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Connes–Karoubi long exact sequence for Fréchet sheaves

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Abstract. In this paper, we prove a long exact sequence involving the algebraic, topological and relative K-theory groups for a sheaf $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ of Fréchet algebras or ultrametric Banach algebras on a scheme X under certain conditions. This extends to sheaves the construction due to A. Connes and M. Karoubi of the relative K-theory group $K_*^{rel}(A)$ and the associated K-theory long exact sequence for a Fréchet algebra A. In doing so, we make use of the generalized sheaf cohomology for simplicial sheaves developed by Brown and Gersten.

1. Introduction

Let A be a unital algebra and let $GL(A)$ denote the direct limit of the general linear groups $\{GL_n(A)\}_{n\geq 1}$ over A. Then, the Quillen K-theory groups of A are defined to be the homotopy groups (see [15]):

$$K_n(A) := \pi_n(BGL(A)^+) \qquad \forall\, n \geq 0, \tag{1}$$

where $BGL(A)$ denotes the classifying space of $GL(A)$ and $BGL(A)^+$ is obtained by applying Quillen's plus construction (see [15]) to the space $BGL(A)$. This definition may be extended to a scheme $(X, \mathcal{O}_X)$ as follows: we consider the presheaf $\mathbb{Z} \times BGL^+$ on X that associates to an open set $U \subseteq X$ the simplicial set $\mathbb{Z} \times BGL(\mathcal{O}_X(U))^+$. Let $\mathbf{Z} \times \mathbf{BGL}^+$ denote the sheafification of $\mathbb{Z} \times BGL^+$. Then, by using hypercohomology (see Soulé [16, § 4.2]), we can define the Quillen K-theory groups of the scheme X.

It is natural to ask if we can similarly extend the definition of K-theory groups to sheaves of topological algebras on a scheme X. For instance, consider a smooth, integral and separated scheme X of finite type over $\mathbb{C}$ such that the complex analytic space X^{an} associated to X is a smooth manifold. Let C_X^∞ denote the sheaf that associates to any Zariski open set $U \subseteq X$ the ring of smooth complex valued continuous functions on $U^{an} \subseteq X^{an}$, where U^{an} denotes the complex analytic subspace of X^{an} associated to U. Then C_X^∞ is a sheaf of Fréchet algebras on X. More generally, in this paper, we consider sheaves of Fréchet algebras or ultrametric Banach algebras on a scheme X

satisfying certain conditions described in Section 2. For sheaves of ultrametric Banach algebras, our motivating example is that of formal schemes (see Example (a) in Section 2).

Let A be a Fréchet algebra. Then, Connes and Karoubi ([7], [8]) have defined a relative K-theory group $K_*^{rel}(A)$ that fits into a long exact sequence with the algebraic and topological K-theory groups of A:

(2)
$$\dots \longrightarrow K_i^{alg}(A) \longrightarrow K_i^{top}(A) \longrightarrow K_{i-1}^{rel}(A) \longrightarrow K_{i-1}^{alg}(A) \longrightarrow \dots$$

In this paper, our objective is to develop similar constructions for sheaves $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ of Fréchet algebras or ultrametric Banach algebras on a noetherian scheme X of finite type over $\mathbb{C}$ (see Definition 2.2). Using the hypercohomology of simplicial sheaves as defined by Brown and Gersten [4], we will define groups $K_*^{alg}(\mathfrak{X})$, $K_*^{top}(\mathfrak{X})$ and $K_*^{rel}(\mathfrak{X})$ that we will respectively call the algebraic, topological and relative K-theory groups of $\mathfrak{X}$. Thereafter, we show that the relative K-theory group $K_*^{rel}(\mathfrak{X})$ fits into a long exact sequence along with the algebraic and topological K-theory groups just as in (2).

(3)
$$\dots \longrightarrow K_i^{alg}(\mathfrak{X}) \longrightarrow K_i^{top}(\mathfrak{X}) \longrightarrow K_{i-1}^{rel}(\mathfrak{X}) \longrightarrow K_{i-1}^{alg}(\mathfrak{X}) \longrightarrow \dots$$

2. Exact Sequence of K-theories for a Fréchet sheaf

Let X be a noetherian scheme of finite type over $\mathbb{C}$. In this paper, we will consider the K-theory of sheaves of Fréchet algebras or ultrametric Banach algebras on X. We recall here the notion of an ultrametric Banach algebra as defined in [14, 5.1].

Definition 2.1. Let A be a $\mathbb{Z}$-algebra provided with a "quasi-norm"
$$p : A \longrightarrow \mathbf{R}^+$$
satisfying the following properties:

(1) For all x, $y \in A$, $p(x+y) \leq \mathrm{Max}(p(x), p(y))$ and $p(xy) \leq p(x)p(y)$.
(2) For any $x \in A$, $p(x) = 0$ if and only if $x = 0$.
(3) For any $x \in A$, $p(-x) = p(x)$.

Then, if A is complete under the metric defined by $d(x,y) := p(y-x)$, $\forall\, x$, $y \in A$, A is said to be an ultrametric Banach algebra.

Definition 2.2. Let X be a noetherian scheme of finite type over $\mathbb{C}$ and let $\mathfrak{X}$ denote a sheaf of rings on X. We will say that the pair $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a Fréchet sheaf (resp. an ultrametric Banach sheaf) if, for each open subset $U \subseteq X$, the associated ring $\mathcal{O}_{\mathfrak{X}}(U)$ is a Fréchet algebra (resp. an ultrametric Banach algebra) and for any open sets $V \subseteq U \subseteq X$, the restriction map $\mathcal{O}_{\mathfrak{X}}(U) \longrightarrow \mathcal{O}_{\mathfrak{X}}(V)$ is a continuous map of topological algebras.

Further, we will say that a Fréchet sheaf (resp. an ultrametric Banach sheaf) $\mathfrak{X}$ is irreducible if the underlying scheme X is irreducible and for any any nonempty open sets $V \subseteq U \subseteq X$, the restriction map $\mathcal{O}_{\mathfrak{X}}(U) \longrightarrow \mathcal{O}_{\mathfrak{X}}(V)$ is an injection.

