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Relative continuous K-theory and cyclic homology

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To Peter Schneider

Introduction

0.1. Let A be a unital associative p-adic ring, p is a prime, so $A = \varprojlim A_i$ where $A_i := A/p^i$. Let $I \subseteq A$ be a two-sided ideal such that the I-adic topology equals the p-adic one, i.e., $p^m \in I$ and $I^m \subseteq pA$ for large enough m. Set $I_i := IA_i$.

The projective system of rings A_i and ideals I_i yields the relative K-theory pro-spectrum $K(A,I)^\wedge$:= "lim"$K(A_i, I_i)$ and the cyclic chain pro-complex $CC(A)^\wedge$:= "lim"$CC(A_i)$. Our main result, see 2.2, is a construction (subject to minor conditions on A) of a natural homotopy equivalence of the corresponding pro-spectra up to quasi-isogeny

$$K(A,I)^\wedge_{\mathbb{Q}} \xrightarrow{\sim} CC(A)^\wedge[1]_{\mathbb{Q}}. \tag{0.1.1}$$

This homotopy equivalence is a continuous version of Goodwillie's isomorphism $K(R,J) \xrightarrow{\sim} CC(R,J)[1]$ valid for any $\mathbb{Q}$-algebra R and a nilpotent two-sided ideal J, see [5] or [9, 11.3] The construction follows closely that of Goodwillie, with a characteristic p simplification (the difference between $CC(A)^\wedge$ and $CC(A,I)^\wedge$, being of torsion, disappears) and the Malcev theory input replaced by a version of Lazard's theory.

0.2. Here is a geometric application (see 2.4): Let E be a p-adic field, O_E its ring of integers, X be a proper O_E-scheme with smooth generic fiber X_E, $Y \subseteq X$ be any closed subscheme whose support equals the closed fiber. Set

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$X_i := X \otimes \mathbb{Z}/p^i$. Then (0.1.1) yields a canonical identification of $\mathbb{Q}$-vector spaces
(0.2.1)
$$\mathbb{Q} \otimes \varprojlim K^B_{n-1}(X_i, Y) \xrightarrow{\sim} \mathbb{Q} \otimes R\varprojlim K^B_{n-1}(X_i, Y) \xrightarrow{\sim} \bigoplus_a H^{2a-n}_{\mathrm{dR}}(X_E)/F^a.$$

Here K^B is the nonconnective K-theory (see [16, § 6]) and $F^\bullet$ is the Hodge filtration on the de Rham cohomology.

The subjects left untouched: (i) comparison of the composition of (0.2.1) and the boundary map $K^B_n(Y) \to \varprojlim K^B_{n-1}(X_i, Y)$ with the crystalline characteristic class map for Y (here Y can be arbitrary singular, see [2] for the basics of the crystalline story in that setting), and (ii) comparison with the p-adic regulators theory (see [14]).

0.3. This article comes from an attempt to understand the work of Bloch, Esnault, and Kerz [1] where an identification similar to (0.2.1) was constructed under extra assumptions on X. The method of [1] is different (and it remains to be checked that the two isomorphisms coincide). Its input is McCarthy's identification of K-theory of an *arbitrary* ring relative to a nilpotent ideal with the relative *topological* cyclic homology, see [3]. One has then to identify the topological cyclic homology with truncated de Rham cohomology. Unlike the classical cyclic homology case, this presents a problem, and the passage taken in [1]—with étale cohomology of X_E as a way station—is not easy.

It would be nice to find conditions on (R, J), where p is nilpotent in R and J nilpotent, that would imply $K_i(R, J) = H_{i-1}CC(R, J)$ for $i < p$, cp. [1, 8.5].

1. Spectral preliminaries

Given a sequence $\cdots \to A_1 \to A_0$ of abelian groups, then $(\varprojlim A_i) \otimes \mathbb{Q}$ is not determined if we know merely $A_i \otimes \mathbb{Q}$, but we are in good shape if A_i are known up to subgroups of torsion whose exponent is bounded by a constant independent of i. We will see that some natural maps known to be isomorphisms in rational homotopy theory are, in fact, invertible up to universally bounded denominators. Thus they remain to be isomorphisms in the rational homotopy theory of pro-spaces.

1.1. **Spectra up to isogeny.** (a) Let $\mathcal{C}$ be an additive category. We say that a nonzero integer n kills an object X of $\mathcal{C}$ if $n\,\mathrm{id}_X = 0$; X is a *bounded torsion* object if it is killed by some n as above. A map $f : X \to Y$ in $\mathcal{C}$ is an *isogeny* if there is $g : Y \to X$ such that $fg = n\,\mathrm{id}_Y$ and $gf = n\,\mathrm{id}_X$ for some nonzero integer n.

We denote by $\mathcal{C} \otimes \mathbb{Q}$ the category equipped with functor $\mathcal{C} \to \mathcal{C}_{\mathbb{Q}}$, $X \mapsto X_{\mathbb{Q}}$, that is bijective on objects and yields identification $\mathrm{Hom}(X_{\mathbb{Q}}, Y_{\mathbb{Q}}) = \mathrm{Hom}(X, Y) \otimes \mathbb{Q}$. We call $X_{\mathbb{Q}}$ the *object up to isogeny* that corresponds to X. Notice that f is an isogeny if and only if $f_{\mathbb{Q}}$ is invertible, and $\mathcal{C} \otimes \mathbb{Q}$ is the localization of $\mathcal{C}$ with respect to isogenies; one has $X_{\mathbb{Q}} = 0$, i.e., X is isogenous to 0, if and only if X is a bounded torsion object.

If $\mathcal{I}$ is a category and $X, Y : \mathcal{I} \to \mathcal{C}$ are functors, then a morphism of functors $f : X \to Y$ is said to be an isogeny if there is a morphism $g : G \to F$ such that $fg = n\,\mathrm{id}_Y$ and $gf = n\,\mathrm{id}_X$ for some nonzero integer n. (If $\mathcal{I}$ is essentially small, this means that f is an isogeny in the category $\mathcal{C}^{\mathcal{I}}$.)

If $\mathcal{C}$ is abelian then $\mathcal{C}\otimes\mathbb{Q}$ is abelian as well, and it coincides with the quotient $\mathcal{C}_{\mathbb{Q}}$ of $\mathcal{C}$ modulo the Serre subcategory of bounded torsion objects. If $\mathcal{C}$ carries a tensor structure then bounded torsion objects form an ideal, hence $\mathcal{C}_{\mathbb{Q}}$ is a tensor category and $\mathcal{C} \to \mathcal{C}_{\mathbb{Q}}$ is a tensor functor. In particular, the category $\mathcal{A}b$ of abelian groups yields the tensor abelian category $\mathcal{A}b_{\mathbb{Q}}$.

Remark. The functor $\mathcal{A}b_{\mathbb{Q}} \to \mathcal{V}ect_{\mathbb{Q}}$, $X_{\mathbb{Q}} \mapsto X \otimes \mathbb{Q}$, is *not* an equivalence of categories (for the category $\mathcal{V}ect_{\mathbb{Q}}$ of $\mathbb{Q}$-vector spaces is the quotient of $\mathcal{A}b$ modulo the Serre subcategory of groups all of whose elements are torsion). One has $\mathrm{Ext}^1(X_{\mathbb{Q}}, Y_{\mathbb{Q}}) = \mathrm{Ext}^1(X, Y) \otimes \mathbb{Q}$, hence $\mathcal{A}b_{\mathbb{Q}}$ has homological dimension 1 (since Ext^1 in $\mathcal{A}b_{\mathbb{Q}}$ is right exact). $\mathcal{A}b_{\mathbb{Q}}$ does not admit infinite direct sums and products.

It would be nice to refine the Quillen–Sullivan theory of rational homotopy types to homotopy types up to isogeny (i.e., modulo Serre's class of abelian groups of bounded torsion). For the present article only stable theory is needed.

(b) For an ∞-category treatment of the theory of spectra, see [11, 1.4.3].

Let $\mathcal{S}p$ be the stable ∞-category of spectra; by abuse of notation, we denote by $\mathcal{S}p$ its homotopy category as well. This is a t-category with heart $\mathcal{A}b$: the subcategory $\mathcal{S}p_{\geq 0}$ is formed by connective spectra, the t-structure homology functor is the (stable) homotopy groups functor $X \mapsto \pi_\bullet(X)$, the t-structure truncations $X \to \tau_{\leq n}X$ are Postnikov's truncations. The t-structure is nondegenerate: if all $\pi_n(X) = 0$ then $X = 0$. $\mathcal{S}p$ is a symmetric tensor ∞-category with respect to the smash product $\wedge$, see [11, 6.3.2]. It is right t-exact and induces the usual tensor product on the heart $\mathcal{A}b$; the unit object is the sphere spectrum S.

Let $\mathcal{D}(\mathcal{A}b)$ be the stable ∞-category of chain complexes of abelian groups, see [11, 1.3.5.3, 8.1.2.8, 8.1.2.9]; we denote again by $\mathcal{D}(\mathcal{A}b)$ its homotopy category which is the derived category of $\mathcal{A}b$ with its standard t-structure. $\mathcal{D}(\mathcal{A}b)$ is a symmetric tensor ∞-category with the usual tensor product $\otimes$.

The Eilenberg–MacLane functor $\mathrm{EM} : \mathcal{D}(\mathcal{A}b) \to \mathcal{S}p$ that identifies complexes with abelian spectra is naturally an ∞-category functor. EM is t-exact and equals $\mathrm{Id}_{\mathcal{A}b}$ on the heart. It sends rings to rings and modules to modules, so the 0th Eilenberg–Maclane spectrum $\mathbb{Z}_{\mathcal{S}p} := \mathrm{EM}(\mathbb{Z})$ is a unital ring spectrum and EM lifts to a functor $\mathcal{D}(\mathcal{A}b) \to \mathbb{Z}_{\mathcal{S}p}$-mod (:= the ∞-category of $\mathbb{Z}_{\mathcal{S}p}$-modules in $\mathcal{S}p$). The latter functor is an equivalence of categories, see [11, 8.1.2.13].

Let $\mathcal{S}p^- := \cup_n \mathcal{S}p_{\geq 0}[-n]$ be the stable ∞-subcategory of eventually connective spectra; this is a tensor subcategory of $\mathcal{S}p$. A map $X \to Y$ in $\mathcal{S}p^-$ is called *quasi-isogeny* if all maps $\pi_n(X) \to \pi_n(Y)$ are isogenies, or, equivalently, all maps $\tau_{\leq n}X \to \tau_{\leq n}Y$ are isogenies in the homotopy category of spectra. Thus $X \in \mathcal{S}p^-$ is quasi-isogenous to 0 if each $\pi_n(X)$ is a bounded torsion group, or, equivalently, every $\tau_{\leq n}X$ is a bounded torsion spectrum; such X form a thick subcategory.

Lemma. *This subcategory is a $\wedge$-ideal in $\mathcal{S}p^-$.*

Proof. Suppose $X, Y \in \mathcal{S}p^-$ and Y quasi-isogenous to 0; let us check that $X \wedge Y$ is quasi-isogenous to 0. We can assume, replacing X, Y by their shifts, that X and Y are connective. Then $\tau_{\leq n}(X \wedge Y) = \tau_{\leq n}(X \wedge \tau_{\leq n}Y)$, and we are done. □

The corresponding Verdier quotient category $\mathcal{S}p^-_{\mathbb{Q}}$ of $\mathcal{S}p^-$ is a symmetric tensor t-category. We call its objects *spectra up to quasi-isogeny*; for a spectrum X we denote by $X_{\mathbb{Q}}$ the corresponding spectrum up to quasi-isogeny.

Consider the derived categories $\mathcal{D}^-(\mathcal{A}b)$, $\mathcal{D}^-(\mathcal{A}b_{\mathbb{Q}})$ of bounded above complexes of abelian groups. Then $\mathcal{D}^-(\mathcal{A}b_{\mathbb{Q}})$ is the quotient of $\mathcal{D}^-(\mathcal{A}b)$ modulo the thick subcategory of complexes with bounded torsion homology. EM sends $\mathcal{D}^-(\mathcal{A}b)$ to $\mathcal{S}p^-$; passing to the quotients, we get a t-exact functor

$$\mathcal{D}^-(\mathcal{A}b_{\mathbb{Q}}) \to \mathcal{S}p^-_{\mathbb{Q}}. \tag{1.1.1}$$

Proposition. *This functor is an equivalence of tensor triangulated categories.*

Proof. Let $\mathbb{Z}_{\mathcal{S}p}$-mod$^-$ be the category of eventually connective $\mathbb{Z}_{\mathcal{S}p}$-modules and $\mathbb{Z}_{\mathcal{S}p}$-mod$^-_{\mathbb{Q}}$ be its quotient modulo objects which are quasi-isogenous to 0. The forgetful functor $\mathbb{Z}_{\mathcal{S}p}\text{-mod}^- \to \mathcal{S}p^-$ admits left adjoint $X \mapsto \mathbb{Z}_{\mathcal{S}p} \wedge X$ which is a tensor functor. The adjoint functors $\mathbb{Z}_{\mathcal{S}p}\text{-mod}^- \leftrightarrows \mathcal{S}p^-$ yield, by passing to the quotients, the adjoint functors $\mathbb{Z}_{\mathcal{S}p}\text{-mod}^-_{\mathbb{Q}} \leftrightarrows \mathcal{S}p^-_{\mathbb{Q}}$. By the above, it is enough to show that the latter functors are mutually inverse, i.e., that for $X \in \mathcal{S}p^-$, $Y \in \mathbb{Z}_{\mathcal{S}p}\text{-mod}^-$ the adjunction maps $a_X : X \to \mathbb{Z}_{\mathcal{S}p} \wedge X$ and $a^\vee_Y : \mathbb{Z}_{\mathcal{S}p} \wedge Y \to Y$ are quasi-isogenies.

One has $a_X = a_S \wedge \mathrm{id}_X$ where $a_S : S \to \mathbb{Z}_{\mathcal{S}p}$ is the unit map. Since $\pi_0(a_S) = \mathrm{id}_{\mathbb{Z}}$, the groups $\pi_i(\mathcal{C}one(a_S))$ (the stable homotopy groups of spheres) are all finite, so $\mathcal{C}one(a_S)$, hence $\mathcal{C}one(a_X) = \mathcal{C}one(a_S) \wedge X$, is quasi-isogenous to 0, i.e., a_X is a quasi-isogeny. Now $a^\vee_Y : \mathbb{Z}_{\mathcal{S}p} \wedge Y \to Y$ is a quasi-isogeny since $a^\vee_Y a_Y = \mathrm{id}_Y$. □

Remark. Let c_n be an integer that kills $\tau_{\leq n}\mathcal{C}one(a_S)$ (e.g. the product of exponents of the first n stable homotopy groups of spheres). Then for every X such that $\tau_{\leq m}X = 0$, c_{n-m} kills $\tau_{\leq n}\mathcal{C}one(a_X) = \tau_{\leq n}((\tau_{\leq n-m}\mathcal{C}one(a_S)) \wedge X)$. Therefore $\pi_n(a_X) : \pi_n X \to \pi_n(\mathbb{Z}_{\mathcal{S}p} \wedge X)$ is an isogeny of $\mathcal{A}b$-valued functors on (every shift of) the category of connective spectra (see 1.1(a)).

(c) We will need an $\mathcal{A}b_{\mathbb{Q}}$-refinement of (a corollary of) the Milnor–Moore theorem:

For a simplicial set (often referred to as topological space below) P we denote by $C(P, \mathbb{Z})$ the chain complex of P and by $\bar{C}(P, \mathbb{Z})$ the reduced chain complex which is the kernel of the augmentation map $C(P, \mathbb{Z}) \to \mathbb{Z}$, so for $p_0 \in P$ one has $\bar{C}(P, \mathbb{Z}) \xrightarrow{\sim} C(P, \{p_0\}; \mathbb{Z})$ (:= the relative chain complex). Suppose P is connected. We say that $a \in H_n C(P, \mathbb{Z})$ is *primitive* if $\Delta_*(a) \in$

$H_nC(P \times P, \mathbb{Z}) = H_n(C(P, \mathbb{Z})^{\otimes 2})$ equals $1 \otimes a + a \otimes 1$, where 1 is the generator of $H_0C(P, \mathbb{Z})$ and $\Delta : P \hookrightarrow P \times P$ is the diagonal map. We denote by $\operatorname{Prim} H_nC(P, \mathbb{Z})$ the subgroup of primitive elements.

Exercise.

(i) Show that the projections $P \times P' \to P, P'$ yield isomorphisms

$$\operatorname{Prim} H_nC(P \times P', \mathbb{Z}) \xrightarrow{\sim} \operatorname{Prim} H_nC(P, \mathbb{Z}) \oplus \operatorname{Prim} H_nC(P', \mathbb{Z}).$$

(ii) If $\pi_i(P) = 0$ for $i > 1$ and $\pi_1(P)$ is abelian then $\operatorname{Prim} H_nC(P, \mathbb{Z}) = 0$ for $n > 1$.

Denote by $\mathcal{T}op_*$ the category of pointed topological spaces. We have the adjoint infinite suspension and infinite loop functors $S^\infty : \mathcal{T}op_* \leftrightarrows \mathcal{S}p^- : \Omega^\infty$. For $X \in \mathcal{S}p^-$ and $P = (P, p_0) \in \mathcal{T}op_*$ let $b_X : S^\infty \Omega^\infty X \to X$ and $c_P : P \to \Omega^\infty S^\infty P$ be the adjunction maps, so one has $\Omega^\infty(b_X)c_{\Omega^\infty X} = \mathrm{id}_{\Omega^\infty X}$ and $b_{S^\infty P} S^\infty(c_P) = \mathrm{id}_{S^\infty P}$. Let h_P be the composition

$$P \to \Omega^\infty S^\infty P \to \Omega^\infty(\mathbb{Z}_{\mathcal{S}p} \wedge S^\infty P) = \Omega^\infty \operatorname{EM}(C(P, \{p_0\}; \mathbb{Z})),$$

the first arrow is c_P, the second arrow is $\Omega^\infty(a_{S^\infty P})$ (see the proof of the proposition in 1.1(b)). Then

$$\pi_n(h_P) : \pi_n(P, p_0) \to H_nC(P, \{p_0\}; \mathbb{Z}) = H_nC(P, \mathbb{Z}),\ n \geq 1,$$

is the classical Hurewicz map; its image lies in $\operatorname{Prim} H_nC(P, \mathbb{Z})$.

