gamma_1(1 \otimes e_2 \otimes 1) = -dv \otimes 1,$$

$$\gamma_2(1 \otimes (e_1 \wedge e_2) \otimes 1) = -dvdu \otimes 1 + \lambda dudv \otimes 1,$$

the rest γ_i are zero;

$$\beta_0 = id_{A^0_\theta \otimes A^0_\theta},$$

$$\beta_1(d(u^{n_1}v^{n_2}) \otimes 1) = -\sum_{i \in [0,n_1]} u^i \otimes e_1 \otimes u^{n_1-1-i} v^{n_2} - \sum_{j \in [0,n_2]} u^{n_1} v^j \otimes e_2 \otimes v^{n_2-1-j},$$

$$\beta_2(d(u^{n_1}v^{n_2}) d(u^{m_1}v^{m_2}) \otimes 1) = - \sum_{i \in [0,m_1]} \sum_{j \in [0,n_2]} \lambda^{n_2 m_1 - (m_1 - i)(j+1)} u^{n_1+i} v^j \otimes (e_1 \wedge e_2) \otimes u^{m_1-1-i} v^{n_2+m_2-1-j},$$

the rest β_i are zero. The notation $\sum\limits_{i\in[n_1,n_2]}$ is explained by (4.12). It follows that

$$\alpha_1(d(u^{n_1}v^{n_2})\otimes 1) = \sum_{i\in[0,n_1]} u^i du\otimes u^{n_1-1-i}v^{n_2} + \sum_{j\in[0,n_2]} u^{n_1}v^j dv\otimes v^{n_2-1-j},$$

$$\alpha_2(d(u^{n_1}v^{n_2})d(u^{m_1}v^{m_2})\otimes 1) = \sum_{i\in[0,m_1]}\sum_{j\in[0,n_2]} \lambda^{n_2m_1-(m_1-i)(j+1)} u^{n_1+i}v^j dvdu\otimes u^{m_1-1-i}v^{n_2+m_2-1-j} - \sum_{i\in[0,m_1]}\sum_{j\in[0,n_2]} \lambda^{n_2m_1-(m_1-i)(j+1)+1} u^{n_1+i}v^j dudv\otimes u^{m_1-1-i}v^{n_2+m_2-1-j},$$

and the rest are zero.

Let us now construct a homotopy between α and $id_{C_\bullet(A^0_\theta)}$. Take $h_0 = 0 : \Omega^0 A^0_\theta\otimes A^0_\theta \to \Omega^1 A^0_\theta\otimes A^0_\theta$; then $\partial_0 h_0 = 0 = id_{A^0_\theta\otimes A^0_\theta} - id_{A^0_\theta\otimes A^0_\theta}$.

Define $h_1 : \Omega^1 A^0_\theta\otimes A^0_\theta \to \Omega^2 A^0_\theta\otimes A^0_\theta$ by

$$h_1(d(u^{n_1}v^{n_2})\otimes 1) = -\sum_{i\in[0,n_1]} du^i du\otimes u^{n_1-1-i}v^{n_2} - \sum_{j\in[0,n_2]} d(u^{n_1}v^j)dv\otimes v^{n_2-1-j}.$$

Check that $h_0\partial_0 + \partial_1 h_1 = id - \alpha_1$.

$$
\begin{aligned}
&(h_0\partial_0+\partial_1 h_1)(d(u^{n_1}v^{n_2})\otimes 1)=\\
&\partial_1\Big(-\sum_{i\in[0,n_1]} du^i du\otimes u^{n_1-1-i}v^{n_2}-\sum_{j\in[0,n_2]} d(u^{n_1}v^j)dv\otimes v^{n_2-1-j}\Big)=\\
&\sum_{i\in[0,n_1]}(du^{i+1}\otimes u^{n_1-1-i}v^{n_2}-u^i du\otimes u^{n_1-1-i}v^{n_2}-du^i\otimes u^{n_1-i}v^{n_2})+\\
&\sum_{j\in[0,n_2]}(d(u^{n_1}v^{j+1})\otimes v^{n_2-1-j}-u^{n_1}v^j dv\otimes v^{n_2-1-j}-d(u^{n_1}v^j)\otimes v^{n_2-j})=\\
&\quad -\sum_{i\in[0,n_1]} u^i du\otimes u^{n_1-1-i}v^{n_2}+du^{n_1}\otimes v^{n_2}-d1\otimes u^{n_1}v^{n_2}-\\
&\quad \sum_{j\in[0,n_2]} u^{n_1}v^j dv\otimes v^{n_2-1-j}+d(u^{n_1}v^{n_2})\otimes 1-du^{n_1}\otimes v^{n_2}=\\
&\quad -\sum_{i\in[0,n_1]} u^i du\otimes u^{n_1-1-i}v^{n_2}-\sum_{j\in[0,n_2]} u^{n_1}v^j dv\otimes v^{n_2-1-j}+\\
&\qquad\qquad d(u^{n_1}v^{n_2})\otimes 1=(id-\alpha_1)(d(u^{n_1}v^{n_2})\otimes 1).
\end{aligned}
$$

Now we evaluate the homomorphism $\varphi:\Omega^2 A\to\Omega^2 A\otimes A$, defined by $\varphi(\omega)=(\alpha_2+h_1\partial_1)(\omega\otimes 1)$, on the basis elements.