Our motivating examples for Definition 2.2 are as follows:

Examples. (a) Let X' be an integral noetherian scheme of finite type over $\mathbb{C}$ and let X be a closed integral subscheme of X', corresponding to a sheaf $\mathcal{I}$ of ideals in the structure sheaf $\mathcal{O}_{X'}$ of X'. Further, suppose that, given any affine open set $U = \mathrm{Spec}(A_U)$ contained in X' in which X is defined by a prime ideal p_U, all powers p_U^i, $i \geq 1$ of p_U are primary ideals. We will refer to such an integral subscheme X as a primary integral subscheme of X' (this would happen, for instance, if the prime ideal corresponding to X in any affine open set were always generated by a regular sequence (see [11])). Then, for any $n \geq 1$, the sheaf $\mathcal{O}_{X'}/\mathcal{I}^n$ is a sheaf of rings supported on X. Then, the scheme X, equipped with the sheaf $\mathcal{O}_{\mathfrak{X}}$ of topological rings $\varprojlim_{n\geq 1} \mathcal{O}_{X'}/\mathcal{I}^n$ is referred to as the formal completion of X' along X, known as a formal scheme (see [10, II.9]). Using the Lemma 2.3, Lemma 2.4(1) and Lemma 2.5, we will show that the formal scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ obtained in this manner is an irreducible ultrametric Banach sheaf on X.

(b) Let X be a smooth, integral, separated scheme of finite type over $\mathbb{C}$ such that the complex analytic space X^{an} associated to X (see, for instance, [10, Appendix B]) is a smooth manifold. Then any Zariski open set $U \subseteq X$ can be associated to an open subset $U^{an} \subseteq X^{an}$ (since the usual topology is finer than the Zariski topology). Then, we let C_X^∞ denote the sheaf of Fréchet algebras on X that associates to the Zariski open subset $U \subseteq X$ the Fréchet algebra $C^\infty(U^{an})$ of smooth complex valued continuous functions on $U^{an} \subseteq X^{an}$. Using Lemma 2.4(2) and Lemma 2.5 below, we will show that C_X^∞ is, in fact, an irreducible Fréchet sheaf on X.

If $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is an ultrametric Banach sheaf on X, for any open set $U \subseteq X$, we let N_U denote the quasi-norm on $\mathcal{O}_{\mathfrak{X}}(U)$. Then, a sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \in \mathcal{O}_{\mathfrak{X}}(U)$ will be said to be Cauchy if for any $\epsilon > 0$, $\exists\, M$ such that for any n, $m > M$, we have $N_U(x_m - x_n) < \epsilon$. The sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \in \mathcal{O}_{\mathfrak{X}}(U)$ will be said to converge to $x \in \mathcal{O}_{\mathfrak{X}}(U)$ if for any $\epsilon > 0$, $\exists\, M$ such that for any $n > M$, we have $N_U(x_m - x) < \epsilon$.

Lemma 2.3. *Let $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ denote the formal scheme obtained by completing an integral noetherian scheme X' of finite type over $\mathbb{C}$ along a closed primary integral subscheme X, as in example (a) above. Then, for each open subset U of X, the ring $\mathcal{O}_{\mathfrak{X}}(U)$ is an ultrametric Banach algebra, i.e., $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ defines an ultrametric Banach sheaf on X.*

Proof. Let the closed subscheme $X \subseteq X'$ be defined by a sheaf $\mathcal{I}$ of ideals in the structure sheaf $\mathcal{O}_{X'}$ of X'. First, suppose that $U = \mathrm{Spec}(A_U/p_U)$ is an affine open subset of X, such that $\mathrm{Spec}(A_U)$ is an affine open subset of X' in which X is defined by the prime ideal p_U. Let $\mathrm{Aff}(X)$ denote the collection of all such affine open subsets U of X. Then, by definition, $\mathcal{O}_{\mathfrak{X}}(U)$ is the completion $\hat{A}_U = \varprojlim_{n\geq 1} A_U/p_U^n$. Since X' is integral, A_U is a noetherian integral domain and from the well known Krull intersection theorem (see, for instance, [2, Thm. 10.17]), it follows that $\cap_{n=1}^{\infty} p_U^n = 0$. We choose any $0 < \lambda < 1$ and define $N_U : A_U \longrightarrow \mathbb{R}_{\geq 0}$ by setting $N_U(x) = \lambda^n$ if $x \in p_U^n$ and $x \notin p_U^{n+1}$ (for $x \neq 0$) and $N_U(0) = 0$. Hence, the completed ring $\mathcal{O}_{\mathfrak{X}}(U) = \hat{A}_U$ is an ultrametric Banach algebra (see [14, 5.1]) with a quasi norm also denoted N_U that makes $\hat{A}_U$ into a Hausdorff topological space.

For an arbitrary open set $U \subseteq X$, we let $\mathrm{Aff}(U)$ denote the collection of affine open sets in $\mathrm{Aff}(X)$ contained in U. We define a quasi norm N_U on $\mathcal{O}_{\mathfrak{X}}(U)$ by the maximum

$$N_U(x) = \mathrm{Max}\{N_V(x_V) \mid V \in \mathrm{Aff}(U)\} \qquad \forall\, x \in \mathcal{O}_{\mathfrak{X}}(U), \tag{4}$$

where $x_V \in \mathcal{O}_{\mathfrak{X}}(V)$ denotes the restriction of $x \in \mathcal{O}_{\mathfrak{X}}(U)$ to $V \subseteq U$. The maximum in (4) exists because all the values $N_V(x_V)$ can take lie in $\{\lambda^0, \lambda^1, \lambda^2, ...\}$ and since $0 < \lambda < 1$, every nonempty subset of $\{\lambda^0, \lambda^1, \lambda^2, ...\}$ has a maximum. In particular, when $U = \mathrm{Spec}(A_U/p_U) \in \mathrm{Aff}(X)$, $x \in p_U^n \subseteq A_U \Rightarrow x_V \in p_V^n \subseteq A_V$ for any $x \in A_U$, $V \in \mathrm{Aff}(U)$ and hence $N_V(x_V) \leq N_U(x)$, i.e., N_U reduces to the previous definition when $U \in \mathrm{Aff}(X)$. Further, given $x,\ y \in \mathcal{O}_{\mathfrak{X}}(U)$, we have $N_V((xy)_V) \leq N_V(x_V)N_V(y_V)$ for each $V \in \mathrm{Aff}(U)$ and hence $N_U(xy) \leq N_U(x)N_U(y)$. Also, if we choose $V \in \mathrm{Aff}(U)$ such that $N_U(x+y) = N_V((x+y)_V)$, we have

$$\begin{aligned} N_U(x+y) = N_V((x+y)_V) &\leq \mathrm{Max}\{N_V(x_V), N_V(y_V)\} \\ &\leq \mathrm{Max}\{N_U(x), N_U(y)\}. \end{aligned} \tag{5}$$

From the maximum in (4), we also note that given any open sets $U' \subseteq U$, since $\mathrm{Aff}(U') \subseteq \mathrm{Aff}(U)$, the restriction maps $\mathcal{O}_{\mathfrak{X}}(U) \longrightarrow \mathcal{O}_{\mathfrak{X}}(U')$ must be continuous.

