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Homology of group von Neumann algebras

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Abstract. The structure of the group von Neumann algebra $\mathcal{N}(G)$ is considered as a module over the group ring $\mathbb{C}G$ for various groups G. In particular, the question of when the group von Neumann algebra will be a flat module is studied. Group homology calculations are used to investigate this question. The main result is that for the class of torsion-free elementary amenable groups, the module $\mathcal{N}(G)$ is flat if and only if G is locally virtually cyclic.

1. Introduction

In [15], Wolfgang Lück makes two conjectures seeking to connect the structure of a group G with properties of the corresponding group von Neumann algebra as a module over the group ring $\mathbb{C}G$. The first conjecture seeks to use $\mathcal{N}(G)$ to classify when a group is amenable.

Conjecture 1.1. *A group G is amenable if and only if $\mathcal{N}(G)$ is dimension-flat over $\mathbb{C}G$.*

It is already known that if G is amenable, then $\mathcal{N}(G)$ is dimension-flat over $\mathbb{C}G$. The converse is still open. However, it is settled in the case when G has a nonabelian free subgroup (see [15, Thm. 6.37]). The second conjecture, which is the primary motivation for this work, guesses for which groups $\mathcal{N}(G)$ will be a flat module.

Conjecture 1.2. *A group G is locally virtually cyclic if and only if $\mathcal{N}(G)$ is flat over $\mathbb{C}G$.*

The forward implication of this conjecture will be proved below. The converse is still open in general, but it will be proved for the class of torsion-free elementary amenable groups.

2. Locally virtually cyclic groups

In this section it is proved that if G is locally virtually cyclic, then $\mathcal{N}(G)$ is a flat $\mathbb{C}G$-module. This class of groups is defined as follows:

Definition 2.1. A group G is called *virtually cyclic* if either it is finite or there exists an infinite cyclic normal subgroup H such that G/H is finite. A group is called *locally virtually cyclic* (LVC) if every finitely generated subgroup is virtually cyclic.

The flatness of $\mathcal{N}(G)$ is readily apparent for certain special cases of LVC groups. If G is finite, then $\mathcal{N}(G) = \mathbb{C}G$ is a free (and hence flat) module. Furthermore, if G is locally finite then the group ring $\mathbb{C}G$ is a "von Neumann regular" ring (as a consequence of Maschke's theorem and the Artin–Wedderburn theorem), which implies every $\mathbb{C}G$-module is flat. The special case of $G = \mathbb{Z}$ requires a lemma regarding modules over PIDs.

Lemma 2.2. *Let R be a PID, and let M be an R-module. If M is R-torsion-free, then M is a flat R-module.*

Proof. Express M as a direct limit of its finitely generated submodules;

$$M \cong \varinjlim M_i.$$

Since M is R-torsion-free, so is every submodule M_i. By the standard structure theorem for finitely generated modules over PIDs, it follows that each M_i is a free module. Since M is the direct limit of free modules, it must be a flat module (see [17, Thm. 3.4]). □

Theorem 2.3. *If $G = \mathbb{Z}$, then the module $\mathcal{N}(G)$ is flat over the ring $\mathbb{C}G$.*

Proof. For the group $G = \mathbb{Z}$ the ring $\mathbb{C}G$ is a PID. By the previous lemma, it suffices to show that $\mathcal{N}(G)$ is $\mathbb{C}G$-torsion-free. This follows from the zero divisor conjecture [16]. □

There is one remaining lemma needed before proving half of Conjecture 1.2.

Lemma 2.4. *If G is a group with an infinite cyclic subgroup H and $1 \leq p \in \mathbb{R}$, then $0 \neq \alpha \in \mathbb{C}H$ and $0 \neq \beta \in \ell^p(G)$ imply $\alpha\beta \neq 0$.*

Proof. Let $0 \neq \alpha \in \mathbb{C}H$ and $0 \neq \beta \in \ell^p(G)$. Let T be a transversal for H in G. Then $\beta = \sum_{t \in T} \alpha_t t$ for some $\alpha_t \in \ell^p(H)$. Suppose, to the contrary, that $\alpha\beta = 0$. Note that $\alpha\beta = \alpha \sum_{t \in T} \alpha_t t = \sum_{t \in T} (\alpha\alpha_t)t = 0$ if and only if $\alpha\alpha_t = 0$ for all $t \in T$. This is true if and only if $\alpha_t = 0$ for all $t \in T$ (see [16]), which is true if and only if $\beta = 0$; a contradiction. Therefore, $\alpha\beta \neq 0$. □

The theorem below establishes half of Conjecture 1.2.

Theorem 2.5. *If G is locally virtually cyclic, then $\mathcal{N}(G)$ is a flat $\mathbb{C}G$-module.*

Proof. Let $M = \mathcal{N}(G)$ and B be any $\mathbb{C}G$-module. It suffices to show that $\mathrm{Tor}_1^{\mathbb{C}G}(M, B) = 0$. Since G is locally virtually cyclic, G can be expressed as a direct limit of virtually cyclic groups: $G = \varinjlim G_i$. Now make the following identifications (see [5, Prop. 2.2 and p. 12]):

$$\mathrm{Tor}_1^{\mathbb{C}G}(M, B) \cong H_1(G, M \otimes B) \cong \varinjlim H_1(G_i, M \otimes B).$$

Hence, it suffices to show $H_1(G_i, M \otimes B) = 0$. This is trivially true if G_i is finite. Suppose G_i is infinite. Since G_i is infinite virtually cyclic, there must be a normal infinite cyclic subgroup of finite index. In other words, there is a short exact sequence $1 \to K_i \to G_i \to Q_i \to 1$, where $K_i \cong \mathbb{Z}$ and Q_i is finite. If $N = M \otimes B$, then there is the following exact sequence (see [5, §7.7]):

$$H_2(Q_i, N_{K_i}) \to H_1(K_i, N)_{Q_i} \to H_1(G_i, N) \to H_1(Q_i, N_{K_i}).$$

Since Q_i is finite, we have

$$H_2(Q_i, N_{K_i}) \cong H_1(Q_i, N_{K_i}) \cong 0 \quad \text{and} \quad H_1(G_i, N) \cong H_1(K_i, N)_{Q_i}.$$

Thus, it suffices to show that $H_1(K_i, N) = 0$. Rewrite this as $\mathrm{Tor}_1^{\mathbb{C}K_i}(M, B)$, which must be trivial since $\mathbb{C}K_i$ is a PID and M does not have any $\mathbb{C}K_i$-torsion. This completes the proof. □

3. Free abelian groups

Next consider the converse of Theorem 2.5; it is conjectured that if G is not locally virtually cyclic, then $\mathcal{N}(G)$ is not flat as a module over $\mathbb{C}G$. The first special case considered below is the free abelian group $\mathbb{Z} \oplus \mathbb{Z}$.

