

Complex analogues of the half-classical geometry

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(Communicated by Joachim Cuntz)

Abstract. Under very strong axioms, there is precisely one real noncommutative geometry between the classical one and the free one, namely the half-classical one, coming from the relations $abc = cba$. We discuss here the complex analogues of this geometry, notably with a study of the geometry coming from the commutation relations between all the variables $\{ab^*, a^*b\}$, that we believe to be the “correct” one.

1. INTRODUCTION

The fact that the behavior of the subatomic particles might be described by some kind of “noncommutative geometry”, with a great deal of probability theory involved, is as old as quantum mechanics. While the weak and strong forces are radically different from gravity and electromagnetism, one hope, however, would be that this noncommutative geometry could simply appear as an “analogue” of the classical geometry.

From this perspective, any exploration of the noncommutative analogues of the various aspects of the classical geometry can only be useful. The subject is, of course, still in its infancy. As a reminder here, the classical geometry itself took about 2000 years to be axiomatized, and applied to basic problems in physics (Kepler, Newton). Noncommutative geometry seems to be on a faster track, but there is no reason to be overly optimistic, and not to remain modest. After all, we are dealing here with phenomena that both our human brains and machinery have big troubles in observing and understanding.

Doing some abstract mathematics, with these ideas in mind, will be our purpose here. We will be interested in noncommutative algebraic geometry, of very elementary type: basic curves and surfaces. Technically speaking, we will use the same formalism and philosophy as Connes [19], a noncommutative space being for us the dual of an operator algebra. For some successful applications of this philosophy, we refer to [18].

Our starting point is a recent discovery, from [10, 11], stating that when imposing the strongest possible axioms, there are only three real geometries, namely the classical one, the free one, and an intermediate one, called “half-classical”. Motivated by this fact, we started in [5] an investigation of the possible half-classical complex geometries. We will finish here the work started in [5], by identifying the “standard” such geometry.

Let us first explain the above-mentioned rigidity phenomenon, from the real case. From an elementary, all-around point of view, the basic objects of the N -dimensional geometry are the unit sphere, the standard cube, and the orthogonal group:

$$\begin{aligned} S_{\mathbb{R}}^{N-1} &= \left\{ x \in \mathbb{R}^N \mid \sum_i x_i^2 = 1 \right\}, \\ T_N &= \left\{ x \in \mathbb{R}^N \mid x_i = \pm \frac{1}{\sqrt{N}} \right\}, \\ O_N &= \left\{ U \in M_N(\mathbb{R}) \mid U^t = U^{-1} \right\}. \end{aligned}$$

Note that we have not included \mathbb{R}^N itself in our list. This is because we would like later on to talk about noncommutative versions of the above objects, and our formalism here requires all the spaces to be compact. Physically speaking, our belief is that for certain key problems, such as those regarding QCD, this restriction is not important.

Quite remarkably, there is a full set of connections between the above objects:

- (i) O_N is the isometry group of $S_{\mathbb{R}}^{N-1}$, and $S_{\mathbb{R}}^{N-1}$ appears as $O_N(\circ)$, where $\circ = (1, 0, \dots, 0)$. In addition, we have an embedding $O_N \subset \sqrt{N} \cdot S_{\mathbb{R}}^{N^2-1}$.
- (ii) T_N appears inside $S_{\mathbb{R}}^{N-1}$ by setting $|x_1| = \dots = |x_N|$. Conversely, $S_{\mathbb{R}}^{N-1}$ appears from $T_N \subset \mathbb{R}^N$ by “deleting” this relation, while still keeping $\sum_i x_i^2 = 1$.
- (iii) $T_N \simeq \mathbb{Z}_2^N$ is a maximal compact abelian subgroup of O_N , and the group O_N itself can be reconstructed from this subgroup, by using various methods.

We are of course a bit vague here, but it is not hard to believe that, with a minimal knowledge of basic algebraic geometry and representation theory, all the $2 \times 3 = 6$ correspondences can indeed be established. This is actually a very good exercise.

Let us discuss now the construction of the noncommutative versions of the above objects. There are several possible choices here, and based on our personal knowledge of quantum mechanics, and of mathematical physics in general, we will use the operator algebra formalism. The idea indeed is that whenever we have a unital C^* -algebra A , we can write $A = C(X)$, with X being a noncommutative compact space. This is supported by a nontrivial theorem of Gelfand, which states that when A is commutative, the formula $A = C(X)$ holds indeed, with X being a classical space, called spectrum of A .

So, let us define the free sphere, free cube, and free orthogonal group, by setting:

$$\begin{aligned}
 C(S_{\mathbb{R},+}^{N-1}) &= C^*\left((x_i)_{i=1,\dots,N} \mid x_i = x_i^*, \sum_i x_i^2 = 1\right), \\
 C(T_N^+) &= C^*\left((x_i)_{i=1,\dots,N} \mid x_i = x_i^*, x_i^2 = \frac{1}{N}\right), \\
 C(O_N^+) &= C^*\left((u_{ij})_{i,j=1,\dots,N} \mid u_{ij} = u_{ij}^*, u^t = u^{-1}\right).
 \end{aligned}$$

Observe that $u_i = \sqrt{N}x_i \in C(T_N^+)$ are subject to the relations $u_i = u_i^* = u_i^{-1}$. Thus, T_N^+ appears as the abstract dual of the discrete group \mathbb{Z}_2^{*N} . As for O_N^+ , this is a compact quantum group in the sense of Woronowicz [29], with structural maps as follows:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \quad \varepsilon(u_{ij}) = \delta_{ij}, \quad S(u_{ij}) = u_{ji}.$$

We refer to the original papers [8, 28] and to the lecture notes [4] for details. In analogy now with what happens in the classical case, we have:

- (i) O_N^+ is the quantum isometry group of $S_{\mathbb{R},+}^{N-1}$, and $S_{\mathbb{R},+}^{N-1}$ appears as an homogeneous space over O_N^+ . In addition, we have an embedding $O_N^+ \subset \sqrt{N} \cdot S_{\mathbb{R},+}^{N^2-1}$.
- (ii) T_N^+ appears inside $S_{\mathbb{R},+}^{N-1}$ by setting $x_1^2 = \dots = x_N^2$. Conversely, $S_{\mathbb{R},+}^{N-1}$ appears from T_N^+ by “deleting” this relation, while still keeping $\sum_i x_i^2 = 1$.
- (iii) $T_N^+ \simeq \widehat{\mathbb{Z}_2^{*N}}$ is a maximal group dual subgroup of O_N^+ , and O_N^+ itself can be reconstructed from this subgroup, via representation theory methods.

To be more precise, having agreed that in the classical case, constructing the $2 \times 3 = 6$ correspondences is a good exercise in basic geometry, the situation is similar here, with all this being a good exercise in basic noncommutative geometry. The only point which is nontrivial is the correspondence $T_N^+ \rightarrow O_N^+$, and we refer here to [2, 7, 11]. Also, we refer to the lecture notes [4] for all the needed details on all this material.

Let us try now to understand the possible “intermediate liberations” of the usual geometry. At the sphere and cube level, there is a lot of freedom in dealing with this question, or at least the known noncommutative geometry theories here do not provide any simple, quick answer. However, at the quantum group level, things are quite rigid. So, as a first good question, we would like to find the intermediate quantum groups, as follows:

$$(1) \quad O_N \subset G \subset O_N^+.$$

In order to deal with this problem, let us recall Brauer’s theorem [17]. Given a pairing $\pi \in P_2(k, l)$, between an upper row of k points, and a lower row of l points, we set

$$T_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_l}.$$

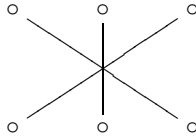
Brauer’s theorem states that the intertwining spaces $\text{Hom}(u^{\otimes k}, u^{\otimes l})$ for the orthogonal group O_N are precisely those spanned by these maps T_π . In addition, a free version of this result is available, stating that for O_N^+ , the intertwining spaces are spanned as well by the maps T_π , but this time with π being a noncrossing pairing, $\pi \in NC_2$. See [7].

Based on these results, let us call a quantum group $O_N \subset G \subset O_N^+$ easy if the following equalities hold for a certain category of pairings $NC_2 \subset D \subset P_2$:

$$\text{Hom}(u^{\otimes k}, u^{\otimes l}) = \text{span}(T_\pi \mid \pi \in D(k, l)).$$

Here the categorical operations are the horizontal and vertical concatenation, and the upside-down turning of the pairings. These operations ensure the fact that the spaces $\text{span}(T_\pi \mid \pi \in D(k, l))$ form a tensor C^* -category, and as a consequence of Woronowicz’s Tannakian duality [30], each such category produces a quantum group. For full details regarding the easy quantum group theory, we refer to [10, 23, 26].

We are now ready to go back to (1). If we restrict the attention to the easy case, we just have to find the intermediate categories $NC_2 \subset D \subset P_2$. And here, there is only one solution, namely the category P_2^* generated by the following “crossing”:



Due to the uniqueness result, we will call this diagram “half-classical crossing”, and the corresponding relation, namely $abc = cba$, “half-commutation relation”. See [11].

Summarizing, we have now an answer to (1), which is in addition unique, in the easy quantum group setting. We should mention that, conjecturally, this solution is actually unique, in the arbitrary compact quantum group setting. See [6].

So, let us go ahead now, and construct our various geometric objects, as follows:

$$\begin{aligned} C(S_{\mathbb{R},*}^{N-1}) &= C(S_{\mathbb{R},+}^{N-1}) / \langle abc = cba \mid \forall a, b, c \in \{x_i\} \rangle, \\ C(T_N^*) &= C(T_N^+) / \langle abc = cba \mid \forall a, b, c \in \{x_i\} \rangle, \\ C(O_N^*) &= C(O_N^+) / \langle abc = cba \mid \forall a, b, c \in \{u_{ij}\} \rangle. \end{aligned}$$

As in the classical and free cases, we have correspondences between these objects, with statements (i)–(iii) as above. For details, we refer to [4, 8, 9, 11].