For any open $U \subseteq X$, we can choose a finite cover $\{V_i\}_{i \in I}$, $V_i \in \mathrm{Aff}(U)$ of U. Then, for any given $V' \in \mathrm{Aff}(U)$ with $V' = \mathrm{Spec}(A_{V'}/p_{V'})$, we can choose a covering $\{V'_j\}_{j \in J}$ of V' with $V'_j \in \mathrm{Aff}(V') \subseteq \mathrm{Aff}(U)$ such that $V'_j \in \cup_{i \in I} \mathrm{Aff}(V_i)$ for all $j \in J$. For a given $\hat{a} \in \mathcal{O}_{\mathfrak{X}}(U)$ restricting to $\hat{a}_{V'} \in \hat{A}_{V'}$, suppose that $\hat{a}_{V'}$ maps to 0 in $A_{V'}/p_{V'}^n$ but not in $A_{V'}/p_{V'}^{n+1}$. Since the restriction of $\mathcal{O}_{X'}/\mathcal{I}^{n+1}$ to V' is also a sheaf, there exists some V'_j such that $\hat{a}_{V'_j}$ does not map to 0 in $A_{V'_j}/p_{V'_j}^{n+1}$. Thus, $N_{V'_j}(\hat{a}_{V'_j}) = N_{V'}(\hat{a}_{V'})$. Hence, if we choose V_i such that $V'_j \in \mathrm{Aff}(V_i)$, $N_{V'}(\hat{a}_{V'}) = N_{V'_j}(\hat{a}_{V'_j}) \leq N_{V_i}(\hat{a}_{V_i})$. It follows that, given the finite open cover $\{V_i\}_{i \in I}$ of U, we may set

$$N_U(x) = \mathrm{Max}\{N_{V_i}(x_{V_i})\} \qquad \forall\, x \in \mathcal{O}_{\mathfrak{X}}(U). \tag{6}$$

It follows from (4) that given a Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$, $x_n \in \mathcal{O}_{\mathfrak{X}}(U)$, $n \in \mathbb{N}$, the restriction $\{x_{n,V}\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{O}_{\mathfrak{X}}(V)$ for each $V \in \mathrm{Aff}(U)$. Since each $\mathcal{O}_{\mathfrak{X}}(V) = \hat{A}_V$ is an ultrametric Banach algebra, each of the sequences $\{x_{n,V}\}_{n \in \mathbb{N}}$ converges to a unique $x_V \in \mathcal{O}_{\mathfrak{X}}(V)$. Further, for any $V'' \subseteq V \subseteq U$, V, $V'' \in \mathrm{Aff}(U)$, it is clear that $x_V \in \mathcal{O}_{\mathfrak{X}}(V)$ restricts to $x_{V''} \in \mathcal{O}_{\mathfrak{X}}(V'')$. Then, since a formal scheme is a sheaf of rings (see [10, II.9]), there exists a unique $x \in \mathcal{O}_{\mathfrak{X}}(U)$ restricting to each $x_V \in \mathcal{O}_{\mathfrak{X}}(V)$,

$V \in \mathrm{Aff}(U)$. Then, using (6), since $N_U(x_n - x)$ is the maximum of the finite set $\{N_{V_i}(x_{n,V_i} - x_{V_i})\}$, $i \in I$ and each of the sequences $\{x_{n,V_i}\}_{n\in\mathbb{N}}$ converges to x_{V_i}, it follows that $\{x_n\}_{n\in\mathbb{N}}$ converges to x. Hence, $\mathcal{O}_{\mathfrak{X}}(U)$ satisfies all the conditions for being an ultrametric Banach algebra (see [14, 5.1]). □

Lemma 2.4.

(1) Let R be a noetherian integral domain and suppose that p is a prime ideal in R such that all powers p^i, $i > 0$ are primary. Then, for any $g \in R - p$, the natural map from $\hat{R} = \varprojlim R/p^n$ to $\hat{R}_g = \varprojlim R_g/p_g^n$ is an injection.

(2) Let X be a smooth, integral, separated scheme of finite type over $\mathbb{C}$ such that the associated analytic space X^{an} is a smooth manifold. Let C_X^∞ denote the Fréchet sheaf associated to X as in example (b) above. Then, there exists a basis $\mathcal{B}$ for the Zariski topology on X such that given any open sets $V \subseteq U \subseteq X$ with U, $V \in \mathcal{B}$, the restriction map $C_X^\infty(U) \longrightarrow C_X^\infty(V)$ is an injection.

Proof. (1) The elements of $\hat{R}$ are, by definition, sequences of the form $(r_1, r_2, ...)$ with $r_i \in R$, $\forall\ i \in \mathbb{N}$, where, for positive integers $i \geq j$, $r_i \equiv r_j \pmod{p^j}$. Suppose that the sequence $(r_1, r_2, ...)$ maps to the zero sequence in $\hat{R}_g$. Then, for each r_i, we must have $r_i \in p_g^i$. Since R is an integral domain, this means that we have an $x_i \in p^i$ and a power g^{k_i} of g such that

$$g^{k_i} r_i = x_i \in p^i \text{ for each } i.$$

By assumption, p^i is a primary ideal and thus $r_i \notin p^i$ would imply that some power of g^{k_i} lies in p^i. This is impossible since $g \notin p$. Hence, each $r_i \in p^i$ and the map is injective.

(2) In the notation of example (b) above, we consider the smooth integral, separated scheme X of finite type over $\mathbb{C}$. Then, we can choose an affine open cover of the form $U_i = \mathrm{Spec}(\mathbb{C}[x_1, ..., x_n]/J_i)$, $i \in I$ of X. Hence, there is a basis $\mathcal{B}_i$ of U_i in Zariski topology, consisting of open sets of the form $U_{i,f} = \mathrm{Spec}((\mathbb{C}[x_1, ..., x_n]/J_i)_f)$ for any $f \in \mathbb{C}[x_1, ..., x_n]$, $f \notin J_i$. Then, each $f \in \mathbb{C}[x_1, ..., x_n]$, $f \notin J_i$, defines a holomorphic function on U_i^{an} and hence the set $U_{i,f}^{an} \subseteq U_i^{an}$ where f does not vanish on U_i^{an} is a dense open subset of U_i^{an} (see, for instance, [9, IV.1.6]). Then, since any function $h \in C_X^\infty(U_{i,g}) = C^\infty(U_{i,g}^{an})$ is continuous on $U_{i,g}^{an}$, for any $U_{i,g} \in \mathcal{B}_i$ such that $U_{i,f} \subseteq U_{i,g}$ (and hence $U_{i,f}^{an} \subseteq U_{i,g}^{an} \subseteq U_i^{an}$), we have an injection $C_X^\infty(U_{i,g}) = C^\infty(U_{i,g}^{an}) \hookrightarrow C^\infty(U_{i,f}^{an}) = C_X^\infty(U_{i,f})$.