Theorem 3.1. *If $G = \mathbb{Z} \oplus \mathbb{Z}$, then $H_1(G, \mathcal{N}(G)) \neq 0$.*

Proof. Identify the torus T^2 with the standard quotient space of the square $[-\pi, \pi] \times [-\pi, \pi]$. By using Fourier transforms, $\mathcal{N}(G)$ can be identified with $L^\infty(T^2)$. Under this identification, elements of $\mathbb{C}G$ are expressed with linear combinations of functions of the form $e^{-inx}e^{-imy}$ for integers m and n. To calculate $H_1(G, \mathcal{N}(G))$, construct a free $\mathbb{C}G$-resolution of $\mathbb{C}$, such as the following:

$$0 \to \mathbb{C}G \xrightarrow{\gamma} \mathbb{C}G^2 \xrightarrow{\beta} \mathbb{C}G \xrightarrow{\epsilon} \mathbb{C} \to 0,$$

where

$$\gamma : x = \sum a_{n,m} \cdot t^n s^m \mapsto (-(s-1)x, (t-1)x)$$

and

$$\beta : (x, y) \mapsto (t-1)x + (s-1)y,$$

for $G = \langle t, s \mid ts = st \rangle$. Next apply the functor $\mathcal{N}(G) \otimes -$ to obtain a deleted complex:

$$0 \to L^\infty(T^2) \otimes_{\mathbb{C}G} \mathbb{C}G \xrightarrow{\gamma'} L^\infty(T^2) \otimes_{\mathbb{C}G} \mathbb{C}G^2 \xrightarrow{\beta'} L^\infty(T^2) \otimes_{\mathbb{C}G} \mathbb{C}G \to 0.$$

This leads to the complex

$$0 \to L^\infty(T^2) \xrightarrow{\gamma'} L^\infty(T^2)^2 \xrightarrow{\beta'} L^\infty(T^2) \to 0,$$

where

$$\gamma' : f \mapsto (f \cdot (1 - e^{-iy}), f \cdot (e^{-ix} - 1))$$

and

$$\beta' : (g, h) \mapsto g \cdot (e^{-ix} - 1) + h \cdot (e^{-iy} - 1).$$

This complex can be used to calculate the first group homology:
$$H_1(G, \mathcal{N}(G)) \cong \ker(\beta')/\operatorname{Im}(\gamma').$$
Suppose $(g,h) \in \ker(\beta')$. Then $g \cdot (e^{-ix}-1) + h \cdot (e^{-iy}-1) = 0$. Define
$$f = \frac{g}{1-e^{-iy}} = \frac{h}{e^{-ix}-1}.$$
The group homology vanishes if every such f must be in $L^\infty(T^2)$. To show the group homology is nontrivial it suffices to show there exists $(g,h) \in \ker(\beta')$ such that $f \notin L^\infty(T^2)$. In other words, it suffices to make the following construction:
$$h \in L^\infty(T^2), \quad f = \frac{h}{e^{-ix}-1} \notin L^\infty(T^2), \quad g = \frac{h \cdot (1-e^{-iy})}{e^{-ix}-1} \in L^\infty(T^2).$$
Define
$$A = \{(x,y) \in T^2 \mid \cos y > \cos x\},$$
i.e., $A = \{(x,y) \in T^2 \mid -|x| < y < |x|\}$. Let $h = \chi_A$. Clearly, $h \in L^\infty(T^2)$. Since A contains open neighborhoods around points with arbitrarily small x-values, it follows that $f \notin L^\infty(T^2)$.

Claim: $g \in L^\infty(T^2)$.

This can be proved by showing the stronger claim: $|g| \leq 1$ for all $(x,y) \in T^2$. Suppose to the contrary that $|g(x,y)| > 1$ for some (x,y). Then (x,y) must be in A. And:
$$\begin{aligned}\left|\frac{1-e^{-iy}}{e^{-ix}-1}\right| > 1 &\implies |e^{-iy}-1| > |e^{-ix}-1| \\ &\implies (\cos y - 1)^2 + (\sin y)^2 > (\cos x - 1)^2 + (\sin x)^2 \\ &\implies 2 - 2\cos y > 2 - 2\cos x \\ &\implies \cos y < \cos x \\ &\implies (x,y) \notin A.\end{aligned}$$
This is a contradiction, which proves the claim and finishes the proof. □

4. Connections: $\mathcal{N}(G)$, $\ell^2(G)$, subgroups, and quotient groups

The following connection between groups and subgroups can be found in [15, Thm. 6.29].

Theorem 4.1. *If $\mathcal{N}(G)$ is flat over $\mathbb{C}G$ and H is a subgroup of G, then $\mathcal{N}(H)$ is flat over $\mathbb{C}H$.*

This theorem will be very useful in the final section below. Conjecture 1.2 will be verified for special examples of elementary amenable groups (such as Baumslag–Solitar groups) using explicit calculations, and then the preceding theorem will be used to expand upon those results. For example, the main result of the last section implies:

Corollary 4.2. *If G is any group with a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, then $\mathcal{N}(G)$ is not flat over $\mathbb{C}G$.*

In an important upcoming calculation, it will be easier to work with the module $\ell^2(G)$ rather than $\mathcal{N}(G)$. Therefore, it is useful to know that $\ell^2(G)$ is a flat $\mathcal{N}(G)$-module, as the remaining results of this section will establish.