In view of the uniqueness results mentioned above, we can stop here the axiomatization work, because this is the third and last possible “geometry”. So, what is left to do now is to start the actual geometric work. We would like to understand the structure and geometry of the various algebraic manifolds $X \subset S_{\mathbb{R},*}^{N-1}$, $X \subset S_{\mathbb{R},+}^{N-1}$, along with their differential geometric aspects, and Riemannian aspects as well. We have as well the important question of finding

explicit matrix models for the coordinates of such manifolds. The whole subject is still very young, and we refer to [3, 5, 15].

The aim of the present paper is to clarify what happens in the complex geometry setting. Available here are the classical theory, having symmetry group U_N , and its free version, with symmetry group U_N^+ . At the level of intermediate geometries, however, the situation is quite complicated, because we have many quantum groups as follows:

$$U_N \subset G \subset U_N^+.$$

Our goal is not to solve this classification problem for the intermediate complex geometries, but to try to identify the “main” solution. For this purpose, we will use an axiomatic approach. We will first axiomatize the triples (S, T, G) which are subject to correspondences (i)–(iii) as above, then we will examine the various complex analogues of the triple $(S_{\mathbb{R},*}^{N-1}, T_N^*, O_N^*)$, and we will identify the “main” solution.

The paper is organized as follows: Sections 2 and 3 contain various preliminaries and generalities. In Sections 4 and 5, we construct and study the complex half-classical geometry.

2. FORMALISM

We agree to call “noncommutative compact spaces” the abstract duals of the unital C^* -algebras. We denote such spaces by X, Y, Z, \dots , with the corresponding C^* -algebras being denoted by $C(X), C(Y), C(Z), \dots$. We use this correspondence for formulating our various findings directly in terms of noncommutative spaces. For instance, we call a morphism $X \rightarrow Y$ injective if the corresponding morphism $C(Y) \rightarrow C(X)$ is surjective, and vice versa. Also, a direct product $X \times Y$ is by definition the noncommutative space corresponding to the C^* -algebra $C(X) \otimes C(Y)$, with \otimes being the minimal tensor product.

In what follows, we are interested in the noncommutative analogues of the real algebraic manifolds $X \subset S_{\mathbb{C}}^{N-1}$. Here we use of course the canonical embedding $S_{\mathbb{C}}^{N-1} \subset \mathbb{C}^N \simeq \mathbb{R}^{2N}$, and by real algebraic manifold we mean as usual the set of zeroes of a certain family of polynomials in the standard coordinates on \mathbb{R}^N , or, equivalently, of a certain family of polynomials in the standard coordinates on \mathbb{C}^N , and their conjugates.

Our starting point is the following well-known fact:

Proposition 2.1. *Consider a real algebraic manifold $X \subset S_{\mathbb{C}}^{N-1}$, appearing as*

$$X = \left\{ x \in \mathbb{C}^N \mid \sum_i |x_i|^2 = 1, P_\alpha(x_1, \dots, x_N) = 0 \right\}.$$

The algebra of continuous functions $f : X \rightarrow \mathbb{C}$ is then given by

$$C(X) = C_{\text{comm}}^* \left(x_1, \dots, x_N \mid \sum_i x_i x_i^* = 1, P_\alpha(x_1, \dots, x_N) = 0 \right),$$

where by C_{comm}^ we mean universal commutative C^* -algebra.*

Proof. Observe first that the universal algebra in the statement is well-defined, because $\sum_i x_i x_i^* = 1$ gives $\|x_i\| \leq 1$ for any i , and so the biggest norm is bounded. If we denote by A this algebra, we have an arrow $A \rightarrow C(X)$. Conversely, by Gelfand duality we have $A = C(X')$ for a certain compact space X' . The coordinates x_i produce an embedding $X' \subset \mathbb{C}^N$, then the condition $\sum_i x_i x_i^* = 1$ gives $X' \subset S_{\mathbb{C}}^{N-1}$. Finally the conditions $P_\alpha(x_1, \dots, x_N) = 0$ give $X' \subset X$. Thus we have $X = X'$, as claimed. \square

The above result suggests to construct a free version X^+ , simply by removing the commutativity assumption from the presentation of $C(X)$. However, this is quite tricky, because the relations $P_\alpha = 0$ must not include, or imply, the commutativity.

In practice, this method works in a number of situations.

Definition 2.2. The free complex sphere and free unitary group are constructed as

$$C(S_{\mathbb{C},+}^{N-1}) = C^*\left(\left(x_i\right)_{i=1,\dots,N} \mid \sum_i x_i x_i^* = \sum_i x_i^* x_i = 1\right),$$

$$C(U_N^+) = C^*\left(\left(u_{ij}\right)_{i=1,\dots,N} \mid u^* = u^{-1}, u^t = \bar{u}^{-1}\right),$$

where on the right we have universal C^* -algebras.

As explained by Wang in [28], the above noncommutative space U_N^+ is a compact quantum group in the sense of Woronowicz [29], with structural maps as follows:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \quad \varepsilon(u_{ij}) = \delta_{ij}, \quad S(u_{ij}) = u_{ji}^*.$$

We have an action $U_N^+ \curvearrowright S_{\mathbb{C},+}^{N-1}$, whose properties are quite similar to that of the action $U_N \curvearrowright S_{\mathbb{C}}^{N-1}$. In order to explain this material, let us introduce a few more notions:

Definition 2.3. Consider an algebraic submanifold $X \subset S_{\mathbb{C},+}^{N-1}$, i.e. a closed subset defined via algebraic relations, and a closed quantum subgroup $G \subset U_N^+$.

(i) We say that we have an affine action $G \curvearrowright X$ when the formula

$$x_i \rightarrow \sum_j u_{ij} \otimes x_j$$

defines a morphism of C^* -algebras $\Phi : C(X) \rightarrow C(G) \otimes C(X)$.

(ii) The biggest quantum subgroup $G \subset U_N^+$ acting affinely on X is denoted by $G^+(X)$ and is called quantum isometry group of X .

Here by ‘‘algebraic relations’’ we mean relations of type $P_\alpha(x_1, \dots, x_N) = 0$, with P_α being noncommutative $*$ -polynomials in N variables. As for the word ‘‘biggest’’, this means ‘‘maximal in the appropriate category’’. We agree in what follows to keep using the language of noncommutative compact spaces and manifolds, and to use more complicated language only when needed and helpful in connection with problems.

Observe that the morphism in (i) above is automatically coassociative, $(\Phi \otimes \text{id})\Phi = (\text{id} \otimes \Delta)\Phi$, and counital as well, $(\text{id} \otimes \varepsilon)\Phi = \text{id}$. When X, G are both classical, such a morphism must appear by transposition from a usual affine group action $G \times X \rightarrow X$.

Regarding (ii), it is routine to check that such a biggest quantum group exists indeed, simply by dividing $C(U_N^+)$ by the appropriate relations. We refer to [2] for details.

We should mention that, in analogy with what happens in the classical case, there are of course several notions of quantum isometries and quantum isometry groups, and those presented above are those that we will need here. As an example, consider the usual sphere $S_{\mathbb{R}}^{N-1}$. The classical isometries of $S_{\mathbb{R}}^{N-1}$ are “obviously” the orthogonal matrices $U \in O_N$, but the meaning of “obvious” can of course vary with the person involved:

- (i) If we agree that the sphere is “something round”, which is basic common sense, the isometry group is quite complicated to compute. We obtain O_N .
- (ii) If we agree that the sphere is the set of solutions of $\sum_i x_i^2 = 1$, which is perhaps a bit unnatural, the isometry group is easy to compute. We obtain O_N too.

In the noncommutative setting, all this becomes considerably more complicated. For a full discussion on these topics, we refer to Goswami’s papers [21, 22].

We will need as well the following constructions:

Definition 2.4. Consider a subspace $S \subset S_{\mathbb{C},+}^{N-1}$ and a subgroup $G \subset U_N^+$.

- (i) The standard torus of S is the subspace $T \subset S$ obtained by setting, at the algebra level, $C(T) = C(S)/\langle x_i x_i^* = x_i^* x_i = 1/N \rangle$.
- (ii) The diagonal torus of G is the subspace $\mathcal{T} \subset G$ obtained by setting, at the algebra level, $C(\mathcal{T}) = C(G)/\langle u_{ij} = 0 \mid \forall i \neq j \rangle$.

Here, as usual, we use the direct language of noncommutative compact spaces, with “subspace” standing for “closed noncommutative subspace” and “subgroup” standing for “closed quantum subgroup”, with all this coming from the Gelfand theorem.

For $S = S_{\mathbb{C}}^{N-1}$, the space constructed in (i) is the usual torus, $T = \mathbb{T}^N$. For $S = S_{\mathbb{C},+}^{N-1}$, the rescaled generators $u_i = \sqrt{N}x_i$ are subject to the relations $u_i^* = u_i^{-1}$, which produce the free group algebra $C^*(F_N)$. Thus, we obtain here a group dual, $T = \widehat{F_N}$.

Regarding (ii), observe that $C(\mathcal{T})$ is generated by the variables $g_i = u_{ii}$, which are group-like. Thus $\mathcal{T} = \widehat{\Gamma}$, where Γ is the group generated by these variables. See [11].

