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Equivariant Homology Theories for Totally Disconnected Groups

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Abstract

The notion of an equivariant family of spectra corresponds to the notion of an equivariant homology theory as used in [Lüc02]. A general principle how to construct equivariant families of spectra will be given. This machine can be used to define many interesting equivariant homology theories. The main examples will be algebraic K- and L-theory for discrete groups and topological K-theory, Hochschild Homology, Cyclic Homology and Periodic Homology for totally disconnected, locally compact groups.

In the appendix equivariant K-theory (cohomology) for proper actions of totally disconnected groups will be considered. We will show that in general it cannot be defined via equivariant vector bundles as for discrete groups, because of the failure of the excision axiom.

Contents

Introduction

In [Lüc02] Lück constructs a Chern character for proper equivariant homology theories. The main result is the following: Let $\mathcal{H}^?_*$ be a proper equivariant homology theory with values in R-modules, where R is a commutative ring containing the rationals. Suppose for every discrete group G that the $R\,\mathrm{Sub}(G,\mathcal{FIN})$-module $\mathcal{H}^G_q(G/?)$ is flat for all $q \geq 0$. Then there is a natural isomorphism

$$ch^G_*(X,A) : \bigoplus_{p+q=*} H^{\mathrm{Or}(G)}_p(X,A;\mathcal{H}^G_q(G/?)) \to \mathcal{H}^G_*(X,A).$$

A *(smooth) equivariant homology theory* $\mathcal{H}^?_*$ is an assignment which associates to every discrete (totally disconnected, locally compact) group G a (smooth) G-homology theory $\mathcal{H}^G_*$ together with the following so called *induction structure*: Let $\alpha : G \to M$ be a (continuous open) group homomorphism, (X,A) a (smooth) G-CW-pair on which $\ker(\alpha)$ acts freely. Then for every $n \in \mathbb{Z}$ there is a natural isomorphism

$$\mathrm{ind}_\alpha : \mathcal{H}^G_n(X,A) \to \mathcal{H}^M_n(\mathrm{ind}_\alpha(X,A))$$

which is compatible with the boundary homomorphisms, functorial in α and satisfies a certain condition if α is conjugation with a group element.

The Chern character in [Lüc02] is explicitly constructed with the help of the induction homomorphisms of $\mathcal{H}^?_*$.

Our main theorem gives a general method how to construct (smooth) equivariant homology theories using $\mathrm{Or}(G)$-spectra. Namely for any discrete (totally disconnected, locally compact) group G let $\mathcal{G}^G$ be the $\mathrm{Or}(G)$-groupoid ($\mathrm{Or}(G,\mathcal{O})$-groupoid) with $Ob(\mathcal{G}^G(G/H)) = G/H$ and $\mathrm{mor}_{\mathcal{G}^G(G/H)}(g_1H, g_2H) = g_2Hg_1^{-1}$. Then the following theorem holds:

Theorem 2.10 *Let $E : \mathbf{Gr} \to \mathbf{Spt}(\mathbf{Top})$ (resp. $E : \mathbf{TGr} \to \mathbf{Spt}(\mathbf{Top})$) be a functor with the property that it maps equivalences of (topological) groupoids to weak equivalences of spectra.*

Then maps i_α can be constructed such that $(E^? := E \circ \mathcal{G}^?, i_?)$ is a (smooth) equivariant family of spectra (and hence $H^?_*(\,\cdot\,; E \circ \mathcal{G}^?)$ a (smooth) equivariant homology theory).

Here $H^G_*(\,\cdot\,; E \circ \mathcal{G}^G)$ is the G-homology theory with coefficients in the $\mathrm{Or}(G)$-spectrum $E \circ \mathcal{G}^G$. In Chapter 1 this and other basic notions concerning the orbit-category and spaces over the Orbit category will be explained. In Chapter 2 we will think about what could or should be an induction structure for equivariant spectra and how this can be mirrored on equivariant groupoids. Finding the right formulation for this is already half of the proof of the main theorem. In the second part of the paper applications for the main theorem are given, including K- and L-theory for discrete groups, topological K-theory, Hochschild Homology, Cyclic Homology and Periodic Homology for totally disconnected, locally compact groups.

Applying the main theorem to get examples of smooth equivariant homology theories, i.e. constructing a functor $E : \mathbf{Gr} \to \mathbf{Spt}(\mathbf{Top})$ often takes two steps. In many cases there exists or can be constructed an interesting functor $F : \mathbf{Cat}_< \to \mathbf{Spt}(\mathbf{Top})$ from some subcategory of the category of small categories, for example the split exact categories, to the category of spectra.

Then we have to choose a suitable functor $\mathbf{Gr} \to \mathbf{Cat}_<$ which assures that we get the right coefficients, namely the group ring for discrete groups and the Hecke algebra resp. the reduced group C^*-algebra for totally disconnected groups. In the case of K-theory for example, this functor assigns to any groupoid $\mathcal{G}$ the symmetric monoidal category $\mathcal{P}(\mathbb{Z}\mathcal{G}_\oplus)$, where $\mathbb{Z}\mathcal{G}$ is the $\mathbb{Z}$-linear category associated to $\mathcal{G}$, $\mathbb{Z}\mathcal{G}_\oplus$ is the category of finite tuples of objects of $\mathbb{Z}\mathcal{G}$, and $\mathcal{P}$ assigns to any category its idempotent completion.

For K- and L-theory we use the construction of Davis and Lück in [DL98], for Hochschild Homology, Cyclic Homology and Periodic Homology we use work of McCarthy, who defines in [McC94] Hochschild Homology, Cyclic Homology and Periodic Homology for k-linear categories with cofibrations, k a commutative ring. This definition leads to the construction of spectra for these homology theories. A functor $K^{top}: \mathbf{C}^*\mathbf{Cat} \to \mathbf{Spt}(\mathbf{Top})$ is constructed in [Joa02]. In both cases we give a detailed construction of the functors $\mathbf{TGr} \to \mathbf{Cat}_<$ mentioned above.

For every totally disconnected, locally compact group G and every G-homology theory $\mathcal{H}_*^G$ the projection $E(G,\mathcal{CO}) \to *$, where $\mathcal{CO}$ is the family of compact open subgroups, gives rise to an *assembly map*

$$a : \mathcal{H}_*^G(E(G,\mathcal{CO})) \to \mathcal{H}_*^G(G/G).$$

The corresponding *isomorphism conjecture* says that this map is an isomorphism. With the above constructed spectra isomorphism conjectures for Hochschild Homology and Cyclic resp. Periodic Homology can be formulated.

In Section 7 we show that the construction of the main theorem can often be extended in the sense that induction homomorphisms can also be defined for G-CW-complexes on which the kernel of the group homomorphism in question doesn't act freely.

The Appendix shows a different aspect of the action of totally disconnected groups. For proper actions of discrete groups, topological K-theory (cohomology) can be defined using equivariant vector-bundles (see [LO01]). We show that the analogous statement for proper smooth actions of totally disconnected groups isn't true in general, an explicit counterexample will be given where excicion doesn't hold. For groups which are an inverse limit of discrete groups, the definition carries through and the Chern character of [LO99] can be extended to these groups.

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1 The Orbit Category and Spaces over the Orbit Category

1.1 Spaces and Spectra over a Category

For an extensive treatment of spaces over a category see [DL98].

Let $\mathcal{C}$ be a small category. A *covariant (contravariant) space over the category* $\mathcal{C}$ or $\mathcal{C}$*-space* is a covariant (contravariant) functor

$$X : \mathcal{C} \to \mathbf{Top}$$

where **Top** is the category of compactly generated topological spaces. A map between $\mathcal{C}$-spaces is a natural transformation of such functors.

A *pointed* $\mathcal{C}$*-space* is a functor from $\mathcal{C}$ to $\mathbf{Top}_+$ and a *weak equivalence* between two $\mathcal{C}$-spaces is a map of $\mathcal{C}$-spaces which at each object is a weak equivalence of topological spaces. A $\mathcal{C}$*-spectrum* (resp. $\mathcal{C}$-Ω*-spectrum*) is a functor from $\mathcal{C}$ to $\mathbf{Spt}(\mathbf{Top})$ (resp. $\mathbf{\Omega Spt}(\mathbf{Top})$).

Remark 1.1 The objects of the category $\mathbf{Spt}(\mathbf{Top})$ are the *spectra*, i.e. families $\{E_n, n \in \mathbb{Z}\}$ of pointed topological spaces together with structure maps $s_n : S^1 \wedge E_n \to E_{n+1}$. The morphisms are the *strict maps* of spectra, i.e. a morphism $f : E \to F$ between two spectra consists of maps $f_n : E_n \to F_n$ which are compatible with the structure maps: $f_{n+1} \circ s_n^E = s_n^F \circ (S^1 \wedge f_n)$. A spectrum is an Ω*-spectrum* if the adjoints $t_n : E_n \to \Omega E_{n+1}$ of the structure maps are weak equivalences. The category $\mathbf{\Omega Spt}(\mathbf{Top})$ is the full subcategory of $\mathbf{Spt}(\mathbf{Top})$ with objects the Ω-spectra.

The *(stable) homotopy groups* $\pi_*(E)$ of a spectrum E are defined to be

$$\pi_n(E) := \varinjlim_{k \in \mathbb{N}} \pi_{n+k}(E_k), \ n \in \mathbb{Z},$$

where the structure maps of the colimit are given by

$$\pi_{n+k}(E_k) \xrightarrow{S} \pi_{n+k+1}(S^1 \wedge E_k) \xrightarrow{\pi_{n+k+1}(s_k)} \pi_{n+k+1} E_{k+1}.$$

A *weak equivalence* of spectra is a map between spectra, which induces an isomorphism on the stable homotopy groups. □

If A is a (pointed) topological space, X a (pointed) $\mathcal{C}$-space, then $A \times X$ (resp. $A \wedge X$) and $\mathrm{map}(A, X)$ are defined objectwise. With this notion a $\mathcal{C}$-spectrum is the same as a family of pointed $\mathcal{C}$-spaces $\{E_n, n \in \mathbb{Z}\}$ together with structure maps $S^1 \wedge E_n \to E_{n+1}$ (or $E_n \to \Omega E_{n+1}$).

A homotopy between two (pointed) $\mathcal{C}$-maps $f, g : X \to Y$ is a $\mathcal{C}$-map $H : X \times I \to Y$ (resp. $H : X \wedge I_+ \to Y$) such that $H_0 = f$ and $H_1 = g$.

Definition 1.2 *Let X be a contravariant and Y a covariant $\mathcal{C}$-space. Define their tensor product to be the space*

$$X \otimes_{\mathcal{C}} Y := \coprod_{c \in Ob(\mathcal{C})} X(c) \times Y(c) / \sim$$

where $\sim$ *is the equivalence relation generated by* $(x\varphi, y) \sim (x, \varphi y)$ *for all morphisms* $\varphi : c \to d$ *and* $x\varphi := X(\varphi)(x), \varphi y := Y(\varphi)(y)$.

Let X *and* Y *be two* $\mathcal{C}$*-spaces of equal variance. Then denote by* $\hom_{\mathcal{C}}(X, Y)$ *the space of* $\mathcal{C}$*-maps between* X *and* Y *with the topology coming from the inclusion into* $\prod_{c \in Ob(\mathcal{C})} \operatorname{map}(X(c), Y(c))$.

For pointed $\mathcal{C}$-spaces $\times$ has to be replaced by $\wedge$ and $\coprod$ by $\vee$.

Definition 1.3 *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a covariant functor,* X *a covariant (resp. contravariant)* $\mathcal{C}$*-space. Then induction of* X *with* F *is defined to be the covariant (resp. contravariant)* $\mathcal{D}$*-space*

$$\operatorname{ind}_F X := \operatorname{mor}_{\mathcal{D}}(F(?), ??) \otimes_{\mathcal{C}} X$$

(resp.

$$\operatorname{ind}_F X := X \otimes_{\mathcal{C}} \operatorname{mor}_{\mathcal{D}}(??, F(?)).)$$

Let Y *be a* $\mathcal{D}$*-space, then the restriction of* Y *with* F *is defined to be the* $\mathcal{D}$*-space*

$$\operatorname{res}_\alpha Y := Y \circ F.$$

Restriction to a subcategory will be denoted by $Y|_{\text{subcategory}}$, sometimes we will just write $Y|$ or even Y if it is clear that and to which subcategory the space has to be restricted.

Remark 1.4 There are natural adjunction homeomorphisms

$$\hom_{\mathcal{D}}(\operatorname{ind}_F X, Y) \xrightarrow{\cong} \hom_{\mathcal{C}}(X, \operatorname{res}_F Y)$$

and

$$\operatorname{ind}_\alpha X \otimes_{\mathcal{D}} Y \xrightarrow{\cong} X \otimes_{\mathcal{C}} \operatorname{res}_\alpha Y$$

(see Lemma 1.9, [DL98]). □

1.2 The Orbit Category

For us the category $\mathcal{C}$ of the last section will most times be the Orbit category of a group G. Let G be a topological group and $\mathcal{F}$ a *family of closed subgroups*, i.e. a non-empty set of closed subgroups of G which is closed under taking conjugates. The *orbit category* $\operatorname{Or}(G, \mathcal{F})$ is the topological category with the homogeneous spaces G/H with $H \in \mathcal{F}$ as objects and the G-maps between them as morphisms. The morphism sets carry the compact-open topology.

Note here that any G-map $G/H \to G/K$ is of the form $gH \mapsto g\gamma^{-1}K$, where $\gamma \in G$ has the property that $\gamma H \gamma^{-1} \subset K$. We will denote this map by R_γ. Two elements γ and γ' of G induce the same G-map $G/H \to G/K$ if and only if $\gamma'\gamma^{-1} \in K$.

Any continuous group homomorphism $\alpha : G \to M$ induces a covariant functor

$$\alpha : \operatorname{Or}(G, \mathcal{F}) \to \operatorname{Or}(M, \alpha(\mathcal{F}))$$

by mapping G/H to $M/\alpha(H)$ and R_γ to $R_{\alpha(\gamma)}$. Here $\alpha(\mathcal{F}) := \{\alpha(H) | H \in \mathcal{F}\}$. If X is an $\operatorname{Or}(M, \mathcal{F})$-space, then $\operatorname{res}_\alpha(X)$ is an $\operatorname{Or}(G, \alpha^{-1}(\mathcal{F}))$-space, where $\alpha^{-1}(\mathcal{F}) := \{H < G | \alpha(H) \in \mathcal{F}\}$.

We will work with the category $\mathrm{Or}(G, \mathcal{O})$, where $\mathcal{O}$ is the family of open (and hence also closed) subgroups. In this situation the objects of the category, the homogeneous spaces, are all discrete and as a morphism is already determined by its value on one element, the topology of each morphism set is discrete. Hence this is the same category as $\mathrm{Or}(G_d, \mathcal{O}_d)$ where G_d is G, but with the discrete topology, and $\mathcal{O}_d$ contains the same subgroups as $\mathcal{O}$. Any continuous *open* group homomorphism $\alpha : G \to M$ then induces a covariant functor $\alpha : \mathrm{Or}(G, \mathcal{O} \cap \mathcal{F}) \to \mathrm{Or}(G, \mathcal{O} \cap \alpha(\mathcal{F}))$ because $\alpha(\mathcal{O}) \subset \mathcal{O}$. If G is discrete, then $\mathrm{Or}(G, \mathcal{O}) = \mathrm{Or}(G)$.

Example 1.5 Any G-space X is a contravariant $\mathrm{Or}(G)$-space by $X(G/H) := X^H$ and

$$X(R_\gamma) : X^K \to X^H, x \mapsto \gamma^{-1}x.$$

A G-space X with isotropy groups in a family $\mathcal{F}$ of closed subgroups becomes an $\mathrm{Or}(G, \mathcal{F})$-space. X is called *smooth* if all its isotropy groups are open, i.e. if it is an $\mathrm{Or}(G, \mathcal{O})$-space, it is called *proper* if all its isotropy groups are compact i.e. if it is an $\mathrm{Or}(G, \mathcal{C})$-space, where $\mathcal{C}$ is the family of compact subgroups.

1.3 G-Homology Theories

Let G be a topological group and R an associative commutative ring with unit. A *G-homology theory $\mathcal{H}_*^G$ with values in R-modules* is a collection of covariant functors $\mathcal{H}_n^G$ from the category of G-CW-pairs to the category of R-modules indexed by $n \in \mathbb{Z}$ together with natural transformations $\partial_n^G(X, A) : \mathcal{H}_n^G(X, A) \to \mathcal{H}_{n-1}^G(A) := \mathcal{H}_{n-1}^G(A, \emptyset)$ for $n \in \mathbb{N}$ such that the following axioms are satisfied:

1. *G-homotopy invariance:* If f_0 and f_1 are G-homotopic maps $(X, A) \to (Y, B)$ of G-CW-pairs, then $\mathcal{H}_n^G(f_0) = \mathcal{H}_n^G(f_1)$ for $n \in \mathbb{Z}$;

2. *Long exact sequence of a pair:* Given a pair (X, A) of G-CW-complexes, there is a long exact sequence

$$\ldots \xrightarrow{\mathcal{H}_{n+1}^G(j)} \mathcal{H}_{n+1}^G(X, A) \xrightarrow{\partial_{n+1}^G} \mathcal{H}_n^G(A) \xrightarrow{\mathcal{H}_n^G(i)} \mathcal{H}_n^G(X) \xrightarrow{\mathcal{H}_n^G(j)} \mathcal{H}_n^G(X, A) \xrightarrow{\partial_n^G} \ldots,$$

   where $i : A \to X$ and $j : X \to (X, A)$ are the inclusions;

3. *Excision:* Let (X, A) be a G-CW-pair and let $f : A \to B$ be a cellular G-map of G-CW-complexes. Equip $(X \cup_f B, B)$ with the induced structure of a G-CW-pair. Then the canonical map $(F, f) : (X, A) \to (X \cup_f B, B)$ induces an isomorphism

$$\mathcal{H}_n^G(F, f) : \mathcal{H}_n^G(X, A) \xrightarrow{\cong} \mathcal{H}_n^G(X \cup_f B, B);$$

4. *Disjoint union axiom:* Let $\{X_i \mid i \in I\}$ be a family of G-CW-complexes. Denote by $j_i : X_i \to \coprod_{i \in I} X_i$ the canonical inclusion. Then the map

$$\bigoplus_{i \in I} \mathcal{H}_n^G(j_i) : \bigoplus_{i \in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G\left(\coprod_{i \in I} X_i\right)$$

   is bijective.

If $\mathcal{H}_*^G$ is defined or considered only for proper (resp. smooth) G-CW-pairs (X, A), we call it a *proper G-homology theory* (resp. *smooth G-homology theory*), or more generally a $\mathcal{F}$-G-homology theory, if it is defined or considered only for G-CW-pairs with isotropy in $\mathcal{F}$.

$\mathcal{F} \cap \mathcal{O}$-$G$-homology theories can be defined using $\mathrm{Or}(G, \mathcal{F} \cap \mathcal{O})$-spectra:

Definition 1.6 *Let G be a topological group and E a covariant $\mathrm{Or}(G, \mathcal{F} \cap \mathcal{O})$-spectrum. For a G-CW-pair (X, A) with isotropy in $\mathcal{F} \cap \mathcal{O}$ define*

$$H_*(X, A; E) := \pi_*((X_+) \cup_{A_+} cone(A_+) \otimes_{\mathrm{Or}(G, \mathcal{F} \cap \mathcal{O})} E).$$

In analogy to the corresponding non-equivariant construction this defines a $\mathcal{F} \cap \mathcal{O}$-$G$-homology theory (see [DL98], Chapter 4).

We will use the following lemma quite often:

Lemma 1.7 *(*[DL98] 4.6*) Let G be a topological group, E and F covariant $\mathrm{Or}(G, \mathcal{F} \cap \mathcal{O})$-spectra, $f : E \to F$ a map between them. It induces a natural transformation of $\mathcal{F} \cap \mathcal{O}$-$G$-homology theories*

$$f_* : H_*(X; E) \to H_*(X; F).$$

If f is a weak equivalence, then f_* is an isomorphism.

1.4 Totally Disconnected Groups

Definition 1.8 *A totally disconnected group is a topological group whose connected components all contain only one point.*

Many interesting groups naturally carry a totally disconnected topology:

Example 1.9

1. All discrete groups.
2. $\mathbb{Q}_p, \mathbb{Z}_p$, p prime.
3. $Gal(L/K)$, L/K a Galois-extension, is compact, totally disconnected (see for example [Bos93], Chapter 4.2).
4. $SL_n(\mathbb{Q}_p), GL_n(\mathbb{Q}_p)$, p prime.
5. $\prod_{\mathbb{Z}} F$, F a finite group.
6. Let G be a topological group and C the component of the unit in G. Then G/C is a totally disconnected group ([HR63], 7.3).
7. Let G be totally disconnected, locally compact, $H < G$ a closed subgroup. Then G/H is totally disconnected (G/H is 0-dimensional by [HR63], 7.11, i.e. the family of all sets that are both open and closed is an open basis. As G/H is again locally compact Hausdorff, every subset containing at least two points can be separated using an element of this basis.).

Every totally disconnected group is Hausdorff ([HR63], 7.7). Let G be a totally disconnected group that is locally compact. Then every neighbourhood of the unit in G contains an open compact subgroup of G ([HR63] II, 7.7). Vice versa let G be a locally compact Hausdorff group with the property that the unit has a neighbourhood base consisting of open compact subgroups. Then G is totally disconnected because the intersection of all open subgroups, which is the connected component of the unit ([HR63], 7.8), is the unit.

Some totally disconnected groups, for example all compact ones, even have a neighbourhood base of the unit consisting of open compact *normal* subgroups. If G has this property, then it can be written as an inverse limit of discrete groups:

Lemma 1.10 *Let G be a totally disconnected group which has a neighbourhood base of the unit consisting of open compact normal subgroups. Then*

$$G \cong \varprojlim_{K \in N(G)} G/K$$

where $N(G)$ is the set of all open compact normal subgroups of G, partially ordered by the opposite of the inclusion.

Proof: By the universal property of the inverse limit there is a continuous group homomorphism $\vartheta : G \to \varprojlim_{K \in N(G)} G/K$. As $\bigcap_{K \in N(G)} K = 1$ (e.g. [HR63], 7.8) this morphism is injective. To see that it is surjective note that any element of $\bigcap_{K \in N(G)} g_K K$ would be a preimage of the element $(g_K K) \in \varprojlim_{K \in N(G)} G/K$. As $\bigcap_{K \in N(G)} g_K K$ can be interpreted as an inverse limit of nonempty compact Hausdorff spaces, it is non-empty (see [RZ00], 1.1.4). □

In the Appendix we'll see that this property is a watershed between totally disconnected groups behaving like discrete groups and such showing different phenomena.

As said above we will work with smooth G-spaces, which have the advantage that the homogeneous spaces $G/G_x, x \in X$ are all discrete. In particular we are interested in the classifying space $E(G, \mathcal{CO})$, where $\mathcal{CO}$ is the family of compact open subgroups. For $G = SL_2(\mathbb{Q}_2)$ the tree with three edges at each vertice (which is the Bruhat-Tits building of $SL_2(\mathbb{Q}_2)$) is a model for it ([LM99], Chapter 3.3).

Lemma 1.11 *Let G be a totally disconnected group. A smooth G space X is a G-CW-complex if and only if G acts cellularly on X (i.e. for $g \in G$ and E an open cell of X, the left translation gE is again an open cell and if $gE = E$ then the induced map $E \to E$ is the identity).*

Proof: As G/G_x is discrete for all $x \in X$, the proof is analogous to the one of the discrete version of this lemma. For this see [tD87], II, 1.15. □

Note that if G is a locally compact Hausdorff (not necessarily totally disconnected) group acting smoothly on a G-space X, then the action factors through an action of the totally disconnected group G/C where C is the component of the unit in G. This is the case because C is the intersection of all open subgroups of G ([HR63], 7.3, 7.8).

In the next chapter the main theorem will be proven for topological groups and smooth actions. Readers only interested in discrete groups may "translate" the proof using the following table:

topological groups	**discrete groups**
open continuous group homomorphism	group homomorphism
smooth G-CW-complex	G-CW-complex
$\mathrm{Or}(G, \mathcal{O} \cap \mathcal{F})$	$\mathrm{Or}(G, \mathcal{F})$
smooth G-homology theory	G-homology theory
TGr	**Gr**

1.5 Categories

The following table is a collection of the categories we use, either with a short explanation (to be read with some good will) and/or the page where to find the definition.

	Objects	Morphisms	
Ab	abelian groups	group homomorphisms	
$\mathbf{Set}_*$	pointed sets	pointed maps	
Top	(compactly generated) topological spaces	continuous maps	
$\mathbf{Spt}(\mathbf{Top})$	spectra	strict maps between spectra	see 1.1
$\mathbf{\Omega Spt}(\mathbf{Top})$	Ω-spectra	strict maps	see 1.1
$\mathbf{Spt}(\mathbf{Ab})$			See 5.5
$\mathbf{Spt_{Kan}}(\mathbf{Ab})$			see p. 44
$\mathbf{Spt}(\mathbf{Ch}_+)$			see p. 45
$\mathbf{Ch}(\mathcal{C})_+$	positive chain complexes with chain objects in $\mathcal{C}$	chain maps	
$\mathbf{S}.\mathcal{C}$	functors $\Delta^{op} \to \mathcal{C}$	natural transformations	"simplicial objects in $\mathcal{C}$"
$R\mathcal{C}$	$c \in Ob(\mathcal{C})$	$R\,\mathrm{mor}_{\mathcal{C}}(c, d)$	
$\mathcal{C}_\oplus$	finite tuples of objects of $\mathcal{C}$	matrices of morphisms in $\mathcal{C}$	
$\mathrm{Or}(G, \mathcal{F})$			see Section 1.1
$\mathrm{Sub}(G, \mathcal{F})$			see p. 80
Gr	groupoids	functors	
TGr			see p. 26
HGr	topological groupoids with fixed Haar system	functors which are **open** on morphism sets	see p. 36
$\mathbf{HGr^{inj}}$		functors which are injective and **open** on morphism sets	see p. 39
$\mathbf{HGr_{td}}$	totally disconnected groupoids with fixed Haar system		see p. 38

	Objects	Morphisms	
Cat	small categories	functors	
$\mathbf{C^*Cat}$	C^*-categories	C^*-functors	see 6.1
kCat	k-categories (k a ring)		sometimes called "k-linear" categories
$\mathbf{Cat_{cof}}$			see p. 47
splexCat	small split exact categories	exact functors	
ν**Cat**	small non-unital categories	non-unital functors	
aCat	small additive categories	additive functors	
ssmCat	small symmetric monoidal categories		
$\{0 \to 1\}$	$0, 1$	$id_0, id_1, 0 \to 1$	
$\{0 \leftrightarrow 1\}$	$0, 1$	$id_0, id_1, 0 \to 1, 1 \to 0$	

We call two functors $F, G : \mathcal{C} \to \mathcal{D}$ *homotopic* if there exists a natural equivalence between them (sometimes a homotopy is only required to be a natural transformation, but as we will be working with groupoids the above notion of homotopy is more adequate). This natural equivalence will be called *homotopy* between F and G.

Lemma 1.12 *Let $\mathcal{C}, \mathcal{D}$ be small categories and $F, G : \mathcal{C} \to \mathcal{D}$ two functors. A natural transformation $\eta : F \Rightarrow G$ is the same as a functor $\eta : \mathcal{C} \times \{0 \to 1\} \to \mathcal{D}$.*

Proof: Given a natural transformation η define $\eta|_{\mathcal{C}\times\{0\}} := F$ and $\eta|_{\mathcal{C}\times\{1\}} := G$. Put $\eta(id_c, 0 \to 1) := \eta(c)$. □

In the same way a natural equivalence $\eta : F \Rightarrow G$ is the same as a functor $\eta : \mathcal{C} \times \{0 \leftrightarrow 1\} \to \mathcal{D}$.

Lemma 1.13 *Let $\mathbf{Cat}_<$ be a subcategory of $\mathbf{Cat}$ such that for every $\mathcal{C} \in Ob(\mathbf{Cat}_<)$ also $\mathcal{C} \times \{0 \leftrightarrow 1\} \in Ob(\mathbf{Cat}_<)$ and $pr : \mathcal{C} \times \{0 \leftrightarrow 1\} \to \mathcal{C}$ and $inc_i : \mathcal{C} \to \mathcal{C} \times \{0 \leftrightarrow 1\}, i = 0, 1$ are morphisms in $\mathbf{Cat}_<$.*
Let $E : \mathbf{Cat}_< \to \mathbf{Spt}(\mathbf{Top})$ be a functor with the property that $E(pr : \mathcal{C} \times \{0 \leftrightarrow 1\} \to \mathcal{C})$ is a weak equivalence of spectra for every category $\mathcal{C} \in Ob(\mathbf{Cat}_<)$.

Then E maps equivalences of categories to weak equivalences of spectra.

Proof: Let $F : \mathcal{C} \to \mathcal{C}$ be a functor which is homotopic to the identity via a homotopy η. We have to show that $\pi_*(E(F)) = id$. The assumption on E assures that $\pi_*(E(inc_0)) = \pi_*(E(inc_1))$, where $inc_i : \mathcal{C} \to \mathcal{C} \times \{0 \leftrightarrow 1\}$ are the inclusions. Hence $\pi_*(E(F)) = \pi_*(E(\eta \circ inc_0)) = \pi_*(E(\eta \circ inc_1)) = \pi_*(E(id))$. □

Lemma 1.14 *Let* $\mathbf{Cat_1}, \mathbf{Cat_2}$ *be subcategories of* $\mathbf{Cat}$ *as in the last lemma and let* $F : \mathbf{Cat_1} \to \mathbf{Cat_2}$ *be a functor such that for every category* $\mathcal{C} \in \mathbf{Cat_1}$ *the two inclusions*

$$F(inc_i) : F(\mathcal{C}) \to F(\mathcal{C} \times \{0 \leftrightarrow 1\}), i = 0, 1$$

extend to a functor

$$\iota_{\mathcal{C}} : F(\mathcal{C}) \times \{0 \leftrightarrow 1\} \to F(\mathcal{C} \times \{0 \leftrightarrow 1\}).$$

Then F *maps homotopic functors to homotopic functors.*

Proof: Let $\eta : \mathcal{C} \times \{0 \leftrightarrow 1\} \to \mathcal{D}$ be a homotopy between two functors A and B. Then $F(\eta) \circ \iota_{\mathcal{C}}$ is a homotopy between $F(A)$ and $F(B)$. □

The situation of the lemma above is given for example if $F(\mathcal{C}) \times \{0 \leftrightarrow 1\}$ is a subcategory of $F(\mathcal{C} \times \{0 \leftrightarrow 1\})$.

2 Smooth Equivariant Families of Spectra

In this chapter we introduce the notion of a smooth equivariant homology theory (recall that we chose the smooth setting because the morphism sets of $\mathrm{Or}(G, \mathcal{O})$ are discrete). In analogy to it we define smooth equivariant families of spectra and prove the main theorem:

Theorem 2.10 *Let* $E : \mathbf{Gr} \to \mathbf{Spt}(\mathbf{Top})$ *(resp.* $E : \mathbf{TGr} \to \mathbf{Spt}(\mathbf{Top})$*) be a functor with the property that it maps equivalences of (topological) groupoids to weak equivalences of spectra.*

Then maps i_α *can be constructed such that* $(E^? := E \circ \mathcal{G}^?, i_?)$ *is a (smooth) equivariant family of spectra (and hence* $H^?_*(\,\cdot\,; E \circ \mathcal{G}^?)$ *a (smooth) equivariant homology theory).*

Using this theorem many interesting examples of (smooth) equivariant homology theories can be constructed, for example equivariant homology theories for algebraic K- and L-theory (Chapter 3), which are studied in [Lüc02], or smooth equivariant homology theories whose coefficients are the Hochschild Homology or Cyclic Homology of the Hecke algebra (Chapter 5), or a smooth equivariant homology for topological K-theory (Chapter 6).

2.1 Smooth Equivariant Homology Theories

We extend the notion of an equivariant homology theory for discrete groups as used in [Lüc02] to that of a *smooth* equivariant homology theory for arbitrary topological groups:

Definition 2.1 *A smooth equivariant homology theory* $\mathcal{H}^?_*$ *(or more exactly* $(\mathcal{H}^?_*, \mathrm{ind}_?)$*) is an assignment which associates to every topological group* G *a smooth* G*-homology theory* $\mathcal{H}^G_*$ *together with the following so called induction structure:*

Let $\alpha : G \to M$ *be a continuous open group homomorphism,* (X, A) *a smooth* G*-CW-pair on which* $\ker(\alpha)$ *acts freely. Then for every* $n \in \mathbb{Z}$ *there is a natural isomorphism*

$$\mathrm{ind}_\alpha : \mathcal{H}^G_n(X, A) \to \mathcal{H}^M_n(\mathrm{ind}_\alpha(X, A))$$

which is compatible with the boundary homomorphisms. We require

1. **Functoriality:** *Let $\beta : M \to L$ be another open group homomorphism. Then*

$$\operatorname{ind}_{\beta\alpha} = \mathcal{H}_n^L(f_1) \circ \operatorname{ind}_\beta \circ \operatorname{ind}_\alpha$$

where $f_1 : \operatorname{ind}_\beta \operatorname{ind}_\alpha(X, A) \to \operatorname{ind}_{\beta\alpha}(X, A)$ is the natural L-homeomorphism.