Lemma 4.3. *Any finitely-generated submodule of a projective $\mathcal{N}(G)$-module is projective.*

Proof. This follows from the fact that $\mathcal{N}(G)$ is a semi-hereditary ring; see [15, Thm. 6.7 (1)]. □

Lemma 4.4. *Any submodule of a free $\mathcal{N}(G)$-module is flat.*

Proof. Let M be a submodule of a free $\mathcal{N}(G)$-module. Then M can be expressed as a direct limit of its finitely generated submodules, each of which must be projective by the previous lemma. Since M is a direct limit of projective modules, it must be a flat module. □

Lemma 4.5. *The $\mathcal{N}(G)$-module $\mathcal{U}(G)$ of affiliated operators can be expressed as a direct limit of free modules.*

Proof. Let $X = \{x \in \mathcal{N}(G) \mid x \text{ is a nonzero divisor}\}$. For every $x \in X$, define $F_x = x^{-1}\mathcal{N}(G)$, which is a free $\mathcal{N}(G)$-module and a submodule of $\mathcal{U}(G)$. In fact, $\mathcal{U}(G) = \bigcup_{x\in X} F_x$. It now suffices to show that $\{F_x\}$ is a directed system. That is, for any $x, y \in X$, it suffices to show there exists a cofinal F_z such that $F_x \subseteq F_z$ and $F_y \subseteq F_z$. Since $\mathcal{N}(G)$ satisfies the Ore condition, there exist $w, \alpha \in \mathcal{N}(G)$ such that $w \in X$ and $wy = \alpha x$. Hence, for any $x^{-1}\beta \in x^{-1}\mathcal{N}(G)$:

$$x^{-1}\beta = y^{-1}yx^{-1}\beta = y^{-1}w^{-1}x\beta \in (wy)^{-1}\mathcal{N}(G).$$

And for any $y^{-1}\beta \in y^{-1}\mathcal{N}(G)$:

$$y^{-1}\beta = y^{-1}w^{-1}w\beta \in (wy)^{-1}\mathcal{N}(G).$$

Thus, define $z = wy$ to produce the desired result. □

Theorem 4.6. *Let G be a group. Then $\ell^2(G)$ is a flat $\mathcal{N}(G)$-module.*

Proof. Since $\ell^2(G)$ can be embedded in $\mathcal{U}(G)$, the module $\ell^2(G)$ is a submodule of $\mathcal{U}(G)$. Hence:

$$\ell^2(G) = \mathcal{U}(G) \cap \ell^2(G) = \Big(\bigcup_{x\in X} F_x\Big) \cap \ell^2(G) = \bigcup_{x\in X} \big(F_x \cap \ell^2(G)\big).$$

Since F_x is a finitely-generated free module, Lemma 4.4 implies that $F_x \cap \ell^2(G)$ is a flat module. Therefore, $\ell^2(G)$ is a direct limit of flat modules, making it a flat module. □

Corollary 4.7. *If $\mathcal{N}(G)$ is flat over $\mathbb{C}G$, then $\ell^2(G)$ is flat over $\mathbb{C}G$.*

5. Baumslag–Solitar groups

5.1. **Introducing the relevant groups.** In a later section, Conjecture 1.2 will be proved for all torsion-free elementary amenable groups. An important building block for that result will be the special case of certain Baumslag–Solitar groups. These groups were first introduced by Gilbert Baumslag and Donald Solitar in 1962 to provide examples of finitely presented Hopfian groups [2]. These groups are defined with the following presentation: for natural numbers m and n, define $B(m,n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle$. If $m \neq 1$ and $n \neq 1$, then $B(m,n)$ has a nonabelian free subgroup [6]. The focus below will be only on the amenable Baumslag–Solitar groups $B(1,n)$. These groups have cohomological dimension 2 (see [4, Thm. 7]). In fact, they are the only finitely-generated elementary amenable groups with this property (see [11, Thm. 3]). The groups $B(1,n)$ can also be expressed as a semi-direct product [7]: $B(1,n) \cong \mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z}$. In particular, $B(1,n)$ has a normal subgroup H isomorphic to $\mathbb{Z}[\frac{1}{n}]$ with quotient $G/H \cong \mathbb{Z}$, and the generator of the quotient acts on H by multiplication by n.

Another class of groups, which contains the groups $B(1,n)$, is defined as follows: for any natural numbers m and n, let $G_{m,n} = \mathbb{Z}[\frac{1}{mn}] \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts by multiplication with $\frac{m}{n}$. The group $G_{m,n}$ has a normal subgroup H isomorphic to $\mathbb{Z}[\frac{1}{mn}]$ and quotient $G/H \cong \mathbb{Z}$. These groups were first introduced within the context of trying to classify all groups of cohomological dimension 2. For $G = G_{2,3}$, Bieri posed the question of whether $\mathrm{cd}(G) = 2$ or $\mathrm{cd}(G) = 3$ (see [4, p. 112]). Gildenhuys provided the answer that $\mathrm{cd}(G) = 3$ (see [7, Thm. 4]).

Before proving Conjecture 1.2 for torsion-free elementary amenable groups, the result will be established for $G_{m,n}$. To see why the groups $B(1,n)$ and $G_{m,n}$ are so relevant to the structure of elementary amenable groups, see [7, Thm. 5] and Lemma 6.2 below. The result for $G_{m,n}$ will be accomplished by investigating the first group homology group $H_1(G, \ell^2(G))$. While it is possible to do this homology calculation for $G = B(1,n)$ directly by using Fox derivatives to build a free $\mathbb{C}G$-resolution of $\mathbb{C}$, the more general case of $G = G_{m,n}$ requires a less direct approach since these groups are typically not finitely presented [3].