We note that $\mathcal{B} = \bigcup_{i\in I} \mathcal{B}_i$ is a basis for X. Further, given any $W' \subseteq W \subseteq X$ with $W \in \mathcal{B}_i$, $W' \in \mathcal{B}_j$, there exists some $W'' \in \mathcal{B}_i$ such that $W'' \subseteq W'$. Then, the composition $C_X^\infty(W) \longrightarrow C_X^\infty(W') \longrightarrow C_X^\infty(W'')$ is an injection and hence $C_X^\infty(W) \longrightarrow C_X^\infty(W')$ is an injection. □

Lemma 2.5. *Let $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ be a sheaf of topological algebras on an irreducible noetherian scheme of finite type over $\mathbb{C}$. Suppose that X has a basis $\mathcal{B}$ such that for any open sets W, $W' \in \mathcal{B}$ with $W' \subseteq W$, the induced map $\mathcal{O}_{\mathfrak{X}}(W) \longrightarrow$*

$\mathcal{O}_{\mathfrak{X}}(W')$ is an injection. Then, given any two nonempty open sets U and V in X with $V \subseteq U$, the natural map $\mathcal{O}_{\mathfrak{X}}(U) \longrightarrow \mathcal{O}_{\mathfrak{X}}(V)$ is an injection.

Proof. Suppose that $V \subseteq U$ are open sets in X such that the restriction map $\mathcal{O}_{\mathfrak{X}}(U) \longrightarrow \mathcal{O}_{\mathfrak{X}}(V)$ is not an injection. Then, we can choose $x \in \mathcal{O}_{\mathfrak{X}}(U)$ such that $x \neq 0$ and x maps to 0 in $\mathcal{O}_{\mathfrak{X}}(V)$. First, we suppose that $U \in \mathcal{B}$. Then, if $V' \in \mathcal{B}$ is an open set contained in V, x restricts to 0 in $V' \subseteq V$. By assumption, we know that we have an injection $\mathcal{O}_{\mathfrak{X}}(U) \hookrightarrow \mathcal{O}_{\mathfrak{X}}(V')$ and hence $0 \neq x \in \mathcal{O}_{\mathfrak{X}}(U)$ cannot restrict to 0 in $\mathcal{O}_{\mathfrak{X}}(V')$, which is a contradiction. Hence, we have an injection $\mathcal{O}_{\mathfrak{X}}(U) \hookrightarrow \mathcal{O}_{\mathfrak{X}}(V)$ whenever $U \in \mathcal{B}$.

In general, let U be any open set and choose an open cover $\{U^i \in \mathcal{B}\}_{i\in I}$ of U. Again, let $x \in \mathcal{O}_{\mathfrak{X}}(U)$ be such that $x \neq 0$ and x maps to 0 in $\mathcal{O}_{\mathfrak{X}}(V)$. Let x^i be the image of x in each $\mathcal{O}_{\mathfrak{X}}(U^i)$. Since $x \neq 0$, we can choose $i_0 \in I$ such that $x^{i_0} \neq 0$. Since X is irreducible, we must have $U^{i_0} \cap V \neq \phi$. Then, since $U^{i_0} \in \mathcal{B}$, it follows from above that we have an injection $\mathcal{O}_{\mathfrak{X}}(U^{i_0}) \hookrightarrow \mathcal{O}_{\mathfrak{X}}(U^{i_0} \cap V)$. Hence x^{i_0} (and hence x) restricts to some $y \neq 0$ in $U^{i_0} \cap V$. But $U^{i_0} \cap V \subseteq V$ and hence, by assumption, x restricts to 0 in $U^{i_0} \cap V$, which is a contradiction. It follows that we have injections $\mathcal{O}_{\mathfrak{X}}(U) \hookrightarrow \mathcal{O}_{\mathfrak{X}}(V)$ whenever $V \subseteq U$. □

From this point onwards, unless otherwise mentioned, we will always let $(\mathfrak{X}, O_{\mathfrak{X}})$ denote an irreducible Fréchet sheaf or an irreducible ultrametric Banach sheaf on an irreducible noetherian scheme X of finite type over $\mathbb{C}$ in the sense of Definition 2.2.

Suppose that for each $n \geq 0$, $C^\infty(\mathbf{\Delta}^n)$ denotes the ring of C^∞-complex functions on the simplex $\mathbf{\Delta}^n$. Doing this for each $n \geq 0$ allows us to construct a simplicial ring $C^\infty(\mathbf{\Delta}^*)$. If $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ denotes an irreducible Fréchet sheaf, for any open set U in X, we now have an injection of simplicial rings

$$\mathcal{O}_{\mathfrak{X}}(U) \hookrightarrow C^\infty(\mathbf{\Delta}^*) \otimes \mathcal{O}_{\mathfrak{X}}(U) \hookrightarrow C^\infty(\mathbf{\Delta}^*)\hat{\otimes}\mathcal{O}_{\mathfrak{X}}(U), \tag{7}$$

where the first inclusion is obtained by treating $\mathcal{O}_{\mathfrak{X}}(U)$ trivially as a simplicial ring. For the sake of brevity, we shall refer to the ring $\mathcal{O}_{\mathfrak{X}}(U)\hat{\otimes}C^\infty(\mathbf{\Delta}^*)$ simply as $\mathcal{O}_{\mathfrak{X}}(U)_*$.

On the other hand, if $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is an irreducible ultrametric Banach sheaf, for each open set $U \subseteq X$ and $n \geq 0$, we consider (see [14, 5.2]) the algebra $\mathcal{O}_{\mathfrak{X}}(U)\langle x_0, ..., x_n\rangle$ of convergent series in $n+1$ variables with coefficients in $\mathcal{O}_{\mathfrak{X}}(U)$, i.e., the algebra of formal series $\sum_I a_I x^I$ (I being a multi-index) with each $a_I \in \mathcal{O}_{\mathfrak{X}}(U)$ and a_I tending to 0 in $\mathcal{O}_{\mathfrak{X}}(U)$ as $|I|$ goes to infinity. The quotient of $\mathcal{O}_{\mathfrak{X}}(U)\langle x_0, ..., x_n\rangle$ by the principal ideal generated by $(x_0 + ... + x_n - 1)$ is denoted $\mathcal{O}_{\mathfrak{X}}(U)_n$. This associates to each open set $U \subseteq X$ a simplicial ring which we denote by $\mathcal{O}_{\mathfrak{X}}(U)_*$, along with an obvious injection $\mathcal{O}_{\mathfrak{X}}(U) \hookrightarrow \mathcal{O}_{\mathfrak{X}}(U)_*$, where $\mathcal{O}_{\mathfrak{X}}(U)$ is treated trivially as a simplicial ring.