Theorem. *For every connected spectrum X the Hurewicz maps*

$$\pi_n(h_{\Omega^\infty X}) : \pi_n(X) = \pi_n(\Omega^\infty X, 0) \to \operatorname{Prim} H_nC(\Omega^\infty X, \mathbb{Z})$$

are isogenies. Moreover, their kernel and cokernel are killed by nonzero integers that depends only on n (and not on X), i.e., $\pi_n(h_{\Omega^\infty})$ are isogenies of $\mathcal{A}b$-valued functors on the category of connected spectra (see 1.1(a)).

Proof. [1] (i) The functor $X \mapsto \operatorname{Prim} H_nC(\Omega^\infty X, \mathbb{Z})$ is additive: To see this, we need to check that for any maps of spectra $f, g : X \to Y$ and $\alpha \in \operatorname{Prim} H_nC(\Omega^\infty X, \mathbb{Z})$ one has $(\Omega^\infty(f+g))_*(\alpha) = (\Omega^\infty f)_*(\alpha) + (\Omega^\infty g)_*(\alpha)$, which follows since $\Omega^\infty(f+g)$ equals the composition

$$\Omega^\infty X \to \Omega^\infty X \times \Omega^\infty X \to \Omega^\infty Y \times \Omega^\infty Y \to \Omega^\infty Y,$$

the first arrow is Δ, the second one is $\Omega^\infty f \times \Omega^\infty g$, the third one is the sum operation $+$.[2]

(ii) Consider a commutative diagram of spectra

(1.1.2)
$$\begin{array}{ccc} S^\infty \Omega^\infty X & \longrightarrow & \mathbb{Z}_{\mathcal{S}p} \wedge S^\infty \Omega^\infty X \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbb{Z}_{\mathcal{S}p} \wedge X, \end{array}$$

[1] I am grateful to Nick Rozenblyum for the help with the proof.

[2] Use the fact that the restriction of $+$ to $\{0\} \times \Omega^\infty Y$ and $\Omega^\infty Y \times \{0\}$ is the identity map.

the horizontal arrows are $a_{S^\infty\Omega^\infty X}$ and a_X, the vertical arrows are b_X and $\mathrm{id}_{\mathbb{Z}_{\mathcal{S}p}} \wedge b_X$. It implies that

$$\begin{aligned}\Omega^\infty(\mathrm{id}_{\mathbb{Z}_{\mathcal{S}p}} \wedge b_X)h_{\Omega^\infty X} &= \Omega^\infty(\mathrm{id}_{\mathbb{Z}_{\mathcal{S}p}} \wedge b_X)\Omega^\infty(a_{S^\infty\Omega^\infty X})c_{\Omega^\infty X}\\ &= \Omega^\infty(a_X)\Omega^\infty(b_X)c_{\Omega^\infty X}\\ &= \Omega^\infty(a_X),\end{aligned}$$

hence $\pi_n(\mathrm{id}_{\mathbb{Z}_{\mathcal{S}p}} \wedge b_X)\pi_n(h_{\Omega^\infty X}) = \pi_n(a_X)$. Let

$$(1.1.3)\qquad r_{nX} : \mathrm{Prim}\, H_n C(\Omega^\infty X, \mathbb{Z}) \to \pi_n(\mathbb{Z}_{\mathcal{S}p} \wedge X)$$

be the restriction of $\pi_n(\mathrm{id}_{\mathbb{Z}_{\mathcal{S}p}} \wedge b_X) : H_n C(\Omega^\infty X, \mathbb{Z}) \to \pi_n(\mathbb{Z}_{\mathcal{S}p} \wedge X)$ to primitive classes. Since $r_{nX}\pi_n(h_{\Omega^\infty X}) = \pi_n(a_X)$ is an isogeny of $\mathcal{A}b$-valued functors on connected spectra (see Remark in 1.1(b)), we see that $\pi_n(h_{\Omega^\infty})$ is an isogeny if and only if r_n is, and to check this it is enough to show that $\mathrm{Ker}\, r_{nX}$ is killed by a nonzero integer e_n that does not depend on X.

(iii) We first prove that r_n is an isogeny on the subcategory of suspension spectra: Let us show that $\mathrm{Ker}\, r_{nS^\infty P}$ for connected $P = (P, p_0) \in \mathcal{T}op_*$ is killed by $n!$.

As in [12], $\Omega^\infty S^\infty P$ identifies naturally with a free E_∞-space F generated by (P, p_0). Let $\Gamma_\bullet$ be our E_∞-operad. Then F is a union of closed subspaces $\{0\} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots$ where $(F_1, F_0) = (P, \{p_0\})$ and F_i is the image of the operad action map $(\Gamma_i \times P^i)/\Sigma_i \to F$ where Σ_i is the symmetric group. Notice that $(\Gamma_i \times P^i)/\Sigma_i$ is the homotopy quotient of P^i modulo the action of Σ_i. The pointed space F_i/F_{i-1} equals $(\Gamma_i \wedge (P, p_0)^{\wedge i})/\Sigma_i$ which is the homotopy quotient of $(P, p_0)^{\wedge i}$ modulo Σ_i, so $C(F_i, F_{i-1}; \mathbb{Z}) = C(\Sigma_i, C(P, \{p_0\}; \mathbb{Z})^{\otimes i})$ where $C(\Sigma_i, \cdot)$ is the complex of group chains. Thus the compositions $C(\Sigma_i, \bar{C}(P, \mathbb{Z})^{\otimes i}) \hookrightarrow C(\Sigma_i, C(P, \mathbb{Z})^{\otimes i}) \to C(F_i, \{0\}; \mathbb{Z})$ split the filtration $C(F_i, \{0\}; \mathbb{Z})$ on $C(F, \{0\}; \mathbb{Z})$ (see the beginning of 1.1(c) for the notation), so we have a canonical quasi-isomorphism

$$(1.1.4)\qquad \bigoplus_{i\geq 1} C(\Sigma_i, \bar{C}(P, \mathbb{Z})^{\otimes i}) \xrightarrow{\sim} \bar{C}(F, \mathbb{Z}) = C(F, \{0\}, \mathbb{Z}).$$

The map $\mathrm{id}_{\mathbb{Z}_{\mathcal{S}p}} \wedge b_{S^\infty P} : \mathbb{Z}_{\mathcal{S}p} \wedge S^\infty\Omega^\infty S^\infty P \to \mathbb{Z}_{\mathcal{S}p} \wedge S^\infty P$ of abelian spectra amounts to a morphism $v_P : C(F, \{0\}; \mathbb{Z}) \to C(P, \{p_0\}; \mathbb{Z})$ of complexes that can be described as follows. Consider the chain complex $C(P, \mathbb{Z})$ as a simplicial abelian group freely generated by P, and $\bar{C}(P, \mathbb{Z})$ as the kernel of the map of simplicial groups $C(P, \mathbb{Z}) \to C(\{\text{point}\}, \mathbb{Z})$; ditto for F, etc. Then v_P is $\mathbb{Z}$-linear extension of the map of simplicial sets $F \to C(P, \mathbb{Z})$ that extends the standard embedding $P \hookrightarrow C(P, \mathbb{Z})$ and transforms the $\Gamma_\bullet$-operations into addition. Therefore v_P is the identity map on the first summand of (1.1.4) and it kills the rest of the sum. Thus it remains to check that $H_n(\bigoplus_{i\geq 2} C(\Sigma_i, \bar{C}(P, \mathbb{Z})^{\otimes i})) \cap \mathrm{Prim}\, H_n(F, \mathbb{Z})$ is killed by $n!$.

Set $\Phi^m := \bigoplus_{i\geq m} C(\Sigma_i, \bar{C}(P, \mathbb{Z})^{\otimes i})$. We view $\Phi^\bullet$ as a split filtration on $C(F, \mathbb{Z})$ via (1.1.4). Let us show that $m(H_n\Phi^m \cap \mathrm{Prim}\, H_n(F, \mathbb{Z})) \subseteq H_n\Phi^{m+1}$ for $m \geq 2$. Since $H_n\Phi^{n+1} = 0$, this proves our assertion.

We equip $C(F \times F, \mathbb{Z}) = C(F, \mathbb{Z}) \otimes C(F, \mathbb{Z})$ with the tensor product of the filtrations $\Phi^\bullet$. Since $\Delta_* : C(F, \mathbb{Z}) \to C(F \times F, \mathbb{Z})$ is compatible with the operad product, Δ_* is compatible with the filtrations and

$$\operatorname{gr}^m_\Phi \Delta_* : C(\Sigma_m, \bar{C}(P, \mathbb{Z})^{\otimes m}) \to \bigoplus_{a+b=m} C(\Sigma_a, \bar{C}(P, \mathbb{Z})^{\otimes a}) \otimes C(\Sigma_b, \bar{C}(P, \mathbb{Z})^{\otimes b})$$

has (a, b)-component equal to the transfer map $\rho_{a,b} : C(\Sigma_m, \bar{C}(P, \mathbb{Z})^{\otimes m}) \to C(\Sigma_a \times \Sigma_b, \bar{C}(P, \mathbb{Z})^{\otimes m})$ for the usual embedding $\Sigma_a \times \Sigma_b \hookrightarrow \Sigma_m$. Now if $\alpha \in \operatorname{Prim} H_n(F, \mathbb{Z})$ lies in Φ^m, $m \geq 2$, then the image $\bar{\alpha}$ of α in $\operatorname{gr}^m_\Phi H_n(F, \mathbb{Z})$ satisfies $\rho_{a,b}(\bar{\alpha}) = 0$ if $a, b \geq 1$. In particular, $\rho_{1,m-1}(\bar{\alpha}) = 0$, and we are done since $\operatorname{Ker} \rho_{1,m-1}$ is killed by $|\Sigma_m / \Sigma_{m-1}| = m$.

(iv) To finish the proof, let us show that $\operatorname{Ker} r_{nX}$ for any connected spectrum X is killed by $e_n = n! c_n^{2n}$ where c_n is as in Remark in 1.1(b).

To that end we will find a map of spectra $s_{Xn} : \tau_{\leq n} X \to \tau_{\leq n} S^\infty \Omega^\infty X$ such that $\pi_i(\tau_{\leq n}(b_X) s_{Xn}) = \ell_n \operatorname{id}_{\pi_i(X)}$ for $i \leq n$ where $\ell_n = c_n^2$. Assuming we have s_{Xn}, let us finish the proof.

One has $\tau_{\leq n}(b_X) s_{Xn} = \ell_n \operatorname{id}_{\tau_{\leq n} X} + \epsilon$ where $\pi_i(\epsilon) = 0$. Since the canonical filtration on $\tau_{\leq n} X$ has length n, one has $\epsilon^n = 0$. So, by (i), $\tau_{\leq n}(b_X) s_{Xn}$ acts on $\operatorname{Prim} H_n C(\Omega^\infty X, \mathbb{Z})$ as the sum of the multiplication by ℓ_n map and an operator ϵ such that $\epsilon^n = 0$. The kernel K of this action is killed by ℓ_n^n (for $\ell_n \operatorname{id}_K = -\epsilon|_K$). We are done since, by (iii) applied to $P = \Omega^\infty X$, the map $n! \tau_{\leq n}(b_X) s_{Xn}$ kills $\operatorname{Ker} r_{nX} = \operatorname{Ker} r_{n \tau_{\leq n} X}$.[3]

It remains to construct the promised s_{Xn}. For any connected spectrum Y let $q_Y : \tau_{\leq n}(\mathbb{Z}_{\mathcal{S}p} \wedge Y) \to \tau_{\leq n} Y$ be a map of spectra such that $q_Y \tau_{\leq n}(a_Y) = c_n \operatorname{id}_{\tau_{\leq n} Y}$.[4] Let u_i, $i \in [1, n]$, be the composition $\pi_i(\mathbb{Z}_{\mathcal{S}p} \wedge X) \to \pi_i(X) \to \pi_i(\mathbb{Z}_{\mathcal{S}p} \wedge S^\infty \Omega^\infty X)$, the first arrow is $\pi_i(q_X)$, the second one is $\pi_i(h_{\Omega^\infty X}) = \pi_i(a_{S^\infty \Omega^\infty X}) \pi_i(c_{\Omega^\infty X})$. Let $u : \tau_{\leq n}(\mathbb{Z}_{\mathcal{S}p} \wedge X) \to \tau_{\leq n}(\mathbb{Z}_{\mathcal{S}p} \wedge S^\infty \Omega^\infty X)$ be a map of spectra such that $\pi_i(u) = u_i$.[5] We define s_{Xn} as the composition of

$$\tau_{\leq n} X \to \tau_{\leq n}(\mathbb{Z}_{\mathcal{S}p} \wedge X) \xrightarrow{u} \tau_{\leq n}(\mathbb{Z}_{\mathcal{S}p} \wedge S^\infty \Omega^\infty X) \to \tau_{\leq n}(S^\infty \Omega^\infty X)$$

where the first arrow is $\tau_{\leq n}(a_X)$ and the last arrow is $q_{S^\infty \Omega^\infty X}$. Finally, for $i \leq n$ one has

$$\begin{aligned}
\pi_i(\tau_{\leq n}(b_X) s_{Xn}) &= \pi_i(b_X) \pi_i(q_{S^\infty \Omega^\infty X}) \pi_i(a_{S^\infty \Omega^\infty X}) \pi_i(c_{\Omega^\infty X}) \pi_i(q_X) \pi_i(a_X) \\
&= c_n^2 \pi_i(b_X) \pi_i(c_{\Omega^\infty X}) \\
&= c_n^2 \operatorname{id}_{\pi_i(X)},
\end{aligned}$$

and we are done. □

Remarks. (i) I do not know if the theorem remains true for connected H-spaces that are not infinite loop spaces.

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[3] Notice that $\operatorname{Prim} H_n C(\Omega^\infty X, \mathbb{Z}) \xrightarrow{\sim} \operatorname{Prim} H_n C(\Omega^\infty \tau_{\leq n} X, \mathbb{Z})$.

[4] Such a q_Y exists since c_n kills $\mathcal{C}one(\tau_{\leq n}(a_Y))$.

[5] Our u exists since $\tau_{\leq n}(\mathbb{Z}_{\mathcal{S}p} \wedge X)$ and $\tau_{\leq n}(\mathbb{Z}_{\mathcal{S}p} \wedge S^\infty \Omega^\infty X)$, being abelian spectra, are homotopy equivalent to the products of the Eilenberg–MacLane spectra corresponding to their homotopy groups.

(ii) The map s_{Xn} from (iv) of the proof was constructed in an artificial manner; it need not commute with morphisms of X. One can look for a natural s_{Xn} (may be, with a better constant ℓ_n). A possible approach: The ideal answer would be to find a natural section $s_X : X \to S^\infty\Omega^\infty X$ of b_X, and take $s_{Xn} := \tau_{\leq n}(s_X)$ with $\ell_n = 1$. Such an s_X does not exist, but we have a natural section $c_{\Omega^\infty X} : \Omega^\infty X \to \Omega^\infty S^\infty \Omega^\infty X$ of $\Omega^\infty(b_X)$ which is *not* an E_∞-map for the standard E_∞-structures on our spaces. Notice though that $\Omega^\infty S^\infty\Omega^\infty X$ is naturally an E_∞-*ring* space with the product operation coming from the usual E_∞-structure on $\Omega^\infty X$, and $c_{\Omega^\infty X}$ is an E_∞-map for the product E_∞-structure. A natural candidate for s_X might be the logarithm of $c_{\Omega^\infty X}$. It is ill defined due (at least) to nonintegrality of the logarithm series, but one can look for $n! \log \tau_{\leq n}(c_{\Omega^\infty X})$ that would be an E_∞-map for the standard E_∞-structures, hence providing a natural s_{Xn} with $\ell_n = n!$.

1.2. **The pro-setting.** (a) The basic reference for pro-objects is [6, § 8].

As in *loc.cit.*, one assigns to any category $\mathcal{C}$ its category of pro-objects pro-$\mathcal{C}$ together with a fully faithful functor $\iota : \mathcal{C} \hookrightarrow$ pro-$\mathcal{C}$, $X \mapsto {}^\iota X$. The category pro-$\mathcal{C}$ is closed under codirected limits (i.e., for any I°-diagram Z_i in pro-$\mathcal{C}$, I a directed category, $\varprojlim Z_i$ exists), and ι is a universal functor from $\mathcal{C}$ to such a category (i.e., every functor $\mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is closed under codirected limits, extends in a unique way to a functor pro-$\mathcal{C} \to \mathcal{D}$ that commutes with codirected limits). For $\mathcal{C}$ small, the Yoneda embedding $\mathcal{C} \to \mathrm{Funct}(\mathcal{C}, \mathcal{S}ets)^\circ$ extends naturally to a functor pro-$\mathcal{C} \to \mathrm{Funct}(\mathcal{C}, \mathcal{S}ets)^\circ$; if $\mathcal{C}$ is closed under finite limits, it identifies pro-$\mathcal{C}$ with the subcategory of functors that commute with finite limits. Each pro-object can be realized as “lim”$X_i := \varprojlim {}^\iota X_i$ for some I°-diagram X_i in $\mathcal{C}$, I is directed, and Hom(“lim”X_i, “lim”Y_j) $=$ $\varprojlim_j \varinjlim_i \mathrm{Hom}(X_i, Y_j)$. If $\mathcal{C}$ is abelian, then pro-$\mathcal{C}$ is abelian and ι is exact. More generally, for any I as above the functor $\mathcal{C}^I \to$ pro-$\mathcal{C}$, $(X_i) \mapsto$ “lim”X_i, is exact. E.g. we have abelian category pro-$\mathcal{A}b$ of pro-abelian groups. The tensor product “lim”$X_i \otimes$ “lim”$Y_j :=$ “lim”$X_i \otimes Y_j$ makes it a symmetric tensor category.

Remarks. (i) If $\mathcal{C}$ is closed under arbitrary limits or colimits, then so is pro-$\mathcal{C}$.

(ii) For $\mathcal{C}$ abelian, an object “lim”X_i vanishes if and only if for every $i \in I$ one of the maps $X_{i'} \to X_i$ of the diagram equals to zero.

(b) An ∞-category version of the story of [6] is explained in Chapter 5 of [10], the ∞-category of pro-objects is treated in Section 5.3.

