A Generalization of the Real Mean Value Inequality

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We propose a Mean Value Inequality concerning functions on a compact interval mapping into an arbitrary Banach space. In the special case of a real-valued function, the statement of our theorem was already formulated by Dale E.Varberg in his paper *On Absolutely Continuous Functions*. Since Varberg's proof is essentially based on the ordered structure of $\mathbb{R}$, it isn't possible to apply this proof to our generalized theorem. Therefore we establish a proper proof which makes use of the well-known *Vitali Covering Theorem*.
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1. INTRODUCTION

Throughout this paper we fix a compact interval $I = [a, b]$, a Banach space $(X, ||\cdot||)$ (over the real or complex numbers), and a map $f : I \to X$. The set of those points in which f is differentiable is denoted by $\mathcal{D}_f$. As usual, Lebesgue measure on $\mathbb{R}$ (Lebesgue outer measure on $\mathbb{R}$ resp.) is denoted by λ (λ^* resp.). In this paper we prove the following

THEOREM 1.1. *Let $A \subset \mathcal{D}_f$ and $K := \sup_{x\in A} ||f'(x)|| < \infty$. Then $\mu_0(f(A)) \leq K \cdot \lambda^*(A)$.*

In this connection, μ_0 is a specific outer measure on X; in particular, $\mu_0 = \lambda^*$ in case $X = \mathbb{R}$. (The construction as well as further properties of μ_0 will be noted down in section 2.) Thus, if $X = \mathbb{R}$ and $\mathcal{D}_f = I$ one concludes from continuity of f and from Theorem 1.1

$$|f(b) - f(a)| \;\leq\; \lambda^*(f(I)) \;\leq\; \sup_{x\in I} |f'(x)| \cdot (b - a) \;,$$

that is the real Mean Value Inequality. In this respect Theorem 1.1 may be called a generalization of the real Mean Value Inequality.

Concerning the special case $X = \mathbb{R}$, the statement of Theorem 1.1 was already formulated some thirty years ago by Dale E.Varberg in his so called

Fundamental Lemma ([3] pp. 832f). Since Varberg makes use of the ordered structure of $\mathbb{R}$, it isn't possible to apply his proof of the *Fundamental Lemma* to the general case concerning functions mapping into an arbitrary Banach space X. Therefore it is necessary to establish a proper proof of Theorem 1.1, which we will lead through in two steps. At first we formulate a preliminary version of Theorem 1.1, which is rather weak (section 3); combining this result with the well-known *Vitali Covering Theorem* one obtains the desired statement of Theorem 1.1 (section 4).

Finally we remark here that already the preliminary version of Theorem 1.1 may be applied to prove the following fact: If f is absolutely continuous and Lebesgue-almost everywhere differentiable with Lebesgue-integrable derivative then f is an indefinite integral. (A complete proof is given in [2], pp. 15ff.)

2. CONSTRUCTING A FAMILY OF OUTER MEASURES ON *X*

This section deals with a family $(\mu_d)_{d\in[0,\infty]}$ of outer measures on X, in special consideration of the outer measure μ_0. Since these so called *d-spherical outer measures* on X are constructed in an analogous way to the well-known Hausdorff 1-dimensional outer measure, standard proofs will be omitted. (For more details see [1], pp. 15ff.)

Let U be a subset of X. If U is non-empty, we define the *diameter* of U as

$$\operatorname{diam}(U) := \sup\{||x-y||; x, y \in U\} ;$$

the diameter of the empty set is defined as 0. U is called a *ball*, if there exist $x \in X$ and $r \geq 0$ such that $B(x,r) \subset U \subset \overline{B}(x,r)$, where $B(x,r)$ ($\overline{B}(x,r)$ resp.) is the open ball (the closed ball resp.) centered at x with radius r. A countable collection $\mathcal{S}$ of subsets of X is called a *d-covering of U*, if $U \subset \bigcup \mathcal{S}$ and $\operatorname{diam}(C) < d$ for every $C \in \mathcal{S}$. We will call a d-covering of U that consists of balls only a *spherical d-covering of U.* For every $d \in]0,\infty]$ one now defines

$$\mu_d(U) \ := \ \inf\left\{\sum_{C\in\mathcal{S}} \operatorname{diam}(C) \ ; \ \mathcal{S} \text{ is a spherical } d\text{-covering of } U\right\} .$$