Lemma 2.6. *Let $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ be an irreducible Fréchet sheaf, or an irreducible ultrametric Banach sheaf on an irreducible noetherian scheme X of finite type over $\mathbb{C}$. Then, for any nonempty open sets $V \subseteq U \subseteq X$, the induced morphism $\mathcal{O}_{\mathfrak{X}}(U)_* \longrightarrow \mathcal{O}_{\mathfrak{X}}(V)_*$ is an injection.*

Proof. If $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is an irreducible Fréchet sheaf, for any open sets $V \subseteq U \subseteq X$, the restriction map $\mathcal{O}_{\mathfrak{X}}(U) \longrightarrow \mathcal{O}_{\mathfrak{X}}(V)$ induces a morphism $\mathcal{O}_{\mathfrak{X}}(U)_* = C^\infty(\mathbf{\Delta}^*)\hat{\otimes}\mathcal{O}_{\mathfrak{X}}(U) \longrightarrow C^\infty(\mathbf{\Delta}^*)\hat{\otimes}\mathcal{O}_{\mathfrak{X}}(V) = \mathcal{O}_{\mathfrak{X}}(V)_*$. Following [8, 3.1], the ring $\mathcal{O}_{\mathfrak{X}}(U)_* = C^\infty(\mathbf{\Delta}^*)\hat{\otimes}\mathcal{O}_{\mathfrak{X}}(U)$ may alternatively be described as follows: for any $n \geq 0$, $\mathcal{O}_{\mathfrak{X}}(U)_n$ is the algebra of C^∞-functions from $\mathbf{\Delta}^n$ with values in the Fréchet algebra $\mathcal{O}_{\mathfrak{X}}(U)$. Since $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is irreducible, given nonempty open sets $V \subseteq U$, the injection $\mathcal{O}_{\mathfrak{X}}(U) \hookrightarrow \mathcal{O}_{\mathfrak{X}}(V)$ induces injections $\mathcal{O}_{\mathfrak{X}}(U)_n \hookrightarrow \mathcal{O}_{\mathfrak{X}}(V)_n$ for each $n \geq 0$.

On the other hand, if $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is an irreducible ultrametric Banach sheaf, for any open set $U \subseteq X$ and $n \geq 0$, following [14, 5.2], we can identify $\mathcal{O}_{\mathfrak{X}}(U)_n$ with the ring $\mathcal{O}_{\mathfrak{X}}(U)\langle t_1, ..., t_n\rangle$ of convergent series in n-variables. Then, for any open sets $V \subseteq U$, the injection $\mathcal{O}_{\mathfrak{X}}(U) \hookrightarrow \mathcal{O}_{\mathfrak{X}}(V)$ induces injections $\mathcal{O}_{\mathfrak{X}}(U)\langle t_1, ..., t_n\rangle \hookrightarrow \mathcal{O}_{\mathfrak{X}}(V)\langle t_1, ..., t_n\rangle$. □

From Lemma 2.6, it follows that, for any $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ (an irreducible Fréchet sheaf or an irreducible ultrametric Banach sheaf) and open sets $V \subseteq U \subseteq X$, we have injections $\mathcal{O}_{\mathfrak{X}}(U)_* \hookrightarrow \mathcal{O}_{\mathfrak{X}}(V)_*$. Since X is irreducible, it follows that the nonempty open sets in X form a filtered inductive system (with open sets $U \leq U'$ if $U' \subseteq U$) , which we denote by I_X. We let $\mathbf{O}_{\mathfrak{X}}$ denote the simplicial ring which is the inductive limit $\mathbf{O}_{\mathfrak{X}} := \underset{U \in I_X}{\operatorname{colim}}\ \mathcal{O}_{\mathfrak{X}}(U)_*$. Further, for any point $p \in X$, we let $I_{X,p}$ denote the filtered inductive system of open sets $U_p \subseteq X$ such that $p \in U_p$. Then, we set

$$\mathcal{O}_{\mathfrak{X},p} := \underset{U_p \in I_{X,p}}{\operatorname{colim}}\ \mathcal{O}_{\mathfrak{X}}(U_p) \qquad \mathcal{O}_{\mathfrak{X},p*} := \underset{U_p \in I_{X,p}}{\operatorname{colim}}\ \mathcal{O}_{\mathfrak{X}}(U_p)_*. \tag{8}$$

Remark 2.7. Since X is irreducible, the system I_X of nonempty open sets in X is a filtered inductive system. By definition, we have $\mathbf{O}_{\mathfrak{X}} = \underset{U \in I_X}{\operatorname{colim}}\ \mathcal{O}_{\mathfrak{X}}(U)_*$, and hence, for any $n \geq 0$, we have $\mathbf{O}_{\mathfrak{X},n} = \underset{U \in I_X}{\operatorname{colim}}\ \mathcal{O}_{\mathfrak{X}}(U)_n$. Further, since I_X is filtered, given $U \in I_X$, the subsystem $I_{X,\geq U}$ of I_X consisting of all open sets V in X such that $V \subseteq U$ is a filtered inductive set cofinal in I_X. Hence, $\mathbf{O}_{\mathfrak{X},n} = \underset{V \in I_{X,\geq U}}{\operatorname{colim}}\ \mathcal{O}_{\mathfrak{X}}(V)_n$. From Lemma 2.6 it follows that, for any $V \in I_{X,\geq U}$, we have an injection $\mathcal{O}_{\mathfrak{X}}(U)_n \hookrightarrow \mathcal{O}_{\mathfrak{X}}(V)_n$ of rings and hence of abelian groups. Further, since the colimit is an exact functor on filtered inductive systems of abelian groups (see [18, Thm. 2.6.15]), the injections $\mathcal{O}_{\mathfrak{X}}(U)_n \hookrightarrow \mathcal{O}_{\mathfrak{X}}(V)_n$ for each $V \in I_{X,\geq U}$ induce an injection $\mathcal{O}_{\mathfrak{X}}(U)_n \hookrightarrow \mathbf{O}_{\mathfrak{X},n}$.