Let pro-$\mathcal{S}p$ be the ∞-category of pro-spectra, i.e., pro-objects of the ∞-category $\mathcal{S}p$. This is a stable ∞-category by [11, 1.1.3.6]. By abuse of notation, we denote its homotopy category by pro-$\mathcal{S}p$ as well. It carries a t-structure with the category of connective objects pro-$\mathcal{S}p_{\geq 0}$ equal to pro-$(\mathcal{S}p_{\geq 0})$. The heart of the t-structure equals pro-$\mathcal{A}b$, the homology functor is $X =$ “lim”$X_i \mapsto$ $\pi_\bullet(X) :=$ “lim”$\pi_\bullet(X_i)$, and the truncations are $\tau_{\geq n}$“lim”$X_i =$ “lim”$\tau_{\geq n} X_i$, $\tau_{\leq n}$“lim”$X_i =$ “lim”$\tau_{\leq n} X_i$. Let pro-$\mathcal{S}p^- := \cup_n$ pro-$\mathcal{S}p_{\geq 0}[-n]$ be the t-subcategory of eventually connective pro-spectra.

The ∞-category space of morphisms of pro-spectra $\mathcal{H}om(\text{"lim"} X_i, \text{"lim"} Y_j)$ is equal to $\text{holim}_j \text{hocolim}_i \mathcal{H}om(X_i, Y_j)$; passing to π_0 we get the set of morphisms in the homotopy category.

By [11, 6.3.1.13], pro-$\mathcal{S}p$ is a symmetric tensor ∞-category with tensor product "lim"$X_i \wedge$ "lim"Y_j = "lim"$X_i \wedge Y_j$. The tensor product is right t-exact; it induces the usual tensor product on the heart pro-$\mathcal{A}b$.

The above discussion can be repeated with "spectrum" replaced by "chain complex of abelian groups": we have stable ∞-category pro-$\mathcal{D}(\mathcal{A}b)$ of pro-complexes, its subcategory pro-$\mathcal{D}(\mathcal{A}b)^-$ of eventually connective pro-complexes, etc. For a pro-complex Z we denote by $H_\bullet Z$ its homology pro-abelian groups. We have the t-exact functor pro-EM : pro-$\mathcal{D}(\mathcal{A}b) \to$ pro-$\mathcal{S}p$ which is identity on the hearts.

A map $X \to Y$ in pro-$\mathcal{S}p^-$ is called *quasi-isomorphism*, resp. *quasi-isogeny*, if the maps $\pi_n(X) \to \pi_n(Y)$ are isomorphisms, resp. isogenies, for all n. So a pro-spectrum X is quasi-isomorphic, resp. quasi-isogenous, to 0 if $\pi_n(X)$, resp. $\pi_n(X)_\mathbb{Q}$, vanish for all n. Such X form thick subcategories of pro-$\mathcal{S}p^-$ that are $\wedge$-ideals (repeat the proof of the lemma in 1.1(b)). We denote by pro-$\mathcal{S}p^-_\pi$ and pro-$\mathcal{S}p^-_\mathbb{Q}$ the corresponding Verdier quotients; these are symmetric tensor t-categories with hearts pro-$\mathcal{A}b$ and pro-$\mathcal{A}b_\mathbb{Q}$, the localizations pro-$\mathcal{S}p^- \to$ pro-$\mathcal{S}p^-_\pi \to$ pro-$\mathcal{S}p^-_\mathbb{Q}$, $X \mapsto X_\pi \mapsto X_\mathbb{Q}$, are t-exact tensor functors. The discussion can be repeated for pro-$\mathcal{D}(\mathcal{A}b)^-$.

Exercise. For X = "lim"X_i and Y = "lim"Y_j in pro-$\mathcal{S}p^-$ one has

$$\mathcal{H}om(X_\pi, Y_\pi) = \text{holim}_n \mathcal{H}om(\tau_{\leq n} X, \tau_{\leq n} Y),$$
$$\mathcal{H}om(X_\mathbb{Q}, Y_\mathbb{Q}) = \text{holim}_n \mathcal{H}om(\tau_{\leq n} X, \tau_{\leq n} Y) \otimes \mathbb{Q}.$$

Remarks. (i) For any $Y \in \mathcal{S}p^-$ the pro-spectrum $X :=$ "lim"$\tau_{\geq n} Y$ is quasi-isomorphic to zero; if Y is not eventually coconnective, then $X \neq 0$.[6]

(ii) Probably pro-$\mathcal{S}p^-_\pi$ coincides with the eventually connective part of the homotopy category of pro-spectra as defined in [4].

Passing to the quotients, pro-EM yields the functor

$$\text{pro-}\mathcal{D}(\mathcal{A}b)^-_\mathbb{Q} \to \text{pro-}\mathcal{S}p^-_\mathbb{Q}. \tag{1.2.1}$$

Proposition. *This functor is an equivalence of tensor triangulated categories.*

Proof. Repeat the proof of the proposition in 1.1(b) using the remark after the proof to see that $\mathcal{C}one(a_X)$ is quasi-isogenous to 0 for every $X \in$ pro-$\mathcal{S}p^-$. □

(c) The theorem in 1.1(c) has an immediate pro-version as well. It will be used in Section 6 in the following manner:

For X = "lim"$X_i \in$ pro-$\mathcal{S}p^-$ one has a canonical map

$$\nu_X : C(\Omega^\infty X, \mathbb{Z})_\mathbb{Q} \to X_\mathbb{Q}$$

[6] I am grateful to Jacob Lurie for that example.

defined as the composition

$$C(\Omega^\infty X, \mathbb{Z}) = \mathbb{Z}_{\mathcal{S}p} \wedge S^\infty \Omega^\infty X \to \mathbb{Z}_{\mathcal{S}p} \wedge X \leftarrow X.$$

Here $C(\Omega^\infty X, \mathbb{Z}) =$ "lim"$C(\Omega^\infty X_i, \mathbb{Z})$, the first arrow is $\mathrm{id}_{\mathbb{Z}_{\mathcal{S}p}} \wedge b_X$, with $b_X =$ "lim"b_{X_i}, the second one is quasi-isogeny $a_X =$ "lim"a_{X_i}, see (1.1.2).

Suppose X is connected, i.e., $\pi_{\leq 0}(X) = 0$. Let V be a pro-complex such that $H_{\leq 0}(V) = 0$ and $\psi : V_{\mathbb{Q}} \to C(\Omega^\infty X, \mathbb{Z})_{\mathbb{Q}}$ be a morphism such that for every n the map $H_n(\psi) : H_n V_{\mathbb{Q}} \to H_n C(\Omega^\infty X, \mathbb{Z})_{\mathbb{Q}}$ is injective with image $\mathrm{Prim}\, H_n C(\Omega^\infty X, \mathbb{Z})_{\mathbb{Q}} := ($"lim" $\mathrm{Prim}\, H_n C(\Omega^\infty X_i, \mathbb{Z}))_{\mathbb{Q}}$.

Proposition. *The map $\nu_X \psi : V_{\mathbb{Q}} \to X_{\mathbb{Q}}$ is a quasi-isogeny.*

Proof. It is enough to check that $a_X \nu_X \psi : V \to \mathbb{Z}_{\mathcal{S}p} \wedge X$ is a quasi-isogeny. One has $H_n(a_X \nu_X \psi) = r_{nX} H_n(\psi)$, see (1.1.3). We are done since r_{nX} is an isogeny (see the proof of the theorem in 1.1(c)). ☐

(d) One has a natural functor of stable ∞-categories holim: pro-$\mathcal{S}p \to \mathcal{S}p$, "lim"$X_i \mapsto \mathrm{holim}\, X_i$, which is left t-exact. Its restriction to the subcategory $\mathrm{pro}^{\aleph_0}$-$\mathcal{S}p^-$ of those "lim"X_i that the set I of indices i is countable, has homological dimension 1,[7] so we have triangulated functor holim: $\mathrm{pro}^{\aleph_0}$-$\mathcal{S}p^- \to \mathcal{S}p^-$ and, passing to the quotient categories, holim: $\mathrm{pro}^{\aleph_0}$-$\mathcal{S}p^-_{\mathbb{Q}} \to \mathcal{S}p^-_{\mathbb{Q}}$. A similar functor holim: $\mathrm{pro}^{\aleph_0}$-$\mathcal{D}(\mathcal{A}b)^- \to \mathcal{D}(\mathcal{A}b)$ is the right derived functor of the projective limit functor "lim"$C_i \mapsto \varprojlim C_i$, and the two holim's are compatible with the (pro-) EM functors.

2. The main theorem and a geometric application

2.1. For an associative unital V-algebra R, V is a commutative ring, we denote by $C(R/V)$ the relative Hochschild complex (see [9, 1.1.3]), so $C(R/V)_n = R^{\otimes n+1}$ for $n \geq 0$ (here $\otimes = \otimes_V$), $C(R/V)_n = 0$ otherwise. If R is V-flat, then $C(R/V)$ equals $R \otimes^L_{R \otimes_V R^\circ} R$. Let $CC(R/V)$ be the cyclic complex as in [9, 2.1.2]: this is the total complex of the cyclic bicomplex $CC(R/R)_{\bullet,\bullet}$; here $CC(R/V)_{m,n} = R^{\otimes n+1}$ for $m, n \geq 0$ and $C(R/V)_{m,n} = 0$ otherwise; the nth row complex is the chain complex for the $\mathbb{Z}/n+1$-action

$$r_0 \otimes \ldots \otimes r_n \mapsto (-1)^n r_n \otimes r_0 \otimes \ldots \otimes r_{n-1}$$

on $R^{\otimes n+1}$; the even column complexes equal $C(R/V)$ and the odd ones are acyclic. We denote by Φ_m the increasing filtration on $CC(R/V)$ that comes from the filtration $CC(R/V)_{\leq 2m,\bullet}$ on the bicomplex; thus $\Phi_{-1} = 0$ and the projection $\mathrm{gr}^\Phi_m CC(R) \twoheadrightarrow CC(R/V)_{2m,\bullet} = C(R/V)[2m]$, $m \geq 0$, is a quasi-isomorphism.

Apart from 2.3–2.4, we consider only Hochschild and cyclic complexes with $V = \mathbb{Z}$ and denote them simply $C(R)$, $CC(R)$.

[7] For a precise homological dimension assertion if I is arbitrary, see [13].

2.2. **The main theorem.** Let A, I be as in 0.1; as in *loc.cit.*, $A_i := A/p^i$ and I_i is the image of I in A_i. Consider the relative K-theory pro-spectrum $K(A,I)^\wedge :=$ "lim"$K(A_i, I_i)$ and the cyclic homology pro-complex $CC(A)^\wedge :=$ "lim"$CC(A_i)$ (see 3.1 below for a reminder and the notation).

Theorem. *Suppose A has bounded p-torsion*[8] *and A_1 has finite stable rank (see* [15]*). Then there is a natural quasi-isogeny*

$$K(A,I)^\wedge_{\mathbb{Q}} \xrightarrow{\sim} CC(A)^\wedge[1]_{\mathbb{Q}}. \tag{2.2.1}$$

The proof takes Sections 3–6. The outline of the construction, which is a continuous version of Goodwillie's story [5] or [9, 11.3.2], is as follows:

- Consider the pro-complex $C(\mathfrak{gl}_r(A)^\wedge) :=$ "lim"$C(\mathfrak{gl}_r(A_i))$ of Lie algebra chains; here $\mathfrak{gl}_r$ is the Lie algebra of matrices. By a version of the Loday–Quillen–Tsygan Theorem [9, 10.2.4], there is a canonical map of pro-complexes $C(\mathfrak{gl}_r(A)^\wedge) \to \mathrm{Sym}(CC(A)^\wedge[1])$ which is a quasi-isogeny in degrees $\leq r$.
- Let $GL_r(A)^{(m)} := \mathrm{Ker}(GL_r(A) \to GL_r(A_m))$ be the congruence subgroup of level $m \geq 2$, and
$$C(GL_r(A)^{(m)\wedge}) := \text{“lim”} C(GL_r(A)^{(m)}/GL_r(A)^{(m+i)}, \mathbb{Z})$$
be the pro-complex of its group chains. A version of Lazard's theory from Ch. V of [8] provides (we use the bounded p-torsion condition here) a canonical quasi-isogeny $C(GL_r(A)^{(m)\wedge})_{\mathbb{Q}} \xrightarrow{\sim} C(\mathfrak{gl}_r(A)^\wedge)_{\mathbb{Q}}$.
- We realize $K(A,I)^\wedge$ using Volodin's construction. By Suslin's results [15] (we use the finiteness of stable rank condition here) one can compute the relative K_i-pro-groups using GL_r's with bounded r depending on i. Viewed up to isogeny, they coincide with the primitive part of $H_i C(GL_r(A)^{(m)\wedge})$.

Combining the above quasi-isogenies and using 1.2(c), we get (2.2.1).

2.3. In the next subsection we explain a geometric application of the main theorem. We need three technical lemmas; the reader can skip them at the moment to return when necessary.

(a) Let R be a ring and I its two-sided ideal. Let K^B be the nonconnective relative K-theory spectrum from [16, § 6], so $K^B_n(R) = K_n(R)$ for $n \geq 0$ and there is a long exact sequence

$$\ldots \to K^B_n(R,I) \to K^B_n(R) \to K^B_n(R/I) \to K^B_{n-1}(R,I) \to \ldots$$

Lemma. *If I is nilpotent then $K^B_n(R,I) = 0$ for $n \leq 0$.*

Proof. The assertion of the lemma amounts to surjectivity of the map $K_1(R) \to K_1(R/I)$ and bijectivity of the maps $K^B_n(R) \to K^B_n(R/I)$ for all $n \leq 0$.

[8] I.e., the ideal of p-torsion elements in A is killed by some p^n.

Every idempotent in $\mathrm{Mat}_n(R/I)$ lifts to an idempotent in $\mathrm{Mat}_n(R)$, and the map $\mathrm{Isom}(P,P') \to \mathrm{Isom}(P/IP, P'/IP')$ is surjective for all projective R-modules P, P'. Therefore $K_1(R) \twoheadrightarrow K_1(R/I)$ and $K_0(R) \xrightarrow{\sim} K_0(R/I)$.

Induction by $-n$: Suppose $K_n^B(R) \xrightarrow{\sim} K_n^B(R/I)$ for all (R,I) with I nilpotent. Then $K_{n-1}^B(R) \xrightarrow{\sim} K_{n-1}^B(R/I)$. Indeed, the map from the canonical exact sequence of [16, 6.6]

$$0 \to K_n^B(R) \to K_n^B(R[t]) \oplus K_n^B(R[t^{-1}]) \to K_n^B(R[t,t^{-1}]) \to K_{n-1}^B(R) \to 0$$

to the similar exact sequence for R/I is an isomorphism at each term but the last one by the induction assumption. So it is an isomorphism everywhere. □

(b) Let $X^\wedge$ be a formal scheme, so $X^\wedge$ is direct limit of a directed family of closed subschemes X_i where the ideals of the embeddings $X_i \hookrightarrow X_j$ are nilpotent. Let Y be a closed subscheme of $X^\wedge$ such that X_i contain Y and the ideal of Y in X_i is nilpotent. One has the projective system of the nonconnective relative K-theory spectra $K^B(X_i,Y)$.

Lemma. *If Y is quasi-compact and separated then the pro-spectrum "lim"$K^B(X_i,Y)$ is eventually connective.*

Proof. Pick a finite open affine Zariski covering $\{Y_\alpha\}_{\alpha\in\mathcal{A}}$ of Y; let $\{X_{i\alpha}\}$ be the corresponding covering of X_i. Let $\mathfrak{S}$ be the set of nonempty subsets S of $\mathcal{A}$ ordered by inclusion. We have an $\mathfrak{S}$-diagram $S \mapsto Y_S := \cap_{\alpha\in S} Y_\alpha$ of affine schemes and open embeddings, and similar $\mathfrak{S}$-diagrams $S \mapsto X_{iS}$. By [16, 8.4], one has $K^B(X_i,Y) \xrightarrow{\sim} \mathrm{holim}_{\mathfrak{S}}\, K^B(X_{iS},Y_S)$. Thus $K^B(X_i,Y)$ has a finite filtration with $\mathrm{gr}^n\, K^B(X_i,Y) = \Pi_{|S|=n} K^B(X_{iS},Y_S)[1-n]$. Since X_{iS} is affine and the ideal of Y_S in X_{iS} is nilpotent, the above lemma shows that the spectrum $K^B(X_{iS},Y_S)$ is connected. Therefore $\pi_{-n}K^B(X_i,Y) = 0$ for $n \geq |\mathcal{A}|-1$, and we are done. □

(c) Let k be a perfect field of characteristic p, $W = W(k)$ be the Witt vectors ring, and R be a unital associative flat W_i-algebra where $W_i := W/p^i$.

Lemma. *The evident map $\tau : CC(R) \twoheadrightarrow CC(R/W_i)$ is a quasi-isomorphism.*

Proof. The map τ is compatible with the filtrations Φ (see 2.1), so it is enough to check that τ is a filtered quasi-isomorphism, i.e., that the map of Hochschild complexes $C(R) \to C(R/W_i)$ is a quasi-isomorphism. Both $C(R)$ and $C(R/W_i)$ are $\mathbb{Z}/p^i$-flat (since R is W_i-flat), so it is enough to check that $C(R)\otimes\mathbb{Z}/p \to C(R/W_i)\otimes\mathbb{Z}/p$ is a quasi-isomorphism. The latter is the map $C(R_1) \to C(R_1/k)$, $R_1 := R/pR$, so, replacing R by R_1, we can assume that $i = 1$.

We check first that $C(k) = C(k/\mathbb{F}_p) \to k$ is a quasi-isomorphism. It is enough to consider the case when k is the perfectization of a field k' finitely generated over $\mathbb{F}_p$. Then k' is a separable extension of a purely transcendental extension of $\mathbb{F}_p$, and $H_iC(k'/\mathbb{F}_p) \xrightarrow{\sim} \Omega^i(k'/\mathbb{F}_p)$ by [9, 3.4.4]. Since Frobenius kills $\Omega^{>0}(k'/\mathbb{F}_p)$, one has $H_{>0}C(k) = 0$, q.e.d.

