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Symplectic dynamics of contact isotropic torus complements

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Abstract. We determine the homotopy type of isotropic torus complements in closed contact manifolds in terms of Reeb dynamics of special contact forms. For that, we utilize holomorphic curve techniques known from symplectic field theory as Gromov–Hofer compactness and localized transversality on noncompact contact manifolds.

1. Introduction

By the isotropic neighborhood theorem, a neighborhood of a closed isotropic submanifold Q in a given contact manifold is determined by the diffeomorphism type of Q and by the isomorphism class of the conformally symplectic normal bundle $\mathrm{CSN}(Q)$ of Q, cf. [9, Theorem 2.5.8]. For instance, if $\mathrm{CSN}(Q)$ is trivial, a trivialization of $\mathrm{CSN}(Q)$ determines a local model given by a neighborhood of Q in the contactisation of $T^*Q \times \mathbb{C}^{n-d}$, $d = \dim Q \leq n$, cf. [19, Section 3.1]. On the other hand, the restriction of any defining contact form to the tangent bundle of a compact hypersurface determines the germ of the contact structure, see [5, Proposition 6.4]. In particular, this applies to the boundary of a disc-like neighborhood of Q. Combined with local contact inversion (see Proposition A.5), it turns out that there is no canonical distinction between inside and outside for this hypersurface if $\mathrm{CSN}(Q)$ is trivial.

In this work we consider the case where Q is a torus T^d and where the complement M of a tubular neighborhood of Q is compact. Assuming $n > d$, we investigate to which extent a choice of a contact form on M that is of model type near the boundary determines the topology of M. As demonstrated by Eliashberg and Hofer [6] for $\dim M = 3$, and Geiges and Zehmisch [15] for $\dim M \geq 5$, M is diffeomorphic to a $(2n+1)$-dimensional ball whenever M does not have any short contractible periodic Reeb orbits and ∂M is a

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sphere. This situation corresponds to $Q = *$. In fact, Eliashberg and Hofer [6] proved a global Darboux theorem in the absence of short periodic Reeb orbits if $\dim M = 3$. In contrast, Geiges, Röttgen and Zehmisch [10] constructed an aperiodic Reeb flow with trapped orbits on $\mathbb{R}^{2n+1}$, $n \geq 2$, that is standard outside a compact set. In order to find a relation between the topology of the isotropic knot complement M and the existence of short periodic Reeb orbits on M, we will utilize holomorphic curves as it is typical in symplectic dynamics as propagated by Bramham and Hofer [4].

1.1. **Main result.** Let us assume that Q is the d-dimensional torus

$$T^d := \mathbb{R}^d / 2\pi\mathbb{Z}^d.$$

We consider a compact, connected $(2n+1)$-dimensional strict contact manifold (M, α) with boundary

$$\partial M = S(T^*Q \oplus \underline{\mathbb{R}}^{2n+1-2d})$$

equal to the unit sphere bundle of the stabilized cotangent bundle $T^*Q \oplus \underline{\mathbb{R}}^{2n+1-2d}$ of T^*Q. In particular, the boundary of M is diffeomorphic to

$$\partial M = T^d \times S^{2n-d}.$$

The aim of this work is to give a criterion for M to be diffeomorphic to the unit disc bundle

$$D(T^*Q \oplus \underline{\mathbb{R}}^{2n+1-2d}) = T^d \times D^{2n+1-d}$$

in terms of the infimum $\inf_0(\alpha)$ of all positive periods of *contractible* closed Reeb orbits of the Reeb vector field of α and the following **embeddability condition**: Write

$$Z := \mathbb{R} \times T^*T^d \times D^2 \times \mathbb{C}^{n-1-d}$$

for the model neighborhood of an isotropic torus T^d with trivial conformally symplectic normal bundle inside a contact manifold equipped with the contact form

$$\alpha_Z := \mathrm{d}b + \sum_{j=1}^{d} p_j \mathrm{d}q_j + \frac{1}{2}(x_0 \mathrm{d}y_0 - y_0 \mathrm{d}x_0) - \sum_{j=1}^{n-1-d} y_j \mathrm{d}x_j,$$

where $b \in \mathbb{R}$, p_j, q_j are coordinates on the cotangent bundle T^*T^d, x_0, y_0 are coordinates on the closed unit disc D^2, and $x_j + \mathrm{i}y_j$ are coordinates on $\mathbb{C}^{n-1-d}$. We will use the following short form of the contact form throughout the text:

$$\alpha_Z = \mathrm{d}b + \mathbf{p}\mathrm{d}\mathbf{q} + \frac{1}{2}(x_0 \mathrm{d}y_0 - y_0 \mathrm{d}x_0) - \mathbf{y}\mathrm{d}\mathbf{x}.$$

We say that ∂M admits a **contact embedding** into (Z, α_Z) if there exists a strict contact embedding φ of a collar neighborhood U of $\partial M \subset M$ into the interior of Z, in the sense that $\varphi^*\alpha_Z = \alpha$, such that

- each flow line of the Reeb vector field ∂_b intersects $\varphi(\partial M) \subset Z$ in at most two points,
- the image $\varphi(U)$ is contained in the bounded component of $Z \setminus \varphi(\partial M)$,
- $\varphi(\partial M)$ is smoothly isotopic to $S(T^*T^d \oplus \underline{\mathbb{R}}^{2n+1-2d})$ inside Z.

Theorem 1.2. *Let (M,α) be a strict contact manifold as described above such that ∂M has a contact embedding into (Z,α_Z) with $n>d$. If $\inf_0(\alpha)\geq\pi$, then M and $T^d\times D^{2n+1-d}$ are homotopy equivalent if $n=2$ and diffeomorphic otherwise.*

Observe the standing assumption $n\geq d$. The extra condition we require in the theorem is $n\neq d$, meaning that Q is **subcritically isotropic**. This assumption allows us the use of holomorphic discs as a $\mathbb{C}$-factor that can be split off in the model situation.

The case $n=1$ is covered by the work of Eliashberg and Hofer [6]; the critical case $n=1$ and $d=1$, without any shortness assumption on $\inf_0(\alpha)$, is proved by Kegel, Schneider and Zehmisch [18] using a different method. Therefore, we assume $n\geq 2$ (and $n>d$) throughout the article. The case $d=0$ is due to the work of Geiges and Zehmisch [15], in which even diffeomorphism can be concluded.

Performing contact connected sum of Z with any contact manifold one obtains a periodic Reeb orbit of period strictly less than (but arbitrarily close to) π contained in the belt sphere, cf. [15, Remark 1.3 (1)]. Hence, the bound π in Theorem 1.2 is optimal. Moreover, the contrapositive of Theorem 1.2 can be used to prove existence of periodic Reeb orbits on noncompact manifolds. Consider a compact contact manifold (M,ξ) whose boundary has precisely two connected components each admitting a contact embedding into (Z,α_Z) individually. Using the Reeb flow on the model (Z,α_Z), the images can be assumed to be not nested so that a gluing of (M,ξ) to (Z,α_Z) along the boundary is possible. The gluing result cannot be homotopy equivalent to $T^d\times D^{2n+1-d}$ so that any contact form on (M,ξ), standard near the boundary, possesses a contractible periodic Reeb orbit, cf. [15, Remark 1.3 (4)].