An easy check establishes that μ_d is an outer measure on X. Taking the supremum

$$\mu_0(U) \ := \ \sup_{d>0} \mu_d(U)$$

we obtain another outer measure μ_0 on X. For every $d \in [0, \infty]$ we call μ_d *d-spherical outer measure* on X. As usual, if $\mu_d(U) = 0$ we call U a *μ_d-null-set.*

According to this construction of μ_0 one immediately realizes the following fact: Using d-coverings instead of spherical d-coverings, one obtains a family $(H^1_d)_{d\in\,]0,\infty]}$ of outer measures on X instead of $(\mu_d)_{d\in\,]0,\infty]}$ and finally the *Hausdorff 1-dimensional outer measure* H^1 on X instead of μ_0. Thus obviously the inequalities $H^1_d(U) \leq \mu_d(U)$ for every $d \in]0; \infty]$ and therefore $H^1(U) \leq \mu_0(U)$ hold. In other words, the spherical outer measures are stronger than their Hausdorff equivalents.

Some basic properties of the d-spherical outer measures are collected in the following four propositions.

Proposition 2.1 (Basic Properties).

(i) If $d, d' \in [0, \infty]$ *and* $d \leq d'$ *then* $\mu_d(U) \geq \mu_{d'}(U)$.
(ii) If $m := \mu_\infty(U) < \infty$ *then* $\mu_d(U) = m$ *for every* $d \in]m, \infty]$.
(iii) $\mu_0(U) = \lim_{d\to 0} \mu_d(U)$.
(iv) $\mu_\infty(U) = \lim_{d\to\infty} \mu_d(U)$.

Proof. *(i)* is trivial. *(ii)* Let $m := \mu_\infty(U) < \infty$ and $d \in]m, \infty]$. Let $\alpha \in]m, d[$. Then there exists a spherical ∞-covering $\mathcal{S}$ of U such that $\sum_{C\in\mathcal{S}} \operatorname{diam}(C) < \alpha < d$. Hence $\operatorname{diam}(C) < d$ for every $C \in \mathcal{S}$. Thus we see that $\mathcal{S}$ is a spherical d-covering of U. It follows $\mu_d(U) \leq \alpha$ and therefore $\mu_d(U) \leq \inf]m, d[= m$. The converse inequality $\mu_d(U) \geq m$ is an immediate consequence of *(i)*. *(iii)* and *(iv)* follow closely by *(i)* and *(ii)*. ▮

As an immediate application of Proposition 2.1 one may prove the following statement already mentioned in the introduction:

Proposition 2.2. *If* $X = \mathbb{R}$ *then* $\mu_d(U) = \lambda^*(U)$ *for every* $d \in [0, \infty]$.

Proof. Let $X = \mathbb{R}$. As a consequence of the construction of μ_∞ one has the identity $\lambda^*(U) = \mu_\infty(U)$. By Proposition 2.1 *(i)*, *(iii)* it is sufficient to prove the inequality $\mu_d(U) \leq \mu_\infty(U)$ for every $d \in]0, \infty[$, so let $d \in]0, \infty[$. Without loss of generality let $m := \mu_\infty(U) < \infty$. Then we can choose $N \in \mathbb{N}$ such that $Nd > m$. By Proposition 2.1 *(ii)* we have $\mu_{Nd}(U) = \mu_\infty(U)$. Therefore it is sufficient to prove $\mu_d(U) \leq \mu_{Nd}(U)$. Let $\mathcal{S}$ be a spherical (Nd)-covering of U, that is, $\mathcal{S}$ is a countable collection of intervals satisfying $\operatorname{diam}(C) < Nd$ for every $C \in \mathcal{S}$. Consequently, the collection

$$\mathcal{S}' \;:=\; \left\{ \inf(C) + [k-1, k]\frac{\operatorname{diam}(C)}{N} \;;\; C \in \mathcal{S}, k \in \{1, ..., N\} \right\}$$

is a spherical d-covering of U and

$$\sum_{C\in\mathcal{S}} \operatorname{diam}(S) \;=\; \sum_{C\in\mathcal{S}'} \operatorname{diam}(S)\;.$$

This proves the desired inequality $\mu_d(U) \le \mu_{Nd}(U)$. ▌

Proposition 2.3 (Null-Sets).

(i) If U is countable then U is a μ_d-null-set for every $d \in [0,\infty]$.

(ii) U is a μ_d-null-set for some $d \in [0,\infty]$ if and only if it is a μ_d-null-set for every $d \in [0,\infty]$.