If R is an arbitrary k-algebra, then let P be an $R \otimes_{\mathbb{F}_p} R^\circ$-flat resolution of R. The terms of P are $k \otimes_{\mathbb{F}_p} k$-flat, hence $P_k := P \otimes_{k \otimes_{\mathbb{F}_p} k} k$ is again a resolution of R (indeed, $P_k \xrightarrow{\sim} R \otimes^L_{k \otimes k} k = R \otimes_k (k \otimes^L_{k \otimes k} k) \xrightarrow{\sim} R \otimes_k C(k)$ which is R by the above). Thus P_k is an $R \otimes_k R^\circ$-flat resolution of R, hence $C(R) \xrightarrow{\sim} P \otimes_{R \otimes_{\mathbb{F}_p} R^\circ} R = P_k \otimes_{R \otimes_k R^\circ} R \xrightarrow{\sim} C(R/k)$, and we are done. □

2.4. Let E be a p-adic field, O_E be its ring of integers, so O_E is a complete mixed characteristic discrete valuation ring with perfect residue field k of characteristic p. We denote by O_E-mod the category of O_E-modules, and by O_E-mod$^{\mathrm{f}}$ its subcategory of finitely generated modules. Consider the functor

$$\mathcal{D}^-(O_E\text{-mod}) \to \text{pro-}\mathcal{D}(\mathcal{A}b)^-,$$

$$M \mapsto \text{“lim”} M \otimes^L \mathbb{Z}/p^i = \text{“lim”} \mathcal{C}one(M \xrightarrow{p^i} M).$$

Notice that for $M \in \mathcal{D}^-(O_E\text{-mod}^{\mathrm{f}})$ the evident map $M \to \operatorname{holim}_i M \otimes^L \mathbb{Z}/p^i$ is a quasi-isomorphism (which is enough to check for $M = O_E$). The restriction of our functor to $\mathcal{D}^-(O_E\text{-mod}^{\mathrm{f}})$ is t-exact (since for any $M \in O_E\text{-mod}^{\mathrm{f}}$ one has “lim”$Tor_1(M, \mathbb{Z}/p^i) = 0$). Since $O_E\text{-mod}^{\mathrm{f}}_{\mathbb{Q}}$ equals the category $\mathrm{Vect}^{\mathrm{f}}_E$ of finite-dimensional E-vector spaces, we get a t-exact functor

$$\mathcal{D}^-(\mathrm{Vect}^{\mathrm{f}}_E) \to \text{pro-}\mathcal{D}(\mathcal{A}b)^-_{\mathbb{Q}} \tag{2.4.1}$$

which is evidently faithful (fully faithful if $E = \mathbb{Q}_p$).

Let X be a proper O_E-scheme with smooth generic fiber X_E. Let $R\Gamma_{\mathrm{dR}}(X_E) := R\Gamma(X_E, \Omega^\bullet_{X_E/E})$ be the de Rham complex of X_E and $F^\bullet$ be its Hodge filtration. One has $R\Gamma_{\mathrm{dR}}(X_E)/F^a \in \mathcal{D}^b(\mathrm{Vect}^{\mathrm{f}}_E)$ (here the latter category is interpreted as the derived category of bounded complexes of E-vector spaces with finite-dimensional cohomology).

Let $Y \subseteq X$ be a closed subscheme whose support equals the closed fiber. Set $X_i := X \otimes \mathbb{Z}/p^i$, so $Y \subseteq X_i$ for large enough i. By 2.3(b) we have the pro-spectrum $K^B(X,Y)^\wedge :=$ “lim”$K^B(X_i, Y) \in \text{pro-}\mathcal{S}p^-$.

Theorem. *There is a natural quasi-isogeny of pro-spectra*

$$K^B(X,Y)^\wedge_{\mathbb{Q}} \xrightarrow{\sim} \bigoplus_a (R\Gamma_{dR}(X_E)/F^a)[2a-1]. \tag{2.4.2}$$

Here the target is seen via (2.4.1) as an object of pro-$\mathcal{D}(\mathcal{A}b)^-_{\mathbb{Q}}$, *hence a pro-spectrum up to quasi-isogeny.*

Applying holim from 1.2(d), we get identification (0.2.1).

Proof. (i) Pick a finite open affine covering $\{X_\alpha\}$ of X; let $\{Y_\alpha := Y \cap X_\alpha\}$ be the induced covering of Y. As in the proof of the lemma in 2.3(b), we write $X_S := \cap_{\alpha \in S} X_\alpha = \operatorname{Spec} R_S$, $Y_S = \operatorname{Spec} R_S/I_S$ for $S \in \mathfrak{S}$, etc. By *loc.cit.*, $K(X,Y)^\wedge =$ “lim” $\operatorname{holim}_{\mathfrak{S}} K(X_{iS}, Y_S) = \operatorname{holim}_{\mathfrak{S}} K(X_S, Y_S)^\wedge$. Passing to pro-spectra up to isogeny, we get $K(X,Y)^\wedge_{\mathbb{Q}} = \operatorname{holim}_{\mathfrak{S}} K(X_S, Y_S)^\wedge_{\mathbb{Q}}$.

Set $CC(X_i) := \mathrm{holim}_{\mathfrak{S}}\, CC(X_{iS})$, $CC(X_S)^{\wedge} := \text{“lim”} CC(X_{iS})$, $CC(X)^{\wedge} := \text{“lim”} CC(X_i) = \mathrm{holim}_{\mathfrak{S}}\, CC(X_S)^{\wedge}$. The rings $A_S = \varprojlim R_{iS}$ satisfy the conditions of the theorem in 2.2, so (2.2.1) applied to A_S and ideals $I_S A_S$ provides then a canonical identification

$$K(X,Y)^{\wedge}_{\mathbb{Q}} \xrightarrow{\sim} CC(X)^{\wedge}_{\mathbb{Q}}[1]. \tag{2.4.3}$$

(ii) The terms of (2.4.2) do not change if we replace E by the field of fractions of $W = W(k)$, so we can assume that $O_E = W$.

Let $X' \subseteq X$ be the closed subscheme whose ideal consists of all torsion sections of $\mathcal{O}_X$. Since X has finite type, the ideal is killed by multiplication by high enough power of p, so the maps $CC(X_{iS})^{\wedge} \to CC(X'_{iS})^{\wedge}$ are quasi-isogenies. Hence, by (i), $K(X,Y)^{\wedge}_{\mathbb{Q}} \xrightarrow{\sim} K(X',Y')^{\wedge}_{\mathbb{Q}}$ where $Y' := Y \cap X'$. Thus the terms of (2.4.2) do not change if we replace X by X', and we can assume that X is flat over W.

(iii) Consider W-complexes $CC(X_S/W) := CC(R_S/W)$; they are W-flat since R_S is W-flat. Set $CC(X/W) := \mathrm{holim}_{\mathfrak{S}}\, CC(X_S/W)$. By 2.3(c) one has $CC(X_{iS}) \xrightarrow{\sim} CC(X_{iS}/W_i) = CC(X_S/W) \otimes \mathbb{Z}/p^i$. Therefore

$$CC(X)^{\wedge} \xrightarrow{\sim} \text{“lim”} CC(X/W) \otimes \mathbb{Z}/p^i. \tag{2.4.4}$$

The complex $CC(X/W)$ lies in $\mathcal{D}^-(W\text{-mod}^{\mathrm{f}})$, i.e., the homology of $CC(X/W)$ are finitely generated W-modules: Indeed, since filtration Φ on CC (see 2.1) is finite on every homology group and $\mathcal{S}$ is finite, it suffices to show that

$$\mathrm{gr}^{\Phi}_m CC(X/W) := \mathrm{holim}_{\mathfrak{S}}\, \mathrm{gr}^{\Phi}_m CC(X_S/W) \in \mathcal{D}^-(W\text{-mod}^{\mathrm{f}}).$$

Since $\mathrm{gr}^{\Phi}_m CC(X_S/W) \xrightarrow{\sim} C(X_S/W)[2m]$, it is enough to check that

$$\mathrm{holim}_{\mathfrak{S}}\, H_n C(X_S/W) \in \mathcal{D}^-(W\text{-mod}^{\mathrm{f}}).$$

Now

$$H_n C(X_S/W) = \Gamma(X_S, \mathcal{H}_n), \quad \mathcal{H}_n := H_n L\Delta^* \Delta_* \mathcal{O}_X$$

where $\Delta : X \hookrightarrow X \times_{\operatorname{Spec} W} X$ is the diagonal embedding. Since $\mathcal{H}_n$ is a coherent $\mathcal{O}_X$-module and X/W is proper, the homology of $\mathrm{holim}_{\mathfrak{S}}\, H_n C(X_S/W) = R\Gamma(X, \mathcal{H}_n)$ are finitely generated W-modules, q.e.d.

(iv) Combining (2.4.3) and (2.4.4) we get a canonical identification

$$K(X,Y)^{\wedge}_{\mathbb{Q}} \xrightarrow{\sim} (\text{“lim”} CC(X/W) \otimes \mathbb{Z}/p^i)_{\mathbb{Q}}[1]. \tag{2.4.5}$$

By (iii) and the definition of (2.4.1), to get (2.4.2) it remains to produce a natural quasi-isomorphism

$$CC(X,W) \otimes \mathbb{Q} \xrightarrow{\sim} \bigoplus_a (R\Gamma_{\mathrm{dR}}(X_E)/F^a)[2a-2]. \tag{2.4.6}$$

Now $CC(X/W) \otimes \mathbb{Q} = \mathrm{holim}_{\mathfrak{S}}\, CC(X_{SE}/E)$, and (2.4.6) comes from a canonical quasi-isomorphism $CC(R/E) \xrightarrow{\sim} \bigoplus_a (\Omega^{\bullet}(R/E)/F^a)[2a-2]$ from [9, 3.4.12][9]

[9]By *loc.cit.*, this quasi-isomorphism is defined as the composition of quasi-isomorphisms $CC(R/E) \hookleftarrow \mathcal{B}(R/E) \twoheadrightarrow \bigoplus_a (\Omega^{\bullet}(R/E)/F^a)[2a-2]$ from [9, 2.1.7, 2.1.8] and [9, 2.3.6, 2.3.7].

valid for any smooth commutative algebra R over $E \supset \mathbb{Q}$, and applied to algebras $R = R_S \otimes \mathbb{Q}$, and we are done. □

3. The Loday–Quillen–Tsygan isomorphism

The Loday–Quillen–Tsygan theorem, see [9, 10.2.4], provides a canonical quasi-isomorphism $C(\mathfrak{gl}(R)) \xrightarrow{\sim} \mathrm{Sym}(CC(R)[1])$ for any $\mathbb{Q}$-algebra R. We adapt the argument of *loc.cit.* for our continuous setting.

3.1. For an associative unital ring R we denote by $C^\lambda(R)$ the Connes complex (see [9, 2.1.4]). This is the quotient complex of the Hochschild complex $C(R)$: namely, $C^\lambda(R)_n$ is the coinvariants of the $\mathbb{Z}/n+1$-action on $R^{\otimes n+1}$ (see 2.1). There is an evident projection $\pi_R : CC(R) \twoheadrightarrow C^\lambda(R)$ that equals the projection $C(R) \twoheadrightarrow C^\lambda(R)$ on $CC(R)_{0,\bullet}$ and kills the rest of the bicomplex $CC(R)_{\bullet,\bullet}$.

Lemma. *The homology group $H_n \operatorname{Ker}(\pi_R)$ is killed by $n!$.*

Proof. Consider the increasing filtration by the bicomplex row number on the cyclic complex $CC(R)_\bullet$ and its subcomplex $\operatorname{Ker}(\pi_R)$. Then $H_n \operatorname{gr}_m \operatorname{Ker}(\pi_R)$ equals $H_{n-m}(\mathbb{Z}/m+1, R^{\otimes m+1})$ if $n > m > 0$ and is 0 otherwise. It is killed by $m+1$, hence the assertion. □

3.2. Let A be any p-adic unital ring, so $A = \varprojlim A_i$ where $A_i := A/p^i$. We have pro-complexes $C(A)^\wedge :=$ “lim”$C(A_i)$ and, similarly, $C^\lambda(A)^\wedge$, $CC(A)^\wedge$. Consider the projection $\pi_{A^\wedge} : CC(A)^\wedge \twoheadrightarrow C^\lambda(A)^\wedge$.

Corollary. *$\pi_{A^\wedge}$ is a quasi-isogeny.*

Proof. Consider the pro-complex $\operatorname{Ker}(\pi_{A^\wedge})$. The pro-group $H_n \operatorname{Ker}(\pi_{A^\wedge})$ equals “lim”$H_n \operatorname{Ker}(\pi_{A_i})$, so it is killed by $n!$ according to the above lemma. We are done by the exact triangle $\operatorname{Ker}(\pi_{A^\wedge}) \to CC(A)^\wedge \to C^\lambda(A)^\wedge$. □

3.3. For a Lie algebra $\mathfrak{g}$ we denote by $C(\mathfrak{g})$ its Chevalley complex with trivial coefficients (see [9, 10.1.3]), so $C(\mathfrak{g})_n = \Lambda^n \mathfrak{g}$ (the exterior power over $\mathbb{Z}$) and $H_n(\mathfrak{g}) := H_n C(\mathfrak{g})$ are the Lie algebra homology groups. Our $C(\mathfrak{g})$ is a cocommutative counital dg coalgebra, the coproduct map

$$\delta : C(\mathfrak{g}) \to C(\mathfrak{g} \times \mathfrak{g}) = C(\mathfrak{g}) \otimes C(\mathfrak{g})$$

comes from the diagonal map $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$.

Consider the symmetric dg algebra $\mathrm{Sym}(C^\lambda(R)[1])$. This is a commutative and cocommutative (unital and counital) Hopf dg algebra; the coproduct comes from the diagonal map $C^\lambda(R)[1] \to C^\lambda(R)[1] \times C^\lambda(R)[1]$.

We recall the key construction of Loday–Quillen and Tsygan, cp. [9, 10.2]. Consider the matrix Lie algebra $\mathfrak{gl}_r(R)$, $r \geq 1$.

Theorem-Construction. *There is a canonical morphism of dg coalgebras*

$$\kappa_R = \kappa_R^r : C(\mathfrak{gl}_r(R)) \to \mathrm{Sym}(C^\lambda(R)[1]). \tag{3.3.1}$$

The maps $C(\mathfrak{gl}_r(R))_n \to (\mathrm{Sym}(C^\lambda(R)[1]))_n$ are surjective for $n \leq r$.

Proof. We need a preliminary construction from [9, 9.2]. Let V be a free abelian group of rank r and I be a finite set of order n. The symmetric group Σ_I acts on $V^{\otimes I}$ transposing the factors. Let $\theta_I^* : \mathbb{Z}[\Sigma_I] \to \mathrm{End}(V)^{\otimes I}$ be the composition $\mathbb{Z}[\Sigma_I] \to \mathrm{End}(V^{\otimes I}) \xleftarrow{\sim} \mathrm{End}(V)^{\otimes I}$ of the Σ_I-action map and the inverse to the tensor product map. The source and target of θ_I^* are naturally self-dual: the elements σ of Σ_I form an orthonormal base of the source, the duality for the target is $f, g \mapsto \mathrm{tr}(fg)$. Thus we have the dual map $\theta_I = \Sigma\theta_\sigma\sigma : \mathrm{End}(V)^{\otimes I} \to \mathbb{Z}[\Sigma_I]$. It commutes with the action of $\mathrm{Aut}(V) \times \Sigma_I$; here Σ_I acts on the target by conjugation.

An explicit formula for θ_σ: Let $\{I_\alpha\}$ be the orbits of σ. So $I = \sqcup I_\alpha$ and I_α are naturally cyclically ordered: we write $I_\alpha = \{i_1^\alpha, \dots, i_{n_\alpha}^\alpha\}$ where $\sigma(i_j^\alpha) = i_{j-1}^\alpha$ for $1 < j \leq n_\alpha$. Then (see [9, 9.2.2])

$$\theta_\sigma(\otimes f_i) = \Pi_\alpha \,\mathrm{tr}(f_{i_1^\alpha} \dots f_{i_{n_\alpha}^\alpha}). \tag{3.3.2}$$

The map θ_I is surjective if $r \geq n$: We need to find for every $\sigma \in \Sigma_I$ some $f_i^\sigma \in \mathrm{End}(V)$, $i \in I$, such that $\theta_I(\otimes f_i^\sigma) = \sigma$. Pick a base $\{v_a\}$ of V indexed by elements a of $I \sqcup J$, and define f_i^σ as $f_i^\sigma(v_i) = v_{\sigma(i)}$, $f_i^\sigma(v_a) = 0$ for $a \neq i$.

The map

$$\theta_{RI} := \theta_I \otimes \mathrm{id}_{(R[1])^{\otimes I}} : \mathrm{End}(V)^{\otimes I} \otimes (R[1])^{\otimes I} \to \mathbb{Z}[\Sigma_I] \otimes (R[1])^{\otimes I}$$

commutes with the diagonal Σ_I-actions (where Σ_I acts on $(R[1])^{\otimes I}$ transposing the factors). One has $\mathrm{End}(V) \otimes R = \mathrm{End}_R(V_R)$, $V_R := V \otimes R$, so $\mathrm{End}(V)^{\otimes I} \otimes (R[1])^{\otimes I} = (\mathrm{End}(V) \otimes R[1])^{\otimes I} = (\mathrm{End}_R(V_R)[1])^{\otimes I}$. So for $V = \mathbb{Z}^r$ we get a Σ_I-equivariant map

$$\theta_{RI} : (\mathfrak{gl}_r(R)[1])^{\otimes I} \to \mathbb{Z}[\Sigma_I] \otimes (R[1])^{\otimes I}. \tag{3.3.3}$$

Lemma. *The coinvariants* $(\mathbb{Z}[\Sigma_I] \otimes (R[1])^{\otimes I})_{\Sigma_I}$ *identify naturally with the degree n component of* $\mathrm{Sym}(C^\lambda(R)[1])$.