2. **Compatibility with Conjugation:** *Let $\gamma \in G, c(\gamma) : G \to G, g \mapsto \gamma g \gamma^{-1}$ and let (X, A) be a smooth G-CW-pair. Then*

$$\operatorname{ind}_{c(\gamma)} : \mathcal{H}_n^G(X, A) \to \mathcal{H}_n^G(\operatorname{ind}_{c(\gamma)}(X, A))$$

agrees with

$$\mathcal{H}_n^G(f_2) : \mathcal{H}_n^G(X, A) \to \mathcal{H}_n^G(\operatorname{ind}_{c(\gamma)}(X, A))$$

where

$$f_2 : (X, A) \to \operatorname{ind}_{c(\gamma)}(X, A), x \mapsto (1, \gamma^{-1}x)$$

is the canonical G-homeomorphism.

Depending on the application we have in mind, we vary the notion of a smooth equivariant homology theory insofar as we restrict to discrete groups ($\rightsquigarrow$ equivariant homology theory as defined in [Lüc02], Chapter 1) or totally disconnected groups, ...

The notion of an equivariant homology theory is needed in [Lüc02] where Lück constructs a Chern-Character using proper equivariant homology theories.

Theorem ([Lüc02], 4.4) *Let $\mathcal{H}_*^?$ be a proper equivariant homology theory with values in R-modules, where R is a commutative ring containing the rationals. Suppose for every discrete group G that the $R\operatorname{Sub}(G, \mathcal{FIN})$-module $\mathcal{H}_*^G(G/?)$ is flat for all $q \geq 0$. Then there is a natural isomorphism*

$$ch_*^G(X, A) : \bigoplus_{p+q=*} H_p^{\operatorname{Or}(G)}(X, A; \mathcal{H}_q^G(G/?)) \xrightarrow{\cong} \mathcal{H}_*^G(X, A)$$

commuting with the boundary maps and the induction homomorphisms.

This isomorphism is explicitly constructed with the help of the induction homomorphisms of $\mathcal{H}_*^?$. An application of this Chern character is the following theorem :

Theorem ([Lüc02], 0.4) *Let R be a commutative ring with $\mathbb{Q} \subset R$. Denote by F the field $\mathbb{R}$ or $\mathbb{C}$. Let G be a (discrete) group. Let J be the set of conjugacy classes (C) of finite cyclic subgroups C of G. Then the rationalized assembly map in the Farrell-Jones Conjecture with respect to the family $\mathcal{FIN}$ of finite subgroups for the algebraic K-groups $K_n(RG)$ and the algebraic L-groups $L_n(RG)$ and in the Baum-Connes Conjecture for the topological K-groups $K_n^{top}(C_r^*(G, F))$ can be identified with the homomorphisms*

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in J} H_p(C_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \theta_C^C \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} K_q(RC)) \to \mathbb{Q} \otimes_{\mathbb{Z}} K_n(RG);$$

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in J} H_p(C_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \theta_C^C \cdot (\mathbb{Q} \otimes_{\mathbb{Z}} L_q(RC)) \to \mathbb{Q} \otimes_{\mathbb{Z}} L_n(RG);$$

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in J} H_p(C_G C;\mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \theta_C^C \cdot (\mathbb{Q}\otimes_{\mathbb{Z}} K_q^{top}(C_r^*(C,F)))$$

$$\to \mathbb{Q} \otimes_{\mathbb{Z}} K_q^{top}(C_r^*(G,F)).$$

In the L-theory case we assume that R comes with an involution $R \to R$, $r \mapsto \overline{r}$ and that we use on RG the involution which sends $\sum_{g\in G} r_g \cdot g$ to $\sum_{g\in G} \overline{r_g} \cdot g^{-1}$.

If the Farrell-Jones Conjecture with respect to $\mathcal{FIN}$ and the Baum-Connes Conjecture are true, then these maps are isomorphisms.

In the special situation that the coefficient ring R is a field F of characteristic zero and one tensors with $\overline{F}\otimes_{\mathbb{Z}}?$ for an algebraic closure $\overline{F}$ of F, these expressions can be simplified further as carried out in Section 8 of [Lüc02]:

Theorem ([Lüc02], 0.5) *Let G be a (discrete) group. Let T be the set of conjugacy classes (g) of elements $g \in G$ of finite order. There is a commutative diagram*

$$\begin{array}{ccc}
\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \longrightarrow & \mathbb{C} \otimes_{\mathbb{Z}} K_n(\mathbb{C}G) \\
\downarrow & & \downarrow \\
\bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(C_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q^{top}(\mathbb{Z}) & \longrightarrow & \mathbb{C} \otimes_{\mathbb{Z}} K_n^{top}(C_r^*(G))
\end{array}$$

where $C_G\langle g\rangle$ is the centralizer of the cyclic group generated by g in G and the vertical arrows come from the obvious change of ring and of K-theory maps $K_q(\mathbb{C}) \to K_q^{top}(\mathbb{C})$ and $K_n(\mathbb{C}G) \to K_n^{top}(C_r^*(G))$. The horizontal arrows can be identified with the assembly maps occurring in the Farrell-Jones Conjecture with respect to $\mathcal{FIN}$ for $K_n(\mathbb{C}G)$ and in the Baum-Connes Conjecture for $K_n^{top}(C_r^*(G))$ after applying $\mathbb{C}\otimes_{\mathbb{Z}}-$. If these conjectures are true for G, then the horizontal arrows are isomorphisms.

The following examples show that equivariant homology theories appear quite naturally.

Example 2.2

1. Let $\mathcal{K}_*$ be a homology theory for (non-equivariant) CW-pairs with values in R-modules. Then we obtain a smooth equivariant homology theory with values in R-modules by defining

   $$\mathcal{H}_n^G(X,A) := \mathcal{K}_n(G\backslash X, G\backslash A)$$

   with induction homomorphisms given by the homeomorphisms

   $$H\backslash X \to G\backslash(G \times_\alpha X), Hx \mapsto G(1,x)$$

   for $\alpha : H \to G$ an open continuous group homomorphism with $\ker(\alpha)$ acting freely on X (compare [Lüc02], 1.3).

2. Again let $\mathcal{K}_*$ be a homology theory for (non-equivariant) CW-pairs with values in R-modules. Then the *equivariant Borel homology associated to* $\mathcal{K}$ defines a discrete equivariant homology theory with values in R-modules. It is given by

   $$\mathcal{H}_n^G(X,A) := \mathcal{K}_n(EG \times_G (X,A)).$$

The induction homomorphisms are given by

$$EH \times_H X \to EG \times_G \times_\alpha X, (e,x) \mapsto (E\alpha(e), 1, x).$$

If $\ker(\alpha)$ acts freely on X then this map is a homotopy equivalence ([Lüc02], 1.3).

3. Equivariant Bordism defines a discrete equivariant homology theory ([Lüc02], 1.4).

4. Let $\mathcal{H}_*^?$ be a proper smooth equivariant homology. Then

$$\mathcal{BH}_n^G(X,A) := \bigoplus_{p+q=n} H_p^{\mathrm{Or}(G,\mathcal{CO})}(X,A;\mathcal{H}_q^G(G/?))$$

where $H_*^{\mathrm{Or}(G,\mathcal{CO})}$ is the *equivariant Bredon homology* defines a smooth proper equivariant homology theory, the *associated Bredon homology* (compare [Lüc02], Chapter 3).

We now want to define a counterpart for this on spectra, meaning that there should be for every topological group G a covariant $\mathrm{Or}(G,\mathcal{O})$-spectrum E^G and maps i_α between these spectra such that the homology theories $H_*^G(\,\cdot\,;E^G)$ form a smooth equivariant homology theory with induction structure induced by the maps i_α.

We first note that if $\alpha : G \to M$ is an open group homomorphism and $\ker(\alpha)$ acts freely on the smooth G-space X, then all the isotropy groups of X are elements of the family $\mathcal{F}_\alpha := \{H \in \mathcal{O} | H \cap \ker(\alpha) = 1\}$. Hence X is a $\mathrm{Or}(G,\mathcal{F}_\alpha)$-space and any map

$$i_\alpha : E^G|_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)} \to (\mathrm{res}_\alpha E^M)|_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)}$$

defines a homomorphism

$$\mathrm{ind}_\alpha(X) : H_*^G(X;E^G) \to H^M(\mathrm{ind}_\alpha(X);E^M)$$

in the following way:

$$\begin{array}{ccc}
H_*^G(X;E^G) = \pi(X_+ \otimes_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)} E^G) & \xrightarrow{(i_\alpha)_*} & \pi_*(X_+ \otimes_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)} (\mathrm{res}_\alpha E^M)) \\
 & & \downarrow \cong \\
\vdots \; \mathrm{ind}_\alpha(X) & & \pi_*(\mathrm{ind}_\alpha(X_+) \otimes_{\mathrm{Or}(M,\alpha(\mathcal{O}\cap\mathcal{F}_\alpha))} E^M) \\
\downarrow & & \downarrow \cong \\
H_*^M(\mathrm{ind}_\alpha(X);E^M) & = & \pi_*(\mathrm{ind}_\alpha(X)_+ \otimes_{\mathrm{Or}(M,\mathcal{O})} E^M)
\end{array}$$

(the spectra have to be restricted to $\mathrm{Or}(G,\mathcal{O})$, we don't write that explicitly). The upper right vertical arrow is the natural adjunction homeomorphism of Remark 1.4. Note that we have

$$X_+ \otimes_{\mathrm{Or}(G,\mathcal{O})} E = X_+|_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)} \otimes_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)} E|_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)}$$

for any covariant $\mathrm{Or}(G,\mathcal{O})$-spectrum E, as $X(G/H) = \emptyset$ for $H \notin \mathcal{F}_\alpha$ by definition. For G-CW-pairs (X,A) we remark that pushout and cone commute with induction.

Now we define:

Definition 2.3 *A smooth equivariant family of spectra $E^?$ (or more exactly $(E^?, i_?)$) is an assignment which associates to every topological group G a covariant* $\mathrm{Or}(G, \mathcal{O})$*-spectrum E^G and to every open continuous group homomorphism $\alpha : G \to M$ a map of* $\mathrm{Or}(G, \mathcal{O} \cap \mathcal{F}_\alpha)$*-spectra*

$$i_\alpha : E^G|_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)} \to (\mathrm{res}_\alpha E_M)|_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)}$$

such that $(H^?_*(\,\cdot\,; E^?), \mathrm{ind}_?)$ as defined above is a smooth equivariant family of spectra.

By Lemma 1.7 naturality and compatibility with the boundary homomorphisms are satisfied always and the morphisms $\mathrm{ind}_\alpha(X)$ are isomorphisms if the i_α are weak equivalences.

Lemma 2.4 *Let $(E^?, i_?)$ be a smooth equivariant family of spectra and let $\alpha : G \to M, \beta : M \to L$ be two open continuous group homomorphisms. Let*

$$i_{\beta\alpha} \simeq (\mathrm{res}_\alpha i_\beta) \circ i_\alpha.$$

Then

$$\mathrm{ind}_{\beta\alpha} = H^L_*(f_1; E^L) \circ \mathrm{ind}_\beta \circ \mathrm{ind}_\alpha .$$

Note that $\mathcal{F}_{\beta\alpha} = \mathcal{F}_\alpha \cap \alpha^{-1}(\mathcal{F}_\beta)$ and $\mathrm{res}_\alpha(E|_{\mathcal{F}_\beta}) = (\mathrm{res}_\alpha E)|_{\alpha^{-1}(\mathcal{F}_\beta)}$.

Proof: The diagram on the next page commutes for any $\mathcal{O} \cap \mathcal{F}_\alpha$-CW-complex X. Note here that $\alpha(\mathcal{F}_{\beta\alpha}) \subset \mathcal{F}_\beta$. The left square commutes by assumption, the other two squares commute because of the naturality of the adjunction homeomorphisms. Hence the outer square commutes. □

There is a very conceptual condition which can be posed on the maps $i_{c(\gamma)}$ to get compatibility with conjugation.

Lemma 2.5 *Let $(E^?, i_?)$ be a smooth equivariant family of spectra, $\gamma \in G, c(\gamma) : G \to G, g \mapsto \gamma g \gamma^{-1}$ and*

$$i_{c(\gamma)} \simeq E_\gamma : E^G \to \mathrm{res}_{c(\gamma)} E^G,$$

where E_γ is the map of $\mathrm{Or}(G, \mathcal{O})$*-spectra given by*

$$E_\gamma(G/H) = E^G(R_\gamma) : E^G(G/H) \to E^G(G/(\gamma H \gamma^{-1})).$$

Then $H^G_*(f_2, E^G) = \mathrm{ind}_{c(\gamma)}$.

Proof: The following diagram commutes ($\mathcal{F}_{c(\gamma)} \cap \mathcal{O} = \mathcal{O}$):

$$\begin{array}{ccc}
X_+ \otimes_{\mathrm{Or}(G,\mathcal{O})} E^G & \xrightarrow{id \otimes E_\gamma} & X_+ \otimes_{\mathrm{Or}(G,\mathcal{O})} \mathrm{res}_{c(\gamma)} E^G \\
& \searrow^{f_2 \otimes id} & \uparrow \cong \\
& & \mathrm{ind}_{c(\gamma)} X_+ \otimes_{\mathrm{Or}(G,\mathcal{O})} E^G
\end{array}$$

where $f_2 : x \mapsto (x, R_\gamma : G/H \to G/(\gamma H \gamma^{-1}))$.

$$
\begin{array}{ccccc}
\pi_*(X_+ \otimes_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_{\beta\alpha})} E^G) & \xrightarrow{(i_\alpha|_{\mathcal{F}_{\beta\alpha}})_*} & \pi_*(X_+ \otimes_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_{\beta\alpha})} \mathrm{res}_\alpha E_M) & \xrightarrow{\cong} & \pi_*(\mathrm{ind}_\alpha X_+ \otimes_{\mathrm{Or}(M,\mathcal{O}\cap\alpha(\mathcal{F}_{\beta\alpha}))} E_M) \\
\big\downarrow{\scriptstyle (i_{\beta\alpha})_*} & & \big\downarrow{\scriptstyle (\mathrm{res}_\alpha i_\beta|_{\mathcal{F}_{\beta\alpha}})_*} & & \big\downarrow{\scriptstyle (i_\beta|_{\alpha\mathcal{F}_{\beta\alpha}})_*} \\
 & & \pi_*(X_+ \otimes_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_{\beta\alpha})} \mathrm{res}_\alpha \mathrm{res}_\beta E_L) & \xrightarrow{\cong} & \pi_*(\mathrm{ind}_\alpha X_+ \otimes_{\mathrm{Or}(M,\mathcal{O}\cap\alpha(\mathcal{F}_{\beta\alpha}))} \mathrm{res}_\beta E_L) \\
 & \diagup\!\!\diagup & & & \big\downarrow{\scriptstyle \cong} \\
\pi_*(X_+ \otimes_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_{\beta\alpha})} \mathrm{res}_\alpha \mathrm{res}_\beta E_L) & \xrightarrow{\cong} & \pi_*(\mathrm{ind}_{\beta\alpha} X_+ \otimes_{\mathrm{Or}(L,\mathcal{O}\cap\beta\alpha(\mathcal{F}_{\beta\alpha}))} E_L) & \xrightarrow{(f_1)_*} & \pi_*(\mathrm{ind}_\beta \mathrm{ind}_\alpha X_+ \otimes_{\mathrm{Or}(L,\mathcal{O}\cap\beta\alpha(\mathcal{F}_{\beta\alpha}))} E_L) \\
 & & \| & & \| \\
 & & \pi_*(\mathrm{ind}_{\beta\alpha} X_+ \otimes_{\mathrm{Or}(L,\mathcal{O})} E_L) & & \pi_*(\mathrm{ind}_\beta \mathrm{ind}_\alpha X_+ \otimes_{\mathrm{Or}(L,\mathcal{O})} E_L)
\end{array}
$$

As $i_\alpha \simeq E_\gamma$ the following diagram commutes up to homotopy

$$\begin{array}{ccc} X_+ \otimes_{\mathrm{Or}(G,\mathcal{O})} E^G & \xrightarrow{id \otimes i_\alpha} & X_+ \otimes_{\mathrm{Or}(G,\mathcal{O})} \mathrm{res}_{c(\gamma)} E^G \\ & \searrow^{f_2 \otimes id} & \uparrow \cong \\ & & \mathrm{ind}_{c(\gamma)} X_+ \otimes_{\mathrm{Or}(G,\mathcal{O})} E^G \end{array}$$

and hence

$$\begin{array}{ccc} \pi_*(X_+ \otimes_{\mathrm{Or}(G,\mathcal{O})} E^G) & \xrightarrow{(id \otimes i_\alpha)_*} & \pi_*(X_+ \otimes_{\mathrm{Or}(G,\mathcal{O})} \mathrm{res}_{c(\gamma)} E^G) \\ & \searrow^{(f_2 \otimes id)_*} & \uparrow \cong \\ & & \pi_*(\mathrm{ind}_{c(\gamma)} X_+ \otimes_{\mathrm{Or}(G,\mathcal{O})} E^G) \end{array}$$

commutes and this proves compatibility with conjugation. □

The last two lemmata motivate the following definition:

Definition 2.6 *A strong smooth equivariant family of spectra $E^?$ is an assignment which associates to every topological group G a covariant $\mathrm{Or}(G,\mathcal{O})$-spectrum E^G and to every open continuous group homomorphism $\alpha : G \to M$ a weak equivalence of $\mathrm{Or}(G, \mathcal{O} \cap \mathcal{F}_\alpha)$-spectra*

$$i_\alpha : E^G|_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)} \to (\mathrm{res}_\alpha E^M)|_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)}$$

such that

$$i_{\beta\alpha} \simeq (\mathrm{res}_\alpha i_\beta) \circ i_\alpha$$

for any second open continuous group homomorphism $\beta : M \to L$ and

$$i_{c(\gamma)} \simeq E_\gamma : E^G \to \mathrm{res}_{c(\gamma)} E^G$$

for any element $\gamma \in G$ and E_γ as above.

By Lemma 2.4 and Lemma 2.5 a strong smooth equivariant family of spectra in particular is a smooth equivariant family of spectra.

Example 2.7 Let $\mathcal{H}_*^?$ be a smooth equivariant homology theory. Then a strong smooth equivariant family of spectra inducing the associated Bredon homology of $\mathcal{H}_*^?$ is given in the following way. Let $K : \mathbf{Ab} \to \mathbf{Spt}(\mathbf{Top})$ be a functor which assigns to an abelian group B an Eilenberg-MacLane-spectrum KB of this group (for the definition of such a functor see for example [Jar97], p.126/127). If $B : \mathrm{Or}(G,\mathcal{O}) \to \mathbf{Ab}$ is a covariant functor, then the $\mathrm{Or}(G,\mathcal{O})$-spectrum E_B defined by

$$E_B(G/H) := K(B(G/H))$$

induces the Bredon homology with coefficients in B (which is an ordinary homology theory). For every topological group G, we define the covariant $\mathrm{Or}(G,\mathcal{O})$-spectrum E^G by

$$E^G(G/H) := \bigvee_{q\in\mathbb{Z}} \Sigma^q K(\mathcal{H}_q^G(G/H)),$$

where $(\Sigma^q F)_n := F_{n+q}$. Then for every smooth G-CW complex X we have

$$\begin{aligned}\pi_*(X_+ \otimes_{\mathrm{Or}(G,\mathcal{O})} E^G) &= \pi_*(X_+ \otimes_{\mathrm{Or}(G,\mathcal{O})} \bigvee_{q\in\mathbb{Z}} \Sigma^q K(\mathcal{H}_q^G(G/?)))\\ &\cong \bigoplus_{q\in\mathbb{Z}} \pi_*(X_+ \otimes_{\mathrm{Or}(G,\mathcal{O})} \Sigma^q K(\mathcal{H}_q^G(G/?)))\\ &= \bigoplus_{q\in\mathbb{Z}} \pi_{*-q}(X_+ \otimes_{\mathrm{Or}(G,\mathcal{O})} K(\mathcal{H}_q^G(G/?)))\\ &= \bigoplus_{q\in\mathbb{Z}} H_{*-q}^{\mathrm{Or}(G,\mathcal{O})}(X; \mathcal{H}_q^G(G/?))\\ &= \mathcal{B}\mathcal{H}_*^G(X)\end{aligned}$$

For every open continuous group homomorphism $\alpha : G \to M$ the induction structure of $\mathcal{H}_*^?$ gives us natural isomorphisms

$$\mathrm{ind}_\alpha(G/H) : \mathcal{H}_*^G(G/H) \to \mathcal{H}_*^M(M/\alpha(H))$$

for every open subgroup H which intersects trivially with the kernel of α. Hence we get maps of spectra

$$i_\alpha : E^G|_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)} \to (\mathrm{res}_\alpha E^M)|_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)}$$

such that

$$i_{\beta\alpha} = (\mathrm{res}_\alpha i_\beta) \circ i_\alpha,$$

as this is true for $\mathrm{ind}_?$. Furthermore for $\gamma \in G$

$$E_\gamma(G/H) = E^G(R_\gamma) : E^G(G/H) \to (\mathrm{res}_{c(\gamma)} E^G)(G/H)$$

is induced by the homomorphism

$$\mathcal{H}_*^G(R_\gamma) : \mathcal{H}_*^G(G/H) \to \mathcal{H}_*^G(G/\gamma H\gamma^{-1})$$

which is equal to $\mathrm{ind}_{c(\gamma)}(G/H)$ as $\mathcal{H}_*^?$ is a smooth equivariant homology theory. Hence $E_\gamma = i_{c(\gamma)}$. □

2.2 Proof of the Main Theorem

We want to construct a smooth equivariant family of spectra, or only a spectrum over the orbit category, say for example for a discrete group G and algebraic K-theory. By this we mean that we look for a spectrum over $\mathrm{Or}(G)$ which at each object G/H has the weak homotopy type of the spectrum $K^{alg}(\mathbb{Z}H)$ and has the property that for $\gamma \in G$ the map of spectra $K^{alg}(R_\gamma : G/H \to G/K)$ induces $K^{alg}(c(\gamma))$ on homotopy, where $c(\gamma) : \mathbb{Z}H \to \mathbb{Z}K$ is induced by conjugation with γ. The naive approach would be to define $K^{alg}(G/H)$ as $K^{alg}(\mathbb{Z}H)$. But this doesn't work, as for every $\gamma \in G$ and $h \in H$ the elements γ and $h\gamma$ induce the same morphism in $\mathrm{Or}(G)$, but not the same map between the spectra $K^{alg}(\mathbb{Z}H)$ and $K^{alg}(\mathbb{Z}K)$, though the two maps induce the same in homotopy. This problem can be solved by factoring the functor $\mathrm{Or}(G) \to \mathbf{Spt}(\mathbf{Top})$ through the category of groupoids (see also [Lüc02], Chapter 2).

Let **Gr** be the category of groupoids and **TGr** the category of topological groupoids, i.e. groupoids with a *locally compact Hausdorff* topology on each morphism set, such that composition and taking the inverse is continuous. A functor between two topological groupoids is required to be continuous on each morphism set. Sometimes a topological groupoid is also required to have a topology on the objects, but in our definition the set of objects is discrete.

Let G be a topological group, H an open subgroup. Let $\mathcal{G}^G(G/H)$ be the groupoid associated to the G-set G/H: The objects are the elements gH of G/H and

$$\mathrm{mor}_{\mathcal{G}^G(G/H)}(g_1H, g_2H) = \{g \in G | gg_1H = g_2H\},$$

composition is given by multiplication. Hence

$$\mathrm{mor}_{\mathcal{G}^G(G/H)}(g_1H, g_2H) = g_2Hg_1^{-1}$$

and $\mathcal{G}^G(G/H)$ becomes a topological groupoid, the topology on the morphism sets being induced by the topology of G. This construction is functorial:

Definition 2.8 *Let G be a topological group. Define*

$$\mathcal{G}^G : \mathrm{Or}(G, \mathcal{O}) \to \mathbf{TGr}$$

to be the functor given by

$$G/H \mapsto \mathcal{G}^G(G/H)$$

$$(G/H \xrightarrow{R_\gamma} G/K) \mapsto \mathcal{G}^G(R_\gamma)$$

where $\mathcal{G}^G(R_\gamma)$ is given by the inclusion

$$gH \mapsto g\gamma^{-1}K$$

$$g_2Hg_1^{-1} \subset g_2\gamma^{-1}K\gamma g_1^{-1}.$$

Remark 2.9 For every topological group G and open subgroup H the inclusion

$$\mathcal{G}^H(H/H) \to \mathcal{G}^G(G/H)$$

is an equivalence of topological groupoids. □

In this section we'll use the notion of an equivariant family of groupoids (which is an auxiliary construction and will appear only here) to prove the following theorem which is the main theorem of this section:

Theorem 2.10 *Let $E : \mathbf{Gr} \to \mathbf{Spt}(\mathbf{Top})$ (resp. $E : \mathbf{TGr} \to \mathbf{Spt}(\mathbf{Top})$) be a functor with the property that it maps equivalences of (topological) groupoids to weak equivalences of spectra.*

Then maps i_α can be constructed such that $(E^? := E \circ \mathcal{G}^?, i_?)$ is a (smooth) equivariant family of spectra (and hence $H_*^?(\,\cdot\,; E \circ \mathcal{G}^?)$ a (smooth) equivariant homology theory).

Remark 2.11 We do the proof for topological groups. The discrete case works just the same, forgetting everything about topology. To translate the proof use the table on page 16. □

Definition 2.12 *A strong smooth equivariant family of topological groupoids $\mathcal{GR}^?$ (or more exactly $(\mathcal{GR}^?, \eta_?)$) is an assignment which associates to every topological group G a covariant topological $\mathrm{Or}(G,\mathcal{O})$-groupoid $\mathcal{GR}^G$ (i.e. a covariant functor $\mathcal{GR}^G$: $\mathrm{Or}(G,\mathcal{O}) \to$ **TGr**) and to every open continuous group homomorphism $\alpha : G \to M$ a morphism of topological $\mathrm{Or}(G, \mathcal{O} \cap \mathcal{F}_\alpha)$-groupoids*

$$\eta_\alpha : \mathcal{GR}^G|_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)} \to (\mathrm{res}_\alpha \, \mathcal{GR}^M)|_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)}$$

such that

$$\eta_{\beta\alpha} \simeq (\mathrm{res}_\alpha \, \eta_\beta) \circ \eta_\alpha$$

for any second open continuous group homomorphism $\beta : M \to L$, and

$$\eta_{c(\gamma)} \simeq \mathcal{GR}_\gamma : \mathcal{GR}^G \to \mathrm{res}_{c(\gamma)} \, \mathcal{GR}^G$$

for any element $\gamma \in G$.

Recall that a homotopy between morphisms $A, B : \mathcal{G}_1 \Rightarrow \mathcal{G}_2$ of topological $\mathrm{Or}(G,\mathcal{F})$-groupoids ($\mathcal{F} \subset \mathcal{O}$ a family of subgroups) is a morphism $\eta : \mathcal{G}_1 \times \{0 \leftrightarrow 1\} \Rightarrow \mathcal{G}_2$ such that $\eta \circ inc_0 = A$ and $\eta \circ inc_1 = B$ where $inc_i : \mathcal{G} \Rightarrow \mathcal{G} \times \{0 \leftrightarrow 1\}, i = 0, 1$ are the inclusions. Note here that a morphism of topological $\mathrm{Or}(G,\mathcal{F})$-groupoids is a natural transformation and in the language of 2-categories a homotopy as above is a modification, as $\mathrm{Or}(G,\mathcal{F})$ has discrete 2-structure. The morphism $\mathcal{GR}_\gamma$ is defined in analogy to E_γ, namely by

$$\mathcal{GR}_\gamma(G/H) := \mathcal{GR}^G(R_\gamma) : \mathcal{GR}^G(G/H) \to \mathrm{res}_{c(\gamma)} \, \mathcal{GR}^G(G/H)$$

for every object G/H of $\mathrm{Or}(G,\mathcal{F})$. Now we can prove the following theorem:

Theorem 2.13 *Let $(\mathcal{GR}^?, \eta_?)$ be a strong smooth equivariant family of topological groupoids and let E : **Gr** $\to$ **Spt**(**Top**) be a functor with the property that it maps equivalences of topological groupoids to weak equivalences of spectra.*

Then maps i_α can be constructed such that $(E \circ \mathcal{GR}^?, i_?)$ is a smooth equivariant family of spectra (and hence $H_*^?(\,\cdot\,; E \circ \mathcal{GR}^?)$ a smooth equivariant homology theory).

Proof: Denote by $\mathcal{E}$ the functor "composition with E" which goes from the category of covariant topological $\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)$-groupoids to the category of covariant $\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)$-spectra. For every open continuous group homomorphism $\alpha : G \to M$ the induction map

$$i_\alpha : \mathcal{E}(\mathcal{G}^G)|_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)} \to (\mathrm{res}_\alpha \, \mathcal{E}(\mathcal{G}^M))|_{\mathrm{Or}(G,\mathcal{O}\cap\mathcal{F}_\alpha)}$$

is defined by

$$i_\alpha = \mathcal{E}(\eta_\alpha).$$

As $\mathcal{E}$ is a functor, i_α is a natural transformation, i.e. a map of $\mathrm{Or}(G, \mathcal{O} \cap \mathcal{F}_\alpha)$-spectra.

i_α is a weak homotopy equivalence because each $\eta_\alpha(G/H)$ is an equivalence of topological groupoids and E maps equivalences of topological groupoids to weak homotopy equivalences of spectra.

Lemma 2.14 *Let T, S be covariant topological $\mathrm{Or}(G,\mathcal{F})$-groupoids ($\mathcal{F} \subset \mathcal{O}$ a family of subgroups) and let $\eta, \zeta : T \Rightarrow S$ be two morphisms between them. Let Θ be a homotopy between η and ζ. Then*

$$\mathcal{E}(\eta)_*(X) = \mathcal{E}(\zeta)_*(X) : H_*^G(X; \mathcal{E}(T)) \to H_*^G(X; \mathcal{E}(S))$$

for any smooth G-CW-complex X.

Proof: Fix a smooth G-CW-complex X. The projection

$$pr : T \times \{0 \leftrightarrow 1\} \Rightarrow T,$$

$$T(G/H) \times \{0 \leftrightarrow 1\} \to T(G/H)$$

and the inclusions

$$inc_i : T \Rightarrow T \times \{0 \leftrightarrow 1\}, i = 0, 1$$

$$T(G/H) \to T(G/H) \times \{0 \leftrightarrow 1\}$$

induce isomorphisms

$$\mathcal{E}(pr)_*(X) : H_*^G(X; \mathcal{E}(T \times \{0 \leftrightarrow 1\})) \to H_*^G(X; \mathcal{E}(T))$$

and

$$\mathcal{E}(inc_i)_*(X) : H_*^G(X; \mathcal{E}(T)) \to H_*^G(X; \mathcal{E}(T \times \{0 \leftrightarrow 1\}))$$

because by assumption on E they induce weak equivalences $\mathcal{E}(pr), \mathcal{E}(inc_i)$ of $\mathrm{Or}(G, \mathcal{F})$-spectra. As $pr \circ inc_0 = pr \circ inc_1$ it follows that $\mathcal{E}(inc_0)_*(X) = \mathcal{E}(inc_1)_*(X)$ and

$$\mathcal{E}(\eta)_*(X) = \mathcal{E}(\Theta \circ inc_0)_*(X) = \mathcal{E}(\Theta \circ inc_1)_*(X) = \mathcal{E}(\zeta)_*(X)$$

□

With this lemma functoriality and compatibility with conjugation follow in analogy to Lemma 2.4 resp. Lemma 2.5. □

So to prove Theorem 2.10 it remains to construct an induction structure making $\mathcal{G}^?$ into a strong smooth equivariant family of topological groupoids:

Let $\alpha : G \to M$ be an open continuous group homomorphism. Then a morphism

$$\eta_\alpha : \mathcal{G}^G|_{\mathrm{Or}(G, \mathcal{O} \cap \mathcal{F}_\alpha)} \to (\mathrm{res}_\alpha \, \mathcal{G}^M)|_{\mathrm{Or}(G, \mathcal{O} \cap \mathcal{F}_\alpha)}$$

of topological $\mathrm{Or}(G, \mathcal{O} \cap \mathcal{F}_\alpha)$-groupoids is defined by

$$\eta_\alpha(G/H) : \mathcal{G}^G(G/H) \to \mathcal{G}^M(M/\alpha(H))$$

$$gH \mapsto \alpha(g)\alpha(H)$$

$$x \in g_2 H g_1^{-1} \mapsto \alpha(x) \in \alpha(g_2 H g_1^{-1})$$

We see immediately that η_α is a natural transformation and $(\mathrm{res}_\alpha \, \eta_\beta) \circ \eta_\alpha = \eta_{\beta\alpha}$. Furthermore each $\eta_\alpha(G/H)$ is an equivalence of topological groupoids, as can be seen using the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{G}^H(H/H) & \xrightarrow{\eta_\alpha(H/H)} & \mathcal{G}^\alpha(H)(\alpha(H)/\alpha(H)) \\
\cong \downarrow \eta_i(H/H) & & \cong \downarrow \eta_i(\alpha(H)/\alpha(H)) \\
\mathcal{G}^G(G/H) & \xrightarrow{\eta_\alpha(G/H)} & \mathcal{G}^M(M/\alpha(H))
\end{array}$$

The upper horizontal map is an isomorphism because $\alpha : H \to \alpha(H)$ is an isomorphism for $H \in \mathcal{F}_\alpha$. Hence it suffices to show that for any $\gamma \in G$ there exists a homotopy

$$\Theta : \mathcal{G}^G \times \{0 \leftrightarrow 1\} \Rightarrow \mathrm{res}_{c(\gamma)} \, \mathcal{G}^G$$

between $\mathcal{G}_\gamma$ and $\eta_{c(\gamma)}$. We define Θ by putting

$$\Theta_{G/H} : \mathcal{G}^G(G/H) \times \{0 \leftrightarrow 1\} \to \operatorname{res}_{c(\gamma)} \mathcal{G}^G(G/H)$$

to be the natural transformation

$$\Theta_{G/H} : \eta_\alpha(G/H) \Rightarrow \mathcal{G}_\gamma(G/H)$$

$$\Theta_{G/H}(gH) = \gamma^{-1} \in \operatorname{mor}(\gamma g \gamma^{-1}(\gamma H \gamma^{-1}), g\gamma^{-1}(\gamma H \gamma^{-1})) = gHg^{-1}\gamma^{-1}$$

(this is a natural transformation because the following diagram commutes

$$\begin{array}{ccc} \gamma g_1 \gamma^{-1}(\gamma H \gamma^{-1}) & \xrightarrow{\gamma g_2 h g_1^{-1} \gamma^{-1}} & \gamma g_2 \gamma^{-1}(\gamma H \gamma^{-1}) \\ \downarrow{\scriptstyle \gamma^{-1}} & & \downarrow{\scriptstyle \gamma^{-1}} \\ g_1 \gamma^{-1}(\gamma H \gamma^{-1}) & \xrightarrow{g_2 h g_1^{-1}} & g_2 \gamma^{-1}(\gamma H \gamma^{-1}) \end{array}$$

for any $g_2 h g_1^{-1} \in \operatorname{mor}_{\mathcal{G}^G(G/H)}(g_1 H, g_2 H)$).