Proof. *(i)* Let U be countable. Then $\{\{x\}; x \in U\}$ is a spherical d-covering of U for every $d \in]0,\infty]$, so $\mu_d(U) = 0$ and hence also $\mu_0(U) = 0$. *(ii)* Let U be a μ_d-null-set for some $d \in [0,\infty]$. By Proposition 2.1 *(i)* $\mu_\infty(U) = 0$. Thus $\mu_d(U) = 0$ for every $d \in [0,\infty]$ by Proposition 2.1 *(ii)* and the definition of μ_0. The converse implication is trivial. ▌

Proposition 2.4 (Diameter).

(i) $\mu_\infty(U) \le 2\operatorname{diam}(U)$. In particular, if U is a ball then $\mu_d(U) \le \operatorname{diam}(U)$ for every $d \in]\operatorname{diam}(U),\infty]$.

(ii) Let $A \subset I$ and let $\mathcal{E}$ be a countable covering of A. Then $\mu_\infty(f(A)) \le 2\sum_{E\in\mathcal{E}} \operatorname{diam}(f(E\cap I))$.

Proof. *(i)* The first inequality is trivial. If U is a ball then $\{U\}$ is a spherical ∞-covering of U. Therfore we obtain $\mu_\infty(U) \le \operatorname{diam}(U) < \infty$. Thus the second assertion follows closely by Proposition 2.1 *(ii)*. *(ii)* Since μ_∞ is monotone and countably subadditive, the inequality $\mu_\infty(f(A)) \le \sum_{E\in\mathcal{E}} \mu_\infty(f(E\cap I))$ holds. Application of *(i)* completes the proof. ▌

In the last proposition of this section we note down some advantageous properties of the outer measure μ_0. Indeed none of the d-spherical outer measures μ_d, $d > 0$, does have these properties either. Since we make no use of this proposition in the following sections, we will omit the non-trivial proof (see [1], pp. 25ff).

Proposition 2.5.

(i) μ_0 is a regular metric outer measure on X.

(ii) If f is continuous then $\operatorname{diam}(f(I)) \le \mu_0(f(I)) \le \operatorname{Var}(f)$, *where* $\operatorname{Var}(f)$ *denotes the total variation of f on I. If, in addition, f is injective then* $\mu_0(f(I)) = \operatorname{Var}(f)$, *that is, μ_0 is measuring the length of a Jordan curve in X correctly .*

3. A PRELIMINARY VERSION OF THEOREM 1.1

In this section a preliminary version of Theorem 1.1 will be established (Proposition 3.1). To this end we need the following lemma; essentially based on the local Lipschitz property the proof of this lemma is quite easy, thus we will omit it.

LEMMA 3.1. *Let A be a subset of $\mathcal{D}_f \cap]a,b[$, and let*

$$K := \sup_{x\in A} ||f\,'(x)|| < \infty \, .$$

Let $\varepsilon > 0$. Then there exist an open set $U \subset]a,b[$ containing A and a family $(\delta_x)_{x\in A}$ of positive numbers satisfying the following conditions:

(C1) It is $\lambda(U) \leq \lambda^*(A) + \varepsilon$.

(C2) For every $x \in A$ the open interval $]x-\delta_x, x+\delta_x[$ is contained in U.

(C3) For every $x \in A$ and for every $y \in]x-\delta_x, x+\delta_x[$ the inequality $||f(y) - f(x)|| \leq (K+\varepsilon)\cdot|y-x|$ holds.

PROPOSITION 3.1. *Under the conditions of Theorem 1.1*

$$\mu_\infty(f(A)) \leq 2\,K \cdot \lambda^*(A) \, .$$

Proof. Without loss of generality let A be non-empty and $a, b \notin A$. Obviously it is sufficient to prove the inequality

$$\mu_\infty(f(A)) \;\leq\; 2\,(K+\varepsilon)\cdot(\lambda^*(A)+\varepsilon) \tag{1}$$

for every $\varepsilon > 0$, so let $\varepsilon > 0$. By Lemma 3.1 we choose an open set $U \subset]a,b[$ containing A and a family $(\delta_x)_{x\in A}$ of positive numbers satisfying conditions (C1)–(C3). For every $x \in A$ we set $E_x :=]x-\delta_x, x+\delta_x[$. Then the open set $E := \bigcup_{x\,\in\,A} E_x$ obviously contains A and is contained in U (cf. condition (C2)). The collection $\mathcal{E}$ of the connected components of E is countable and consists of open intervals. Combining these facts with Proposition 2.4 *(ii)* and condition (C1) we obtain

$$\mu_\infty(f(A)) \;\leq\; 2 \sum_{W\in\mathcal{E}} \mathrm{diam}(f(W)) \tag{2}$$

and

$$\sum_{W\in\mathcal{E}} \lambda(W) \;=\; \lambda(E) \;\leq\; \lambda(U) \;\leq\; \lambda^*(A)+\varepsilon\,. \tag{3}$$