Proof. Since Σ_I acts on $\mathbb{Z}[\Sigma_I]$ by conjugation, $(\mathbb{Z}[\Sigma_I] \otimes (R[1])^{\otimes I})_{\Sigma_I}$ equals the direct sum, labeled by conjugacy classes of $\sigma \in \Sigma_I$, of coinvariants $((R[1])^{\otimes I})_{S_\sigma}$; here S_σ is the centralizer of σ in Σ_I. For an orbit I_α of σ let $\sigma_\alpha \in S_\sigma$ be equal to σ on I_α and identity outside; set $n_\alpha := |I_\alpha|$. The subgroup $N_\sigma = \Pi_\alpha \mathbb{Z}/n_\alpha$ generated by σ_α's is normal in S_σ. The action of S_σ on the set of I_α's yields an identification $S_\sigma/N_\sigma = \Pi_\ell \Sigma_{J_\ell}$; here J_ℓ is the set of orbits I_α with $n_\alpha = \ell$. One has $((R[1])^{\otimes I})_{N_\sigma} = \bigotimes_\alpha (C^\lambda_{n_\alpha - 1}(R)[n_\alpha])$, hence $((R[1])^{\otimes I})_{S_\sigma} = \bigotimes_\ell ((C^\lambda_{\ell-1}(R)[\ell])^{\otimes J_\ell})_{\Sigma_{J_\ell}}$, q.e.d. □

Since $((\mathfrak{gl}_r(R)[1])^{\otimes I})_{\Sigma_I} = \Lambda^n(\mathfrak{gl}_r(R))[n]$ and by the lemma, the Σ_I-coinvariants map $(\theta_{RI})_{\Sigma_I}$ can be rewritten as

$$\kappa^r_{Rn} : C(\mathfrak{gl}_r(R))_n \to (\mathrm{Sym}(C^\lambda(R)[1]))_n.$$

One checks using (3.3.2) that $\kappa^r_R : C(\mathfrak{gl}_r(R)) \to \mathrm{Sym}(C^\lambda(R)[1])$ commutes with the differential; it commutes with the coproducts by construction. The surjectivity assertion follows from surjectivity of θ_I. We are done. □

3.4. Consider the subcomplex $\mathrm{Ker}(\kappa_R^r)$ of $C(\mathfrak{gl}_r(R))$.

Theorem. *For every $r \geq 0$ there is a nonzero integer c_r that kills the homology group $H_n \mathrm{Ker}(\kappa_R^r)$ for every ring R and $n \leq r$.*

Proof. As in 3.3, set $V = \mathbb{Z}^r$. Consider the action of the Lie algebra $\mathfrak{g} := \mathfrak{gl}(V) = \mathfrak{gl}_r(\mathbb{Z})$ on $\mathrm{End}(V)^{\otimes n}$. The map $\theta_n : \mathrm{End}(V)^{\otimes n} \twoheadrightarrow \mathbb{Z}[\Sigma_n]$ from 3.3 commutes with the $\mathfrak{g}$-actions, so $\mathrm{Ker}(\theta_n)$ is a free $\mathbb{Z}$-module with $\mathfrak{g}$-action. By the invariant theory [9, 9.2.8] and reductivity of $\mathfrak{g}_{\mathbb{Q}}$, $\mathrm{Ker}(\theta_n)_{\mathbb{Q}}$ is a direct sum of finitely many *nontrivial* irreducible $\mathfrak{g}_{\mathbb{Q}}$-modules. Let $\mathfrak{c} \in U(\mathfrak{g})$ be the Casimir element. Recall that $\mathfrak{c}$ lies in the center of the enveloping algebra $U(\mathfrak{g})$, it kills the trivial representation, i.e., $\mathfrak{c} \in \mathfrak{g}U(\mathfrak{g})$, and its action on any nontrivial irreducible finite-dimensional representation of $\mathfrak{g}_{\mathbb{Q}}$ is nontrivial. So we can find $h(t) \in t\mathbb{Q}[t]$ such that $h(\mathfrak{c})$ acts as identity on $\mathrm{Ker}(\theta_n)_{\mathbb{Q}}$. Let c be an integer such that $cg(t) \in t\mathbb{Z}[t]$.

We check that $(n!c)^2$ kills $H_n \mathrm{Ker}(\kappa_R^r)$; then $c_r := (c^r \Pi_{1\leq n\leq r} n!)^2$ satisfies the condition of the theorem. Consider the projection

$$\pi : (\mathfrak{gl}_r(R)[1])^{\otimes n} \twoheadrightarrow ((\mathfrak{gl}_r(R)[1])^{\otimes n})_{\Sigma_n} = C(\mathfrak{gl}_r(R))_n[n].$$

Let $s : C(\mathfrak{gl}_r(R))_n[n] \to (\mathfrak{gl}_r(R)[1])^{\otimes n}$ be the map such that $s\pi = \Sigma_{\sigma\in\Sigma_n}\sigma$. Both π and s commute with $\mathfrak{g}$-action, and πs is multiplication by $n!$. Since θ_n is surjective, $s(\mathrm{Ker}(\kappa_{Rn}^r)) \subseteq \mathrm{Ker}(\theta_{Rn}) = \mathrm{Ker}(\theta_n) \otimes (R[1])^{\otimes n}$. Thus $h(\mathfrak{c})s$ equals cs on $\mathrm{Ker}(\kappa_{Rn}^r)$; composing with π, we see that $n!h(\mathfrak{c})$ equals $n!c$ on $\mathrm{Ker}(\kappa_{Rn}^r)$. The adjoint action of $\mathfrak{gl}_r(R)$ on the Chevalley complex $C(\mathfrak{gl}_r(R))$ is homotopically trivial, hence such is the action of the subalgebra $\mathfrak{g}$. So, since $h(\mathfrak{c}) \in \mathfrak{g}U(\mathfrak{g})$, its action on $C(\mathfrak{gl}_r(R))$ is homotopic to zero; since $h(\mathfrak{c})$ sends $C(\mathfrak{gl}_r(R))$ to $\mathrm{Ker}(\kappa_R^r)$, the action of $h(\mathfrak{c})^2$ on $\mathrm{Ker}(\kappa_R^r)$ is homotopic to zero. The multiplication by $(n!c)^2$ on $\mathrm{Ker}(\kappa_R^r)_n$ equals $(n!)^2 h(c)^2$, hence it kills the homology. □

3.5. For a p-adic Lie algebra $\mathfrak{g} = \varprojlim \mathfrak{g}/p^i\mathfrak{g}$ we have the Chevalley pro-complex $C(\mathfrak{g}^{\wedge}) :=$ "lim" $C(\mathfrak{g}/p^i\mathfrak{g})$, etc. So for a p-adic A as in 3.2 we have pro-complexes $C(\mathfrak{gl}_r(A)^{\wedge}) :=$ "lim" $C(\mathfrak{gl}_r(A_i))$, $\mathrm{Sym}(C^\lambda(A)^{\wedge}[1]) :=$ "lim" $\mathrm{Sym}(C^\lambda(A_i)[1])$ and a morphism $\kappa_{A^\wedge}^r =$ "lim" $\kappa_{A_i}^r : C(\mathfrak{gl}_r(A)^{\wedge}) \to \mathrm{Sym}(C^\lambda(A)^{\wedge}[1])$.

Corollary. *The map $\tau_{\leq r}\kappa_{A^\wedge}^r : \tau_{\leq r} C(\mathfrak{gl}_r(A)^{\wedge}) \to \tau_{\leq r} \mathrm{Sym}(C^\lambda(A)^{\wedge}[1])$ is a quasi-isogeny.* □

Remark. Suppose A has bounded p-torsion. Then the terms of both complexes $C(\mathfrak{gl}_r(A)^{\wedge})$ and $\mathrm{Sym}(C^\lambda(A)^{\wedge}[1])$ have bounded p-torsion as well, so the evident maps $C(\mathfrak{gl}_r(A)^{\wedge}) \otimes^L C(\mathfrak{gl}_r(A)^{\wedge}) \to C(\mathfrak{gl}_r(A)^{\wedge}) \otimes C(\mathfrak{gl}_r(A)^{\wedge})$ and $\mathrm{Sym}(C^\lambda(A)^{\wedge}[1]) \otimes^L \mathrm{Sym}(C^\lambda(A)^{\wedge}[1]) \to \mathrm{Sym}(C^\lambda(A)^{\wedge}[1]) \otimes \mathrm{Sym}(C^\lambda(A)^{\wedge}[1])$ are isogenies. Therefore the coproducts on $C(\mathfrak{gl}_r(A)^{\wedge})$ and $\mathrm{Sym}(C^\lambda(A)^{\wedge}[1])$ yield the coproduct maps $\Delta_C : C(\mathfrak{gl}_r(A)^{\wedge})_{\mathbb{Q}} \to C(\mathfrak{gl}_r(A)^{\wedge})_{\mathbb{Q}} \otimes^L C(\mathfrak{gl}_r(A)^{\wedge})_{\mathbb{Q}}$ and $\Delta_S : \mathrm{Sym}(C^\lambda(A)^{\wedge}[1])_{\mathbb{Q}} \to \mathrm{Sym}(C^\lambda(A)^{\wedge}[1])_{\mathbb{Q}} \otimes^L \mathrm{Sym}(C^\lambda(A)^{\wedge}[1])_{\mathbb{Q}}$ in pro-$\mathcal{D}(\mathcal{A}b)_{\mathbb{Q}}^-$, and one has $(\kappa_{A^\wedge}^r)^{\otimes 2}\Delta_C = \Delta_S \kappa_{A^\wedge}^r$.

3.6. We want to pass to the limit $r \to \infty$ in the above corollary to get rid of the truncation $\tau_{\leq r}$. Since the torsion exponent c_r of the theorem in 3.4 depends on r, we cannot pass to the limit directly, and proceed as follows:

In the setting of 3.3 consider the standard embeddings

$$\mathfrak{gl}_1(R) \subseteq \mathfrak{gl}_2(R) \subseteq \dots$$

and set $\mathfrak{gl}(R) := \cup \mathfrak{gl}_r(R)$. Maps κ_R^r are mutually compatible, so they yield a morphism $\kappa_R : C(\mathfrak{gl}(R)) \to \mathrm{Sym}(C^\lambda(R)[1])$. The complexes $C(\mathfrak{gl}_r(R))$ form an increasing filtration on $C(\mathfrak{gl}(R))$. Denote by $C(\mathfrak{gl}(R))_{(*)}$ the shift of that filtration, so the nth component of $C(\mathfrak{gl}(R))_{(a)}$ equals

$$\{c \in C(\mathfrak{gl}_{a+n}(R))_n : \partial(c) \in C(\mathfrak{gl}_{a+n-1}(R))_{n-1}\}.$$

Let κ_{aR} be the restriction of κ_R to $C(\mathfrak{gl}(R))_{(a)}$.

Corollary. *For $a \geq 0$ the map $\kappa_{aR} : C(\mathfrak{gl}(R))_{(a)} \to \mathrm{Sym}(C^\lambda(R)[1])$ is surjective and the groups $H_n \mathrm{Ker}(\kappa_{aR})$ are killed by nonzero integers that depend on a and n but not on R.* □

Notice that $C(\mathfrak{gl}(R))_{(a)}$ are *not* subcoalgebras of $C(\mathfrak{gl}(R))$.

3.7. For a p-adic A set $C(\mathfrak{gl}(A)^\wedge)_{(a)} :=$ "lim"$C(\mathfrak{gl}(A_i))_{(a)}$. We have the maps

$$\kappa_{aA^\wedge} := \text{“lim”} \kappa_{aA_i} : C(\mathfrak{gl}(A)^\wedge)_{(a)} \to \mathrm{Sym}(C^\lambda(A)^\wedge[1]) \tag{3.7.1}$$

that are compatible with the embeddings $C(\mathfrak{gl}(A)^\wedge)_{(a)} \hookrightarrow C(\mathfrak{gl}(A)^\wedge)_{(a+1)}$.

Corollary. *The maps (3.7.1) are quasi-isogenies for $a \geq 0$.* □

3.8. Consider the adjoint action of $GL_r(R)$ on $\mathfrak{gl}_r(R)$ and the corresponding $GL_r(R)$-action on $C(\mathfrak{gl}_r(R))$.

Corollary. *For every $r \geq 0$ there is a nonzero integer d_r such that for each ring R and $n \leq r$ the $GL_r(R)$-action on the subgroup $d_r H_n C(\mathfrak{gl}_r(R))$ of $H_n C(\mathfrak{gl}_r(R))$ is trivial.*

Proof. Consider the standard embedding $\mathfrak{gl}_r(R) \times \mathfrak{gl}_r(R) \hookrightarrow \mathfrak{gl}_{2r}(R)$ and the corresponding two embeddings $i_1, i_2 : \mathfrak{gl}_r(R) \hookrightarrow \mathfrak{gl}_{2r}(R)$. Consider the adjoint action of $GL_r(R)$ on $\mathfrak{gl}_{2r}(R)$ via the standard embedding $GL_r(R) \hookrightarrow GL_{2r}(R)$. The restriction of this action on the image of i_1 is the adjoint action on $\mathfrak{gl}_r(R)$, and on the image of i_2 it is trivial. The theorem in 3.4 implies that the kernel and cokernel of both maps $i_1, i_2 : H_n C(\mathfrak{gl}_r(R)) \to H_n C(\mathfrak{gl}_{2r}(R))$ are killed by some nonzero integer b_r, so we can take b_r^2 for d_r. □

4. The Lazard isomorphism

In chapter V of [8] (see [7] for a digest) Lazard identifies the continuous group and Lie algebra cohomology for saturated groups of finite rank. We adapt his argument to the setting of pro-complexes where the finite rank condition becomes irrelevant.

4.1. Let G be a topological group such that open normal subgroups $\{G_\alpha\}$ of G form a base of the topology and the topology is complete, i.e., $G = \varprojlim G/G_\alpha$. We denote by $C(G^\wedge)$ the group chain pro-complex "lim"$C(G/G_\alpha, \mathbb{Z})$. The diagonal embedding $G \hookrightarrow G \times G$ yields a cocommutative coproduct on $C(G^\wedge)$.

The example we need: Let $A = \varprojlim A_i$, $A_i := A/p^i$, be a p-adic unital ring. Then $GL_r(A)$ carries a topology as above whose base are congruence subgroups $GL_r(A)^{(i)} := \mathrm{Ker}(GL_r(A) \twoheadrightarrow GL_r(A_i))$. Thus for any open subgroup G of $GL_r(A)$ we have the pro-complex $C(G^\wedge)$. As in 3.5, we have the Chevalley pro-complex $C(\mathfrak{gl}_r(A)^\wedge) :=$ "lim"$C(\mathfrak{gl}_r(A_i))$.

Theorem. *Suppose A has bounded p-torsion. Then for any open G that lies in $GL_r(A)^{(1)}$ if p is odd and in $GL_r(A)^{(2)}$ if $p = 2$ there is a natural quasi-isogeny*

$$\zeta = \zeta_G : C(G^\wedge)_{\mathbb{Q}} \xrightarrow{\sim} C(\mathfrak{gl}_r(A)^\wedge)_{\mathbb{Q}}. \tag{4.1.1}$$

These ζ_G are compatible with the coproduct, embeddings of G, the adjoint action of $GL_r(A)$, the embeddings $GL_r(A) \hookrightarrow GL_{r+1}(A)$, and the morphisms of A.

The proof occupies Sections 4.2–4.10. We start with general remarks about the homology of topological groups of relevant kind (see 4.2–4.4), then we recast the theorem, following Drinfeld's suggestion, as a general assertion about p-adic Lie algebras (see 4.5–4.6). The map ζ is constructed in 4.7, and we check that it is a quasi-isogeny in 4.8–4.10.

4.2. Let K be a discrete group. Its finite filtration $K = K_n \supset \ldots \supset K_1 \supset K_0 = \{1\}$ by normal subgroups is called *p-special* if every successive quotient K_i/K_{i-1} is an abelian group killed by p. We say that K is *p-special* if it admits a p-special filtration.

Lemma. *Suppose that K admits an n-step p-special filtration. Let F be an abelian group viewed as a K-module with trivial K-action. Then $H_a(K, F)$, $a > 0$, is killed by $f(a, n) = p^{\binom{a+n-1}{n-1}}$.*

Proof. Since $C(K, F) = C(K, \mathbb{Z}) \otimes^L_{\mathbb{Z}} F$, the group $H_a(K, F)$ is isomorphic to a direct sum of $H_a(K, \mathbb{Z}) \otimes F$ and $\mathrm{Tor}_1^{\mathbb{Z}}(H_{a-1}(K, \mathbb{Z}), F)$. So it is enough to consider the case $F = \mathbb{Z}$.

Let $K' := K_{n-1}$ be the next to the top term of the filtration, so the lemma for K' is known by induction. Let us compute $H_a(K, \mathbb{Z})$ using Hochschild-Serre spectral sequence $E^2_{i,j} = H_i(K/K', H_j(K', \mathbb{Z}))$. Term $E^2_{i,j}$ is killed by $p^{\binom{j+n-2}{n-2}}$ if $i + j > 0$: indeed, for $j > 0$ this follows by the induction assumption, and $E^2_{i,0} = H_i(K/K', \mathbb{Z})$ is killed by p for $i > 0$ since K/K' is a direct sum of copies of $\mathbb{Z}/p$ (use Künneth formula and the standard computation of $H_\bullet(\mathbb{Z}/p, \mathbb{Z})$). Thus $H_a(K, \mathbb{Z})$, $a > 0$, is killed by $p^{\Sigma_{0 \le j \le a} \binom{j+n-2}{n-2}} = p^{\binom{a+n-1}{n-1}}$. □

Remarks. (i) If K' is a normal subgroup of K, then K is p-special if and only if both K' and K/K' are p-special.

(ii) The assertion of the lemma need not be true if F is an arbitrary K-module, as follows from the next exercise:

Exercise. For any finite group K and any $a > 0$ find a K-module F such that $H_a(K, F) = \mathbb{Z}/|K|\mathbb{Z}$.

4.3. We say that a topological group $G = \varprojlim G/G_\alpha$ as in 4.1 is *p-special* if every G/G_α is a p-special discrete group.

Example. $GL_r(A)^{(1)}$, hence every of its open subgroups, is p-special.

Remark. If G' is a closed normal subgroup of G then G is p-special if and only if both G' and G/G' are p-special.

Corollary. *Suppose G is p-special. If the map $C(G'^{\wedge}) \to C(G^{\wedge})$ is a quasi-isogeny for all G' in some base of open normal subgroups of G, then the same is true for every open $G' \subseteq G$.*

Proof. Pick a normal open $G'' \subseteq G'$ such that $C(G''^{\wedge}) \to C(G^{\wedge})$ is a quasi-isogeny. Our G' is an extension of $K := G'/G''$ by G'', and we compute $H_\bullet C(G'^{\wedge})$ using the Hochschild-Serre spectral sequence. It is enough to check that $H_a(K, H_b C(G''^{\wedge})) =$ "lim" $H_a(K, H_b(G''/G'' \cap G_\alpha, \mathbb{Z}))$ is killed by a power of p if $a > 0$ and that the map $H_b C(G''^{\wedge}) \to H_0(K, H_b C(G''^{\wedge}))$ is an isogeny. We can replace $H_b C(G''^{\wedge})$ by $H_b C(G^{\wedge})$ since the map $H_b C(G''^{\wedge}) \to H_b C(G^{\wedge})$ is an isogeny. Now the action of K on $H_b(G/G_\alpha, \mathbb{Z})$ is trivial, and we are done by the lemma in (4.2). □

4.4. It will be convenient to replace integral chains by p-adic ones: For G as in 4.1 set $C(G)^{\wedge} :=$ "lim"$_{\alpha,i} C(G/G_\alpha, \mathbb{Z}/p^i)$. There is an evident map $C(G^{\wedge}) \to C(G)^{\wedge}$.