5.2. **Setting up the calculation.** For notation, let $G = G_{m,n}$, let H be the normal subgroup $\mathbb{Z}[\frac{1}{mn}]$, and let Q denote the quotient $G/H \cong \mathbb{Z}$. Let M denote the left $\mathbb{C}G$-module $\ell^2(G)$. Let $t \in G$ be such that $\bar{t}$ generates Q. For a natural number i let $h_i = (\frac{1}{mn})^i$, and define $H_i = \langle h_i \rangle \leq H$. Then $H_i \cong \mathbb{Z}$ and $H = \bigcup H_i$. For the group H, let $\Delta(H)$ denote the augmentation ideal of H, and let M_H be the quotient $M/\Delta(H)M$.

Lemma 5.3. *If $H_1(Q, M_H) \neq 0$, then $H_1(G, M) \neq 0$.*

Proof. The short exact sequence of groups $1 \to H \to G \to Q \to 1$ leads to an exact sequence of group homology (see [5, §7.7]):

$$H_1(H, M)_Q \to H_1(G, M) \to H_1(Q, M_H) \to 0.$$

It now suffices to show $H_1(H, M)$ is trivial. Since

$$H_1(H, M) \cong \varinjlim H_1(H_i, M)$$

(see [5, p. 121]), it suffices to show $H_1(H_i, M)$ is trivial for every $i \in \mathbb{N}$. Since $H = \langle h_i \rangle \cong \mathbb{Z}$, there is the following free $\mathbb{C}H_i$-resolution of $\mathbb{C}$:

$$0 \to \mathbb{C}H_i \xrightarrow{f} \mathbb{C}H_i \xrightarrow{\epsilon} \mathbb{C}.$$

The map ϵ represents the augmentation map, and f is multiplication by $h_i - 1$. This induces the complex

$$0 \to M \xrightarrow{f_*} M \to 0.$$

To show $H_1(H_i, M) = 0$, it suffices to show $\ker(f_*) = 0$, and this follows from [13, Thm. 2]. □

Lemma 5.4. *If there exists $\alpha \in M \setminus \Delta(H)M$ such that $(t-1)\alpha \in \Delta(H)M$, then $H_1(G, M) \neq 0$.*

Proof. By the previous lemma, it suffices to show that $H_1(Q, M_H) \neq 0$. Since $Q = \langle \overline{t} \rangle \cong \mathbb{Z}$, there is the following free $\mathbb{C}Q$-resolution of $\mathbb{C}$:

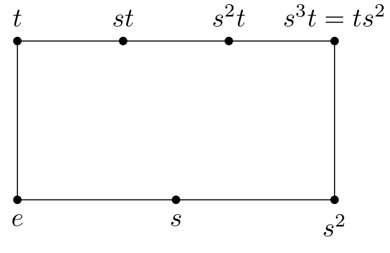
$$0 \to \mathbb{C}Q \xrightarrow{g} \mathbb{C}Q \xrightarrow{\epsilon} \mathbb{C}.$$

The map ϵ represents the augmentation map, and g is multiplication by $\overline{t} - 1$. This induces the complex

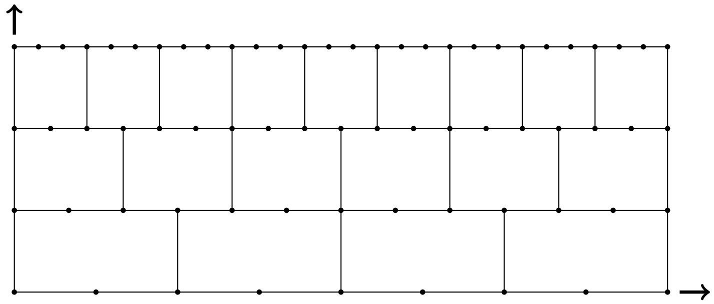
$$0 \to M_H \xrightarrow{g_*} M_H \to 0.$$

It suffices to show that $\ker(g_*) \neq 0$. That is, it will suffice to find a nontrivial $\overline{\alpha} \in M_H$ such that $(t-1)\overline{\alpha} = 0$. This is true if there exists $\alpha \in M \setminus \Delta(H)M$ such that $(t-1)\alpha \in \Delta(H)M$, which establishes the result. □

5.5. **Introducing Cayley graphs.** The previous two lemmas establish a straight-forward way to demonstrate that $H_1(G, \ell^2(G)) \neq 0$. To help visualize the support of the element $\alpha \in \ell^2(G)$ that will be constructed, consider the Cayley graph of G with respect to the generating set $\{s, t\}$, where $s = h_1$. In the Cayley graph, let rightward edges correspond to left-multiplication by s, and let upward edges correspond to left-multiplication by t. One of the defining relations on the generating set is $ts^n t^{-1} = t^m$. In the case of $G_{3,2}$, this corresponds to a subgraph of the Cayley graph:



Let Γ be the subgraph of the Cayley graph generated from the vertices $\{s^j t^i \mid i, j \geq 0\}$, a portion of which is pictured below.



See [3, p. 49] for a full presentation of $G_{m,n}$ with respect to the generators s and t. In particular, every relator other than $ts^n t^{-1} = t^m$ is of the form $[s, t^i s t^{-i}] = e$ for natural numbers i, which will not create any duplications or further edges between the vertices pictured.

5.6. **Constructing $\boldsymbol{\alpha}$.** The goal is to construct an element α in $\ell^2(G)$ that satisfies the requirements of Lemma 5.4. The support of α will be contained within the subgraph Γ of the Cayley graph. In particular, other than e and s^{mn} (pictured on the bottom row of Γ), we will restrict our attention to the vertices $V = \{v \in \Gamma \mid tv \in \Gamma \text{ and } t^{-1}v \in \Gamma\}$. That is, V is the collection of vertices in Γ with both upward and downward edges. To build the subset of V that will be featured in the support of α, we will construct sets denoted by X_i. Whenever "distance" is mentioned below, it will refer to the standard Cayley graph distance restricted to individual rows of Γ.