Now to see that Θ assembles to a homotopy, we have to show that for every $R_\lambda : G/H \to G/K, \lambda \in G, \lambda H \lambda^{-1} \subset K$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G}^G(G/H) \times \{0 \leftrightarrow 1\} & \xrightarrow{\Theta_{G/H}} & (\operatorname{res}_{c(\gamma)} \mathcal{G}^G)(G/H) \\ \downarrow{\scriptstyle \mathcal{G}^G(R_\lambda) \times id} & & \downarrow{\scriptstyle (\operatorname{res}_{c(\gamma)} \mathcal{G}^G)(R_\lambda)} \\ \mathcal{G}^G(G/K) \times \{0 \leftrightarrow 1\} & \xrightarrow{\Theta_{G/K}} & (\operatorname{res}_{c(\gamma)} \mathcal{G}^G)(G/K) \end{array}$$

The following diagram (where $\{0,1\}$ is the category with two objects and only the identity morphisms) is just saying that $\mathcal{G}^G_\gamma$ and $\eta_{c(\gamma)}$ are natural transformations.

$$\begin{array}{ccc} \mathcal{G}^G(G/H) \times \{0, 1\} & \xrightarrow{\Theta_{G/H}} & (\operatorname{res}_{c(\gamma)} \mathcal{G}^G)(G/H) \\ \downarrow{\scriptstyle \mathcal{G}^G(R_\lambda) \times id} & & \downarrow{\scriptstyle (\operatorname{res}_{c(\gamma)} \mathcal{G}^G)(R_\lambda)} \\ \mathcal{G}^G(G/K) \times \{0, 1\} & \xrightarrow{\Theta_{G/K}} & (\operatorname{res}_{c(\gamma)} \mathcal{G}^G)(G/K) \end{array}$$

Hence it suffices to show that

$$((\operatorname{res}_{c(\gamma)} \mathcal{G}^G)(R_\lambda) \circ \Theta_{G/H})(id_{gH}, 0 \to 1) = (\Theta_{G/K} \circ \mathcal{G}^G(R_\lambda))(id_{gH}, 0 \to 1).$$

The left hand side is $(\operatorname{res}_{c(\gamma)} \mathcal{G}^G)(R_\lambda)(\gamma^{-1}) = \gamma^{-1}$ and the right hand side is $\Theta_{G/K}(id_{g\lambda^{-1}}, 0 \to 1) = \gamma^{-1}$. And this finishes the proof of Theorem 2.10. $\square$

We can play the game with the equivariant families still further: Often we want to construct equivariant homology theories with the help of functors $\mathbf{Cat}_< \to \mathbf{Spt}(\mathbf{Top})$ starting at some subcategory of **Cat** as in Lemma 1.13. Looking at the proof of Theorem 2.13 we see that we get the analogous result for *strong smooth equivariant families of categories* which would be defined analogously.

Remark 2.15 We have seen that a strong smooth equivariant family of topological groupoids induces a smooth equivariant family of spectra given we have a functor $E : \mathbf{TGr} \to \mathbf{Spt}(\mathbf{Top})$ which maps equivalences of topological groupoids to weak equivalences of spectra. When do we get a *strong* smooth equivariant family of spectra meaning that $i_{\beta\alpha} \simeq (\mathrm{res}_\alpha\, i_\beta) \circ i_\alpha$ and $i_{c(\gamma)} \simeq E_\gamma$? For this it would be necessary that homotopic maps of $\mathrm{Or}(G,\mathcal{F})$-groupoids are mapped to homotopic maps of $\mathrm{Or}(G,\mathcal{F})$-spectra by $\mathcal{E}$ (*). Looking at the proof above we see that it would suffice that equivalences of $\mathrm{Or}(G,\mathcal{F})$-groupoids get mapped to homotopy equivalences between $\mathrm{Or}(G,\mathcal{F})$-spectra (**) ($\mathcal{F}$ any family of open subgroups). It doesn't suffice that E maps homotopic functors between groupoids to homotopic maps between spectra, it has to happen naturally. A warning for those who are familiar with the language of 2-categories: It isn't the right condition that E is 2-functor with the usual 2-structures on $\mathbf{Gr}$ and $\mathbf{Spt}(\mathbf{Top})$, because then natural transformations are only mapped to homotopy classes of homotopies instead of homotopies as one could think at the first glance.

Remark 2.16 With the same arguments as in Lemma 1.14 we see that condition (*) is satisfied by a functor $E : \mathbf{TGr} \to \mathbf{Spt}(\mathbf{Top})$ if for every $\mathrm{Or}(G,\mathcal{F})$-groupoid $\mathcal{G}$ there exists a map of $\mathrm{Or}(G,\mathcal{F})$-spectra

$$\mathcal{E}(\mathcal{G}) \wedge I_+ \to \mathcal{E}(\mathcal{G} \times \{0 \leftrightarrow 1\})$$

extending the two inclusions

$$\mathcal{E}(inc_i) : \mathcal{E}(\mathcal{G}) \to \mathcal{E}(G \times \{0 \leftrightarrow 1\}), i = 0, 1.$$

Condition (**) can be reached if $\mathcal{E}$ maps to subcategory of $\mathrm{Or}(G,\mathcal{F})$-$\mathbf{Spt}(\mathbf{Top})$ where weak equivalences are already $\mathrm{Or}(G,\mathcal{F})$-homotopy equivalences. □

3 Algebraic K- and L-theory

A first application of the theorem proved in the last chapter is the construction of an induction structure for algebraic K- and L-theory as used in [Lüc02] to construct a Chern-character for these homology theories.

The equivariant homology theories used there are characterized by saying that on coefficients they are the rationalized algebraic K-groups $\mathbb{Q} \otimes_{\mathbb{Z}} K_n(RH)$ or the rationalized algebraic L-groups $\mathbb{Q} \otimes_{\mathbb{Z}} L_n^{\langle j \rangle}(RH)$ for some commutative ring R with unit containing the rationals. They are constructed in [DL98] using functors

$$\mathbf{K}^{alg} : \mathbf{Gr} \to \Omega\mathbf{Spt}(\mathbf{Top})$$

and

$$\mathbf{L}^{\langle j \rangle} : \mathbf{Gr} \to \Omega\mathbf{Spt}(\mathbf{Top})$$

such that

$$\pi_n(\mathbf{K}^{alg}(G/H)) \cong K_n^{alg}(RH)$$
$$\pi_n(\mathbf{L}^{\langle j \rangle}(G/H)) \cong L_n^{\langle j \rangle}(RH).$$

These functors have the property that two morphisms in $\mathbf{Gr}$ which are homotopic (i.e. there exists a natural equivalence between them) induce homotopic maps on spectra, in particular natural equivalences induce weak equivalences.

Hence by Theorem 2.10 we have equivariant homology theories $\mathbf{K}_*^{alg}$ and $\mathbf{L}_*^{\langle j \rangle}$. Rationalizing them by either rationalizing the spectra (this is possible in a functorial way) or just tensoring with $\mathbb{Q}$ (which is the same as rationalizing the spectrum as we speak of homology theories) we get equivariant homology theories as needed in [Lüc02].

We'll take a closer look at the definition of K^{alg} (the one of $L^{\langle j \rangle}$ is analogous) because we will do an analogous construction later for totally disconnected, locally compact groups.

The main input is the functor

$$\mathbb{K}^{-\infty} : \mathbf{aCat} \to \Omega\mathbf{Spt}(\mathbf{Top})$$

of Pedersen and Weibel which assigns to any small additive category the non-connective K-theory spectrum of [PW85] (in [BFJR01], 2.1 an overview over the properties of $\mathbb{K}^{-\infty}$ is given).

The K-theory of a ring with unit is given by the homotopy groups of the K-theory spectrum associated to the category of finitely generated projective modules over this ring (i.e. to a small skeleton of it). $\mathbf{K}^{alg}$ is defined by

$$\mathbf{K}^{alg} := \mathbb{K}^{-\infty}(\mathcal{P}((R-)_{\oplus})) : \mathbf{Gr} \to \mathbf{\Omega Spt}(\mathbf{Top}).$$

Here for every groupoid $\mathcal{G}$ the category $R\mathcal{G}$ has the same objects as $\mathcal{G}$ and as morphism sets the free modules over R generated by the respective morphism set of $\mathcal{G}$: $\mathrm{mor}_{R\mathcal{G}}(x,y) = R\,\mathrm{mor}_{\mathcal{G}}(x,y)$.

For any R-category $\mathcal{C}$ the category $\mathcal{C}_{\oplus}$, the *symmetric monoidal R-category associated to $\mathcal{C}$*, has as objects finite tuples of objects of $\mathcal{C}$ and as morphisms matrices of the right size with entries in the morphism sets of $\mathcal{C}$. The category $\mathcal{C}_{\oplus}$ has an associative and commutative sum, the block sum, which makes $\mathcal{C}_{\oplus}$ into a symmetric monoidal category.

The last thing to explain is the functor $\mathcal{P} : \mathbf{Cat} \to \mathbf{Cat}$ which assigns to a category $\mathcal{C}$ its *idempotent completion*, i.e. the objects of $\mathcal{P}(\mathcal{C})$ are the idempotent morphisms $p : x \to x$ of $\mathcal{C}$ and the morphisms $f : (p : x \to x) \to (q : y \to y)$ are the morphisms $f : x \to y$ of $\mathcal{C}$ such that $q \circ f \circ p = f$. The functor $\mathbb{K}^{-\infty}$ doesn't see the difference between an additive category and its idempotent completion, the inclusion $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ induces a weak equivalence (cf.[BFJR01], 2.1), but the idempotent completion has a feature that will be important for us later, namely the idempotent completion of a non unital category is a unital category.

The category $\mathcal{P}((R\mathcal{G})_{\oplus})$ is equivalent to the category of isomorphism classes of finitely generated $R\mathcal{G}$-modules. As $R-$ and $\mathcal{P}$ commute with '$\times\{0 \leftrightarrow 1\}$' and $(-)_{\oplus}$ satisfies the assumptions of Lemma 1.14, the functor $\mathcal{P}((R-)_{\oplus})$ maps equivalences of groupoids to equivalences of symmetric monoidal R-categories.

4 Topological Groupoids with Haar Systems

The notion of the Haar system on a groupoid generalizes that of the Haar measure on a group. To prepare and motivate its definition we give a short introduction to Haar Measures and the Hecke-Algebra

4.1 Haar Measures

A Haar measure on a locally compact, Hausdorff group is - very vaguely speaking - a left invariant “nice” Borel measure, where Borel measure means that the measure lives on the Borel-σ-algebra $\mathcal{B}$ of the group and is locally finite in the sense that every element has an open neighbourhood of finite measure. But there are two different conventions as what "nice" should mean. Some authors ([HR63], [Coh93]) want the measure to be "regular" in the sense that

$$A \in \mathcal{B}(G) \Rightarrow \mu(A) = inf\{\mu(U)|A \subset U, U \, \text{open}\}$$

and

$$U \subset G \, \text{open} \Rightarrow \mu(U) = sup\{\mu(K)|K \subset U, K \, \text{compact}\}.$$

Others ([Els99]) define a Haar measure as a left-invariant Radon measure, i.e. a Borel measure fulfilling

$$A \in \mathcal{B}(G) \Rightarrow \mu(A) = sup\{\mu(K)|K \subset A, K \text{compact}\}.$$

The theorem that on a locally compact Hausdorff group there exists up to a positive scalar factor only one left Haar measure holds true for both versions. Behind this ambiguity stands the fact that this is really a theorem about a left-invariant linear form:

Theorem 4.1 *(A.Haar, J.v.Neumann, A.Weil)*
If G is a locally compact Hausdorff group, then there exists a left-invariant positive linear form

$$I : C_c(G) \to \mathbb{R}, I \neq 0$$

and I is unique up to a positive scalar factor.(cf. [Els99],3.11, p.354*)*

Here $C_c(G)$ is the space of continuous functions $f : G \to \mathbb{C}$ with compact support. As the Darstellungssatz by F.Riesz exists for regular Borel measures as well as for Radon measures, both kinds of Haar measures exist uniquely up to a positive scalar on G (cf [Els99] p.334).

The two measures (if normed at the same compact set) coincide on compact sets and on open sets and also on sets which are finite with respect to the "regular" measure, but it can happen that a Borel set A is finite with respect to the Radon measure but infinite with respect to the "regular" measure (see [Bau90] pp. 204)

Important for us is that both measures generate the same Hecke-algebra $\mathcal{H}(G)$ which will be defined in a moment. The measure enters in the definition only for the multiplication. But this can be written in terms of the corresponding linear form:

$$p, q \in \mathcal{H}(G) \Rightarrow (q * p)(\gamma) = I((q \circ L_\gamma) \cdot (p \circ Inv)) \, \forall \gamma \in G$$

where $Inv : G \to G, g \mapsto g^{-1}$. So for our purposes one measure is as good as the other, but the Radon measure has one advantage, namely that the following lemma can be proven easily:

Lemma 4.2 *Let G, M be two locally compact Hausdorff groups, μ a Radon measure on G. Let $\alpha : G \to M$ be a continuous open group homomorphism with compact kernel. Then the measure $\alpha(\mu)$ defined by $\alpha(\mu)(B) := \mu(\alpha^{-1}(B))$ is a Radon measure.*

Proof: First we show that $\alpha(\mu)$ is locally finite: Let $m = \alpha(g) \in \alpha(G) \subset M$. Let U be an open neighbourhood of g whose closure is compact (G is locally compact). Then $\alpha(U)$ is an open neighbourhood of m and $\alpha(\mu)(\alpha(U)) = \mu(\alpha^{-1}(U)) = \mu(U \ker(\alpha)) \leq \mu(\overline{U} \ker(\alpha)) < \infty$ as $\overline{U}$ and $\ker(\alpha)$ are compact. Note that $\alpha(G)$ is an open subgroup of M and is hence also closed in M. Therefore if $m \in M$ is not in the image of G, then there exists an open neighbourhood of m which lies in the complement of $\alpha(G)$ and has therefore measure zero. Hence $\alpha(\mu)$ is locally finite.

It remains to show that $\mu(\alpha)$ can be approximated from below by compact sets: Let $A \in \mathcal{B}(M)$. Then $\alpha(\mu)(A) \leq sup\{\alpha(\mu)(K)|K \subset A, K \text{ compact}\}$. Let $\alpha(\mu)(A) < \infty$ and $\epsilon > 0$, then as μ is a Radon measure there exists a compact set $L \subset \alpha^{-1}(A)$ such that $\mu(\alpha^{-1}(A)) \leq \epsilon + \mu(L) \leq \epsilon + \mu(\alpha^{-1}(\alpha(L)))$. As $\alpha(L) \subset A$ is compact this shows that $\alpha(\mu)(A) = sup\{\alpha(\mu)(K)|K \subset A, K \text{ compact}\}$. Similarly this is shown if $\alpha(\mu)(A) = \infty$. □

The condition that the kernel of α is compact is even necessary, if there exists a compact subgroup $H < G$ with measure greater zero. Because then $\alpha^{-1}(\alpha(H))$ is the disjunct sum of as much translates of H as there are elements in $(\ker(\alpha) \cap H) \backslash \ker(\alpha)$. If α is surjective, then left-invariance of μ implies left-invariance of $\alpha(\mu)$.

The uniqueness of the left Haar measure leads to the existence of the *modular function*

$$\Delta : G \rightarrow]0, \infty[$$

with the property that

$$\mu(Ag^{-1}) = \Delta(g)\mu(A) \,\forall\, A \in \mathcal{B}(G), g \in g, \mu \text{ any left Haar measure on } G.$$

Δ depends only on G.

Remark 4.3 See [Els99] pp. 363.

1. Δ is a continuous group homomorphism into the multiplicative group $]0, \infty[$
2. $\int_G f(xg)d\mu(x) = \Delta(g) \int_G f(x)d\mu(x), f \in \mathcal{L}^1(\mu)$
3. $\int_G f(x^{-1})\Delta(x)d\mu(x) = \int_G f(x)d\mu(x), f \in \mathcal{L}^1(\mu)$
4. If $\Delta = 1$, then G is called unimodular. All compact groups are unimodular.
5. $\int_G f(gx)d\mu(x) = \int_G f(x)d\mu(x)$

Recall that for $0 < p < \infty$ $\mathcal{L}^p(\mu)$ is the space of all measurable functions $f : G \rightarrow \mathbb{C}$ which have finite p-norm: $\|f\|_p := (\int |f|^p d\mu)^{\frac{1}{p}} < \infty$ and $L^p(\mu) = \mathcal{L}^p(\mu)/\mathcal{N}$ where $\mathcal{N}$ is the space of all functions in $\mathcal{L}^p(\mu)$ which are zero almost everywhere.

Remark 4.4 Let G be a locally compact Hausdorff group and μ a left Haar measure on G. Then there exists a real number ζ, such that $\mu(H) \in \mathbb{Q}\zeta$ for all open compact subgroups H of G. Namely put $\zeta = \mu(H)$ for some open compact subgroup H of G. Then $\nu = \frac{1}{\zeta}\mu$ is a Haar measure such that $\nu(H) = 1$. Now let L be another open compact subgroup, then $\nu(L) = \frac{[L:(L\cap H)]}{[H:(L\cap H)]} \in \mathbb{Q}$. □

4.2 The Hecke-Algebra

Let G be a totally disconnected, locally compact group. Then the Hecke algebra $\mathcal{H}(G) = C_c^\infty(G)$ of G is defined to be the ring of all locally constant, compactly supported functions $G \rightarrow \mathbb{C}$. The product is given by the convolution product:

$$p, q \in \mathcal{H}(G) \Rightarrow q * p(\gamma) := \int_G q(\gamma g)p(g^{-1})d\mu(g) \,\forall \gamma \in G$$

where μ is a fixed left Haar measure. The function $q * p$ is again an element of $\mathcal{H}(G)$ as $supp(q * p) \subset supp(p)supp(q)$ and the following lemma holds:

Lemma 4.5 *Let G be totally disconnected, locally compact and let $r : G \to \mathcal{C}$ be a locally constant function with compact support. Then there exists a compact open subgroup K of G such that r is K-bi-invariant.* $\square$

As the Haar measure is unique up to a scalar factor the Hecke algebra up to isomorphism is independent of the choice of the Haar measure.

Remark 4.6 Let $p, q \in (G), \gamma \in G$. Then

1. $q * (p \circ R_\gamma) = (q * p) \circ R_\gamma$
2. $(q \circ L_\gamma) * p = (q * p) \circ L_\gamma$
3. $(q \circ R_\gamma) * (p \circ L_\gamma) = \Delta(\gamma^{-1}) q * p$

where R_γ is right multiplication with γ^{-1} and L_γ is left multiplication with γ.

Lemma 4.7 *Let $K < G$ be an open compact subgroup and let $\mathcal{H}(G/\!/K) \subset \mathcal{H}(G)$ be the set of the left and right K-invariant functions in $\mathcal{H}(G)$. This is a subring with unit.*

Proof: Let $p, q \in \mathcal{H}(G/\!/K)$. Then $p + q \in \mathcal{H}(G/\!/K)$ and

$$\begin{aligned} q * p(\gamma k) &= \int_G q(\gamma k g) p(g^{-1}) d\mu(g) = \int_G q(\gamma g) p(g^{-1} k) d\mu(g) \\ &= \int_G q(\gamma g) p(g^{-1}) d\mu(g) = q * p(\gamma) \\ &= \int_G q(k \gamma g) p(g^{-1}) d\mu(g) = q * p(k\gamma) \end{aligned}$$

Hence $q * p \in \mathcal{H}(G/K)$. The function $e_K := \frac{1}{vol_G(K)} \chi_K \in \mathcal{H}(G/K)$ is a unit for $\mathcal{H}(G/\!/K)$:

1. $(p * e_K)(\gamma) = \int_G p(\gamma g) e_K(g^{-1}) d\mu(g) = \frac{1}{vol_G(K)} \int_K p(\gamma) d\mu(g) = p(\gamma)$
2. $$\begin{aligned} (e_K * p)(\gamma) &= \int_G e_K(\gamma g) p(g^{-1}) d\mu(g) = \int_G e_K(g) p(g^{-1}\gamma) d\mu(g) \\ &= \frac{1}{vol_G(K)} \int_K p(\gamma) d\mu(g) = p(\gamma) \end{aligned}$$

$\square$

As every function in $\mathcal{H}(G)$ is K-bi-invariant for some open compact subgroup $K < G$, we have that

$$\mathcal{H}(G) = \bigcup_{I(G)} \mathcal{H}(G/\!/K) = \varinjlim_{I(G)} \mathcal{H}(G/\!/K)$$

where $I(G) = \{K < G | K \text{ open, compact}\}$ ordered by the reverse of the inclusion.

The Hecke algebra encodes information about the smooth representations of the totally disconnected, locally compact group G. A complex smooth representation of G is a topological $\mathbb{C}$-vector space with a continuous linear action of G such that all isotropy groups are open. The category of non-degenerate $\mathcal{H}(G)$-modules (i.e. $\mathcal{H}(G)V = V$) can be identified with the category of smooth representations of G, namely let (V, π) be a smooth G-representation, then V is a non-degenerate $\mathcal{H}(G)$-module via

$$fv := \int_G f(g)(\pi(g)v) d\mu(g)$$

for every $f \in \mathcal{H}(G)$ and $v \in V$ (cf. for example [Kut98], Thm 1.1 or [Car79], p. 118).

An n-dimensional complex smooth representation of G is the same as a continuous group homomorphism $\varphi : G \to U(n)$ with open kernel. As the only compact totally disconnected subgroups of $U(n)$ are the finite ones, a complex finite dimensional representation of a *compact* totally disconnected group is always smooth (because $G/\ker(\varphi)$ has to be discrete).

For any totally disconnected, locally compact group G let $R(G)$ denote the Grothendieck group of the isomorphism classes of finite dimensional smooth G-representations.

Lemma 4.8 *Let G be a totally disconnected, locally compact group which has a neighbourhood base of the unit consisting of open compact normal subgroups. Then*

$$R(G) \cong \varinjlim_{L \in N(G)} R(G/L)$$

where $N(G) := \{L < G | L \text{ open, compact, normal}\}$.

Proof: The isomorphism is given in one direction by restricting with $G \to G/L$ and in the other direction by dividing out the kernel of the group homomorphism $G \to U(n)$ associated to the representation. □

The Periodic Homology of the Hecke algebra $\mathcal{H}(G)$ is linked with the homology of the classifying space $E(G, \mathcal{CO})$, where $\mathcal{CO}$ is the family of compact open subgroups of G:

Theorem 4.9 [HN96] *Let G be a totally disconnected, locally compact group. Then*

$$\bigoplus_{ev/odd} HP_*(\mathcal{H}(G)) \cong \bigoplus_{ev/odd} H_*^G(E(G, \mathcal{CO}), R(?) \otimes_{\mathbb{Z}} \mathbb{C}).$$

4.3 Topological Groupoids with Haar Systems

We want to replace the functor $\mathcal{P}(R(-)_\oplus) : \mathbf{Gr} \to \mathbf{ssmRCat}$ of Chapter 3 by some analogue functor $\mathbf{TGr} \to \mathbf{ssmRCat}$. For the correct formulation of it we need the notion of a Haar system on a topological groupoid.

A *left Haar system* on a topological groupoid $\mathcal{G}$ (i.e. a groupoid with discrete set of objects and a locally compact Hausdorff topology on each morphism set, such that composition and taking the inverse is continuous) consists of Radon measures $\lambda_v^u, u, v \in$

$Ob(\mathcal{G})$ on $\mathrm{mor}_{\mathcal{G}}(v,u)$ such that for any morphism $x : u_1 \to u_2$ the following identity holds:

$$\lambda_v^{u_2} = L_x(\lambda_v^{u_1}),$$

i.e. $\lambda_v^{u_2}(B) = \lambda_v^{u_1}(x^{-1}B)$ for every Borel set B. In particular $\lambda_u^u, u \in Ob(\mathcal{G})$ is a left Haar measure.

Remark 4.10 This definition of a Haar system on a topological groupoid is a special case of a more general notion as for example in [Ren80]. There the whole morphism set of the groupoid is topologized, thereby giving also the set of objects, identified with the identity morphisms, a topology. A Haar system consists of 'left invariant' Radon measures λ^u on $\coprod_{v \in Ob(\mathcal{G})} \mathrm{mor}(v,u)$ satisfying a 'continuity condition', which is empty in our case.

Now let **HGr** be the following category: The objects are pairs $(\mathcal{G}, \{\lambda_v^u\})$ where $\mathcal{G}$ is a topological groupoid and $\{\lambda_v^u\} =: \lambda$ is a left Haar system on it; morphisms are functors $F : \mathcal{G}_1 \to \mathcal{G}_2$ such that $\mathrm{mor}_{\mathcal{G}_1}(u,v) \to \mathrm{mor}_{\mathcal{G}_2}(Fu, Fv)$ is an **open** continuous map. Often we'll omit the Haar system in the notation.

By the uniqueness of the Haar measure it follows that a Haar system on a connected groupoid $\mathcal{G}$ (i.e. $\mathrm{mor}_{\mathcal{G}}(x,y) \neq \emptyset$ for all $x, y \in Ob(\mathcal{G})$) is unique up to a scalar factor and is determined by a Haar measure on one of the endomorphism sets.

Example 4.11 Let G be a totally disconnected, locally compact group, μ a left Haar measure on it. Then for every open subgroup H we defined $\mathcal{G}^G(G/H)$ to be the following topological groupoid: The objects are the elements gH of G/H and

$$\mathrm{mor}_{\mathcal{G}^G(G/H)}(g_1H, g_2H) = g_2Hg_1^{-1}.$$

A left Haar system on it is given by $\mu_{uH}^{gH} = \mu|_{\mathrm{mor}(uH,gH)}$. We have to show that for $x \in \mathrm{mor}(g_1H, g_2H)$ and $B \subset \mathrm{mor}_{\mathcal{G}}(uH, g_2H)$ a Borel set the following equation holds:

$$\mu_{uH}^{g_2H}(B) = \mu_{uH}^{g_1H}(x^{-1}B).$$

This is clear as μ is a left Haar measure. □

Example 4.12 Let (E, λ) be an object of **HGr**. Then $E \times \{0 \leftrightarrow 1\}$ is also an object of **HGr**. Here $\{0 \leftrightarrow 1\}$ is the category consisting of two objects 0 and 1 and four morphism, namely the two identities and two morphisms $0 \to 1$ respectively $1 \to 0$ which are inverse to each other. $E \times \{0 \leftrightarrow 1\}$ is obviously a topological groupoid, a Haar system on it is given by $\lambda_{(v,y)}^{(u,x)} := \lambda_v^u, u, v \in Ob(\mathcal{G}), x, y \in \{0,1\}$. □

For a topological space X let $C_c^\infty(X)$ be the $\mathbb{C}$-vector space of locally constant functions $X \to \mathbb{C}$ with compact kernel.

Remark 4.13 Let $(\mathcal{G}, \lambda)$ be an object of **HGr**, $u_1, u_2, v \in Ob(\mathcal{G})$, $f \in C_c^\infty(\mathrm{mor}_{\mathcal{G}}(v, u_2))$. Then

$$\int f(y) d\lambda_v^{u_2}(y) = \int f(xy) d\lambda_v^{u_1}(y)$$

because this holds for the characteristic functions by definition. □

For a topological groupoid with Haar system the analogue of the modular function Δ can be defined. We'll need this notion in Chapter 6.

Lemma 4.14 *Let $\mathcal{G}$ be an object of* **HGr**. *Then there exists a functor*

$$\Delta_{\mathcal{G}} : \mathcal{G} \to \mathrm{cat}(]0, \infty[)$$

such that

$$R_\varphi(\lambda_{v'}^u) = \Delta_{\mathcal{G}}(\varphi^{-1}) \lambda_v^u$$

for any $u, v, v' \in Ob(\mathcal{G}), \varphi \in \mathrm{mor}_{\mathcal{G}}(v', v)$, where $R_\varphi : \mathrm{mor}_{\mathcal{G}}(v', u) \to \mathrm{mor}_{\mathcal{G}}(v, u)$ is right multiplication by φ^{-1}. Here $\mathrm{cat}([0, \infty])$ is the category with one object and morphism set the multiplicative group $]0, \infty[$.

Proof: It suffices to consider $u = v$. Then λ_v^v is a Haar measure on $\mathrm{mor}_{\mathcal{G}}(v, v)$ and so is $R_\varphi(\lambda_{v'}^v)$ (For $a \in \mathrm{mor}_{\mathcal{G}}(v, v)$ we have $R_\varphi(\lambda_{v'}^v)(aB) = \lambda_{v'}^v(aB\varphi) = \lambda_{v'}^v(B\varphi) = \lambda_{v'}^v$).

Hence $R_\varphi(\lambda_{v'}^v) = \frac{R_\varphi(\lambda_{v'}^v)(B)}{\lambda_v^v(B)} \lambda_v^v(B)$ for any Borel set $B \subset \mathrm{mor}_{\mathcal{G}}(v, v)$ of finite non-zero measure and the proposition follows with

$$\Delta_{\mathcal{G}}(\varphi^{-1}) := \frac{R_\varphi(\lambda_{v'}^v)(B)}{\lambda_v^v(B)}$$

(independent of the choice of B). □

The following corollary is an analogue to remark 4.3

Corollary 4.15 *Let $f \in C_c^\infty(\mathrm{mor}_{\mathcal{G}}(u, v))$ and $\varphi : w \to u$. Then $f \circ R_\varphi \in C_c^\infty(\mathrm{mor}_{\mathcal{G}}(w, v))$ and*

$$\Delta_{\mathcal{G}}(\varphi^{-1}) \int f(y) d\lambda_u^v(y) = \int f \circ R_\varphi(y) d\lambda_w^v(y).$$

Proof: It suffices to show this for $f = \chi_B$, $B \subset \mathrm{mor}_{\mathcal{G}}(u, v)$ a Borel set of finite measure. In this case

$$\begin{aligned}
\Delta_{\mathcal{G}}(\varphi^{-1}) \int \chi_B(y) d\lambda_u^v(y) &= \Delta_{\mathcal{G}}(\varphi^{-1}) \lambda_u^v(B) \\
&= (R_\varphi(\lambda_w^v))(B) \\
&= \lambda_w^v(B\varphi) \\
&= \int \chi_{B\varphi}(y) d\lambda_w^v(y) \\
&= \int \chi_B \circ R_\varphi(y) d\lambda_w^v(y)
\end{aligned}$$

□

Notation 4.16 Let $(\mathcal{G}, \lambda), (\mathcal{M}, \mu) \in Ob(\mathbf{HGr}), F : \mathcal{G} \to \mathcal{M}$ a morphism between them such that F_v^v has compact kernel for all $v \in Ob(\mathcal{G})$. We define

$$c_F(v) := \frac{\lambda_v^v((F_v^v)^{-1}(B))}{\mu_{Fv}^{Fv}(B)},$$

where $B \subset \mathrm{mor}_{\mathcal{M}}(Fv, Fv)$ has finite non zero measure.