We can now formulate our main definition, as follows:

Definition 2.5. A noncommutative geometry consists of three spaces,

- (i) an intermediate algebraic manifold $S_{\mathbb{R}}^{N-1} \subset S \subset S_{\mathbb{C},+}^{N-1}$, called sphere,
- (ii) an intermediate space $\widehat{\mathbb{Z}}_2^N \subset T \subset \widehat{F_N}$, called torus,

(iii) and an intermediate quantum group, $O_N \subset G \subset U_N^+$, such that the following two conditions are satisfied:

- (i) G is the quantum isometry group of S ,
- (ii) T is the standard torus of S , as well as the diagonal torus of G ,

where, for the needs of the second axiom, we think of full and reduced group algebras as representing the same quantum space.

This definition is something technical and temporary. Further improving it, comparing it with the other known definitions of noncommutative manifolds and geometries, and looking for applications too, is of course something that we have in mind. In what follows, we will focus on this definition as it is, and present our results on the subject, as they are, and of course we will be back to all this in our future papers.

Regarding the terminology, the torus T will be sometimes called “cube” of the geometry, because in the real case, as explained below, what we have here is rather a cube. It is also useful to think of the discrete dual $\Gamma = \widehat{T}$, which appears as an intermediate quotient group $F_N \rightarrow \Gamma \rightarrow \mathbb{Z}_2^N$, as being the “structural group” of the geometry.

Finally, regarding the identification made at the end, this is something standard in noncommutative geometry. The point indeed is that there is a “problem” with Gelfand duality, coming from the fact that a discrete group dual $T = \widehat{\Gamma}$ can be represented by several C^* -algebras, including the maximal one, $C^*(\Gamma)$, and the minimal one, $C_{\text{red}}^*(\Gamma)$. The standard way of fixing this issue is that of identifying all these algebras, and this is what we do here too. For a full discussion on all this, see Woronowicz [29].

Here is now a useful reformulation of the axioms, in terms of S only:

Proposition 2.6. *An algebraic manifold $S_{\mathbb{R}}^{N-1} \subset S \subset S_{\mathbb{C},+}^{N-1}$, with standard torus denoted by $T \subset S$, produces a noncommutative geometry precisely if*

- (i) *we have an affine action $O_N \curvearrowright S$,*
- (ii) *$\delta(x_i) = \sqrt{N}x_i \otimes x_i$ defines a morphism of algebras*

$$\delta : C(S) \rightarrow C(T) \otimes C(S).$$

If these conditions are satisfied, we say that S is a noncommutative sphere.

Proof. Given $S_{\mathbb{R}}^{N-1} \subset S \subset S_{\mathbb{C},+}^{N-1}$, consider its quantum isometry group $G \subset U_N^+$, and consider as well the standard torus $T \subset S$ and the diagonal torus $\mathcal{T} \subset G$.

Assuming that (S, T, G) is a noncommutative geometry, we have $O_N \subset G$, so condition (i) is clear. Also, since we have $T = \mathcal{T}$, the morphism in (ii) simply appears by composing the universal coaction with the diagonal torus quotient map:

$$\begin{aligned} C(S) &\rightarrow C(G) \otimes C(S) \rightarrow C(\mathcal{T}) \otimes C(S) = C(T) \otimes C(S), \\ x_i &\rightarrow \sum_j u_{ij} \otimes x_j \rightarrow g_i \otimes x_i = \sqrt{N}x_i \otimes x_i. \end{aligned}$$

Conversely, assuming that (i) and (ii) are satisfied, we must prove that we have $T = \mathcal{T}$. For this purpose, observe first that the map δ induces a morphism as follows:

$$\bar{\delta} : C(T) \rightarrow C(T) \otimes C(T), \quad x_i \rightarrow \sqrt{N}x_i \otimes x_i.$$

Thus $C(T)$ is a cocommutative Hopf algebra, and its elements $\sqrt{N}x_i$ are group-like.

With this picture in mind, the map δ in the statement corresponds to an action $T \curvearrowright S$, and by universality of the quantum isometry group G , we obtain a map

$$\theta : C(\mathcal{T}) \rightarrow C(T), \quad u_{ii} \rightarrow \sqrt{N}x_i.$$

In order to construct an inverse map, we will use a composition as follows:

$$\begin{aligned} C(S) &\rightarrow C(G) \otimes C(S) \rightarrow C(G) \rightarrow C(G) && \rightarrow C(\mathcal{T}), \\ x_i &\rightarrow \sum_j u_{ij} \otimes x_j \rightarrow u_{ij} && \rightarrow \frac{1}{\sqrt{N}} \sum_j u_{ij} \rightarrow \frac{1}{\sqrt{N}} u_{ii}. \end{aligned}$$

Here the first map is the universal coaction, and the second map comes from the evaluation at $(1, 0, \dots, 0) \in S_{\mathbb{R}}^{N-1} \subset S$, which gives a map

$$\varepsilon : C(S) \rightarrow \mathbb{C}, \quad x_i \rightarrow \delta_{i1}.$$

Regarding the third map, this is the algebra automorphism $u_{ij} \rightarrow \sum_k m_{kj} u_{ik}$ constructed by using the inclusion $O_N \subset G$ coming from our assumption (i), along with an orthogonal matrix $M = (m_{ij})$ having $1/\sqrt{N}$ entries on its first column. Observe that such orthogonal matrices M exist indeed, for instance by using the Gram–Schmidt orthogonalization procedure. Finally, the fourth map is the canonical quotient map.

Now since the elements $X_i = 1/\sqrt{N} u_{ii}$ satisfy the relations

$$X_i X_i^* = X_i^* X_i = \frac{1}{N},$$

the composition that we constructed factorizes into a map

$$\rho : C(T) \rightarrow C(\mathcal{T}), \quad x_i \rightarrow \frac{1}{\sqrt{N}} u_{ii}.$$

It is clear that θ, ρ are inverse to each other, and this finishes the proof. \square

At the level of basic examples, we have:

Proposition 2.7. *We have the following examples of geometries:*

- (i) *Real geometry:* $S = S_{\mathbb{R}}^{N-1}, T = \widehat{\mathbb{Z}}_2^N, G = O_N.$
- (ii) *Complex geometry:* $S = S_{\mathbb{C}}^{N-1}, T = \widehat{\mathbb{Z}}^N, G = U_N.$
- (iii) *Free real geometry:* $S = S_{\mathbb{R},+}^{N-1}, T = \widehat{\mathbb{Z}}_2^{*N}, G = O_N^+.$
- (iv) *Free complex geometry:* $S = S_{\mathbb{C},+}^{N-1}, T = \widehat{F}_N, G = U_N^+.$

Proof. These results are well known. For details, we refer to [1, 8]. \square

Based on these examples, here are now a few more general notions:

Definition 2.8. A noncommutative geometry is called

- (i) real if $S \subset S_{\mathbb{R},+}^{N-1}, T \subset \widehat{\mathbb{Z}_2^{*N}}, G \subset O_N^+$;
- (ii) complex if $S \subset S_{\mathbb{C}}^{N-1} \subset S, \widehat{\mathbb{Z}^N} \subset T, U_N \subset G$;
- (iii) classical if $S \subset S_{\mathbb{C}}^{N-1}, T \subset \widehat{\mathbb{Z}^N}, G \subset U_N$;
- (iv) free if $S_{\mathbb{R},+}^{N-1} \subset S, \widehat{\mathbb{Z}_2^{*N}} \subset T, O_N^+ \subset G$.

We will illustrate these notions in what follows, with several other examples.

Let us introduce now a few more notions. First, a geometry which is neither real, nor complex, will be called “hybrid”. Also, a geometry which is neither classical, nor free, will be called “intermediate”. Basic examples come from the following definition.

Definition 2.9. We have spheres, tori, and quantum groups as follows:

- (i) $S_{\mathbb{R},*}^{N-1}, \widehat{\mathbb{Z}_2^{\circ N}}, O_N^*$, obtained respectively from $S_{\mathbb{R},+}^{N-1}, \widehat{\mathbb{Z}_2^{*N}}, O_N^+$ by imposing to the standard coordinates the relations $abc = cba$.
- (ii) $S_{\mathbb{C},**}^{N-1}, \widehat{\mathbb{Z}^{\circ\circ N}}, U_N^{**}$, obtained respectively from $S_{\mathbb{C},+}^{N-1}, \widehat{F_N}, U_N^+$ by imposing to the standard coordinates and their adjoints the relations $abc = cba$.

Here (i) is a well-established definition, coming from the work in [10, 11]. In (ii), we have some temporary objects coming from [16], which are somehow “minimal” and which will be replaced with the correct, maximal ones later on.

We can now formulate our first result:

Theorem 2.10. *We have geometries whose unitary groups are as follows:*

$$\begin{array}{ccccc}
 U_N & \longrightarrow & U_N^{**} & \longrightarrow & U_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbb{T}O_N & \longrightarrow & \mathbb{T}O_N^* & \longrightarrow & \mathbb{T}O_N^+ \\
 \uparrow & & \uparrow & & \uparrow \\
 O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+
 \end{array}$$

with the middle row spaces obtained from the upper ones via the relations $ab^* = a^*b$.