Theorem 5.7. *Let $p = p_{m,n}$ be the smallest natural number such that $(\frac{m}{n})^p > 2$. For all $i \geq 0$, there exist subsets $X_i \subseteq V$ that satisfy the following properties:*

(i) *Vertices in X_i have the form $s^j t^i$. That is, vertices in X_i are on the i-th row of Γ. The notation g_{ir} will represent the r-th element of X_i, moving from left to right in Γ.*

(ii) *The cardinalities $|X_i|$ are non-decreasing powers of two. In particular, for any $k \geq 0$, if $kp \leq i < (k+1)p$, then $|X_i| = 2^{k+1}$.*

(iii) *If $k \geq 0$, $kp < i < (k+1)p$, and $1 \leq r \leq |X_i|$, then the distance between g_{ir} and $t \cdot g_{i-1,r}$ is no larger than $(m^2 + mn)/2$.*

(iv) *If $k \geq 0$, $i = (k+1)p$, $1 \leq r \leq |X_i|$, and r is odd, then the distance between g_{ir} and $t \cdot g_{i-1,\frac{r+1}{2}}$ is no larger than $(m^2 + mn)/2$.*

Note 5.8. Before proving the theorem above, some comments are in order. Condition (ii) hints at the relevance of the constant p defined in the theorem; this constant is related to the rate at which $|X_i| \to \infty$. Condition (iii) is relevant to sets X_i that have the same size as X_{i-1}. For these sets, the r-th element in X_i is approximately above the r-th vertex in X_{i-1}. Condition (iv) is relevant to sets X_i that have twice as many elements as X_{i-1}. For these sets, the odd-numbered vertices are approximately above vertices in X_{i-1}.

Proof. For any point v in Γ, let $F(v)$ denote the point that is "directly above" v. More precisely, if v is on row i of Γ with a distance of d from t^i, then let $F(v)$ be the point on row $i+1$ of Γ of distance $\frac{m}{n}\cdot d$ from t^{i+1}. If v is a vertex in Γ, then $F(v)$ will also be a vertex if $t\cdot v$ is in Γ. However, even in that case, $F(v)$ will probably not be in V because it may not have an upward edge within Γ. Thus, define $T(v)$ to be nearest element of V to the point $F(v)$. More generally, for any $q\geq 1$, let $T_q(v)$ be the nearest element in V to $F^q(v)$, which will be q rows higher than v. If $F^q(v)$ is midway between two vertices in V, then choose the one on the right.

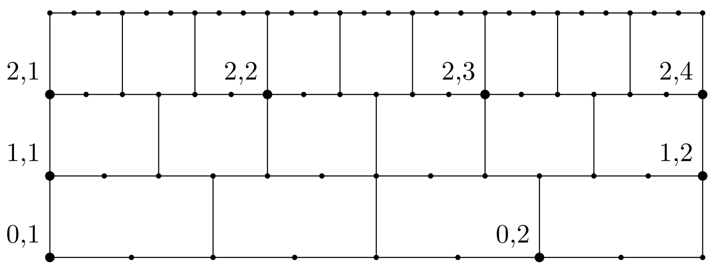
To begin the inductive definition of the sets X_i, let $X_0=\{e,s^{mn}\}$. For any nonnegative integer k and any integer i between $kp+1$ and $(k+1)p-1$, let $X_i:=\{T_{i-kp}(v)\mid v\in X_{kp}\}$. At the next multiple of p, there is a cardinality jump:

$$\begin{aligned}X_{(k+1)p}&=A_{(k+1)p}\cup B_{(k+1)p}\\&:=\{T_p(v)\mid v\in X_{kp}\}\cup\{s^{mn}T_p(v)\mid v\in X_{kp}\}.\end{aligned}$$

It is important to note that on any row of Γ the vertices in V are spaced a distance of mn apart. For consecutive elements v and w of X_{kp}, the distance between $F^p(v)$ and $F^p(w)$ will be at least $(\frac{m}{n})^p\cdot mn>2mn$, which guarantees that $s^{mn}T_p(v)$ will be between $T_p(v)$ and $T_p(w)$ in $X_{(k+1)p}$. That is, the elements of $X_{(k+1)p}$, moving from left to right in Γ, will alternate between $A_{(k+1)p}$ and $B_{(k+1)p}$.

To paraphrase the definition of $\{X_i\}$ informally, after X_{kp} the next $p-1$ sets X_i will include only the i-th row vertices in V "approximately above" the vertices in X_{kp}. However, for the p-th row after X_{kp} we will include not only the vertices closest to being above those in X_{kp}, but we will also include other vertices in V that are in between. There will not necessarily be these "in between" vertices available at every step, which is why it takes p steps before the cardinality can safely be doubled.

For example, if $m=3$ and $n=2$, then $p=2$, and the following picture shows the first three rows of Γ with elements g_{ir} labeled as i,r:



The definition of $\{X_i\}$ clearly satisfies condition (i). Since the vertices specified above are all distinct, the definition also satisfies condition (ii). For condition (iii), let $k\geq 0$, $kp<i<(k+1)p$, and $1\leq r\leq|X_i|$. Then, there exists $v\in X_{kp}$ such that $g_{i-1,r}=T_{i-1-kp}(v)$ and $g_{i,r}=T_{i-kp}(v)$. Since vertices in V are spaced a distance of mn apart, the vertex $g_{i-1,r}$ is no farther than $\frac{mn}{2}$ from $F^{i-1-kp}(v)$. Looking up one row, the vertices $t \cdot g_{i-1,r}$ and $F^{i-kp}(v)$ are at most a distance of $\frac{m}{n} \cdot \frac{mn}{2}$ apart. Also, the distance between $g_{i,r}$ and $F^{i-kp}(v)$ is no greater than $\frac{mn}{2}$. By the triangle inequality, the vertices $g_{i-1,r}$ and $g_{i,r}$ are at most a distance of $\frac{m}{n} \cdot \frac{mn}{2} + \frac{mn}{2}$ apart. Condition (iv) is a similar application of the triangle inequality. □

Now that the sets X_i are established, α can be defined.