For existence results of periodic Reeb orbits on noncompact contact manifolds with asymptotic and periodic boundary conditions, we refer to the work of Suhr and Zehmisch [22] and Bae, Wiegand and Zehmisch [1] (cf. [23]), respectively.

1.3. Filling by holomorphic discs.

The basic idea of the proof of Theorem 1.2 is the same as described in [15, Section 1.2], invoking *filling by holomorphic discs* techniques as worked out in [2, 11, 13, 14, 15, 24]. Using the contact embedding φ of ∂M into (Z,α_Z), we form a new strict contact manifold $(\hat{M},\hat{\alpha})$ by replacing the bounded component of $\mathbb{R}\times T^*T^d\times\mathbb{C}^{n-d}\setminus\varphi(\partial M)$ by M. The contact form $\hat{\alpha}$ equals α on M and

$$\mathrm{d}b+\mathbf{p}\mathrm{d}\mathbf{q}+\frac{1}{2}(x_0\mathrm{d}y_0-y_0\mathrm{d}x_0)-\mathbf{y}\mathrm{d}\mathbf{x}$$

on the unbounded component of $\mathbb{R}\times T^*T^d\times\mathbb{C}^{n-d}\setminus\varphi(\partial M)$. Observe that the latter contact form coincides with α_Z on Z. We define $\hat{Z}$ similarly by gluing M into Z.

Further, we will consider holomorphic maps

$$u = (a, f)\colon \mathbb{D} \to W$$

defined on the closed unit disc $\mathbb{D} \subset \mathbb{C}$ and taking values in the symplectization W of $(\hat{M}, \hat{\alpha})$ subject to varying Lagrangian boundary conditions. The moduli space $\mathcal{W}$ of all such holomorphic discs carries an evaluation map

$$\mathrm{ev}\colon \mathcal{W} \times \mathbb{D} \to \hat{Z}, \quad ((a, f), z) \mapsto f(z).$$

A priori, ev takes values in $\hat{M}$, but we will show that ev indeed takes values in the smaller set $\hat{Z} \subset \hat{M}$. It will turn out that either the evaluation map ev is proper and surjective of degree one, in which case we can draw conclusions with the s-cobordism theorem as in the work of Barth, Geiges and Zehmisch [3], or the moduli space $\mathcal{W}$ is not locally compact in the sense that there will be breaking off of finite energy planes. By a result of Hofer [16, 17], this in turn results in the existence of short contractible periodic Reeb orbits of α, as the Reeb flow of α_Z is linear, given by ∂_b.

Observe that a contact embedding of ∂M into (Z, α_Z) yields an embedding of ∂M into $\mathbb{R} \times T^*T^d \times D^2_r \times \mathbb{C}^{n-1-d}$ for some slightly smaller radius $r \in (0, 1)$. The proof of Theorem 1.2 that we are going to present in this work will show that being short for a contractible periodic Reeb orbit should mean to have period less than or equal to πr^2. Therefore, the above mentioned second alternative will be excluded by requiring $\inf_0(\alpha) > \pi r^2$ as an Arzelà–Ascoli argument shows. For ease of notation, we will assume $r = 1$ so that we assume the stronger condition $\inf_0(\alpha) > \pi$ during the proof.

1.4. **Relevance of the torus.** Large parts of the argument work under considerably weaker assumptions – mainly the topological part, which is similar to [3]. In order to set up the holomorphic disc analysis, we use a foliation by Lagrangian submanifolds of T^*Q as a parameterized boundary condition. Moreover, we use a choice of strictly plurisubharmonic potential for the Liouville form on T^*Q that, together with the maximum principle, ensures C^0-bounds in the compactness argument, see Section 3.3.2. This together with the Niederkrüger map, which we use to construct holomorphic discs, works particularly well in global (periodic) coordinates on T^*Q. It is not clear how to change the set-up to enlarge the class of examples.

2. Standard holomorphic discs

The model contact manifold (Z, α_Z) is the contactisation of the Liouville manifold

$$(V, \lambda_V) := \Big(T^*T^d \times D^2 \times \mathbb{C}^{n-1-d}, \, \mathbf{p}\mathrm{d}\mathbf{q} + \frac{1}{2}(x_0\mathrm{d}y_0 - y_0\mathrm{d}x_0) - \mathbf{y}\mathrm{d}\mathbf{x}\Big),$$

which contains the holomorphic discs $\{\mathbf{w}\} \times D^2 \times \{\mathbf{s} + \mathrm{i}\mathbf{t}\}$. The aim of this section is to describe a lift of these holomorphic discs to the symplectization of (Z, α_Z). These holomorphic discs will appear as the standard discs of the moduli space $\mathcal{W}$ and serve as a description of the end of $\mathcal{W}$. In order to lift, we proceed in two steps. The first will be a lift to $\mathbb{C} \times T^*T^d \times D^2 \times \mathbb{C}^{n-1-d}$; the second is a transformation along a biholomorphic map Φ from $\mathbb{R} \times \mathbb{R} \times T^*T^d \times D^2 \times \mathbb{C}^{n-1-d}$ to $\mathbb{C} \times T^*T^d \times D^2 \times \mathbb{C}^{n-1-d}$, the Niederkrüger map from [21, Proposition 5].

2.1. **The contactization.** Following the explanations from [15, Section 2], we denote the Liouville manifold form the beginning of Section 2 by (V, λ_V) so that its contactization $(\mathbb{R} \times V, \mathrm{d}b + \lambda_V)$ is equal to the strict contact manifold (Z, α_Z). The corresponding contact structure ξ_Z is given by the set of tangent vectors $v - \lambda_V(v)\partial_b$ for all $v \in TV$.