Proof. The results in the upper and lower row are well known, see, e.g., [1]. Regarding the middle row, consider the quotient algebra

$$C(\mathbb{T}O_N^+) = C(U_N^+) / \langle ab^* = a^*b \mid \forall a, b \in \{u_{ij}\} \rangle.$$

Inside this algebra, observe that with $U_{ij} = \sum_a u_{ia} \otimes u_{aj}$ we have

$$U_{ij}U_{kl}^* = \sum_{ab} u_{ia}u_{kb}^* \otimes u_{aj}u_{bl}^* = \sum_{ab} u_{ia}^*u_{kb} \otimes u_{aj}^*u_{bl} = U_{ij}^*U_{kl}.$$

Thus we can construct a comultiplication morphism Δ , by mapping $u_{ij} \rightarrow U_{ij}$, and the existence of the counit ε and of the antipode S is clear too. Now with

$\mathbb{T}O_N^+$ constructed as above, we can construct the other quantum groups as well:

$$\mathbb{T}O_N = \mathbb{T}O_N^+ \cap U_N, \quad \mathbb{T}O_N^* = \mathbb{T}O_N^+ \cap U_N^{**}.$$

For the spheres and tori, the discussion here parallels the one from the quantum group case. To be more precise, the definition of these objects is as follows:

$$C(\mathbb{T}S_{\mathbb{R}, \times}^{N-1}) = C(S_{\mathbb{C}, \times}^{N-1}) / \langle ab^* = a^*b \mid \forall a, b \in \{x_i\} \rangle,$$

$$C(\widehat{\mathbb{T}\mathbb{Z}_2^{\times N}}) = C(\widehat{\mathbb{Z}_2^{\times N}}) / \langle ab^* = a^*b \mid \forall a, b \in \{g_i\} \rangle.$$

Regarding the axioms, let us prove now that the standard action $U_N^\times \curvearrowright S_{\mathbb{C}, \times}^{N-1}$ restricts to an action $\mathbb{T}O_N^\times \curvearrowright \mathbb{T}S_{\mathbb{R}, \times}^{N-1}$. With $X_i = \sum_a u_{ia} \otimes x_a$ we have

$$X_i X_j^* = \sum_{ab} u_{ia} u_{jb}^* \otimes x_a x_b^* = \sum_{ab} u_{ia}^* u_{jb} \otimes x_a^* x_b = X_i^* X_j.$$

Thus we can indeed define our coaction map, via $x_i \rightarrow X_i$. In order to prove now the universality, assume that we have an action $G \curvearrowright \mathbb{T}S_{\mathbb{R}, \times}^{N-1}$. With $X_i = \sum_a u_{ia} \otimes x_a$ as above we have $X_i X_j^* = X_i^* X_j$. From this we obtain, via the standard method from [14], that $u_{ia} u_{jb}^* = u_{ia}^* u_{jb}$ for any i, j, a, b , and so $G \subset \mathbb{T}O_N^\times$. We refer to [1] for the classical case, the proof in the half-classical and free cases being similar. \square

3. EASINESS, AMENABILITY

We will need the notion of easy quantum group from [10, 20, 26, 27].

We denote by $P(k, l)$ the set of partitions between an upper row of k points and a lower row of l points, with each leg colored black or white, and with k, l standing for the corresponding ‘‘colored integers’’.

Definition 3.1. A category of partitions is a collection of sets $D = \bigcup_{kl} D(k, l)$, with $D(k, l) \subset P(k, l)$, which contains the identity and is stable under

- (i) the horizontal concatenation operation \otimes ;
- (ii) the vertical concatenation \circ , after deleting closed strings in the middle;
- (iii) the upside-down turning operation $*$ (with reversing of the colors).

As explained in [27], such categories produce quantum groups. To be more precise, associated to any partition $\pi \in P(k, l)$ is the following linear map:

$$T_\pi(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \delta_\pi \begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_l \end{pmatrix} e_{j_1} \otimes \cdots \otimes e_{j_l}.$$

Here the Kronecker type symbol $\delta_\pi \in \{0, 1\}$ is by definition 1 if all the strings of π join pairs of equal indices, and is 0 otherwise. With this notion in hand, we have:

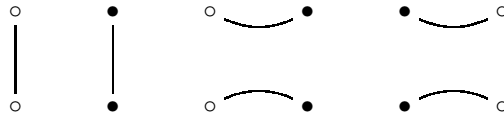
Definition 3.2. A compact quantum group $G \subset U_N^+$ is called easy if

$$\text{Hom}(u^{\otimes k}, u^{\otimes l}) = \text{span}(T_\pi \mid \pi \in D(k, l))$$

for any k, l , for a certain category of partitions $D \subset P$.

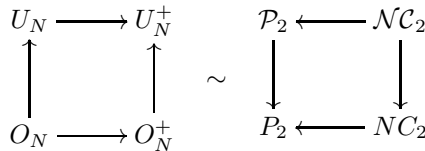
In other words, the easiness condition states that the Tannakian dual of G , also called Schur–Weyl dual, should come in the simplest possible way: from partitions.

In order to discuss some basic examples, consider the categories of pairings and of noncrossing pairings, $NC_2 \subset P_2$. Consider as well the “color-matching” versions of these categories, $\mathcal{NC}_2 \subset \mathcal{P}_2$, the color-matching condition stating that the various types of strings (upper, lower, through) of our pairings must be colored as follows:



With these notions in hand, we have the following result:

Proposition 3.3. *We have easy quantum groups as follows:*



The diagram on the right-hand side describes the corresponding categories of partitions.

Proof. The results on the right are Brauer’s theorem for O_N and for U_N . The results on the left are free versions of Brauer’s theorem, discussed in [7]. As a quick, partly heuristic explanation here, all these results follow from Tannakian duality:

(i) U_N^+ is defined via the relations $u^* = u^{-1}$, $u^t = \bar{u}^{-1}$, which tell us that the operators T_π , with $\pi = \begin{smallmatrix} \circ & \\ \circ & \end{smallmatrix}$ and $\pi = \begin{smallmatrix} \circ & \\ \bullet & \end{smallmatrix}$, must be in the associated Tannakian category C . We therefore obtain, as claimed,

$$C = \text{span}(T_\pi \mid \pi \in D) \quad \text{with } D = \langle \begin{smallmatrix} \circ & \\ \circ & \end{smallmatrix}, \begin{smallmatrix} \circ & \\ \bullet & \end{smallmatrix} \rangle = \mathcal{NC}_2.$$

(ii) $O_N^+ \subset U_N^+$ is defined by imposing the relations $u_{ij} = \bar{u}_{ij}$, which tell us that the operators T_π , with $\pi = \begin{smallmatrix} \circ & \\ \circ & \end{smallmatrix}$ and $\pi = \begin{smallmatrix} \circ & \\ \bullet & \end{smallmatrix}$, must be in the associated Tannakian category C . We therefore obtain, as claimed,

$$C = \text{span}(T_\pi \mid \pi \in D) \quad \text{with } D = \langle \mathcal{NC}_2, \begin{smallmatrix} \circ & \\ \bullet & \end{smallmatrix}, \begin{smallmatrix} \bullet & \\ \bullet & \end{smallmatrix} \rangle = NC_2.$$

(iii) $U_N \subset U_N^+$ is defined via the relations $[u_{ij}, u_{kl}] = 0$ and $[u_{ij}, \bar{u}_{kl}] = 0$, which tell us that the operators T_π , with $\pi = \begin{smallmatrix} \circ & \circ \\ \circ & \circ \end{smallmatrix}$ and $\pi = \begin{smallmatrix} \circ & \circ \\ \bullet & \bullet \end{smallmatrix}$, must be in the associated Tannakian category C . Thus, as claimed,

$$C = \text{span}(T_\pi \mid \pi \in D) \quad \text{with } D = \langle \mathcal{NC}_2, \begin{smallmatrix} \circ & \circ \\ \circ & \circ \end{smallmatrix}, \begin{smallmatrix} \circ & \circ \\ \bullet & \bullet \end{smallmatrix} \rangle = \mathcal{P}_2.$$

(iv) In order to deal with O_N , we can use the formula $O_N = O_N^+ \cap U_N$. At the categorical level, this tells us that the associated Tannakian category is

given by

$$C = \text{span}(T_\pi \mid \pi \in D) \quad \text{with } D = \langle NC_2, \mathcal{P}_2 \rangle = P_2,$$

as claimed. □

There are many other examples of easy quantum groups, as for instance the permutation group S_N and its free analogue S_N^+ , the corresponding categories being here the category of all partitions P and the category of noncrossing partitions $NC \subset P$. See [10].

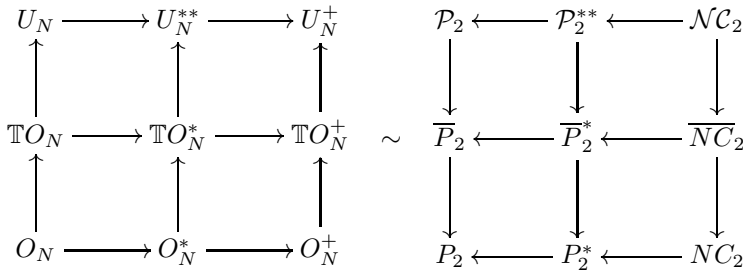
In the pairing case, however, the main examples remain those in Proposition 3.3. Now observe that the four quantum groups there are precisely the unitary groups of the four main geometries from Proposition 2.7. We are therefore led to the following notion:

Definition 3.4. A noncommutative geometry is called easy if its associated unitary group $O_N \subset G \subset U_N^+$ is easy, coming from a category of pairings $NC_2 \subset D \subset P_2$.

Our first task is to prove that the nine geometries from Theorem 2.10 are all easy. For this purpose, let $P_2^* \subset P_2$ be the category of pairings having the property that when flattening the pairing (which means rotating, as for the resulting pairing to have only lower legs), each string has an even number of points between its legs. Let also $\overline{P}_2 \subset P_2$ be the category of pairings having the property that when flattening the pairing, the number of \circ symbols equals the number of \bullet symbols. Finally, let us define as well categories $\mathcal{P}_2^{**}, \overline{\mathcal{P}}_2^*, \overline{NC}_2$ in the obvious way, by taking intersections.

With these notions in hand, we have the following result:

Proposition 3.5. *The basic nine geometries are easy, with quantum groups as follows:*



The diagram on the right-hand side describes the corresponding categories of partitions.