This definition is independent of the choice of B because of the uniqueness of the Haar measure and we have

$$F_v^v(\lambda_v^v) = c_F(v)\mu_{Fv}^{Fv}.$$

Now let $u \in Ob(\mathcal{G})$ be another object. Then

$$F_u^v(\lambda_u^v) = c_F(u)\mu_{Fu}^{Fv}$$

because for any $B \subset \mathrm{mor}_{\mathcal{M}}(Fu, Fv)$ and any $p \in \mathrm{mor}_{\mathcal{G}}(v, u)$ we have $\lambda_u^v((F_u^v)^{-1}(B)) = \lambda_u^u(p(F_u^v)^{-1}(B)) = \lambda_u^u((F_u^u)^{-1}(F(p)B)) = c_F(u)\mu_{Fu}^{Fu}(F(p)B) = c_F(u)\mu_{Fu}^{Fv}(B)$.

Example 4.17 For $\eta_\alpha : \mathcal{G}^G(G/H) \to \mathcal{G}^M(M/\alpha(H))$ with $H \cap \ker(\alpha) = 1$ (and hence $\ker((\eta_\alpha)_{g_1 H}^{g_2 H}) = 1$) we get

$$c_{\eta_\alpha}(gH) = \frac{\mu_G(K)}{(\alpha|_{gHg^{-1}})(\mu_M)(K)} = \frac{\mu_G(K)}{\mu_M(\alpha^{-1}(K))}$$

for any open compact subgroup $K < gHg^{-1}$. For the same K

$$c_{\eta_\alpha}(eH) = \frac{\mu_G(g^{-1}Kg)}{\mu_M(\alpha^{-1}(g^{-1}Kg))} = c_{\eta_\alpha}(gH)$$

because of the uniqueness of the Haar measure.

Now we can define the functor we are looking for. Let $\nu\mathbf{Cat}$ be the category of small categories which don't necessarily have units, the morphisms being non-unital functors, i.e. functors without the "identity condition". Let $\mathbf{HGr_{td}}$ be the full subcategory of $\mathbf{HGr}$ with objects the totally disconnected groupoids, i.e. those whose morphism sets are totally disconnected.

Definition 4.18 *Let $(\mathcal{G}, \lambda)$ be an object of $\mathbf{HGr_{td}}$. Then $C_c^\infty(\mathcal{G})$ is the $\mathbb{C}$-category without unit that has the same objects as $\mathcal{G}$, but*

$$\mathrm{mor}_{C_c^\infty(\mathcal{G})}(u, v) = C_c^\infty(\mathrm{mor}_{\mathcal{G}}(u, v)).$$

The composition is given by

$$q \circ p(x) = q * p(x) = \int q(xy)p(y^{-1})d\lambda_v^u$$

for $p \in \mathrm{mor}_{C_c^\infty(\mathcal{G})}(u, v), q \in \mathrm{mor}_{C_c^\infty(\mathcal{G})}(v, w)$. (In the language of [Ren80] *this is the convolution product with the twococycle $\sigma = 1$) With the same arguments as for the Hecke algebra $q * p$ is an element of $\mathrm{mor}_{C_c^\infty(\mathcal{G})}(u, w)$.*

If $\mathcal{G}$ has only one object then the only morphism set is a group G and $C_c^\infty(\mathcal{G})$ is the category with one object and morphism set $\mathcal{H}(G)$.

Using the function $\Delta_{\mathcal{G}}$ we get an analogue to Remark 4.6 which will be used in the definition of a spectrum for topological K-theory:

Corollary 4.19 *Let $\mathcal{G} \in Ob(\mathbf{HGr_{td}})$, $u_1, u_2, u_3, v \in Ob(\mathcal{G})$, $p \in C_c^\infty(\mathrm{mor}_{\mathcal{G}}(u_1, u_2))$, $q \in C_c^\infty(\mathrm{mor}_{\mathcal{G}}(u_2, u_3))$, $\varphi \in \mathrm{mor}_{\mathcal{G}}(v, u_3)$, $\psi \in \mathrm{mor}_{\mathcal{G}}(v, u_1)$ and $\rho \in \mathrm{mor}_{\mathcal{G}}(v, u_2)$. Then*

1. $(q \circ L_\varphi) * p = (q * p) \circ L_\varphi$.
2. $q * (p \circ R_\psi) = (q * p) \circ R_\psi$
3. $(q \circ R_\rho) * (p \circ L_\rho) = \Delta_{\mathcal{G}}(\rho^{-1}) q * p$.

Proof: The first two parts follow directly from the definition of the convolution product. For the third part we use Lemma 4.14: Let $t \in \mathrm{mor}_{\mathcal{G}}(u_1, u_3)$, then

$$\begin{aligned}
(q \circ R_\rho) * (p \circ L_\rho)(t) &= \int (q \circ R_\rho)(ts)(p \circ L_\rho)(s^{-1}) d\lambda_v^{u_1}(s) \\
&= \int q(ts\rho^{-1}) p(\rho s^{-1}) d\lambda_v^{u_1}(s) \\
&= \int (q \circ L_t \circ R_\rho(s))(p \circ Inv \circ R_\rho(s)) d\lambda_v^{u_1}(s) \\
&= \int q \circ L_t(r) p(r^{-1}) R_\rho(d\lambda_v^{u_1})(r) \\
&= \Delta_{\mathcal{G}}(\rho^{-1}) \int q(tr) p(r^{-1}) d\lambda_{u_2}^{u_1}(r) \\
&= \Delta_{\mathcal{G}}(\rho^{-1}) q * p(t).
\end{aligned}$$

□

Let $\mathbf{HGr^{inj}}$ be the subcategory of topological groupoids with Haar system which allows only those functors which are injective on each morphism set. With this restriction we can show that C_c^∞ is a functor:

Lemma 4.20 *Let $F : \mathcal{G} \to \mathcal{M}$ a morphism in $\mathbf{HGr_{td}^{inj}}$. Then we get a non-unital functor*

$$\begin{aligned}
C_c^\infty(F) : C_c^\infty(\mathcal{G}) &\to C_c^\infty(\mathcal{M}) \\
u &\mapsto F(u) \\
p \in \mathrm{mor}_{C_c^\infty(\mathcal{G})}(u, v) &\mapsto (x \mapsto \begin{cases} c_F(v) p \circ (F_u^v)^{-1}(x) & x \in \mathrm{im}(F_u^v) \\ 0 & else \end{cases}
\end{aligned}$$

Proof: Let $p \in \mathrm{mor}_{C_c^\infty(\mathcal{G})}(u,v), q \in \mathrm{mor}_{C_c^\infty(\mathcal{G})}(v,w), x \in F(\mathrm{mor}_{\mathcal{G}}(u,w))$. Then

$$\begin{aligned}(C_c^\infty(q) *_\mu C_c^\infty(p))(x) &= \int C_c^\infty(q)(xy)C_c^\infty(p)(y^{-1})d\mu_{Fv}^{Fu}(y)\\ &= \frac{1}{c_F(v)}\int C_c^\infty(q)(xy)C_c^\infty(p)(y^{-1})dF_v^u(\lambda_v^u)(y)\\ &= \frac{1}{c_F(v)}\int C_c^\infty(q)(xF_v^u(t))C_c^\infty(p)(F_v^u(t^{-1}))d\lambda_v^u(t)\\ &= \frac{1}{c_F(v)}\int c_F(w)q((F_u^w)^{-1}(x)t)c_F(v)p(t^{-1})d\lambda_v^u(t)\\ &= c_F(w)(q*p)((F_u^w)^{-1}(x))\\ &= C_c^\infty(q*p)(x)\end{aligned}$$

□

Now let $G : \mathcal{M} \to \mathcal{N}$ be a second morphism in $\mathbf{HGr_{td}^{inj}}$. Then

$$\begin{aligned}c_F(v)c_G(F(v)) &= \frac{\lambda_v^v(F^{-1}G^{-1}(K))}{\mu_{Fv}^{Fv}(G^{-1}(K))}\frac{\mu_{Fv}^{Fv}(G^{-1}(K))}{\nu_{GFv}^{GFv}(K)}\\ &= \frac{\nu_{GFv}^{GFv}(GFK)}{\lambda_v^v(K)}\\ &= c_{G\circ F}(v)\end{aligned}$$

Hence we get a functor

$$\begin{aligned}C_c^\infty : \mathbf{HGr_{td}^{inj}} &\to \nu\mathbb{C}\mathbf{Cat}\\ \mathcal{G} &\mapsto C_c^\infty(\mathcal{G})\\ (F:\mathcal{G}\to\mathcal{M}) &\mapsto C_c^\infty(F)\end{aligned}$$

This is the totally disconnected analogue of the linearization functor $k-$ in Chapter 3 for $k = \mathbb{C}$. We continue as in Chapter 3 using the functor

$$\mathcal{P}(-_\oplus) : \nu\mathbf{kCat} \to \mathbf{ssmkCat}$$

$$\begin{aligned}\mathcal{C} &\mapsto \mathcal{P}(\mathcal{C}_\oplus)\\ F &\mapsto \mathcal{P}(F_\oplus),\end{aligned}$$

k a commutative ring with unit. As mentiond before, the idempotent completion of a non unital category is unital.

Lemma 4.21 *The two inclusions*

$$\mathcal{P}(C_c^\infty(\mathcal{G})_\oplus) \to \mathcal{P}(C_c^\infty(\mathcal{G}\times\{0\leftrightarrow 1\})_\oplus)$$

extend to a functor

$$\mathcal{P}(C_c^\infty(\mathcal{G})_\oplus)\times\{0\leftrightarrow 1\} \to \mathcal{P}(C_c^\infty(\mathcal{G}\times\{0\leftrightarrow 1\})_\oplus).$$

Proof: We have $C_c^\infty(\mathcal{G} \times \{0 \leftrightarrow 1\}) = C_c^\infty(\mathcal{G}) \times \{0 \leftrightarrow 1\}$ for any groupoid $\mathcal{G}$. Now compare with page 31 $\square$

Together with Lemma 1.14 we get that $\mathcal{P}(-_\oplus) \circ C_c^\infty$ maps equivalences of categories to equivalences of categories.

Lemma 4.22 *Define*

$$E := \mathbb{K}^{-\infty} \circ \mathcal{P}(-_\oplus) \circ C_c^\infty.$$

Then with the same argument as in Chapter 3 we get an equivariant family of spectra and hence a smooth equivariant homology theory $K_*^?$ with coefficients $K_*(\mathcal{H}(?))$.

Recall that the algebraic K-theory of a ring R without unit is defined as

$$K_*(R) := K_*(R_+, R) = \ker(K_*(R_+) \to K_*(\mathbb{Z}))$$

where R_+ is the ring obtained by adjoining a unit element to R, [Ros94], 1.5.7.

Remark 4.23 For any ring R we write $\mathrm{cat}(R)$ for the (perhaps non-unital) category with one object e and R as morphism set. If $\mathcal{G} \in Ob(\mathbf{HGr}_{\mathbf{td}}^{\mathbf{inj}})$ has only one object, then $\mathcal{P}(C_c^\infty(\mathcal{G})_\oplus) = \mathcal{P}(\mathrm{cat}(\mathcal{H}(G))_\oplus)$, where $G = \mathrm{mor}_{\mathcal{G}}(e, e)$. As seen in Section 4.2

$$\mathcal{H}(G) = \bigcup_{I(G)} \mathcal{H}(G/\!\!/K) = \varinjlim_{I(G)} \mathcal{H}(G/\!\!/K)$$

where $I(G) = \{K < G | K \text{ open, compact}\}$ is ordered by the opposite of the inclusion. Hence

$$\mathcal{P}(\mathrm{cat}(\mathcal{H}(G))_\oplus) = \bigcup_{I(G)} \mathcal{P}(\mathrm{cat}(\mathcal{H}(G/\!\!/K))_\oplus).$$

As the ring $\mathcal{H}(G/\!\!/K)$ is unital, the category $\mathcal{P}(\mathrm{cat}(\mathcal{H}(G/\!\!/K))_\oplus)$ is equivalent to the category of finitely generated projective $\mathcal{H}(G/\!\!/K)$-modules. $\square$

The coefficients of K_*^G are

$$K_*^G(G/H) = \pi_*(\mathbb{K}^{-\infty}(\mathcal{P}(C_c^\infty(\mathcal{G}^G(G/H))_\oplus)) \cong \pi_*(\mathbb{K}^{-\infty}(\mathcal{P}(\mathrm{cat}(\mathcal{H}(H))_\oplus)))$$

the last isomorphism being induced by the weak equivalence

$$\mathbb{K}^{-\infty}(\mathcal{P}(\mathcal{G}^H(H/H)_\oplus)) \overset{\mathbb{K}^{-\infty}(\mathcal{P}(i_\oplus))}{\to} \mathbb{K}^{-\infty}(\mathcal{P}(\mathcal{G}^G(G/H)_\oplus)).$$

Now

$$\pi_*(\mathbb{K}^{-\infty}(\mathcal{P}(\mathrm{cat}(\mathcal{H}(H)))_\oplus)) \cong \varinjlim_{I(H)} (\pi_*(\mathbb{K}^{-\infty}(\mathcal{P}(\mathrm{cat}(\mathcal{H}(H/\!\!/K)))_\oplus))$$

by the last remark and [BFJR01], 2.1. Also by the last remark

$$\pi_*(\mathbb{K}^{-\infty}(\mathcal{P}(\mathrm{cat}(\mathcal{H}(H/\!\!/K)))_\oplus)) \cong K_*(\mathcal{H}(H/\!\!/K))$$

for every $K \in I(H)$. As algebraic K-theory is compatible with colimits (the non unital case follows from the unital with the long exact sequence induced by $R \to R_+ \to \mathbb{Z}$.) it follows that

$$K_*^G(G/H) \cong K_*(\mathcal{H}(H)).$$

If H is compact, then $K_0(\mathcal{H}(H))$ is isomorphic to the Grothendieck group $R(H)$ of isomorphism classes of finite dimensional smooth representations of H as this is true for finite groups (cf. Lemma 4.8).

Remark 4.24 In the situation above the following diagram commutes for any $\gamma \in G$ and $\gamma H \gamma^{-1} \subset K$ because $\mathrm{ind}_{c(\gamma)}(G/H) = K_*^G(f : G/H \to G/(\gamma H \gamma^{-1}))$:

$$\begin{array}{ccccc}
K_*^G(G/H) & \xrightarrow{\mathrm{ind}_{c(\gamma)}(G/H)} & K_*^G(G/(\gamma H\gamma^{-1})) & \xrightarrow{K_*^G(pr)} & K_*^G(G/K) \\
\uparrow{\scriptstyle \mathrm{ind}_i(H/H)} & & \uparrow{\scriptstyle \mathrm{ind}_i(K/(\gamma H\gamma^{-1}))} & & \uparrow{\scriptstyle \mathrm{ind}_i(K/K)} \\
K_*^H(H/H) & \xrightarrow{\mathrm{ind}_{c(\gamma)}(H/H)} & K_*^K(K/(\gamma H\gamma^{-1})) & \xrightarrow{K_*^K(pr)} & K_*^K(K/K) \\
\| & & & & \| \\
K_*(\mathcal{H}(H)) & & \xrightarrow{\qquad\qquad\varphi\qquad\qquad} & & K_*(\mathcal{H}(K))
\end{array}$$

Looking at the definition of φ we see that is just the homomorphism induced by the ring homomorphism $\mathcal{H}(H) \to \mathcal{H}(K)$ which comes from conjugation with γ. $\square$

For every totally disconnected, locally compact group G the projection $E(G, \mathcal{CO}) \to *$ gives rise to an assembly map

$$a : K_*^G(E(G, \mathcal{CO})) \to K_*(\mathcal{H}(G)).$$

The corresponding isomorphism conjecture says that this map is an isomorphism (see [DL98] for an introduction to assembly maps).

For any G-homology theory $\mathcal{H}_*^G$ with values in R-modules for a commutative ring R there is an equivariant version of the Atiyah-Hirzebruch spectral sequence. It converges to $\mathcal{H}_{p+q}^G(X, A)$ and its E^2-term is $E_{p,q}^2 = H_p^{\mathrm{Or}(G)}(X, A; \mathcal{H}_q^G(G/?))$ (cf [Lüc02], 3.9.).

Because $K_*(\mathbb{C}H) = 0$ for $* < 0$ and H finite as $\mathbb{C}H$ is regular ([Ros94] 3.3.1) also $K_*(\mathcal{H}(H)) = 0$ for $* < 0$ and H compact totally disconnected (because K-theory is compatible with direct limits). Hence it follows that $K_*(\mathcal{H}(G)) = 0$ for $* < 0$ if the above isomorphism conjecture holds for G.

To get other examples of smooth equivariant homology theories we now concentrate on Hochschild Homology, Cyclic and Periodic Homology and later we define topological K-theory.

5 Hochschild, Cyclic and Periodic Homology

Using the functor $\mathcal{P}(C_c^\infty(-)_\oplus)$ constructed in Chapter 4 we will construct smooth equivariant families of spectra with coefficients $HH_*(\mathcal{H}(K)), HC_*(\mathcal{H}(K))$ resp. $HP_*(\mathcal{H}(K))$, where $\mathcal{H}(K)$ is the Hecke algebra of the totally disconnected, locally compact group K. For this we will use the functors

$$HH_*, HC_*, HP_* : \mathbb{C}\mathbf{Cat_{cof}} \to \mathbf{Ab}$$

constructed by McCarthy in [McC94] which induce functors $\mathbb{C}\mathbf{Cat_{cof}} \to \mathbf{Spt}(\mathbf{Top})$ using a delooping theorem.

5.1 Simplicial Objects

In this section we recall some facts about simplicial objects, which we need for the construction of the equivariant families of spectra we are aiming at. For the basics on simplicial homotopy theory see [GJ99], for the theory of simplicial spectra and spectrum objects see [Jar97].

Let Δ be the category whose objects are the finite ordered sets $[n] = XS\{0 < 1 < ... < n\}$ for $n \in \mathbb{N}$, and whose morphisms are the nondecreasing monotone functions. Let $\mathcal{C}$ be a category, then a *simplicial object* A in $\mathcal{C}$ is a functor $A : \Delta^{op} \to \mathcal{C}$. The category $\mathbf{S}.\mathcal{C}$ of simplicial objects in $\mathcal{C}$ has these as objects and the natural transformations as morphisms. Equivalently a simplicial object A in $\mathcal{C}$ can be described as a family of objects $A_n, n \geq 0$ of $\mathcal{C}$ together with morphisms

$$d_i : A_n \to A_{n-1}, \qquad i = 0, ..., n, \qquad \text{called face maps}$$
$$s_i : A_n \to A_{n+1}, \qquad i = 0, ..., n, \qquad \text{called degeneracy maps}$$

satisfying certain identities (see for example [Lod92], 1.6.1).

A *cyclic object* C in $\mathcal{C}$ is a simplicial object together with an automorphism t_n of order $n+1$ on each C_n such that $d_i t_n = -t_{n-1} d_{i-1}$ and $s_i t_n = -t_{n+1} s_{i-1}$ for $1 \leq i \leq n$, $d_0 t_n = (-1)^n d_n$ and $s_0 t_n = (-1)^n t_{n+1}^2 s_n$.

Notation 5.1 Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. This functor induces a functor $\mathbf{S}.\mathcal{C} \to \mathbf{S}.\mathcal{D}$ between the category of simplicial objects in $\mathcal{C}$ and the category of simplicial objects in $\mathcal{D}$. We call this functor again F, or F^s if this clarifies the situation.

Notation 5.2 Let $\mathcal{C}$ be the category $\mathbf{Set}_*, \mathbf{Ab}$ or $\mathbf{k\text{-}Mod}$, k a commutative ring with unit and let $X \in \mathbf{S}.\mathcal{C}$. Then by $|X|$ we denote the realization of X as pointed simplicial set, i.e. after forgetting the extra structure.

For a pointed bisimplicial set Y the realization of Y is defined as the realization of the diagonal simplicial set $d(Y)$. It is naturally homeomorphic to space that is obtained by first realizing in one direction and then in the other.

Let X, Y be two pointed simplicial sets. The function space $\mathcal{S}(X, Y)$ is the pointed simplicial set $[n] \mapsto \hom_{\mathbf{S.set}_*}(\Delta^n_+ \wedge X, Y)$ with $\Delta^n_+ = \hom([?], [n])_+$.

Lemma 5.3 *If Y is a Kan complex (e.g. a simplicial abelian group), then there is a natural weak equivalence*

$$|\mathcal{S}(X, Y)| \to map(|X|, |Y|).$$

Proof: Use all possible adjunctions and the fact that the map $Y \to \operatorname{sing}(|Y|)$ is a homotopy equivalence because Y is a Kan complex. $\square$

As the definition of Hochschild Homology and Cyclic Homology is chain-complex based, an important tool for us is the Dold-Kan-correspondence:

Theorem 5.4 *For any abelian category $\mathcal{A}$, the normalized chain complex functor N is an equivalence of categories between $\mathbf{S}.\mathcal{A}$ and $\mathbf{Ch}_+(\mathcal{A})$.*

Under this correspondence simplicial homotopy corresponds to homology (i.e. $\pi_*(A) \cong H_*(NA)$) and simplicially homotopic morphisms correspond to chain homotopic maps. (see [Wei94]*).*

Here $N : \mathbf{S}.\mathcal{A} \to \mathbf{Ch}(\mathcal{A})_+$ is given by

$$N(A)_n = \bigcap_{i=0}^{n-1} \ker(d_i : A_n \to A_{n-1})$$

with differential $d = (-1)^n d_n$. An inverse functor $K : \mathbf{Ch}_+(\mathcal{A}) \to \mathbf{S}.\mathcal{A}$ to N is given by

$$K(C)_n := \bigoplus_{p \leq n} \bigoplus_{\eta} C_p,$$

where the index η ranges over all surjections $[p] \to [n]$ in Δ, [Wei94], 8.4.4. We also need the the *unnormalized chain complex functor* $U : \mathbf{S}.\mathcal{A} \to \mathbf{Ch}(\mathcal{A})_+$ with $U(A)_n = A_n$ and differentials

$$d = \sum (-1)^i d_i : U(A)_n \to U(A)_{n-1}$$

(U is called C in [Wei94], but we want to keep this letter for Cyclic Homology). By [Wei94] 8.3.8 there is a natural transformation $N \Rightarrow U$ consisting of chain homotopy equivalences.

As we will be dealing with spectra we want to link simplicial spectra (i.e. families of pointed simplicial sets X^n and pointed structure maps $s^n : S^1 \wedge X^n \to X^{n+1}$, with $S^1 := \Delta^1/\partial\Delta^1$) with something similar consisting of chain complexes. For this we note that in our applications the spectra will really consist of simplicial abelian groups (note here that simplicial abelian groups and simplicial modules are canonically pointed), and use the notion of a *spectrum object in the category of simplicial abelian groups*:

Definition 5.5 *A spectrum object B in the category of simplicial abelian groups consists of simplicial abelian groups $B^n, n \in \mathbb{N}$ and structure morphisms $S^1 \otimes B^n \to B^{n+1}$, where $S^1 \otimes B^n := (\mathbb{Z}[S^1] \otimes B^{n+1})/(\mathbb{Z}[*] \otimes B^{n+1})$ replaces the smash product. The corresponding category (with strict morphisms) will be denoted* $\mathbf{Spt}(\mathbf{Ab})$.

For every simplicial abelian group X, there is a canonical map

$$can : S^1 \wedge X \to S^1 \otimes X$$

which makes a simplicial spectrum out of a spectrum object in simplicial abelian groups. In particular a morphism of spectrum objects in simplicial abelian groups is called a weak equivalence if the morphism between the underlying spectra is one.

To link spectrum objects in simplicial abelian groups with chain complexes we need to vary the notion a bit: A *Kan spectrum object C in the category of simplicial abelian groups* consists of simplicial abelian groups $C^n, n \in \mathbb{N}$ and structure morphisms $\overline{W}C^n \to C^{n+1}$, where $\overline{W} : \mathbf{S}.\mathbf{Ab} \to \mathbf{S}.\mathbf{Ab}$ is the *Kan suspension*, a functor which is naturally homotopy equivalent to the functor $S^1 \otimes ?$ by a natural map $S^1 \otimes ? \to \overline{W}?$. For the definition of $\overline{W}$ see [Jar97], p. 117. The category of Kan spectrum objects and strict maps will be denoted $\mathbf{Spt_{Kan}}(\mathbf{Ab})$.

The main feature of the Kan suspension is that for any simplicial abelian group A there is a natural isomorphism of chain complexes $N\overline{W}A \cong NA[-1]$, where $C[-1]_* = C_{*-1}$.

Because of this the Dold-Kan correspondence is in fact a "stable" correspondence: N and K induce inverse equivalences of the categories $\mathbf{Spt_{Kan}}(\mathbf{Ab})$ and $\mathbf{Spt}(\mathbf{Ch}_+(\mathbf{Ab}))$, where the latter is the category of *spectrum objects in chain complexes*. These consist of positive chain complexes of abelian groups $C^n, n \in \mathbb{N}$ and structure maps $C^n[-1] \to C^{n+1}$ ([Jar97], p. 130). A stable equivalence in $\mathbf{Spt}(\mathbf{Ch}_+(\mathbf{Ab}))$ is a morphism $f : C^* \to D^*$ such that the induced morphism

$$f : colim(C^0 \to C^1[1] \to C^2[2] \to ...) \to colim(D^0 \to D^1[1] \to D^2[2] \to ...)$$

(the colimit being defined using the adjoint of the structure maps) is a homology isomorphism.

Remark 5.6 If C^* is an object of $\mathbf{Spt}(\mathbf{Ch}(\mathbf{Ab})_+)$ we define ΣC^* to be the spectrum object $(\Sigma C)^n = C^{n+1}$. Using the structure maps we get a map $C^*[-1] \to \Sigma C^*$. This map is a stable equivalence as it induces the identity on the above limit. □

Under the stable Dold-Kan correspondence stable equivalences of spectrum objects in chain complexes correspond to stable equivalences of Kan spectrum objects (resp. to stable equivalences of spectrum objects) in simplicial abelian groups.

5.2 Simplicial Spectra and Quasi-Fibrations

Next we look at the construction of simplicial spectra and of spectrum objects in simplicial abelian groups. Let

$$X \xrightarrow{a} Y \xrightarrow{f} Z$$

be a sequence of pointed topological spaces or pointed simplicial sets such that $f \circ a = *$ and Y is contractible. Then a continuous map $l : S^1 \wedge X \to Z$ is given by

$$t \wedge x \mapsto f(H(a(x), t)),$$

where $H : Y \wedge I_+ \to Y$ be a homotopy between the constant map and the identity.

The map l is natural in the following sense: Let the following diagram be commutative with $f \circ a = *$, $f' \circ a' = *$ and Y and Y' contractible with contraction homotopies H resp. H':

$$\begin{array}{ccccc} X & \xrightarrow{a} & Y & \xrightarrow{f} & Z \\ \downarrow \varphi & & \downarrow \chi & & \downarrow \psi \\ X' & \xrightarrow{a'} & Y' & \xrightarrow{f'} & Z' \end{array}$$

Let furthermore $\chi \circ H = H' \circ (\chi \wedge id)$. Then the following diagram commutes:

$$\begin{array}{ccc} S^1 \wedge X & \xrightarrow{l} & Z \\ \downarrow \varphi & & \downarrow \psi \\ S^1 \wedge X' & \xrightarrow{l'} & Z'. \end{array}$$

With this construction we get a spectrum: Let $X^n, n \in \mathbb{N}$ be a family of pointed topological spaces or pointed simplicial sets. For every $n \in \mathbb{N}$ let there be a space Y^n of the same kind, which is contractible, and let there be a sequence

$$X^n \xrightarrow{a^n} Y^n \xrightarrow{f^n} X^{n+1}$$

such that $f^n \circ a^n = *$. Then the structure maps are to be the maps l^n.

Remark 5.7 We can do the same construction with simplicial abelian groups replacing $\wedge$ by $\otimes$. Then we get a spectrum object in simplicial groups whose underlying simplicial spectrum is just the one we would get if we forgot the group structure beforehand. $\square$

If we deal with pointed topological spaces, then the adjoint s of this map fits into a diagram

$$\begin{array}{ccccc} X & \xrightarrow{a} & Y & \xrightarrow{f} & Z \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle j} & & \| \\ \Omega Z & \longrightarrow & PZ & \longrightarrow & Z \end{array}$$

where $j : Y \to PZ$ is given by $j(y) = (t \mapsto f(H(y,t)))$. As Y and PZ are contractible this map is a weak equivalence and hence so is s if the upper row is a quasi-fibration (an analogous statement holds for pointed simplicial sets with the right definition of ΩZ and PZ).

Definition 5.8 *A sequence*

$$X. \stackrel{a}{\to} Y. \stackrel{b}{\to} Z.$$

of pointed simplicial sets is a quasi-fibration of simplicial sets if

$$\begin{array}{ccc} X. & \xrightarrow{a} & Y. \\ \downarrow & & \downarrow{\scriptstyle b} \\ * & \longrightarrow & Z. \end{array}$$

is homotopy cartesian, which means in this special case that there exists a pointed simplicial set W_* and a trivial cofibration (= cofibration + weak equivalence) $i : Y. \to W.$ and a fibration $f : W. \to Z.$ such that the following diagram, with $F.$ the fiber of f, commutes and $j : X. \to F., j_n(x) = i_n a_n(x)$ is a weak equivalence.

$$\begin{array}{ccc} X. & \xrightarrow{j} & F. \\ \downarrow{\scriptstyle a} & & \downarrow \\ Y. & \xrightarrow{i} & W. \\ \downarrow{\scriptstyle b} & \swarrow{\scriptstyle f} & \\ Z. & & \end{array}$$

Note that if $X. \to Y. \to Z.$ is a quasi-fibration of simplicial sets, then

$$|X.| \to |Y.| \to |Z.|$$

is a quasi-fibration of topological spaces. This is true because $|f|$ is a fibration with fiber $|F|$ (See [GJ99] I,10.10 and 10.9) and $|j|$ and $|i|$ are weak equivalences, $|i|$ a cofibration and $|b| \circ |a| = *$. Hence if in the construction of the spectrum we assume that all the sequences are quasi-fibrations, then we get an Ω-spectrum.

In Section 5.5 we will be dealing with bisimplicial sets. In this context the theorem of Bousfield-Friedlander will come handy:

Theorem 5.9 ([GJ99] IV,4.9) *Let*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ W & \longrightarrow & Z \end{array}$$

be a pointwise homotopy cartesian square of bisimplicial sets, i.e.

$$\begin{array}{ccc} X_{n.} & \longrightarrow & Y_{n.} \\ \big\downarrow & & {\scriptstyle n.}\big\downarrow{\scriptstyle p} \\ W_{n.} & \longrightarrow & Z_{n.} \end{array}$$

is a homotopy cartesian square of simplicial sets for ever $n \geq 0$*, such that* Y *and* Z *satisfy the* π_**-Kan condition ([GJ99], p.224, 226) and the induced map* $\pi_0^v(p) : \pi_0^v(Y) \to \pi_0^v(Z)$ *of vertical path components is a fibration. Then the associated commutative square*

$$\begin{array}{ccc} d(X) & \longrightarrow & d(Y) \\ \big\downarrow & & \big\downarrow{\scriptstyle d(p)} \\ d(W) & \longrightarrow & d(Z) \end{array}$$

of the diagonal simplicial sets is homotopy cartesian.

Here $\pi_0^v(Y)$ is the simplicial set $\pi_0^v(Y)_m = \pi_0(Y_{m,*})$. The π_*-Kan condition is automatically satisfied if Y and Z are bisimplicial abelian groups (see [BF78], p. 120).

5.3 Hochschild Homology, Cyclic Homology and Periodic Homology for Categories with Cofibrations

Defining a spectrum for Hochschild Homology is just a short step when using the results of [McC94]. There, for any k-linear category with cofibrations its *n-th Hochschild (Cyclic, Periodic) Homology* is defined. For this the following notions are needed (see [McC94]):

A *category with cofibrations* $\mathcal{C}$ is a category with a distinguished zero object together with a subcategory *co*$\mathcal{C}$ satisfying the following three axioms. The morphisms in *co*$\mathcal{C}$ will be called the cofibrations in $\mathcal{C}$ and will be denoted with arrows $\hookrightarrow$.

1. The isomorphisms in $\mathcal{C}$ are cofibrations.
2. For every $A \in Ob(\mathcal{C})$, $0 \to A$ is a cofibration.
3. Cofibrations admit cobase change. That is if $A \hookrightarrow B$ is a cofibration and $A \to C$ any arrow, then the pushout $C \coprod_A B$ exists in $\mathcal{C}$ and the canonical arrow $C \to C \coprod_A B$ is a cofibration.

The category $\mathbf{Cat_{cof}}$ is to be the category of small categories with cofibrations and exact functors (i.e. functors which preserve all structure).