2.2. **Liouville manifold and Kähler potential.** The Liouville manifold (V, λ_V) admits a complex structure

$$J_V := (-\mathrm{i}) \oplus \mathrm{i} \oplus \mathrm{i},$$

where $-\mathrm{i}$ is meant to be the negative of the complex structure on T^d obtained by the quotient of $\mathbb{R}^d$ by $2\pi\mathbb{Z}^d$ and $T^*\mathbb{R}^d \equiv \mathbb{R}^{2n} \equiv \mathbb{C}^d$ such that $-\mathrm{i}$ is an almost complex structure on T^*T^d compatible with $\mathrm{d}\mathbf{p} \wedge \mathrm{d}\mathbf{q}$, cf. [21, Appendix B]. A strictly plurisubharmonic potential ψ in the sense of [12, Section 3.1], so that J_V is compatible with the symplectic form $\mathrm{d}\lambda_V$ and $\lambda_V = -\mathrm{d}\psi \circ J_V$, is given by

$$\psi(\mathbf{w}, z_0, \mathbf{z}) := \frac{1}{2}\sum_{j=1}^{d} p_j^2 + \frac{1}{4}|z_0|^2 + \frac{1}{2}\sum_{j=1}^{n-1-d} y_j^2,$$

where the point $\mathbf{w} \in T^*T^d$ is written in coordinates as $(q_1, p_1, \dots, q_d, p_d)$ and $\mathbf{z} = z_1, \dots, z_{n-1-d}$, where $z_j = x_j + \mathrm{i}y_j$, $j = 0, 1, \dots, n-1-d$, denote coordinates on $D^2 \times \mathbb{C}^{n-1-d}$.

2.3. **The symplectization.** For any positive, strictly increasing smooth function $\tau \equiv \tau(a)$ on $\mathbb{R}$ the symplectization of (Z, α_Z) is defined to be the symplectic manifold

$$(\mathbb{R} \times Z, \mathrm{d}(\tau\alpha_Z)).$$

A compatible and translation invariant almost complex structure that preserves the contact hyperplanes ξ_Z on all slices $\{a\} \times Z$ is determined by $\partial_a \mapsto \partial_b$ and the requirement that for all $v \in TV$, the tangent vectors $v - \lambda_V(v)\partial_b$ get mapped to $J_V v - \lambda_V(J_V v)\partial_b$. With that choice of an almost complex structure on $\mathbb{R} \times \mathbb{R} \times V$ the **Niederkrüger map**

$$\Phi(a, b; p) = \big(a - \psi(p) + \mathrm{i}b, p\big)$$

is a biholomorphism onto $\mathbb{C} \times V$, equipped with the almost complex structure $\mathrm{i} \oplus J_V$.

2.4. **The Niederkrüger transform.** The resulting holomorphic discs maps

$$\mathbb{D} \to \mathbb{R} \times \mathbb{R} \times T^*T^d \times D^2 \times \mathbb{C}^{n-1-d},$$

to which we refer as being **standard**, can be parameterized by

$$u^{\mathbf{t},\mathbf{w}}_{\mathbf{s},b}(z) = \Big(\frac{1}{4}(|z|^2 - 1), b; \mathbf{w}, z, \mathbf{s} + \mathrm{i}\mathbf{t}\Big)$$

for parameters $b \in \mathbb{R}$, $\mathbf{w} \in T^*T^d$, and $\mathbf{s}, \mathbf{t} \in \mathbb{R}^{n-1-d}$, cf. [15, Section 2.2]. Natural Lagrangian boundary conditions for the restrictions of the standard holomorphic disc maps to $\partial\mathbb{D}$ are given by Lagrangian cylinders

$$L^{\mathbf{t}}_{\mathbf{p}} := \{0\} \times \mathbb{R} \times T^d \times \{\mathbf{p}\} \times \partial D^2 \times \mathbb{R}^{n-1-d} \times \{\mathbf{t}\},$$

parameterized by $\mathbf{t} \in \mathbb{R}^{n-1-d}$ and $\mathbf{p} \in \mathbb{R}^d$, which foliate $\{0\} \times \partial Z$. In order to verify $L^{\mathbf{t}}_{\mathbf{p}}$ to be Lagrangian, observe that the restriction of $\mathrm{d}(\tau\alpha_Z)$ to the tangent bundle of $\{0\} \times Z$ equals $\tau(0)\mathrm{d}\alpha_Z$, which is a positive multiple of

$$\mathrm{d}\mathbf{p} \wedge \mathrm{d}\mathbf{q} + \mathrm{d}x_0 \wedge \mathrm{d}y_0 + \mathrm{d}\mathbf{x} \wedge \mathrm{d}\mathbf{y},$$

and that $L^{\mathbf{t}}_{\mathbf{p}}$ is of dimension $n+1$.

3. A boundary value problem

Let (W, ω) be the symplectization

$$(W, \omega) := (\mathbb{R} \times \hat{M}, \mathrm{d}(\tau\hat{\alpha}))$$

of the glued strict contact manifold $(\hat{M}, \hat{\alpha})$ introduced in Section 1.3, where τ is a positive, strictly increasing smooth function on $\mathbb{R}$ such that $\tau(a) = \mathrm{e}^a$ for all $a \geq 0$.

3.1. **An almost complex structure.** Let J be a compatible almost complex structure on (W, ω) that is invariant under translations in $\mathbb{R}$-direction, sends the coordinate vector field ∂_a to the Reeb vector field of the contact from $\hat{\alpha}$, and restricts to a compatible complex bundle structure on $(\hat{\xi}, \mathrm{d}\hat{\alpha})$, where $\hat{\xi}$ denotes the contact structure defined by $\hat{\alpha}$. Observe that the required conditions for J are satisfied simultaneously for all admissible τ.

We would like to specify a choice of an almost complex structure J in order to deal with the non-compactness of $\hat{M}$. For positive real numbers b_0, r, R, we define the **box** by

$$B := [-b_0, b_0] \times D_R T^*T^d \times D^2_r \times D^{2n-2-2d}_R,$$

where $D^{2\ell}_\rho \subset \mathbb{C}^\ell$ denotes the closed 2ℓ-disc of radius ρ and $D_\rho T^*T^d$ is the closed ρ-disc subbundle of T^*T^d. We choose $r < 1$ so that the box is contained in Z and require that the interior of the box contains $\varphi(\partial M)$, i.e.,

$$\varphi(\partial M) \subset \mathrm{Int}(B) \subset Z.$$

Similarly to the use of the symbols $\hat{M}$ and $\hat{Z}$, we write $\hat{B}$ for the result of gluing M into B. Observe the chain of strict inclusions

$$M \subset \hat{B} \subset \hat{Z} \subset \hat{M}.$$

On the complement of $\mathbb{R} \times \mathrm{Int}(\hat{B})$, we require the almost complex structure J to be the one defined in Section 2, with the obvious modification of the construction by taking the contactization of $\mathbb{R} \times T^*T^d \times \mathbb{C} \times \mathbb{C}^{n-1-d}$ instead of Z. On $\mathbb{R} \times \mathrm{Int}(\hat{B})$, the choice of J will be subject to genericity considerations specified in Section 3.5.3.