From Lemma 2.6, we know that given open sets $V \subseteq U$ with $V \neq \phi$, we have injections $\mathcal{O}_{\mathfrak{X}}(U) \hookrightarrow \mathcal{O}_{\mathfrak{X}}(V)$ and $\mathcal{O}_{\mathfrak{X}}(U)_* \hookrightarrow \mathcal{O}_{\mathfrak{X}}(V)_*$. Then, it follows from Remark 2.7 that, for each open set U in X, we have injections:

$$\mathcal{O}_{\mathfrak{X}}(U) \hookrightarrow \mathcal{O}_{\mathfrak{X}}(U)_* \hookrightarrow \mathbf{O}_{\mathfrak{X}} \tag{9}$$

into the inductive limit $\mathbf{O}_{\mathfrak{X}}$, where $\mathcal{O}_{\mathfrak{X}}(U)$ is treated trivially as a simplicial ring. Using this, we get injections of simplicial groups:

$$GL(\mathcal{O}_{\mathfrak{X}}(U)) \hookrightarrow GL(\mathcal{O}_{\mathfrak{X}}(U)_*) \hookrightarrow GL(\mathbf{O}_{\mathfrak{X}}), \tag{10}$$

where, once again, $GL(\mathcal{O}_{\mathfrak{X}}(U))$ is treated trivially as a simplicial group. Further, using (9), for any given point $p \in X$, we get injections $\mathcal{O}_{\mathfrak{X}}(U_p) \hookrightarrow \mathcal{O}_{\mathfrak{X}}(U_p)_* \hookrightarrow \mathbf{O}_{\mathfrak{X}}$ of simplicial rings for any open set U_p containing p. Again, as in Remark 2.7, using the fact that the colimit is an exact functor on filtered inductive systems of abelian groups, it follows that we have injections

$$\mathcal{O}_{\mathfrak{X},p} := \operatorname*{colim}_{U_p \in I_{X,p}} \mathcal{O}_{\mathfrak{X}}(U_p) \hookrightarrow \mathcal{O}_{\mathfrak{X},p*} := \operatorname*{colim}_{U_p \in I_{X,p}} \mathcal{O}_{\mathfrak{X}}(U_p)_* \hookrightarrow \mathbf{O}_{\mathfrak{X}} \tag{11}$$

of filtered colimits.

We will now introduce the classifying spaces $BGL(\mathcal{O}_{\mathfrak{X}}(U))$ and $BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)$ of the simplicial groups $GL(\mathcal{O}_{\mathfrak{X}}(U))$ and $GL(\mathcal{O}_{\mathfrak{X}}(U)_*)$ in (10) that will be used to define the K-theory groups of $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$. Therefore, we mention here some standard notation for simplicial groups. An ordinary group G may be treated as a category whose objects are the elements of G and whose morphisms are as follows: for g, $g' \in G$, the set of morphisms $\mathrm{Mor}(g, g')$ from g to g' is defined as

$$\mathrm{Mor}(g, g') := \{h \in G \mid hg = g'\}. \tag{12}$$

The simplicial nerve of this category is referred to as EG. The simplicial set EG is contractible and we consider the classifying space $BG := EG/G$. If G injects into a group G', it follows that G has a free action on the contractible space EG'. Then, it is well known (see, for instance, [6, § 1]) that, up to homotopy type (of the geometric realization), we might as well set $BG := EG'/G$. Further, the injection of groups $G \hookrightarrow G'$ leads to a Kan fibration of simplicial sets (see, for instance, [6, § 1])

$$BG := EG'/G \longrightarrow BG := EG'/G'. \tag{13}$$

We recall that a bisimplicial set $X = \{X_{n,k}\}_{n,k \geq 0}$ is a simplicial object in the category of simplicial sets, i.e. a collection of simplicial sets $X_n = \{X_{n,k}\}_{k \geq 0}$ connected by face maps and degeneracies which are themselves morphisms of simplicial sets. Given a bisimplicial set $X = \{X_{n,k}\}_{n,k \geq 0}$, it is well known that its diagonal $d(X) := \{X_{n,n}\}_{n \geq 0}$ is a simplicial set and by abuse of notation, for any bisimplicial set $X = \{X_{n,k}\}_{n,k \geq 0}$, we shall often refer to its diagonal as the "simplicial set X". For a simplicial group $G = \{G_n\}_{n \geq 0}$, EG is actually a bisimplicial set with $(EG)_n = EG_n$. If we have an injection $G = \{G_n\}_{n \geq 0} \hookrightarrow G' = \{G'_n\}_{n \geq 0}$ of simplicial groups, we can define $BG := EG'/G$ (see, for instance, [1, § 5]). Then, we have a Kan fibration (see, for instance, [6, § 1])

$$BG \longrightarrow BG' \tag{14}$$

of simplicial sets. Continuing from (10), we now consider the bisimplicial set $EGL(\mathbf{O}_{\mathfrak{X}})$ associated to the simplicial group $GL(\mathbf{O}_{\mathfrak{X}})$. For each open set U in X, we define:

$$\begin{aligned} BGL(\mathcal{O}_{\mathfrak{X}}(U)) &:= EGL(\mathbf{O}_{\mathfrak{X}})/GL(\mathcal{O}_{\mathfrak{X}}(U)) \\ BGL(\mathcal{O}_{\mathfrak{X}}(U)_*) &:= EGL(\mathbf{O}_{\mathfrak{X}})/GL(\mathcal{O}_{\mathfrak{X}}(U)_*) \end{aligned} \tag{15}$$

Let us denote by BGL the presheaf on X that associates to each open set U the simplicial set $BGL(\mathcal{O}_{\mathfrak{X}}(U))$ and by BGL^{top} the presheaf on X that associates to each open set U the simplicial set $BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)$. The presheaf that associates $BGL(\mathcal{O}_{\mathfrak{X}}(U))^+$ to the open set U will be denoted BGL^+, where $BGL(\mathcal{O}_{\mathfrak{X}}(U))^+$ is obtained by applying Quillen's plus construction (see [15]) to $BGL(\mathcal{O}_{\mathfrak{X}}(U))$. We will use $\mathbf{BGL}$, $\mathbf{BGL}^+$ and $\mathbf{BGL}^{top}$ to denote the sheafification of the presheaves BGL, BGL^+ and BGL^{top} respectively.