Let $\mathcal{C}$ be a category with cofibrations. Then the so called *S-construction* $S.\mathcal{C}$ is to be the following simplicial category with cofibrations: $Ob(S_n\mathcal{C})$ is the class of all diagrams

$$\begin{array}{ccccccccccc}
0 = A_{0,0} & \hookrightarrow & A_{0,1} & \hookrightarrow & A_{0,2} & \hookrightarrow & A_{0,3} & \hookrightarrow & ... & \hookrightarrow & A_{0,n} \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
 & & 0 & \hookrightarrow & A_{1,2} & \hookrightarrow & A_{1,3} & \hookrightarrow & ... & \hookrightarrow & A_{1,n} \\
 & & & & \downarrow & & \downarrow & & & & \downarrow \\
 & & & & 0 & \hookrightarrow & A_{2,3} & \hookrightarrow & ... & \hookrightarrow & A_{2,n} \\
 & & & & & & \downarrow & & & & \downarrow \\
 & & & & & & 0 & \hookrightarrow & ... & \hookrightarrow & A_{3,n} \\
 & & & & & & & & & & \downarrow
\end{array}$$

such that all squares are pushouts. The morphisms $\mathrm{mor}_{S_n\mathcal{C}}(A, B)$ are collections $(\varphi_{i,j} : A_{i,j} \to B_{i,j})$ of morphisms commuting with the structure maps of A and B. $S_n\mathcal{C}$ is again a category with cofibrations. The cofibrations are those morphisms $(\varphi_{i,j})$, where all the $\varphi_{i,j}$ and the induced maps $B_{0,j} \coprod_{A_{0,j}} A_{0,j+1} \to B_{0,j+1}$ are cofibrations.

The simplicial structure maps of $S.\mathcal{C}$ are given by inserting identities respectively composing adjacent morphisms. Only d_0 is defined differently:

$$d_0(0 \hookrightarrow A_{0,1} \hookrightarrow A_{0,2} \hookrightarrow A_{0,3} \hookrightarrow ... \hookrightarrow A_{0,n}) = 0 \hookrightarrow A_{1,2} \hookrightarrow A_{1,3} \hookrightarrow ... \hookrightarrow A_{1,n}$$

Let $\mathcal{C}$ be a small k-linear category, k a commutative ring with unit. The *additive cyclic nerve* $CN.\mathcal{C}$ of $\mathcal{C}$ is defined to be the following cyclic k-module:

$$CN_n\mathcal{C} := \bigoplus_{(c_o,c_1,\dots,c_n)\in Ob(\mathcal{C}^{n+1})} \mathrm{mor}_{\mathcal{C}}(c_1, c_0) \otimes_k \mathrm{mor}_{\mathcal{C}}(c_2, c_1) \otimes_k ... \otimes_k \mathrm{mor}_{\mathcal{C}}(c_0, c_n)$$

The structure maps are defined by

$$d_i(f_0 \otimes ... \otimes f_n) = \begin{cases} (f_0 \otimes f_1 \otimes ... \otimes f_{i+1} \circ f_i \otimes ... \otimes f_n) & \text{if } 0 \le i \le n-1 \\ (f_0 \circ f_n \otimes f_1 \otimes ... \otimes f_{n-1}) & \text{if } i = n \end{cases}$$

$$s_i(f_0 \otimes ... \otimes f_n) = \begin{cases} (f_0 \otimes f_1 \otimes ... \otimes f_i \otimes id_{c_{i+1}} \otimes f_{i+1} \otimes ... \otimes f_n) & \text{if } 0 \le i \le n-1 \\ (f_0 \otimes f_1 \otimes ... \otimes f_n \otimes id_{c_0}) & \text{if } i = n \end{cases}$$

$$t(f_0 \otimes ... \otimes f_n) = (f_n \otimes f_1 \otimes ... \otimes f_{n-1})$$

See [McC94], 2.1.1. The additive cyclic nerve is a covariant functor form the category of small k-categories to the category of cyclic k-modules (we'll often use that in particular it is a functor to cyclic abelian groups).

Let k be a commutative ring with unit and A a cyclic k-module. Then the bicomplex $CC^{per}(A)$ of k-modules is used to define Hochschild Homology, Cyclic Homology and

Periodic Homology (cf. [Lod92], 2.5.3 and 5.1.1):

$$\begin{array}{lccccc}
CC^{per}(A) & -2 & -1 & 0 & 1 & 2
\end{array}$$

$$\begin{array}{ccccccccccc}
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & \\
 & \xleftarrow{N} & A_2 & \xleftarrow{1-t} & A_2 & \xleftarrow{N} & A_2 & \xleftarrow{1-t} & A_2 & \xleftarrow{N} & A_2 & \xleftarrow{1-t} \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & \\
 & \xleftarrow{N} & A_1 & \xleftarrow{1-t} & A_1 & \xleftarrow{N} & A_1 & \xleftarrow{1-t} & A_1 & \xleftarrow{N} & A_1 & \xleftarrow[1-t]{} \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & \\
 & \xleftarrow{N} & A_0 & \xleftarrow{1-t} & A_0 & \xleftarrow{N} & A_0 & \xleftarrow{1-t} & A_0 & \xleftarrow{N} & A_0 & \xleftarrow{1-t}
\end{array}$$

Here t is the cyclic operator of A, $b = \sum_{i=o}^{n}(-1)^i d_i$, $b' = \sum_{i=0}^{n-1}(-1)^i d_i$ and $N = \sum_{i=0}^{n} t^i$.

Now the Periodic Homology $HP_*(A)$ is defined to be the homology of the chain complex $\text{Tot}\prod(CC^{per}(A))$, where $\text{Tot}\prod$ is the total complex taken with products instead of direct sums. The Hochschild Homology $HH_*(A)$ is defined to be the homology of the zero-column of $CC^{per}(A)$, which is just the unnormalized chain complex of the underlying simplicial k-module A, or equivalently the homology of $\text{Tot}(CC^{\{2\}}(A))$, where $CC^{\{2\}}(A)$ is the bicomplex consisting of the zeroth and the first column of $CC^{per}(A)$ ([Lod92], 2.2.1). The Cyclic Homology $HC_*(A)$ is defined to be the homology of $\text{Tot}(CC(A))$, where $CC(A)$ is the first quadrant-part of $CC^{per}(A)$.

Now we can define the Hochschild Homology, Cyclic Homology and Periodic Homology of a k-category with cofibrations:

Definition 5.10 *(*[McC94], 3.2.1) *Let k be a commutative ring with unit, $\mathcal{C}$ a k-linear category with cofibrations. The the n-th Hochschild (Cyclic, Periodic) Homology of $\mathcal{C}$ is defined to be the $(n{+}1)$st Hochschild (Cyclic, Periodic) Homology of the simplicial$\times$cyclic k-module $CN.S.\mathcal{C}$ (via the unnormalized chain complex functor applied to the simplicial direction and $CC^{per}, CC, \ldots$ to the cyclic direction, a simplicial$\times$cyclic k-module gives rise to triplecomplexes, whose total complexes define $HP_*, HC_*, \ldots$ as above).*

5.4 A Spectrum for Hochschild Homology

For any k-category with cofibrations $\mathcal{C}$ a spectrum $HH(\mathcal{C})^*$ whose homotopy groups are the Hochschild Homology of $\mathcal{C}$ can be constructed in analogy to the Waldhausen-K-theory-spectrum. The Dennis trace map of [McC94], 4.4.1 then becomes a map of spectra (for details see [LMR]). Namely we define

$$HH(\mathcal{C})^0 := \Omega|CN.S.\mathcal{C}|$$

$$HH(\mathcal{C})^n := |CN.S^n.\mathcal{C}|, n > 0$$

meaning that we forget the cyclic structure and realize as a bisimplicial set. To get the structure maps we note that the middle term of the sequence

$$|CN.S^n.\mathcal{C}| \to |CN.PS.S^n.\mathcal{C}| \to |CN.S^{n+1}.\mathcal{C}|$$

is contractible and that the composition is trivial. As seen above such a sequence always gives a map

$$s^n : |CN.S^n.\mathcal{C}| \to \Omega|CN.S^{n+1}.\mathcal{C}|$$

and using the delooping theorem of [McC94], 3.6.3 which tells us that the sequence above is a quasi-fibration we get that s^n is a weak homotopy equivalence and hence $HH(\mathcal{C})^*$ an Ω-spectrum. Its homotopy groups are

$$\pi_*(HH(\mathcal{C})^*) \cong \pi_*(\Omega|CN.S.\mathcal{C}|) \cong H_{*+1}(\mathrm{Tot}(CN.S.\mathcal{C})) = HH_*(\mathcal{C}).$$

Here the last but one step follows with the theorem of Eilenberg-Zilber [Wei94], p. 275. and "Tot" of a bisimplicial abelian group means the total complex of the associated bicomplex.

Now the details: To see that we really get a functor

$$HH^* : \mathbf{kCat_{cof}} \to \mathbf{\Omega Spt(Top)}$$

we show that there is a general principle behind this construction, which later will enable us to give a functorial definition of spectra for Cyclic Homology.

Lemma 5.11 *Let $PS.$ be the Path object of $S.$ (i.e. $PS. : \mathbf{kCat_{cof}} \to \mathbf{S.kCat_{cof}}, \mathcal{C} \mapsto (n \mapsto S_{n+1}\mathcal{C})$) and let $F : \mathbf{kCat_{cof}} \to \mathbf{S.set}$ be any functor such that $F(0)_n = *$ for all $n \geq 0$ where 0 is the category with one object and one morphism.*

Then there is a $\mathbf{kCat_{cof}}$-contraction of $d(F^s \circ PS.)$, i.e. a natural transformation

$$H : d(F^s \circ PS.) \wedge (\mathcal{C} \mapsto \hom_{\Delta^{op}}(?, [1]))_+ \Rightarrow d(F^s \circ PS.)$$

such that $H \circ (id \times d_0) = id$ and $(H \circ (id \times d_1))_n = *$. Here $d(X)$ is the diagonal simplicial set.

Proof: For every category with cofibrations $\mathcal{C}$ the bisimplicial set $F^s \circ PS.\mathcal{C}$ is the same as the simplicial simplicial-set (=bisimplicial set) $P(F^s \circ S.\mathcal{C})$ where the P is taken with respect to the simplicial coordinate of $S.$ not of F. Hence by [Wal85], 1.5.1 there is a simplicial homotopy between $id_{P(F^s \circ S.\mathcal{C})}$ and the map of simplicial simplicial-sets $P(F^s \circ S.\mathcal{C}) \to F^s \circ S_0\mathcal{C} \to P(F^s \circ S.\mathcal{C})$:

$$h_{\mathcal{C}} : P(F^s \circ S.\mathcal{C}) \wedge \hom_{\Delta^{op}}([k], [1])_+ \to P(F^s \circ S.\mathcal{C})$$

$$(x, a : [k] \to [1]) \mapsto (P(F^s \circ S.\mathcal{C})(\varphi_a))(x)$$

where $\varphi_a : [k+1] \to [k+1]$ is given by $\varphi_a(j) = 0$ if $j = 0$ or $a(j-1) = 0$ and $\varphi_a(j) = j$ if $a(j-1) = 1$. Hence we also get a homotopy $H_{\mathcal{C}}$ between the diagonals of these maps, i.e. between the identity and the map which for every $n \in \mathbb{N}$ is the map

$$(F(S_{n+1}\mathcal{C}))_n \to (F(S_0\mathcal{C}))_n \to (F(S_{n+1}\mathcal{C}))_n.$$

This is the constant map, because the middle term is a point as $S_0\mathcal{C} = 0$. By construction the homotopy is natural in the category $\mathcal{C}$. □

Let

$$S.^n : \mathbf{kCat_{cof}} \to \{\Delta^{op} \to \mathbf{kCat_{cof}}\}$$

$$\mathcal{C} \mapsto S.^n\mathcal{C}$$

be the diagonal of n-times the S-construction and let $PS.S.^n : \mathbf{kCat_{cof}} \to \{\Delta^{op} \to \mathbf{kCat_{cof}}\}$ be $(n+1)$-times the S-construction, then shifting one simplicial direction and then taking the diagonal: $(PS.S.^n\mathcal{C})_k = S_{k+1}S_kS_k...S_k\mathcal{C}$. There is a sequence

$$([n] \mapsto \mathcal{C}) \overset{\eta}{\Rightarrow} PS. \overset{\zeta}{\Rightarrow} S.$$

of natural transformations such that $\zeta \circ \eta = const$. Namely for every k-category with cofibrations $\mathcal{C}$ and any $m \in \mathbb{N}$ this is given by

$$\mathcal{C} = S_1\mathcal{C} \overset{PS.([m]\to[0])}{\longrightarrow} PS_m\mathcal{C} \overset{d_0}{\longrightarrow} S_m\mathcal{C}.$$

Let $F : \mathbf{kCat_{cof}} \to \mathbf{S.set}_+$ be any functor such that $F(0)_n = *$. Then the $\mathbf{kCat_{cof}}$-contraction of $d \circ F^s \circ PS.$ of Lemma 5.11 induces a $\mathbf{S.kCat_{cof}}$-contraction H of the functor $(d \circ F^s \circ PS.)^s$ Hence the middle term of the sequence

$$|d \circ (d \circ F^s \circ ([n] \mapsto \mathcal{C}))^s| \overset{\eta}{\to} |d \circ (d \circ F^s \circ PS.)^s| \overset{\zeta}{\to} |d \circ (d \circ F^s \circ S.)^s|$$

is $\mathbf{S.kCat_{cof}}$-contractible. This means that for every simplicial k-category with cofibrations $\mathcal{C}$ (in our case $S.^n\mathcal{D}$) we get a contraction

$$H_{\mathcal{C}} : |d((d \circ F^s \circ PS.)^s\mathcal{C})| \wedge I_+ \to |d((d \circ F^s \circ PS.)^s\mathcal{D})|$$

such that for any functor $T : \mathcal{C} \to \mathcal{D}$ the diagram

$$\begin{array}{ccc} |d((d \circ F^s \circ PS.)^s\mathcal{C})| \wedge I_+ & \xrightarrow{T \times id} & |d((d \circ F^s \circ PS.)^s\mathcal{D})| \wedge I_+ \\ \downarrow{\scriptstyle H_{\mathcal{C}}} & & \downarrow{\scriptstyle H_{\mathcal{D}}} \\ |d((d \circ F^s \circ PS.)^s\mathcal{C})| & \xrightarrow{T} & |d((d \circ F^s \circ PS.)^s\mathcal{D})| \end{array}$$

commutes. Hence by the construction of Section 5.1 we can define

Definition 5.12 *For a functor $F : \mathbf{kCat_{cof}} \to \mathbf{S.set}$ with $F(0)_n = *$ let*

$$Spt_F : \mathbf{kCat_{cof}} \to \mathbf{Spt(Top)}$$

be the functor which associates to a k-category with cofibrations $\mathcal{C}$ the spectrum $Spt_F^*(\mathcal{C})$, which is given by

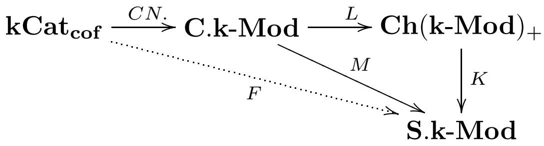
$$Spt_F^0(\mathcal{C}) := \Omega|(d \circ F^s)(S.\mathcal{C}))|$$

$$Spt_F^n(\mathcal{C}) := |(d \circ F^s)(S.^n\mathcal{C}))|, n > 0$$

with structure maps given by the sequence above.

$$\begin{array}{ccccc} \mathbf{kCat_{cof}} & \xrightarrow{S.^n} & \mathbf{S.kCat_{cof}} & \xrightarrow{F^s} & \mathbf{S.S.Ab} \\ \vdots{\scriptstyle Spt_F^n} & & & & \downarrow{\scriptstyle d} \\ \mathbf{Top} & & \xleftarrow{\quad |.| \quad} & & \mathbf{S.Ab} \end{array}$$

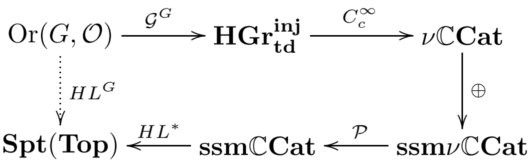
In our application we define F as in the following diagram starting either with a functor $L : \mathbf{C.k\text{-}Mod} \to \mathbf{Ch}(\mathbf{k\text{-}Mod})_+$ or a functor $M : \mathbf{C.k\text{-}Mod} \to \mathbf{S.k\text{-}Mod}$ with $L(0) = 0$ resp. $M(0) = 0$:



So we get functors

$$HL^* \text{ resp. } HM^* : \mathbf{kCat_{cof}} \to \mathbf{Spt}(\mathbf{Top}).$$

Definition 5.13 *Let $k = \mathbb{C}$, L resp. M as above, G a totally disconnected, locally compact group. Then we define HL^G resp. HM^G to be the* $\mathrm{Or}(G, \mathcal{O})$*-spectra which we get by composing HL^* resp. HM^* with the functor $\mathcal{P}(C_c^\infty(\mathcal{G}^G)_\oplus) : \mathrm{Or}(G, \mathcal{O}) \to \mathbb{C}\mathbf{Cat_{cof}}$.*



Remark 5.14 If $F, G : \mathbf{C.k\text{-}Mod} \to \mathbf{S.k\text{-}Mod}$ are functors as above and $\eta : F \Rightarrow G$ a natural transformation, then η induces a map of **Cat**-spectra $H\eta : HF^* \to HG^*$. If η consists of weak equivalences, then $H\eta$ is a weak equivalence. Note here that a morphism of bisimplicial groups is already a weak equivalence if it is a pointwise weak equivalence ([GJ99], IV, 2.6). □

As seen in Section 5.1 there are natural transformations $\zeta : KN \Rightarrow id, \eta : KN \to KU$ which consist of weak equivalences (here we suppress the forgetful functor $\mathbf{C.k\text{-}Mod} \to \mathbf{S.k\text{-}Mod}$ in the notation). By remark 5.14

$$HH^* := H\,\mathrm{id}^* \overset{H\zeta}{\leftarrow} HN^* \overset{H\eta}{\to} HU^*$$

are weak equivalences of **Cat**-spectra.

Let $T : \mathbf{C.k\text{-}Mod} \to \mathbf{Ch}(\mathbf{k\text{-}Mod})_+$ be given by $C \mapsto \mathrm{Tot}(CC^{\{2\}}(C))$. There is a natural transformation $U \Rightarrow T$ consisting of homology isomorphisms. Hence there is a natural transformation $\theta : KU \Rightarrow KT$ consisting of weak equivalences and we get a weak equivalence $H\theta : HU^* \to HT^*$.

By Lemma 1.7 the associated $\mathrm{Or}(G, \mathcal{O})$-spectra HH^G, HU^G, HT^G and HN^G all induce the same G-homology theory on G-CW-complexes, which we call HH_*^G. By the delooping theorem [McC94], 3.6.3, which says that the sequence

$$|CN.S.^n\mathcal{C}| \to |CN.PS.S.^n\mathcal{C}| \to |CN.S.^{n+1}\mathcal{C}|$$

is a quasi-fibration, HH^* is a functor to Ω-spectra and hence

$$\pi_*(HH(\mathcal{C})) \cong \pi_{*+1}(|CN.S.\mathcal{C}|) = HH_*(\mathcal{C}).$$

Lemma 5.15 *Let $k = \mathbb{C}$. Then $\{HH_*^?(\ \cdot\ )\}$ is a smooth equivariant homology theory on totally disconnected, locally compact groups and has coefficients $HH_*^G(G/H) \cong HH_*(\mathcal{H}(H))$ for every open subgroup H of G.*

Proof: The functor $\mathcal{P}(C_c^\infty(-)_\oplus)$ maps a totally disconnected groupoid to a symmetric monoidal $\mathbb{C}$-category which gets the structure of a $\mathbb{C}$-category with cofibrations if we define the cofibration-subcategory to be generated by the inclusions $A \to A \oplus B$. By Lemma 4.21 $\mathcal{P}(C_c^\infty(-)_\oplus)$ maps equivalences of totally disconnected groupoids to equivalences of categories. These are even exact equivalences. By [McC94], 3.2.6. HH^* maps exact equivalences to weak equivalences of spectra. Hence we can apply Theorem 2.10.

For the coefficients of HH_*^G note that $\mathcal{G}^G(G/H)$ is equivalent to $\mathcal{G}^H(H/H)$ and that $\mathcal{P}(C_c^\infty(\mathcal{G}^H(H/H))_\oplus)$ is the category of idempotent matrices over $\mathcal{H}$. As HH_* is compatible with colimits ([McC94], 3.2.4) the proposition follows with remark 4.23 $\square$

5.5 A Spectrum for Cyclic Homology

To get a spectrum for Cyclic Homology we have to take the cyclic structure of $CN.S.\mathcal{C}$ into consideration. Define

$$Z : \mathbf{C.k\text{-}Mod} \to \mathbf{Ch(k\text{-}Mod)}_+, C \mapsto \mathrm{Tot}(CC(C)).$$

Then for any cyclic k-module C its Cyclic Homology is given as $HC_*(C) = H_*(Z(A))$. Now

$$HC^* := HZ^* : \mathbf{Cat_{cof}} \to \mathbf{Spt(Top)}$$

Lemma 5.16 *For every k-category $\mathcal{C}$ with cofibrations $HC(\mathcal{C})$ is an Ω-spectrum.*

For the proof we can't use the delooping theorem [McC94], 3.6.3. directly, but will prove a slightly different version of it which isn't proved in [McC94] but seems to be used there:

Theorem 5.17 *Let $\mathcal{C}$ and $\mathcal{D}$ be k-categories with cofibrations, $F : \mathcal{C} \to \mathcal{D}$ an exact functor. Let $|X|_{HC} := |KZ(X)|$ for any simplicial$\times$cyclic k-module, then*

$$|CN.S.\mathcal{D}|_{HC} \to |CN.S.S.(F : \mathcal{C} \to \mathcal{D})|_{HC} \to |CN.S.S.\mathcal{C}|_{HC}$$

is a quasi-fibration.

Here $S.(F : \mathcal{C} \to \mathcal{D})$ is the pullback of

$$\begin{array}{ccc} & & PS.\mathcal{D} \\ & & \downarrow{\scriptstyle d_0} \\ S.\mathcal{C} & \xrightarrow{S.F} & S.\mathcal{D} \end{array}$$

which for $F = id$ is just $PS.\mathcal{C}$ (see [McC94], 3.6.1).

Proof: We go along the proof of [McC94], 3.6.3. We have to show that

$$\begin{array}{ccc} d((p,n) \mapsto K(ZCN.S_n\mathcal{D})_n) & \longrightarrow & d((p,n) \mapsto K(ZCN.S_nS_p(F))_n) \\ \downarrow & & \downarrow \\ * & \longrightarrow & d((p,n) \mapsto K(ZCN.S_nS_p\mathcal{C})_n) \end{array}$$

is a homotopy cartesian square. For this we use the theorem of Bousfield-Friedlander 5.9, i.e we have to show that the above square is pointwise homotopy cartesian before taking the diagonal .

Looking at the proof of 3.6.3 we see that there is for every m an exact functor $\Psi : \mathcal{D} \times S_m\mathcal{C} \to S_m(F : \mathcal{C} \to \mathcal{D})$ such that the following diagram commutes and both vertical maps are weak equivalences of cyclic $\times$ simplicial k-modules.

$$\begin{array}{ccccc}
CN.S.\mathcal{D} & \longrightarrow & CN.S.\mathcal{D} \times CN.S.S_m\mathcal{C} & \longrightarrow & CN.S.S_m\mathcal{C} \\
\Vert & & \uparrow & & \Vert \\
CN.S.\mathcal{D} & \longrightarrow & CN.S.(\mathcal{D} \times S_m\mathcal{C}) & \longrightarrow & CN.S.S_m\mathcal{C} \\
\Vert & & \downarrow{\scriptstyle CN.S.\Psi} & & \Vert \\
CN.S.\mathcal{D} & \longrightarrow & CN.S.S_m(F : \mathcal{C} \to \mathcal{D}) & \longrightarrow & CN.S.S_m\mathcal{C}
\end{array}$$

Remark 5.18 Let $f : X \to Y$ be a map of cyclic$\times$simplicial k-modules such that

$$\pi_*(f) : \pi_*(|X|) \to \pi_*(|Y|)$$

is an isomorphism. Then

$$\pi_*(f) : \pi_*(|X|_{HC}) \to \pi_*(|Y|_{HC})$$

is an isomorphism. For this note that by Lemma 5.19

$$\pi_*(|X|_{HC}) \cong HC_*(X)$$

where $X_{\bullet *}$ is considered as a cyclic complex. Now use 2.5.12 [Lod92] (which also holds true for cyclic complexes). $\square$

Hence the following diagram commutes and the vertical maps are weak equivalences of simplicial groups:

$$\begin{array}{ccccc}
K(ZCN.S_*\mathcal{D})_* & \longrightarrow & K(ZCN._*\mathcal{D})_* \times K(ZCN.S_*S_m\mathcal{C})_* & \xrightarrow{\pi} & K(ZCN.S_*S_m\mathcal{C})_* \\
\Vert & & \uparrow & & \Vert \\
K(ZCN.S_*\mathcal{D})_* & \longrightarrow & K(ZCN.S_*(\mathcal{D} \times S_m\mathcal{C}))_* & \longrightarrow & K(ZCN.S_*S_m\mathcal{C})_* \\
\Vert & & \downarrow{\scriptstyle K(ZCN.S_*\Psi)_*} & & \Vert \\
K(ZCN.S_*\mathcal{D})_* & \longrightarrow & K(ZCN.S_*S_m(F : \mathcal{C} \to \mathcal{D}))_* & \xrightarrow{pr_m} & K(ZCN.S_*S_m\mathcal{C})_*
\end{array}$$

Note that $Z(X. \times Y.) = Z(X.) \oplus Z(Y.)$ and $K(C_* \oplus D_*) = K(C_*) \times K(D_*)$. First we remark that the upper row is a trivial fibration. Hence taking π_0 of the diagram we see that $\pi_0(\pi)$ is surjective and hence $\pi_0(pr_m)$ is surjective. Hence $\pi_0^v(pr)$ is surjective and therefore a fibration (as π_0^v of a bisimplicial abelian group is a simplicial abelian group. Now use [GJ99] III, 2.9.).

As π is a fibration it follows that the middle row is a quasi-fibration and hence the lower row is also a quasi-fibration.

Now we can apply the theorem of Bousfield-Friedlander and the proposition follows. □

As said above $S.(id : \mathcal{C} \to \mathcal{C}) = PS.\mathcal{C}$. Hence we have a quasi-fibration

$$|CN.S.\mathcal{D}|_{HC} \to |CN.PS.S.\mathcal{C}|_{HC} \to |CN.S.S.\mathcal{C}|_{HC}.$$

Replacing $\mathcal{C}$ by $S^n_k\mathcal{C}$, we get that

$$d(CN.S.S^n_k\mathcal{D}) \to d(CN.PS.S.S^n_k\mathcal{C}) \to d(CN.S.S.S^n_k\mathcal{C}).$$

is a quasi-fibration for every k. Again by the theorem of Bousfield-Friedlander

$$|CN.S.S.^n\mathcal{D}|_{HC} \to |CN.PS.S.^{n+1}\mathcal{C}|_{HC} \to |CN.S.S.^{n+1}\mathcal{C}|_{HC}.$$

is a quasi-fibration. And this proves Lemma 5.16.

To calculate the homotopy groups of $HC(\mathcal{C})$, we now need only calculate the homotopy groups of the zeroth space. Here we have to be a bit careful, because there is a dummy variable coming from the simplicial direction $S.^n$. The following lemma makes it precise, that "taking care of a dummy variable in chain complexes" is the same as "taking care of a dummy variable in simplicial abelian groups".

Lemma 5.19 *Let $X \in Ob(\mathbf{S.Ch_+(Ab)})$. Then*

$$H_*(\mathrm{Tot}(\varphi \circ U(X))) \cong \pi_*((d \circ K^s)(X))$$

where φ is the functor from complexes of complexes to bicomplexes which reverts the sign of every other vertical differential.

Proof: By the Theorem of Eilenberg-Zilber (cf. [Wei94], p. 275) we have

$$\pi_*((d \circ K^s)(X)) \cong H_*(\mathrm{Tot}(C(K^s(X))))$$

where $C : \mathbf{S.S.Ab} \to \mathbf{Bi\text{-}Ch(Ab)_+}$ is the functor

$$\begin{array}{ccccc}
\mathbf{S.S.Ab} & \xrightarrow{ad} & \{\Delta^{op} \to \mathbf{S.Ab}\} & \xrightarrow{U^s} & \{\Delta^{op} \to \mathbf{Ch_+(Ab)}\} \\
\downarrow{\scriptstyle C} & & & & \| \\
\mathbf{Bi\text{-}Ch(Ab)_+} & \xleftarrow[\varphi]{} & \mathbf{Ch(Ch(Ab)_+)_+} & \xleftarrow[U]{} & \mathbf{S.Ch(Ab)_+}
\end{array}$$

Comparing

$$\mathbf{S.Ch(Ab)_+} \stackrel{K^s}{\to} \{\Delta^{op} \to \mathbf{S.Ab}\} \stackrel{U^s}{\to} \{\Delta^{op} \to \mathbf{Ch(Ab)_+}\} \stackrel{\varphi \circ U}{\to} \mathbf{Bi\text{-}Ch(Ab)_+}$$

with $\varphi \circ U$ we see that there is a natural transformation between them, which consists of morphisms of bicomplexes which are column-wise quasi-isomorphisms (there is a natural equivalence $NK \Rightarrow id$ consisting of isomorphisms and a natural transformation $U \Rightarrow N$ consisting of quasi-isomorphisms). It follows that it is a quasi-isomorphism on the total complexes (cf. [Lod92] 1.0.12). □

For $X = Z(CN.S.\mathcal{C})$ it follows that

$$\pi_*(HC^0(\mathcal{C})) \cong HC^*(\mathcal{C}).$$

As in the last chapter we get:

Lemma 5.20 *Let $k = \mathbb{C}$. Then $\{HC_*^?(\ \cdot\ )\}$ is a smooth equivariant homology theory on totally disconnected, locally compact groups and has coefficients $HC_*^G(G/H) \cong HC_*(\mathcal{H}(H))$ for any open subgroup H of G.*

Proof: As above, only note that Z is compatible with colimits and maps weak equivalences to homology isomorphisms. $\square$

5.6 A Spectrum for Periodic Homology

To construct a spectrum for Periodic Homology we will argue a bit differently to simplify the arguments. The price we have to pay, is that the definition of the spectrum isn't in strict analogy to that of the spectrum for Waldhausen K-theory.

The functor $\mathcal{P}(C_c^\infty(\mathcal{G}^?)_\oplus)$ is a functor $\mathrm{Or}(G) \to \mathbf{splex\mathbb{C}Cat}$, where $\mathbf{splex\mathbb{C}Cat}$ is the category of small split exact $\mathbb{C}$-categories. Hence it suffices to define a functor

$$HP^* : \mathbf{splex\mathbb{C}Cat} \to \mathbf{Spt}(\mathbf{Top}).$$

Put

$$P : \mathbf{C.k\text{-}Mod} \to \mathbf{Ch}(\mathbf{k\text{-}Mod})_+, C \mapsto \mathrm{Trc}(\mathrm{Tot}^{\prod}(CC^{per}(C))).$$

where $\mathrm{Trc} : \mathbf{Ch}(\mathbf{k\text{-}Mod}) \to \mathbf{Ch}(\mathbf{k\text{-}Mod})_+$ truncates a chain complex C at the zeroth chain group and replaces the zeroth chain group by the image of ∂_0. This means that the new chain complex has the same homology as the old one in non-negative degrees and homology zero in negative degrees.

We define $HP^0(\mathcal{M}) := |K\mathcal{P}CN.\mathcal{M}|$ for any split exact category $\mathcal{M}$. Then $\pi_*(HP^0(\mathcal{M})) = HP_*^s(\mathcal{M})$ as defined in [McC94], 2.1.2.. By 3.3.4 this equals the Periodic homology of $\mathcal{M}$.

A natural map $HP^0(\mathcal{M}) \to \Omega^2 HP^0(\mathcal{M})$ is defined in the following way, forgetting about the k-module structure, keeping only the abelian group structures:

$$\begin{array}{ccc} K\mathcal{P}CN.\mathcal{M} & \xrightarrow[\simeq]{can_*} & \Omega(S^1 \otimes K\mathcal{P}CN.\mathcal{M}) \\ \Big\downarrow{\scriptstyle \varphi} & & \simeq \Big\downarrow{\scriptstyle \Omega can_*} \\ \Omega^2 K\mathcal{P}CN.\mathcal{M} & \xleftarrow{\Omega^2 f} & \Omega^2(S^1 \otimes (S^1 \otimes K\mathcal{P}CN.\mathcal{M})) \end{array}$$

where for every simplicial abelian group A the map

$$can_* : A \to \mathcal{S}(S^1, S^1 \otimes A) =: \Omega(S^1 \otimes A)$$

is the adjoint of the canonical map $can : S^1 \wedge A \to S^1 \otimes A$ (the simplicial mapping space has the structure of a simplicial abelian group). It is a weak equivalence by [Jar97], 4.53. The map f is given by

$$\begin{array}{ccccc} S^2 \otimes K\mathcal{P}CN.\mathcal{M} & \xrightarrow{\simeq} & \overline{W}^2 K\mathcal{P}CN.\mathcal{M} & \xrightarrow{\simeq} & KN(\overline{W}^2 K\mathcal{P}CN.\mathcal{M}) \\ & & & & \Big\downarrow{\scriptstyle \simeq} \\ \Big\downarrow{\scriptstyle f} & & & & K(NK\mathcal{P}CN.\mathcal{M})[-2] \\ & & & & \Big\downarrow{\scriptstyle \simeq} \\ K\mathcal{P}CN.\mathcal{M} & & \xleftarrow{K(incl)} & & K(\mathcal{P}CN.\mathcal{M})[-2] \end{array}$$

$\pi_*(K(incl))$ is an isomorphism for $* \geq 2$ as in $K(incl)$ is the identity for $* \geq 3$ and and the identity on homotopy for $* = 2$ by definition of the truncation. Hence it follows that $\varphi : |K\mathcal{P}CN.\mathcal{M}| \to |\Omega^2 K\mathcal{P}CN.\mathcal{M}|$ is a natural weak equivalence. So we define

$$HP^n(\mathcal{M}) := \Omega^n HP^0(\mathcal{M})$$

with structure maps $\Omega^n(\nu \circ \varphi)$, where $\nu : |\Omega^2 K\mathcal{P}CN.\mathcal{M}| \to \Omega^2 |K\mathcal{P}CN.\mathcal{M}|$ is the natural weak equivalence of Lemma 5.3, and get a functor

$$HP^* : \mathbf{splex}\mathbb{C}\mathbf{Cat} \to \Omega\text{-}\mathbf{Spt}(\mathbf{Top}).$$

The spectrum $HP^*(\mathcal{M})$ is non-connective and has as homotopy groups the Periodic Homology of $\mathcal{M}$. As above we get for every totally disconnected, locally compact group G an $\mathrm{Or}(G, \mathcal{O})$-spectrum HP^G.