3.2. **The moduli space.** Let $\mathcal{W}$ be the **moduli space** of all holomorphic discs

$$u = (a, f)\colon \mathbb{D} \to (W, J),$$

for which there exists a **level** $(\mathbf{p}, \mathbf{t}) \in \mathbb{R}^d \times \mathbb{R}^{n-1-d}$ selecting the Lagrangian boundary cylinder $L^{\mathbf{t}}_{\mathbf{p}}$ in $\{0\} \times \partial Z$ such that the following boundary condition is satisfied:

$$u(\partial\mathbb{D}) \subset L^{\mathbf{t}}_{\mathbf{p}}.$$

In particular, the map $\Psi \circ f$, which is defined in a neighborhood of $\partial\mathbb{D}$, is constant along $\partial\mathbb{D}$, setting $\Psi(b, \cdot\,) = \psi$ for all $b \in \mathbb{R}$. Additionally, we require that for all $u \in \mathcal{W}$, there exist sufficiently large parameters $b \in \mathbb{R}$, $\mathbf{w} = (\mathbf{q}, \mathbf{p}) \in T^*T^d$, and $\mathbf{s} \in \mathbb{R}^{n-1-d}$ such that the standard disc $u^{\mathbf{t},\mathbf{w}}_{\mathbf{s},b}$ – $(\mathbf{p}, \mathbf{t})$ being the level of u – can be regarded as holomorphic disc in (W, J) and is homologous to u in W relative $L^{\mathbf{t}}_{\mathbf{p}}$, i.e.,

$$[u] = [u^{\mathbf{t},\mathbf{w}}_{\mathbf{s},b}] \quad \text{in } H_2(W, L^{\mathbf{t}}_{\mathbf{p}}).$$

Because of $n \geq 2$, all standard holomorphic discs of the same level are homotopic relative boundary so that the homological condition is well posed. The reparametrization group – the group of biholomorphic diffeomorphisms of $\mathbb{D}$ – is divided out by the requirement

$$f(\mathrm{i}^k) \in \mathbb{R} \times T^d \times \{\mathbf{p}\} \times \{\mathrm{i}^k\} \times \mathbb{R}^{n-1-d} \times \{\mathbf{t}\} \quad \text{for } k = 0, 1, 2,$$

i.e., u is required to map the marked points $1, \mathrm{i}, -1$ to the characteristic leaves $L^{\mathbf{t}}_{\mathbf{p}} \cap \{z_0 = 1\}$, $L^{\mathbf{t}}_{\mathbf{p}} \cap \{z_0 = \mathrm{i}\}$, and $L^{\mathbf{t}}_{\mathbf{p}} \cap \{z_0 = -1\}$, respectively.

3.3. **Convergence.** We will study C^∞-compactness properties of holomorphic discs in $\mathcal{W}$. In the following we list elementary properties that all $u = (a, f) \in \mathcal{W}$ share.

3.3.1. *Uniform energy bounds.* The L^2-norm of the gradient is uniformly bounded in the sense that the **symplectic energy** $\int_{\mathbb{D}} u^*\omega$, which is equal to the **action** $\int_{\partial\mathbb{D}} f^*\hat{\alpha}$ of the boundary circle, is equal to π. This follows as in [15, Lemma 3.2] because u is homologous to a certain standard disc.

3.3.2. C^0*-bounds and maximum principle.* As it is the case for any holomorphic curve $u = (a, f)$ in symplectizations, the function a is subharmonic, cf. [15, Lemma 3.6 (i)]. In the situation at hand, we conclude with the arguments from [15, Lemma 3.6 (i)] that $a < 0$ on $\mathrm{Int}(\mathbb{D})$ for all $u = (a, f) \in \mathcal{W}$.

In order to describe the behavior of $u = (a, f) \in \mathcal{W}$ in the direction of $\hat{M}$, we denote by G the f-preimage of $\hat{M} \setminus \mathrm{Int}(\hat{B})$. Namely, on G we can introduce coordinate functions

$$f = (b, \mathbf{w}, h_0, \mathbf{h}) \quad \text{on } \mathbb{R} \times T^*T^d \times \mathbb{C} \times \mathbb{C}^{n-1-d},$$

according to the indicated splitting. By the properties of the Niederkrüger map, the coordinate function b is harmonic and the $\mathbf{w}, h_0, \mathbf{h}$ are holomorphic.

If $G = \mathbb{D}$, then u will be one of the standard discs $u^{\mathbf{t},\mathbf{w}}_{\mathbf{s},b}$ sitting in the complement of $\mathbb{R} \times \mathrm{Int}(\hat{B})$. The argument for that is the same as for [15, Lemma 3.7] with the following additional observation: The holomorphic map $\mathbf{w}\colon \mathbb{D} \to T^*T^d$ lifts to a holomorphic map to the universal cover resulting into an anti-holomorphic disc map into $\mathbb{C}^d$, with boundary circle $\partial\mathbb{D}$ mapped into a totally real affine plane $\mathbb{R}^d \times \{\mathbf{p}\}$. Hence, as in [15, Lemma 3.7] or by Schwarz reflection, $\mathbf{w}$ must be constant. An alternative argument is based on the fact that the symplectic energy of $\mathbf{w}$, which is equal to the Dirichlet energy, vanishes, so that, again, $\mathbf{w}$ must be constant. Denoting the level of u by $(\mathbf{p}, \mathbf{t})$, we use a retraction of T^*T^d to $T^d \times \{\mathbf{p}\}$ to homotope the disc $\mathbf{w}$ into the Lagrangian submanifold $T^d \times \{\mathbf{p}\}$ relative boundary. As the homotoped disc has vanishing symplectic energy, the symplectic energy of $\mathbf{w}$ vanishes by Stoke's theorem too.

In the situation that G is a proper subset of $\mathbb{D}$, we will make the following observations: By construction, G contains a neighborhood of $\partial\mathbb{D}$ so that the strong maximum principle and the boundary lemma of E. Hopf apply to h_0. Indeed, as in [15, Lemma 3.6 (ii)], we conclude that $f(\mathrm{Int}(\mathbb{D}))$ is contained in $\mathrm{Int}(\hat{Z})$. Moreover, by the comments on [15, p. 669 and p. 671], we see that h_0 restricts to an immersion on $\partial\mathbb{D}$ so that $u(\partial\mathbb{D})$ is positively transverse to each of the characteristic leaves $L^{\mathbf{t}}_{\mathbf{p}} \cap \{z_0 = \mathrm{e}^{\mathrm{i}\theta}\}$, $\theta \in [0, 2\pi)$, denoting the level of u by $(\mathbf{p}, \mathbf{t})$. By the homological condition posed by the boundary value problem for $u = (a, f) \in \mathcal{W}$, we infer that h_0 restrict in fact to an embedding on $\partial\mathbb{D}$.