Proof. The idea is to convert the defining relations for the quantum group into statements regarding certain operators of type T_π , and then compute the category of partitions generated by these defining partitions π . More precisely:

(i) First of all, the four results at the corners are known from Proposition 3.3. Also, the result for O_N^* is known from [11], and its unitary version,

for U_N^{**} , is known from [1]. We are therefore left with proving the three results corresponding to the middle rows.

(ii) As a first observation here, the result is clear for $\mathbb{T}O_N$, by using an elementary approach. Indeed, if we denote the standard corepresentation by $u = zv$, with $z \in \mathbb{T}$ and with $v = \bar{v}$, then in order to have $\text{Hom}(u^{\otimes k}, u^{\otimes l}) \neq \emptyset$, the z variables must cancel, and in the case where they cancel, we obtain the same Hom-space as for O_N . Now since the cancelling property for the z variables corresponds precisely to the fact that k, l must have the same numbers of \circ symbols minus \bullet symbols, the associated Tannakian category must come from the category of pairings $\bar{P}_2 \subset P_2$, as claimed.

(iii) In order to deal now with the free version $\mathbb{T}O_N^+$, no such shortcut is available here, and we must use the regular, abstract method. So, observe that the defining relations for this quantum group, namely $ab^* = a^*b$, correspond to the following diagram:



Thus the associated category of partitions is

$$D = \langle \mathcal{NC}_2, \begin{array}{c} \circ \\ | \\ \bullet \end{array} \rangle = \overline{NC}_2,$$

as claimed.

(iv) Finally, since we have $\mathbb{T}O_N^* = \mathbb{T}O_N^+ \cap U_N^{**}$, here the associated category of partitions follows to be

$$D = \langle \overline{NC}_2, \mathcal{P}_2^{**} \rangle = \bar{P}_2^*.$$

This finishes the proof. □

Now back to the general case, our claim is that, for an easy geometry, there are a few simplifications in the axioms. We first have the following result:

Proposition 3.6. *For a geometry which is easy, coming from a category of pairings $\mathcal{NC}_2 \subset D \subset P_2$, the associated quantum group is given by*

$$C(G) = C(U_N^+) / \langle T_\pi \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) \mid \forall k, l, \forall \pi \in D(k, l) \rangle$$

and the associated noncommutative torus is $T = \hat{\Gamma}$ with

$$\Gamma = F_N / \left\langle g_{i_1} \dots g_{i_k} = g_{j_1} \dots g_{j_l} \mid \forall i, j, k, l, \exists \pi \in D(k, l), \delta_\pi \binom{i}{j} \neq 0 \right\rangle.$$

Moreover, in both cases, we can just use partitions π which generate the category D .

Proof. The first assertion is well known, see [10, 23]. If we denote by $g_i = u_{ii}$ the standard coordinates on the associated torus T , then

$$\begin{aligned} C(T) &= [C(U_N^+) / \langle T_\pi \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) \mid \forall \pi \in D \rangle] / \langle u_{ij} = 0 \mid \forall i \neq j \rangle \\ &= [C(U_N^+) / \langle u_{ij} = 0 \mid \forall i \neq j \rangle] / \langle T_\pi \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) \mid \forall \pi \in D \rangle \\ &= C^*(F_N) / \langle T_\pi \in \text{Hom}(g^{\otimes k}, g^{\otimes l}) \mid \forall \pi \in D \rangle, \end{aligned}$$

where $g = \text{diag}(g_1, \dots, g_N)$.

The associated discrete group, $\Gamma = \widehat{T}$, is therefore given by

$$\Gamma = F_N / \langle T_\pi \in \text{Hom}(g^{\otimes k}, g^{\otimes l}) \mid \forall \pi \in D \rangle.$$

Now observe that, again with $g = \text{diag}(g_1, \dots, g_N)$, we have

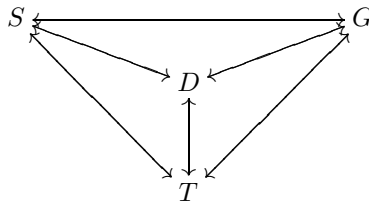
$$\begin{aligned} T_\pi g^{\otimes k}(e_{i_1} \otimes \dots \otimes e_{i_k}) &= \sum_{j_1 \dots j_l} \delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_l} \cdot g_{i_1} \dots g_{i_k}, \\ g^{\otimes l} T_\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) &= \sum_{j_1 \dots j_l} \delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} e_{j_1} \otimes \dots \otimes e_{j_l} \cdot g_{j_1} \dots g_{j_l}. \end{aligned}$$

We conclude that the relation $T_\pi \in \text{Hom}(g^{\otimes k}, g^{\otimes l})$ reformulates as follows:

$$\delta_\pi \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} \neq 0 \implies g_{i_1} \dots g_{i_k} = g_{j_1} \dots g_{j_l}.$$

Thus we obtain the formula in the statement. Finally, the last assertion follows from Tannakian duality in the quantum group case, and then in the torus case as well. \square

We conjecture that, in the case of an easy geometry, the category D determines everything, and is determined by everything. Thus, in this case we should have full correspondences, between all the objects involved, with D being the central object:



Observe the similarity with the usual real and complex geometries, which have as well a central object, namely \mathbb{R}^N and \mathbb{C}^N , respectively. Thus, in a certain abstract sense, for an easy geometry, D is the analogue of the ambient space $\mathbb{R}^N, \mathbb{C}^N$, which cannot be axiomatized.

The following definition provides two more abstract notions:

Definition 3.7. A noncommutative geometry is called

- (i) amenable if the discrete quantum group \widehat{G} is amenable;
- (ii) weakly amenable if the discrete group $\Gamma = \widehat{T}$ is amenable.

Here we use the usual amenability notion for the discrete groups and for the discrete quantum groups. For the general theory regarding this latter notion, see [24].

We should mention that, for all the known examples of noncommutative geometries in our sense, the coamenability of G is equivalent to the coamenability of T . We conjecture that this should be true in general, but have no idea on how to prove this.

Let us discuss now more in detail the half-classical real geometry, associated to O_N^* . We recall that the projective version of a quantum subgroup $G \subset O_N^+$ is the quotient quantum group $G \rightarrow PG$ having $w_{ij,kl} = u_{ik}u_{jl}$ as fundamental corepresentation. In the classical case, $G \subset O_N$, we recover in this way the usual projective version.

Theorem 3.8. *The following uniqueness results hold true:*

- (i) O_N^* is the unique easy quantum group $O_N \subset G \subset O_N^+$.
- (ii) O_N^* is maximal coamenable, in the easy framework.
- (iii) O_N^* is the biggest quantum group whose projective version PO_N^* is classical.

Proof. The results are well known. We recall the ideas of their proofs because they will serve as inspiration for various unitary generalizations, to be done below.

(i) This result is from [11], the idea being that P_2^* is the unique intermediate category of partitions $NC_2 \subset D \subset P_2$. We should mention that, conjecturally, O_N^* is the unique quantum group $O_N \subset G \subset O_N^+$, even without the easiness assumption. See [6].

(ii) The precise claim is that O_N^* is coamenable and, in addition, maximal with this coamenability property, in the easy quantum group framework. The coamenability is known from [11]; the maximality claim follows from (i).

(iii) This is well known too, see again [11]. Indeed, the relations $abc = cba$ are equivalent to the relations $abcd = cdab$, as shown by the following two computations:

$$[abc = cba] \implies [abcd = cbad = cdab],$$

$$[abcd = cdab] \implies \left[abc = \sum_d abcdd = \sum_d cdabd = \sum_d cbdda = cba \right].$$

Here we assume that all the variables are standard coordinates, and we have used the quadratic condition relating these coordinates, namely $\sum_d d^2 = 1$. \square

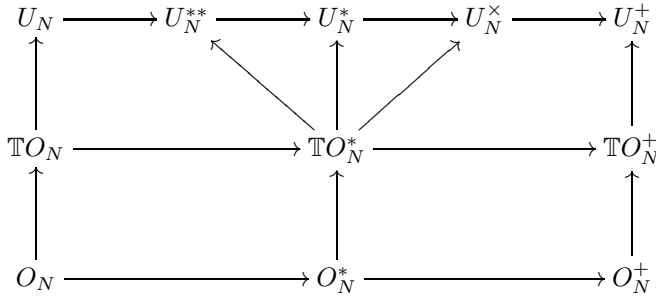
4. COMPLEX GEOMETRIES

In this and the next section, we discuss the construction of the complex half-classical geometry. We will proceed in two steps:

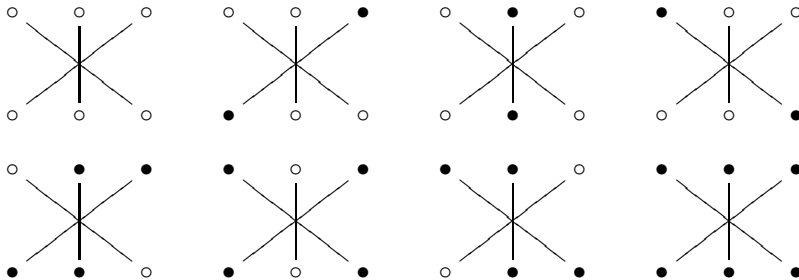
(i) In this section, we discuss a first extension of the U_N^{**} geometry, with unitary group denoted by U_N^\times . This quantum group is the one constructed

in [12, 13], and denoted there by U_N^* . The problem, however, is that this geometry is not amenable.