Corollary. *If G is p-special then $\tau_{>0} C(G^{\wedge}) \xrightarrow{\sim} \tau_{>0} C(G)^{\wedge}$.*

Proof. Replacing $\mathbb{Z}/p^i$ by $\mathcal{C}one(p^i : \mathbb{Z} \to \mathbb{Z})$, we see that the homotopy fiber of the arrow is a pro-complex "lim"$_{\alpha,i} \tau_{>0} C(G/G_\alpha, \mathbb{Z}_i)$ where $\mathbb{Z}_i$ are copies of $\mathbb{Z}$ arranged into a projective system $\ldots \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z}$. We want to show that its homology "lim"$_{\alpha,i} H_a(G/G_\alpha, \mathbb{Z}_i) =$ "lim"$_\alpha$ "lim"$_i H_a(G/G_\alpha, \mathbb{Z}_i)$, $a > 0$, vanish. It is enough to check that "lim"$_i H_a(K, \mathbb{Z}_i)$ vanishes for every $K = G/G_\alpha$ and $a > 0$. By the lemma in 4.2, $H_a(K, \mathbb{Z})$, $a > 0$, is killed by some power of p, and we are done by Remark (ii) in 1.2. □

4.5. We recast the theorem in 4.1 as a general result about p-adic Lie algebras:

Let $\mathfrak{g}$ be any p-adic Lie algebra, so $\mathfrak{g} = \varprojlim \mathfrak{g}/p^i\mathfrak{g}$. Suppose $\mathfrak{g}$ has no p-torsion and its Lie bracket is divisible by p if $p \neq 2$ and by 4 if $p = 2$. Then the Campbell–Hausdorff series converges and defines a topological group structure on $\mathfrak{g}$. Denote the corresponding group by $G_\mathfrak{g}$, and let

$$\log : G_\mathfrak{g} \rightleftarrows \mathfrak{g} : \exp \tag{4.5.1}$$

be the mutually inverse "identity" identifications. Consider the pro-complexes $C(G_\mathfrak{g}^{\wedge})$, $C(\mathfrak{g}^{\wedge})$ of group and Lie algebra chains (see 4.1, 3.5).

Theorem. *There is a canonical quasi-isogeny*

$$\zeta = \zeta_{\mathfrak{g}} : C(G_{\mathfrak{g}}^{\wedge})_{\mathbb{Q}} \xrightarrow{\sim} C(\mathfrak{g}^{\wedge})_{\mathbb{Q}} \tag{4.5.2}$$

functorial with respect to morphisms of $\mathfrak{g}$*'s.*

Remarks. (i) The topological group $G_{\mathfrak{g}}$ is p-special. More precisely, set $\mathfrak{g}^{(i)} := p^{i-m}\mathfrak{g}$ where $m = 1$, $i \geq 1$ if p is odd, and $m = 2$, $i \geq 2$ if $p = 2$ (the reason for that normalization will be clear later). Then $G_{\mathfrak{g}}^{(i)} := \exp(\mathfrak{g}^{(i)})$ are normal subgroups that form a basis of the topology on $G_{\mathfrak{g}}$, the successive quotients $\mathrm{gr}^{(\bullet)} G_{\mathfrak{g}}$ are abelian, and one has a canonical identification of abelian groups $\mathrm{gr}^{(\bullet)} \exp : \mathrm{gr}^{(\bullet)} \mathfrak{g} \xrightarrow{\sim} \mathrm{gr}^{(\bullet)} G_{\mathfrak{g}}$.

(ii) Since $\exp(pa) = \exp(a)^p$, the map $g \mapsto g^p$ yields bijections $G_{\mathfrak{g}}^{(i)} \xrightarrow{\sim} G_{\mathfrak{g}}^{(i+1)}$ and $\mathbb{F}_p$-vector space isomorphisms $\mathrm{gr}^{(i)} G_{\mathfrak{g}} \xrightarrow{\sim} \mathrm{gr}^{(i+1)} G_{\mathfrak{g}}$.

(iii) Consider the p-adic Chevalley pro-complex $C(\mathfrak{g})^{\wedge} :=$ "lim"$C(\mathfrak{g}) \otimes \mathbb{Z}/p^i =$ "lim"$C(\mathfrak{g}/p^i\mathfrak{g}, \mathbb{Z}/p^i)$. One has $\tau_{>0}C(\mathfrak{g}^{\wedge}) = \tau_{>0}C(\mathfrak{g})^{\wedge}$. Since the chain complexes in (4.5.2) are direct sums of $\mathbb{Z}$ and their $\tau_{>0}$ truncations, the corollary in 4.4 implies that (4.5.2) amounts to a natural quasi-isogeny

$$\zeta = \zeta_{\mathfrak{g}} : C(G_{\mathfrak{g}})_{\mathbb{Q}}^{\wedge} \xrightarrow{\sim} C(\mathfrak{g})_{\mathbb{Q}}^{\wedge}. \tag{4.5.3}$$

Question (Drinfeld)**.** Is there an assertion for p-torsion Lie algebras behind the theorem? More precisely, let $\mathfrak{g}$ be a p-torsion Lie algebra, and suppose we have another Lie bracket on $\mathfrak{g}$ such that the original bracket equals p times the new one if p is odd or 4 times the new one if $p = 2$. Then we have the group $G_{\mathfrak{g}}$ as above, and a natural map $C(G_{\mathfrak{g}}) \to C(\mathfrak{g}')$, where $\mathfrak{g}'$ is $\mathfrak{g}$ with the new Lie bracket, defined as in 4.7 below. What can one say about this map?

4.6. **Lemma.** The theorem in 4.5 implies that in 4.1.

Proof. (a) It is enough to consider the case when A has no torsion: Indeed, for an arbitrary A let $A_{\mathrm{tors}} \subseteq A$ be the ideal of p-torsion elements, $\bar{A} := A/A_{\mathrm{tors}}$, and let $\bar{G}$ be the image of G in $GL_r(\bar{A})$. Then G is an extension of $\bar{G}$ by $K := G \cap (1 + \mathrm{Mat}_r(A_{\mathrm{tors}}))$, and K is discrete since A has bounded p-torsion. Since K is p-special (see the remark in 4.3), the Hochschild-Serre spectral sequence and 4.2 show that the map $C(G^{\wedge}) \to C(\bar{G}^{\wedge})$ is a quasi-isogeny. The map $C(\mathfrak{gl}_r(A)^{\wedge}) \to C(\mathfrak{gl}_r(\bar{A})^{\wedge})$ is a quasi-isogeny as well, and we are done.

(b) It is enough to construct quasi-isogeny ζ_G of (4.1.1) when G is a congruence subgroup: Indeed, by the corollary in 4.3, we know then that the embeddings of G's yield quasi-isogenies of pro-complexes $C(G^{\wedge})$, which gives ζ_G for every G.

(c) Suppose that $m \geq 1$ if $p \neq 2$ and $m \geq 2$ if $p = 2$. The logarithm and exponential series converge p-adically on $GL_r(A)^{(m)}$ and $p^m\mathfrak{gl}_r(A)$ and define mutually inverse continuous bijections

$$\log : GL_r(A)^{(m)} \rightleftarrows p^m\mathfrak{gl}_r(A) : \exp \tag{4.6.1}$$

that satisfy the Campbell–Hausdorff formula. One can interpret this saying that for $\mathfrak{g} = p^m\mathfrak{gl}_r(A)$ the corresponding $G_{\mathfrak{g}}$ identifies canonically with $GL_r(A)^{(m)}$ preserving the logarithm and exponent maps.

So (4.5.2) for $\mathfrak{g} = p^m\mathfrak{gl}_r(A)$ can be rewritten as a quasi-isogeny

$$C(GL_r(A)^{(m)^\wedge})_{\mathbb{Q}} \xrightarrow{\sim} C(p^m\mathfrak{gl}_r(A)^\wedge)_{\mathbb{Q}}.$$

Now the map $C(p^m\mathfrak{gl}_r(A)^\wedge) \to C(\mathfrak{gl}_r(A)^\wedge)$ that comes from the embedding $p^m\mathfrak{gl}_r(A) \hookrightarrow \mathfrak{gl}_r(A)$ is a quasi-isogeny as well (which is clear since $C(p^m\mathfrak{gl}_r(A)^\wedge)_n = p^{mn}C(\mathfrak{gl}_r(A)^\wedge)_n$). We define $\zeta_{GL_r(A)^{(m)}}$ from (4.1.1) as composition of the above quasi-isogenies, and we are done. The compatibility assertions in the theorem 4.1 follow since $\zeta_{\mathfrak{g}}$ is functorial with respect to morphisms of $\mathfrak{g}$'s. □

4.7. **Proof of the theorem in 4.5.** From now on $\mathfrak{g}$ is a Lie algebra as in 4.5 and $G = G_{\mathfrak{g}}$. Let us construct the map $\zeta_{\mathfrak{g}} : G(G)^\wedge_{\mathbb{Q}} \to C(\mathfrak{g})^\wedge_{\mathbb{Q}}$ of (4.5.3).

Let $U(\mathfrak{g})^\wedge := \varprojlim U(\mathfrak{g})_i$, where $U(\mathfrak{g})_i = U(\mathfrak{g})/p^iU(\mathfrak{g}) = U_{\mathbb{Z}/p^i}(\mathfrak{g}/p^i\mathfrak{g})$, be the p-adic completion of the enveloping algebra $U(\mathfrak{g})$. One has

$$C(\mathfrak{g})^\wedge = \mathbb{Z}_p \widehat{\otimes}^L_{U(\mathfrak{g})^\wedge} \mathbb{Z}_p := \text{“lim”}(\mathbb{Z}/p^i) \otimes^L_{U(\mathfrak{g})_i} (\mathbb{Z}/p^i). \tag{4.7.1}$$

Set $\mathfrak{g}' := p^{-m}\mathfrak{g} \subseteq \mathfrak{g} \otimes \mathbb{Q}$ where $m = 1$ if $p \neq 2$ and $m = 2$ if $p = 2$. The conditions on the Lie bracket on $\mathfrak{g}$ mean that it extends to the Lie bracket on $\mathfrak{g}' \supset \mathfrak{g}$. We have same objects as above for $\mathfrak{g}'$, and

$$C(\mathfrak{g}')^\wedge = \mathbb{Z}_p \widehat{\otimes}^L_{U(\mathfrak{g}')^\wedge} \mathbb{Z}_p := \text{“lim”}(\mathbb{Z}/p^i) \otimes^L_{U(\mathfrak{g}')_i} (\mathbb{Z}/p^i). \tag{4.7.2}$$

Similarly, consider the Iwasawa algebra $\mathbb{Z}_p[[G]] := \varprojlim(\mathbb{Z}/p^i)[G/G^{(i)}]$ where $G^{(i)}$ are as in Remark (i) in 4.5. Then

$$C(G)^\wedge = \mathbb{Z}_p \widehat{\otimes}^L_{\mathbb{Z}_p[[G]]} \mathbb{Z}_p := \text{“lim”}(\mathbb{Z}/p^i) \otimes^L_{(\mathbb{Z}/p^i)[G/G^{(i)}]} (\mathbb{Z}/p^i). \tag{4.7.3}$$

One has natural maps of topological algebras

$$\mathbb{Z}_p[[G]] \xrightarrow{\eta} U(\mathfrak{g}')^\wedge \xleftarrow{\iota} U(\mathfrak{g})^\wedge \tag{4.7.4}$$

where ι comes from the embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}'$ and η is defined as follows. The exponential series in $U(\mathfrak{g}')^\wedge$ converges on $p^mU(\mathfrak{g}')^\wedge$ and yields a map

$$\exp_U : p^mU(\mathfrak{g}')^\wedge \to (1 + p^mU(\mathfrak{g}')^\wedge) \subseteq U(\mathfrak{g}')^{\wedge\times} \tag{4.7.5}$$

that satisfies the Campbell–Hausdorff formula. We get a continuous group homomorphism (here $\log(g) \in \mathfrak{g} \subseteq \mathfrak{g}'$ is as in (4.5.1))

$$\eta : G \to U(\mathfrak{g}')^{\wedge\times}, \quad \xi(g) := \exp_U(\log(g)). \tag{4.7.6}$$

We define η in (4.7.4) as the morphism of topological rings that extends (4.7.6). Notice that $\eta = \varprojlim \eta_i$ where

$$\eta_i : (\mathbb{Z}/p^i)[G/G^{(i)}] \to U(\mathfrak{g}')_i = U_{\mathbb{Z}/p^i}(\mathfrak{g}'/p^i\mathfrak{g}').$$

Applying (4.7.4) to the Tor pro-complexes of (4.7.1)–(4.7.3) we get the maps

$$G(G)^{\wedge} \xrightarrow{\xi} C(\mathfrak{g}')^{\wedge} \xleftarrow{i} C(\mathfrak{g})^{\wedge}. \tag{4.7.7}$$

The map $i : C(\mathfrak{g})^{\wedge} \to C(\mathfrak{g}')^{\wedge}$ is evidently a quasi-isogeny since it identifies $C(\mathfrak{g})^{\wedge}_n$ with $p^{mn}C(\mathfrak{g}')^{\wedge}_n$. We define $\zeta_{\mathfrak{g}} : G(G)^{\wedge}_{\mathbb{Q}} \to C(\mathfrak{g})^{\wedge}_{\mathbb{Q}}$ of (4.5.3) as the composition $i^{-1}\xi$ of (4.7.7).

It remains to prove that the Tor map $\xi : G(G)^{\wedge} \to C(\mathfrak{g}')^{\wedge}$ for η is a quasi-isogeny.

4.8. Let $I_p := \operatorname{Ker}(\mathbb{Z}[G] \to \mathbb{Z}/p)$ be the p-augmentation ideal; set $\mathbb{Z}_p[[G]]^- := \varprojlim \mathbb{Z}[G]/I_p^i$. Remark (ii) in 4.5 implies that the I_p-adic topology is weaker than the Iwasawa algebra topology from 4.7,[10] so we have a continuous map $\alpha : \mathbb{Z}_p[[G]] \to \mathbb{Z}_p[[G]]^-$, $\alpha(g) = g$. Since $\eta(I_p) \subseteq pU(\mathfrak{g}')^{\wedge}$, η is continuous for the I_p-adic topology, i.e., we have a continuous map $\bar\eta : \mathbb{Z}_p[[G]]^- \to U(\mathfrak{g}')^{\wedge}$ such that $\eta = \bar\eta\alpha$. Set

$$C(G)^- = \mathbb{Z}_p \widehat{\otimes}^L_{\mathbb{Z}_p[[G]]^-} \mathbb{Z}_p := \text{“}\varprojlim\text{”} (\mathbb{Z}/p^i) \otimes^L_{\mathbb{Z}[G]/I_p^i} (\mathbb{Z}/p^i), \tag{4.8.1}$$

and let τ, $\bar\xi$ be the Tor maps for α and $\bar\eta$. Since $\xi = \bar\xi\tau$, the theorem in 4.5, hence that in 4.1, follows from the next result to be proven in 4.9–4.10 below:

Theorem.

(i) The map $\bar\xi : C(G)^- \to C(\mathfrak{g}')^{\wedge}$ is a quasi-isogeny.

(ii) The map $\tau : C(G)^{\wedge} \to C(G)^-$ is a quasi-isomorphism.

Remark. The map $\alpha : \mathbb{Z}_p[[G]] \to \mathbb{Z}_p[[G]]^-$ is *not* an isomorphism unless $\mathfrak{g}$ is a finitely generated $\mathbb{Z}_p$-module: Indeed, $\mathbb{Z}_p[[G]]$ has discrete quotient $\mathbb{F}_p[G/G^{(2)}] = \mathbb{F}_p[\mathfrak{g}/p\mathfrak{g}]$ (we assume p is odd), while the corresponding quotient of $\mathbb{Z}_p[[G]]^-$ is the completion of that group algebra with respect to powers of the augmentation ideal.

4.9. **Proof of 4.8(i).** We adapt Lazard's argument from chapter V of [8]:

Below we identify the graded ring $\operatorname{gr}^\bullet \mathbb{Z}_p$ for the p-adic filtration with $\mathbb{F}_p[t]$, $\deg t = 1$, where t is class of p in $p\mathbb{Z}/p^2\mathbb{Z} = \operatorname{gr}^1 \mathbb{Z}_p$. For every flat $\mathbb{Z}_p$-algebra B the associated graded ring $\operatorname{gr}^\bullet B$ for the p-adic filtration on B identifies in the evident manner with $B_1 \otimes \mathbb{F}_p[t] = B_1[t]$ where $B_1 := B/pB$. For example, we have $\operatorname{gr}^\bullet U(\mathfrak{g}')^{\wedge} = U_{\mathbb{F}_p}(\mathfrak{h})[t] = U_{\mathbb{F}_p[t]}(\mathfrak{h}[t])$ where $\mathfrak{h} := \mathfrak{g}'/p\mathfrak{g}'$. Recall that $\mathfrak{g} = p^m\mathfrak{g}'$.

Let R be the closure of the image of $\bar\eta$ (or η) in $U(\mathfrak{g}')^{\wedge}$. Let $R^{[n]} := R \cap p^nU(\mathfrak{g}')^{\wedge}$ be the ring filtration on R induced by the p-adic filtration on $U(\mathfrak{g}')^{\wedge}$ and $\operatorname{gr}^{[\bullet]} R \subseteq \operatorname{gr}^\bullet U(\mathfrak{g}')^{\wedge}$ be the associated graded ring.

[10] By *loc.cit.*, it is enough to find for each $n > 0$ some $a = a(n)$ such that g^{p^a} equals 1 in $\mathbb{Z}[G]/I_p^n$ for every $g \in G$. If $p^b > n$ then g^{p^b} equals 1 in $\mathbb{Z}[G]/I_p^n + p\mathbb{Z}[G]$, so we can take $a = b + n$.

Lemma.