Continuing the discussions on the case $G \neq \mathbb{D}$, we observe that, by the arguments in [15, Lemma 3.8], the coordinate function b of u, as u cannot be a standard disc, takes values in $[-b_0, b_0]$. For that, recall that $\Psi \circ f$ is constant along $\partial\mathbb{D}$. Similarly, there exists a real number $R_0 > R$ such that the intersection of the non-standard disc $f(\mathbb{D})$ with

$$\mathbb{R} \times (T^*T^d \setminus D_{R_0}T^*T^d) \times \mathbb{C} \times (\mathbb{C}^{n-1-d} \setminus D^{2n-2-2d}_{R_0})$$

is empty. The projection $\mathbf{h}$ to the $\mathbb{C}^{n-1-d}$-factor can be treated similar to [15, Lemma 3.9]. For the cotangent factor, notice that a composition of local lifts of $\mathbf{w}$ with complex conjugation on the universal cover results in local holomorphic maps with respect to the standard complex structure on $\mathbb{C}^d$. Therefore, the maximum principle implies then that $\mathbf{w}$ takes values in the codisc bundle of radius $|\mathbf{p}|$. In view of the uniform energy bounds stated in Section 3.3.1, the monotonicity lemma gives an upper bound on the level $|\mathbf{p}|$ of u analogously to the arguments on [15, p. 674] (by possibly using several geodesic balls of the same radius less than or equal to π to cut area out of the holomorphic disc $\mathbf{w}$).

In conclusion, we obtain uniform C^0-bounds on the $\hat{M}$-part of all non-standard holomorphic discs in $\mathcal{W}$.

3.3.3. *Compactness.* The space of non-standard holomorphic discs in $\mathcal{W}$ is C^∞-compact under the assumption that all contractible periodic Reeb orbits of $\hat{\alpha}$ have action greater than π. This follows with the arguments in [15, Section 4] and [7, 8], and an identification of T^*T^d with $T^d \times \mathbb{R}^d$ so that the variation of the boundary condition $T^d \times \{\mathbf{p}\}$ in the cotangent factor of $L^{\mathbf{t}}_{\mathbf{p}}$ can be described with help of translations in $\mathbb{R}^d$. What remains to show is the following indecomposability statement.

Lemma 3.4. *The homology class* $[u]$ *of all* $u \in \mathcal{W}$ *in* $H_2(W, L^{\mathbf{t}}_{\mathbf{p}})$*, where* $(\mathbf{p}, \mathbf{t})$ *denotes the level of* u*, is* J*-indecomposable.*

Proof. If not, we could find non-constant holomorphic discs $u^1, \dots, u^N$ with boundary on $L^{\mathbf{t}}_{\mathbf{p}}$ such that $[u]$ can be decomposed into the sum

$$[u] = m_1[u^1] + \cdots + m_N[u^N]$$

for natural numbers N and $m_1, \dots, m_N$, where at least one of them is greater than 1. By exactness of the symplectic form ω, none of the holomorphic maps $u^1, \dots, u^N$ can be defined on a sphere. Denote the restrictions to the boundary by γ and $\gamma^1, \dots, \gamma^N$, and observe that

$$[\gamma] = m_1[\gamma^1] + \cdots + m_N[\gamma^N]$$

in $H_1(L^{\mathbf{t}}_{\mathbf{p}})$. As $L^{\mathbf{t}}_{\mathbf{p}}$ is contained in the complement of $\mathrm{Int}(\hat{B})$ the boundary loops admit a splitting with respect to

$$\{0\} \times \mathbb{R} \times T^d \times \{\mathbf{p}\} \times \partial D^2 \times \mathbb{R}^{n-1-d} \times \{\mathbf{t}\}.$$

The symplectic energy of holomorphic discs in (W, ω) is equal to the action of the boundary loops so that by positive transversality with respect to the characteristic leaves mentioned in Section 3.3.2, we obtain

$$\pi = \sum_{j=1}^{N} m_j [\gamma^j_{T^d}] + \sum_{j=1}^{N} m_j n_j \pi$$

for natural numbers $n_1, \dots, n_N$. The first summand equals the total action of the projections of $u^j|_{\partial\mathbb{D}}$ to $T^d \times \{\mathbf{p}\}$ and so is in turn equal to the action of the corresponding projection γ_{T^d} of γ. Because $[u]$ can be represented by a standard disc $u^{\mathbf{t},\mathbf{w}}_{\mathbf{s},b}$, $\mathbf{w} = (\mathbf{q}, \mathbf{p})$, whose boundary map equals

$$e^{i\theta} \mapsto (0, b; \mathbf{w}, e^{i\theta}, \mathbf{s} + i\mathbf{t}),$$

the action of γ_{T^d} vanishes. In total, we reach the inequality $\pi \geq N\pi$. Therefore, $N = 1$, proving J-indecomposability. $\square$

3.5. **Transversality.** Each standard holomorphic disc in $\mathcal{W}$ admits a neighborhood that can be parameterized by

$$(b;\mathbf{w},\mathbf{s}+\mathrm{i}\mathbf{t})\in\mathbb{R}\times T^*T^d\times\mathbb{C}^{n-1-d}.$$

We will show that a similar parametrization near each of the non-standard holomorphic discs in $\mathcal{W}$ exists so that $\mathcal{W}$ will be a smooth manifold of dimension $2n-1$.

3.5.1. *Maslov index.* The Maslov index of all $u\in\mathcal{W}$ is equal to 2. By the considerations in [15, Lemma 3.1], it is enough to compute the Maslov index for all standard discs $u\in\mathcal{W}$, which lift up to standard holomorphic discs in

$$\mathbb{R}\times\mathbb{R}\times T^*\mathbb{R}^d\times D^2\times\mathbb{C}^{n-1-d}\equiv\mathbb{R}\times\mathbb{R}\times D^2\times\mathbb{C}^{n-1}$$

in the sense of [15].

3.5.2. *Simplicity.* Using Lemma 3.4 and [20, Theorem A], we see that all holomorphic discs $u\in\mathcal{W}$ are simple, cf. [15, Lemma 3.4]. Based on that, one shows as in [15, Lemma 3.5] that for all $u=(a,f)\in\mathcal{W}$, the set of all f-injective points is open and dense in $\mathbb{D}$. One only has to observe that the projection h_0 of f to the D^2-factor is an embedding along $\partial\mathbb{D}$ so that $u|_{\partial\mathbb{D}}$ is positively transverse to the characteristic leaves – as done in Section 3.3.2.

3.5.3. *Linearized Cauchy–Riemann operator.* Based on Section 3.5.2, one chooses a regular almost complex structure J as in [15, Section 5.2] so that the linearized Cauchy–Riemann operator D_u is onto for all $u\in\mathcal{W}$. Taking variations of the level parameters $(\mathbf{p},\mathbf{t})\in\mathbb{R}^d\times\mathbb{R}^{n-1-d}$ for the Lagrangian boundary conditions $L_{\mathbf{p}}^{\mathbf{t}}$ induced by translations similar to [15, Section 4.1], one computes, using Section 3.5.1, the Fredholm index of D_u to be $2n+2$ as in [15, Section 5.1]. Subtracting 3 for the marked points fixed by three characteristic leaves yields $2n-1$, which turns out to be the dimension of $\mathcal{W}$. In fact, $\mathcal{W}$ is a smooth manifold that admits a natural orientation obtained by the orientation of D_u described in [15, Section 5.3] – observe that $L_{\mathbf{p}}^{\mathbf{t}}$ admits a canonical parallelization and and the above mentioned variations of the Lagrangian boundary conditions $L_{\mathbf{p}}^{\mathbf{t}}$.