(ii) In Section 5, we construct and study the “correct” complex analogue of the O_N^* geometry, with unitary group denoted by U_N^* . This quantum group is the one constructed in [5], and denoted there by $U_{N,\infty}$. The enlarged picture will look as follows:

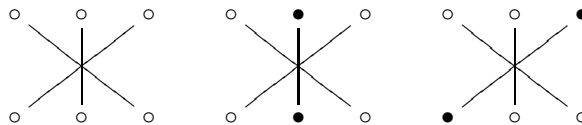


In order to get started, our first task is to look for extensions of the U_N^{**} geometry. This geometry is by definition easy, coming from the following diagrams:



These diagrams stand for the relations $abc = cba$, $abc^* = c^*ba$, and so on, up to $a^*b^*c^* = c^*b^*a^*$. For more about such pictures and relations, we refer to [27].

There are some obvious equivalences between these relations, and by erasing the corresponding diagrams, we are led to three diagrams, namely:



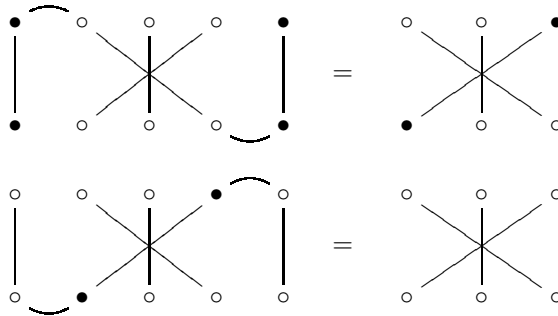
In order to extend now the U_N^{**} geometry, the idea would be that of picking one of these diagrams and using the corresponding relations. But here, we have:

Proposition 4.1. Consider the following types of relations between abstract variables $a, b, c \in \{x_i\}$ subject to the relations $\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1$:

- (i) $abc = cba$.
- (ii) $ab^*c = cb^*a$.
- (iii) $abc^* = c^*ba$.

We have then (i) \iff (iii) \implies (ii).

Proof. The equivalence (i) \iff (iii) follows from the computations



As for (i) + (iii) \implies (ii), this is best worked out at the algebraic level:

$$ab^*c = \sum_d ab^*cdd^* = \sum_d adcb^*d^* = \sum_d cdab^*d^* = \sum_d cb^*add^* = cb^*a.$$

Thus we have indeed (i) \iff (iii) \implies (ii), as claimed. □

Now, by getting back to our problem and, more specifically, to our above-mentioned idea of using one diagram out of three possible ones, we can see, as a consequence of Proposition 4.1, that we have only one good choice. We are led to the following definition:

Definition 4.2. We have a sphere, a torus, and a quantum group as follows:

- (i) $S_{\mathbb{C}, \times}^{N-1} \subset S_{\mathbb{C}, +}^{N-1}$, obtained via $ab^*c = cb^*a$, with $a, b, c \in \{x_i\}$.
- (ii) $T_N^\times \subset \widehat{F}_N$, obtained via $ab^{-1}c = cb^{-1}a$, with $a, b, c \in \{g_i\}$.
- (iii) $U_N^\times \subset U_N^+$, obtained via $ab^*c = cb^*a$, with $a, b, c \in \{u_{ij}\}$.

Observe that U_N^\times is indeed a quantum group, containing U_N^{**} , and this because we are imposing to the standard coordinates of U_N^+ certain easy relations. It is of course possible to check Woronowicz’s axioms in [29] as well, directly. We will prove later on that we have indeed a noncommutative geometry, in the sense of Definition 2.5.

Let us clarify now a few more algebraic issues. The most elegant approach to the U_N^\times geometry is in fact via projective space theory, using the following notions:

Definition 4.3. Given a subspace $X \subset S_{\mathbb{C},+}^{N-1}$, we define quotients as follows:

- (i) Left projective version: $X \rightarrow PX$, with coordinates $p_{ij} = x_i x_j^*$.
- (ii) Right projective version: $X \rightarrow P'X$, with coordinates $q_{ij} = x_j^* x_i$.
- (iii) Full projective version: $X \rightarrow \mathcal{P}X$, with coordinates p_{ij}, q_{ij} .

We say that X is left/right/full half-classical if these spaces are classical.

Observe that in the classical case, $X \subset S_{\mathbb{C}}^{N-1}$, the three projective versions coincide and equal the usual projective version obtained by dividing under the action of \mathbb{T} .

In the real case, $X \subset S_{\mathbb{R},+}^{N-1}$, the three projective versions coincide as well, and X is left or right half-classical if $X \subset S_{\mathbb{R},*}^{N-1}$. This follows indeed from Theorem 3.8.

In relation now with $S_{\mathbb{C},\times}^{N-1}, S_{\mathbb{C},**}^{N-1}$, we can use the following simple fact from [5].

Proposition 4.4. Let $X \subset S_{\mathbb{C},+}^{N-1}$, with coordinates x_1, \dots, x_N .

- (i) $X \subset S_{\mathbb{C},\times}^{N-1}$ precisely when $\{x_i x_j^*\}$ commute, and $\{x_i^* x_j\}$ commute as well.
- (ii) $X \subset S_{\mathbb{C},**}^{N-1}$ precisely when the variables $\{x_i x_j, x_i x_j^*, x_i^* x_j, x_i^* x_j^*\}$ all commute.

Proof. Regarding the first assertion, the implication \implies follows from

$$ab^*cd^* = cb^*ad^* = cd^*ab^*, \quad a^*bc^*d = c^*ba^*d = c^*da^*b.$$

The implication \impliedby is obtained as follows, by using the commutation assumptions in the statement, and by summing over $e = x_i$:

$$ae^*eb^*c = ab^*ce^*e = ce^*ab^*e = cb^*ee^*a \implies ab^*c = cb^*a.$$

The proof of the second assertion is similar, because we can remove all the $*$ signs, except for those concerning e^* , and use the above computations with $a, b, c, d \in \{x_i, x_i^*\}$. □

The assertions in the next proposition follow from Proposition 4.4.

Proposition 4.5. We have the following results:

- (i) $S_{\mathbb{C},\times}^{N-1}$ is left and right half-classical; it is maximal with this property.
- (ii) $S_{\mathbb{C},**}^{N-1}$ is fully half-classical.
- (iii) $S_{\mathbb{R},*}^{N-1}$ is fully half-classical; it is maximal inside $S_{\mathbb{R},+}^{N-1}$ with this property.

We still have an issue to be clarified, namely to prove that, in the diagram drawn in the beginning of this section, U_N^\times sits indeed above $\mathbb{T}O_N^*$. This comes from:

Proposition 4.6. We have $\mathbb{T}O_N^+ \cap U_N^\times = \mathbb{T}O_N^*$, as quantum subgroups of U_N^+ .

Proof. According to the definition of $\mathbb{T}O_N^*$, from Section 2, this quantum group appears as $\mathbb{T}O_N^* = \mathbb{T}O_N^+ \cap U_N^{**}$. Thus, we must prove $\mathbb{T}O_N^+ \cap U_N^\times \subset U_N^{**}$.

In terms of defining relations, we must prove that $ab^* = a^*b$ and $ab^*c = cb^*a$ for any $a, b, c \in \{u_{ij}\}$ implies $abc = cba$ for any $a, b, c \in \{u_{ij}, u_{ij}^*\}$. But this is clear, because using $ab^* = a^*b$, we can first obtain $a^*bc = cba^*$, and then, by using Proposition 4.1, we can obtain from this the other relations as well.

Here we have used the fact that what we know about abstract variables satisfying $\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1$ applies to the coordinates to any closed subgroup $G \subset U_N^+$, simply because these coordinates, when rescaled by \sqrt{N} , do satisfy these relations. □

We recall that the free complexification of a compact quantum group G , with standard coordinates denoted by v_{ij} , is the compact quantum group \tilde{G} corresponding to the subalgebra $C(\tilde{G}) \subset C(\mathbb{T}) * C(G)$ generated by the variables $u_{ij} = zv_{ij}$, where z is the standard generator of $C(\mathbb{T})$. Observe that \tilde{G} is indeed a quantum group, because it appears as a subgroup of $\mathbb{T} \hat{*} G$, the quantum group associated to $C(\mathbb{T}) * C(G)$. See [25].

Following [3], we can formulate the next result.

Proposition 4.7. *The quantum groups $O_N^*, U_N, U_N^{**}, U_N^\times$ have the following properties:*

- (i) *They have the same left projective version, equal to PU_N .*
- (ii) *They have the same free complexification, equal to U_N^\times .*

Proof. With terminology and notations from [3], the idea is as follows:

(i) The equality $PO_N^* = PU_N$ is well known from [11]. It is clear as well that $PU_N \subset PU_N^{**} \subset PU_N^\times$. Using Proposition 4.4 (i), we conclude that PU_N^\times is classical; so we must have $PU_N^\times \subset (PU_N^+)_{\text{class}}$. But this latter space $(PU_N^+)_{\text{class}}$ is known to be equal to PU_N , which finishes the proof. See [3, 12, 13].

(ii) If we denote by v_{ij} the standard coordinates on $G = O_N^*, U_N, U_N^{**}, U_N^\times$, and by z the generator of a copy of $C(\mathbb{T})$, free from $C(G)$, then with $a, b, c \in \{v_{ij}\}$ we have

$$(za)(zb)^*(zc) = zab^*c = zcb^*a = (zc)(zb)^*(za).$$

Thus we have $\tilde{G} \subset U_N^\times$. Conversely now, it follows from the general theory of the free complexifications of easy quantum groups [25] that both $K = \tilde{G}, U_N^\times$ should appear as free complexifications of certain intermediate easy quantum groups $O_N \subset H \subset O_N^+$. On the other hand, since $PH = \tilde{P}\tilde{H} = PK = PU_N$, the only choice here is $H = O_N^*$. Thus we have $\tilde{G} = U_N^\times = \tilde{O}_N^*$, and this finishes the proof. See [3]. □

Let us prove the quantum isometry group result. This is known from [3], but we have now a much simpler proof. We use the following notion:

Definition 4.8. Given a closed subgroup $G \subset U_N^+$ and a closed subset $X \subset S_{\mathbb{C},+}^{N-1}$, we say that G acts projectively on X , and write $PG \curvearrowright PX$, if

$$\Phi(x_i x_j^*) = \sum_{ab} u_{ia} u_{jb}^* \otimes x_a x_b^*$$

defines a morphism of C^* -algebras $\Phi : C(PX) \rightarrow C(PG) \otimes C(PX)$.