If, in addition, we show for every $W \in \mathcal{E}$

$$\operatorname{diam}(f(W)) \;\leq\; (K+\varepsilon)\cdot\lambda(W)\,, \tag{4}$$

the inequalities (2) and (3) lead to the desired inequality (1).

Proof of (4): Let $W \in \mathcal{E}$ and $x, y \in W$, $x \leq y$. Then the closed interval $[x, y]$ is contained in W, since W is connected. Because of

$$W \;=\; \bigcup_{v\in A\cap W} E_v \tag{5}$$

the collection $\{E_v; v \in A\cap W\}$ of open sets is a covering of $[x, y]$. By compactness we can choose a finite subset P of $A\cap W$ such that $\{E_v; v \in P\}$ is also a covering of $[x, y]$. Without loss of generality we may assume that $E_v \cap [x, y] \neq \emptyset$ for every $v \in P$ and $E_v \not\subset E_w$ for all $v, w \in P$, $v \neq w$. Let now n be the number of elements in P, and denote these elements in ascending order by $x_1, ..., x_n$. At last we set $E_j := E_{x_j}$ and $\delta_j := \delta_{x_j}$ for every $j \leq n$. Then one verifies that $x \in E_1$, $y \in E_n$ and $E_j \cap E_{j+1} \neq \emptyset$ for every $j < n$. Thus we can choose $p_j \in E_j \cap E_{j+1} \cap [x_j, x_{j+1}]$ for every $j < n$. Using condition (C3) we obtain:

$$\begin{aligned}
||f(x)-f(y)|| \;\leq\;& ||f(x)-f(x_1)||\,+\\
&+\sum_{j=1}^{n-1}\left(||f(x_j)-f(p_j)||+||f(p_j)-f(x_{j+1})||\right)\;+\;||f(x_n)-f(y)||\\
\leq\;& (K+\varepsilon)\cdot\left(|x-x_1|\;+\;\sum_{j=1}^{n-1}((p_j-x_j)+(x_{j+1}-p_j))\;+\;|x_n-y|\right)\\
\leq\;& (K+\varepsilon)\cdot\left(\delta_1+(x_n-x_1)+\delta_n\right)\\
=\;& (K+\varepsilon)\cdot\left((x_n+\delta_n)-(x_1-\delta_1)\right)\,.
\end{aligned}$$

Since x_1 and x_n are elements of $P \subset A\cap W$, the interval $]x_1-\delta_1, x_n+\delta_n[$ is contained in the connected set W (cf. (5)). Thus finally

$$||f(x)-f(y)|| \;\leq\; (K+\varepsilon)\cdot(\sup W-\inf W) \;=\; (K+\varepsilon)\cdot\lambda(W)\,,$$

and the proof of (4) is complete. ▮

Corollary 3.1. *Let A be a Lebesgue-null-set contained in $\mathcal{D}_f$. Then $f(A)$ is a μ_∞-null-set.*

Proof. For every $k \in \mathbb{N}$ the set $A_k := \{x \in A; ||f'(x)|| \leq k\}$ is a Lebesgue-null-set, hence $f(A_k)$ is a μ_∞-null-set by Proposition 3.1. Since $f(A)$ is contained in $\bigcup_{k\in\mathbb{N}} f(A_k)$, the proof is complete. ▌

4. THE PROOF OF THEOREM 1.1

As mentioned above, we will make use of the famous *Vitali Covering Theorem* within the proof of Theorem 1.1. For this reason we recall that a collection $\mathcal{V}$ of intervals is called a *Vitali covering* of a subset A of $\mathbb{R}$, if for given $\varepsilon > 0$ and $x \in A$ there exists $I \in \mathcal{V}$ such that $x \in I$ and $\lambda(I) < \varepsilon$.