Definition 5.9. Let the element $\alpha = \sum_{g \in G} a_g g \in \ell^\infty(G)$ have the following coefficients. If $g = g_{ir} \in X_i$, then let

$$a_g = a_{ir} = \begin{cases} \frac{1}{|X_i|} & \text{if } 1 \le r \le \frac{1}{2}|X_i|, \\ \frac{-1}{|X_i|} & \text{if } \frac{1}{2}|X_i| + 1 \le r \le |X_i|. \end{cases}$$

For all other elements g of the group, let $a_g = 0$.

For example, if $|X_i|$ had four elements, then the coefficients would be $a_{i1} = \frac{1}{4}$, $a_{i2} = \frac{1}{4}$, $a_{i3} = \frac{-1}{4}$, and $a_{i4} = \frac{-1}{4}$.

5.10. **Nontrivial first homology group.**

Lemma 5.11. *For the element $\alpha \in \ell^\infty(G)$ constructed above, $\alpha \in M$.*

Proof. The support of α is $\operatorname{supp}(\alpha) = \bigcup_{i \ge 0} X_i$. For every $q \in \mathbb{N}$, p of the sets X_i have cardinality 2^q. And for each set X_i of cardinality 2^q, the corresponding coefficients are all $\pm 2^{-q}$. Therefore,

$$\|\alpha\|_2^2 = p \sum_{q=1}^{\infty} 2^q (2^{-q})^2 < \infty. \qquad \square$$

Lemma 5.12. *For the element $\alpha \in M$ constructed above, $(t-1)\alpha \in \Delta(H)M$.*

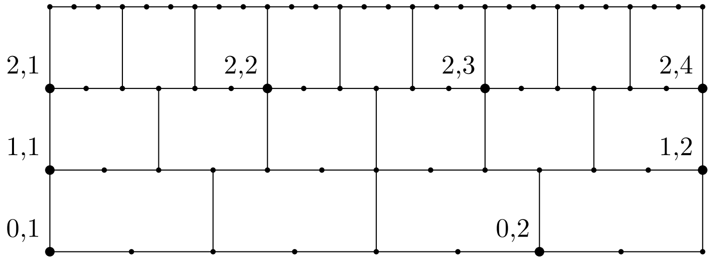
Proof. Let $\beta = \sum_{g \in G} b_g g \in \ell^\infty(G)$ be such that $(t-1)\alpha = (s-1)\beta$. If $\alpha = \sum_{g \in G} a_g g$, then

$$\sum_{g \in G} (a_{t^{-1}g} - a_g) g = \sum_{g \in G} (b_{s^{-1}g} - b_g) g.$$

It follows that $b_g = (a_g - a_{t^{-1}g}) + b_{s^{-1}g}$ for every $g \in G$, and therefore

$$b_g = \sum_{k=0}^{\infty} (a_{s^{-k}g} - a_{t^{-1}s^{-k}g}).$$

The claim can be verified by showing that $\beta \in M$. This is where it is important that the support of α was restricted to V, since it guarantees the support of β is contained in Γ. Looking at a vertex g in Γ, the terms $a_{s^{-k}g} - a_{t^{-1}s^{-k}g}$ in the summation above correspond to differences of α's coefficients of vertices to the left of g with α's coefficients of vertices directly beneath those. For vertices on row i of Γ to the right of the rightmost vertex in X_i, the coefficients of α were designed precisely so that this summation would equal 0. For example, consider $G_{3,2}$, once again with selected vertices g_{ir} labeled as i, r:



Consider the vertex $g = g_{12}$ or any vertex to the right of that. The coefficient would be

$$b_g = \Big(\frac{-1}{2} - 0\Big) + \Big(0 - \frac{-1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{2}\Big) = 0.$$

Vertices in the support of β can be found between certain vertices in X_i. Back to the example, if $g = s^{-1}g_{12}$, then

$$b_g = \Big(0 - \frac{-1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{2}\Big) = \frac{-1}{2}.$$

To bound $\|\beta\|_2$ it will be helpful to split the coefficients into two categories. First consider the rows of Γ for which $|X_i| = |X_{i-1}|$. If $kp + 1 \leq i < (k+1)p$ and g is on the i-th row of Γ, then $b_g = 0$ unless g is between g_{ij} and $tg_{i-1,j}$ for some j. In this case, $b_g = \pm\frac{1}{2^k}$. Since $j \leq 2^{k+1}$, the distance between g_{ij} and $tg_{i-1,j}$ is no larger than $\frac{m^2}{2} + \frac{mn}{2}$, and there are $p-1$ rows of this type, the total contribution of these rows to $\|\beta\|_2^2$ can be bounded by

$$\Big(\frac{m^2}{2} + \frac{mn}{2}\Big)(p-1)\sum_{k=1}^{\infty} 2^{k+1}\Big(\frac{1}{2^k}\Big)^2 < \infty.$$

Next consider the rows of Γ for which $|X_i| = 2|X_{i-1}|$. That is, suppose $i = (k+1)p$. Vertices in $X_i \cap \operatorname{supp}(\beta)$ could lie between g_{ij} and $g_{i,j+1}$ if j is odd. Or, similar to the first category, they could occur between g_{ij} and $tg_{i-1,(j+1)/2}$ if j is odd. As before, there are at most $\frac{m^2}{2} + \frac{mn}{2}$ vertices in each of these (at most) 2^{k+1} gaps. The coefficients of these vertices must be $\pm\frac{1}{2^k}$ or $\pm\frac{1}{2^{k-1}}$. This is true because each such coefficient is a sum of finitely many nonzero terms of the form $(a_g - a_{t^{-1}g})$, which in this case mostly cancel; each term for $g \in B_i$ has the form $\pm(\frac{1}{2^k} - 0)$, which is then followed in the sum by a term $\pm(\frac{1}{2^k} - 0)$ for $g \in A_i$ and a term $\pm(0 - \frac{1}{2^{k-1}})$ for $g \in tX_{i-1}$. (The cancellation is not complete if the sum does not start with $g \in B_i$, thus leaving behind either $\pm\frac{1}{2^k}$ or $\pm\frac{1}{2^{k-1}}$ uncanceled.) Therefore, the contribution of these remaining vertices in $\operatorname{supp}(\beta)$ to $\|\beta\|_2^2$ can be bounded by

$$\Big(\frac{m^2}{2} + \frac{mn}{2}\Big)\sum_{k=1}^{\infty} 2^{k+1}\Big(\frac{1}{2^{k-1}}\Big)^2 < \infty.$$