(i) The map $\bar\eta : \mathbb{Z}_p[[G]]^- \to R$ *is a homeomorphism of topological rings.*
(ii) $\mathrm{gr}^{[\bullet]} R$ *is equal to the* $\mathbb{F}_p[t]$*-subalgebra of* $U_{\mathbb{F}_p}(\mathfrak{h})[t]$ *generated by* $t^m\mathfrak{h}$ *which is* $U_{\mathbb{F}_p[t]}(t^m\mathfrak{h}[t]) \subseteq U_{\mathbb{F}_p[t]}(\mathfrak{h}[t])$.

Proof. Let $I := \mathrm{Ker}(\mathbb{Z}[G] \to \mathbb{Z})$ be the augmentation ideal, and let $\mathbb{Z}[G]^{[n]}$ be the sum of the ideals $p^a I^b$ where $a, b \geq 0$ and $a + mb \geq n \geq 0$. This is a ring filtration on $\mathbb{Z}[G]$, and the topology it defines equals the I_p-adic topology (for $\mathbb{Z}[G]^{[mn]} \subseteq I_p^n \subseteq \mathbb{Z}[G]^{[n]}$). The ring $\mathrm{gr}^{[\bullet]}\mathbb{Z}[G]$ is a graded $\mathbb{F}_p[t]$-algebra generated by $\mathcal{G} := I/(I^2 + pI) \subseteq \mathrm{gr}^{[m]}\mathbb{Z}[G]$.

One has $\eta(\mathbb{Z}[G]^{[\bullet]}) \subseteq R^{[\bullet]}$, so we have the map of graded rings $\mathrm{gr}^{[\bullet]}\eta : \mathrm{gr}^{[\bullet]}\mathbb{Z}[G] \to \mathrm{gr}^{\bullet} U(\mathfrak{g}')^{\wedge}$. To prove the lemma it is enough to show that $\mathrm{gr}^{[\bullet]}\eta$ is injective.

Recall that $I/I^2 = G/[G,G]$, so $\mathcal{G} = G/G^p[G,G]$ that identifies canonically with $\mathfrak{g}/p\mathfrak{g}$ by Remarks (i), (ii) in 4.5. The map $\mathrm{gr}^{[m]}\eta|_{\mathcal{G}}$ is the embedding $\mathfrak{g}/p\mathfrak{g} = t^m\mathfrak{h} \hookrightarrow U_{\mathbb{F}_p}(\mathfrak{h})[t]$. Therefore the $\mathbb{F}_p[t]$-submodule of $\mathrm{gr}^{[\bullet]}\mathbb{Z}[G]$ generated by $\mathcal{G}$ equals $\mathcal{G}[t] = \mathbb{F}_p[t] \otimes \mathcal{G}$ and $\mathrm{gr}^{[\bullet]}\eta$ identifies it with $t^m\mathfrak{h}[t]$.

Notice that $[\mathcal{G},\mathcal{G}]$ lies in $t^m\mathcal{G} \subseteq \mathrm{gr}^{[2m]}\mathbb{Z}[G]$. Thus $\mathcal{G}[t]$ is a graded Lie $\mathbb{F}_p[t]$-subalgebra of $\mathrm{gr}^{[\bullet]}\mathbb{Z}[G]$, so we have a surjective map of graded associative algebras $\nu : U_{\mathbb{F}_p[t]}(\mathcal{G}[t]) \twoheadrightarrow \mathrm{gr}^{[\bullet]}\mathbb{Z}[G]$. The composition $(\mathrm{gr}^{[\bullet]}\eta)\nu : U_{\mathbb{F}_p[t]}(\mathcal{G}[t]) \to U_{\mathbb{F}_p}(\mathfrak{h})[t]$ is the morphism of enveloping algebras that corresponds to the embedding of Lie algebras $\mathcal{G}[t] \xrightarrow{\sim} t^m\mathfrak{h}[t] \hookrightarrow \mathfrak{h}[t]$. It is injective, hence $\mathrm{gr}^{[\bullet]}\eta$ is injective. □

Set $C(R)^- :=$ "lim"$(\mathbb{Z}/p^i) \otimes^L_{R/R^{[i]}} (\mathbb{Z}/p^i)$, so the embedding $R \hookrightarrow U(\mathfrak{g}')^{\wedge}$ yields a map of pro-complexes $C(R)^- \to C(\mathfrak{g}')^{\wedge}$ (see (4.7.2)). By (i) of the lemma, 4.8(i) amounts to the next assertion:

Proposition. *The morphism* $C(R)^- \to C(\mathfrak{g}')^{\wedge}$ *is a quasi-isogeny.*

Proof. Let K be the Chevalley chain complex of the Lie $\mathbb{F}_p[t]$-algebra $t^m\mathfrak{h}[t]$ with coefficients in the free module $U_{\mathbb{F}_p[t]}(t^m\mathfrak{h}[t])$. This is a graded free $U_{\mathbb{F}_p[t]}(t^m\mathfrak{h}[t])$-resolution of $\mathbb{F}_p[t]$ (with trivial Lie algebra action) with terms $K_n = t^{mn}\Lambda^n_{\mathbb{F}_p}(\mathfrak{h}) \otimes_{\mathbb{F}_p} U_{\mathbb{F}_p[t]}(t^m\mathfrak{h}[t])$. Our K can be lifted to a *complete filtered* R-resolution P of $\mathbb{Z}_p$,[11] so one has

$$K \xrightarrow{\sim} \mathrm{gr}^{[\bullet]} P. \tag{4.9.1}$$

Thus every component P_n is a complete filtered free R-module with generators in filtered degree mn.

Let S be the induced complete filtered $U(\mathfrak{g}')^{\wedge}$-complex, so $S = \varprojlim S/S^{[i]}$ where $S/S^{[i]} = U_{\mathbb{Z}/p^i}(\mathfrak{g}'/p^i\mathfrak{g}') \otimes_R (P/P^{[i]})$. Notice that $S_n^{[mn+i]} = p^i S_n$ for

[11]One constructs silly truncations of P by induction using $\mathrm{gr}^{[\bullet]}\mathrm{Ker}(\partial) = \mathrm{Ker}(\mathrm{gr}^{[\bullet]}\partial)$ where ∂ is the differential of P.

$i \geq 0$. Set $T_n := p^{-mn} S_n \subseteq \mathbb{Q}_p \otimes_{\mathbb{Z}_p} S_n$; then T is a subcomplex of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} S$. We equip T with the p-adic filtration, $T^i = p^i T$. Then (4.9.1) yields isomorphism

$$L[t] \xrightarrow{\sim} \operatorname{gr}^\bullet T \tag{4.9.2}$$

where L is the Chevalley complex of $\mathfrak{h}$ with coefficients in $U_{\mathbb{F}_p}(\mathfrak{h})$.

Using resolutions P and T, one can realize the homotopy map of the proposition as the composition of maps

$$C(R)^- \xleftarrow{\sim} \text{“lim”}(\mathbb{Z}/p^i) \otimes^L_{R/R^{[i]}} (P/P^{[i]}) \xrightarrow{\alpha} \text{“lim”}(\mathbb{Z}/p^i) \otimes_R (P/P^{[i]}) =$$

$$\text{“lim”}(\mathbb{Z}/p^i) \otimes_{U(\mathfrak{g}')} (S/S^{[i]}) \xrightarrow{\beta} \text{“lim”}(\mathbb{Z}/p^i) \otimes_{U(\mathfrak{g}')} (T/T^i) \xrightarrow{\sim} C(\mathfrak{g}')^\wedge .$$

Here the first quasi-isomorphism $\xleftarrow{\sim}$ comes since $P/P^{[i]}$ is a resolution of $\mathbb{Z}/p^i$ by (4.9.1), and the last quasi-isomorphism $\xrightarrow{\sim}$ comes since $\mathbb{Z}/p^i \otimes_{U(\mathfrak{g}')} T/T^i = \mathbb{Z}/p^i \otimes^L_{U(\mathfrak{g}'/p^i\mathfrak{g}')} \mathbb{Z}/p^i = C(\mathfrak{g}'/p^i\mathfrak{g}', \mathbb{Z}/p^i)$ by (4.9.2). The map β is a quasi-isogeny by the construction of T.

To finish the proof, we show that α is a quasi-isomorphism. It is enough to check that “lim”$_i$ $\operatorname{Tor}_a^{R/R^{[i]}}(\mathbb{Z}/p^j, P_n/P_n^{[i]})$, $a > 0$, vanishes for each n and j. This follows since $P_n/P_n^{[mn+i]}$ is a free $R/R^{[i]}$-module and the map

$$\operatorname{Tor}_a^{R/R^{[mn+i]}}(\mathbb{Z}/p^j, P_n/P_n^{[mn+i]}) \to \operatorname{Tor}_a^{R/R^{[i]}}(\mathbb{Z}/p^j, P_n/P_n^{[i]})$$

factors through $\operatorname{Tor}_a^{R/R^{[i]}}(\mathbb{Z}/p^j, P_n/P_n^{[mn+i]})$. □

4.10. **Proof of 4.8(ii).** (a) Consider the pro-complexes $C(G, \mathbb{Z}/p^j)^\wedge :=$ “lim”$_i$ $(\mathbb{Z}/p^j) \otimes^L_{(\mathbb{Z}/p^i)[G/G^{(i)}]} (\mathbb{Z}/p^i) =$ “lim”$_i$ $C(G/G^{(i)}, \mathbb{Z}/p^j)$ and $C(G, \mathbb{Z}/p^j)^- :=$ “lim”$_i$ $(\mathbb{Z}/p^j) \otimes^L_{\mathbb{Z}[G]/I_p^i} (\mathbb{Z}/p^i)$. Since $C(G)^\wedge$ and $C(G)^-$ are homotopy limits of pro-complexes $C(G, \mathbb{Z}/p^j)^\wedge$ and $C(G, \mathbb{Z}/p^j)^-$, see (4.7.3) and (4.8.1), assertion 4.8(ii) would follow if we show that the natural maps $\tau : C(G, \mathbb{Z}/p^j)^\wedge \to C(G, \mathbb{Z}/p^j)^-$ are quasi-isomorphisms. By dévissage, it is enough to check that $\tau : C(G, \mathbb{Z}/p)^\wedge \to C(G, \mathbb{Z}/p)^-$ is a quasi-isomorphism.

(b) We compute $C(G, \mathbb{Z}/p)^-$ using resolution P from the proof of the proposition in 4.9: As in *loc.cit.*, we have quasi-isomorphisms

$$C(G, \mathbb{Z}/p)^- \xleftarrow{\sim} \text{“lim”}(\mathbb{Z}/p) \otimes^L_{R/R^{[i]}} (P/P^{[i]}) \xrightarrow{\alpha} \text{“lim”}(\mathbb{Z}/p) \otimes_R (P/P^{[i]})$$

where α is a quasi-isomorphism by the argument at the end of 4.9. By the construction, “lim”$(\mathbb{Z}/p) \otimes_R (P/P^{[i]})$ equals $(\mathbb{Z}/p) \otimes_{U_{\mathbb{F}_p[t]}(t^m \mathfrak{h}[t])} K$ which is the same as the Chevalley complex of the *abelian* Lie $\mathbb{F}_p$-algebra $t^m\mathfrak{h}[t]/t^{m+1}\mathfrak{h}[t]$ with coefficients in $\mathbb{F}_p$. Thus, using the multiplication by t^m isomorphism $\mathfrak{h} \xrightarrow{\sim} t^m\mathfrak{h}[t]/t^{m+1}\mathfrak{h}[t]$, we get identification $H_n C(G, \mathbb{Z}/p)^- = \Lambda^n_{\mathbb{F}_p} \mathfrak{h}$.

(c) Let us compute $C(G, \mathbb{Z}/p)^\wedge =$ “lim”$C(G/G^{(i)}, \mathbb{Z}/p)$. This is a cocommutative counital $\mathbb{F}_p$-coalgebra in the usual way.

Proposition. *There is a canonical isomorphism of coalgebras*

$$H_\bullet C(G, \mathbb{Z}/p)^\wedge \xrightarrow{\sim} \Lambda^\bullet_{\mathbb{F}_p}(\mathfrak{h}). \tag{4.10.1}$$

Proof. Set $G_{i,n} := G^{(n)}/G^{(n+i)}$. Below we write $H_\bullet(?)$ for $H_\bullet(?, \mathbb{Z}/p)$.

(i) By Remark (ii) in 4.5 the projection $G_{i,n} \twoheadrightarrow G_{1,n} = \mathfrak{h}$, $\exp(p^n a) \mapsto a \bmod p\mathfrak{g}'$, yields an isomorphism $\epsilon_{i,n} : H_1(G_{i,n}) \xrightarrow{\sim} H_1(\mathfrak{h}) = \mathfrak{h}$; here we view $\mathfrak{h}$ as mere abelian group.

(ii) The kernel of the projection $G_{i+1,n} \twoheadrightarrow G_{i,n}$ is $G_{1,n+i} = \mathfrak{h}$, and the adjoint action of $G_{i+1,n}$ on it is trivial. Thus the corresponding Hochschild-Serre spectral sequence converging to $H_\bullet(G_{i+1,n})$ has $E^2_{p,q} = H_p(G_{i,n}) \otimes H_q(G_{1,n+i})$. By (i) for $G_{i+1,n}$, the component $\nu_{i,n} : H_2(G_{i,n}) \to H_1(G_{1,n+i}) = \mathfrak{h}$ of the differential in E^2 is surjective.

(iii)[12] Case $p \neq 2$: Then $\Lambda^\bullet_{\mathbb{F}_p}(\mathfrak{h})$ is the free cocommutative counital graded $\mathbb{F}_p$-coalgebra cogenerated by a copy of $\mathfrak{h}$ in degree 1. Since $H_\bullet(G_{i,n})$ are cocommutative counital graded $\mathbb{F}_p$-coalgebras, we have compatible morphisms of graded coalgebras $\tilde{\epsilon}_{i,n} : H_\bullet(G_{i,n}) \to \Lambda^\bullet_{\mathbb{F}_p}(\mathfrak{h})$ that equal $\epsilon_{i,n}$ in degree 1. We define (4.10.1) as $\varprojlim \tilde{\epsilon}_{i,n}$. To see that it is an isomorphism, let us compute the homology of $G_{i,n}$.

The free cocommutative counital graded $\mathbb{F}_p$-coalgebra cogenerated by a copy of $\mathfrak{h}$ in degree 2 equals $\Gamma^{\bullet/2}_{\mathbb{F}_p}(\mathfrak{h})$.[13] Thus $\nu_{i.n}$ extends to a morphism of graded coalgebras $\tilde{\nu}_{i,n} : H_\bullet(G_{i,n}) \to \Gamma^{\bullet/2}_{\mathbb{F}_p}(\mathfrak{h})$. Let us show that the morphism

$$\tilde{\epsilon}_{i,n} \otimes \tilde{\nu}_{i,n} : H_\bullet(G_{i,n}) \to \Lambda^\bullet_{\mathbb{F}_p}(\mathfrak{h}) \otimes \Gamma^{\bullet/2}_{\mathbb{F}_p}(\mathfrak{h}) \tag{4.10.2}$$

of graded coalgebras is an isomorphism.

For $i = 1$ one has $G_{1,n} = \mathfrak{h}$, and the assertion follows from the standard calculation of $H_\bullet(\mathbb{Z}/p)$ combined with the Künneth formula.[14] We proceed by induction by i using the spectral sequence from (ii) to pass from i to $i+1$: By the induction assumption and since we know that (4.10.2) is an isomorphism for $G_{1,n+i}$, one has $E^2_{\bullet,\bullet} = H_\bullet(G_{i,n}) \otimes H_\bullet(G_{1,n+i}) = (\Lambda^\bullet_{\mathbb{F}_p}(\mathfrak{h}) \otimes \Gamma^{\bullet/2}_{\mathbb{F}_p}(\mathfrak{h})) \otimes (\Lambda^\bullet_{\mathbb{F}_p}(\mathfrak{h}) \otimes \Gamma^{\bullet/2}_{\mathbb{F}_p}(\mathfrak{h}))$. By the construction of $\nu_{i,n}$, the differential in E^2 identifies the second copy of $\mathfrak{h}$ (which lies in $H_2(G_{i,n})$) with the third copy of $\mathfrak{h}$ (which is in $H_1(G_{1,n+i})$). Since the composition of $\mathfrak{h} \hookrightarrow H_2(G_{1,n+i}) \to H_2(G_{i+1,n})$ is injective (indeed, its composition with $\nu_{i+1,n}$ equals $\mathrm{id}_\mathfrak{h}$), the map from the fourth copy of $\mathfrak{h}$ (which lies in $H_2(G_{1,n+i})$) to $E^\infty_{0,2}$ is injective. The compatibility of the differentials with the coproduct implies then that $E^3_{p,q} = \Lambda^p(\mathfrak{h}) \otimes \Gamma^{q/2}_{\mathbb{F}_p}(\mathfrak{h}) = E^\infty_{p,q}$, therefore $\tilde{\epsilon}_{i+1,n} \otimes \tilde{\nu}_{i+1,n}$ is an isomorphism.

Now, since $\nu_{i,n}$ vanishes on the image of $H_2(G_{i+1,n}) \to H_2(G_{i,n})$, the transition map $H_\bullet(G_{i+1,n}) \to H_\bullet(G_{i,n})$ rewritten in terms of (4.10.2) kills the copy of $\mathfrak{h}$ in degree 2. Thus, since the transition map is a map of coalgebras, it

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[12]The reader may find it more convenient to follow the computation in (iii) and (iv) using the dual language of cohomology (the coalgebra structure turns then into the algebra one).

[13]Here $\Gamma^\bullet_{\mathbb{F}_p}(\mathfrak{h})$ is the divided powers Hopf $\mathbb{F}_p$-algebra generated by a copy of $\mathfrak{h}$.

[14]Use the fact that for every homomorphism $\iota_a : \mathbb{Z}/p \to G_{1,n}$, $\iota_a(x) = (1 + ap^n)^x$, $a \in \mathfrak{h}$, one has $\nu_{1,n}(\iota_a(\xi)) = a$, where ξ is the standard generator of $H_2(\mathbb{Z}/p)$ (which follows since the map $x \mapsto x^p$ on $G_{2,n}$ yields an isomorphism $G_{1,n} \xrightarrow{\sim} G_{1,n+1}$).

equals the composition $\Lambda^\bullet_{\mathbb{F}_p}(\mathfrak{h}) \otimes \Gamma^{\bullet/2}_{\mathbb{F}_p}(\mathfrak{h}) \twoheadrightarrow \Lambda^\bullet_{\mathbb{F}_p}(\mathfrak{h}) \hookrightarrow \Lambda^\bullet_{\mathbb{F}_p}(\mathfrak{h}) \otimes \Gamma^{\bullet/2}_{\mathbb{F}_p}(\mathfrak{h})$. This implies that (4.10.1) is an isomorphism, q.e.d.