Lemma 5.21 *$\{HP_*^?(\ \cdot\ )\}$ is a smooth equivariant homology theory on totally disconnected, locally compact groups and has coefficients $HP_*^G(G/H) \cong HP_*(\mathcal{H}(H))$ for any open subgroup H of G.*

Proof: As before. □

5.7 The Long Exact Sequence of HH and HC

There is a long exact sequence connecting Hochschild Homology and Cyclic Homology.

Lemma 5.22 *For every G-CW-complex X there is a long exact sequence*

$$\ldots \to HC_{n-1}(X) \to HH_n(X) \to HC_n(X) \to HC_{n-2}(X) \to \ldots$$

Proof: Let $0 \Rightarrow A_1 \Rightarrow A_2 \Rightarrow A_3 \Rightarrow 0$ be a sequence of functors $\mathbf{C.k\text{-}Mod} \to \mathbf{S.k\text{-}Mod}$ with $A_i(0)_n = 0$, which object-wise form a short exact sequence of simplicial k-modules (our example: $KT \Rightarrow KZ \Rightarrow KZ[-2]$). Then we get a sequence

$$HA_1^G \stackrel{i}{\Rightarrow} HA_2^G \stackrel{p}{\Rightarrow} HA_3^G$$

of functors $\mathrm{Or}(G, \mathcal{O}) \to \mathbf{Spt}(\mathbf{S.Set})$, which has the property that $i(G/H)$ is a degreewise cofibration (i.e. in each degree of the spectrum an inclusion of simplicial sets) and $p(G/H)$ is degreewise the cofiber.

Hence by realizing we get a sequence

$$HA_1^G \stackrel{i}{\Rightarrow} HA_2^G \stackrel{p}{\Rightarrow} HA_3^G$$

of functors $\mathrm{Or}(G, \mathcal{O}) \to \mathbf{Spt}(\mathbf{Top})$, which has the property that $i(G/H)$ is a degreewise cofibration (i.e. in each degree of the spectrum a cofibration, even an inclusion of subcomplexes) and $p(G/H)$ is degreewise the cofiber. Now we use the following lemma:

Lemma 5.23 *Let $\mathcal{C}$ be a category and*

$$A \stackrel{f}{\to} B \stackrel{g}{\to} C$$

a cofiber sequence of pointed $\mathcal{C}$-spaces, i.e. $g \circ f = *$ and the vertical maps of the following commutative diagram are weak equivalences of $\mathcal{C}$-spaces:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \| & & \uparrow & & \uparrow \\ A & \longrightarrow & Cyl(f) & \longrightarrow & Cone(f) \end{array}$$

Then for every free $\mathcal{C}$-CW-complex X

$$X \otimes_{\mathcal{C}} A \stackrel{id \otimes f}{\to} X \otimes_{\mathcal{C}} B \stackrel{id \otimes g}{\to} X \otimes_{\mathcal{C}} C$$

is a cofiber sequence of spaces. (For the definition of free $\mathcal{C}$-CW-complexes see [DL98], 3.2. If X is a smooth G-CW-complex, then it is a free $\mathrm{Or}(G, \mathcal{O})$*-CW-complex.)*

Proof: Because the functor $X \otimes_{\mathcal{C}} -$ has a right adjoint it commutes with pushouts. Hence $X \otimes_{\mathcal{C}} Cyl(f) \cong Cyl(id_X \otimes_{\mathcal{C}} f)$ and $X \otimes_{\mathcal{C}} Cone(f) \cong Cone(id_X \otimes_{\mathcal{C}} f)$. Using Lemma 3.11 of [DL98] it follows that the vertical maps of the following diagram are weak equivalences.

$$\begin{array}{ccccc} X \otimes_{\mathcal{C}} A & \xrightarrow{id \otimes f} & X \otimes_{\mathcal{C}} B & \xrightarrow{g} & X \otimes_{\mathcal{C}} C \\ \| & & \uparrow & & \uparrow \\ X \otimes_{\mathcal{C}} A & \longrightarrow & Cyl(id \otimes f) & \longrightarrow & Cone(id \otimes f) \end{array}$$

Using this lemma we get that for every smooth G-CW-complex X the sequence

$$X \otimes_{\mathcal{C}} HA_1^G \stackrel{id \otimes i}{\Rightarrow} X \otimes_{\mathcal{C}} HA_2^G \stackrel{id \otimes p}{\Rightarrow} X \otimes_{\mathcal{C}} HA_3^G$$

is a cofibration sequence of spectra (the definition of mapping cylinder and mapping cone for spectra is degree-wise). Hence we get a long exact sequence

$$... \to H(A_3)_{n+1}(X) \to H(A_1)_n(X) \to H(A_2)_n(X) \to H(A_3)_n(X) \to ...$$

In our example we get

$$... \to H(Z[-2])_{n+1}(X) \to HH_n(X) \to HZ_n(X) \to H(Z[-2])_n(X) \to ...$$

It remains to identify $H(Z[-2])_n^G(X)$ with $HZ_{n-2}^G(X)$. For this note that there are natural equivalences

$$\begin{array}{ccccl} K \circ [-1] \circ Z & \Longrightarrow & \overline{W}KZ & \Longrightarrow & (S^1 \otimes ?) \circ KZ : \mathbf{C.Ab} \to \mathbf{S.Ab} \\ \Downarrow & & \Uparrow & & \\ K \circ [-1] \circ NKZ & \Longrightarrow & KN\overline{W}KZ & & \end{array}$$

Hence $H(K(Z[-1]))^*$ is $\mathbf{Cat_{cof}}$-equivalent to $H((S^1 \otimes ?)KZ)^*$ (see 5.14) As

$$d \circ ad \circ (S^1 \otimes ?) = (S^1 \otimes ?) \circ d \circ ad$$

we get that $H(K(Z[-1]))^*$ is $\mathbf{Cat_{cof}}$-equivalent to $S^1 \otimes H(KZ)^*$, which is $\mathbf{Cat_{cof}}$-equivalent via two equivalences going in different directions to the shifted spectrum $\Sigma H(KZ)^*$, because for every simplicial spectrum X the spectra ΣX and $S^1 \wedge X$ are naturally isomorphic in the stable homotopy category (see [GJ99], 1.9). Applying this twice shows that $H(Z[-2])_n^G(X) \cong HZ_{n-2}^G(X)$. □

6 Topological K-theory

In [Joa02] functors

$$A^f : \nu\mathbf{C^*Cat} \to \mathbf{C^*Alg}, \mathcal{C} \mapsto A^f_{\mathcal{C}}$$

and

$$\mathbb{K} : \mathbf{C^*Alg} \to \mathbf{Spt^\Sigma}$$

are constructed. The composition of these two will be again called $\mathbb{K}$:

$$\mathbb{K} : \mathbf{C^*Cat} \to \mathbf{Spt^\Sigma}.$$

Here $\mathbf{Spt^\Sigma}$ is the category of symmetric spectra. This functor has the following properties:

1. If the C^*-category $\mathcal{C}$ has only one object, then

$$\pi_n(\mathbb{K}(\mathcal{C})) = \pi_n(\mathbb{K}(A^f_{\mathcal{C}})) \cong K_n(A^f_{\mathcal{C}}) \cong K_n(\mathrm{mor}_{\mathcal{C}}(x,x))$$

   is the K-theory of the C^*-algebra $\mathrm{mor}_{\mathcal{C}}(x,x)$.

2. If $\mathcal{C} = \overline{\bigcup \mathcal{C}_i}, \mathcal{C}_i \subset \mathcal{C}_j$for $i \leq j$, $Ob(C_i) = Ob(C_j)$, then $A^f_{\mathcal{C}} \cong \varinjlim_{i \in I} A^f_{\mathcal{C}_i}$. Hence we have

$$\begin{aligned}\pi_n(\mathbb{K}(\mathcal{C})) &= \pi_n(\mathbb{K}(A^f_{\mathcal{C}})) \cong K_n(A^f_{\mathcal{C}}) \\ &\cong \varinjlim_{i \in I} K_n(A^f_{\mathcal{C}_i}) \\ &\cong \varinjlim_{i \in I} \pi_n(\mathbb{K}(A^f_{\mathcal{C}_i})) \\ &\cong \varinjlim_{i \in I} \pi_n(\mathbb{K}(\mathcal{C}_i))\end{aligned}$$

3. A C^*-equivalence φ between two C^*-categories induces an isomorphism $\pi_*(\mathbb{K}(\varphi))$.

As before we want to construct for every totally disconnected, locally compact group G an $\mathrm{Or}(G, \mathcal{O})$-Spectrum K^{top}_G whose homotopy groups on G/H are the K-theory of the reduced C^*-algebra $C^*_r(H)$. For this we need a functor

$$C^*_r : \mathbf{HGr^{inj}_{td}} \to \mathbf{C^*Cat}$$

which generalizes the notion of the reduced C^*-algebra of a group G. This functor will be a generalization of the construction done in [DL98].

6.1 C^*-Categories

In this section we give the definition of a C^*-category (also see [Joa02]) and look at the example that is the most important one for us.

Definition 6.1 *1. A small category $\mathcal{C}$ is called a Banach category if for any two objects $x, y \in Ob(\mathcal{C})$ the morphism set* $\mathrm{mor}_{\mathcal{C}}(x,y)$ *is a complex Banach space with norm $\| \ \|_{xy}$ such that composition is bilinear and satisfies*

$$\|g \circ f\|_{xz} \leq \|g\|_{yz} \|f\|_{xy}$$

for all $x, y, z \in Ob(\mathcal{C})$ and all morphisms $f : x \to y, g : y \to z$.

2. *An involution $*$ on a Banach category $\mathcal{C}$ consists of maps*

$$*_{xy} : \mathrm{mor}_{\mathcal{C}}(x,y) \to \mathrm{mor}_{\mathcal{C}}(y,x)$$

for $x, y \in Ob(\mathcal{C})$, such that for all $x, y, z \in Ob(\mathcal{C})$ and morphisms $f, h : x \to y, g : y \to z$

   (a) $*_{xy}(\lambda f + \mu h) = \overline{\lambda} \cdot *_{xy}(f) + \overline{\mu} \cdot *_{xy}(g)$ *for all* $\lambda, \mu \in \mathbb{C}$,

   (b) $*_{xy} \circ *_{yx} = id_{\mathrm{mor}_{\mathcal{C}}(x,y)}$,

   (c) $*_{xy}(g \circ f) = *_{xy}(f) \circ *_{yz}(g)$.

The element $f^* := *_{xy}(f)$ is called the adjoint of f.

3. *A Banach category $\mathcal{C}$ with involution $*$ is called a C^*-category if*

   (a) $\|f^* \circ f\|_{xx} = \|f\|^2_{xy}$ *for all $x, y \in Ob(\mathcal{C})$ and all $f : x \to y$,*

   (b) *for all morphisms $f : x \to y$ there is a morphism $g : x \to x$ such that $f^* \circ f = g^* \circ g$, i.e. $f^* \circ f$ is positive in* $\mathrm{mor}_{\mathcal{C}}(x,x)$.

A C^*-functor $F : \mathcal{C} \to \mathcal{D}$ between two C^*-categories is a covariant functor such that the induced maps

$$F^y_x : \mathrm{mor}_{\mathcal{C}}(x,y) \to \mathrm{mor}_{\mathcal{D}}(Fx, Fy), x, y \in Ob(\mathcal{C})$$

are additive, $\mathbb{C}$-linear and adjoint preserving, i.e. $F(f)^* = F(f^*)$. The category of C^*-categories ("small" is part of the definition) and C^*-functors will be called $\mathbf{C^*Cat}$

4. *A non-unital C^*-category is a small non-unital category which satisfies the analogs of points 1,2,3. The corresponding category of non-unital C^*-categories will be called* $\nu\mathbf{C^*Cat}$.

Now an example: Let $\mathcal{G} \in Ob(\mathbf{HGr^{inj}_{td}})$. Then we define the non-unital C^*-category $C^*_r\mathcal{G}$ in the following way: Its objects are the objects of $\mathcal{G}$ and each morphism set $\mathrm{mor}_{C^*_r\mathcal{G}}(g_1, g_2)$ is the completion of $C^\infty_c(\mathrm{mor}_{\mathcal{G}}(g_1, g_2))$ with respect to the regular representation. This has to be explained:

Lemma 6.2 *Let $\gamma \in Ob(\mathcal{G})$ such that $\mathrm{mor}_{\mathcal{G}}(\gamma, g_1) \neq \emptyset$ then*

$$i_{g_1,g_2,\gamma} : C^\infty_c(\mathrm{mor}_{\mathcal{G}}(g_1, g_2)) \to \mathcal{B}(L^2(\mathrm{mor}_{\mathcal{G}}(\gamma, g_1)), L^2(\mathrm{mor}_{\mathcal{G}}(\gamma, g_2))$$

$$f \mapsto \{\vartheta_f : p \mapsto f * p\}$$

is a $\mathbb{C}$-linear injective map. Here $\mathcal{B}$ are the bounded linear operators.

Proof: The map is injective, because for $f \neq g$ there exists an open compact subgroup $K \subset \mathrm{mor}_{\mathcal{G}}(g_1, g_1)$ such that $f * \frac{1}{\lambda^{g_1}_{g_1}(K)}\chi_K = f$ and $g * \frac{1}{\lambda^{g_1}_{g_1}(K)}\chi_K = g$.

Next we have to show that $i_{g_1,g_2,\gamma}$ is well defined, i.e. that ϑ_f is really a bounded operator. For this we will show that

$$\|i_{g_1,g_2,\gamma}(f)\| = \|i_{g_1,g_2,\eta}(f)\|(*)$$

for any $f \in C_c^\infty(\mathrm{mor}_\mathcal{G}(g_1, g_2)), \gamma, \eta \in Ob(\mathcal{G})$ such that $\mathrm{mor}_\mathcal{G}(\gamma, \eta) \neq \infty$. Then we take $\gamma = g_1$ an get

$$\begin{aligned}
\|\vartheta_f\| &= \sup_{r \in L^2(\mathrm{mor}(g_1,g_1)), \|r\|_2 \le 1} \|f * r\|_2 \\
&= \sup_{r \in L^2(\mathrm{mor}(g_1,g_1)), \|r\|_2 \le 1} \|(f \circ L_\varphi) * r\|_2 \\
&\le \sup_{r \in L^2(\mathrm{mor}(g_1,g_1)), \|r\|_2 \le 1} \|f \circ L_\varphi\|_1 \|r\|_2 \\
&\le \|f\|_1
\end{aligned}$$

for any $\varphi \in \mathrm{mor}_\mathcal{G}(g_2, g_1)$. Here we used that left multiplication doesn't change the norm (cf. remark 4.3). The third step follows from [Loo53] 31A.

It remains to prove equation $(*)$. Choose an element $\psi \in \mathrm{mor}_\mathcal{G}(\gamma, \eta)$. Then

$$\begin{aligned}
\|i_{g_1,g_2,\gamma}(f)\| &= \sup_{r \in L^2(\mathrm{mor}(\gamma,g_1)), \|r\|_2 \le 1} \|f * r\|_2 \\
&= \Delta_\mathcal{G}(\psi) \sup_{r \in L^2(\mathrm{mor}(\gamma,g_1)), \|r\|_2 \le 1} \|(f * r) \circ R_\psi\|_2 \\
&= \Delta_\mathcal{G}(\psi) \sup_{r \in L^2(\mathrm{mor}(\gamma,g_1)), \|r\|_2 \le 1} \|f * (r \circ R_\psi)\|_2 \\
&= \Delta_\mathcal{G}(\psi) \sup_{r \in L^2(\mathrm{mor}(\eta,g_1)), \|r \circ R_{\psi^{-1}}\|_2 \le 1} \|f * r\|_2 \\
&= \Delta_\mathcal{G}(\psi) \sup_{r \in L^2(\mathrm{mor}(\eta,g_1)), \Delta_\mathcal{G}(\psi)\|r\|_2 \le 1} \|f * r\|_2 \\
&= \Delta_\mathcal{G}(\psi)\Delta_\mathcal{G}(\psi^{-1}) \sup_{r \in L^2(\mathrm{mor}(\eta,g_1)), \|r\|_2 \le 1} \|f * r\|_2 \\
&= \|i_{g_1,g_2,\eta}(f)\|
\end{aligned}$$

Here we used corollary 4.15 which also holds because $C_c^\infty(X)$ is dense in $L^1(X)$ if X is a topological space with a Radon measure (see [Els99] VI, 2.28) and hence, again using [Loo53] 31A, r can be approximated by elements of $C_c^\infty(\mathrm{mor}(\gamma, g_1))$. (See also p. 63) □

With this lemma the operator norm on the target defines a norm on $C_c^\infty(\mathrm{mor}_\mathcal{G}(g_1, g_2))$:

$$\|f\|_{g_1,g_2} := \|i_{g_1,g_2,\gamma}(f)\| \text{ for } f \in C_c^\infty(\mathrm{mor}_\mathcal{G}(g_1, g_2).$$

Definition 6.3 *Let $\mathcal{G} \in \mathbf{HGr}_{\mathbf{td}}^{\mathbf{inj}}$. Then the category $C_r^*\mathcal{G}$ has the same objects as $\mathcal{G}$ and*

$$\mathrm{mor}_{C_r^*\mathcal{G}}(g_1, g_2) := \overline{C_c^\infty(\mathrm{mor}_\mathcal{G}(g_1, g_2))}^{\|\cdot\|_{g_1,g_2}}$$

For composition in $C_r^*\mathcal{G}$ note that the map

$$\mathcal{B}(L^2(\mathrm{mor}_\mathcal{G}(\gamma, g_1)), L^2(\mathrm{mor}_\mathcal{G}(\gamma, g_2))) \times \mathcal{B}(L^2(\mathrm{mor}_\mathcal{G}(\gamma, g_2)), L^2(\mathrm{mor}_\mathcal{G}(\gamma, g_3)))$$
$$\to \mathcal{B}(L^2(\mathrm{mor}_\mathcal{G}(\gamma, g_1)), L^2(\mathrm{mor}_\mathcal{G}(\gamma, g_3)))$$

restricts to

$$\mathrm{mor}_{C_r^*\mathcal{G}}(g_1, g_2) \times \mathrm{mor}_{C_r^*\mathcal{G}}(g_2, g_3) \to \mathrm{mor}_{C_r^*\mathcal{G}}(g_1, g_3)$$

because
$$\vartheta_g \circ \vartheta_f = \vartheta_{g*f}.$$

So up to now we have a non-unital category, whose morphism sets are all Banach spaces and where composition is a continuous $\mathbb{C}$-bilinear map. An involution on the non-unital category is defined by taking the adjoint in $\mathcal{B}(L^2(\mathrm{mor}_{\mathcal{G}}(\gamma, g_1)), L^2(\mathrm{mor}_{\mathcal{G}}(\gamma, g_2)))$ Note that
$$(\vartheta_f)^* = \vartheta_{f^*}, \text{where } f^*(x) = \overline{f(x^{-1})}$$
for $f \in C_c^\infty(\mathrm{mor}_{\mathcal{G}}(g_1, g_2)), g \in C_c^\infty(\mathrm{mor}_{\mathcal{G}}(g_2, g_3))$. The adjoint $*$ is an isometry as this is true in the space of bounded operators. It has the required properties $(*(\lambda f + \mu g) = \overline{\lambda} * (f) + \overline{\mu} * (g), *(*(f)) = f, *(g \circ f) = *(f) \circ *(g))$ as can be easily checked for $* : C_c^\infty(\mathrm{mor}_{\mathcal{G}}(g_1, g_2)) \to C_c^\infty(\mathrm{mor}_{\mathcal{G}}(g_2, g_1))$ and then follows immediately. Also using that the norm is given by the norm on a space of bounded operators we see that
$$\|g \circ f\| \le \|g\| \cdot \|f\| \text{ and } \|f^* \circ f\| = \|f\|^2$$
for all $g_1, g_2, g_3 \in Ob(\mathcal{G}), g \in \mathrm{mor}_{C_r^*\mathcal{G}}(g_2, g_3), f \in \mathrm{mor}_{C_r^*\mathcal{G}}(g_1, g_2)$.

So $\| \, \|$ imposes a C^*-structure on each endomorphism set. We exploit this fact to show that $f^* \circ f (f \in \mathrm{mor}_{C_r^*\mathcal{G}}(g_1, g_2))$ is positive, i.e. there exists a $g \in \mathrm{mor}_{C_r^*\mathcal{G}}(g_1, g_1)$ such that $g^* \circ g = f^* \circ f$: For $f \in C_c^\infty(\mathrm{mor}_{\mathcal{G}}(g_1, g_2))$ the map $g = \Delta_{\mathcal{G}}(\varphi^{-1})^{\frac{1}{2}} f \circ L_\varphi$, $\varphi \in \mathrm{mor}_{\mathcal{G}}(g_1, g_2)$ any morphism, does the thing. The general statement follows from the fact that the set of positive elements in a C^*-algebra is closed. All in all it follows that $C_r^*\mathcal{G}$ is a non-unital C^*-category.

$C_r^*\mathcal{G}$ has the property that every endomorphism set $\mathrm{mor}_{C_r^*\mathcal{G}}(g, g)$ is the reduced C^*-algebra $C_r^* \mathrm{mor}_{\mathcal{G}}(g, g)$.

Next we show that every morphism $F : \mathcal{G} \to \mathcal{M}$ in $\mathbf{HGr}_{\mathbf{td}}^{\mathbf{inj}}$ induces a C^*-functor
$$C_r^* F : C_r^*\mathcal{G} \to C_r^*\mathcal{M}.$$
As seen in Chapter 4.3 we get a functor
$$C_c^\infty F : C_c^\infty \mathcal{G} \to C_c^\infty \mathcal{M}$$
so we just have to show that the map
$$C_c^\infty F_u^v : C_c^\infty(\mathrm{mor}_{\mathcal{G}}(u, v)) \to C_c^\infty(\mathrm{mor}_{\mathcal{M}}(Fu, Fv))$$
is bounded if $\mathrm{mor}_{\mathcal{G}}(u, v) \neq \emptyset$. For this it suffices to show this in the case $u = v$, as left multiplication is an isometry.

Let $H := \mathrm{mor}_{\mathcal{G}}(u, u)$ and $G := \mathrm{mor}_{\mathcal{M}}(Fu, Fu)$. We have to show that
$$C_c^\infty(F) : C_c^\infty(H) \to C_c^\infty(G)$$
is bounded. We can assume that H is an open subgroup of G. Let $\{g_i\}$ be a system of representatives of $H\backslash G$. As $C_c^\infty(G)$ is dense in $L^2(G)$ in such a way that for every $f \in L^2(G)$ and every $\epsilon > 0$ there exists a $g \in C_c^\infty(G)$ with $|g| \le |f|$ and $\|f - g\|_2 < \epsilon$ (see [Els99] VI,2.28, as our measures are Radon measures, the characteristic functions of compact sets suffice) and because $\|f * g\| \le \|f\|_1 \|g\|_2$ (see [Loo53] 31A), we can use

$C_c^\infty(G)$ instead of $L^2(G)$ to calculate the norm of θ_f, which we will denote by $\|f\|_G$ to emphasize that even though our f is already an element of $C_c^\infty(H)$, we take the norm with respect to G.

$$\begin{aligned}
\|f\|_G^2 &= \sup_{r\in C_c^\infty(G),\|r\|_2\le 1} \|f*r\|_2^2 \\
&= \sup_{r\in C_c^\infty(G),\|r\|_2\le 1} \|\sum_{Hg_i}(f*r)\chi_{Hg_i}\|_2^2 \\
&= \sup_{r\in C_c^\infty(G),\|r\|_2\le 1} \sum_{Hg_i}\|(f*r)\chi_{Hg_i}\|_2^2 \\
&= \sup_{r\in C_c^\infty(G),\|r\|_2\le 1} \sum_{Hg_i}\Delta_G(g_i^{-1})\|((f*r)\chi_{Hg_i})\circ R_{g_i^{-1}}\|_2^2 \\
&= \sup_{r\in C_c^\infty(G),\|r\|_2\le 1} \sum_{Hg_i}\Delta_G(g_i^{-1})\|(f*(r\circ R_{g_i^{-1}}))\chi_H\|_2^2 \\
&= \sup_{r\in C_c^\infty(G),\|r\|_2\le 1} \sum_{Hg_i}\Delta_G(g_i^{-1})\|f*(r\circ R_{g_i^{-1}}\chi_H)\|_2^2 \\
&\le \sup_{r\in C_c^\infty(G),\|r\|_2\le 1} \sum_{Hg_i}\Delta_G(g_i^{-1})\|f\|_H^2\|(r\chi_{Hg_i})\circ R_{g_i^{-1}}\|_2^2 \\
&= \|f\|_H^2 \sup_{r\in C_c^\infty(G),\|r\|_2\le 1} \sum_{Hg_i}\|r\chi_{Hg_i}\|_2^2 \\
&= \|f\|_H^2
\end{aligned}$$

Hence C_r^* is a C^*-functor. We now see that C_r^* is a functor $\mathbf{HGr_{td}^{inj}} \to \mathbf{C^*Cat}$.

Remark 6.4 Let $\mathcal{G} \in Ob(\mathbf{HGr_{td}^{inj}})$, $g \in Ob(\mathcal{G})$ and K an open compact subgroup of $\mathrm{mor}_\mathcal{G}(g,g)$. Define $u(g)_K := \{\frac{1}{vol(K)}\chi_K\}$. Then $(u(g)_K)_{K\in I(G)}$ where $I(G)$ is the family of open compact subgroups of G, is an approximative unit of $\mathrm{mor}_\mathcal{G}(g,g)$ consisting of selfadjoint idempotent elements.

Define $C_r^*(\mathcal{G})_K$ to be the C^*-subcategory of $C_r^*(\mathcal{G})$ which has the same objects, but

$$\mathrm{mor}_{C_r^*(\mathcal{G})_K}(g_1,g_2) = u(g_2)_K \circ \mathrm{mor}_{C_r^*(\mathcal{G})}(g_1,g_2)\circ u(g_1)_K.$$

$C_r^*(\mathcal{G})_K$ is a unital Banach-$*$-category, because the elements of the approximative unit are selfadjoint. To see that it also is a C^*-category, we note that an element $f^*f, f \in \mathrm{mor}_{C_r^*(\mathcal{G})_K}(g_1,g_2)$ is positive in $\mathrm{mor}_{C_r^*(\mathcal{G})}(g_1,g_1)$ and hence also in the sub-C^*-algebra $\mathrm{mor}_{C_r^*(\mathcal{G})_K}(g_1,g_1)$. By definition

$$\mathrm{mor}_{C_r^*(\mathcal{G})}(g_1,g_1) = \overline{\bigcup_K u(g_2)_K \circ \mathrm{mor}_{C_r^*(\mathcal{G})}(g_1,g_2)\circ u(g_1)_K}^{\|\cdot\|}$$

for all $g_1, g_2 \in Ob(\mathcal{G})$. □

Lemma 6.5 *Let $E \in Ob(\mathbf{HGr_{td}^{inj}})$ $(\Rightarrow E\times\{0\leftrightarrow 1\} \in Ob(\mathbf{HGr_{td}^{inj}}))$.*
Then $C_r^*(E\times\{0\leftrightarrow 1\}) = C_r^*(E)\times\{0\leftrightarrow 1\}$.

Proof: We see at once that both categories have the same objects. Let (e,x) and $(f,y), e,f \in Ob(E), x,y \in \{0,1\}$ be two such objects. Then

$$\mathrm{mor}_{C^*_r(E)\times\{0\leftrightarrow 1\}}((e,x),(f,y)) = \mathrm{mor}_{C^*_r(E)}(e,f)$$

and

$$\begin{aligned}\mathrm{mor}_{C^*_r(E\times\{0\leftrightarrow 1\})}((e,x),(f,y)) &= \overline{C^\infty_c(\mathrm{mor}_{E\times\{0\leftrightarrow 1\}}((e,x),(f,y)))}^{\|\cdot\|_{E\times\{0\leftrightarrow 1\}}} \\ &= \overline{C^\infty_c(\mathrm{mor}_E(e,f))}^{\|\cdot\|_E} = \mathrm{mor}_{C^*_r(E)}(e,f)\end{aligned}$$

as the norm with respect to which the set is completed is defined by the morphism

$$\begin{aligned} i : C^\infty_c(&\mathrm{mor}_{E\times\{0\leftrightarrow 1\}}((e,x),(f,y))) \to \\ &\mathcal{B}(L^2(\mathrm{mor}_{E\times\{0\leftrightarrow 1\}}((\gamma,z),(e,x))), L^2(\mathrm{mor}_{E\times\{0\leftrightarrow 1\}}((\gamma,z),(f,y)))) \\ \varphi \mapsto \{&\theta_\varphi : p \mapsto \varphi * p\}\end{aligned}$$

for any object (γ,z) in $E\times\{0\leftrightarrow 1\}$. But this is the same as the map

$$i : C^\infty_c(\mathrm{mor}_E(e,f)) \to \mathcal{B}(L^2(\mathrm{mor}_E(\gamma,e)), L^2(\mathrm{mor}_e(\gamma,f))),$$

which defines $\|\cdot\|_E$. The composition is in both cases given by the composition in $C^*_r E$. □

6.2 A Spectrum for Topological K-theory

Definition 6.6 *Let G be a totally disconnected, locally compact group. Then*

$$K^{top}_G := \mathbb{K} \circ C^*_r \circ \mathcal{G}^G.$$

Lemma 6.7 *$K^{top}_?(\,\cdot\,)$ is a smooth equivariant homology theory on totally disconnected locally compact groups and has coefficients $K_*(C^*_r(H))$.*

For the proof we have to look closer at the definition of the C^*-algebra $A^f_\mathcal{C}$. Let $\mathcal{C}$ be a non-unital C^*-category and A a C^*-algebra. A *$*$-homomorphism $\tau : \mathcal{C} \to A$* is a C^*-functor between $\mathcal{C}$ and A considered as a non-unital C^*-category with one object. Now $(A^f_\mathcal{C}, \tau^f_\mathcal{C})$ is defined to be a pair consisting of a natural choice of a C^*-algebra $A^f_\mathcal{C}$ and a $*$-homomorphism $\tau^f_\mathcal{C} : \mathcal{C} \to A^f_\mathcal{C}$ with the following universal property: Given a $*$-homomorphism $\tau : \mathcal{C} \to A$ into a C^*-algebra A, then there is a unique $*$-homomorphism $\alpha_\tau : A^f_\mathcal{C} \to A$ such that $\tau = \alpha_\tau \circ \tau^f_\mathcal{C}$ (See [Joa02], 2.4). $A^f_\mathcal{C}$ is explicitly given in [Joa02] and leads to a functor

$$A^f : \nu\mathbf{C^*Cat} \to \mathbf{C^*Alg},\ \mathcal{C} \mapsto A^f_\mathcal{C}.$$

The universal property of the C^*-algebra $A^f_\mathcal{C}$ assures that this construction is compatible with colimits:

Lemma 6.8 *Let $\mathcal{C}$ be a non-unital C^*-category and $\mathcal{C}_i, i \in I$, I a directed system, subcategories such that $Ob(\mathcal{C}_i) = Ob(\mathcal{C})$, $\mathcal{C}_i \subset \mathcal{C}_j$ if $i \le j$ and $\overline{\bigcup \mathcal{C}_i} = \mathcal{C}$. Then*

$$A^f_\mathcal{C} \cong \varinjlim_I (A^f_{\mathcal{C}_i}, \alpha_{ij})$$

Proof: For an introduction to limits of C^*-algebras see for example [WO93], App L. The structure maps α_{ij} are defined by the universal property of the algebra $A^f_{\mathcal{C}_i}$:

$$\begin{array}{ccc} \mathcal{C}_i & \xrightarrow{\tau^f_{\mathcal{C}_i}} & A^f_{\mathcal{C}_i} \\ \downarrow & & \downarrow{\scriptstyle \alpha_{ij}} \\ \mathcal{C}_j & \xrightarrow{\tau^f_{\mathcal{C}_j}} & A^f_{\mathcal{C}_j} \end{array}$$

The right vertical arrow exists uniquely by the universal property. The uniqueness assures that the α_{ij} is an inductive system of C^*-homomorphisms. To prove the lemma we show that $\varinjlim_I A^f_{\mathcal{C}_i}$ has the universal property of $A^f_{\mathcal{C}}$: We get a $*$-homomorphism $\tau : \mathcal{C} \to \varinjlim_I A^f_{\mathcal{C}_i}$ by putting

$$\tau_{cd} = \varinjlim_I \tau^f_{\mathcal{C}_i cd} : \mathrm{mor}_{\mathcal{C}}(c,d) = \varinjlim_I \mathrm{mor}_{\mathcal{C}_i}(c,d) \to \varinjlim_I A^f_{\mathcal{C}_i}$$

By restriction we get $*$-homomorphisms $\rho_i : \mathcal{C}_i \to A$ and hence by the universal property of $A^f_{\mathcal{C}_i}$ unique C^*-homomorphisms $\sigma_i : A^f_{\mathcal{C}_i} \to A$ such that $\sigma_i \circ \tau^f_{\mathcal{C}_i} = \rho_n$ and $\sigma_i = \sigma_j \circ \alpha_{ij}$ and hence a unique C^*-homomorphism $\sigma : \varinjlim_I A^f_{\mathcal{C}_i} \to A$ such that $\sigma \circ \tau^f_{\mathcal{C}} = \rho$. □

Remark 6.9 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of non-unital C^*-categories and $F_n : \mathcal{C}_i \to \mathcal{D}_i$ the induced functor. Then the following diagram commutes:

$$\begin{array}{ccc} \varinjlim_I A^f_{\mathcal{C}_i} & \xrightarrow{\varinjlim_I F^f_i} & \varinjlim_I A^f_{\mathcal{D}_i} \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ A^f_{\mathcal{C}} & \xrightarrow{F^f} & A^f_{\mathcal{D}} \end{array}$$

□

Proof of 6.7: It remains to show that $\mathbb{K} \circ C^*_r$ maps equivalences in $\mathbf{HGr^{inj}_{td}}$ to weak equivalences of spectra. Then we can apply Theorem 2.10. For this it suffices to show that for every $E \in Ob(\mathbf{HGr^{inj}_{td}})$ the projection $pr : E \times \{0 \leftrightarrow 1\} \to E$ induces an isomorphism in K-theory, i.e. that $\pi_*((\mathbb{K} \circ C^*_r)pr)$ is an isomorphism.