This accounts for all the elements of $\operatorname{supp}(\beta)$, so it follows that $\beta \in M$. ☐

Lemma 5.13. *For the element $\alpha \in M$ constructed above, $\alpha \notin \Delta(H)M$.*

Proof. Suppose $\alpha \in \Delta(H)M$. There exists $q \in \mathbb{N}$ such that $\alpha \in \Delta(H_q)M$. Since $H_q = \langle h_q \rangle$, there exists $\gamma \in M$ such that $\alpha = (h_q - 1)\gamma$. If $\gamma = \sum_{g\in G} c_g g$, then

$$c_g = -\sum_{k=0}^{\infty} a_{h_q^{-k} g}.$$

Since there exists $N \in \mathbb{N}$ such that $h_q^N = s$, it follows that

$$c_g = -\sum_{k=0}^{\infty} a_{s^{-k} g}.$$

For any $i \in \mathbb{N}$, if $|X_i| = 2^r$, then consider $g = g_{ij}$ for $j = 2^{r-1}$. For this choice of g, we have $c_g = \frac{-1}{2}$. Since every row of Γ has at least one vertex such that $c_g = \frac{-1}{2}$, it follows that $\gamma \notin M$, producing a contradiction. Hence, $\alpha \notin \Delta(H)M$. $\square$

Theorem 5.14. *Let $m > n$ be natural numbers, $G = G_{m,n}$, and $M = \ell^2(G)$. Then $H_1(G, M) \neq 0$.*

Proof. Lemma 5.4 reduced this theorem to the construction of an element $\alpha \in \ell^\infty(G)$ that satisfied sufficient properties. The choice of α in Definition 5.9 was shown to satisfy those properties in Lemmas 5.11, 5.12, and 5.13. $\square$

Corollary 5.15. *If $G = G_{m,n}$ for natural numbers m and n, then $\mathcal{N}(G)$ is not flat over $\mathbb{C}G$.*

Proof. If $m \neq n$, then the result follows from Theorem 5.14 and Lemma 4.7. If $m = n$, then G has a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and the result follows from Theorem 3.1 and Theorem 4.1. $\square$

6. Elementary amenable groups

The next objective is to use the previous results to prove Conjecture 1.2 for the class of torsion-free elementary amenable groups. The proof will be by induction, utilizing a standard inductive construction of the class of elementary amenable groups which can be found in [12, §3]. Let $\mathcal{X}_0$ consist of only the trivial group, and let $\mathcal{X}_1$ be the class of finitely generated abelian-by-finite groups. For any successor ordinal α, define $\mathcal{X}_\alpha = (L\mathcal{X}_{\alpha-1})\mathcal{X}_1$; i.e., any group in $\mathcal{X}_\alpha$ has a normal subgroup that is locally in $\mathcal{X}_{\alpha-1}$ with quotient in $\mathcal{X}_1$. Define $\mathcal{X}_\alpha = \bigcup_{\beta<\alpha} \mathcal{X}_\beta$ if α is a limit ordinal. Using this notation, the class of all elementary amenable groups can be expressed as $\bigcup_{\alpha\geq 0} \mathcal{X}_\alpha$. It will be useful to note that the notion of "Hirsch length" originally defined only for polycyclic groups can be extended to the class of elementary amenable groups [9]. The notation $h(G)$ will be used to denote the Hirsch length of an elementary amenable group G.

In the upcoming proof of Conjecture 1.2 for torsion-free elementary amenable groups, the next two lemmas will be useful for connecting some of these groups back to the groups G_{mn}.

Lemma 6.1. *Let H be an additive subgroup of the rationals $\mathbb{Q}$, and let $\varphi \in \operatorname{Aut}(H)$. Then $\varphi(x) = rx$ for some $r \in \mathbb{Q}$.*

Proof. For any $n \in \mathbb{N}$, define $X_n = \{k \in \mathbb{N} \mid \frac{k}{n} \in H\}$, and let $p_n = \min(X_n)$. Then $X_n = \{kp_n \mid k \in \mathbb{N}\}$. There exists $r_n \in \mathbb{Q}$ such that $\varphi(\frac{p_n}{n}) = r_n(\frac{p_n}{n})$. Since φ is additive, $\varphi(x) = r_n x$ for every $x \in H$ such that $x = \frac{kp_n}{n}$. Since every nonempty X_n intersects nontrivially with X_1, it follows that $r_n = r_1$ for all n. Therefore, $\varphi(x) = rx$ for $r = r_1$. □

Lemma 6.2. *Let G be a group. Suppose H is a normal subgroup of G, H is an additive subgroup of $\mathbb{Q}$, and $G/H \cong \mathbb{Z}$. Then G has a subgroup isomorphic to G_{pq} for some natural numbers p and q.*

Proof. Let $x \in G$ be such that $G/H = \langle \overline{x} \rangle$. Define $\varphi \in \operatorname{Aut}(H)$ by $\varphi(h) = xhx^{-1}$. By Lemma 6.1, there exists $r \in \mathbb{Q}$ such that $\varphi(h) = rh$ for all $h \in H$. Express r as a reduced fraction of integers $\frac{p}{q}$. Then $\varphi^i(1) = \frac{p^i}{q^i} \in H$ for all $i \in \mathbb{Z}$. Thus,

$$a\frac{p^i}{q^i} + b\frac{q^i}{p^i} = \frac{ap^{2i} + bq^{2i}}{p^i q^i} \in H$$

for all $a, b, i \in \mathbb{Z}$. Since $\gcd(p, q) = 1$, it follows that $\gcd(p^{2i}, q^{2i}) = 1$. Hence, $\frac{1}{p^i q^i} \in H$ for all $i \in \mathbb{Z}$. It follows that $K = \mathbb{Z}[\frac{1}{pq}] \leq H$.