4. THE HOMOTOPY TYPE

In Section 3 we showed that the moduli space $\mathcal{W}$ is a smooth, naturally oriented manifold of dimension $2n-1$ such that the evaluation map

$$\mathrm{ev}\colon\mathcal{W}\times\mathbb{D}\to\hat{Z},\quad((a,f),z)\mapsto f(z)$$

is proper and of degree one. In this section we will use the evaluation map in order to draw conclusions on the homotopy type of the contact manifold M.

4.1. **Homology type and fundamental group.** With the argumentation used in [3, Sections 2.3 and 2.5] and [15, Section 6], we obtain that the evaluation map

$$\mathrm{ev}\colon \mathcal{W}\times\mathbb{D}\to\hat{Z}$$

is surjective in homology and π_1-surjective. The restriction to $z_0=1$ is given by

$$\mathrm{ev}\colon \mathcal{W}\times\{1\}\to\bigcup_{(\mathbf{p},\mathbf{t})\in\mathbb{R}^d\times\mathbb{R}^{n-1-d}} L^{\mathbf{t}}_{\mathbf{p}}\cap\{z_0=1\},$$

the target being equal to

$$\mathbb{R}\times T^*T^d\times\{1\}\times\mathbb{C}^{n-1-d}.$$

Both evaluation maps complete to a commutative square

$$\begin{array}{ccc} \mathcal{W}\times\mathbb{D} & \xrightarrow{\mathrm{ev}} & \hat{Z} \\ \uparrow\subset & & \uparrow\subset \\ \mathcal{W}\times\{1\} & \xrightarrow{\mathrm{ev}} & \mathbb{R}\times T^*T^d\times\{1\}\times\mathbb{C}^{n-1-d} \end{array}$$

via the homotopy equivalence

$$\mathcal{W}\times\{1\}\subset\mathcal{W}\times\mathbb{D}$$

and the inclusion

$$\mathbb{R}\times T^*T^d\times\{1\}\times\mathbb{C}^{n-1-d}\subset\hat{Z}.$$

Therefore, the induced map

$$T^d\subset\hat{Z}\simeq M$$

is surjective in homology and π_1-surjective too.

Similar to [3, Section 2.4], one shows, by simply replacing $2n$ by $2n+1$, that $H_kM=0$ for all higher degrees $k\geq d+1$ and that the inclusion of

$$\partial M\to M$$

induces an isomorphism for the homology groups H_k of low degree

$$k=0,1,\dots,2n-1-d.$$

Therefore,

$$H_*M=H_*(T^d\times D^{2n+1-d}).$$

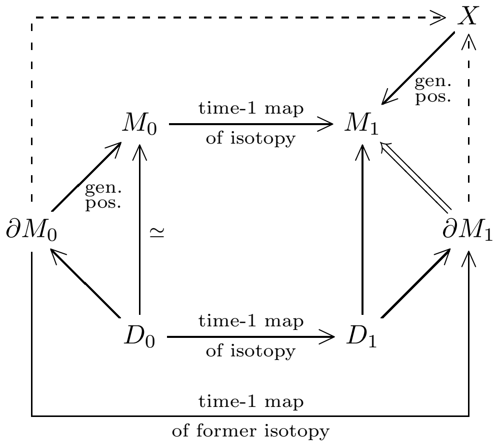
Because the fundamental group of ∂M is abelian, we infer with [3, Section 2.5] that the inclusion $\partial M\subset M$ is π_1-isomorphic. In particular,

$$\pi_1M=\pi_1(T^d\times D^{2n+1-d}).$$

4.2. **A cobordism.** Recall that $\hat{Z}$ is obtained by removing the bounded component of the complement of $\varphi(\partial M) \subset \operatorname{Int}(Z)$ in Z and gluing with M along the boundaries via φ. By assumption, $\varphi(\partial M)$ is isotopic to the sphere bundle $S(T^*T^d \oplus \underline{\mathbb{R}}^{2n+1-2d})$ inside Z viewed as a subset of $\mathbb{R} \times T^*T^d \times D^2 \times \mathbb{C}^{n-1-d}$. Choosing a suitable bundle metric for $T^*T^d \oplus \underline{\mathbb{R}}^{2n+1-2d}$, we assume that $\varphi(\partial M)$ is contained in the interior of the corresponding disc bundle $D(T^*T^d \oplus \underline{\mathbb{R}}^{2n+1-2d})$. After gluing M to the unbounded component in the total space, we obtain a manifold denoted by M_1, which deformation retracts onto M and is homotopy equivalent to $\hat{Z}$. Furthermore, we denote by M_0 a possibly rescaled copy of $D(T^*T^d \oplus \underline{\mathbb{R}}^{2n+1-2d})$ to which a nowhere vanishing section is added – before gluing – so that M_0 is contained in $M_1 \setminus M$ – after gluing. In the following we will study homotopical properties of the cobordism

$$X := M_1 \setminus \operatorname{Int} M_0.$$

Denote by $D_0 \subset \partial M_0$ and $D_1 \subset \partial M_1$ in M_1 the isotopic copies of the disc bundle of $\mathbb{R} \times T^*T^d \times \{1\} \times \mathbb{C}^{n-1-d}$, which strongly deformation retracts to T^d. The isotopy between D_0 and D_1 can be chosen to be the restriction of the above used shift and rescaling isotopy between ∂M_0 and ∂M_1, and extends to an isotopy between M_0 and M_1. Therefore, we obtain the following homotopy commutative diagram:



where all indicated maps are obtained by inclusion with the exception of the diffeomorphisms $D_0 \to D_1$ and $\partial M_0 \to \partial M_1$. The arrow $M_0 \to M_1$ can be alternatively understood to mean the time-1 map of the described isotopy besides the meaning of the inclusion.

As in [3, Lemma 5.1 and 5.2], one shows that the inclusions

$$\partial M_0, \partial M_1 \to X$$

are π_1- and H_*-isomorphic. Indeed, in low degrees $k = 0, 1, \ldots, 2n-1-d$, this follows with the results stated in Section 4.1 and general position arguments, which are available whenever $k+d < 2n+1$, using $n > d$ and $n \geq 2$; the arrows labeled by *gen. pos.* are homotopy (resp. homology) isomorphisms in those degrees in view of the induced long exact sequences. In higher degrees $k \geq d+1$, in which the homology groups of M_0 and M_1 vanish by Section 4.1, this follows with the induced long exact sequence of the pair (M_1, M_0), with excision and Poincaré duality applied to the compact cobordism X in combination with the universal coefficient theorem.

Remark 4.3. The above arguments show that the inclusion of M_0 into M_1 induces an isomorphism in homology. This is *a priori* not clear even if the involved homology groups are isomorphic.