Note the similarity to Definition 2.3 which deals with the affine case. As in the affine case, such a morphism is automatically coassociative and counital. Observe also that any affine action $G \curvearrowright X$ produces a projective action $PG \curvearrowright PX$. See [9].

We can now formulate our quantum isometry group results:

Proposition 4.9. *The following implications hold true:*

- (i) $PG \curvearrowright P_{\mathbb{C}}^{N-1}$ implies $G \subset U_N^{\times}$.
- (ii) $G \curvearrowright S_{\mathbb{C}, \times}^{N-1}$ implies $G \subset U_N^{\times}$.

Proof. Since $G \curvearrowright S_{\mathbb{C}, \times}^{N-1}$ implies $PG \curvearrowright PS_{\mathbb{C}, \times}^{N-1} = P_{\mathbb{C}}^{N-1}$, we just have to prove the first assertion. For this purpose, we use an old method from [14], as in [4].

Let us consider a coaction map, written

$$\Phi(p_{ij}) = \sum_{ab} u_{ia} u_{jb}^* \otimes p_{ab},$$

with $p_{ab} = z_a \bar{z}_b$. The idea is to use the formula $p_{ab} p_{cd} = p_{ad} p_{cb}$. We have

$$\begin{aligned} \Phi(p_{ij} p_{kl}) &= \sum_{abcd} u_{ia} u_{jb}^* u_{kc} u_{ld}^* \otimes p_{ab} p_{cd}, \\ \Phi(p_{il} p_{kj}) &= \sum_{abcd} u_{ia} u_{ld}^* u_{kc} u_{jb}^* \otimes p_{ad} p_{cb}. \end{aligned}$$

The left terms being equal, and the last terms on the right being equal too, we deduce, with $[a, b, c] = abc - cba$, the following equality:

$$\sum_{abcd} u_{ia} [u_{jb}^*, u_{kc}, u_{ld}^*] \otimes p_{ab} p_{cd} = 0.$$

In order to exploit this equality, we use basic tensor product theory. The trick when having a formula of type $\sum_i a_i \otimes b_i = 0$ is to compact the elements on the right, by using linear dependence, and then to conclude that the elements on the left must vanish.

In our case, since the quantities $p_{ab} p_{cd} = z_a \bar{z}_b z_c \bar{z}_d$ on the right depend only on the numbers $|\{a, c\}|, |\{b, d\}| \in \{1, 2\}$, and this dependence produces the only possible linear relations between the variables $p_{ab} p_{cd}$, we are led to the following $2 \times 2 = 4$ equations:

- (2) $u_{ia} [u_{jb}^*, u_{ka}, u_{lb}^*] = 0$ for all a, b ,
- (3) $u_{ia} [u_{jb}^*, u_{ka}, u_{ld}^*] + u_{ia} [u_{jd}^*, u_{ka}, u_{lb}^*] = 0$ for all a and $b \neq d$,
- (4) $u_{ia} [u_{jb}^*, u_{kc}, u_{lb}^*] + u_{ic} [u_{jb}^*, u_{ka}, u_{lb}^*] = 0$ for all $a \neq c$ and b ,
- (5) $u_{ia} ([u_{jb}^*, u_{kc}, u_{ld}^*] + [u_{jd}^*, u_{kc}, u_{lb}^*])$
 $+ u_{ic} ([u_{jb}^*, u_{ka}, u_{ld}^*] + [u_{jd}^*, u_{ka}, u_{lb}^*]) = 0$ for all $a \neq c$ and $b \neq d$.

Let us process now all these formulae. (We will not need (4) and (5).) From (2) and (3) we conclude that (3) holds with no restriction on the indices.

Multiplying now this formula to the left by u_{ia}^* , and then summing over i , we obtain

$$[u_{jb}^*, u_{ka}, u_{ld}^*] + [u_{jd}^*, u_{ka}, u_{lb}^*] = 0.$$

By applying now the antipode, then the involution, and finally by suitably relabelling all the indices, we successively obtain

$$\begin{aligned} & [u_{dl}, u_{ak}^*, u_{bj}] + [u_{bl}, u_{ak}^*, u_{dj}] = 0 \\ \implies & [u_{dl}^*, u_{ak}, u_{bj}^*] + [u_{bl}^*, u_{ak}, u_{dj}^*] = 0 \\ \implies & [u_{ld}^*, u_{ka}, u_{jb}^*] + [u_{jd}^*, u_{ka}, u_{lb}^*] = 0. \end{aligned}$$

Comparing with the original relation above, we conclude that

$$[u_{jb}^*, u_{ka}, u_{ld}^*] = [u_{jd}^*, u_{ka}, u_{lb}^*] = 0.$$

Thus we have reached the formulae defining U_N^\times , and we are done. □

We have now all the needed ingredients for proving the following result.

Theorem 4.10. *The sphere $S_{\mathbb{R}, \times}^{N-1}$, the torus T_N^\times and the quantum group U_N^\times form a noncommutative geometry. This geometry is easy and not amenable.*

Proof. The verification of all the axioms is standard. The only nontrivial fact is the universality of the action $U_N^\times \curvearrowright S_{\mathbb{C}, \times}^{N-1}$, coming from Proposition 4.9.

Regarding the non-amenability claim for the geometry that we have, here we must prove that the quantum group U_N^\times is not coamenable. But this can be checked by using a free complexification trick. We know that the discrete group $\Gamma_N^\times = \widehat{T_N^\times}$ is given by

$$\Gamma_N^\times = \langle g_1, \dots, g_N \mid ab^{-1}c = cb^{-1}a, \forall a, b, c \in \{g_i\} \rangle.$$

If we denote by h_1, \dots, h_N the standard generators of \mathbb{Z}^N , and by z the generator of a copy of \mathbb{Z} , which is free from \mathbb{Z}^N , then with $a, b, c \in \{h_i\}$ we have

$$(za)(zb)^{-1}(zc) = zab^{-1}c = zcb^{-1}a = (zc)(zb)^{-1}(za).$$

We therefore have a group morphism $\Gamma_N^\times \rightarrow \mathbb{Z} * \mathbb{Z}^N$, given by $g_i \rightarrow zh_i$. Now observe that the image of this morphism contains the following two elements:

$$(zh_1)^{-1}(zh_2) = h_1^{-1}h_2, \quad (zh_1)(zh_2)^{-1} = zh_1h_2^{-1}z^*.$$

These elements being free, we obtain a copy of F_2 inside the image. Thus the image, and then Γ_N^\times , and then $\widehat{U_N^\times}$ itself, follow to be non-amenable as well. □

The U_N^\times geometry can be further investigated by using various algebraic tricks, notably the free complexification formula $U_N^\times = \widehat{U_N}$. We refer to [3].

5. HALF-CLASSICAL GEOMETRY

In this section, we introduce and study the U_N^* geometry, which is the “correct” complex half-classical one, in the sense that it is the biggest half-classical geometry.

In view of Definition 4.3 above, we can indeed formulate:

Definition 5.1. We have a sphere, a torus, and a quantum group as follows:

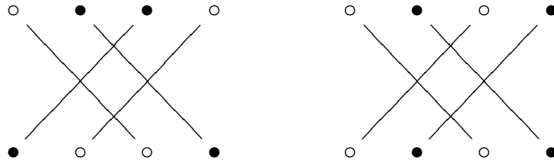
- (i) $S_{\mathbb{C},*}^{N-1} \subset S_{\mathbb{C},+}^{N-1}$, obtained via “ ab^*, a^*b all commute”, with $a, b, c \in \{x_i\}$.
- (ii) $T_N^* \subset \widehat{F}_N$, obtained via “ $ab^{-1}, a^{-1}b$ all commute”, with $a, b, c \in \{g_i\}$.
- (iii) $U_N^* \subset U_N^+$, obtained via “ ab^*, a^*b all commute”, with $a, b, c \in \{u_{ij}\}$.

In other words, these are the biggest half-classical sphere, torus, and quantum group.

As a first remark, the real version of $S_{\mathbb{C},*}^{N-1}$, obtained by imposing the conditions $x_i = x_i^*$ to the standard coordinates, is the half-classical real sphere $S_{\mathbb{R},*}^{N-1}$. Observe also that we have inclusions as follows, coming from the various results in Proposition 4.4:

$$S_{\mathbb{C},**}^{N-1} \subset S_{\mathbb{C},*}^{N-1} \subset S_{\mathbb{C},x}^{N-1}.$$

Similar inclusions are valid for the tori and for the quantum groups. Finally, observe that U_N^* is by definition easy, coming from the following two diagrams:



In order to better understand the construction $X \rightarrow \mathcal{P}X$, and the definition of $S_{\mathbb{C},*}^{N-1}$ itself, let us perform now some explicit computations. By $P_{\mathbb{C}}^{N-1}$ we denote the usual complex projective space.