Define $A = \langle x \rangle K \leq G$. Then K is a normal subgroup of A and $A/K \cong \mathbb{Z}$. Furthermore, conjugation in K by x is equivalent to multiplication by $\frac{p}{q}$. Therefore, $A \cong G_{pq}$. □

The proof of the conjecture for torsion-free elementary groups will use induction. To complete the induction step the following lemma about locally virtually cyclic groups will be needed.

Lemma 6.3. *If a group G has a locally virtually cyclic subgroup of finite index, then G is locally virtually cyclic.*

Proof. Let G be a group, and let H be a locally virtually cyclic subgroup of finite index. Let G_1 be an arbitrary finitely generated subgroup of G; it will be proved that G_1 is virtually cyclic. Define $H_1 = G_1 \cap H$. Since $H_1 \leq H$, it follows that H_1 is locally virtually cyclic. Moreover, we have

$$[G_1 : H_1] = [G_1 : H \cap G_1] \leq [G : H] < \infty$$

(see [10, Prop. 4.8]). Since G_1 is finitely generated and $[G_1 : H_1] < \infty$, it follows that H_1 is finitely generated. Therefore, H_1 is virtually cyclic. If H_1 is finite, then G_1 is finite. If H_1 is infinite, then there exists an infinite cyclic subgroup K of H_1 of finite index. Then K is also an infinite cyclic subgroup of G_1 of finite index, and so G_1 is virtually cyclic. □

The theorem below establishes Conjecture 1.2 for the class of torsion-free elementary amenable groups.

Theorem 6.4. *Let G be a torsion-free elementary amenable group. If $\mathcal{N}(G)$ is flat over $\mathbb{C}G$, then G is locally virtually cyclic.*

Proof. First note that if G is virtually cyclic, then the group must be either finite, finite-by-(infinite cyclic), or finite-by-(infinite dihedral) by [8, Lem. 11.4]. And since G is torsion-free by assumption, the terms "virtually cyclic" and "cyclic" will be used interchangeably. Using the description $\bigcup \mathcal{X}_\alpha$ of elementary amenable groups described above, the proof will proceed by induction.

Base case: Suppose $G \in \mathcal{X}_1$. Then there exists a subgroup H of G such that H is finitely generated abelian and $[G : H] < \infty$. Since $\mathcal{N}(G)$ is flat over $\mathbb{C}G$, H cannot contain $\mathbb{Z} \oplus \mathbb{Z}$ as a subgroup by Theorem 3.1. Therefore, $H \cong \mathbb{Z}$, and G is virtually cyclic.

Induction hypothesis: Suppose the result is true for all torsion-free groups in $\mathcal{X}_\alpha$ for some ordinal α.

Induction step: Let G be a torsion-free group in $\mathcal{X}_{\alpha+1}$. Then there exists a normal subgroup H of G such that $H \in L\mathcal{X}_\alpha$ and $G/H \in \mathcal{X}_1$. Since $\mathcal{N}(G)$ is flat over $\mathbb{C}G$, $\mathcal{N}(H)$ must be flat over $\mathbb{C}H$ by Theorem 4.1. By the induction hypothesis, H must be locally cyclic. The rest of the induction step will be split into two cases.

Case 1: G/H is finite. Since H is locally virtually cyclic and $[G : H] < \infty$, Lemma 6.3 implies that G is locally virtually cyclic.

Case 2: G/H is infinite. Then there exists an infinite cyclic subgroup of G/H. Let $x \in G$ be such that $\overline{x}$ generates that infinite cyclic subgroup. Define $\tilde{G} = \langle x \rangle H$. Then $\tilde{G} \leq G$, H is a normal subgroup of $\tilde{G}$, and $\tilde{G}/H \cong \mathbb{Z}$. Hence $h(\tilde{G}) = h(H) + h(\tilde{G}/H) = 2$ by [9, Thm. 1]. By [9, Thm. 2], $\tilde{G}$ must be solvable. The lowest nontrivial member of the derived series for $\tilde{G}$ is a torsion-free abelian normal subgroup K of $\tilde{G}$. Since $\mathcal{N}(G)$ is flat over $\mathbb{C}G$, K cannot have $\mathbb{Z} \oplus \mathbb{Z}$ as a subgroup. Hence, either $K \cong \mathbb{Z}$ or K is a subgroup of $\mathbb{Q}$ which is not finitely generated [1]. Pick any $g \in \tilde{G} \setminus K$, and define $\hat{G} = \langle g \rangle K \leq \tilde{G}$. Then K is a normal subgroup of $\hat{G}$, and $\hat{G}/K \cong \mathbb{Z}$. If $K \cong \mathbb{Z}$, then $\hat{G}$ is an elementary amenable group of cohomological dimension two, and [7, Thm. 5] implies $\hat{G} \cong B(1, n)$ for some $n \in \mathbb{Z}$. If K is isomorphic to some other additive subgroup of $\mathbb{Q}$, then Lemma 6.2 implies $\hat{G}$ has a subgroup isomorphic to G_{mn} for some $m, n \in \mathbb{N}$. In either event, this implies $\mathcal{N}(G)$ is not flat over $\mathbb{C}G$ by Corollary 5.15, which is a contradiction. Therefore, G/H cannot be infinite. □

Conjecture 1.2 is still open. While it can be verified for some elementary amenable groups with torsion, the induction technique above runs into problems for the class of all elementary amenable groups. In particular, difficulty arises if the torsion of the group includes infinite locally finite subgroups. The conjecture's assumption of "torsion-free" could be removed if there was a relationship analogous to Theorem 4.1 for groups and quotient groups by locally finite subgroups. However, in some special cases of groups with infinite locally finite subgroups the conjecture has been established, such as the Lamplighter group (see [14, Lem. 5]).

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