4.4. **Being an *h*-cobordism.** In order to prove that the inclusions ∂M_0, $\partial M_1 \to X$ are in fact homotopy equivalences, one shows that the topological pairs $(X, \partial M_0)$ and $(X, \partial M_1)$ are homotopically trivial. As these are homotopically trivial, an application of the relative Hurewicz theorem shows that the quotients of the relative homotopy groups by the action of the fundamental group of ∂M, which is isomorphic to $\mathbb{Z}^d$, are trivial.

To conclude with the vanishing of the relative homotopy groups, we will employ universal coverings as done in the argumentation in [3, Section 6]. The following proposition is based on the triviality of $\pi_1(X, \partial M_0)$ and $\pi_1(X, \partial M_1)$ obtained in Section 4.2.

Proposition 4.5. *X is an h-cobordism.*

In order to prove Proposition 4.5, we consider the universal covering $\pi\colon \widetilde{M} \to M$, provided with the contact form $\tilde{\alpha} := \pi^*\alpha$, which satisfies $\inf_0(\tilde{\alpha}) > \pi$. Observe that the covering π is infinite as $\pi_1 M = \mathbb{Z}^d$, see Section 4.1. The induced covering on the boundary $\pi|_{\partial\widetilde{M}}\colon \partial\widetilde{M} \to \partial M$ is given by the universal covering $\mathbb{R}^d \times S^{2n-d} \to T^d \times S^{2n-d}$ caused by the π_1-isomorphicity of the inclusion $\partial M \subset M$, see again Section 4.1. Furthermore, the universal covering space of $\hat{Z}$ is made up by the analog of the gluing construction of $\hat{Z}$ involving this time $\widetilde{M}$ and

$$\widetilde{Z} = \mathbb{R} \times T^*\mathbb{R}^d \times D^2 \times \mathbb{C}^{n-1-d} = D^2 \times \mathbb{R}^{2n-1},$$

with the gluing map being a lift of $\varphi \circ \pi$. The universal covering map of $\hat{Z}$ restricts to π on $\widetilde{M}$.

Similar to Section 3 and [3, Section 6], one defines a moduli space $\mathcal{W}'$ of holomorphic discs in the covering space of $\hat{Z}$ with respect to the lift of the almost complex structure J. This results into a covering $\mathcal{W}' \to \mathcal{W}$ of moduli spaces. As in [3, Lemma 6.1], one shows that the evaluation map

$$\mathrm{ev}\colon \mathcal{W}' \times \mathbb{D} \to \widetilde{\hat{Z}}$$

is proper of degree 1 because the projections of all holomorphic discs in $\mathcal{W}'$ to the $\mathbb{R}$-factor of the symplectization are contained in a uniform compact interval. Alternatively, one can compensate the non-compactness caused by $\widetilde{M}$ with the results in [1] applied to the *trivial* virtually contact structure given by the universal covering of $\hat{Z}$. Here, by a **virtually contact structure** we mean a Riemannian covering together with a contact form primitive of the pull back of an odd-symplectic form on the base that is uniformly bounded from below and above. The virtually contact structure is **trivial** if the contact form is obtained from a primitive of the odd-symplectic from on the base by pull back.

As in Section 4.1, one considers a diagram

$$\begin{array}{ccc} \mathcal{W}' \times \mathbb{D} & \xrightarrow{\quad\mathrm{ev}\quad} & \widetilde{\hat{Z}} \\ \uparrow\subset & & \uparrow\subset \\ \mathcal{W}' \times \{1\} & \xrightarrow{\quad\mathrm{ev}\quad} & \mathbb{R} \times T^*\mathbb{R}^d \times \{1\} \times \mathbb{C}^{n-1-d} \end{array}$$

and concludes that $\widetilde{M}$ is contractible, cf. [3, Proposition 6.2] and its preceding remarks in [3].

Proof of Proposition 4.5. With the above shown contractibility of $\widetilde{M}$, the claim follows with the purely topological argumentation used in [3, Theorem 9.1]. Alternatively, one can follow the reasoning in [3, Section 8] or [2, Section 2.5] invoking simplicity of the topological space ∂M and the cobordism diagram from Section 4.2, similar to [3, Lemma 6.3]. □

Proof of Theorem 1.2. With Proposition 4.5, the claim follows with standard arguments based on Whitehead's theorem, $\mathrm{Wh}(\mathbb{Z}^d) = 0$, and the s-cobordism theorem as done in [2, 3]. □

APPENDIX A. LOCAL CONTACT INVERSION

As in knot theory, we call the complement of the interior of a tubular neighborhood of a submanifold Q the **exterior** of Q. We show that a collar extension of the exterior of a subcritically isotropic torus in a contact manifold admits a positive contact inversion along the boundary of the exterior. This allows surgerial constructions for contact manifolds near subcritically isotropic tori similar to the considerations in Section 1.3.

A.1. **Hypersurfaces transverse to a Liouville flow.**

Let (V, λ) be a Liouville manifold with symplectic form $\omega = \mathrm{d}\lambda$ and Liouville vector field Y defined by $i_Y\omega = \lambda$. Let M_0 and M_1 be hypersurfaces in V that are transverse to Y such that $\alpha_i := \lambda|_{TM_i}$, $i = 0, 1$, is a contact form. We assume that each flow line of Y intersects each of M_0 and M_1 in a single point so that the resulting correspondence forms a bijection $M_0 \to M_1$.

Under the stated assumptions, we find a domain D in $\mathbb{R} \times M_0$ that contains $\{0\} \times M_0$ such that for all $p \in M_0$, the intersection of D with each line $\mathbb{R} \times \{p\}$ is the maximal interval on which the flow line $t \mapsto \varphi_t(p)$ of Y is defined. This defines an embedding $\Phi\colon D \to V$ of Liouville manifolds (i.e., $\Phi^*\lambda = \mathrm{e}^t\alpha_0$), via $\Phi(t, p) = \varphi_t(p)$ for all $(t, p) \in D$, such that $M_1 \subset \Phi(D)$ is the Φ-image of the graph of a smooth function $f\colon M_0 \to \mathbb{R}$ and α_1 corresponds to the contact form $\mathrm{e}^f\alpha_0$ on $\Phi^{-1}(M_1)$. Setting $\psi(0, p) = (f(p), p)$ for all $p \in M_0$, we obtain a strict contactomorphism $\Phi \circ \psi\colon (M_0, \mathrm{e}^f\alpha_0) \to (M_1, \alpha_1)$.