As $C^*_r(E \times \{0 \leftrightarrow 1\}) = C^*_r(E) \times \{0 \leftrightarrow 1\}$ it suffices to show that the functor $pr : C^*_r(E) \times \{0 \leftrightarrow 1\} \to C^*_r(E)$ induces an isomorphism in K-theory.

By remark 6.4 there exist unital C^*-categories $\mathcal{C}_i, i \in I$ such that $Ob(\mathcal{C}_i) = Ob(C^*_r(E))$, $\mathcal{C}_i \subset \mathcal{C}_j \subset C^*_r(E)$ for $j \geq i$ and $\overline{\bigcup_{\mathbb{I}} \mathcal{C}_i} = C^*_r(E)$. The functors $pr_i : \mathcal{C}_i \times \{0 \leftrightarrow 1\} \to \mathcal{C}_i$ are equivalences of C^*-categories and hence induce isomorphisms in K-theory. Also $\overline{\bigcup_I (\mathcal{C}_i \times \{0 \leftrightarrow 1\})} = C^*_r(E) \times \{0 \leftrightarrow 1\}$ and $\overline{\bigcup_I pr_i} = pr$. Hence (cf. Lemma 6.8)

$$\pi_*(\mathbb{K}pr) \cong \varinjlim_I \pi_*(\mathbb{K}pr_i)$$

is an isomorphism.

For every $G/H \in Ob(G, \mathcal{O})$ the homotopy groups are

$$\pi_*(K_G^{top}(G/H)) = \pi_*(\mathbb{K}(C_r^*(\mathcal{G}^G(G/H)))).$$

As $\mathcal{G}^G(G/H)$ is equivalent to the groupoid with one object and morphism set H, it follows that

$$\pi_*(K_G^{top}(G/H)) \cong K_*^{top}(C_r^*(H)).$$

□

7 Induction Maps for Spaces on which the Kernel doesn't act freely

In this section we construct for every smooth equivariant homology theory $\mathcal{H}^?$ which is constructed using a functor $E : \mathbf{ssm}\mathbb{C}\mathbf{Cat} \to \mathbf{Spt}(\mathbf{Top})$ and the functor $\mathcal{P}(C_c^\infty(\mathcal{G}^?)_\oplus)$ natural maps

$$\mathrm{ind}_\alpha(X, A) : \mathcal{H}_n^G(X, A) \to \mathcal{H}_n^M(\mathrm{ind}_\alpha(X, A))$$

for every open continuous group homomorphism $\alpha : G \to M$ and **every** smooth G-CW-complex X. These maps are compatible with the boundary homomorphisms, natural in α and agree with the ones defined before if $\ker(\alpha)$ acts freely on X.

It suffices to construct a natural transformation

$$\zeta_\alpha : C_c^\infty(\mathcal{G}^G) \to C_c^\infty(\mathrm{res}_\alpha\, \mathcal{G}^M)$$

and show that it is natural in α. To do this we use the following lemma:

Lemma 7.1 *For every open group homomorphism $\alpha : G \to M$ between totally disconnected, locally compact groups the following map is a ring homomorphism:*

$$\zeta_\alpha : \mathcal{H}(G) \to \mathcal{H}(M)$$
$$p \mapsto \{m \mapsto \frac{1}{vol_M(\alpha(K_p))} \int_{\alpha^{-1}(m\alpha(K_p))} p(g)dg\}$$

where $K_p < G$ is an open compact subgroup such that p is K_p-bi-invariant (we will later omit the index p) and the definition is independent of the choice of the K_p.

Proof: We start with the independency of the choice of the K: Let $K' < K$ be another open compact subgroup as above. Then $\#(K/K') = n < \infty$ and $K = \coprod_{i=1}^{n} k_i K'$ for some $k_i \in K$. Now

$$\alpha^{-1}(\alpha(K)) = \alpha^{-1}(\alpha(\coprod_{i=1}^{n} k_i K')) = \bigcup_{i=1}^{n} k_i \alpha^{-1}(\alpha(K')).$$

As $\alpha^{-1}(\alpha(K))$ is a group, there exist $i_1, ..., i_l$ such that $\alpha^{-1}(\alpha(K)) = \coprod_{j=1}^{l} k_{i_j} \alpha^{-1}(\alpha(K'))$, where

$$l = \#(\alpha^{-1}(\alpha(K))/\alpha^{-1}(\alpha(K'))) = \#(\alpha(K)/\alpha(K')) = \frac{vol_M(\alpha(K))}{vol_M(\alpha(K'))}.$$

Now for $m = \alpha(g_0)$ we have

$$\begin{aligned}
\frac{1}{vol_M(\alpha(K))}\int_{\alpha^{-1}(\alpha(g_0)\alpha(K))} p(g)dg &= \frac{1}{vol_M(\alpha(K))}\int_{\alpha^{-1}(\alpha(K))} p(g_0g)dg \\
&= \frac{1}{vol_M(\alpha(K))}\sum_{j=1}^{l}\int_{k_{i_j}\alpha^{-1}(\alpha(K'))} p(g_0g)dg \\
&= \frac{1}{vol_M(\alpha(K))}\sum_{j=1}^{l}\int_{\alpha^{-1}(\alpha(K'))} p(k_{i_j}g_0g)dg \\
&= \frac{1}{vol_M(\alpha(K))}\sum_{j=1}^{l}\int_{\alpha^{-1}(\alpha(K'))} p(g_0g)dg \\
&= \frac{1}{vol_M(\alpha(K))} l \int_{\alpha^{-1}(\alpha(K'))} p(g_0g)dg \\
&= \frac{1}{vol_M(\alpha(K))}\frac{vol_M(\alpha(K))}{vol_M(\alpha(K'))}\int_{\alpha^{-1}(\alpha(K'))} p(g_0g)dg
\end{aligned}$$

Note that $\alpha^{-1}(\alpha(g_0)\alpha(K)) = g_0\alpha^{-1}(\alpha(K))$ and

$$\int_G \chi_{g_0\alpha^{-1}(\alpha(K))}(g)p(g)dg = \int_G \chi_{g_0\alpha^{-1}(\alpha(K))}(g_0g)p(g_0g)dg = \int_G \chi_{\alpha^{-1}(\alpha(K))}(g)p(g_0g)dg$$

So the definition is independent of the choice of K.

Also $\zeta_\alpha(p)$ is compactly supported, as $\mathrm{supp}(\zeta_\alpha(p)) \subset \alpha(\mathrm{supp}(p))$. This is the case because for $m \notin \alpha(\mathrm{supp}(p)), m = \alpha(g_0) \in \alpha(G)$ (else $\zeta_\alpha(p)(m) = 0$) we have

$$\int_{m\alpha^{-1}(\alpha(K))} p(g)dg = 0$$

because

$$p(\alpha^{-1}(m\alpha(K))) = p(g_0 \ker(\alpha)K) = p(g_0 \ker(\alpha)) = 0$$

as $\alpha(g_0 \ker(\alpha)) = \alpha(g_0) \notin \alpha(\mathrm{supp}(p))$

Furthermore $\zeta_\alpha(p)$ is locally constant, as it is $\alpha(K_p)$-bi-invariant: For $k \in K$ we have:

$$\begin{aligned}
\zeta_\alpha(p)(\alpha(k)m) &= \frac{1}{vol_M(\alpha(K))}\int_{\alpha^{-1}(\alpha(k)m\alpha(K))} p(g)dg \\
&= \frac{1}{vol_M(\alpha(K))}\int_{\alpha^{-1}(m\alpha(K))} p(kg)dg \\
&= \frac{1}{vol_M(\alpha(K))}\int_{\alpha^{-1}(m\alpha(K))} p(g)dg \\
&= \zeta_\alpha(p)(m).
\end{aligned}$$

and $\zeta_\alpha(p)(m\alpha(k)) = \frac{1}{vol_M(\alpha(K))}\int_{\alpha^{-1}(m\alpha(k)\alpha(K))} p(g)dg = \zeta_\alpha(p)(m)$.

It remains to show that ζ_α is a ring homomorphism: Let $p, q \in \mathcal{H}(G)$ and choose an open compact subgroup $K < G$ such that both p and q are K-bi-invariant. Then

$$\zeta_\alpha(q) * \zeta_\alpha(p) = \zeta_\alpha(q * p),$$

as

$$\begin{aligned}
\zeta_\alpha(q*p)(m_0) &= \frac{1}{vol_M(\alpha(K))}\int_{\alpha^{-1}(m_0\alpha(K))}(q*p)(\overline{g})d\overline{g}\\
&= \frac{1}{vol_M(\alpha(K))}\int_{\alpha^{-1}(m_0\alpha(K))}\int_G q(g)p(g^{-1}\overline{g})dgd\overline{g}\\
&= \frac{1}{vol_M(\alpha(K))}\int_G\int_{\alpha^{-1}(m_0\alpha(K))} q(g)p(g^{-1}\overline{g})d\overline{g}dg\\
&\overset{3}{=} \frac{vol_M(\alpha(K))}{vol_M(\alpha(K))}\sum_{i=1}^{n}\int_{\alpha^{-1}(\alpha(g_i)\alpha(K))}\zeta_\alpha(p)(\alpha(g)^{-1}m_0)q(g)dg\\
&\overset{4}{=} \sum_{i=1}^{n}\zeta_\alpha(p)(\alpha(g_i)^{-1}m_0)\int_{\alpha^{-1}(\alpha(g_i)\alpha(K))}q(g)dg\\
&= vol_M(\alpha(K))\sum_{i=1}^{n}\zeta_\alpha(p)(\alpha(g_i)^{-1}m_0)\zeta_\alpha(q)(\alpha(g_i))\\
&= vol_M(\alpha(K))\sum_{i=1}^{n}\zeta_\alpha(p)(m_i^{-1}m_0)\zeta_\alpha(q)(m_i)\\
&\overset{7}{=} vol_M(\alpha(K))\frac{1}{vol_M(\alpha(K))}\int_M\zeta_\alpha(p)(m^{-1}m_0)\zeta_\alpha(q)(m)dm\\
&= \int_M\zeta_\alpha(p)(m^{-1}m_0)\zeta_\alpha(q)(m)dm\\
&= \zeta_\alpha(q)*\zeta_\alpha(p)(m_0)
\end{aligned}$$

In step 3 we use that $\mathrm{supp}(p) \subset \coprod_{i=1}^{n} g_i\alpha^{-1}(\alpha(K))$. For step 4 note that

$$\alpha^{-1}(\alpha(g_i)\alpha(K)) = g_i\ker(\alpha)K$$

and hence $\zeta_\alpha(p)(\alpha(g)^{-1}m_0) = \zeta_\alpha(p)(\alpha(g_i)^{-1}m_0)$ on $\alpha^{-1}(\alpha(g_i)\alpha(K))$. In step 7 we use that $\zeta_\alpha(q)(m) = \zeta_\alpha(q)(m_i)\,\forall m \in m_i\alpha(K)$ as p is K-bi-invariant, analogously for $\zeta_\alpha(p)(m_i^{-1}m_0)$. □

Now we can define the natural transformation ζ_α:

Definition 7.2 *For each $G/H \in Ob(\mathrm{Or}(G,\mathcal{O}))$ the map*

$$\zeta_\alpha : C_c^\infty(\mathcal{G}^G) \to \mathrm{res}_\alpha\, C_c^\infty(\mathcal{G}^M)$$

is given by the following functor of categories without units:

$$\zeta_\alpha(G/H) : C_c^\infty(\mathcal{G}^G(G/H)) \to C_c^\infty(\mathcal{G}^M(M/\alpha(H)))$$

$$gH \mapsto \alpha(g)\alpha(H)$$

$$p \in \mathcal{H}(g_2Hg_1^{-1}) = C_c^\infty(g_2Hg_1^{-1}) \mapsto \zeta_\alpha(p) \in \mathcal{H}(\alpha(H)) = C_c^\infty(\alpha(g_2Hg_1^{-1}))$$

where $\zeta_\alpha : \mathcal{H}(G) \to \mathcal{H}(M)$ is the ring homomorphism defined above.

This definition coincides with the old definition if $H \cap \ker(\alpha) = 1$:

Lemma 7.3 *If $H \cap \ker(\alpha) = 1$, then $\zeta_\alpha(G/H) = C_c^\infty(\eta_\alpha(G/H))$.*

Proof: Let $p \in C_c^\infty(g_2 H g_1^{-1})$ and $K < H$ a compact open subgroup such that p is K-bi-invariant and let $m = \alpha(\gamma) \in \alpha(g_2 H g_1^{-1})$ (else $\zeta_\alpha(p)(m) = 0$). Then

$$\begin{aligned} \zeta_\alpha(p)(m) &= \frac{1}{vol_M(\alpha(K))} \int_{\alpha^{-1}(m\alpha(K))} p(g)dg \\ &= \frac{1}{vol_M(\alpha(K))} \int_{\alpha^{-1}(\alpha(K))} p(\gamma g)dg \\ &= \frac{1}{vol_M(\alpha(K))} \int_{K} p(\gamma g)dg \\ &= \frac{vol_G(K)}{vol_M(\alpha(K))} p(\gamma). \end{aligned}$$

On the other hand we have

$$C_c^\infty(\eta_\alpha(G/H))(p) = c_{\eta_\alpha(g_2 H)} p \circ (\alpha|_{g_2 H g_1^{-1}})^{-1}.$$

In Section 4.3 we saw that $c_{\eta_\alpha}(g_2 H) = \frac{vol_G(K)}{vol_M(\alpha(K))}$ for any open compact subgroup $K < H$. $\square$

We still have to show that ζ_α is well defined: $\zeta_\alpha(p)$ is an element of $C_c^\infty(\alpha(g_2 H g_1^{-1}))$ as $\mathrm{supp}(\zeta_\alpha(p)) \subset \alpha(\mathrm{supp}(p))$. Each $\zeta_\alpha(G/H)$ is a functor of categories without units as ζ_α is a ring homomorphism. ζ_α is a natural transformation because the following diagram commutes for every morphism R_γ in $\mathrm{Or}(G, \mathcal{O})$:

$$\begin{array}{ccc} C_c^\infty(\mathcal{G}^G)(G/H) & \xrightarrow{C_c^\infty(\mathcal{G}^G)(R_{\gamma^{-1}})} & C_c^\infty(\mathcal{G}^G)(G/K) \\ \downarrow{\scriptstyle \zeta_\alpha(G/H)} & & \downarrow{\scriptstyle \zeta_\alpha(G/K)} \\ C_c^\infty(\mathcal{G}^M)(M/\alpha(H)) & \xrightarrow{C_c^\infty(\mathcal{G}^M(R_{\alpha(\gamma)^{-1}}))} & C_c^\infty(\mathcal{G}^M)(M/\alpha(K)) \end{array}$$

This is clear on objects, for morphisms the following diagram shows it:

$$\begin{array}{ccc} p \in C_c^\infty(g_2 H g_1^{-1}) & \longmapsto & p \in C_c^\infty(g_2 \gamma^{-1} H \gamma g_1^{-1}) \\ \downarrow & & \downarrow \\ \left(\begin{smallmatrix} \zeta_\alpha(G/H)(p)(m) = \\ \frac{1}{vol_M(\alpha(L))} \int_{\alpha^{-1}(m\alpha(L))} p(g)dg \end{smallmatrix}\right) & & \left(\begin{smallmatrix} \zeta_\alpha(G/K)(p)(m) = \\ \frac{1}{vol_M(\alpha(L))} \int_{\alpha^{-1}(m\alpha(L))} p(g)dg \end{smallmatrix}\right) \\ & \searrow & \\ & & \left(\begin{smallmatrix} \zeta_\alpha(G/H)(p)(m) = \\ \frac{1}{vol_M(\alpha(L))} \int_{\alpha^{-1}(m\alpha(L))} p(g)dg \end{smallmatrix}\right) \end{array}$$

$\square$

It remains to show that $\zeta_?$ is functorial:

Lemma 7.4 *Let $\alpha : G \to M$ and $\beta : M \to L$ be two open group homomorphisms. Then*

$$(\mathrm{res}_\alpha \zeta_\beta) \circ \zeta_\alpha = \zeta_{\beta\alpha}.$$

Proof: For any G/H we have to show that

$$(\mathrm{res}_\alpha \zeta_\beta) \circ \zeta_\alpha(G/H) = \zeta_{\beta\alpha}(G/H).$$

This follows at once on objects. Hence let p be a morphism $p \in C_c^\infty(g_2 H g_1^{-1}) = \mathrm{mor}_{C_c^\infty(\mathcal{G}^G(G/H))}(g_1 H, g_2 H)$:

$(\mathrm{res}_\alpha \zeta_\beta)(\zeta_\alpha(p))$ and $\zeta_{\beta\alpha}(p)$ are functions on $\beta\alpha(g_2 H g_1^{-1})$. Let $l = \beta\alpha(g_0)$. Then

$$\begin{aligned}
(\mathrm{res}_\alpha \zeta_\beta)(\zeta_\alpha(p))(l) \mapsto & \frac{1}{vol_L(\beta\alpha(K))} \int_{\beta^{-1}(l\beta(\alpha(K)))} \zeta_\alpha(p)(m)dm \\
=& \frac{1}{vol_L(\beta\alpha(K))} \int_{\beta^{-1}(l\beta(\alpha(K)))} \frac{1}{vol_M(\alpha(K))} \int_{\alpha^{-1}(m\alpha(K))} p(g)dgdm \\
=& \frac{1}{vol_L(\beta\alpha(K))} \frac{1}{vol_M(\alpha(K))} \int_{\beta^{-1}(\beta(\alpha(K)))} \int_{\alpha^{-1}(m\alpha(K))} p(g_0 g)dgdm \\
\overset{3}{=}& \frac{1}{vol_L(\beta\alpha(K))} \frac{1}{vol_M(\alpha(K))} \int_{\beta^{-1}(\beta(\alpha(K)))} \int_G \chi_{\alpha(g)\alpha(K)}(m) p(g_0 g)dgdm \\
=& \frac{1}{vol_L(\beta\alpha(K))} \frac{1}{vol_M(\alpha(K))} \int_G p(g_0 g) \int_{\beta^{-1}(\beta\alpha(K))} \chi_{\alpha(g)\alpha(K)}(m)dmdg \\
\overset{5}{=}& \frac{1}{vol_L(\beta\alpha(K))} \frac{1}{vol_M(\alpha(K))} \int_G p(g_0 g) vol_M(\alpha(K)) \chi_{(\beta\alpha)^{-1}(\beta\alpha(K))}(g)dg \\
=& \frac{1}{vol_L(\beta\alpha(K))} \frac{vol_M(\alpha(K))}{vol_M(\alpha(K))} vol_L(\beta\alpha(K)) \zeta_{\beta\alpha}(p)(\beta\alpha(g_0)) \\
=& \zeta_{\beta\alpha}(p)(m)
\end{aligned}$$

In the third step we use that $g \in \alpha^{-1}(m\alpha(K)) \Leftrightarrow \alpha(g) \in m\alpha(K) \Leftrightarrow m \in \alpha(g)\alpha(K)$ and in the fifth step that $\beta^{-1}(\beta\alpha(K)) \cap \alpha(g)\alpha(K) = \ker(\beta)\alpha(K) \cap \alpha(g)\alpha(K)$

$$= \begin{cases} \emptyset & \text{if } \beta(\ker(\beta)\alpha(K) \cap \alpha(g)\alpha(K)) \subset \beta\alpha(K) \cap \beta\alpha(g)\beta\alpha(K) = \emptyset \\ & \text{i.e. if } \beta(\alpha(g)) \notin \beta\alpha(K) \\ & \text{i.e. if } g \notin (\beta\alpha)^{-1}(\beta\alpha(K)) \\ \alpha(K)\alpha(g) & \text{if } g \in (\beta\alpha)^{-1}(\beta\alpha(K)) \end{cases}$$

□

Appendix: K-Theory for Proper Actions of a Totally Disconnected Group

In this appendix we look at equivariant K-theory (cohomology) for proper smooth actions of a totally disconnected group G. We would like to define it via G-vector bundles as can be done for discrete groups (See [LO01] along the lines of which we follow closely). Almost everything can be translated at once from the discrete setting to the totally disconnected one, only when it comes to excision a further assumption must be posed on G. In general excision doesn't hold, an explicit example for this phenomenon will be given.

The material in this appendix was the starting point for our interest in totally disconnected groups. Also here it was noticed for the first time that even in the smooth setting totally disconnected groups can behave quite differently from discrete groups.

8.1 G-vector bundles

Let G be a totally disconnected, locally compact group. A *(smooth) G-vector bundle* over a (smooth) G-CW-complex X consists of a complex vector bundle $p : E \to X$, together with a (smooth) G-action on E such that p is G-equivariant and every $g \in G$ acts on E and X via a bundle isomorphism.

The following lemmata are stated in [LO01] for proper actions of Lie groups. They also hold for smooth G-vector bundles as then the isotropy groups of X (and E) are open and hence G/G_x discrete. This assures that for every G-map $p : X \to G/G_x$ the canonical map $G \times_{G_x} p^{-1}(eG_x) \to X$ is a G-homeomorphism (See [tD87], I,4.4).

Lemma 8.1 *Let G be a totally disconnected, locally compact group, $H < G$ open. Then any G-vector bundle over $G/H \times D^n$ is G-isomorphic to $G \times_H (V \times D^n)$ for some H-representation V,* [LO01] 1.1. □

If H is compact, then we saw (p. 35) that any H-representation V is smooth. It follows that the isotropy groups of the total space of a G-vector bundle over a proper smooth G-CW-complex are automatically all open.

Theorem 8.2 *Let G be a totally disconnected, locally compact group, X a proper smooth G-CW-complex and $p : E \to X \times I$ a G-vector bundle. Then $E_0 := E|_{X\times\{0\}}$ and E_1 are isomorphic as G-vector bundles over X,* [LO01] 1.2. □

And

Lemma 8.3 *Let G be a totally disconnected, locally compact group, $\varphi : (X_1, X_0) \to (X, X_2)$ a map of smooth G-CW-pairs. Set $\varphi_0 = \varphi_{|X_0}$ and assume that $X \cong_G X_2 \cup_{\varphi_0} X_1$. Let $p_1 : E_1 \to X_1$ and $E_2 \to X_2$ be G-vector bundles, let $\overline{\varphi_0} : E_1|_{X_0} \to E_2$ be a strong map (i.e. a map restricting to linear isomorphisms on the fibers) covering φ_0 and set $E = E_2 \cup_{\overline{\varphi_0}} E_1$. Then $p = p_1 \cup p_2 : E \to X$ is a G-vector bundle over X,* [LO01] 1.5. □

8.2 Equivariant K-Theory for proper smooth G-CW-complexes: Definition and First Properties

In this section we define for any totally disconnected, locally compact group G a K-theory functor

$$\mathbb{K}_G^{-*} : \mathcal{CO}\text{-}\mathbf{G\text{-}CW\text{-}complexes} \to \mathbf{Ab},\ * \geq 0$$

in analogy to the vector bundle construction for discrete groups (see [LO01]). In the next section we'll see that this doesn't define a G-cohomology theory in general, because excision doesn't hold.

Definition 8.4 *For any totally disconnected, locally compact group G and any proper smooth G-CW-complex X, let* $\mathbb{K}_G(X) = \mathbb{K}_G^0(X)$ *be the Grothendieck group of the monoid of isomorphism classes of G-vector bundles over X. Define* $\mathbb{K}_G^{-n}(X)$*, for all* $n > 0$*, by setting*

$$\mathbb{K}_G^{-n}(X) = \ker[\mathbb{K}_G(X \times S^n) \stackrel{incl^*}{\to} \mathbb{K}_G(X)].$$

For any proper smooth G-CW-pair (X, A)*, set*

$$\mathbb{K}_G^{-n}(X, A) = \ker[\mathbb{K}_G(X \cup_A X) \stackrel{i_2^*}{\to} \mathbb{K}_G^{-n}(X)].$$

When G has a neighbourhood base of the unit consisting of open compact normal subgroups and (X, A) *is a finite proper smooth G-CW-pair write*

$$K_G^{-n}(X, A) = \mathbb{K}_G^{-n}(X, A).$$

The first properties of these functors like homotopy invariance or induction hold for any totally disconnected, locally compact group G and are proven exactly as in [LO01]. Homotopy invariance for instance follows immediately from 8.2:

Lemma 8.5 (Homotopy invariance) *Let G be a totally disconnected, locally compact group. If* $f_0, f_1 : (X, A) \to (Y, B)$ *are G-homotopic G-maps between proper smooth G-CW-pairs, then*

$$f_0^* = f_1^* : \mathbb{K}_G^{-n}(Y, B) \to \mathbb{K}_G^{-n}(X, A)$$

for all $n \geq 0$ [LO01] 3.3. □

Lemma 8.6 (Induction) *Let* $H < G$ *be an inclusion of totally disconnected, locally compact groups, H open in G and let* (X, A) *be a proper smooth H-CW-pair. Then* $G \times_H (X, A)$ *is a proper smooth G-CW-pair, and there are isomorphisms*

$$i_H^G : \mathbb{K}_H^{-n}(X, A) \stackrel{\cong}{\to} \mathbb{K}_G^{-n}(G \times_H (X, A))$$

for all $n \geq 0$*, defined by sending* $[E]$ *to* $[G \times_H E]$*,* [LO01] 3.4. □

For the proof of this note that as $G \to G/H$ has sections, any G-vector bundle $p : E \to G \times_H X$ is of the form $G \times_H p^{-1}(X) \to G \times_H X$.

Lemma 8.7 (Restriction) *Let H, G be two totally disconnected, locally compact groups, $\varphi : H \to G$ an open continuous group homomorphism with compact kernel and (X, A) a proper smooth G-CW-pair. Then there is a natural homomorphism.*

$$\mathrm{res}_\varphi(X, A) : \mathbb{K}_G^{-n}(X, A) \to \mathbb{K}_H^{-n}(\mathrm{res}_\varphi(X, A))$$

for all $n \geq 0$.

Note that any proper smooth G-CW-complex X becomes a proper smooth H-CW-complex by restriction with φ, because X is a CW-complex, H acts cellularly on X and the isotropy groups of X as an H-space are open and compact because φ is open and has compact kernel (cf. Lemma 1.11). In the same way any G-vector bundle $E \to X$ becomes an H-vector bundle. If the image of φ has finite index in G, then a finite proper smooth G-CW-complex becomes a finite proper smooth H-CW-complex by restriction with φ. We get homomorphisms

$$\mathrm{res}_\varphi(X, A) : \mathbb{K}_G^*(X, A) \to \mathbb{K}_H^*(X, A)$$

and

$$\mathrm{res}_\varphi(X, A) : K_G^*(X, A) \to K_H^*(X, A).$$

Lemma 8.8 (Free quotients) *Let G be a totally disconnected, locally compact group, H an closed normal subgroup in G and let (X, A) be a proper smooth G-CW-pair on which H acts freely. Then the projection $pr : X \to X/H$ induces an isomorphism*

$$pr^* : \mathbb{K}_{G/H}^{-n}(X/H, A/H) \xrightarrow{\cong} \mathbb{K}_G^{-n}(X, A)$$

for all $n \geq 0$, the inverse of which is defined by sending $[E] \in \mathbb{K}_G(X)$ to $[E/H] \in \mathbb{K}_{G/H}(X/H)$ in the absolute case, [LO01] *3.5.* □

8.3 Excision

When turning to the Mayer-Vietoris sequence and excision, the main lemma that is used for the proof in the discrete setting, namely that a G-vector bundle E_0 over a G-subspace $A \subset X$ can be embedded as a direct summand of some G-vector bundle E over X, fails for totally disconnected, locally compact groups in general and so does excision. Hence in this case the functors $\mathbb{K}_*^G$ cannot be extended to a G-cohomology theory. We will give an example for these phenomena following along the lines of the one given in [LO01], Section 5, for proper actions of Lie groups.

Let $T := \prod_{\mathbb{Z}} F$, $F = \mathbb{Z}/2$ and $G := T \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts on T via translation.

Then G is a totally disconnected, locally compact group and T is a open compact normal subgroup of G. T is the only open subgroup of T which is normal in G. This is the case because every open subset U of T is a union of finitely many intersections of sets $\pi_z^{-1}(U_z)$, where π_z is the projection onto the z-coordinate and $U_z \subset F_z = \mathbb{Z}/2$, i.e. $U \cong U_{z_1,\ldots,z_n} \times \prod_{z \neq z_1,\ldots,z_n} F_z$, where $U_{z_1,\ldots,z_n} \subset F_{z_1} \times \ldots \times F_{z_n}$. But conjugation with an element $((f_z), t) \in G$ is just translation by t, and therefore $tUt^{-1} \neq U$ if $t \neq 0$.

G acts on $\mathbb{R}$ via the translation of $\mathbb{Z}$ and $\mathbb{R}$ has the structure of a G-CW-complex with two 0-cells and two 1-cells, the one and only isotropy group of which is T.