$$
\begin{aligned}
&h_1\partial_1\left(d(u^{n_1}v^{n_2})d(u^{m_1}v^{m_2})\otimes 1\right)=-h_1(\lambda^{n_2m_1}d(u^{n_1+m_1}v^{n_2+m_2})\otimes 1)-\\
&\qquad h_1(-u^{n_1}v^{n_2}d(u^{m_1}v^{m_2})\otimes 1-d(u^{n_1}v^{n_2})\otimes u^{m_1}v^{m_2})=
\end{aligned}
$$

$$
\begin{aligned}
&\lambda^{n_2m_1}\sum_{i\in[0,n_1+m_1]} du^i du\otimes u^{n_1+m_1-1-i}v^{n_2+m_2}+\\
&\qquad\lambda^{n_2m_1}\sum_{j\in[0,n_2+m_2]} d(u^{n_1+m_1}v^j)dv\otimes v^{n_2+m_2-1-j}-\\
&\qquad\qquad\sum_{i\in[0,m_1]} u^{n_1}v^{n_2}du^i du\otimes u^{m_1-1-i}v^{m_2}-
\end{aligned}
$$

$$\sum_{j\in[0,m_2]} u^{n_1}v^{n_2}d(u^{m_1}v^j)dv\otimes v^{m_2-1-j}-$$

$$\sum_{i\in[0,n_1]} \lambda^{n_2m_1}du^i du\otimes u^{n_1+m_1-1-i}v^{n_2+m_2}-$$

$$\sum_{j\in[0,n_2]} \lambda^{(n_2-1-j)m_1}d(u^{n_1}v^j)dv\otimes u^{m_1}v^{n_2+m_2-1-j}$$

and it follows that

$$\varphi\left(d(u^{n_1}v^{n_2})d(u^{m_1}v^{m_2})\right)=$$

$$\sum_{i\in[0,m_1]}\sum_{j\in[0,n_2]} \lambda^{n_2m_1-(m_1-i)(j+1)}u^{n_1+i}v^j dvdu\otimes u^{m_1-1-i}v^{n_2+m_2-1-j}-$$

$$\sum_{i\in[0,m_1]}\sum_{j\in[0,n_2]} \lambda^{n_2m_1-(m_1-i)(j+1)+1}u^{n_1+i}v^j dudv\otimes u^{m_1-1-i}v^{n_2+m_2-1-j}+$$

$$\sum_{i\in[0,n_1+m_1]} \lambda^{n_2m_1}du^i du\otimes u^{n_1+m_1-1-i}v^{n_2+m_2}+$$

$$\sum_{j\in[0,n_2+m_2]} \lambda^{n_2m_1}d(u^{n_1+m_1}v^j)dv\otimes v^{n_2+m_2-1-j}-$$

$$\sum_{i\in[0,m_1]} u^{n_1}v^{n_2}du^i du\otimes u^{m_1-1-i}v^{m_2}-\sum_{j\in[0,m_2]} u^{n_1}v^{n_2}d(u^{m_1}v^j)dv\otimes v^{m_2-1-j}-$$

$$\sum_{i\in[0,n_1]} \lambda^{n_2m_1}du^i du\otimes u^{n_1+m_1-1-i}v^{n_2+m_2}-$$

$$\sum_{j\in[0,n_2]} \lambda^{(n_2-1-j)m_1}d(u^{n_1}v^j)dv\otimes u^{m_1}v^{n_2+m_2-1-j}.$$

The desired connection $\nabla = m(1\otimes d)(\alpha_2 + h_1\partial_1)$ is then given explicitly by

$$\nabla(du^{n_1}v^{n_2}du^{m_1}v^{m_2}) =$$

$$\sum_{i\in[0,m_1]}\sum_{j\in[0,n_2]}\lambda^{n_2m_1-(m_1-i)(j+1)}u^{n_1+i}v^jdvdud(u^{m_1-1-i}v^{n_2+m_2-1-j})-$$

$$\sum_{i\in[0,m_1]}\sum_{j\in[0,n_2]}\lambda^{n_2m_1-(m_1-i)(j+1)+1}u^{n_1+i}v^jdudvd(u^{m_1-1-i}v^{n_2+m_2-1-j})+$$

$$\sum_{i\in[0,n_1+m_1]}\lambda^{n_2m_1}du^idud(u^{n_1+m_1-1-i}v^{n_2+m_2})+$$

$$\sum_{j\in[0,n_2+m_2]}\lambda^{n_2m_1}d(u^{n_1+m_1}v^j)dvdv^{n_2+m_2-1-j}-$$

$$\sum_{i\in[0,m_1]}u^{n_1}v^{n_2}du^idud(u^{m_1-1-i}v^{m_2})-\sum_{j\in[0,m_2]}u^{n_1}v^{n_2}d(u^{m_1}v^j)dvdv^{m_2-1-j}-$$

$$\sum_{i\in[0,n_1]}\lambda^{n_2m_1}du^idud(u^{n_1+m_1-1-i}v^{n_2+m_2})-$$

$$\sum_{j\in[0,n_2]}\lambda^{(n_2-1-j)m_1}d(u^{n_1}v^j)dvd(u^{m_1}v^{n_2+m_2-1-j}).$$

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