THEOREM 4.1 (Vitali Covering Theorem). *Let A be a subset of the real numbers such that $\lambda^*(A)$ is finite, and let $\mathcal{V}$ be a Vitali covering of A. Then there exists a countable collection $\mathcal{W}$ consisting of mutually disjoint elements of $\mathcal{V}$ such that $\lambda^*(A \setminus \bigcup \mathcal{W}) = 0$.*

Proof of Theorem 1.1. Without loss of generality let A be non-empty and $a, b \notin A$. By the definiton of μ_0 it is sufficient to prove the inequality

$$\mu_d(f(A)) \;\leq\; (K+\varepsilon)\cdot(\lambda^*(A)+\varepsilon) \tag{6}$$

for every $d > 0$ and $\varepsilon > 0$, so let $d > 0$ and $\varepsilon > 0$. By Lemma 3.1 we choose an open set $U \subset\,]a, b[$ containing A and a family $(\delta_x)_{x\in A}$ of positive numbers satisfying conditions (C1)–(C3); without loss of generality we may assume that

$$\sup_{x\in A} \delta_x \;<\; \frac{d}{2\,(K+\varepsilon)}\;. \tag{7}$$

Then obviously the collection

$$\mathcal{V} \;:=\; \left\{ \left]\, x - \frac{\delta_x}{n}\,,\, x + \frac{\delta_x}{n} \,\right[\;;\; x \in A,\; n \in \mathbb{N} \right\}$$

is a Vitali covering of A. Hence by the Vitali Covering Theorem we choose a countable collection $\mathcal{W}$ consisting of mutually disjoint elements of $\mathcal{V}$ such that $\lambda^*(A \setminus \bigcup \mathcal{W}) = 0$. Applying Proposition 2.3 *(ii)* and Corollary 3.1 we obtain immediately $\mu_d(f(A \setminus \bigcup \mathcal{W})) = 0$, thus

$$\begin{aligned}
\mu_d(f(A)) \;&\leq\; \mu_d\left(f\left(A \cap \bigcup \mathcal{W}\right)\right) + \mu_d\left(f\left(A \setminus \bigcup \mathcal{W}\right)\right) \\
&\leq\; \mu_d\left(f\left(\bigcup \mathcal{W}\right)\right) \;\leq\; \sum_{W\in\mathcal{W}} \mu_d(f(W))\;.
\end{aligned} \tag{8}$$

We now fix $W \in \mathcal{W}$. Referring to the definition of $\mathcal{V}$ we choose $x_W \in A$ and $\delta_W \in]0, \delta_{x_W}]$ such that $W =]x_W - \delta_W, x_W + \delta_W[$. Applying condition (C3) we see that $f(W)$ is a subset of the open ball $B(f(x_W), (K+\varepsilon)\delta_W)$, hence the inequality

$$\mu_d(f(W)) \;\leq\; \mu_d\left(B(f(x_W), (K+\varepsilon)\delta_W)\right) \tag{9}$$

holds. By (7) we have $\operatorname{diam}(B(f(x_W), (K+\varepsilon)\delta_W)) = 2(K+\varepsilon)\delta_W \leq d$. Thus by Proposition 2.4 *(i)* we obtain

$$\mu_d(B(f(x_W), (K+\varepsilon)\delta_W)) \;\leq\; 2\,(K+\varepsilon)\,\delta_W\ . \tag{10}$$

Combination of (7)–(10) leads immediately to

$$\mu_d(f(A)) \;\leq\; \sum_{W\in\mathcal{W}} 2\,(K+\varepsilon)\,\delta_W \;=\; (K+\varepsilon) \sum_{W\in\mathcal{W}} \lambda(W)\ . \tag{11}$$

By choice the countable collection $\mathcal{W}$ consists of mutually disjoint elements of $\mathcal{V}$, hence $\sum_{W\in\mathcal{W}} \lambda(W) = \lambda\left(\bigcup \mathcal{W}\right)$. Applying this fact and conditions (C2), (C3) we obtain from (11)

$$\begin{aligned} \mu_d(f(A)) \;&\leq\; (K+\varepsilon)\,\lambda\left(\bigcup \mathcal{W}\right) \;\leq\; (K+\varepsilon)\,\lambda\left(\bigcup_{x\in A}]x-\delta_x, x+\delta_x[\right) \\ &\leq\; (K+\varepsilon)\,\lambda(U) \;\leq\; (K+\varepsilon)\cdot(\lambda^*(A)+\varepsilon)\ , \end{aligned}$$

that is the desired inequality (6). ▌

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