(iv) Case $p = 2$: The free cocommutative counital graded $\mathbb{F}_2$-coalgebra cogenerated by a copy of $\mathfrak{h}$ in degree 1 equals $\Gamma^\bullet_{\mathbb{F}_2}(\mathfrak{h})$. The morphism of graded coalgebras $\tilde{\epsilon}_{1,m} : H_\bullet(G_{1,n}) \to \Gamma^\bullet_{\mathbb{F}_2}(\mathfrak{h})$ that extends $\epsilon_{1,n}$ is an isomorphism (by the standard computation of $H_\bullet(\mathbb{Z}/2)$ and Künneth formula).

As in (iii), the free cocommutative counital graded $\mathbb{F}_2$-coalgebra cogenerated by a copy of $\mathfrak{h}$ in degree 2 equals $\Gamma^{\bullet/2}_{\mathbb{F}_2}(\mathfrak{h})$, and $\nu_{i,n}$ extends to a morphism of graded coalgebras $\tilde{\nu}_{i,n} : H_\bullet(G_{i,n}) \to \Gamma^{\bullet/2}_{\mathbb{F}_2}(\mathfrak{h})$. Since $\nu_{1,m}$ vanishes on the image of $H_2(G_{2,n}) \to H_2(G_{1,n})$ and $\Lambda^\bullet_{\mathbb{F}_2}(\mathfrak{h}) \subseteq \Gamma^\bullet_{\mathbb{F}_2}(\mathfrak{h})$ is the largest graded subcoalgebra of $H_\bullet(G_{1,n})$ on which $\nu_{1,n}$ vanishes, we see that $\tilde{\epsilon}_{i,n}$ takes values in $\Lambda_{\mathbb{F}_2}(\mathfrak{h})$ if $i \geq 2$, i.e., $\tilde{\epsilon}_{i,n}$ yields $\bar{\epsilon}_{i,n} : H_\bullet(G_{i,n}) \to \Lambda^\bullet_{\mathbb{F}_2}(\mathfrak{h})$. We define (4.10.1) as $\varprojlim \bar{\epsilon}_{i,n}$.

Now for $i \geq 1$ we have the map $\bar{\epsilon}_{i,n} \otimes \tilde{\nu}_{i,n} : H_\bullet(G_{i,n}) \to \Lambda^\bullet_{\mathbb{F}_2}(\mathfrak{h}) \otimes \Gamma^{\bullet/2}_{\mathbb{F}_2}(\mathfrak{h})$ of graded coalgebras. One checks that it is an isomorphism in the same way as we did it in (iii), and this implies that (4.10.1) is an isomorphism, q.e.d. □

(d) To finish the proof of 4.8(ii), hence of the theorems in 4.5 and in 4.1, it remains to notice that $H_\bullet C(G, \mathbb{Z}/p)^-$ is a cocommutative coalgebra in the same way as $H_\bullet C(G, \mathbb{Z}/p)^\wedge$ is, and our map $\tau : H_\bullet C(G, \mathbb{Z}/p)^\wedge \to H_\bullet C(G, \mathbb{Z}/p)^-$ is a morphism of graded coalgebras. We have identified both coalgebras with $\Lambda^\bullet_{\mathbb{F}_p}(\mathfrak{h})$ in (b) and (c). Since τ equals $\mathrm{id}_{\mathfrak{h}}$ in degree 1, it equals $\mathrm{id}_{\Lambda^\bullet_{\mathbb{F}_p}(\mathfrak{h})}$ in all degrees, q.e.d. □

4.11. We are in the setting of 4.1. The theorem in 4.1 can be partially extended to some other subgroups of $GL_r(A)$ due to the next proposition:

Let $G' \subseteq GL_r(A)$ be any open p-special subgroup (see 4.3). E.g. the latter condition holds if the image of G' in $GL_r(A_1)$ consists of upper triangular matrices with 1 on the diagonal. Let $G \subseteq G'$ be a small enough congruence subgroup.

Proposition. *The map $H_nC(G^\wedge) \to H_nC(G'^\wedge)$ is an isogeny if $n \leq r$.*

Proof. Consider the adjoint action of $GL_r(A)$ on $H_nC(G^\wedge)$. Combining the theorem in 4.1 with the corollary in 3.8, we see that for some nonzero integer c_r this action is trivial on $c_rH_nC(G^\wedge)$. Now compute $H_nC(G')$ using the Hochschild-Serre spectral sequence for $G \subseteq G'$, and use the lemma in 4.2. □

Question. Is it true that the adjoint action of $GL_r(A)$ on $C(\mathfrak{gl}_r(A_i))$ is trivial? If yes, this can replace the reference to 3.8 in the above proof, and the assumption $n \leq r$ can be dropped.

5. RELATIVE K-GROUPS AND SUSLIN'S STABILIZATION

5.1. We identify the symmetric group Σ_r with a subgroup of GL_r in the usual way (the transpositions of standard base vectors). For every $\sigma \in \Sigma_r$ let $U^\sigma \subseteq$

GL_r be the σ-conjugate of the subgroup of unipotent upper triangular matrices. For a ring R we have Volodin's simplicial set $X_r(R) := \cup_{\sigma\in\Sigma_r} B_{U^\sigma(R)} \subseteq B_{GL_r(R)}$. It is clear that $X_r(R)$ is connected, $\pi_1(X_r(R)) = St_r(R)$. As usually, $GL(R)$ is the union of $GL_1(R) \subseteq GL_2(R) \subseteq \ldots$, $X(R) := \cup X_r(R)$, etc.

For a cell complex P let $P \to \tau_{\leq n}P$ be the nth Postnikov truncation of P, so $\pi_i(P) \xrightarrow{\sim} \pi_i(\tau_{\leq n}P)$ for $i \leq n$ and $\pi_{>n}(\tau_{\leq n}P) = 0$. Below $^+$ is Quillen's $+$-construction. Let $\mathfrak{r}$ be the stable rank of R. The next result is due to Suslin [15]:

Theorem.

(i) The reduced homology $\tilde{H}_n(X_r(R), \mathbb{Z})$ vanish for $r \geq 2n+1$. Therefore $X(R)$ is acyclic.
(ii) $X(R)$ identifies naturally with the homotopy fiber of $B_{GL(R)} \to B^+_{GL(R)}$.
(iii) If $r \geq \max(2n+1, \mathfrak{r}+n)$, then $H_n(GL_r(R), \mathbb{Z}) \xrightarrow{\sim} H_n(GL(R), \mathbb{Z})$ and $\tau_{<n}X_r(R)$ identifies naturally with the homotopy fiber of $B_{GL_r(R)} \to \tau_{\leq n}B^+_{GL_r(R)}$.

Proof. (i) is [15, 7.1]. (ii) The composition $X(R) \to B_{GL(R)} \to B^+_{GL(R)}$ factors through $X(R)^+$ which is contractible by (i). The resulting map from $X(R)$ to the homotopy fiber of $B_{GL(R)} \to B^+_{GL(R)}$ is a homotopy equivalence by [9, 11.3.6]. (iii) The composition of maps $\tau_{\leq n}X_r(R) \to B_{GL_r(R)} \to \tau_{\leq n}B^+_{GL_r(R)}$ factors through $\tau_{\leq n}X_r(R)^+$ which is contractible if $r \geq 2n+1$ by (i), hence comes the map from $\tau_{<n}X_r(R)$ to the homotopy fiber of $B_{GL_r(R)} \to \tau_{\leq n}B^+_{GL_r(R)}$; it is a homotopy equivalence by [15, 8.1]. The rest is [15, 8.2]. □

5.2. If $J \subseteq R$ is a two-sided ideal, then the relative Volodin's simplicial set $X_r(R, J)$ is defined as the preimage of $X_r(R/J)$ by the projection $B_{GL_r(R)} \to B_{GL_r(R/I)}$, i.e., $X_r(R, J) = \cup_{\sigma\in\Sigma_r} B_{U^\sigma(R,J)} \subseteq B_{GL_r(R)}$ where $U^\sigma(R, J)$ is the preimage of $U^\sigma(R/J)$ in $GL_r(R)$. So $X_r(R, J)$ is connected, $\pi_1(X_r(R, J)) = St_r(R/J) \times_{GL_r(R/J)} GL_r(R)$.

From now on we assume that J is nilpotent. Then $K_0(R) \xrightarrow{\sim} K_0(R/J)$ and $GL_r(R) \twoheadrightarrow GL_r(R/J)$, hence $K_1(R) \twoheadrightarrow K_1(R/J)$. Thus the relative K-spectrum $K(R, J)$ is connected and $\Omega^\infty K(R, J)$ is the homotopy fiber of $B^+_{GL(R)} \to B^+_{GL(R/J)}$. Let $\mathfrak{r}$ be the stable rank of R or of R/J (they coincide since J is nilpotent).

Proposition.

(i) One has $X(R, J)^+ \xrightarrow{\sim} \Omega^\infty K(R, J)$, hence $C(X(R, J), \mathbb{Z}) \xrightarrow{\sim} C(\Omega^\infty K(R, J), \mathbb{Z})$.
(ii) $H_nC(X_r(R, J), \mathbb{Z}) \xrightarrow{\sim} H_nC(X(R, J), \mathbb{Z})$ for $r \geq \max(2n+1, \mathfrak{r}+n)$.

Proof. (i) (see [9, 11.3.6]) The projection $B_{GL(R)} \to B_{GL(R/J)}$ is Kan's fibration. So, by 5.1(ii) applied to R/J, $X(R, J)$ is the homotopy fiber of $B_{GL(R)} \to B^+_{GL(R/J)}$. The map $B_{GL(R)} \to B^+_{GL(R)}$ over $B^+_{GL(R/J)}$ yields the map of the homotopy fibers $X(R, J) \to \Omega^\infty K(R, J)$. By 5.1(ii), the homotopy fiber of the latter map equals $X(R)$. Thus, by 5.1(i), $C(X(R,J),\mathbb{Z}) \xrightarrow{\sim} C(\Omega^\infty K(R,J),\mathbb{Z})$, hence the assertion.

(ii) Repeat the argument of (i) with $B^+_{GL(?)}$ replaced by $\tau_{\leq n} B^+_{GL_r(?)}$ and $X(R,J)$ replaced by $\tau_{\leq n} X_r(R,J)$. $\square$

Question. Can it be that (ii) actually holds for $r > n$?[15] If yes, this would eliminate the assumption of finiteness of stable rank from the corollary in 5.4, hence from the theorem in 2.2.

5.3. Suppose A and I are as in 2.2 and A has bounded p-torsion. Choose m such that $p^m \in I$. Then $GL_r(A_i)^{(m)} := \mathrm{Ker}(GL_r(A_i) \to GL_r(A_{\min(i,m)})) \subseteq U^\sigma(A_i, I_i)$ for any σ, hence $B_{GL_r(A_i)^{(m)}} \subseteq X_r(A_i, I_i) \subseteq B_{GL_r(A_i)}$. Consider the maps of chain complexes $C(GL_r(A_i)^{(m)},\mathbb{Z}) \to C(X_r(A_i,I_i),\mathbb{Z})$. Applying "lim", we get a map of pro-complexes

$$C(GL_r(A)^{(m)^\wedge}) \to C(X_r(A,I)^\wedge,\mathbb{Z}). \tag{5.3.1}$$

Proposition. *For $n \leq r$ the map $H_n C(GL_r(A)^{(m)^\wedge}) \to H_n C(X_r(A,I)^\wedge,\mathbb{Z})$ is an isogeny.*

Proof. $X_r(A_i,I_i)$ is covered by simplicial subsets $B_{U^\sigma(A_i,I_i)}$. Let $B_{U(A_i,I_i)}$ be the intersection of several of these subsets. One has $GL_r(A_i)^{(m)} \subseteq U(A_i,I_i)$, so $G := GL_r(A)^{(m)} \subseteq G' := \varprojlim U(A_i,I_i)$. The proposition follows since the maps $H_n C(G^\wedge) \to H_n C(G'^\wedge)$ for $n \leq r$ are isogenies by 4.11. $\square$

5.4. For a ring R consider the standard embeddings $GL_1(R) \subseteq GL_2(R) \subseteq \ldots$. The complexes $C(GL_r(R),\mathbb{Z})$ form an increasing filtration on $C(GL(R),\mathbb{Z})$. As in 3.6, we denote by $C(GL(R),\mathbb{Z})_{(*)}$ the shift of that filtration, the nth component of $C(GL(R),\mathbb{Z})_{(a)}$ is formed by those $c \in C(GL_{a+n}(R),\mathbb{Z})_n$ that $\partial(c) \in C(GL_{a+n-1}(R),\mathbb{Z})_{n-1}$. Replacing GL_r by its congruence subgroup we get projective systems of complexes $C(GL(A_i)^{(m)},\mathbb{Z})_{(a)}$. Set $C(GL(A)^{(m)^\wedge})_{(a)} :=$ "lim"$C(GL(A_i)^{(m)},\mathbb{Z})_{(a)}$.

Maps (5.3.1) for different r's are compatible; by 5.2(i), they produce maps

$$C(GL(A)^{(m)^\wedge})_{(a)} \to C(\Omega^\infty K(A,I)^\wedge,\mathbb{Z}) \tag{5.4.1}$$

that are compatible with embeddings $C(GL(A)^{(m)^\wedge})_{(a)} \hookrightarrow C(GL(A)^{(m)^\wedge})_{(a+1)}$ and $C(GL(A)^{(m+1)^\wedge})_{(a)} \hookrightarrow C(GL(A)^{(m)^\wedge})_{(a)}$. Notice that the theorem in 4.1 yields quasi-isogenies (see 3.7)

$$C(GL(A)^{(m)^\wedge})_{(a)\mathbb{Q}} \xrightarrow{\sim} C(\mathfrak{gl}(A)^\wedge)_{(a)\mathbb{Q}}. \tag{5.4.2}$$

Corollary. *Suppose the stable rank of A_1 is finite. Then maps (5.4.1) are quasi-isogenies for $a \geq 0$.*

[15]If R is a $\mathbb{Q}$-algebra, the positive answer follows from Goodwillie's theorem [5], [9, 11.3] combined with the Loday–Quillen stabilization [9, 10.3.2].

Proof. The stable rank of A_1 equals that of A/I, so, by 5.3 and 5.2(ii), we know that for given n the map $H_nC(GL_r(A)^{(m)^\wedge}) \to H_nC(\Omega^\infty K(A,I)^\wedge, \mathbb{Z})$ is an isogeny if r is large enough. Now (5.4.2) and the corollary in 3.7 imply that $C(GL(A)^{(m)^\wedge})_{(0)} \to C(GL(A)^{(m)^\wedge})_{(1)} \to \dots$ are all quasi-isogenies. □

6. Coda

It remains to tie up the segments of the tail:

6.1. Let A, I be as in the theorem in 2.2. Denote by χ be the composition of the chain of quasi-isogenies $\mathrm{Sym}(CC(A^\wedge)[1])_\mathbb{Q} \xrightarrow{3.2} \mathrm{Sym}(C^\lambda(A^\wedge)[1])_\mathbb{Q} \xleftarrow{(3.7.1)}$ $C(\mathfrak{gl}(A)^\wedge)_{(a)\mathbb{Q}} \xleftarrow{(5.4.2)} C(GL(A)^{(m)^\wedge})_{(a)\mathbb{Q}} \xrightarrow{(5.4.1)} C(\Omega^\infty K(A,I)^\wedge, \mathbb{Z})_\mathbb{Q}$; it is independent of the choice of $a \geq 0$, $m \geq 1$ involved. Consider the composition

$$CC(A^\wedge)[1]_\mathbb{Q} \hookrightarrow \mathrm{Sym}(CC(A^\wedge)[1])_\mathbb{Q} \xrightarrow{\sim} C(\Omega^\infty K(A,I)^\wedge, \mathbb{Z})_\mathbb{Q} \to K(A,I)^\wedge_\mathbb{Q}$$

where the first arrow is the evident embedding, the second arrow is χ, and the third one is $\nu_{K(A,I)^\wedge}$ from 1.2(c).

Proposition. *The composition $CC(A^\wedge)[1]_\mathbb{Q} \to K(A,I)^\wedge_\mathbb{Q}$ is a quasi-isogeny.*

The promised quasi-isogeny (2.2.1) is its inverse.

Proof. Let $\psi : CC(A^\wedge)[1]_\mathbb{Q} \to C(\Omega^\infty K(A,I)^\wedge, \mathbb{Z})_\mathbb{Q}$ be the composition of the first two arrows. By the proposition in 1.2(c), it is enough to check that $H_n(\psi)$ identifies $H_n(CC(A^\wedge)[1]_\mathbb{Q})$ with $\mathrm{Prim}\, H_nC(\Omega^\infty K(A,I)^\wedge, \mathbb{Z})_\mathbb{Q}$. Since $H_n(CC(A^\wedge)[1]_\mathbb{Q})$ equals $\mathrm{Prim}\, H_n \,\mathrm{Sym}(CC(A^\wedge)[1]_\mathbb{Q})$, we need to show that χ identifies the primitive parts of the homology. The only problem is that the terms of the segment $C(\mathfrak{gl}(A)^\wedge)_{(a)\mathbb{Q}} \xleftarrow{(5.4.2)} C(GL(A)^{(m)^\wedge})_{(a)\mathbb{Q}}$ of the chain that defines χ are not coalgebras. To solve it, pick any $a \geq r \geq n$. By 3.5, 3.7 the embedding $C(\mathfrak{gl}_r(A)^\wedge) \hookrightarrow C(\mathfrak{gl}(A)^\wedge)_{(a)}$ yields isogenies between the homology in degrees $\leq n$, so $\tau_{\leq n}\chi$ can be computed replacing the above segment by $C(\mathfrak{gl}_r(A)^\wedge)_\mathbb{Q} \xleftarrow{(4.1.1)} C(GL_r(A)^\wedge)_\mathbb{Q}$. Now all terms of the chain are coalgebras, the maps are compatible with the coproducts (see Remark in 3.5), and we are done. □

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