A.2. **A model involution.** We consider the Liouville manifold

$$\left(T^*T^d \times \mathbb{C}^{n+1-d}, \mathbf{p}\mathrm{d}\mathbf{q} + \frac{1}{2}(\mathbf{x}\mathrm{d}\mathbf{y} - \mathbf{y}\mathrm{d}\mathbf{x})\right)$$

with symplectic form

$$\mathrm{d}\mathbf{p} \wedge \mathrm{d}\mathbf{q} + \mathrm{d}\mathbf{x} \wedge \mathrm{d}\mathbf{y}$$

and Liouville vector field

$$Y_0 = \mathbf{p}\partial_{\mathbf{p}} + \frac{1}{2}(\mathbf{x}\partial_{\mathbf{x}} + \mathbf{y}\partial_{\mathbf{y}}),$$

which is transverse to

$$M = \left\{|\mathbf{p}|^2 + |\mathbf{x}|^2 + |\mathbf{y}|^2 = 1\right\},$$

defining a contact form α_0 on M. The involution

$$\iota(\mathbf{q}, \mathbf{p}; \mathbf{x}, \mathbf{y}) = \big(\mathbf{q}, \mathbf{p}; -x_1, x_2, \dots, x_{n+1-d}, -y_1, y_2, \dots, y_{n+1-d}\big)$$

preserves the Liouville form and the M-defining distance function inducing a strict contactomorphism of (M, α_0). Observe that ι interchanges

$$M \cap \left\{\pm x_1 \geq 0\right\} \cong T^d \times D^{2n+1-d}$$

as well as the isotropic tori

$$T_\pm = \left\{(\mathbf{q}, \mathbf{0}; \pm 1, 0, \dots, 0) \mid \mathbf{q} \in T^d\right\}.$$

We remark that the (by a Hamiltonian vector field) shifted Liouville vector field

$$Y_1 = Y_0 + \frac{1}{4}\partial_{x_1}$$

defines the standard (rotationally invariant) contact form

$$\alpha_1 = \mathrm{d}t + \mathbf{p}\mathrm{d}\mathbf{q} + \frac{1}{2}(\mathbf{x}\mathrm{d}\mathbf{y} - \mathbf{y}\mathrm{d}\mathbf{x})$$

on

$$M_1 = T^*T^d \times \left\{x_1 = \frac{2}{3}\right\} \times \mathbb{R}_{y_1} \times \mathbb{C}^{n-d} \equiv \mathbb{R}_t \times T^*T^d \times \mathbb{C}^{n-d},$$

where y_1 is renamed in t and where now $\mathbf{x}$ and $\mathbf{y}$ stand for the corresponding tuples with x_1 and y_1 deleted. With [9, Example 2.1.3], α_1 can be brought to α_Z by a strict contactomorphism.

A.3. **Interpolating Liouville vector fields.** We continue the discussion from Section A.2. Let $\chi \equiv \chi(\mathbf{q}, \mathbf{p}; \mathbf{x}, \mathbf{y})$ be the cut off function $\tilde{\chi}(|\mathbf{p}|^2 + |\mathbf{x}|^2 + |\mathbf{y}|^2)$ induced by a smooth function $\tilde{\chi}\colon [0, \infty) \to [0, 1/4]$ that is equal to 0 on $[0, 1]$, strictly increasing on $(1, 3/2)$, and equal to $1/4$ on $[3/2, \infty)$. With respect to the Hamiltonian function $H = -\chi y_1$, we define the Liouville vector field $Y = Y_0 + X_H$, where

$$X_H = 2\tilde{\chi}' y_1\big(-\mathbf{p}\partial_{\mathbf{q}} + \mathbf{y}\partial_{\mathbf{x}} - \mathbf{x}\partial_{\mathbf{y}}\big) + \chi\partial_{x_1}$$

denotes the Hamiltonian vector field of H. The Liouville vector field Y equals Y_0 on $\{|\mathbf{p}|^2+|\mathbf{x}|^2+|\mathbf{y}|^2\leq 1\}$ and Y_1 on $\{|\mathbf{p}|^2+|\mathbf{x}|^2+|\mathbf{y}|^2\geq 3/2\}$; each flow line of Y connects M with $\{|\mathbf{p}|^2+|\mathbf{x}|^2+|\mathbf{y}|^2=3/2\}$ as

$$\mathrm{d}(|\mathbf{p}|^2+|\mathbf{x}|^2+|\mathbf{y}|^2)(Y)=|\mathbf{p}|^2+\frac{1}{2}(|\mathbf{x}|^2+|\mathbf{y}|^2)+\chi x_1,$$

defining a bijection between the two hypersurfaces. Denote by M_0 the set of intersection points of M with those flow lines of Y that intersect $\{|\mathbf{p}|^2+|\mathbf{x}|^2+|\mathbf{y}|^2=3/2\}$ along $\{x_1>-1/2\}$. Observe that M_0 is an open neighborhood of $M\cap\{x_1=0\}$ in M as

$$\mathrm{d}x_1(Y)=\frac{1}{2}x_1+2\tilde{\chi}'y_1^2+\chi$$

is positive on $\{1<|\mathbf{p}|^2+|\mathbf{x}|^2+|\mathbf{y}|^2\}$ along $\{x_1=0\}$. With the considerations from Section A.2, we obtain a strict contactomorphism $\varphi\colon(M_0,\mathrm{e}^f\alpha_0)\to(M_1,\alpha_1)$ defined in a neighborhood of the invariant set $M\cap\{x_1=0\}$ of the involution ι.

A.4. **Local inversion.** Continuing the discussion from Section A.3, we observe that $\varphi\circ\iota\circ\varphi^{-1}$ defines a strict contactomorphism of α_1 on $\varphi(M_0\cap\iota(M_0))$ that leaves $\varphi(M_0\cap\{x_1=0\})$ invariant changing (co-)orientations. We call such a positive (not necessarily strict) contactomorphism a **contact inversion** along $\varphi(M_0\cap\{x_1=0\})$.

Observe that $\varphi(M_0\cap\{x_1=0\})$, as the boundary of $\varphi(M_0\cap\{x_1\geq 0\})$, can be brought into any neighborhood of the zero section $\varphi(M_0\cap T_+)$ of $T^*T^d\oplus\underline{\mathbb{R}}^{2n+1-2d}$ using the contact vector field

$$X=t\partial_t+\mathbf{p}\partial_{\mathbf{p}}+\frac{1}{2}(\mathbf{x}\partial_{\mathbf{x}}+\mathbf{y}\partial_{\mathbf{y}})$$

on $(\mathbb{R}\times T^*T^d\times\mathbb{C}^{n-d},\alpha_1)$. Conjugating the contact inversion $\varphi\circ\iota\circ\varphi^{-1}$ with the flow of X, we obtain, combined with the isotropic neighborhood theorem, the following proposition.

Proposition A.5. *Any isotropic submanifold of a given contact manifold, whose conformally symplectic normal bundle possesses a non-vanishing section, admits a tubular neighborhood U together with a contact inversion along the boundary ∂U.*

Indeed, this is because the submanifold, being a subcritically isotropic torus with trivial conformally symplectic normal bundle, is not (really) used in the above construction.

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