Lemma 8.9 *Let G be a totally disconnected, locally compact group, X a proper smooth connected G-CW-complex. Let the normal compact open subgroup $H < G$ be the one and only isotropy group of X. Then for any G-vector bundle $p : E \to X$ we have that E_x and E_y are isomorphic as H-representations for all $x, y \in X$ and that the open subgroup $\ker(E_x) = \{h \in H \mid he = e\ \forall e \in E_x\}$ is normal in G.*

Proof: We have $X = X^H$. Let $\alpha : I \to X^H$ be a path between the elements $x, y \in X$. This is the same as a G-map $\alpha : G/H \times I \to X$ along which we can pull the G-vector bundle p back:

$$\begin{array}{ccc} \alpha^*E & \longrightarrow & E \\ \downarrow & & \downarrow p \\ G/H \times I & \xrightarrow{\alpha} & X \end{array}$$

With 8.1 and 8.2 we get that:

$$G \times_H V_0 \cong (\alpha^*E)_0 \cong (\alpha^*E)_1 \cong G \times_H V_1$$

as G-vector bundles over G/H, where V_0 is the H-representation given by $(\alpha^*E)_{eH\times 0} \cong_H E_x$ and $V_1 \cong_H E_y$. Hence E_x and E_y are isomorphic as H-representations and therefore also $\ker(E_x) = \ker(E_y)$. On the other hand for any $g \in G$ we have:

$$\begin{aligned} \ker(E_x) = \ker(E_{gx}) &= \{h \in H \mid he = e\ \forall e \in E_{gx}\} \\ &= \{h \in H \mid hge = ge\ \forall e \in E_x\} \\ &= \{ghg^{-1} \in H \mid ghg^{-1}ge = ge\ \forall e \in E_x\} \\ &= g\{h \in H \mid ghe = ge\ \forall e \in E_x\}g^{-1} \\ &= g \ker(E_x)g^{-1}. \end{aligned}$$

It follows that $\ker(E_x)$ is normal in G. It is open because $\ker(E_x) = \bigcap_{i=1}^n G_{v_i}$ where $(v_1, ..., v_n)$ is a basis of E_x. □

As T is the only open compact normal subgroup of G it follows that $\ker(E_x) = T$ for any G-vector bundle E over $\mathbb{R}$ and any $x \in \mathbb{R}$, which means that T acts trivially on E. So we see that a G-vector bundle over the subcomplex $\mathbb{Z} = G/T$ (=one of the zero-cells), which is the same as a representation of T, can only be a direct summand of a G-vector bundle over $\mathbb{R}$ if it is a multiple of the trivial representation. But $R(T) \ncong \mathbb{Z}$ in general as $R(T) \cong \underset{L\in N(T)}{colim}\, R(T/L)$ where $N(T)$ is the family of all open compact normal subgroups of T. Namely $R(\mathbb{Z}/2) \cong \mathbb{Z} \oplus \mathbb{Z}$. Set $T_n := 1 \times ... \times 1 \times \prod_{z\neq 0,...n-1} \mathbb{Z}/2$, then $\{T_n \in \mathbb{N}_0\}$ is a cofinal subsystem of $N(T)$ and therefore

$$\begin{aligned} R(T) &\cong \underset{L\in T_n}{colim} R(T/L) \\ &\cong \underset{n\in\mathbb{N}_0}{colim}\{R(T/T_0) \to R(T/T_1) \to R(T/T_2) \to ...\} \\ &\cong \underset{n\in\mathbb{N}_0}{colim}\{R(1) \to R(\mathbb{Z}/2) \to R(\mathbb{Z}/2 \times \mathbb{Z}/2) \to ...\} \\ &\cong \underset{n\in\mathbb{N}_0}{colim}\{\mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \hookrightarrow ...\} \end{aligned}$$

The example above can also be used to show that excision doesn't hold. We have two proper smooth G-CW pairs, namely $(\mathbb{R}, \mathbb{Z})$ where $\mathbb{Z}$ is one of the 0-cells (and $\mathbb{Z} + \frac{1}{2}$ is the other) and $(\mathbb{Z} \times I, \mathbb{Z} \times \delta I)$. As $(\mathbb{R}, \mathbb{Z}) \simeq_G (\mathbb{R}, \mathbb{Z} + [0, \frac{1}{2}])$ and $(\mathbb{Z} \times I, \mathbb{Z} \times \delta I) \simeq_G (\mathbb{Z} + [\frac{1}{2}, 1], \mathbb{Z} + \{0, \frac{1}{2}\})$, we get by homotopy invariance

$$\mathbb{K}_G^*(\mathbb{R}, \mathbb{Z}) \cong \mathbb{K}_G^*(\mathbb{R}, \mathbb{Z} + [0, \frac{1}{2}])$$

and

$$\mathbb{K}_G^*(\mathbb{Z} \times I, \mathbb{Z} \times \delta I) \cong \mathbb{K}_G^*(\mathbb{Z} + [\frac{1}{2}, 1], \mathbb{Z} + \{0, \frac{1}{2}\}).$$

On the other hand we have the pushout

$$\begin{array}{ccc} \mathbb{Z} + \{\frac{1}{2}, 1\} & \longrightarrow & \mathbb{Z} + [\frac{1}{2}, 1] \\ \downarrow & & \downarrow \\ \mathbb{Z} + [0, \frac{1}{2}] & \longrightarrow & \mathbb{R}, \end{array}$$

hence if excision held for $\mathbb{K}_G^*$ then $\mathbb{K}_G^*(\mathbb{R}, \mathbb{Z})$ and $\mathbb{K}_G^*(\mathbb{Z} \times I, \mathbb{Z} \times \delta I)$ would have to be isomorphic. But these groups can be calculated:

$$\begin{aligned} \mathbb{K}_G^{-n}(\mathbb{R}, \mathbb{Z}) &\underset{(1)}{\cong} \mathbb{K}_{G/T}^{-n}(\mathbb{R}, \mathbb{Z}) \\ &\underset{(2)}{\cong} \mathbb{K}_1^{-n}(\mathbb{R}/\mathbb{Z}, \mathbb{Z}/\mathbb{Z}) \\ &\cong K_1^{-n} = (S^1, pt) \\ &= \begin{cases} \mathbb{Z} & n \text{ odd,} \\ 0 & n \text{ even} \end{cases} n \geq 0 \end{aligned}$$

and

$$\begin{aligned} \mathbb{K}_G^{-n}(\mathbb{Z} \times I, \mathbb{Z} \times \delta I) &\underset{(3)}{\cong} \mathbb{K}_T^{-n}(I, \delta I) \\ &\cong K_T^{-n-1}(pt) \\ &= \begin{cases} R(T) & n \text{ odd,} \\ 0 & n \text{ even} \end{cases} n \geq 0 \end{aligned}$$

(1) By 8.9 T acts trivially on any smooth G-vector bundle $p : E \to \mathbb{R}$ and hence $\mathbb{K}_G^{-n}(\mathbb{R}, \mathbb{Z}) \cong \mathbb{K}_{G/T}^{-n}(\mathbb{R}, \mathbb{Z})$
(2) By 8.8.
(3) By 8.6 as $\mathbb{Z} \times I \cong_G G/T \times I \cong_G G \times_T I$.

Hence excision doesn't hold for $\mathbb{K}_G^*$ in general.

8.4 Equivariant K-theory for proper actions of prodiscrete groups

The main result of this section will be that when G has a neighbourhood base of the unit consisting of open compact *normal* subgroups then G-vector bundles define a $\mathbb{Z}/2$-graded multiplicative G-cohomology theory $K_G^*(-)$ on the category of finite proper G-CW-pairs. In analogy to the name *profinite* for groups which are an inverse limit of finite groups, we call groups which have a neighbourhood base of the unit consisting of open compact normal subgroups (which is the same as saying that they are an inverse limit of discrete groups, cf. 1.10) *prodiscrete*.

Lemma 8.10 (Mayer-Vietoris sequence) *Let G be a prodiscrete group and let*

$$\begin{array}{ccc} A & \xrightarrow{i_1} & X_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow{j_2} & X \end{array}$$

be a pushout square of finite proper smooth G-CW-complexes, where i_1 and j_2 are inclusions of subcomplexes. Then there is a natural exact sequence, infinite to the left

$$\ldots \xrightarrow{d^{-n-1}} K_G^{-n}(X) \xrightarrow{j_1^* \oplus j_2^*} K_G^{-n}(X_1) \oplus K_G^{-n}(X_2) \xrightarrow{i_1^* - i_2^*} K_G^{-n}(A) \xrightarrow{d^{-n}} \ldots$$
$$\ldots \longrightarrow K_G^{-1}(A) \xrightarrow{d^{-1}} K_G^0(X) \xrightarrow{j_1^* \oplus j_2^*} K_G^0(X_1) \oplus K_G^0(X_2) \xrightarrow{i_1^* - i_2^*} K_G^0(A).$$

Looking at the proof of the Mayer-Vietoris sequence in the discrete setting we see that the following lemma is its key point. We have seen above that it doesn't hold for totally disconnected groups in general, but for prodiscrete groups it follows from the discrete case:

Lemma 8.11 *Let G be a prodiscrete group and let $\varphi : X \to Y$ be an equivariant map between finite proper smooth G-CW-complexes and $p' : E' \to X$ a G-vector bundle.*

Then there exists a G-vector bundle $p : E \to Y$ such that E' is a summand of $\varphi^* E$.

Proof: Pulling back p' to a cell $e_i, i \in I =$ (set of cells of X) yields a G-vector bundle $G \times_{H_i} (V_i \times D^n) \to G/H_i \times D^n$, where V_i is an H_i-representation. As $\ker(V_i) < G$ is compact and open, there is by assumption on G an open compact subgroup $K_i < \ker(V_i)$ which is normal in G. Hence $K := \bigcap_{i \in I} K_i$ is an open compact and normal subgroup of G such that $K < G_x$ and $K < \ker(E'_x) \, \forall x \in X$ (every $\ker(E'_x)$ is conjugated to some $\ker(V_i)$ and K_i is normal.).

As Y is a finite G-CW-complex we can find an open normal compact subgroup G_0 of G such that $X = X^{G_0}$ (For every cell $G/H \times D^n$ choose $H' < H$, H' normal in G.).

Define $K_0 := K \cap G_0$. Then K_0 is open compact normal in G and $X = X^{G_0}, Y = Y^{K_0}, E' = E'^{K_0}$ and we can consider them as finite proper G/K_0-CW-complexes and p' as a G/K_0-vector bundle. By [LO01] 3.7 (the discrete version of this lemma) it follows that there exists a G/K_0-vector bundle $p : E \to Y$ such that E' is a direct summand of $\varphi^* E$. Now restriction of p with $G \to G/K_0$ already is a G-vector bundle with the required properties. □

Excision and the long exact sequence for a pair follow as immediate consequences of the Mayer-Vietoris sequence exactly as in the discrete case.

Lemma 8.12 (Excision) *Assume that G is prodiscrete and let*

$$\varphi : (X, A) \longrightarrow (Y, B)$$

be a map of finite proper smooth G-CW-pairs, such that $Y \cong B \cup_{\varphi|A} X$. Then

$$\varphi^* : K_G^{-n}(Y, B) \longrightarrow K_G^{-n}(X, A)$$

is an isomorphism for all $n \geq 0$. □

Lemma 8.13 (Exactness) *Assume that G is prodiscrete and let (X, A) be a finite proper smooth G-CW-pair. Then the following sequence, extending infinitely far to the left, is natural and exact:*

$$... \xrightarrow{\partial^{-n-1}} K_G^{-n}(X, A) \xrightarrow{i^*} K_G^{-n}(X) \xrightarrow{j^*} K_G^{-n}(A) \xrightarrow{\partial^{-n}} K_G^{-n+1}(X, A) \xrightarrow{i^*} ...$$
$$... \xrightarrow{\partial^{-1}} K_G^0(X, A) \xrightarrow{i^*} K_G^0(X) \xrightarrow{j^*} K_G^0(A).$$

□

If G is a compact totally disconnected group, the classical definition of K-theory in [Seg68] coincides with the one given here.

Before we come to equivariant Bott periodicity, we consider products on $K_G^*(X, A)$. As pointed out in [Seg68] for any group G the tensor product of two G-vector bundles over a G-space X is again a G-vector bundle. So for G prodiscrete we get — analogously to the discrete case — the structure of a graded commutative ring on $K_G^*(X)$ and of a $K_G^*(X)$-module on $K_G^*(X, A)$ for any finite proper smooth G-CW-pair (X, A) and the induced maps $f^* : K_G^*(X) \to K_G^*(Y)$ etc. respect these structures. Also the boundary maps of the Mayer-Vietoris sequence and of the long exact sequence are $K_G^*(X)$-linear.

It remains to prove Bott periodicity in this situation. Apart from the fact that the external product of two equivariant vector bundles is again an equivariant vector bundle, the construction of the Bott homomorphism requires no group-specific knowledge, but for the proof that it is an isomorphism we need to know this fact for the case where G is a compact group.

Recall that $\tilde{K}(S^2) = \ker[K(S^2) \to K(pt)] \cong \mathbb{Z}$, and is generated by the Bott element $B \in \tilde{K}(S^2)$: the element $[S^2 \times \mathbb{C}] - [H] \in \tilde{K}(S^2)$, where H is the canonical complex line bundle over $S^2 = \mathbb{CP}^1$. If G is prodiscrete, then for any finite proper smooth G-CW-complex there is a pairing

$$K_G^{-n}(X) \otimes \tilde{K}(S^2) \xrightarrow{\otimes} \ker[K_G^{-n}(X \times S^2) \to K_G^{-n}(X \times pt)] \cong K_G^{-n-2}(X),$$

induced by the external tensor product of bundles. For any finite proper smooth G-CW-pair (X, A) and all $n \geq 0$ the evaluation at the Bott element now defines a homomorphism

$$b = b(X) : K_G^{-n}(X) \longrightarrow K_G^{-n-2}(X)$$

which by construction is natural in X and extends to a homomorphism

$$b = b(X, A) : K_G^{-n}(X, A) \longrightarrow K_G^{-n-2}(X, A).$$

Theorem 8.14 (Equivariant Bott periodicity) *Let G be prodiscrete and let (X, A) be a finite proper smooth G-CW-pair. Then the Bott homomorphism*

$$b = b(X, A) : K_G^{-n}(X, A) \longrightarrow K_G^{-n-2}(X, A)$$

is an isomorphism for all $n \geq 0$.

Proof: It is sufficient to prove this for $X = Y \cup_\varphi (G/H \times D^m)$, where $H < G$ is open and compact, Y is a finite proper smooth G-CW-complex, $\varphi : G/H \times S^{m-1} \to Y$ is a G-map and $b(Y)$ an isomorphism, because the Bott homomorphism is compatible with the boundary operators of the Mayer-Vietoris-sequence. For $b(X, A)$ the proposition then follows because the Bott homomorphism is also compatible with the boundary maps of the long exact sequence.

Since

$$K_G^{-n}(G/H \times S^{m-1}) \cong K_H^{-n}(S^{m-1}) \,\text{and}\, K_G^{-n}(G/H \times D^m) \cong K_H^{-n}(D^m),$$

the Bott homomorphisms $b(G/H \times S^{m-1})$ and $b(G/H \times D^m)$ are isomorphisms by the equivariant Bott periodicity theorem for actions of compact groups as shown in [Seg68]. □

We can now redefine our K-groups to get a $\mathbb{Z}/2$-graded G-cohomology theory:

$$K_G^n(X, A) := \begin{cases} K_G^0(X, A) & \text{if } n \text{ is even} \\ K_G^1(X, A) & \text{if } n \text{ is odd} \end{cases}$$

for any prodiscrete group G and any finite proper smooth G-CW-pair (X, A). The boundary operator $\partial^n : K_G^n(A) \to K_G^{n+1}(X, A)$ is to be the old one if n is odd and the composite $K_G^0(A) \underset{\cong}{\xrightarrow{b}} K_G^{-2}(A) \xrightarrow{\partial} K_G^{-1}(X, A)$ if n is even.

Collecting all the results together, we get for any prodiscrete group G a multiplicative $\mathbb{Z}/2$-graded G-cohomology theory $K_G^*(-)$ on the category of finite proper G-CW-pairs. If G is compact and X a finite proper smooth G-CW-complex, this construction agrees with the classical definition. Its coefficients are given by

$$K_G^*(G/H) = K_G^*(G \times_H pt) \cong K_H^*(pt) = \begin{cases} R(H) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

for any open compact subgroup H of G, where $R(H)$ is the Grothendieck group of the isomorphism classes of finite dimensional H-representations.

8.5 The Chern Character

Let G be a prodiscrete group and for every finite proper smooth G-CW-complex X let $I_G(X) := N(G) \cap \{L < G_0 := \bigcap G_x, x \in X\}$, where $N(G)$ is the family of all open compact normal subgroups of G. The group G_0 is open in G, because as G is prodiscrete it has a neighbourhood base of the unit consisting of open compact normal subgroups and hence for every $x \in X$ an open compact normal subgroup of G_x can be chosen. As this subgroup also lies in every conjugate $gG_xg^{-1} = G_{gx}$ of G_x the above intersection becomes a finite intersection because X is finite. Let (X, A) be a proper smooth G-CW-pair. Then for every $L \in I_G(X)$ we can consider (X, A) as a G/L-CW-pair and define

$$\mathcal{K}_G^*(X, A) := \varinjlim_{L \in I_G(X)} K_{G/L}^*(X, A),$$

where $I_G(X)$ is partially ordered by the opposite of the inclusion and the structure maps of the colimit are given by restriction with the homomorphisms $G/L \to G/K, K > L, K, L \in I_G(X)$, whose kernels K/L are compact.

As $I_G(X)$ is cofinal in $I_G(A)$ and taking the colimit is exact, this defines a G-cohomology theory on finite proper smooth G-CW-complexes.

Lemma 8.15 *Let G be prodiscrete. Then there is a natural equivalence of G-cohomology theories on finite proper smooth G-CW-complexes*

$$\Phi : \mathcal{K}_G^* \to K_G^*.$$

Proof: We define $\Phi_L(X, A) := \mathrm{res}_{G \to G/L}(X, A)$. Because of functoriality of restriction, these homomorphisms induce an homomorphism $\Phi(X, A) : \mathcal{K}_G^*(X, A) \to K_G^*(X, A)$. This homomorphism is natural and commutes with the boundary homomorphisms (as this is the case for restriction).

It now suffices to show that $\Phi(G/H)$ is an isomorphism for $H < G$ any compact open normal subgroup. For any $L \in I_G(G/H) = N(G) \cap \{L < H\}$

$$\begin{aligned} K^*_{G/L}(G/H) &= K^*_{G/L}((G/L)/(H/L)) \\ &= \begin{cases} R(H/L) & \text{if } * \text{ is even} \\ 0 & \text{if } * \text{ is odd} \end{cases} \end{aligned}$$

and

$$K_G^*(G/H) = \begin{cases} R(H) & \text{if } * \text{ is even} \\ 0 & \text{if } * \text{ is odd.} \end{cases}$$

If $*$ is even, then $\Phi(G/H)$ is the isomorphism of Lemma 4.8. □

We now want to exploit this lemma to show that for a prodiscrete group G and any finite proper smooth G-CW-complex X we get a Chern character

$$ch_X^* : K_G^*(X) \to H_G^*(X; \mathbb{Q} \otimes R(-))$$

which is rationally an isomorphism. Here we regard K_G^* as being $\mathbb{Z}/2$-graded, ch_X^* sending $K_G^0(X)$ to $H_G^{ev}(X; \mathbb{Q} \otimes R(-))$ and $K_G^1(X)$ to $H_G^{odd}(X; \mathbb{Q} \otimes R(-))$.

We want to define ch_G^* for G prodiscrete as the colimit over all $ch^*_{G/L}, L \in I_G(X)$ (note here that Lemma 8.15 only holds for finite G-CW-complexes. So this method will only give a Chern character for finite G-CW-complexes).

Remark 8.16 Let G, M be topological groups, $\pi : G \to M$ an open continuous surjective group homomorphism. Let X be a smooth G-CW-complex on which the kernel of π acts trivially. Then we can regard X also as a smooth M-CW-complex and for every open subgroup H of G we have that $X^H = X^{\pi(H)}$. Hence the singular chain complex $\underline{C}_*^G(X) = \mathrm{res}_\pi \underline{C}_*^M(X)$ ($\underline{C}_*^M(X)(M/H) := C_*(X^H)$, cf. [LO99], Chapter 5) and

$$\begin{aligned} \hom_{\mathrm{Or}(M,\mathcal{O})}(\underline{C}_*^M(X), \mathbb{Q} \otimes R(-)) &= \hom_{\mathrm{Or}(G,\mathcal{O})}(\mathrm{res}_\pi \underline{C}_*^M(X), \mathrm{res}_\pi(\mathbb{Q} \otimes R(-))) \\ &= \hom_{\mathrm{Or}(G,\mathcal{O})}(\underline{C}_*^G(X), \mathrm{res}_\pi(\mathbb{Q} \otimes R(-))). \end{aligned}$$

Together with the homomorphism

$$\hom_{\mathrm{Or}(G,\mathcal{O})}(\underline{C}_*^G(X), \mathrm{res}_\pi(\mathbb{Q} \otimes R(-))) \overset{\hom(id, \mathrm{res}_\pi)}{\to} \hom_{\mathrm{Or}(G,\mathcal{O})}(\underline{C}_*^G(X), \mathbb{Q} \otimes R(-)))$$

this defines a restriction homomorphism

$$\mathrm{res}_\pi : H_*^M(X;\mathbb{Q}\otimes R(-)) \to H_*^G(X;\mathbb{Q}\otimes R(-))$$

which is natural in X, compatible with the boundary homomorphisms and functorial in π. $\square$

As before we get for any prodiscrete group G and any finite proper smooth G-CW-pair (X,A) a morphism

$$\Psi(X,A) : \underset{L\in I_G(X)}{colim}\, H^*_{G/L}(X,A;\mathbb{Q}\otimes R(-)) \to H^*_G(X,A;\mathbb{Q}\otimes R(-))$$

by putting $\Psi(X,A)_L = \mathrm{res}_{\pi:G\to G/L}(X,A)$. As in Lemma 8.15 it follows that $\Psi(X,A)$ is an isomorphism.

We want to define the Chern character by the following diagram:

$$\begin{array}{ccc} \varinjlim_{L\in I_G(X)} K^*_{G/L}(X,A) & \xrightarrow{\varinjlim ch^*_{G/L}} & \varinjlim_{L\in I_G(X,A)} H^*_{G/L}(X,A;\mathbb{Q}\otimes R(-)) \\ \downarrow{\scriptstyle \Phi(X,A)} & & \downarrow{\scriptstyle \Psi(X,A)} \\ K^*_G(X,A) & \overset{ch^*_G}{\dashrightarrow} & H^*_G(X,A;\mathbb{Q}\otimes R(-)) \end{array}$$

Hence we have to show that for any discrete groups H,H' (above: $G/L, G/L', L < L', L, L' \in I_G(X)$) and any surjective group homomorphism $\pi : H \to H'$ with finite kernel (above: $G/L \to G/L'$) and any finite proper H'-CW-complex X the following formula holds:

$$\mathrm{res}_{\pi:H\to H'} \circ ch^*_{H'} = ch^*_H \circ \mathrm{res}_{\pi:H\to H'} .$$

Remark 8.17 Let G be a discrete group. By [LO99], 5.3 there is a natural isomorphism

$$H^*_G(X;\mathbb{Q}\otimes R(-)) \to \hom_{\mathrm{Sub}(G,\mathcal{FIN})}(H_*(X^?/C_G?),\mathbb{Q}\otimes R(-)),$$

where C_GH is the centralizer of H in G and $\mathrm{Sub}(G,\mathcal{FIN})$ is the category with objects the finite subgroups of G and

$$\mathrm{mor}_{\mathrm{Sub}(G,\mathcal{FIN})}(H,K) \subset \hom(H,K)/Inn(K)$$

is the subset consisting of the monomorphisms induced by conjugation and inclusion in G.

Let G, M be discrete groups, $\pi : G \to M$ a surjective group homomorphism with finite kernel, X a proper finite G-CW-complex. Then for every finite subgroup $H < G$ we get that $C_M(\pi H) \supset \pi(C_GH)$ and $X^H = X^{\pi H}$ and hence a morphism

$$proj_* : H_*(X^H/C_GH) \to H_*(X^{\pi H}/C_M(\pi H)),$$

which induces a morphism

$$\hom_{\mathrm{Sub}(M)}(H_*(X^?/C_M?),\mathbb{Q}\otimes R(-)) \overset{\hom(proj_*,\mathrm{res}_\pi)}{\to} \hom_{\mathrm{Sub}(G)}(H_*(X^?/C_G?),\mathbb{Q}\otimes R(-))$$

Namely let $\varphi : K \to H$ be a morphism in $\mathrm{Sub}(G, \mathcal{FIN})$, then $\varphi_\pi : \pi(K) \to \pi(H)$ is a morphism in $\mathrm{Sub}(M, \mathcal{FIN})$ (and every morphism there is of this form) and the following diagram commutes:

$$\begin{array}{ccccccc} H_*(\mathrm{res}_\pi X^H/C_G H) & \longrightarrow & H_*(X^{\pi H}/C_M(\pi H)) & \longrightarrow & R(\pi H) & \xrightarrow{\mathrm{res}_\pi} & R(H) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{res}_{\varphi_\pi}} & & \downarrow{\scriptstyle \mathrm{res}_\varphi} \\ H_*(\mathrm{res}_\pi X^K/C_G K) & \longrightarrow & H_*(X^{\pi K}/C_M(\pi K)) & \longrightarrow & R(\pi K) & \xrightarrow{\mathrm{res}_\pi} & R(K) \end{array}$$

The following diagram commutes

$$\begin{array}{ccc} H^*_M(X; \mathbb{Q} \otimes R(-)) & \xrightarrow{\mathrm{res}_\pi} & H^*_G(X; \mathbb{Q} \otimes R(-)) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ \hom_{\mathrm{Sub}(M,\mathcal{FIN})}(H_*(X^?/C_M?), \mathbb{Q} \otimes R(-)) & \xrightarrow{\hom(proj_*, \mathrm{res}_\pi)} & \hom_{\mathrm{Sub}(G,\mathcal{FIN})}(H_*(X^?/C_G?), \mathbb{Q} \otimes R(-)). \end{array}$$

□

Using the above remark, the Chern character for a discrete group H and a proper H-CW-complex X is defined by homomorphisms

$$ch^L_H : K_H(X) \to \hom(H_*(X^L/C_H L), \mathbb{Q} \otimes R(L)), L < H \text{ finite.}$$

Hence to extend it to prodiscrete groups it suffices to show that the following diagram commutes for any discrete groups H, H', $\pi : H \to H'$ a surjective group homomorphism with finite kernel, L a finite subgroup of H and X a finite proper H-CW-complex on which $\ker(\pi)$ acts trivially:

$$\begin{array}{ccc} K_{H'}(X) & \xrightarrow{\mathrm{res}_\pi} & K_H(X) \\ \downarrow{\scriptstyle ch^{\pi L}_{H'}} & & \downarrow{\scriptstyle ch^L_H} \\ \hom(H_*(X^{\pi L}/C_{H'}(\pi L)), \mathbb{Q} \otimes R(-)) & \xrightarrow{\hom(proj_*, \mathrm{res}_\pi)} & \hom(H_*(X^H/C_H L), \mathbb{Q} \otimes R(-)). \end{array}$$

To see that this diagram commutes first note that $C_{H'}(\pi L) \supset \pi(C_H L)$ and $N_{H'}(\pi L) \supset \pi(N_H L)$. Furthermore we need the H-map $\varphi : EH \to \mathrm{res}_\pi E(H')$. The following diagram is the same as the one above, but with the definition of the Chern character for discrete groups as in [LO99] 5.4 fully given.

$$
\begin{array}{ccccc}
K^*_{H'}(X) & \xrightarrow{\qquad\qquad\qquad\qquad\mathrm{res}_\pi\qquad\qquad\qquad\qquad} & & & K^*_H(X) \\
\Big\downarrow{\scriptstyle \mathrm{res}^{H'}_{N_{H'}(\pi L)}\otimes id} & & & & \Big\downarrow{\scriptstyle \mathrm{res}^{H}_{N_H L}\otimes id} \\
K_{N^*_{H'}(\pi L)}(X^{\pi L}) & \xrightarrow{\mathrm{res}^{N_{H'}(\pi L)}_{\pi(N_H L)}} & K^*_{\pi(N_H L)}(X^{\pi L}) & \xrightarrow{\mathrm{res}_\pi} & K^*_{N_H L}(X^L) \\
\Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \psi} \\
K^*_{C_{H'}(\pi L)}(X^{\pi L})\otimes R(\pi L) & \xrightarrow{\mathrm{res}^{C_{H'}(\pi L)}_{\pi(C_H L)}\otimes id} & K^*_{\pi(C_H L)}(X^{\pi L})\otimes R(\pi L) & \xrightarrow{\mathrm{res}_\pi\otimes\mathrm{res}_\pi} & K^*_{C_H L}(X^L)\otimes R(L) \\
\Big\downarrow{\scriptstyle proj^*\otimes id} & & \Big\downarrow{\scriptstyle proj^*\otimes id} & & \Big\downarrow{\scriptstyle proj^*\otimes id} \\
K^*_{C_{H'}(\pi L)}(E(H')\times X^{\pi L})\otimes R(\pi L) & \xrightarrow{\mathrm{res}^{C_{H'}(\pi L)}_{\pi(C_H L)}\otimes id} & K^*_{\pi(C_H L)}(E(H')\times X^{\pi L})\otimes R(\pi L) & \xrightarrow{((\varphi\times id)^*\circ\mathrm{res}_\pi)\otimes\mathrm{res}_\pi} & K^*_{C_H L}(EH\times X^L)\otimes R(L) \\
\Big\downarrow{\scriptstyle infl^{-1}\otimes id} & & \Big\downarrow{\scriptstyle infl^{-1}\otimes id} & & \Big\downarrow{\scriptstyle infl^{-1}\otimes id} \\
K^*(E(H')\times_{C_{H'}(\pi L)} X^{\pi L})\otimes R(\pi L) & \xrightarrow{proj^*\otimes id} & K^*(E(H')\times_{\pi(C_H L)} X^{\pi L})\otimes R(\pi L) & \xrightarrow{(\varphi\times id)^*\otimes\mathrm{res}_\pi} & K^*(EH\times_{C_H L} X^L)\otimes R(L) \\
\Big\downarrow{\scriptstyle ch\otimes id} & & \Big\downarrow{\scriptstyle ch\otimes id} & & \Big\downarrow{\scriptstyle ch\otimes id} \\
H^*(E(H')\times_{C_{H'}(\pi L)} X^{\pi L};\mathbb{Q})\otimes R(\pi L) & \xrightarrow{proj^*\otimes id} & H^*(E(H')\times_{\pi(C_H L)} X^{\pi L};\mathbb{Q})\otimes R(\pi L) & \xrightarrow{(\varphi\times id)^*\otimes\mathrm{res}_\pi} & H^*(EH\times_{C_H L} X^L;\mathbb{Q})\otimes R(L) \\
\Big\uparrow{\scriptstyle proj^*\otimes id} & & \Big\uparrow{\scriptstyle proj^*\otimes id} & & \Big\uparrow{\scriptstyle proj^*\otimes\mathrm{res}_\pi} \\
H^*(X^{\pi L}/C_{H'}(\pi L);\mathbb{Q})\otimes R(\pi L) & \xrightarrow{proj^*\otimes id} & H^*(X^{\pi L}/\pi(C_H L);\mathbb{Q})\otimes R(\pi L) & \xrightarrow{proj^*\otimes id} & H^*(X^L/C_H L;\mathbb{Q})\otimes R(L) \\
\Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \cong} \\
\hom(H_*(X^{\pi L}/C_{H'}(\pi L)),\mathbb{Q}\otimes R(\pi L)) & \xrightarrow{\hom(proj_*,id)} & \hom(H_*(X^{\pi L}/\pi(C_H L)),\mathbb{Q}\otimes R(\pi L)) & \xrightarrow{\hom(proj_*,\mathrm{res}_\pi)} & \hom(H_*(X^L/C_H L),\mathbb{Q}\otimes R(L))
\end{array}
$$

Here ch is the ordinary Chern character. For any discrete group G and any normal finite subgroup $N < G$ and $K < G$ such that $[K, N] = 1$ and any proper G/N-CW-complex Y the map

$$\Psi : K^*_G(X) \to K^*_H(X) \otimes R(N)$$

is given by

$$[E] \mapsto \sum_{V \in Irr(N)} [\hom_N(V, E)] \otimes [V].$$

It is natural in X and commutes with restriction (see [LO99], 3.4).

For K a discrete group, N a normal subgroup and Y a proper K-CW-complex the inflation map $\mathrm{infl}^K_{K/N}(Y) : K^*_{K/N}(Y/N) \to K^*_K(Y)$ is defined by $\mathrm{infl}^K_{K/N}(Y) = f^* \circ \mathrm{res}_{K \to K/N}$, where $f : Y \to Y/N$ is the projection ([LO99], 3.3).

The uppermost square commutes because of functoriality of restriction and the left square of the next row commutes because of [LO99] 3.4(b). For the right hand square of the second row we have to take a closer look: Let $p : E \to X^{\pi L}$ be a $\pi(N_H L)$-vector bundle. Then we have

$$\begin{array}{ccc} [E] & & \\ \downarrow & & \\ \sum_{V \in Irr(\pi L)} [\hom_{\pi(L)}(V, E)] \otimes [V] & \longmapsto & \sum_{V \in Irr(\pi L)} [\mathrm{res}_\pi(\hom_{\pi(L)}(V, E))] \otimes [\mathrm{res}_\pi V] \end{array}$$

and on the other hand

$$\begin{array}{ccc} [E] & \longmapsto & [\mathrm{res}_\pi E] \\ & & \downarrow \\ & & \sum_{W \in Irr(L)} [\hom_L(W, \mathrm{res}_\pi E)] \otimes [W] \\ & & \| \\ & & \sum_{W \in Irr(L)} [\mathrm{res}_\pi \hom_{\pi L}(\mathrm{ind}_\pi W, E)] \otimes [W] \\ & & \| \\ & & \sum_{W \in Irr(L), W^K = W} [\mathrm{res}_\pi \hom_{\pi L}(W^K, E)] \otimes [W] \\ & & \| \\ & & \sum_{v \in Irr(\pi L)} [\mathrm{res}_\pi \hom_{\pi L}(V, E)] \otimes [\mathrm{res}_\pi V] \end{array}$$

Note that $\mathrm{ind}_\pi W \cong W^{\ker(\pi)}$ (See [Ser96] ex. 7.1) and if $W \in Irr(L)$, then $W^K = W$ or $W^K = 0$. But $\{W \in Irr(L) | W^K = W\} = \{W \in R(L) | W = \mathrm{res}_\pi V, V \in Irr(\pi L)\}$ as K is normal in L.

Collecting everything together we get

Theorem 8.18 *Let G be a prodiscrete group. For any finite proper smooth G-CW-complex X there is a Chern character*

$$ch^*_X : K^*_G(X) \to H^*(X; \mathbb{Q} \otimes R(-))$$

which is natural in X and commutes with the boundary homomorphisms of the Mayer-Vietoris sequence.

Rationally it is an isomorphism.

Proof: Naturality and compatibility with the boundary homomorphisms follow because they hold if G is discrete and restriction has these two properties.

Rationally the Chern character is an isomorphism because this holds for discrete groups ([LO99], 5.5). □

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