We now recall that a morphism $p : E \longrightarrow B$ of sheaves of simplicial sets on X is said to be a local fibration if, for each point $x \in X$, the induced morphism $p_x : E_x \longrightarrow B_x$ of stalks is a Kan fibration of simplicial sets. The morphism p is said to be a global fibration, if for any open sets $V \subseteq U \subseteq X$, we have a Kan fibration

$$E(U) \longrightarrow B(U) \times_{B(V)} E(V) \tag{16}$$

of simplicial sets. If E is a sheaf of simplicial sets on X such that the morphism $p_E : E \longrightarrow *$ is a global fibration, E is said to be flasque. It is well known that given any sheaf E of simplicial sets on X, there exists a flasque sheaf $R(E)$ and a weak equivalence

$$i_R : E \longrightarrow R(E) \tag{17}$$

of sheaves of simplicial sets. The association $E \mapsto R(E)$ can be shown to be functorial and R is referred to as the flasque resolution functor. Then, the generalized cohomology of Brown and Gersten [4] is defined to be

$$H^n(X, K) := \pi_{-n}(\Gamma(X, R(K))). \tag{18}$$

We mention here that simplicial sheaves carry the structure of a model category (see Jardine [13]). Hence, in the language of model categories, the sheaf $R(E)$ can be described as the "fibrant replacement" of E and the morphism $i_R : E \longrightarrow R(E)$ is a "trivial cofibration" (see Hovey [12, § 1.1] for definitions).

Proposition 2.8. *Let $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ be an irreducible Fréchet sheaf or an irreducible ultrametric Banach sheaf on an irreducible noetherian scheme of finite type over $\mathbb{C}$. Let U be an open subset of X and consider the simplicial sets*

$$BGL(\mathcal{O}_{\mathfrak{X}}(U)) := EGL(\mathbf{O}_{\mathfrak{X}})/GL(\mathcal{O}_{\mathfrak{X}}(U))$$
$$\text{and } BGL(\mathcal{O}_{\mathfrak{X}}(U)_*) := EGL(\mathbf{O}_{\mathfrak{X}})/GL(\mathcal{O}_{\mathfrak{X}}(U)_*).$$

Then, there exists a fibration

$$BGL(\mathcal{O}_{\mathfrak{X}}(U))^+ \longrightarrow BGL(\mathcal{O}_{\mathfrak{X}}(U)_*), \tag{19}$$

where $BGL(\mathcal{O}_{\mathfrak{X}}(U))^+$ is obtained from $BGL(\mathcal{O}_{\mathfrak{X}}(U))$ by applying Quillen's plus construction.

Proof. For each open set U in X, it follows from the injection $GL(\mathcal{O}_{\mathfrak{X}}(U)) \hookrightarrow GL(\mathcal{O}_{\mathfrak{X}}(U)_*)$ that we have a Kan fibration

$$BGL(\mathcal{O}_{\mathfrak{X}}(U)) \longrightarrow BGL(\mathcal{O}_{\mathfrak{X}}(U)_*). \tag{20}$$

If $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is an irreducible Fréchet sheaf, for each open set $U \subseteq X$, the ring $\mathcal{O}_{\mathfrak{X}}(U)$ is a Fréchet algebra. Then, from the work of Connes and Karoubi [8, 3.1], we know that applying the plus construction, we have an induced fibration:

$$BGL(\mathcal{O}_{\mathfrak{X}}(U))^+ \longrightarrow BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)^+ = BGL(\mathcal{O}_{\mathfrak{X}}(U)_*). \tag{21}$$

On the other hand, if $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is an irreducible ultrametric Banach sheaf, for each open set $U \subseteq X$, the ring $\mathcal{O}_{\mathfrak{X}}(U)$ is an ultrametric Banach algebra. Then, by the construction of the simplicial ring $\mathcal{O}_{\mathfrak{X}}(U)_*$ and [14, 5.3], we note that $\pi_1(BGL(\mathcal{O}_{\mathfrak{X}}(U)_*))$ is equal to the abelian group $K_1^{top}(\mathcal{O}_{\mathfrak{X}}(U)) = \pi_1(BGL(\mathcal{O}_{\mathfrak{X}}(U)_*))$, where K_1^{top} refers to the topological K-theory group. Since the group $\pi_1(BGL(\mathcal{O}_{\mathfrak{X}}(U)_*))$ is abelian for each open set U, its maximal perfect subgroup $\mathcal{P}\pi_1(BGL(\mathcal{O}_{\mathfrak{X}}(U)_*))$ is trivial. Hence, it follows (see Berrick [3]) that applying the Quillen plus construction to both $BGL(\mathcal{O}_{\mathfrak{X}}(U))$ and $BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)$ preserves the fibration in (20) Further, the fact that the maximal perfect subgroup $\mathcal{P}\pi_1(BGL(\mathcal{O}_{\mathfrak{X}}(U)_*))$ is trivial for each open set U also implies that $BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)^+ = BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)$. Hence, applying the plus construction to both $BGL(\mathcal{O}_{\mathfrak{X}}(U))$ and $BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)$ in (20) yields a fibration:

$$BGL(\mathcal{O}_{\mathfrak{X}}(U))^+ \longrightarrow BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)^+ = BGL(\mathcal{O}_{\mathfrak{X}}(U)_*). \tag{22}$$ □

Our next objective is to prove that the fibrations in Proposition 2.8 induce a local fibration $\mathbf{BGL}^+ \longrightarrow \mathbf{BGL}^{top}$ of sheaves. In order to do that, we will need the following Lemma, the proof of which is indicated in [12, § 7.4] (or see [17, Prop. 2.2]).

Lemma 2.9. *Let $\{f_i : E_i \longrightarrow B_i\}_{i \in I}$ be a family of fibrations of simplicial sets indexed over a filtered inductive system I. Then, setting $E = \operatorname{colim}_{i \in I} E_i$ and $B = \operatorname{colim}_{i \in I} B_i$, the induced morphism $f : E \longrightarrow B$ of filtered colimits is also a fibration.*

Proof. We know (see [12, 3.2.1]) that in the category **SSet** of simplicial sets, a morphism is a fibration if and only if it satisfies the right lifting property with respect to all canonical inclusions $\Lambda^r[n] \hookrightarrow \Delta[n]$ for $n > 0$ and $0 \leq r \leq n$ ($\Lambda^r[n]$ being the r-horn of the n-simplex $\Delta[n]$). Further, we note that $\Delta[n]$ and $\Lambda^r[n]$ are finite simplicial sets and hence they have the following important property: for every filtered inductive system $\{D_j\}_{j \in J}$ in **SSet** the natural maps

$$\begin{aligned} \operatorname{colim}_{j \in J} \mathbf{SSet}(\Delta[n], D_j) &\longrightarrow \mathbf{SSet}(\Delta[n], \operatorname{colim}_{j \in J} D_j) \\ \operatorname{colim}_{j \in J} \mathbf{SSet}(\Lambda^r[n], D_j) &\longrightarrow \mathbf{SSet}(\Lambda^r[n], \operatorname{colim}_{j \in J} D_j) \end{aligned} \tag{23}$$

are isomorphisms.