Proposition 5.2. *The projective versions of $S_{\mathbb{C},**}^{N-1} \subset S_{\mathbb{C},*}^{N-1} \subset S_{\mathbb{C},x}^{N-1}$ are given by*

$$\begin{array}{ccccc}
 S_{\mathbb{C},**}^{N-1} & \longrightarrow & S_{\mathbb{C},*}^{N-1} & \longrightarrow & S_{\mathbb{C},x}^{N-1} \\
 \downarrow & & \downarrow & & \downarrow \\
 P_{\mathbb{C}}^{N-1} \times P_{\mathbb{C}}^{N-1} & \longrightarrow & P_{\mathbb{C}}^{N-1} \times P_{\mathbb{C}}^{N-1} & \longrightarrow & P_{\mathbb{C}}^{N-1} \circ P_{\mathbb{C}}^{N-1}
 \end{array}$$

where the product on the bottom right is constructed by conjugating by a free unitary.

Proof. We use the following presentation result, which comes from the Gelfand theorem and from the fact that $P_{\mathbb{C}}^{N-1}$ is the space of rank 1 projections in $M_N(\mathbb{C})$:

$$C(P_{\mathbb{C}}^{N-1}) = C_{\text{comm}}^* \{ (p_{ij})_{i,j=1,\dots,N} \mid p = p^2 = p^*, \text{Tr}(p) = 1 \}.$$

Let us first discuss the computation of the spaces $\mathcal{P}S_{\mathbb{C},**}^{N-1} \subset \mathcal{P}S_{\mathbb{C},*}^{N-1}$. We know that these spaces are both classical. We also know that the left and right components of these spaces, in the sense of Definition 4.3, are all equal to $P_{\mathbb{C}}^{N-1}$, for instance because their standard generators satisfy the above defining relations for $C(P_{\mathbb{C}}^{N-1})$.

In order to finish, it remains to prove that the subspaces $PS_{\mathbb{C},**}^{N-1}, P'S_{\mathbb{C},**}^{N-1} \subset \mathcal{P}S_{\mathbb{C},**}^{N-1}$, which are both isomorphic to $P_{\mathbb{C}}^{N-1}$, are in generic position. For this purpose, we can use a suitable matrix model, coming from [5]. Let indeed u_i, v_i be the standard coordinates of two independent copies of $S_{\mathbb{C}}^{N-1}$, and consider the following matrices:

$$X_i = \begin{pmatrix} 0 & u_i \\ v_i & 0 \end{pmatrix}, \quad X_i^* = \begin{pmatrix} 0 & \bar{v}_i \\ \bar{u}_i & 0 \end{pmatrix}.$$

We have then $\sum_i X_i X_i^* = \sum_i X_i^* X_i = 1$ and the relations $abc = cba$ hold as well, for any $a, b, c \in \{X_i, X_i^*\}$. Thus we have the following matrix model:

$$C(S_{\mathbb{C},**}^{N-1}) \rightarrow M_2(C(S_{\mathbb{C}}^{N-1} \times S_{\mathbb{C}}^{N-1})), \quad x_i \rightarrow X_i.$$

The point now is that, in this model, we have the following formulae:

$$X_i X_j^* = \begin{pmatrix} u_i \bar{u}_j & 0 \\ 0 & v_i \bar{v}_j \end{pmatrix}, \quad X_j^* X_i = \begin{pmatrix} v_j \bar{v}_i & 0 \\ 0 & u_j \bar{u}_i \end{pmatrix}.$$

Since these matrices are conjugated by an order 2 automorphism, the algebra that they generate is isomorphic to $C(P_{\mathbb{C}}^{N-1} \times P_{\mathbb{C}}^{N-1})$, and this finishes the proof.

Finally, regarding the computation for $S_{\mathbb{C},\times}^{N-1}$, let us denote by $p_{ij} = z_i \bar{z}_j$ the standard coordinates on $P_{\mathbb{C}}^{N-1} = PS_{\mathbb{C}}^{N-1}$. In the usual free complexification model for $S_{\mathbb{C},\times}^{N-1}$, namely $\tilde{S}_{\mathbb{C}}^{N-1}$, we have then

$$x_i x_j^* = p_{ij}, x_j^* x_i = z p_{ij} z^*,$$

and this gives the result. □

In order to verify the axioms, we follow the proof for U_N^{\times} .

Definition 5.3. Given a closed subgroup $G \subset U_N^+$, and a closed subset $X \subset S_{\mathbb{C},+}^{N-1}$, we say that G acts fully projectively on X , and write $\mathcal{P}G \curvearrowright \mathcal{P}X$, if the formulae

$$\Phi(x_i x_j^*) = \sum_{ab} u_{ia} u_{jb}^* \otimes x_a x_b^*, \quad \Phi(x_i^* x_j) = \sum_{ab} u_{ia}^* u_{jb} \otimes x_a^* x_b$$

define a morphism of C^* -algebras $\Phi : C(\mathcal{P}X) \rightarrow C(\mathcal{P}G) \otimes C(\mathcal{P}X)$.

As in the affine case, such a morphism is automatically coassociative and counital. Observe also that any affine action $G \curvearrowright X$ produces a projective action $\mathcal{P}G \curvearrowright \mathcal{P}X$.

Proposition 5.4. *The following quantum isometry group results hold true:*

- (i) *We have an affine action $U_N^* \curvearrowright S_{\mathbb{C},*}^{N-1}$.*
- (ii) *$\mathcal{P}G \curvearrowright P_{\mathbb{C}}^{N-1} \times P_{\mathbb{C}}^{N-1}$ implies $G \subset U_N^*$.*

(iii) $G \curvearrowright S_{\mathbb{C},*}^{N-1}$ implies $G \subset U_N^*$.

(iv) U_N^* is the quantum isometry group of $S_{\mathbb{C},*}^{N-1}$.

Proof. Our first claim is that it is enough to prove (ii). Indeed, in order to prove (i), observe that with $X_i = \sum_a u_{ia} \otimes x_a$, we have the following formulae:

$$X_i X_j^* = \sum_{ab} u_{ia} u_{jb}^* \otimes x_a x_b^*, \quad X_j^* X_i = \sum_{ab} u_{jb}^* u_{ia} \otimes x_b^* x_a.$$

Now since the various variables on the right pairwise commute, the variables on the left commute as well, and so we can define the action map by $x_i \rightarrow X_i$.

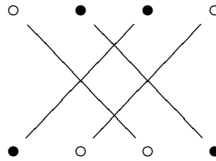
The other remark is that since $G \curvearrowright S_{\mathbb{C},*}^{N-1}$ implies

$$\mathcal{P}G \curvearrowright PS_{\mathbb{C},\times}^{N-1} = P_{\mathbb{C}}^{N-1} \times P_{\mathbb{C}}^{N-1},$$

we have (ii) \implies (iii). Finally, the implication (i) + (iii) \implies (iv) is trivial.

In order to prove now (ii), let us observe that $\mathcal{P}G \curvearrowright P_{\mathbb{C}}^{N-1} \times P_{\mathbb{C}}^{N-1}$ implies $\mathcal{P}G \curvearrowright P_{\mathbb{C}}^{N-1}$. Thus, we can use Proposition 4.9 (i), and we obtain $G \subset U_N^*$.

We are therefore left with proving that $\mathcal{P}G \curvearrowright P_{\mathbb{C}}^{N-1} \times P_{\mathbb{C}}^{N-1}$ implies that the following diagram belongs to the Tannakian category of G :



For this purpose, consider a coaction map, written as in Definition 5.3. By multiplying the two relations there, we obtain

$$\begin{aligned} \Phi(x_i x_j^* x_k^* x_l) &= \sum_{abcd} u_{ia} u_{jb}^* u_{kc}^* u_{ld} \otimes x_a x_b^* x_c^* x_d, \\ \Phi(x_k^* x_l x_i x_j^*) &= \sum_{abcd} u_{kc}^* u_{ld} u_{ia} u_{jb}^* \otimes x_c^* x_d x_a x_b^*. \end{aligned}$$

Assuming now that x_1, \dots, x_N are the standard coordinates on $S_{\mathbb{C},*}^{N-1}$, the products of x variables at left are equal, and so are the products at right. Thus, we have

$$\sum_{abcd} [u_{ia} u_{jb}^*, u_{kc}^* u_{ld}] \otimes x_a x_b^* x_c^* x_d = 0.$$

Now recall that, in view of Proposition 5.2, we can write

$$x_a x_b^* x_c^* x_d = p_{ab} \otimes q_{dc},$$

where p_{ab}, q_{cd} are the standard coordinates on $P_{\mathbb{C}}^{N-1}$. Thus, our formula becomes

$$\sum_{abcd} [u_{ia} u_{jb}^*, u_{kc}^* u_{ld}] \otimes p_{ab} \otimes q_{dc} = 0.$$

Since the variables on the right are linearly independent, we obtain that all the commutators vanish, and this finishes the proof. \square

We can now formulate our main result:

Theorem 5.5. *We have a noncommutative geometry, with sphere $S_{\mathbb{C},*}^{N-1}$, torus T_N^* and quantum group U_N^* . This is the biggest geometry having a classical projective version.*

Proof. The verification of all the axioms is standard. The only nontrivial fact is the universality of the action $U_N^* \curvearrowright S_{\mathbb{C},*}^{N-1}$, coming from Proposition 5.4. The last assertion, where “biggest” means as usual “maximal”, is clear. \square

In view of the above result, the U_N^* geometry as constructed above seems to be the “correct” complex half-classical geometry. This geometry waits of course to be developed, with the potential questions concerning the submanifolds $X \subset S_{\mathbb{C},*}^{N-1}$ being a priori as many as the questions concerning the submanifolds $X \subset S_{\mathbb{C}}^{N-1}$, or perhaps $X \subset S_{\mathbb{R}}^{N-1}$, with the remark of course that there are whole books written on these latter manifolds. As a very first question here, it would be interesting to work out the analogues of [15, 16] in the present setting. Finally, we conjecture that the U_N^* geometry is amenable. Moreover, it is maximal with this amenability property, at least in the easy framework.

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Received March 11, 2017; accepted August 10, 2017

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