Therefore, we consider a canonical inclusion $\Lambda^r[n] \hookrightarrow \Delta[n]$ and a commutative diagram

$$\begin{array}{ccc} \Lambda^r[n] & \xrightarrow{g} & E = \operatorname{colim}_{i\in I} E_i \\ {\scriptstyle i}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ \Delta[n] & \xrightarrow{h} & B = \operatorname{colim}_{i\in I} B_i. \end{array} \tag{24}$$

Using the property in (23) and the fact that the indexing set I is filtered, we can choose some $i_0 \in I$ such that $g : \Lambda^r[n] \longrightarrow E$ (resp. $h : \Delta[n] \longrightarrow B$) factors through $g' : \Lambda^r[n] \longrightarrow E_{i_0}$ (resp. $h' : \Delta[n] \longrightarrow B_{i_0}$). Since $f_{i_0} : E_{i_0} \longrightarrow B_{i_0}$ is a fibration, there exists a lifting morphism $q : \Delta[n] \longrightarrow E_{i_0}$ such that $f_{i_0} q = h'$ and $g' = qi$. It follows that there exists a lifting $\Delta[n] \longrightarrow E$ in the commutative square (24) and hence $f : E \longrightarrow B$ is a fibration. □

Proposition 2.10. *The morphism of sheaves*

$$\mathbf{BGL}^+ \longrightarrow \mathbf{BGL}^{top} \tag{25}$$

is a local fibration.

Proof. We have to check that the induced morphism on stalks at each point of X is a Kan fibration. Choose a point p in X and let $I_{X,p}$ denote the filtered inductive system of open sets of X containing p. From Proposition 2.8, we know that for each open set $U_p \in I_{X,p}$, we have a fibration

$$BGL(\mathcal{O}_{\mathfrak{X}}(U_p))^+ \longrightarrow BGL(\mathcal{O}_{\mathfrak{X}}(U_p)_*)^+ = BGL(\mathcal{O}_{\mathfrak{X}}(U_p)_*). \tag{26}$$

Since the filtered colimit of fibrations is a fibration (using Lemma 2.9 above) we have a fibration

$$\operatorname*{colim}_{U_p\in I_{X,p}} BGL(\mathcal{O}_{\mathfrak{X}}(U_p))^+ \longrightarrow \operatorname*{colim}_{U_p\in I_{X,p}} BGL(\mathcal{O}_{\mathfrak{X}}(U_p)_*) \tag{27}$$

for each point $p \in X$. Since $\mathbf{BGL}^+$ and $\mathbf{BGL}^{top}$ are the sheafifications of BGL^+ and BGL^{top} respectively (and therefore have the same stalks as BGL^+ and BGL^{top} respectively), it is clear from (27) that there is a local fibration $\mathbf{BGL}^+ \longrightarrow \mathbf{BGL}^{top}$ of sheaves. □

Definition 2.11. Let $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ be an irreducible Fréchet sheaf or an irreducible ultrametric Banach sheaf on an irreducible noetherian scheme X of finite type over $\mathbb{C}$. Let $\mathbf{Z}$ denote the constant sheaf on X given by $\mathbb{Z}$. Then, the algebraic K-theory of $\mathfrak{X}$ is defined to be

$$K_n^{alg}(\mathfrak{X}) := H^{-n}(X, \mathbf{Z} \times \mathbf{BGL}^+). \tag{28}$$

Consider the sheafification $\mathbf{GL}^{rel+}$ of the presheaf which associates to an open set U in X the homotopy fiber

$$U \mapsto [\mathbb{Z} \times BGL(\mathcal{O}_{\mathfrak{X}}(U)^+), \mathbb{Z} \times BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)] \tag{29}$$

of the fibration $\mathbb{Z} \times BGL(\mathcal{O}_{\mathfrak{X}}(U))^+ \longrightarrow \mathbb{Z} \times BGL(\mathcal{O}_{\mathfrak{X}}(U)_*)$. We define the topological and relative K theories of $\mathfrak{X}$ to be

$$K_n^{top}(\mathfrak{X}) := H^{-n}(X, \mathbf{Z} \times \mathbf{BGL}^{top}), \quad K_n^{rel}(\mathfrak{X}) := H^{-n}(X, \mathbf{GL}^{rel+}). \tag{30}$$

Proposition 2.12. *Let $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ be an irreducible Fréchet sheaf or an irreducible ultrametric Banach sheaf on an irreducible noetherian scheme X of finite type over $\mathbb{C}$. Then there is a long exact sequence of K-theory groups*

$$\ldots \longrightarrow K_n^{rel}(\mathfrak{X}) \longrightarrow K_n^{alg}(\mathfrak{X}) \longrightarrow K_n^{top}(\mathfrak{X}) \longrightarrow K_{n-1}^{rel}(\mathfrak{X}) \longrightarrow \ldots \tag{31}$$

Proof. Using the result of Proposition 2.10, we have a local fibration of sheaves:

$$\mathbf{BGL}^+ \longrightarrow \mathbf{BGL}^{top} \tag{32}$$

with fiber $\mathbf{GL}^{rel+}$. Therefore, we have a local fibration of sheaves:

$$\mathbf{Z} \times \mathbf{BGL}^+ \longrightarrow \mathbf{Z} \times \mathbf{BGL}^{top} \tag{33}$$

with fiber $\mathbf{GL}^{rel+}$. Using [5, Thm. 7], the local fibration (33) gives rise to a long exact sequence

$$\begin{aligned} \ldots \longrightarrow H^m(X, \mathbf{GL}^{rel+}) \longrightarrow H^m(X, \mathbf{Z} \times \mathbf{BGL}^+) \longrightarrow \\ H^m(X, \mathbf{Z} \times \mathbf{BGL}^{top}) \longrightarrow H^{m+1}(X, \mathbf{GL}^{rel+}) \longrightarrow \ldots \end{aligned} \tag{34}$$

By Definition 2.11, $K_m^{alg}(\mathfrak{X}) = H^{-m}(X, \mathbf{Z} \times \mathbf{BGL}^+)$, $K_m^{top}(\mathfrak{X}) = H^{-m}(X, \mathbf{Z} \times \mathbf{BGL}^{top})$ and $K_m^{rel}(\mathfrak{X}) = H^{-m}(X, \mathbf{GL}^{rel+})$ and hence the required long exact sequence (31) is a restatement of (34). □

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